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TAHRI MOHAMMED

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Intitulée

Stabilisation de l'équation des ondes à coefficients variables avec un feedback fractionnaire frontière.

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Devant le jury composé de :

Président :	*Mr HAKEM ALI*	Professeur	Univ. de Sidi Bel Abbès
Directeur de thèse :	*Mr BENAISSA Abbes*	Professeur	Univ. de Sidi Bel Abbès
Examinateurs :			
	Mr BAHLIL Mounir	MC(A)	Univ. de Mascara
	Mr BENIANI Abderrahmene	MC(A)	Univ. d'Ain Temouchent

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Mohammed Tahri

List of symbols

H^2, H^1 The sobolev spaces,
H^{-1}, The dual space of H_0^1,
C^0 The space of continuous functions,
C^1, The space of continuously differentiable functions,
$L(X, Y)$ The space of bounded linear operators from X into$Y$,
X', The dual space of X,
B_X The unit ball in$X$,
$|.|$, The modulus,
$\|.\|$, The norm,
$\langle ., . \rangle$, The scalar product,
sup, The supreme,
inf, The infimum,
∂_t, The partial derivative with respect of t,
∂_{tt}, The second partial derivative with respect of t,
$\partial_t^{\alpha,\eta}$, The generalized fractional derivative,
$I^{\alpha,\eta}$, The generalized fractional itegral,
$\mathbb{R}$, The set of real numbers,
$\mathbb{C}$, The set of complex numbers,
$\mathbb{Z}$, The set of integer numbers,
$\mathbb{N}$, The set of natural numbers,
$\Re e$, The real part,
$\Im m$, The imaginary part,
$\bar{\gamma}$, The conjugate of a complex number γ,
$D(\mathcal{A})$, The domain of $\mathcal{A}$,
$R(\mathcal{A})$, The range of $\mathcal{A}$,
$N(\mathcal{A})$, The kernel of $\mathcal{A}$,
C^T, The Transpose of a matrix C,
$\mathcal{A}^*$, The adjoint operator of $\mathcal{A}$,
$\sigma(\mathcal{A})$, The spectrum of $\mathcal{A}$,
$\sigma_p(\mathcal{A})$, The ponctuel spectrum of $\mathcal{A}$,

$\sigma_r(\mathcal{A})$, The residual spectrum of $\mathcal{A}$,
$\sigma_c(\mathcal{A})$, The The continuous spectrum of $\mathcal{A}$,
$R(.,\mathcal{A})$, The resolvent of $\mathcal{A}$,
o, The little $o: f(x) = og(x)$ for $x \to \infty$ if $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 0$,
O, The big $O: f(x) = O(g(x))$ for $x \to \infty$ if $|f(x)| \leqslant C|g(x)|$ for all $x \geq x_0$,
$\sim$ The asymptotically equivalent: $f \sim g$ for $x \to \infty$ if $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 1$,
Γ, The Gamma function:$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$,
B, The Beta function $: B(u,\upsilon) = \int_0^1 t^{u-1}(1-t)^{\upsilon-1}dt = \frac{\Gamma(u)\Gamma(\upsilon)}{\Gamma(u+\upsilon)}$,
$\overline{\lim\limits_{|\beta|\to\infty}} \|(i\beta I - \mathcal{A})^{-1}\| = \inf\limits_{|\beta|}(\sup\limits_{k\geqslant|\beta|} \|(ikI - \mathcal{A})^{-1}\|)$.

Introduction

In recent years, boundary control of systems represented by PDEs has become an important area of research because it improve the performance of the systems. A control system is defined as a system of devices that manages, commands, directs, or regulates the behavior of other devices or systems to achieve a desired result. Its application ranges widely from earthquake engineering and seismology to fluid transfer, cooling water and noise reduction in cavities, Acoustics, aeronautics, hydraulics, are also some of the diverse disciplines where control theory is applied. This thesis is devoted to the study of the stabilisation of some hyperbolic evolution system with a fractional dissipation. We are concerned with the nondegenerate wave equation with a fractional boundary control.

$$w_{tt}(x,t) - (a(x)w_x(x,t))_x = 0 \text{ in } (0,1) \times (0,+\infty), \tag{1}$$

where the coefficient a is a positive function on $[0,1]$.
Up to now, there are many works concerning the stabilization and controllability of nondegenerate wave equation with different types of dampings (see e.g. [52], [21], [24], [26] and the references therein). In [26], for $a(x) = a_1(x) + a_0$: the authors have established asymptotics stabilization under boundary conditions of the form

$$\begin{cases} (aw_x)(0,t) = 0, \\ (aw_x)(1,t) = -kw(1,t) - w_t(1,t), k > 0. \end{cases}$$

It has been shown in [52], for $a \in H^1(0,1), a(x) \geqslant a_0 > 0$, that the feedback law

$$\begin{cases} (aw_x)(0,t) = -cw(0,t) - Fw_t(0,t), c > 0, \\ Mw_{tt}(1,t) + (aw_x)(1,t) = 0. \end{cases}$$

exponentially stabilizes equation (1) under appropriate assumptions on the function F. In [21] the authors considered the following modelization of a flexible torque arm controlled by two feedbacks depending only on the boundary velocities:

$$\begin{cases} w_{tt}(x,t) - (a(x)w_x)_x + \alpha w_t(x,t) + \beta w(x,t) = 0, 0 < x < 1, t > 0, \\ (a(x)w_x)(0) = k_1\omega_t(0,t), t > 0, \\ (a(x)w_x)(1) = -k_2 w_t(1,t), t > 0, \end{cases}$$

where

$$\begin{cases} \alpha \geqslant 0, \beta > 0, k_1, k_2 \geqslant 0, k_1 + k_2 \neq 0, \\ a \in W^{1,\infty}(0,1), a(x) \geqslant a_0 > 0 \quad \text{for all} \quad x \in [0,1]. \end{cases}$$

They proved the exponential decay of the solutions. In [44] Mbodje studies the energy decay of the wave equation ($a \equiv 1$). with a boundary fractional derivative control. He considered the following system

$$\begin{cases} w_{tt}(x,t) - \ w_{xx}(x,t) = 0 & \text{in } (0,1) \times (0,+\infty), \\ w(0,t) = 0 & \text{on } (0,+\infty) \\ w_x(1,t) = -\gamma \partial_t^{\alpha,\eta} \ w(1,t) & \text{on } (0,+\infty), \\ w(x,0) = \ w_0(x), \ w_t(x,0) = \ w_1(x) & \text{on } (0,1), \end{cases}$$

and used a diffusive representation and the semi-group theory to establish the strong asymptotic stability of solutions when $\eta = 0$ and a polynomial type decay rate $E(t) \leqslant \frac{C}{t}$ if $\eta \neq 0$. Our purpose in this thesis is to give a global solvability in Sobolev spaces and energy decay estimates of the solutions to the problem (1) for linear damping and to show that system (1) is not exponentially stable for a general nondegenerate function a. Furthermore, we prove that the solution decays to zero polynomially when t goes to infinity for general initial data taken in the domain of $\mathcal{A}$ and for a general nondegenerate function a for both cases $\eta > 0$ and $\eta = 0$. The boundary feedback under the consideration in this thesis are of fractional type and are described by the fractional derivatives

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s) ds, \eta \geq 0.$$

The order of our derivatives is between 0 and 1. Very little is known in the literature. In addition to being nonlocal, fractional derivatives involve singular and non-integrable kernels ($t^{\alpha}, 0 < t < 1$). This makes the problem more delicate. It has been shown (see [43]) that, as ∂_t the fractional derivative ∂_t^{α} forces the system to become dissipative and the solution to approach the equilibrium state. Therefore, when applied on the boundary, we can consider them as controllers which help to reduce the vibrations. This thesis is divided into three Chapters :

CHAPTER 1: Preliminaries

Firstly, in this Chapter, we present some well known results on Sobolev spaces and some basic definitions and theorems. Secondly, we recall some results on a C_0-semigroup, including some theorems on strong, exponential and polynomial stability of a C_0-semigroup. Next, we display a brief historical introduction to fractional derivatives and we define the fractional derivative operator and we present some physical interpretations. After that, we present the Bessel functions and their basic definitions. Finally, we present an appendix that contains almost all the secondary calculations used in this Thesis.

CHAPTER 2: Energy decay for a nondegenerate wave equation with a fractional boundary control.

In this chapter, we are concerned with the system

$$\begin{cases} w_{tt}(x,t) - \left(a(x)w_x(x,t)\right)_x = 0 & \text{in } (0,1)\times(0,+\infty), \\ w(0,t) = 0 & \text{on } (0,+\infty), \\ \left(aw_x\right)(1,t) = -\varrho \partial_t^{\alpha,\eta} w(1,t) & \text{on } (0,+\infty), \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x) & \text{on } (0,1), \end{cases} \tag{P}$$

where $\varrho > 0$. The notation $\partial^{\alpha,\eta}$, stands for the generalized Caputo's fractional derivative (see [11] and [23]) defined by the following formula:

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)ds, \ \eta \geq 0,$$

where Γ is the usual Euler gamma function and ($0 < \alpha < 1$). We show that the problem is not uniformly stable by a spectrum method and we study the polynomial stability using the semigroup theory of linear operators. using a frequency domain approach, we estabish an optimal polynomial energy decay depending with the parameter for smooth solution.

CHAPTER 3: Global existence and stabilization of nondegenerate wave equation with a dynamic boundary dissipation.

In this chapter we investigate the existence and decay properties of solutions for the following initial boundary value problem :

$$\begin{cases} w_{tt}(x,t) - \left(a(x)w_x(x,t)\right)_x = 0 & \text{in } (0,L)\times(0,+\infty), \\ w(0,t) = 0 & \text{on } (0,+\infty), \\ mw_{tt}(L,t) + \left(aw_x\right)(L,t) = -\varrho \partial_t^{\alpha,\eta} w(L,t) & \text{on } (0,+\infty), \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x) & \text{on } (0,L). \end{cases} \tag{Q}$$

we study the existence, uniqueness and stability of solutions for the nondegenerate wave equation with a dynamic boundary dissipation of fractional derivative type, and we proved optimal polynomial decay estimates in appropriate spaces. The results are obtained through an estimate on the resolvent of the generator associated with the semigroup.

Chapter 1

PRELIMINARIES

In this chaptres, we recall some basic definitions and theorems which will be used in the following chapters. We refer to [1, 4, 13, 12, 14, 24, 30, 51].

1.1 Sobolev spaces

In many problems of mathematical physics it is not sufficient to deal with the classical solutions of partial differential equations(PDE). It is necessary to introduce the notion of weak derivatives and to work in the so called Sobolev spaces. We denote by Ω an open domain in $\mathbb{R}^n, n \geqslant 1$. We will also use the following multi-index notation for partial differential derivatives of a function

$$\partial_i^k u = \frac{\partial^k u}{\partial x_i^k} \text{ for all } k \in \mathbb{N} \text{ and } i = 1, ..., n, ...$$

$$D^\alpha u = \partial_1^{\alpha_1} \partial_2^{\alpha_2} ... \partial_n^{\alpha_n} u = \frac{\partial^{\alpha_1 + ... + \alpha_n} u}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha n}}$$

$$\alpha = (\alpha_1, \alpha_2, ... \alpha_n) \in \mathbb{N}^n, |\alpha| = \alpha_1 + ... + \alpha_n.$$

Definition 1.1.1. For $1 \geq p \geq \infty$, we call $L^p(\Omega)$ the space of measurable functions f on Ω such that

$$||f||_{L_p(\Omega)} = \left(\int_\Omega |f(x)|^p dx \right) < +\infty \text{ for } p < +\infty$$

$$||f||_{L^\infty(\Omega)} = \sup_\Omega |f(x)| < +\infty \text{ for } p = +\infty$$

The space $L^p(\Omega)$ equipped with the norm $f \to ||f||_{L_p}$ is a Banach space: it is reflexive and separable for $1 < p < \infty$ (its dual is $L^{\frac{p}{p-1}}(\Omega)$), separable but nor reflexive for $p = 1$ (its dual is $L^\infty(\Omega)$), and not separable, not reflexive for $p = \infty$ (its dual contains stricty $L^1(\Omega)$). In particular the space $L^2(\Omega)$ is a Hilber space equipped wiht the scalar product defined by

$$(f, g)_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx$$

Definition 1.1.2. The Sobolev space $W^{m,p}(\Omega)$ is defined to be the subset of L^p such that function f and its weak derivatives up to some order m have a finite L^p norm, For given $p \geq 1$.

$$W^{m,p}(\Omega) = \{f \in L^p(\Omega), D^\alpha f \in L^p(\Omega). \text{ for all } \alpha, |\alpha| \geq m\}$$

With this definition, the Sobolev spaces admit a natural norm. and

$$f \to ||f||_{w^{m,p}(\Omega)} = \Big(\sum_{|\alpha| \leq m} ||D^\alpha f||^P_{L^P(\Omega)} \Big)^{1/p}, \quad \text{for } p < +\infty$$

and

$$f \to ||f||_{w^{m,p}(\Omega)} = \sum_{|\alpha| \leq m} ||D^\alpha f||_{L^\infty(\Omega)}, \quad \text{for } p = +\infty$$

The space $W^{m,p}(\Omega)$ equipped with the norm $||.||_{w^{m,p}}$ is a Banach space. Moreover is a reflexive space for $1 < p < \infty$ and a separable space for $1 \leq p < \infty$.

Remark 1.1.1. Sobolev spaces $W^{m,p}(\Omega)$ with $p = 2$ are especially important because of their connection with Fourier series and because they form a Hilbert space. A special notation has arisen to cover this case:

$$W^{m,2}(\Omega) = H^m(\Omega)$$

the H^m inner product is defined in terms of the L^2 inner product:

$$(f, g)_{H^m(\Omega)} = \sum_{|\alpha| \leq m} (D^\alpha f, D^\alpha g)_{L^2(\Omega)}$$

1.2 M-Dissipative operators

In this section we introduce unbounded operators and put together some properties which will be frequently used.

1.2.1 Unboubded Linear Operators on Banach space

Let X and Y be two Banach spaces.

Definition 1.2.1. An unbounded linear operator from X into Y is linear map $\mathcal{A} : D(\mathcal{A}) \subset X \to Y$ defined on a subspace $D(\mathcal{A}) \subset X$ with values in Y. The set $D(\mathcal{A})$ is called the domain of the operateur $\mathcal{A}$. If $X = Y$, we shall simply say that $\mathcal{A}$ is an unbounded linear operator on X.

Definition 1.2.2. One says that $\mathcal{A}$ is bounded if $D(\mathcal{A}) = X$ and if there is a constant $C \geq 0$

$$||\mathcal{A}x||_Y \leq C||x||_X \quad \forall x \in X$$

The set of all bounded linear operators from X into Y is denoted by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$, Moreover, the set of all bounded linear operators from X into X is denoted by $\mathcal{L}(\mathcal{X})$. The norm of a bounded linear operator is define by

$$||\mathcal{A}||_{\mathcal{L}(X,Y)} = \sup_{x\neq 0} \frac{||\mathcal{A}x||_Y}{||x||_X}$$

Definition 1.2.3. Let $\mathcal{A} : D(\mathcal{A}) \subset X \to Y$ be an unbounded linear operator. We define
Graph of $\mathcal{A}$: $G(\mathcal{A}) = \{(x, \mathcal{A}x) : x \in D(\mathcal{A})\} \subset X \times Y$,
Range of $\mathcal{A} : R(\mathcal{A}) = \{\mathcal{A}x : x \in D(\mathcal{A})\} \subset Y$,
Kernal of $\mathcal{A} : N(\mathcal{A}) = \{x \in D(\mathcal{A}) : \mathcal{A}x = 0\} \subset X$.

Definition 1.2.4. An unbounded linear operator $\mathcal{A}$ is a closed operateur if its graph $G(\mathcal{A})$ is closed in $X \times Y$

Definition 1.2.5. Let $\mathcal{A} : D(\mathcal{A}) \subset X \to Y$ be an unbounded linear operator . We say that $\mathcal{A}$ is a densely defined operateur in X, or $\mathcal{A}$ is an operateur with dense domaine in $\mathcal{A}$, if $D(\mathcal{A})$ is dense in $X, i.e., \overline{D(\mathcal{A})} = X$.

Definition 1.2.6. Let $\mathcal{A} : D(\mathcal{A}) \subset X \to Y$ be a densely defined operator in X. The adjoint operator of $\mathcal{A}$ is the operator $\mathcal{A}^* : D(\mathcal{A}^*) \subset Y' \to X'$ defined by

$$D(\mathcal{A}^*) = \{y \in Y' : \exists C \geq 0 \text{ such that } \langle \mathcal{A}x, y\rangle_{Y\times Y'} \leq C||x||_X \text{ for all } x \in D(\mathcal{A})\}$$

and

$$\langle x, \mathcal{A}^*y\rangle_{X\times X'} = \langle \mathcal{A}x, y\rangle_{Y\times Y'} \text{ for all } x \in D(\mathcal{A}), \text{ for all } y \in D(\mathcal{A}^*)$$

Definition 1.2.7. $\mathcal{A}$ bounded linear operator $\mathcal{A} : X \to Y$ is said to be compact if $T(B_X)$ has compact closure in Y. The set of all compact operators from X into Y is denoted by $K(X, Y)$. Moreover, the set $K(X, X)$ is denoted by $K(X)$.

Theorem 1.2.1. ***(Fredholm alternative)*** *. Let $\mathcal{A} \in K(X)$. Then:*
1. $N(I - \mathcal{A})$ is finite-dimensionel.
2. $N(I - \mathcal{A})$ is closed and $R(I - \mathcal{A}) = N(I - \mathcal{A}^*)^{\perp}$,
3. $N(I - \mathcal{A}) = 0 \Leftrightarrow R(I - \mathcal{A}) = X$,
4. dim $N(I - \mathcal{A}) =$ dim $N(I - \mathcal{A}^*)$.

Remark 1.2.1. The Fredholm alternative deals with solvability of the equation $u - \mathcal{A}u = f$.

1.2.2 The Resolvent set and the Spectrum of Linear Operators

Let X be a Banach space, and $\mathcal{A}$ be a closed unbounded operator on X.

Definition 1.2.8. The resolvent set of $\mathcal{A}$ is given by

$$\rho(\mathcal{A}) = \{\lambda \in \mathbb{C}; \lambda I - \mathcal{A} : D(\mathcal{A}) \to X \text{ is bijective } \}$$

and its spectrum by

$$\sigma(\mathcal{A}) = \mathbb{C} \setminus \rho(\mathcal{A})$$

if $\lambda \in \rho(\mathcal{A})$, then $R(\lambda, \mathcal{A}) = (\lambda I - \mathcal{A})^{-1}$ is called the resolvent of $\mathcal{A}$.

Remark 1.2.2. The numbers in $\rho(\mathcal{A})$ are called regular values of $\mathcal{A}$.

Theorem 1.2.2. *The sets $\rho(\mathcal{A})$ and $\sigma(\mathcal{A})$ are open and closed, respectively.*

Definition 1.2.9. The point spectrum or ponctuel spectrum of $\mathcal{A}$ is defined by

$$\sigma_p(\mathcal{A}) = \{\lambda \in \mathbb{C} : \text{ there exists some } v \in D(\mathcal{A}) \setminus \{0\} \text{ with } \mathcal{A}v = \lambda v\}$$
$$= \{\lambda \in \mathbb{C} : N(\lambda I - \mathcal{A}) \neq \{0\}\} \subset \sigma(\mathcal{A}).$$

Remark 1.2.3. If $\lambda \in \sigma_p(\mathcal{A})$, then there exists a vecteur $v \neq 0$ such that $(\lambda I - \mathcal{A})v = 0$, i.e., $\mathcal{A}v = \lambda v$. Such a vector is called un eigenvector of $\mathcal{A}$ and the corresponding number λ an eigenvalue of $\mathcal{A}$.

Definition 1.2.10. The continuous spectrum $\sigma_c(\mathcal{A})$ is the set of all numbers $\lambda \in \mathbb{C}$ such that $N(\lambda I - \mathcal{A}) = 0, R(\lambda I - \mathcal{A}) \neq X,$ but $\overline{R(\lambda I - \mathcal{A})} = X$.

Definition 1.2.11. The residual spectrum $\sigma_r(\mathcal{A})$ is the set of all numbers $\lambda \in \mathbb{C}$ such that $N(\lambda I - \mathcal{A}) = 0,$ and$\overline{R(\lambda I - \mathcal{A})} \neq X$.

Remark 1.2.4. It is apparent that the sets $\sigma_p(\mathcal{A}), \sigma_c(\mathcal{A}), \sigma_r(\mathcal{A})$ are disjoint, and that

$$\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) \cup \sigma_c(\mathcal{A}) \cup \sigma_r(\mathcal{A})$$

Proposition 1.2.1. (**Spectrum of the adjoint operator**). Let $\mathcal{H}$ be a Hilbert space, and $\mathcal{A} \in \mathcal{L}(\mathcal{H})$. Then:
(i) $\lambda \in \rho(\mathcal{A}) \Leftrightarrow \bar{\lambda} \in \rho(\mathcal{A}^*)$.
(ii) $\lambda \in \sigma_p(\mathcal{A}) \Rightarrow \bar{\lambda} \in \sigma_p(\mathcal{A}^*) \cup \sigma_r(\mathcal{A}^*)$.
(iii) $\lambda \in \sigma_r(A) \Rightarrow \bar{\lambda} \in \sigma_p(\mathcal{A}^*)$.
(iv) $\lambda \in \sigma_c(\mathcal{A}) \Rightarrow \bar{\lambda} \in \sigma_c(\mathcal{A}^*)$.

1.2.3 M-Dissipative Operators on Hilbert spaces

Let $\mathcal{H}$ Hilbert space equiped with the inner product $\langle ., . \rangle_{\mathcal{H}}$.

