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List of Publications

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Contents

Introduction

The theory of functional differential equations has emerged as an important branch of nonlinear analysis. Differential delay equations, and functional differential equations, have been used in modeling the evolution of some physical, biological and economic systems, in which the response of the system depends purely on the current state of the system. However, in many applications the response of the system can be delayed, or depend on the past history of the system in more complicated way. Often, it has been assumed that the delay is either a fixed constant or is given as an integral in which case it is called a distributed delay; see for instance the books by Hale and Verduyn Lunel [58], Kolmanovskii and Myshkis [75], and Wu [100], and the references therein. An extensive theory is developed for evolution equations [6, 50].

In 1806 Poisson [90] published one of the first papers on functional differential equations and studied a geometric problem leading to an example with a state-dependent delay (see also [97]). However, complicated situations in which the delay depends on the unknown functions have been proposed in modeling in recent years. These equations are frequently called equations with state-dependent delay, see, for instance [48, 68, 71, 98]. Existence results and among other things were derived recently for functional differential equations when the solution is depending on the delay on a bounded interval $[0, b]$ for impulsive problems. We refer the reader to the papers by Abada *et al.* [1], Ait Dads and Ezzinbi [9], Anguraj *et al.* [10], Hernandez *et al.* [69] and Li *et al.* [78]. Uniqueness and existence results have been established recently for different evolution problems in the papers by Baghli and Benchohra for finite and infinite delay in [13, 14, 15]. In [86, 87, 88, 30, 31, 29, 35], the authors considered the existence of mild solutions for evolution equations on unbounded intervals.

Differential equations on infinite intervals frequently occur in mathematical modelling of various applied problems. For example, in the study of unsteady flow of a gas through a semi-infinite porous medium [4, 73], analysis of the mass transfer on a rotating disk in a non-Newtonian fluid [5], heat transfer in the radial flow between parallel circular disks [82], investigation of the temperature distribution in the problem of phase change of solids with temperature dependent thermal conductivity [82], as well as numerous problems arising in the study of circular membranes [3, 44, 45], plasma physics [5], nonlinear mechanics, and non-Newtonian fluid flows [3]. Over the past several years it has become apparent that equations with state-dependent delay arise also in several areas such as in classical electrodynamics [49], in population models [8, 25, 37, 38], in models of commodity price fluctuations [23, 79], in models of blood cell productions [24, 39, 41, 80], and in drilling [83]. the differential inclusions is a generalization of the notion of an ordinary differential equation. Therefor all problems considered for differential equations, that is, existence of solutions, continuations of solutions, dependence on initial and parameters, are present in the theory of differential inclusions.

Partial neutral differential equation with finite delay arise, for instance, from the transmission line theory [99]. Wu and Xia have shown in [100] that a ring array of identical resistibly coupled lossless transmission lines leads to a system of neutral functional differential equations with discrete diffusive coupling which exhibits various types of discrete waves. For more results on partial neutral functional-differential equations and related issues we refer to Adimy and Ezzinbi [2], Hale [56], Wu and Xia [99, 100] for finite delay equations, and Hernández and Henriquez [66, 67] for unbounded delays. Functional-differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equations has received a significant amount of attention in the last years, see for instance [1, 2, 8, 11, 25, 37, 77] and the references therein. We also cite [9, 78, 38, 48, 69, 83, 101, 32, 33] for the case neutral differential equations with state-dependent delay.

The cosine function theory is related to abstract linear second order differential equations in the same manner that the semigroup theory of bounded linear operators is related to first order partial differential equations and it's equally appealing devoted their generality and simplicity. For basic concepts and applications of this theory, we refer to the reader to Fattorini [51] , Travis and Weeb [96]. Our purpose in this work is consider a simultaneous generalization of the classical second order abstract Cauchy problem studied by Travis and Weeb in [95, 96] . Additionally, we observe that the ideas and techniques in this paper permit the reformulation of the problems studied in [17, 26, 76, 84, 85] to the context of " partial " second order differential equations, see [95] and the referred papers for details, we also cite [34]

In this thesis, we shall be concerned by global existence some classes of first and second order of partial functional and neutral functional differential evolutions equations and inclusions with finite, infinite and state-dependent delay on a positive real semiinfinite interval. Our results are based upon fixed point theorems and using semigroups theory. This thesis is arranged as follows:

In **Chapter 1**, we introduce notations and definitions, lemmas and notions of semigroup, fixed point theorem which are used throughout this thesis.

In **Chapter 2**, we give the global existence of mild solution on a semiinfinite positive real interval for partial functional differential evolution equations with delay.

In the section 2.2.1 the delay is finite i.e. on a bounded historical interval $\mathcal{H} = [-d, 0]$ for $d > 0$. we consider the following problem

$$y'(t) = Ay(t) + f(t, y_t), \quad \text{a.e. } t \in J := [0, +\infty) \tag{1}$$

$$y(t) = \phi(t), \quad t \in [-d, 0], \tag{2}$$

where $f : J \times C([-d, 0], E) \to E$ is given function, $\phi : [-d, 0] \to E$ is given continuous function, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \in J$, and $(E, |.|)$ is a real Banach space.

In the section 2.3.2 the delay is infinite we introduce the notion of phase space $\mathcal{B}$ who plays an important role in the study of both qualitative and quantitative theory. we consider the following problem

$$y'(t) = Ay(t) + f(t, y_{\rho(t,y_t)}), \quad \text{a.e. } t \in J := [0, +\infty) \tag{3}$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0], \tag{4}$$

where $f : J \times \mathcal{B} \to E$, $\phi \in \mathcal{B}$, $\rho : J \times \mathcal{B} \to (-\infty, +\infty)$ are given functions, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \in J$, $\mathcal{B}$ is the phase space to be specified later, $(E, |.|)$ is a real Banach space.

In **Chapter 3** is devoted to the existence of solutions for semiinfinite positive real interval for neutral functional differential evolution equations with state-dependent delay

$$\frac{d}{dt}[y(t) - g(t, y_{\rho(t,y_t)})] = A[y(t) - g(t, y_{\rho(t,y_t)})] + f(t, y_{\rho(t,y_t)}), \quad \text{a.e. } t \in J := [0, +\infty) \tag{5}$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0], \tag{6}$$

where $f, g : J \times \mathcal{B} \to E$, $\phi \in \mathcal{B}$, $\rho : J \times \mathcal{B} \to (-\infty, +\infty)$ are given functions, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \in J$, $\mathcal{B}$ is the phase space to be specified later, $(E, |.|)$ is a real Banach space.

In **Chapter 4** we give the global existence of mild solution on a semiinfinite positive real interval $J = [0, +\infty)$ for partial functional differential evolution inclusions with delay.

In the section 4.2.1 the delay is finite, we consider the following problem

$$y'(t) - Ay(t) \in F(t, y_t), \quad \text{a.e. } t \in J := [0, +\infty) \tag{7}$$

$$y(t) = \phi(t), \quad t \in [-d, 0], \tag{8}$$

where $F : J \times C([-d,0], \to \mathcal{P}(E)$ is a multivalued map with nonempty compact values, $\phi : [-d,0] \to E$ is given continuous function, $\mathcal{P}(E)$ is the family of all nonempty subsets of E, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \in J$, and $(E, |.|)$ is a real Banach space.

In the section 4.3.1 we will consider the following problem :

$$y'(t) - Ay(t) \in F(t, y_{\rho(t,y_t)}), \quad \text{a.e. } t \in J := [0,+\infty) \tag{9}$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0], \tag{10}$$

where $F : J \times \mathcal{B} \to \mathcal{P}(E)$ is a multivalued map with nonempty compact values, $\mathcal{P}(E)$ is the family of all nonempty subsets of E, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \in J$, and $(E, |.|)$ is a real Banach space. $\mathcal{B}$ is the phase space to be specified later, $\phi \in \mathcal{B}$, $\rho : J \times \mathcal{B} \to (-\infty, +\infty)$.

In **Chapter 5**, is devoted to the existence of mild solution on $J = [0, +\infty)$ for neutral functional differential evolution inclusions with state-dependent delay, More precisely we will consider the following problem :

$$\frac{d}{dt}[y(t) - g(t, y_{\rho(t,y_t)})] - A[y(t) - g(t, y_{\rho(t,y_t)})] \in F(t, y_{\rho(t,y_t)}), \quad \text{a.e. } t \in J := [0,+\infty) \tag{11}$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0], \tag{12}$$

where $F : J \times \mathcal{B} \to \mathcal{P}(E)$ is a multivalued map with nonempty compact values, $\mathcal{P}(E)$ is the family of all nonempty subsets of E, $g : J \times \mathcal{B} \to E$ is given function, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \in J$, $\mathcal{B}$ is the phase space to be specified later, $\phi \in \mathcal{B}$, $\rho : J \times \mathcal{B} \to (-\infty, +\infty)$ and $(E, |.|)$ is a real Banach space.

In **Chapter 6**, we give the global existence of mild solution on $J = [0, +\infty)$ for partial functional differential evolution inclusions of second order with delay.

In the section 6.2.1 we consider the following problem

$$y''(t) = Ay(t) + f(t, y_t), \quad \text{a.e. } t \in J := [0,+\infty) \tag{13}$$

$$y(t) = \phi(t), \quad t \in [-d, 0], \quad y'(0) = \varphi \in E, \tag{14}$$

where $f : J \times C([-d,0], E) \to E$ is given function, $\phi : [-d,0] \to E$ is given continuous function, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous cosine function of bounded linear operators $(C(t))_{t\in\mathbb{R}}$, on E, and $(E, |.|)$ is a real Banach space.

In the section 6.3.1 we will consider the following problem More precisely, we will consider the following problem

$$y''(t) = Ay(t) + f(t, y_{\rho(t,y_t)}), \quad \text{a.e. } t \in J := [0,+\infty) \tag{15}$$

$$y(t) = \phi(t) \in \mathcal{B}, \quad y'(0) = \varphi \in E, \tag{16}$$

where $f : J \times \mathcal{B} \to E$ $\phi \in \mathcal{B}$, $\rho : J \times \mathcal{B} \to (-\infty, +\infty)$, are given functions, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous cosine function of bounded linear operators $(C(t))_{t \in \mathbb{R}}$, on E, and $(E, |.|)$ is a real Banach space.

Chapter 1

Preliminaries

In this chapter, we introduce notations, definitions, lemmas and fixed point theorems which are used throughout this thesis.

1.1 Notations and definitions

Let $J = [0, +\infty)$ be a real interval, $\mathcal{H} = [-d, 0]$ interval be the historical for $d > 0$ and $(E, \|\cdot\|)$ be a real Banach space.

Let $\mathcal{C}([-d, 0]; E)$ be the Banach space of continuous functions with the norm

$$\|y\| = \sup\{|y(t)| : t \in [-d, 0]\}.$$

Let $B(E)$ be the space of all bounded linear operators from E into E, with the norm

$$\|N\|_{B(E)} = \sup\{|N(y) : |y| = 1\}.$$

A measurable function $y : J \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For properties of the Bochner integral, see for instance, Yosida[102]).
Let $L^1(J; E)$ be the space of measurable functions $y : J \to E$ which are Bochner integrable normed by

$$\|y\|_{l^1} = \int_0^{+\infty} |y(t)|dt.$$

Definition 1.1.1 *A map $f : J \times E \to E$ is said to be Carathéodory if*

(i) $t \to f(t, y)$ is measurable for all $y \in E$.

(ii) $y \to f(t, y)$ is continuous for almost each $t \in J$.

1.2 Some properties of set-valued maps

Let (E, d) be a metric space and Y be a subset of E. We denote:

$$\mathcal{P}_{cl}(E) = \{Y \in \mathcal{P}(E) : Y \text{closed}\}, \quad \mathcal{P}_{cv}(E) = \{Y \in \mathcal{P}(E) : Y \text{convex}\},$$

$$\mathcal{P}_b(E) = \{Y \in \mathcal{P}(E) : Y \text{bounded}\}.$$

A multivalued map (multimap) F of a set E into a set Y is a correspondence which associates to very $x \in E$ a non- empty subset $F(x) \subset Y$, called the value of x.

We will write this correspondence as

$$F : E \to \mathcal{P}(Y).$$

Consider $H_d : \mathcal{P}(E) \times \mathcal{P}(E) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$, given by

$$H_d(\mathcal{A}, \mathcal{B}) = \max \left\{ \sup_{a \in \mathcal{A}} d(a, \mathcal{B}) \ , \ \sup_{b \in \mathcal{B}} d(\mathcal{A}, b) \right\},$$

where $d(\mathcal{A}, b) = \inf_{a \in \mathcal{A}} d(a, b)$, $d(a, \mathcal{B}) = \inf_{b \in \mathcal{B}} d(a, b)$.

Definition 1.2.1 *A multivalued map $F : E \to \mathcal{P}(E)$ has convex (closed) values if $F(x)$ is convex (closed) for all $x \in E$. We say that F is bounded on bounded sets if $F(B) = \bigcup_{x \in B} F(x)$ is bounded in E for each bounded set B of E, i.e.,*

$$\sup_{x \in B} \{\sup\{ \ \|y\| \ : \ y \in F(x)\}\} < \infty.$$

The set $\Gamma_F \subset E \times Y$, defined by

$$\Gamma_F = \{(x, y) : x \in E, y \in F(x)\}$$

is said to be graph of F.
F is called closed graph if Γ_F is closed $E \times Y$

Definition 1.2.2 *Let X, Y be Hausdorff topological spaces and $F : X \to \mathcal{P}(Y)$ is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $F(x)$ is a nonempty closed subset of X and if for each open set N of X containing $F(x)$, there exists an open neighborhood N_0 of x_0 such that $F(N_0) \subseteq N$.*

F is said to be completely continuous if $F(B)$ is relatively compact for every $B \in \mathcal{P}_b(E)$. If the multivalued map F is completely continuous with non empty values, then F is u.s.c. if an only if F has a closed graph ($i.e. x_n \to x_*, y_n \to y_*, y_n \in F(x_n)$ implies $y_* \in F(x_*)$). F has a fixed point if there is $x \in E$ such that $x \in F(x)$. The fixed point set of the multivalued operator F will be denote by $FixF$.

Definition 1.2.3 *A multivalued map $F : J \to \mathcal{P}(E)$ is said to be measurable if for every $y \in E$, the function $t \to d(y, F(t)) = \inf\{|y - z| : z \in F(t)\}$ is Lebesgue measurable.*

Definition 1.2.4 *A function $F : J \times E \longrightarrow \mathcal{P}(E)$ is said to be an L^1- Carathéodory multivalued map if it satisfies :*

(i) $y \mapsto F(t, y)$ is upper semicontinuous for almost all $t \in J$;

(ii) $t \mapsto F(t, y)$ is measurable for each $y \in E$;

(iii) for every positive constant l there exists $h_l \in L^1(J, \mathbb{R}^+)$

$$\|F(t, y)\| = \sup\{|v| : v \in F(t, y)\} \leq h_l$$

for all $|y| \leq l$ for almost all $t \in J$.

Definition 1.2.5 *A function $F : J \times E \longrightarrow \mathcal{P}(E)$ is said to be an Carathéodory multivalued map if it satisfies (i) and (ii).*

Lemma 1.2.6 *([[12], Theorem 1.4.13]).*
If $G : X \to \mathcal{P}(X)$ is u.s.c, then for any $x_0 \in X$,

$$\lim_{x \to x_0} \sup G(x) = G(x_0).$$

Lemma 1.2.7 *([[12], Lemma 1.1.9]).*
Let $(K_n)_{n \in \mathbb{N}} \subset K \subset X$ be a sequence of subsets where K is compact in the separable Banach space X. Then

$$\bar{co}(\lim_{x \to \infty} \sup K_n) = \bigcap_{N>0} \bar{co}(\bigcup_{n \geq N} K_n),$$

where $\bar{co}A$ refers to the closure of the convex hull of A.

Lemma 1.2.8 *(Mazur's Lemma [[81], Theorem 21.4]).*
Let E be a normed space and $\{x_k\}_{k \in \mathbb{N}} \subset E$ be a sequence weakly converging to a limit $x \in E$. Then there exists a sequence of convex combinations

$$y_m = \sum_{k=1}^{m} \alpha_m k x_k$$

with $\alpha_k m > 0$ for $k = 1, 2, ..., m$ and $\sum_{k=1}^{m} \alpha_m k = 1$, which converges strongly to x.

Lemma 1.2.9 *Let E be a Banach space. Let $F : J \times E \to \mathcal{P}_{cl,cv}(E)$ be a L^1-Carathéodory multivalued map ; and let Γ be a linear continuous from $L^1(J;E)$ into $C(J;E)$, then the operator*

$$\begin{aligned} \Gamma \circ S_F : C(J,E) \longrightarrow & \quad \mathcal{P}_{cp,cv}(C(J,X)), \\ y \longmapsto & \quad (\Gamma \circ S_F)(y) := \Gamma(S_{F,y}) \end{aligned}$$

is a closed graph operator in $C(J;X) \times C(J;X)$.

We say that A has a *fixed point* if there exists $x \in E$ such that $x \in A(x)$.

For each $y : (-\infty, +\infty) \to E$ let the set $S_{F,y}$ known as the set of selectors from F defined by

$$S_{F,y} = \{v \in L^1(J;E) : v(t) \in F(t, y_t) \ , \ a.e. \ t \in J\}.$$

For more details on multivalued maps we refer to the books of Deimling [42], Denkowski *et al.* [43], Djebali *et al.* [46], Górniewicz [54] and Hu and Papageorgiou [72].

1.3 Semigroups

Let E be a Banach space and $B(E)$ be the Banach space of linear bounded operators.

Definition 1.3.1 *A semigroup of class C_0 is a one parameter family $\{T(t)/\ t \geq 0\} \subset B(E)$ satisfying the conditions:*

(i) $T(t) \circ T(s) = T(t+s)$, *for* $t, s \geq 0$,

(ii) $T(0) = I$,

(iii) the map $t \to T(t)x$ is strongly continuous, for each $x \in E$, i.e;

$$T(t)x = x, \forall x \in E.$$

A semigroup of bounded linear operators $T(t$, is uniformly continuous if

$$\lim_{t \to 0} \|T(t) - I\| = 0.$$

Where I denotes the identity operator in E.

We note that if a semigroup $T(t)$ is of class (C_0) then the following growth conditions is satisfied $\|T(t)\|_{B(E)} \leq M. \exp(\beta t)$, for $0 \leq t < \infty$, with some constants $M > 0$ and β.
In particular, if $M = 1$ and $\beta = 0$, i.e; $\|T(t)\|_{B(E)} \leq 1$, for $t \geq 0$, then the semigroup $T(t)$ is called a contraction C_0-semigroup.

Definition 1.3.2 *Let $T(t)$ be a semigroup of class (C_0) defined on E. The infinitesimal generator A of $T(t)$ is the linear operator defined by*

$$A(x) = \lim_{h\to 0} \frac{T(h)x - x}{h}, \text{ for } x \in D(A),$$

where

$$D(A) = \{x \in E | \lim_{h\to 0} \frac{T(h)x - x}{h} \text{ exists in } E\}.$$

Let us recall the following property:

Proposition 1.3.3 *The infinitesimal generator A is a closed linear and densely defined operator in E. If $x \in D(A)$, then $T(t)(x)$ is a C^1-map and*

$$\frac{d}{dt}T(t)x = A(T(t)(x)) = T(t)(A(x)), \quad \text{on } [0, \infty).$$

Theorem 1.3.4 *Hille and Yosida[89].*
Let A be a densely defined linear operator with domain and range in a Banach space E. Then A is the infinitesimal generator of uniquely determined semigroup $T(t)$ of class (C_0) satisfying

$$\|T(t)\|_{B(E)} \leq M \exp(\omega t), \ t \geq 0,$$

where $M > 0$ and $\omega \in \mathbb{R}$ if and only if

$$(\lambda I - A)^{-1} \in B(E) \text{and } \|(\lambda I - A)^{-n}\| \leq M/(\lambda - \omega)^n, \text{ for } \lambda > \omega n = 1, 2, ..., \text{for all } \lambda \in \mathbb{R}.$$

For more details on strongly continuous operator, we refer the reader to the books of Goldstien[53], Fattorini[51], and the paper of Travis and Webb [95, 96], and for properties on semigroup theory we refer the intersected reader to the books of Ahmed [7],Goldstien[53] ,Pazy[89].

1.4 Cosine and sine families

In this section, we recall briefly some notations, definitions and lemmas needed to establish our main results. For second order differential equations A is the infinitesimal generator of a strongly continuous cosine function of bounded linear operators $(C(t))_{t\in\mathbb{R}}$ on Banach space $(E, \|\cdot\|)$.

Definition 1.4.1 *A one parameter family $(C(t))_{t\in\mathbb{R}}$ of bounded linear operators mapping the Banach space E into itself is called a strongly continuous cosine family if and only if*

(i) $C(t+s) + C(s-t) = 2C(s)C(t)$, *for all* $s, t \in \mathbb{R}$,

(ii) $C(0) = I$;

(iii) $C(t)x$ *is continuous in* t *on* $\mathbb{R}$ *for each fixed* $X \in E$.

We denote by $(S(t))_{t\in\mathbb{R}}$ the sine function associated with $(C(t))_{t\in\mathbb{R}}$ which is defined by $S(t)x = \displaystyle\int_0^t C(s)xds, x \in E, t \in \mathbb{R}$ and we always assume that M and M' are positive constants such that $\|C(t)\| \leq M$ and $\|S(t)\| \leq M'$, for every $t \in J$. The infinitesimal generator of a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ is the operator $A : E \to E$ defined by

$$Ax = \frac{d^2}{dt^2}C(t)x|_{t=0}, \quad x \in D(A)$$

where $D(A) = \{x \in E : C(t)x$ is twice differentiable in $t\}$. Define $X = \{x \in E : C(t)x$ is once continuously differentiable in $t\}$.

