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Une contribution aux systèmes d'équations et inclusions différentielles stochastique impulsive d'ordre fractionnaire.

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Declaration

I hereby declare that the work contained in this thesis entitled ”A contribution to systems of impulsive stochastic differential equations and inclusions of fractional order ” is my own work under the supervision and direction of Pr. Abdelghani OUAHAB, professor at Djillali Liabes University, Department of Mathematics, Sidi Bel Abess for the award of the degree of Doctor of Philosophy in mathematics. To the best of my knowledge and belief, it contains no material previously published or written by another person.

Dedicace

It is with great emotion and immense pleasure that I dedicate this modest work:

To my dear parents Mebarka and Ghaouti may God have mercy on him who supported me throughout my studies, thank you for all your love and support, and for taking just the right amount of interest in my research.

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To all my friends.

To all those who love me and whom I love...

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Contents

Introduction

In 1695, a dialogue took place in the form of correspondence (see [87]) between G.W. Leibniz and J. de L'Hospital about the possibility to generalized the derivatives with integer order to derivatives with non-integer (fractional) orders, where the derivative of 1/2-order was discussed, this famous correspondence was the reason for the creation of a new aspect in pure mathematics called Fractional Calculus (FC). The latter attracted the interest of many well-known mathematicians as they developed it. For example, according to the glimpse history mentioned in the book [88], we mention the studies of L. Euler (1730), J.L. Lagrange (1772), P.S. Laplace (1812), J.B.J. Fourier (1822), N.H. Abel (1823), J. Liouville (1832), B. Riemann (1847), H.L. Greer (1859), H. Holmgren (1865), A.K. Grunwald (1867), A.V. Letnikov (1868), N.Ya. Sonin (1869), H. Laurent (1884), P.A. Nekrassov (1888), A. Krug (1890), J. Hadamard (1892), O. Heaviside (1892), S. Pincherle (1902), G.H. Hardy and J.E. Littlewood (1917), H. Weyl (1919), P. L´evy (1923), A. Marchaud (1927), H.T. Davis (1924), A. Zygmund (1935), E.R. Love (1938), A. Erd´elyi (1939), H. Kober (1940), D.V. Widder (1941), M. Riesz (1949)and W. Feller (1952). In addition, the book published by M. Caputo in 1969 see [63], systematically using his innovative definition of fractional differentiation for solving viscoelasticity problems, and don't forget his lectures on seismology see [64]. This historical reading led us to say in an extension sense that the FC is the differentiation and the integration of any real order.

In recent years, the subject of differential and integral equations via different types of fractional derivatives has received much attention because its applications in various areas of sciences. For more information on applications we refer the reader to [2, 20, 40, 51, 74, 88] and references therein.

The theory of stochastic differential equations has become an active area of investigation due to their applications in the fields such as chemistry, mechanics, electrical engineering, medical biology, economical systems, finance and several fields in engineering, etc. One can find detailed information in [39, 38, 55, 28, 68, 31, 57, 67] and references therein.

In natural ecosystems, the dynamic interaction between the predator and the prey has long been and will continue to be one of the most attractive field in mathematics due to its existence and importance in mathematical ecology. The preservation of the balance in an ecosystem is necessary for the ecologists. It depends on different relationships between organisms in nature, which can be divided into several forms such as competition, symbiosis, predator-prey interactions and so on. One of the first models describing the interaction between species was developed in the 1920s, independently by the American Alfred Lotka [59] $(1880-1949)$ and the Italian Vito Volterra [81] $(1860-1940)$, and is known as Lotka-Volterra or predator-prey model. Throughout the last century, several researchers are interested in the mathematical ecology area [1, 14, 70, 71, 76, 79]. They have proposed and studied several ecological phenomena between species through models of ordinary or partial differential equations which describe the interactions between these species in nature. The results of these studies can determine and predict the behavior of the living beings in nature which provides enough time to ecologists to give an appropriate control strategy that yields to avoid extinction of the living beings.

The thesis is divided into four chapters. In **Chapter 1** and **Chapter 2** introduces preliminary facts from fractional calculus and stochastic calculus which are used in our main results.

In **Chapter 3**, we establish the existence and uniqueness of solutions for a fractional stochastic differential equation driven by countably many Brownian motions on bounded and unbounded intervals. Also, we study the continuous dependence of solutions on initial data. Finally, we establish the transportation quadratic cost inequality for some classes of fractional stochastic equations and continuous dependence of solutions with respect Wasserstein distance.

In **Chapter 4**, a new approach of a stochastic predator-prey interaction with protection zone for the prey is developed and studied. The considered mathematical model consists of a system of two stochastic differential equations, SDEs, describing the interaction between the prey and predator populations where the prey exhibits a social behavior called also by “herd behavior.” First, according to the theory of the SDEs, some properties of the solution are obtained, including: the existence and uniqueness of the global positive solution and the stochastic boundedness of the solutions. Then, the sufficient conditions for the persistence in the mean and the extinction of the species are established, where the extinction criteria are discussed in two different cases, namely, the firstcase is the survival of the prey population, while the predator population goes extinct; the second case is the extinction of all prey and predator populations. Next, by constructing a suitable stochastic Lyapunov function and under certain parametric restrictions, it has been proved that the system has a unique stationary distribution which is ergodic. Finally, some numerical simulations based on the Milstein’s higher-order scheme are performed to illustrate the theoretical predictions.

Chapter 1

Fractional calculus

1.1 Integrable Functions

1.1.1 Variation, Quadratic variation of a Function

Let a, b be two real numbers such that $-\infty < a < b < \infty$ and let f be a real function defined on $[a,b]$. The total variation of f on $[a,b]$ is defined by

$$V(f;[a,b]) = \sup_{P \in \mathcal{P}} \left\{ \sum_{k=1}^{n} |f(t_k) - f(t_{k-1})| \right\},$$

where $\mathcal{P}$ is the set of all partitions of $[a,b]$. The quantity on the right-hand side of the above formula increases by adding points to partitions. Therefore

$$V(f;[a,b]) = \lim_{|P_n| \longrightarrow 0} \sum_{k=1}^{n} |f(t_k) - f(t_{k-1})|,$$

where $|P_n| = \max_{1 \leq k \leq n}(t_k - t_{k-1})$ is the largest mesh size of a sequence P_n of nested subdivisions of $[a,b]$. If $V_t(f;[a,b]) < \infty$ then f is said to be a function of bounded variations. In particular, if f is defined on $[0,\infty)$, we define the non-decreasing function V_f on $[0,\infty)$ by

$$V_f(t) = V(f;[0,t]).$$

V_f is called the variation function of f on $[0,\infty)$.
Similarly, let $t \in [0,\infty)$. If

$$\lim_{|P_n|\longrightarrow 0} \sum_{k=1}^{n} |f(t_k) - f(t_{k-1})|^2,$$

exists and finite over all partitions of $[0,t]$. Then the quadratic variation of f on $[0,t]$ denoted by $[f]_t$, is given by

$$[f]_t = \lim_{|P_n|\longrightarrow 0} \sum_{k=1}^{n} (f(t_k) - f(t_{k-1}))^2.$$

We now give an interesting result in this context that any real continuous function on [0,t] with bounded variation, its quadratic variation is zero. Indeed,

$$\begin{aligned} [f]_t &= \lim_{|P_n|\longrightarrow 0} \sum_{k=1}^{n} (f(t_k) - f(t_{k-1}))^2 \\ &\leq \lim_{|P_n|\longrightarrow 0} \max_{1\leq k\leq n} |f(t_k) - f(t_{k-1}| \sum_{k=1}^{n} |f(t_k) - f(t_{k-1})|. \end{aligned}$$

Moreover,

$$\lim_{|P_n|\longrightarrow 0} \sum_{k=1}^{n} |f(t_k) - f(t_{k-1})| = V_f(t) < \infty,$$

and the continuity of f on $[0,t]$ implies that it is uniformly continuous on $[0,t]$, hence

$$\lim_{|P_n|\longrightarrow 0} \max_{1\leq k\leq n} |f(t_k) - f(t_{k-1}| = 0,$$

the result is proved. ∎

1.1.2 Riemann-Stieltjes Integral

Let f a bounded real functions defined on $[a,b]$ and let g be a real function of bounded variations over $[a,b]$. The Riemann-Stieltjes Integral of f over $[a,b]$ with respect to g is defined as the limit of Riemann-Stieltjes sums

$$\int_a^b f dg = \int_a^b f(t)dg(t) = \lim_{\delta \longrightarrow 0} \sum_{i=1}^{n} f(\xi_i^n)(g(t_i^n) - g(t_{i-1}^n)), \tag{1.1.1}$$

where $\xi_i^n \in [t_{i-1}^n, t_i^n]$ and $\delta = \max_{1 \leq i \leq n}(t_i^n - t_{i-1}^n)$ is the biggest mesh size of the subdivisions

$$a = t_0^n < t_1^n < \cdots < t_n^n = b.$$

In particular, if $g(t) = t$ then the Riemann-Stieltjes Integral is called the Riemann Integral and is defined by

$$\int_a^b f(t)dt = \lim_{\delta \longrightarrow 0} \sum_{i=1}^{n} f(\xi_i^n)(t_i^n - t_{i-1}^n). \tag{1.1.2}$$

One can check easily that if f is differentiable on $[a,b]$ and $f^{'}$ is Riemann integrable on $[a,b]$ then

$$f(b) - f(a) = \int_a^b f^{'}(s)ds.$$

This result is called the fundamental theorem of calculus.

we will denote by $\mathbb{L}([a,b];\mathbb{R})$ the space of all real-valued functions f defined on $[a,b]$ such that

$$\int_a^b |f(t)|dt < \infty.$$

If $f \in \mathbb{L}([a,b];\mathbb{R})$, then f is said to be Riemann-integrable (or simply integrable) function.

1.2 Special Functions

1.2.1 Gamma Function

The Gamma Function Γ is defined by the integral form

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \quad z \in \mathbb{C}_{Re>0}, \tag{1.2.1}$$

where $\mathbb{C}_{Re>0} = \{z \in \mathbb{C} : Re(z) > 0\}$.

Now, we give the basic properties of the Gamma Function and some brief steps for its proof. Firstly, the recursion formula property, which is as follows

$$\Gamma(z+1) = z\Gamma(z), \quad z \in \mathbb{C}_{Re>0}.$$

to prove this property, we used the formula of integration by parts. In particular, for all $n \in \mathbb{N}$, we have

$$\Gamma(n+1) = n!, \tag{1.2.2}$$

and

$$\Gamma(n+\frac{1}{2}) = \frac{\sqrt{\pi}(2n)!}{2^{2n}n!},$$

these two particular cases can easily be demonstrated by the nth iteration of the previous property and take $\Gamma(1) = 1, \Gamma(\frac{1}{2}) = \sqrt{\pi}$.

The second property called the Limit representation of the gamma function,

$$\Gamma(z) = \lim_{n\longrightarrow\infty} \frac{n!n^z}{z(z+1)...(z+n)}, \quad z \in \mathbb{C}\backslash\mathbb{Z},$$

initially, we show that its holds for any $z \in \mathbb{C}_{Re>0}$. Indeed, we consider the auxiliary function

$$\Gamma_n(z) = \int_0^n (1-\frac{t}{n})^n t^{z-1} dt,$$

changing the variable $\alpha = \frac{t}{n}$ and repeating the integration by parts, we get

$$\begin{aligned}
\Gamma_n(z) &= n^z \int_0^1 (1-\alpha)^n \alpha^{z-1} d\alpha \\
&= \frac{n!n^z}{z(z+1)\dots(z+n-1)} \int_0^1 \alpha^{z+n-1} d\alpha \\
&= \frac{n!n^z}{z(z+1)\dots(z+n)},
\end{aligned}$$

since the interchange of the limit and the integral i.e.,

$$\lim_{n\longrightarrow\infty} \Gamma_n(z) = \int_0^\infty \lim_{n\longrightarrow\infty} (1-\frac{t}{n})^n t^{z-1} dt,$$

are justified in [52], then by the well-known limit

$$\lim_{n\longrightarrow\infty}(1-\frac{t}{n})^n = e^{-t},$$

the Limit representation property holds for any $z \in \mathbb{C}_{Re>0}$. the proof is complete. ■

The third property of Gamma Function as a relationship to another function is as follow

$$\beta(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)},$$

where the Function β is called the Beta function defined for any $u, v \in \mathbb{C}_{Re>0}$ by the integral form

$$\beta(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1}dt.$$

With the help of the Beta function we can establish the following two important relationships for the gamma function. The first one is called reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad z \in \mathbb{C}\backslash\mathbb{Z}.$$

In particular, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. The second is called Legendre duplication formula

$$\Gamma(z)\Gamma(z+\frac{1}{2}) = \frac{\sqrt{\pi}\Gamma(2z)}{2^{2z-1}}, \quad 2z \neq 0,-1,-2,-3,......$$

In particular, we can make sure again that $\Gamma(n+\frac{1}{2}) = \frac{\sqrt{\pi}(2n)!}{2^{2n}n!}$ holds for all $n \in \mathbb{N}$

1.2.2 Mittag-Leffler Function

The two parameter Mittag-Leffler Function $E_{\alpha,\beta}$ is defined for any complex number z by the following series

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \tag{1.2.3}$$

where $\alpha, \beta \in \mathbb{C}$ and $Re(\alpha) > 0$. Taking $\alpha = 1$ and $\beta = n+1$ in (1.2.3) we get a set of particular functions of Mittag-Leffler type

$$E_{1,n+1}(z) = \begin{cases} \frac{1}{z^n}\left(e^z - \sum_{k=0}^{n-1} \frac{z^k}{k!}\right), & n \in \mathbb{N}^*, \\ \\ e^z, & \text{n=0.} \end{cases}$$

Indeed, if $n = 0$ (i.e., taking $\alpha = \beta = 1$ in (1.2.3)), by (1.2.2), we deduce

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z,$$

and if $n \in \mathbb{N}^*$, by (1.2.2), we have

$$E_{1,n+1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+n+1)} = \sum_{k=0}^{\infty} \frac{z^k}{(k+n)!},$$

put $m = k+n$, then

$$\begin{aligned} E_{1,n+1}(z) = &= \sum_{m=n}^{\infty} \frac{z^{m-n}}{m!} \\ &= \frac{1}{z^n} \sum_{m=n}^{\infty} \frac{z^m}{m!} \\ &= \frac{1}{z^n}\left(\sum_{m=0}^{\infty} \frac{z^m}{m!} - \sum_{m=0}^{n-1} \frac{z^m}{m!}\right) \\ &= \frac{1}{z^n}\left(e^z - \sum_{k=0}^{n-1} \frac{z^k}{k!}\right). \end{aligned}$$

■

1.3 Fractional Derivatives and Integrals

1.3.1 Riemann-Liouville Fractional Derivatives

Let a, b be two real numbers such that $-\infty < a < b < \infty$ and let $\alpha \in (0,\infty)$. Consider that $f \in \mathbb{L}([a,b];\mathbb{R})$. Then the left and right Riemann-Liouville fractional integrals $I^{\alpha}_{a+}f(x), I^{\alpha}_{b-}f(x)$ of order α, are defined by

$$I^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a, \tag{1.3.1}$$

and

$$I^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad x < b, \tag{1.3.2}$$

respectively. When $\alpha = n \in \mathbb{N}$, the definitions (1.3.1) and (1.3.2) coincide with the n-th integrals of the form

$$I^{n}_{a+}f(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}f(t)dt, \tag{1.3.3}$$

and

$$I^{n}_{b-}f(x) = \frac{1}{(n-1)!}\int_x^b (t-x)^{n-1}f(t)dt. \tag{1.3.4}$$

In particular, if $f(x) = (x-a)^{\beta-1}$ with $\beta > 0$, we get

$$I^{\alpha}_{a+}(x-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1},$$

and, if $f(x) = (b-x)^{\beta-1}$ with $\beta > 0$, we get

$$I^{\alpha}_{b-}(b-x)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(b-x)^{\beta+\alpha-1}.$$

The Riemann-Liouville fractional integrals has the following properties

$$I^{\alpha}_{a+}(I^{\beta}_{a+}f(x)) = I^{\alpha+\beta}_{a+}f(x);$$

$$I^{\alpha}_{b-}(I^{\beta}_{b-}f(x)) = I^{\alpha+\beta}_{b-}f(x).$$

We now give the definition of the The left and right Riemann-Liouville fractional Derivatives $D^{\alpha}_{a+}f(x)$ and $D^{\alpha}_{b-}f(x)$ of order α as follows

$$\begin{aligned} D^{\alpha}_{a+}f(x) &= \left(\frac{d}{dx}\right)^{n}\left(I^{n-\alpha}_{a+}f(x)\right) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^{n}\int_{a}^{x}\frac{f(t)dt}{(x-t)^{\alpha-n+1}}, \quad x>a. \end{aligned} \tag{1.3.5}$$

And

$$\begin{aligned} D^{\alpha}_{b-}f(x) &= \left(-\frac{d}{dx}\right)^{n}\left(I^{n-\alpha}_{b-}f(x)\right) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^{n}\int_{x}^{b}\frac{f(t)dt}{(t-x)^{\alpha-n+1}}, \quad x<b, \end{aligned} \tag{1.3.6}$$

respectively, where $n=[\alpha]+1$, $[\alpha]$ means the integer part of α. In particular,

1. if $0<\alpha<1$ (i.e., $n=1$), then

$$D^{\alpha}_{a+}f(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{a}^{x}\frac{f(t)dt}{(x-t)^{\alpha}}, \quad x>a,$$

$$D^{\alpha}_{b-}f(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x}^{b}\frac{f(t)dt}{(t-x)^{\alpha}}, \quad x<b.$$

2. If $\alpha=m\in\mathbb{N}$ (i.e., $n=m+1$) then

$$D^{m}_{a+}f(x)=\begin{cases} f^{(m)}(x), & m\neq 0 \\ f(x), & m=0, \end{cases}$$

and

$$D^{m}_{b-}f(x)=\begin{cases} (-1)^{m}f^{(m)}(x), & m\neq 0 \\ f(x), & m=0. \end{cases}$$

Where $f^{(m)}$ is the usual derivative of f of order m.

3. If $f(x) = (x-a)^{\beta-1}$ with $\beta > 0$, then

$$D^{\alpha}_{a+}(x-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1},$$

and, if $f(x) = (b-x)^{\beta-1}$ with $\beta > 0$, we get

$$D^{\alpha}_{b-}(b-x)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-x)^{\beta-\alpha-1}.$$

In particular, if $\beta = 1$, we get that the Riemann-Liouville fractional Derivatives of a constant function $x \longmapsto C$

$$D^{\alpha}_{a+}C = \frac{C(x-a)^{-\alpha}}{\Gamma(1-\alpha)},$$

$$D^{\alpha}_{b-}C = \frac{C(b-x)^{-\alpha}}{\Gamma(1-\alpha)}.$$

1.3.2 Caputo Fractional Derivatives

As needed in our result, we will only define the left Caputo fractional derivatives ${}^cD^{\alpha}f(x)$ of order α for $f \in AC^n([a,b];\mathbb{R})$ via the above Riemann-Liouville fractional integrals, as follows

$$ {}^cD^{\alpha}f(x) := I^{n-\alpha}_{a+}f^{(n)}(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1}f^{(n)}(t)dt, \qquad (1.3.7)$$

where $n = [\alpha]+1$. In particular, the Caputo fractional derivatives of a constant function $x \longmapsto C$ is null since $\frac{d^nC}{dx^n} = 0$ for all $n = [\alpha]+1 \geq 1$. And also, if $f(x) = (x-a)^{\beta-1}$ with $\beta > 0$, then

$$ {}^cD^{\alpha}(b-x)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-n)}(x-a)^{\beta-\alpha-1}.$$

The relationship between the Caputo fractional derivatives and the Riemann-Liouville fractional Derivatives is given by the formula

$$ {}^cD^{\alpha}f(x) = D^{\alpha}_{a+}f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha}.$$

In particular, if $0 < \alpha < 1$ (i.e., $n = 1$), then

$$ {}^cD^{\alpha}f(x) = D^{\alpha}_{a+}(f(x)-f(a)).$$

1.4 Fractional Cauchy Problem

Let α be a non-integer numbers with $\alpha > 0$. Consider the fractional Cauchy problem of the form

$$\begin{cases} {}^{c}D^{\alpha}y(x) = f(x, y(x)), \quad x \in [a,b], \\ \\ y^{(k)}(a) = b_k \in \mathbb{R}, k = 0, 1, \dots, n-1, n. \end{cases} \tag{1.4.1}$$

Where ${}^{c}D^{\alpha}$ is Caputo fractional derivatives, $n = [\alpha] + 1$ and $f(\cdot, y) : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function with respect to $x \in [a;b]$ for each $y \in \mathbb{R}$.

Let $C^{n-1}[a,b]$ be Banach space such that

$$C^{n-1}[a,b] = \left\{ g : [a,b] \to \mathbb{R} : \|g\|_{C^{n-1}[a,b]} = \sum_{k=1}^{n-1} \|g^{(k)}\|_{C[a;b]} \right\}.$$

Where $C^0[a,b] = C[a,b]$. We say that $y \in C^{n-1}[a,b]$ is a solution of the fractional Cauchy problem (1.4.1) if y is a solution of the following Volterra integral equation

$$y(x) = \sum_{k=1}^{n-1} \frac{b_k}{k!}(x-a)^k + \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t, y(t))}{(x-t)^{1-\alpha}} dt. \tag{1.4.2}$$

We now give the existence and uniqueness result of the fractional Cauchy problem (1.4.1), it is as follows

Theorem 1. *Let G be an open set of $\mathbb{R}$. Suppose that $f(\cdot, y) : [a,b] \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions*

i. For any fixed $y \in G$, $f(\cdot, y) \in C[a,b]$.

ii. (Lipschitz condition) for all $y_1, y_2 \in G$ and $x \in [a,b]$, there exist $L > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2.$$

If

$$L \sum_{k=1}^{n-1} \frac{(b-a)^{\alpha-k}}{\Gamma(\alpha - k + 1)} < 1.$$

Then, the fractional Cauchy problem (1.4.1) has a unique solution

Lemma 1. *For all $\alpha \in (0,1]$ and $\gamma > 0$, the following inequality holds:*

$$\frac{\gamma}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} E_\alpha(\gamma s^\alpha) ds \leq E_\alpha(\gamma t^\alpha).$$

Proof. Let $0 < \gamma \leq 1$. We consider first the linear problem

$$^cD^\alpha y(t) = \gamma y(t), \quad t \in \mathbb{R}_+. \tag{1.4.3}$$

From [20, Theorem 7.2 and Remark 7.1], the function $y(t) = E(\tau t^\alpha)$ is solution of (1.4.3), and for any $t \in \mathbb{R}_+$, we have

$$E(\gamma t^\alpha) = 1 + \frac{\tau}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} E_\alpha(\gamma s^\alpha) ds.$$

This concludes the proof of lemma. □

We recall Gronwall's lemma for singular kernels, whose proof can be found in [86, Lemma 7.1.1].

Lemma 2. *Let $v : [0,b) \to [0,\infty)$ be a real function and $w(\cdot)$ be a nonnegative, locally integrable function on $[0,b)$,(some $b \leq +\infty$)) and $a(t)$ be a nonnegative, nondecreasing continuous function defined on $0 \leq t < b$, with $a(t) \leq M$ (constant), and suppose $v(t)$ is nonnegative and locally integrable on $0 \leq t < b$. Assume $\gamma > 0$ such that*

$$v(t) \leq w(t) + a(t) \int_0^t \frac{v(s)}{(t-s)^{1-\gamma}} ds.$$

Then

$$v(t) \leq w(t) + \int_0^t \sum_{n=1}^{\infty} \frac{(a(t)\Gamma(\gamma))^n}{\Gamma(n\gamma)} (t-s)^{n\gamma-1} w(s) ds,$$

for every $t \in [0,b)$.

Chapter 2

Stochastic calculus

2.1 Stochastic basis-Random variables

2.1.1 Stochastic basis

Mathematically, the set of all possible outcomes of trials whose outcomes depend on chance is denoted by Ω. With a sample point ω of Ω, some of them can be grouped together under a common feature as a subset of Ω is called an event. Any family $\mathcal{F}$, from subsets of Ω which satisfies the following conditions

(i) $\emptyset$ and $\Omega \in \mathcal{F}$,

(ii) Any union (optional) of elements of $\mathcal{F}$ belongs to $\mathcal{F}$,

(iii) The intersection of any two elements of $\mathcal{F}$ belongs to $\mathcal{F}$,

is called a topology over a set Ω. The topology over a set Ω is henceforth denoted by $\mathcal{T}_\Omega$, the pair $(\Omega, \mathcal{T}_\Omega)$ is called topological space and its elements are called open sets. And if $\mathcal{F}$ has the following properties

(i) $\Omega \in \mathcal{F}$,

(ii) $A \in \mathcal{F} \Longrightarrow A^c \in \mathcal{F}$, A^c is the complement of A in Ω,

(iii) $\{A_n\}_{n\geq 1} \subset \mathcal{F} \Longrightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$,

we say about $\mathcal{F}$ that it is σ-algebra (or σ-field), the pair $(\Omega, \mathcal{F})$ is called measurable space and its elements are called measurable sets. From the definition of σ-algebra and topology it is clear that every σ-algebra is a topology and the opposite is not necessarily true. Therefore, we will present two famous examples about σ-algebra (respectively, topology) over a set Ω, first ones $\{\emptyset, \Omega\}$ is the smallest possible σ-algebra (respectively, topology) and the second is the largest possible σ-algebra (respectively, topology) denoted by $\mathcal{P}(\Omega)$ (the family of all possible subsets of Ω). Moreover, if $\mathcal{C} \subset \mathcal{P}(\Omega)$, then the intersection of all σ-algebra which contain $\mathcal{C}$ is the smallest σ-algebra $\sigma_\Omega(\mathcal{C})$ on Ω which contains $\mathcal{C}$. This $\sigma_\Omega(\mathcal{C})$ is called the σ-algebra generated by $\mathcal{C}$. If $\mathcal{C} = \mathcal{T}_\Omega$, then $\mathcal{B}(\Omega) = \sigma_\Omega(\mathcal{T}_\Omega)$ is called the Borel σ-algebra which containing all open sets of Ω and its elements are called the Borel sets.

