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Zohra BOUTEFFAL

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Abdelghani OUAHAB	Pr	U.Sidi Bel-Abbes	Président
Mouffak BENCHOHRA	Pr	U. Sidi Bel-Abbes	Promoteur
Mohamed BELMEKKI	Pr	U.Saida	Examinateur
Said ABBAS	Pr	U.Saida	Examinateur
Naima HAMIDI	Pr	U.Sidi Bel-Abbes	Examinateur

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Publications

1. M. Benchohra and **Z. Boutefal**, Impulsive Differential Equations of Fractional Order with Infinit Delay, *Journal of Fractional Calculus and Applications* **5** (5) (2013), 1-15.

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3. M. Benchohra, **Z. Boutefal**, Impulsive Partial Hyperbolic Differential Equations of Fractional Order with State Dependent Delay (submitted).

4. M. Benchohra and **Z. Boutefal**, Global Uniqueness Results for Impulsive Fractional Partial Hyperbolic Differential Equations with Delay (submitted).

Contents

Introduction

The idea of fractional calculus and fractional order differential equations has been a subject of interest not only among mathematicians, but also among physicists and engineers. During the last decade, it was found to play a fundamental role in the modeling of a considerable number of phenomena, in particular, the modeling of memory dependent phenomena and complex media such as porous media. Fractional calculus emerged as an important and efficient tool for the study of dynamical systems where classical methods reveal strong limitations.

Fractional order models are found to be more adequate than integer order models in some real world problems. In fact, fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes; see the books by Baleanu *et al.* [13], Hilfer [40], Tarasov [53], and the references therein. Recent developments on fractional differential equations from theoretical point of view are given in the books by Abbas *et al.* [7], and Lakshmikantham *et al.* [45].

The theory of impulsive integer order differential equations and inclusions has become important in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology, and economics. At present the foundations of the general theory are already laid, and many of them are investigated in detail the papers of Abbas and Benchohra [3, 4], Agarwal *et al.* [5],Lakshmikantham et al[44], Samoilenko and Peresyuk [51], and the references therein. There was an intensive development of the impulse theory, especially in the area of impulsive differential equations and inclusions with fixed moments.

This thesis is devoted to the existence and uniqueness of solutions for various classes of Darboux problem for hyperbolic differential equations involving the Caputo fractional derivative, the best fractional derivative of the time. Some equations present delay which may be finite, infinite, or state-dependent. The tools used include classical fixed point theorems in Banach and Fréchet spaces. Each chapter concludes with a section devoted to notes and all abstract results are illustrated by examples.

In the following we give an outline of our thesis organization, Consists of six chapters defining the work contributed. The first chapter gives some notations, definitions, lemmas and fixed point theorems which are used throughout this thesis. In Sect. 1.1, we give some notations from the theory of Banach spaces. Section 1.2 is concerned to recall some basic definitions and facts on partial fractional calculus theory. In Sect. 1.3, we give definition and examples of phase space. In Sect. 1.4, we give some properties in Fréchet space. Section. 1.5, is devoted to fixed-points theory, here we give the main theorems that will be used in the following chapters.

In **Chapter 2**, we study a system of impulsive partial hyperbolic differential equation. Our results are based on fixed point theorem due to Burton and Kirk for the sum of contraction and completely continuous operators. The first result is for impulsive partial hyperbolic differential equation of the form

$$({}^cD^r_{z_k}u)(t,x) = f(t,x,u(t,x)), \text{ if } (t,x) \in J_k, \ k = 0,\ldots,m, \tag{1}$$

$$u(t_k^+,x) = u(t_k^-,x) + I_k(u(t_k^-,x)), \text{ if } x \in [0,b]; \ k = 1,\ldots,m, \tag{2}$$

$$u(t,0) = \varphi(t), \ u(0,x) = \psi(x), \ t \in [0,a], \ x \in [0,b], \tag{3}$$

where $J_0 = [0,t_1] \times [0,b]$, $J_k := (t_k,t_{k+1}] \times [0,b]$, $k = 1,\ldots,m, z_k = (t_k,0)$, $k = 0,\ldots,m$, $J = [0,a] \times [0,b]$, $a,b > 0$, ${}^cD^r_0$ is the Caputo fractional derivative of order $r = (r_1,r_2) \in (0,1] \times (0,1]$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = a$, $f : J \times \mathbb{R}^n \to \mathbb{R}^n$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 0,1,\ldots,m$ are given functions, $\varphi : [0,a] \to \mathbb{R}^n$ and $\psi : [0,b] \to \mathbb{R}^n$ are given absolutely continuous functions with $\varphi(0) = \psi(0)$.

In Sect. 2.3, we give similar result to the following nonlocal initial value problem

$$({}^cD^r_{z_k}u)(t,x) = f(t,x,u(t,x)), \text{ if } (t,x) \in J_k, \ k = 0,\ldots,m, \tag{4}$$

$$u(t_k^+,x) = u(t_k^-,x) + I_k(u(t_k^-,x)), \text{ if } x \in [0,b]; \ k = 1,\ldots,m, \tag{5}$$

$$u(t,0) + Q(u) = \varphi(t), \ u(0,x) + K(u) = \psi(x), \ t \in [0,a], \ x \in [0,b], \tag{6}$$

where f, φ, ψ, I_k; $k = 1,\ldots,m$, are as in problem (2.1)-(2.3) and $Q,K : PC(J,\mathbb{R}^n) \to \mathbb{R}^n$ are continuous functions. $PC(J,\mathbb{R}^n)$ is a Banach space to be specified in section 2.2 of Chapter 2.
An example will be presented in the last illustrating the abstract theory.

In Chapter 3, we shall be concerned by functional differential equations.
In Sect. 3.2, we investigate the existence of solutions for impulsive partial hyperbolic functional differential equations of fractional order with finite delay of the form

$$({}^cD^r_{z_k}u)(t,x) = f(t,x,u_{(t,x)}); \text{ if } (t,x) \in J_k, \ \ k = 0,\ldots,m, \tag{7}$$

$$u(t_k^+,x) = u(t_k^-,x) + I_k(u(t_k^-,x)), \text{ if } x \in [0,b], \ k = 1,\ldots,m, \tag{8}$$

$$u(t,x) = \phi(t,x); \text{ if } (t,x) \in \tilde{J}, \tag{9}$$

$$u(t,0) = \varphi(t), \ t \in [0,a], \ u(0,x) = \psi(x); \ x \in [0,b], \tag{10}$$

where $J_0 = [0,t_1] \times [0,b]$, $J_k := (t_k,t_{k+1}] \times [0,b]$, $k = 1,\ldots,m, z_k = (t_k,0)$, $k = 0,\ldots,m$, $a,b,\alpha,\beta > 0, J = [0,a] \times [0,b]$, $\tilde{J} = [-\alpha,a] \times [-\beta,b] \backslash (0,a] \times (0,b]$, ${}^cD^r_0$ is the Caputo fractional derivative of order $r = (r_1,r_2) \in (0,1] \times (0,1]$, $\varphi : [0,a] \to \mathbb{R}^n$,

$\psi : [0,b] \to \mathbb{R}^n$ are given continuous functions with $\varphi(t) = \phi(t,0),\ \psi(x) = \phi(0,x)$ for each $(t,x) \in J,\ \ 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = a,\ f : J \times C \to \mathbb{R}^n,\ I_k : \mathbb{R}^n \to \mathbb{R}^n,\ k = 1,\ldots,m,\ \phi : \tilde{J} \to \mathbb{R}^n$, are given functions and $C := C([-\alpha,0] \times [-\beta,0],\mathbb{R}^n)$ is the space of continuous functions on $[-\alpha,0] \times [-\beta,0]$.
If $u : [-\alpha,0] \times [-\beta,0] \longrightarrow \mathbb{R}^n$, then for any $(t,x) \in J$ define $u_{(t,x)}$ by

$$u_{(t,x)}(s,\tau) = u(t+s,x+\tau)$$

An example is presented in the last part of this section
In Sect. 3.3, we prove a existence of solutions for the following impulsive functional partial hyperbolic differential equations with infinite delay

$$({}^cD^r_{z_k}u)(x,y) = f(x,y,u_{(x,y)});\ \text{ if } (x,y) \in J_k, \quad k = 0,\ldots,m, \tag{11}$$

$$u(x_k^+,y) = u(x_k^-,y) + I_k(u(x_k^-,y)),\ \text{ if } y \in [0,b],\ k = 1,\ldots,m, \tag{12}$$

$$u(x,y) = \phi(x,y);\ \text{if } (x,y) \in \tilde{J}, \tag{13}$$

$$u(x,0) = \varphi(x),\ x \in [0,a],\ u(0,y) = \psi(y);\ y \in [0,b], \tag{14}$$

where $J_0 = [0,x_1] \times [0,b],\ J_k := (x_k,x_{k+1}] \times [0,b];\ k = 1,\ldots,m,\ z_k = (x_k,0),\ k = 0,\ldots,m,\ a,b > 0,\ J = [0,a] \times [0,b],\ \tilde{J} = (-\infty,a] \times (-\infty,b] \backslash (0,a] \times (0,b],\ {}^cD^r_0$ is the Caputo fractional derivative of order $r = (r_1,r_2) \in (0,1] \times (0,1],\ \varphi : [0,a] \to \mathbb{R}^n$, $\psi : [0,b] \to \mathbb{R}^n$ are given continuous functions with $\varphi(x) = \phi(x,0),\ \psi(y) = \phi(0,y)$ for each $(x,y) \in J,\ \ 0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = a,\ f : J \times \mathcal{B} \to \mathbb{R}^n,\ I_k : \mathbb{R}^n \to \mathbb{R}^n,\ k = 1,\ldots,m,\ \phi : \tilde{J} \to \mathbb{R}^n$, are given functions. $\mathcal{B}$ is called a phase space that will be specified in the next Section. If $u : (-\infty,a] \times (-\infty,b] \to \mathbb{R}^n$, then for any $(x,y) \in J$ define $u_{(x,y)}$ by

$$u_{(x,y)}(s,t) = u(x+s,y+t),\ \text{ for } (s,t) \in [-\alpha,0] \times [-\beta,0].$$

An example is presented in the last part of this section

In Chapter 4, we study the existence of solutions for fractional impulsive hyperbolic differential equations with state-dependent delay. Section 4.2 deals with the existence of solutions to fractional impulsive hyperbolic differential equations with finite delay

$$({}^cD^r_{z_k}u)(t,x) = f(t,x,u_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))});\ \text{ if } (t,x) \in J_k,\ k = 0,\ldots,m, \tag{15}$$

$$u(t_k^+,x) = u(t_k^-,x) + I_k(u(t_k^-,x)),\ \text{ if } x \in [0,b],\ k = 1,\ldots,m, \tag{16}$$

$$u(t,x) = \phi(t,x);\ \text{if } (t,x) \in \tilde{J} := [-\alpha,a] \times [-\beta,b] \setminus (0,a] \times (0,b], \tag{17}$$

$$u(t,0) = \varphi(t),\ t \in [0,a],\ u(0,x) = \psi(x);\ x \in [0,b], \tag{18}$$

where $J_0 = [0,t_1] \times [0,b],\ J_k := (t_k,t_{k+1}] \times [0,b],\ k = 1,\ldots,m, z_k = (t_k,0),\ k = 0,\ldots,m, J = [0,a] \times [0,b],\ a,b,\alpha,\beta > 0,\ 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = a,\ {}^cD^r_0$ is the Caputo fractional derivative of order $r = (r_1,r_2) \in (0,1] \times (0,1],\ \varphi : [0,a] \to \mathbb{R}^n$,

$\psi : [0,b] \to \mathbb{R}^n$ are given continuous functions with $\varphi(t) = \phi(t,0),\ \psi(x) = \phi(0,x)$ for each $(t,x) \in J$, $f : J \times C \to \mathbb{R}^n,\ \rho_1, \rho_2 : J \times C \to \mathbb{R},\ I_k : \mathbb{R}^n \to \mathbb{R}^n,\ k = 1, \dots, m,\ \phi : \tilde{J} \to \mathbb{R}^n$, are given functions and C is the Banach space defined in section 4.2.
An example is presented in the last part of this section.

In Section 4.3, we give our second main result concerning system of impulsive partial hyperbolic differential equation of fractional order with infinite delay

$$({}^cD^r_{z_k}u)(t,x) = f(t,x,u_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))});\ \text{ if } (t,x) \in J_k,\ k = 0, \dots, m, \tag{19}$$

$$u(t_k^+, x) = u(t_k^-, x) + I_k(u(t_k^-, x)),\ \text{ if } x \in [0,b],\ k = 1, \dots, m, \tag{20}$$

$$u(t,x) = \phi(t,x);\ \text{if } (t,x) \in \tilde{J}' := (-\infty, a] \times (-\infty, b] \setminus (0,a] \times (0,b], \tag{21}$$

$$u(t,0) = \varphi(t),\ t \in [0,a],\ u(0,x) = \psi(x);\ x \in [0,b], \tag{22}$$

where φ, ψ, I_k are as in problem (5.1)–(5.4), $f : J \times \mathcal{B} \to \mathbb{R}^n, \rho_1, \rho_2 : J \times \mathcal{B} \to \mathbb{R}, \phi : \tilde{J}' \to \mathbb{R}^n$ and $\mathcal{B}$ is a phase space.
Also, we present an example illustrating the applicability of the imposed conditions.

In Chapter 5 we shall be concerned by impulsive initial value problem for differential equations of fractional order with fixed time impulses.

Section 5.2 deals with the existence of solutions of impulsive differential equations of fractional order with finite delay given by

$$({}^cD^r_{z_k}u)(t,x) = f(t,x,u_{(t,x)}),\ \text{ if } (t,x) \in J_k,\ k = 0, \dots, m, \tag{23}$$

$$u(t_k^+, x) = u(t_k^-, x) + I_k(u(t_k^-, x)),\ \text{if } x \in [0,b];\ k = 1, \dots, m, \tag{24}$$

$$u(t,x) = \phi(t,x);\ \text{if } (t,x) \in \tilde{J}, \tag{25}$$

$$u(t,0) = \varphi(t),\ u(0,x) = \psi(x),\ t \in [0,a],\ x \in [0,b], \tag{26}$$

where $J_0 = [0,t_1] \times [0,b],\ J_k := (t_k, t_{k+1}] \times [0,b],\ k = 1, \dots, m, z_k = (t_k, 0),\ k = 0, \dots, m,\ \varphi(0) = \psi(0),\ J := [0,\infty) \times [0,\infty),\ \tilde{J} := [-\alpha,\infty) \times [-\beta,\infty) \setminus [0,\infty) \times [0,\infty),\ \alpha, \beta > 0,\ {}^cD^r_0$ is the standard Caputo's fractional derivative of order $r = (r_1, r_2) \in (0,1] \times (0,1]$, $f : J \times C([-\alpha,0] \times [-\beta,0], \mathbb{R}^n) \to \mathbb{R}^n,\ I_k : \mathbb{R}^n \to \mathbb{R}^n,\ k = 0, 1, \dots, m$ are given functions, for each $(t,x) \in J, \varphi, \psi : [0,\infty) \to \mathbb{R}^n$, are given absolutely continuous functions and $C([-\alpha,0] \times [-\beta,0], \mathbb{R}^n)$ is the space of continuous functions on $[-\alpha,0] \times [-\beta,0]$. We denote by $u_{(t,x)}$ the element of $C([-\alpha,\infty) \times [-\beta,\infty), \mathbb{R}^n)$ defined by

$$u_{(t,x)}(s,\tau) = u(t+s, x+\tau);\ (s,\tau) \in [-\alpha,0] \times [-\beta,0],$$

here $u_{(t,x)}(.,.)$ represents the history of the state from time $t-\alpha$ up to the present time t and from time $x-\beta$ up to the present time x.

In Section 5.3, we investigate the existence of solutions for impulsive hyperbolic differential equations with infinite delay

$$({}^cD^r_{z_k}u)(t,x) = f(t,x,u(t,x)),\ \text{ if } (t,x) \in J_k,\ k = 0, \dots, m, \tag{27}$$

$$u(t_k^+, x) = u(t_k^-, x) + I_k(u(t_k^-, x)), \text{ if } x \in [0, b];\ k = 1, \dots, m, \tag{28}$$

$$u(t, x) = \phi(t, x);\ \text{ if } (t, x) \in \tilde{J}', \tag{29}$$

$$u(t, 0) = \varphi(t),\ u(0, x) = \psi(x),\ t \in [0, a],\ x \in [0, b], \tag{30}$$

where $\varphi,\ \psi$ are as in problem (5.1)-(5.3), $\tilde{J}' =: (-\infty, +\infty)\times(-\infty, +\infty)\backslash[0, \infty)\times[0, \infty)$, $f : J \times \mathcal{B} \to \mathbb{R},\ \phi : \tilde{J}' \to \mathbb{R}^n$ and $\mathcal{B}$ is called a phase space that will be specified in
Finally, we present an illustrative example.

Chapter 1

Preliminaries

We introduce in this Chapter notations, definitions, fixed point theorems and preliminary facts that will be used in the remainder of this thesis.

1.1 Some Notations and definitions

Let $J := [0, a] \times [0, b],\ a, b > 0$. Denote L^1 the space of Lebegue integrable functions $u : J \to \mathbb{R}^n$ with the norm

$$||u||_{L^1} = \int_0^a \int_0^b ||u(t, x)||dxdt,$$

Let $L^\infty(J, \mathbb{R}^n)$ be the Banach space of mesurable functions $u : J \to \mathbb{R}^n$ wich are bounded, equiped with the norm

$$||u||_{L^\infty} = \inf\{c > 0 :\ ||u(t, x)|| \leq c,\ a.e.\ (t, x) \in J\}$$

As usual, by $AC(J, \mathbb{R}^n)$ we denote the space of absolutely continuous functions from J into $\mathbb{R}^n$, and $C(J, \mathbb{R}^n)$ is the Banach space of all continuous functions from J into $\mathbb{R}^n$ with the norm

$$||u||_\infty = \sup_{(t,x)\in J} ||u(t, x)||.$$

Also $C(J, \mathbb{R})$ is endowed with norm $||.||_\infty$ defined by

$$||u||_\infty = \sup_{(t,x)\in J} |u(t, x)|$$

If $u \in C([-\alpha, a] \times [-\beta, b], \mathbb{R}^n);\ a, b, \alpha, \beta > 0$ then for any $(t, x) \in J$ define $u_{(t,x)}$ by

$$u_{(t,x)}(s, \theta) = u(t + s, x + \theta)$$

for $(s, \theta) \in C([-\alpha, 0] \times [-\beta, 0], \mathbb{R}^n)$. Here $u_{(t,x)}(., .)$ represents the history of the state from time $t - \alpha$ up to the present time t and from time $x - \beta$ uo to the present time x.

1.2 Some properties of Partial Fractional Calculs

In this section, we introduce notations, definitions and preliminary Lemmas concerning to partial fractional calculs theory

Definition 1.1 [55] *Let* $r = (r_1, r_2) \in (0, \infty) \times (0, \infty)$ *and* $u \in L^1(J_k, \mathbb{R}^n)$*. The left-sided mixed Riemann-Liouville integral of order* r *of* u *is defined by*

$$(I^r_{z_k} u)(t, x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t} \int_0^x (t - s)^{r_1 - 1}(x - \tau)^{r_2 - 1} u(s, \tau) d\tau ds.$$

In particular,

$$(I^\sigma_{z_k} u)(t, x) = \int_0^t \int_0^x u(s, \tau) d\tau ds; \text{ for almost all } (t, x) \in J_k,$$

where $\sigma = (1, 1)$.
For instance, $I^r_{z_k} u$ exists for all $r_1, r_2 \in (0, \infty) \times (0, \infty)$, when $u \in L^1(J_k, \mathbb{R}^n)$. Note also that when $u \in C(J_k, \mathbb{R}^n)$, then $(I^r_{z_k} u) \in C(J_k, \mathbb{R}^n)$, moreover

$$(I^r_{z_k} u)(t, 0) = (I^r_{z_k} u)(0, x) = 0; \ (t, x) \in J_k.$$

By $1 - r$ we mean $(1 - r_1, 1 - r_2) \in (0, 1] \times (0, 1]$. Denote by $D^2_{tx} := \frac{\partial^2}{\partial t \partial x}$, the mixed second order partial derivative.

Definition 1.2 *[55] Let* $r \in (0, 1] \times (0, 1]$ *and* $u \in L^1(J_k, \mathbb{R}^n)$*. The mixed fractional Riemann-Liouville derivative of order* r *of* u *is defined by the expression*

$$D^r_{z_k} u(t, x) = (D^2_{tx} I^{1-r}_{z_k} u)(t, x)$$

and the Caputo fractional-order derivative of order r *of* u *is defined by the expression*

$$({}^c D^r_{z_k} u)(t, x) = (I^{1-r}_{z_k} \frac{\partial^2}{\partial t \partial x} u)(t, x).$$

and the mixed fractional Riemann-Liouville derivative of order R *of* u *defined by the expression* $({}^{RL} D^r_{z_k} u)(t, x) = D^2_{tx} I^{1-r}_{z_k} u)(t, x)$

The case $\sigma = (1, 1)$ is included and we have

$$(D^\sigma_{z_k} u)(t, x) = ({}^c D^\sigma_{z_k} u)(t, x) = (D^2_{tx} u)(t, x), \text{ for almost all } (t, x) \in J_k.$$

Remark 1.3 *(Relation between* ${}^{RL} D_{z_k}$ *and* ${}^c D_{z_k}$*) Let* $u \in L^1(J,)$ *and* $\varphi : [0, a] \to \mathbb{R}^n$*,* $\psi : [0, b] \to \mathbb{R}^n$ *be given absolutely continuous functions such that* $u(t, 0) = \varphi(t);\ t \in [0, a], u(0, x) = \psi(x);\ x \in [0, b]$ *and* $\varphi(0) = \psi(0)$*. Then we have for* $(t, x) \in J$

$$({}^{RL} D_{z_k} u)(t, x) = \lambda(t, x) + ({}^c D_{z_k} u)(t, x),$$

where

$$\lambda(t,x) = \frac{t^{-r_1}}{\Gamma(r_2)\Gamma(1-r_1)}\int_0^x (x-s)^{-r_2}\psi(s)ds + \frac{x^{-r_2}}{\Gamma(r_1)\Gamma(1-r_2)}\int_0^t (t-s)^{-r_1}\varphi(s)ds + \frac{t^{-r_1}x^{-r_2}\varphi(0)}{\Gamma(1-r_1)\Gamma(1-r_2)}$$

and the dot denotes differentiation

In the sequel we will make use of the following generalization of Gronwall's lemma for two independent variables and singular kernel.

Lemma 1.4 *([37]) Let $\upsilon : J \to [0,\infty)$ be a real function and $\omega(.,.)$ be a nonnegative, locally integrable function on J. If there are constants $c > 0$ and $0 < r_1, r_2 < 1$ such that*

$$\upsilon(t,x) \le \omega(t,x) + c\int_0^t\int_0^x \frac{\upsilon(s,\tau)}{(t-s)^{r_1}(x-\tau)^{r_2}}d\tau ds,$$

then there exists a constant $\delta = \delta(r_1, r_2)$ such that

$$\upsilon(t,x) \le \omega(t,x) + \delta c\int_0^t\int_0^x \frac{\omega(s,\tau)}{(t-s)^{r_1}(x-\tau)^{r_2}}d\tau ds,$$

for every $(t,x) \in J$.

1.3 Phase spaces

The notion of the phase space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory for functional differential equations. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato [32] (see [33, 41, 46]).