The following properties are well known[95].

(i) If $x \in E$ then $S(t)x \in X$ for every $t \in \mathbb{R}$.

(ii) If $x \in X$ then $S(t)x \in D(A), (\frac{d}{dt})C(t)x = AS(t)x$ and $(\frac{d^2}{dt^2})S(t)x = S(t)x$ for every $t \in \mathbb{R}$.

(iii) If $x \in D(A)$ then $C(t)x \in D(A)$, and $(\frac{d^2}{dt^2})C(t)x = AC(t)x = C(t)Ax$ for every $t \in \mathbb{R}$.

(iv) If $x \in D(A)$ then $S(t)x \in D(A)$, and $(\frac{d^2}{dt^2})S(t)x = AS(t)x = S(t)Ax$ for every $t \in \mathbb{R}$.

The notation $[D(A)]$ stands for the domain of the operator A endowed with the graph norm $\|y\|_A = \|y\| + \|Ay\|, y \in D(A)$. Moreover, in this work, X is the space formed by the vector $y \in E$ for which $C(\cdot)y$ is of class C^1 on $\mathbb{R}$. it was proved by Kisinsky[74] that X endowed with the norm

$$\|y\|_X = \|y\| + \sup_{0\leq t\leq 1} \|AS(t)y\|, y \in X,$$

is a Banach space. The operator valued function

$$G(t) = \begin{pmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{pmatrix}$$

is a strongly continuous group of bounded linear operators on the space $X \times E$ generated by the operator

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$$

defined on $D(A)\times X$. It follows this that $AS(t):X\to E$ is a bounded linear operator and that $AS(T)y\to 0,\ t\longrightarrow 0$, for each $y\in X$. Furthermore, if $y:[0,+\infty)\to E$ is a locally integrable function, then $z(t)=\displaystyle\int_0^t S(t-s)y(s)ds$ defined an $X-$valued continuous function. This is a consequence of the fact that:

$$\int_0^t G(t-s)\begin{pmatrix}0\\ y(s)\end{pmatrix}ds=\begin{pmatrix}\displaystyle\int_0^t S(t-s)y(s)ds\\ \displaystyle\int_0^t C(t-s)y(s)ds\end{pmatrix}$$

defines an $X\times E-$ valued continuous function. The existence of solutions for the second order abstract Cauchy problem.

$$\begin{cases} y''(t)=Ay(t)+h(t), \quad t\in J:=[0,+\infty) & ;\\ y(0)=y_0, \quad y'(0)=y_1 & . \end{cases} \tag{1.1}$$

where $h:J\to E$ is an intergrable function has been discussed in [95]. Similarly, the existence of solutions of the semilinear second order abstract Cauchy problem it has been treated in [96].

Definition 1.4.2 *The function $y(\cdot)$ given by:*

$$y(t)=C(t)y_0+S(t)y_1+\int_0^t S(t-s)h(s)ds, t\in J,$$

is called mild solution of (1.1).

Remark 1.4.3 *When $y_0\in X, y(\cdot)$ is continuously differentiable we have:*

$$y'(t)=AS(t)y_0+C(t)y_1+\int_0^t C(t-s)h(s)ds.$$

For additional details about cosine function theory, we refer to the reader to [95, 96].

1.5 Some fixed point theorems

In this section we give some fixed point theorems that will be used in the sequel.
At first we give this lemma concerning the notion of subset relatively compact.

Lemma 1.5.1 *(Corduneanu [40])*
Let $D\subset BC([0,+\infty),E)$. Then D is relatively compact if the following conditions hold:

(a) D is bounded in BC;

(b) The functions belonging to D are almost equicontinuous on $[0,+\infty)$, i.e., equicontinuous on every compact of $[0,+\infty)$;

(c) The set $D(t) := \{y(t) : y \in D\}$ is relatively compact on every compact of $[0,+\infty)$.

(d) The functions from D are equiconvergent, that is, given $\epsilon > 0$, there exists $T(\epsilon) > 0$ such that $|u(t) - \lim\limits_{t\to+\infty} u(t)| < \epsilon$, for any $t \geq T(\epsilon)$ and $u \in D$.

The following is due to Schauder.

Theorem 1.5.2 *(Schauder's fixed point [55])*
Let B be a closed, convex and nonempty subset of a Banach space E. Let $N : B \to B$ be a continuous mapping such that $N(B)$ is a relatively compact subset of E. Then N has at least one fixed point in B. That is, there exists $y \in B$ such that $Ny = y$.

We also need the following form of fixed point theorem of Bohnenblust-Karlin.

Theorem 1.5.3 *(Bohnenblust-Karlin fixed point [36])*
Let $B \in \mathcal{P}_{cl,cv}(E)$. And $N : B \to \mathcal{P}_{cl,cv}(B)$ be a upper semicontinuous operator and $N(B)$ is a relatively compact subset of E. Then N has at least one fixed point in B.

1.6 Some examples of phase spaces

In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [57] and follow the terminology used in [71]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into E, and satisfying the following axioms :

(A_1) If $y : (-\infty, b) \to E, b > 0$, is continuous on J and $y_0 \in \mathcal{B}$, then for every $t \in J$ the following conditions hold :
(i) $y_t \in \mathcal{B}$;
(ii) There exists a positive constant H such that $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$;
(iii) There exist two functions $L(\cdot), M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of y with L continuous and bounded, and M locally bounded such that :

$$\|y_t\|_{\mathcal{B}} \leq L(t) \sup\{\ |y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) For the function y in (A_1), y_t is a $\mathcal{B}-$valued continuous function on J.

(A_3) The space $\mathcal{B}$ is complete.

Assume that:

$$l = \sup\{L(t) : t \in J\},\ m = \sup\{M(t) : t \in J\}.$$

Remark 1.6.1 1. *(ii) is equivalent to* $|\phi(0)| \leq H\|\phi\|_{\mathcal{B}}$ *for every* $\phi \in \mathcal{B}$.

2. *Since* $\|\cdot\|_{\mathcal{B}}$ *is a seminorm, two elements* $\phi, \psi \in \mathcal{B}$ *can verify* $\|\phi - \psi\|_{\mathcal{B}} = 0$ *without necessarily* $\phi(\theta) = \psi(\theta)$ *for all* $\theta \leq 0$.

3. *From the equivalence of in the first remark, we can see that for all* $\phi, \psi \in \mathcal{B}$ *such that* $\|\phi - \psi\|_{\mathcal{B}} = 0$ *: We necessarily have that* $\phi(0) = \psi(0)$.

Example 1.6.2 *The spaces* $BC; BUC; C^{\infty}$ *and* C^0 .
Let :

$$BC \ \text{ the space of bounde continuous functions defined from } (-\infty, 0] \text{ to } E$$

$$BUC \ \text{ the space of bounde uniformly continuous functions defined from } (-\infty, 0] \text{ to } E$$

$$C^{\infty} := \{\phi \in BC : \lim_{\theta \to -\infty} (\theta) \ \text{ exist in } E\}$$

$$C^0 := \{\phi \in BC : \lim_{\theta \to -\infty} (\theta) = 0\}, \ \text{ endowed with the uniform norm}$$

$\|\phi\| = \sup\{|\phi(\theta)| : \theta \leq 0\}$.

Then we have that the spaces BUC, C^{∞}, and C^0 satisfy conditions $(A1) - (A_3)$. BC satisfy conditions (A_3) and (A_2) but (A_1) is not satisfied.

Example 1.6.3 *The spaces* C_g, UC_g, C_g^{∞} *and* C^0. *Let* g *be a positive continuous function on* $(-\infty, 0]$. *We define:*

$$C_g := \{\phi \in C((-\infty, 0]; E) : \frac{\phi(\theta)}{g(\theta)} \ \text{ is bounded on}(-\infty, 0]\};$$

$$C_g^0 := \{\phi \in C_g : \lim_{\theta \to -\infty} \frac{\phi(\theta)}{g(\theta)} = 0\}, \ \text{ endowed with the uniform norm}$$

$$\|\phi\| = \sup\{\frac{|\phi(\theta)|}{g(\theta)} : \theta \leq 0\}.$$

We consider the following condition on the function g.

(g_1) For all $a > 0$, $\sup_{0 \leq t \leq a} \sup\{\frac{g(t+\theta)}{g(\theta) : -\infty \leq \theta \leq -t}\} < \infty$.

Then we have that the spaces C_g and C_g^0 satisfy condition (A_3). They satisfy conditions (A_1) and (A_2) if (g_1) holds.

Example 1.6.4 *(The phase space $(\mathbf{C_r} \times \mathbf{L^p}(\mathbf{g}, \mathbf{E}))$.)*

Let $g : (-\infty, -r) \to \mathbb{R}$ be a positive Lebesgue integrable function and assume that there exists a non-negative and locally bounded function γ on $(-\infty, 0]$ such that

$$g(\xi + \theta) \leq \gamma(\xi)g(\theta), \text{for all } \xi \leq 0 \text{and } \theta \in (-\infty, -r) \setminus N_\xi,$$

where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero.

The space $C_r \times L^p(g, E)$ consists of all classes of functions $\varphi(-\infty, 0] \to \mathbb{R}$ such that ϕ is continuous on $[-r, 0]$, Lebesgue-measurable and $g\|\phi\|^p$ is Lebesgue integrable on $(-\infty, -r)$.

The seminorm in $C_r \times L^p(g, E)$ is defined by

$$\|\phi\|_{\mathcal{B}} := \sup\{\|\phi(\theta)\| : -r \leq \theta \leq 0\} + (\int_{-\infty}^{-r} g(\theta)\|\phi(\theta)\|^p d\theta)^{\frac{1}{p}}$$

Assume that $g(\cdot)$ verifies the condition $(g-5), (g-6)$ and $(g-7)$ in the nomenclature [71]. In this case, $\mathcal{B} = C_r \times L^p(g, E)$ verifies assumptions $(A_1), (A_2), (A_3)$ see ([71] Theorem 1.3.8) for details.

Moreover, when $r = 0$ and $p = 2$ we have that $H = 1, M(t) = \gamma(-t)^{\frac{1}{2}}$ and $L(t) = 1 + (\int_{-t}^{0} g(\theta)d\theta)^{\frac{1}{2}}$ for $t \geq 0$.

Set $\mathcal{R}(\rho^-) = \{\rho(s, \phi) : (s, \phi) \in J \times \mathcal{B},\ \rho(s, \phi) \leq 0\}$, we always assume that $\rho : J \times \mathcal{B} \to \mathbb{R}$ is continuous. Additionally, we introduce following hypothesis:

(H_ϕ) *The function $t \to \phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $\mathcal{L}^\phi : \mathcal{R}(\rho^-) \to (0, \infty)$ such that*

$$\|\phi_t\| \leq \mathcal{L}^\phi(t)\|\phi\| \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

Remark 1.6.5 *The condition (H_ϕ), is frequently verified by functions continuous and bounded. For more details, see for instance [71].*

Lemma 1.6.6 *([69], Lemma 2.4) If $y : (-\infty, +\infty) \to E$ is a function such that $y_0 = \phi$, then*

$$\|y_s\|_{\mathcal{B}} \leq (M + \mathcal{L}^\phi)\|\phi\|_{\mathcal{B}} + l \sup\{|y(\theta)|; \theta \in [0, max\{0, s\}]\},\ \ s \in \mathcal{R}(\rho^-) \cup J,$$

where $\mathcal{L}^\phi = \sup\limits_{t \in \mathcal{R}(\rho^-)} \mathcal{L}^\phi(t)$.

Chapter 2

Functional Differential Equations With Delay

2.1 Introduction

In this Chapter, we study some first order classes of partial functional, evolution equation on $J = [0, +\infty)$ with finite and state-dependent delay.

In the literature devoted to equations with finite delay, the phase space is much of time the space of all continuous functions on $\mathcal{H}$ for $d > 0$, endowed with the uniform norm topology. we mention, for instance, the books by Hale and Verduyn Lunel [58], Kolmanovskii and Myshkis [75], and Wu [100], the reference therein. An extensive theory is developed for evolution equations [6, 7, 20, 50, 58, 75, 89].

When the delay is infinite, we introduce the notion of phase space $\mathcal{B}$ who plays an important role in the study of both qualitative and quantitative theory. However the complicated situations in which the delay depends on the unknown functions have been considered in recent years. An extensive theory is developed for evolution equations [6, 50]. These equations are frequently called equations with state-dependent delay, see, for instance [48, 68, 71, 98]. We also refer the reader to the papers by Abada *et al.* [1], Ait Dads and Ezzinbi [9], Anguraj *et al.* [10], Hartung *et al.* [60, 63], Hernandez *et al.* [69] and Li *et al.* [78].

2.2 Functional differential equations with finite delay

In this section, we study some first order class of semilinear functional evolution equations with finite delay. Our investigations will be situated in the Banach space of real continuous and bounded functions on the real half axis . We will use Schauder's fixed point theorem combined with the semigroup theory to have the existence of solutions of the following functional differential equation with delay:

$$y'(t) = Ay(t) + f(t, y_t), \quad \text{a.e. } t \in J := [0, +\infty) \tag{2.1}$$

$$y(t) = \phi(t), \quad t \in [-d, 0], \tag{2.2}$$

where $f : J \times C([-d, 0], E) \to E$ is given function, $\phi : [-d, 0] \to E$ is given continuous function, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \in J$, and $(E, |.|)$ is a real Banach space. For any function y defined on $[-d, +\infty)$ and any $t \in J$, we denote by y_t the element of $C([-d, 0], E)$ defined by

$$y_t(\theta) = y(t + \theta), \theta \in [-d, 0].$$

Here $y_t(.)$ represents the history of the state from time $t - d$, up to the present time t.

In order to define a mild solution of problem (2.1)-(2.2), we shall consider the space $BC := BC([-d, +\infty))$ which is the Banach space of all bounded and continuous functions from $[-d, +\infty)$ into $\mathbb{R}$ equipped with the standard norm

$$\|y\|_{BC} = \sup_{t \in [-d, +\infty)} |y(t)|.$$

2.2.1 Existence of mild solutions

In this section, we give our main existence results for problem (2.1)-(2.2). Before starting and proving this result, we give the definition of its mild solution.

Definition 2.2.1 *We say that a continuous function $y : [-d, +\infty) \to E$ is a mild solution of problem (2.1)-(2.2) if $y(t) = \phi(t),\ t \in [-d, 0]$ and*

$$y(t) = T(t)\phi(0) + \int_0^t T(t - s) f(s, y_s) ds,\ t \in J.$$

Let us introduce the following hypotheses:

(H_1) $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \in J$ which is compact for $t > 0$ in the Banach space E. Let $M = \sup\{\|T\|_{B(E)} : t \geq 0\}$.

(H_2) The function $f : J \times C([-d, 0], E) \to E$ is Carathéodory.

(H_3) There exists a continuous function $k : J \to [0, +\infty)$ such that:

$$|f(t, u) - f(t, v)| \leq k(t)\|u - v\|,\ t \in J,\ u, v \in C([-d, 0], E)$$

and

$$k^* := \sup_{t \in J} \int_0^t k(s) ds < \infty.$$

(H_4) The function $t \to f(t,0) = f_0 \in L^1(J,[0,+\infty))$ with $F^* = \|f_0\|_{L^1}$.

Theorem 2.2.2 *Assume that $(H_1)-(H_4)$ hold. If $k^*M < 1$, then the problem (2.1)-(2.2) has at least one mild solution on BC.*

Proof. We transform the problem (2.1)-(2.2) into a fixed point problem. Consider the operator: $N : BC \to BC$ define by:

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [-d,0], \\ \\ T(t)\ \phi(0) + \displaystyle\int_0^t T(t-s)\ f(s,y_s)\ ds, & \text{if } t \in J. \end{cases}$$

The operator N maps BC into BC; indeed the map $N(y)$ is continuous on $[-d,+\infty)$ for any $y \in BC$, and for each $t \in J$, we have

$$\begin{aligned} |N(y)(t)| &\leq M\|\phi\| + M\int_0^t |f(s,y_s) - f(s,0) + f(s,0)|ds \\ &\leq M\|\phi\| + M\int_0^t |f(s,0)|ds + M\int_0^t k(s)\|y_s\|ds \\ &\leq M\|\phi\| + MF^* + M\int_0^t k(s)\|y_s\|ds \\ &\leq M\|\phi\| + MF^* + M\|y\|_{BC}k^* := c. \end{aligned}$$

Hence, $N(y) \in BC$.

Moreover, let $r > 0$ be such that $r \geq \frac{M\|\phi\|+MF^*}{1-Mk^*}$, and B_r be the closed ball in BC centered at the origin and of radius r. Let $y \in B_r$ and $t \in [0,+\infty)$. Then,

$$|N(y)(t)| \leq M\|\phi\| + MF^* + Mk^*r.$$

Thus,

$$\|N(y)\|_{BC} \leq r,$$

which means that the operator N transforms the ball B_r into itself.

Now we prove that $N : B_r \to B_r$ satisfies the assumptions of Schauder's fixed theorem. The proof will be given in several steps.

Step 1: N is continuous in B_r.
Let $\{y_n\}$ be a sequence such that $y_n \to y$ in B_r. We have

$$|N(y_n)(t) - N(y)(t)| \leq M\int_0^t |f(s,(y_s)_n) - f(s,y_s)|ds.$$

Then by (H_2) we have $f(s,(y_s)_n) \to f(s,y_s)$, as $n \to \infty$, for a.e. $s \in J$, and by the Lebesgue dominated convergence theorem we have

$$\|N(y_n) - N(y)\|_{BC} \to 0, \text{ as } n \to \infty.$$

Thus, N is continuous.

Step 2 : $N(B_r) \subset B_r$ this is clear.

Step 3: $N(B_r)$ is equicontinuous on every compact interval $[0, b]$ of $[0, +\infty)$ for $b > 0$. Let $\tau_1, \tau_2 \in [0, b]$ with $\tau_2 > \tau_1$, we have

$$\begin{aligned}
|N(y)(\tau_2) - N(y)(\tau_1)| \;\leq\;& \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)}\|\phi\| \\
+\;& \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)}|f(s, y_s)|ds \\
+\;& \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)}|f(s, y_s)|ds \\
\leq\;& \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)}\|\phi\| \\
+\;& \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)}|f(s, y_s) - f(s, 0) + f(s, 0)|ds \\
+\;& \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)}|f(s, y_s) - f(s, 0) + f(s, 0)|ds \\
\leq\;& \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)}\|\phi\| \\
+\;& r\int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)}k(s)ds \\
+\;& \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)}|f(s, 0)|ds \\
+\;& r\int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)}k(s)ds \\
+\;& \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)}|f(s, 0)|ds.
\end{aligned}$$

When $\tau_2 \longrightarrow \tau_2$, the right-hand side of the above inequality tends to zero, since $T(t)$ is a strongly continuous operator and the compactness of $T(t)$ for $t > 0$, implies he continuity in the uniform operator topology (see [20, 89]). This proves the equicontinuity.

Step 4: $N(B_r)$ is relatively compact on every compact interval of $[0, +\infty)$.

Let $t \in [0, b]$ for $b > 0$ and let ε be a real number satisfying $0 < \varepsilon < t$. For $y \in B_r$, let $h \in N(y)$ we define

$$N_\varepsilon(t) = T(t)\phi(0) + T(\varepsilon)\int_0^{t-\varepsilon} T(t - s - \varepsilon)f(s)ds.$$

the set,

$$\{N_\varepsilon(y)(t), y \in B_r\}$$

is precompact in E for every ε, $0 < \varepsilon < t$. Moreover, for every $y \in B_r$ we have

$$|N(y)(t) - N_\varepsilon(y)(t)| \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Therefore, the set $\{N(y)(t) : y \in B_r\}$ is precompact, i.e., relatively compact. Hence the set $Y(t) = \{h(t) : h \in N(B_r)\}$ is relatively compact.

Step 5:$N(B_r)$ is equiconvergent.
Let $t \in [0, +\infty)$ and $y \in B_r$; we have

$$\begin{aligned} |N(y)(t)| &\leq M\|\phi\| + M\int_0^t |f(s, y_s)|ds \\ &\leq M\|\phi\| + MF^* + Mr\int_0^t k(s)ds. \end{aligned}$$

Then,

$$|N(y)(t)| \to l \leq, \quad \text{as} \ \ t \to +\infty,$$

where $l \leq M\|\phi\| + MF^* + Mrk^*$ since $\lim\limits_{t\to+\infty}\int_0^t k(s)ds = k^*$. Hence,

$$|N(y)(t) - N(y)(+\infty)| \to 0, \quad \text{as} \ \ t \to +\infty.$$

As a consequence of Steps $1-4$, with Lemma 1.5.1, we can conclude that $N : B_r \to B_r$ is continuous and compact. From Schauder's theorem, we deduce that N has a fixed point y that is a mild solution of the problem (2.1)-(2.2).