Let $(\Omega, \mathcal{F})$ be a measurable space. We define on $(\Omega, \mathcal{F})$ a probability measure $\mathbb{P}$, i.e. $\mathbb{P} : \mathcal{F} \longrightarrow [0,1]$ such that

(i) $\mathbb{P}(\Omega) = 1$,

(ii) $\mathbb{P}$ is σ-additivity, i.e., for any disjoint sequence $\{A_n\}_{n\geq 1} \subset \mathcal{F}$,

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space. If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, we set

$$\overline{\mathcal{F}} = \{A \subset \Omega : \exists B, C \in \mathcal{F} \quad such \quad that \quad B \subset A \subset C, \quad \mathbb{P}(B) = \mathbb{P}(C)\}.$$

$\overline{\mathcal{F}}$ is a σ-algebra, called the completion of $\mathcal{F}$. If $\mathcal{F} = \overline{\mathcal{F}}$, then the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be complete.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with an increasing family $(\mathcal{F}_t)_{t\geq 0}$ of sub-σ-algebra of $\mathcal{F}$ (increasing means: $\mathcal{F}_s \subset \mathcal{F}_t$ for all $0 \leq s < t$). The family $(\mathcal{F}_t)_{t\geq 0}$ is called a filtration of Ω. The quadruple $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is called a stochastic basis (or a filtered probability space). The "physical" meaning of $\mathcal{F}_t$ is the collection of events occurring up to time t (or the collection of the "past information up to t"). We can replace $\mathcal{F}$ when is not specified, with the σ-algebra generated by $\bigcup_{s\geq 0} \mathcal{F}_s$, is denoted by $\mathcal{F}_\infty$, i.e.

$$\mathcal{F}_\infty = \sigma\left(\bigcup_{s\geq 0} \mathcal{F}_s\right).$$

Given a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. With the filtration $(\mathcal{F}_t)_{t\geq 0}$ are associated the following families $(\mathcal{F}_{t^+})_{t\geq 0}$ and $(\mathcal{F}_{t^-})_{t\geq 0}$ of σ-algebras

$$\mathcal{F}_{t^+} = \bigcap_{s>t} \mathcal{F}_s,$$

$$\mathcal{F}_{t^-} = \sigma\left(\bigcup_{s<t} \mathcal{F}_s\right), \quad for \quad t = 0, \quad \mathcal{F}_{0^-} = \mathcal{F}_0.$$

The filtration $(\mathcal{F}_t)_{t\geq 0}$ is said to be

- a right-continuous if, $\mathcal{F}_t = \mathcal{F}_{t^+}$ for all $t \geq 0$;
- a left-continuous if, $\mathcal{F}_t = \mathcal{F}_{t^-}$ for all $t \geq 0$.

The filtration $(\mathcal{F}_t)_{t\geq 0}$ is said to fulfill the "usual hypotheses" (or satisfies the usual conditions) if

(i) the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous;

(ii) $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets of $\mathcal{F}$, i.e., $\mathcal{F}_0$ contains all $A \in \mathcal{F}$ such that $\mathbb{P}(A) = 0$.

2.1.2 Random variables

Let $(\Omega, \mathcal{F})$ and $(E, \mathcal{G})$ be two measurable spaces. A function X from $(\Omega, \mathcal{F})$ into the state space $(E, \mathcal{G})$ is said to be $(\mathcal{F}, \mathcal{G})$-measurable (or simply, $\mathcal{F}$-measurable) if

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F} \quad for \quad all \quad B \in \mathcal{G}.$$

The function X is then called an E-valued random variable. In particular,

- We can define the smallest σ-algebra which makes X measurable (E-valued random variable), is denoted by $\sigma(X)$ and given by

$$\sigma(X) = \sigma(\{X^{-1}(B) : B \in \mathcal{G}\}) = \sigma\left(\{\omega \in \Omega : X(\omega) \in B\}, B \in \mathcal{G}\right).$$

  $\sigma(X)$ is called the σ-algebra generated by X.

- The indicator function I_A of a set $A \subset \Omega$ is defined by

$$I_A(\omega) = \begin{cases} 1 & for \quad \omega \in A, \\ \\ 0 & for \quad \omega \notin A. \end{cases}$$

  I_A is E-valued random variable (or $\mathcal{F}$-measurable) if and only if A is an $\mathcal{F}$-measurable set, i.e. $A \in \mathcal{F}$.

- If X be an E-valued random variable and takes only a finite number of values of E, then X is called a simple random variable and has the following form

$$X = \sum_{i=1}^{n} x_i I_{A_i}, \quad \bigcup_{i=1}^{n} A_i = \Omega, \quad x_i \in E, \quad n \in \mathbb{N}^*.$$

Let X be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ and takes its values in $(E, \mathcal{G})$. Then by $\mathbb{P}_X$ we will denote the measure image of $\mathbb{P}$ by X

$$\mathbb{P}_X(B) := \mathbb{P}\{\omega \in \Omega : X(\omega) \in B\}, B \in \mathcal{G}.$$

The measure image $\mathbb{P}_X$ is called the probability law (or the probability distribution) of X. If the random variables have the same distribution, we say that they are identically distributed.

We will now present four important properties on which the expectation of a random variable is based, for more details see [68]. Let E be a separable Banach space (we shall denote its norm by $\|\cdot\|$ and its topological dual by E^*) and let X and Y be E-valued random variables defined on $(\Omega, \mathcal{F})$. Then

1. $\mathcal{B}(E)$ is the smallest σ-algebra of subsets of E containing all sets of the form

$$\{x \in E : \quad \varphi(x) \leq \alpha\}, \quad \varphi \in E^*, \quad \alpha \in \mathbb{R},$$

2. $\alpha X + \beta Y$ is an E-valued random variable for any $\alpha, \beta \in \mathbb{R}$,

3. $\|X(\cdot)\|$ is a real valued random variable,

4. there exists a sequence (X_n) of simple E-valued random variables such that, the sequence $(\|X(\omega) - X_n(\omega)\|)$ is monotonically decreasing to 0 for all $\omega \in \Omega$.

The expectation $\mathbb{E}$: Let $(\Omega, \mathcal{F}, \mathbb{P})$ and $(E, \mathcal{B}(E))$ be, respectively, a probability space and measurable space such that E is a separable Banach space. In the same way as the Lebesgue integral, we set

$$\int_F X d\mathbb{P} = \int_F X(\omega)\mathbb{P}(d\omega) := \sum_{i=1}^{n} x_i \mathbb{P}(A_i \cap F), \quad for \quad all \quad F \in \mathcal{F},$$

for a simple E-valued random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$. The properties of the measure $\mathbb{P}$ ensure that the simple E-valued random variable is integrable and the finite value of the integral calculated by the above definition is independent of the representation of X. Moreover the usual properties of additivity and linearity of the integral hold true and

$$\left\| \int_B X(\omega)\mathbb{P}(d\omega) \right\| \leq \int_B \|X(\omega)\|\mathbb{P}(d\omega) = \int_B \|X\| d\mathbb{P}.$$

In general, let X be an E-valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ such that the real valued random variable $\|X\|$ is a Lebesgue integrable, i.e.

$$\int_\Omega \|X\| d\mathbb{P} < \infty.$$

Then, by the existence of a sequence (X_n) of simple E-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ such that the sequence $(\|X(\omega) - X_n(\omega)\|)$ decreases to 0 for all $\omega \in \Omega$. We define the Bochner integral of an E-valued random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\int_\Omega X d\mathbb{P} = \int_\Omega X(\omega)\mathbb{P}(d\omega) := \lim_{n \longrightarrow \infty} \int_\Omega X_n(\omega)\mathbb{P}(d\omega).$$

Furthermore, the limit is independent of the approximating sequence (X_n) of simple random variables satisfying

$$\int_\Omega \|X_n - X\| d\mathbb{P} \longrightarrow 0, \quad as \quad n \longrightarrow 0.$$

The integral $\int_\Omega X d\mathbb{P}$ is called the expectation valued (or mean valued) of X and is denoted by $\mathbb{E}(X)$ or $\mathbb{E}X$, i.e.

$$\mathbb{E}X = \int_\Omega X d\mathbb{P}.$$

The Bochner integral has many properties of the Lebesgue integral, the most important being that if ψ is a measurable mapping from $(E, \mathcal{B}(E))$ into $(G, \mathcal{B}(G))$ integrable with respect to $\mathbb{P}_X$ then, by a standard limit argument, we have

$$\mathbb{E}(\psi(X)) = \int_G \psi(x)\mathbb{P}_X(dx).$$

Assume that $E = \mathbb{R}^d$, for any $p \in [1, \infty)$, $\mathbb{L}^p(\Omega; \mathbb{R}^d)$ stands for the Banach space of R^d-valued random variable X from $(\Omega, \mathcal{F}, \mathbb{P})$ into $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that

$$\|X\|_{\mathbb{L}^p(\Omega;\mathbb{R}^d)} := (\mathbb{E}\|X\|^p)^{\frac{1}{p}} < \infty.$$

Let $X \in \mathbb{L}^p(\Omega; \mathbb{R})$, then the number $\mathbb{E}\|X\|^p$ is called the pth moment of the real-valued random variable X. If $Y \in \mathbb{L}^p(\Omega; \mathbb{R})$ be another real-valued random variable, then the covariance of X and Y is defined as

$$Cov(X,Y) = \mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)] = \mathbb{E}(XY) - \mathbb{E}X\mathbb{E}Y,$$

in particular, X and Y are called uncorrelated if $Cov(X,Y) = 0$. The number

$$\mathbb{V}(X) := Cov(X,X) = \mathbb{E}(X - \mathbb{E}X)^2 = \mathbb{E}(X^2) - (\mathbb{E}X)^2,$$

is called the variance of X. For an R^d-valued random variable $X = (X_1, ..., X_d)^T$, define $\mathbb{E}X = (\mathbb{E}X_1, ..., \mathbb{E}X_d)^T$. If X and Y in $\mathbb{L}^p(\Omega; \mathbb{R}^d)$, the covariance matrix of X and Y is given by

$$Cov(X,Y) = \mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)^T] = \mathbb{E}(XY^T) - \mathbb{E}X(\mathbb{E}Y)^T.$$

We will mention some results as theorems and inequalities, Through which it is possible to distinguish and observe the elements of the space $\mathbb{L}^p(\Omega;\mathbb{R}^d)$. For any $X \in \mathbb{L}^p(\Omega;\mathbb{R}^d)$, we have

1. $\|\mathbb{E}X\| \leq \mathbb{E}\|X\| = \|X\|_{\mathbb{L}^1(\Omega;\mathbb{R}^d)}$, for $p = 1$.

2. Holder's inequality: if $p > 1$, then

$$\|\mathbb{E}(X^T Y\| \leq \|X\|_{\mathbb{L}^p(\Omega;\mathbb{R}^d)} \times \|Y\|_{\mathbb{L}^q(\Omega;\mathbb{R}^d)},$$

   for all $Y \in \mathbb{L}^q(\Omega;\mathbb{R}^d)$, such that $\frac{1}{p} + \frac{1}{q} = 1$. in particular,

$$\|X\|_{\mathbb{L}^r(\Omega;\mathbb{R}^d)} \leq \|X\|_{\mathbb{L}^p(\Omega;\mathbb{R}^d)},$$

   if $1 < r < p$.

3. Minkovski's inequality: if $p > 1$, then

$$\|X + Y\|_{\mathbb{L}^p(\Omega;\mathbb{R}^d)} \leq \|X\|_{\mathbb{L}^p(\Omega;\mathbb{R}^d)} + \|Y\|_{\mathbb{L}^p(\Omega;\mathbb{R}^d)},$$

   for all $Y \in \mathbb{L}^p(\Omega;\mathbb{R}^d)$.

4. Chebyshev's inequality:

$$\mathbb{P}\{\omega \in \Omega : \|X(\omega)\| \geq \lambda\} \leq \frac{\mathbb{E}\|X\|^p}{\lambda^p}, \quad \lambda > 0.$$

5. Monotonic convergence theorem: if (X_n) is an increasing sequence of nonnegative random variables, then

$$\lim_{n \longrightarrow \infty} \mathbb{E}X_n = \mathbb{E}\left(\lim_{n \longrightarrow \infty} X_n\right).$$

6. Dominated convergence theorem: Let (X_n) be sequence of random variable such that $(X_n) \subset \mathbb{L}^p(\Omega;\mathbb{R}^d)$ and (X_n) converges to X in probability. If there exist a

nonnegative real-valued random variable Y satisfies $Y \in \mathbb{L}^p(\Omega;\mathbb{R})$ and $\|X_n\| \leq Y$ a.s., then $X \in \mathbb{L}^p(\Omega;\mathbb{R}^d)$ and (X_n) converges to X in $\mathbb{L}^p(\Omega;\mathbb{R}^d)$, i.e.

$$\lim_{n \longrightarrow \infty} \|X_n - X\|_{\mathbb{L}^p(\Omega;\mathbb{R}^d)} = 0,$$

and

$$\lim_{n \longrightarrow \infty} \mathbb{E}X_n = \mathbb{E}X.$$

In measure theory, let μ and ν two measures on the same measurable space $(E, \mathcal{B}(E))$. A measure ν is called absolutely continuous with respect μ, if we have

$$\forall B \in \mathcal{B}(E), \quad \mu(B) = 0 \Longrightarrow \nu(B) = 0.$$

We denoted by $\nu \ll \mu$. The Radon-Nikodym theorem ensure that if μ and ν are two finite positive measures on $(E, \mathcal{B}(E))$ such that $\nu \ll \mu$, then there exists a unique nonnegative $\mathcal{B}(E)$-measurable function $f : E \longrightarrow \mathbb{R}$ such that for any $B \in \mathcal{B}(E)$, we have

$$\nu(B) = \int_B f d\mu.$$

The function f is called the Radon-Nikodym derivative of ν, with respect μ, is denoted by $f = \frac{d\nu}{d\mu}$. The Radon-Nikodym theorem is very important because it tells us how to move from one measure to another that has more applied properties that may contribute to expanding the ideas of applied mathematics in various applied scientific fields. Therefore, if X is R^d-valued continuous random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{P}_X$ its distribution (or law), in which is absolutely continuous with respect to the Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Then the unique Radon-Nikodym derivative function f is called the joint (or multivariate) probability density function of X, whereupon $\mathbb{P}_X$ of X is called the joint probability distribution (or joint probability law) and is given by

$$\mathbb{P}_X(B) = \mathbb{P}(X \in B) = \int_B f(x)dx, \quad B \in \mathcal{B}(\mathbb{R}^d).$$

Moreover, for every measurable function $\psi : \mathbb{R}^d \longrightarrow \mathbb{R}$ such that $\psi(X)$ is integrable, then

$$\mathbb{E}(\psi(X)) = \int_{\mathbb{R}^d} \psi(x) f(x) dx,$$

note that $\int_B$ is the multiple integral (d times), and its marginal probability distribution denoted by $\mathbb{P}_{X_i}$ for $i=1,..,d$

$$\mathbb{P}_{X_i}(B) = \mathbb{P}(X_i \in B) = \int_B f_{X_i}(x_i)dx_i, \quad B \in \mathcal{B}(\mathbb{R}),$$

where

$$f_{X_i}(x_i) = \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} f(x_1,...,x_d) \prod_{1\leq k\neq i\leq n} dx_k$$

f_{X_i} is called the marginal probability density function of each coordinate X_i and the joint cumulative distribution function of X is the function $F_X : \mathbb{R}^d \longrightarrow [0,1]$ defined for all $x = (x_1,...,x_d)^T$ such that $x^T \in \mathbb{R}^d$ by

$$F_X(x) := \mathbb{P}(X \leq x) = \int_{-\infty}^{x_1} ... \int_{-\infty}^{x_d} f(t_1,...,t_d)dt_1...dt_d,$$

where $X \leq x$ in $\mathbb{R}^d$ means that $X_i \leq x_i$ for all $i = 1,..,d$. If X is a real-valued random variable, avoid saying "joint" in previous terminologies and we have an only integral for calculate instead of a multiple integral.

Below we give a typical measure that played an important role in applied probability and statistics, especially in modeling and simulating the distributions of the outcomes for some random phenomena.

The Gaussian (or The Normal) measure: Let $\sigma > 0$, $m \in \mathbb{R}$. Any measure ν has a density function

$$\mathcal{N}_{(m,\sigma)}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\Big(-\frac{1}{2\sigma^2}(x-m)^2\Big), \quad x \in \mathbb{R},$$

with respect to the Lebesgue measure on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ is called The Gaussian measure and are denoted by $\mathcal{N}(m,\sigma)$. In particular, if $\sigma = 1$, $m = 0$ is called the Standard Gaussian measure. A real-valued random variable X is said to be Gaussian random variable if its distribution $\mathbb{P}_X$ is a Gaussian measure, i.e.

$$\mathbb{P}_X = \mathcal{N}(m,\sigma), \quad we \quad write \quad X \sim \mathcal{N}(m,\sigma).$$

And X be Standard Gaussian random variable if $X \sim \mathcal{N}(0,1)$. In general, the density function of Multivariate Gaussian measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is denoted by $\mathcal{N}(m,\Sigma)$ and has the following form

$$\mathcal{N}_{(m,\Sigma)}(x) = \frac{1}{(2\pi)^{\frac{d}{2}}(det\Sigma)^{\frac{1}{2}}} exp\left(-\frac{1}{2}(x-m)^T\Sigma^{-1}(x-m)\right), \quad x \in \mathbb{R}^d,$$

where $m \in \mathbb{R}^d$ and Σ is a symmetric positive definite $d \times d$ matrix. A multivariate random vector $X = (X_1, \ldots, X_d)^T$ is said to be Gaussian random vector if its distribution $\mathbb{P}_X$ is a Multivariate Gaussian measure, i.e.

$$\mathbb{P}_X = \mathcal{N}(m,\Sigma), \quad we \quad write \quad X \sim \mathcal{N}(m,\Sigma).$$

One can check easily that

$$\mathbb{E}X = m, \quad Cov(X) = \Sigma,$$

and the random variables $X_1, \ldots, X_d$ are independent if and only if the matrix Σ is diagonal. If the covariance matrix Σ is not invertible (or degeneracy), we need to define the Gaussian distribution via characteristic functions (see Section 1.9).

Conditional Expectation: Let $X = (X_1, \ldots, X_d)^T \in \mathbb{L}^1(\Omega;\mathbb{R}^d)$ and let $\mathcal{A}$ be a sub-σ-algebra of $\mathcal{F}$. There exists a unique integrable $\mathbb{R}^d$-valued random variable denoted by $\mathbb{E}(X|\mathcal{A}) = (\mathbb{E}(X_1|\mathcal{A}), \ldots, \mathbb{E}(X_d|\mathcal{A}))^T$, which almost surely satisfies the following conditions:

(i) $\mathbb{E}(X|\mathcal{A})$ is $\mathcal{A}$-measurable,

(ii) $\int_A \mathbb{E}(X|\mathcal{A})d\mathbb{P} = \int_A X d\mathbb{P}, \quad \forall A \in \mathcal{A}$.

Moreover, if Y is any $\mathcal{A}$-measurable random variable satisfying

$$\int_A Y d\mathbb{P} = \int_A X d\mathbb{P}, \quad \forall A \in \mathcal{A},$$

then $Y = \mathbb{E}(X|\mathcal{A})$ a.s., in $(\Omega, \mathcal{F}, \mathbb{P})$. For any $\mathbb{R}^d$-valued random variable Z on $(\Omega, \mathcal{F}, \mathbb{P})$, the insert $\mathbb{E}(X|Z)$ its means $\mathbb{E}(X|\sigma(Z))$ (the conditional expectation of X given $\sigma(Z)$).

For any $X, Y \in \mathbb{L}^1(\Omega;\mathbb{R}^d)$. The basic properties mentioned in the following of $\mathbb{E}(X|\mathcal{A})$ hold almost surely

1. $\mathbb{E}(X|\mathcal{A}) = X$ if X is $\mathcal{A}$-measurable. In particular, $\mathbb{E}(\mathbb{E}(X|\mathcal{A})) = \mathbb{E}X$,

2. $\mathbb{E}(X|\mathcal{A}) = \mathbb{E}X$ if $\mathcal{A}, \sigma(X)$ are independent or $\mathcal{A} = \{\phi, \Omega\}$. In particular,

$$\mathbb{E}(X|Y) = \mathbb{E}X \quad if \quad X,Y \quad are \quad independent,$$

3. $\mathbb{E}(\mathbb{E}(X|\mathcal{A})|\mathcal{K}) = \mathbb{E}(\mathbb{E}(X|\mathcal{K})|\mathcal{A}) = \mathbb{E}(X|\mathcal{K})$ if $\mathcal{K}$ is a sub-σ-algebra of $\mathcal{A}$,

4. $\mathbb{E}((aX + bY)|\mathcal{A}) = a\mathbb{E}(X|\mathcal{A}) + b\mathbb{E}(Y|\mathcal{A})$ if a, b are constants,

5. $\mathbb{E}(X|\mathcal{A}) \leq \mathbb{E}(Y|\mathcal{A})$ if $X \leq Y$,

6. $\mathbb{E}(XY|\mathcal{A}) = X\mathbb{E}(Y|\mathcal{A})$ if X is $\mathcal{A}$-measurable and XY is integrable.

2.2 Stochastic processes

2.2.1 General concepts

A family $X = (X_t)_{t \in I}$ of random variables defined on a sample space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in a state space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is called a stochastic process with index set I (where $I = [0,T], \quad T > 0$ or $I = [0, \infty)$). Or a stochastic process $X = (X_t)_{t \in I}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with index set I and state space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is a bivariate function $X(.,.)$ defined as follows

$$X : I \times \Omega \longrightarrow \mathbb{R}^d, \quad (t, \omega) \longmapsto X(t, \omega) := X_t(\omega),$$

such that for any fixed $t \in I$ the function

$$X_t := X_t(\cdot) : \Omega \longrightarrow \mathbb{R}^d, \quad \omega \longmapsto X_t(\omega),$$

is an $\mathbb{R}^d$-valued random variable and for any fixed $\omega \in \Omega$ the function

$$X_{.}(\omega) : I \longrightarrow \mathbb{R}^d, \quad t \longmapsto X_t(\omega),$$

is not necessary measurable with respect to any σ-algebra on I. The functions $X_{.}(\omega), \quad \omega \in \Omega$ are called the sample paths (realizations, trajectories). In particular,

- If X as a bivariate function is $(\mathcal{B}(I)\times\mathcal{F},\mathcal{B}(\mathbb{R}^d))$-measurable then X is said to be measurable stochastic process.

- If X is defined on a stochastic basis $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in I},\mathbb{P})$. Then X is said to be

    - adapted (or $\mathcal{F}_t$-adapted) if, for any $t\in I$, X_t is $\mathcal{F}_t$-measurable,
    - progressively measurable if, for any $t\in I$, X as a bivariate function is $(\mathcal{B}([0,t])\times\mathcal{F}_t,\mathcal{B}(\mathbb{R}^d))$-measurable.

Let $Y=(Y_t)_{t\in I}$ be another stochastic process on $(\Omega,\mathcal{F},\mathbb{P})$. By the probability measure $\mathbb{P}$, we can weaken the sameness property between two processes X and Y, we list in the following three related concepts

1. Y is a version or modification of X if

$$\mathbb{P}\{\omega\in\Omega : X_t(\omega)=Y_t(\omega)\}=1,\quad \forall t\in I.$$

2. X and Y have the same finite-dimensional distributions if, for any integer $n\geq 1$, $t_1,...,t_n\in I$, $B\in\mathcal{B}(\mathbb{R}^{dn})$,

$$\mathbb{P}\{(X_{t_1},...,X_{t_n})\in B\}=\mathbb{P}\{(Y_{t_1},...,Y_{t_n})\in B\}.$$

3. X and Y are called indistinguishable if

$$\mathbb{P}\{\omega\in\Omega : X_t(\omega)=Y_t(\omega),\quad \forall t\in I\}=1.$$

We now pick several definitions of regularity properties for a $\mathbb{R}^d$-valued stochastic process X when $\mathbb{R}^d$ be a normed or metric space, it is as follows

- X is mean square continuous at $t_0\in I$ if

$$\lim_{t\longrightarrow t_0}\mathbb{E}\|X_t-X_{t_0}\|^2=0.$$

  If X is mean square continuous for any $t\in I$ then X is said to be mean square continuous on I.

- X is continuous in probability (or stochastically continuous) at $t_0 \in I$ if for all $\epsilon > 0$ and all $\eta > 0$ there exists $\rho > 0$ such that

$$\mathbb{P}(\|X_t - X_{t_0}\| \geq \epsilon) \leq \eta, \quad \forall t \in [t_0 - \rho, t_0 + \rho] \cap I.$$

If X is continuous in probability for any $t \in I$ then X is said to be continuous in probability on I. In particular, if

$$\mathbb{P}(\|X_t - X_{t_0}\| \geq \epsilon) \leq \eta, \quad \forall t, \quad t_0 \in I, \quad such \quad that \quad |t - t_0| < \rho,$$

then X is said to be uniformly continuous in probability on I.

- A stochastically continuous process on $[0,T]$ is uniformly stochastically continuous on $[0,T]$.

- X is continuous (resp. right continuous, left continuous) if $\mathbb{P}$-a.s, its trajectories are continuous (resp. right continuous, left continuous) on I.

- X is said to be regular process if X is adapted and its trajectories have right and left limits for any $t \in I$, i.e.,

$$\lim_{s \downarrow t} X_s \quad and \quad \lim_{s \uparrow t} X_s, \quad for \quad any \quad t \in I$$

exists and is finite $\mathbb{P}$-a.s.

- X is said to be cadlag process if X is right continuous and its trajectories have left limits.

2.2.2 Variation, quadratic variation of a process

Let X be a real cadlag stochastic process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with an index set I and let $t \in I$. Now we show the process X as a bivariate function $X(.,.)$ defined on $I \times \Omega$. When we fix the second variable $\omega \in \Omega$ of $X(.,.)$ we get a trajectory of X as a real function

$X_{.}(\omega)$ defined on I, therefore we can define the total variance and quadratic variance of a trajectory of X as we defined them with a real function. It is as follow

The total variation of its trajectories on $[0,t]$ is given by

$$V_{X_{.}(\omega)}(t) = V(X_{.}(\omega);[0,t]) = \lim_{|P_n| \longrightarrow 0} \sum_{k=1}^{n} \Big| X_{t_k}(\omega) - X_{t_{k-1}}(\omega) \Big|.$$

A process X is a process of bounded variation on I if all its trajectories have bounded variation on I.

Similarly, For a fixed $\omega \in \Omega$. If

$$\lim_{|P_n| \longrightarrow 0} \sum_{k=1}^{n} \Big| X_{t_k}(\omega) - X_{t_{k-1}}(\omega) \Big|^2,$$

exists and finite. Then the quadratic variation of a trajectory of the process X on $[0,t]$ is given by

$$[X_{.}(\omega)]_t = \lim_{|P_n| \longrightarrow 0} \sum_{k=1}^{n} \Big| X_{t_k}(\omega) - X_{t_{k-1}}(\omega) \Big|^2.$$

Moreover, if there exists a finite process $[X]$ such that

$$\sum_{k=1}^{n} \Big| X_{t_k}(\omega) - X_{t_{k-1}}(\omega) \Big|^2 \quad in \quad probability \quad as \quad |P_n| \longrightarrow 0, \quad \forall \quad t \in I,$$

then $[X]$ is called the quadratic variation process of X. One can check easily that $[X]$ is increasing process on I, as well as if X be a continuous process with a bounded variation on I then its quadratic variation is zero on I.