For any $(x,y) \in J$ denote $E_{(x,y)} := [0,x] \times \{0\} \cup \{0\} \times [0,y]$, furthermore in case $x = a$, $y = b$ we write simply E. Consider the space $(\mathcal{B}, \|(.,.)\|_{\mathcal{B}})$ is a seminormed linear space of functions mapping $(-\infty,0] \times (-\infty,0]$ into $\mathbb{R}^n$, and satisfying the following fundamental axioms which were adapted from those introduced by Hale and Kato for ordinary differential functional equations:

(A_1) If $z : (-\infty,a] \times (-\infty,b] \to \mathbb{R}^n$ and $z_{(x,y)} \in \mathcal{B}$, for all $(x,y) \in E$, then there are constants $H, K, M > 0$ such that for any $(x,y) \in J$ the following conditions hold:

- (i) $z_{(x,y)}$ is in $\mathcal{B}$;
- (ii) $\|z(x,y)\| \le H\|z_{(x,y)}\|_{\mathcal{B}}$,
- (iii) $\|z_{(x,y)}\|_B \le K \sup_{(s,t)\in[0,x]\times[0,y]} \|z(s,t)\| + M \sup_{(s,t)\in E_{(x,y)}} \|z_{(s,t)}\|_{\mathcal{B}}$,

(A_2) The space $\mathcal{B}$ is complete.

Now, we present some examples of phase spaces ([26, 27]).

Example 1.5 *Let $\mathcal{B}$ be the set of all functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ which are piece-wise continuous on $[-\alpha, 0] \times [-\beta, 0]$, $\alpha, \beta \geq 0$, with the seminorm*

$$\|\phi\|_{\mathcal{B}} = \sup_{(s,t)\in[-\alpha,0]\times[-\beta,0]} \|\phi(s,t)\|.$$

Then we have $H = K = M = 1$. The quotient space $\widehat{\mathcal{B}} = \mathcal{B}/\|.\|_{\mathcal{B}}$ is isometric to the space $C([-\alpha, 0] \times [-\beta, 0], \mathbb{R}^n)$ of all continuous functions from $[-\alpha, 0] \times [-\beta, 0]$ into $\mathbb{R}^n$ with the supremum norm, this means that partial differential functional equations with finite delay are included in our axiomatic model.

Example 1.6 *Let $\gamma \in \mathbb{R}$, and C_γ be the set of all piece-wise continuous functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ for which a limit $\lim_{\|(s,t)\|\to\infty} e^{\gamma(s+t)}\phi(s,t)$ exists, with the norm*

$$\|\phi\|_{C_\gamma} = \sup_{(s,t)\in(-\infty,0]\times(-\infty,0]} e^{\gamma(s+t)}\|\phi(s,t)\|.$$

Then we have $H = 1$ and $K = M = \max\{e^{-\gamma(a+b)}, 1\}$.

Example 1.7 *Let $\alpha, \beta, \gamma \geq 0$ and let*

$$\|\phi\|_{CL_\gamma} = \sup_{(s,t)\in[-\alpha,0]\times[-\beta,0]} \|\phi(s,t)\| + \int_{-\infty}^{0}\int_{-\infty}^{0} e^{\gamma(s+t)}\|\phi(s,t)\|dtds.$$

be the seminorm for the space CL_γ of all functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ which are piece-wise continuous on $[-\alpha, 0] \times [-\beta, 0]$ measurable on $(-\infty, -\alpha] \times (-\infty, 0] \cup (-\infty, 0] \times (-\infty, -\beta]$, and such that $\|\phi\|_{CL_\gamma} < \infty$. Then

$$H = 1,\ K = \int_{-\alpha}^{0}\int_{-\beta}^{0} e^{\gamma(s+t)}dtds,\ M = 2.$$

1.4 Some properties in Fréchet spaces

Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies :

$$\|u\|_1 \leq \|u\|_2 \leq \|u\|_3 \leq ... \quad for\ every\ u \in X.$$

Let $Y \subset X$, we say that Y is bounded if for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that

$$\|y\|_n \leq \overline{M}_n \quad for\ all\ v \in Y.$$

To X we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows : For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by : $u \sim_n v$ if and only if $\|u-v\|_n = 0$ for $u, v \in X$. We denote $X^n = (X|_{\sim_n}, \|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y \subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows : For every $u \in X$, we denote $[u]_n$ the equivalence class of u of subset X^n and we defined $Y^n = \{[u]_n : u \in Y\}$. We denote $\overline{Y^n}$, $int_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|_n$ in X^n. For more information about this subject see [29].

Definition 1.8 *Let X be a Fréchet space. A function $N : X \longrightarrow X$ is said to be a contraction if for each $n \in \mathbb{N}$ there exists $k_n \in (0,1)$ such that*

$$\|N(u) - N(v)\|_n \leq k_n \|u - v\|_n \quad \textit{for all } u, v \in X.$$

1.5 Some fixed point theorems

Theorem 1.9 *(Nonlinear Alternative of Frigon and Granas, [29]).*
Let X be a Fréchet space *and $Y \subset X$ a closed subset in Y and let $N : Y \to X$ be* a contraction *such that $N(Y)$ is bounded.*
Then one of the following statements holds :

$(S1)$ *N has a unique fixed point ;*

$(S2)$ *There exists $\lambda \in [0,1)$, $n \in \mathbb{N}$ and $x \in \partial_n Y^n$ such that $\|x - \lambda\ N\ (x)\|_n = 0$.*

Theorem 1.10 *(Burton-Kirk)([25]) Let X be a Banach space, and $A, B : X \to X$ two operators satisfying:*

(i) *A is completely continuous, and*

(ii) *B is a contraction.*

Then either

(a) *the operator equation $u = A(u) + B(u)$ has a solution, or*

(b) *the set $\mathcal{E} = \{u \in X : u = \lambda A(u) + \lambda B(\frac{u}{\lambda}), for \lambda \in (0,1)\}$ is unbounded .*

Chapter 2

Impulsive Partial Hyperbolic Differential Equations of Fractional Order

2.1 Introduction

This chapter deals with the existence of solutions for impulsive initial value problem for differential equations of fractional order given by

$$({}^cD^r_{z_k}u)(t,x) = f(t,x,u(t,x)), \ \text{ if } (t,x) \in J_k, \ k = 0,\dots,m, \tag{2.1}$$

$$u(t_k^+,x) = u(t_k^-,x) + I_k(u(t_k^-,x)), \text{ if } x \in [0,b]; \ k = 1,\dots,m, \tag{2.2}$$

$$u(t,0) = \varphi(t), \ u(0,x) = \psi(x), \ t \in [0,a], \ x \in [0,b], \tag{2.3}$$

where $J_0 = [0,t_1] \times [0,b]$, $J_k := (t_k,t_{k+1}] \times [0,b]$, $k = 1,\dots,m, z_k = (t_k,0)$, $k = 0,\dots,m$, $J = [0,a] \times [0,b]$, $a,b > 0$, ${}^cD^r_0$ is the Caputo fractional derivative of order $r = (r_1,r_2) \in (0,1] \times (0,1]$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = a$, $f : J \times \mathbb{R}^n \to \mathbb{R}^n$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 0,1,\dots,m$ are given functions, $\varphi : [0,a] \to \mathbb{R}^n$ and $\psi : [0,b] \to \mathbb{R}^n$ are given absolutely continuous functions with $\varphi(0) = \psi(0)$.

Next we consider the following nonlocal initial value problem

$$({}^cD^r_{z_k}u)(t,x) = f(t,x,u(t,x)), \ \text{ if } (t,x) \in J_k, \ k = 0,\dots,m, \tag{2.4}$$

$$u(t_k^+,x) = u(t_k^-,x) + I_k(u(t_k^-,x)), \ \text{ if } \ x \in [0,b]; \ k = 1,\dots,m, \tag{2.5}$$

$$u(t,0) + Q(u) = \varphi(t), \ u(0,x) + K(u) = \psi(x), \ t \in [0,a], \ x \in [0,b], \tag{2.6}$$

where f, φ, ψ, I_k; $k = 1,\dots,m$, are as in problem (2.1)-(2.3) and $Q, K : PC(J,\mathbb{R}^n) \to \mathbb{R}^n$ are continuous functions. $PC(J,\mathbb{R}^n)$ is a Banach space to be specified later.

2.2 Existence of Solutions

First of all, we define what we mean by a solution of the initial value problem (2.1)–(2.3).

$$PC(J,\mathbb{R}^n) = \big\{u : J \to \mathbb{R}^n : u \in C((t_k,t_{k+1}] \times [0,b],\mathbb{R}^n);\ k = 1,\dots,m, \text{ and there exist } u(t_k^-,x) \text{ and } u(t_k^+,x); k = 1,\dots,m, \text{ with } u(t_k^-,x) = u(t_k,x)\big\}.$$

This set is a Banach space with the norm

$$\|u\|_{PC} = \sup_{(t,x)\in J} |u(t,x)|.$$

Definition 2.1 *A function $u \in PC(J,\mathbb{R}^n)$ is said to be a solution of (2.1)–(2.3) if u satisfies the equation $(^cD^r u)(t,x) = f(t,x,u(t,x))$ on J', and conditions (2.2), (2.3) are satisfied.*

Let $h(t,x) \in C((t_k,t_{k+1}] \times [0,b],\mathbb{R}^n)$, $\quad z_k = (t_k,0)$, and

$$\mu_k(t,x) = u(t,0) + u(t_k^+,x) - u(t_k^+,0),\ k = 0,\dots,m.$$

For the existence of solutions for the problem (2.1)–(2.2), we need the following lemma:

Lemma 2.2 *A function $u \in C((t_k,t_{k+1}] \times [0,b],\mathbb{R}^n)$, $k = 0,\dots,m$ is a solution of the differential equation*

$$(^cD^r_{z_k} u)(t,x) = h(t,x);\ (t,x) \in (t_k,t_{k+1}] \times [0,b], \tag{2.7}$$

if and only if $u(t,x)$ satisfies

$$u(t,x) = \mu_k(t,x) + (I^r_{z_k} h)(t,x);\ (t,x) \in (t_k,t_{k+1}] \times [0,b]. \tag{2.8}$$

Proof. Let $u(t,x)$ be a solution of (2.7). Then, from the definition of $(^cD^r_{z_k^+} u)(x,x)$, we have

$$I^{1-r}_{z_k^+}(D^2_{tx} u)(t,x) = h(t,x).$$

Which yield

$$I^r_{z_k^+}(I^{1-r}_{z_k} D^2_{tx} u)(t,x) = (I^r_{z_k^+} h)(t,x),$$

then

$$I^1_{z_k^+} D^2_{tx} u(t,x) = (I^r_{z_k^+} h)(t,x).$$

Since

$$I^1_{z_k^+}(D^2_{tx} u)(t,x) = u(t,x) - u(t,0) - u(t_k^+,x) + u(t_k^+,0),$$

we have

$$u(t,x) = \mu_k(t,x) + (I^r_{z_k^+} h)(t,x).$$

Now let $u(t,x)$ satisfies (2.8). It is clear that $u(t,x)$ satisfies

$$(^cD^r_{z_k} u)(t,x) = h(t,x), \quad \text{on} \quad (t_k,t_{k+1}] \times [0,b].$$

In all what follows set

$$\mu(t,x) = \mu_0(t,x),\ (t,x) \in J.$$

Lemma 2.3 *Let $0 < r_1, r_2 \leq 1$ and let $h : J \to \mathbb{R}^n$ be continuous. A function u is a solution of the fractional integral equation*

$$u(t,x) = \begin{cases} \mu(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} h(s,y) ds dy & \text{if } (t,x) \in [0,t_1] \times [0,b], \\ \mu(t,x) + \sum_{i=1}^{k} \Big(I_i(u(t_i^-, x)) - I_i(u(t_i^-, 0)) \Big) & \text{if } (t,x) \in (t_k, t_{k+1}] \times [0,b], \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \int_0^x (t_i - s)^{r_1-1}(x-y)^{r_2-1} h(s,y) ds dy & k = 1, \dots, m, \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} h(s,y) ds dy & \end{cases} \tag{2.9}$$

if and only if u is a solution of the fractional initial value problem

$$^cD^r u(t,x) = h(t,x), \quad (t,x) \in J', \tag{2.10}$$

$$u(t_k^+, x) = u(t_k^-, x) + I_k(u(t_k^-, x)), \quad k = 1, \dots, m. \tag{2.11}$$

Proof. Assume u satisfies (2.10)-(2.11). If $(t,x) \in [0,t_1] \times [0,b]$ then

$$^cD^r u(t,x) = h(t,x).$$

Lemma 2.2 implies

$$u(t,x) = \mu(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} h(s,y) ds dy.$$

If $(t,x) \in (t_1, t_2] \times [0,b]$ then Lemma 2.2 implies

$$\begin{aligned} u(t,x) &= \mu_1(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} h(s,y) ds dy \\ &= \varphi(t) + u(t_1^+, x) - u(t_1^+, 0) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} h(s,y) ds dy \\ &= \varphi(t) + u(t_1^-, x) - u(t_1^-, 0) + I_1(u(t_1^-, x)) - I_1(u(t_1^-, 0)) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} h(s,y) ds dy \\ &= \varphi(t) + u(t_1, x) - u(t_1, 0) + I_1(u(t_1^-, x)) - I_1(u(t_1^-, 0)) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} h(s,y) ds dy \\ &= \mu(t,x) + I_1(u(t_1^-, x)) - I_1(u(t_1^-, 0)) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{t_1} \int_0^x (t_1-s)^{r_1-1}(x-y)^{r_2-1} h(s,y) ds dy \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} h(s,y) ds dy. \end{aligned}$$

If $(t,x)\in(t_2,t_3]\times[0,b]$ then again from Lemma 2.2 we get

$$\begin{aligned}
u(t,x) &= \mu_2(t,x)+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_2}^{t}\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}h(s,y)dsdy\\
&= \varphi(t)+u(t_2^+,x)-u(t_2^+,0)\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_2}^{t}\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}h(s,y)dsdy\\
&= \varphi(t)+u(t_2^-,x)-u(t_2^-,0)+I_2(u(t_2^-,x))-I_2(u(t_2^-,0))\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_2}^{t}\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}h(s,y)dsdy\\
&= \varphi(t)+u(t_2,x)-u(t_2,0)+I_2(u(t_2^-,x))-I_2(u(t_2^-,0))\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_2}^{t}\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}h(s,y)dsdy\\
&= \mu(t,x)+I_2(u(t_2^-,x))-I_2(u(t_2^-,0))+I_1(u(t_1^-,x))-I_1(u(t_1^-,0))\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{t_1}\int_0^x (t_1-s)^{r_1-1}(x-y)^{r_2-1}h(s,y)dsdy\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_0^x (t_2-s)^{r_1-1}(x-y)^{r_2-1}h(s,y)dsdy\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_2}^{t}\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}h(s,y)dsdy.
\end{aligned}$$

If $(t,x)\in(t_k,t_{k+1}]\times[0,b]$ then from Lemma 2.2 we get (2.9).

Conversely, assume that u satisfies the impulsive fractional integral equation (2.9). If $(t,y)\in[0,t_1]\times[0,b]$ and using the fact that ${}^cD^r$ is the left inverse of I^r we get

$${}^cD^r u(t,x)=h(t,x),\quad \text{for each } (t,x)\in[0,t_1]\times[0,b].$$

If $(t,x)\in[t_k,t_{k+1})\times[0,b]$, $k=1,\dots,m$ and using the fact that ${}^cD^rC=0$, where C is a constant, we get

$${}^cD^r u(t,x)=h(t,x),\text{for each } (t,x)\in[t_k,t_{k+1})\times[0,b].$$

Also, we can easily show that

$$u(t_k^+,x)=u(t_k^-,x)+I_k(u(t_k^-,x)),\quad x\in[0,b],k=1,\dots,m.$$

In this section, we give our main existence result for problem (2.1)-(2.3).

Definition 2.4 *A function $u\in PC(J,\mathbb{R}^n)$ with its mixed derivative D^2_{xy} exists and is integrable is said to be a solution of (2.1)–(2.3) if u satisfies the equation $({}^cD^r u)(t,x)=f(t,x,u(t,x))$, and conditions (2.2), (2.3) are satisfied.*

Our result is based upon the fixed point theorem due to Burton and Kirk.

Let us introduce the following hypotheses which are assumed after:

$(H1)$ The functions $I_k : \mathbb{R}^n \to \mathbb{R}^n$ are continuous.

$(H2)$ There exist $p, q \in C(J, \mathbb{R}_+)$ such that

$$\|f(t,x,u)\| \leq p(t,x) + q(t,x)\|u\|, \text{ for } (t,x) \in J \text{ and each } u \in \mathbb{R}^n.$$

$(H3)$ There exists $l > 0$ such that

$$\|I_k(u) - I_k(v)\| \leq l\|u - v\| \text{ for each } \ u, v \in \mathbb{R}^n$$

Theorem 2.5 *Assume that hypotheses* $(H1)$*-*$(H3)$ *hold. If*

$$2ml + \frac{2a^{r_1}b^{r_2}p^*}{\Gamma(r_1+1)\Gamma(r_2+1)} < 1, \tag{2.12}$$

then the IVP *(2.1)-(2.3) has at least one solution on* J

Proof: We shall reduce the existence of solutions of (2.1)-(2.3) to a fixed point problem. Consider the operator $N : PC(J, \mathbb{R}^n) \longrightarrow PC(J, \mathbb{R}^n)$ defined by

$$\begin{aligned} N(u)(t,x) &= \mu(t,x) + \sum_{0<t_k<t} (I_k(u(t_k^-,x)) - I_k(u(t_k^-,0))) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} \int_0^x (t_k - s)^{r_1-1}(x-y)^{r_2-1} f(s,y,u(s,y))dyds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t} \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} f(s,y,u(s,y))dyds, \end{aligned}$$

and the operators $F, G : PC(J, \mathbb{R}^n) \longrightarrow PC(J, \mathbb{R}^n)$

$$F(u)(t,x) = \mu(t,x) + \sum_{0<t_k<t} (I_k(u(t_k^-,x)) - I_k(u(t_k^-,0)))$$

$$\begin{aligned} G(u)(t,x) &= \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} \int_0^x (t_k - s)^{r_1-1}(x-y)^{r_2-1} f(s,y,u(s,y))dyds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t} \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} f(s,y,u(s,y))dyds. \end{aligned}$$

Then the problem of finding the solution of the IVP (2.1)–(2.3) is reduced to finding the solutions of the operator equation $F(u)+G(u) = u$. We shall show that the operators

F and G satisfy the conditions of Theorem (2.5). The proof will be given by several steps.

Step 1: G is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $PC(J, \mathbb{R}^n)$, then for each $(t,x) \in J$

$$\begin{aligned}
&\|G(u_n)(t,x) - G(u)(t,x)\| \\
&\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^x (t_k - s)^{r_1-1}(x-y)^{r_2-1} \|f(s,y,u_n(s,y)) - f(s,y,u(s,y))\| dyds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t} \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \|f(s,y,u_n(s,y)) - f(s,y,u(s,y))\| dyds. \\
&\leq \frac{\|f(.,.,u_{n(.,.)}) - f(.,.,u_{(.,.)})\|}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^x (t_k - s)^{r_1-1}(x-y)^{r_2-1} dyds \\
&+ \frac{\|f(.,.,u_{n(.,.)}) - f(.,.,u_{(.,.)})\|}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t} \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} dyds.
\end{aligned}$$

Since f is continuous function, we have

$$\|G(u_n) - G(u)\|_{PC} \leq \frac{2a^{r_1}b^{r_2}\|f(.,.,u_{n(.,.)}) - f(.,.,u_{(.,.)})\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)} \to 0 \text{ as } n \to \infty$$

Thus G is continuous.

Step 2: G maps bounded sets into bounded sets in $PC(J, \mathbb{R}^n)$. Indeed, it is enough to show that for any positive real number η^*, there exists a positive constant l such that, for each $u \in B_{\eta^*} = \{u \in PC(J, \mathbb{R}^n) : \|u\|_{PC} \leq \eta^*\}$ we have $\|G(u)\|_{PC} \leq l$
By $(H2)$ we have for each $(t,x) \in J$,

$$\begin{aligned}
\|G(u)(t,x)\| &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^x (t_k - s)^{r_1-1}(x-y)^{r_2-1} \|f(s,y,u(s,y))\| dsdy \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t} \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \|f(s,y,u(s,y))\| dsdy \\
&\leq \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^x (t_k - s)^{r_1-1}(x-y)^{r_2-1} dsdy \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \int_{t_k}^{t} \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} dsdy
\end{aligned}$$

Thus

$$\|G(u)\|_{PC} \leq \frac{2a^{r_1}b^{r_2}(\|p\|_\infty + \|q\|_\infty \eta^*)}{\Gamma(r_1+1)\Gamma(r_2+1)} := l^*$$

Hence $\|G(u)\|_{PC} \leq l^*$.

Step 3: G maps bounded sets into equicontinuous sets in PC.
Let $(t_1, x_1), (t_2, x_2) \in (0, a] \times (0, b]$, $t_1 < t_2$, $x_1 < x_2$, B_η be a bounded set as in step 2,

let $u \in B_{\eta^*}$ be a bounded set of $PC(J, \mathbb{R}^n)$ as in Step 2. Then

$$
\begin{aligned}
&\|G(u)(t_2,x_2) - G(u)(t_1,x_1)\| \\
&\le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^{x_1} (t_k - s)^{r_1-1}[(x_2-\tau)^{r_2-1} - (x_1-\tau)^{r_2-1}] \times |f(s,\tau,u(s,\tau))| d\tau ds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_{x_1}^{x_2} (t_k - s)^{r_1-1}(x_2-\tau)^{r_2-1} |f(s,\tau,u(s,\tau))| d\tau ds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{t_1} \int_0^{x_1} [(t_2 - s)^{r_1-1}(x_2-\tau)^{r_2-1} - (t_1 - s)^{r_1-1}(x_1-\tau)^{r_2-1}] \times |f(s,\tau,u(s,\tau))| d\tau dx \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^{t_2} \int_{x_1}^{x_2} (t_2 - s)^{r_1-1}(x_2-\tau)^{r_2-1} |f(s,\tau,u(s,\tau))| d\tau ds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{t_1} \int_{x_1}^{x_2} (t_2 - s)^{r_1-1}(x_2-\tau)^{r_2-1} |f(s,\tau,u(s,\tau))| d\tau ds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^{t_2} \int_0^{x_1} (t_2 - s)^{r_1-1}(x_2-\tau)^{r_2-1} |f(s,\tau,u(s,\tau))| d\tau ds \\
&\le \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^{x_1} (t_k - s)^{r_1-1}[(x_2-\tau)^{r_2-1} - (x_1-\tau)^{r_2-1}] d\tau ds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_{x_1}^{x_2} (t_k - s)^{r_1-1}(x_2-\tau)^{r_2-1} d\tau ds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \int_0^{t_1} \int_0^{x_1} [(t_2 - s)^{r_1-1}(x_2-\tau)^{r_2-1} - (t_1 - s)^{r_1-1}(x_1-\tau)^{r_2-1}] d\tau dx \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \int_{t_1}^{t_2} \int_{x_1}^{x_2} (t_2 - s)^{r_1-1}(x_2-\tau)^{r_2-1} d\tau ds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \int_0^{t_1} \int_{x_1}^{x_2} (t_2 - s)^{r_1-1}(x_2-\tau)^{r_2-1} d\tau ds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \int_{t_1}^{t_2} \int_0^{x_1} (t_2 - s)^{r_1-1}(x_2-\tau)^{r_2-1} d\tau ds \\
&\le \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^{x_1} (t_k - s)^{r_1-1}[(x_2-\tau)^{r_2-1} - (x_1-\tau)^{r_2-1}] d\tau ds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_{x_1}^{x_2} (t_k - s)^{r_1-1}(x_2-\tau)^{r_2-1} d\tau ds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} [2x_2^{r_2}(t_2-t_1)^{r_1} + 2t_2^{r_1}(x_2-x_1)^{r_2} \\
&+ t_1^{r_1}x_1^{r_2} - t_2^{r_1}x_2^{r_2} - 2(t_2-t_1)^{r_1}(x_2-x_1)^{r_2}].
\end{aligned}
$$

As $t_1 \to t_2$, $x_1 \to x_2$ the right-hand side of the above inequality tends to zero.