Definition 1.2.12. An unbounded linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is said to be dissipative if

$$\forall x \in D(\mathcal{A}), \ \langle \mathcal{A}x, x \rangle_{\mathcal{H}} \leq 0.$$

Remark 1.2.5. For a complex Hilbert space the previous condition is replaced by

$$\forall x \in D(\mathcal{A}), \ \Re e \langle \mathcal{A}x, x \rangle_{\mathcal{H}} \leq 0.$$

Definition 1.2.13. An unbounded linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is m- dissipative (or maximal dissipative) if 1. $\mathcal{A}$ is dissipative. 2. $\lambda I - \mathcal{A}$ is surjective fo every $\lambda > 0, i.e., \forall y \in \mathcal{H}, \forall \lambda > 0, \exists x \in D(\mathcal{A})$. such that

$$\lambda x - \mathcal{A}x = y.$$

Theorem 1.2.3. *Let $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ be an unbounded linear dissipative operator. The operator $\mathcal{A}$ is m-dissipative if and onlysuch that if $\exists \lambda_0 > 0$ such that $\lambda_0 I - \mathcal{A}$, i.e.,$R(\lambda_0 I - \mathcal{A}) = \mathcal{H}$.*

Theorem 1.2.4. *$If \mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is an m-dissipative operator, then*
1. $\mathcal{A}$ is closed operator,
2. $D(\mathcal{A})$ is dense in $\mathcal{H}, i., e., \overline{D(\mathcal{A})} = \mathcal{H}$,
3. $]0, +\infty[\subseteq \rho(\mathcal{A})$.

1.3 Semigroups of Linear Operators in Banach space

In this section we introduce semigroups and their generators. newline Let X be a Banach space, and $\mathcal{H}$ be a Hilbert space equiped with the inner product $(.,.)_{\mathcal{H}}$ and the iduced norm $||\ .\ ||_{\mathcal{H}}$.

1.3.1 Strongly Continuous Semigroups Generated by Dissipative Operator

We consider the linear Cauchy problem

$$(C) \qquad \begin{cases} u'(t) = \mathcal{A}u(t) \\ \quad u(0) = u_0 \end{cases}$$

where $\mathcal{A}$ is an unbounded operator on X. By using operator semigroup theory, we establish some results about the existence and uniqueness of solution of (C).

Definition 1.3.1. A family of bounded linear operators $(S(t))_{t\geq 0}$ on X is a semigroup of bounded linear operators on X if
1. $S(0) = I$,
2. $S(t+s) = S(t)S(s)$ for every $s, t \geq 0$

Remark 1.3.1. It follows immediately from the definition that

$$S(t)S(s) = S(s)S(t), \text{ for all } t, s \geq 0$$

Definition 1.3.2. A semigroup $((S(t))_{t\geq 0}$ is uniformly continuous if

$$\lim_{t\to 0+} ||S(t) - I||_{\mathcal{L}(\mathcal{X})} = 0$$

Definition 1.3.3. A semigroup $((S(t))_{t\geq 0}$ is a C_0 -semigroup (or a strongly continuous semigroup) if

$$\lim_{t\to 0+} ||S(t)x - x||_{\mathcal{X}} = 0$$

Theorem 1.3.1. *Let $(S(t))_{t\geq 0}$ be a C_0-semigroup. Then there exist two constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that*

$$||S(t)||_{\mathcal{L}(\mathcal{X})} \leqslant Me^{\omega t}, \forall t \geq 0$$

Remark 1.3.2. If $\omega = 0, i.e.,$

$$||S(t)||_{\mathcal{L}(\mathcal{H})} \leqslant M, \forall t \geq 0$$

then $(S(t))_{t\geq 0}$ is called a uniformly bounded C_0-semigroup.
If $\omega = 0$, and $M = 1, i.e.,$

$$||S(t)||_{\mathcal{L}(\mathcal{H})} \leqslant 1, \forall t \geq 0$$

then $(S(t))_{t\geq 0}$ is called a strongly continuous semigroup (or C_0-semigroup) of contractions.
We now define the generator of semigroup.

Definition 1.3.4. Let $(S(t))_{t\geq 0}$ be a C_0-semigroup. The infinitesimal generator of the semigroup $(S(t))_{t\geq 0}$ is the linear operator $\mathcal{A}$ defined by

$$D(\mathcal{A}) = \{x \in X : \lim_{t\to 0+} \frac{S(t) - x}{t} \text{ exists in X } \}$$

and

$$\mathcal{A}x = \lim_{t\to 0+} \frac{S(t) - x}{t}, \forall x \in D(\mathcal{A})$$

Remark 1.3.3. Sometimes we also denote $S(t)$ by $e^{\mathcal{A}t}$

Theorem 1.3.2. *Let $(S(t))_{t\geq 0}$ be a C_0-semigroup and let $\mathcal{A}$ be its infinitesimal generator. Then*

$$S(t)x \in D(\mathcal{A})$$

and

$$\frac{d}{dt}S(t)x = \mathcal{A}S(t)x = S(t)\mathcal{A}x$$

for $x \in D(\mathcal{A})$ and $t \geq 0$

Remark 1.3.4. From the above theorem, the solution to the initial value problem (C) admits the following representation

$$u(t) = S(t)u_0 = e^{\mathcal{A}t}u_0 \;\forall t \geq 0$$

The following theorems (Theorem 1.3.3 and Theorem 1.3.4)gives a necessary and sufficient condition for an operator to be the generator of a C_0-semigroup (see Pazy [49]).

Theorem 1.3.3. ***(Hill-Yosida Theorem in Banach spaces)*** *An unbounded linear operator $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ is the infinitesimal generator of a semigroup of contractions if and only if*
1. $D(\mathcal{A})$ is dense in $X, i, e., \overline{D(\mathcal{A})} = X$,
2. $\mathcal{A}$ is a closed operator,
3. The resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$ contains $\mathbb{R}_+$ and for every $\lambda > 0$,

$$||R(\lambda, \mathcal{A})||_{\mathcal{L}(X)} \leq \frac{1}{\lambda}$$

Theorem 1.3.4. ***(Lumer-Phillips Theorem in Hilbert spaces)*** *An unbounded linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is the infinitesimal generator of a semigroup of contractions if and only if $\mathcal{A}$ is m-dissipative operator.*

The existence and uniqueness of the solution of the initial value problem (C) is justified by the following theorem.

Theorem 1.3.5. ***(Hill-Yosida Theorem)*** *Let $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ be an unbounded linear operator. If $\mathcal{A}$) is the infinitesimall generator of $(S(t))_{t\geq 0}$ a C_0-semigroup of contraction, (or $\mathcal{A}$ is m-dissipative operator), then 1. if $U_0 \in D(\mathcal{A})$, then the initial value problem (C) has a unique strong solution*

$$U \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

2. if $U_0 \in \mathcal{A}$, then the initial value problem (C) has a unique weak solution

$$U \in C^0(\mathbb{R}_+, \mathcal{H})$$

1.3.2 Stability of Semigroups

The stability theory of semigroups provides powerful tools for the investigation of the convergence to 0 of weak and strong solutions of linear Cauchy problem

$$(C) \qquad \begin{cases} u'(t) = \mathcal{A}u(t) \\ \quad u(0) = u_0 \end{cases}$$

where $\mathcal{A}$ generates the C_0-semigroup of contraction $(S(t))_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. In this section, we introduce the notions of stability that will be used throughout this thesis. Let $(S(t))_{t\geq 0}$ be a C_0-semigroup of contractions on a $\mathcal{H}$ and let $\mathcal{A}$ be its infinitesimal generator.

Definition 1.3.5. (Strong stability) We say that the semigoup $(S(t))_{t\geq 0}$ is strongly (or asymptotically) stable if for all $x \in \mathcal{H}$

$$\lim_{t\to+\infty} ||e^{\mathcal{A}t}x||_{\mathcal{H}} = 0$$

Definition 1.3.6. (Exponential stability) We say that the semigoup $(S(t))_{t\geq 0}$ is exponentially (or uniformly) stable if there exist $\alpha, M > 0$ such that

$$||S(t)x||_{\mathcal{H}} \leq Me^{-\alpha t}, \ \forall t \geq 0, \forall x \in \mathcal{H}$$

Definition 1.3.7. (Polynomial stability) We say that the semigoup $(S(t))_{t\geq 0}$ is is polynomially stable if there exist $\beta, C > 0$ such that

$$||S(t)x||_{\mathcal{H}} \leq \frac{C}{t^\beta}||x||_{\mathcal{H}}, \ \forall t \geq 0, \forall x \in \mathcal{H}$$

The following theorem (a general criteria of Arendt-Batty) gives a necessary conditions for a strong stability of the C_0-semigroup (see [4]).

Theorem 1.3.6. ***(Arendt-Batty)*** *Let $\mathcal{A}$ be the generator of a uniformly bounded C_0-semigroup $(S(t))_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. If: (i) $\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.*
(ii) The intersection of the spectrum $\sigma(\mathcal{A})$ with $i\mathbb{R}$ is at most a countable set.

Then the semigroup $(S(t))_{t\geq 0}$ is strongly (or asymptotically) stable, i.e, $||S(t)z||_{\mathcal{H}} \to 0$ as $t \to \infty$ for any $z \in \mathcal{H}$.
When the C_0-semigroup is asymptotically, we look the type of stability (exponential or polynomial) of the semigroup (see [51], [9] and [12]).

Theorem 1.3.7. ***(Huang-Pruss)*** *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on Hilbert space $\mathcal{H}$. Then $(S(t))_{t\geq 0}$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\beta : \beta \in \mathbb{R}\} = i\mathbb{R}$$

and

$$\overline{\lim_{|\beta|\to\infty}} ||(i\beta I - \mathcal{A})^{-1}||_{\mathcal{L}(\mathcal{H})} < \infty$$

This theorem is equivalent to the following theorem:

Theorem 1.3.8. *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on Hilbert space $\mathcal{H}$. Then $(S(t))_{t\geq 0}$ is exponentially stable if and only if*

$$sup\{\Re e\ \lambda, \lambda \in \sigma(\mathcal{A})\} < 0$$

and

$$\sup_{\Re e\lambda \geqslant 0} ||(\lambda I - \mathcal{A})^{-1}||_{\mathcal{L}(\mathcal{H})} < \infty$$

Theorem 1.3.9. ***(Borichev-Tomilov)*** *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup on a Hilbert space $\mathcal{H}$. If*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \quad and \quad \sup_{|\beta|\geqslant 1} \frac{1}{\beta^{l'}} ||(i\beta I - \mathcal{A})^{-1}||_{\mathcal{L}(\mathcal{H})} \leqslant M.$$

for some l', then there exist c such that

$$||e^{\mathcal{A}t}u_0||^2 \leq \frac{c}{t^{\frac{2}{l'}}} ||u_0||^2_{D(\mathcal{A})}, \forall t > 0, \forall u_0 \in D(\mathcal{A}).$$

1.4 Lax-Milgrame Theorem

Let $\mathcal{A}$ be a Hilbert space equiped with the inner product $(.,.)_{\mathcal{H}}$ and the iduced norm $||.||_{\mathcal{H}}$.

Definition 1.4.1. A bilinear form

$$a : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$$

is said to be
(i) continuous if there is a constant C such that

$$|a(u,v)| \leq C||u||||v||, \forall u, v \in \mathcal{H}$$

(ii) coercive if there is a constant $\alpha > 0$ such that

$$a(u, v) \geq \alpha ||u||^2, \forall u \in \mathcal{H}$$

Theorem 1.4.1. ***(Lax-Milgrame Theorem)*** *Assume that $a(.,.)$ is a continuous coercive bilinear form on $\mathcal{H}$. Then, given any $L \in \mathcal{L}(\mathcal{H}, \mathbb{C})$, there exists a unique element $u \in \mathcal{H}$ such that*

$$a(u, v) = L(v), \forall v \in \mathcal{H}.$$

1.5 Fractional Derivatives:

Basic definitions

Fractional derivative, or more precisely derivative of non-integer order, is a generalization of ordinary derivation. The fractional derivatives have been used in various fields of science and engineering, for example in electronics, wave propagation, mechanics, biology, biophysics and viscoelasticity (see [6], [7], [8], [30], [39],[50] and [56]). In this part, we recall some basic notations and definitions for the fractional derivative (see [10], [40]).

1.5.1 A brief historical introduction to fractional derivatives

In a letter to the French mathematician L'Hospital (1659), Leibniz raised the following question: "Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?" L'Hospital was some what curious about that question and replied by another question to Leibniz: "What if the order will be 1/2?" Leibnitz in a letter dated September 30, replied: "It will lead to a paradox, from which one day useful consequences will be drawn. Many known mathematicians contributed to this theory over the years. Thus, September 30, 1695 is the exact date of birth of the fractional calculus. Therefore, the fractional calculus it its origin in the works by Leibnitz, L'Hopital (1695), Bernoulli (1697), Euler (1730), and Lagrange (1772). Some years later, Laplace (1812), Fourier (1822), Abel (1823), Liouville (1832), Riemann (1847), Grunwald(1867),Letnikov (1868), Nekrasov (1888), Hadamard (1892), Heaviside (1892), Hardy (1915), Weyl (1917), Riesz (1922), P.Levy (1923), Davis (1924), Kober (1940), Zygmund (1945), Kuttner (1953), J. L. Lions (1959), and Liverman (1964)...have developed the basic concept of fractional derivatives. In 1783, Leonhard Euler made his first comments on fractional order derivative. He worked on progressions of numbers and introduced first time the generalization of factorials to Gamma function. A little more than fifty year after the death of Leibniz, Lagrange, in 1772, indirectly contributed to the development of exponents law for differential operators of integer order, which can be transferred to arbitrary order under certain conditions. In 1812, Laplace has provided the first detailed definition for fractional derivative. Laplace states that fractional derivative can be defined for functions with representation by an integral, in modern notation it can be written as $\int f(t)t^{-x}dt$. Few years after, Lacroix worked on generalizing the integer order derivative of function $f(t) = t^m$ to fractional order, where m is some natural number. In modern notations, integer order n^{th} derivative derived by Lacroix can be given as

$$\frac{d^n f}{dt^n} = \frac{m!}{(m-n)!}t^{m-n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}t^{m-n}, m > n$$

where , Γ is the Gamma function. Thus, for $n = \frac{1}{2}$ and $m = 1$, one obtains the derivative of order $\frac{1}{2}$ of the function $f(t) = t$

$$\frac{d^{\frac{1}{2}} f(t)}{dt^{\frac{1}{2}}} = \frac{\Gamma(2)}{\Gamma(\frac{3}{2})}t^{\frac{1}{2}} = \frac{2}{\sqrt{\pi}}\sqrt{t}$$

In the period 1900-1970 a modest amount of published work appeared on the subject of the fractional derivative. The year 1974 saw the first international conference on fractional calculus held at the University of New Haven.
In the period 1975 to the present, many papers have been published relating to the application of the fractional derivative to ordinary and partial diferential equations.

1.5.2 Some notations and denitions of Fractional derivatives

In this section, we give the definition of the generalized Caputo's fractional derivative and the generalized fractional integral.

Definition 1.5.1. The Gamma function, denoted by Γ, is given by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$$

The exponential provides the convergence of this integral in ∞, the convergence at zero obviously occurs for all complex z from the right half of the complex plane ($\Re e(z) > 0$).
The Gamma function is generalization of a factorial in the following form

$$\Gamma(n) = (n-1)!$$

Remark 1.5.1. **(Some usefull identities)** We have

$$\Gamma(z+1) = z\Gamma(z)$$

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}$$

Definition 1.5.2. The fractional derivative of order $\alpha, 0 < \alpha < 1$, in sens of Caputo, is defined by

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{df}{ds}(s)ds$$

Definition 1.5.3. The fractional integral of order $\alpha, 0 < \alpha < 1$, in sens Riemann-Liouville, is defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds$$

Remark 1.5.2. From the above denitions, clearly

$$D^\alpha f = I^{1-\alpha} Df, 0 < \alpha < 1$$

Lemma 1.5.1.

$$I^\alpha D^\alpha f(t) = f(t) - f(0), 0 < \alpha < 1$$

Lemma 1.5.2. *If*

$$D^\beta f(0) = 0$$

then

$$D^\alpha D^\beta f = D^{\alpha+\beta} f,\ 0 < \alpha < 1,\ 0 < \beta < 1.$$

Now, we give the definitions of the generalized Caputo's fractional derivative and the generalized fractional integral. These exponentially modified fractional integro-differential operators were first proposed in [20].

Definition 1.5.4. The generalized Caputo's fractional derivative is given by

$$D^{\alpha,\eta} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{df}{ds}(s) ds,\ 0 < \alpha < 1,\ \eta \geq 0$$

Remark 1.5.3. The operators D^α and $D^{\alpha,\eta}$ differ just by their kernels.

Definition 1.5.5. The generalized fractional integral is given by

$$I^{\alpha,\eta} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\eta(t-s)} f(s) ds,\ 0 < \alpha < 1,\ \eta \geq 0$$

Remark 1.5.4. We have

$$D^{\alpha,\eta} f = I^{1-\alpha,\eta} Df,\ 0 < \alpha < 1,\ \eta \geq 0$$

Chapter 2

Energy decay of solutions to a nondegenerate wave equation with a fractional boundary control

2.1 Introduction

In this chapter, we are concerned with the boundary stabilization of convolution type for nondegenerate wave equation of the form

$$w_{tt}(x,t) - (a(x)w_x(x,t))_x = 0 \text{ in } (0,1) \times (0,\infty), \tag{2.1}$$

where the coefficient a is a positive function on $[0,1]$.
Up to now,there are many works concerning the stabilization and controllability of nondegenerate wave equation with different types of dampings (see e.g. [52], [21], [24], [26] and the references therein). In[26], for $a(x) = a_1(x) + a_0$: the authors have established asymptotics stabilization under boundary conditions of the form

$$\begin{cases} (aw_x)(0,t) = 0, \\ (aw_x)(1,t) = -kw(1,t) - w_t(1,t), k > 0. \end{cases}$$

It has been shown in[52], for $a \in H^1(0,1), a(x) \geqslant a_0 > 0$, that the feedback law

$$\begin{cases} (aw_x)(0,t) = -cw(0,t) - Fw_t(0,t), c > 0, \\ Mw_{tt}(1,t) + (aw_x)(1,t) = 0. \end{cases}$$

exponentially stabilizes equation (2.1) under appropriate assumptions on the function F. Another stabilization result for equation (2.1) has also been established in [24] via the action of the following feedback:

$$\begin{cases} (aw_x)(0,t) = -cw(0,t) - Fw_t(0,t), \\ (aw_x)(1,t) = -cw(1,t) - Fw_t(1,t), c > 0. \end{cases}$$

In [21] the authors considered the following modelization of a flexible torque arm controlled by two feedbacks depending only on the boundary velocities:

$$\begin{cases} w_{tt}(x,t) - (a(x)w_x)_x + \alpha w_t(x,t) + \beta w(x,t) = 0, 0 < x < 1, t > 0, \\ (a(x)w_x)(0) = k_1 w_t(0,t), t > 0, \\ (a(x)w_x)(1) = -k_2 w_t(1,t), t > 0, \end{cases}$$

where

$$\begin{cases} \alpha \geqslant 0, \beta > 0, k_1, k_2 \geqslant 0, k_1 + k_2 \neq 0, \\ a \in W^{1,\infty}(0,1), a(x) \geqslant a_0 > 0 \quad \text{for all} \quad x \in [0,1]. \end{cases}$$

They proved the exponential decay of the solutions. Motivated by the work of [18] a feedback control depending only on the velocity has been proposed in [25] for the system (2.1) and an asymptotic convergence result has been established (see also [2], [3] and [22]). In this chapter, we are concerned with the system

$$\begin{cases} w_{tt}(x,t) - \left(a(x)w_x(x,t)\right)_x = 0 & \text{in } (0,1) \times (0,+\infty), \\ w(0,t) = 0 & \text{on } (0,+\infty), \\ (aw_x)(1,t) = -\varrho \partial_t^{\alpha,\eta} w(1,t) & \text{on } (0,+\infty), \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x) & \text{on } (0,1), \end{cases} \tag{P}$$

where $\varrho > 0$. The notation $\partial^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative (see [11] and [23]) defined by the following formula:

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s) ds, \ \eta \geq 0,$$

where Γ is the usual Euler gamma function and $(0 < \alpha < 1)$. Although there is quite a bit of work on damping mechanisms for beam models of this kind, there does not seem to be much about damping involving fractional derivatives. In [44] Mbodje studies the energy decay of the wave equation $(a \equiv 1)$. with a boundary fractional derivative control. He used a diffusive representation and the semigroup theory to establish the strong asymptotic stability under the condition $\eta = 0$ and a polynomial type decay rate $E(t) \leqslant \frac{C}{t}$ if $\eta \neq 0$.
The main result of this chapter is to show that system (P) is not exponentially stable for a general nondegenerate function a. Furthermore,we prove that the solution decays to zero polynomially when t goes to infinity for general initial data taken in the domain of $\mathcal{A}$ and for a general nondegenerate function a for both cases $\eta > 0$ and $\eta = 0$. Fractional Boundary dissipations can be encountered in many physical, chemical, biological, and economical phenomena (see [38], [56] and [57]). In recent years, the control of PDEs with boundary control of convolution type has become an active area of research because it improves the performance of the systems.This work is divided into five sections. In section 2, we give preliminary results and we reformulate the system (P) into an augmented system by coupling the nondegenerate wave equation with a suitable diffusion equation. In section 3, we convert the system into an evolution equation in an appropriate Hilbert space,and then prove the well-posedness of our problem by semigroup

theory. In section 4, we prove lack of exponential stability by spectral analysis. In section 5, we study asymptotic stability of above model and we establish an optimal polynomial energy decay depending with the parameter α for smooth solution.