In general, the joint quadratic variation (or the cross quadratic variation) of X and Y is defined by

$$[X,Y]_t = \lim_{|P_n| \longrightarrow 0} \sum_{k=1}^{n} \Big(X_{t_k} - X_{t_{k-1}})(Y_{t_k} - Y_{t_{k-1}} \Big),$$

and the cross-variance has the following property

$$[X,Y]_t = \frac{1}{2}([X+Y]_t - [X]_t - [Y]_t).$$

2.2.3 Gaussian, Martingale processes

A stochastic process $X = (X_t)_{t\in I}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with state space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and index set I, is said to be

- Gaussian if for any integer $n \geq 1$, $t_1, \ldots, t_n \in I$, the $d\times n$-dimensional random vector $(X_{t_1}, \ldots, X_{t_n})^T$ is a Gaussian vector.
- Martingale if for all $s,t \in I$ such that $s \leq t$, we have, $\mathbb{P}$-a.s:

  (i) X is $\mathcal{F}_t$-adapted,

  (ii) X is integrable, i.e., $\mathbb{E}\|X_t\| < \infty, \quad \forall t \in I$,

  (iii) $\mathbb{E}(X_t|\mathcal{F}_s) = M_s$,

  where $(\mathcal{F}_t)_{t\in I}$ is any given filtration on $(\Omega, \mathcal{F}, \mathbb{P})$.

If X is real valued integrable and $\mathcal{F}_t$-adapted process. Then X is said to be supermartingale (resp. submartingale) if for all $s,t \in I$ such that $s \leq t$, we have

$$\mathbb{E}(M_t|\mathcal{F}_s) \leq M_s \quad (resp. \quad \mathbb{E}(M_t|\mathcal{F}_s) \geq M_s), \quad \mathbb{P}-a.s.$$

Now we give some fundamental result for a $\mathbb{R}^d$-valued continuous martingale $M = (M_t)_{t\in I}$, it is as follow

1. If $\psi : \mathbb{R}^d \longrightarrow \mathbb{R}$ is a convex function such that

$$\mathbb{E}(\psi(\|M_t\|)) < \infty, \quad \forall \quad t \in I,$$

   then $\psi(M) = (\psi(M_t))_{t\in I}$ is a submartingale.

2. Doob's martingale inequalities:

   (i) If M is integrable, then for all $t \in I$ and $\lambda > 0$, we have

$$\lambda \mathbb{P}\left(\sup_{0\leq s\leq t} \|M_s\| \geq \lambda\right) \leq \mathbb{E}\|M_t\|.$$

(ii) If M is p-integrable for $p>1$, then for all $t \in I$, we have

$$\mathbb{E}\left(\sup_{0\leq s\leq t} \|M_s\|^p\right) \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\|M_t\|^p.$$

3. Doob-Meyer decomposition: Let M and N be two a real-valued square-integrable continuous martingales, then there exists a unique continuous integrable adapted increasing process denoted by $[M,N]$ such that $MN-[M,N]$ is a continuous martingale vanishing at $t=0$. Where the process $[M,N]=([M,N]_t)_{t\in I}$ is the joint quadratic variation of M and N. In particular, $M^2-[M]$ is a continuous martingale vanishing at $t=0$. Where the process $[M]=([M]_t)_{t\in I}$ is the quadratic variation of M.

4. Burkholder-Davis-Gundy (BDG) Inequality: If M is a real-valued continuous martingale and $M_0=0$. Then, for all $p>0$, there exist two c_p and C_p constants such that for all $t \in I$, we have

$$c_p \mathbb{E}[M]_t^{\frac{p}{2}} \leq \mathbb{E}\left(\sup_{0\leq s\leq t} \|M_s\|^p\right) \leq C_p \mathbb{E}[M]_t^{\frac{p}{2}}.$$

2.2.4 Wiener processes

We now discuss the most important example of stochastic processes with continuous paths, the Wiener process, commonly referred to as the Brownian motion in the physics literature.

Definition 1. *A Wiener process is a real-valued adapted process $W=(W_t)_{t\geq 0}$ defined on some stochastic basis $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ with the following properties*

(i) W has continuous trajectories;

(ii) for $0\leq s<t$, the increment W_t-W_s is independent of $\mathcal{F}_s$ and $W_t-W_s \sim \mathcal{N}(0,\sigma^2(t-s))$, where σ^2 is a nonnegative constant.

In particular, if $W_0=0$ a.s. and $\sigma^2=1$ then W is called standard Wiener process.

Some times we used the natural filtration $\mathcal{F}_t^W = \sigma(B_s, 0 \leq s < t), \quad t \geq 0$ of W when a filtration $(\mathcal{F}_t)_{t\geq 0}$ does not specified. A standard d-dimensional Wiener process is a vector-valued stochastic process

$$W = (W_1, \ldots, W_d)^T,$$

whose components are independent and standard Wiener processes. A standard Wiener process has an other equivalently definition is given by the following proposition

Proposition 1. *A stochastic process $W = (W_t)_{t\geq 0}$ is a standard Wiener process if and only if it is a continuous centered Gaussian process whose covariance is given by*

$$Cov(W_s, W_t) = \mathbb{E}(W_s W_t) = s \wedge t, \quad s, t \geq 0.$$

Proof. Let W be standard Wiener process, from the definition of a standard Wiener process W is continuous centered Gaussian process. Moreover, for $s \leq t$, we have

$$\mathbb{E}(W_s W_t) = \mathbb{E}(W_s(W_t - W_s)) + \mathbb{E}W_s^2 = s,$$

since W_s and $W_t - W_s$ are independent.
Reciprocally, W has continuous trajectories and $W_0 = 0$ a.s., sine W is a Gaussian process then to show that $W_t - W_s$ is independent of $\mathcal{F}_s^W$, it is sufficiently to show that $Cov(W_t - W_s, W_r) = 0$ for $r \leq s \leq t$, indeed

$$\begin{aligned} Cov(W_t - W_s, W_r) &= Cov(W_t, W_r) - Cov(W_s, W_r) \\ &= \mathbb{E}(W_t W_r) - \mathbb{E}(W_s W_r) \\ &= t \wedge r - s \wedge r \\ &= 0, \end{aligned}$$

finally, for $0 \leq s \leq t$, $W_t - W_s$ is a centered Gaussian random variable and its variance $\mathbb{E}((W_t - W_s)^2) = t + s - 2s = t - s$ i.e.,

$$W_t - W_s \sim \mathcal{N}(0, (t-s)).$$

And the result follows. □

In the following we give some important properties of a Wiener process

Lemma 3. *(Scaling invariance): Let $W = (W_t)_{t\geq 0}$ be a standard Wiener process and $a > 0$. Then, the process $B = (B_t)_{t\geq 0}$ such that*

$$B_t = \frac{1}{a} W_{a^2 t}, \quad t \geq 0,$$

is a standard Wiener process.

Theorem 2. *(Time-inversion) Let $W = (W_t)_{t\geq 0}$ be a standard Wiener process. Then, the process $B = (B_t)_{t\geq 0}$ such that*

$$B_t = \begin{cases} 0 & if \quad t = 0, \\ \\ tW_{\frac{1}{t}} & if \quad t > 0, \end{cases}$$

is a standard Wiener process.

Corollary 1. *(Law of large numbers) Let $W = (W_t)_{t\geq 0}$ be a standard Wiener process. Then, we have*

$$\lim_{t\longrightarrow\infty} \frac{W_t}{t} = 0, \quad a.s.$$

Theorem 3. *(Quadratic variation of a Wiener process) If $W = (W_t)_{t\geq 0}$ is a standard Wiener process. Then, for all $t \geq 0$, we get*

$$[W]_t = t, \quad a.s.$$

2.3 The stochastic integral

2.3.1 The Itô Integral

Before constructing the Itô stochastic integral, we will denote by $\mathcal{M}^2_{u,v}(\mathbb{R})$ the appropriate family of integrands which makes the integral in (1,1) exists and well defined. The elements of $\mathcal{M}^2_{u,v}(\mathbb{R})$ are all real-valued measurable $\mathcal{F}_t$-adapted processes $X=(X_t)_{t\in[u,v]}$ defined on $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ such that

$$\|X\|_{u,v} = \left(\mathbb{E}\int_u^v \|X_t\|^2 dt\right)^{\frac{1}{2}} < \infty.$$

We say that two elements $X=(X_t)_{t\in[u,v]}$ and $Y=(Y_t)_{t\in[u,v]}$ of $\mathcal{M}^2_{u,v}(\mathbb{R})$ are equivalent if

$$\|X-Y\|_{u,v} = 0.$$

If $u=0$ and $v=T$ where T is a positive real number we will abbreviate and write $\mathcal{M}^2_T(\mathbb{R})$ (resp. $\|\cdot\|_T$) instead of $\mathcal{M}^2_{0,T}(\mathbb{R})$ (resp. $\|\cdot\|_{0,T}$). Clearly, the space $\mathcal{M}^2_{u,v}(\mathbb{R})$ equipped with the norm $\|\cdot\|_{u,v}$ is a Banach space. Moreover, it is also a complete metric space under the metric associated to the norm $\|\cdot\|_{u,v}$.

As promised, we can now show in two steps how to construct the Itô integral for a process.

Step 1. Let $W=(W_t)_{t\geq 0}$ be a Wiener process defined on a complete stochastic basis $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$, with $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. A real-valued stochastic process $S=(S_t)_{t\in[u,v]}$ is said to be simple (elementary or step) process if there exists a partition $u=t_0<t_1<...<t_k=v$ of $[u,v]$, and bounded random variables s_i, $i=0,1,\dots,k-1$, such that s_i is $\mathcal{F}_{t_i}$-measurable and

$$S_t = \sum_{i=0}^{k-1} s_i I_{[t_i,t_{i+1})}(t),$$

where $I_{[t_i,t_{i+1})}$ is the indicator function of $[t_i,t_{i+1})$. For a simple process S we define the stochastic Itô integral of S with respect to the Wiener process W by the formula

$$\int_u^v S_t dW_t = \sum_{i=0}^{k-1} s_i (W_{t_{i+1}} - W_{t_i}).$$

Moreover, $\int_u^v S_t dW_t$ is a real-valued random variable, also it is $\mathcal{F}_v$-measurable and has the properties given by the following lemma.

Lemma 4. *If S be simple process, then we have*

1. $\mathbb{E}\left(\int_u^v S_t dW_t\right) = 0$,

2. (Itô isometry) $\mathbb{E}\left(\int_u^v S_t dW_t\right)^2 = \mathbb{E}\left(\int_u^v S_t^2 dt\right)$.

Proof. The first property is clearly due to the independence between $W_{t_{i+1}} - W_{t_i}$ and $\mathcal{F}_{t_i}$, and the measurability of s_i with respect to $\mathcal{F}_{t_i}$. For the second property, we have

$$\begin{aligned}
\mathbb{E}\left(\int_u^v S_t dW_t\right)^2 &= \mathbb{E}\left(\sum_{0\le i,j\le k-1} s_i s_j (W_{t_{i+1}} - W_{t_i})(W_{t_{j+1}} - W_{t_j})\right) \\
&= \mathbb{E}\left(\sum_i s_i^2 (W_{t_{i+1}} - W_{t_i})^2 + 2\sum_{i<j} s_i s_j (W_{t_{i+1}} - W_{t_i})(W_{t_{j+1}} - W_{t_j})\right) \\
&= \sum_i \mathbb{E}s_i^2 \mathbb{E}(W_{t_{i+1}} - W_{t_i})^2 + 2\sum_{i<j} \mathbb{E}(s_i s_j (W_{t_{i+1}} - W_{t_i}))\mathbb{E}(W_{t_{j+1}} - W_{t_j}) \\
&= \sum_i \mathbb{E}s_i^2 (t_{i+1} - t_i) \\
&= \mathbb{E}\left(\int_u^v S_t^2 dt\right),
\end{aligned}$$

where we have used the independence between $W_{t_{j+1}} - W_{t_j}$ and $s_i s_j (W_{t_{i+1}} - W_{t_i})$ for $i < j$, and $W_{t_{i+1}} - W_{t_i} \sim \mathcal{N}(0, t_{i+1} - t_i)$. □

Step 2. We can now extend the definition of the Itô integral established over simple processes to also include the processes in $\mathcal{M}^2_{u,v}(\mathbb{R})$. This is due to the following approximation theorem, for more details about its proof see [60].

Theorem 4. *For any $X \in \mathcal{M}^2_{u,v}(\mathbb{R})$, there exists a sequence $(S_n)_{n\ge 1}$ of simple processes such that*

$$\lim_{n\longrightarrow\infty} \mathbb{E}\left(\int_u^v \|X_t - S_n(t)\|^2 dt\right) = 0.$$

Furthermore, the limit

$$\lim_{n\longrightarrow\infty} \int_u^v S_n(t) dW_t,$$

is well-defined. Moreover, by the Itô isometry for a simple process we have $\int_u^v S_n(t) dW_t$

is in $\mathbb{L}^2(\Omega,\mathbb{R})$ for any simple process $S_n(t)$. And

$$\begin{aligned}\lim_{n,m\longrightarrow\infty}\mathbb{E}\left\|\int_u^v S_n(t)dW_t-\int_u^v S_m(t)dW_t\right\|^2 &= \lim_{n,m\longrightarrow\infty}\mathbb{E}\left\|\int_u^v (S_n(t)-S_m(t))dW_t\right\|^2\\ &= \lim_{n,m\longrightarrow\infty}\mathbb{E}\int_u^v \|S_n(t)-S_m(t)\|^2dt\\ &= 0.\end{aligned}$$

Thus, $(\int_u^v S_n(t)dW_t)_{n\geq 1}$ is a Cauchy sequence in $\mathbb{L}^2(\Omega,\mathbb{R})$ so that its limit exists in $\mathbb{L}^2(\Omega,\mathbb{R})$. Finally, we will define this limit as the Itô stochastic integral of X with respect to the Wiener process W, i.e.

$$\int_u^v X_t dW_t=\lim_{n\longrightarrow\infty}\int_u^v S_n(t)dW_t \quad in \quad \mathbb{L}^2(\Omega,\mathbb{R}).$$

Moreover, The above limit is independent of the choice of the approximating sequence $(S_n)_{n\geq 1}$. The Itô stochastic integral $\int_u^v X_t dW_t$ is $\mathcal{F}_v$-measurable and for any $A\subset[u,v]$, we have

$$\int_A X_t dW_t=\int_u^v X_t I_A(t)dW_t.$$

In particular, if X is deterministic function, then the random variable $\int_u^v X_t dW_t$ is Gaussian, i.e.

$$\int_u^v X_t dW_t\sim\mathcal{N}\left(0,\int_u^v X_t^2dt\right).$$

In the following, we summarize without proof some perfect properties of an Itô stochastic integral. For any $X,Y\in\mathcal{M}^2_{u,v}(\mathbb{R})$ and $a,b\in\mathbb{R}$, we have

1. $\mathbb{E}\left(\int_u^v X_t dW_t\right)=0.$

2. (Itô isometry) $\mathbb{E}\left(\int_u^v X_t dW_t\right)^2=\mathbb{E}\left(\int_u^v X_t^2dt\right)$. Furthermore,

$$\mathbb{E}\left(\int_u^v X_t dW_t\int_u^v Y_t dW_t\right)=\mathbb{E}\left(\int_u^v X_tY_tdt\right).$$

3. (Linearity) $\int_u^v(aX_t+bY_t)dW_t=a\int_u^v X_t dW_t+b\int_u^v Y_t dW_t.$

4. (Additivity) $\int_u^v X_t dW_t=\int_u^c X_t dW_t+\int_c^v X_t dW_t, \quad for \quad all \quad c\in(u,v).$

5. If Z is a real-valued bounded $\mathcal{F}_u$-measurable random variable, then $ZX \in \mathcal{M}^2_{u,v}(\mathbb{R})$ and
$$\int_u^v ZX_t dW_t = Z\int_u^v X_t dW_t.$$

Let $X \in \mathcal{M}^2_T(\mathbb{R})$. We define the indefinite Itô stochastic integral as follows
$$I_t = \int_0^t X_s dW_s, \quad for \quad t \in [0,T],$$
where $I_0 = \int_0^0 X_s dW_s = 0$. We call the stochastic process $I = (I_t)_{t\in[0,T]}$ the Itô process it is the special case of Itô processes, which will be presented in the following subsection. And some of its important properties is given by the following theorem

Theorem 5. *If $X \in \mathcal{M}^2_T(\mathbb{R})$, then the special Itô process $I = (I_t)_{t\in[0,T]}$ is a square-integrable continuous martingale and its quadratic variation is given by*
$$[I]_t = \int_0^t X_s^2 ds, \quad t \in [0,T].$$
In particular,
$$\mathbb{E}\left(\sup_{0\le t\le T} \|I_t\|^2\right) = \mathbb{E}\left[\sup_{0\le t\le T}\left\|\int_0^t X_s dW_s\right\|^2\right] \le 4\mathbb{E}\int_0^T \|X_s\|^2 ds.$$

Now we will see an applied example of Itô's integral

Example 1. *Let $W = (W_t)_{t\ge 0}$ be a Wiener process such that $W_0 = 0$, then for all $t \ge 0$ we get*
$$\int_0^t W_s dW_s = \frac{1}{2}W_t^2 - \frac{1}{2}t, \quad a.s.$$
Indeed, let P be partition of $[0,t]$ with mesh size $|P|$. For almost surely all $\omega \in \Omega$ we have
$$W_{t_i}(W_{t_{i+1}} - W_{t_i}) = \frac{1}{2}(W^2_{t_{i+1}} - W^2_{t_i}) - \frac{1}{2}(W_{t_{i+1}} - W_{t_i})^2,$$
moreover,
$$\begin{aligned}\sum_i W_{t_i}(W_{t_{i+1}} - W_{t_i}) &= \frac{1}{2}\sum_i(W^2_{t_{i+1}} - W^2_{t_i}) - \frac{1}{2}\sum_i(W_{t_{i+1}} - W_{t_i})^2 \\ &= \frac{W_t^2}{2} - \frac{1}{2}\sum_i(W_{t_{i+1}} - W_{t_i})^2,\end{aligned}$$

since, $\sum_i (W_{t_{i+1}} - W_{t_i})^2 \longrightarrow [W]_t = t$ *in* $\mathbb{L}^2(\Omega, \mathbb{R})$ *as* $|P| \longrightarrow 0$ *and by the definition of Itô integral which is*

$$\int_0^t W_s dW_s = \lim_{|P| \longrightarrow 0} \sum_i W_{t_i} \left(W_{t_{i+1}} - W_{t_i} \right),$$

we get

$$\int_0^t W_s dW_s = \frac{1}{2} W_t^2 - \frac{1}{2} t, \quad a.s,$$

As required. ■

We shall now show the multi-dimensional indefinite Itô stochastic integral (or multi-dimensional special Itô process) as

$$\int_0^t \sigma_s dW_s, \quad for \quad t \in [0,T],$$

where $W = (W^1, \ldots, W^m)^T$ is an m-dimensional Wiener process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and $\sigma = \{(\sigma_{ij}(t))_{d \times m}\}_{t \in [0,T]}$ is $n \times m$-matrix-valued measurable $\mathcal{F}_t$-adapted process such that

$$\mathbb{E} \int_0^T |\sigma_t|^2 dt < \infty,$$

where $|A|$ denoted the trace norm for matrix $A \in \mathbb{R}^{n \times m}$ ($\mathbb{R}^{n \times m}$ denotes the space of real $n \times m$-matrices), i.e.

$$|A| := \sqrt{\sum_{i,j} a_{ij}^2}, \quad for \quad A = (a_{ij})_{ij}.$$

The n-column-vector-valued process whose i'th component is the following sum of 1-dimensional Itô integrals

$$\sum_{j=1}^m \int_0^t \sigma_{ij}(s) dW_s^j, \quad for \quad t \in [0,T].$$

Clearly, the Itô integral is an $\mathbb{R}^d$-valued continuous martingale with respect to $(\mathcal{F}_t)_{t \geq 0}$.

2.3.2 The Itô's Formula

Let $W = (W_1, ..., W_m)^T$ be an m-dimensional Wiener process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Let $\mathcal{I}^1(\mathbb{R}_+, \mathbb{R}^n)$ and $\mathcal{I}^2(\mathbb{R}_+, \mathbb{R}^{n\times m})$ denotes, respectively, the family of all $\mathbb{R}^n$-valued measurable $\mathcal{F}_t$-adapted processes $b = (b_t)_{t\geq 0}$ defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}$ and the family of all $n \times m$-matrix-valued measurable $\mathcal{F}_t$-adapted processes $\sigma = (\sigma_t)_{t\geq 0}$ defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ such that

$$\int_0^T \|b_t\| dt < \infty, \quad a.s. \quad for \quad every \quad T > 0,$$

and

$$\int_0^T |\sigma_t|^2 dt < \infty, \quad a.s. \quad for \quad every \quad T > 0.$$

If $b \in \mathcal{I}^1(\mathbb{R}_+, \mathbb{R}^n)$ and $\sigma \in \mathcal{I}^2(\mathbb{R}_+, \mathbb{R}^{n\times m})$. Then a $\mathbb{R}^n$-valued continuous adapted process $X = (X_t)_{t\geq 0}$ of the form

$$X_t = X_0 + \int_0^s b_s ds + \int_0^t \sigma_s dW_s,$$

is called a n-dimensional Itô process associated to the stochastic differential form

$$dX_t = b_t dt + \sigma_t dW_t, \quad \text{for} \quad t \geq 0. \tag{2.3.1}$$

In particular, if $n = m = 1$ then $X = (X_t)_{t\geq 0}$ is called one-dimensional Itô process.

We now give without proof the multi-dimensional Itô formula (see [60]). Let $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R})$ Denote the family of all real-valued functions $V(x,t)$ defined on $\mathbb{R}^n \times \mathbb{R}_+$ such that they are continuously twice differentiable in x (where $x = (x_1, \ldots, x_n)^T$) and one in t. If $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R})$, we set

$$V_t(x,t) = \frac{\partial V(x,t)}{\partial t}, \quad V_x(x,t) = \left(\frac{\partial V(x,t)}{\partial x_1}, \ldots, \frac{\partial V(x,t)}{\partial x_d} \right), \quad V_{xx}(x,t) = \left(\frac{\partial^2 V(x,t)}{\partial x_i \partial x_j} \right)_{n\times n}.$$

Lemma 5. *(Itô's formula [60]) Let $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R})$ and consider the n-dimensional Itô process $X = (X_t)_{t\geq 0}$ which takes the stochastic differential form in (2.3.1). Then $V(X_t, t)$ is again an Itô's process with the stochastic differential given as*

$$dV(X_t, t) = LV(X_t, t)dt + V_x(X_t, t)\sigma_t dW_t, \quad for \quad t \geq 0, \tag{2.3.2}$$

where L is the differential operator of Eq. (2.3.1) defined in [60] as

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{n} b_i(t) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} \left[(\sigma_t)^T \sigma_t \right]_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}. \tag{2.3.3}$$

Then, we have

$$LV(X_t, t) = V_t(X_t, t) + V_x(X_t, t) b_t + \frac{1}{2} trace \left[(\sigma_t)^T V_{xx}(X_t, t) \sigma_t \right]. \tag{2.3.4}$$

2.4 Stochastic Differential Equations

Let $W = (W_1, ..., W_m)^T$ be an m-dimensional Wiener process defined on a complete stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, with $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions. Let b and σ be two measurable functions defined on $\mathbb{R}^n \times \mathbb{R}_+$ and they take their values, respectively in $\mathbb{R}^n$ and $\mathbb{R}_{n \times m}$. Consider the n-dimensional stochastic differential equation (SDE) of the form

$$\begin{cases} dX_t = b(X_t, t)dt + \sigma(X_t, t)dW_t, \quad t \in [0, T], \\ \\ X_0 = x, \end{cases} \tag{2.4.1}$$

where x is a $\mathbb{R}^n$-valued random variable such that

$$x \quad is \quad independent \quad of \quad \mathcal{F}_\infty^W = \sigma\left(\bigcup_{t \geq 0} \mathcal{F}_t^W \right) \quad and \quad \mathbb{E}\|x\|^2 < \infty.$$

Let us first give the definition of the solution of a stochastic cauchy problem (2.4.1)

Definition 2. *An $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted process $X = (X_t)_{t \in [0,T]}$ is called a solution of problem (2.4.1) if it has the following properties*

(i) $\{b(X_t, t)\} \in \mathcal{I}^1([0, T]; \mathbb{R}^n)$ and $\{\sigma(X_t, t)\} \in \mathcal{I}^2([0, T]; \mathbb{R}^{n \times m})$;

(ii) for all $t \in [0, T]$ the following integral stochastic equation holds,

$$X_t = x + \int_0^t b(X_s, s)ds + \int_0^t \sigma(X_s, s)dW_s.$$

In the following we give without proof the existence and uniqueness theorem of the stochastic cauchy problem (2.4.1)

Theorem 6. *Suppose that there exist a positive constant K such that the coefficients b and σ satisfies the following conditions*

(i) (Lipschitz condition) for all $x, y \in \mathbb{R}^n$ and $t \in [0,T]$

$$\|b(x,t) - b(y,t)\| + |\sigma(x,t) - \sigma(y,t)| \leq K\|x - y\|;$$

(ii) (Linear growth condition) for all $(x,t) \in \mathbb{R}^n \times [0,T]$

$$\|b(x,t)\|^2 + |\sigma(x,t)|^2 \leq K^2(1 + \|x\|^2).$$

Then the problem (2.4.1) has a unique solution $X = (X_t)_{t\in[0,T]}$ that satisfies $X \in \mathcal{M}_T^2(\mathbb{R})$

Let the following n-dimensional stochastic differential equation

$$dX_t = b(X_t,t)dt + \sigma(X_t,t)dW_t, \quad t \in [0,\infty), \tag{2.4.2}$$

with initial value $X_0 = x$. If the assumptions of the existence and uniqueness theorem hold on every finite subinterval $[0,T]$ of $[0,\infty)$, then equation (2.4.2) has a unique global solution X_t on the entire interval $[0,\infty)$. Moreover, the global solution X_t of (2.4.2) is a Markov process that is, the following Markov property holds for all $0 \leq s \leq t < \infty$ and $N \in \mathcal{B}(\mathbb{R}^n)$,

$$\mathbb{P}(X_t \in N|\mathcal{F}_s) = \mathbb{P}(X_t \in N|X_s).$$

The previous Markov property is equivalent to the following formulation, for any bounded Borel measurable function $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $0 \leq s \leq t < \infty$,

$$\mathbb{E}(\psi(X_t)|\mathcal{F}_s) = \mathbb{E}(\psi(X_t)|X_s).$$

And its transition probability is the function $\mathbb{P}(s,y,t,N)$, defined on $0 \leq s \leq t < \infty$, $y \in \mathbb{R}^n$ and $N \in \mathcal{B}(\mathbb{R}^n)$, with the following properties:

(i) For every $0 \leq s \leq t < \infty$ and $N \in \mathcal{B}(\mathbb{R}^n)$,

$$\mathbb{P}(s, X_s, t, N) = \mathbb{P}(X_t \in N | \mathcal{F}_s) = \mathbb{P}(X_t \in N | X_s).$$

We can also use the notation

$$\mathbb{P}(s, y, t, N) = \mathbb{P}(X_t \in N | X_s = y),$$

which is the probability that the process will be in the set N at time t given the condition that the process was in the state y at time $s \leq t$.