2.2.2 An example

Consider the functional partial differential equation

$$\frac{\partial}{\partial t}z(t,x) - \frac{\partial^2}{\partial x^2}z(t,x) = f(t, z(t-d,x)), \ x \in [0,\pi], \ t \in J := [0, +\infty), \tag{2.3}$$

$$z(t,0) = z(t,\pi) = 0, \ t \in [0, +\infty), \tag{2.4}$$

$$z(t,x) = \phi(t), \ t \in [-d, 0], \ x \in [0,\pi], \tag{2.5}$$

where

$$f(t, z(t-d,x)) = \exp(-t)\frac{|z(t-d,x)|}{1 + |z(t-d,x)|}.$$

Take $E = L^2[0,\pi]$ and define $A : E \to E$ by $A\omega = \omega''$ with domain

$$D(A) = \{\omega \in E; \omega, \omega' \text{are absolutely continuous}, \ \omega'' \in E, \ \omega(0) = \omega(\pi) = 0\}.$$

Then,

$$A\omega = \sum_{n=1}^{\infty} n^2(\omega, \omega_n)\omega_n, \ \omega \in D(A)$$

where $\omega_n(s) = \sqrt{\frac{2}{\pi}} \sin ns, n = 1, 2, \ldots$, is the orthogonal set of eigenvectors in A. It is well know (see [89]) that A is the infinitesimal generator of an analytic semigroup $T(t), t \geq 0$ in E and is given by

$$T(t)\omega = \sum_{n=1}^{\infty} \exp(-n^2 t)(\omega, \omega_n)\omega_n, \ \omega \in E.$$

Since the analytic semigroup $T(t)$ is compact, there exists a positive constant M such that

$$\|T(t)\|_{B(E)} \leq M.$$

The function $f(t, z(t-d, x)) = e^{-t} \frac{|z(t-d,x)|}{1+|z(t-d,x)|}$ is Carathéodory, and

$$|f(t, z_1(t-d, x)) - f(t, z_2(t-d, x))| \leq e^{-t}|z_1(t-d, x) - z_2(t-d, x)|;$$

thus $k(t) = e^{-t}$. Moreover, we have

$$K^* = \sup\left\{\int_0^t e^{-s} ds, \ t \in [0, +\infty)\right\} = 1, \ f_0 = 0.$$

Then the problem (2.1)-(2.2) in an abstract formulation of the problem (2.3)-(2.5), and conditions $(H_1)-(H_4)$ are satisfied. Theorem 2.2.2 implies that the problem (2.3)-(2.5) has at least one mild solution on BC.

2.3 Functional differential equations with state-dependent delay

2.3.1 Introduction

In this section we prove the existence of solutions of a class of functional differential equations. Our investigations will be situated in the Banach space of real functions which are defined, continuous and bounded on $\mathbb{R}$. We will use Schauder's fixed point theorem combined with the semigroup theory to have the existence of solutions of the following functional differential equation with state-dependent delay:

$$y'(t) = Ay(t) + f(t, y_{\rho(t,y_t)}), \quad \text{a.e. } t \in J := [0, +\infty) \tag{2.6}$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0], \tag{2.7}$$

where $f : J \times \mathcal{B} \to E$, $\phi \in \mathcal{B}$, $\rho : J \times \mathcal{B} \to (-\infty, +\infty)$ are given functions and $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \in J$, $\mathcal{B}$ is the phase space to be specified later, $(E, |.|)$ is a real Banach space.

For any function y defined on $(-\infty,+\infty)$ and any $t\in J$ we denote by y_t the element of $\mathcal{B}$ defined by

$$y_t(\theta)=y(t+\theta),\theta\in(-\infty,0].$$

We assume that the histories y_t to some abstract phases $\mathcal{B}$.

In order to define a mild solution of problem (2.6)-(2.7), we shall consider the space $BC:=BC(-\infty,+\infty)$ is the Banach space of all bounded and continuous functions from $(-\infty,+\infty)$ into E equipped with the standard norm

$$\|y\|_{BC}=\sup_{t\in(-\infty,+\infty)}|y(t)|.$$

And $BC':=BC'([0,+\infty))$ is the Banach space of all bounded and continuous functions from $[0,+\infty)$ into E equipped with the standard norm

$$\|y\|_{BC'}=\sup_{t\in[0,+\infty)}|y(t)|.$$

2.3.2 Existence of mild solutions

In this section we give our main existence result for problem (2.6)-(2.7). Before starting and proving this result, we give the definition of the mild solution.

Definition 2.3.1 *We say that a continuous function $y:(-\infty,+\infty)\to E$ is a mild solution of problem (2.6)-(2.7) if $y(t)=\phi(t),\ t\in(-\infty,0]$ and the restriction of $y(.)$ to the interval $[0,+\infty)$ is continuous and satisfies the following integral equation:*

$$y(t)=T(t)\phi(0)+\int_0^t T(t-s)f(s,y_{\rho(s,y_s)})ds,\ t\in J. \tag{2.8}$$

Let us introduce the following hypotheses:

(H_1) $A:D(A)\subset E\to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t),t\in J$ which is compact for $t>0$ in the Banach space E. Let $M'=\sup\{\|T\|_{B(E)}:t\geq 0\}$.

(H_2) The function $f:J\times\mathcal{B}\to E$ is Carathéodory.

(H_3) There exists a continuous function $k:J\to[0,+\infty)$ such that:

$$|f(t,u)-f(t,v)|\leq k(t)\|u-v\|_{\mathcal{B}},\ t\in J,\ u,v\in\mathcal{B},$$

and

$$k^*:=\sup_{t\in J}\int_0^t k(s)ds<\infty.$$

(H_4) The function $t \to f(t,0) = f_0 \in L^1(J,[0,+\infty))$ with $F^* = \|f_0\|_{L^1}$.

Theorem 2.3.2 *Assume that $(H_1)-(H_4),(H_\phi)$ hold. If $k^*M'l < 1$, then the problem (2.6)-(2.7) has at least one mild solution on BC.*

Proof. Transform the problem (2.6)-(2.7) into a fixed point problem. Consider the operator $N : BC \to BC$ defined by:

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty,0], \\ T(t)\ \phi(0) + \displaystyle\int_0^t T(t-s)\ f(s, y_{\rho(s,y_s)})\ ds, & \text{if } t \in J. \end{cases}$$

Let $x(.) : (-\infty,+\infty) \to E$ be the function defined by:

$$x(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty,0]; \\ T(t)\ \phi(0), & \text{if } t \in J, \end{cases}$$

Then $x_0 = \phi$. For each $z \in BC$ with $z(0) = 0$, we denote by $\overline{z}$ the function

$$\overline{z}(t) = \begin{cases} 0, & \text{if } t \in (-\infty,0]; \\ z(t), & \text{if } t \in J. \end{cases}$$

If y satisfies (2.8), we can decompose it as $y(t) = z(t) + x(t),\ t \in J$, which implies $y_t = z_t + x_t$ for every $t \in J$ and the function $z(.)$ satisfies

$$z(t) = \int_0^t T(t-s) f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})ds, t \in J.$$

Set

$$BC'_0 = \{z \in BC' : z(0) = 0\}$$

and let

$$\|z\|_{BC'_0} = \sup\{|z(t)| : t \in J\},\ z \in BC'_0.$$

BC'_0 is a Banach space with the norm $\|.\|_{BC'_0}$. We define the operator $\mathcal{A} : BC'_0 \to BC'_0$ by:

$$\mathcal{A}(z)(t) = \int_0^t T(t-s) f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})ds,\ t \in J.$$

We shall show that the operator $\mathcal{A}$ satisfies all conditions of Schauder's fixed point theorem. The operator A maps BC'_0 into BC'_0, indeed the map $\mathcal{A}(z)$ is continuous on

$[0,+\infty)$ for any $z \in BC_0'$, and for each $t \in J$ we have

$$\begin{aligned}
|\mathcal{A}(z)(t)| &\leq M' \int_0^t |f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) - f(s,0) + f(s,0)| ds \\
&\leq M' \int_0^t |f(s,0)| ds + M' \int_0^t k(s) \|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} ds \\
&\leq M'F^* + M' \int_0^t k(s)(l|z(s)| + (m + \mathcal{L}^\phi + lM'H)\|\phi\|_{\mathcal{B}}) ds.
\end{aligned}$$

Set

$$C := (m + \mathcal{L}^\phi + lM'H)\|\phi\|_{\mathcal{B}}.$$

Then, we have

$$\begin{aligned}
|\mathcal{A}(z)(t)| &\leq M'F^* + M'C \int_0^t k(s) ds + M' \int_0^t l|z(s)|k(s) ds \\
&\leq M'F^* + M'Ck^* + M'l\|z\|_{BC_0'} k^*.
\end{aligned}$$

Hence, $\mathcal{A}(z) \in BC_0'$.

Moreover, let $r > 0$ be such that

$$r \geq \frac{M'F^* + M'Ck^*}{1 - M'k^*l},$$

and B_r be the closed ball in BC_0' centered at the origin and of radius r. Let $z \in B_r$ and $t \in [0,+\infty)$. Then

$$|\mathcal{A}(z)(t)| \leq \quad M'F^* + M'Ck^* + M'k^*lr.$$

Thus

$$\|\mathcal{A}(z)\|_{BC_0'} \leq r,$$

which means that the operator $\mathcal{A}$ transforms the ball B_r into itself.

Now we prove that $\mathcal{A} : B_r \to B_r$ satisfies the assumptions of Schauder's fixed theorem. The proof will be given in several steps.

Step 1: $\mathcal{A}$ is continuous in B_r.
Let $\{z_n\}$ be a sequence such that $z_n \to z$ in B_r. At the first, we study the convergence of the sequences $(z^n_{\rho(s,z^n_s)})_{n\in\mathbb{N}}$, $s \in J$.
If $s \in J$ is such that $\rho(s, z_s) > 0$, then we have,

$$\begin{aligned}
\|z^n_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\|_{\mathcal{B}} &\leq \|z^n_{\rho(s,z^n_s)} - z_{\rho(s,z^n_s)}\|_{\mathcal{B}} + \|z_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\|_{\mathcal{B}} \\
&\leq L\|z_n - z\|_{B_r} + \|z_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\|_{\mathcal{B}},
\end{aligned}$$

which proves that $z^n_{\rho(s,z^n_s)} \to z_{\rho(s,z_s)}$ in $\mathcal{B}$ as $n \to \infty$ for every $s \in J$ such that $\rho(s, z_s) > 0$. Similarly, is $\rho(s, z_s) < 0$, we get

$$\|z^n_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\|_{\mathcal{B}} = \|\phi^n_{\rho(s,z^n_s)} - \phi_{\rho(s,z_s)}\|_{\mathcal{B}} = 0$$

which also shows that $z^n_{\rho(s,z^n_s)} \to z_{\rho(s,z_s)}$ in $\mathcal{B}$ as $n \to \infty$ for every $s \in J$ such that $\rho(s, z_s) < 0$. Combining the pervious arguments, we can prove that $z^n_{\rho(s,z_s)} \to \phi$ for every $s \in J$ such that $\rho(s, z_s) = 0$. Finally,

$$\begin{aligned} |\mathcal{A}(z_n)(t) - \mathcal{A}(z)(t)| &\leq M' \int_0^t |f(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)}) - f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|ds \\ &\leq M' \int_0^t |f(s, z_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)}) - f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|ds. \end{aligned}$$

Then by (H_2) we have

$$f(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)}) \to f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}), \text{ as } n \to \infty,$$

and by the Lebesgue dominated convergence theorem we get,

$$\|\mathcal{A}(z_n) - \mathcal{A}(z)\|_{BC'_0} \to 0, \text{ as } n \to \infty.$$

Thus $\mathcal{A}$ is continuous.

Step 2 : $\mathcal{A}(B_r) \subset B_r$ this is clear.

Step 3: $\mathcal{A}(B_r)$ is equicontinuous on every compact interval $[0,b]$ of $[0,+\infty)$ for $b>0$. Let $\tau_1,\tau_2\in[0,b]$ with $\tau_2>\tau_1$, we have:

$$\begin{aligned}
|\mathcal{A}(z)(\tau_2)-\mathcal{A}(z)(\tau_1)| &\leq \int_0^{\tau_1}\|T(\tau_2-s)-T(\tau_1-s)\|_{B(E)}|f(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})|ds\\
&+ \int_{\tau_1}^{\tau_2}\|T(\tau_2-s)\|_{B(E)}|f(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})|ds\\
&\leq \int_0^{\tau_1}\|T(\tau_2-s)-T(\tau_1-s)\|_{B(E)}|f(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})-f(s,0)|ds\\
&+ \int_0^{\tau_1}\|T(\tau_2-s)-T(\tau_1-s)\|_{B(E)}|f(s,0)|ds\\
&+ \int_{\tau_1}^{\tau_2}\|T(\tau_2-s)\|_{B(E)}|f(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})-f(s,0)|ds\\
&+ \int_{\tau_1}^{\tau_2}\|T(\tau_2-s)\|_{B(E)}|f(s,0)|ds\\
&\leq C\int_0^{\tau_1}\|T(\tau_2-s)-T(\tau_1-s)\|_{B(E)}k(s)ds\\
&+ rL\int_0^{\tau_1}\|T(\tau_2-s)-T(\tau_1-s)\|_{B(E)}k(s)ds\\
&+ \int_0^{\tau_1}\|T(\tau_2-s)-T(\tau_1-s)\|_{B(E)}|f(s,0)|ds\\
&+ C\int_{\tau_1}^{\tau_2}\|T(\tau_2-s)\|_{B(E)}k(s)ds\\
&+ rL\int_{\tau_1}^{\tau_2}\|T(\tau_2-s)\|_{B(E)}k(s)ds\\
&+ \int_{\tau_1}^{\tau_2}\|T(\tau_2-s)\|_{B(E)}|f(s,0)|ds.
\end{aligned}$$

When $\tau_2\to\tau_1$, the right-hand side of the above inequality tends to zero, since $T(t)$ is a strongly continuous operator and the compactness of $T(t)$ for $t>0$, implies the continuity in the uniform operator topology (see [89]), this proves the equicontinuity.

Step 4: $\mathcal{A}(B_r)(t)$ is relatively compact on every compact interval of $t\in[0,\infty)$. Let $t\in[0,b]$ for $b>0$ and let ε be a real number satisfying $0<\varepsilon<t$. For $z\in B_r$ we define

$$\mathcal{A}_\varepsilon(z)(t)=T(\varepsilon)\int_0^{t-\varepsilon}T(t-s-\varepsilon)f(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})ds.$$

Note that the set

$$\left\{\int_0^{t-\varepsilon}T(t-s-\varepsilon)f(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})ds : z\in B_r\right\}$$

is bounded.

$$|\int_0^{t-\varepsilon} T(t-s-\varepsilon) f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) ds| \leq r$$

Since $T(t)$ is a compact operator for $t > 0$, the set,

$$\{\mathcal{A}_\varepsilon(z)(t) : z \in B_r\}$$

is precompact in E for every ε, $0 < \varepsilon < t$. Moreover, for every $z \in B_r$ we have

$$\begin{aligned} |\mathcal{A}(z)(t) - \mathcal{A}_\varepsilon(z)(t)| &\leq \int_{t-\varepsilon}^t T(t-s) f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) ds \\ &\leq M' F^* \varepsilon + M' C \int_{t-\varepsilon}^t k(s) ds + r M' \int_{t-\varepsilon}^t l k(s) ds, \\ &\to 0 \quad \text{as} \quad \varepsilon \to 0. \end{aligned}$$

Therefore, the set $\{\mathcal{A}(z)(t) : z \in B_r\}$ is precompact, i.e., relatively compact.

Step 5: $\mathcal{A}(B_r)$ is equiconvergent.
Let $t \in [0, +\infty)$ and $z \in B_r$, we have,

$$\begin{aligned} |\mathcal{A}(z)(t)| &\leq M' \int_0^t |f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})| ds \\ &\leq M' F^* + M' C \int_0^t k(s) ds + M' r \int_0^t L k(s) ds \\ &\leq M' F^* + M' C \int_0^t k(s) ds + M' r l \int_0^t k(s) ds. \end{aligned}$$

We have

$$|\mathcal{A}(z)(t)| \to l, \quad \text{as} \quad t \to +\infty.$$

Where $l \leq M'F^* + M'Ck^* + M'rlk^*$ Hence,

$$|\mathcal{A}(z)(t) - \mathcal{A}(z)(+\infty)| \to 0, \quad \text{as} \quad t \to +\infty.$$

As a consequence of Steps 1-4, with Lemma 1.5.1, we can conclude that $\mathcal{A} : B_r \to B_r$ is continuous and compact. From Schauder's theorem, we deduce that $\mathcal{A}$ has a fixed point z^*. Then $y^* = z^* + x$ is a fixed point of the operator N, which is a mild solution of the problem (2.6)-(2.7).

2.3.3 An example

Consider the following functional partial differential equation

$$\frac{\partial}{\partial t} z(t,x) - \frac{\partial^2}{\partial x^2} z(t,x) = e^{-t} \int_{-\infty}^0 z\left(s - \sigma_1(t)\sigma_2\left(\int_0^\pi a(\theta)|z(t,\theta)|^2 d\theta\right), x\right) ds, \quad (2.9)$$

$$x \in [0,\pi],\ t \in [0,+\infty)$$

$$z(t,0) = z(t,\pi) = 0,\ t \in [0,+\infty), \tag{2.10}$$

$$z(\theta,x) = z_0(\theta,x),\ t \in (-\infty,0],\ x \in [0,\pi], \tag{2.11}$$

where $z_0 \neq 0$. Set

$$f(t,\psi)(x) = \int_{-\infty}^{0} e^{-t}\psi(s,x)ds,$$

and

$$\rho(t,\psi) = t - \sigma_1(t)\sigma_2\left(\int_0^{\pi} a_2(\theta)|\psi(t,\theta)|^2 d\theta\right),$$

$\sigma_i : \mathrm{I\!R}^+ \to \mathrm{I\!R}^+, i = 1,2$ and $a : \mathrm{I\!R} \to \mathrm{I\!R}$ are continuous functions.
Take $E = L^2[0,\pi]$ and define $A : E \to E$ by $A\omega = \omega''$ with domain

$$D(A) = \{\omega \in E, \omega, \omega' \text{are absolutely continuous},\ \omega'' \in E,\ \omega(0) = \omega(\pi) = 0\}.$$

Then

$$A\omega = \sum_{n=1}^{\infty} n^2(\omega,\omega_n)\omega_n, \omega \in D(A)$$

where $\omega_n(s) = \sqrt{\frac{2}{\pi}}\sin ns, n = 1,2,\ldots$ is the orthogonal set of eigenvectors in A. It is well know (see [89]) that A is the infinitesimal generator of an analytic semigroup $T(t), t \geq 0$ in E and is given by

$$T(t)\omega = \sum_{n=1}^{\infty} \exp(-n^2 t)(\omega,\omega_n)\omega_n, \omega \in E.$$

Since the analytic semigroup $T(t)$ is compact, there exists a positive constant M such that

$$\|T(t)\|_{B(E)} \leq M.$$

Let $\mathcal{B} = BCU(\mathrm{I\!R}^-;E)$ and $\phi \in \mathcal{B}$, then (H_ϕ).
The function $f(t,\psi)(x)$ is Carathéodory, and

$$|f(t,\psi_1)(x) - f(t,\psi_2)(x)| \leq e^{-t}|\psi_1(t,x) - \psi_2(t,x)|,$$

thus $k(t) = e^{-t}$, moreover we have

$$k^* = \sup\left\{\int_0^t e^{-s}ds,\ t \in [0,+\infty)\right\} = 1,\ f_0 \equiv 0.$$

Then the problem (2.6)-(2.7) in an abstract formulation of the problem (2.9)-(2.11), and conditions $(H_1)-(H_4), (H_\phi)$ are satisfied. Theorem 2.3.2 implies that the problem (2.9)-(2.11) has at least one mild solutions on BC.

Chapter 3

Neutral Differential Equations with State-Dependent Delay

3.1 Introduction

In this Chapter, we study some first order classes of partial neutral functional evolution equation with infinite state-dependent delay, Our investigations will be situated in the Banach space of real functions which are defined, continuous and bounded on $\mathbb{R}$.

The literature relative to ordinary neutral functional differential equations is very extensive and refer to [14, 15, 26, 28, 84, 85]. For more results on partial neutral functional-differential equations and related issues we refer to Adimy and Ezzinbi [2], Hale [56], Wu and Xia [99, 100] for finite delay equations, and Hern'andez and Henriquez [66, 67] for unbounded delays.

Functional-differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equations has received a significant amount of attention in the last years, see for instance [1, 2, 8, 11, 25, 37, 77] and the references therein. We also cite [9, 78, 38, 48, 58, 83, 101] for the case neutral differential equations with State-dependent delay. In this chaptre we are going to study an extension of case finite delay and state-dependent delay for functional differential equations see for instance [30, 31].

We will use Schauder's fixed point theorem combined with the semigroup theory to have the existence of solutions of the following functional differential equation with state-dependent delay:

$$\frac{d}{dt}[y(t)-g(t,y_{\rho(t,y_t)})] = A[y(t)-g(t,y_{\rho(t,y_t)})]+f(t,y_{\rho(t,y_t)}), \quad \text{a.e. } t \in J := [0,+\infty) \tag{3.1}$$

$$y(t) = \phi(t), \quad t \in (-\infty,0], \tag{3.2}$$

where $f, g : J \times \mathcal{B} \to E$, $\phi \in \mathcal{B}$, $\rho : J \times \mathcal{B} \to (-\infty, +\infty)$ are given functions, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \in J$, $\mathcal{B}$ is the phase space to be specified later, $(E, |.|)$ is a real Banach space. For any function y defined on $(-\infty, +\infty)$ and any $t \in J$ we denote by y_t the element of $\mathcal{B}$ defined by

$$y_t(\theta) = y(t + \theta), \theta \in (-\infty, 0].$$

We assume that the histories y_t belongs to some abstract phases $\mathcal{B}$, to be specified later.