As a consequence of Steps 1 to 3, together with the Arzela-Ascoli theorem, we can conclude that $G : PC(J, \mathbb{R}^n) \to PC(J, \mathbb{R}^n)$ is continuous and completely continuous.

Step 4: F is a contraction.
Let $u, v \in PC(J, \mathbb{R}^n)$, then we have for each $(t, x) \in J$

$$\begin{aligned}
&|F(u)(t,x) - F(v)(t,x)| \\
&\leq \sum_{k=1}^{m} (|I_k(u(t_k^-, x)) - I_k(v(t_k^-, x))| + |I_k(u(t_k^-, 0)) - I_k(v(t_k^-, 0))|) \\
&\leq \sum_{k=1}^{m} l(\|u - v\|_{PC} + \|u - v\|_{PC}) \\
&\leq 2ml\|u - v\|_{PC}.
\end{aligned}$$

Thus

$$\|F(u) - F(v)\|_{PC} \leq 2ml\|u - v\|_{PC}.$$

Hence by (2.12), F is a contraction.

Step 5: (A priori bounds)

Now it remains to show that the set

$$\mathcal{E} = \{u \in PC(J, \mathbb{R}^n) : u = \lambda F(\frac{u}{\lambda}) + \lambda G(u), \text{ for some } \lambda \in (0, 1)\}$$

is bounded. Let $u \in \mathcal{E}$, then $u = \lambda F(\frac{u}{\lambda}) + \lambda G(u)$. Thus, for each $(t, x) \in J$ we have

$$\begin{aligned}
u(t,x) &= \lambda\mu(t,x) + \sum_{k=1}^{m} \lambda(|I_k \frac{u(t_k^-, y)}{\lambda}| + |I_k \frac{u(t_k^-, 0)}{\lambda}|) \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s, \tau, u(s, \tau)) d\tau ds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t} \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s, \tau, u(s, t)) d\tau ds.
\end{aligned}$$

This implies by $(H2)$ and $(H3)$ that, for each $(t,x)\in J$, we have

$$\begin{aligned}
|u(t,x)| &\leq \lambda|\mu(t,x)| + \sum_{k=1}^{m}\lambda(|I_k\frac{u(t_k^-,x)}{\lambda}| - |I_k(0)| + |I_k\frac{u(t_k^-,0)}{\lambda}| - |I_k(0)|) \\
&+ 2\lambda\sum_{k=1}^{m}|I_k(0)| + \frac{\|p\|_\infty}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}|u(s,\tau)|_{PC}d\tau ds \\
&+ \frac{\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}d\tau ds \\
&+ \frac{\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}|u(s,\tau)|_{PC}d\tau ds \\
&+ \frac{\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}d\tau ds \\
&\leq \|\mu(t,x)\| + l\sum_{k=1}^{m}(|u(t_k^-,x)| + |u(t_k^-,0)|) + 2I^* \\
&+ \frac{\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}|u(s,\tau)|_{PC}d\tau ds \\
&+ \frac{\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}d\tau ds \\
&+ \frac{\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}|u(s,\tau)|_{PC}d\tau ds \\
&+ \frac{\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}d\tau ds,
\end{aligned}$$

where

$$I^* = \sum_{k=1}^{m}|I_k(0)|.$$

We have for $(t,x)\in J$

$$\begin{aligned}
\|u(t,x)\| &\leq \|\mu\| + 2ml\|u\|_{PC} + 2I^* \\
&+ \frac{2a^{r_1}b^{r_2}\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)} + \frac{2a^{r_1}b^{r_2}\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\|u\|_{PC}.
\end{aligned}$$

Then

$$\|u\|_{PC}\Big(1 - 2ml - \frac{2a^{r_1}b^{r_2}\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\Big) \leq |\mu\|_\infty + 2I^* + \frac{2a^{r_1}b^{r_2}\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)} := M.$$

Thus

$$\|u\|_{PC} \leq \frac{M}{1 - 2ml - \frac{2a^{r_1}b^{r_2}p^*}{\Gamma(r_1+1)\Gamma(r_2+1)}} := M^*.$$

This shows that the set $\mathcal{E}$ is bounded. As a consequence of Theorem 2.5 we deduce that $F+G$ has a fixed point which is a solution of problem (2.1)-(2.3).

2.3 Nonlocal impulsive partial differential equations

Now we present (without proof) an existence result for the nonlocal initial value problem (2.4)–(2.6).

Definition 2.6 *A function $u \in PC(J, \mathbb{R}^n)$ is said to be a solution of integrable is said to be a solution of (4.10)–(4.12) if u satisfies the equations (2.4)–(2.6) on J, and conditions (2.5), (2.6) are satisfied.*

Theorem 2.7 *Assume $(H1)$-$(H3)$ hold, and moreover we assume that*

$(H'1)$ *There exists $\tilde{k} > 0$ such that*

$$\|Q(u) - Q(v)\| \leq \tilde{k}\|u - v\|_{PC}, \text{ for any } u,\ v \in PC(J, \mathbb{R}^n)$$

$(H'2)$ *There exists $k^* > 0$ such that*

$$\|K(u) - K(v)\| \leq k^*\|u - v\|_{PC}, \text{ for any } u,\ v \in PC(J, \mathbb{R}^n)$$

hold. If

$$\tilde{k} + k^* + 2ml + \frac{2a^{r_1}b^{r_2}p^*}{\Gamma(r_1+1)\Gamma(r_2+1)} < 1,$$

then there exists at least one solution for IVP (2.4)–(2.6) on J.

2.4 An Example

In this section we give an example to illustrate the usefulness of our main results. Let us consider the following impulsive partial hyperbolic differential equations of the form

$$({}^cD_0^r u)(t,x) = \frac{1}{8e^{t+x+3}} \frac{|u(t,x)|}{(1+|u(t,x)|)}, \quad \text{if } (t,x) \in J_0 \cup J_1, \tag{2.13}$$

$$u(\frac{1}{3}^+, x) = u(\frac{1}{3}^-, x) + \frac{1}{6e^{t+x+4}} \frac{|u(\frac{1}{3}^-, x)|}{(15+|u(\frac{1}{3}^-, x)|)}, \text{ if } x \in [0,1], \tag{2.14}$$

$$u(t,0) = t,\ u(0,x) = x^2,\ t \in [0,1],\ x \in [0,1], \tag{2.15}$$

where $J_0 = [0, \frac{1}{3}] \times [0, 1]$, $J_1 = (\frac{1}{3}, 1] \times [0, 1]$. Set

$$f(t, x, u) = \frac{1}{8e^{t+x+3}} \frac{|u(t, x)|}{(1 + |u(t, x)|)}, \; (t, x) \in [0, 1] \times [0, 1].$$

Hence (H_2) is satisfied with

$$p(t, x) = \frac{1}{8e^{t+x+3}} \quad \textit{and} \quad q(t, x) = \frac{2}{8e^{t+x+3}}$$

and

$$I_k(u(t_k^-, x)) = \frac{1}{6e^{t+x+4}} \frac{|u(t_k{}^-, x)|}{(15 + |u(t_k{}^-, x)|)}, \; x \in [0, 1].$$

For each u, v and $(t, x) \in [0, 1] \times [0, 1]$ we have

$$|I_k(u) - I_k(v)| \leq \frac{1}{6e^4} |u - v|.$$

Hence (H_3) is satisfied. We shall show that condition (4.15) holds with $a = b = 1$ and $m = 1$. Indeed,

$$2ml + \frac{2a^{r_1} b^{r_2} p^*}{\Gamma(r_1 + 1)\Gamma(r_2 + 1)} = \frac{1}{3e^4} + \frac{1}{4\Gamma(r_1 + 1)\Gamma(r_2 + 1)} < 1,$$

which is satisfied for each $(r_1, r_2) \in (0, 1] \times (0, 1]$. Consequently Theorem 2.5 implies that problem (5.10)-(5.12) has a solution defined on $[0, 1] \times [0, 1]$.

Chapter 3

Impulsive Partial Hyperbolic Functional Differential Equations of Fractional Order with Delay

3.1 Introduction

In this Chapter, we prove sufficient conditions for the existence solutions for fractional impulsive partial differential equations with delay,

3.2 Impulsive Partial Differential Equations with Finite Delay

3.2.1 Introduction

In this section, we study the existence of solutions for the following impulsive partial hyperbolic differential equations:

$$({}^cD^r_{z_k}u)(t,x) = f(t,x,u_{(t,x)}); \quad \text{if } (t,x) \in J_k, \quad k = 0,\dots,m, \tag{3.1}$$

$$u(t_k^+,x) = u(t_k^-,x) + I_k(u(t_k^-,x)), \quad \text{if } x \in [0,b], \ k = 1,\dots,m, \tag{3.2}$$

$$u(t,x) = \phi(t,x); \ \text{if } (t,x) \in \tilde{J}, \tag{3.3}$$

$$u(t,0) = \varphi(t), \ t \in [0,a], \ u(0,x) = \psi(x); \ x \in [0,b], \tag{3.4}$$

where $J_0 = [0,t_1] \times [0,b]$, $J_k := (t_k,t_{k+1}] \times [0,b]$, $k = 1,\dots,m, z_k = (t_k,0)$, $k = 0,\dots,m$, $a,b,\alpha,\beta > 0, J = [0,a] \times [0,b]$, $\tilde{J} = [-\alpha,a] \times [-\beta,b] \backslash (0,a] \times (0,b]$, ${}^cD^r_{z_k}$ is the Caputo fractional derivative of order $r = (r_1,r_2) \in (0,1] \times (0,1]$, $\varphi : [0,a] \to \mathbb{R}^n$, $\psi : [0,b] \to \mathbb{R}^n$ are given continuous functions with $\varphi(t) = \phi(t,0)$, $\psi(x) = \phi(0,x)$ for each $(t,x) \in J$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = a$, $f : J \times C \to \mathbb{R}^n$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k =$

$1,\ldots,m,\ \phi:\tilde{J}\to\mathbb{R}^n$, are given functions and $C := C([-\alpha,0]\times[-\beta,0],\mathbb{R}^n)$ is the space of continuous functions on $[-\alpha,0]\times[-\beta,0]$.
If $u:[-\alpha,0]\times[-\beta,0]\longrightarrow\mathbb{R}^n$, then for any $(t,x)\in J$ define $u_{(t,x)}$ by

$$u_{(t,x)}(s,\tau)=u(t+s,x+\tau)$$

3.2.2 Existence Results

In this section, we give our main existence result for problem (5.1)-(5.4).
Set $J_k=(t_k,t_{k+1}]\times(0,b]$. Consider the Banach space

$$\begin{aligned}PC &:= PC(J,\mathbb{R}^n)\\ &= \big\{u:J\to\mathbb{R}^n : u\in C(J_k,\mathbb{R}^n), k=1,\ldots,m,\ \text{and there exist } u(t_k^-,x) \text{ and}\\ &\quad u(t_k^+,x), k=1,\ldots,m,\ \text{with } u(t_k^-,x)=u(t_k,x)\big\}.\end{aligned}$$

with the norm

$$\|u\|_{PC}=\sup_{(t,x)\in J}|u(t,x)|.$$

Set

$$\Omega=\{u:[-\alpha,a]\times[-\beta,b]\longrightarrow\mathbb{R}^n),u|_{\tilde{J}}\in C \text{ and } u|_{[0,a]\times[0,b]}\in PC\}.$$

Definition 3.1 A function $u\in\Omega$ such that its mixed derivative D^2_{tx} exists on J' is said to be a solution of (5.1)-(5.4) if u satisfies the condition (5.3) on $\tilde{J}$, the equation (5.1) on J' and conditions (5.2), (5.4) are satisfied on J.

Lemma 3.2 *[7] Let $0<r_1,r_2\le 1$ and let $h:J\to\mathbb{R}^n$ be continuous. A function u is a solution of the fractional integral equation*

$$u(t,x)=\begin{cases}\phi(t,x) & \text{if } (t,x)\in\tilde{J},\\ \mu(t,x)+\sum\limits_{0<t_k<t}\Big(I_k(u(t_k^-,x))-I_k(u(t_k^-,0))\Big) & \text{if } (t,x)\in J,\\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum\limits_{0<t_k<t}\int_{t_{k-1}}^{t_k}\int_0^x(t_k-s)^{r_1-1}(x-y)^{r_2-1}h(s,y)dyds & k=1,\ldots,m,\\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_k}^{t}\int_0^x(t-s)^{r_1-1}(x-y)^{r_2-1}h(s,y)dyds, & \end{cases}\tag{3.5}$$

if and only if u is a solution of the fractional initial value problem

$$^cD^r u(t,x)=h(t,x),\quad (t,x)\in J_k,\ k=0,\ldots,m,\tag{3.6}$$

$$u(t_k^+,x)=u(t_k^-,x)+I_k(u(t_k^-,x)),\quad k=1,\ldots,m.\tag{3.7}$$

Our result is based upon the fixed point theorem due to Burton and Kirk. Let us introduce the following hypotheses which are assumed to hold in the sequel.

$(H1)$ The functions $I_k:\mathbb{R}^n\to\mathbb{R}^n$ are continuous.

$(H2)$ There exist $p, q \in C(J, \mathbb{R}_+)$ such that

$$\|f(t,x,u)\| \leq p(t,x) + q(t,x)\|u\|_C, \text{ for } (t,x) \in J \text{ and each } u \in C.$$

$(H3)$ There exists $l > 0$ such that

$$\|I_k(u) - I_k(v)\| \leq l\|u - v\| \text{ for each } u, v \in \mathbb{R}^n, \ k = 1, \ldots, m.$$

Theorem 3.3 *Assume that hypotheses $(H1)$-$(H3)$ hold. If*

$$2ml < 1, \tag{3.8}$$

then the IVP (5.1)-(5.4) has at least one solution on J.

Proof. We shall reduce the existence of solutions of (5.1)-(5.4) to a fixed point problem. Consider the operator $N : \Omega \longrightarrow \Omega$ defined by

$$N(u)(t,x) = \begin{cases} \phi(t,x) & \text{if } (t,x) \in \tilde{J}, \\ \mu(t,x) + \sum\limits_{0<t_k<t} \Big(I_k(u(t_k^-,x)) - I_k(u(t_k^-,0))\Big) & \text{if } (t,x) \in J, \quad k = 1,\ldots,m, \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum\limits_{0<t_k<t} \int_{t_{k-1}}^{t_k} \int_0^x (t_k - s)^{r_1-1}(x-y)^{r_2-1} & \\ \times f(s,y,u_{(s,y)})dyds & \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t} \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} & \\ \times f(s,y,u_{(s,y)})dyds & \end{cases}$$

Consider the operators $F, G : \Omega \to \Omega$ defined by,

$$G(u)(t,x) = \begin{cases} \phi(t,x), & (t,x) \in \tilde{J}, \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum\limits_{0<t_k<t} \int_{t_{k-1}}^{t_k} \int_0^x (t_k - s)^{r_1-1}(x-y)^{r_2-1} & k = 1,\ldots,m \\ \times f(s,y,u_{(s,y)})dyds & \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t} \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} & \\ \times f(s,y,u_{(s,y)})dyds, & (t,x) \in J. \end{cases}$$

and

$$F(u)(t,x) = \begin{cases} 0, & (t,x) \in \tilde{J}, \\ \mu(t,x) + \sum\limits_{0<t_k<t} (I_k(u(t_k^-,x)) - I_k(u(t_k^-,0))), & (t,x) \in J. \end{cases}$$

Then the problem of finding the solution of the IVP (5.1)–(5.3) is reduced to finding the solutions of the operator equation $F(u) + G(u) = u$. We shall show that the operators F

and G satisfy the conditions of Theorem 3.3. The proof will be given by several steps.

Step 1: G is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in C, then for each $(t,x) \in J$

$$\begin{aligned}
&\|G(u_n)(t,x) - G(u)(t,x)\| \\
&\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^x (t_k - s)^{r_1-1}(x-y)^{r_2-1}\|f(s,y,u_{n(s,y)}) - f(s,y,u_{(s,y)})\|dyds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t} \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}\|f(s,y,u_{n(s,y)}) - f(s,y,u_{(s,y)})\|dyds. \\
&\leq \frac{\|f(.,.,u_{n(.,.)}) - f(.,.,u_{(.,.)})\|}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^x (t_k - s)^{r_1-1}(x-y)^{r_2-1}dyds \\
&+ \frac{\|f(.,.,u_{n(.,.)}) - f(.,.,u_{(.,.)})\|}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t} \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}dyds.
\end{aligned}$$

Since f is a continuous function, we have

$$\|G(u_n) - G(u)\|_\infty \leq \frac{2a^{r_1}b^{r_2}\|f(.,.,u_{n(.,.)}) - f(.,.,u_{(.,.)})\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)} \to 0 \text{ as } n \to \infty$$

Thus G is continuous.

Step 2: G maps bounded sets into bounded sets in C.

Indeed, it is enough show that for any η^*, there exists a positive constant l such that, for each $u \in B_{\eta^*} = \{u \in C : \|u\|_\infty \leq \eta^*\}$ we have $\|G(u)\|_C \leq l$.
By $(H2)$ we have for each $(t,x) \in J$,

$$\begin{aligned}
\|G(u)(t,x)\| &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^x (t_k - s)^{r_1-1}(x-y)^{r_2-1}\|f(s,y,u_{(s,y)})\|dyds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t} \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}\|f(s,y,u_{(s,y)})\|dyds \\
&\leq \frac{\|p\|_\infty + \|q\|_\infty\eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^x (t_k - s)^{r_1-1}(x-y)^{r_2-1}dyds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty\eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \int_{t_k}^{t} \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}dyds.
\end{aligned}$$

Thus

$$\|G(u)\|_C \leq \frac{2a^{r_1}b^{r_2}(\|p\|_\infty + \|q\|_\infty\eta^*)}{\Gamma(r_1+1)\Gamma(r_2+1)} := l^*.$$

Hence $\|G(u)\|_{\widetilde{PC}} \leq l^*$.

Step 3: G maps bounded sets into equicontinuous sets in C.
Let $(t_1, x_1), (t_2, x_2) \in (0, a] \times (0, b]$, $t_1 < t_2$, $x_1 < x_2$, B_η be a bounded set as in Step 2, let $u \in B_{\eta^*}$ be a bounded set of C as in Step 2. Then

$$
\begin{aligned}
&\|G(u)(t_2, x_2) - G(u)(t_1, x_1)\| \\
&\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^{x_1} (t_k - s)^{r_1-1}[(x_2-\tau)^{r_2-1} - (x_1-\tau)^{r_2-1}] \times \|f(s, \tau, u_{(s,\tau)})\| d\tau ds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_{x_1}^{x_2} (t_k - s)^{r_1-1}(x_2-\tau)^{r_2-1} \|f(s, \tau, u_{(s,\tau)})\| d\tau ds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{t_1} \int_0^{x_1} [(t_2 - s)^{r_1-1}(x_2-\tau)^{r_2-1} - (t_1-s)^{r_1-1}(x_1-\tau)^{r_2-1}] \times \|f(s, \tau, u_{(s,\tau)})\| d\tau ds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^{t_2} \int_{x_1}^{x_2} (t_2 - s)^{r_1-1}(x_2-\tau)^{r_2-1} |f(s, \tau, u_{(s,\tau)})| d\tau ds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{t_1} \int_{x_1}^{x_2} (t_2 - s)^{r_1-1}(x_2-\tau)^{r_2-1} \|f(s, \tau, u_{(s,\tau)})\| d\tau ds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^{t_2} \int_0^{x_1} (t_2 - s)^{r_1-1}(x_2-\tau)^{r_2-1} \|f(s, \tau, u_{(s,\tau)})\| d\tau ds \\
&\leq \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^{x_1} (t_k - s)^{r_1-1}[(x_2-\tau)^{r_2-1} - (x_1-\tau)^{r_2-1}] d\tau ds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2)+1} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_{x_1}^{x_2} (t_k - s)^{r_1-1}(x_2-\tau)^{r_2-1} d\tau ds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \int_0^{t_1} \int_0^{x_1} [(t_2 - s)^{r_1-1}(x_2-\tau)^{r_2-1} - (t_1-s)^{r_1-1}(x_1-\tau)^{r_2-1}] d\tau ds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \int_{t_1}^{t_2} \int_{x_1}^{x_2} (t_2 - s)^{r_1-1}(x_2-\tau)^{r_2-1} d\tau ds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \int_0^{t_1} \int_{x_1}^{x_2} (t_2 - s)^{r_1-1}(x_2-\tau)^{r_2-1} d\tau ds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \int_{t_1}^{t_2} \int_0^{x_1} (t_2 - s)^{r_1-1}(x_2-\tau)^{r_2-1} d\tau ds \\
&\leq \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^{x_1} (t_k - s)^{r_1-1}[(x_2-\tau)^{r_2-1} - (x_1-\tau)^{r_2-1}] d\tau ds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_{x_1}^{x_2} (t_k - s)^{r_1-1}(x_2-\tau)^{r_2-1} d\tau ds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} [2x_2^{r_2}(t_2-t_1)^{r_1} + 2t_2^{r_1}(x_2-x_1)^{r_2} \\
&+ t_1^{r_1}x_1^{r_2} - t_2^{r_1}x_2^{r_2} - 2(t_2-t_1)^{r_1}(x_2-x_1)^{r_2}].
\end{aligned}
$$

As $t_1 \to t_2$, $x_1 \to x_2$ the right-hand side of the above inequality tends to zero.

As a consequence of Steps 1 to 3, together with the Arzela-Ascoli theorem, we can conclude that $G : \widetilde{PC} \to \widetilde{PC}$ is continuous and completely continuous.

Step 4: F is a contraction.
Let $u, v \in C$, then we have for each $(t, x) \in J$

$$\begin{aligned}
&\|F(u)(t,x) - F(v)(t,x)\| \\
&\leq \sum_{k=1}^{m}(\|I_k(u(t_k^-, x)) - I_k(v(t_k^-, x))\| + \|I_k(u(t_k^-, 0)) - I_k(v(t_k^-, 0))\|) \\
&\leq \sum_{k=1}^{m} l(\|u - v\|_C + \|u - v\|_C) \\
&\leq 2ml\|u - v\|_C.
\end{aligned}$$

Thus

$$\|F(u) - F(v)\|_C \leq 2ml\|u - v\|_C.$$

Hence by (3.8), F is a contraction.