(ii) $\mathbb{P}(s, y, t, .)$ is a probability measure on $\mathcal{B}(\mathbb{R}^n)$ for every $0 \leq s \leq t < \infty$, $y \in \mathbb{R}^n$. And $\mathbb{P}(s, ., t, N)$ is Borel measurable for every $0 \leq s \leq t < \infty$, $N \in \mathcal{B}(\mathbb{R}^n)$.

(iii) The Chapman-Kohnogorov equation

$$\mathbb{P}(s, y, t, N) = \int_{\mathbb{R}^n} \mathbb{P}(r, z, t, N)\mathbb{P}(s, y, r, dz),$$

holds for any $0 \leq s \leq r \leq t < \infty$, $y \in \mathbb{R}^n$ and $N \in \mathcal{B}(\mathbb{R}^n)$.

In particular, if the transition probability function $\mathbb{P}(s, y, t, N)$ is stationary, we give the following definition

Definition 3. *[33] The transition probability function $\mathbb{P}(s, y, t, N)$ of a given Markov process is said to be time-homogeneous (and the corresponding Marcov process is called time-homogeneous) if the function $\mathbb{P}(s, y, t+s, N)$ is independent of the variable s, where $0 \leq s \leq t < \infty$, $y \in \mathbb{R}^n$ and $N \in \mathcal{B}(\mathbb{R}^n)$. i.e, the transition probability $\mathbb{P}(s, y, t, N)$ is stationary, namely*

$$\mathbb{P}(s+r, y, t+r, N) = \mathbb{P}(s, y, t, N),$$

for all $0 \leq s \leq t < \infty$, $r \geq 0$, $y \in \mathbb{R}^n$ and $N \in \mathcal{B}(\mathbb{R}^n)$.

Putting

$$dZ(t) = f(Z(t), t)dt + \sum_{\varsigma=1}^{k} g_{\varsigma}(Z(t), t)dW_{\varsigma}(t)dW(t), \tag{2.4.3}$$

where $Z(t)$ is a regular time-homogeneous Markov process in $\mathbb{R}^n$. The diffusion matrix associated with the process $Z(t)$ is given as

$$A(z) = (b_{i,j}(z)), b_{i,j}(z) = \sum_{\varsigma=1}^{k} g_\varsigma^i g_\varsigma^j. \tag{2.4.4}$$

In the theory of ordinary differential equations (ODE), the Lyapunov function is one of the powerful tools in the stability theory of dynamical systems and their control theory, it is used for dealing with the stability of a point of equilibrium of an ODE from a global and not only a local point of view. The existence of Lyapunov functions is a necessary and sufficient condition for stability in some classes of ODE. However there is no generic approach for creating Lyapunov functions for ODE, the construction of Lyapunov functions is known in many specific cases. For instance, for systems with only one state, quadratic functions suffice; for linear systems, the solution of a certain linear matrix inequality yields Lyapunov functions; and for physical systems, conservation rules are frequently employed to generate Lyapunov functions. Mathematically speaking, a Lyapunov function is a scalar function defined on a region D that is continuous, positive definite, and has continuous first-rder partial derivatives at every point of D. A similar concept of this kind of function is appears in the theory of the stochastic differential equations where it is used to prove the global existence and uniqueness for stochastic boundedness of positive solution and discuss the existence and uniqueness of an ergodic stationary distribution of the positive solutions for a system of stochastic differential equations. Then we have the following lemma

Lemma 6. *[33] We said that the Markov process $Z(t)$ has a unique ergodic stationary distribution $\chi(.)$ if there is a bounded domain $E \subset \mathbb{R}^n$ with regular boundary Γ and the following properties hold:*

$(\mathbf{P_1})$: *There exists a positive number $\tilde{c}$ such that*

$$\sum_{i,j=1}^{n} b_{i,j}(z)\xi_i\xi_j \geq \tilde{c}|\xi|^2,\ z \in E,\ \xi \in \mathbb{R}^n.$$

$(\mathbf{P_2})$: *There is a nonnegative C^2-function denoted by V such that LV is negative for any $\mathbb{R}^n_+ \setminus E$.*

Define the following n-dimensional Euclidean space

$$\mathbb{R}^n_+ = \{U = (u_1, \ldots, u_n) \in \mathbb{R}^n : u_j > 0, \quad 1 \leq j \leq n\},$$

and

$$\overline{\mathbb{R}}^n_+ = \{U = (u_1, \ldots, u_n) \in \mathbb{R}^n : u_j \geq 0, \quad 1 \leq j \leq n\}.$$

And consider the n-dimensional Markov prosess take the following stochastic differential equation

$$dU(t) = f(U(t))dt + g(U(t))dW(t), \quad \text{for} \quad t > t_0, \tag{2.4.5}$$

where, $U(0) = U_0 \in \mathbb{R}^n$ is the initial value and $W(t)$ represents the n-dimensional standard Browrian motion defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t>0}, \mathbb{P})$. $f \in L^2(\mathbb{R}_+; \mathbb{R}^n), g \in L^2(\mathbb{R}_+; \mathbb{R}^{n\times m})$ are measurable functions. Denote by $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \overline{\mathbb{R}}_+)$ the family of all non-negative functions $V(U,t)$ defined on $\mathbb{R}^n \times \mathbb{R}_+$ such that they are continuously twice differentiable in U and one in t. The following lemma is due to the Itô's formula given in (5)

Lemma 7. *(Itô's formula [60]) If $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \overline{\mathbb{R}}_+)$, then $V(U(t),t)$ is again an Itô's process with the stochastic differential equation given as*

$$dV(U(t),t) = LV(U(t),t)dt + V_U(U(t),t)g(U(t))dW(t), \quad \textit{for} \quad t > t_0, \tag{2.4.6}$$

where L is the differential operator of Eq. (2.4.5) defined in [60] as

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i(U)\frac{\partial}{\partial U_i} + \frac{1}{2}\sum_{i,j=1}^{n}\left[g^T(U)g(U)\right]_{i,j}\frac{\partial^2}{\partial U_i \partial U_j}. \tag{2.4.7}$$

Then, we have

$$LV(U(t),t) = V_t(U(t),t) + V_U(U(t),t)f(U) + \frac{1}{2}trace\left[g^T(U)V_{UU}(U(t),t)g(U)\right], \tag{2.4.8}$$

where

$$V_t(U,t) = \frac{\partial V(U,t)}{\partial t}, \quad V_U(U,t) = \left(\frac{\partial V(U,t)}{\partial U_1}, \ldots, \frac{\partial V(U,t)}{\partial U_n}\right), \quad V_{UU}(U,t) = \left(\frac{\partial^2 V(U,t)}{\partial U_i \partial U_j}\right)_{n\times n}.$$

Chapter 3

Fractional stochastic differential equations

In this chapter, because the modeling of a great many problems in real situations is described by stochastic differential equations, rather than deterministic equations, it is of great importance to study fractional differential equations with stochastic effects. The remainder of this chapter is organized as follows: The presentation of the model to be studied and giving some auxiliary results from stochastic analysis and fractional calculus are gathered together in **Section 3.1**. In **Section 3.2**, we present results on the existence and continuous dependence of solutions on initial data. We end the paper with a transportation inequality of some classes of fractional stochastic differential equations.

3.1 Cauchy problem

Consider the following stochastic fractional differential equations:

$$\begin{cases} {}^cD^\alpha X_t & = g(t,X_t)dt + \sum_{l=1}^{\infty} f_l(t,X_t)dW_t^l, \quad t \in [0,\infty), \\ & \\ X_0 & = x \in \mathbb{R}, \end{cases} \tag{3.1.1}$$

where $\alpha \in (\frac{1}{2},1)$, $(f_l)_{l\in\mathbb{N}}, g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are given functions and $(W_t^i)_{i\in\mathbb{N}}$ is an infinite sequence of independent standard Brownian motions defined on a complete filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$, with $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions (i.e. right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets). An $\mathbb{R}-$valued random variable is an $\mathcal{F}_t-$measurable function $X_t : \Omega \to \mathbb{R}$ and the collection of random variables,

$$S = \{X(t,\omega) : \Omega \to \mathbb{R} |\ t \in [0,\infty)\},$$

is called a stochastic process. Generally, we just write X_t instead of $X(t,\omega)$.

Set

$$\begin{cases} f(\cdot,x) = (f_1(\cdot,x), f_2(\cdot,x), \ldots), \\ \|f(\cdot,x)\| = \Big(\sum_{l=1}^{\infty} f_i^2(\cdot,x)\Big)^{\frac{1}{2}} \end{cases} \tag{3.1.2}$$

where $f(\cdot,x) \in \ell^2$ for all $x \in \mathbb{R}$ and

$$\ell^2 = \{\phi = (\phi_l)_{l\geq 1} : \mathbb{R}_+ \to \mathbb{R} \quad : \|\phi(t)\|^2 = \sum_{l=1}^{\infty} |\phi_l(t)|^2 < \infty\}.$$

Some existence results of solutions for differential equations and inclusions with infinite Brownian or fractional Brownian motion were obtained in [6, 7, 16, 29, 56].

For each $t \in \mathbb{R}_+$, $\mathbb{L}^2(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ denote the space of all $\mathcal{F}_t-$measurable, mean square integrable functions $X : \Omega \to \mathbb{R}$, i.e.

$$\mathbb{E}\|X\|^2 < \infty, \quad for \quad all \quad X \in \mathbb{L}^2(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P}).$$

We shall write $\mathbb{L}^2(\mathcal{F}_t)$ instead of $\mathbb{L}^2(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$.

Definition 4. *A $(\mathcal{F}_t)_{t\geq 0}-$adapted process $X:\mathbb{R}_+\times\Omega\longrightarrow\mathbb{R}$ is called solution of equation (3.1.1) with initial condition $X_0=x\in\mathbb{R}$ if, for all $t\geq 0$, the following integral stochastic equation holds,*

$$X_t = x+\frac{1}{\Gamma(\alpha)}\sum_{l=1}^{\infty}\int_0^t (t-s)^{\alpha-1}f_l(s,X_s)dW_s^l+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s,X_s)ds,$$

where $\Gamma(\alpha):=\int_0^\infty s^{\alpha-1}e^{-s}ds$ is the Gamma function.

Let $T>0$. $\mathbb{H}_2$ stands for the Banach space of adapted processes X, equiped with the norm $\|\cdot\|_{\mathbb{H}_2}$ such that

$$\|X\|_{\mathbb{H}_2}=\sup_{t\in[0,T]}(E\|X_t\|^2)^{1/2}<\infty.$$

We define for all $\gamma>0$ the weighted norm $\|\cdot\|_\gamma$ by

$$\|X\|_\gamma := \sup_{t\in[0,T]}\sqrt{\frac{\mathbb{E}\|X_t\|^2}{E_{2\alpha-1}(\gamma t^{2\alpha-1})}} \quad for \quad all \quad X\in\mathbb{H}_2,$$

where $E_{2\alpha-1}(.)$ is the Mittag-Leffler function such that

$$E_{2\alpha-1}(t) := \sum_{k=1}^{\infty}\frac{t^k}{\Gamma((2\alpha-1)k+1)} \quad for \quad all \quad t\in\mathbb{R}.$$

For more details about the Mittag-Leffler functions, see [20]. Obviously, $(\mathbb{H}_2,\|\cdot\|_\gamma)$ is a Banach space, since the norms $\|\cdot\|_{\mathbb{H}_2}$ and $\|\cdot\|_\gamma$ are equivalent.

In the following we state standing hypotheses holding for the coefficients f and g in our model of this paper.

(H_1) There exists $K>0$ such that for all $x,y\in\mathbb{R}$ and $t\in[0,\infty)$

$$|f(t,x)-f(t,y)|+\|g(t,x)-g(t,y)\|\leq K\|x-y\|, \quad x,y\in\mathbb{R}$$

(H_2)

$$|f(\cdot,0)|_\infty := ess \sup_{s\in[0,\infty)}|f(s,0)|<\infty \quad and \quad \int_0^\infty \|g(s,0)\|^2 ds<\infty.$$

3.2 Existence,uniqueness and dependence on initial conditions

The following result is one of the elementary properties of square-integrable stochastic processes [55].

Lemma 8. *(Itô Isometry for Elementary Processes) Let $(\mathcal{F}_t)_{t\geq 0}$ satisfy the usual conditions and be generated by $(W_t^i)_{i\in\mathbb{N}^\star}$. Given two sequences of measurable $(\mathcal{F}_t)_{t\geq 0}-$adapted processes X_i and Y_i, set*

$$\begin{cases} M_t = \displaystyle\sum_{i=1}^{\infty}\int_0^t (t-s)^{\alpha-1}X_i(s)dW_s^i, \\ \\ N_t = \displaystyle\sum_{i=1}^{\infty}\int_0^t (t-s)^{\alpha-1}Y_i(s)dW_s^i. \end{cases}$$

If $\sum_{i=1}^{\infty}\|X_i\|_{\mathbb{H}_2}^2 < \infty$, then M is a continuous $\mathbb{L}^2(\mathcal{F}_t)-$martingale. The quadratic variation of M denoted by $[M]_t$ is

$$[M]_t = \int_0^t (t-s)^{2\alpha-2}|X(s)|^2 ds, \quad for \quad all \quad t\geq 0,$$

where $|X(s)|^2 = \sum_{i=1}^{\infty}X_i^2(s)$. And the cross variation of M and N, denoted by $[M,N]_t$, is

$$[M,N]_t = \sum_{i=1}^{\infty}\int_0^t (t-s)^{2\alpha-2}X_i(s)Y_i(s)ds, \quad for \quad all \quad t\geq 0.$$

Proof. Let $n\geq 1$, we put

$$M_t^n = \sum_{i=1}^{n}\int_0^t (t-s)^{\alpha-1}X_i(s)dW_s^i.$$

For all $t\geq 0$ we have

$$\begin{aligned}\mathbb{E}\sum_{i=1}^{n}\int_0^t (t-s)^{2\alpha-2}\|X_i(s)\|^2 ds &= \textstyle\int_0^t (t-s)^{2\alpha-2}\sum_{i=1}^{n}\mathbb{E}\|X_i(s)\|^2 ds\\ &= \textstyle\int_0^t J(s)\displaystyle\sum_{i=1}^{n}\left(\sqrt{\frac{\mathbb{E}\|X_i(s)\|^2}{E_{2\alpha-1}(\gamma s^{2\alpha-1})}}\right)^2 ds,\end{aligned}$$

where

$$J(s) = (t-s)^{2\alpha-2}E_{2\alpha-1}(\gamma s^{2\alpha-1})$$

Then, by the definition of $\|\cdot\|_\gamma$, we have

$$\mathbb{E}\sum_{i=1}^{n}\int_0^t (t-s)^{2\alpha-2}\|X_i(s)\|^2 ds \leq \sum_{i=1}^{n}\|X_i\|_\gamma^2\int_0^t (t-s)^{2\alpha-2}E_{2\alpha-1}(\gamma s^{2\alpha-1})ds.$$

Then Lemma 2.1 implies that

$$\mathbb{E}\sum_{i=1}^{n}\int_0^t (t-s)^{2\alpha-2}\|X_i(s)\|^2 ds \leq \frac{\Gamma(2\alpha-1)E_{2\alpha-1}(\gamma T^{2\alpha-1})}{\gamma}\sum_{i=1}^{n}\|X_i\|_\gamma^2.$$

Choose and fix a positive constant γ such that

$$\gamma = \Gamma(2\alpha-1)E_{2\alpha-1}(\gamma T^{2\alpha-1}).$$

Then

$$\mathbb{E}\sum_{i=1}^{n}\int_0^t (t-s)^{2\alpha-2}\|X_i(s)\|^2 ds \leq \sum_{i=1}^{n}\|X_i\|_\gamma^2.$$

Since $\sum_{i=1}^{n}\|X_i\|_\gamma^2 < \infty$ for all $n \geq 1$, then

$$\mathbb{E}\sum_{i=1}^{n}\int_0^t (t-s)^{2\alpha-2}\|X_i(s)\|^2 ds < \infty.$$

Consequently, M^n is a continuous $\mathbb{L}^2(\mathcal{F}_t)-$martingale for all $n \geq 1$, and its quadratic variation is $[M^n]_t$ such that

$$[M^n]_t = \int_0^t \sum_{i=1}^{n}(t-s)^{2\alpha-2}X_i^2(s)ds.$$

By Burkholder-Davis-Gundy inequality [11, 12], we have, for some positive real constant C,

$$\mathbb{E}\sup_{s\in[0,t]}(M_s^n - M_s^m)^2 \le C\int_0^t (t-s)^{2\alpha-2}\sum_{i=n\wedge m+1}^{n\vee m}\mathbb{E}\|X_i(s)\|^2 ds.$$

By the definition of $\|\cdot\|_\gamma$ and Lemma 8, we obtain that

$$\mathbb{E}\sup_{s\in[0,t]}(M_s^n - M_s^m)^2 \le \frac{C\Gamma(2\alpha-1)E_{2\alpha-1}(\gamma T^{2\alpha-1})}{\gamma}\sum_{i=n\wedge m+1}^{n\vee m}\|X_i\|_\gamma^2.$$

Choose and fix a positive constant γ such that

$$\gamma = C\Gamma(2\alpha-1)E_{2\alpha-1}(\gamma T^{2\alpha-1}).$$

Then

$$\mathbb{E}\sup_{s\in[0,t]}(M_s^n - M_s^m)^2 \le \sum_{i=n\wedge m+1}^{n\vee m}\|X_i\|_\gamma^2.$$

Since $\sum_{i=1}^{\infty}\|X_i\|_\gamma^2 < \infty$, then we have

$$\mathbb{E}\sup_{s\in[0,t]}(M_s^n - M_s^m)^2 \le \sum_{i=n\wedge m+1}^{n\vee m}\|X_i\|_\gamma^2 \longrightarrow 0.$$

as $n,m \longrightarrow \infty$, where $n\wedge m = \min(n,m)$ and $n\vee m = \max(n,m)$, and so M^n is a Cauchy sequence with respect to the norm $(\mathbb{E}\sup_{t\in[0,T]}(\cdot)^2)^{\frac{1}{2}}$ for any bounded time interval $[0,T]$. Denote its limit by M. Consequently, by the continuity of M^n, we obtain

$$\lim_{n\longrightarrow\infty}\mathbb{E}(\mathbb{E}(M_t^n/\mathcal{F}_s) - \mathbb{E}(M_t/\mathcal{F}_s))^2 = 0, \quad for \quad all \quad s<t,$$

and

$$\lim_{n\longrightarrow\infty}\mathbb{E}(M_s^n - M_s)^2 = 0.$$

Since $\mathbb{E}(M_t^n/\mathcal{F}_s) = M_s^n$ for all $s<t$ and $n\ge 1$, by the two previous limits, we have

$$\mathbb{E}(M_t/\mathcal{F}_s) = M_s, \quad for \quad all \quad s<t.$$

Hence, M is a continuous $\mathbb{L}^2(\mathcal{F}_t)-$martingale. Moreover $[M^n]_t$ converges to $[M]_t$ as $n \longrightarrow \infty$ in probability, for all $t \geq 0$, i.e.

$$[M]_t = \int_0^t (t-s)^{2\alpha-2}|X(s)|^2 ds, \quad where \quad |X(s)|^2 = \sum_{i=1}^{\infty} X_i^2(s).$$

Similarly, the cross variation of M and N for all $t \geq 0$ is

$$[M,N]_t = \sum_{i=1}^{\infty} \int_0^t (t-s)^{2\alpha-2} X_i(s) Y_i(s) ds.$$

□

Now, we define the operator L on $\mathbb{H}_2$ by

$$(LX)(t) = x + \frac{1}{\Gamma(\alpha)} \sum_{l=1}^{\infty} \int_0^t (t-s)^{\alpha-1} f_l(s, X_s) dW_s^l + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, X_s) ds.$$

Lemma 9. *The operator L is well-defined on $\mathbb{H}_2([0,T])$.*

Proof. Let $X \in \mathbb{H}_2$, then for all $t \in [0,T]$, we get

$$\begin{aligned} \mathbb{E}\|(LX)(t)\|^2 &\leq 3\mathbb{E}\|x\|^2 + \tfrac{3}{\Gamma^2(\alpha)}\mathbb{E}\left\|\textstyle\sum_{l=1}^{\infty}\int_0^t (t-s)^{\alpha-1} f_l(s,X_s) dW_s^l\right\|^2 \\ &\quad + \tfrac{3}{\Gamma^2(\alpha)}\mathbb{E}\left\|\textstyle\int_0^t (t-s)^{\alpha-1} g(s,X_s) ds\right\|^2. \end{aligned}$$

By Lemma 8 and Hölder's inequality, we obtain

$$\begin{aligned} \mathbb{E}\|(LX)(t)\|^2 &\leq 3\mathbb{E}\|x\|^2 + \tfrac{3}{\Gamma^2(\alpha)}\mathbb{E}\textstyle\int_0^t (t-s)^{2\alpha-2}|f(s,X_s)|^2 ds \\ &\quad + \tfrac{3}{\Gamma^2(\alpha)}\mathbb{E}\left\|\textstyle\int_0^t (t-s)^{\alpha-1} g(s,X_s) ds\right\|^2 \\ &\leq 3\mathbb{E}\|x\|^2 + \tfrac{3}{\Gamma^2(\alpha)}\mathbb{E}\textstyle\int_0^t (t-s)^{2\alpha-2}|f(s,X_s)|^2 ds \\ &\quad + \tfrac{3t^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\mathbb{E}\textstyle\int_0^t \|g(s,X_s)\|^2 ds. \end{aligned}$$

From (H_1) and (H_2), we derive

$$\begin{aligned} \mathbb{E}\|(LX)(t)\|^2 &\leq 3\mathbb{E}\|x\|^2 + \tfrac{3}{\Gamma^2(\alpha)}\mathbb{E}\textstyle\int_0^t (t-s)^{2\alpha-2} 2(|f(s,X_s)-f(s,0)|^2 + |f(s,0)|^2) ds \\ &\quad + \tfrac{3t^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\mathbb{E}\textstyle\int_0^t 2(\|g(s,X_s)-g(s,0)\|^2 + \|g(s,0)\|^2) ds \\ &\leq 3\mathbb{E}\|x\|^2 + \tfrac{3}{\Gamma^2(\alpha)}\mathbb{E}\textstyle\int_0^t (t-s)^{2\alpha-2} 2(K^2\|X_s\|^2 + |f(.,0)|_\infty^2) ds \\ &\quad + \tfrac{3t^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\mathbb{E}\textstyle\int_0^t 2(K^2\|X_s\|^2 + \|g(s,0)\|^2) ds \\ &\leq 3\mathbb{E}\|x\|^2 + \tfrac{6t^{2\alpha-1}|f(.,0)|_\infty^2}{(2\alpha-1)\Gamma^2(\alpha)} + \tfrac{6t^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\textstyle\int_0^t \|g(s,0)\|^2 ds \\ &\quad + \tfrac{6K^2t^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)}\|X\|_{\mathbb{H}_2}^2 + \tfrac{6K^2t^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\|X\|_{\mathbb{H}_2}^2. \end{aligned}$$

Moreover,

$$\begin{aligned}\sup_{t\in[0,T]} \mathbb{E}\|(LX)(t)\|^2 &\leq 3\mathbb{E}\|x\|^2 + \frac{6T^{2\alpha-1}|f(.,0)|_\infty^2}{(2\alpha-1)\Gamma^2(\alpha)} + \frac{6T^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\int_0^T \|g(s,0)\|^2 ds \\ &\quad + \left(\frac{6K^2T^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)} + \frac{6K^2T^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\right)\|X\|_{\mathbb{H}_2}^2.\end{aligned}$$

Therefore, $\|LX\|_{\mathbb{H}_2} < \infty$. Hence, the operator L is well-defined. □

Theorem 7. *Assume that* (H_1) *and* (H_2) *hold. Then the problem (1.1) has a unique global solution on* $[0,\infty)$.