In order to define a mild solution of problem (3.1)-(3.2), we shall consider the space $BC := BC(-\infty, +\infty)$ we denote the Banach space of all bounded and continuous functions from $(-\infty, +\infty)$ into E equipped with the standard norm

$$\|y\|_{BC} = \sup_{t \in (-\infty, +\infty)} |y(t)|.$$

By BUC we denote the space of bounded uniformly continuous functions defined from $(-\infty, 0]$ to E.
Finally, by $BC' := BC'([0, +\infty))$ we denote the Banach space of all bounded and continuous functions from $[0, +\infty)$ into E equipped with the standard norm

$$\|y\|_{BC'} = \sup_{t \in [0, +\infty)} |y(t)|.$$

3.2 Existence of mild solutions

Now we give our main existence result for problem (3.1)-(3.2). Before starting and proving this result, we give the definition of the mild solution.

Definition 3.2.1 *We say that a continuous function $y : (-\infty, +\infty) \to E$ is a mild solution of problem (3.1)-(3.2) if $y(t) = \phi(t),\ t \in (-\infty, 0]$ and the restriction of $y(.)$ to the interval $[0, +\infty)$ is continuous and satisfies the following integral equation:*

$$y(t) = T(t)[\phi(0) - g(0, \phi(0))] + g(t, y_{\rho(t, y_t)}) + \int_0^t T(t-s) f(s, y_{\rho(s, y_s)}) ds,\ t \in J. \quad (3.3)$$

Let us introduce the following hypotheses:

(H_1) $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \in J$ which is compact for $t > 0$ in the Banach space E. Let $M' = \sup\{\|T\|_{B(E)} : t \geq 0\}$.

(H_2) The function $f : J \times \mathcal{B} \to E$ is Carathéodory.

(H_3) There exists a continuous function $k : J \to [0, +\infty)$ such that:

$$|f(t,u) - f(t,v)| \leq k(t)\|u - v\|_{\mathcal{B}}, \ t \in J, \ u, v \in \mathcal{B}$$

and

$$k^* := \sup_{t \in J} \int_0^t k(s)ds < \infty.$$

(H_4) The function $t \to f(t,0) = f_0 \in L^1(J, [0, +\infty))$ with $F^* = \|f_0\|_{L^1}$.

(H_5) The function $g(t, \cdot)$ is continuous on J and there exists a constant $k_g > 0$ such that

$$|g(t,u) - g(t,v)| \leq k_g \|u - v\|_{\mathcal{B}}, \text{for each}, \ u, v \in \mathcal{B}$$

and

$$g^* := \sup_{t \in J} |g(t,0)| < \infty.$$

(H_6) For each $t \in J$ and any bounded set $B \subset \mathcal{B}$, the set $\{g(t,u) : \ u \in B\}$ is relatively compact in E

(H_7) For any bounded set $B \subset \mathcal{B}$, the function $\{t \to g(t, y_t) : y \in B\}$ is equicontinuous on each compact interval of $[0, +\infty)$.

Remark 3.2.2 *By the condition* $(H_3), (H_4)$ *we deduce that*

$$|f(t,y)| \leq k(t)\|u\|_{\mathcal{B}} + F^*, \ t \in J, \ u \in \mathcal{B},$$

and by (H_5) *we deduce that :*

$$|g(t,u)| \leq k_g \|u\|_{\mathcal{B}} + g^* \ t \in J, \ u \in \mathcal{B}.$$

Theorem 3.2.3 *Assume that* $(H_1) - (H_7)$ *and* (H_ϕ) *hold. If* $l(M'k^* + k_g) < 1$, *then the problem (3.1)-(3.2) has at least one mild solution on* BC.

Proof. Transform the problem (3.1)-(3.2) into a fixed point problem. Consider the operator $N : BC \to BC$ defined by:

$$(Ny)(t) = \begin{cases} \phi(t); & \text{if } t \in (-\infty, 0], \\ T(t) \ [\phi(0) - g(0, \phi(0))] & \\ +g(t, y_{\rho(t,y_t)}) + \displaystyle\int_0^t T(t-s) \ f(s, y_{\rho(s,y_s)}) \ ds; & \text{if } t \in J. \end{cases}$$

Let $x(.) : (-\infty, +\infty) \to E$ be the function defined by:

$$x(t) = \begin{cases} \phi(t); & \text{if } t \in (-\infty, 0]; \\ T(t) \ \phi(0); & \text{if } t \in J, \end{cases}$$

then $x_0 = \phi$. For each $z \in BC$ with $z(0) = 0$, we denote by $\overline{z}$ the function

$$\overline{z}(t) = \begin{cases} 0; & \text{if } t \in (-\infty, 0]; \\ z(t); & \text{if } t \in J. \end{cases}$$

If y satisfies (3.3), we can decompose it as $y(t) = z(t) + x(t),\ t \in J$, which implies $y_t = z_t + x_t$ for every $t \in J$ and the function $z(.)$ satisfies

$$z(t) = g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) - T(t)g(0, \phi(0))$$

$$+ \int_0^t T(t-s)f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})ds, t \in J.$$

Set

$$BC'_0 = \{z \in BC' : z(0) = 0\}$$

and let

$$\|z\|_{BC'_0} = \sup\{|z(t)| : t \in J\},\ z \in BC'_0.$$

BC'_0 is a Banach space with the norm $\|.\|_{BC'_0}$. We define the operator $\mathcal{A} : BC'_0 \to BC'_0$ by:

$$\mathcal{A}(z)(t) = g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) - T(t)g(0, \phi(0))$$

$$+ \int_0^t T(t-s)f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})ds,\ t \in J.$$

We shall show that the operator $\mathcal{A}$ satisfies all conditions of Schauder's fixed point theorem. The operator A maps BC'_0 into BC'_0, indeed the map $\mathcal{A}(z)$ is continuous on $[0, +\infty)$ for any $z \in BC'_0$, and for each $t \in J$ we have

$$\begin{aligned} |\mathcal{A}(z)(t)| &\leq |g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)})| + M'|g(0, \phi(0))| \\ &\quad + M' \int_0^t |f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) - f(s, 0) + f(s, 0)|ds \\ &\leq M'(k_g\|\phi\|_{\mathcal{B}} + g^*) + k_g\|z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\|_{\mathcal{B}} + g^* \\ &\quad + M' \int_0^t |f(s, 0)|ds + M' \int_0^t k(s)\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}ds \\ &\leq M'(k_g\|\phi\|_{\mathcal{B}} + g^*) + k_g(l|z(t)| + (m + \mathcal{L}^\phi + lM'H)\|\phi\|_{\mathcal{B}}) + g^* \\ &\quad + M'F^* + M' \int_0^t k(s)(l|z(s)| + (m + \mathcal{L}^\phi + lM'H)\|\phi\|_{\mathcal{B}})ds. \end{aligned}$$

Set

$$C_1 := (m + \mathcal{L}^\phi + lM'H)\|\phi\|_{\mathcal{B}}.$$

$$C_2 := M'(k_g\|\phi\|_{\mathcal{B}} + g^*) + k_g(m + \mathcal{L}^\phi + lM'H)\|\phi\|_{\mathcal{B}} + g^* + M'F^*.$$

Then, we have

$$\begin{aligned} |\mathcal{A}(z)(t)| &\leq C_2 + k_g l |z(t)| + M'C_1 \int_0^t k(s)ds + M' \int_0^t l|z(s)|k(s)ds \\ &\leq C_2 + k_g l \|z\|_{BC'_0} + M'C_1 k^* + M'l\|z\|_{BC'_0} k^*. \end{aligned}$$

Hence, $\mathcal{A}(z) \in BC'_0$.

Moreover, let $r > 0$ be such that

$$r \geq \frac{C_2 + M'Ck^*}{1 - l(M'k^* + k_g)},$$

and B_r be the closed ball in BC'_0 centered at the origin and of radius r. Let $z \in B_r$ and $t \in [0, +\infty)$. Then

$$|\mathcal{A}(z)(t)| \leq \; C_2 + k_g lr + M'Ck^* + M'k^*lr.$$

Thus

$$\|\mathcal{A}(z)\|_{BC'_0} \leq r,$$

which means that the operator $\mathcal{A}$ transforms the ball B_r into itself.

Now we prove that $\mathcal{A} : B_r \to B_r$ satisfies the assumptions of Schauder's fixed theorem. The proof will be given in several steps.

Step 1: $\mathcal{A}$ is continuous in B_r.
Let $\{z_n\}$ be a sequence such that $z_n \to z$ in B_r. At the first, we study the convergence of the sequences $(z^n_{\rho(s,z^n_s)})_{n\in\mathbb{N}}$, $s \in J$.
If $s \in J$ is such that $\rho(s, z_s) > 0$, then we have,

$$\begin{aligned} \|z^n_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\|_{\mathcal{B}} &\leq \|z^n_{\rho(s,z^n_s)} - z_{\rho(s,z^n_s)}\|_{\mathcal{B}} + \|z_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\|_{\mathcal{B}} \\ &\leq l\|z_n - z\|_{B_r} + \|z_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\|_{\mathcal{B}}, \end{aligned}$$

which proves that $z^n_{\rho(s,z^n_s)} \to z_{\rho(s,z_s)}$ in $\mathcal{B}$ as $n \to \infty$ for every $s \in J$ such that $\rho(s, z_s) > 0$. Similarly, is $\rho(s, z_s) < 0$, we get

$$\|z^n_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\|_{\mathcal{B}} = \|\phi^n_{\rho(s,z^n_s)} - \phi_{\rho(s,z_s)}\|_{\mathcal{B}} = 0$$

which also shows that $z^n_{\rho(s,z^n_s)} \to z_{\rho(s,z_s)}$ in $\mathcal{B}$ as $n \to \infty$ for every $s \in J$ such that $\rho(s, z_s) < 0$. Combining the pervious arguments, we can prove that $z^n_{\rho(s,z_s)} \to \phi$ for every $s \in J$ such that $\rho(s, z_s) = 0$. Finally,

$$\begin{aligned} |\mathcal{A}(z_n)(t) - \mathcal{A}(z)(t)| &\leq |g(t, z^n_{\rho(t,z^n_t+x_t)} + x_{\rho(t,z^n_t+x_t)}) - g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)})| \\ &+ \; M' \int_0^t |f(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)}) - f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|ds \\ &\leq |g(t, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)}) - g(t, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})| \\ &+ \; M' \int_0^t |f(s, z_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)}) - f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|ds. \end{aligned}$$

Then by $(H_2), (H_5)$ we have

$$f(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)}) \to f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}), \text{ as } n \to \infty,$$

$$g(t, z^n_{\rho(t,z^n_t+x_t)} + x_{\rho(t,z^n_t+x_t)}) \to g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}), \text{ as } n \to \infty,$$

and by the Lebesgue dominated convergence theorem we get,

$$\|\mathcal{A}(z_n) - \mathcal{A}(z)\|_{BC'_0} \to 0, \text{ as } n \to \infty.$$

Thus $\mathcal{A}$ is continuous.

Step 2 : $\mathcal{A}(B_r) \subset B_r$. This is clear.

Step 3: $\mathcal{A}(B_r)$ is equicontinuous on every compact interval $[0,b]$ of $[0,+\infty)$ for $b>0$. Let $\tau_1,\tau_2\in[0,b]$ with $\tau_2>\tau_1$, we have:

$$\begin{aligned}
|\mathcal{A}(z)(\tau_2)-\mathcal{A}(z)(\tau_1)| &\leq |g(\tau_2,z_{\rho(\tau_2,z_{\tau_2}+x_{\tau_2})}+x_{\rho(\tau_2,z_{\tau_2}+x_{\tau_2})})-g(\tau_1,z_{\rho(\tau_1,z_{\tau_1}+x_{\tau_1})}+x_{\rho(\tau_1,z_{\tau_1}+x_{\tau_1})})| \\
&+ \|T(\tau_2)-T(\tau_1)\|_{B(E)}|g(0,\phi(0))| \\
&+ \int_0^{\tau_1}\|T(\tau_2-s)-T(\tau_1-s)\|_{B(E)}|f(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})|ds \\
&+ \int_{\tau_1}^{\tau_2}\|T(\tau_2-s)\|_{B(E)}|f(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})|ds \\
&\leq |g(\tau_2,z_{\rho(\tau_2,z_{\tau_2}+x_{\tau_2})}+x_{\rho(\tau_2,z_{\tau_2}+x_{\tau_2})})-g(\tau_1,z_{\rho(\tau_1,z_{\tau_1}+x_{\tau_1})}+x_{\rho(\tau_1,z_{\tau_1}+x_{\tau_1})})| \\
&+ \|T(\tau_2)-T(\tau_1)\|_{B(E)}(k_g\|\phi\|_{\mathcal{B}}+g^*) \\
&+ \int_0^{\tau_1}\|T(\tau_2-s)-T(\tau_1-s)\|_{B(E)}|f(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})-f(s,0)|ds \\
&+ \int_0^{\tau_1}\|T(\tau_2-s)-T(\tau_1-s)\|_{B(E)}|f(s,0)|ds \\
&+ \int_{\tau_1}^{\tau_2}\|T(\tau_2-s)\|_{B(E)}|f(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})-f(s,0)|ds \\
&+ \int_{\tau_1}^{\tau_2}\|T(\tau_2-s)\|_{B(E)}|f(s,0)|ds \\
&\leq k_g|g(\tau_2,z_{\rho(\tau_2,z_{\tau_2}+x_{\tau_2})}+x_{\rho(\tau_2,z_{\tau_2}+x_{\tau_2})})-g(\tau_1,z_{\rho(\tau_1,z_{\tau_1}+x_{\tau_1})}+x_{\rho(\tau_1,z_{\tau_1}+x_{\tau_1})})| \\
&+ \|T(\tau_2)-T(\tau_1)\|_{B(E)}(k_g\|\phi\|_{\mathcal{B}}+g^*) \\
&+ C_1\int_0^{\tau_1}\|T(\tau_2-s)-T(\tau_1-s)\|_{B(E)}k(s)ds \\
&+ rL\int_0^{\tau_1}\|T(\tau_2-s)-T(\tau_1-s)\|_{B(E)}k(s)ds \\
&+ \int_0^{\tau_1}\|T(\tau_2-s)-T(\tau_1-s)\|_{B(E)}|f(s,0)|ds \\
&+ C_1\int_{\tau_1}^{\tau_2}\|T(\tau_2-s)\|_{B(E)}k(s)ds \\
&+ rL\int_{\tau_1}^{\tau_2}\|T(\tau_2-s)\|_{B(E)}k(s)ds \\
&+ \int_{\tau_1}^{\tau_2}\|T(\tau_2-s)\|_{B(E)}|f(s,0)|ds.
\end{aligned}$$

When $\tau_2\longrightarrow\tau_1$, the right-hand side of the above inequality tends to zero, since (H_7) and $T(t)$ is a strongly continuous operator and the compactness of $T(t)$ for $t>0$, implies the continuity in the uniform operator topology (see [89]), this proves the equicontinuity.

Step 4: The set $\mathcal{A}(B_r)(t)$ is relatively compact on every compact interval of $[0,\infty)$. Let $t\in[0,b]$ for $b>0$ and let ε be a real number satisfying $0<\varepsilon<t$. For $z\in B_r$ we

define

$$\mathcal{A}_\varepsilon(z)(t) = g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)})) - T(\varepsilon)(T(t-\varepsilon)g(0,\phi(0)))$$
$$+T(\varepsilon)\int_0^{t-\varepsilon} T(t-s-\varepsilon)f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})ds.$$

Note that the set

$$\{g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) - T(t-\varepsilon)g(0,\phi(0))$$
$$+\int_0^{t-\varepsilon} T(t-s-\varepsilon)f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})ds : z \in B_r\}$$

is bounded.

$$|g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) - T(t-\varepsilon)g(0,\phi(0))$$
$$+\int_0^{t-\varepsilon} T(t-s-\varepsilon)f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})ds| \leq r$$

Since $T(t)$ is a compact operator for $t > 0$, and (H_6) we have that the set,

$$\{\mathcal{A}_\varepsilon(z)(t) : z \in B_r\}$$

is precompact in E for every ε, $0 < \varepsilon < t$. Moreover, for every $z \in B_r$ we have

$$\begin{aligned}
|\mathcal{A}(z)(t) - \mathcal{A}_\varepsilon(z)(t)| &\leq \int_{t-\varepsilon}^t T(t-s)f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})ds \\
&\leq M'F^*\varepsilon + M'C\int_{t-\varepsilon}^t k(s)ds + rM'\int_{t-\varepsilon}^t lk(s)ds, \\
&\to 0 \quad \text{as} \quad \varepsilon \to 0.
\end{aligned}$$

Therefore, the set $\{\mathcal{A}(z)(t) : z \in B_r\}$ is precompact, i.e., relatively compact.

Step 5: $\mathcal{A}(B_r)$ is equiconvergent.
Let $t \in [0, +\infty)$ and $z \in B_r$, we have,

$$\begin{aligned}
|\mathcal{A}(z)(t)| &\leq |g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}))| + M'|g(0,\phi(0))| \\
&\quad +M'\int_0^t |f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|ds \\
&\leq C_2 + k_g lr + M'C\int_0^t k(s)ds + M'rl\int_0^t k(s)ds.
\end{aligned}$$

Set

$$C_3 = C_2 + k_g lr + M'Ck^* + M'lrK^*.$$

Then we have

$$|\mathcal{A}(z)(t)| \to l, \quad \text{as} \;\; t \to +\infty,$$

where $l \leq C_3$ Hence,

$$|\mathcal{A}(z)(t) - \mathcal{A}(z)(+\infty)| \to 0, \quad \text{as} \;\; t \to +\infty.$$

As a consequence of Steps 1-5, with Lemma 1.5.1, we can conclude that $\mathcal{A} : B_r \to B_r$ is continuous and compact. From Schauder's theorem, we deduce that $\mathcal{A}$ has a fixed point z^*. Then $y^* = z^* + x$ is a fixed point of the operators N, which is a mild solution of the problem (3.1)-(3.2).

3.3 An example

Consider the following neutral functional partial differential equation:

$$\begin{aligned}\frac{\partial}{\partial t}[z(t,x) - g(t, z(t - \sigma(t, z(t,0)), x))] &= \frac{\partial^2}{\partial x^2}[z(t,x) - g(t, z(t - \sigma(t, z(t,0)), x))] \\ &+ f(t, z(t - \sigma(t, z(t,0)), x)), \; x \in [0,\pi], \; t \in [0, +\infty) \end{aligned} \tag{3.4}$$

$$z(t,0) = z(t,\pi) = 0, \; t \in [0, +\infty), \tag{3.5}$$

$$z(\theta, x) = z_0(\theta, x), \; t \in (-\infty, 0], \; x \in [0,\pi], \tag{3.6}$$

where f, g is a given functions , and $\sigma : \mathbb{R} \to \mathbb{R}^+$. Take $E = L^2[0,\pi]$ and define $A : E \to E$ by $A\omega = \omega''$ with domain

$$D(A) = \{\omega \in E, \omega, \omega' \text{are absolutely continuous}, \; \omega'' \in E, \; \omega(0) = \omega(\pi) = 0\}.$$

Then

$$A\omega = \sum_{n=1}^{\infty} n^2(\omega, \omega_n)\omega_n, \omega \in D(A),$$

where $\omega_n(s) = \sqrt{\frac{2}{\pi}} \sin ns, n = 1, 2, \ldots$ is the orthogonal set of eigenvectors in A. It is well know (see [89]) that A is the infinitesimal generator of an analytic semigroup $T(t), t \geq 0$ in E and is given by

$$T(t)\omega = \sum_{n=1}^{\infty} \exp(-n^2 t)(\omega, \omega_n)\omega_n, \omega \in E.$$

Since the analytic semigroup $T(t)$ is compact for $t > 0$, there exists a positive constant M such that

$$\|T(t)\|_{B(E)} \leq M.$$

Let $\mathcal{B} = BCU(\mathbb{R}^-; E)$ and $\phi \in \mathcal{B}$, then (H_ϕ), where $\rho(t, \varphi) = t - \sigma(\varphi)$.
Hence, the problem (3.1)-(3.2) in an abstract formulation of the problem (3.4)-(3.6), and if the conditions $(H_1) - (H_6)$, (H_ϕ) are satisfied. Theorem 3.2.3 implies that the problem (3.4)-(3.6) has at least one mild solutions on BC.

Chapter 4

Functional Differential Inclusions With Delay

4.1 Introduction

In this Chapter, we study some first order classes of partial functional, evolution inclusion on $J = [0, +\infty)$ with finite and infinite state-dependent delay.

For modeling scientific phenomena where the delay is either a fixed constant or is given as an integral in which case is called distributed delay, we use differential delay equations or functional differential equations ; see for instance the books [58, 75, 100]. An extensive theory is developed for evolution equations [6, 7, 52]. Uniqueness and existence results have been established recently for various classes of evolution problems in the papers by Baghli and Benchohra for finite and infinite delay in the Fréchet space setting in [13, 14, 15, 16].

However, complicated situations in which the delay depends on the unknown functions have been proposed in modeling in recent years. These equations are frequently called equations with state-dependent delay. Existence results and among other things were derived recently for functional differential equations when the solution is depending on the delay on a bounded interval $[0, b]$ for impulsive problems. We refer the reader to the papers by Abada *et al.* [1], Ait Dads and Ezzinbi [9], Anguraj *et al.* [10], Hernandez *et al.* [69] and Li *et al.* [78].

