N° d'ordre :

REPUBLIQUE ALGERIENNE DEMOCRATIQUE & POPULAIRE

MINISTERE DE L'ENSEIGNEMENT SUPERIEUR & DE LA RECHERCHE SCIENTIFIQUE

UNIVERSITE DJILLALI LIABES
FACULTE DES SCIENCES EXACTES
SIDI BEL ABBES

THESE DE DOCTORAT EN SCIENCES

Présentée par: **TOUMI Fatima**

Spécialité : Mathématiques.
Option : équations différentielles.

Intitulée

Stabilité de certaines équations différentielles impulsives.

Soutenue le : 13/07/2022
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Année universitaire : 2021/2022

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Dédicaces

A mes chers parents,

Remerciements

C'est une habitude saine que de remercier, après Dieu, au début d'une thèse tous ceux qui ont contribué à la rendre agréable. C'est avec mon enthousiasme le plus vif le plus sincère que je voudrais rendre mérite à tous ceux qui plus ou moins à leur manière m'ont aidée à mener à bien cette thèse.

Je ne saurai assez remercier le Professeur Abderrahmane OUMANSOUR qui a accepté de diriger patiemment ma thèse, pour son soutien constant, pour sa disponibilité et sa générosité exceptionnelle, et pour toutes ses remarques. Je le remercie également de m'avoir encouragée à faire connaître mes travaux.

Je tiens à remercier très chaleureusement Monsieur le Professeur Mouffak Benchohra de m'avoir fait l'honneur de présider ce jury de thèse.

Mes remerciements s'adressent au même titre à Messieurs le Professeur Ahmed HAMMOUDI , Dr Djamila BELDJERD pour m'avoir fait l'honneur de faire partie du jury, pour avoir accepté la charge d'examiner mon travail et qui m'ont fait l'honneur de rapporter sur cette thèse.

J'exprime ma gratitude et remercie vivement Monsieur le Professeur Abdelghani OUAHAB pour m'avoir mise sur la piste de ces intéressants problèmes de recherches.

Je souhaite également exprimer mes sincères remerciements à toutes les personnes qui ont contribué directement ou indirectement au bon déroulement de cette thèse au travers de leurs aides, soutiens et encouragements. Je réserve une reconnaissance particulière à Djamila ma trés chère soeur pour son aide précieuse.
J'aimerai conclure en remerciant particulièrement mes parent, pour m'avoir toujours encouragée à poursuivre mes études.

Abstract

The necessity to study impulsive functional differential equations is due to the fact that these equations are useful in modeling many real processes and phenomena studied in optimal control, biology, mechanics, medicine, bio-technologies, electronics, economics, etc. This thesis consists of three parts. The first part is a reminder on the fundamentals concepts of stability.

The purpose of the second part is devoted to the study of the practical stability of impulsive cascade systems of the form

$$\begin{cases} \dot{x}_1 = f_1(t, x_1) + h(t, x)x_2, t \neq t_k \\ \Delta x_1 = I_k(x_1), t = t_k, \\ \dot{x}_2 = f_2(t, x_2), \ t \neq \tau_\sigma, \\ \Delta x_2 = J_\sigma(x_2), \ t = \tau_\sigma. \end{cases} \tag{1}$$

where $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^m$, and $x := col(x_1, x_2)$. The functions f_1, f_2 and h are continuous, locally Lipschitz in x uniformly in t, and f_1 is continuously differentiable in both arguments.

A generalization of the work of V. Lakshmikantham [40] is established.

Finally, the third part deals with the exponential practical stability of the solutions of the same cascade systems. Precisely, by a similar study we develop the exponential practical stability with weaker assumptions for cascades impulsive systems of the form (1) using technique which permits us to eliminate some restrictions that are usually imposed.

Résumé

La nécessité d'étudier les équations différentielles fonctionnelles impulsives est due au fait que ces équations sont utiles pour modéliser de nombreux processus et phénomènes réels étudiés en contrôle optimal, biologie, mécanique, médecine, biotechnologies, électronique, économie, etc. Cette thèse se compose de trois parties. La première partie est un rappel sur les concepts fondamentaux de la stabilité.

L'objet de la deuxième partie est consacré à l'étude de la stabilité pratique des systèmes impulsifs en cascade du type

$$\begin{cases} \dot{x}_1 = f_1(t, x_1) + h(t, x)x_2, t \neq t_k \\ \Delta x_1 = I_k(x_1), t = t_k, \\ \dot{x}_2 = f_2(t, x_2), \ t \neq \tau_\sigma, \\ \Delta x_2 = J_\sigma(x_2), \ t = \tau_\sigma. \end{cases}$$

où $x_1 \in \mathbb{R}^n, \ x_2 \in \mathbb{R}^m$, La fonction $f_1, \ f_2$ et h sont continues, localement Lipschitzienne en x uniformément en t, et f_1 est continûment différentiable. Une généralisation des travaux de V. Lakshmikantham [40] est établie.