Proof. We show that, for every $T > 0$, the operator L is a contractive map with respect to some Bielecki-type norm on $\mathbb{H}_2$ which will be defined later. Let $X, Y \in \mathbb{H}_2$ and $t \in [0,T]$. Then

$$\begin{aligned}\mathbb{E}\|(LX)(t)-(LY)(t)\|^2 &= \mathbb{E}\Bigg\|\frac{1}{\Gamma(\alpha)}\sum_{l=1}^{\infty}\int_0^t (t-s)^{\alpha-1}(f_l(s,X_s)-f_l(s,Y_s))dW_s^l \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(g(s,X_s)-g(s,Y_s))ds\Bigg\|^2 \\ &\leq \frac{2}{\Gamma^2(\alpha)}\mathbb{E}\left\|\sum_{l=1}^{\infty}\int_0^t (t-s)^{\alpha-1}(f_l(s,X_s)-f_l(s,Y_s))dW_s^l\right\|^2 \\ &\quad + \frac{2}{\Gamma^2(\alpha)}\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1}(g(s,X_s)-g(s,Y_s))ds\right\|^2.\end{aligned}$$

By Lemma 8 and Hölder's inequality, we obtain

$$\begin{aligned}\mathbb{E}\|(LX)(t)-(LY)(t)\|^2 &\leq \tfrac{2}{\Gamma^2(\alpha)}\textstyle\int_0^t (t-s)^{2\alpha-2}\mathbb{E}\sum_{l=1}^{\infty}(f_l(s,X_s)-f_l(s,Y_s))^2 ds \\ &\quad + \tfrac{2t}{\Gamma^2(\alpha)}\textstyle\int_0^t (t-s)^{2\alpha-2}\mathbb{E}\|g(s,X_s)-g(s,Y_s)\|^2 ds \\ &\leq \tfrac{2}{\Gamma^2(\alpha)}\textstyle\int_0^t (t-s)^{2\alpha-2}\mathbb{E}|f(s,X_s)-f(s,Y_s)|^2 ds \\ &\quad + \tfrac{2t}{\Gamma^2(\alpha)}\textstyle\int_0^t (t-s)^{2\alpha-2}\mathbb{E}\|g(s,X_s)-g(s,Y_s)\|^2 ds.\end{aligned}$$

From (H_1), we derive

$$\begin{aligned}\mathbb{E}\|(LX)(t)-(LY)(t)\|^2 &\leq \tfrac{2K^2}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}\mathbb{E}\|X_s-Y_s\|^2 ds \\ &\quad +\tfrac{2tK^2}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}\mathbb{E}\|X_s-Y_s\|^2 ds \\ &= \tfrac{2K^2}{\Gamma^2(\alpha)}(t+1)\int_0^t (t-s)^{2\alpha-2}\mathbb{E}\|X_s-Y_s\|^2 ds.\end{aligned}$$

Moreover

$$\begin{aligned}\tfrac{\mathbb{E}\|(LX)(t)-(LY)(t)\|^2}{E_{2\alpha-1}(\gamma t^{2\alpha-1})} &\leq \tfrac{2(t+1)K^2}{E_{2\alpha-1}(\gamma t^{2\alpha-1})\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}E_{2\alpha-1}(\gamma s^{2\alpha-1})\times \\ &\quad \left(\sup_{s\in[0,T]}\sqrt{\frac{\mathbb{E}\|X_s-Y_s\|^2}{E_{2\alpha-1}(\gamma s^{2\alpha-1})}}\right)^2 ds.\end{aligned}$$

If we choose $\|\cdot\| = \|\cdot\|_\gamma$ for the Bielecki-type norm on $\mathbb{H}_2$, then definition of $\|\cdot\|_\gamma$ and the Lemma 8 imply that

$$\begin{aligned}\tfrac{\mathbb{E}\|(LX)(t)-(LY)(t)\|^2}{E_{2\alpha-1}(\gamma t^{2\alpha-1})} &\leq \tfrac{2(t+1)K^2}{\Gamma^2(\alpha)}\|X-Y\|_\gamma^2\left(\tfrac{\int_0^t (t-s)^{2\alpha-2}E_{2\alpha-1}(\gamma s^{2\alpha-1})ds}{E_{2\alpha-1}(\gamma t^{2\alpha-1})}\right) \\ &\leq \tfrac{2(t+1)K^2\Gamma(2\alpha-1)}{\gamma\Gamma^2(\alpha)}\|X-Y\|_\gamma^2.\end{aligned}$$

In particular,

$$\|L(X)-L(Y)\|_\gamma \leq \lambda\|X-Y\|_\gamma, \quad \textit{where} \quad \lambda = \sqrt{\frac{2(T+1)K^2\Gamma(2\alpha-1)}{\gamma\Gamma^2(\alpha)}}.$$

Choose and fix a positive constant γ such that

$$\gamma > \frac{3(T+1)K^2\Gamma(2\alpha-1)}{\Gamma^2(\alpha)}.$$

Then $\lambda < 1$, and therefore L is a contraction mapping. According to the Banach fixed point theorem, the unique fixed point of this map is the unique solution on $\mathbb{H}_2$ of the problem (3.1.1). □

We are now in the position to prove the continuous dependence of solutions on the initial data on bounded intervals for the problem (3.1.1).

Theorem 8. *Assume that* (H_1) *holds. Then for any bounded time interval* $[0,T]$ *the solution of problem* (3.1.1) *depends continuously on* x*, i.e.*

$$\lim_{x \longrightarrow \eta} \|X^x - X^\eta\|_{\mathbb{H}_2} = 0.$$

Proof. Fix $T>0$ and $x,\eta \in \mathbb{R}$. Let X_t^x and X_t^η be two solutions of problem (3.1.1), i.e.

$$X_t^x = x + \frac{1}{\Gamma(\alpha)} \sum_{l=1}^{\infty} \int_0^t (t-s)^{\alpha-1} f_l(s, X_s^x) dW_s^l + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, X_s^x) ds,$$

$$X_t^\eta = \eta + \frac{1}{\Gamma(\alpha)} \sum_{l=1}^{\infty} \int_0^t (t-s)^{\alpha-1} f_l(s, X_s^\eta) dW_s^l + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, X_s^\eta) ds.$$

It follows that

$$\begin{aligned}\mathbb{E}\|X_t^x - X_t^\eta\|^2 &= \mathbb{E}\Big\|x-\eta+\tfrac{1}{\Gamma(\alpha)}\textstyle\sum_{l=1}^{\infty}\int_0^t (t-s)^{\alpha-1}(f_l(s,X_s^x)-f_l(s,X_s^\eta))dW_s^l \\ &\quad +\tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t (t-s)^{\alpha-1}(g(s,X_s^x)-g(s,X_s^\eta))ds\Big\|^2.\end{aligned}$$

Then

$$\begin{aligned}\mathbb{E}\|X_t^x - X_t^\eta\|^2 &\le 3\mathbb{E}\|x-\eta\|^2 \\ &\quad +\tfrac{3}{\Gamma^2(\alpha)}\mathbb{E}\Big\|\textstyle\sum_{l=1}^{\infty}\int_0^t (t-s)^{\alpha-1}(f_l(s,X_s^x)-f_l(s,X_s^\eta))dW_s^l\Big\|^2 \\ &\quad +\tfrac{3}{\Gamma^2(\alpha)}\mathbb{E}\Big\|\textstyle\int_0^t (t-s)^{\alpha-1}(g(s,X_s^x)-g(s,X_s^\eta))ds\Big\|^2.\end{aligned}$$

By Lemma 8 and Hölder's inequality, from (H_1), we get

$$\mathbb{E}\|X_t^x - X_t^\eta\|^2 \le 3\mathbb{E}\|x-\eta\|^2 + \frac{3K^2}{\Gamma^2(\alpha)}(t+1)\int_0^t (t-s)^{2\alpha-2}\mathbb{E}\|X_s^x - X_s^\eta\|^2 ds.$$

By the definition of $\|\cdot\|_\gamma$, we have

$$\frac{\mathbb{E}\|X_t^x - X_t^\eta\|^2}{E_{2\alpha-1}(\gamma t^{2\alpha-1})} \le 3\mathbb{E}\|x-\eta\|^2 + \frac{3(t+1)K^2}{\Gamma^2(\alpha)}\|X^x - X^\eta\|_\gamma^2 \left(\frac{\int_0^t (t-s)^{2\alpha-2}E_{2\alpha-1}(\gamma s^{2\alpha-1})ds}{E_{2\alpha-1}(\gamma t^{2\alpha-1})}\right).$$

Therefore, using Lemma 8, we obtain

$$\|X^x - X^\eta\|_\gamma^2 \le 3\mathbb{E}\|x-\eta\|^2 + \frac{3(T+1)K^2\Gamma(2\alpha-1)}{\gamma\Gamma^2(\alpha)}\|X^x - X^\eta\|_\gamma^2.$$

Since $\gamma > \frac{3(T+1)K^2\Gamma(2\alpha-1)}{\Gamma^2(\alpha)}$, then we have

$$\left(1-\frac{3(T+1)K^2\Gamma(2\alpha-1)}{\gamma\Gamma^2(\alpha)}\right)\|X^x-X^\eta\|_\gamma^2 \leq 3\mathbb{E}\|x-\eta\|^2.$$

We conclude

$$\lim_{x\longrightarrow\eta}\|X^x-X^\eta\|_{\mathbb{H}_2}=0.$$

The proof is complete. □

3.2.1 Transportation inequality

Let $(X,\mathcal{A})$, $(Y,\mathcal{Y})$ be a two measurable spaces, where $\mathcal{A}$ and $\mathcal{Y}$ is any given Borel σ-algebras, respectively on X and Y. We denote by $\mathcal{P}(X)$, $\mathcal{P}(Y)$ and $\mathcal{P}(X\times Y)$ the spaces of probability measures, respectively on X, Y and $X\times Y$. Before starting this study, we have to make clear what a way of transportation, or a transport plan (in the probability literature it is also called the coupling between probability measures). It is the work required to move mass from a location A on X ($A\in\mathcal{A}$) to a location B on Y ($B\in\mathcal{Y}$) taking into account the transportation cost as a Borel function $c: X\times Y \longrightarrow [0,\infty]$ which tells us the cost of such transport. Hence Kantorovich's formulation of the optimal transport problem asks to find

$$\inf_{\pi\in\Theta(\mu,\nu)}\left(\int_{X\times Y} c(x,y)d\pi(x,y)\right),$$

where $\Theta(\mu,\nu)$ is the set of all probability measures on the product space $X\times Y$ with marginals μ on X and ν on Y. More explicitly, given $\mu\in\mathcal{P}_p(X)$ and $\nu\in\mathcal{P}_p(Y)$ then

$$\Theta(\mu,\nu)=\{\pi\in\mathcal{P}(X\times Y):\pi(A\times Y)=\mu(A),\quad \pi(X\times B)=\nu(B)\quad \forall (A,B)\in\mathcal{A}\times\mathcal{Y}\}.$$

If $\Theta(\mu,\nu)$ is nonempty, convex set then, $\pi\in\Theta(\mu,\nu)$ if and only if it is a nonnegative measure on $X\times Y$ such that, for all measurable functions $(\varphi,\psi)\in L^1(d\mu)\times L^1(d\nu)$ (or in $L^\infty(d\mu)\times L^\infty(d\nu)$),

$$\int_{X\times Y}(\varphi(x)+\psi(y))d\pi(x,y)=\int_X(\varphi d\mu+\int_Y\psi d\nu.$$

And Kantorovich's optimal transportation problem has a dual formulation is given by the following theorem

Theorem 9. *(Kantorovich duality [80]) Let X and Y be Polish spaces, let $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$, and let $c : X \times Y \longrightarrow [0,\infty]$ be a lower semi-continuous cost function. Whenever $\pi \in \Theta(\mu,\nu)$ and $(\varphi,\psi) \in L^1(d\mu) \times L^1(d\nu)$, define*

$$I[\pi] = \int_{X\times Y} c(x,y)d\pi(x,y), \quad J(\varphi,\psi) = \int_X \varphi d\mu + \int_Y \psi d\nu.$$

Define Φ_c to be the set of all measurable functions $(\varphi,\psi) \in L^1(d\mu) \times L^1(d\nu)$ satisfying

$$\varphi(x) + \psi(y) \leq c(x,y),$$

for μ-almost all $x \in X$, ν-almost all $y \in Y$.

Then

$$\inf_{\pi\in\Theta(\mu,\nu)} I[\pi] = \sup_{\Phi_c} J(\varphi,\psi).$$

Note that, a topological space (X,τ) is said to be Polish if there exists a distance d on X inducing τ such that (X,d) is complete and separable.

Let (X,d) be a Polish metric space. We denote by $\mathcal{P}_p(X)$ the space of probability measures with finite moments of order p, i.e.

$$\mathcal{P}_p(X) = \{\mu \in \mathcal{P}(X) : \int_X d(x_0,x)^p d\mu(dx) < \infty \quad for \quad some \quad (and \quad thus \quad for \quad all) \quad x_0 \in X\}.$$

For every $p \in [1,\infty)$ we define the Wasserstein distance $W_p : \mathcal{P}_p(X) \times \mathcal{P}_p(X) \to \mathbb{R}_+$ by

$$W_p(\mu,\nu) = \inf_{\pi\in\prod(\mu,\nu)} \left(\int_{X\times X} d(x,y)^p d\pi(x,y)\right)^{\frac{1}{p}},$$

where $\prod(\mu,\nu)$ is the set of all probability measures on the product space $X \times X$ with marginals μ and ν. If (X,d) be a complete separable metric space, then $(\mathcal{P}_p(X), W_p)$ is also a complete separable metric space. For investigation of the Monge-Kantorovich optimal transportation problem, this distance plays an important role in the minimal cost to transport distribution μ into ν at the cost rate (cost function) d. Proprieties and some applications of the Wasserstein distance can be find in the important contribution by Ambrosio *et al.* [4] and Villani [80].

The Wasserstein distance exponent 1, is given by

$$W_1(\mu,\nu) = \inf_{\pi \in \prod(\mu,\nu)} \left(\int_{X \times X} d(x,y) d\pi(x,y) \right),$$

will be called the Kantorovich-Rubinstein distance. Its can also be defined in an alternative way by the following Kantorovich-Rubinstein duality formula,

$$W_1(\mu,\nu) = \sup_{\|\psi\|_{Lip} \leq 1} \left\{ \int_X \psi d\mu - \int_X \psi d\nu \right\},$$

where $\|\psi\|_{Lip}$ is the best admissible Lipschitz constant of a Lipschitz function ψ on X.

For $p=2$, the relative entropy of the probability measure ν with respect to μ is defined by

$$H(\mu|\nu) = \begin{cases} \int_X \ln \frac{d\nu}{d\mu} d\nu, & \nu \ll \mu, \\ \\ \infty, & \text{otherwise.} \end{cases}$$

Definition 5. *Given probability measure μ, if there exists $C > 0$ such that for every probability measure ν,*

$$W_2(\mu,\nu) \leq \sqrt{CH(\mu|\nu)},$$

then we say μ satisfies the transportation and entropy inequality.

Finally, we give some property of the space $(\mathcal{P}_2(X), W_2)$, it is as

1. If X is a Polish space endowed with a distance d, then $(\mathcal{P}_p(X), W_p)$ is separable.

2. The space $(\mathcal{P}_p(X), W_p)$ is compact, if (X,d) is compact.

3. Let μ be a probability measure on a Hilbert space X and let a be an element of X. Then

$$W_2(\mu,\delta_a)^2 = \int_X \|x-a\|^2 d\mu(x),$$

where δ_a is the dirac measure on $\mathcal{P}_p(X)$. In particular, the mean of μ, which is defined as $m = \int_X x d\mu(x)$, is the unique solution of the minimization problem

$$\inf_{a\in X} W_2(\mu,\delta_a),$$

and the corresponding cost is just the variance of μ.

In 1996 Talagrand [77] estimated the transportation distance (or Wasserstein distance) with a quadratic cast of the standard Guassian measure by the entropy functional. Transportation-cost inequalities have been recently deeply studied, because of their connection between the concentrations of measure phenomenon, or for deviation inequalities for Markov processes [30, 49]. The Talagrand inequality was generalized by Otto and Villani [58].

By means of Girsanov's formula, Djellout *et al.* [22] obtained a direct proof of Talagrand's transportation inequality for the law of a diffusion process. This idea was used for stochastic differential equations [5, 8, 9, 10, 50].

Now, we will establish the transport inequality for the solution of the following problem:

$$\begin{cases} {}^cD^\alpha X_t &= f(t,X_t)dW_t + g(t,X_t)dt, \quad t\in[0,T], \\ \\ X_0 &= x \in \mathbb{R}, \end{cases} \tag{3.2.1}$$

where $\alpha \in (\frac{1}{2},1)$, $f,g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Theorem 10. *Assume that the conditions (H_1) and that there exists $M>0$ such that*

$$|f(t,x)| \le M, \quad \textit{for all } (t,x)\in[0,T]\times\mathbb{R},$$

hold, and let $\mathbb{P}_x$ be a law of the proesses $X_t(x,\cdot)$ solution of the problem (3.2.1). Then

$$W_2(\mathbb{P}_x,\mathbb{Q}) \le \sqrt{2CH(\mathbb{P}|\mathbb{Q})}.$$

Proof. Let $\mathbb{Q} \in \mathcal{P}(C([0,T],\mathbb{R}))$ such that $\mathbb{Q} \ll \mathbb{P}_x$. Consider

$$\widehat{\mathbb{Q}} = \frac{d\mathbb{Q}}{d\mathbb{P}_x}(X_\cdot(x,\cdot))\mathbb{P}.$$

Then

$$\begin{aligned}
H(\widehat{\mathbb{Q}}|\mathbb{P}) &= \int_\Omega \ln\left(\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}\right) d\widehat{\mathbb{Q}} \\
&= \int_\Omega \ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}_x}(X_\cdot(x,\cdot))\right)\frac{d\mathbb{Q}}{d\mathbb{P}_x}(X_\cdot(x,\cdot))d\mathbb{P} \\
&= \int_{C([0,T],\mathbb{R})} \ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}_x}\right)\frac{d\mathbb{Q}}{d\mathbb{P}_x}d\mathbb{P}_x \\
&= \int_{C([0,T],\mathbb{R})} \ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}_x}\right) d\mathbb{Q} \\
&= H(\mathbb{Q}|\mathbb{P}_x).
\end{aligned}$$

As in [22], there exists $\widehat{f} \in L^2([0,T],\mathbb{R})$ with $\int_0^T |\widehat{f}(s)|^2 ds < \infty$, $\mathbb{P}$-almost surely, such that

$$H(\widehat{\mathbb{Q}}|\mathbb{P}) = H(\mathbb{Q}|\mathbb{P}_x) = \frac{1}{2}\mathbb{E}_{\widehat{\mathbb{Q}}}\left(\int_0^T |\widehat{f}(s)|^2 ds\right).$$

By the Girsanov theorem, the following process $\widehat{W}_t$ defined by

$$\widehat{W}_t = W_t - \int_0^t \widehat{f}(s)ds$$

is a Brownian motion with respect the filtration $(\mathcal{F}_t)_{t\geq 0}$ on the probability space $(\Omega,\mathcal{F},\widehat{\mathbb{Q}})$. We consider the following problem for the fractional stochastic differential equation

$$\begin{cases} {}^cD^\alpha Y_t &= f(t,Y_t)d\widehat{W}_t + g(t,Y_t)dt, \quad t \in [0,T], \\ \\ Y_0 &= x \in \mathbb{R}. \end{cases} \tag{3.2.2}$$

From Theorem 7 there exists a unique solution $Y \in \mathbb{H}_2([0,T])$ such that

$$Y_t = x + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,Y_s)d\widehat{W}_s + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s,Y_s)ds.$$

Under $\widehat{\mathbb{Q}}$, the law of $(Y_t)_{t\in[0,T]}$ is exactly $\mathbb{P}_x$. Hence (X,Y), under $\widehat{\mathbb{Q}}$, is a coupling of $(\mathbb{Q},\mathbb{P}_x)$. This implies that

$$W_2(\mathbb{Q},\mathbb{P}_x)^2 \leq \mathbb{E}_{\widehat{\mathbb{Q}}}\|X-Y\|_\infty^2.$$

Let $t\in[0,T]$. Then

$$\begin{aligned}
\mathbb{E}_{\widehat{\mathbb{Q}}}|X_t-Y_t|^2 &= \mathbb{E}_{\widehat{\mathbb{Q}}}\left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(f(s,X_s)-f(s,Y_s))dW_s\right. \\
&\quad +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,Y_s)\widehat{f}(s)ds \\
&\quad \left.+\int_0^t (t-s)^{\alpha-1}(g(s,X_s)-g(s,Y_s))ds\right|^2 \\
&\leq \frac{3}{\Gamma^2(\alpha)}\mathbb{E}_{\widehat{\mathbb{Q}}}\left|\int_0^t (t-s)^{\alpha-1}(f(s,X_s)-f(s,Y_s))dW_s\right|^2 \\
&\quad +\frac{3}{\Gamma^2(\alpha)}\mathbb{E}_{\widehat{\mathbb{Q}}}\left|\int_0^t (t-s)^{\alpha-1} f(s,Y_s)\widehat{f}(s)ds\right|^2 \\
&\quad +\frac{3}{\Gamma^2(\alpha)}\mathbb{E}_{\widehat{\mathbb{Q}}}\left|\int_0^t (t-s)^{\alpha-1}(g(s,X_s)-g(s,Y_s))ds\right|^2.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbb{E}_{\widehat{\mathbb{Q}}}|X_t-Y_t|^2 &\leq \frac{3K^2}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}\mathbb{E}_{\widehat{\mathbb{Q}}}|X_s-Y_s|^2 ds \\
&\quad +\frac{3}{\Gamma^2(\alpha)}\mathbb{E}_{\widehat{\mathbb{Q}}}\left|\int_0^t (t-s)^{\alpha-1} f(s,Y_s)\widehat{f}(s)ds\right|^2 \\
&\quad +\frac{3}{\Gamma^2(\alpha)}\mathbb{E}_{\widehat{\mathbb{Q}}}\left|\int_0^t (t-s)^{\alpha-1}(g(s,X_s)-g(s,Y_s))ds\right|^2.
\end{aligned}$$

It follows from Hölder's inequality,

$$\begin{aligned}\mathbb{E}_{\widehat{\mathbb{Q}}}|X_t-Y_t|^2 &\le \frac{3K^2}{\Gamma^2(\alpha)}\int_0^t(t-s)^{2\alpha-2}\mathbb{E}_{\widehat{\mathbb{Q}}}|X_s-Y_s|^2ds\\ &\quad+\frac{3}{\Gamma^2(\alpha)}\mathbb{E}_{\widehat{\mathbb{Q}}}\left|\int_0^t(t-s)^{\alpha-1}f(s,Y_s)\widehat{f}(s)ds\right|^2\\ &\quad+\frac{3K^2}{\Gamma^2(\alpha)}\int_0^t(t-s)^{2\alpha-2}ds\int_0^t\mathbb{E}_{\widehat{\mathbb{Q}}}|X_s-Y_s|^2ds\\ &\le \frac{3K^2}{\Gamma^2(\alpha)}\int_0^t(t-s)^{2\alpha-2}\mathbb{E}_{\widehat{\mathbb{Q}}}|X_s-Y_s|^2ds\\ &\quad+\frac{3K^2M^2}{\Gamma^2(\alpha)}\int_0^t(t-s)^{2\alpha-2}ds\mathbb{E}_{\widehat{\mathbb{Q}}}\|\widehat{f}\|^2_{L^2}\\ &\quad+\frac{3K^2}{\Gamma^2(\alpha)}\int_0^t(t-s)^{2\alpha-2}ds\int_0^t\mathbb{E}_{\widehat{\mathbb{Q}}}|X_s-Y_s|^2ds.\end{aligned}$$

Then

$$\begin{aligned}\mathbb{E}_{\widehat{\mathbb{Q}}}|X_t-Y_t|^2 &\le \frac{3K^2}{\Gamma^2(\alpha)}\int_0^t(t-s)^{2\alpha-2}\mathbb{E}_{\widehat{\mathbb{Q}}}|X_s-Y_s|^2ds\\ &\quad+\frac{3K^2M^2}{(2\alpha-1)\Gamma^2(\alpha)}t^{2\alpha-1}\mathbb{E}_{\widehat{\mathbb{Q}}}\|\widehat{f}\|^2_{L^2}\\ &\quad+\frac{3K^2}{\Gamma^2(\alpha)}\int_0^t(t-s)^{2\alpha-2}ds\int_0^t\mathbb{E}_{\widehat{\mathbb{Q}}}|X_s-Y_s|^2ds.\end{aligned}$$

Further, we have

$$\begin{aligned}\mathbb{E}_{\widehat{\mathbb{Q}}}|X_t-Y_t|^2 &\le \frac{3K^2M^2T^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\mathbb{E}_{\widehat{\mathbb{Q}}}\|\widehat{f}\|^2_{L^2}+\frac{3K^2T^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\int_0^t\mathbb{E}_{\widehat{\mathbb{Q}}}|X_s-Y_s|^2ds\\ &\quad+\frac{3K^2}{\Gamma^2(\alpha)}\int_0^t(t-s)^{2\alpha-2}\mathbb{E}_{\widehat{\mathbb{Q}}}|X_s-Y_s|^2ds.\end{aligned}$$

Let

$$\mathcal{V}(t)=C_1+C_2\int_0^t\mathbb{E}_{\widehat{\mathbb{Q}}}|X_s-Y_s|^2ds,\quad t\in[0,b].$$

Then

$$\mathcal{V}'(t)=C_2\mathbb{E}_{\widehat{\mathbb{Q}}}|X_t-Y_t|^2,\quad \mathcal{V}(0)=C_1$$

and

$$\mathbb{E}_{\widehat{\mathbb{Q}}}|X_t-Y_t|^2\le\mathcal{V}(t)+C_3\int_0^t(t-s)^{\beta-1}\mathbb{E}_{\widehat{\mathbb{Q}}}|X_s-Y_s|^2ds,$$

where

$$C_1 = \frac{3K^2M^2T^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\mathbb{E}_{\widehat{\mathbb{Q}}}\|\widehat{f}\|_{L^2}^2,\ C_2 = \frac{3K^2T^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)},\ C_3 = \frac{3K^2}{\Gamma^2(\alpha)},\ \beta = 2\alpha-1.$$

Furthermore, by Lemma 2

$$\begin{aligned}\mathbb{E}_{\widehat{\mathbb{Q}}}|X_t-Y_t|^2 &\leq \mathcal{V}(t)+\int_0^t \sum_{n=1}^{\infty}\frac{(C_3\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}\mathcal{V}(s)ds\\ &\leq \mathcal{V}(t)+\int_0^t \sum_{n=1}^{\infty}\frac{(C_3\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}\mathcal{V}(t)ds.\end{aligned}$$

Therefore

$$\mathbb{E}_{\widehat{\mathbb{Q}}}|X_t-Y_t|^2 \leq \left[1+\sum_{n=1}^{\infty}\frac{(C_3\Gamma(\alpha)T^{\beta})^n}{\Gamma(n\beta+1)}\right]\mathcal{V}(t).$$

Then

$$\mathbb{E}_{\widehat{\mathbb{Q}}}|X_t-Y_t|^2 \leq C_1E(C_3\Gamma(\beta)T^{\alpha})+C_2E(C_3\Gamma(\beta)T^{\alpha})\int_0^t \mathbb{E}_{\widehat{\mathbb{Q}}}|X_s-Y_s|^2ds.$$

By Gronwall's lemma

$$\mathbb{E}_{\widehat{\mathbb{Q}}}|X_t-Y_t|^2 \leq C_1E(C_3\Gamma(\beta)T^{\alpha})e^{tC_2E(C_3\Gamma(\beta)T^{\alpha})} \quad t\in[0,T].$$

This means that

$$\mathbb{E}_{\widehat{\mathbb{Q}}}|X_t-Y_t|^2 \leq \frac{C}{2}\mathbb{E}_{\widehat{\mathbb{Q}}}\|\widehat{f}\|_{L^2}^2 \quad t\in[0,T],$$

where

$$C = \frac{6K^2M^2T^{2\alpha-1}e^{TC_2E(C_3\Gamma(\beta)T^{\alpha})}}{(2\alpha-1)\Gamma^2(\alpha)}E(C_3\Gamma(\beta)T^{\alpha}).$$

Thus it follows,

$$W_2^2(\mathbb{P}_x,\mathbb{Q}) \leq CH(\mathbb{P}_x|\mathbb{Q}).$$

The proof of this lemma is complete. □

Now, we give the continuity dependance result via the Wasserstein distance.