2.2 Preliminary results

Let $a \in C([0,1]) \cap C^1(]0,1])$ be a function satisfying the following assumptions:

$$a \in W^{1,\infty}(0,1), a(x) \geq a_0 > 0 \text{ for all } x \in [0,1]. \tag{2.2}$$

2.2.1 Augmented model

Theorem 2.2.1. (*see*[41]).*Let κ be the function:*

$$\kappa(\xi) = |\xi|^{\frac{(2\alpha-1)}{2}}, \quad -\infty < \xi < +\infty, \ 0 < \alpha < 1. \tag{2.3}$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t \theta(\xi,t) + \left(\xi^2 + \eta\right)\theta(\xi,t) - U(t)\kappa(\xi) = 0, \quad -\infty < \xi < +\infty, \ \eta \geq 0, \ t > 0 \tag{2.4}$$

$$\theta(\xi,0) = 0 \tag{2.5}$$

$$O(t) = (\pi)^{-1}\sin(\alpha\pi)\int_{-\infty}^{+\infty} \kappa(\xi)\theta(\xi,t)d\xi \tag{2.6}$$

is given by

$$O(t) = I^{1-\alpha,\eta}U(t) \tag{2.7}$$

where

$$\left[I^{\alpha,\eta}f\right](t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}e^{-\eta(t-\tau)}f(\tau)d\tau$$

Lemma 2.2.2. *(see[10]).If $\gamma \in D_\eta = \mathbb{C}\setminus]-\infty,\eta]$ then*

$$\int_{-\infty}^{+\infty} \frac{\kappa^2(\xi)}{\xi^2+\eta+\gamma}d\xi = \frac{\pi}{\sin\alpha\pi}(\gamma+\eta)^{\alpha-1}.$$

We are now in a position to reformulate system (P). Indeed, by using Theorem 2.2.1, system

(P) may be recast into the augmented model:

$$\begin{cases} w_{tt}(x,t) - (a(x)w_x(x,t))_x = 0 & \text{in } (0,1)\times(0,+\infty), \\ \theta_t(\xi,t) + (\xi^2+\eta)\,\theta(\xi,t) - w_t(1,t)\kappa(\xi) = 0 & \text{in } (-\infty,+\infty)\times(0,+\infty), \\ w(0,t) = 0 & \text{on } (0,+\infty), \\ (aw_x)(1,t) = -\zeta\int_{-\infty}^{+\infty}\kappa(\xi)\theta(\xi,t)d\xi & \text{in } (-\infty,+\infty)\times(0,+\infty), \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x) & \text{on } (0,1), \\ \theta(\xi,0) = 0 & \text{on } (-\infty,+\infty). \end{cases} \tag{P'}$$

where $\zeta = \varrho(\pi)^{-1}\sin(\alpha\pi)$ We define the energy associated to the solution of the problem (P') by the following formula:

$$E(t) = \frac{1}{2}\int_0^1 \left(|w_t|^2 + a(x)\,|w_x|^2\right)dx + \frac{\zeta}{2}\int_{-\infty}^{+\infty}|\theta(\xi,t)|^2 d\xi. \tag{2.8}$$

Differentiating E in a formal way, using (P') and integrating by parts, we obtain after a straightforward computation the following Lemma.

Lemma 2.2.3. *Let (w,θ) be a regular solution of the problem (P'). Then,the energy functional defined by (2.8) satisfies*

$$E'(t) = -\zeta\int_{-\infty}^{+\infty}\left(\xi^2+\eta\right)(\theta(\xi,t))^2 d\xi \le 0. \tag{2.9}$$

Remark 2.2.1. For an initial datum in $D(\mathcal{A})$ (see Theorem 2.3.1 below), we know that (w,θ) is of class in time,thus we can defferentiate the energy E(t).

2.3 Well-Posedness

The energy space associated to system (P') is $\mathcal{H} = H_L^1(0,1)\times L^2(0,1)\times L^2(-\infty,+\infty)$, $H_L^1(0,1) = \{w \in H^1(0,1), w(0)=0\}$
with the inner product induced norm

$$\|(w,\upsilon,\theta)\|_{\mathcal{H}}^2 = \int_0^1\left[a(x)\,|w_x|^2 + |\upsilon|^2\right]dx + \zeta\int_{-\infty}^{+\infty}|\theta|^2 d\xi.$$

The system (P') can be written as

$$\begin{cases} \partial_t U = \mathcal{A}U, U = (w, w_t, \theta) \\ U(0) = U_0 = (w_0, w_1, 0) \end{cases} \tag{2.10}$$

where the associated system operator

$$\mathcal{A}(w,\upsilon,\theta) = \left(\upsilon, (a(x)w_x)_x, -\left(\xi^2+\eta\right)\theta + \upsilon(1)\kappa(\xi)\right) \tag{2.11}$$

$$D(\mathcal{A}) = \left\{ \begin{array}{c} (w, \upsilon, \theta) \in \mathcal{H}, w \in H^2(0,1) \cap H_L^1(0,1), \upsilon \in H_L^1(0,1) \\ -\left(\xi^2 + \eta\right)\theta + \upsilon(1)\kappa(\xi) \in L^2(-\infty, +\infty) \\ \left(a w_x\right)(1) + \zeta \int_{-\infty}^{+\infty} \kappa(\xi)\theta(\xi) d\xi = 0 \\ |\xi|\theta \in L^2(-\infty, +\infty) \end{array} \right\} \tag{2.12}$$

We have the following existence and uniqueness result.

Theorem 2.3.1. *(Existence and uniqueness). (1) If $U_0 \in D(\mathcal{A})$, then system (3.1) has a unique strong solution*

$$U \in C^0(\mathbb{R}_+, D(\mathcal{A}) \cap C^1(\mathbb{R}_+, \mathcal{H}),$$

(2) If $U_0 \in \mathcal{H}$,then system (3.1) has a unique weak solution

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

Proof. We use the semigroup approach. First, we prove that $\mathcal{A}$ is dissipative. Indeed, for $U \in D(\mathcal{A})$ and using (2.10), (2.9) and the fact that

$$E(t) = \frac{1}{2}\|U\|_{\mathcal{H}}^2. \tag{2.13}$$

we have

$$\Re e\langle \mathcal{A}U, U\rangle = -\zeta \int_{-\infty}^{+\infty} \left(\xi^2 + \eta\right)|\theta(\xi)|^2 d\xi. \tag{2.14}$$

Hence, $\mathcal{A}$ is dissipative. Next,we show that $\gamma I - \mathcal{A}$ is surjective for $\gamma > 0$. That is, for $G = (g_1, g_2, g_3)^T \in \mathcal{H}$, we have to find $U = (w, \upsilon, \theta)^T \in D(\mathcal{A})$, such that

$$(\gamma I - \mathcal{A})U = G \tag{2.15}$$

$$\left\{ \begin{array}{c} \gamma w - \upsilon = g_1 \\ \gamma \upsilon - \left(a(x) w_x\right)_x = g_2 \\ \gamma\theta + \left(\xi^2 + \eta\right)\theta - \upsilon(1)\kappa(\xi) = g_3 \end{array} \right. \tag{2.16}$$

Suppose w is found with the appropriate regularity. Then $(2.16)_1$ and $(2.16)_3$ yield

$$\upsilon = \gamma w - g_1 \in H_L^1(0,1), \tag{2.17}$$

and

$$\theta = \frac{g_3(\xi) + \kappa(\xi)\upsilon(1)}{\xi^2 + \eta + \gamma}. \tag{2.18}$$

Also, substituting the equation (2.17) into the equation $(2.16)_2$ we get

$$\gamma^2 w - \left(a(x) w_x\right)_x = g_2 + \gamma g_1. \tag{2.19}$$

Solving equation (2.10) is equivalent to finding $w \in H^2(0,1) \cap H_L^1(0,1)$ such that

$$\int_0^1 \left(\gamma^2 w\bar{z} - (a(x)w_x)_x\,\bar{z}\right) dx = \int_0^1 (g_2 + \gamma g_1)\,\bar{z}dx \tag{2.20}$$

for all $z \in H_L^1(0,1)$. By using (2.20), the boundary condition $(2.12)_3$ and (2.18) the function w satisfies the following equation

$$\begin{aligned} &\int_0^1 \left(\gamma^2 w\bar{z} + (a(x)w_x)\,\bar{z}_x\right) dx + \tilde{\zeta}v(1)\bar{z}(1) \\ &= \int_0^1 (g_2 + \gamma g_1)\,\bar{z}dx - \zeta \int_{-\infty}^{+\infty} \frac{\kappa(\xi)}{\xi^2+\eta+\gamma} g_3(\xi)d\xi\bar{z}(1) \end{aligned} \tag{2.21}$$

where

$$\tilde{\zeta} = \zeta \int_{-\infty}^{+\infty} \frac{\kappa^2(\xi)}{\xi^2+\eta+\gamma} d\xi.$$

Using again (2.17), we deduce that

$$v(1) = \gamma w(1) - g_1(1) \tag{2.22}$$

$$\left\{ \begin{array}{l} \int_0^1 \left(\gamma^2 w\bar{z} + a(x)w_x\bar{z}_x\right) dx + \gamma\tilde{\xi}w(1)\bar{z}(1) \\ = \int_0^1 (g_2 + \gamma g_1)\,\bar{z}dx - \zeta \int_{-\infty}^{+\infty} \frac{\kappa(\xi)}{\xi^2+\eta+\gamma} g_3(\xi)d\bar{\xi}\bar{z}(1) + \tilde{\zeta}g_1(1)\bar{z}(1). \end{array} \right. \tag{2.23}$$

Inserting (2.22) into (2.21), we get Problem (2.23) is of the form

$$\mathcal{B}(w,z) = \mathcal{L}(z), \tag{2.24}$$

where $\mathcal{B} : [H_L^1(0,1) \times H_L^1(0,1)] \to \mathbb{C}$ is the sesquilinear form defined by

$$\mathcal{B}(w,z) = \int_0^1 \left(\gamma^2 w\bar{z} + a(x)w_x\bar{z}_x\right) dx + (\widetilde{\gamma\zeta})w(1)\bar{z}(1)$$

and $\mathcal{L} : H_L^1(0,1) \to \mathbb{C}$ is the antilinear functional given by

$$\mathcal{L}(z) = \int_0^1 (g_2 + \gamma g_1)\,\bar{z}dx - \zeta\bar{z}(1) \int_{-\infty}^{+\infty} \frac{\kappa(\xi)}{\xi^2+\eta+\gamma} g_3(\xi)d\xi + \widetilde{\zeta}g_1(1)\bar{z}(1).$$

It is easy to verify that $\mathcal{B}$ is continuous and coercive, and $\mathcal{L}$ is continuous. Therefore, Lax-Milgram says that $\exists! w \in H_L^1(0,1)$ satisfying (2.24). Now, by the regularity theory for the linear elliptic equations, it follows that $w \in H^2(0,1)$. Thus, $\gamma - \mathcal{A}$ is surjective for any $\gamma > 0$. Consequently, using Hille-Yosida theorem, the result of Theorem 3.1 follows. ■

2.4 Lack of exponential stability

In this section we prove the lack of exponential decay of the solutions of system (3.1). Inorder to state and prove our stability results, we need the following Theorem.

Theorem 2.4.1. *([16]). Let S(t) be a C_0-semigroup of contractions on Hilbert space with generator $\mathcal{A}$. Then S(t) is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R}$$

and

$$\overline{\lim_{|\beta|\to\infty}} ||(i\beta I - \mathcal{A})^{-1}||_{\mathcal{L}(\mathcal{H})} < \infty.$$

Our main result is stated as follows:

Theorem 2.4.2. *The semigroup generated by the operator $\mathcal{A}$ is not exponentially stable*

Proof. We will examine two cases.
• **Case** $\eta = 0$: We shall show that $i\gamma = 0$ is not in the resolvent set of the operator $\mathcal{A}$. Indeed, noting that $(\sin x, 0, 0)^T \in \mathcal{H}$, and denoting by $(w, \upsilon, \theta)^T$ the image of $(\sin x, 0, 0)^T \in \mathcal{H}$ by $\mathcal{A}^{-1}$ we see that $\theta\,(\xi) = |\xi|^{\frac{2\alpha-5}{2}} \sin 1$. But, then $\theta \notin L^2(-\infty, +\infty)$, since $\alpha \in]0, 1]$. So $(w, \upsilon, \theta)^T \notin D(\mathcal{A})$.
• **Case** $\eta \neq 0$:
We aim to show that an infinite number of eigenvalues of $\mathcal{A}$ approach the imaginary axis which prevents the system (P) from being exponentially stable. Indeed we first compute the characteristic equation that gives the eigenvalues of $\mathcal{A}$. Let γ be an eigenvalue of $\mathcal{A}$ with associated eigenvector $(w, \upsilon, \theta)^T$. Then $\mathcal{A}U = \gamma U$ is equivalent to

$$\begin{cases} \gamma w - \upsilon = 0, \\ \gamma \upsilon - (a(x)w_x)_x = 0, \\ \gamma\theta + (\xi^2 + \eta)\,\theta - \upsilon(1)\kappa(\xi) = 0 \end{cases} \tag{2.25}$$

From $(2.25)_1, (2.25)_2$ for such, γ we find

$$\gamma^2 w - (a(x)w_x)_x = 0. \tag{2.26}$$

Using the boundary conditions and $(2.25)_3$, we deduce that

$$\begin{cases} \gamma^2 w - (a(x)w_x)_x = 0 \\ w(0) = 0 \\ (aw_x)(1) + \zeta\upsilon(1)\int_{-\infty}^{+\infty} \frac{\kappa^2(\xi)}{\xi^2+\eta+\gamma} d\xi = (aw_x)(1) + \varrho\gamma(\gamma+\eta)^{\alpha-1}w(1) = 0. \end{cases} \tag{2.27}$$

Our purpose is to prove, thanks to Rouche's Theorem, that there is a subsequence of eigenvalues for which their real part tends to 0.
In the sequel, since $\mathcal{A}$ is dissipative, we study the asymptotic behavior of the large eigenvalues γ of $\mathcal{A}$ in the strip $-\alpha_0 \leq \Re e(\gamma) \leq 0,$ for some $\alpha_0 > 0,$ large enough. ■

Lemma 2.4.3. *There exists $N \in \mathbb{N}$ such that*

$$\{\gamma_k\}_{k\in\mathbf{Z}^*,|k|\geq N} \subset \sigma(\mathcal{A}), \tag{2.28}$$

where

$$\gamma_k = i\frac{(k+1/2)\pi}{\int_0^1 \frac{1}{\sqrt{a(x)}}dx} + \frac{\widetilde{\alpha}}{k^{1-\alpha}} + \frac{\beta}{k^{1-\alpha}} + o\left(\frac{1}{k^{1-\alpha}}\right), k \geq N, \widetilde{\alpha} \in i\mathbb{R}, \beta < 0.$$

$$\gamma_k = \overline{\gamma - k} \text{ if } k \leq -N.$$

Moreover for all $|k| \geq N$, the eigenvalues γ_k are simple.

Proof. The proof is decomposed in three steps: Writing (2.27) in the standard form of a linear differential operator with homogeneous boundary conditions, we obtain

$$\begin{cases} w_{xx} + \frac{a_x(x)}{a(x)}w_x - \frac{\gamma^2}{a(x)}w = 0 \\ w(0) = 0 \\ (aw_x)(1) + \varrho\gamma(\gamma+\eta)^{\alpha-1}w(1) = 0. \end{cases} \tag{2.29}$$

In order to simplify the computations, we introduce a spatial-scale transformation in x

$$\phi(y) = w(x), y = \frac{1}{h}\int_0^x \frac{1}{\sqrt{a(s)}}ds, y \in (0,1), \tag{2.30}$$

$$h = \int_0^1 \frac{1}{\sqrt{a(s)}}ds.$$

Then Eq. (2.29) has the form

$$\begin{cases} \phi''(y) + \frac{h}{2}\frac{a_x(x)}{\sqrt{a(x)}}\phi'(y) - h^2\gamma^2\phi(y) = 0 \\ \phi(0) = 0 \\ \frac{1}{h}\phi'(1) + \varrho\gamma(\gamma+\eta)^{\alpha-1}\phi(1) = 0 \end{cases} \tag{2.31}$$

Equation (2.31) can be further simplified by applying another invertible transformation (see [45]):

$$\varphi(y) = e^{\frac{1}{2}\int_0^y \tilde{a}(s)ds}\phi(y), \quad y \in (0,1). \tag{2.32}$$

where

$$\tilde{a}(y) = \frac{h}{2}\frac{a_x(x)}{\sqrt{a(x)}}.$$

(2.32) allows one to cancel the term $\frac{h}{2}\frac{a_x(x)}{\sqrt{a(x)}}\phi'(y)$ in (2.31). Hence we arrive at an equivalent eigenvalue problem

$$\begin{cases} \varphi''(y) - \left(\frac{1}{2}\tilde{a}'(y) + \frac{1}{4}\tilde{a}^2(y) + h^2\gamma^2\right)\varphi(y) = 0 \\ \varphi(0) = 0 \\ \varphi'(1) + \left(-\frac{1}{2} + \rho h\gamma(\gamma+\eta)^{\alpha-1}\right)\varphi(1) = 0 \end{cases} \tag{2.33}$$

To asymptotically estimate the solutions to the eigenvalue problem (2.33), we proceed as in [48]. The equation

$$\varphi''(y) - \left(\frac{1}{2}\tilde{a}'(y) + \frac{1}{4}\tilde{a}^2(y) + h^2\gamma^2\right)\varphi(y) = 0,$$

has two linearly independent asymptotic fundamental solutions:

$$\begin{aligned} \varphi_1(y) &= e^{h\gamma y}\left(1 + \frac{\varphi_{10}(y)}{ih\gamma} + 0\left(\frac{1}{\gamma^2}\right)\right) \\ \varphi_2(y) &= e^{-h\gamma y}\left(1 + \frac{\varphi_{20}(y)}{ih\gamma} + 0\left(\frac{1}{\gamma^2}\right)\right) \end{aligned}$$

and hence their derivatives are given by

$$\begin{aligned} \frac{d}{dy}\varphi_1(y) &= h\gamma e^{h\gamma y}\left(1 + \frac{\varphi_{10}(y)}{ih\gamma} + 0\left(\frac{1}{\gamma^2}\right)\right) \\ \frac{d}{dy}\varphi_2(y) &= -h\gamma e^{-h\gamma y}\left(1 + \frac{\varphi_{20}(y)}{ih\gamma} + 0\left(\frac{1}{\gamma^2}\right)\right) \end{aligned}$$

where

$$\begin{aligned} \varphi_{10}(y) &= -\frac{i}{2}\int_0^y \left(\frac{1}{2}\tilde{a}'(s) + \frac{1}{4}\tilde{a}^2(s)\right) ds, \\ \varphi_{20}(y) &= \frac{i}{2}\int_0^y \left(\frac{1}{2}\tilde{a}'(s) + \frac{1}{4}\tilde{a}^2(s)\right) ds. \end{aligned}$$

For simplicity, we introduce the following notation $[a]_i := a + \mathcal{O}(\gamma^{-i})$ for $i = 1, 2$. From Lemma 4.2, one can write the asymptotic solution of (2.33) as follows:

$$\varphi(y) = \sum_{i=1}^{2} c_i\varphi_i. \tag{2.34}$$

where c_i is chosen so that φ satisfies the boundary conditions, i.e.,

$$\tilde{M}(\gamma)C(\gamma) = \begin{pmatrix} [1]_2 & [1]_2 \\ \left[(\gamma+\rho\gamma^\alpha)e^{\gamma h}\right]_0 & \left[(-\gamma+\rho\gamma^\alpha)e^{-\gamma h}\right]_0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.35}$$

Hence a non-trivial solution w exists if and only if the determinant of $\tilde{M}(\gamma)$ vanishes. Set

$f(\gamma) = det\tilde{M}(\gamma)$, thus the characteristic equation is $f(\gamma) = 0$

$$\begin{aligned} f(\gamma) &= \left(-\gamma + \rho\gamma^{\alpha}\right) e^{-\gamma h} - \left(\gamma + \rho\gamma^{\alpha}\right) e^{\gamma h} + \mathcal{O}(1) \\ &= -\gamma e^{-\gamma h} \left(e^{2\gamma h} + 1 + \frac{\rho}{\gamma^{1-\alpha}} \left(e^{2\gamma h} - 1\right) + \mathcal{O}\left(\gamma^{-1}\right) \right). \end{aligned} \tag{2.36}$$

We set

$$\tilde{f}(\gamma) = e^{2\gamma h} + 1 + \frac{\rho}{\gamma^{1-\alpha}} \left(e^{2\gamma h} - 1\right) + \mathcal{O}\left(\gamma^{-1}\right) \quad = f_0(\gamma) + \frac{f_1(\gamma)}{\gamma^{1-\alpha}} + o\left(\frac{1}{\gamma^{1-\alpha}}\right) \tag{2.37}$$

where

$$f_0(\gamma) = e^{2\gamma h} + 1, \tag{2.38}$$

$$f_1(\gamma) = \rho\left(e^{2\gamma h} - 1\right). \tag{2.39}$$

Note that f_0 and f_1 remain bounded in the strip $-\alpha_0 \leq \Re e(\gamma) \leq 0$.

• **Step2**. We look at the roots of f_0. From (2.38), f_0 has one familie of roots that we denote γ_k^0.

$$f_0(\gamma) = 0 \Longleftrightarrow \exp(2\gamma h) = -1.$$

Hence

$$2h\gamma = i(2k+1)\pi, \quad k \in \mathbb{Z},$$

i.e.

$$\gamma_k^0 = \frac{i(2k+1)\pi}{2h}, \quad k \in \mathbb{Z}.$$

Now with the help of Rouche's Theorem, we will show that the roots of $\tilde{f}$ are close to those of f_0. Let us start with the first family. Changing in (2.37) the unknown γ by $u = 2h\gamma$ then (2.37) becomes

$$\tilde{f}(u) = (e^u + 1) + O\left(\frac{1}{u^{(1-\alpha)}}\right) = f_0(u) + O\left(\frac{1}{u^{(1-\alpha)}}\right).$$

The roots of f_0. are $u_k = \frac{i(k+1/2)}{h}\pi, \quad k \in \mathbb{Z}$, and setting $u = u_k + re^{it}, t \in [0, 2\pi]$, we can easily check that there exists a constant $C > 0$ independent of k such that $|e^u + 1| \geqslant Cr$ for r small enough.This allows to apply Rouche's Theorem. Consequently, there exists a subsequence of roots of $\tilde{f}$ which tends to the roots u_k of f_0. Equivalently, it means that there exist $N \in \mathbb{N}$ and a subsequence $\{\gamma_k\}_{|k|\geqslant N}$ of roots of $f(\gamma)$, such that $\gamma_k = \gamma_k^0 + o(1)$ which tends to the roots $\frac{i(k+1/2)}{h}\pi$ of f_0. Finally for $|k| \geqslant N, \gamma_k$ is simple since γ_k^0 is.