Step 5: (A priori bounds)

Now it remains to show that the set

$$\mathcal{E} = \{u \in C : u = \lambda F(\frac{u}{\lambda}) + \lambda G(u) \text{ for some } \lambda \in (0, 1)\}$$

is bounded. Let $u \in \mathcal{E}$, then $u = \lambda F(\frac{u}{\lambda}) + \lambda G(u)$. Thus, for each $(t, x) \in J$ we have

$$\begin{aligned}
u(t,x) &= \lambda\mu(t,x) + \sum_{k=1}^{m} \lambda \left(I_k \left(\frac{u(t_k^-, x)}{\lambda} \right) + I_k \left(\frac{u(t_k^-, 0)}{\lambda} \right) \right) \\
&+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s, \tau, u_{(s,\tau)}) d\tau ds \\
&+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t} \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s, \tau, u_{(s,\tau)}) d\tau ds.
\end{aligned}$$

This implies by $(H2)$ and $(H3)$ that, for each $(t,x) \in J$, we have

$$\begin{aligned}
\|u(t,x)\| &\leq \|\mu(t,x)\| + \sum_{k=1}^{m}(\|I_k u(t_k^-,x)\| - \|I_k(0)\| + \|I_k u(t_k^-,0)\| - \|I_k(0)\|) \\
&+ 2\sum_{k=1}^{m}\|I_k(0)\| + \frac{\|p\|_\infty}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\|u_{(s,\tau)}\|_C d\tau ds \\
&+ \frac{\|q\|_\infty}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} d\tau ds \\
&+ \frac{\|p\|_\infty}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\|u_{(s,\tau)}\|_C d\tau ds \\
&+ \frac{\|q\|_\infty}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} d\tau ds \\
&\leq \|\mu(t,x)\| + l\sum_{k=1}^{m}(\|u(t_k^-,x)\| + \|u(t_k^-,0)\|) + 2I^* \\
&+ \frac{\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\|u_{(s,\tau)}\|_C d\tau ds \\
&+ \frac{\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} d\tau ds \\
&+ \frac{\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\|u_{(s,\tau)}\|_C d\tau ds \\
&+ \frac{\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} d\tau ds,
\end{aligned}$$

where

$$I^* = \sum_{k=1}^{m}\|I_k(0)\|.$$

We consider the function γ defined by

$$\gamma(t,x) = sup\{|u(s,\tau)| : -\alpha \leq s \leq t, -\beta \leq \tau \leq x, 0 \leq t \leq a, 0 \leq x \leq b\}.$$

Let $(t^*,x^*) \in [-\alpha,t] \times [-\beta,x]$ be such that $\gamma(t,x) = |u(t^*,x^*)|$. If $(t^*,x^*) \in J$, then by the previous inequality, we have for $(t,x) \in J$

$$\begin{aligned}
\gamma(t,x) &\leq \|\mu(t,x)\| + l\sum_{k=1}^{m}(\|u(t_k^-,x)\| + \|u(t_k^-,0)\|) + 2I^* \\
&+ \frac{\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\Big(\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\gamma(s,\tau) d\tau ds \\
&+ \int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\gamma(s,\tau) d\tau ds\Big) + \frac{2a^{r_1}b^{r_2}\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}.
\end{aligned}$$

If $(t^*, x^*) \in \tilde{J}$, then $\gamma(t,x) = \|\phi\|_C$ and the previous inequality holds. If $(t,x) \in J$, by Lemma 1.4 implies that there exists $\tilde{k} = \tilde{k}(r_2, r_2)$ such that

$$\begin{aligned}
\gamma(t,x) &\leq \left(\|\mu(t,x)\| + l\sum_{k=1}^{m}(\|u(t_k^-,x)\| + \|u(t_k^-,0)\|) + 2I^* + \frac{2a^{r_1}b^{r_2}\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\right) \\
&\times \left(1 + \tilde{k}\frac{\|p\|_\infty}{\Gamma(r_2+1)\Gamma(r_2+1)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}d\tau ds\right) \\
&\leq \left(\|\mu(t,x)\| + l\sum_{k=1}^{m}(\|u(t_k^-,x)\| + \|u(t_k^-,0)\|) + 2I^* + \frac{2a^{r_1}b^{r_2}\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\right) \\
&\times \left(1 + \tilde{k}\frac{\|p\|_\infty a^{r_1}b^{r_2}}{\Gamma(r_2+1)\Gamma(r_2+1)}\right) := \tilde{R}.
\end{aligned}$$

Since for every $(t,x) \in J$, $\|u(t,x)\|_C \leq \gamma(t,x)$. This shows that the set $\mathcal{E}$ is bounded. As a consequence of Theorem 4.2 we deduce that $F+G$ has a fixed point u which is a solution of problem (5.1)-(5.4).

3.2.3 An Example

As an application of our results we consider the following impulsive partial hyperbolic differential equations of the form

$$({}^cD_0^r u)(t,x) = \frac{1}{(10e^{t+x+2})(1+|u(t-1,x-2)|)}; \quad \text{if } (t,x) \in (t_k, t_{k+1}]\times[0,1],\ k=1,\dots,m, \tag{3.9}$$

$$u(t_k^+,x) = u(t_k^-,x) + \frac{1}{(6e^{t+x+4})(1+|u(t_k^-,x)|)}; \ x \in [0,1],\ k=1,\dots,m, \tag{3.10}$$

$$u(t,x) = t + x^2,\ (t,x) \in [-1,1]\times[-2,1]\backslash(0,1]\times(0,1], \tag{3.11}$$

$$u(t,0) = t,\ t \in [0,1],\ u(0,x) = x^2,\ x \in [0,1]. \tag{3.12}$$

Set

$$f(t,x,u) = \frac{1}{(10e^{t+x+2})(1+|u|)}; \quad \text{if } (t,x) \in [0,1]\times[0,1]$$

$$I_k(u(t_k^-,x)) = \frac{1}{(6e^{t+x+4})(1+|u(t_k^-,x)|)}; \ x \in [0,1].$$

For each $u, \bar{u} \in \mathbb{R}^n$ and $(t,x) \in [0,1]\times[0,1]$ we have

$$\|I_k(u) - I_k(v)\| \leq \frac{1}{6e^4}\|u-v\|_{\widetilde{PC}}.$$

Hence (H_3) is satisfied with $l = \frac{1}{6e^4}$. We shall show that condition (4.5) holds with $a = b = 1$ and $m = 1$. Indeed,

$$2ml = \frac{1}{3e^4} < 1,$$

which is satisfied for each $(r_1, r_2) \in (0,1] \times (0,1]$. Consequently Theorem 3.3 implies that problem (3.9)-(3.12) has a solution defined on $[-1,1] \times [-2,1]$.

3.3 Impulsive Partial Differential Equations with Infinite Delay

3.3.1 Introduction

In this section, we shall be concerned with the existence of solutions for the following impulsive partial hyperbolic differential equations:

$$({}^cD^r_{z_k}u)(x,y) = f(x,y,u_{(x,y)}); \text{ if } (x,y) \in J_k, \quad k = 0,\ldots,m, \tag{3.13}$$

$$u(x_k^+, y) = u(x_k^-, y) + I_k(u(x_k^-, y)), \text{ if } y \in [0,b], \ k = 1,\ldots,m, \tag{3.14}$$

$$u(x,y) = \phi(x,y); \text{ if } (x,y) \in \tilde{J}, \tag{3.15}$$

$$u(x,0) = \varphi(x), \ x \in [0,a], \ u(0,y) = \psi(y); \ y \in [0,b], \tag{3.16}$$

where $J_0 = [0,x_1] \times [0,b]$, $J_k := (x_k, x_{k+1}] \times [0,b]$; $k = 1,\ldots,m$, $z_k = (x_k, 0)$, $k = 0,\ldots,m$, $a, b > 0$, $J = [0,a] \times [0,b]$, $\tilde{J} = (-\infty, a] \times (-\infty, b] \backslash (0,a] \times (0,b]$, ${}^cD^r_{z_k}$ is the Caputo fractional derivative of order $r = (r_1, r_2) \in (0,1] \times (0,1]$, $\varphi : [0,a] \to \mathbb{R}^n$, $\psi : [0,b] \to \mathbb{R}^n$ are given continuous functions with $\varphi(x) = \phi(x,0)$, $\psi(y) = \phi(0,y)$ for each $(x,y) \in J$, $0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = a$, $f : J \times \mathcal{B} \to \mathbb{R}^n$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1,\ldots,m$, $\phi : \tilde{J} \to \mathbb{R}^n$, are given functions. $\mathcal{B}$ is called a phase space that will be specified in the next Section. If $u : (-\infty, a] \times (-\infty, b] \to \mathbb{R}^n$, then for any $(x,y) \in J$ define $u_{(x,y)}$ by

$$u_{(x,y)}(s,t) = u(x+s, y+t), \text{ for } (s,t) \in [-\alpha, 0] \times [-\beta, 0].$$

3.3.2 Existence Results

Our main result in this section is based upon the fixed point theorem due to Burton and Kirk. To define the solutions of problems (3.13)-(3.16), we shall consider the space

$$\begin{aligned}\Omega = \big\{ & u : (-\infty, a] \times (-\infty, b] \to \mathbb{R}^n : u_{(x,y)} \in B \text{ for } (x,y) \in E \text{ and there exist} \\ & u(x_k^-, .), \ u(x_k^+, .) \text{ exist with } u(x_k^-, .) = u(x_k, .); \ k = 1,\ldots,m, \text{ and} \\ & u \in C(J_k, \mathbb{R}^n); k = 0,\ldots,m \big\},\end{aligned}$$

where $J_k = (x_k, x_{k+1}] \times (0,b]$
Let us define what we mean by a solution of problem (3.13)-(3.16).

Definition 3.4 A function $u \in \Omega$ is said to be a solution of (3.13)-(3.16) if u satisfies $({}^cD^r_{z_k}u)(x,y) = f(x,y,u(x,y))$ on J' and conditions (3.14), (3.15) and (3.16) are satisfied.

Let $h \in C([x_k, x_{k+1}] \times [0,b], \mathbb{R}^n)$, $z_k = (x_k, 0)$, and

$$\mu_k(x,y) = u(x,0) + u(x_k^+, y) - u(x_k^+, 0), \quad k = 0, \dots, m.$$

For the existence of solutions for the problem (3.13)−(3.16), we need the following lemma:

Lemma 3.5 *A function $u \in C([x_k, x_{k+1}] \times [0,b], \mathbb{R}^n)$; $k = 0, \dots, m$ is a solution of the differential equation*

$$({}^cD^r_{z_k}u)(x,y) = h(x,y); \ (x,y) \in [x_k, x_{k+1}] \times [0,b],$$

if and only if $u(x,y)$ satisfies

$$u(x,y) = \mu_k(x,y) + (I^r_{z_k}h)(x,y); \ (x,y) \in [x_k, x_{k+1}] \times [0,b]. \tag{3.17}$$

Lemma 3.6 *[7] Let $0 < r_1, r_2 \leq 1$ and let $h : J \to \mathbb{R}^n$ be continuous. A function u is a solution of the fractional integral equation*

$$u(x,y) = \begin{cases} \phi(x,y) & \text{if } (x,y) \in \tilde{J}, \\ \mu(x,y) + \sum\limits_{0<x_k<x} \Big(I_k(u(x_k^-, y)) - I_k(u(x_k^-, 0)) \Big) & \text{if } (x,y) \in J, \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum\limits_{0<x_k<x} \int_{x_{k-1}}^{x_k} \int_0^y (x_k - s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds & k = 1, \dots, m, \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds & \end{cases} \tag{3.18}$$

if and only if u is a solution of the fractional initial value problem

$${}^cD^r u(x,y) = h(x,y), \quad (x,y) \in J_k, \ k = 0, \dots, m, \tag{3.19}$$

$$u(x_k^+, y) = u(x_k^-, y) + I_k(u(x_k^-, y)), \quad k = 1, \dots, m. \tag{3.20}$$

Let us introduce the following hypotheses which are assumed hereafter.

$(H1)$ The functions $I_k : \mathbb{R}^n \to \mathbb{R}^n$ are continuous.

$(H2)$ There exist $p, q \in C(J, \mathbb{R}_+)$ such that

$$\|f(t,x,u)\| \leq p(t,x) + q(t,x)\|u\|_{\mathcal{B}}, \ \text{for } (t,x) \in J \text{ and each } u \in \mathcal{B}.$$

$(H3)$ There exists $l > 0$ such that

$$\|I_k(u) - I_k(v)\| \leq l\|u - v\| \ \text{ for each } \ u, v \in \mathbb{R}^n.$$

Theorem 3.7 *Assume that hypotheses (H1)-(H3) hold. If*

$$2ml < 1, \tag{3.21}$$

then the IVP (3.13)-(3.16) has at least one solution on J.

Proof. We shall reduce the existence of solutions of (3.13)-(3.16) to a fixed point problem. Consider the operator $N : \Omega \longrightarrow \Omega$ defined by

$$N(u)(x,y) = \begin{cases} \phi(x,y) & \text{if } (x,y) \in \tilde{J}, \\ \mu(x,y) + \sum_{0<x_k<x} \Big(I_k(u(x_k^-,y)) - I_k(u(x_k^-,0)) \Big) & \text{if } (x,y) \in J, \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{0<x_k<x} \int_{x_{k-1}}^{x_k} \int_0^y (x_k-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds & k=1,\dots,m, \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds. & \end{cases}$$

Consider the operators $A, B : \Omega \to \Omega$ defined by,

$$A(u)(x,y) = \begin{cases} \phi(x,y), & (x,y) \in \tilde{J}, \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{0<x_k<x} \int_{x_{k-1}}^{x_k} \int_0^y (x_k-s)^{r_1-1}(y-t)^{r_2-1} & k=1,\dots,m \\ \times f(s,t,u(s,t))dtds & \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} & \\ \times f(s,t,u(s,t))dtds, & (x,y) \in J. \end{cases}$$

and

$$B(u)(x,y) = \begin{cases} 0, & (x,y) \in \tilde{J}, \\ \mu(x,y) + \sum_{0<x_k<x} (I_k(u(x_k^-,y)) - I_k(u(x_k^-,0))), & (x,y) \in J. \end{cases}$$

Let $v(.,.) : (-\infty, a] \times (-\infty, b] \to \mathbb{R}^n$ be a function defined by,

$$v(x,y) = \begin{cases} \phi(x,y), & (x,y) \in \tilde{J}. \\ \mu(x,y), & (x,y) \in J. \end{cases}$$

Then $v_{(x,y)} = \phi$ for all $(x,y) \in E$.
For each $w \in (J, \mathbb{R}^n)$ with $w(x,y) = 0$ for each $(x,y) \in E$, we denote by $\overline{w}$ the function defined by

$$\overline{w}(t,x) = \begin{cases} 0, & (x,y) \in \tilde{J}, \\ w(x,y) & (x,y) \in J. \end{cases}$$

If $u(.,.)$ satisfies the integral equation,

$$u(x,y) \quad = \quad \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u_{(s,t)}) dtds$$

we can decompose $u(.,.)$ as $u(x,y) = \overline{w}(x,y) + v(x,y);\ (x,y) \in (x_k, x_{k+1}] \times [0,b]$, which implies $u_{(x,y)} = \overline{w}_{(x,y)} + v_{(x,y)}$, for every $(x,y) \in J$ and the function $w(.,.)$ satisfies

$$\begin{aligned} w(x,y) = & \sum_{0<x_k<x} (I_k(u(x_k^-,y)) - I_k(u(_k^-,0))) \\ & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{0<x_k<x} \int_{x_{k-1}}^{x_k} \int_0^y (x_k-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,\overline{w}_{(s,t)} + v_{(s,t)}) dtds \\ & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1}(t-t)^{r_2-1} f(s,t,\overline{w}_{(s,t)} + v_{(s,t)}) dtds. \end{aligned}$$

Set

$$C_0 = \{w \in \Omega :\ w(x,y) = 0 \ \text{ for } (x,y) \in E\},$$

and let $\|.\|_{C_0}$ be the norm in C_0 defined by

$$\|w\|_{C_0} = \sup_{(x,y)\in E} \|w_{(x,y)}\|_{\mathcal{B}} + \sup_{(x,y)\in J} \|w(x,y)\| = \sup_{(x,y)\in J} \|w(x,y)\|,\ w \in C_0.$$

C_0 is a Banach space with norm $\|.\|_{C_0}$. Let the operators $A, B:\ C_0 \to C_0$ defined by

$$(Aw)(x,y) = \begin{cases} \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{0<x_k<x} \int_{x_{k-1}}^{x_k} \int_0^y (x_k-s)^{r_1-1}(y-t)^{r_2-1}, \quad k = 1,\dots,m \\ \times f(s,t,\overline{w}_{(s,t)} + v_{(s,t)}) dtds \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\ \times f(s,t,\overline{w}_{(s,t)} + v_{(s,t)}) dtds, \qquad (x,y) \in J. \end{cases}$$

and

$$(Bw)(x,y) = \mu(x,y) + \sum_{0<x_k<x} (I_k(u(x_k^-,y)) - I_k(u(x_k^-,0))), (x,y) \in J.$$

Then the problem of finding the solution of the IVP (3.13)–(3.16) is reduced to finding the solutions of the operator equation $A(w) + B(w) = w$. We shall show that the operators A and B satisfy the conditions of Theorem 3.7. The proof will be given by a couple of steps.

Step 1: A is continuous.

Let $\{w_n\}$ be a sequence such that $w_n \to w$ in C_0, then for each $(x,y) \in J$

$$
\begin{aligned}
&\|A(w_n)(x,y) - A(w)(x,y)\| \\
&\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^y (x_k-s)^{r_1-1}(y-t)^{r_2-1} \\
&\times \|f(s,t,\overline{w}_{n(s,t)}+v_{n(s,t)}) - f(s,t,\overline{w}_{(s,t)}+v_{(s,t)})\|dtds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\
&\times \|f(s,t,\overline{w}_{n(s,t)}+v_{n(s,t)}) - f(s,t,\overline{w}_{(s,t)}+v_{(s,t)})\|dtds. \\
&\leq \frac{\|f(.,.,\overline{w}_{n(.,.)}+v_{n(.,.)}) - f(.,.,\overline{w}_{(.,.)}+v_{(.,.)})\|}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^y (x_k-s)^{r_1-1}(y-t)^{r_2-1}dtds \\
&+ \frac{\|f(.,.,\overline{w}_{n(.,.)}+v_{n(.,.)}) - f(.,.,\overline{w}_{(.,.)}+v_{(.,.)})\|}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}dtds.
\end{aligned}
$$

Since f is continuous function, we have

$$
\|A(w_n) - A(w)\|_{C_0} \leq \frac{2a^{r_1}b^{r_2}\|f(.,.,w_{n(.,.)}) - f(.,.,w_{(.,.)})\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)} \to 0 \ \text{as} \ n\to\infty
$$

Thus A is continuous.

Step 2: A maps bounded sets into bounded sets in C_0.
Indeed, it is enough show that for any η^*, there exists a positive constant l such that, for each $w \in B_{\eta^*} = \{w \in C_0 : \|w\|_{(a,b)} \leq \eta^*\}$ we have $\|A(w)\|_\infty \leq l$
By $(H2)$ we have for each $(x,y) \in (x_k, x_{k+1}] \times [0,b]$,

$$
\begin{aligned}
\|A(w)(x,y)\| &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^y (x_k-s)^{r_1-1}(y-t)^{r_2-1}\|f(s,t,\overline{w}_{(s,t)}+v_{(s,t)})\|dtds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\|f(s,t,\overline{w}_{(s,t)}+v_{(s,t)})\|dtds \\
&\leq \frac{\|p\|_\infty + \|q\|_\infty\eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^y (x_k-s)^{r_1-1}(y-t)^{r_2-1}dtds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty\eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)}\int_{x_k}^{x}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}dtds
\end{aligned}
$$

Thus

$$
\|A(w)\|_B \leq \frac{2a^{r_1}b^{r_2}(\|p\|_\infty + \|q\|_\infty\eta^*)}{\Gamma(r_1+1)\Gamma(r_2+1)} := l
$$

where

$$
\begin{aligned}
\|\overline{w}_{(s,t)} + v_{(s,t)}\|_{\mathcal{B}} &\leq \|\overline{w}_{(s,t)}\|_{\mathcal{B}} + \|v_{(s,t)}\|_{\mathcal{B}} \\
&\leq K\eta^* + K\|\phi(0,0)\| + M\|\phi\|_{\mathcal{B}} := \eta.
\end{aligned}
$$

Hence $\|A(w)\|_{C_0} \leq l$.

Step 3: A maps bounded sets into equicontinuous sets in C_0.
Let $(x_1, y_1), (x_2, y_2) \in (0, a] \times (0, b]$, $x_1 < x_2$, $y_1 < y_2$, B_{η^*} be a bounded set as in Step 2.

Let $w \in B_{\eta^*}$, then

$$\begin{aligned}
&\|A(w)(x_2,y_2) - A(w)(x_1,y_1)\| \\
&\le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \int_0^{y_1} (x_k - s)^{r_1-1}[(y_2-t)^{r_2-1} - (y_1-t)^{r_2-1}] \\
&\times f(s,t,\overline{w}_{(s,t)} + v_{(s,t)})dtds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \int_{y_1}^{y_2} (x_k - s)^{r_1-1}(y_2-t)^{r_2-1}\|f(s,t,\overline{w}_{(s,t)} + v_{(s,t)})\|dtds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1} \int_0^{y_1} [(x_2-s)^{r_1-1}(y_2-t)^{r_2-1} - (x_1-s)^{r_1-1}(y_1-t)^{r_2-1}] \\
&\times f(s,t,\overline{w}_{(s,t)} + v_{(s,t)})dtdx \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|f(s,t,\overline{w}_{(s,t)} + v_{(s,t)})\|dtds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1} \int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|f(s,t,\overline{w}_{(s,t)} + v_{(s,t)})\|dtds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_0^{y_1} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|f(s,t,\overline{w}_{(s,t)} + v_{(s,t)})\|dtds \\
&\le \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \int_0^{y_1} (x_k - s)^{r_1-1}[(y_2-t)^{r_2-1} - (y_1-t)^{r_2-1}]dtds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \int_{y_1}^{y_2} (x_k - s)^{r_1-1}(y_2-t)^{r_2-1}dtds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \int_0^{x_1} \int_0^{y_1} [(x_2-s)^{r_1-1}(y_2-t)^{r_2-1} - (x_1-s)^{r_1-1}(y_1-t)^{r_2-1}]dtds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}dtds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \int_0^{x_1} \int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}dtds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \int_{x_1}^{x_2} \int_0^{y_1} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}dtds \\
&\le \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \int_0^{y_1} (x_k - s)^{r_1-1}[(y_2-t)^{r_2-1} - (y_1-t)^{r_2-1}]dtds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \int_{y_1}^{y_2} (x_k - s)^{r_1-1}(y_2-t)^{r_2-1}dtds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)}[2y_2^{r_2}(x_2-x_1)^{r_1} + 2x_2^{r_1}(y_2-y_1)^{r_2} \\
&+ x_1^{r_1}y_1^{r_2} - x_2^{r_1}y_2^{r_2} - 2(x_2-x_1)^{r_1}(y_2-y_1)^{r_2}].
\end{aligned}$$

As $x_1 \to x_2$, $y_1 \to y_2$ the right-hand side of the above inequality tends to zero.

As a consequence of Steps 1 to 3, together with the Arzela-Ascoli theorem, we can conclude that $A : C_0 \to C_0$ is continuous and completely continuous.

Step 4: B is a contraction.
Let $w, w^* \in C_0$, then we have for each $(x, y) \in J$

$$\begin{aligned}
&\|B(w)(x,y) - B(w^*)(x,y)\| \\
&\leq \sum_{k=1}^{m}(\|I_k(w(x_k^-,y)) - I_k(w^*(x_k^-,y))\| + \|I_k(w(x_k^-,0)) - I_k(w^*(x_k^-,0))\|) \\
&\leq \sum_{k=1}^{m} l(\|w-w^*\|_{C_0} + \|w-w^*\|_{C_0}) \\
&\leq 2ml\|w-w^*\|_{C_0}.
\end{aligned}$$

Thus

$$\|B(w) - B(w^*)\|_{C_0} \leq 2ml\|w-w^*\|_{C_0}.$$

Hence by (3.21), B is a contraction.