4.2 Functional differential inclusions with delay

In this section we are going to prove the existence of solutions of a class of semilinear functional evolution inclusion with delay. Our investigations will be situated in the Banach space of real continuous and bounded functions on $[0, +\infty)$. We will use Bohnenblust-Karlin's fixed theorem, combined with the Corduneanu's compactness criteria. More precisely, we will consider the following problem

$$y'(t) - Ay(t) \in F(t, y_t), \quad \text{a.e. } t \in J := [0, +\infty) \tag{4.1}$$

$$y(t) = \phi(t), \quad t \in [-d, 0], \tag{4.2}$$

where $F : J \times C([-d, 0], E) \to \mathcal{P}(E)$ is a multivalued map with nonempty compact values, $\mathcal{P}(E)$ is the family of all nonempty subsets of E, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \in J$, $\phi : [-d, 0] \to E$ is given continuous function, and $(E, |.|)$ is a real Banach space. For any function y defined on $[-d, +\infty)$ and any $t \in J$, we denote by y_t the element of $C([-d, 0], E)$ defined by

$$y_t(\theta) = y(t + \theta), \theta \in [-d, 0].$$

Here $y_t(.)$ represents the history of the state from time $t - d$, up to the present time t.

In order to define a mild solution of problem (4.1)-(4.2), we shall consider the space $BC := BC([-d, +\infty))$ which is the Banach space of all bounded and continuous functions from $[-d, +\infty)$ into $\mathbb{R}$ equipped with the standard norm

$$\|y\|_{BC} = \sup_{t \in [-d, +\infty)} |y(t)|.$$

4.2.1 Existence of mild solutions

In this section we give our main existence result for problem (4.1)-(4.2). Before starting and proving this result, we give the definition of a mild solution.

Definition 4.2.1 *we say that a continuous $y \in [-d, +\infty)$ is a mild solution of (4.1)-(4.2) if there exist function $f \in L^1(J, E)$ such that $f(t) \in F(t, y_t)$, a.e. J, $y(t) = \phi(t)$, $t \in [-d, 0]$, and*

$$y(t) = T(t)\phi(t) - \int_0^t T(t - s) f(s) ds, \ t \in J.$$

Let us introduce the following hypotheses:

(H_1) $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \in J$ which is compact for $t > 0$ in the Banach space E. Let $M = \sup\{\|T\|_{B(E)} : t \geq 0\}$.

(H_2) The multifunction $F : J \times C([-d, 0]; E) \longrightarrow \mathcal{P}(E)$ is Carathéodory with compact and convex values.

(H_3) There exists a continuous function $k : J \to [0, +\infty)$ such that:

$$H_d(F(t,u), F(t,v)) \le k(t)\|u - v\|,$$

for each $t \in J$ and for all $u, v \in C([-d, 0]; E)$ and

$$d(0, F(t, 0)) \le k(t),$$

with

$$k^* := \sup_{t \in J} \int_0^t k(s)ds < \infty.$$

Theorem 4.2.2 *Assume that $(H_1) - (H_3)$ hold. If $k^*M < 1$, then the problem (4.1)-(4.2) has at least one mild solution on BC.*

Proof. Transform the problem (4.1)-(4.2) into a fixed point problem. Consider the multivalued operator $N : BC \to \mathcal{P}(BC)$ defined by :

$$N(y) := \left\{ h \in BC : \ h(t) = \begin{cases} \phi(t), & \text{if } t \in [-d, 0]; \\ T(t)\ \phi(0) & \\ + \displaystyle\int_0^t T(t-s)\ f(s)\ ds, \quad f \in S_{F,y} & \text{if } t \in J. \end{cases} \right\}$$

The operator N maps BC into BC; for any $y \in BC$,and $h \in N(y)$ and for each $t \in J$, we have

$$\begin{aligned} |h(t)| &\le M\|\phi\| + M \int_0^t |f(s)|ds \\ &\le M\|\phi\| + M \int_0^t (k(s)\|y_s\| + \|F(s,0)\|)ds \\ &\le M\|\phi\| + M \int_0^t k(s)(\|y_s\| + 1)ds \\ &\le M\|\phi\| + M(\|y\|_{BC} + 1)k^* := c. \end{aligned}$$

Hence, $h(t) \in BC$.

Moreover, let $r > 0$ be such that $r \ge \frac{M\|\phi\| + Mk^*}{1 - Mk^*}$, and B_r be the closed ball in BC centered at the origin and of radius r. Let $y \in B_r$ and $t \in [0, +\infty)$. Then,

$$|h(t)| \le \ M\|\phi\| + Mk^* + Mk^*r.$$

Thus,

$$\|h\|_{BC} \le r,$$

which means that the operator N transforms the ball B_r into itself.

Now we prove that $N : B_r \to B_r$ satisfies the assumptions of Bohnenblust-Karlin's fixed theorem. The proof will be given in several steps.

Step 1: We shall show that the operator N is closed and convex. This will be given in two claims.

Claim 1 : $N(y)$ is closed for each $y \in B_r$.

Let $(h_n)_{n\geq 0} \in N(y)$ such that $h_n \to \tilde{h}$ in B_r. Then for $h_n \in B_r$ there exists $f_n \in S_{F,y}$ such that:

$$h_n(t) = T(t)\phi(0) + \int_0^t T(t-s)f_n(s)ds.$$

Using the fact that F has compact values and from hypotheses $(H_2), (H_3)$. An application of Mazur's theorem [102] we may pass a subsequence if necessary to get that f_n converges to $f \in L^1(J,E)$ and hence $f \in S_{F,y}$. Then for each $t \in J$,

$$h_n(t) \to \tilde{h}(t) = T(t)\phi(0) + \int_0^t T(t-s)f(s)ds.$$

So, $\tilde{h} \in N(y)$.

Claim 2 :$N(y)$ is convex for each $y \in B_r$.
Let $h_1, h_2 \in N(y)$, the there exists $f_1, f_2 \in S_{F,y}$ such that, for each $t \in J$ we have :

$$h_i(t) = T(t)\phi(0) + \int_0^t T(t-s)f_i(s)ds, i = 1, 2.$$

Let $0 \leq \delta \leq 1$. Then, we have for each $t \in J$:

$$(\delta h_1 + (1-\delta)h_2)(t) = T(t)\phi(0) + \int_0^t T(t-s)[\delta f_1(s) + (1-\delta)f_2(s)]ds.$$

Since $F(t,y)$ is convex, one has

$$\delta h_1 + (1-\delta)h_2 \in N(y).$$

Step 2 : $N(B_r) \subset B_r$ this is clear.

Step 3: $N(B_r)$ is equicontinuous on every compact interval $[0,b]$ of $[0,+\infty)$ for $b > 0$.

Let $\tau_1, \tau_2 \in [0, b]$ with $\tau_2 > \tau_1$, we have

$$\begin{aligned}
|h(\tau_2) - h(\tau_1)| &\leq \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)}\|\phi\| \\
&+ \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)}|f(s)|ds \\
&+ \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)}|f(s)|ds \\
&\leq \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)}\|\phi\| \\
&+ \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)}(k(s)\|y_s\| + |F(s,0)|)ds \\
&+ \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)}(k(s)\|y_s\| + |F(s,0)|)ds \\
&\leq \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)}\|\phi\| \\
&+ (r+1)\int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)}k(s)ds \\
&+ (r+1)\int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)}k(s)ds.
\end{aligned}$$

When $\tau_2 \to \tau_1$, the right-hand side of the above inequality tends to zero, since $T(t)$ is a strongly continuous operator and the compactness of $T(t)$ for $t > 0$, implies he continuity in the uniform operator topology (see [89]). This proves the equicontinuity.

Step 4: $N(B_r)$ is relatively compact on every compact interval of $[0, +\infty)$.

Let $t \in [0, b]$ for $b > 0$ and let ε be a real number satisfying $0 < \varepsilon < t$. For $y \in B_r$, let $h \in N(y)$ and $f \in S_{F,y}$ we define

$$h_\varepsilon(t) = T(t)\phi(0) + T(\varepsilon)\int_0^{t-\varepsilon} T(t - s - \varepsilon)f(s)ds.$$

Note that the set

$$\left\{T(t)\phi(0) + \int_0^{t-\varepsilon} T(t - s - \varepsilon)f(s)ds : y \in B_r\right\}$$

is bounded.

$$|T(t)\phi(0) + \int_0^{t-\varepsilon} T(t - s - \varepsilon)f(s)ds| \leq r.$$

Since $T(t)$ is a compact operator for $t > 0$, the set,

$$H_\varepsilon(t) = \{h_\varepsilon(t) : h_\varepsilon \in N(y), y \in B_r\}$$

is precompact in E for every ε, $0 < \varepsilon < t$. Moreover, for every $y \in B_r$ we have

$$\begin{aligned} |h(t) - h_\varepsilon(t)| &\leq M \int_{t-\varepsilon}^{t} |f(s)|ds \\ &\leq M \int_{t-\varepsilon}^{t} (k(s)\|y_s\| + |F(s,0|)ds \\ &\leq M(1+r) \int_{t-\varepsilon}^{t} k(s)ds \\ &\to 0 \quad \text{as} \quad \varepsilon \to 0. \end{aligned}$$

Therefore, the set $H(t) = \{h(t) : h \in N(y), y \in B_r\}$ is precompact, i.e., relatively compact. Hence the set $H(t) = \{h(t) : h \in N(B_r)\}$ is relatively compact.

Step 5: N has closed graph.
Let $\{y_n\}$ be a sequence such that $y_n \to y_*, h_n \in N(y_n)$ and $h_n \to h_*$. We shall show that $h_* \in N(y_*)$. $h_n \in N(y_n)$ means that there exists $f_n \in S_{F,y_n}$ such that

$$h_n(t) = T(t)\ \phi(0) + \int_0^t T(t-s)\ f_n(s)\ ds,\ t \in J.$$

We must prove that there exists f_*

$$h_*(t) = T(t)\ \phi(0) + \int_0^t T(t-s)\ f_*(s)\ ds,\ t \in J.$$

Consider the linear and continuous operator $K : L^1(J, E) \to BC$ defined by

$$K(v)(t) = \int_0^t T(t-s)v(s)ds.$$

We have

$$\begin{aligned} |K(f_n)(t) - K(f_*)(t)| &= \\ |(h_n(t) - T(t)\ \phi(0)) - (h_*(t) - T(t)\ \phi(0))| &= |h_n(t) - h_*(t)| \\ &\leq \|h_n - h_*\|_\infty \to 0,\ as\ n \to \infty. \end{aligned}$$

From Lemma 1.2.9 it follows that $K \circ S_F$ is a closed graph operator and from the definition of K has

$$h_n(t) - T(t)\phi(0) \in K \circ S_{F,y_n}.$$

As $y_n \to y_*$ and $h_n \to h_*$, there exist $f_* \in S_{F,y_*}$ such that:

$$h_*(t) - T(t)\phi(0) = \int_0^t T(t-s)\ f_*(s).$$

Hence the multivalued operator N has closed graph, which implies that it is upper semi-continuous.

Step 6: $N(B_r)$ is equiconvergent.

Let $h \in N(y)$, there exists $f \in S_{F,y}$ such that for each $t \in [0, +\infty)$ and $y \in B_r$ we have

$$\begin{aligned} |h(t)| &\leq M\|\phi\| + M\int_0^t |f(s)|ds \\ &\leq M\|\phi\| + Mk^* + Mr\int_0^t k(s)ds \\ &\leq M\|\phi\| + Mk^* + Mrk^*. \end{aligned}$$

Then,

$$|h(t)| \to l, \quad \text{as } \ t \to +\infty.$$

Where $l \leq M\|\phi\| + Mk^*(1+r)$ Hence,

$$|h(t) - h(+\infty)| \to 0, \quad \text{as } \ t \to +\infty.$$

As a consequence of Steps $1-6$, and Lemma 1.5.1, we conclude from Bohnenblust-Karlin's theorem that N has a fixed point y which is a mild solution of the problem (4.1)-(4.2).

4.2.2 An example

Consider the functional partial differential equation

$$\frac{\partial}{\partial t}z(t,x) - \frac{\partial^2}{\partial x^2}z(t,x) \in F(t, z(t-d,x)),\ x \in [0,\pi],\ t \in J := [0,+\infty), \tag{4.3}$$

$$z(t,0) = z(t,\pi) = 0,\ t \in J, \tag{4.4}$$

$$z(t,x) = \phi(t),\ t \in [-d,0],\ x \in [0,\pi], \tag{4.5}$$

where F is a given multivalued map. Take $E = L^2[0,\pi]$ and define $A : E \to E$ by $A\omega = \omega''$ with domain

$$D(A) = \{\omega \in E; \omega, \omega' \text{are absolutely continuous},\ \omega'' \in E,\ \omega(0) = \omega(\pi) = 0\}.$$

Then,

$$A\omega = \sum_{n=1}^{\infty} n^2(\omega, \omega_n)\omega_n,\ \omega \in D(A)$$

where $\omega_n(s) = \sqrt{\frac{2}{\pi}}\sin ns, n = 1, 2, \ldots$, is the orthogonal set of eigenvectors in A. It is well know (see [89]) that A is the infinitesimal generator of an analytic semigroup $T(t), t \geq 0$ in E and is given by

$$T(t)\omega = \sum_{n=1}^{\infty} \exp(-n^2 t)(\omega, \omega_n)\omega_n, \ \omega \in E.$$

Since the analytic semigroup $T(t)$ is compact, there exists a positive constant M such that

$$\|T(t)\|_{B(E)} \leq M.$$

Then the problem (4.1)-(4.2) is the abstract formulation of the problem (4.3)-(4.5). If conditions $(H_1)-(H_3)$ are satisfied, Theorem 4.2.2 implies that the problem (4.3)-(4.5) has at least one global mild solution on BC.

4.3 Functional differential inclusions with state-dependent delay

In this section, we are going to prove the existence of solutions of a functional differential inclusion . Our investigations will be situated in the Banach space of real functions which are defined, continuous and bounded on $\mathbb{R}$. We will use Bohnenblust-Karlin's fixed theorem, combined with the Corduneanu's compactness criteria. More precisely we will consider the following problem :

$$y'(t) - Ay(t) \in F(t, y_{\rho(t,y_t)}), \quad \text{a.e. } \ t \in J := [0, +\infty) \tag{4.6}$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0], \tag{4.7}$$

where $F : J \times \mathcal{B} \to \mathcal{P}(E)$ is a multivalued map with nonempty compact values, $\mathcal{P}(E)$ is the family of all nonempty subsets of E, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \in J$, and $(E, |.|)$ is a real Banach space. $\mathcal{B}$ is the phase space to be specified later, $\phi \in \mathcal{B}$, $\rho : J \times \mathcal{B} \to (-\infty, +\infty)$. For any function y defined on $(-\infty, +\infty)$ and any $t \in J$ we denote by y_t the element of $\mathcal{B}$ defined by

$$y_t(\theta) = y(t + \theta), \theta \in (-\infty, 0].$$

We assume that the histories y_t to some abstract phases $\mathcal{B}$, to be specified later.

By BUC we denote the space of bounded uniformly continuous functions defined from $(-\infty, 0]$ to E.

By $BC := BC(-\infty, +\infty)$ we denote the Banach space of all bounded and continuous functions from $(-\infty, +\infty)$ into E equipped with the standard norm

$$\|y\|_{BC} = \sup_{t \in (-\infty, +\infty)} |y(t)|.$$

And we denote by $BC' := BC'([0, +\infty))$ the Banach space of all bounded and continuous functions from $[0, +\infty)$ into E equipped with the standard norm

$$\|y\|_{BC'} = \sup_{t \in [0, +\infty)} |y(t)|.$$

4.3.1 Existence of mild solutions

Now we give our main existence result for problem (4.6)-(4.7). Before starting and proving this result, we give the definition of the mild solution.

Definition 4.3.1 *We say that a continuous function $y : (-\infty, +\infty) \to E$ is a mild solution of problem (4.6)-(4.7)if $y(t) = \phi(t)$ for all $t \in (-\infty, 0]$, and the restriction of $y(\cdot)$ to the interval J is continuous and there exists $f(\cdot) \in L^1(J; E) : f(t) \in F(t, y_{\rho(t, y_t)})$ a.e. in J such that y satisfies the following integral equation*

$$y(t) = T(t)\phi(t) - \int_0^t T(t-s)\ f(s)\ ds \qquad for\ each\ t \in J. \tag{4.8}$$

Let us introduce the following hypotheses:

(H_1) $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \in J$ which is compact for $t > 0$ in the Banach space E. Let $M' = \sup\{\|T\|_{B(E)} : t \geq 0\}$.

(H_2) The multifunction $F : J \times \mathcal{B} \longrightarrow \mathcal{P}(E)$ is Carathéodory with compact and convex values.

(H_3) There exists a continuous function $k : J \to [0, +\infty)$ such that:

$$H_d(F(t, u), F(t, v)) \leq k(t)\ \|u - v\|_{\mathcal{B}}$$

for each $t \in J$ and for all $u, v \in \mathcal{B}$ and

$$d(0, F(t, 0)) \leq k(t)$$

with

$$k^* := \sup_{t \in J} \int_0^t k(s) ds < \infty.$$

Theorem 4.3.2 *Assume that $(H_1)-(H_3), (H_\phi)$ hold. If $k^*M'L < 1$, then the problem (4.6)-(4.7) has at least one mild solution on BC.*

Proof. Transform the problem (4.6)-(4.7) into a fixed point problem. Consider the multivalued operator $N : BC \to \mathcal{P}(BC)$ defined by :

$$N(y) := \left\{ h \in BC : \ h(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0]; \\ T(t)\ \phi(0) + \displaystyle\int_0^t T(t-s)\ f(s)\ ds, & \text{if } t \in J, \end{cases} \right\}$$

where $f \in S_{F,y_{\rho(s,y_s)}}$.

Let $x(\cdot) : (-\infty, +\infty) \to E$ be the function defined by:

$$x(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0]; \\ T(t)\ \phi(0), & \text{if } t \in J. \end{cases}$$

Then $x_0 = \phi$. For each $z \in BC$ with $z(0) = 0$, we denote by $\overline{z}$ the function

$$\overline{z}(t) = \begin{cases} 0, & \text{if } t \in (-\infty, 0]; \\ z(t), & \text{if } t \in J, \end{cases}$$

if $y(\cdot)$ satisfies (4.8), we can decompose it as $y(t) = z(t) + x(t),\ t \in J$, which implies $y_t = z_t + x_t$ for every $t \in J$ and the function $z(\cdot)$ satisfies

$$z(t) = \int_0^t T(t-s)f(s)ds, \quad t \in J,$$

where $f \in S_{F,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}}$.

Set

$$BC'_0 = \{z \in BC' : z(0) = 0\}$$

and let

$$\|z\|_{BC'_0} = \sup\{|z(t)| : t \in J\},\ z \in BC'_0.$$

BC'_0 is a Banach space with the norm $\|\cdot\|_{BC'_0}$.

We define the operator $\mathcal{A} : BC'_0 \to \mathcal{P}(BC'_0)$ by:

$$\mathcal{A}(z) := \left\{ h \in BC'_0 : \ h(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ \displaystyle\int_0^t T(t-s)\ f(s)\ ds, & \text{if } t \in J, \end{cases} \right\}$$

where $f \in S_{F,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}}$.

The operator A maps BC'_0 into BC'_0, indeed the map $\mathcal{A}(z)$ is continuous on $[0, +\infty)$ for any $z \in BC'_0$, $h \in \mathcal{A}(z)$ and for each $t \in J$ we have

$$\begin{aligned}
|h(t)| &\leq M' \int_0^t |f(s)| ds \\
&\leq M' \int_0^t (k(s)\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} + |F(s,0)|) ds \\
&\leq M' \int_0^t k(s) ds + M' \int_0^t k(s)(L|z(s)| + (M + \mathcal{L}^\phi + LM'H)\|\phi\|_{\mathcal{B}}) ds \\
&\leq M'k^* + M' \int_0^t k(s)(L|z(s)| + (M + \mathcal{L}^\phi + LM'H)\|\phi\|_{\mathcal{B}}) ds.
\end{aligned}$$

Set

$$C := (M + \mathcal{L}^\phi + LM'H)\|\phi\|_{\mathcal{B}}.$$

Then, we have

$$\begin{aligned}
|h(t)| &\leq M'k^* + M'C \int_0^t k(s) ds + M' \int_0^t L|z(s)|k(s) ds \\
&\leq M'k^* + M'Ck^* + M'L\|z\|_{BC'_0} k^*.
\end{aligned}$$

Hence, $\mathcal{A}(z) \in BC'_0$.

Moreover, let $r > 0$ be such that

$$r \geq \frac{M'k^* + M'Ck^*}{1 - M'k^*L},$$

and B_r be the closed ball in BC'_0 centered at the origin and of radius r. Let $z \in B_r$ and $t \in [0, +\infty)$. Then

$$|h(t)| \leq \quad M'k^* + M'Ck^* + M'k^*Lr.$$

Thus

$$\|h\|_{BC'_0} \leq r,$$

which means that the operator $\mathcal{A}$ transforms the ball B_r into itself.

Now we prove that $\mathcal{A} : B_r \to \mathcal{P}(B_r)$ satisfies the assumptions of Bohnenblust-Karlin's fixed theorem. The proof will be given in several steps.

Step 1 We shall show that the operator $\mathcal{A}$ is closed and convex. This will be given in several claims.

Claim 1 : $\mathcal{A}(z)$ is closed for each $z \in B_r$.

Let $(h_n)_{n\geq 0} \in \mathcal{A}(z)$ such that $h_n \to \tilde{h}$ in B_r. Then for $h_n \in B_r$ there exists $f_n \in S_{F,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}}$ such that for each $t \in J$,

$$h_n(t) = \int_0^t T(t-s)f_n(s)ds.$$

Using the fact that F has compact values and from hypotheses $(H_2), (H_3)$ we may pass a subsequence if necessary to get that f_n converges to $f \in L^1(J, E)$ and hence $f \in S_{F,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}}$. Then for each $t \in J$,

$$h_n(t) \to \tilde{h}(t) = \int_0^t T(t-s)f(s)ds.$$

So, $\tilde{h} \in \mathcal{A}(z)$.