• **Step 3**. From Step 2, we can write

$$\gamma_k = i\frac{1}{h}(k+1/2)\pi + \varepsilon_k. \tag{2.40}$$

Using (2.40), we get

$$e^{2h\gamma_k L} = -1 - 2h\varepsilon_k + O(\varepsilon_k). \tag{2.41}$$

Substituting (2.41) into (2.37), using that $\tilde{f} = 0$, we get:

$$\tilde{f} = -2h\varepsilon_k - \frac{2\rho}{(\gamma_k^0)^{1-\alpha}} + O(\varepsilon_k) = 0. \tag{2.42}$$

and hence

$$\varepsilon_k = -\frac{\rho}{h^\alpha (ki\pi)^{1-\alpha}} + o(\frac{1}{k^{1-\alpha}}) = \begin{cases} -\frac{\rho}{h^\alpha (k\pi)^{1-\alpha}} \left(\cos(1-\alpha)\frac{\pi}{2} - i\sin(1-\alpha)\frac{\pi}{2}\right) + o\left(\frac{1}{k^{1-\alpha}}\right) & \text{for } k \succeq 0, \\ -\frac{\rho}{h^\alpha (-k\pi)^{1-\alpha}} \left(\cos(1-\alpha)\frac{\pi}{2} + i\sin(1-\alpha)\frac{\pi}{2}\right) + o\left(\frac{1}{k^{1-\alpha}}\right) & \text{for } k \preceq 0 \end{cases} \tag{2.43}$$

From (2.43) we have in that case

$$|k|^{1-\alpha}\mathcal{R}\gamma_k \sim \beta,$$

with

$$\beta = -\frac{\rho}{h^\alpha \pi^{1-\alpha}} \cos(1-\alpha)\frac{\pi}{2}.$$

■

The operator $\mathcal{A}$ has a non exponential decaying branche of eigenvalues. Thus the proof is complete.

2.5 Asymptotic behavior

2.5.1 Strong stability of the system

To prove that the semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ is strongly asymptotically stable, we shall apply a version of the Arendt-Batty and Lyubich-Vu for Hilbert spaces [4],[37].

Theorem 2.5.1. *([4],[37]). Let $\mathcal{A}$ be the generator of a uniformly bounded C_0 -semigroupe $S(t)_{t\geqslant 0}$ on a Hilbert space $\mathcal{H}$. If:*
(i) $\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.
(ii) The intersection of the spectrum $\sigma(\mathcal{A})$ with $i\mathbb{R}$. is at most a countable set, then the semigroup is asymptotically stable, i.e, $\|S(t)z\|_{\mathcal{H}} \to 0$ as $t \to +\infty$, for any $z \in \mathcal{H}$.

Our next main result in this part is the following theorem.

Theorem 2.5.2. *The C_0 -semigroupe $(e^{t\mathcal{A}})_{t\geq 0}$ is strongly stable in $\mathcal{H}$, i.e, for all $U_0 \in \mathcal{H}$, the solution of (2.10) satisfies*

$$\lim_{t\to+\infty} \left\|e^{t\mathcal{A}}U_0\right\|_{\mathcal{H}} = 0$$

For the proof of Theorem (2.5.2) we need the following two lemmas.

Lemma 2.5.3. *$\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.*

Proof. We will argue by contraction. Suppose that there is $\gamma \in \mathbb{R}$. such that $i\gamma$ is an eigenvalue for $\mathcal{A}$ and hence one can find a corresponding eigenfunction $U = (w, v, \theta) \in \mathcal{D}(\mathcal{A})$. Consequently, we have

$$\mathcal{A}U = i\gamma U \tag{2.44}$$

Our immediate aim is to prove that this equation has only $U = 0$ as a solution, which,contradicts the definition of an eigenfunction. Firstly, the equation (2.44) is equivalent to consider the following system

$$\begin{cases} i\gamma w - v = 0 \\ i\gamma v - (a(x)w_x)_x = 0 \\ i\gamma\theta + (\xi^2 + \eta)\,\theta - v(1)\kappa(\xi) = (aw_x)(1) + \varrho\gamma(\gamma+\eta)^{\alpha-1}w(1) = 0. \end{cases} \tag{2.45}$$

Secondly, we will consider two cases:

• **case** $\gamma \neq 0$: Taking the $L^2(0,1)$-inner product with U of both sides of (2.44) and using (2.14), we immediately obtain

$$0 = \Re e\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\zeta \int_{-\infty}^{+\infty} \left(\xi^2 + \eta\right) |\theta(\xi)|^2 d\xi. \tag{2.46}$$

Hence

$$\theta = 0 \tag{2.47}$$

Then $(2.45)_3$ gives

$$v(1) = 0, \tag{2.48}$$

then using the first equation of (2.45) and the boundary condition $(2.45)_4$, we deduce

$$w(1) = 0 \text{ and } w_x(1) = 0. \tag{2.49}$$

We deduce that w satisfies the boundary value problem:

$$\begin{cases} \gamma^2 w + (a(x)w_x)_x = 0 \\ w(1) = w_x(1) = 0 \\ w(0) = 0 \end{cases} \tag{2.50}$$

Next, let

$$\psi(x) = \int_0^x \exp\left(\int_s^x \left|\frac{a_x}{a}\right| dv\right). \text{ ds } \forall x \in [0,1]$$

It is easy to see that

$$\begin{cases} \psi(0) = 0, \psi(x) > 0,\ \forall x \in (0,1] \\ \psi_x \geq 1, a\left(\frac{\psi}{a}\right)_x \geq 1. \end{cases}$$

Multiplying equation $(2.50)_1$ by $\psi\bar{w}_x$, we get

$$\gamma^2 \int_0^1 \psi(x) w\bar{w}_x dx + \int_0^1 \psi(x)\bar{w}_x \left(a(x)w_x\right)_x dx = 0. \tag{2.51}$$

$U \in \mathcal{D}(\mathcal{A})$ then the regularity is sufficiently for applying an integration on the second integral in the left hand side in equation (2.51). Then we obtain

$$\frac{\gamma^2}{2}\int_0^1 \psi(x)\frac{d}{dx}|w|^2dx - \int_0^1 \psi_x(x)a(x)\left|w_x\right|^2 dx - \frac{1}{2}\int_0^1 \psi(x)a(x)\frac{d}{dx}\left|w_x\right|^2 dx = 0. \tag{2.52}$$

Using Green formula and the boundary conditions, we get

$$\gamma^2\int_0^1 \psi_x(x)|w|^2dx + \int_0^1 \left(\psi_x(x)a(x) - \psi(x)a_x(x)\right)\left|w_x\right|^2 dx = 0, \tag{2.53}$$

We deduce that

$$w = 0 \tag{2.54}$$

Using equation $(2.45)_1$, we obtain

$$\upsilon = 0 \tag{2.55}$$

Consequently, using equations (2.54), (2.55) and (2.47), we deduce that the only solution of (2.44) is the null one.

• **case** $\gamma = 0$: In this case, by $(2.45)_1$, we have $\upsilon = 0$ which gives that $\theta = 0$ by $(2.45)_3$. Multiplying equation $(2.45)_2$ by $\bar{w}$, using Green formula and the boundary conditions, we get

$$\int_0^1 a(x)\left|w_x\right|^2 dx = 0. \tag{2.56}$$

Then

$$w_x(x) = 0 \quad \forall x \in (0,1). \tag{2.57}$$

Hence w is constant in (0,1). As $w(1) = 0$, then

$$w \equiv 0.$$

Hence U must be the trivial solution of (2.44), which is the desired result. The proof has been completed. ■

Lemma 2.5.4. *We have*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \text{ if } \eta \neq 0,$$
$$i\mathbb{R}^* \subset \rho(\mathcal{A}) \text{ if } \eta = 0^{\cdot}$$

where

$$\mathbb{R}^* = \mathbb{R} \setminus \{0\}.$$

Proof. • **case** $\gamma \neq 0$: We will prove that the operator $i\gamma I - \mathcal{A}$ is surjective for $\gamma \neq 0$. For this purpose, let $G = (g_1, g_2, g_3)^T \in \mathcal{H}$, we seek $U = (w, \upsilon, \theta)^T \in D(\mathcal{A})$, solution of the following equation

$$(i\gamma I - \mathcal{A})X = G. \tag{2.58}$$

Equivalently, we have

$$\begin{cases} i\gamma w - \upsilon = g_1 \\ i\gamma \upsilon - (a(x)w_x)_x = g_2 \\ i\gamma\theta + (\xi^2+\eta)\,\theta - \upsilon(1)\kappa(\xi) = g_3. \end{cases} \tag{2.59}$$

From $(2.59)_1$ and $(2.59)_2$, we have

$$-\gamma^2 w - (a(x)w_x)_x = (g_2 + i\gamma g_1)\,. \tag{2.60}$$

Let $z \in H_L^1(0,1)$. Multiplying the equation (2.60) by $\bar{z}$ and integrating in (0,1), we obtain

$$\int_0^1 \left(-\gamma^2 w\bar{z} - (a(x)w_x)_x\,\bar{z}\right) dx = \int_0^1 (g_2 + i\gamma g_1)\,\bar{z}dx. \tag{2.61}$$

From the boundary conditions and the fact that $w(0) = 0$, we get

$$\begin{cases} \int_0^1 (-\gamma^2 w\bar{z} + a(x)w_x\bar{z}_x)\,dx + i\gamma\tilde{\zeta}w(1)\bar{z}(1) \\ = \int_\Omega (g_2 + i\gamma g_1)\,\bar{z}dx - \zeta\int_{-\infty}^{+\infty} \frac{\kappa(\xi)}{\tilde{\zeta}^2+\eta+i\gamma} g_3(\tilde{\xi})\bar{z}d\tilde{\xi} + \tilde{\zeta}g_1(1)\bar{z}(1) \end{cases} \tag{2.62}$$

We can rewrite (2.62) as

$$-\left(L_\gamma w, z\right)_{H_L^1} + (w,z)_{H_L^1} = \mathcal{L}(z), \tag{2.63}$$

with the inner product defined by

$$(w,z)_{H_L^1} = \int_0^1 a(x)w_x\bar{z}_x dx,$$

and

$$(L_\gamma w, z)_{H_L^1} = \int_0^1 \gamma^2 w\bar{z}dx - i\gamma\tilde{\zeta}w(1)\bar{z}(1).$$

Using the compactness embedding from $L^2(0,1)$ into $H^{-1}(0,1)$ and from $H^1(0,1)$ into $L^2(0,1)$ we deduce that the operator L_γ is compact from $L^2(0,1)$ into $L^2(0,1)$. Consequently, by Fredholm alternative, proving the existence of w solution of (2.63) reduces to proving that 1 is not an eigenvalue of L_γ Indeed if 1 is an eigenvalue, then there exists $w \neq 0$ such that

$$(L_\gamma w, z)_{H_L^1} = (w,z)_{H_L^1} \quad \forall z \in H_L^1. \tag{2.64}$$

In particular for $z = w$, it follows that

$$\gamma^2\|w(x)\|^2_{L^2(0,1)} - i\gamma\tilde{\zeta}|w(1)|^2 = \left\|\sqrt{a(x)}w_x(x)\right\|^2_{L^2(0,1)}.$$

Hence, we have

$$w(1) = 0. \tag{2.65}$$

From (2.65), we obtain

$$(aw_x)(1) = 0 \tag{2.66}$$

Then

$$\begin{cases} \gamma^2 w + (a(x)w_x)\, x = 0 \\ w(1) = w_x(1) = 0 \\ w(0) = 0 \end{cases}$$

We deduce that U =0.
• **Case** $\gamma = 0$ **and** $\eta \neq 0$. Using Lax-milgram theorem, we obtain the result. ∎

2.5.2 Polynomial Stability (For $\eta \neq 0$)

In order to establish the polynomial energy decay rate, we need the following theorem.

Theorem 2.5.5. *([12]). Let S(t) be a bounded C_0-semigroup on a Hilbert space $\mathcal{H}$ with generator $\mathcal{A}$. If*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \text{ and } \overline{\lim_{|s|\to\infty}} \frac{1}{s^l} \left\| (isI - \mathcal{A})^{-1} \right\|_{\mathcal{L}(\mathcal{H})} < \infty,$$

for some $l > 0$, then there exist a positive constant C such that

$$\left\| e^{\mathcal{A}t} U_0 \right\|^2 \leq \frac{\mathcal{C}}{t^{\frac{2}{l}}} \left\| U_0 \right\|^2_{D(\mathcal{A})}.$$

Our main result is the following.

Theorem 2.5.6. *The semigroup $S_{\mathcal{A}}(t)$ is polynomially stable and*

$$E(t) = \| S_{\mathcal{A}}(t) U_0 \|^2_{\mathcal{H}} \leq \frac{1}{t^{\frac{2}{1-a}})} \| U_0 \|^2_{D(\mathcal{A})}.$$

Moreover, the rate of energy decay $t^{2/1-\alpha}$ is optimal for general initial data in $D(\mathcal{A})$.

Proof. Given $G = (g_1, g_2, g_3)^T \in \mathcal{H}, let\, U = (w, \upsilon, \theta)^T \in D(\mathcal{A})$. be the solution of the resolvent equation $(i\gamma I - \mathcal{A})U = G$, for $\gamma \in \mathbb{R}$, i.e.,

$$\begin{cases} i\gamma w - \upsilon = g_1 \\ i\gamma \upsilon - (a(x)w_x)_x = g_2 \\ i\gamma\theta + (\xi^2 + \eta)\,\theta - \upsilon(1)\kappa(\xi) = g_3 \end{cases} \tag{2.67}$$

• **Step 1** Taking the real part of the inner product of $(i\gamma I - \mathcal{A})U = G$, with U in $\mathcal{H}$ and using (2.14), we get

$$|\Re e\langle \mathcal{A}U, U\rangle_{\mathcal{H}}| \leq \|U\|_{\mathcal{H}} \|G\|_{\mathcal{H}}, \tag{2.68}$$

This implies that

$$\zeta \int_{-\infty}^{+\infty} \left(\xi^2 + \eta\right) |\theta(\xi, t)|^2 d\xi \leq \|U\|_{\mathcal{H}} \|G\|_{\mathcal{H}} \tag{2.69}$$

and, applying $(2.67)_1$, we obtain

$$\Big| |\gamma|\,|w(1)| - |g_1| \Big|^2 \leqslant |\upsilon(1)|^2.$$

We conclude that

$$|\gamma|^2|w(1)|^2 \leqslant c|g_1|^2 + c|\upsilon(1)|^2. \tag{2.70}$$

From the boundary condition

$$(aw_x)(1) = -\zeta \int_{-\infty}^{+\infty} \kappa(\xi)\theta(\xi)d\xi.$$

we deduce that

$$\begin{aligned} |(aw_x)(1)|^2 &\leqslant 2\zeta^2 \left| \int_{-\infty}^{+\infty} \kappa(\xi)\theta(\xi)d\xi \right|^2 \\ &\leqslant 2\zeta^2 \left(\int_{-\infty}^{+\infty} (\xi^2+\eta)^{-1}|\kappa(\xi)|^2 d\xi \right) \left(\int_{-\infty}^{+\infty} (\xi^2+\eta)|\theta(\xi)|^2 d\xi \right) \leqslant c\|U\|_{\mathcal{H}}\|G\|_{\mathcal{H}}. \end{aligned} \tag{2.71}$$

Now, from $(2.67)_3$, we obtain

$$\upsilon(1)\kappa(\xi) = \left(i\gamma + \xi^2 + \eta\right)\theta - g_3(\xi). \tag{2.72}$$

By multiplying (2.72) by $\left(i\gamma + \xi^2 + \eta\right)^{-2}|\xi|$, we get

$$\left(i\gamma + \xi^2 + \eta\right)^{-2} \upsilon(1)\kappa(\xi)|\xi| = \left(i\gamma + \xi^2 + \eta\right)^{-1}|\xi|\theta - \left(i\gamma + \xi^2 + \eta\right)^{-2}|\xi|g_3(\xi). \tag{2.73}$$

Hence, by taking absolute values of both sides of (2.73), integrating over the interval $]-\infty, +\infty[$ with respect to the variable ξ and applying Cauchy-Schwartz inequality, we obtain

$$S|\upsilon(1)| \leqslant \sqrt{2}\mathcal{U}\left(\int_{-\infty}^{+\infty} \xi^2|\theta|^2 d\xi \right)^{\frac{1}{2}} + 2\nu\left(\int_{-\infty}^{+\infty} |g_3(\xi)|^2 d\xi \right)^{\frac{1}{2}}, \tag{2.74}$$

where

$$\mathcal{S} = \left| \int_{-\infty}^{+\infty} \left(i\gamma + \xi^2 + \eta\right)^{-2}|\xi|\kappa(\xi)d\xi \right| = \frac{|1-2\alpha|}{4} \frac{\pi}{|\sin\frac{(2\alpha+3)}{4}\pi|}|i\gamma+\eta|^{\frac{2\alpha-5}{4}},$$

$$\mathcal{U} = \left(\int_{-\infty}^{+\infty} \left(|\gamma| + \xi^2 + \eta\right)^{-2} d\xi \right)^{\frac{1}{2}} = (\frac{\pi}{2})^{\frac{1}{2}} \left||\gamma|+\eta\right|^{\frac{-3}{4}},$$

$$\nu = \left(\int_{-\infty}^{+\infty} \left(|\gamma| + \xi^2 + \eta\right)^{-4}|\xi|^2 d\xi \right)^{\frac{1}{2}} = \left(\frac{\pi}{16}\left||\gamma|+\eta\right|^{\frac{-5}{2}}\right)^{\frac{1}{2}}.$$

Thus, by using the inequality

$$2PQ \leqslant P^2 + Q^2, P \geqslant 0, Q \geqslant 0,$$

again, we get

$$S^2|\upsilon(1)|^2 \leqslant 2U^2\left(\int_{-\infty}^{+\infty}(\xi^2+\eta)|\theta|^2 d\xi\right) + 4\nu^2\left(\int_{-\infty}^{+\infty}|g_3(\xi)|^2 d\xi\right). \tag{2.75}$$

We deduce that

$$|\upsilon(1)|^2 \leq c|\gamma|^{1-\alpha}\|U\|_{\mathcal{H}}\|G\|_{\mathcal{H}} + c\|G\|^2_{\mathcal{H}}. \tag{2.76}$$

• **Step 2** Let us introduce the following notation

$$\mathcal{I}_u(x) = \left|\sqrt{a(x)}w_x(x)\right|^2 + |\upsilon(x)|^2$$

$$\mathcal{E}_u = \int_0^1 \mathcal{I}_u(x)dx$$

Lemma 2.5.7. *We have that*

$$c_1\int_0^1 \psi_x(x)|\upsilon(x)|^2 dx + c_0\int_0^1 (\psi_x(x)-1)|\upsilon(x)|^2 dx\,(c_0+c_1)\int_0^1\left(\frac{\psi(x)}{a(x)}\right)_x |aw_x|^2 dx$$
$$+c_0\int_0^1 a(x)|w_x|^2 dx = (c_0+c_1)\left[\psi(x)|\upsilon|^2\right]_0^1 + (c_0+c_1)\left[\left(\frac{\psi(x)}{a(x)}\right)|aw_x|^2\right]_0^1 + c_0[a(x)w_x\bar{w}]_0^1 + R, \tag{2.77}$$

for every, $c_0, c_1 > 0$ and R satisfies

$$|R| \leqslant C\|U\|_{\mathcal{H}}\|G\|_{\mathcal{H}}.$$

for a positive constant C.