Step 5: (A priori bounds)

Now it remains to show that the set

$$\mathcal{E} = \{w \in C_0 : w = \lambda B\left(\frac{w}{\lambda}\right) + \lambda A(w), \text{ for some } \lambda \in (0,1)\}$$

is bounded. Let $w \in \mathcal{E}$, then $w = \lambda B\left(\frac{w}{\lambda}\right) + \lambda A(w)$. Thus, for each $(x, y) \in J$ we have

$$\begin{aligned}
w(x,y) &= \lambda \sum_{k=1}^{m}(\|I_k\frac{(u(x_k^-,y))}{\lambda})\| + \|I_k\frac{(u(x_k^-,0))}{\lambda}\|) \\
&+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,\overline{w}_{(s,t)} + v_{(s,t)})dtds \\
&+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,\overline{w}_{(s,t)} + v_{(s,t)})dtds.
\end{aligned}$$

This implies by $(H2)$ and $(H3)$ that, for each $(x,y)\in J$, we have

$$\begin{aligned}
\|w(x,y)\| &\leq \sum_{k=1}^{m}\lambda(\|I_k\frac{u(x_k^-,y)}{\lambda}\| - \|I_k(0)\| + \|I_k\frac{u(x_k^-,0)}{\lambda}\| - \|I_k(0)\|)\\
&+ 2\lambda\sum_{k=1}^{m}\|I_k(0)\| + \frac{\|p\|_\infty}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\\
&\times \|\overline{w}_{(s,t)}+v_{(s,t)}\|_B dtds\\
&+ \frac{\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}dtds\\
&+ \frac{\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\|\overline{w}_{(s,t)}+v_{(s,t)}\|_B dtds\\
&+ \frac{\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}dtds\\
&\leq l\sum_{k=1}^{m}(\|u(t_k^-,x)\| + \|u(t_k^-,0)\|) + 2I^*\\
&+ \frac{\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\|\overline{w}_{(s,t)}+v_{(s,t)}\|_B dtds\\
&+ \frac{\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}dtds\\
&+ \frac{\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\|\overline{w}_{(s,t)}+v_{(s,t)}\|_B dtds\\
&+ \frac{\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}dtds,
\end{aligned}$$

where

$$I^* = \sum_{k=1}^{m}\|I_k(0)\|.$$

and

$$\begin{aligned}
\|\overline{w}_{(s,t)}+v_{(s,t)}\|_{\mathcal{B}} &\leq \|\overline{w}_{(s,t)}\|_{\mathcal{B}} + \|v_{(s,t)}\|_{\mathcal{B}}\\
&\leq K\sup\{w(\tilde{s},\tilde{t}) : (\tilde{s},\tilde{t})\in[0,s]\times[0,t]\}\\
&\quad +M\|\phi\|_{\mathcal{B}} + K\|\phi(0,0)\|. \qquad (3.22)
\end{aligned}$$

If we name $\gamma(s,t)$ the right hand side of (3.22), then we have

$$\|\overline{w}_{(s,t)}+v_{(s,t)}\|_{\mathcal{B}} \leq \gamma(x,y),$$

and therefore, for $\gamma(x,y) \in J$ we obtain

$$\begin{aligned}\|w(x,y)\| &\leq l\sum_{k=1}^{m}(\|u(x_k^-,y)\|+\|u(x_k^-,0)\|)+2I^*+\frac{2a^{r_1}b^{r_2}\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\\ &+ \frac{\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\Big(\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^y(x-s)^{r_1-1}(y-\tau)^{r_2-1}\gamma(s,\tau)d\tau ds\\ &\quad +\int_0^x\int_0^y(x-s)^{r_1-1}(x-\tau)^{r_2-1}\gamma(s,\tau)d\tau ds\Big). \qquad (3.23)\end{aligned}$$

Using the above inequality and the definition of γ for each $(x,y)\in J$ we have

$$\begin{aligned}\gamma(x,y) &\leq M\|\phi\|_{\mathcal{B}}+K\|\phi(0,0)\|+l\sum_{k=1}^{m}(\|u(x_k^-,y)\|+\|u(x_k^-,0)\|)+2I^*\\ &+ \frac{\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\Big(\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^y(x-s)^{r_1-1}(y-\tau)^{r_2-1}\gamma(s,\tau)d\tau ds\\ &+ \int_0^x\int_0^y(x-s)^{r_1-1}(y-\tau)^{r_2-1}\gamma(s,\tau)d\tau ds\Big)+\frac{2a^{r_1}b^{r_2}\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\\ &\leq M\|\phi\|_{\mathcal{B}}+K\|\phi(0,0)\|+2ml\gamma(x,y)+2I^*\\ &+ \frac{\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\Big(\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^y(x-s)^{r_1-1}(y-\tau)^{r_2-1}\gamma(s,\tau)d\tau ds\\ &+ \int_0^x\int_0^y(x-s)^{r_1-1}(y-\tau)^{r_2-1}\gamma(s,\tau)d\tau ds\Big)+\frac{2a^{r_1}b^{r_2}\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\end{aligned}$$

Thus

$$\begin{aligned}\gamma(x,y)(1-2ml) &\leq M\|\phi\|_{\mathcal{B}}+K\|\phi(0,0)\|+2I^*+\frac{2a^{r_1}b^{r_2}\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\\ &+ \frac{\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\Big(\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^y(x-s)^{r_1-1}(y-\tau)^{r_2-1}\gamma(s,\tau)d\tau ds\\ &+ \int_0^x\int_0^y(x-s)^{r_1-1}(y-\tau)^{r_2-1}\gamma(s,\tau)d\tau ds\Big)\end{aligned}$$

Then

$$\begin{aligned}\gamma(x,y) &\leq \Big(\frac{1}{1-2ml}\Big)\times\Big(M\|\phi\|_{\mathcal{B}}+K\|\phi(0,0)\|+2I^*+\frac{2a^{r_1}b^{r_2}\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\\ &+ \frac{\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\Big(\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^y(x-s)^{r_1-1}(y-\tau)^{r_2-1}\gamma(s,\tau)d\tau ds\\ &+ \int_0^x\int_0^y(x-s)^{r_1-1}(y-\tau)^{r_2-1}\gamma(s,\tau)d\tau ds\Big)\Big)\end{aligned}$$

If $(x^*, y^*) \in \tilde{J}$, then $\gamma(x,y) = \|\phi\|_C$ and the previous inequality holds. If $(x,y) \in J$, by Lemma 1.4 implies that there exists $\tilde{k} = \tilde{k}(r_2, r_2)$ such that

$$\begin{aligned}\gamma(x,y) &\leq \Big(\frac{1}{1-2ml}\Big) \times \Big(M\|\phi\|_{\mathcal{B}} + K\|\phi(0,0)\| + 2I^* + \frac{2a^{r_1}b^{r_2}\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\Big) \\ &\times \Big(1 + \tilde{k}\frac{\|p\|_\infty}{\Gamma(r_2)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-\tau)^{r_2-1}d\tau ds\Big)\end{aligned}$$

Then

$$\begin{aligned}\gamma(x,y) &\leq \Big(\frac{1}{1-2ml}\Big) \times \Big(M\|\phi\|_{\mathcal{B}} + K\|\phi(0,0)\| + 2I^* + \frac{2a^{r_1}b^{r_2}\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\Big) \\ &\times \Big(1 + \tilde{k}\frac{a^{r_1}b^{r_2}\|p\|_\infty}{\Gamma(r_2+1)\Gamma(r_2+1)}\Big) := \tilde{R}.\end{aligned}$$

Since for every $(x,y) \in J$, $\|w_{(x,y))}\|_\infty \leq \gamma(x,y)$.
This shows that the set $\mathcal{E}$ is bounded. As a consequence of Theorem 3.7 we deduce that $A + B$ has a fixed point which is a solution of problem (3.13)-(3.16).

3.3.3 An Example

In this section we give an example as an application of our results. We consider the following impulsive partial hyperbolic functional differential equations of the form

$$({}^cD^r_{z_k}u)(x,y) = \frac{e^{-x-y}}{9+e^{x+y}}\frac{|u(x,y)|}{(1+|u(x,y)|)},\ (x,y) \in J = [0,\frac{1}{2}] \times [0,1] \cup (\frac{1}{2},1] \times [0,1], \quad (3.24)$$

$$u(\frac{1}{2}^+, y) = u(\frac{1}{2}^-, y) + \frac{|u((\frac{1}{2})^-, y)|}{\frac{1}{4} + |u((\frac{1}{2})^-, y)|},\ \text{if } y \in [0,1], \quad (3.25)$$

$$u(x,y) = x + y^2,\ \text{if } (x,y) \in [-1,1] \times [-2,1] \setminus (0,1] \times (0,1], \quad (3.26)$$

$$u(x,0) = x,\ u(0,y) = y^2,\ x \in [0,1],\ y \in [0,1]. \quad (3.27)$$

where $z_0 = (0,0),\ z_1 = (\frac{1}{2}, 0)$. Let $\gamma \in \mathbb{R}$, and C_γ be the set of all piece-wise continuous functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ for which a limit $lim_{||(s,t)|| \to \infty} e^{\gamma(s+t)}\phi(s,t)$ exists, with the norm

$$||\phi||_{C_\gamma} = \sup_{(s,t)\in(-\infty,0]\times(-\infty,0]} e^{\gamma(s+t)}||\phi(s,t)||$$

Set

$$f(x,y,\varphi) = \frac{e^{-x-y}|\varphi|}{(9+e^{x+y})(1+|\varphi|)}, (x,y) \in [0,1] \times [0,1], \varphi \in C,$$

and

$$I_1(u) = \frac{|u|}{\frac{1}{4}+|u|}, u \in \mathbb{R}_+.$$

It is clear that the functions f and I_1 are continuous, and for $(x, y) \in [0, 1] \times [0, 1]$ and $\varphi \in C$, we have

$$f(x, y, \varphi) = \frac{e^{-x-y}}{9 + e^{x+y}}(2 + |\varphi|)$$

Hence $(H2)$is satisfied with

$$p(x, y) = \frac{2e^{-x-y}}{9 + e^{x+y}} \quad \text{and} \quad q(x, y) = \frac{e^{-x-y}}{9 + e^{x+y}}$$

Also, for $u_1, u_2 \in \mathbb{R}$, we have

$$|I(u_1) - I(u_2)| = \left| \frac{|u_1|}{\frac{1}{4} + |u_1|} - \frac{|u_2|}{\frac{1}{4} + |u_2|} \right| \leq \frac{1}{4}|u_1 - u_2|$$

Thus $(H3)$ is satisfied with $l = \frac{1}{4}$. Finally conditions of Theorem 3.7 are satisfied, which implies that problem (3.13)-(3.16) has at least one solution defined on $(-\infty, 1] \times (-\infty, 1]$

Chapter 4

Impulsive Partial Hyperbolic Functional Differential Equations of Fractional Order with State Dependent Delay

4.1 Introduction

In this chapter, we shall be concerned to the existence of solutions for impulsive hyperbolic differential equations of fractional order with state dependent delay.

4.2 Impulsive Partial Differential Equations with Finite Delay

4.2.1 Introduction

In this section, we shall be concerned with the existence of solutions for the following impulsive partial hyperbolic differential equations:

$$({}^cD^r_{z_k}u)(t,x) = f(t,x,u_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))});\ \text{ if } (t,x)\in J_k, k=0,\dots,m,\ t\neq t_k, \quad (4.1)$$

$$u(t_k^+,x) = u(t_k^-,x) + I_k(u(t_k^-,x)),\ \text{ if } x\in[0,b],\ k=1,\dots,m, \quad (4.2)$$

$$u(t,x)=\phi(t,x);\ \text{if } (t,x)\in\tilde{J} := [-\alpha,a]\times[-\beta,b]\setminus(0,a]\times(0,b], \quad (4.3)$$

$$u(t,0)=\varphi(t),\ t\in[0,a],\ u(0,x)=\psi(x);\ x\in[0,b], \quad (4.4)$$

where $J=[0,a]\times[0,b]$, $a,b,\alpha,\beta>0$, $0=t_0<t_1<\cdots<t_m<t_{m+1}=a$, ${}^cD^r_{z_k}$ is the Caputo fractional derivative of order $r=(r_1,r_2)\in(0,1]\times(0,1]$, $\varphi:[0,a]\to\mathbb{R}^n$, $\psi:[0,b]\to\mathbb{R}^n$ are given continuous functions with $\varphi(t)=\phi(t,0)$, $\psi(x)=\phi(0,x)$ for each

$(t,x) \in J$, $f : J \times C \to \mathbb{R}^n$, $\rho_1, \rho_2 : J \times C \to \mathbb{R}$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, \dots, m$, $\phi : \tilde{J} \to \mathbb{R}^n$, are given functions and C is the Banach space defined by

$$C = C_{(\alpha,\beta)} = \big\{u : [-\alpha, 0] \times [-\beta, 0] \to \mathbb{R}^n : \text{ continuous and there exist } \tau_k \in (-\alpha, 0) \text{ with } u(\tau_k^-, \tilde{x}) \text{ and } u(t_k^+, \tilde{x}), k = 1, \dots, m, \text{ exist for any } \tilde{y} \in [-\beta, 0] \text{ with } u(\tau_k^-, \tilde{x}) = u(\tau_k, \tilde{x})\big\}.$$

This set is a Banach space with the norm

$$\|u\|_C = \sup_{(t,x)\in[-\alpha,0]\times[-\beta,0]} \|u(t,x)\|.$$

4.2.2 Existence Results

Set $J_k = (t_k, t_{k+1}] \times (0, b]$. Consider the Banach space

$$\begin{aligned} PC &:= PC(J, \mathbb{R}^n) \\ &= \big\{u : J \to \mathbb{R}^n : u \in C(J_k, \mathbb{R}^n); k = 1, \dots, m, \text{ and there exist } u(t_k^-, x) \text{ and } \\ &\quad u(t_k^+, x); k = 1, \dots, m, \text{ with } u(t_k^-, x) = u(t_k, x)\big\}. \end{aligned}$$

with the norm

$$\|u\|_{PC} = \sup_{(t,x)\in J} \|u(t,x)\|$$

Set

$\widetilde{PC} := PC([-\alpha, 0] \times [-\beta, 0], \mathbb{R}^n)$, which is a Banach space with the norm

$$\|u\|_{\widetilde{PC}} = sup\{\|u(t,x)\| : (t,x) \in [-\alpha, a] \times [-\beta, b]\}.$$

Definition 4.1 A function $u \in \widetilde{PC}$ such that its mixed derivative D^2_{tx} exists on J' is said to be a solution of (5.1)-(5.4) if u satisfies the condition (5.3) on $\tilde{J}$, the equation (5.1) on J' and conditions (5.2) and (5.4) are satisfied on J.

Set $\mathcal{R} := \mathcal{R}_{(\rho_1^-, \rho_2^-)}$

$$= \{(\rho_1(s,y,u), \rho_2(s,y,u)) : (s,y,u) \in J \times C, \rho_i(s,y,u) \leq 0; i = 1, 2\}$$

We always assume that $\rho_i : J \times C \longrightarrow \mathbb{R}$; $i = 1, 2$ are continuous and the function $(s,y) \longmapsto u_{(s,y)}$ is continuous from $\mathcal{R}$ into C.
Our first existence result for the IVP (5.1)-(5.4) is based upon the fixed point theorem due to Burton and Kirk.
Let us introduce the following hypotheses which are assumed after.

$(H1)$ The functions $I_k : \mathbb{R}^n \to \mathbb{R}^n$ are continuous.

$(H2)$ There exist $p, q \in C(J, \mathbb{R}_+)$ such that

$$\|f(t,x,u)\| \leq p(t,x) + q(t,x)\|u\|_C, \text{ for } (t,x) \in J \text{ and each } u \in C.$$

$(H3)$ There exists $l > 0$ such that

$$\|I_k(u) - I_k(v)\| \leq l\|u - v\| \text{ for each } u, v \in \mathbb{R}^n.$$

Theorem 4.2 *Assume that hypotheses* $(H1)$*-*$(H3)$ *hold. If*

$$2ml < 1, \tag{4.5}$$

then the IVP (5.1)-(5.4) has at least one solution on $[-\alpha, a] \times [-\beta, b]$.

Proof. We shall reduce the existence of solutions of (5.1)-(5.4) to a fixed point problem. Consider the operator $N : \widetilde{PC} \longrightarrow \widetilde{PC}$ defined by

$$N(u)(t,x) = \begin{cases} \phi(t,x) & \text{if } (t,x) \in \tilde{J}, \\ \mu(t,x) + \sum_{0<t_k<t} \Big(I_k(u(t_k^-, x)) - I_k(u(t_k^-, 0)) \Big) & \text{if } (t,x) \in J, \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} \int_0^x (t_k - s)^{r_1-1}(x-y)^{r_2-1} & k = 1, \dots, m, \\ \times f(s, y, u_{(\rho_1(s,y,u_{(s,y)}), \rho_2(s,y,u_{(s,y)}))}) dy ds & \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t} \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} & \\ \times f(s, y, u_{(\rho_1(s,y,u_{(s,y)}), \rho_2(s,y,u_{(s,y)}))}) dy ds & \end{cases}$$

Consider the operators $F, G : \widetilde{PC} \to \widetilde{PC}$ defined by,

$$G(u)(t,x) = \begin{cases} \phi(t,x), & (t,x) \in \tilde{J}, \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{0<t_k<tx} \int_{t_{k-1}}^{t_k} \int_0^x (t_k - s)^{r_1-1}(x-y)^{r_2-1} & k = 1, \dots, m \\ \times f(s, y, u_{(\rho_1(s,y,u_{(s,y)}), \rho_2(s,y,u_{(s,y)}))}) dy ds & \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t} \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} & \\ \times f(s, y, u_{(\rho_1(s,y,u_{(s,y)}), \rho_2(s,y,u_{(s,y)}))}) dy ds, & (t,x) \in J. \end{cases}$$

and

$$F(u)(t,x) = \begin{cases} 0, & (t,x) \in \tilde{J}, \\ \mu(t,x) + \sum_{0<t_k<t} (I_k(u(t_k^-, x)) - I_k(u(t_k^-, 0))), & (t,x) \in J. \end{cases}$$

Then the problem of finding the solution of the IVP (5.1)–(5.3) is reduced to finding the solutions of the operator equation $F(u) + G(u) = u$. We shall show that the operators F and G satisfy the conditions of Theorem 4.2. The proof will be given by several steps.

Step 1: G is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $\widetilde{PC}$, then for each $(t,x) \in J$

$$\begin{aligned}
&\|G(u_n)(t,x) - G(u)(t,x)\| \\
&\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^x (t_k - s)^{r_1-1}(x-y)^{r_2-1} \|f(s,y,u_{n(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))}) \\
&- f(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))})\| dyds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t} \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \|f(s,y,u_{n(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))}) \\
&- f(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))})\| dyds. \\
&\leq \frac{\|f(.,.,u_{n(.,.)}) - f(.,.,u_{(.,.)})\|}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^x (t_k - s)^{r_1-1}(x-y)^{r_2-1} dyds \\
&+ \frac{\|f(.,.,u_{n(.,.)}) - f(.,.,u_{(.,.)})\|}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t} \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} dyds.
\end{aligned}$$

Since f is continuous function, we have

$$\|G(u_n) - G(u)\|_\infty \leq \frac{2a^{r_1}b^{r_2}\|f(.,.,u_{n(.,.)}) - f(.,.,u_{(.,.)})\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)} \to 0 \ \text{ as } \ n \to \infty$$

Thus G is continuous.

Step 2: G maps bounded sets into bounded sets in $\widetilde{PC}$. Indeed, it is enough show that for any η^*, there exists a positive constant l such that, for each $u \in B_{\eta^*} = \{u \in \widetilde{PC} : \|u\|_\infty \leq \eta^*\}$ we have $\|G(u)\|_{\widetilde{PC}} \leq l$
By $(H2)$ we have for each $(t,x) \in J$,

$$\begin{aligned}
&\|G(u)(t,x)\| \\
&\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^x (t_k - s)^{r_1-1}(x-y)^{r_2-1} \|f(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))})\| dyds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t} \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \|f(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))})\| dyds \\
&\leq \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^x (t_k - s)^{r_1-1}(x-y)^{r_2-1} dyds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \int_{t_k}^{t} \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} dyds
\end{aligned}$$

Thus

$$\|G(u)\|_{\widetilde{PC}} \leq \frac{2a^{r_1}b^{r_2}(\|p\|_\infty + \|q\|_\infty \eta^*)}{\Gamma(r_1+1)\Gamma(r_2+1)} := l^*$$

Hence $\|G(u)\|_{\widetilde{PC}} \leq l^*$.
Step 3: G maps bounded sets into equicontinuous sets in $\widetilde{PC}$.
Let $(t_1, x_1), (t_2, x_2) \in (0, a] \times (0, b]$, $t_1 < t_2$, $x_1 < x_2$, B_η be a bounded set as in step 2,

let $u \in B_{\eta^*}$ be a bounded set of $\widetilde{PC}$ as in Step 2. Then

$$\begin{aligned}
&\|G(u)(t_2,x_2) - G(u)(t_1,x_1)\| \\
&\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^{x_1} (t_k - s)^{r_1-1}[(x_2-\tau)^{r_2-1} - (x_1-\tau)^{r_2-1}] \\
&\times f(s,\tau,u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))})d\tau ds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_{x_1}^{x_2} (t_k - s)^{r_1-1}(x_2-\tau)^{r_2-1}\|f(s,\tau,u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))})\|d\tau ds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{t_1}\int_0^{x_1} [(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1} - (t_1-s)^{r_1-1}(x_1-\tau)^{r_2-1}] \\
&\times f(s,\tau,u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))})d\tau ds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^{t_2}\int_{x_1}^{x_2} (t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}\|f(s,\tau,u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))})\|d\tau ds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{t_1}\int_{x_1}^{x_2} (t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}\|f(s,\tau,u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))})\|d\tau ds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^{t_2}\int_0^{x_1} (t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}\|f(s,\tau,u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))})\|d\tau ds \\
&\leq \frac{\|p\|_\infty + \|q\|_\infty\eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^{x_1} (t_k - s)^{r_1-1}[(x_2-\tau)^{r_2-1} - (x_1-\tau)^{r_2-1}]d\tau ds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty\eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_{x_1}^{x_2} (t_k - s)^{r_1-1}(x_2-\tau)^{r_2-1}d\tau ds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty\eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \int_0^{t_1}\int_0^{x_1} [(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1} - (t_1-s)^{r_1-1}(x_1-\tau)^{r_2-1}]d\tau ds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty\eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \int_{t_1}^{t_2}\int_{x_1}^{x_2} (t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}d\tau ds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty\eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \int_0^{t_1}\int_{x_1}^{x_2} (t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}d\tau ds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty\eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \int_{t_1}^{t_2}\int_0^{x_1} (t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}d\tau ds \\
&\leq \frac{\|p\|_\infty + \|q\|_\infty\eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^{x_1} (t_k - s)^{r_1-1}[(x_2-\tau)^{r_2-1} - (x_1-\tau)^{r_2-1}]d\tau ds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty\eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_{x_1}^{x_2} (t_k - s)^{r_1-1}(x_2-\tau)^{r_2-1}d\tau ds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty\eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)}[2x_2^{r_2}(t_2-t_1)^{r_1} + 2t_2^{r_1}(x_2-x_1)^{r_2} \\
&+ t_1^{r_1}x_1^{r_2} - t_2^{r_1}x_2^{r_2} - 2(t_2-t_1)^{r_1}(x_2-x_1)^{r_2}].
\end{aligned}$$

As $t_1 \to t_2$, $x_1 \to x_2$ the right-hand side of the above inequality tends to zero.