Theorem 11. *Assume that the condition* (H_1) *holds. Then, for every pair of solutions* X_t, Y_t *to* (3.2.1)*, with respective laws* $\mu_t, \nu_t \in P_2(\mathbb{H}_2([0,T]))$*, such that the initial data* $X_0, Y_0 \in L^2(\Omega, \mathbb{P})$*, we have*

$$W_2(\mu_t, \nu_t) \leq C(t) W_2(\mu_0, \nu_0),$$

where μ_0, ν_0 *are laws of* X_0, Y_0*, respectively, and* $C \in C([0,T], \mathbb{R})$*.*

Proof. From [80], it is clear that we can rewrite W_2 in the following form,

$$W_2(\mu_t, \nu_t) = \inf\{\left[E\|X_\cdot - Y_\cdot\|_\infty^2\right]^2 : \quad \text{law}(X_t) = \mu_t,\ \text{law}(Y_t) = \nu_t\}.$$

Since X_t, Y_t are solutions of (3.2.1), then

$$X_t = X_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, X_s) dW_s + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, X_s) ds$$

and

$$Y_t = Y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, Y_s) dW_s + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, Y_s) ds.$$

Thus

$$\begin{aligned}
\mathbb{E}|X_t - Y_t|^2 &= \mathbb{E}\left|X_0 - Y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(f(s, X_s) - f(s, Y_s)) dW_s \right. \\
&\quad \left. + \int_0^t (t-s)^{\alpha-1}(g(s, X_s) - g(s, Y_s)) ds \right|^2 \\
&\leq 3\mathbb{E}|X_0 - Y_0|^2 \\
&\quad + \frac{3}{\Gamma^2(\alpha)} \mathbb{E}\left|\int_0^t (t-s)^{\alpha-1}(f(s, X_s) - f(s, Y_s)) dW_s\right|^2 \\
&\quad + \frac{3}{\Gamma^2(\alpha)} \mathbb{E}\left|\int_0^t (t-s)^{\alpha-1}(g(s, X_s) - g(s, Y_s)) ds\right|^2 \\
&\leq 3\mathbb{E}|X_0 - Y_0|^2 + \frac{3K^2}{\Gamma^2(\alpha)} \int_0^t (t-s)^{2\alpha-2} \mathbb{E}|X_s - Y_s|^2 ds \\
&\quad + \frac{3K^2 t^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)} \int_0^t \mathbb{E}|X_s - Y_s|^2 ds.
\end{aligned}$$

By the same argument of Theorem 8, we can prove that there exist $M_1, M_2 \geq 0$ such that

$$\mathbb{E}|X_t - Y_t|^2 \leq M_1 e^{M_2 t} \mathbb{E}|X_0 - Y_0|^2, \quad t \in [0,T].$$

Since $\mathrm{law}(X_t) = \mu_t$ and $\mathrm{law}(X_t) = \mu_t$, then

$$W_2^2(\mu_t, \nu_t) \leq \mathbb{E}|X_t - Y_t|^2.$$

By taking the infimum over X_0 and Y_0, we obtain

$$W_2^2(\mu_t, \nu_t)^2 \leq M_1 e^{M_2 t} W_2^2(\mu_0, \nu_0), \quad t \in [0,T].$$

$\square$

Chapter 4

Rich dynamics in a stochastic predator-prey model

In terms of mathematical modeling, the global dynamics of predator-prey systems can be affected by many factors such as death rate, birth rate, time delay, and so on. One crucial component to describe the relationship between the prey and predator populations is the predator-prey interaction (also called functional response). This latter one can be classified into many different types such as Holling $I - IV$ types, Hassell-Varley type, Crowley-Martin type, and Beddington-DeAnglis type, and so on. In savanna forests, most domestic species live in huge groups permanently and establish stable social relationships, such as elephants, zebras, buffaloes, bees, deers and others. This behavior gives them various advantages, where the weakest preys will be inserted in the interior of the herd and the strongest ones take the position in the exterior corridor of the herd. This strategy may reduce the predation rate thanks to the protection zone formed by the prey. In addition, its increases the vigilance for the prey against the predator, which causes confusion for the predator and distracts the predator from his target. Furthermore, the social behavior improves the method of locating food, also it contributes to the process of promoting feeding to different herds through the exchange of information regarding the location of food or how to get it. The first mathematical approach of the social behavior has been offered by Venturino *et.al* [1], where they have supposed that the interaction between the prey population and the predator population is done only on the outermost of the herd formed by the prey. It is equivalent to say that the number of the captured prey by a successful predator attack will be proportional to the density which is on the boundary of the herd. This latter leads to a new functional response in terms of square root of the prey density. Later, Braza [14] takes in the account the average time for the predator to process the hunted prey, where he investigate with a new interaction functional with a square root by using an approximation of a classical Holling II functional response. The phenomenon of the herds for the animals tempted many researchers, which enriched the environmental and ecological field. We refer the readers to papers [13, 15, 70, 71, 76, 78, 79, 82]

Besides, the prey herd's shape changes from one species to another depending on the physiological and sociological characteristics that control the behaviors of living beings. However, the way at which the animals interact with their environment, the number of their individuals as well as their individual efficiency, all of these factors and more will determine how the prey form their herd. The concept of herd shape for animals was modeled and introduced for the first time by Venturino *et al* in [79], where they generalized the interaction between the prey and the predator in both cases $2D$ and $3D$ of herd's forms with a new functional response in term of a new parameter which modeling the shape of the herd. For better explanation, we consider the following deterministic model that been introduced in [79]

$$\begin{cases} \dfrac{d}{dt}u(t) = \rho u(t)\left(1 - \dfrac{u(t)}{k}\right) - \delta u^{\alpha}(t)v(t), \\ \dfrac{d}{dt}v(t) = -\eta v(t) + e\delta u^{\alpha}(t)v(t), \end{cases} \tag{4.0.1}$$

where $u(t)$ and $v(t)$ stands for prey and predator density at time t, respectively. ρ is the intrinsic growth rate. k is the environment carrying capacity for the prey. η represents the natural mortality rate for the predators. δ stands for the predation rate of the prey population. e is the conversion rate of the prey density to a predator density and $0 < \alpha < 1$

represents the rate of the prey herd's shape. For the biological relevance of the parameter α, we consider a simple example for the case of $2D$ herd shape. We assume that the prey forms a group in $\mathbb{R}^2$ with some regular shape such as the circles or the squares, we find that the number of the captured preys by the predator will be proportional to the square root of the prey population density (i.e. $\alpha = 1/2$). We consider of course the interaction between the two species that affects mainly the prey individuals which are in the boundary of the herd. Clearly, the regular forms do not only exist in the case of $2D$. However, in the case of $3D$ such as birds or sardines, where the prey forms a regular form (cube, sphere and so on). Then, the consumed prey by a predaor will be proportional to $u^{2/3}$. Obviously, for $\alpha = 1$, The model (4.0.1) turn into the classical predator-prey model of Lotka and Volterra [59, 81]. In these last years, the model (4.0.1) is widely studied by several researchers. In [82], the authors obtained the global dynamics of the model (4.0.1). They discussed the singularity near the original equilibrium point. Further, the dynamical behavior of the model (4.0.1) has been investigated in the presence of spatial diffusion in [26]. More recently, the author in [25] has proposed a new approach of the system (4.0.1) with Holling II functional response as follows

$$\begin{cases} \dfrac{d}{dt}u(t) = \rho u(t)\left(1 - \dfrac{u(t)}{k}\right) - \dfrac{\delta u^{\alpha}(t)v(t)}{1+\delta t_h u^{\alpha}(t)}, \\ \dfrac{d}{dt}v(t) = -\eta v(t) + \dfrac{e\delta u^{\alpha}(t)v(t)}{1+\delta t_h u^{\alpha}(t)}, \end{cases} \tag{4.0.2}$$

we mention that the parameters of model (4.0.1) remain the same for model (4.0.2) and the new parameter t_h represents the time spent by predator in handling with the prey (please see [14, 25]). The main interest in [25] is to study the impact of the herd shape rate α on the global dynamics of the model (4.0.2) with the presence of the time delay. In addition, the author has proved that the time delay plays an important role on the stability of the equilibria which gives a rich dynamics such as Hopf bifurcation and transcritical bifurcation.

In the real life situations, All ecological processes are inevitably affected by environmental noise which represent an important parameter in an ecosystem, however, the

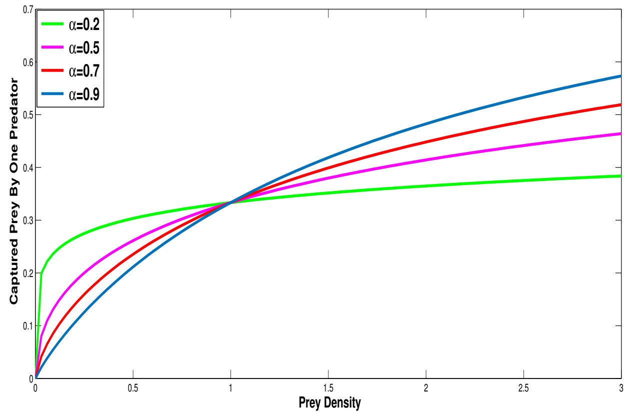

FIGURE 4.1: Impact of the prey herd's shape rate α on the quantity of the captured prey by one predator for different values of α where $\delta = 0.5,\ t_h = 1$.

mathematical modeling of ecological phenomena by a deterministic approach gives limitations in term of results, which leads to difficulties in fitting of data and predicting the future dynamics of the system precisely. Up to now, a large number of researchers have introduced a stochastic environmental variation using the Brownian motion into parameters in the deterministic model to construct a stochastic predator-prey models, which has been considered as a stochastic fluctuations. For more details on the stochastic predator-prey models, May [62] emphasizes out that due to continuous environmental fluctuation, the parameters in a systems such as the birth rates, carrying capacity, death rates and so on exhibited random fluctuations to a great or lesser extent. X. Zhang *et.al* [89] considered a stochastic predator-prey model with hyperbolic mortality and Holling type II functional response in which they founded sufficient conditions for the existence and uniqueness of an ergodic stationary distribution and derived sufficient conditions for extinction of the predator populations. LV. Jingliang and Ke. Wang [35] have deeply discussed the persistence, permanent and extinction of a stochastic model of a predator-prey system with Holling-type II functional response. Sengupta *et.al* [72] examined a stochastic non autonomous predator-prey system with Holling-type III functional response and predator's intra-specific competition where they obtained the stochastic permanence. M. Liu and C. Bai [43] estabilished the Sufficient and necessary criteria for the existence of optimal harvesting strategy for a stochastic predator-prey model. In the literature, the kind of stochastic predator-prey interaction was widely used, we refer the readers to [18, 27, 34, 36, 37, 44, 45, 46, 47, 48, 61, 83, 84, 89, 90, 91].

The present chapter is organized as follows. Sec.4.1 is devoted to the formulation of the mathematical model and gives some results on the stochastic differential equations which have been used in the rest of the sections. In Sec.4.2, the properties of the stochastic predator-prey model (4.1.1) have been established including: the global existence and uniqueness for stochastic boundedness of positive solution by using the Itô's formula and the comparison theorem of stochastic equations. The persistence and extinction criteria of the species have been discussed in Sec.4.3, where the sufficient conditions for extinction in two case as well as the persistence of the species have been obtained. In Sec.4.4, the existence and uniqueness of an ergodic stationary distribution of the positive solutions for the system (4.1.1) have been proved under certain parametric restrictions. Several numerical simulations have been offered in Sec.4.5 to support the theoretical results. Finally, conclusions and discussions ended this paper in Sec.4.6.

4.1 The mathematical model

Motivated by the above referred works and inspired by the work in [25], we introduce a random fluctuations to system (4.0.2). Our principal topic is to prove that the random fluctuations can completely change the dynamics generated by the model (4.0.2), where in this case the extinction of both species occur. There are many methods to establish the stochastic fluctuations into dynamical systems. One of the most used approach was adopted in [35, 46, 47, 69]. We suppose that the intrinsic growth rate of prey and the death rate of predator are mainly affected by environmental noise such that

$$\rho \longrightarrow \rho + \beta dW_1(t), \quad -\eta \longrightarrow -\eta + \gamma dW_2(t),$$

where $W_i(t)(i=1,2)$ are the mutually independent standard Browrian motions with $W_i(0)=0$. β and γ are positive and represent the intensities of the white noise. The stochastic predator-prey version corresponding to the model (4.0.2) takes the following form

$$\begin{cases} du(t) = \left[\rho u(t)\left(1-\dfrac{u(t)}{k}\right) - \dfrac{\delta u^{\alpha}(t)v(t)}{1+\delta t_h u^{\alpha}(t)}\right]dt + \beta u(t)dW_1(t), \\ dv(t) = \left[-\eta v(t) + \dfrac{e\delta u^{\alpha}(t)v(t)}{1+\delta t_h u^{\alpha}(t)}\right]dt + \gamma v(t)dW_2(t), \end{cases} \tag{4.1.1}$$

where $W_i(t)$ for $i=1,2$ be a mutually independent standard Browrian motions defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t>0}, \mathbb{P})$, with $\{\mathcal{F}_t\}_{t>0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}-$null sets). For our best of knowledge, the dynamics of the stochastic predator-prey model (4.1.1) have never been studied.

4.2 Properties of the solution

In this section, according to the best result in [34], we prove that the model (4.1.1) is well-posed in the sense that for any pair of positive initial value $(u(0), v(0))$, the system (4.1.1) has a unique global solution which remains positive and bounded. By using the Lypunov analysis method [19, 48], we show that the solution is global. Next, we analysis the Boundedness of the state variables u and v.

4.2.1 Existence and uniqueness of the global positive solution

Since $u(t)$ and $v(t)$ denote the population densities of the prey and the predator, respectively then, we are only interested in the positive solutions. Thus, we have the following theorem

Theorem 12. *For each initial values* $(u(0),v(0)) \in \mathbb{R}^2_+$*, there exists a unique positive local solution* $(u(t),v(t))$ *of system (4.1.1) for all* $t \in [0;\tau_e)$ *almost surely (a.s.), and the solution remains in* $\mathbb{R}^2_+$ *with probability* 1 *where* τ_e *is the explosion time.*

Proof. Putting

$X(t) = \ln u(t), \quad Y(t) = \ln v(t),$

then from the Itô's formula in Lemma.7, the system (4.1.1) can be written as

$$\begin{cases} dX(t) = \left[X(t)\left(\rho - \frac{\beta^2}{2} - \frac{e^{X(t)}}{k}\right) - \frac{\delta e^{\alpha X(t)}}{1+\delta t_h e^{\alpha X(t)}} e^{Y(t)}\right] dt + \beta dW_1(t), \\ \\ dY(t) = \left[-\eta e^{Y(t)} - \frac{\gamma^2}{2} + \frac{e\delta e^{\alpha X(t)}}{1+\delta t_h e^{\alpha X(t)}}\right] dt + \gamma dW_2(t), \end{cases} \tag{4.2.1}$$

with the initial values $X(0) = \ln u(0)$, $Y(0) = \ln v(0)$. It is easy to seen that the right hand side of the above system satisfy the local Lipschitz condition, then for any given initial values $X(0) > 0, Y(0) > 0$ there is a unique maximal local solution $(X(t),Y(t))$ for all $t \in [0;\tau_e)$ where τ_e is the explosion time of the solution. Now, using the Itô's formula in Lemma.7, we obtain $u(t) = e^{X(t)}$ and $v(t) = e^{Y(t)}$ as the positive local solution of the system (4.1.1) with the initial value $u(0) > 0$, $v(0) > 0$. The proof is completed. □

Now, we focus on proving the global existence of the solution for our proposed model (4.1.1). For this task, we only need to prove that τ_e goes to the infinity (i.e. $\tau_e = \infty$), then we have the following theorem

Theorem 13. *For each* $(u(0),v(0)) \in \mathbb{R}^2_+$*, there exists a unique positive global solution* $(u(t),v(t))$ *of system (4.1.1) for all* $t > 0$ *almost surly (a.s.), and the solution remains in* $\mathbb{R}^2_+$ *with probability* 1*.*

Proof. Let m_0, be a sufficiently large nonnegative integer number, such that $u(0)$ and $v(0)$ lie inside in the interval $[\frac{1}{m_0}, m_0]$. For any integer $m > m_0$, we can define the following stopping times as [60]

$$\tau_m = \inf\left\{t \in [0;\tau_e) : u(t) \notin \left(\frac{1}{m}, m\right) \quad \text{or} \quad v(t) \notin \left(\frac{1}{m}, m\right)\right\}. \tag{4.2.2}$$

Obviously, τ_m increases when $m \longrightarrow \infty$. Set $\tau_\infty = \lim_{m\to+\infty} \tau_m$, with $\tau_\infty < \tau_e$ a.s.. Next, we only need to prove that $\tau_\infty = \infty$, then $\tau_e = \infty$ for which we obtain $(u(t), v(t)) \in \mathbb{R}^2_+$ a.s. for all $t \geq 0$. If this statement is not verified, then there exist $T > 0$ and $\epsilon \in (0,1)$ such that

$$\mathbb{P}(\tau_\infty \leq T) > \epsilon. \tag{4.2.3}$$

Consequently, there is an integer $m_1 > m_0$ such that

$$\mathbb{P}(\tau_m \leq T) > \epsilon, \quad \text{for all} \quad m \geq m_1. \tag{4.2.4}$$

Now, let

$$V(u,v) = u - 1 - \ln u + \frac{1}{e}(v - 1 - \ln v) \tag{4.2.5}$$

be a C^2 function. It is not difficult to prove that $V(u,v) \geq 0$ for all $(u,v) \in \mathbb{R}^2_+$. This statement comes from the following inequality

$$u - 1 - \ln u \geq 0, \ \forall u > 0. \tag{4.2.6}$$

Using the Itô's formula in Lemma.7 yields

$$dV(u,v) = LV(u,v)dt + \beta(u-1)dW_1(t) + \gamma\frac{1}{e}(v-1)dW_2(t), \tag{4.2.7}$$

from the definition of the operator L given above, a straight forward calculation gives

$$\begin{aligned}
LV &= \rho u - \frac{\rho}{k}u^2 - \rho + \frac{\rho}{k}u + \frac{\delta u^\alpha v}{1+\delta t_h u^\alpha} - \frac{\delta v}{u^{1-\alpha}(1+\delta t_h u^\alpha)} - \frac{\eta}{e}v + \frac{\eta}{e} \\
&- \frac{\delta u^\alpha v}{1+\delta t_h u^\alpha} + \frac{\delta u^\alpha}{1+\delta t_h u^\alpha} + \frac{1}{2}\beta^2 + \frac{1}{2e}\gamma^2 \\
&\leq \frac{\rho_(k+1)}{k}u - \frac{\rho}{k}u^2 + \frac{\delta u^\alpha}{1+\delta t_h u^\alpha} + \frac{\eta}{e} + \frac{1}{2}\beta^2 + \frac{1}{2e}\gamma^2 \\
&\leq \frac{\rho(k+1)}{k}u - \frac{\rho}{k}u^2 + \frac{1}{t_h} + \frac{\eta}{e} + \frac{1}{2}\beta^2 + \frac{1}{2e}\gamma^2 \\
&\leq \frac{\rho(k+1)^2}{4k} + \frac{1}{t_h} + \frac{\eta}{e} + \frac{1}{2}\beta^2 + \frac{1}{2e}\gamma^2 \\
&\leq M,
\end{aligned}$$

where

$$\frac{\rho(k+1)^2}{4k} = \sup_{u\in\mathbb{R}_+}\left\{\frac{\rho(k+1)}{k}u - \frac{\rho}{k}u^2\right\},$$

and M is a positive constant. Then we have

$$dV(u,v) \leq Mdt + \beta(u-1)dW_1(t) + \gamma(v-1)dW_2(t). \tag{4.2.8}$$

Now, integrating both sides of the above inequality from 0 to $\tau_m \wedge T$ and take the expectation on both sides leads to

$$\mathbb{E}V(u(\tau_m \wedge T), V(\tau_m \wedge T)) \leq V(u(0), v(0)) + M\mathbb{E}(\tau_m \wedge T), \tag{4.2.9}$$

which gives

$$\mathbb{E}V(u(\tau_m \wedge T), V(\tau_m \wedge T)) \leq V(u(0), v(0)) + MT, \tag{4.2.10}$$

where, $\tau_m \wedge T = \min\{\tau_m, T\}$. Taking $\Omega_m = \{\tau_m \leq T\}$ for $m \geq m_1$ and from (4.2.4) we obtain $\mathbb{P}(\Omega_m) \geq \epsilon$. Note that for any $\omega \in \Omega_m$ there exists $u(\tau_m, \omega)$ or $v(\tau_m, \omega)$ equals either m or $\frac{1}{m}$. Hence $V(u(\tau_m, \omega), v(\tau_m, \omega))$ is no less than

$$\min\left\{m - 1 - \ln m\ ,\ \frac{1}{m} - 1 - \ln\frac{1}{m}\right\}. \tag{4.2.11}$$

Therefore

$$V(u(\tau_m, \omega), v(\tau_m, \omega)) \geq (m - 1 - \ln m) \wedge \left(\frac{1}{m} - 1 - \ln\frac{1}{m}\right). \tag{4.2.12}$$

So, using inequality (4.2.2), we obtain

$$V(u(0), v(0)) + MT \geq \mathbb{E}[I_{\Omega_m(\omega)} V(u(\tau_m, \omega), v(\tau_m, \omega))] \geq \epsilon(m - 1 - \ln m) \wedge \left(\frac{1}{m} - 1 - \ln\frac{1}{m}\right), \tag{4.2.13}$$

where I_{Ω_m} represents the indicator function of Ω_m. Taking $m \longrightarrow \infty$, we get

$$\infty > V(u(0), v(0)) + MT = \infty, \tag{4.2.14}$$

which gives a contradiction. Hence, we must have $\tau_\infty = \infty$ and consequently the solution of the system (4.1.1) exists for all $t \geq 0$. This completes the proof of Theorem.13. □

4.2.2 Stochastic boundedness

Biological validity of a mathematical model is decided by its boundedness. The nonexplosion property in a population dynamical system is often not sufficient. However, the ultimate boundedness property is more desired. Now, we establish the theorem which gives us almost sure eventual boundedness of the solutions. To this end, we first give the definition of stochastic ultimate boundedness which is one of the most important topics in population dynamics

Definition 6. *(see [53]) The solution $U(t) = (u(t), v(t))$ of the system (4.1.1) is said to be stochastically ultimately bounded, if for all $a \in (0;1)$, there exists a positive constant $\lambda = \lambda(a)$, such that for each initial value $U(0) \in \mathbb{R}^2_+$, the solution $U(t)$ satisfying the following property*

$$\lim_{t\to\infty} \sup \mathbb{P}\{|U(t)| \ > \lambda\} < a. \tag{4.2.15}$$

Theorem 14. *For all initial value $(u(0), v(0)) \in \mathbb{R}^2_+$, the solutions of the system (4.1.1) are stochastically ultimately bounded.*

Proof. Let $(u(t), v(t))$ be any solution of the system (4.1.1). From Theorem.13, we know that the solution $(u(t), v(t))$ will remains in $\mathbb{R}^2_+$ for all $t \geq 0, a.s.$. Now define the two Lyaponov functions

$$F(u,v) = e^t u^p, \quad G(u,v) = e^t v^p, \text{ with } (u,v) \in \mathbb{R}^2_+ \text{ and } p > 0. \tag{4.2.16}$$

From the Itô's formula in Lemma.7 and system (4.1.1), one can obtain

$$\begin{aligned}
d(e^t u^p) &= e^t u^p dt + pe^t u^{p-1} du + \frac{p(1-p)e^t u^{p-1}}{2}(du)^2, \\
&= e^t u^p dt + pe^t u^{p-1}\left[\rho - \frac{\rho u}{k} - \frac{\delta u^{\alpha-1} v}{1+\delta t_h u^\alpha}\right] dt + \frac{1}{2}p(p-1)e^t u^p \beta^2 dt + pe^t u^p \beta dW_1(t), \\
&= e^t u^p \left\{1 + p\left[\rho - \frac{\rho u}{k} - \frac{\delta u^{\alpha-1} v}{1+\delta t_h u^\alpha}\right] + \frac{p(p-1)}{2}\beta^2\right\} dt + pe^t u^p \beta dW_1(t),
\end{aligned}$$

with a simalary calculate, we get

$$d(e^t v^p) = e^t v^p \left\{1 + p\left[-\eta + \frac{e\delta u^\alpha v}{1+\delta t_h u^\alpha}\right] + \frac{p(p-1)}{2}\gamma^2\right\} dt + pe^t v^p \gamma dW_2(t).$$

Then, we have

$$LF = e^t u^q \left\{ 1 + p\left[\rho - \frac{\rho u}{k} - \frac{\delta u^{\alpha-1} v}{1+\delta t_h u^\alpha}\right] + \frac{p(p-1)}{2}\beta^2 \right\} \leq M_1 e^t,$$

and

$$LG = e^t v^q \left\{ 1 + p\left[-\eta + \frac{e\delta u^\alpha v}{1+\delta t_h u^\alpha}\right] + \frac{p(p-1)}{2}\gamma^2 \right\} \leq M_2 e^t,$$

where

$$M_1 = \left(\frac{k}{\rho}\right)^p \left(\frac{1+\rho p + t_h^{-\alpha} + \frac{1}{2}p(p-1)\beta^2}{p+1}\right)^{p+1}, \; M_2 = \left(\frac{1}{m}\right)^p \left(\frac{1+\rho p + \left(\frac{e}{t_h}\right)^\alpha + \frac{1}{2}p(p-1)\beta^2}{p+1}\right)^{p+1}.$$

Hence, we have

$$e^t \mathbb{E}[u^p] - \mathbb{E}[u^p(0)] \leq M_1 e^t \text{ and } e^t \mathbb{E}[v^p] - \mathbb{E}[v^p(0)] \leq M_2 e^t.$$

This leads to

$$\lim_{t\to\infty} \sup \mathbb{E}[u^p] \leq M_1 < \infty, \quad \lim_{t\to\infty} \sup \mathbb{E}[v^p] \leq M_2 < \infty.$$

Now, for $U(t) = (u(t), v(t)) \in \mathbb{R}_+^2$, we have $|U(t)|^p \leq 2^{p/2}(u^p + v^p)$ which gives

$$\lim_{t\to\infty} \sup \mathbb{E}[|U(t)|^p] \leq M_3 < \infty,$$

where, $M_3 = 2^{p/2}(M_1 + M_2)$.