Proof. To get (2.77), let us multiply the equation $(2.67)_2$ by $2(c_0+c_1)\psi\bar{w}_x + c_0\bar{w}$. Integrating on (0,1) we obtain

$$i\gamma\int_0^1 \upsilon(2(c_0+c_1)\psi\bar{w}_x + c_0\bar{w})dx - \int_0^1 (a(x)w_x)_x(2(c_0+c_1)\psi\bar{w}_x + c_0\bar{w})dx$$
$$= \int_0^1 g_2(2(c_0+c_1)\psi\bar{w}_x + c_0\bar{w})dx$$

or

$$-2(c_0+c_1)\int_0^1 \upsilon\psi(x)(\overline{i\gamma w})dx - c_0\int_0^1 \upsilon(\overline{i\gamma w})dx - 2(c_0+c_1)\int_0^1 \psi(x)(a(x)w_x)_x\bar{w}_x dx$$

$$-c_0\int_0^1 (a(x)w_x)_x\bar{w}dx = \int_0^1 g_2(2\left(c_0+c_1\right)\psi\bar{w}_x + c_0\bar{w})dx$$

Since $i\gamma w = \upsilon + g_1$ and $i\gamma w_x = \upsilon_x + g_{1x}$ taking the real part in the above equality, it follows that

$$-(c_0+c_1)\int_0^1 \psi(x)\frac{d}{dx}|\upsilon|^2 - (c_0+c_1)\int_0^1 \frac{\psi(x)}{a(x)}\frac{d}{dx}\left|aw_x\right|^2 dx - -c_0\int_0^1 (a(x)w_x)_x\bar{w}dx$$

$$-c_0\int_0^1 |\upsilon|^2dx = 2(c_0+c_1)\Re e\int_0^1 \upsilon\psi(x)\bar{g_{1x}}dx + \Re e\int_0^1 g_2(2\left(c_0+c_1\right)\psi\bar{w}_x + c_0\bar{w})dx + c_0\Re e\int_0^1 \upsilon\bar{g}_1dx$$

and integrating by parts, we get

$$\begin{aligned} c_0\int_0^1 (\psi_x(x)-1)|\upsilon(x)|^2dx + c_1\int_0^1 \psi_x(x)|\upsilon(x)|^2dx + (c_0+c_1)\int_0^1 \left(\frac{\psi(x)}{a(x)}\right)_x \left|aw_x\right|^2 dx \\ + c_0\int_0^1 a(x)|w_x|^2dx = (c_0+c_1)\left[\psi(x)|\upsilon|^2\right]_0^1 + (c_0+c_1)\left[\left(\frac{\psi(x)}{a(x)}\right)\left|aw_x\right|^2\right]_0^1 \\ + c_0[a(x)w_x\bar{w}]_0^1 + R, \end{aligned} \quad (2.78)$$

where

$$R = 2(c_0+c_1)\Re e\int_0^1 \upsilon\psi(x)\bar{g_{1x}}dx + \Re e\int_0^1 g_2(2\left(c_0+c_1\right)\psi\bar{w}_x + c_0\bar{w})dx + c_0\Re e\int_0^1 \upsilon\bar{g}_1dx$$

Moreover

$$\left|\int_0^1 \psi(x)g_2\bar{w}_xdx\right| \leqslant C\|g_2\|_{L^2(0,1)}\|w_x\|_{L^2(0,1)} \leqslant C\|G\|_{\mathcal{H}}\|U\|_{\mathcal{H}},$$

$$\left|\int_0^1 \psi(x)\upsilon\bar{f_{1x}}dx\right| \leqslant C\|\upsilon\|_{L^2(0,1)}\|g_{1x}\|_{L^2(0,1)} \leqslant C\|G\|_{\mathcal{H}}\|U\|_{\mathcal{H}},$$

$$\left|\int_0^1 \upsilon\bar{g}_1dx\right| \leqslant C\|G\|_{\mathcal{H}}\|U\|_{\mathcal{H}},$$

and

$$\left|\int_0^1 g_2\bar{w}dx\right| \leqslant C\|G\|_{\mathcal{H}}\|U\|_{\mathcal{H}},$$

Hence, we deduce that

$$|R| \leqslant C\|G\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \quad (2.79)$$

■

- **Step 3** We have

$$(a(x)w_x\bar{w})_{x=0} = 0, (\psi(x)|\upsilon(x)|^2)_{x=0} = 0, (\psi(x)a(x)|w_x|^2)_{x=0} = 0.$$

It holds that

$$\begin{aligned} c_1 \int_0^1 \left(a(x)\left|w_x\right|^2+|v|^2\right) dx \leq & \left(c_0+c_1\right) \psi(1)|v(1)|^2+\left(c_0+c_1\right) \frac{\psi(1)}{a(1)}\left|\left(a w_x\right)(1)\right|^2 \\ & +c_0 a(1)\left|w_x(1) \| w(1)\right|+C\|U\|_{\mathcal{H}}\| G \|_{\mathcal{H}} \\ & \leq c|v(1)|^2+c^{\prime}(\varepsilon)\left|\left(a w_x\right)(1)\right|^2+\varepsilon|w(1)|^2+C\|U\|_{\mathcal{H}}\|G\|_{\mathcal{H}} . \end{aligned} \tag{2.80}$$

for any $\varepsilon > 0$. Moreover, using the Sobolev injection, we have

$$|w(1)| \leqslant \|w\|_{H^1(0.1)} \leqslant c\|w_x\|_{L^2(0.1)} \leqslant c\|\sqrt{a}w_x\|_{L^2(0.1)}.$$

Then

$$\mathcal{E}_w \leq c\left|\left(a w_x\right)(1)\right|^2+c^{\prime}|v(1)|^2+c^{\prime\prime}\|U\|_{\mathcal{H}}\|G\|_{\mathcal{H}} \tag{2.81}$$

Since that

$$\int_{-\infty}^{+\infty} |\theta(\xi)|^2 d\xi \leq C \int_{-\infty}^{+\infty} \left(\xi^2+\eta\right)|\theta(\xi)|^2 d\xi \leq C\|U\|_{\mathcal{H}}\|G\|_{\mathcal{H}}.$$

Hence

$$\|U\|_{\mathcal{H}}^2 \leq c\left|\left(a w_x\right)(1)\right|^2+c^{\prime}|v(1)|^2+c^{\prime\prime}\|U\|_{\mathcal{H}}\|G\|_{\mathcal{H}}. \tag{2.82}$$

Substitution of inequalities (2.71) and (2.76) into (2.82), we obtain that

$$\|U\|_{\mathcal{H}}^2 \leq c|\gamma|^{1-\alpha}\|U\|_{\mathcal{H}}\|G\|_{\mathcal{H}}+c^{\prime}\|G\|_{\mathcal{H}}^2+c^{\prime\prime}\|U\|_{\mathcal{H}}\|G\|_{\mathcal{H}}. \tag{2.83}$$

Then

$$\|U\|_{\mathcal{H}} \leq c|\gamma|^{1-\alpha}\|G\|_{\mathcal{H}}.$$

The conclusion then follows by applying the Theorem 2.5.5 ■

2.5.3 Polynomial Stability (For $\eta = 0$)

By theorem 2.4.2 (see case 1) 0 is a spectral point. Therefore it is convenient to have the following generalization of theorem 5.2.2 at hand:

Theorem 2.5.8. *([15]). Let $S(t)$ be a bounded C_0-semigroup on a Hilbert space $\mathcal{H}$ with generator $\mathcal{A}$. Assume that $\sigma(A) \cap i\mathbb{R} = \{0\}$ and that there exist $\vartheta > 1$ et $\upsilon > 0$ such that*

$$\left\|(isI-\mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})}=\left\{\begin{array}{ll} O\left(|s|^{-\vartheta}\right), & s \rightarrow 0 \\ O\left(|s|^{\upsilon}\right), & |s| \rightarrow \infty \end{array}\right.$$

Then there exist constants $C, t_0 > 0$ such that for all $t \geqslant t_0$ and $U_0 \in D(\mathcal{A}) \cap R(\mathcal{A})$

$$\left\|e^{\mathcal{A}t}U_0\right\|^2 \leq C\frac{1}{t^{\frac{2}{\varsigma}}}\left\|U_0\right\|_{D(\mathcal{A})\cap R(\mathcal{A}),}$$

where $\varsigma = \max(\vartheta, \upsilon)$.

Our main result is the following.

Theorem 2.5.9. *The semigroup $S_{\mathcal{A}}(t)$ is polynomially stable and*

$$E(t) = \|S_{\mathcal{A}}(t)U_0\|_{\mathcal{H}}^2 \leq \frac{1}{t^{\frac{2}{\max\left\{\frac{2-\alpha}{2},\frac{3}{4}\right\}}}} \|U_0\|_{D(\mathcal{A})\cap R(\mathcal{A})}^2 .$$

Proof. First for γ large enough, from the estimation in the proof of Theorem 2.5.6, we have

$$\|(i\gamma I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leqslant C|\gamma|^{1-\alpha}.$$

For γ near 0, we have from (2.76)

$$|\upsilon(1)|^2 \leqslant c|\gamma|^{1-\alpha}\|U\|_{\mathcal{H}}\|G\|_{\mathcal{H}} + c|\gamma|^{-\alpha}\|G\|^2. \tag{2.84}$$

Now, from the boundary conditions, we have

$$\begin{aligned}
|aw_x(1)|^2 = \left|\int_{-\infty}^{+\infty}\kappa(\xi)\theta(\xi)d\xi\right|^2 &= \left|\upsilon(1)\int_{-\infty}^{+\infty}\frac{\kappa^2(\xi)}{i\gamma+\xi^2}d\xi + \int_{-\infty}^{+\infty}\frac{\kappa(\xi)g_3(\xi)}{i\gamma+\xi^2}d\xi\right|^2 \\
&\leqslant 2|\upsilon(1)|^2\left|\int_{-\infty}^{+\infty}\frac{\kappa^2(\xi)}{i\gamma+\xi^2}d\xi\right|^2 + 2\int_{-\infty}^{+\infty}\frac{\kappa^2(\xi)}{|i\gamma+\xi^2|^2}d\xi\|g_3\|_{L^2(-\infty,+\infty)} \\
&\leqslant 2\frac{\pi}{\sin\alpha\pi}|\upsilon(1)|^2|\gamma|^{2(\alpha-1)} + 4\int_{-\infty}^{+\infty}\frac{\kappa^2(\xi)}{|i\gamma+\xi^2|^2}d\xi\|g_3\|_{L^2(-\infty,+\infty)} \\
&\leqslant 2\frac{\pi}{\sin\alpha\pi}|\upsilon(1)|^2|\gamma|^{2(\alpha-1)} + 4(1-\alpha)\frac{\pi}{\sin\alpha\pi}|\gamma|^{\alpha-2}\|g_3\|_{L^2(-\infty,+\infty)},
\end{aligned} \tag{2.85}$$

$$\begin{aligned}
\|\theta\|^2 = &\left|\upsilon(1)\int_{-\infty}^{+\infty}\frac{\kappa^2(\xi)}{i\gamma+\xi^2}d\xi\int_{-\infty}^{+\infty}\frac{g_3(\xi)}{i\gamma+\xi^2}d\xi\right|^2 \\
&\leqslant 2|\upsilon(1)|^2\left|\int_{-\infty}^{+\infty}\frac{\kappa(\xi)}{i\gamma+\xi^2}d\xi\right|^2 + 2\int_{-\infty}^{+\infty}\frac{1}{|i\gamma+\xi^2|^2}d\xi\|g_3\|_{L^2(-\infty,+\infty)} \\
&\leq 2\frac{\pi}{\sin\frac{2\alpha+1}{4}\pi}|\upsilon(1)|^2|\gamma|^{(2\alpha-3)/2} + 2\pi|\gamma|^{-3/2}\|g_3\|_{L^2(-\infty,+\infty)}^2 .
\end{aligned} \tag{2.86}$$

Substitution of inequalities (2.84) into (2.85) and (2.86), we obtain that

$$|aw_x(1)|^2 \leq c|\gamma|^{\alpha-1}\|U\|_{\mathcal{H}}\|G\|_{\mathcal{H}} + c|\gamma|^{\alpha-2}\|G\|_{\mathcal{H}}^2, \tag{2.87}$$

$$|\theta|^2 \leq c|\gamma|^{\frac{-1}{2}}U\|_{\mathcal{H}}\|G\|_{\mathcal{H}} + c|\gamma|^{\frac{-3}{2}}\|G\|_{\mathcal{H}}^2, \tag{2.88}$$

Substitution of inequalities (2.84) and (2.87) into (2.82) and using (2.88), we obtain that

$$\|\|U_{\mathcal{H}}^2 \leqslant (c|\gamma|^{\alpha-1} + c|\gamma|^{-1/2})U\|_{\mathcal{H}}\|G\|_{\mathcal{H}} + (c'|\gamma|^{\alpha-2} + c''|\gamma|^{-3/2})\|G\|_{\mathcal{H}}. \tag{2.89}$$

Then

$$\|U\|_{\mathcal{H}} \leqslant c' \frac{1}{|\gamma|^{\max\left(\frac{2-\alpha}{2}, \frac{3}{4}\right)}}. \tag{2.90}$$

Applying theorem 2.5.8, we obtain that

$$E(t) \leq C \frac{\|U_0\|^2_{D(\mathcal{A}) \cap R(\mathcal{A})}}{t^{\frac{2}{\max\left\{\frac{2-\alpha}{2}, \frac{3}{4}\right\}}}}.$$

■

Chapter 3

Energy decay of solutions to a nondegenerate wave equation with a fractional dynamic feedback

3.1 Introduction

In this chapter we investigate the existence and decay properties of solutions for the initial boundary value problem of the wave equation of the type

$$w_{tt}(x,t) - \left(a(x)w_x\right)_x(x,t) = 0 \text{ in } (0,L)\times(0,+\infty), \tag{Q}$$

where $(x,t) \in (0,L)\times(0,+\infty)$. This system is subject to the boundary conditions

$$w(0,t) = 0 \quad \text{ in } (0,+\infty),$$

$$mw_{tt}(L,t) + \left(a(x)w_x\right)(L,t) = -\varrho\partial_t^{\alpha,\eta}w(L,t) \quad \text{ in } (0,+\infty),$$

where $m > 0$ and $\varrho > 0$. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo fractional derivative of order $\alpha, 0 < \alpha < 1$, with respect to the time variable (see [11] and [23]). It is defined as follows

$$\partial_t^{\alpha,\eta}w(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}e^{-\eta(t-s)}\frac{dw}{ds}(s)ds, \ \eta \geq 0.$$

The system is finally completed with initial conditions

$$w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x)$$

where the initial data (w_0, w_1) belong to a suitable function space. The problem (Q) describes the motion of a pinched vibration cable with tip mass $m > 0$. In [44] B. Mbodje studies the decay rate of the energy of the wave equation with a boundary fractional derivative control as in this chapter (with m=0). Using energy methods, he proves strong asymptotic stability under the condition $\eta = 0$ and a polynomial type decay rate $E(t) \leqslant \frac{C}{t}$ if $\eta \neq 0$. Recently in

[10], A.Benaissa and Benkheda considered the stabilisation for the following wave equation with dynamic boundary control of fractional derivative type

$$\begin{cases} w_{tt}(x,t) - w_{xx}(x,t) = 0 & \text{in } (0,L)\times(0,+\infty), \\ w(0,t) = 0 & \text{on } (0,+\infty), \\ mw_{tt}(L,t) + w_x(L,t) = -\varrho\partial_t^{\alpha,\eta} w(L,t) & \text{on } (0,+\infty), \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x) & \text{on } (0,L), \end{cases} \tag{PF}$$

where $\varrho > 0$. They proved that the decay of the energy is not exponential but polynomial, that is, $E(t) \leqslant C_1/t^{(2-\alpha)}$. Very recently in [55], A.Benaissa, M.Tahri and H.Benkhedda considered the stabilization of the following problem

$$\begin{cases} w_{tt}(x,t) - (a(x)w_x(x,t))_x = 0 & \text{in } (0,1)\times(0,+\infty), \\ w(0,t) = 0 & \text{on } (0,+\infty), \\ (aw_x)(1,t) = -\varrho\partial_t^{\alpha,\eta} w(1,t) & \text{on } (0,+\infty), \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x) & \text{on } (0,1), \end{cases} \tag{P}$$

where $\varrho > 0$. They proved that system (P) is not exponentially stable for a general nondegenerate function a and they established an optimal polynomial energy decay depending with the parameter α for smooth solution. The boundary feedback under the consideration here are of fractional type and are described by the fractional derivatives

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)ds, \ \eta \geq 0.$$

The order of our derivatives is between 0 and 1. Very little is known in the literature. In addition to being nonlocal, fractional derivatives involve singular and non integrable kernels ($t^{-\alpha}, 0 < \alpha < 1$). This makes the problem more delicate. It has been shown (see[45]) that, as ∂_t the fractional derivative ∂_t^{α} forces the system to become dissipative and the solution to approach the equilibrium state. Therefore, when applied on the boundary, we can consider them as controllers which help to reduce the vibrations. In the recent years, fractional calculus has been applied successfully in various areas to modify many existing models of physical processes such as heat conduction, diffusion, viscoelasticity, wave propagation, electronics etc. Caputo and Mainardi [18] have established the relation between fractional derivative and theory of viscoelasticity. The generalization of the concept of derivative and integral to a non-integer order has been subjected to several approaches and some various alternative definition of fractional derivative appeared in [30, 32]. Our purpose in this chapter is to give a global solvability in Sobolev spaces and energy decay estimates of the solutions to the problem (Q) with a dynamic boundary control of fractional derivative type. The organization of this chapter is as follows. In Section 2, we show that the above system can be replaced by an augmented one obtained by coupling the nondegenarate wave equation with a suitable diffusion equation (as in [45]). In Section3, we introduce our functional analytic setting with a view of tackling the problem later on. In Section 4, existence and uniqueness of strong and weak solutions of the system are proved, using the Hille - Yosida theorem. In Section 5, we show the lack of exponential stability by spectral analysis. In Section 6, we study asymptotic stability of the above model, and we establish an optimal polynomial energy decay depending with the parameter α for smooth solution. Finally, Section 7 is devoted to conclusions on the problems treated in this chapter and future works, including some possible generalizations and interesting open questions.

3.2 Augmented model

This section is concerned with the reformulation of the model (Q) into an augmented system. For that, we need the following claims.

Theorem 3.2.1. (*see*[41]). *Let κ be the function:*

$$\kappa(\xi) = |\xi|^{\frac{(2\alpha-1)}{2}}, \ -\infty < \xi < +\infty, \ 0 < \alpha < 1. \tag{3.1}$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t \theta(\xi, t) + \left(\xi^2 + \eta\right) \theta(\xi, t) - U(t)\kappa(\xi) = 0, \quad -\infty < \xi < +\infty, \ \eta \geq 0, \ t > 0 \tag{3.2}$$

$$\theta(\xi, 0) = 0 \tag{3.3}$$

$$O(t) = (\pi)^{-1} \sin(\alpha\pi) \int_{-\infty}^{+\infty} \kappa(\xi)\theta(\xi, t) d\xi \tag{3.4}$$

is given by

$$O(t) = I^{1-\alpha,\eta} U(t), \tag{3.5}$$

where

$$\left[I^{\alpha,\eta} f\right](t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{-\eta(t-\tau)} f(\tau) d\tau.$$

Proof. From (3.2) and (3.3), we have

$$\theta(\xi, t) = \int_0^t \kappa(\xi) e^{-(\xi^2+\eta)(t-\tau)} U(\tau) d\tau. \tag{3.6}$$

Hence, by using (3.4), we get

$$O(t) = (\pi)^{-1} \sin(\alpha\pi) e^{-\eta t} \int_0^t [2 \int_0^{+\infty} |\xi|^{2\alpha-1} e^{-\xi^2(t-s)} d\xi] e^{\eta\tau} U(\tau) d\tau. \tag{3.7}$$

Thus,

$$\begin{aligned} O(t) &= (\pi)^{-1} \sin(\alpha\pi) e^{-\eta t} \int_0^t [(t-s)^{-\alpha} \Gamma(\alpha)] e^{\eta\tau} U(\tau) d\tau \\ &= (\pi)^{-1} \sin(\alpha\pi) \int_0^t [(t-s)^{-\alpha} \Gamma(\alpha)] e^{-\eta(t-\tau)} U(\tau) d\tau, \end{aligned} \tag{3.8}$$

which completes the proof. Indeed, we know that

$$(\pi)^{-1}\sin(\alpha\pi) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}$$

■

Lemma 3.2.2. *If* $\gamma \in D_\eta = \mathbb{C}\setminus]-\infty, \eta]$ *then*

$$\int_{-\infty}^{+\infty} \frac{\kappa^2(\xi)}{\gamma+\eta+\xi^2} d\xi = \frac{\pi}{\sin\alpha\pi}(\gamma+\eta)^{\alpha-1}.$$

Proof. Let us set

$$f_\gamma(\xi) = \frac{\kappa^2(\xi)}{\xi^2+\eta+\gamma}$$

We have

$$\left|\frac{\kappa^2(\xi)}{\gamma+\eta+\xi^2}\right| \leqslant \begin{cases} \frac{\kappa^2(\xi)}{\Re e(\gamma)+\eta+\xi^2} \quad \text{or} \\ \\ \frac{\kappa^2(\xi)}{\Im m(\gamma)+\eta+\xi^2} \end{cases}$$

Then the function f_γ is integrable. Moreover

$$\left|\frac{\kappa^2(\xi)}{\gamma+\eta+\xi^2}\right| \leqslant \begin{cases} \frac{\kappa^2(\xi)}{\eta_0+\eta+\xi^2} \quad \text{for all } \Re e(\gamma) \geq \eta_0 > -\eta \\ \\ \frac{\kappa^2(\xi)}{\tilde{\eta_0}+\xi^2} \text{ for all } |\Im m(\gamma)| \geq \tilde{\eta_0} > 0 \end{cases}$$

From [53, Theorem 1.16.1], the function $f_\gamma : D_\eta \longrightarrow \mathbb{C}$ is holomorphic. For a real number $\gamma > -\eta$ we have

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{\kappa^2(\xi)}{\gamma+\eta+\xi^2} d\xi &= \int_{-\infty}^{+\infty} \frac{|\xi|^{2\alpha-1}}{\gamma+\eta+\xi^2} d\xi \\ &= \int_{-\infty}^{+\infty} \frac{x^{\alpha-1}}{\gamma+\eta+x} dx \\ &= (\gamma+\eta)^{\alpha-1}\int_1^{+\infty} y^{-1}(y-1)^{\alpha-1}dy, \text{ with } y = \frac{x}{\gamma+\eta}+1 \\ &= (\gamma+\eta)^{\alpha-1}\int_0^1 z^{-\alpha}(1-z)^{\alpha-1}dz, \text{ with } z = \frac{1}{y} \\ &= (\gamma+\eta)^{\alpha-1}B(1-\alpha,\alpha) \\ &= (\gamma+\eta)^{\alpha-1}\Gamma(1-\alpha)\Gamma(\alpha) \\ &= (\gamma+\eta)^{\alpha-1}\frac{\pi}{\sin\alpha\pi}. \end{aligned}$$

Both holomorphic functions f_γ and $\gamma \longmapsto (\gamma+\eta)^{\alpha-1}\frac{\pi}{\sin\alpha\pi}$ coincide on the half line $]-\eta, +\infty[$, hence on D_η following the principe of isolated zeroes. ■