As a consequence of Steps 1 to 3, together with the Arzela-Ascoli theorem, we can conclude that $G : \widetilde{PC} \to \widetilde{PC}$ is continuous and completely continuous.

Step 4: F is a contraction.
Let $u, v \in \widetilde{PC}$, then we have for each $(t, x) \in J$

$$\begin{aligned}
&\|F(u)(t,x) - F(v)(t,x)\| \\
&\leq \sum_{k=1}^{m}(\|I_k(u(t_k^-,x)) - I_k(v(t_k^-,x))\| + \|I_k(u(t_k^-,0)) - I_k(v(t_k^-,0))\|) \\
&\leq \sum_{k=1}^{m} l(\|u-v\| + \|u-v\|) \\
&\leq 2ml\|u-v\|.
\end{aligned}$$

Thus

$$\|F(u) - F(v)\| \leq 2ml\|u - v\|.$$

Hence by (4.5), F is a contraction.

Step 5: (A priori bounds)

Now it remains to show that the set

$$\mathcal{E} = \{u \in \widetilde{PC} : u = \lambda F(\frac{u}{\lambda}) + \lambda G(u) \text{ for some } \lambda \in (0,1)\}$$

is bounded. Let $u \in \mathcal{E}$, then $u = \lambda F(\frac{u}{\lambda}) + \lambda G(u)$. Thus, for each $(t, x) \in J$ we have

$$\begin{aligned}
u(t,x) &= \lambda\mu(t,x) + \sum_{k=1}^{m}\lambda(\|I_k\frac{(u(t_k^-,x))}{\lambda})\| + \|I_k\frac{(u(t_k^-,0))}{\lambda}\|) \\
&+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}f(s,\tau,u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))})d\tau ds \\
&+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\int_{t_k}^{t}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}f(s,\tau,u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))})d\tau ds.
\end{aligned}$$

This implies by $(H2)$ and $(H3)$ that, for each $(t,x)\in J$, we have

$$
\begin{aligned}
&\|u(t,x)\| \\
&\leq \|\mu(t,x)\| + \sum_{k=1}^{m}(\|I_k u(t_k^-,x)\| - \|I_k(0)\| + \|I_k u(t_k^-,0)\| - \|I_k(0)\|) + 2\sum_{k=1}^{m}\|I_k(0)\| \\
&+ \frac{\|p\|_\infty}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\|u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\|_C d\tau ds \\
&+ \frac{\|q\|_\infty}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} d\tau ds \\
&+ \frac{\|p\|_\infty}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\|u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\|_C d\tau ds \\
&+ \frac{\|q\|_\infty}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} d\tau ds \\
&\leq \|\mu(t,x)\| + l\sum_{k=1}^{m}(\|u(t_k^-,x)\| + \|u(t_k^-,0)\|) + 2I^* \\
&+ \frac{\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\|u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\|_C d\tau ds \\
&+ \frac{\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} d\tau ds \\
&+ \frac{\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\|u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\|_C d\tau ds \\
&+ \frac{\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} d\tau ds,
\end{aligned}
$$

where

$$I^* = \sum_{k=1}^{m}\|I_k(0)\|.$$

We consider the function γ defined by

$$\gamma(t,x) = sup\{|u(s,\tau)| : -\alpha \leq s \leq t, -\beta \leq \tau \leq x, 0 \leq t \leq a, 0 \leq x \leq b\}.$$

Let $(t^*, x^*) \in [-\alpha, t] \times [-\beta, x]$ be such that $\gamma(t,x) = |u(t^*, x^*)|$. If $(t^*, x^*) \in J$, then by the previous inequality, we have for $(t,x) \in J$

$$\begin{aligned}
\gamma(t,x) &\leq \|\mu(t,x)\| + l\sum_{k=1}^{m}(\|u(t_k^-,x)\| + \|u(t_k^-,0)\|) + 2I^* \\
&+ \frac{\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\Big(\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\gamma(s,\tau)d\tau ds \\
&+ \int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\gamma(s,\tau)d\tau ds\Big) + \frac{2a^{r_1}b^{r_2}\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}.
\end{aligned}$$

Since for every $(t,x) \in J$, $\|u_{(t,x))}\|_C \leq \gamma(t,x)$ This shows that the set $\mathcal{E}$ is bounded. As a consequence of Theorem 4.2 we deduce that $F+G$ has a fixed point u which is a solution of problem (5.1)-(5.3)

4.2.3 An Example

As an application of our results we consider the following impulsive fractional order partial hyperbolic functional differential equations with finite delay of the form

$$({}^cD_0^r u)(t,x) = \frac{e^{-t-x}}{9+e^{t+x}}$$

$$\times\frac{|u(t-\sigma_1(u(t,x)), x-\sigma_2(u(t,x)))|}{1+|u(t-\sigma_1(u(t,x)), x-\sigma_2(u(t,x)))|}, \text{ if } (t,x) \in [0,1]\times[0,1],\ t \neq \frac{1}{2} \tag{4.6}$$

$$u((\tfrac{1}{2})^+, x) = u((\tfrac{1}{2})^-, x) + \frac{|u((\frac{1}{2})^-, x)|}{3+|u((\frac{1}{2})^-, x)|},\ x \in [0,1], \tag{4.7}$$

$$u(t,x) = t + x^2,\ (t,x) \in [-1,1]\times[-2,1] \setminus (0,1]\times(0,1], \tag{4.8}$$

$$u(t,0) = t,\ u(0,x) = x^2,\ (t,x) \in [0,1]\times[0,1], \tag{4.9}$$

where $\sigma_1 \in C(\mathbb{R}, [0,1]),\ \sigma_2 \in C(\mathbb{R}, [0,2])$.

$$\rho_1(t,x,\varphi) = t - \sigma_1(\varphi(0,0)), \quad (t,x,\varphi) \in J \times C,$$

$$\rho_2(t,x,\varphi) = x - \sigma_2(\varphi(0,0)), \quad (t,x,\varphi) \in J \times C,$$

where $C := C_{(1,2)}$. Set

$$f(t,x,\varphi) = \frac{e^{-t-x}|\varphi|}{(9+e^{t+x})(1+|\varphi|)},\ (t,x) \in [0,1]\times[0,1],\ \varphi \in C,$$

and

$$I_k(u) = \frac{u}{3+u},\ u \in \mathbb{R}$$

A simple computations show that conditions of Theorem 4.2 are satisfied which implies that problem (5.10)-(5.13) has a unique solution defined on $[-1,1]\times[-2,1]$

4.3 Impulsive Partial Differential Equations with Infinite Delay

4.3.1 Introduction

Next we consider the following system of partial hyperbolic differential equation of fractionnal order with infinite delay

$$({}^cD^r_{z_k}u)(t,x) = f(t,x,u_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))}); \quad \text{if } (t,x) \in J_k, k = 0,\dots,m; \tag{4.10}$$

$$u(t_k^+,x) = u(t_k^-,x) + I_k(u(t_k^-,x)), \quad \text{if } x \in [0,b], \ k = 1,\dots,m, \tag{4.11}$$

$$u(t,x) = \phi(t,x); \ \text{if } (t,x) \in \tilde{J}' := (-\infty,a] \times (-\infty,b] \setminus (0,a] \times (0,b], \tag{4.12}$$

$$u(t,0) = \varphi(t), \ t \in [0,a], \ u(0,x) = \psi(x); \ x \in [0,b], \tag{4.13}$$

where φ, ψ, I_k are as in problem (5.1)–(5.4), $f : J \times \mathcal{B} \to \mathbb{R}^n$, $\rho_1, \rho_2 : J \times \mathcal{B} \to \mathbb{R}$, $\phi : \tilde{J}' \to \mathbb{R}^n$ and $\mathcal{B}$ is a phase space.

4.3.2 Existence Results

Let the space

$$\Omega := \left\{u : (-\infty,a] \times (-\infty,b] \to \mathbb{R}^n : u_{(t,x)} \in \mathcal{B} \text{ for } (t,x) \in E \ \text{ and } u|_J \in PC\right\}$$

Definition 4.3 A function $u \in \Omega$ such that its mixed derivative $D^2_{\underset{\sim}{t}x}$ exists on J is said to be a solution of (4.10)-(4.13) if u satisfies the condition (4.12) on $\tilde{J}$, the equation (4.10) on J and conditions (4.13) and (4.13) are satisfied on J.

Set $\mathcal{R}' := \mathcal{R}'_{(\rho_1^-,\rho_2^-)}$

$$= \{(\rho_1(s,y,u),\rho_2(s,y,u)) : (s,y,u) \in J \times \mathcal{B}, \rho_i(s,y,u) \le 0; i = 1,2\}$$

We always assume that $\rho_i : J \times \mathcal{B} \longrightarrow \mathbb{R}$; $i = 1,2$ are continuous and the function $(s,y) \longmapsto u_{(s,y)}$ is continuous from $\mathcal{R}'$ into $\mathcal{B}$. We will need to introduce the following hypothesis:

(H_ϕ) There exists a continuous bounded function $L : \mathcal{R}'_{(\rho_1^-,\rho_1^-)} \longrightarrow (0,\infty)$ such that

$$\|\phi_{(s,y)}\|_{\mathcal{B}} \le L(s,y)\|\phi\|_{\mathcal{B}}, \quad for\ any\ (s,y) \in \mathcal{R}'$$

In the sequel we will make use of the following generalization of a consequence of the phase space axioms.

Lemma 4.4 *If $u \in \Omega$, then*

$$\|u_{(s,y)}\|_{\mathcal{B}} = (M + L')\|\phi\|_{\mathcal{B}} + K \sup_{(\theta,\eta)\in[0,max\{0,s\}]\times[0,max\{0,y\}]} \|u(\theta,\eta)\|$$

where

$$L' = \sup_{(s,y)\in\mathcal{R}'} L(s,y).$$

Our main result for the IVP (4.10)-(4.13) is based upon the fixed point theorem due to Burton and Kirk. Let us introduce the following hypotheses which are assumed hereafter.

$(H1)$ The functions $I_k : \mathbb{R}^n \to \mathbb{R}^n$ are continuous.

$(H2)$ There exist $p, q \in C(J, \mathbb{R}_+)$ such that

$$\|f(t,x,u)\| \le p(t,x) + q(t,x)\|u\|_{\mathcal{B}}, \text{ for } (t,x) \in J \text{ and each } u \in \mathcal{B}.$$

$(H3)$ There exists $l > 0$ such that

$$\|I_k(u) - I_k(v)\| \le l\|u - v\| \text{ for each } u, v \in \mathbb{R}^n.$$

Theorem 4.5 *Assume that hypotheses $(H1)$-$(H3)$ hold. If*

$$2ml < 1, \tag{4.14}$$

then the IVP (4.10)-(4.13) has at least one solution on $(-\infty, a] \times (-\infty, b]$.

Proof. We shall reduce the existence of solutions of (5.1)-(5.4) to a fixed point problem. Consider the operator $N : \Omega \longrightarrow \Omega$ defined by

$$N(u)(t,x) = \begin{cases} \phi(t,x) & \text{if } (t,x) \in \tilde{J}, \\ \mu(t,x) + \sum\limits_{0<t_k<t} \Big(I_k(u(t_k^-,x)) - I_k(u(t_k^-,0))\Big) & \text{if } (t,x) \in J, \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum\limits_{0<t_k<t} \int_{t_{k-1}}^{t_k}\int_0^x (t_k - s)^{r_1-1}(x-y)^{r_2-1} & k = 1,\dots,m, \\ \times f(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))})dyds & \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t}\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} & \\ f(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))})dyds. & \end{cases}$$

Consider the operators $A, B : \Omega \to \Omega$ defined by,

$$A(u)(t,x) = \begin{cases} \phi(t,x), & (t,x) \in \tilde{J}, \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum\limits_{0<t_k<t} \int_{t_{k-1}}^{t_k}\int_0^x (t_k - s)^{r_1-1}(x-y)^{r_2-1} & k = 1,\dots,m \\ \times f(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))})dyds & \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t}\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} & \\ \times f(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))})dyds, & (t,x) \in J. \end{cases}$$

and

$$B(u)(t,x)=\begin{cases}0, & (t,x)\in\tilde{J},\\ \mu(t,x)+\sum\limits_{0<t_k<t}(I_k(u(t_k^-,x))-I_k(u(t_k^-,0))), & (t,x)\in J.\end{cases}$$

Let $v(.,.):(-\infty,a]\times(-\infty,b]\to\mathbb{R}^n$ be a function defined by,

$$v(t,x)=\begin{cases}\phi(t,x), & (t,x)\in\tilde{J}.\\ \mu(t,x), & (t,x)\in J.\end{cases}$$

Then $v_{(t,x)}=\phi$ for all $(t,x)\in E$.
For each $w\in(J,\mathbb{R}^n)$ with $w(t,x)=0$ for each $(t,x)\in E$, we denote by $\overline{w}$ the function defined by

$$\overline{w}(t,x)=\begin{cases}0, & (t,x)\in\tilde{J},\\ w(t,x) & (t,x)\in J.\end{cases}$$

If $u(.,.)$ satisfies the integral equation,

$$u(t,x) \;=\; \mu(t,x)+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-y)^{r_2-1}f(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))})dyds$$

we can decompose $u(.,.)$ as $u(t,x)=\overline{w}(t,x)+v(t,x);\;(t,x)\in(t_k,t_{k+1}]\times[0,b]$, which implies $u_{(t,x)}=\overline{w}_{(t,x)}+v_{(t,x)}$, for every $(t,x)\in J\times[0,b]$ and the function $w(.,.)$ satisfies

$$\begin{aligned}w(t,x)=&\sum_{0<t_k<t}(I_k(u(t_k^-,x))-I_k(u(t_k^-,0)))\\ &+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}\int_0^x(t_k-s)^{r_1-1}(x-y)^{r_2-1}\\ &\times f(s,y,\overline{w}_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))})}+v_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))})})dyds\\ &+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_k}^t\int_0^x(t-s)^{r_1-1}(x-y)^{r_2-1}\\ &\times f(s,y,\overline{w}_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))})}+v_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))})})dyds.\end{aligned}$$

Set

$$C_0=\{w\in\Omega:\;w(t,x)=0\;\text{ for }(t,x)\in E\},$$

and let $\|.\|_{C_0}$ be the norm in C_0 defined by

$$\|w\|_{C_0}=\sup_{(t,x)\in E}\|w_{(t,x)}\|_{\mathcal{B}}+\sup_{(t,x)\in J}\|w(t,x)\|=\sup_{(t,x)\in J}\|w(t,x)\|,\;w\in C_0.$$

C_0 is a Banach space with norm $\|.\|_{C_0}$. Let the operators $A, B: C_0 \to C_0$ defined by

$$(Aw)(t,x) = \begin{cases} \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} \int_0^x (t_k - s)^{r_1-1}(x-y)^{r_2-1} & k = 1, \dots, m \\ \times f(s, y, \overline{w}_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)})))}} + v_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)})))}})dyds & \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t} \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} & (t,x) \in J. \\ \times f(s, y, \overline{w}_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)})))}} + v_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)})))}})dyds & \end{cases}$$

and

$$(Bw)(t,x) = \mu(t,x) + \sum_{0<t_k<t} (I_k(u(t_k^-, x)) - I_k(u(t_k^-, 0))), (t,x) \in J.$$

Then the problem of finding the solution of the IVP (4.10)–(4.13) is reduced to finding the solutions of the operator equation $A(w) + B(w) = w$. We shall show that the operators A and B satisfy the conditions of Theorem 4.5. The proof will be given by a couple of steps.

Step 1: A is continuous.

Let $\{w_n\}$ be a sequence such that $w_n \to w$ in C_0, then for each $(t,x) \in J$

$$\begin{aligned}
&\|A(w_n)(t,x) - A(w)(t,x)\| \\
&\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^x (t_k - s)^{r_1-1}(x-y)^{r_2-1} \\
&\times \|f(s, y, \overline{w}_{n(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)})))}} + v_{n(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)})))}}) \\
&- f(s, y, \overline{w}_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)})))}} + v_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)})))}})\|dyds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t} \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \\
&\times \|f(s, y, \overline{w}_{n(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)})))}} + v_{n(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)})))}}) \\
&- f(s, y, \overline{w}_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)})))}} + v_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)})))}})\|dyds. \\
&\leq \frac{\|f(.,.,\overline{w}_{n(.,.)} + v_{n(.,.)}) - f(.,.,\overline{w}_{(.,.)} + v_{(.,.)})\|}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^x (t_k - s)^{r_1-1}(x-y)^{r_2-1} dyds \\
&+ \frac{\|f(.,.,\overline{w}_{n(.,.)} + v_{n(.,.)}) - f(.,.,\overline{w}_{(.,.)} + v_{(.,.)})\|}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t} \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} dyds.
\end{aligned}$$

Since f is continuous function, we have

$$\|A(w_n) - A(w)\|_{C_0} \leq \frac{2a^{r_1}b^{r_2}\|f(.,.,w_{n(.,.)}) - f(.,.,w_{(.,.)})\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)} \to 0 \text{ as } n \to \infty$$

Thus A is continuous.

Step 2: A maps bounded sets into bounded sets in C_0.
Indeed, it is enough show that for any η^*, there exists a positive constant l such that, for each $w \in B_{\eta^*} = \{w \in C_0 : \|w\|_{(a,b)} \leq \eta^*\}$ we have $\|A(w)\|_\infty \leq l$
By $(H2)$ we have for each $(x, y) \in (x_k, x_{k+1}] \times [0, b]$,

$$\begin{aligned}
\|A(w)(t,x)\| \leq & \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^x (t_k - s)^{r_1-1}(x-y)^{r_2-1} \\
& \times \|f(s, y, \overline{w}_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)})))}} + v_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)})))}})\|dyds \\
& + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t} \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \\
& \times \|f(s, y, \overline{w}_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)})))}} + v_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)})))}})\|dyds \\
& \leq \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^x (t_k - s)^{r_1-1}(x-y)^{r_2-1} dyds \\
& + \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \int_{t_k}^{t} \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} dyds
\end{aligned}$$

Thus

$$\|A(w)\|_B \leq \frac{2a^{r_1}b^{r_2}(\|p\|_\infty + \|q\|_\infty \eta^*)}{\Gamma(r_1+1)\Gamma(r_2+1)} := l$$

where

$$\begin{aligned}
\|\overline{w}_{(s,y)} + v_{(s,y)}\|_{\mathcal{B}} & \leq \|\overline{w}_{(s,y)}\|_{\mathcal{B}} + \|v_{(s,y)}\|_{\mathcal{B}} \\
& \leq K\eta^* + K\|\phi(0,0)\| + M\|\phi\|_{\mathcal{B}} := \eta.
\end{aligned}$$

Hence $\|A(w)\|_{C_0} \leq l$.

Step 3: A maps bounded sets into equicontinuous sets in C_0.
Let $(t_1, x_1), (t_2, x_2) \in (0, a] \times (0, b]$, $t_1 < t_2$, $x_1 < x_2$, B_{η^*} be a bounded set as in Step 2.

Let $w \in B_{\eta^*}$, then

$$\begin{aligned}
&\|A(w)(t_2,x_2) - A(w)(t_1,x_1)\| \\
&\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^{x_1} (t_k - s)^{r_1-1}[(x_2-y)^{r_2-1} - (x_1-y)^{r_2-1}] \\
&\times f(s,y,\overline{w}_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))})} + v_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))})})dyds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_{x_1}^{x_2} (t_k - s)^{r_1-1}(x_2-y)^{r_2-1} \\
&\times \|f(s,y,\overline{w}_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))})} + v_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))})})\|dyds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{t_1} \int_0^{x_1} [(t_2-s)^{r_1-1}(x_2-y)^{r_2-1} - (t_1-s)^{r_1-1}(x_1-y)^{r_2-1}] \\
&\times f(s,y,\overline{w}_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))})} + v_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))})})dydx \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^{t_2} \int_{x_1}^{x_2} (t_2-s)^{r_1-1}(x_2-y)^{r_2-1} \\
&\times \|f(s,y,\overline{w}_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))})} + v_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))})})\|dyds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{t_1} \int_{x_1}^{x_2} (t_2-s)^{r_1-1}(x_2-y)^{r_2-1} \\
&\times \|f(s,y,\overline{w}_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))})} + v_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))})})\|dyds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^{t_2} \int_0^{x_1} (t_2-s)^{r_1-1}(x_2-y)^{r_2-1} \\
&\times \|f(s,y,\overline{w}_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))})} + v_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))})})\|dyds \\
&\leq \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^{x_1} (t_k-s)^{r_1-1}[(x_2-y)^{r_2-1} - (x_1-y)^{r_2-1}]dyds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_{x_1}^{x_2} (t_k-s)^{r_1-1}(x_2-y)^{r_2-1}dyds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \int_0^{t_1} \int_0^{x_1} [(t_2-s)^{r_1-1}(x_2-y)^{r_2-1} - (t_1-s)^{r_1-1}(x_1-y)^{r_2-1}]dyds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \int_{t_1}^{t_2} \int_{x_1}^{x_2} (t_2-s)^{r_1-1}(x_2-y)^{r_2-1}dyds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \int_0^{t_1} \int_{x_1}^{x_2} (t_2-s)^{r_1-1}(x_2-y)^{r_2-1}dyds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \int_{t_1}^{t_2} \int_0^{x_1} (t_2-s)^{r_1-1}(x_2-y)^{r_2-1}dyds \\
&\leq \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^{x_1} (t_k-s)^{r_1-1}[(x_2-y)^{r_2-1} - (x_1-y)^{r_2-1}]dyds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_{x_1}^{x_2} (t_k-s)^{r_1-1}(x_2-y)^{r_2-1}dyds \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} [2x_2^{r_2}(t_2-t_1)^{r_1} + 2t_2^{r_1}(x_2-x_1)^{r_2} \\
&+ t_1^{r_1}x_1^{r_2} - t_2^{r_1}x_2^{r_2} - 2(t_2-t_1)^{r_1}(x_2-x_1)^{r_2}].
\end{aligned}$$

As $t_1 \to t_2$, $x_1 \to x_2$ the right-hand side of the above inequality tends to zero.

As a consequence of Steps 1 to 3, together with the Arzela-Ascoli theorem, we can conclude that $A : C_0 \to C_0$ is continuous and completely continuous.

Step 4: B is a contraction.
Let $w, w^* \in C_0$, then we have for each $(t, x) \in J$

$$\begin{aligned}
&\|B(w)(t,x) - B(w^*)(t,x)\| \\
&\leq \sum_{k=1}^{m}(\|I_k(w(t_k^-,x)) - I_k(w^*(t_k^-,x))\| + \|I_k(w(t_k^-,0)) - I_k(w^*(t_k^-,0))\|) \\
&\leq \sum_{k=1}^{m} l(\|w - w^*\|_{C_0} + \|w - w^*\|_{C_0}) \\
&\leq 2ml\|w - w^*\|_{C_0}.
\end{aligned}$$

Thus

$$\|B(w) - B(w^*)\|_{C_0} \leq 2ml\|w - w^*\|_{C_0}.$$

Hence by (4.14), B is a contraction.