Claim 2 : $\mathcal{A}(z)$ is convex for each $z \in B_r$.
Let $h_1, h_2 \in \mathcal{A}(z)$, the there exists $f_1, f_2 \in S_{F,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}}$ such that, for each $t \in J$ we have :

$$h_i(t) = \int_0^t T(t-s)f_i(s)ds, i = 1, 2.$$

Let $0 \leq \delta \leq 1$. Then, we have for each $t \in J$:

$$(\delta h_1 + (1-\delta)h_2)(t) = \int_0^t T(t-s)[\delta f_1(s) + (1-\delta)f_2(s)]ds.$$

Since F has convex values, one has

$$\delta h_1 + (1-\delta)h_2 \in \mathcal{A}(z)$$

Step 2 : $\mathcal{A}(B_r) \subset B_r$ this is clear.

Step 3: $\mathcal{A}(B_r)$ is equicontinuous on every compact interval $[0,b]$ of $[0,+\infty)$ for $b>0$. Let $\tau_1,\tau_2\in[0,b], h\in\mathcal{A}(z)$ with $\tau_2>\tau_1$, we have:

$$\begin{aligned}
|h(\tau_2)-h(\tau_1)| &\leq \int_0^{\tau_1}\|T(\tau_2-s)-T(\tau_1-s)\|_{B(E)}|f(s)|ds\\
&+ \int_{\tau_1}^{\tau_2}\|T(\tau_2-s)\|_{B(E)}|f(s)|ds\\
&\leq \int_0^{\tau_1}\|T(\tau_2-s)-T(\tau_1-s)\|_{B(E)}(k(s)\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}+|F(s,0)|)ds\\
&+ \int_{\tau_1}^{\tau_2}\|T(\tau_2-s)\|_{B(E)}(k(s)\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}+|F(s,0)|)ds\\
&\leq C\int_0^{\tau_1}\|T(\tau_2-s)-T(\tau_1-s)\|_{B(E)}k(s)ds\\
&+ rL\int_0^{\tau_1}\|T(\tau_2-s)-T(\tau_1-s)\|_{B(E)}k(s)ds\\
&+ \int_0^{\tau_1}\|T(\tau_2-s)-T(\tau_1-s)\|_{B(E)}k(s)ds\\
&+ C\int_{\tau_1}^{\tau_2}\|T(\tau_2-s)\|_{B(E)}k(s)ds\\
&+ rL\int_{\tau_1}^{\tau_2}\|T(\tau_2-s)\|_{B(E)}k(s)ds\\
&+ \int_{\tau_1}^{\tau_2}\|T(\tau_2-s)\|_{B(E)}k(s)ds.
\end{aligned}$$

When $\tau_2\to\tau_2$, the right-hand side of the above inequality tends to zero, since $T(t)$ is a strongly continuous operator and the compactness of $T(t)$ for $t>0$, implies he continuity in the uniform operator topology (see [89]), this proves the equicontinuity.

Step 4: $\mathcal{A}(B_r)$ is relatively compact on every compact interval of $[0,\infty)$.

Let $t\in[0,b]$ for $b>0$ and let ε be a real number satisfying $0<\varepsilon<t$. For $z\in B_r$ we define

$$h_\varepsilon(t)=T(\varepsilon)\int_0^{t-\varepsilon}T(t-s-\varepsilon)f(s)ds.$$

Note that the set

$$\left\{\int_0^{t-\varepsilon}T(t-s-\varepsilon)f(s)ds : z\in B_r\right\}$$

is bounded.

$$|\int_0^{t-\varepsilon}T(t-s-\varepsilon)f(s)ds|\leq r.$$

Since $T(t)$ is a compact operator for $t>0$, the set,

$$\{h_\varepsilon(t) : z\in B_r\}$$

is precompact in E for every ε, $0 < \varepsilon < t$. Moreover, for every $z \in B_r$ we have

$$\begin{aligned}
& |h(t) - h_\varepsilon(t)| \\
\leq\ & M' \int_{t-\varepsilon}^{t} |f(s)| ds \\
\leq\ & M' \int_{t-\varepsilon}^{t} k(s) ds + M'C \int_{t-\varepsilon}^{t} k(s) ds + rM' \int_{t-\varepsilon}^{t} Lk(s) ds, \\
& \to 0 \quad \text{as} \quad \varepsilon \to 0.
\end{aligned}$$

Therefore, the set $\{h(t) : z \in B_r\}$ is precompact, i.e., relatively compact.

Step 5: $\mathcal{A}$ has closed graph.
Let $\{z_n\}$ be a sequence such that $z_n \to z_*, h_n \in \mathcal{A}(z_n)$ and $h_n \to h_*$. We shall show that $h_* \in \mathcal{A}(z_*)$.

$h_n \in \mathcal{A}(z_n)$ means that there exists $f_n \in S_{F, z^n_{\rho(s, z^n_s + x_s)} + x_{\rho(s, z^n_s + x_s)}}$ such that

$$h_n(t) = \int_0^t T(t-s)\ f_n(s)\ ds,$$

we must prove that there exists f_*

$$h_*(t) = \int_0^t T(t-s)\ f_*(s)\ ds.$$

Consider the linear and continuous operator $K : L^1(J, E) \to B_r$ defined by

$$K(v)(t) = \int_0^t T(t-s) v(s) ds.$$

we have

$$|K(f_n)(t) - K(f_*)(t)| = |h_n(t) - h_*(t)| \leq \|h_n - h_*\|_\infty \to 0, as\ n \to \infty$$

From Lemma 1.2.9 it follows that $K \circ S_F$ is a closed graph operator and from the definition of K has

$$h_n(t) \in K \circ S_{F, z^n_{\rho(s, z^n_s + x_s)} + x_{\rho(s, z^n_s + x_s)}}.$$

As $z_n \to z_*$ and $h_n \to h_*$, there exist $f_* \in S_{F, z^*_{\rho(s, z^*_s + x_s)} + x_{\rho(s, z^* + x_s)}}$ such that:

$$h_*(t) = \int_0^t T(t-s)\ f_*(s) ds.$$

Hence the mutivalued operator $\mathcal{A}$ is upper semi-continuous.

Step 5: $\mathcal{A}(B_r)$ is equiconvergent.
Let $z \in B_r$, we have, for $h \in \mathcal{A}(z)$:

$$\begin{aligned}
|h(t)| &\leq M' \int_0^t |f(s)| ds \\
&\leq M'k^* + M'C \int_0^t k(s)ds + M'r \int_0^t Lk(s)ds \\
&\leq M'k^* + M'C \int_0^t k(s)ds + M'rL \int_0^t k(s)ds.
\end{aligned}$$

Then by (4), we have

$$|h(t)| \to l, \quad \text{as} \ \ t \to +\infty.$$

Where $l \leq M'k^*(1 + C + rL)$ Hence,

$$|h(t) - h(+\infty)| \to 0, \quad \text{as} \ \ t \to +\infty.$$

As a consequence of Steps 1-4, with Lemma 1.5.1, we can conclude that $\mathcal{A} : B_r \to \mathcal{P}(B_r)$ is continuous and compact. From Schauder's theorem, we deduce that $\mathcal{A}$ has a fixed point z^*. Then $y^* = z^* + x$ is a fixed point of the operators N, which is a mild solution of the problem (4.6)-(4.7).

4.3.2 An example

Consider the following functional partial differential equation

$$\frac{\partial}{\partial t} z(t,x) - \frac{\partial^2}{\partial x^2} z(t,x) \in F(t, z(t - \sigma(t, z(t,0)), x)) \tag{4.9}$$

$$x \in [0, \pi], \ \ t \in [0, +\infty)$$

$$z(t, 0) = z(t, \pi) = 0, \ \ t \in [0, +\infty), \tag{4.10}$$

$$z(\theta, x) = z_0(\theta, x), \ \ t \in (-\infty, 0], \ \ x \in [0, \pi], \tag{4.11}$$

where F is a given multivalued map, and $\sigma : \mathbb{R} \to \mathbb{R}^+$ is continuous.
Take $E = L^2[0, \pi]$ and define $A : E \to E$ by $A\omega = \omega''$ with domain

$$D(A) = \{\omega \in E, \omega, \omega' \text{are absolutely continuous}, \ \ \omega'' \in E, \ \ \omega(0) = \omega(\pi) = 0\}.$$

Then

$$A\omega = \sum_{n=1}^{\infty} n^2 (\omega, \omega_n)\omega_n, \omega \in D(A)$$

where $\omega_n(s) = \sqrt{\frac{2}{\pi}} \sin ns, n = 1, 2, \ldots$ is the orthogonal set of eigenvectors in A. It is well know (see [89]) that A is the infinitesimal generator of an analytic semigroup $T(t), t \geq 0$ in E and is given by

$$T(t)\omega = \sum_{n=1}^{\infty} \exp(-n^2 t)(\omega, \omega_n)\omega_n, \omega \in E.$$

Since the analytic semigroup $T(t)$ is compact, there exists a positive constant M such that

$$\|T(t)\|_{B(E)} \leq M.$$

Let $\mathcal{B} = BCU(\mathrm{I\!R}^{-}; E)$ and $\phi \in \mathcal{B}$, then (H_ϕ), where $\rho(t, \varphi) = t - \sigma(\varphi)$.

Then the problem (4.6)-(4.7) in an abstract formulation of the problem (4.9)-(4.11), and if the conditions $(H_1) - (H_3), (H_\phi)$ are satisfied. Theorem 4.3.2 implies that the problem (4.9)-(4.11) has at least one mild solutions on BC .

Chapter 5

Neutral Functional Differential Inclusions with State-Dependent Delay

5.1 Introduction

In this Chapter, we study some first order classes of partial neutral functional evolution inclusion with infinite state-dependent delay, Our investigations will be situated in the Banach space of real functions which are defined, continuous and bounded on $\mathbb{R}$.

Neutral functional differential equations arise in many areas of applied mathematics and for this reason these equations have received much attention in the last decades. The literature relative to ordinary neutral functional differential equations is very extensive and refer to [14, 15, 26, 28, 84, 85]. Partial neutral differential equation with finite delay arise, for instance, from the transmission line theory [99]. Wu and Xia have shown in [100] that a ring array of identical resistibly coupled lossless transmission lines leads to a system of neutral functional differential equations with discrete diffusive coupling which exhibits various types of discrete waves. For more results on partial neutral functional-differential equations and related issues we refer to Adimy and Ezzinbi [2], Hale [56], Wu and Xia [99, 100] for finite delay equations, and Hern'andez and Henriquez [66, 67] for unbounded delays. We also cite [9, 78, 38, 48, 69, 83, 101] for the case neutral differential equations with State-dependent delay.

We will use Bohnenblust-Karlin's fixed theorem, combined with the Corduneanu's compactness criteria, for solution of the following problem :

$$\frac{d}{dt}[y(t) - g(t, y_{\rho(t,y_t)})] - A[y(t) - g(t, y_{\rho(t,y_t)})] \in F(t, y_{\rho(t,y_t)}), \quad \text{a.e. } t \in J := [0, +\infty) \tag{5.1}$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0], \tag{5.2}$$

where $F : J \times \mathcal{B} \to \mathcal{P}(E)$ is a multivalued map with nonempty compact values, $\mathcal{P}(E)$ is the family of all nonempty subsets of E, $g : J \times \mathcal{B} \to E$ is given function, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \in J$, $\mathcal{B}$ is the phase space to be specified later, $\phi \in \mathcal{B}$, $\rho : J \times \mathcal{B} \to (-\infty, +\infty)$ and $(E, |.|)$ is a real Banach space. For any function y defined on $(-\infty, +\infty)$ and any $t \in J$ we denote by y_t the element of $\mathcal{B}$ defined by

$$y_t(\theta) = y(t + \theta), \theta \in (-\infty, 0].$$

We assume that the histories y_t belongs to some abstract phases $\mathcal{B}$, to be specified later.

By BUC we denote the space of bounded uniformly continuous functions defined from $(-\infty, 0]$ to E.
By $BC := BC(-\infty, +\infty)$ we denote the Banach space of all bounded and continuous functions from $(-\infty, +\infty)$ into E equipped with the standard norm

$$\|y\|_{BC} = \sup_{t \in (-\infty, +\infty)} |y(t)|.$$

Finally, by $BC' := BC'([0, +\infty))$ we denote the Banach space of all bounded and continuous functions from $[0, +\infty)$ into E equipped with the standard norm

$$\|y\|_{BC'} = \sup_{t \in [0, +\infty)} |y(t)|.$$

5.2 Existence of mild solutions

Now we give our main existence result for problem (5.1)-(5.2). Before starting and proving this result, we give the definition of the mild solution.

Definition 5.2.1 *We say that a continuous function $y : (-\infty, +\infty) \to E$ is a mild solution of problem (5.1)-(5.2)if $y(t) = \phi(t)$ for all $t \in (-\infty, 0]$, and the restriction of $y(\cdot)$ to the interval J is continuous and there exists $f(\cdot) \in L^1(J; E) : f(t) \in F(t, y_{\rho(t, y_t)})$ a.e. in J such that y satisfies the following integral equation*

$$y(t) = T(t)[\phi(0) - g(0, \phi(0))] + g(t, y_{\rho(t, y_t)}) + \int_0^t T(t - s) f(s) ds, \ t \in J. \tag{5.3}$$

Let us introduce the following hypotheses:

(H_1) $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \in J$ which is compact for $t > 0$ in the Banach space E. Let $M' = \sup\{\|T\|_{B(E)} : t \geq 0\}$.

(H_2) The multifunction $F : J\times\mathcal{B} \longrightarrow \mathcal{P}(E)$ is Carathéodory with compact and convex, closed values.

(H_3) There exists a continuous function $k : J \to [0, +\infty)$ such that:

$$H_d(F(t,u), F(t,v)) \leq k(t)\ \|u - v\|_{\mathcal{B}}$$

for each $t \in J$ and for all $u, v \in \mathcal{B}$ and

$$d(0, F(t,0)) \leq k(t)$$

with

$$k^* := \sup_{t\in J} \int_0^t k(s)ds < \infty.$$

(H_4) The function $g(t,\cdot)$ is continuous on J ant there exists a constant $k_g > 0$ such that

$$|g(t,u) - g(t,v)| \leq k_g\|u - v\|_{\mathcal{B}}, \text{for each},\ u, v \in \mathcal{B}$$

and

$$g^* := \sup_{t\in J} |g(t,0)| < \infty.$$

(H_6) For each $t \in J$ and any bounded set $B \subset \mathcal{B}$, the set $\{g(t,u) :\ u \in B\}$ is relatively compact in E

(H_7) For any bounded set $B \subset \mathcal{B}$, the function $\{t \to g(t, y_t) : y \in B\}$ is equicontinuous on each compact interval of $[0, +\infty)$.

Remark 5.2.2 *By the condition (H_4) we deduce that*

$$|g(t,u)| \leq k_g\|u\|_{\mathcal{B}} + g^*,\ \ t \in J,\ \ u \in \mathcal{B}.$$

Theorem 5.2.3 *Assume that $(H_1) - (H_7)$ and (H_ϕ) hold. If $l(M'k^* + k_g) < 1$, then the problem (5.1)-(5.2) has at least one mild solution on BC.*

Proof. Transform the problem (5.1)-(5.2) into a fixed point problem. Consider the operator $N : BC \to \mathcal{P}(BC)$ defined by:

$$N(y) := \left\{ h \in BC :\ h(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0]; \\ T(t)\ [\phi(0) - g(0, \phi(0))] & \\ +g(t, y_{\rho(t,y_t)}) + \displaystyle\int_0^t T(t-s)\ f(s)\ ds, & \text{if } t \in J, \end{cases} \right\}$$

where $f \in S_{F,y_{\rho(t,y_t)}}$.

Let $x(\cdot) : (-\infty, +\infty) \to E$ be the function defined by:

$$x(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0]; \\ T(t)\ \phi(0), & \text{if } t \in J. \end{cases}$$

Then $x_0 = \phi$. For each $z \in BC$ with $z(0) = 0$, we denote by $\overline{z}$ the function

$$\overline{z}(t) = \begin{cases} 0, & \text{if } t \in (-\infty, 0]; \\ z(t), & \text{if } t \in J, \end{cases}$$

if $y(\cdot)$ satisfies (5.3), we can decompose it as $y(t) = z(t) + x(t),\ t \in J$, which implies $y_t = z_t + x_t$ for every $t \in J$ and the function $z(\cdot)$ satisfies

$$\begin{aligned} z(t) &= g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) - T(t)g(0, \phi(0)) \\ &\quad + \int_0^t T(t-s)f(s)ds, t \in J. \end{aligned}$$

where $f \in S_{F,z_{\rho(t,z_t+x_t)}+x_{\rho(t,z_t+x_t)}}$.

Set

$$BC_0' = \{z \in BC' : z(0) = 0\}$$

and let

$$\|z\|_{BC_0'} = \sup\{|z(t)| : t \in J\},\ z \in BC_0'.$$

BC_0' is a Banach space with the norm $\|\cdot\|_{BC_0'}$.
We define the operator $\mathcal{A} : BC_0' \to \mathcal{P}(BC_0')$ by:

$$\mathcal{A}(z) := \left\{ h \in BC_0' :\ h(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) - T(t)g(0, \phi(0)) & \\ + \displaystyle\int_0^t T(t-s)\ f(s)\ ds, & \text{if } t \in J, \end{cases} \right\}$$

where $f \in S_{F,z_{\rho(t,z_t+x_t)}+x_{\rho(t,z_t+x_t)}}$.

. The operator A maps BC_0' into BC_0', indeed the map $\mathcal{A}(z)$ is continuous on $[0, +\infty)$ for any $z \in BC_0'$, $h \in \mathcal{A}(z)$ and for each $t \in J$ we have

$$\begin{aligned} |h(t)| &\leq |g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)})| + M'|g(0, \phi(0))| \\ &\quad + M'\int_0^t |f(s)|ds \\ &\leq M'(k_g\|\phi\|_{\mathcal{B}} + g^*) + k_g\|z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\|_{\mathcal{B}} + g^* \\ &\quad + M'\int_0^t |F(s, 0)|ds + M'\int_0^t k(s)\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}ds \\ &\leq M'(k_g\|\phi\|_{\mathcal{B}} + g^*) + k_g(l|z(t)| + (m + \mathcal{L}^\phi + lM'H)\|\phi\|_{\mathcal{B}}) + g^* \\ &\quad + M'k^* + M'\int_0^t k(s)(l|z(s)| + (m + \mathcal{L}^\phi + lM'H)\|\phi\|_{\mathcal{B}})ds. \end{aligned}$$

Set

$$C_1 := (m + \mathcal{L}^{\phi} + lM'H)\|\phi\|_{\mathcal{B}}.$$
$$C_2 := M'(k_g\|\phi\|_{\mathcal{B}} + g^*) + k_gC_1 + g^* + M'k^*.$$

Then, we have

$$\begin{aligned}|h(t)| &\leq C_2 + k_gl|z(t)| + M'C_1\int_0^t k(s)ds + M'\int_0^t l|z(s)|k(s)ds \\ &\leq C_2 + k_gl\|z\|_{BC_0'} + M'C_1k^* + M'l\|z\|_{BC_0'}k^*.\end{aligned}$$

Hence, $\mathcal{A}(z) \in BC_0'$.

Moreover, let $r > 0$ be such that

$$r \geq \frac{C_2 + M'C_1k^*}{1 - l(M'k^* + k_g)},$$

and B_r be the closed ball in BC_0' centered at the origin and of radius r. Let $z \in B_r$ and $t \in [0, +\infty)$. Then

$$|h(t)| \leq \quad C_2 + k_glr + M'C_1k^* + M'k^*lr.$$

Thus

$$\|h\|_{BC_0'} \leq r,$$

which means that the operator $\mathcal{A}$ transforms the ball B_r into itself.

Now we prove that $\mathcal{A} : B_r \to \mathcal{P}(B_r)$ satisfies the assumptions of Bohnenblust-Karlin's fixed theorem. The proof will be given in several steps.

Step 1 We shall show that the operator $\mathcal{A}$ is closed and convex. This will be given in several claims.

Claim 1 : $\mathcal{A}(z)$ is closed for each $z \in B_r$.

Let $(h_n)_{n\geq 0} \in \mathcal{A}(z)$ such that $h_n \to \tilde{h}$ in B_r. Then for $h_n \in B_r$ there exists $f_n \in S_{F,z_{\rho(t,z_t+x_t)}+x_{\rho(t,z_t+x_t)}}$ such that for each $t \in J$,

$$h_n(t) = g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) - T(t)g(0, \phi(0)) + \int_0^t T(t-s)f_n(s)ds.$$

Using the fact that F has compact values and from hypotheses $(H_2), (H_3)$. An application of Mazur's lemma 1.2.8 we may pass a subsequence if necessary to get that f_n converges to $f \in L^1(J, E)$ and hence $f \in S_{F,y}$.
It remains to prove that $f \in F(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)})$, for each $t \in J$. Lemma 1.2.8 yields the existence of $\alpha_i^n \geq 0, i = n, ..., k-n)$ such that $\sum_{i=1}^{k(n)} \alpha_i^n = 1$ and the sequence

of convex combinations $g_n(\cdot) = \sum_{i=1}^{k(n)} \alpha_i^n f_i(\cdot)$ converges strongly to $f \in L^1$. Since F takes convex values, using Lemma 1.2.7, we obtain that

$$
\begin{aligned}
f(t) \in \bigcap_{n\geq 1} \{\bar{g_n}(t)\}, a.e. t \in J \subset \bigcap_{n\geq 1} \bar{co}\{f_k(t), k \geq n\} \subset \\
\subset \bigcap_{n\geq 1} \bar{co}\{\bigcup_{k\geq n} F(t, z^k_{\rho(z^k_t+x_t)} + x_{\rho(t,z^k_t+x_t)})\} = \qquad (5.4)\\
= \bar{co}(\lim_{k\to\infty} \sup F(t, z^k_{\rho(z^k_t+x_t)} + x_{\rho(t,z^k_t+x_t)}).
\end{aligned}
$$

Since F is u.s.c with compact values, then by Lemma 1.2.6, we have

$$
\lim_{n\to\infty} \sup F(t, z^n_{\rho(z^n_t+x_t)} + x_{\rho(t,z^n_t+x_t)}) = F(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}), \text{for a.e. } t \in J
$$

This with (5.4) imply that $f(t) \in \bar{co}F(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)})$. Since $F(\cdot,\cdot)$ has closed, convex values, we deduce that $f(t) \in F(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)})$ for a.e. $t \in J$.
Then for each $t \in J$, $f \in S_{F,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}}$. Then for each $t \in J$,

$$
h_n(t) \to \tilde{h}(t) = g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) - T(t)g(0,\phi(0)) + \int_0^t T(t-s)f(s)ds.
$$

So, $\tilde{h} \in \mathcal{A}(z)$.