We are now in a position to reformulate system (Q). Indeed, by using Theorem 3.2.1, system (Q) is equivalent to the following:

$$\begin{cases} w_{tt}(x,t) - (a(x)w_x(x,t))_x = 0 & \text{in } (0,L)\times(0,+\infty),\\ \theta_t(\xi,t) + (\xi^2+\eta)\,\theta(\xi,t) - w_t(L,t)\kappa(\xi) = 0 & \text{in } (-\infty,+\infty)\times(0,+\infty),\\ w(0,t) = 0 & \text{on } (0,+\infty),\\ mw_{tt}(L,t) + (aw_x)(L,t) = -\zeta\int_{-\infty}^{+\infty}\kappa(\xi)\theta(\xi,t)d\xi & \text{in } (-\infty,+\infty)\times(0,+\infty),\\ w(x,0) = w_0(x), w_t(x,0) = w_1(x) & \text{on } (0,L),\\ \theta(\xi,0) = 0 & \text{on } (-\infty,+\infty). \end{cases} \tag{Q'}$$

where $\zeta = \varrho(\pi)^{-1}\sin(\alpha\pi)$. For the solution of problem (Q'), we define the energy functional

$$E(t) = \frac{1}{2}\|w_t\|_2^2 + \frac{1}{2}\left\|\sqrt{a(x)}w_x\right\|_2^2 + \frac{m}{2}|w_t(L,t)|^2 + \frac{\zeta}{2}\int_{-\infty}^{+\infty}|\theta(\xi,t)|^2 d\xi. \tag{3.9}$$

Lemma 3.2.3. *Let (w,θ) be a solution of the problem (Q'). Then, the energy functional defined by (3.9) satisfies*

$$E'(t) = -\zeta\int_{-\infty}^{+\infty}\left(\xi^2+\eta\right)|\theta(\xi,t)|^2 d\xi \le 0. \tag{3.10}$$

Proof. Multiplying the first equation in (Q'), by $\bar{w}_t$, integrating over (0, L) and using integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}\|w_t\|_2^2 - \Re e\int_0^L (a(x)w_x)_x\bar{w}_t dx = 0.$$

Then

$$\frac{d}{dt}\left(\frac{1}{2}\|w_t\|_2^2 + \frac{1}{2}\left\|\sqrt{a(x)}w_x\right\|_2^2 + \frac{m}{2}|w_t(L,t)|^2\right) + \zeta\Re e\bar{w}_t(L,t)\int_{-\infty}^{+\infty}\kappa(\xi)\theta(\xi,t)d\xi = 0. \tag{3.11}$$

Multiplying the second equation in (Q'), by $\zeta\bar{\theta}_t$, and integrating over $(-\infty,+\infty)$ to obtain:

$$\frac{\zeta}{2}\frac{d}{dt}\|\theta\|_2^2 + \zeta\int_{-\infty}^{+\infty}(\xi^2+\eta)|\theta(\xi,t)|^2 d\xi - \zeta\Re e w_t(L,t)\int_{-\infty}^{+\infty}\kappa(\xi)\bar{\theta}(\xi,t)d\xi = 0. \tag{3.12}$$

From (3.9), (3.11) and (3.12) we get

$$E'(t) = -\zeta\int_{-\infty}^{+\infty}\left(\xi^2+\eta\right)|\theta(\xi,t)|^2 d\xi.$$

This completes the proof of the lemma. ■

3.3 Functional analytic setting

Let us introduce the semigroup representation of the system (Q'). We consider the following condition of the right end contour of wave

$$\upsilon(t) = w_t(L,t), \text{ for } t > 0, \tag{3.13}$$

were υ solve the equation

$$m\upsilon_t(t) + (a(x)w_x)(L,t) + \zeta \int_{-\infty}^{+\infty} \kappa(\xi)\theta(\xi,t)d\xi = 0. \tag{3.14}$$

Let $U = (w, w_t, \theta, \upsilon)^T$ and rewrite (Q') as

$$\begin{cases} U' = \mathcal{A}U \\ U(0) = (w_0, w_1, \theta_0, \upsilon_0) \end{cases} \tag{3.15}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} w \\ u \\ \theta \\ \upsilon \end{pmatrix} = \begin{pmatrix} u \\ (a(x)w_x)_x \\ -\left(\xi^2+\eta\right)\theta + u(L)\kappa(\xi) \\ -\frac{1}{m}\left(a(x)w_x\right)(L) - \frac{\zeta}{m}\int_{-\infty}^{+\infty}\kappa(\xi)\theta(\xi)d\xi \end{pmatrix} \tag{3.16}$$

We consider the following space

$$H^1_L(0,L) = \left\{ w \in H^1(0,L), w(0) = 0 \right\},$$

and the Hilbert space

$$\mathcal{H} = H^1_L(0,L) \times L^2(0,L) \times L^2(-\infty,+\infty) \times \mathbb{C},$$

equipped with the inner product

$$\langle U, \tilde{U}\rangle_{\mathcal{H}} = \int_0^L \left(u\overline{\tilde{u}} + a(x)w_x\overline{\tilde{w}_x}\right)dx + \zeta\int_{-\infty}^{+\infty}\theta\overline{\tilde{\theta}}d\xi + m\upsilon\overline{\tilde{\upsilon}}$$

The domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{ (w,u,\theta,\upsilon)^T \in \mathcal{H} \;\middle|\; \begin{array}{l} w \in H^2(0,L) \cap H^1_L(0,L) \\ u \in H^1_L(0,L), \upsilon \in \mathbb{C} \\ -\left(\xi^2+\eta\right)\theta + u(L)\kappa(\xi) \in L^2(-\infty,+\infty) \\ u(L) = \upsilon, |\xi|\theta \in L^2(-\infty,+\infty) \end{array} \right\} \tag{3.17}$$

3.4 Global existence

In this section we will give well-posedness results for problem (Q') using semigroup theory. We show that the operator $\mathcal{A}$ generates a C_0-semigroup in $\mathcal{H}$. We prove that $\mathcal{A}$ is a maximal dissipative operator. For this purpose we need the following two lemmas.

Lemma 3.4.1. *The operator $\mathcal{A}$ is dissipative and satisfies, for any $U \in D(\mathcal{A})$.*

$$\Re e\langle \mathcal{A}U, U\rangle = -\zeta \int_{-\infty}^{+\infty} \left(\xi^2+\eta\right) |\theta(\xi)|^2 d\xi, \tag{3.18}$$

Proof. For any $U = (w, w_t, \theta, \upsilon)^T \in D(\mathcal{A})$, using (3.15), (3.10) and the fact that

$$E(t) = \frac{1}{2}\|U\|_{\mathcal{H}}^2,$$

estimate (3.18) easily follows. ∎

Lemma 3.4.2. *The operator $\gamma I - \mathcal{A}$ is surjective for all $\gamma > 0$.*

Proof. We need to show that for all $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$, there exists $U = (w, w_t, \theta, \upsilon)^T \in D(\mathcal{A})$, such that

$$\gamma U - \mathcal{A}U = F. \tag{3.19}$$

Then, in terms of components, the above equation reads

$$\begin{cases} \gamma w - u = f_1 \\ \gamma u - (a(x)w_x)_x = f_2 \\ \gamma\theta + (\xi^2+\eta)\theta - u(L)\kappa(\xi) = f_3 \\ \gamma\upsilon + \frac{1}{m}(a(x)w_x)(L) + \frac{\zeta}{m}\int_{-\infty}^{+\infty}\kappa(\xi)\theta(\xi)d\xi = f_4 \end{cases}. \tag{3.20}$$

Suppose w is found with the appropriate regularity. Then, $(3.20)_1$ yields

$$u = \gamma w - f_1. \tag{3.21}$$

It is clear that $u \in H_L^1(0, L)$ Furthermore, by $(3.20)_3$ we can find θ as

$$\theta = \frac{f_3(\xi) + \kappa(\xi)u(L)}{\xi^2+\eta+\gamma}. \tag{3.22}$$

By using $(3.20)_2$ and (3.21) the function w satisfying the following system

$$\gamma^2 w - (a(x)w_x)_x = f_2 + \gamma f_1. \tag{3.23}$$

Solving equation (3.23) is equivalent to finding $w \in H^2 \cap H_L^1(0, L)$ such that

$$\int_0^L (\gamma^2 w - (a(x)w_x)_x)\bar{z}dx = \int_0^L (f_2 + \gamma f_1)\bar{z}dx, \tag{3.24}$$

for all $z \in H_L^1(0,L)$. Using integration by parts in (3.24) and taking into account (3.22), we obtain

$$\begin{cases} \int_0^L (\gamma^2 w\bar{z} + a(x)w_x\bar{z}_x)\,dx + (\gamma m + \tilde{\zeta})u(L)\bar{z}(L) \\ = \int_0^L (f_2 + \gamma f_1)\,\bar{z}dx - \zeta\int_{-\infty}^{+\infty} \frac{\kappa(\xi)}{\xi^2+\eta+\gamma} f_3(\xi)d\xi.\bar{z}(L) + mf_4\bar{z}(L) \end{cases} \tag{3.25}$$

where $\tilde{\zeta} = \zeta\int_{-\infty}^{+\infty} \frac{\kappa(\xi)}{\xi^2+\eta+\gamma}$. Using again (3.21), we deduce that

$$u(L) = \gamma.w(L) - f_1(L). \tag{3.26}$$

Inserting (3.26) into (3.25), we get

$$\begin{cases} \int_0^L (\gamma^2 w\bar{z} + a(x)w_x\bar{z}_x)\,dx + \gamma(\gamma m + \tilde{\zeta})w(L)\bar{z}(L) \\ = \int_0^L (f_2 + \gamma f_1)\,.\bar{z}dx - \zeta.\bar{z}(L)\int_{-\infty}^{+\infty} \frac{\kappa(\xi)}{\xi^2+\eta+\gamma} f_3(\xi)d\xi \\ +(\gamma m + \tilde{\zeta})f_1(L).\bar{z}(L) + mf_4\bar{z}(L) \end{cases} \tag{3.27}$$

Consequently, problem (3.27) is equivalent to the problem

$$\mathcal{B}(w,z) = \mathcal{L}(z), \tag{3.28}$$

where the bilinear form $\mathcal{B} :\ H_L^1(0,L) \times H_L^1(0,L) \longrightarrow \mathbb{C}$
and the linear form $\mathcal{L} : H_L^1(0,L) \longrightarrow \mathbb{C}$, are defined by

$$\mathcal{B}(w,z) = \int_0^L \left(\gamma^2 w\bar{z} + a(x)w_x\bar{z}_x\right)dx + \gamma(\gamma m + \tilde{\zeta})w(L)\bar{z}(L)$$

and

$$\mathcal{L}(w) = \int_0^L (f_2 + \gamma f_1)\,.\bar{z}dx - \zeta.\bar{z}(L)\int_{-\infty}^{+\infty} \frac{\kappa(\xi)}{\xi^2+\eta+\gamma} f_3(\xi)d\xi + (\gamma m + \tilde{\zeta})f_1(L).\bar{z}(L) + mf_4\bar{z}(L).$$

It is easy to verify that $\mathcal{B}$ is continuous and coercive, and $\mathcal{L}$ is continuous. So applying the Lax-Milgram theorem, we deduce that for all $z \in H_L^1(0,L)$ problem (3.28) admits a unique solution $w \in H_L^1(0,L)$. Applying the classical elliptic regularity, it follows from (3.27) that $w \in H^2(0,L)$. Therefore, the operator $\gamma I - \mathcal{A}$ is surjective for any $\gamma > 0$. Consequently, using Hille-Yosida Theorem, we have the following existence and uniqueness result. ■

Theorem 3.4.3. *Let $U_0 \in \mathcal{H}$, then there exists a unique solution $U \in C^0(\mathbb{R}_+, \mathcal{H})$, of problem (3.15). Moreover if $U_0 \in D(\mathcal{A})$, then*

$$U \in C^0(\mathbb{R}_+, D(\mathcal{A}) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

3.5 Lack of exponential stability

In order to state and prove our stability results, we need some Theorems.

Theorem 3.5.1. *([51],[31]) Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on Hilbert space $\mathcal{H}$. Then S(t) is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\beta : \beta \in \mathbb{R}\} = i\mathbb{R}$$

and

$$\overline{\lim_{|\beta|\to\infty}} ||(i\beta I - \mathcal{A})^{-1}||_{\mathcal{L}(\mathcal{H})} < \infty$$

Theorem 3.5.2. *([12]) Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup on Hilbert space $\mathcal{H}$. If*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \quad and \quad \sup_{|\beta|\geqslant 1} \frac{1}{\beta^{l'}} ||(i\beta I - \mathcal{A})^{-1}||_{\mathcal{L}(\mathcal{H})} \leqslant M.$$

for some l', then there exist c such that

$$||e^{At}u_0||^2 \leq \frac{c}{t^{\frac{2}{l'}}} ||u_0||^2_{D(A)}, \forall t > 0, \forall u_0 \in D(\mathcal{A}).$$

Theorem 3.5.3. *([4] -[37]) Let $\mathcal{A}$ be the generator of a uniformly bounded C_0-semigroup $(S(t))_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. If:*
(i) $\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.
(ii) The intersection of the spectrum $\sigma(\mathcal{A})$ with $i\mathbb{R}$ is at most a countable set.
Then the semigroup $(S(t))_{t\geq 0}$ is strongly (or asymptotically) stable,i.e, $||S(t)_z||_{\mathcal{H}} \to 0$ as $t \to \infty$ for any $z \in \mathcal{H}$.

Our main result is the following

Theorem 3.5.4. *The semigroup generated by the operator $\mathcal{A}$ is not exponentially stable.*

Proof. We will examine two cases.
• **Case 1**. $\eta = 0$: We shall show that $i\gamma = 0$ is not in the resolvent set of the operator $\mathcal{A}$. Indeed, noting that $(\sin x, 0, 0, 0)^T \in \mathcal{H}$, and denoting by $(w, u, \theta, \upsilon)^T$ the image of $(\sin x, 0, 0, 0)^T$ by $\mathcal{A}^{-1}$, we see that $\theta\,(\xi)$=$|\xi|^{\frac{2\alpha-5}{2}} \sin L$. But, then $\theta \notin L^2(-\infty, +\infty)$, since $\alpha \in]0, 1]$. And so $(w, u, \theta, \upsilon)^T \notin D(\mathcal{A})$.
• **Case 2**. $\eta \neq 0$: We aim to show that an infinite number of eigenvalues of $\mathcal{A}$ approach the imaginary axis which prevents the wave system (Q') from being exponentially stable. Indeed we first compute the characteristic equation that gives the eigenvalues of $\mathcal{A}$. Let γ be an eigenvalue of $\mathcal{A}$ with associated eigenvector $(w, u, \theta, \upsilon)^T$. Then $\mathcal{A}U = \gamma U$ is equivalent to

$$\begin{cases} \gamma w - u = 0 \\ \gamma u - (a(x)w_x)_x = 0 \\ \gamma\theta + (\xi^2 + \eta)\theta - u(L)\kappa(\xi) = 0 \\ \gamma\upsilon + \frac{1}{m}(a(x)w_x)(L) + \frac{\zeta}{m}\int_{-\infty}^{+\infty} \kappa(\xi)\theta(\xi)d\xi = 0 \end{cases} . \tag{3.29}$$

From $(3.29)_1, (3.29)_2$ for such γ, we find

$$\gamma^2 w - (a(x)w_x)_x = 0. \tag{3.30}$$

Since $\upsilon = u(L)$. using $(3.29)_3$ and $(3.29)_4$, we get

$$\begin{cases} w(0) = 0, \\ \left(\gamma + \dfrac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty} \dfrac{\kappa^2(\xi)}{\xi^2+\gamma+\eta} d\xi\right) u(L) + \dfrac{1}{m} a(L) w_x(L) = \\ \left(\gamma + \dfrac{\varrho}{m}(\gamma+\eta)^{\alpha-1}\right)\gamma w(L) + \dfrac{1}{m} a(L) w_x(L) = 0. \end{cases} \tag{3.31}$$

Writing (3.30) and (3.31) in the standard form of a linear differential operator with homogeneous boundary conditions, we obtain

$$\begin{cases} w_{xx} + \frac{a_x(x)}{a(x)} w_x - \frac{\gamma^2}{a(x)} w = 0 \\ w(0) = 0 \\ \left(\gamma + \frac{\varrho}{m}(\gamma+\eta)^{\alpha-1}\right)\gamma w(L) + \frac{1}{m} a(L) w_x(L) = 0. \end{cases} \tag{3.32}$$

In order to simplify the computations, we introduce a spatial-scale transformation in x

$$\phi(y) = w(x), y = \frac{1}{h}\int_0^x \frac{1}{\sqrt{a(s)}} ds, y \in (0,1), \tag{3.33}$$

where

$$h = \int_0^L \frac{1}{\sqrt{a(s)}} ds$$

Then Eq. (3.32) has the form

$$\begin{cases} \phi''(y) + \frac{h}{2}\frac{a_x(x)}{\sqrt{a(x)}}\phi'(y) - h^2\gamma^2\phi(y) = 0, \\ \phi(0) = 0, \\ \frac{\sqrt{a(L)}}{mh}\phi'(L) + \left(\gamma + \frac{\varrho}{m}(\gamma+\eta)^{\alpha-1}\right)\gamma\phi(L) = 0. \end{cases} \tag{3.34}$$

Equation (3.34) can be further simplified by applying another invertible transformation (see [48]):

$$\varphi(y) = e^{\frac{1}{2}\int_0^y \tilde{a}(s)ds}\phi(y), y \in (0,1), \tag{3.35}$$

where

$$\tilde{a}(y) = \frac{h}{2}\frac{a_x(x)}{\sqrt{a(x)}}.$$

(3.35) allows one to cancel the term $\frac{h}{2}\frac{a_x(x)}{\sqrt{a(x)}}\phi'(y)$ in (3.34). Hence we arrive at an equivalent eigenvalue problem

$$\begin{cases} \varphi''(y) - \left(\frac{1}{2}\tilde{a}'(y) + \frac{1}{4}\tilde{a}^2(y) + h^2\gamma^2\right)\varphi(y) = 0 \\ \varphi(0) = 0 \\ \varphi'(L) + \left(-\frac{h}{4}\frac{a'(L)}{\sqrt{a(L)}} + \frac{mh}{\sqrt{a(L)}}\gamma + \frac{\gamma h}{\sqrt{a(L)}}\gamma(\gamma+\eta)^{\alpha-1}\right)\varphi(L) = 0 \end{cases} \tag{3.36}$$

To asymptotically estimate the solutions to the eigenvalue problem (3.36), we proceed as in [45]. ■

Lemma 3.5.5. *The equation*

$$\varphi''(y) - \left(\frac{1}{2}\tilde{a}'(y) + \frac{1}{4}\tilde{a}^2(y) + h^2\gamma^2\right)\varphi(y) = 0,$$

has two linearly independent asymptotic fundamental solutions:

$$\varphi_1(y) = e^{h\gamma y}(1 + \frac{\varphi_{10}(y)}{ih\gamma}) + O(\frac{1}{\gamma^2}),$$

$$\varphi_2(y) = e^{-h\gamma y}(1 + \frac{\varphi_{20}(y)}{ih\gamma}) + O(\frac{1}{\gamma^2})$$

and hence their derivatives are given by

$$\frac{d}{dy}\varphi_1(y) = h\gamma e^{h\gamma y}(1 + \frac{\varphi_{10}(y)}{ih\gamma}) + O(\frac{1}{\gamma^2}),$$

$$\frac{d}{dy}\varphi_2(y) = -h\gamma e^{-h\gamma y}(1 + \frac{\varphi_{20}(y)}{ih\gamma}) + O(\frac{1}{\gamma^2}),$$

where

$$\varphi_{10}(y) = -\frac{i}{2}\int_0^y \left(\frac{1}{2}\tilde{a}'(s) + \frac{1}{4}\tilde{a}^2(s)\right) ds,$$

$$\varphi_{20}(y) = \frac{i}{2}\int_0^y \left(\frac{1}{2}\tilde{a}'(s) + \frac{1}{4}\tilde{a}^2(s)\right) ds.$$

For simplicity, we introduce the following notation: $[a]_i := a + \mathcal{O}(\gamma^{-i})$ for $i = 1, 2$. From Lemma 3.5.5, one can write the asymptotic solution of (3.36) as follows:

$$\varphi(y) = \sum_{i=1}^{2} c_i\varphi_i, \tag{3.37}$$

where c_i is chosen so that φ satisfies the boundary conditions, i.e.,

$$\tilde{M}(\gamma)C(\gamma) = \left(\begin{array}{cc} [1]_2 & [1]_2 \\ \left[\left(\gamma + \frac{m}{\sqrt{a(L)}}\gamma^2 + \frac{\varrho}{\sqrt{a(L)}}\gamma^\alpha\right) e^{\gamma h}\right]_0 & \left[\left(-\gamma + \frac{m}{\sqrt{a(L)}}\gamma^2 + \frac{\varrho}{\sqrt{a(L)}}\gamma^\alpha\right) e^{-\gamma h}\right]_0 \end{array} \right) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (3.38)$$

Hence a non-trivial solution w exists if and only if the determinant of $\tilde{M}(\gamma)$ vanishes. Set $f(\gamma) = det\tilde{M}(\gamma)$, thus the characteristic equation is $f(\gamma) = 0$. Our purpose in the sequel is to prove, thanks to Rouche's Theorem, that there is a subsequence of eigenvalues for which their real part tends to 0. In the sequel, since $\mathcal{A}$ is dissipative, we study the asymptotic behavior of the large eigenvalues γ of $\mathcal{A}$ in the strip $-\alpha_0 \leqslant \Re e(\gamma) \leqslant 0,$ for some $\alpha_0 > 0$ large enough and for such γ we remark that $e^{ti}, i = 1, 2$ remains bounded.