Step 5: (A priori bounds)

Now it remains to show that the set

$$\mathcal{E} = \{w \in C_0 : w = \lambda B\left(\frac{w}{\lambda}\right) + \lambda A(w), \text{ for some } \lambda \in (0,1)\}$$

is bounded. Let $w \in \mathcal{E}$, then $w = \lambda B\left(\frac{w}{\lambda}\right) + \lambda A(w)$. Thus, for each $(x, y) \in J$ we have

$$\begin{aligned}
w(t,x) &= \lambda \sum_{k=1}^{m}(\|I_k \frac{(u(t_k^-,x))}{\lambda})\| + \|I_k \frac{(u(t_k^-,0))}{\lambda}\|) \\
&+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \\
&\times f(s, y, \overline{w}_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))})} + v_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))})})dyds \\
&+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t} \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \\
&\times f(s, y, \overline{w}_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))})} + v_{(s,y,u_{(\rho_1(s,y,u_{(s,y)}),\rho_2(s,y,u_{(s,y)}))})})dyds.
\end{aligned}$$

This implies by $(H2)$ and $(H3)$ that, for each $(t,x)\in J$, we have

$$\begin{aligned}
\|w(t,x)\| &\leq \sum_{k=1}^{m}\lambda(\|I_k\frac{u(t_k^-,x)}{\lambda}\|-\|I_k(0)\|+\|I_k\frac{u(t_k^-,0)}{\lambda}\|-\|I_k(0)\|)\\
&+ 2\lambda\sum_{k=1}^{m}\|I_k(0)\|+\frac{\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x(t-s)^{r_1-1}(x-y)^{r_2-1}\\
&\times \|\overline{w}_{(s,y)}+v_{(s,y)}\|_B dyds\\
&+ \frac{\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x(t-s)^{r_1-1}(x-y)^{r_2-1}dyds\\
&+ \frac{\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-y)^{r_2-1}\|\overline{w}_{(s,y)}+v_{(s,y)}\|_B dyds\\
&+ \frac{\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-y)^{r_2-1}dyds\\
&\leq l\sum_{k=1}^{m}(\|u(t_k^-,x)\|+\|u(t_k^-,0)\|)+2I^*\\
&+ \frac{\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x(t-s)^{r_1-1}(x-y)^{r_2-1}\|\overline{w}_{(s,y)}+v_{(s,y)}\|_B dyds\\
&+ \frac{\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x(t-s)^{r_1-1}(x-y)^{r_2-1}dyds\\
&+ \frac{\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-y)^{r_2-1}\|\overline{w}_{(s,y)}+v_{(s,y)}\|_B dyds\\
&+ \frac{\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-y)^{r_2-1}dyds,
\end{aligned}$$

where

$$I^*=\sum_{k=1}^{m}\|I_k(0)\|.$$

and

$$\begin{aligned}
\|\overline{w}_{(s,y)}+v_{(s,y)}\|_{\mathcal{B}} &\leq \|\overline{w}_{(s,y)}\|_{\mathcal{B}}+\|v_{(s,y)}\|_{\mathcal{B}}\\
&\leq K\sup_{\{w(\tilde{s},\tilde{y}):(\tilde{s},\tilde{y})\in[0,s]\times[0,t]\}}\\
&+M\|\phi\|_{\mathcal{B}}+K\|\phi(0,0)\|. \qquad (4.15)
\end{aligned}$$

If we name $\gamma(s,y)$ the right hand side of (4.15), then we have

$$\|\overline{w}_{(s,y)}+v_{(s,y)}\|_{\mathcal{B}}\leq\gamma(t,x),$$

and therefore, for $\gamma(t,x) \in J$ we obtain

$$\begin{aligned}
\|w(t,x)\| &\leq l\sum_{k=1}^{m}(\|u(t_k^-,x)\| + \|u(t_k^-,0)\|) + 2I^* + \frac{2a^{r_1}b^{r_2}\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)} \\
&+ \frac{\|p\|_\infty}{\Gamma(r_1)\Gamma(r_2)}\Big(\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\gamma(s,\tau)d\tau ds \\
&+ \int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\gamma(s,\tau)d\tau ds\Big). \qquad (4.16)
\end{aligned}$$

Using the above inequality and the definition of γ for each $(t,x) \in J$ we have

$$\begin{aligned}
\gamma(t,x) &\leq M\|\phi\|_{\mathcal{B}} + K\|\phi(0,0)\| + l\sum_{k=1}^{m}(\|u(t_k^-,x)\| + \|u(t_k^-,0)\|) + 2I^* \\
&+ \frac{\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\Big(\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\gamma(s,\tau)d\tau ds \\
&+ \int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\gamma(s,\tau)d\tau ds\Big) + \frac{2a^{r_1}b^{r_2}\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}.
\end{aligned}$$

If $(t,x) \in J$, then Lemma 1.4 implies that there exists $\tilde{k} = \tilde{k}(r_2, r_2)$ such that

$$\begin{aligned}
\gamma(t,x) &\leq \Big(M\|\phi\|_{\mathcal{B}} + K\|\phi(0,0)\| + l\sum_{k=1}^{m}(\|u(t_k^-,x)\| + \|u(t_k^-,0)\|) \\
&+ 2I^* + \frac{2a^{r_1}b^{r_2}\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\Big) \\
&\times \Big(1 + \tilde{k}\frac{\|p\|_\infty}{\Gamma(r_2+1)\Gamma(r_2+1)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}d\tau ds\Big) \\
&\leq \Big(M\|\phi\|_{\mathcal{B}} + K\|\phi(0,0)\| + l\sum_{k=1}^{m}(\|u(t_k^-,x)\| + \|u(t_k^-,0)\|) \\
&+ 2I^* + \frac{2a^{r_1}b^{r_2}\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\Big) \\
&\times \Big(1 + \tilde{k}\frac{\|p\|_\infty a^{r_1}b^{r_2}}{\Gamma(r_2+1)\Gamma(r_2+1)}\Big) := \tilde{R}.
\end{aligned}$$

Since for every $(t,x) \in J$, $\|w_{(t,x))}\|_\infty \leq \gamma(t,x)$.
This shows that the set $\mathcal{E}$ is bounded. As a consequence of Theorem 4.5 we deduce that $A+B$ has a fixed point which is a solution of problem (4.10)-(4.13).

4.3.3 An Example

We consider now the following impulsive fractional order partial hyperbolic functional equations with infinite delay of the form

$$({}^cD_0^r u)(t,x) = \frac{ce^{t+x-\gamma(t+x)}|u(t-\sigma_1(u(t,x)), x-\sigma_2(u(t,x)))|}{(e^{t+x}+e^{-t-x})(1+|u(t-\sigma_1(u(t,x)), x-\sigma_2(u(t,x)))|)},$$

$$\text{if } (t,x) \in [0,1]\times[0,1],\ t \neq \frac{k}{k+1};\ k=1,\dots,m. \tag{4.17}$$

$$u((\frac{k}{k+1})^+, x) = u((\frac{k}{k+1})^-, x) + \frac{|u((\frac{k}{k+1})^-, x)|}{3mk+|u((\frac{k}{k+1})^-, x)|},\ x\in[0,1],\ k=1,\dots,m, \tag{4.18}$$

$$u(t,x) = t + x^2,\ (t,x) \in (-\infty,1]\times(-\infty,1]\setminus(0,1]\times(0,1], \tag{4.19}$$

$$u(t,0) = t,\ u(0,x) = x^2,\ (t,x)\in[0,1]\times[0,1], \tag{4.20}$$

where $c = \frac{10}{\Gamma(r_r+1)\Gamma(r_2+1)}$, γ a positive real constant and $\sigma_1, \sigma_2 \in C(\mathbb{R}, [0,\infty))$. Let $\mathcal{B}_\gamma$ be tha phase space. Set

$$\rho_1(t,x,\varphi) = t - \sigma_1(\varphi(0,0)),\quad (t,x,\varphi)\in J\times\mathcal{B}_\gamma,$$

$$\rho_2(t,x,\varphi) = x - \sigma_2(\varphi(0,0)),\quad (t,x,\varphi)\in J\times\mathcal{B}_\gamma,$$

$$f(t,x,\varphi) = \frac{ce^{t+x-\gamma(t+x)}|\varphi|}{(e^{t+x}+e^{-t-x})(1+|\varphi|)}, (t,x)\in[0,1]\times[0,1], \varphi\in\mathcal{B}_\gamma$$

and

$$I_k(u) = \frac{u}{3mk+u};\ u\in\mathbb{R}, k=1,\dots,m$$

We easily show that conditions of Theorem 4.5 are satisfied, and hence problem (4.17)-(4.20) has a unique solution defined on $(-\infty,1]\times(-\infty,1]$

Chapter 5

Global Uniqueness Results for Impulsive Partial Hyperbolic Functional Differential Equations of Fractional Order

5.1 Introduction

This chapter deals with the global existence and uniqueness of solutions for impulsive partial functional differential equations with delay, involving the Caputo fractional derivative. Our works will be conducted by using a nonlinear alternative of Leray-Schauder due to Frigon-Granas type for contraction maps on Fréchet spaces.

5.2 Existence Results for the Finite Delay Case

5.2.1 Introduction

This section deals with the existence of solutions for impulsive initial value problem for differential equations of fractional order with fixed time impulses given by

$$({}^cD^r_{z_k}u)(t,x) = f(t,x,u_{(t,x)}), \text{ if } (t,x) \in J_k, \; k = 0,\dots,m, \tag{5.1}$$

$$u(t_k^+,x) = u(t_k^-,x) + I_k(u(t_k^-,x)), \text{ if } x \in [0,b]; \; k = 1,\dots,m, \tag{5.2}$$

$$u(t,x) = \phi(t,x); \text{ if } (t,x) \in \tilde{J}, \tag{5.3}$$

$$u(t,0) = \varphi(t), \; u(0,x) = \psi(x), \; t \in [0,a], \; x \in [0,b], \tag{5.4}$$

where $J_0 = [0,t_1] \times [0,b]$, $J_k := (t_k,t_{k+1}] \times [0,b]$, $k = 1,\dots,m, z_k = (t_k,0)$, $k = 0,\dots,m$, $\varphi(0) = \psi(0)$, $J := [0,\infty)\times[0,\infty)$, $\tilde{J} := [-\alpha,\infty)\times[-\beta,\infty)\backslash[0,\infty)\times[0,\infty)$, $\alpha,\beta > 0$, ${}^cD^r_{z_k}$ is the standard Caputo's fractional derivative of order $r = (r_1,r_2) \in (0,1] \times$

$(0,1]$, $f : J \times C([-\alpha,0]\times[-\beta,0],\mathbb{R}^n) \to \mathbb{R}^n$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 0,1,\ldots,m$ are given functions, for each $(t,x) \in J, \varphi,\psi : [0,\infty) \to \mathbb{R}^n$, are given absolutely continuous functions and $C([-\alpha,0]\times[-\beta,0],\mathbb{R}^n)$ is the space of continuous functions on $[-\alpha,0]\times[-\beta,0]$. We denote by $u_{(t,x)}$ the element of $C([-\alpha,\infty)\times[-\beta,\infty),\mathbb{R}^n)$ defined by

$$u_{(t,x)}(s,\tau) = u(t+s,x+\tau);\ (s,\tau) \in [-\alpha,0]\times[-\beta,0],$$

here $u_{(t,x)}(.,.)$ represents the history of the state from time $t-\alpha$ up to the present time t and from time $x-\beta$ up to the present time x.

5.2.2 Main Results

In this section we present a global existence and uniqueness result for the problem (5.1)-(5.4).

We shall consider the space

$$PC = \big\{u : J \to \mathbb{R}^n : u \in C(J_k,\mathbb{R}^n); k = 1,\ldots,m, \text{ and there exist } u(t_k^-,x) \text{ and } u(t_k^+,x); k = 1,\ldots,m, \text{ with } u(t_k^-,x) = u(t_k,x)\big\}.$$

For each $p,q \in \mathbb{N}$ we consider following set,

$$C_{(p,q)} = C([-\alpha,p]\times[-\beta,q],\mathbb{R}^n)$$

and we define in $C_0 := C([-\alpha,\infty)\times[-\beta,\infty),\mathbb{R}^n)$ the semi-norms by:

$$\|u\|_{(p,q)} = \{\sup\|u(t,x)\| : -\alpha \le t \le p, -\beta \le x \le q\}.$$

Let

$$\Omega = \{u : J \to \mathbb{R}^n : u \in PC \cap C_0\}$$

Then Ω is a Fréchet space with the family of semi-norms $\{\|u\|_{(p,q)}\}$.

Let us start by defining what we mean by a solution of the problem (5.1)-(5.4)

Definition 5.1 *A function $u \in \Omega$ is said to be a solution of (5.1)-(5.4) if u satisfies equations (5.1)-(5.4) on J and the condition (5.3) on $\tilde{J}$.*

For the existence of solutions for the problem (5.1)-(5.4), we need the following lemma:

Lemma 5.2 *[7] Let $0 < r_1, r_2 \le 1$ and let $h : J \to \mathbb{R}^n$ be continuous. A function u is a solution of the fractional integral equation*

$$u(t,x) = \begin{cases} \phi(t,x) & \text{if } (t,x) \in \tilde{J}, \\ \mu(t,x) + \sum\limits_{0<t_k<t}\Big(I_k(u(t_k^-,x)) - I_k(u(t_k^-,0))\Big) & \text{if } (t,x) \in J, \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum\limits_{0<t_k<t}\int_{t_{k-1}}^{t_k}\int_0^x (t_k-s)^{r_1-1}(x-y)^{r_2-1}h(s,y)dyds & k = 1,\ldots,m, \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_k}^{t}\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}h(s,y)dyds & \end{cases} \tag{5.5}$$

if and only if u is a solution of the fractional initial value problem

$$^{c}D^{r}u(t,x)=h(t,x), \quad (t,x)\in J_k,\ k=0,\dots,m, \tag{5.6}$$

$$u(t_k^{+},x)=u(t_k^{-},x)+I_k(u(t_k^{-},x)), \quad k=1,\dots,m. \tag{5.7}$$

Further, we present conditions for the existence and uniqueness of a solution of problem (5.1)-(5.3).

Theorem 5.3 *Assume*

$(H1)$ *The functions $I_k:\mathbb{R}^n\to\mathbb{R}^n$ are continuous.*

$(H2)$ *For each $p,q\in\mathbb{N}$, there exists $\ell_{(p,q)}\in C(J_0,\mathbb{R}^n)$ such that for each $(t,x)\in\widetilde{J}_0$*

$$\|f(t,x,u)-f(t,x,v)\|\le \ell_{(p,q)}(t,x)\|u-v\|_C,\ \textit{for each } u,v\in\mathbb{R}^n.$$

$(H3)$ *For each $p,q\in\mathbb{N}$, there exists $\overline{\ell}_{(p,q)}\in C(J_0,\mathbb{R}^n)$ such that for each $(t,x)\in\widetilde{J}_0$*

$$\|I_k(u)-I_k(v)\|\le \overline{\ell}_{(p,q)}\|u-v\|_C,\ \textit{for each } u,v\in\mathbb{R}^n.$$

If

$$2m\overline{\ell}_{(p,q)}+\frac{2\ell^{*}_{(p,q)}p^{r_1}q^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}<1, \tag{5.8}$$

where

$$\ell^{*}_{(p,q)}=\sup_{(t,x)\in\widetilde{J_0}}\ell_{(p,q)}(t,x),$$

then there exists a unique solution for IVP (5.1)-(5.4) on $[-\alpha,\infty)\times[-\beta,\infty)$.

Proof: Transform the problem (5.1)-(5.4) into a fixed point problem. Consider the operator $N:\Omega\to\Omega$ defined by,

$$(Nu)(t,x)=\begin{cases}\phi(t,x) & \text{if } (t,x)\in\tilde{J},\\ \mu(t,x)+\sum\limits_{0<t_k<t}\Big(I_k(u(t_k^{-},x))-I_k(u(t_k^{-},0))\Big) & \text{if } (t,x)\in J,\\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum\limits_{0<t_k<t}\int_{t_{k-1}}^{t_k}\int_0^x(t_k-s)^{r_1-1}(x-y)^{r_2-1}f(s,y,u_{(s,y)})dyds & k=1,\dots,m,\\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_k}^{t}\int_0^x(t-s)^{r_1-1}(x-y)^{r_2-1}f(s,y,u_{(s,y)})dyds & \end{cases}$$

Let u be a possible solution of the problem $u = \lambda N(u)$ for some $0 < \lambda < 1$. Thus for each $(t, x) \in \widetilde{J}_0$,

$$\begin{aligned} u(t,x) &= \lambda\mu(t,x) + \lambda\sum_{k=1}^{m}\Big(I_k(u(t_k^-,x)) - I_k(u(t_k^-,0))\Big) \\ &+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=0}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t_k-s)^{r_1-1}(x-y)^{r_2-1}f(s,y,u_{(s,y)})dyds \\ &+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\int_{t_k}^{t}\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}f(s,y,u_{(s,y)})dyds \end{aligned}$$

This implies by $(H2)$ and $(H3)$ that

$$\begin{aligned} ||u(t,x)|| &= \lambda||\mu(t,x)|| + \lambda\sum_{k=1}^{m}(\|I_k(u(t_k^-,x))\| - \|I_k(0)\| + I_k(u(t_k^-,0))\| - \|I_k(0)\|) \\ &+ 2\lambda\sum_{k=1}^{m}\|I_k(0)\| + \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \\ &\times ||f(s,\tau,u_{(s,\tau)}) - f(s,\tau,0)||d\tau ds \\ &+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}||f(s,\tau,0)||d\tau ds \\ &+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\int_{t_k}^{t}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}||f(s,\tau,u_{(s,\tau)}) - f(s,\tau,0)||d\tau ds \\ &+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\int_{t_k}^{t}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}||f(s,\tau,0)||d\tau ds \\ &\leq ||\mu(t,x)|| + \bar{l}\sum_{k=1}^{m}(\|u(t_k^-,x)\| + \|u(t_k^-,0)\|) + 2I^* + \frac{2f^*p^{r_1}q^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\Big(\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\ell_{(p,q)}(s,\tau)\|u_{(s,\tau)}\|_C d\tau ds \\ &+ \int_{t_k}^{t}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\ell_{(p,q)}(s,\tau)\|u_{(s,\tau)}\|_C d\tau ds\Big), \end{aligned}$$

where $f^* = \sup_{(s,\tau)\in J_0}||f(s,\tau,0)||, I^* = \sum_{k=1}^{m}\|I_k(0)\|$

We consider the function y defined by

$$y(t,x) = \sup\{\|u(s,\tau)\| : -\alpha \leq s \leq t, \ -\beta \leq \tau \leq x\}, \ 0 \leq t \leq p, \ 0 \leq x \leq q.$$

Let $(t^*, x^*) \in [-\alpha, t] \times [-\beta, x]$ be such that $y(t,x) = \|u(t^*, x^*)\|$. If $(t^*, x^*) \in \widetilde{J}_0$, then by the previous inequality, we have for $(t, x) \in J_0$,

$$\begin{aligned}||u(t,x)|| &\leq ||\mu(t,x)|| + \bar{l}\sum_{k=1}^{m}(\|u(t_k^-,x)\| + \|u(t_k^-,0)\|) + 2I^* + \frac{2f^*p^{r_1}q^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\Big(\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\ell_{(p,q)}(s,\tau)y(s,\tau)d\tau ds \\ &+ \int_{t_k}^{t}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\ell_{(p,q)}(s,\tau)y(s,\tau)d\tau ds\Big). \end{aligned} \tag{5.9}$$

If $(t^*,x^*) \in \tilde{J}$, then $y(t,x) = \|\phi\|_C$ and the previous inequality holds. By (5.9) obtain that

$$\begin{aligned}||y(t,x)|| &= ||\mu(t,x)|| + \bar{l}\sum_{k=1}^{m}(\|u(t_k^-,x)\| + \|u(t_k^-,0)\|) + 2I^* + \frac{2f^*p^{r_1}q^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\Big(\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\ell_{(p,q)}(s,\tau)y(s,\tau)d\tau ds \\ &+ \int_{t_k}^{t}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\ell_{(p,q)}(s,\tau)y(s,\tau)d\tau ds\Big) \\ &\leq ||\mu(t,x)|| + \bar{l}\sum_{k=1}^{m}(\|u(t_k^-,x)\| + \|u(t_k^-,0)\|) + 2I^* + \frac{2f^*p^{r_1}q^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\Big(\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}y(s,\tau)\tau ds \\ &+ \int_{t_k}^{t}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}y(s,\tau)\tau ds\Big),\end{aligned}$$

and Lemma 1.4 implies that there exists a constant $\delta = \delta(r_1, r_2)$ such that we have

$$\begin{aligned} y(t,x) &\leq \left[||\mu||_{(p,q)} + \bar{l}\sum_{k=1}^{m}(\|u(t_k^-,x)\| + \|u(t_k^-,0)\|) + 2I^* + \frac{2f^*p^{r_1}q^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\right] \\ &\times\left[1 + \frac{\delta\ell^*_{(p,q)}}{\Gamma(r_1+1)\Gamma(r_2+1)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}d\tau ds\right] \\ &:= M_{(p,q)}.\end{aligned}$$

Then from (5.9) we have

$$\begin{aligned}||u||_{(p,q)} &\leq ||\mu||_{(p,q)} + \bar{l}\sum_{k=1}^{m}(\|u(t_k^-,x)\| + \|u(t_k^-,0)\|) + 2I^* \\ &+ \frac{2f^*p^{r_1}q^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} + \frac{M_{(p,q)}\ell^*_{(p,q)}}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ &:= M^*_{(p,q)}.\end{aligned}$$

Since for every $(t,x) \in \widetilde{J}_0$, $\|u_{(t,x)}\|_C \leq y(t,x)$, we have

$$\|u\|_{(p,q)} \leq \max(\|\phi\|_C, M^*_{(p,q)}) := R_{(p,q)}.$$

Set

$$U = \{u \in \Omega : \|u\|_{(p,q)} \leq R_{(p,q)} + 1 \ for\ all\ p,q \in \mathbb{N}\}.$$

We shall show that $N : U \to C_{(p,q)}$ is a contraction maps. Indeed, consider $v, w \in U$. Then for each $(t,x) \in \widetilde{J}_0$, we have

$$\begin{aligned}
\|(Nv)(t,x) - (Nw)(t,x)\| &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^x |(t-s)^{r_1-1}||(x-\tau)^{r_2-1}| \\
&\quad \times \|f(s,\tau,v_{(s,\tau)}) - f(s,\tau,w_{(s,\tau)})\| d\tau ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t} \int_0^x |(t-s)^{r_1-1}||(x-\tau)^{r_2-1}| \\
&\quad \times \|f(s,\tau,v_{(s,\tau)}) - f(s,\tau,w_{(s,\tau)})\| d\tau ds \\
&\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \\
&\quad \times \ell_{(p,q)} \|v_{(s,\tau)} - w_{(s,\tau)}\|_C d\tau ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t} \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \\
&\quad \times \ell_{(p,q)} \|v_{(s,\tau)} - w_{(s,\tau)}\|_C d\tau ds \\
&\leq \frac{2\ell^*_{(p,q)} p^{r_1} q^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \|v - w\|_{(p,q)}.
\end{aligned}$$

Thus

$$\|(Nv)(t,x) - (Nw)(t,x)\|_{(p,q)} \leq \frac{2\ell^*_{(p,q)} p^{r_1} q^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \|v - w\|_{(p,q)}.$$

Hence by (5.9), $N : U \to C_{(p,q)}$ is a contraction.
By the choice of U, there is no $u \in \partial_n U^n$ such that $u = \lambda N(u)$, for $\lambda \in (0,1)$. As a consequence of Theorem 5.3, we deduce that N has a unique fixed point u in U which is a solution to problem (5.1)-(5.4).