Claim 2 : $\mathcal{A}(z)$ is convex for each $z \in B_r$.
Let $h_1, h_2 \in \mathcal{A}(z)$, the there exists $f_1, f_2 \in S_{F,z_{\rho(t,z_t+x_t)}+x_{\rho(t,z_t+x_t)}}$ such that, for each $t \in J$ we have :

$$
h_i(t) = g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) - T(t)g(0,\phi(0)) + \int_0^t T(t-s)f_i(s)ds, i = 1, 2.
$$

Let $0 \leq \delta \leq 1$. Then, we have for each $t \in J$:

$$
(\delta h_1+(1-\delta)h_2)(t) = g(t, z_{\rho(t,z_t+x_t)}+x_{\rho(t,z_t+x_t)})-T(t)g(0,\phi(0))+\int_0^t T(t-s)[\delta f_1(s)+(1-\delta)f_2(s)]ds.
$$

Since F has convex values, one has

$$
\delta h_1 + (1-\delta)h_2 \in \mathcal{A}(z)
$$

Step 2 : $\mathcal{A}(B_r) \subset B_r$ this is clear.

Step 3: $\mathcal{A}(B_r)$ is equicontinuous on every compact interval $[0,b]$ of $[0,+\infty)$ for $b>0$. Let $\tau_1,\tau_2\in[0,b], h\in\mathcal{A}(z)$ with $\tau_2>\tau_1$, we have:

$$\begin{aligned}
|h(\tau_2)-h(\tau_1)| &\leq |g(\tau_2, z_{\rho(\tau_2,z_{\tau_2}+x_{\tau_2})}+x_{\rho(\tau_2,z_{\tau_2}+x_{\tau_2})})-g(\tau_1, z_{\rho(\tau_1,z_{\tau_1}+x_{\tau_1})}+x_{\rho(\tau_1,z_{\tau_1}+x_{\tau_1})})| \\
&+ \|T(\tau_2)-T(\tau_1)\|_{B(E)}|g(0,\phi(0))| \\
&+ \int_0^{\tau_1}\|T(\tau_2-s)-T(\tau_1-s)\|_{B(E)}|f(s)|ds \\
&+ \int_{\tau_1}^{\tau_2}\|T(\tau_2-s)\|_{B(E)}|f(s)|ds \\
&\leq |g(\tau_2, z_{\rho(\tau_2,z_{\tau_2}+x_{\tau_2})}+x_{\rho(\tau_2,z_{\tau_2}+x_{\tau_2})})-g(\tau_1, z_{\rho(\tau_1,z_{\tau_1}+x_{\tau_1})}+x_{\rho(\tau_1,z_{\tau_1}+x_{\tau_1})})| \\
&+ \|T(\tau_2)-T(\tau_1)\|_{B(E)}(k_g\|\phi\|_{\mathcal{B}}+g^*) \\
&+ \int_0^{\tau_1}\|T(\tau_2-s)-T(\tau_1-s)\|_{B(E)}(k(s)\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}+|F(s,0)|)ds \\
&+ \int_{\tau_1}^{\tau_2}\|T(\tau_2-s)\|_{B(E)}(k(s)\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}+|F(s,0)|)ds \\
&\leq |g(\tau_2, z_{\rho(\tau_2,z_{\tau_2}+x_{\tau_2})}+x_{\rho(\tau_2,z_{\tau_2}+x_{\tau_2})})-g(\tau_1, z_{\rho(\tau_1,z_{\tau_1}+x_{\tau_1})}+x_{\rho(\tau_1,z_{\tau_1}+x_{\tau_1})})| \\
&+ C_1\int_0^{\tau_1}\|T(\tau_2-s)-T(\tau_1-s)\|_{B(E)}k(s)ds \\
&+ rl\int_0^{\tau_1}\|T(\tau_2-s)-T(\tau_1-s)\|_{B(E)}k(s)ds \\
&+ \int_0^{\tau_1}\|T(\tau_2-s)-T(\tau_1-s)\|_{B(E)}k(s)ds \\
&+ C_1\int_{\tau_1}^{\tau_2}\|T(\tau_2-s)\|_{B(E)}k(s)ds \\
&+ rl\int_{\tau_1}^{\tau_2}\|T(\tau_2-s)\|_{B(E)}k(s)ds \\
&+ \int_{\tau_1}^{\tau_2}\|T(\tau_2-s)\|_{B(E)}k(s)ds.
\end{aligned}$$

When $\tau_2\to\tau_1$, the right-hand side of the above inequality tends to zero, since (H_7) and $T(t)$ is a strongly continuous operator and the compactness of $T(t)$ for $t>0$, implies the continuity in the uniform operator topology (see [89]), this proves the equicontinuity.

Step 4: $\mathcal{A}(B_r)$ is relatively compact on every compact interval of $[0,\infty)$.

Let $t\in[0,b]$ for $b>0$ and let ε be a real number satisfying $0<\varepsilon<t$. For $z\in B_r$ we define

$$h_\varepsilon(t)=g(t,z_{\rho(t,z_t+x_t)}+x_{\rho(t,z_t+x_t)}))-T(\varepsilon)(T(t-\varepsilon)g(0,\phi(0)))$$

$$+T(\varepsilon)\int_0^{t-\varepsilon}T(t-s-\varepsilon)f(s)ds.$$

Note that the set

$$\{g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) - T(t-\varepsilon)g(0,\phi(0))$$
$$+ \int_0^{t-\varepsilon} T(t-s-\varepsilon)f(s)ds : z \in B_r\}$$

is bounded.

$$|g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) - T(t-\varepsilon)g(0,\phi(0))$$
$$+ \int_0^{t-\varepsilon} T(t-s-\varepsilon)f(s)ds| \leq r.$$

Since $T(t)$ is a compact operator for $t > 0$,and (H_6) we have that the set,

$$\{h_\varepsilon(t) : z \in B_r\}$$

is precompact in E for every ε, $0 < \varepsilon < t$. Moreover, for every $z \in B_r$ we have

$$\begin{aligned}
& |h(t) - h_\varepsilon(t)| \\
\leq\ & M' \int_{t-\varepsilon}^{t} |f(s)|ds \\
\leq\ & M' \int_{t-\varepsilon}^{t} k(s)ds + M'C_1 \int_{t-\varepsilon}^{t} k(s)ds + rM' \int_{t-\varepsilon}^{t} lk(s)ds, \\
& \to 0 \quad \text{as} \quad \varepsilon \to 0.
\end{aligned}$$

Therefore, the set $\{h(t) : z \in B_r\}$ is precompact, i.e., relatively compact.

Step 5: $\mathcal{A}$ has closed graph.
Let $\{z_n\}$ be a sequence such that $z_n \to z_*, h_n \in \mathcal{A}(z_n)$ and $h_n \to h_*$. We shall show that $h_* \in \mathcal{A}(z_*)$.

$h_n \in \mathcal{A}(z_n)$ means that there exists $f_n \in S_{F,z^n_{\rho(t,z^n_t+x_t)}+x_{\rho(t,z^n_t+x_t)}}$ such that

$$h_n(t) = g(t, z^n_{\rho(t,z^n_t+x_t)} + x_{\rho(t,z^n_t+x_t)}) - T(t)g(0,\phi(0))$$
$$+ \int_0^t T(t-s)\ f_n(s)\ ds,$$

we must prove that there exists f_*

$$h_*(t) = g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) - T(t)g(0,\phi(0))$$
$$+ \int_0^t T(t-s)\ f_*(s)\ ds.$$

Consider the linear and continuous operator $K : L^1(J,E) \to B_r$ defined by

$$K(v)(t) = g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) - T(t)g(0,\phi(0))$$

$$+\int_0^t T(t-s)v(s)ds.$$

we have

$$\begin{array}{rcl}|K(f_n)(t)-K(f_*)(t)| & = & |(h_n(t)-g(t,z_{\rho(t,z_t+x_t)}+x_{\rho(t,z_t+x_t)})+T(t)g(0,\phi(0))) \\ & & -(h_*(t)-g(t,z_{\rho(t,z_t+x_t)}+x_{\rho(t,z_t+x_t)})+T(t)g(0,\phi(0)))| \\ & \leq & \|h_n-h_*\|_\infty \to 0, as\ n\to\infty\end{array}$$

From Lemma 1.2.9 it follows that $K\circ S_F$ is a closed graph operator and from the definition of K has

$$h_n(t)\in K\circ S_{F,z^n_{\rho(t,z^n_t+x_t)}+x_{\rho(t,z^n_t+x_t)}}.$$

As $z_n\to z_*$ and $h_n\to h_*$, there exist $f_*\in S_{F,z^*_{\rho(t,z^*_t+x_t)}+x_{\rho(t,z^*+x_t)}}$ such that:

$$h_*(t)=g(t,z_{\rho(t,z_t+x_t)}+x_{\rho(t,z_t+x_t)})-T(t)g(0,\phi(0))$$

$$+\int_0^t T(t-s)\ f_*(s)ds.$$

Hence the mutivalued operator $\mathcal{A}$ is upper semi-continuous.

Step 5: $\mathcal{A}(B_r)$ is equiconvergent.
Let $z\in B_r$, we have, for $h\in\mathcal{A}(z)$:

$$\begin{array}{rcl}|h(t)| & \leq & |g(t,z_{\rho(t,z_t+x_t)}+x_{\rho(t,z_t+x_t)})|+M'|g(0,\phi(0))| \\ & & +M'\int_0^t |f(s)|ds \\ & \leq & M'(k_g\|\phi\|_{\mathcal{B}}+g^*)+k_g(l|z(t)|+(m+\mathcal{L}^\phi+lM'H)\|\phi\|_{\mathcal{B}})+g^* \\ & & +M'k^*+M'\int_0^t k(s)(l|z(s)|+(m+\mathcal{L}^\phi+lM'H)\|\phi\|_{\mathcal{B}})ds \\ & \leq & C_2+k_gl\|z\|_{BC'_0}+M'C_1k^*+M'l\|z\|_{BC'_0}k^*.\end{array}$$

Then we have

$$|h(t)|\to l,\quad \text{as}\ \ t\to+\infty.$$

Where $l\leq C_2+k_glr+M'C_1k^*+M'lrk^*$ Hence,

$$|h(t)-h(+\infty)|\to 0,\quad \text{as}\ \ t\to+\infty.$$

As a consequence of Steps 1-4, with Lemma 1.5.1, we can conclude that $\mathcal{A}:B_r\to\mathcal{P}(B_r)$ is continuous and compact. From Schauder's theorem, we deduce that $\mathcal{A}$ has a fixed point z^*. Then $y^*=z^*+x$ is a fixed point of the operators N, which is a mild solution of the problem (5.1)-(5.2).

5.3 An example

Consider the following neutral functional partial differential equation:

$$\frac{\partial}{\partial t}[z(t,x) - g(t, z(t-\sigma(t,z(t,0)),x))] - \frac{\partial^2}{\partial x^2}[z(t,x) - g(t, z(t-\sigma(t,z(t,0)),x))]$$
$$+ \in F(t, z(t-\sigma(t,z(t,0)),x)),\ x \in [0,\pi],\ t \in [0,+\infty) \tag{5.5}$$

$$z(t,0) = z(t,\pi) = 0,\ t \in [0,+\infty), \tag{5.6}$$

$$z(\theta,x) = z_0(\theta,x),\ t \in (-\infty,0],\ x \in [0,\pi], \tag{5.7}$$

where F is a given multivalued map g is a given functions , and $\sigma : \mathbb{R} \to \mathbb{R}^+$. Take $E = L^2[0,\pi]$ and define $A : E \to E$ by $A\omega = \omega''$ with domain

$$D(A) = \{\omega \in E, \omega, \omega' \text{are absolutely continuous},\ \omega'' \in E,\ \omega(0) = \omega(\pi) = 0\}.$$

Then

$$A\omega = \sum_{n=1}^{\infty} n^2(\omega,\omega_n)\omega_n, \omega \in D(A),$$

where $\omega_n(s) = \sqrt{\frac{2}{\pi}}\sin ns, n = 1,2,\ldots$ is the orthogonal set of eigenvectors in A. It is well know (see [89]) that A is the infinitesimal generator of an analytic semigroup $T(t), t \geq 0$ in E and is given by

$$T(t)\omega = \sum_{n=1}^{\infty} \exp(-n^2 t)(\omega,\omega_n)\omega_n, \omega \in E.$$

Since the analytic semigroup $T(t)$ is compact for $t > 0$, there exists a positive constant M such that

$$\|T(t)\|_{B(E)} \leq M.$$

Let $\mathcal{B} = BCU(\mathbb{R}^-; E)$ and $\phi \in \mathcal{B}$, then (H_ϕ), where $\rho(t,\varphi) = t - \sigma(\varphi)$.
Hence, the problem (5.1)-(5.2) in an abstract formulation of the problem (5.5)-(5.7), and if the conditions $(H_1) - (H_7)$, (H_ϕ) are satisfied. Theorem 5.2.3 implies that the problem (5.5)-(5.7) has at least one mild solutions on BC .

Chapter 6

Second Order Functional Differential Equations With Delay

6.1 Introduction

In this Chapter, we study some second order classes of partial functional evolution equation on $J = [0, +\infty)$ with finite and state-dependent delay.

The cosine function theory is related to abstract linear second order differential equations in the same manner that the semigroup theory of bounded linear operators is related to first order partial differential equations and it's equally appealing devoted their generality and simplicity. For basic concepts and applications of this theory, we refer to the reader to Fattorini [51] , Travis and Weeb [96].

Our purpose in this work is consider a simultaneous generalization of the classical second order abstract Cauchy problem studied by Travis and Weeb in [95, 96] . Additionally, we observe that the ideas and techniques in this paper permit the reformulation of the problems studied in [17, 26, 76, 84, 85] to the context of " partial " second order differential equations, see [95] pp. 557 and the referred papers for details.

The problem of the existence of solutions for first and second order partial functional differential with state-dependent delay have treated recently in [10, 68, 70, 66, 67, 64, 69, 78, 92, 93]. The literature relative second order impulsive differential system with state-dependent delay is very restrict, and related this matter we only cite [94] for ordinary differential system and [65] for abstract partial differential systems.

6.2 Second order functional differential equations with finite delay

In this section we are going at the first time to prove the existence of solutions of a class of semilinear functional evolution equations of second order with finite delay .

Our investigations will be situated in the Banach space of real continuous and bounded functions on the real half axis $[0, +\infty)$. More precisely, we will consider the following problem

$$y''(t) = Ay(t) + f(t, y_t), \quad \text{a.e. } t \in J := [0, +\infty) \tag{6.1}$$

$$y(t) = \phi(t), \quad t \in [-d, 0], \quad y'(0) = \varphi \in E, \tag{6.2}$$

where $f : J \times C([-d, 0], E) \to E$ is given function, $\phi : [-d, 0] \to E$ is given continuous function, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous cosine function of bounded linear operators $(C(t))_{t\in\mathbb{R}}$, on E, $(E, |.|)$ is a real Banach space. For any function y defined on $[-d, +\infty)$ and any $t \in J$, we denote by y_t the element of $C([-d, 0], E)$ defined by

$$y_t(\theta) = y(t + \theta), \theta \in [-d, 0].$$

Here $y_t(.)$ represents the history of the state from time $t - d$, up to the present time t.

In this section by $BC := BC([-d, +\infty))$ we denote the Banach space of all bounded and continuous functions from $[-d, +\infty)$ into $\mathbb{R}$ equipped with the standard norm

$$\|y\|_{BC} = \sup_{t\in[-d,+\infty)} |y(t)|.$$

6.2.1 Existence of mild solutions

Now we give our main existence result for problem (6.1)-(6.2). Before starting and proving this result, we give the definition of a mild solution.

Definition 6.2.1 *We say that a continuous function $y : [-d, +\infty) \to E$ is a mild solution of problem (6.1)-(6.2) if $y(t) = \phi(t),\ t \in [-d, 0],\ y(.)$ and $y'(0) = \varphi$,and*

$$y(t) = C(t)\phi(0) + S(t)\varphi + \int_0^t S(t - s)f(s, y_s)ds,\ t \in J.$$

Let us introduce the following hypotheses:

(H_1) $C(t)$ is compact for $t > 0$ in the Banach space E. Let $M = \sup\{\|C\|_{B(E)} : t \geq 0\}$, and $M' = \sup\{\|S\|_{B(E)} : t \geq 0\}$.

(H_2) The function $f : J \times C([-d, 0], E) \to E$ is Carathéodory.

(H_3) There exists a continuous function $k : J \to [0, +\infty)$ such that:

$$|f(t, u) - f(t, v)| \leq k(t)\|u - v\|,\ t \in J,\ u, v \in C([-d, 0], E)$$

and

$$k^* := \sup_{t\in J} \int_0^t k(s)ds < \infty.$$

(H_4) The function $t \to f(t,0) = f_0 \in L^1(J,[0,+\infty))$ with $F^* = \|f_0\|_{L^1}$.

(H_5) For each bounded $B \subset BC$ and $t \in J$ the set :

$$\{C(t)\phi(0) + S(t)\varphi + \int_0^t S(t-s)f(s,y_t)ds : y \in B\}$$

is relatively compact in E.

Theorem 6.2.2 *Assume that $(H_1)-(H_5)$ hold. If $k^*M' < 1$, then the problem (6.1)-(6.2) has at least one mild solution on BC.*

Proof. We transform the problem (6.1)-(6.2) into a fixed point problem. Consider the operator: $N : BC \to BC$ define by:

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [-d,0], \\ C(t)\ \phi(0) + S(t)\varphi + \displaystyle\int_0^t S(t-s)\ f(s,y_s)\ ds, & \text{if } t \in J. \end{cases}$$

The operator N maps BC into BC; indeed the map $N(y)$ is continuous on $[-d,+\infty)$ for any $y \in BC$, and for each $t \in J$, we have

$$\begin{aligned} |N(y)(t)| &\leq M\|\phi\| + M'\|\varphi\| + M'\int_0^t |f(s,y_s) - f(s,0) + f(s,0)|ds \\ &\leq M\|\phi\| + M'\|\varphi\| + M'\int_0^t |f(s,0)|ds + M'\int_0^t k(s)\|y_s\|ds \\ &\leq M\|\phi\| + M'\|\varphi\| + M'F^* + M'\int_0^t k(s)\|y_s\|ds \\ &\leq M\|\phi\| + M'\|\varphi\| + M'F^* + M'\|y\|_{BC}k^* := c. \end{aligned}$$

Let

$$C = M\|\phi\| + M'\|\varphi\|.$$

Hence, $N(y) \in BC$.

Moreover, let $r > 0$ be such that $r \geq \frac{C+M'F^*}{1-M'k^*}$, and B_r be the closed ball in BC centered at the origin and of radius r. Let $y \in B_r$ and $t \in [0,+\infty)$. Then,

$$|N(y)(t)| \leq \ C + M'F^* + M'k^*r.$$

Thus,

$$\|N(y)\|_{BC} \leq r,$$

which means that the operator N transforms the ball B_r into itself.

Now we prove that $N : B_r \rightarrow B_r$ satisfies the assumptions of Schauder's fixed theorem. The proof will be given in several steps.

Step 1: N is continuous in B_r.
Let $\{y_n\}$ be a sequence such that $y_n \rightarrow y$ in B_r. We have

$$|N(y_n)(t) - N(y)(t)| \leq M' \int_0^t |f(s, y_{s_n}) - f(s, y_s)| ds.$$

Then by (H_2) we have $f(s, y_{s_n}) \rightarrow f(s, y_s)$, as $n \rightarrow \infty$, for a.e. $s \in J$, and by the Lebesgue dominated convergence theorem we have

$$\|N(y_n) - N(y)\|_{BC} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Thus, N is continuous.

Step 2 : $N(B_r) \subset B_r$ this is clear.

Step 3: $N(B_r)$ is equicontinuous on every compact interval $[0, b]$ of $[0, +\infty)$ for $b > 0$. Let $\tau_1, \tau_2 \in [0, b]$ with $\tau_2 > \tau_1$, we have

$$\begin{aligned}
|N(y)(\tau_2) - N(y)(\tau_1)| &\leq \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)}\|\phi\| + \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}\|\varphi\| \\
&+ \int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}|f(s, y_s)|ds \\
&+ \int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|_{B(E)}|f(s, y_s)|ds \\
&\leq \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)}\|\phi\| + \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}\|\varphi\| \\
&+ \int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}|f(s, y_s) - f(s, 0) + f(s, 0)|ds \\
&+ \int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|_{B(E)}|f(s, y_s) - f(s, 0) + f(s, 0)|ds \\
&\leq \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)}\|\phi\| + \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}\|\varphi\| \\
&+ r\int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}k(s)ds \\
&+ \int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}|f(s, 0)|ds \\
&+ r\int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|_{B(E)}k(s)ds \\
&+ \int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|_{B(E)}|f(s, 0)|ds.
\end{aligned}$$

When $\tau_2 \rightarrow \tau_2$, the right-hand side of the above inequality tends to zero, since $C(t), S(t)$ are a strongly continuous operator and the compactness of $C(t), S(t)$ for $t > 0$, implies he continuity in the uniform operator topology (see [95, 96]). This proves the equicontinuity.