Lemma 3.5.6. *There exists $N \in \mathbb{N}$ such that*

$$\{\gamma_k\}_{k\in\mathbf{Z}^*,|k|\geq N} \subset \sigma(\mathcal{A}), \quad (3.39)$$

where

$$\gamma_k = i\left(\frac{k\pi}{L} + \frac{1}{mk\pi}\right) + \frac{\tilde{\alpha}}{k^{3-\alpha}} + \frac{\beta}{k^{(3-\alpha)}} + o\left(\frac{1}{k^{3-\alpha}}\right), \quad k \geq N, \tilde{\alpha} \in i\mathbb{R}, \beta \in \mathbb{R}, \beta < 0,$$

$\gamma_k = \overline{\gamma - k}$ *if $k \leq -N$, Moreover for all $|k| \geq N$, the eigenvalues γ_k are simple.*

Proof.
• **Step1** :

$$\begin{aligned} f(\gamma) &= e^{t_2}h(t_2) - e^{t_1}h(t_1) \\ &= -e^{-\gamma L}h(-\gamma)\left(e^{2\gamma h} - \frac{-\gamma + \frac{m}{\sqrt{a(L)}}\gamma^2 + \frac{\varrho}{\sqrt{a(L)}}\gamma(\gamma+\eta)^{\alpha-1}}{\gamma + \frac{mh}{\sqrt{a(L)}}\gamma^2 + \frac{\varrho}{\sqrt{a(L)}}\gamma(\gamma+\eta)^{\alpha-1}}\right). \\ &= -e^{-\gamma L}h(-\gamma)\left(e^{2\gamma h} - 1 + \frac{2}{1 + \frac{m}{\sqrt{a(L)}}\gamma + \frac{\varrho}{\sqrt{a(L)}}(\gamma+\eta)^{\alpha-1}}\right). \end{aligned} \quad (3.40)$$

We set

$$\begin{aligned} \tilde{f}(\gamma) &= e^{2\gamma h} - 1 + \frac{2}{1 + \frac{m}{\sqrt{a(L)}}\gamma + \frac{\varrho}{\sqrt{a(L)}}(\gamma+\eta)^{\alpha-1}} \\ &= f_0(\gamma) + \frac{f_1(\gamma)}{\gamma} + \frac{f_2(\gamma)}{\gamma^2} + + \frac{f_3(\gamma)}{\gamma^{3-\alpha}} + o\left(\frac{1}{\gamma^{3-\alpha}}\right), \end{aligned} \quad (3.41)$$

where

$$f_0(\gamma) = e^{2\gamma h} - 1, \tag{3.42}$$

$$f_1(\gamma) = \frac{2\sqrt{a(L)}}{m}, \tag{3.43}$$

$$f_2(\gamma) = \frac{-2a(L)}{m^2}, \tag{3.44}$$

$$f_3(\gamma) = \frac{2\gamma\sqrt{a(L)}}{m^2}. \tag{3.45}$$

Note that f_0, f_1, f_2 and f_3 remain bounded in the strip $-\alpha_0 \leqslant \Re e(\gamma) \leqslant 0$.

• **Step2** : We look at the roots of f_0. From (3.42), f_0 has one familie of roots that we denote γ_k^0.

$$f_0(\gamma) = 0 \Longleftrightarrow e^{2\gamma h} = 1.$$

Hence

$$2\gamma h = i2k\pi, i.e., \gamma_k^0 = \frac{ik\pi}{h}, k \in \mathbb{Z}.$$

Now with the help of Rouche's Theorem, we will show that the roots of $\tilde{f}$ are close to those of f_0. Changing in (3.41) the unknown γ by $u = 2\gamma h$ then (3.41) becomes

$$\tilde{f}(u) = (e^u - 1) + O\left(\frac{1}{u}\right) = f_0(u) + O\left(\frac{1}{u}\right).$$

The roots of f_0 are $u_k = \frac{ik}{h}\pi, k \in \mathbb{Z}$, and setting $u = u_k + re^{it}, t \in [0, 2\pi]$, we can easily check that there exists a constant $C > 0$ independent of k such that $|e^u - 1| \geqslant C.r$ for r small enough. This allows to apply Rouche's Theorem. Consequently, there exists a subsequence of roots of ˜f which tends to the roots u_k of f_0. Equivalently, it means that there exists $N \in \mathbb{N}$ and a subsequence $(\gamma_K)_{|k|\geqslant N}$ of roots of $f(\gamma)$, such that $\gamma_k = \gamma_k^0 + o(1)$ which tends to the roots $\frac{ik}{h}\pi$ of f_0 Finally for $|k| \geqslant N, \gamma_k$ is simple since γ_k^0 is.

• **Step3** : From Step 2, we can write

$$\gamma_k = \frac{ik}{h}\pi + \varepsilon_k. \tag{3.46}$$

Using (3.46), we get

$$e^{2\gamma_k h} = 1 + 2h\varepsilon_k + 2h^2\varepsilon_k^2 + o(\varepsilon_k^2). \tag{3.47}$$

Substituting (3.47) into (3.41), using that $\tilde{f}(\gamma_k) = 0$, we get:

$$\tilde{f}(\gamma_k) = 2h\varepsilon_k + 2h^2\varepsilon_k^2 + \frac{\frac{2\sqrt{a(L)}}{m}}{\frac{ik\pi}{h} + \varepsilon_k} - \frac{\frac{2a(L)}{m^2}}{(\frac{ik\pi}{h} + \varepsilon_k)^2} + o(\varepsilon_k^2)$$

$$= 2h\varepsilon_k + \frac{\frac{2L}{m}}{k\pi i} + o\left(\varepsilon_k\right) + o\left(\frac{1}{k}\right) = 0,$$

and hence

$$\varepsilon_k = \frac{\sqrt{a(L)}}{mk\pi} i.$$

• **Step4** : From Step 3, we can write

$$\gamma_k = i\frac{1}{h}k\pi + \frac{\sqrt{a(L)}}{mk\pi}i + \varepsilon_k. \tag{3.48}$$

Using (3.48), we get

$$e^{2\gamma_k h} = 1 + \left(\frac{2\sqrt{a(L)}h}{mk\pi}i + 2h\varepsilon_k\right) + \frac{1}{2}\left(\frac{2\sqrt{a(L)}h}{mk\pi}i + 2h\varepsilon_k\right)^2 + o\left(\varepsilon_k^3\right). \tag{3.49}$$

Substituting (3.49) into (3.41), using that $\tilde{f}(\gamma_k) = 0$, we get:

$$\begin{aligned}
\tilde{f}(\lambda_k) = &\left(\frac{2\sqrt{a(L)}h}{mk\pi}i + 2h\varepsilon_k\right) + \frac{1}{2}\left(\frac{2\sqrt{a(L)}h}{mk\pi}i + 2h\varepsilon_k\right)^2 + \frac{\frac{2\sqrt{a(L)}}{m}}{\frac{k\pi i}{h} + \frac{\sqrt{a(L)}}{mk\pi}i + \varepsilon_k} \\
&- \frac{\frac{2a(L)}{m^2}}{\left(\frac{k\pi i}{h} + \frac{\sqrt{a(L)}}{mk\pi}i + \varepsilon_k\right)^2} - \frac{\frac{2\sqrt{a(L)}\gamma}{m^2}}{\left(\frac{k\pi i}{h} + \frac{\sqrt{a(L)}}{mk\pi}i + \varepsilon_k\right)^{(3-\alpha)}} + O\left(\varepsilon_k^3\right) + O\left(\frac{1}{k^3}\right) \\
&= 2h\varepsilon_k - \frac{2\sqrt{a(L)}\gamma}{m^2}\left(\frac{h}{k\pi i}\right)^{3-\alpha} + o\left(\varepsilon_k^3\right) + o\left(\frac{1}{k^3}\right) = 0.
\end{aligned} \tag{3.50}$$

$$\varepsilon_k = \frac{\sqrt{a(L)}\gamma}{m^2 h^{\alpha-2}(k\pi i)^{3-\alpha}} + o\left(\frac{1}{k^{3-\alpha}}\right)$$

$$= \begin{cases} -\frac{\sqrt{a(L)}\gamma}{m^2 h^{\alpha-2}(k\pi)^{3-\alpha}}\left(\cos(1-\alpha)\frac{\pi}{2} - i\sin(1-\alpha)\frac{\pi}{2}\right) + o\left(\frac{1}{k^{3-\alpha}}\right) & \text{for } k \succeq 0 \\ -\frac{\sqrt{a(L)}\gamma}{m^2 h^{\alpha-2}(-k\pi)^{3-\alpha}}\left(\cos(1-\alpha)\frac{\pi}{2} - i\sin(1-\alpha)\frac{\pi}{2}\right) + o\left(\frac{1}{k^{3-\alpha}}\right) & \text{for } k \preceq 0 \end{cases}$$

From this equation we obtain $|k|^{3-\alpha}\mathcal{R}\gamma_k \sim \beta$ in that case, with

$$\beta = -\frac{\sqrt{a(L)}\gamma}{m^2 h^{\alpha-2}\pi^{3-\alpha}}\cos(1-\alpha)\frac{\pi}{2}.$$

The operator $\mathcal{A}$ has a non exponential decaying branch of eigenvalues. Thus the proof is complete. ■

3.6 Asymptotic stability

Because of the unboundedness of the ξ-domain for the diffusive equation, the resolvent of $\mathcal{A}$ is not compact, and a major difficulty arises in the use of LaSalle's invariance principle to prove asymptotic stability. A refined analysis of the spectrum of generator of the semigroup can be performed, which allows for the use of the stability results of [4, 37]. A direct application of this result on the pseudo-differentially damped linearized pendulum, can be found in [42].

3.6.1 Strong stability of the system

In this part, we use a general criteria of Lemma 3.5.3 to show the strong stability of the C_0-semigroup $e^{t\mathcal{A}}$ associated to the wave system (Q') in the absence of the compactness of the resolvent of $\mathcal{A}$. Our main result is the following theorem:

Theorem 3.6.1. *The C_0-semigroup $e^{t\mathcal{A}}$ is strongly stable in $\mathcal{H}$, i.e., for all $U_0 \in \mathcal{H}$, the solution of (3.15) satisfies*

$$\lim_{t\to+\infty} \|e^{t\mathcal{A}}U_0\|.$$

For the proof of Theorem 3.6.1, we need the following two lemmas.

Lemma 3.6.2. *$\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.*

Proof. We make a distinction between $i\gamma = 0,$ and $i\gamma \neq 0.$
Step1 : Solving for $\mathcal{A}U = 0$ leads to $U = 0$, thanks to the boundary conditions in (3.17). Hence, $i\gamma = 0$ is not is not an eigenvalue of $\mathcal{A}$.
Step2: We will argue by contradiction. Let us suppose that there
$\gamma \in \mathbb{R}, \gamma \neq 0$, such that

$$\mathcal{A}U = i\gamma U. \tag{3.51}$$

Firstly, the equation (3.51) is equivalent to the following system

$$\begin{cases} i\gamma w - u = 0 \\ i\gamma u - (a(x)w_x)_x = 0 \\ i\gamma\theta + (\xi^2+\eta)\theta - u(L)\kappa(\xi) = 0 \\ i\gamma\upsilon + \frac{1}{m}(a(x)w_x)(L) + \frac{\zeta}{m}\int_{-\infty}^{+\infty}\kappa(\xi)\theta(\xi)d\xi = 0 \end{cases}. \tag{3.52}$$

Secondly, we will consider two cases:
• **Case 1**. $\gamma \neq 0$: Taking the $L^2(0,L)$-inner product with U of both sides of (3.51) and using (3.18), we immediately obtain

$$0 = \Re e\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\zeta\int_{-\infty}^{+\infty}\left(\xi^2+\eta\right)|\theta(\xi)|^2 d\xi. \tag{3.53}$$

Hence

$$\theta \equiv 0. \tag{3.54}$$

From $(3.52)_3$, we have

$$u(L) = 0. \tag{3.55}$$

Hence, from $(3.52)_1$ and $(3.52)_4$ we obtain

$$w(L) = 0 \text{ and } w_x(L) = 0. \tag{3.56}$$

from $(3.52)_1$ and $(3.52)_2$

$$-\gamma^2 w - (a(x)w_x)_x = 0. \tag{3.57}$$

We deduce that w satisfies the boundary value problem:

$$\begin{cases} \gamma^2 w + (a(x)w_x)_x = 0 \\ w(L) = w_x(L) = 0 \\ w(0) = 0 \end{cases} \tag{3.58}$$

Next, let

$$\psi(x) = \int_0^x \exp\left(\int_s^x \left|\frac{a_x}{a}\right| dv\right) ds \quad \text{for all } x \in [0,1]$$

It is easy to see that

$$\begin{cases} \psi(0) = 0, \psi(x) > 0, \ \forall x \in (0,1] \\ \psi_x \geq 1, a\left(\frac{\psi}{a}\right)_x \geq 1. \end{cases}$$

Multiplying equation $(3.58)_1$ by $\psi \bar{w}_x$, we get

$$\gamma^2 \int_0^L \psi(x) w \bar{w}_x dx + \int_0^L \psi(x) \bar{w}_x \left(a(x)w_x\right)_x dx = 0. \tag{3.59}$$

$U \in D(\mathcal{A})$, then the regularity is sufficiently for applying an integration on the second integral in the left hand side in equation (3.59). Then we obtain

$$\frac{\gamma^2}{2} \int_0^L \psi(x) \frac{d}{dx} |w|^2 dx - \int_0^L \psi_x(x) a(x) \left|w_x\right|^2 dx - \frac{1}{2} \int_0^L \psi(x) a(x) \frac{d}{dx} \left|w_x\right|^2 dx = 0. \tag{3.60}$$

Using Green formula and the boundary conditions, we get

$$\gamma^2 \int_0^L \psi_x(x) |w|^2 dx + \int_0^L \left(\psi_x(x) a(x) - \psi(x) a_x(x)\right) \left|w_x\right|^2 dx = 0, \tag{3.61}$$

We deduce that

$$w = 0. \tag{3.62}$$

Using equation $(3.52)_1$ we obtain

$$u = 0 \text{ and } v = 0. \tag{3.63}$$

Consequently, using equations (3.63), (3.62) and (3.54), we deduce that the only solution of (3.52) is the null one.

• **Case 2**. $\gamma = 0$:

In this case, by $(3.52)_1$ we have $u = 0$, which gives that $\theta = 0$ by $(3.52)_3$.

Multiplying equation $(3.52)_2$ by $\bar{w}$ using Green formula and the boundary conditions, we get

$$\int_0^L a(x) \left|w_x\right|^2 dx = 0. \tag{3.64}$$

Then

$$w_x(x) = 0 \quad \forall x \in (0, L), \tag{3.65}$$

Hence w is constant in (0,L). As $w(L) = 0$, then

$$w = 0.$$

Hence U must be the trivial solution of (5.1), which is the desired result. The proof has been completed. ■

Lemma 3.6.3. *We have*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \text{ if } \eta \neq 0,$$
$$i\mathbb{R}^* \subset \rho(\mathcal{A}) \text{ if } \eta = 0,$$

. where

$$\mathbb{R}^* = \mathbb{R} \setminus \{0\}$$

Proof. • **Case 1**. $\gamma \neq 0$: We will prove that the operator $i\gamma I - \mathcal{A}$ is surjective for $\gamma \neq 0$. For this purpose, let $G = (g_1, g_2, g_3, g_4)^T \in \mathcal{H}$, we seek $X = (w, u, \theta, v)^T \in D(\mathcal{A})$ solution of the following equation

$$(i\gamma I - \mathcal{A})X = G. \tag{3.66}$$

Equivalently, we have

$$\begin{cases} i\gamma w - u = g_1 \\ i\gamma u - (a(x)w_x)_x = g_2 \\ i\gamma\theta + (\xi^2 + \eta)\theta - u(L)\kappa(\xi) = g_3 \\ i\gamma v + \frac{1}{m}(a(x)w_x)(L) + \frac{\zeta}{m}\int_{-\infty}^{+\infty} \kappa(\xi)\theta(\xi)d\xi = g_4 \end{cases} . \tag{3.67}$$

From $(3.67)_1$ and $(3.67)_2$, we have

$$-\gamma^2 w - (a(x)w_x)_x = (g_2 + i\gamma g_1.)\,. \tag{3.68}$$

Let $z \in H_L^1(0, L)$. Multiplying the equation (3.68) by $\bar{z}$ and integrating in (0,L), we obtain

$$\int_0^L \left(-\gamma^2 w\bar{z} - (a(x)w_x)_x\,\bar{z}\right) dx = \int_0^L (g_2 + i\gamma g_1)\,\bar{z}dx. \tag{3.69}$$

From the boundary conditions and the fact that $w(0) = 0$, we get

$$\begin{cases} \int_0^L (-\gamma^2 w\bar{z} + a(x)w_x\bar{z}_x)\,dx + i\gamma(i\gamma m + \tilde{\zeta})w(L)\bar{z}(L) \\ \\ = \int_0^L (g_2 + i\gamma g_1)\,.\bar{z}dx - \zeta.\bar{z}(L)\int_{-\infty}^{+\infty} \frac{\kappa(\xi)}{\xi^2+\eta+i\gamma} g_3(\xi)d\xi \\ \\ +(i\gamma m + \tilde{\zeta})g_1(L).\bar{z}(L) + mg_4\bar{z}(L) \end{cases} \tag{3.70}$$

We can rewrite (3.70) as

$$-\left(L_\gamma w, z\right)_{H^1_L} + (w, z)_{H^1_L} = \mathcal{L}(z), \tag{3.71}$$

with the inner product defined by

$$(w, z)_{H^1_L} = \int_0^1 a(x) w_x \bar{z}_x dx$$

and

$$\left(L_\gamma w, z\right)_{H^1_L} = \int_0^L \gamma^2 w \bar{z} dx - i\gamma(i\gamma m + \tilde{\zeta}) w(L) \bar{z}(L).$$

Using the compactness embedding from $L^2(0, L)$ into $H^1_L(0, L)$ and from $H^{-1}_L(0, L)$ into $L^2(0, L)$ we deduce that the operator L_γ is compact from $L^2(0, L)$ into $L^2(0, L)$. Consequently, by Fredholm alternative, proving the existence of w solution of (3.71) reduces to proving that 1 is not an eigenvalue of L_γ. Indeed if 1 is an eigenvalue, then there exists $w \neq 0$ such that

$$\left(L_\gamma w, z\right)_{H^1_L} = (w, z)_{H^1_L} \quad \forall z \in H^1_L. \tag{3.72}$$

In particular for $z = w$, it follows that

$$\gamma^2 \|w(x)\|^2_{L^2(0,1)} - i\gamma(i\gamma m + \tilde{\zeta}) |w(L)|^2 = \left\| \sqrt{a(x)} w_x(x) \right\|^2_{L^2(0,1)}.$$

Hence, we have

$$w(L) = 0, \tag{3.73}$$

From (3.72), we obtain

$$(a w_x)(L) = 0. \tag{3.74}$$

Then

$$\begin{cases} \gamma^2 w + \left(a(x) w_x\right)_x = 0 \\ w(L) = w_x(L) = 0 \\ w(0) = 0. \end{cases} \tag{3.75}$$

We deduce that U = 0.