5.2.3 An Example

As an application of our results we consider the following fractional order partial hyperbolic functional differential equations with finite delay of the form

$$({}^cD^r_{z_k} u)(t,x) = \frac{c_{(p,q)}}{e^{t+x+2}(1+|u(t,x)|)}, \quad \text{if } (t,x) \in J := [0,\infty) \times [0,\infty), \tag{5.10}$$

$$u(\frac{1}{2}^+, y) = u(\frac{1}{2}^-, y) + \frac{|u((\frac{1}{2})^-, y)|}{3 + |u((\frac{1}{2})^-, y)|}, \text{ if } y \in [0, 1], \tag{5.11}$$

$$u(t, x) = t + x^2, \ (t, x) \in \tilde{J} := [-1, \infty) \times [-2, \infty) \backslash [0, \infty) \times [0, \infty), \tag{5.12}$$

$$u(t, 0) = t, \ u(0, x) = x^2, \ (t, x) \in J, \tag{5.13}$$

where $z_0 = (0, 0), z_1 = (\frac{1}{2}, 0)$ and

$$c_{(p,q)} = \frac{\Gamma(r_1 + 1)\Gamma(r_2 + 1)}{p^{r_1} q^{r_2}}, \quad p, q \in \mathbb{N}^*.$$

$$f(t, x, \varphi) = \frac{c_{(p,q)}}{(e^{t+x+2})(1 + |\varphi|)}, \ (t, x) \in J, \ \varphi \in C([-1, 0] \times [-2, 0], \mathbb{R}),$$

and

$$I_1(u) = \frac{|u|}{\frac{1}{4} + |u|}$$

It is clear that the functions f and I_1 are continuous, and for each $p, q \in \mathbb{N}^*$ and $\varphi, \ \overline{\varphi} \in C([-1, 0] \times [-2, 0], \mathbb{R})$ and $(t, x) \in J_0 := [0, p] \times [0, q]$ we have

$$|f(t, x, \varphi) - f(t, x, \overline{\varphi})| \leq \frac{c_{(p,q)}}{e^2} \|\varphi - \overline{\varphi}\|_C.$$

Also, for $u, v \in \mathbb{R}$, we have

$$|I_1(u) - I_1(v)| \leq \frac{1}{4}|u - v|$$

Hence conditions $(H2)$ and $(H3)$ are satisfied with $\bar{l} = \frac{1}{4}$ and $\ell^*_{(p,q)} = \frac{c_{(p,q)}}{e^2}$. We shall show that condition (5.4) holds for all $p, q \in \mathbb{N}^*$. Indeed

$$2m\bar{l} + \frac{\ell^*_{(p,q)} p^{r_1} q^{r_2}}{\Gamma(r_1 + 1)\Gamma(r_2 + 1)} = 2\frac{1}{4} + \frac{1}{e^2 \Gamma(r_1 + 1)\Gamma(r_2 + 1)} < 1,$$

which is satisfied for each $(r_1, r_2) \in (0, 1] \times (0, 1]$. Consequently Theorem 5.3 implies that problem (5.10)-(5.12) has a unique solution defined on $[-1, \infty) \times [-2, \infty)$.

5.3 Existence Results for the Infinite Delay Case

5.3.1 Introduction

Next result deals with the existence of solutions to fractional order partial differential equations

$$({}^c D^r_{z_k} u)(t, x) = f(t, x, u(t, x)), \text{ if } (t, x) \in J_k, \ k = 0, \ldots, m, \tag{5.14}$$

$$u(t_k^+, x) = u(t_k^-, x) + I_k(u(t_k^-, x)), \text{ if } x \in [0, b]; \ k = 1, \ldots, m, \tag{5.15}$$

$$u(t,x) = \phi(t,x); \text{ if } (t,x) \in \tilde{J}', \tag{5.16}$$

$$u(t,0) = \varphi(t),\ u(0,x) = \psi(x),\ t \in [0,a],\ x \in [0,b], \tag{5.17}$$

where $\varphi,\ \psi$ are as in problem (**??**)-(**??**), $\tilde{J}' =: (-\infty,+\infty)\times(-\infty,+\infty)\backslash[0,\infty)\times[0,\infty)$, $f : J \times \mathcal{B} \to \mathbb{R},\ \phi : \tilde{J}' \to \mathbb{R}^n$ and $\mathcal{B}$ is called a phase space that will be specified in .

5.3.2 Main Results

In this section we present a global existence and uniqueness result for the problem (5.14)-(5.17).

Let the space

$$\begin{aligned}\widetilde{PC} =& \big\{u : (-\infty,a] \times (-\infty,b] \to \mathbb{R}^n : u_{(t,x)} \in B \text{ for } (t,x) \in E \text{ and there exist}\\ & u(t_k^-,.),\ u(t_k^+,.) \text{ exist with } u(t_k^-,.) = u(t_k,.);\ k = 1,\dots,m, \text{ and}\\ & u \in C(J_k,\mathbb{R}^n); k = 0,\dots,m\big\},\end{aligned}$$

Definition 5.4 *A function $u \in \widetilde{PC}$ is said to be a solution of (5.14)-(5.17) if u satisfies equations (5.14) and (5.17) on J and the condition (5.15) on $\tilde{J}'$.*

For each $p,q \in \mathbb{N}$ we consider following set,

$$C'_{(p,q)} = \{u : \mathbb{R}^2 \to \mathbb{R}^n : u_{(t,x)} \in \mathcal{B}\bigcap C(\widetilde{J}_0,\mathbb{R}^n), u_{(t,x)} = 0 \text{ for } (t,x) \in E\},$$

and we define in

$$C'_0 = \{u : \mathbb{R}^2 \to \mathbb{R}^n : u_{(t,x)} \in \mathcal{B}\bigcap C(J,\mathbb{R}^n), u_{(t,x)} = 0 \text{ for } (t,x) \in E\}$$

the semi-norms by:

$$\begin{aligned}\|u\|_{(p,q)} &= \sup_{(t,x)\in E} \|u_{(t,x)}\| + \sup_{(t,x)\in\widetilde{J_0}} \|u(t,x)\|\\ &= \sup_{(t,x)\in\widetilde{J_0}} \|u(t,x)\|, \quad u \in C'_{(p,q)}.\end{aligned}$$

Then C'_0 is a Fréchet space with the family of semi-norms $\{\|u\|_{(p,q)}\}$.

Further, we present conditions for the existence and uniqueness of a solution of problem (4.10)-(4.13)

Theorem 5.5 *Assume the following hypothesis holds:*

$(H'1)$ *The functions $I_k : \mathbb{R}^n \to \mathbb{R}^n$ are continuous.*

$(H'2)$ *For each $p,q \in \mathbb{N}$, there exists $\ell_{(p,q)} \in C(\widetilde{J}_0,\mathbb{R}^n)$ such that for each $(t,x) \in \widetilde{J}_0$*

$$\|f(t,x,u) - f(t,x,v)\| \le \ell_{(p,q)}(t,x)\|u - v\|_{\mathcal{B}},\ \textit{for each } u,v \in \mathcal{B}.$$

$(H'3)$ *For each* $p, q \in \mathbb{N}$, *there exists* $\widetilde{\ell}_{(p,q)} \in C(\widetilde{J}_0, \mathbb{R}^n)$ *such that for each* $(t,x) \in \widetilde{J}_0$

$$\|I_k(u) - I_k(v)\| \le \widetilde{\ell}_{(p,q)} \|u - v\|, \ for\ each\ u, v \in \mathbb{R}^n.$$

If

$$2m\bar{l} + \frac{k\widetilde{\ell}^*_{(p,q)} p^{r_1} q^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} < 1, \tag{5.18}$$

where

$$\widetilde{\ell}^*_{(p,q)} = \sup_{(t,x)\in\widetilde{J}_0} \ell_{(p,q)}(t,x),$$

then there exists a unique solution for IVP (5.14)-(5.17) on $(-\infty,\infty)\times(-\infty,\infty)$.

Proof: Transform the problem (5.14)-(5.17) into a fixed point problem. Consider the operator $N' : \widetilde{PC} \to \widetilde{PC}$ defined by,

$$(N'u)(t,x) = \begin{cases} \phi(t,x), & (t,x) \in \tilde{J}, \\ \mu(t,x) + \sum\limits_{0<t_k<t} \Big(I_k(u(t_k^-, x)) - I_k(u(t_k^-, 0)) \Big) & \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_{k-1}}^{t_k} \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} & \\ \times f(s,\tau,u_{(s,\tau)}) d\tau ds & \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^{t} \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau,u_{(s,\tau)}) d\tau ds, & (t,x) \in J. \end{cases}$$

Let $v(.,.) : (-\infty, a] \times (-\infty, b] \to \mathbb{R}^n$ be a function defined by,

$$v(t,x) = \begin{cases} \mu(t,x), & (t,x) \in J. \\ \phi(t,x), & (t,x) \in \tilde{J}, \end{cases}$$

Then $v_{(t,x)} = \phi$ for all $(t,x) \in E$.
For each $w \in C(J, \mathbb{R}^n)$ with $w(t,x) = 0$ for each $(t,x) \in E$ we denote by $\overline{w}$ the function defined by

$$\overline{w}(t,x) = \begin{cases} w(t,x) & (t,x) \in J. \\ 0, & (t,x) \in \tilde{J}, \end{cases}$$

If $u(.,.)$ satisfies the integral equation,

$$u(t,x) \;=\; \mu(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau,u_{s,\tau}) d\tau ds$$

we can decompose $u(.,.)$ as

$$u(t,x) = \overline{w}(t,x) + v(t,x); \ (t,x) \in J,$$

which implies

$$u_{(t,x)} = \overline{w}_{(t,x)} + v_{(t,x)}, \ \ (t,x) \in J,$$

and the function $w(.,.)$ satisfies

$$\begin{aligned} w(t,x) &= \sum_{0<t_k<t} (I_k(u(t_k^-,x)) - I_k(u(t_k^-,0))) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau,\overline{w}_{(s,\tau)} + v_{(s,\tau)}) d\tau ds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau,\overline{w}_{(s,\tau)} + v_{(s,\tau)}) d\tau ds \end{aligned}$$

Let the operator $P : C_0' \to C_0'$ be defined by

$$\begin{aligned} (Pw)(t,x) &= \sum_{0<t_k<t} (I_k(u(t_k^-,x)) - I_k(u(t_k^-,0))) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau,\overline{w}_{(s,\tau)} + v_{(s,\tau)}) d\tau ds \end{aligned}$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau,\overline{w}_{(s,\tau)} + v_{(s,\tau)}) d\tau ds \quad (t,x) \in J. \quad (5.19)$$

Obviously, the operator N' has a fixed point is equivalent to P has a fixed point, and so we turn to prove that P has a fixed point. We shall use the nonlinear alternative of Leray-Schauder due to Frigon-Granas type to prove that P has a fixed point. Let w be a possible solution of the problem $w = \lambda P(w)$ for some $0 < \lambda < 1$. This implies that for each $(t,x) \in J_0$, we have

$$\begin{aligned} w(t,x) &= \lambda \sum_{0<t_k<t} (I_k(u(t_k^-,x)) - I_k(u(t_k^-,0))) \\ &+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau,\overline{w}_{(s,\tau)} + v_{(s,\tau)}) d\tau ds \\ &+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k}^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau,\overline{w}_{(s,\tau)} + v_{(s,\tau)}) d\tau ds \end{aligned}$$

This implies by $(H'2)$ and $(H'3)$ that

$$\begin{aligned}
||u(t,x)|| &\leq \lambda\sum_{k=1}^{m}(\|I_k(u(t_k^-,x))\| - \|I_k(0)\| + \|I_k(u(t_k^-,0))\| - \|I_k(0)\|) + 2\lambda\sum_{k=1}^{m}\|I_k(0)\| \\
&+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \\
&\times ||f(s,\tau,\overline{w}_{(s,\tau)} + v_{(s,\tau)}) - f(s,\tau,0)||d\tau ds \\
&+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}||f(s,\tau,0)||d\tau ds \\
&+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\int_{t_k}^{t}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}||f(s,\tau,\overline{w}_{(s,\tau)} + v_{(s,\tau)}) - f(s,\tau,0)||d\tau ds \\
&+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\int_{t_k}^{t}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}||f(s,\tau,0)||d\tau ds \\
&\leq \bar{l}\sum_{k=1}^{m}(\|u(t_k^-,x)\| + \|u(t_k^-,0)\|) + 2I^* + \frac{2f^*p^{r_1}q^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\Big(\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\ell_{(p,q)}(s,\tau)\|\overline{w}_{(s,\tau)} + v_{(s,\tau)}\|_{\mathcal{B}}d\tau ds, \\
&\qquad \int_{t_k}^{t}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\ell_{(p,q)}(s,\tau)\|u_{(s,\tau)}\|_{\mathcal{B}}d\tau ds\Big) \qquad (5.20)
\end{aligned}$$

where $f^* = \sup\limits_{(s,\tau)\in J_0}||f(s,\tau,0)||$, $I^* = \sum\limits_{k=1}^{m}\|I_k(0)\|$

and

$$\begin{aligned}
\|\overline{w}_{(s,\tau)} + v_{(s,\tau)}\|_{\mathcal{B}} &\leq \|\overline{w}_{(s,\tau)}\|_{\mathcal{B}} + \|v_{(s,\tau)}\|_{\mathcal{B}} \\
&\leq K\sup\{w(\tilde{s},\tilde{\tau}) : (\tilde{s},\tilde{\tau}) \in [0,s]\times[0,\tau]\} \\
&\quad +(M)\|\phi\|_{\mathcal{B}} + K\|\phi(0,0)\|. \qquad (5.21)
\end{aligned}$$

If we name $y(s,\tau)$ the right hand side of (5.21), then we have

$$\|\overline{w}_{(s,\tau)} + v_{(s,\tau)}\|_{\mathcal{B}} \leq y(t,x). \qquad (5.22)$$

Therefore, from (5.20) and (5.22), then we get

$$\begin{aligned}
||w(t,x)|| &\leq \bar{l}\sum_{k=1}^{m}(\|u(t_k^-,x)\| + \|u(t_k^-,0)\|) + 2I^* + \frac{2f^*p^{r_1}q^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\Big(\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\ell_{(p,q)}(s,\tau)y(s,\tau)d\tau ds \\
&+ \int_{t_k}^{t}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\ell_{(p,q)}(s,\tau)y(s,\tau)d\tau ds\Big),
\end{aligned}$$

Using the above inequality and the defintion of y, we have that

$$\begin{aligned} y(t,x) &\leq (M)\|\phi\|_{\mathcal{B}} + K\|\phi(0,0)\| + \bar{l}\sum_{k=1}^{m}(\|u(t_k^-,x)\| + \|u(t_k^-,0)\|) + 2I^* \\ &+\frac{2Kf^*_{(p,q)}p^{r_1}q^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} + \frac{K\ell^*_{(p,q)}}{\Gamma(r_1)\Gamma(r_2)}\Big(\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}y(s,t)d\tau ds \\ &+\int_{t_k}^{t}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}y(s,t)d\tau ds\Big). \end{aligned}$$

By Lemma 1.4, there exists a costant $\delta = \delta(r_1, r_2)$ such that we have

$$\begin{aligned} \|y\|_{(p,q)} &\leq \left[(M)\|\phi\|_{\mathcal{B}} + K\|\phi(0,0)\| + \bar{l}\sum_{k=1}^{m}(\|u(t_k^-,x)\| + \|u(t_k^-,0)\|) + 2I^* + \frac{2Kf^*_{(p,q)}p^{r_1}q^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\right] \\ &\times \left[1 + \delta\frac{2K\ell^*_{(p,q)}}{\Gamma(r_1+1)\Gamma(r_2+1)}\right] \\ &:= \widetilde{M}. \end{aligned}$$

Then from (5.20) we have

$$\begin{aligned} ||w||_{(p,q)} &\leq \bar{l}\sum_{k=1}^{m}(\|u(t_k^-,x)\| + \|u(t_k^-,0)\|) + 2I^* + \frac{2f^*_{(p,q)}p^{r_1}q^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} + \widetilde{M}\frac{2\ell^*_{(p,q)}p^{r_1}q^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ &:= \widetilde{M}^* \end{aligned}$$

Set

$$U' = \{w \in C_0 : \|w\|_{(p,q)} \leq \widetilde{M}^* + 1 \ for\ all\ p, q \in \mathbb{N}\}.$$

We shall show that $P : U' \to C_{(p,q)}$ is a contraction maps. Indeed, consider $v, w \in U'$.

Then for each $(t,x)\in J_0$, we have

$$\begin{aligned}\|(Pv)(t,x)-(Pw)(t,x)\| &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x |(t-s)^{r_1-1}||(x-\tau)^{r_2-1}| \\ &\quad \times\|f(s,\tau,v_{(s,\tau)})-f(s,\tau,w_{(s,\tau)})\|d\tau ds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_k}^{t}\int_0^x |(t-s)^{r_1-1}||(x-\tau)^{r_2-1}| \\ &\quad \times\|f(s,\tau,v_{(s,\tau)})-f(s,\tau,w_{(s,\tau)})\|d\tau ds \\ &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \\ &\quad \times\ell_{(p,q)}\|v_{(s,\tau)}-w_{(s,\tau)}\|_C d\tau ds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_k}^{t}\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \\ &\quad \times\ell_{(p,q)}\|v_{(s,\tau)}-w_{(s,\tau)}\|_C d\tau ds \\ &\leq \frac{2\ell^*_{(p,q)}p^{r_1}q^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\|v-w\|_{(p,q)}.\end{aligned}$$

Thus

$$\|(Pv)(t,x)-(Pw)(t,x)\|_{(p,q)}\leq \frac{2\ell^*_{(p,q)}p^{r_1}q^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\|v-w\|_{(p,q)}.$$

Hence by (5.18), $P:U'\to C_{(p,q)}$ is a contraction.
By our choice of U', there is no $w\in\partial_n U'^n$ such that $w=\lambda P(w)$, for $\lambda\in(0,1)$. As a consequence of Theorem 5.5, we deduce that P has a unique fixed point w in U' which is a solution to problem (5.14)-(5.17).

5.3.3 An Example

As an application of our results we consider the following fractional order partial hyperbolic functional differential equations with infinite delay of the form

$$({}^cD^r_{z_k}u)(t,x)=\frac{|u(t,x)|}{c_{(p,q)}e^{t+x}(1+|u(t,x)|)},\quad \text{if } (t,x)\in J:=[0,\infty)\times[0,\infty), \tag{5.23}$$

$$u(t_k^+,x)=u(t_k^-,x)+\frac{1}{(6e^{t+x+4})(1+|u(t_k^-,x)|)};\ x\in[0,1],\ k=1,\dots,m, \tag{5.24}$$

$$u(t,0)=t,\ u(0,x)=x^2,\ (t,x)\in J, \tag{5.25}$$

$$u(t,x)=t+x^2,\ (t,x)\in\tilde{J}:=\mathrm{I\!R}^2\backslash[0,\infty)\times[0,\infty), \tag{5.26}$$

$$c_{(p,q)}=\frac{3p^{r_1}q^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}$$

$$\mathcal{B}_\gamma = \{u \in C((-\infty,0]\times(-\infty,0],\mathbb{R}) : \lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta+\eta)}u(\theta,\eta) \text{ exists } \in \mathbb{R}\}.$$

The norm of $\mathcal{B}_\gamma$ is given by

$$\|u\|_\gamma = \sup_{(\theta,\eta)\in(-\infty,0]\times(-\infty,0]} e^{\gamma(\theta+\eta)}|u(\theta,\eta)|.$$

Let

$$E := [0,1]\times\{0\}\cup\{0\}\times[0,1],$$

and $u : (-\infty,1]\times(-\infty,1]\to\mathbb{R}$ such that $u_{(t,x)}\in\mathcal{B}_\gamma$ for $(t,x)\in E$, then

$$\lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta+\eta)}u_{(t,x)}(\theta,\eta) = \lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta-t+\eta-x)}u(\theta,\eta) = e^{\gamma(t+x)}\lim_{\|(\theta,\eta)\|\to\infty} u(\theta,\eta) < \infty.$$

Hence $u_{(t,x)}\in\mathcal{B}_\gamma$. Finally we prove that

$$\|u_{(t,x)}\|_\gamma = K\sup\{|u(s,\tau)| : (s,\tau)\in[0,t]\times[0,x]\} + M\sup\{\|u_{(s,\tau)}\|_\gamma : (s,\tau)\in E_{(t,x)}\},$$

where $K = M = 1$ and $H = 1$,
If $t+\theta\le 0,\ x+\eta\le 0$ we get

$$\|u_{(t,x)}\|_\gamma = \sup\{|u(s,\tau)| : (s,\tau)\in(-\infty,0]\times(-\infty,0]\},$$

and if $t+\theta\ge 0,\ x+\eta\ge 0$ then we have

$$\|u_{(t,x)}\|_\gamma = \sup\{|u(s,\tau)| : (s,\tau)\in[0,t]\times[0,x]\}.$$

Thus for all $(t+\theta,x+\eta)\in[0,1]\times[0,1]$, we get

$$\begin{aligned}\|u_{(t,x)}\|_\gamma &= \sup\{|u(s,\tau)| : (s,\tau)\in(-\infty,0]\times(-\infty,0]\}\\ &\quad+\sup\{|u(s,\tau)| : (s,\tau)\in[0,t]\times[0,x]\}.\end{aligned}$$

Then

$$\|u_{(t,x)}\|_\gamma = \sup\{\|u_{(s,\tau)}\|_\gamma : (s,\tau)\in E\} + \sup\{|u(s,\tau) : (s,\tau)\in[0,t]\times[0,x]|\}.$$

$(\mathcal{B}_\gamma, \|.\|_\gamma)$ is a Banach space. We conclude that $\mathcal{B}_\gamma$ is a phase space.

$$f(t,x,\varphi) = \frac{|\varphi|}{c_{(p,q)}e^{t+x}(1+|\varphi|)},\ (t,x)\in J,\ \varphi\in\mathcal{B}_\gamma,$$

and

$$I_1(u) = \frac{1}{(6e^{t+x+4})(1+|u|)};\ x\in[0,1].$$

For each $\varphi,\ \overline{\varphi}\in\mathcal{B}_\gamma$ we have

$$|f(t,x,\varphi)-f(t,x,\overline{\varphi})| \le \frac{1}{c_{(p,q)}e^{t+x}}\|\varphi-\overline{\varphi}\|_{\mathcal{B}_\gamma}$$

For each $u, v \in \mathbb{R}^n$ and $(t,x) \in [0,1] \times [0,1]$ we have

$$\|I_1(u) - I_1(v)\| \leq \frac{1}{6e^4}\|u - v\|.$$

Hence condition $(H'2)$ is satisfied with $\ell_{(p,q)}e^{t+x} = \dfrac{1}{c_{(p,q)}e^{t+x}}$. Since

$$\ell^*_{(p,q)} = \sup\{\frac{1}{c_{(p,q)}e^{t+x}}, \quad (t,x) \in J\} \leq \frac{1}{c}$$

and $K = 1,\ m = 1$ we get

$$2m\widetilde{l} + \frac{k\ell^*_{(p,q)}p^{r_1}q^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} = \frac{1}{3e^4} + \frac{1}{3} < 1.$$

Hence condition (5.18) holds for each $(r_1, r_2) \in (0,1] \times (0,1]$ and all $p, q \in \mathbb{N}^*$. Consequently Theorem 5.5 implies that problem (5.23)-(5.25) has a unique solution defined on $\mathbb{R}^2$.

Conclusion

In this thesis, we have considered the problem of existence and uniqueness results of solutions for fractional order partial hyperbolic functional differential with fixed time impulses. Sufficient conditions for existence and uniqueness od solutions for initial value problems for partial differential equations involving the Caputo fractional derivative were given.

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