Step 4:$N(B_r)$ is relatively compact on every compact interval of $[0,\infty)$ by (H_5).

Step 5: $N(B_r)$ is equiconvergent.
Let $y \in B_r$, we have:

$$\begin{aligned}
|N(y)(t)| &\leq M\|\phi\| + M'\|\varphi\| + M' \int_0^t |f(s, y_s)| ds \\
&\leq C + M'F^* + M'r \int_0^t k(s) ds \\
&\leq C + M'F^* + M'r \int_0^t k(s) ds.
\end{aligned}$$

Then

$$|N(y)(t)| \to l, \quad \text{as } \; t \to +\infty.$$

Where $l \leq C_1 := C + M'F^* + M'k^*r$ Hence,

$$|N(y)(t) - N(y)(+\infty)| \to 0, \quad \text{as } \; t \to +\infty.$$

As a consequence of Steps 1-5, with Lemma 1.5.1, we can conclude that $N : B_r \to B_r$ is continuous and compact. From Schauder's theorem, we deduce that N has a fixed point y^* which is a mild solution of the problem (6.1)-(6.2).

6.2.2 An example

Consider the functional partial differential equation of second order:

$$\frac{\partial^2}{\partial t^2} z(t,x) = \frac{\partial^2}{\partial x^2} z(t,x) + f(t, z(t-d, x)), \; x \in [0,\pi], \; t \in J := [0,+\infty), \tag{6.3}$$

$$z(t,0) = z(t,\pi) = 0, \; t \in [0,+\infty), \tag{6.4}$$

$$z(t,x) = \phi(t,x), \quad \frac{\partial z(0,x)}{\partial t} = w(x), \; t \in [-d,0], \; x \in [0,\pi], \tag{6.5}$$

where f is a given map. Take $E = L^2[0,\pi]$ and define $A : E \to E$ by $A\omega = \omega''$ with domain

$$D(A) = \{\omega \in E; \omega, \omega' \text{are absolutely continuous}, \; \omega'' \in E, \; \omega(0) = \omega(\pi) = 0\}.$$

It is well known that A is the infinitesimal generator of a strongly continuous cosine function $(C(t))_{t\in\mathbb{R}}$on E, respectively. Moreover, A has discrete spectrum, the eigenvalues are $-n^2, n \in \mathbb{N}$ with corresponding normalized eigenvectors

$$z_n(\tau) := (\frac{2}{\pi})^{\frac{1}{2}} \sin n\tau,$$

and the following properties hold:

(a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of E.

(b) If $y \in E$, then $Ay = -\sum_{n=1}^{\infty} n^2 < y, z_n > z_n$.

(c) For $y \in E, C(t)y = \sum_{n=1}^{\infty} \cos(nt) < y, z_n > z_n$, and the associated sine family is

$$S(t)y = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} < y, z_n > z_n$$

which implies that the operator S(t) is compact, for all $t > 0$and that $\|C(t)\| = \|S(t)\| \leq 1$, for all $t \geq 0$.

(d) If Φ denotes the group of translation on E defined by $\Phi(t)y(\xi) = \tilde{y}(\xi + t)$ where $\tilde{y}$ is the extension of y with period 2π, then $C(t) = \frac{1}{2}(\Phi(t) + \Phi(-t)); A = B^2$, where B is the infinitesimal generator of the group Φ on

$$X = \{y \in H^1(0, \pi) : y(0) = x(\pi) = 0\}.$$

For more details, see [51].

Then the problem (6.1)-(6.2) in an abstract formulation of the problem (6.3)-(6.5). If conditions $(H_1) - (H_5)$ are satisfied. Theorem 6.2.2 implies that the problem (6.3)-(6.5) has at least one mild solution on BC.

6.3 Second order functional differential equations with state-dependent delay

At the second time in the case of state-dependent delay, our investigations will be situated in the Banach space of real continuous and bounded functions on $\mathbb{R}$. More precisely, we will consider the following problem

$$y''(t) = Ay(t) + f(t, y_{\rho(t,y_t)}), \quad \text{a.e. } t \in J := [0, +\infty) \tag{6.6}$$

$$y(t) = \phi(t) \in \mathcal{B}, \quad y'(0) = \varphi \in E, \tag{6.7}$$

where $f : J \times \mathcal{B} \to E$ is given function, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous cosine function of bounded linear operators $(C(t))_{t \in \mathbb{R}}$, on E, $\phi \in \mathcal{B}$, $\rho : J \times \mathcal{B} \to (-\infty, +\infty)$, and $(E, |.|)$ is a real Banach space. For any function y defined on $[-\infty, +\infty)$ and any $t \in J$, we denote by y_t the element of $\mathcal{B}$ defined by

$$y_t(\theta) = y(t + \theta), \theta \in (\infty-, 0].$$

We assume that the histories y_t belongs to some abstract phases $\mathcal{B}$.
In the both cases We will use Schauder's fixed theorem, combined with the Corduneanu's compactness criteria. In this section by $BC := BC(-\infty, +\infty)$ we denote

the Banach space of all bounded and continuous functions from $(-\infty, +\infty)$ into E equipped with the standard norm

$$\|y\|_{BC} = \sup_{t\in(-\infty,+\infty)} |y(t)|.$$

Finally, by $BC' := BC'([0, +\infty))$ we denote the Banach space of all bounded and continuous functions from $[0, +\infty)$ into E equipped with the standard norm

$$\|y\|_{BC'} = \sup_{t\in[0,+\infty)} |y(t)|.$$

By BUC we denote the space of bounded uniformly continuous functions defined from $(-\infty, 0]$ to E.

6.3.1 Existence of mild solutions

Now we give our main existence result for problem (6.6)-(6.7). Before starting and proving this result, we give the definition of a mild solution.

Definition 6.3.1 *We say that a continuous function $y : (-\infty, +\infty) \to E$ is a mild solution of problem (6.6)-(6.7) if $y(t) = \phi(t),\ t \in (-\infty, 0],\ y(.)$ is continuously differentiable and $y'(0) = \varphi$, and*

$$y(t) = C(t)\phi(0) + S(t)\varphi + \int_0^t S(t-s)f(s, y_{\rho(t,y_t)})ds,\ t \in J. \tag{6.8}$$

Let us introduce the following hypotheses:

(H_1) $C(t), S(t)$ are compact for $t > 0$ in the Banach space E. Let $M = \sup\{\|C\|_{B(E)} : t \geq 0\}$, and $M' = \sup\{\|S\|_{B(E)} : t \geq 0\}$.

(H_2) The function $f : J \times \mathcal{B} \to E$ is Carathéodory.

(H_3) There exists a continuous function $k : J \to [0, +\infty)$ such that:

$$|f(t, u) - f(t, v)| \leq k(t)\|u - v\|,\ t \in J,\ u, v \in \mathcal{B}$$

and

$$k^* := \sup_{t\in J} \int_0^t k(s)ds < \infty.$$

(H_4) The function $t \to f(t, 0) = f_0 \in L^1(J, [0, +\infty))$ with $F^* = \|f_0\|_{L^1}$.

(H_5) For each bounded $B \subset BC'$ and $t \in J$ the set :

$$\{S(t)\varphi + \int_0^t S(t-s)f(s, y_{\rho(t,y_t)})ds : y \in B\}$$

is relatively compact in E.

Theorem 6.3.2 *Assume that* $(H_1)-(H_5),(H_\phi)$ *hold. If* $k^*Ml<1$, *then the problem (6.6)-(6.7) has at least one mild solution on* BC.

Proof. We transform the problem (6.6)-(6.7) into a fixed point problem. Consider the operator: $N: BC \to BC$ define by:

$$N(y)(t)=\begin{cases}\phi(t), & \text{if } t\in(-\infty,0],\\[2ex] C(t)\ \phi(0)+S(t)\varphi+\displaystyle\int_0^t S(t-s)\ f(s,y_{\rho(t,y_t)})\ ds, & \text{if } t\in J.\end{cases}$$

Let $x(.):(-\infty,+\infty)\to E$ be the function defined by:

$$x(t)=\begin{cases}\phi(t); & \text{if } t\in(-\infty,0];\\[1ex] C(t)\ \phi(0); & \text{if } t\in J,\end{cases}$$

then $x_0=\phi$. For each $z\in BC$ with $z(0)=0$,$y'(0)=\varphi=z'(0)=\varphi_1$, we denote by $\overline{z}$ the function

$$\overline{z}(t)=\begin{cases}0; & \text{if } t\in(-\infty,0];\\[1ex] z(t); & \text{if } t\in J.\end{cases}$$

If y satisfies (6.8), we can decompose it as $y(t)=z(t)+x(t),\ t\in J$, which implies $y_t=z_t+x_t$ for every $t\in J$ and the function $z(.)$ satisfies

$$z(t)=S(t)\varphi_1+\int_0^t S(t-s)\ f(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})ds, t\in J.$$

Set

$$BC_0'=\{z\in BC':z(0)=0\}$$

and let

$$\|z\|_{BC_0'}=\sup\{|z(t)|:t\in J\},\ z\in BC_0'.$$

BC_0' is a Banach space with the norm $\|.\|_{BC_0'}$. We define the operator $\mathcal{A}:BC_0'\to BC_0'$ by:

$$\mathcal{A}(z)(t)=S(t)\varphi_1+\int_0^t S(t-s)\ f(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})ds,\ t\in J.$$

We shall show that the operator $\mathcal{A}$ satisfies all conditions of Schauder's fixed point theorem. The operator A maps BC_0' into BC_0', indeed the map $\mathcal{A}(z)$ is continuous on $[0,+\infty)$ for any $z\in BC_0'$, and for each $t\in J$ we have

$$\begin{aligned}|\mathcal{A}(z)(t)| &\le M'\|\varphi_1\|+M'\int_0^t|f(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})-f(s,0)+f(s,0)|ds\\ &\le M'\|\varphi_1\|+M'\int_0^t|f(s,0)|ds+M'\int_0^t k(s)\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}ds\\ &\le M'\|\varphi_1\|+M'F^*+M'\int_0^t k(s)(l|z(s)|+(m+\mathcal{L}^\phi+lMH)\|\phi\|_{\mathcal{B}})ds.\end{aligned}$$

Let
$$C = (m + \mathcal{L}^{\phi} + lMH)\|\phi\|_{\mathcal{B}}.$$
Then, we have:
$$\begin{aligned} |\mathcal{A}(z)(t)| &\leq M'\|\varphi_1\| + M'F^* + M'C\int_0^t k(s)ds + M'l\int_0^t k(s)|z(s)|ds \\ &\leq M'\|\varphi_1\| + M'F^* + M'Ck^* + M'l\|z\|_{BC_0'}k^* \end{aligned}$$

Hence, $\mathcal{A}(z) \in BC_0'$.

Moreover, let $r > 0$ be such that $r \geq \frac{M'\|\varphi_1\|+M'F^*+M'Ck^*}{1-M'lk^*}$, and B_r be the closed ball in BC_0' centered at the origin and of radius r. Let $y \in B_r$ and $t \in [0, +\infty)$. Then,
$$|\mathcal{A}(z)(t)| \leq \ M'\|\varphi_1\| + M'F^* + M'Ck^* + M'lk^*r.$$
Thus,
$$\|\mathcal{A}(z)\|_{BC_0'} \leq r,$$
which means that the operator N transforms the ball B_r into itself.

Now we prove that $\mathcal{A} : B_r \to B_r$ satisfies the assumptions of Schauder's fixed theorem. The proof will be given in several steps.

Step 1: $\mathcal{A}$ is continuous in B_r.
Let $\{z_n\}$ be a sequence such that $z_n \to z$ in B_r. At the first, we study the convergence of the sequences $(z^n_{\rho(s,z^n_s)})_{n\in\mathbb{N}}$, $s \in J$.
If $s \in J$ is such that $\rho(s, z_s) > 0$, then we have,
$$\begin{aligned} \|z^n_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\|_{\mathcal{B}} &\leq \|z^n_{\rho(s,z^n_s)} - z_{\rho(s,z^n_s)}\|_{\mathcal{B}} + \|z_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\|_{\mathcal{B}} \\ &\leq l\|z_n - z\|_{B_r} + \|z_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\|_{\mathcal{B}}, \end{aligned}$$
which proves that $z^n_{\rho(s,z^n_s)} \to z_{\rho(s,z_s)}$ in $\mathcal{B}$ as $n \to \infty$ for every $s \in J$ such that $\rho(s, z_s) > 0$. Similarly, is $\rho(s, z_s) < 0$, we get
$$\|z^n_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\|_{\mathcal{B}} = \|\phi^n_{\rho(s,z^n_s)} - \phi_{\rho(s,z_s)}\|_{\mathcal{B}} = 0$$
which also shows that $z^n_{\rho(s,z^n_s)} \to z_{\rho(s,z_s)}$ in $\mathcal{B}$ as $n \to \infty$ for every $s \in J$ such that $\rho(s, z_s) < 0$. Combining the pervious arguments, we can prove that $z^n_{\rho(s,z_s)} \to \phi$ for every $s \in J$ such that $\rho(s, z_s) = 0$. Finally,
$$|\mathcal{A}(z_n)(t) - \mathcal{A}(z)(t)| \leq M'\int_0^t |f(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)}) - f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|ds.$$
Then by (H_2) we have
$$f(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)}) \to f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}), \text{ as } n \to \infty,$$

and by the Lebesgue dominated convergence theorem we get,

$$\|\mathcal{A}(z_n) - \mathcal{A}(z)\|_{BC'_0} \to 0, \text{ as } n \to \infty.$$

Thus $\mathcal{A}$ is continuous.

Step 2 : $\mathcal{A}(B_r) \subset B_r$ this is clear.

Step 3: $\mathcal{A}(B_r)$ is equicontinuous on every compact interval $[0, b]$ of $[0, +\infty)$ for $b > 0$. Let $\tau_1, \tau_2 \in [0, b]$ with $\tau_2 > \tau_1$, we have

$$\begin{aligned}
|\mathcal{A}(z)(\tau_2) - \mathcal{A}(z)(\tau_1)| &\leq \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}\|\varphi_1\| \\
&+ \int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}|f(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)})|ds \\
&+ \int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|_{B(E)}|f(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)})|ds \\
&\leq \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}\|\varphi_1\| \\
&+ \int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}|f(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)}) - f(s, 0)|ds \\
&+ \int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}f(s, 0)|ds \\
&+ \int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|_{B(E)}|f(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)}) - f(s, 0)|ds \\
&+ \int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|_{B(E)}|f(s, 0)|ds \\
&\leq \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}\|\varphi_1\| \\
&+ C\int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}k(s)ds \\
&+ lr\int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}k(s)ds \\
&+ \int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}|f(s, 0)|ds \\
&+ C\int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|_{B(E)}k(s)ds \\
&+ lr\int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|_{B(E)}k(s)ds \\
&+ \int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|_{B(E)}|f(s, 0)|ds.
\end{aligned}$$

When $\tau_2 \to \tau_2$, the right-hand side of the above inequality tends to zero, since $C(t)$ are a strongly continuous operator and the compactness of $C(t)$ for $t > 0$, implies he continuity in the uniform operator topology (see [95, 96]). This proves the equicontinuity.

Step 4:$N(B_r)$ is relatively compact on every compact interval of $[0,\infty)$. This is satisfied from (H_5).

Step 5: $N(B_r)$ is equiconvergent.
Let $y \in B_r$, we have:

$$\begin{aligned}|\mathcal{A}(z)(t)| &\leq M'\|\varphi_1\| + M'\int_0^t |f(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)})|ds \\ &\leq M'\|\varphi_1\| + M'F^* + M'Ck^* + M'rl\int_0^t k(s)ds.\end{aligned}$$

Then

$$|\mathcal{A}(z)(t)| \to l, \quad \text{as } t \to +\infty.$$

Where $l \leq C_1 := M'\|\varphi_1\| + M'F^* + M'k^*(C + lr)$ Hence,

$$|\mathcal{A}(z)(t) - \mathcal{A}(z)(+\infty)| \to 0, \quad \text{as } t \to +\infty.$$

As a consequence of Steps 1-5, with Lemma 1.5.1, we can conclude that $\mathcal{A} : B_r \to B_r$ is continuous and compact. we deduce that $\mathcal{A}$ has a fixed point z^*. Then $y^* = z^* + x$ is a fixed point of the operators N, which is a mild solution of the problem (6.6)-(6.7).

6.3.2 An example

Take $E = L^2[0,\pi]; \mathcal{B} = C_0 \times L^2(g, E)$ and define $A : E \to E$ by $A\omega = \omega''$ with domain

$$D(A) = \{\omega \in E; \omega, \omega' \text{are absolutely continuous, } \omega'' \in E, \ \omega(0) = \omega(\pi) = 0\}.$$

It is well known that A is the infinitesimal generator of a strongly continuous cosine function $(C(t))_{t\in\mathbb{R}}$on E, respectively. Moreover, A has discrete spectrum, the eigenvalues are $-n^2, n \in \mathbb{N}$ with corresponding normalized eigenvectors

$$z_n(\tau) := (\frac{2}{\pi})^{\frac{1}{2}} \sin n\tau,$$

and the following properties hold:

(a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of E.

(b) If $y \in E$, then $Ay = -\sum_{n=1}^{\infty} n^2 < y, z_n > z_n$.

(c) For $y \in E, C(t)y = \sum_{n=1}^{\infty} \cos(nt) < y, z_n > z_n$, and the associated sine family is

$$S(t)y = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} < y, z_n > z_n$$

which implies that the operator S(t) is compact, for all $t \in J$and that

$$\|C(t)\| = \|S(t)\| \leq 1, \text{ for all } t \in \mathrm{I\!R}.$$

(d) If Φ denotes the group of translations on E defined by $\Phi(t)y(\xi) = \tilde{y}(\xi + t)$ where $\tilde{y}$ is the extension of y with period 2π, then $C(t) = \frac{1}{2}(\Phi(t) + \Phi(-t)); A = B^2$, where B is the infinitesimal generator of the group Φ on

$$X = \{y \in H^1(0, \pi) : y(0) = x(\pi) = 0\}.$$

Consider the functional partial differential equation of second order:

$$\frac{\partial^2}{\partial t^2} z(t,x) = \frac{\partial^2}{\partial x^2} z(t,x) + \int_{-\infty}^{0} a(s-t) z(s - \rho_1(t)\rho_2(\|z(t)\|), x) ds, \ x \in [0, \pi], \ t \in J := [0, +\infty), \tag{6.9}$$

$$z(t, 0) = z(t, \pi) = 0, \ t \in [0, +\infty), \tag{6.10}$$

$$z(t, x) = \phi(t, x), \quad \frac{\partial z(0, x)}{\partial t} = \omega(x), \ t \in [-r, 0], \ x \in [0, \pi], \tag{6.11}$$

where $\phi \in \mathcal{B}, \ \rho_i : [0, \infty) \to [0, \infty), a; \mathrm{I\!R} \to \mathrm{I\!R}$ be continuous, and $L_f = \left(\int_{-\infty}^{0} \frac{a^2(s)}{g(s)} ds \right)^{\frac{1}{2}} < \infty$. Under these conditions, we define the function $f : J \times \mathcal{B} \to E, \rho : J \times \mathcal{B} \to \mathrm{I\!R}$ by

$$f(t, \psi)(x) = \int_{-\infty}^{0} a(s)\psi(s, x) ds,$$

$$\rho(s, \psi) = s - \rho_1(s)\rho_2(\|\psi(0)\|,$$

we have $\|f(t, .)\|_{\mathfrak{B}(\mathcal{B},E)} \leq L_f$.

Then the problem (6.6)-(6.7) in an abstract formulation of the problem (6.9)-(6.11). If conditions $(H_1) - (H_5)$ are satisfied. Theorem 6.3.2 implies that the problem (6.9)-(6.11) has at least one mild solution on BC.

Conclusion and perspective

In this thesis we have considered the global existence of mild solutions for some classes of first and second order functional and neutral functional differential evolutions equations and inclusions with finite, infinite and state-dependent delay on a positive real line. Our tool is based on the evolution system and appropriate fixed point theorems. In the future we shall look for the asymptotic behavior of the solutions.

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Résumé : Dans ce travail, l’objectif est d’apporter une contribution à l’étude de l’existence des solutions faibles globales pour diverses classes d’équations et d’inclusions d’évolution fonctionnelles et de type neutre à retard fini et infini, et dépendant de l’état définies sur des intervalles non bornés. Les résultats principaux sont basés sur l’approche du point fixe et la théorie des semi groupes.

Abstract: In this work, we give a contribution to the study of the existence of global mild solutions of various classes of first and second order of partial functional and neutral functional evolution equations and inclusions with finite, infinite and state-dependent delay defined on unbounded intervals. The main results are based on the fixed point theorem approach and the semi group theory.

ملخص :

الهدف في هذه الرسالة هو تقديم مساهمة لدراسة مختلف فئات من معادلات واحتواءات تفاضلية و تطورية و دالية جزئية من نوع حيادي ذي تأ خر منته و غير منته و متعلق بالحالة على النتائج تعتمد على مقاربة النقطة الثابتة ونظرية مجالات غير محدودة . Semi groups.