• **Case** $\gamma = 0$ and $\eta \neq 0$: Using Lax-milgram theorem, we obtain the result. ■

3.6.2 Residual spectrum of $\mathcal{A}$

Lemma 3.6.4. *Let $\mathcal{A}$ be defined by (3.16) Then*

$$\mathcal{A}^* \begin{pmatrix} w \\ u \\ \theta \\ v \end{pmatrix} = \begin{pmatrix} -u \\ -\left(a(x) w_x\right)_x \\ -\left(\xi^2 + \eta\right)\theta - u(L)\kappa(\xi) \\ \frac{1}{m}\left(a(x) w_x\right)(L) + \frac{\zeta}{m} \int_{-\infty}^{+\infty} \kappa(\xi) \theta(\xi) d\xi \end{pmatrix}, \tag{3.76}$$

with domain

$$D\left(\mathcal{A}^*\right)=\left\{\begin{array}{c|l}(w,u,\theta,\upsilon)^T\in\mathcal{H} & \begin{array}{l}w\in H^2(0,L)\cap H_L^1(0,L)\\ u\in H_L^1(0,L),\upsilon\in\mathbb{C}\\ -\left(\xi^2+\eta\right)\theta-u(L)\kappa(\xi)\in L^2(-\infty,+\infty)\\ u(L)=\upsilon,|\xi|\theta\in L^2(-\infty,+\infty)\end{array}\end{array}\right\}. \tag{3.77}$$

Proof. Let $U=(w,u,\theta,\upsilon)^T$ and $V=(\tilde{w},\tilde{u},\tilde{\theta},\tilde{\upsilon})^T$. We have

$$<\mathcal{A}U,V>_{\mathcal{H}}=<U,\mathcal{A}^*V>_{\mathcal{H}}.$$

$$\begin{aligned}<\mathcal{A}U,V>_{\mathcal{H}}=&\int_0^L a(x)u_x\bar{\tilde{w}}_xdx+\int_0^L(a(x)w_x)_x\bar{\tilde{u}}dx+\zeta\int_{-\infty}^{+\infty}[-\left(\xi^2+\eta\right)\theta+u(L)\kappa(\xi)]\bar{\tilde{\theta}}d\xi\\ &+m(\frac{1}{m}\left(a(x)w_x\right)(L)+\frac{\zeta}{m}\int_{-\infty}^{+\infty}\kappa(\xi)\theta(\xi)d\xi)\bar{\tilde{\upsilon}}\\ =&-\int_0^L u(a(x)\bar{\tilde{w}}_x)_xdx-\int_0^L a(x)w_x\bar{\tilde{u}}_x+a(L)w_x(L)\bar{\tilde{u}}(L)+a(L)\bar{\tilde{w}}_x(L)u(L)\\ &-\zeta\int_{-\infty}^{+\infty}\theta[(\xi^2+\eta)\bar{\tilde{\theta}}]d\xi+\zeta u(L)\int_{-\infty}^{+\infty}\kappa(\xi)\bar{\tilde{\theta}}d\xi-a(L)w_x(L)\bar{\tilde{\upsilon}}-\zeta\int_{-\infty}^{+\infty}\kappa(\xi)\theta(\xi)d\xi.\bar{\tilde{\upsilon}}\end{aligned}$$

As $\upsilon=u(L)$ and if we set $\tilde{\upsilon}=\tilde{u}(L)$, we find

$$\begin{aligned}<\mathcal{A}U,V>_{\mathcal{H}}=&-\int_0^L u(a(x)\bar{\tilde{w}}_x)_xdx-\int_0^L a(x)w_x\bar{\tilde{u}}_xdx-\zeta\int_{-\infty}^{+\infty}\theta(\xi)[(\xi^2+\eta)\bar{\tilde{\theta}}+\kappa(\xi)\bar{\tilde{u}}(L)]d\xi\\ &+\upsilon(a(L)\bar{\tilde{w}}_x(L)+\zeta\int_{-\infty}^{+\infty}\kappa(\xi)\bar{\tilde{\theta}}d\xi).\end{aligned}$$

■

Theorem 3.6.5. *$\sigma_r(\mathcal{A})=\emptyset$ where $\sigma_r(\mathcal{A})$ denotes the set of residual spectrum of $\mathcal{A}$.*

Proof. Since $\gamma\in\sigma_r(\mathcal{A}),\bar{\gamma}\in\sigma_p(\mathcal{A}^*)$ the proof will be accomplished if we can show that $\sigma_p(\mathcal{A})=\sigma_p(\mathcal{A}^*)$. This is because obviously the eigenvalues of $\mathcal{A}$ are symmetric on the real axis. From (3.76), the eigenvalue problem $\mathcal{A}^*\mathtt{Z}=\gamma\mathtt{Z}$ for $\gamma\in\mathbb{C}$ and $0\neq\mathtt{Z}=(w,u,\theta,\upsilon)\in D(\mathcal{A}^*)$ we have

$$\begin{cases}\gamma w+u=0\\ \gamma u+(a(x)w_x)_x=0\\ \gamma\theta+(\xi^2+\eta)\theta+u(L)\kappa(\xi)=0\\ \gamma\upsilon-\frac{1}{m}(a(x)w_x)(L)-\frac{\zeta}{m}\int_{-\infty}^{+\infty}\kappa(\xi)\theta(\xi)d\xi=0\end{cases}. \tag{3.78}$$

From $(3.78)_1$ and $(3.78)_2$, we find

$$\gamma^2w-(a(x)w_x)_x=0. \tag{3.79}$$

As $\upsilon = u(L) = -\gamma w(L)$, we deduce from $(3.68)_3$ and $(3.68)_4$ that

$$\left(\gamma + \frac{\varrho}{m}(\gamma+\eta)^{\alpha-1}\right)\gamma w(L) + \frac{1}{m}a(L)w_x(L) = 0, \tag{3.80}$$

with the following conditions

$$w(0) = 0 \tag{3.81}$$

System (3.79)–(3.80) is the same as (3.30)-(3.31). Hence $\mathcal{A}^*$ has the same eigenvalues with $\mathcal{A}$. The proof is complete. ∎

3.6.3 Polynomial stability for $\eta \neq 0$

In this part, we prove that the system (P') is polynomially stable when $\eta > 0$. Note that in [41], an early example of such refined decay estimate had been proved for Webster-Lokshin model with constant coefficients in the case $\alpha = \frac{1}{2}$ and inferred for other values of α by using a modal decomposition on a Riesz basis and the asymptotic of the eigenfunctions of the ∂_t^α operator.

Theorem 3.6.6. *The semigroup $(S_{\mathcal{A}}(t))_{t\geq 0}$ is polynomially stable and*

$$\|(S_{\mathcal{A}}(t))U_0\| \leqslant \frac{1}{t^{3-\alpha}}\|U_0\|_{D(\mathcal{A})}.$$

Proof. An early example of such refined decay estimate had been proved for the case $\alpha = \frac{1}{2}$ and inferred for other values of α in [41]. We will need to study the resolvent equation $(i\gamma - \mathcal{A})U = F$, for $\gamma \in \mathbb{R}$, namely

$$\begin{cases} i\gamma w - u = f_1 \\ i\gamma u - (a(x)w_x)_x = f_2 \\ i\gamma\theta + (\xi^2+\eta)\theta - u(L)\kappa(\xi) = f_3 \\ i\gamma\upsilon + \frac{1}{m}(a(x)w_x)(L) + \frac{\zeta}{m}\int_{-\infty}^{+\infty}\kappa(\xi)\theta(\xi)d\xi = f_4 \end{cases}. \tag{3.82}$$

• **Step 1** Taking the real part of the inner product of $(i\gamma I - \mathcal{A})U = F$, with U in $\mathcal{H}$ and using (3.18), we get

$$|\Re\mathrm{e}\langle \mathcal{A}U, U\rangle_{\mathcal{H}}| \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}, \tag{3.83}$$

This implies that

$$\zeta\int_{-\infty}^{+\infty}\left(\xi^2+\eta\right)|\theta(\xi,t)|^2 d\xi \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{3.84}$$

and, applying $(3.81)_1$, we obtain

$$\Big|\,|\gamma|\,|w(L)| - |f_1(L)|\,\Big|^2 \leqslant |u(L)|^2. \tag{3.85}$$

We deduce that

$$|\gamma|^2|w(L)|^2 \leqslant c|f_1(L)|^2 + c|u(L)|^2. \tag{3.86}$$

From $(3.81)_4$, we have

$$(aw_x)(L) = -i\gamma m.u(L) - \zeta \int_{-\infty}^{+\infty} \kappa(\xi)\theta(\xi)d\xi + mf_4.$$

Then

$$\Big|a(L)w_x(L)\Big|^2 \leqslant 2m^2|\gamma|^2|u(L)|^2 + 2m^2 f_4^2 + 2\zeta^2 \left| \int_{-\infty}^{+\infty} \kappa(\xi)\theta(\xi)d\xi \right|^2$$

$$\leqslant 2m^2|\gamma|^2|u(L)|^2 + 2m^2 f_4^2 + 2\zeta^2 \left(\int_{-\infty}^{+\infty} (\xi^2+\eta)^{-1}\kappa(\xi)d\xi \right) \left(\int_{-\infty}^{+\infty} (\xi^2+\eta)\theta(\xi)d\xi \right)$$

$$\leqslant 2m^2|\gamma|^2|u(L)|^2 + c\|U\|_{\mathcal{H}}\|F\|_H + c'\|F\|_H^2. \tag{3.87}$$

From $(3.86)_3$, we obtain

$$u(L)\kappa(\xi) = (i\gamma + \xi^2 + \eta)\theta - f_3(\xi). \tag{3.88}$$

By multiplying (3.86) by $(i\gamma + \xi^2 + \eta)^{-2}|\xi|$, we get

$$(i\gamma + \xi^2 + \eta)^{-2}u(L)\kappa(\xi)|\xi| = (i\gamma + \xi^2 + \eta)^{-1}|\xi|\theta - (i\gamma + \xi^2 + \eta)^{-2}|\xi|g_3(\xi). \tag{3.89}$$

Hence, by taking absolute values of both sides of (3.87) integrating over $]-\infty, +\infty[$ respect to the variable ξ and applying Cauchy Schwartz inequality, we obtain

$$S|u(L)| \leqslant \sqrt{2}\mathcal{U} \left(\int_{-\infty}^{+\infty} \xi^2|\theta|^2 d\xi \right)^{\frac{1}{2}} + 2\nu \left(\int_{-\infty}^{+\infty} |f_3(\xi)|^2 d\xi \right)^{\frac{1}{2}}, \tag{3.90}$$

where

$$\mathcal{S} = \left| \int_{-\infty}^{+\infty} \left(i\gamma + \xi^2 + \eta\right)^{-2} u(L)|\xi|\kappa(\xi)d\xi \right| = \frac{|1-2\alpha|}{4} \frac{\pi}{|\sin \frac{(2\alpha+3)}{4}\pi|} |i\gamma + \eta|^{\frac{2\alpha-5}{4}},$$

$$\mathcal{U} = \left(\int_{-\infty}^{+\infty} \left(|\gamma| + \xi^2 + \eta\right)^{-2} d\xi \right)^{\frac{1}{2}} = (\frac{\pi}{2})^{\frac{1}{2}} \big||\gamma| + \eta\big|^{\frac{-3}{4}},$$

$$\nu = \left(\int_{-\infty}^{+\infty} \left(|\gamma| + \xi^2 + \eta\right)^{-4} |\xi|^2 d\xi \right)^{\frac{1}{2}} = (\frac{\pi}{16}\big||\gamma| + \eta\big|^{\frac{-5}{2}})^{\frac{1}{2}}.$$

Thus, by using the inequality

$$2PQ \leqslant P^2 + Q^2, P \geqslant 0, \ Q \geqslant 0,$$

again, we get

$$S^2|u(L)|^2 \leqslant 2\mathcal{U}^2\left(\int_{-\infty}^{+\infty}(\xi^2+\eta)|\theta|^2 d\xi\right)+4\nu^2\left(\int_{-\infty}^{+\infty}|f_3(\xi)|^2 d\xi\right). \tag{3.91}$$

We deduce that

$$|u(L)|^2 \leq c|\gamma|^{1-\alpha}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+c\|F\|^2_{\mathcal{H}}. \tag{3.92}$$

- **Step 2** : Now, we use the classical multiplier method. Let us introduce the following notation

$$\mathcal{I}_\varphi(x)=\left|\sqrt{a(x)}w_x(x)\right|^2+|u(x)|^2$$

$$\mathcal{E}_\varphi=\int_0^L \mathcal{I}_\varphi(x)dx$$

Lemma 3.6.7. *We have that*

$$c_1\int_0^L \psi_x(x)|u(x)|^2dx+c_0\int_0^L(\psi_x(x)-1)|u(x)|^2dx\,(c_0+c_1)\int_0^L\left(\frac{\psi(x)}{a(x)}\right)_x|aw_x|^2\,dx$$

$$+c_0\int_0^L a(x)\,|w_x|^2\,dx=(c_0+c_1)\left[\psi(x)|u|^2\right]_0^1+(c_0+c_1)\left[\left(\frac{\psi(x)}{a(x)}\right)|aw_x|^2\right]_0^1$$

$$+c_0[a(x)w_x\bar{w}]_0^1+R, \tag{3.93}$$

for every, $c_0,c_1>0$, and R satisfies

$$|R|\leqslant C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

for a positive constant C.

Proof. To get (3.91), let us multiply the equation $(3.81)_2$ by $2\,(c_0+c_1)\,\psi\bar{w}_x+c_0\bar{w}$. Integrating on (0,L) we obtain

$$i\gamma\int_0^L u(2\,(c_0+c_1)\,\psi\bar{w}_x+c_0\bar{w})dx-\int_0^L(a(x)w_x)_x(2\,(c_0+c_1)\,\psi\bar{w}_x+c_0\bar{w})dx$$

$$=\int_0^L f_2(2\,(c_0+c_1)\,\psi\bar{w}_x+c_0\bar{w})dx$$

or

$$-2(c_0+c_1)\int_0^L u\psi(x)(\overline{i\gamma w_x})dx-c_0\int_0^L u(\overline{i\gamma w})dx-2(c_0+c_1)\int_0^L\psi(x)(a(x)w_x)_x\bar{w}_xdx$$

$$-c_0\int_0^L(a(x)w_x)_x\bar{w}dx=\int_0^L f_2(2\,(c_0+c_1)\,\psi\bar{w}_x+c_0\bar{w})dx$$

Since $i\gamma w = u + f_1$ and $i\gamma w_x = u_x + f_{1x}$ taking the real part in the above equality, it follows that

$$-(c_0+c_1)\int_0^L \psi(x)\frac{d}{dx}|u|^2 - (c_0+c_1)\int_0^L \frac{\psi(x)}{a(x)}\frac{d}{dx}\left|aw_x\right|^2 dx - c_0\int_0^L (a(x)w_x)_x\bar{w}dx$$

$$-c_0\int_0^L |u|^2dx = 2(c_0+c_1)\Re e\int_0^L u\psi(x)\bar{f_{1x}}dx + \Re e\int_0^L f_2(2\left(c_0+c_1\right)\psi\bar{w}_x + c_0\bar{w})dx + c_0\Re e\int_0^L u\bar{f}_1dx$$

and integrating by part, we get

$$\begin{aligned} c_0\int_0^L (\psi_x(x)-1)|u(x)|^2dx + c_1\int_0^L \psi_x(x)|u(x)|^2dx + (c_0+c_1)\int_0^L \left(\frac{\psi(x)}{a(x)}\right)_x \left|aw_x\right|^2 dx \\ + c_0\int_0^L a(x)|w_x|^2dx = (c_0+c_1)\left[\psi(x)|u|^2\right]_0^L + (c_0+c_1)\left[\left(\frac{\psi(x)}{a(x)}\right)\left|aw_x\right|^2\right]_0^L \\ + c_0[a(x)w_x\bar{w}]_0^L + R, \end{aligned} \tag{3.94}$$

where

$$R = 2(c_0+c_1)\Re e\int_0^L u\psi(x)\bar{f_{1x}}dx + \Re e\int_0^L f_2(2\left(c_0+c_1\right)\psi\bar{w}_x + c_0\bar{w})dx + c_0\Re e\int_0^L u\bar{f}_1dx$$

Moreover

$$\left|\int_0^L \psi(x)f_2\bar{w}_xdx\right| \leqslant C\|f_2\|_{L^2(0,L)}\|w_x\|_{L^2(0,L)} \leqslant C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}},$$

$$\left|\int_0^L \psi(x)u\bar{f_{1x}}dx\right| \leqslant C\|u\|_{L^2(0,L)}\|f_{1x}\|_{L^2(0,L)} \leqslant C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}},$$

$$\left|\int_0^L u\bar{f}_1dx\right| \leqslant C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}},$$

and

$$\left|\int_0^L f_2\bar{w}dx\right| \leqslant C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}},$$

Hence, we deduce that

$$|R| \leqslant C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{3.95}$$

■

- **Step 3** We have

$$(a(x)w_x\bar{w})_{x=0} = 0, (\psi(x)|u(x)|^2)_{x=0} = 0, (\psi(x)a(x)|w_x|^2)_{x=0} = 0.$$

It holds that

$$c_1 \int_0^L \left(a(x)\,|w_x|^2 + |u|^2\right) dx \leq (c_0 + c_1)\,\psi(1)|u(L)|^2 + (c_0 + c_1)\,\frac{\psi(1)}{a(L)}\,|(aw_x)\,(L)|^2$$
$$+ c_0 a(L)\,|w_x(L)\|w(L)\,| + C\|U\,\|_{\mathcal{H}}\|\,F\|_{\mathcal{H}}$$
$$\leq c|u(L)|^2 + c'(\varepsilon)\,|(aw_x)\,(L)|^2 + \varepsilon|w(L)|^2 + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \quad (3.96)$$

for any $\varepsilon > 0$. Moreover, using the Sobolev injection, we have

$$|w(L)| \leqslant \|w\|_{H^1(0.L)} \leqslant c\|w_x\|_{L^2(0.L)} \leqslant c\|\sqrt{a}w_x\|_{L^2(0.L)}.$$

Then

$$\mathcal{E}_w \leq c\,|(aw_x)\,(L)|^2 + c'|u(L)|^2 + c''\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \quad (3.97)$$

Since that

$$\int_{-\infty}^{+\infty} |\theta(\xi)|^2 d\xi \leq C \int_{-\infty}^{+\infty} \left(\xi^2 + \eta\right) |\theta(\xi)|^2 d\xi \leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

Hence

$$\|U\|_{\mathcal{H}}^2 \leq c\,|(aw_x)\,(L)|^2 + c'|u(L)|^2 + c''\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \quad (3.98)$$

Substitution of inequalities (3.86) and (3.91) into (3.97), we obtain that

$$\|U\|_{\mathcal{H}}^2 \leq c|\gamma|^{2-2\alpha}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c'\|F\|_{\mathcal{H}}^2 + c''\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \quad (3.99)$$

Then

$$\|U\|_{\mathcal{H}} \leq c|\gamma|^{2-2\alpha}\|F\|_{\mathcal{H}}.$$

Then, using Theorem 3.5.2 with $\delta = 4 - 2\alpha$ one has conclusion of Theorem. The proof is now complete. ■

3.7 Conclusions and future works

3.7.1 Conclusions

We have studied the dynamic boundary stabilization of the wave system with dissipation law of fractional derivative type. Using a spectral analysis we have proved a non-uniform stability. Using Arendt-Batty Theorem, we have proved the strong asymptotic stability. If $\eta > 0$, using a frequency domain approach, we prove some polynomial energy decay rate depending on parameter α.

3.7.2 Future works

In Theorems 3.4.3, 3.6.1, 3.6.5, 3.6.6, our approach can be generalized to multi-dimensional spaces. But it is difficult to use spectral analysis to generalize Theorem 3.5.4. Instead we can show the lack of exponential stability by proving that the second condition in Theorem 3.5.1

does not hold. We can extend (paper in preparation) the results of this paper to more general measure density instead of (3.1). Indeed we can consider $\int_{-\infty}^{+\infty} \frac{\kappa(\xi)}{\xi^{2+\eta\gamma}} d\xi$, as Stieltjes function. By the help of Abelian/Tauberian theorem of Karamata, we obtain many interesting cases that is resolvent growth slower or faster. We use a general Borichev–Tomilov theorem (see [6]). It seems to be interesting to study a global decaying solutions of hyperbolic systems (strong and weakly) under control of fractional derivative type. We think that the interaction of the hyperbolicity (order of multiplicity) and the number of dissipative terms have an effect on the result. It seems to be interesting to develop some energy methods to treat nonlinear evolution under control of fractional derivative type. The problem of global existence and energy decay for the following wave equation of Kirchhoff type is open

$$\begin{cases} w_{tt}(x,t) - M\Big(\|w_x\|^2_{L^2(0,L)}\Big) w_{xx}(x,t) = 0 & \text{in } (0,L)\times(0,+\infty), \\ w(0,t) = 0 & \text{in } (0,+\infty) \\ M\Big(\|w_x\|^2_{L^2(0,L)}\Big) w_x(x,t) = -\varrho \partial_t^{\alpha,\eta} & \text{in } (0,+\infty) \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x) & \text{on } (0,L). \end{cases} .$$

Abstract

In recent years, the stability of PDEs has attracted the attention of many authors and become an active area of research. the stabilization problem we are interested in amounts to determining the asymptotic behavior of the energy, denoted by E(t), to study its limit in order to determine if the latter is null or not, and if this limit is null give an estimate of the rate of its decay towards zero. In this thesis, we consider the non-degenerate wave equation with the presence of dissipative terms of fractional type. we have focused our study on the global existence and asymptotic behavior of solutions. For the global existence, we used the argument combining the semigroup theory with the energy estimation method and with the help of a spectral analysis we proved a non-uniform stability. Using the Arendt-Batty theorem, we proved the strong asymptotic stability. For the polynomial stability, we succeeded to establish a polynomial decay rate of the energy which depends on a parameter by an estimation of the resolvent of the generator associated with the semi-group and the Borichev-Tomilov theorem.

Key Words: Nondegenerate wave equation, fractional boundary control, Polynomial stability, C_0 -semigroup, frequency domain approach .

Résumé

Au cours des dernières années, la stabilité des EDPs a attiré l'attention de nombreux auteurs et est devenue un domaine de recherche actif. le problème de stabilisation auquel nous nous intéressons revient à déterminer le comportement asymptotique de l'énergie, notée par E(t), étudier sa limite afin de déterminer si cette dernière est nulle ou pas, et si cette limite est nulle, donner une estimation de la vitesse de sa décroissance vers zéro. Dans cette thèse, nous considérons l'équation des ondes non dégénérée avec la présence des termes dissipatifs de type fractionnaire. nous avons concentré notre étude sur l'existence globale et le comportement asymptotique des solutions. Pour l'existence globale, nous avons utilisé l'argument combinant la théorie des semi-groupe avec la méthode d'estimation de l'énergie et à l'aide d'une analyse spectrale nous avons prouvé une stabilité non uniforme. En utilisant le théorème d'Arendt-Batty, nous avons prouvé la stabilité asymptotique forte. Pour la stabilité polynomiale, nous avons réussi à établir un taux de décroissance polynomiale de l'énergie qui dépends d'un paramètre

par une estimation de la résolvante du générateur associé au semi-groupe et le théorème de Borichev-Tomilov.

Mots Clés: Equation des ondes non dégénérée, contrôle de frontière fractionnaire, Stabilité polynomiale, C_0 -semi-groupe, approche domaine fréquentielle.

ملخص :

مسألة استقرار المعادلات التفاضلية كانت ولا تزال السؤال الأول في نظرية النظم الديناميكية وفي السنوات الأخيرة أصبحت مسألة الاستقرار تجذب انتباه العديد من المتخصصين. هنا نتحدث عن استقرار الطاقة و هو موضوع اهتمامنا. في هذه الأطروحة درسنا معادلة الموجة غير المولدة بوجود حدود للتبديد من الصنف الكسرى. ركزنا في هذه الدراسة على الوجود الشامل والسلوك المقارب للحلول حيث جمعنا بين نظرية شبه الزمر و طريقة تقدير الطاقة وباستخدام التحليل الطيفي أثبتنا استقرارا غير منتظم ثم بالاعتماد على نظرية Arendt-Batty برهنا على الاستقرار المقارب القوي. كما اثبتنا نتيجة اضمحلال متعدد الحدود باستخدام طريقة مجال التردد ونظرية Borichev-Tomilov .

كلمات مفتاحية: *معادلة الموجة غير المولدة، تحكم حدودي من الصنف الكسرى، نظرية شبه الزمر، استقرار متعدد الحدود*، طريقة مجال التردد ونظرية Borichev-Tomilov .

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