For any $a > 0$, taking $\lambda(a) = \left(\frac{M_3}{a}\right)^{1/p}$ and applying the Chebyshev inequality yields

$$\mathbb{P}\{|U(t)| \; > \lambda\} \leq \frac{\mathbb{E}[|U(t)|^q]}{\lambda^p(a)}.$$

Thus

$$\lim_{t\to\infty} \mathbb{P}\{|U(t)| \; > \lambda\} \leq \frac{M_3}{\lambda^p(a)} = a.$$

This lead to the required assertion (4.2.15). The proof of Theorem.14 is completed. □

4.3 Persistence and Extinction criteria

In this section, we shall discuss the case of the persistence and the extinction of the two population (the prey and the predator) for our proposed model (4.1.1), where we shall try to give the sufficient conditions which determines the extinction and the persistence of the stochastic predator-prey model (4.1.1). Firstly, we study the extinction scenario in two situations, the first case is the prey population survival where the predator population goes to extinction, the second case is all the two species will die out. Before proceeding with the analysis, we give the following definitions.

Definition 7. *[89]*

(i) *If $\lim_{t\to\infty} u(t) = 0, a.s.$, then the prey density $u(t)$ is said to be extinctive almost surely.*

(ii) *If $\lim_{t\to\infty} v(t) = 0, a.s.$, then the predator density $v(t)$ is said to be extinctive almost surely.*

.

Now, we give the fundamental lemma which will be used in the following

Lemma 10. *[34, 36] Define the following one-dimensional stochastic differential equation*

$$dU(t) = \rho U(t)\left(1 - \frac{U(t)}{k}\right)dt + \beta U(t) dW_1(t), \tag{4.3.1}$$

where ρ, k and β are positive, and $W_1(t)$ is standard Brownian motion. Then we have the following assertions

- *if $\rho < \dfrac{\beta^2}{2}$, then we have $\lim_{t\to\infty} U(t) = 0$.*
- *if $\rho > \frac{\beta^2}{2}$, then we have*

$$\lim_{t\to\infty} \frac{\ln U(t)}{t} = 0, a.s., \ \lim_{t\to\infty} \frac{1}{t}\int_0^t U(s)\,\mathrm{d}s = k - \frac{k\beta^2}{2\rho}.$$

Theorem 15. *Assuming that*

$$(\mathbf{H}):\, \rho > \frac{\beta^2}{2},$$

and let $(u(t), v(t))$ be a positive solution of the system (4.1.1) with the initial condition $(u(0), v(0)) \in \mathbb{R}^2_+$.

Putting

$$A = -\eta - \frac{\gamma^2}{2} + e\delta \int_0^\infty \frac{u^\alpha}{1+\delta t_h u^\alpha} \chi(u)\,\mathrm{d}u.$$

If $A < 0$, then we have

$$\lim_{t\to\infty} v(t) = 0, \text{ a.s.},$$

which means that the predator density goes to extinction with probability one. In addition, the distribution of $u(t)$ converges weakly a.s. to the measure which has the density

$$\chi(u) = \frac{\Theta}{\beta^2} u^{-2+\frac{2\rho}{\beta^2}} e^{-\frac{2\rho}{k\beta^2}},\ u \in (0;\infty),$$

where

$$\Theta = \left[\frac{1}{\beta^2} \left(\frac{k\beta^2}{2\rho} \right)^{\frac{2\rho}{\beta^2}-1} \Gamma\left(\frac{2\rho}{\beta^2} - 1 \right) \right]^{-1},$$

is a constant satisfying $\int_0^\infty \chi(u)\,\mathrm{d}u = 1$ and Γ is the gamma function.

Proof. Consider the following 1-dimensional stochastic differential equation

$$\begin{cases} dU(t) = \rho U(t) \left(1 - \frac{U(t)}{k}\right) dt + \beta U(t) dW_1(t), \\ U(0) = u(0). \end{cases} \tag{4.3.2}$$

Putting

$$g(u) = \rho u \left(1 - \frac{u}{k}\right),\ \beta(u) = \beta u,\ u \in (0;\infty).$$

By a straight forward calculation we get:

$$\int \frac{g(s)}{\beta(s)^2} \mathrm{d}s = \int \left(\frac{\rho}{s\beta^2} - \frac{\rho}{k\beta^2} \right) \mathrm{d}s = \frac{\rho}{\beta^2} \ln s - \frac{\rho}{k\beta^2} + \Theta.$$

Therefore,

$$e^{\int \frac{g(s)}{\beta(s)^2} \mathrm{d}s} = e^{\left(\Theta_s \frac{\rho}{\beta^2}\right)} e^{\left(-\frac{\rho}{k\beta^2} s\right)}.$$

Then, from the Theorem 1.16 in [42] it follows that the equation (4.3.2) has the ergodic property and the invariant density given as

$$\chi(u) = \frac{\Theta}{\beta^2} u^{-2+\frac{2\rho}{\beta^2}} e^{\left(-\frac{2\rho}{k\beta^2}\right)},\ u \in (0;\infty), \tag{4.3.3}$$

where

$$\Theta = \left[\frac{1}{\beta^2} \left(\frac{k\beta^2}{2\rho} \right)^{\frac{2\rho}{\beta^2}-1} \Gamma\left(\frac{2\rho}{\beta^2} - 1 \right) \right]^{-1},$$

satisfying

$$\int_0^\infty \chi(x)\,\mathrm{d}x = 1,$$

with

$$\frac{1}{t}\int_0^t u(s)\,\mathrm{d}s = \int_0^\infty u\chi(u)\,\mathrm{d}u,\ a.s. \tag{4.3.4}$$

Using the comparison theorem of 1-dimensional stochastic differential equation [66], we obtain

$$u(t) \le U(t),\ \forall t > 0,\ a.s. \tag{4.3.5}$$

Now, Applying the Itô's formula in Lemma.7 to $\ln v(t)$ for the second equation of the system (4.1.1) and using (4.3.5), then we obtain

$$\begin{aligned} d\ln v(t) &= \left(-\eta - \frac{\gamma^2}{2} + \frac{e\delta u^\alpha(t)}{1+\delta t_h u^\alpha(t)} \right) dt + \gamma dW_2(t) \\ &\le \left(-\eta - \frac{\gamma^2}{2} + \frac{e\delta U^\alpha(t)}{1+\delta t_h U^\alpha(t)} \right) dt + \gamma dW_2(t). \end{aligned}$$

For the both sides, integrating the above equation from 0 to t and dividing by t gives

$$\frac{\ln v(t) - \ln v(0)}{t} \le -\eta - \frac{\gamma^2}{2} + e\delta \frac{1}{t}\int_0^t \frac{U^\alpha(s)}{1+\delta t_h U^\alpha(s)}\,\mathrm{d}s + \frac{N_2(t)}{t}, \tag{4.3.6}$$

where $N_2(t) = \int_0^t \gamma dW_2(s)\,\mathrm{d}s$ is a real-valued continuous local martingales. According to [54], we have $\lim\limits_{t\to\infty} \frac{N_2(t)}{t} = 0, a.s.$. Next, taking the superior limit on both sides of (4.3.6) and using Lemma.10 together with (4.3.4), we obtain

$$\begin{aligned}
\limsup_{t\to\infty} \frac{\ln v(t)}{t} &\leq -\eta - \frac{\gamma^2}{2} + \limsup_{t\to\infty} e\delta \frac{1}{t}\int_0^t \frac{U^{\alpha}(s)}{1+\delta t_h U^{\alpha}(s)}\,\mathrm{d}s, \\
&\leq -\eta - \frac{\gamma^2}{2} + e\delta \int_0^{\infty} \frac{u^{\alpha}}{1+\delta t_h u^{\alpha}}\chi(u)\,\mathrm{d}u, \\
&= A < 0, \quad a.s.,
\end{aligned}$$

which leads to the extinction of the predator spacie *i.e.* $\lim_{t\to\infty} v(t) = 0, a.s.$.

Now, for a sufficiently small $\epsilon_1 > 0$ there exists $\tilde{t}$ and a set $\Omega_{\epsilon_1} \subset \Omega$ such that $\mathbb{P}(\Omega_{\epsilon_1}) > 1-\epsilon$ and

$$\frac{\delta u^{\alpha} v}{1+\delta t_h u^{\alpha}} \leq \delta u^{\alpha} v \leq \delta \epsilon_1 u^{\alpha}, \text{ for } t \geq \tilde{t} \text{ and } \omega \in \Omega_{\epsilon_1}.$$

From

$$\left[\rho u\left(1-\frac{u}{k}\right) - \delta\epsilon_1 u^{\alpha}\right] dt + \beta u dW_1(t) \leq du \leq \rho u\left(1-\frac{u}{k}\right) dt + \beta u dW_1(t),$$

we obtain that the distribution of the process $u(t)$ converges weakly to the measure with the density μ. The proof is complete. □

Theorem 16. *Assume that $\rho < \frac{\beta^2}{2}$ and $e\delta < \frac{\gamma^2}{2}$ hold. Then for any initial condition $(u(0), v(0)) \in \mathbb{R}^2_+$ the two species die out, where the solution $(u(t), v(t))$ of the system (4.1.1) will be extinct exponentially with probability one.*

Proof. Applying the Itô's formula in Lemma.7 to the first equation of the system (4.1.1), implies that

$$d\ln u(t) = \left[-\frac{\beta^2}{2} + \rho\left(1-\frac{u(t)}{k}\right) - \frac{\delta u^{\alpha-1}(t)v(t)}{1+\delta t_h u^{\alpha}(t)}\right] dt + \beta dW_1(t). \tag{4.3.7}$$

Integrating the above equation from 0 to t and dividing by t on both sides of (4.3.7), we obtain

$$\frac{\ln u(t) - \ln u(0)}{t} = \rho - \frac{\beta^2}{2} - \frac{\rho}{k}\frac{1}{t}\int_0^t u(s)\,\mathrm{d}s - \delta\frac{1}{t}\int_0^t \frac{u^{\alpha-1}(s)v(s)}{1+\delta t_h u^{\alpha}(s)}\,\mathrm{d}s + \beta\frac{W_1(t)}{t}. \tag{4.3.8}$$

Using the strong law of large numbers for local martingales [54], we get

$$\lim_{t\to\infty}\frac{W_1(t)}{t}=0, a.s..$$

Taking the superior limit on both sides of the equation (4.3.9) gives

$$\lim_{t\to\infty}\sup\frac{\ln u(t)}{t}\leq \rho-\frac{\beta^2}{2}<0, a.s.,$$

which leads to

$$\lim_{t\to\infty}u(t)=0,\ a.s.$$

Application of Itô's formula in Lemma.7 to $\ln v(t)$ yields

$$d\ln v(t)=\left[-\eta-\frac{\gamma^2}{2}+\frac{e\delta u^{\alpha}(t)}{1+\delta t_h u^{\alpha}(t)}\right]dt+\gamma dW_2(t). \tag{4.3.9}$$

From $\lim_{t\to\infty}u(t)=0,\ a.s.$, there exists $T_0>0$ such that $u(t)<\epsilon$ for $t>T_0$. Hence, we get

$$\begin{aligned} d\ln v(t) &\leq \left[-\frac{\gamma^2}{2}+\frac{e\delta\epsilon^{\alpha}}{1+\delta t_h\epsilon^{\alpha}}\right]dt+\gamma dW_2(t),\\ &\leq \left(-\frac{\gamma^2}{2}+e\delta\right)dt+\gamma dW_2(t). \end{aligned}$$

Integrating the above inequality from 0 to t and dividing by t on both sides, we obtain

$$\frac{\ln v(t)-\ln v(0)}{t}=-\frac{\gamma^2}{2}+e\delta+\gamma\frac{W_2(t)}{t}. \tag{4.3.10}$$

Applying the strong law of large numbers for local martingales [54], we obtain

$$\lim_{t\to\infty}\frac{W_2(t)}{t}=0, a.s.$$

Taking the superior limit on both sides of (4.3.10), then we have

$$\lim_{t\to\infty}\sup\frac{\ln v(t)}{t}\leq -\frac{\gamma^2}{2}+e\delta<0,\ a.s.,$$

which gives

$$\lim_{t\to\infty}v(t)=0,\ a.s.$$

This completes the proof of Theorem.16. □

Remark 1.

(i) *According to Theorem.15, one can easly show that A is the critical value between the extinction and the persistence in the mean for the predator specie. Moreover, from Lemme.10, if $A < 0$, we obtain*

$$\lim_{t\to\infty} v(t) = 0, a.s., \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t u(s)\,\mathrm{d}s = k - \frac{k\beta^2}{2\rho}, a.s..$$

(ii) *Theorem.16 show that if the white noise intensities take a big values, then all both species are die out. On the other hand, the stochastic predator-prey model (4.1.1) will be persistent if the white noise disturbances are small enough. This assertion can easly be seen from Theorem.15 and Theorem.15.*

4.4 Existence of ergodic stationary distribution

In this part, according to the theory of Has'minskii [33] and using the Lyapunov function method we try to prove that under certains sufficient conditions, the stochastic predator-prey model (4.1.1) has a unique stationary distribution which is ergodic.

Theorem 17. *Suppose that*

$$\frac{\beta^2}{2} + \frac{\gamma^2}{2} < \rho - \eta, \text{ and } \rho > \eta,$$

then for any initial condition $(u(0), v(0)) \in \mathbb{R}_+^2$, the system (4.1.1) has a unique stationary distribution $\chi(.)$ which has the ergodic property.

Proof. In order to prove Theorem.17, we only need to verify the two assumption $(\mathbf{P_1})$ and $(\mathbf{P_2})$ in Lemma.6. We first begin by proving the validation of the first condition, then the diffusion matrix of the system (4.1.1) is

$$B = \begin{pmatrix} \beta^2 u^2 & 0 \\ 0 & \gamma^2 v^2 \end{pmatrix}.$$

It is not difficult to see that there exists a positive constants $\tilde{b}$, $\tilde{c}$ such that

$$\sum_{i,j=1}^{2} \tilde{b}_{i,j}(z)\xi_i\xi_j = \beta^2 u^2\xi_1^2 + \gamma^2 v^2 \xi_2^2 \geq \tilde{c}|\xi|^2,\ (u,v) \in \overline{D},\ \xi = (\xi_1, \xi_2) \in \mathbb{R}_+^2,$$

that is to say that B is a positive definite matrix for any compact subset of $\mathbb{R}_+^2$. Thus the assertion $(\mathbf{P_1})$ of Lemma.6 holds. Now, focusing on proving the second assertion in Lemma.6. From the system (4.1.1), we get

$$L(-\ln u) = -\frac{1}{u}\left(\rho u\left(1-\frac{u}{k}\right) - \frac{\delta u^\alpha v}{1+\delta t_h u^\alpha}\right) + \frac{\beta^2}{2} = -\left(\rho - \frac{\beta^2}{2}\right) + \rho\frac{u}{k} + \frac{\delta u^{\alpha-1}v}{1+\delta t_h u^\alpha} \quad (4.4.1)$$

and

$$L(-\ln v) = \eta - \frac{e\delta u^\alpha}{1+\delta t_h u^\alpha} + \frac{\gamma^2}{2} \leq \eta + \frac{\gamma^2}{2}. \quad (4.4.2)$$

Define

$$V_1(u,v) = -\ln u - \ln v,$$

then, from (4.4.1) and (4.4.2), we have

$$LV_1 = -\left(\rho - \eta - \frac{\beta^2}{2} - \frac{\gamma^2}{2}\right) + \rho\frac{u}{k} + \frac{\delta u^{\alpha-1}v}{1+\delta t_h u^\alpha}. \quad (4.4.3)$$

Now, we denote

$$V^*(u,v) = \Pi V_1(u,v) + v^{-\tau} + u + \frac{v}{e},$$

where $0 < \tau < 1$ is a sufficiently small constant satisfying the following assertion

$$\rho - \eta > \frac{\tau+1}{2}(\beta^2+\gamma^2),$$

with

$$\Pi = \frac{2}{\left(\rho-\eta-\frac{\beta^2}{2}-\frac{\gamma^2}{2}\right)} \max_{(u,v)\in\mathbb{R}_+^2}\left\{2, -\frac{\rho}{2k}u^2 - \tau v^{-\tau}\left(-\eta - \frac{1+\tau}{2}\gamma^2\right) + \rho u + \tau v^{-\tau-1} - \frac{\eta}{e}v\right\}.$$

We claim that $V^*(u,v)$ is not only continuous, but also tends to ∞ as (u,v) approaches the boundary of $\mathbb{R}_+^2$ and as $||(u,v)|| \to \infty$, where $||.||$ is the Euclidean norm of a point in $\mathbb{R}_+^2$. Therefore, it must be lower bounded and achieve this lower bound at a point $(u(0), v(0))$

in the interior of $\mathbb{R}^2_+$. Thus, we can define a nonnegative C^2-function $V : \mathbb{R}^2_+ \to \mathbb{R}_+ \cup \{0\}$ as

$$V(u,v) = \Pi V_1(u,v) + V_2(v) + V_3(u,v),$$

where

$$V_2(v) = v^{-\tau}, \quad V_3(u,v) = u + \frac{v}{e} - V^*(u(0), v(0)).$$

By applying the Itô's formula in Lemma.7 to $V_2(v)$ and $V_3(u,v)$, we obtain

$$\begin{array}{rcl} LV_2 & = & -\tau v^{-\tau-1}\left(-\eta v + \frac{e\delta u^\alpha v}{1+\delta t_h u^\alpha}\right) + \frac{\tau(1+\tau)}{2}\gamma^2 v^{-\tau} \\ & \leq & -\tau v^{-\tau}\left(-\eta - \frac{1+\tau}{2}\gamma^2\right) + \tau v^{-\tau-1}, \end{array} \tag{4.4.4}$$

and

$$LV_3 = \rho u\left(1 - \frac{u}{k}\right) - \frac{\eta}{e}v = -\frac{\rho}{k}u^2 + \rho u - \frac{\eta}{e}v. \tag{4.4.5}$$

Then, according to (4.4.4) and (4.4.5), we get

$$\begin{array}{rcl} LV & \leq & \Pi\left\{-\left(\rho - \eta - \frac{\beta^2}{2} - \frac{\gamma^2}{2}\right) + \rho\frac{u}{k} + \frac{\delta u^{\alpha-1} v}{1+\delta t_h u^\alpha}\right\} \\ & - & \tau v^{-\tau}\left(-\eta - \frac{1+\tau}{2}\gamma^2\right) + \tau v^{-\tau-1} - \frac{\rho}{k}u^2 + \rho u - \frac{\eta}{e}v. \end{array} \tag{4.4.6}$$

To complete the prove, we need to construct a bounded open domain E_ϵ for which the assumption p_2 of Lemma.6 holds. Let's define the following bounded open set

$$E_\epsilon = \left\{(u,v) \in \mathbb{R}^2_+ : \epsilon < u < \frac{1}{\epsilon},\ \epsilon < v < \frac{1}{\epsilon}\right\},$$

where $0 < \epsilon < 1$ is a sufficiently small number which satisfying the following conditions in $\mathbb{R}^2_+ \setminus E_\epsilon$

$$\epsilon \leq \left(\frac{\Pi\left(\rho - \eta - \frac{\beta^2}{2} - \frac{\gamma^2}{2}\right)}{4\delta}\right)^{\frac{1}{1-\alpha}}, \tag{4.4.7}$$

$$\epsilon \leq \frac{\rho}{2k\Pi\delta}, \tag{4.4.8}$$

$$\epsilon \leq \frac{1}{4\delta}\left(\rho - \eta - \frac{\beta^2}{2} - \frac{\gamma^2}{2}\right), \tag{4.4.9}$$

$$-\Pi\left(\rho - \eta - \frac{\beta^2}{2} - \frac{\gamma^2}{2}\right) + \frac{\rho}{2k\epsilon^2} + \Pi_1 \leq -1, \tag{4.4.10}$$

$$-\Pi\left(\rho-\eta-\frac{\beta^2}{2}-\frac{\gamma^2}{2}\right)+\Pi_2. \tag{4.4.11}$$

Now, we divide the set $\mathbb{R}^2_+ \setminus E_\epsilon$ into four subsets defined as

$$E^1_\epsilon=\left\{(u,v)\in\mathbb{R}^2_+ : u\le\epsilon\right\},\quad E^2_\epsilon=\left\{(u,v)\in\mathbb{R}^2_+ : v\le\epsilon\right\}$$
$$E^3_\epsilon=\left\{(u,v)\in\mathbb{R}^2_+ : u\ge\frac{1}{\epsilon}\right\},\quad E^4_\epsilon=\left\{(u,v)\in\mathbb{R}^2_+ : v\ge\frac{1}{\epsilon}\right\}.$$

clearly, $E_\epsilon=E^1_\epsilon\cup E^2_\epsilon\cup E^3_\epsilon\cup E^4_\epsilon$. Our objective in the next, is to prove that $LV(u,v)\le -1$ for any $(u,v)\in E^i_\epsilon,\ i\in\{1,2,3,4\}$.

Case 1 : If $(u,v)\in E^1_\epsilon$ and from $u^{1-\alpha}\le\epsilon^{1-\alpha}$, we have

$$u^{1-\alpha}v\le\epsilon^{1-\alpha}(1+v^2).$$

Then, it follows that

$$\begin{aligned}
LV(u,v) &\le -\frac{\Pi}{4}\left(\rho-\eta-\frac{\beta^2}{2}-\frac{\gamma^2}{2}\right)+\left[-\frac{\Pi}{4}\left(\rho-\eta-\frac{\beta^2}{2}-\frac{\gamma^2}{2}\right)+\delta\Pi\epsilon^{1-\alpha}\right]-\frac{\rho}{2k}u^2-\delta\Pi\epsilon^{1-\alpha}v^2\\
&+\left[-\frac{\Pi}{2}\left(\rho-\eta-\frac{\beta^2}{2}-\frac{\gamma^2}{2}\right)-\frac{\rho}{2k}u^2-\delta\epsilon^{1-\alpha}v^2-\tau v^{-\tau}\left(-\eta-\frac{1+\tau}{2}\gamma^2\right)+\rho u+\tau v^{-\tau-1}-\frac{\eta}{e}v\right],\\
&\le -\frac{\Pi}{4}\left(\rho-\eta-\frac{\beta^2}{2}-\frac{\gamma^2}{2}\right)+\left[-\frac{\Pi}{4}\left(\rho-\eta-\frac{\beta^2}{2}-\frac{\gamma^2}{2}\right)+\epsilon^{1-\alpha}\delta\right]-\frac{\rho}{2k}u^2-\delta\Pi\epsilon^{1-\alpha}v^2\\
&+\left[-\frac{\Pi}{2}\left(\rho-\eta-\frac{\beta^2}{2}-\frac{\gamma^2}{2}\right)+\sup_{(u,v)\in\mathbb{R}^2_+}\left\{-\frac{\rho}{2k}u^2-\tau v^{-\tau}\left(-\eta-\frac{1+\tau}{2}\gamma^2\right)+\rho u+\tau v^{-\tau-1}-\frac{\eta}{e}v\right\}\right].
\end{aligned}$$

Since

$$\Pi=\frac{2}{\left(\rho-\eta-\frac{\beta^2}{2}-\frac{\gamma^2}{2}\right)}\max_{(u,v)\in\mathbb{R}^2_+}\left\{2,-\frac{\rho}{2k}u^2-\tau v^{-\tau}\left(-\eta-\frac{1+\tau}{2}\gamma^2\right)+\rho u+\tau v^{-\tau-1}-\frac{\eta}{e}v\right\},$$

we obtain that

$$\Pi\left(\rho-\eta-\frac{\beta^2}{2}-\frac{\gamma^2}{2}\right)\le-\frac{\Pi}{4}\left(\rho-\eta-\frac{\beta^2}{2}-\frac{\gamma^2}{2}\right)\le -1. \tag{4.4.12}$$

Hence

$$LV(u,v)\le-\frac{\Pi}{4}\left(\rho-\eta-\frac{\beta^2}{2}-\frac{\gamma^2}{2}\right)-\frac{\rho}{2k}u^2-\delta\Pi\epsilon^{1-\alpha}v^2\le-\frac{\Pi}{4}\left(\rho-\eta-\frac{\beta^2}{2}-\frac{\gamma^2}{2}\right)\le -1.$$

From (4.4.7), we have

$$LV(u,v)\le -1,\ \forall (u,v)\in E^1_\varepsilon. \tag{4.4.13}$$

Case 2 : If $(u,v) \in E_\epsilon^2$, we have $v \le \epsilon$. Since

$$u^{1-\alpha}v \le \epsilon(1+u^2),$$

we obtain

$$\begin{aligned}
LV(u,v) &\le -\frac{\Pi}{4}\left(\rho-\eta-\frac{\beta^2}{2}-\frac{\gamma^2}{2}\right)+\left[-\frac{\Pi}{4}\left(\rho-\eta-\frac{\beta^2}{2}-\frac{\gamma^2}{2}\right)+\Pi\delta\epsilon\right]+\left(\Pi\delta\epsilon-\frac{\rho}{2k}\right)u^2\\
&+ \left[-\frac{\Pi}{2}\left(\rho-\eta-\frac{\beta^2}{2}-\frac{\gamma^2}{2}\right)-\frac{\rho}{2k}u^2-\tau v^{-\tau}\left(-\eta-\frac{1+\tau}{2}\gamma^2\right)+\rho u+\tau v^{-\tau-1}-\frac{\eta}{e}v\right],\\
&\le -\frac{\Pi}{4}\left(\rho-\eta-\frac{\beta^2}{2}-\frac{\gamma^2}{2}\right)+\left[-\frac{\Pi}{4}\left(\rho-\eta-\frac{\beta^2}{2}-\frac{\gamma^2}{2}\right)+\Pi\delta\epsilon\right]-\left(\Pi\delta\epsilon-\frac{\rho}{2k}\right)u^2\\
&+ \left[-\frac{\Pi}{2}\left(\rho-\eta-\frac{\beta^2}{2}-\frac{\gamma^2}{2}\right)+\sup_{(u,v)\in\mathbb{R}_+^2}\left\{-\frac{\rho}{2k}u^2-\tau v^{-\tau}\left(-\eta-\frac{1+\tau}{2}\gamma^2\right)+\rho u+\tau v^{-\tau-1}-\frac{\eta}{e}v\right\}\right].
\end{aligned}$$

According to (4.4.8) and (4.4.9), it follows that

$$LV(u,v) \le -\frac{\Pi}{4}\left(\rho-\eta-\frac{\beta^2}{2}-\frac{\gamma^2}{2}\right) \le -1, \text{ for any } (u,v)\in E_\epsilon^2. \tag{4.4.14}$$

Case 3 : If $(u,v) \in E_\epsilon^3$, we get $u \le \frac{1}{\epsilon}$. Then we have

$$\begin{aligned}
LV(u,v) &\le -\Pi\left(\rho-\eta-\frac{\beta^2}{2}-\frac{\gamma^2}{2}\right)-\frac{\rho}{2k}u^2+-\tau v^{-\tau}\left(-\eta-\frac{1+\tau}{2}\gamma^2\right)\\
&+ \tau v^{-\tau-1}-\frac{\rho}{2k}u^2+\rho u-\frac{\eta}{e}v+\Pi\left(\rho\frac{u}{k}+\frac{\delta u^{\alpha-1}v}{1+\delta t_h u^\alpha}\right)\\
&\le -\Pi\left(\rho-\eta-\frac{\beta^2}{2}-\frac{\gamma^2}{2}\right)-\frac{\rho}{2k}u^2+\Pi_1\\
&\le -\Pi\left(\rho-\eta-\frac{\beta^2}{2}-\frac{\gamma^2}{2}\right)-\frac{\rho}{2k\epsilon^2}+\Pi_1.
\end{aligned}$$

Using (4.4.10) and (4.4.12), then we obtain

$$LV(u,v) \le -1, \text{ for any } (u,v)\in E_\epsilon^3, \tag{4.4.15}$$

where

$$\Pi_1 = \sup_{(u,v)\in\mathbb{R}_+^2}\left\{-\tau v^{-\tau}\left(-\eta-\frac{1+\tau}{2}\gamma^2\right)+\tau v^{-\tau-1}-\frac{\rho}{2k}u^2+\rho u-\frac{\eta}{e}v+\Pi\left(\rho\frac{u}{k}+\frac{\delta u^{\alpha-1}v}{1+\delta t_h u^\alpha}\right)\right\}.$$

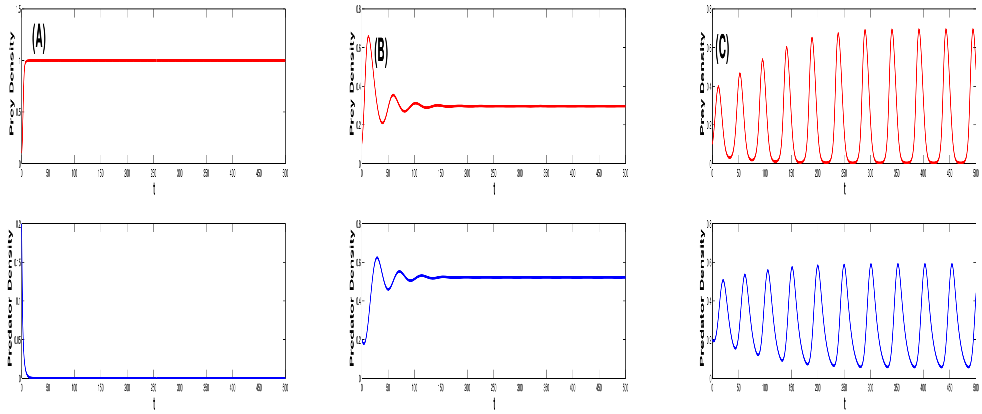

FIGURE 4.2: Numerical simulation of the deterministic system (4.0.2) with the parameter values $\rho = 0.5,\ k = 1,\ \delta = 0.5,\ t_h = 2,\ e = 0.4,\ \alpha = 0.8$ and different value of the parameter η. In (A), we take $\eta = 0.9$. In (B), we have $\eta = 0.1$ and for (C), we put $\eta = 0.04$.

Case 4 : If $(u,v) \in E_\epsilon^4$, we have $v \leq \frac{1}{\epsilon}$. Which gives

$$\begin{aligned} LV(u,v) &\leq -\Pi\left(\rho - \eta - \frac{\beta^2}{2} - \frac{\gamma^2}{2}\right) + -\tau v^{-\tau}\left(-\eta - \frac{1+\tau}{2}\gamma^2\right) \\ &+ \tau v^{-\tau-1} - \frac{\rho}{k}u^2 + \rho u - \frac{\eta}{e}v + \Pi\left(\rho\frac{u}{k} + \frac{\delta u^{\alpha-1}v}{1+\delta t_h u^\alpha}\right) \\ &\leq -\Pi\left(\rho - \eta - \frac{\beta^2}{2} - \frac{\gamma^2}{2}\right) + \Pi_2 \\ &\leq -\Pi\left(\rho - \eta - \frac{\beta^2}{2} - \frac{\gamma^2}{2}\right) + \Pi_2. \end{aligned}$$

Therefore, from (4.4.11) and (4.4.12) we get

$$LV(u,v) \leq -1, \text{ for any } (u,v) \in E_\epsilon^4, \tag{4.4.16}$$

with

$$\Pi_2 = \sup_{(u,v)\in\mathbb{R}_+^2}\left\{-\tau v^{-\tau}\left(-\eta - \frac{1+\tau}{2}\gamma^2\right) + \tau v^{-\tau-1} - \frac{\rho}{k}u^2 + \rho u - \frac{\eta}{e}v + \Pi\left(\rho\frac{u}{k} + \frac{\delta u^{\alpha-1}v}{1+\delta t_h u^\alpha}\right)\right\}.$$

Thus, if we combine the results (4.4.13), (4.4.14), (4.4.15) and (4.4.16), we can deduce

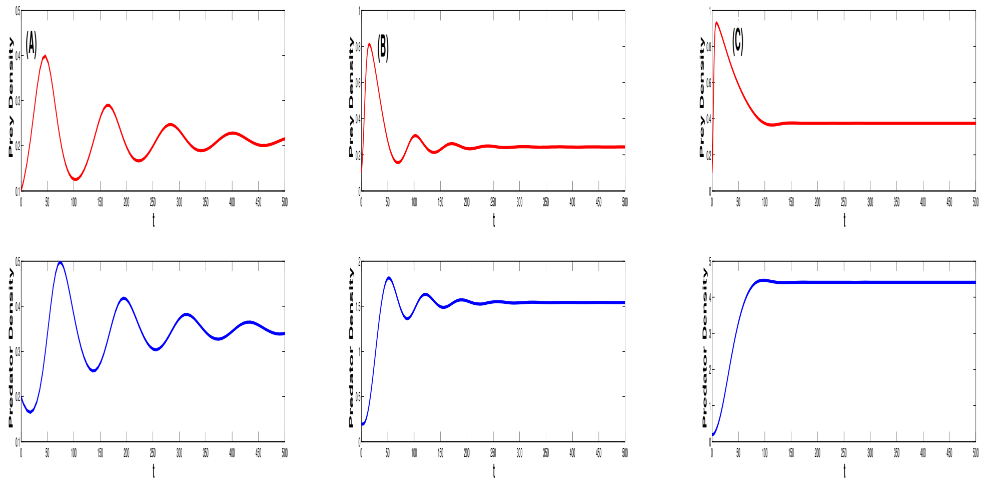

FIGURE 4.3: Numerical simulation of the deterministic system (4.0.2) with the parameter values $\delta = 0.5,\ k = 1,\ t_h = 2,\ e = 0.4,\ \alpha = 0.8,\ \eta = 0.1$ and multi values of ρ. In (A), we take $\rho = 0.09$. In (B), we have $\rho = 0.4$ and for (C), we put $\rho = 0.9$.

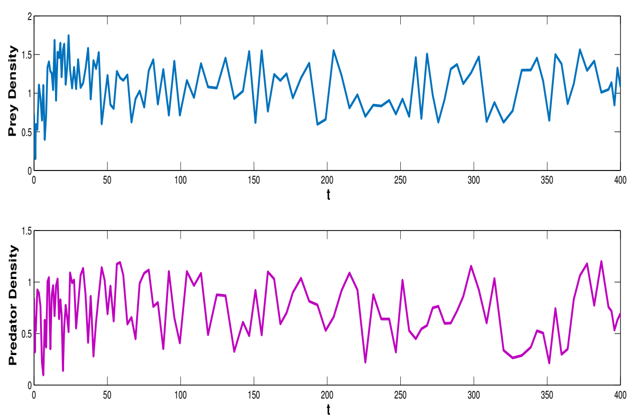

FIGURE 4.4: Numerical simulation of the stochastic predator-prey system (4.1.1) for the parameter values $\rho = 0.55,\ k = 1,\ \delta = 0.26,\ t_h = 0.71,\ \eta = 0.09,\ e = 0.39,\ \alpha = 1/3$ and the noise intensities $\beta^2/2 = 0.2,\ \gamma^2/2 = 0.168$. Here the initial data is $u(0) = 0.1,\ v(0) = 0.25$.

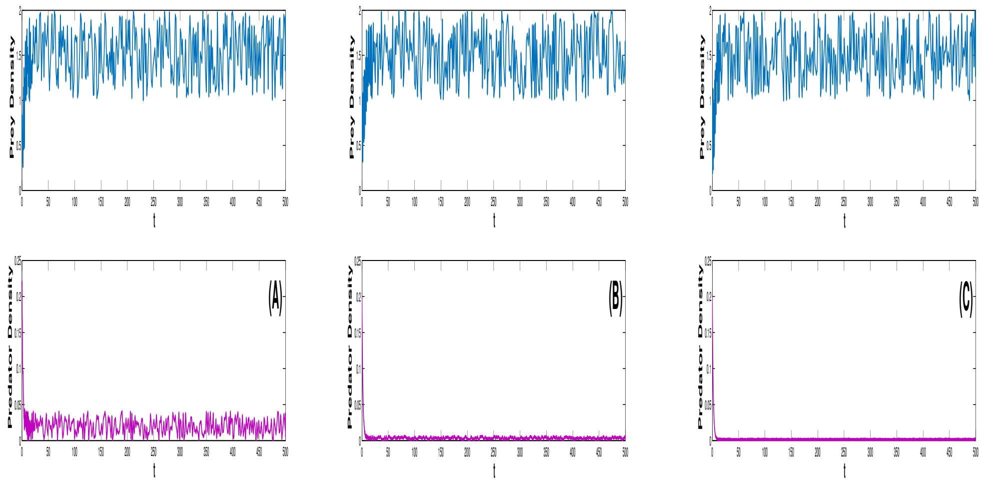

FIGURE 4.5: Numerical simulation of the stochastic predator-prey system (4.1.1) for the parameter values $\rho = 0.55,\ k = 1,\ \delta = 0.26,\ t_h = 0.71,\ \eta = 0.09,\ e = 0.39,\ \alpha = 1/3$. In (A), we choose $\beta^2/ = 0.14,\ \gamma^2/2 = 0.223$. In (B), we have $\beta^2/2 = 0.14,\ \gamma^2/2 = 0.322$ and for the last case (C), we put $\beta^2/2 = 0.14,\ \gamma^2/2 = 0.655$. The initial data is $u(0) = 0.2,\ v(0) = 0.25$.

that for a sufficiently small ϵ we have

$$LV(u,v) \leq -1, \text{ for any } (u,v) \in \mathbb{R}^2_+ \setminus E_\epsilon. \tag{4.4.17}$$

Hence, the assertion $(\mathbf{P_2})$ of Lemma.6 holds. Consequently, the stochastic predator-prey model (4.1.1) has a unique stationary distribution. The proof is completed. $\square$

Remark 2. *Theorem.17 shows that when the noises are small enough, then the model (4.1.1) has a unique stationary distribution which is ergodic. The presence of the fractal term "u^α" in our proposed model (4.1.1) makes the difficulties when we prove the Theorem.17. Here we construct a new Lyapunov function and a rectongular set which do not depend on the equilibrium point (u^*, v^*) of the deterministic model (4.0.2). The ergodic property in Theorem.17 means that the solution of the stochastic predator-prey model (4.1.1) tends to a fixed positive point in the sens of time average with probability one, which implies that the system (4.1.1) is permanent.*

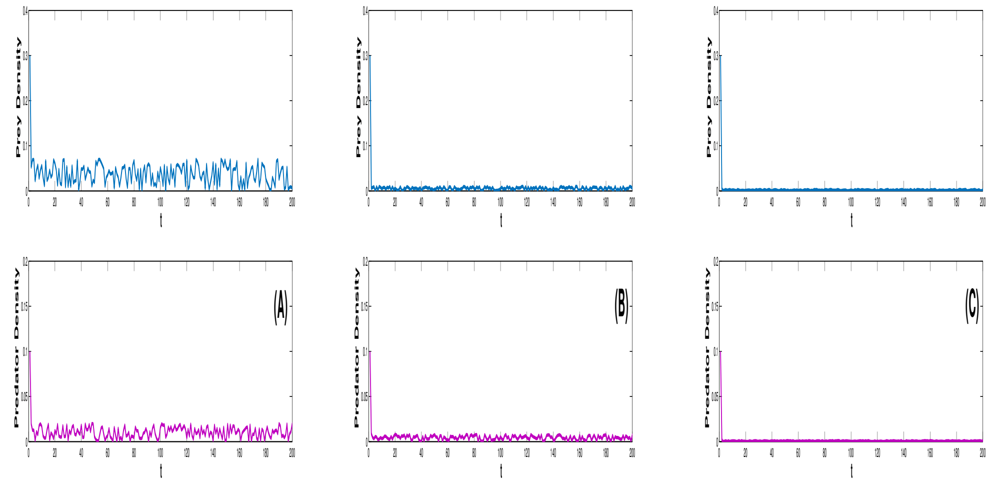

FIGURE 4.6: Numerical simulation of the stochastic predator-prey system (4.1.1) for the parameter values $\rho = 0.55,\ k = 1,\ \delta = 0.26,\ t_h = 0.71,\ \eta = 0.09,\ e = 0.39,\ \alpha = 1/3$. In (A), we choose $\beta^2/2 = 0.54,\ \gamma^2/2 = 0.09$. In (B), we take $\beta^2/2 = 0.551,\ \gamma^2/2 = 0.11$ and for the last case (C), we put $\beta^2/2 = 0.81,\ \gamma^2/2 = 0.52$. Here the initial value $u(0) = 0.2,\ v(0) = 0.25$.

4.5 Numerical simulations

In order to substantiate the analytical findings, we give some numerical examples. Using the semi-implicit Milstein's higher method described in [32], then we obtain the following discretization system as

$$\begin{cases} u_{i+1} = u_i + \left[\rho u_i \left(1 - \frac{u_i}{k}\right) - \frac{\delta u_i^{\alpha} v_i}{1+\delta t_h u_i^{\alpha}}\right] \Delta t + \beta u_i a_i \sqrt{\Delta t} + \frac{\beta^2}{2} u_i (a_i^2 - 1)\Delta t, \\ \\ v_{i+1} = \left[-\eta v_i + \frac{e \delta u_i^{\alpha} v_i}{1+\delta t_h u_i^{\alpha}}\right] \Delta t + \gamma v_i b_i \sqrt{\Delta t} + \frac{\gamma^2}{2} v_i (b_i^2 - 1)\Delta t. \end{cases} \tag{4.5.1}$$

where the time increment $\Delta t > 0$, a_i and b_i are $N(0,1)$ independent Gaussian random variables.

In Fig.4.2, we show the numerical simulation of the deterministic system (4.0.2) with the parameter values $\rho = 0.5,\ k = 1,\ \delta = 0.5,\ t_h = 2,\ e = 0.4,\ \alpha = 0.8$ and different value of the parameter η. For **(A)**, we choose $\eta = 0.9$, then we obtain the extinction of the predator species. In **(B)**, we put $\eta = 0.1$, which gives the coexistence of all both spaces. Next, we fix $\eta = 0.04$ then, the system transits to an oscillatory regime where a limit cycle appears.

In Fig.4.3, we display the graphical representation of the impact of the intrinsic growth rate ρ on both prey and predator densities equilibrium for the same values of the fixed parameters in Fig.4.2 and multi values of ρ. In **(A)**, we choose $\rho = 0.09$ which gives $(u^*, v^*) = (0.343, 0.221)$. In **(B)**, we take $\rho = 0.4$ implies that $(u^*, v^*) = (0.233, 1.621)$. Finally, in **(C)** we set $\rho = 0.9$, yielding $(u^*, v^*) = (0.389, 4.489)$. Here, we denote that (u^*, v^*) represents the positive equilibrium associated with the deterministic system (4.0.2) such that

$$u^* = \left[\frac{\eta}{\delta(e - t_h\eta)}\right]^{\frac{1}{\alpha}}, \ v^* = \frac{e\rho}{\eta}\left(1 - \frac{u^*}{k}\right), \ e > t_h\eta \text{ and } 0 < u^* < k.$$

Clearly, one can see the massive impact of the parameters η and ρ on the dynamical behavior of the deterministic system (4.0.2), especially on the predator density equilibrium. The large value of the death rate of the predator population η may result the extinction of the predator specie. On the other hand, as the parameter ρ increases as the predator density increases with a considerable values. This means that ρ has a positive and significant impact on the predator density. Biologically speaking, The increase in the number of prey individuals within the herd may result a high rate of infection with various diseases, as well as conflicts between males for mating; all of this leads to herd destabilization, which reduces the defensive effectiveness of the pack and thus facilitates the predator's task during the hunting process.

In order to verify the result obtained in Theorem.17, we choose the parameter values $\beta^2/2 = 0.2$, $\gamma^2/2 = 0.168$ and the other parameter values are pointed out in Tab.4.1. Then we obtain $\beta^2/2 + \gamma^2/2 = 0.368 < \rho - \eta = 0.46$ and according to Theorem.17, we can conclude that the stochastic predator-prey system (4.1.1) has a unique ergodic stationary distribution $\chi(.)$ and ergodic property which means that both prey and predator are persistent $a.s.$ This result is depicted in Fig.4.4.

In Fig.4.5 our aim is to examine the case of the extinction of the predator population. To this end, we take $\beta^2/2 = 0.14 < \rho = 0.55$ which means that the condition (**H**) of Theorem.15 is satisfied. Recall that the second condition of Theorem.15 is that $A < 0$ where

$$A = -\eta - \frac{\gamma^2}{2} + e\delta \int_0^\infty \frac{u^\alpha}{1+\delta t_h u^\alpha}\chi(u)\,\mathrm{d}u.$$

Using the parameters given in Tab.4.1 and a simple integral, we obtain

$$e\delta \int_0^\infty \frac{u^\alpha}{1+\delta t_h u^\alpha}\chi(u)\,\mathrm{d}u \approx 0.312.$$

Now, from Fig.4.5 we have tree cases. In **(A)**, we choose $\gamma^2/2 = 0.223$, then we obtain immediately $A = -0.001 < 0$. Next, in **(B)** we put $\gamma^2/2 = 0.322$ which gives $A = -0.1 < 0$. In the last case **(C)**, we take $\gamma^2/2 = 0.655$, it follows that $A = -0.433 < 0$. By comparing the three cases in Fig.4.5, one can easily observe that the predator population goes more and more towards extinction while the prey population persist. which means that the noise associated with the predator population can change the properties of the model greatly. More precisely, comparing Fig.4.5(A) with Fig.4.5(C), we can easily see that with the increase of γ^2 the density of the predator population $v(t)$ tends to the extinction while the the prey population $u(t)$ persist in mean.

In Fig.4.6, we examine numerically the result obtained in Theorem.16. For the first case **(A)** in Fig.4.6, we choose $\beta^2/2 = 0.54$ and $\gamma^2/2 = 0.09$, then $\rho = 0.55 > \beta^2/2 = 0.54$ and $e\delta = 0.10 > \gamma^2/2 = 0.09$. Next, in **(B)** we choose $\beta^2/2 = 0.551$ and $\gamma^2/2 = 0.11$, we obtain $\rho = 0.55 < \beta^2/2 = 0.551$ and $e\delta = 0.10 < \gamma^2/2 = 0.11$. In the last case **(C)**, we take $\beta^2/2 = 0.81$ and $\gamma^2/2 = 0.52$, then we have $\rho = 0.55 < \beta^2/2 = 0.81$ and $e\delta = 0.10 < \gamma^2/2 = 0.52$. Other values of the system parameters can be seen from Tab.4.1. For the two last cases, we can easily see that the conditions of Theorem.16 hold, which explains the extinction of both populations u and v (please see fig.4.6(B) and Fig.4.6(C)). in other words, if the noise intensities β^2 and γ^2 increase, the prey and the predator populations die out exponentially with probability one.

Fig.4.7 represents the impact of the prey herd's shape rate α on the ergodic stationary

Parameters	Description	Values	Source
ρ	The intrinsic growth rate of the prey	0.55	[14, 25]
k	The carrying capacity for the prey	1	[14, 25]
δ	The search efficiency of the predator for the prey	0.26	[14, 25]
t_h	The average handling time for the prey by the predator	0.71	[14, 25]
η	The death rate of the predator in the absence of prey	0.09	[14, 25]
e	The biomass conversion or consumption	0.39	[14, 25]
α	The prey herd's shape rate	1/3	[14, 25]

TABLE 4.1: Lists of parameters used in the simulations of Fig.4.4,4.5,4.6 and Fig.4.7

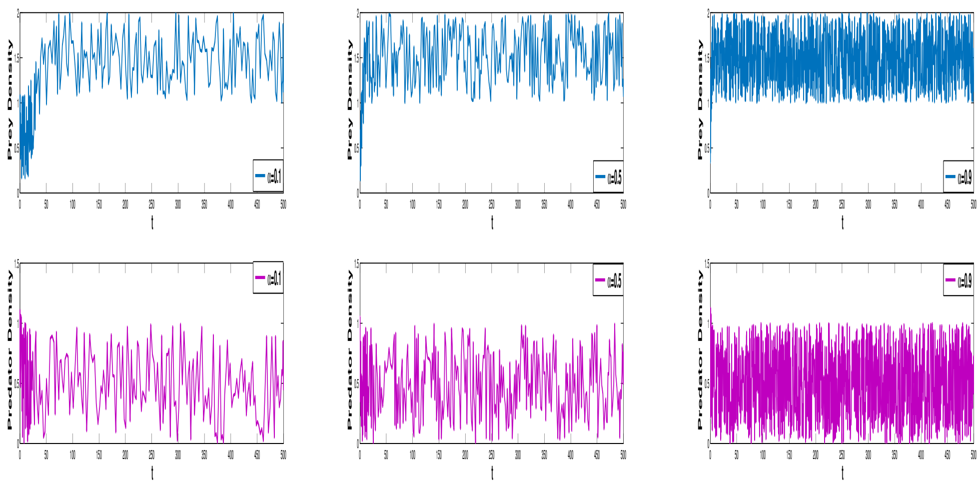

FIGURE 4.7: Impact of the herd shape rate α on the numerical ergodic stationary distribution associated with the system (4.1.1) for the parameter values $\rho = 0.55,\ k = 1,\ \delta = 0.26,\ t_h = 0.71,\ \eta = 0.09,\ e = 0.39,\ \beta^2 = 0.3,\ \gamma^2 = 0.2$ and different values of the parameter α.

distribution associated with the stochastic predator-prey model (4.1.1). We choose $\beta^2/2 = 0.3$ and $\gamma^2/2 = 0.2$, then we obtain $\beta^2/2 + \gamma^2/2 = 0.4 < \rho - \eta = 0.46$. The other parameter values are given in detail in Tab.4.1.

4.6 Discussion

In order to understand the dynamics induced by environmental driving forces, we explain the effect of the environmental noises on the predator-prey interaction in the presence of social behavior for the prey and multiplicative noise. A new approach of a stochastic predator-prey model is obtained. In the great savanna, many living beings gather to together in huge herds. This provides a protection zone and a useful strategy for defending against predators. On the other hand, as it has been mentioned in the introduction section, the prey population can forms several shape of herd, this kind of phenomena has been modeled in [79] and widely studied in literature. Consequently, a new functional responses have been introduced into the interface which are modeled by using a new parameter α represents the prey's herd shape rate. Further, the real life situations are often subject to environmental noises. This gives the necessary and the importance of studying the environmental fluctuations impact on the population systems in ecology. In this work, we consider the predator-prey model (4.0.2) of [25] subject to environmental noises. Our aim is to studies how the intensities of environmental noises affect the stochastic predator-prey model (4.1.1) by revealing the relationships between the coefficients of the population model and the intensities of environmental noise. From the stochastic model analysis, a rich properties have been deduced. First, the existence of the global positive solution as well as the stochastic uniform boundedness of the solution have been successfully confirmed by using conventional methods. Next, the sufficient conditions for the extinction and persistence of the predator and the prey populations have been established where, the extinction criteria is discussed in two cases, the first case is the prey population survival where the predator population die out; the second case is both the prey and predator populations extinction. Moreover, by constructing a suitable stochastic Lyapunov function, it has been proved that the stochastic predator-prey model (4.1.1) has a unique stationary distribution which is ergodic. Theorem.17 show that the staionary distribution exists if the white noise is small. But the large amplitude enviromental fluctuations may destabilize the stochastic system and consequently no stationary distribution can exist. Mathematically speaking, the ergodic stationary distribution can be considered as a stability of system in weak sense, that appears as a solution fluctuating near the positive equilibrium of the corresponding deterministic system (4.0.2). From an biological point of view, this means that both prey and preator populations coexist in the long run. which leads to said that the system is permanent.

By comparing the stochastic predator-prey system with the corresponding deterministic system (4.0.2) which has been studied in [25], two interesting facts have been revealed, the first one is the high environmental noise intensity could drive two species to extinct. In our model, this can be seen in two different cases; the first case is the prey population persist while the predator extinct. This situation was graphically represented in Fig.4.5. The second case is both the two species die out (please see Fig.4.6). Here, it has been remarked that A which defined in Theorem.15, is the crucial parameter for the persistence in the mean and extinction of the model (4.1.1). The second fact, is that the term of herd behavior cannot avoid the extinction of the prey population when the nature presents significant environmental fluctuations although the prey herd's shape has a significant impact on the solution of the stochastic system (4.1.1) (please see Fig.4.7). In the deterministic model (4.0.2), the situation of the extinction of both species is absolutely impossible Fig.4.2). Consequently, we can conclude that the survival of living beings is related to the environmental fluctuations more than the nature of their behaviors.

Prospect and Future Directions

Finally, we would like to mention that some meaningful problems deserve further investigation. For one side, one can propose some more realistic models, such as considering the effects of the prey herd aggressiveness on the predator population, nonlocal prey competition or the harvesting on the populations and so.on. On the other side, it is interesting to introduce the telegraph noise in our model, such as continuous-time Markov chain. The motivation for investigating this is that the living beings suffer from unexpected environmental changes such as global warming, temperature increase, humidity, precipitation changes and so.on. It has been confirmed that animals have specific responses to climate changes. All living beings respond to climate change either through migration or adaptation. But they extinct if they do not reached one of the two options. So, it is interesting to study the impact of all these factors on the predator-prey interaction in order to improve the condition of living beings and avoid the extinction of species to keep the ecosystem balanced. In the next works, we will try to consider more realistic situations in term of mathematical modeling.

Finally, we hope that this thesis can help to further educate students about the interest of this theme.

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