Enfin, la troisième partie traite la stabilité pratique exponentielle des solutions des mêmes systèmes en cascade. Précisément, par une étude similaire on développe l'exponentielle stabilité pratique avec des hypothèses plus générales pour les systèmes impulsifs en cascades de la forme (1) en utilisant une technique qui nous permet d'éliminer certaines restrictions qui sont habituellement imposées.

General introduction

The concept of impulsive control and its mathematical foundation called impulsive differential equations, or differential equations with impulse effects have a long history. But the long history of impulsive differential equations and impulsive control systems did not mean that we already had a good understanding of impulsive systems. This is because for many years, the study of impulsive problems had been restricted to only a few kinds of special problems such as mechanical systems with impacts and the optimal control of spacecraft.

There was Publication of many books on impulsive differential equations since 1982 to 1995, but the control community still saw nothing exciting about these mathematical tools because the well-known plants that can be studied by these mathematical tools seem to be too limited. For example, mechanical systems with impacts are not a main focus of control community, predator-prey systems can not attract serious attention of control engineers. Unfortunately, mathematicians only know the above few kinds of real examples that fall into the scope of impulsive differential equations. And to make things worse, the existing monographs on impulsive differential equations target mainly mathematicians as potential readers.Fortunately, this slow developing of impulsive control system had been changed at the end of last century because of the theory of impulsive differential equations had been gradually diffused into control community; much more new plants, which can be modeled by impulsive differential equations.

In applied mathematics, it is now recognized that real-world phenomena that are subject to short-term perturbations whose duration is negligible in comparison with the duration of the process are more accurately described using impulsive differential equations. In the modeling process, it is natural to assume that these perturbations act instantaneously or in the form of impulses.

Applied impulsive mathematical models have become an active research topic in nonlinear science and have attracted further attention in many diverse fields. For example, important impulsive mathematical models have recently been introduced in population dynamics, such as vaccination [21, 51], population ecology [3, 4, 5, 7, 11, 23, 30, 35, 43, 44, 58, 60, 62, 63, 66, 67, 68, 70, 72], drug treatment[42, 69], the chemostat [64], the tumor–normal cell interaction[24], plankton allelopathy [28], in mechanics [8, 14, 15], in radio engineering [8, 31], in communication security [32, 33], in neural networks [6, 48, 57, 59, 61, 63, 74], etc. In addition, in optimal control of economic systems, frequency-modulated signal processing systems, and the motion of some flying objects, many systems are characterized by abrupt changes in their states at certain instants. This type of impulsive phenomenon can also be found in the fields of information science, electronics, automatic control systems, computer networks, artificial intelligence, robotics, and telecommunications. Many sudden and sharp changes occur instantaneously in these systems, in the form of impulses which cannot be well described by pure continuous-time or discrete-time models [41, 47, 71, 73].

To analyze the stability of dynamical. systems with impulsive effects, Lyapunov stability results have been presented in the literature. local and global asymptotic stability conclusions of an equilibrium point of a given impulsive dynamical system are provided if a smooth (at least continuously differentiable) positive-definite function of the nonlinear system states (Lyapunov function) can be constructed for which its time rate of change over the continuous-time dynamics is strictly negative and its difference across the resetting times is negative.
In particular, the notion of practical stability, which includes as a special case stability in the sense of Lyapunov, is one of the most important notions in the stability theory.

In [13] Benabdallah et al. investigated global practical uniform exponential stability of cascade dynamical systems by using a known result by Corless which appeared in [20]. Some good results related to the subject have been obtained, See [1-16]. Recently, Beldjerd et al [26] studied the practical exponential stability of the cascade system under some hypotheses.

In [22] M. Dlala, and M. A. Hammami give certain results of practical stability under some sufficient conditions for a certain class of perturbed impulsive systems

In this thesis we study the practical stability and practical exponential stability of nonlinear impulsive time-varying system of the form

$$\begin{cases} \dot{x}_1 = f_1(t, x_1) + h(t, x)x_2, t \neq t_k \\ \Delta x_1 = I_k(x_1), t = t_k, \\ \dot{x}_2 = f_2(t, x_2),\ t \neq \tau_\sigma, \\ \Delta x_2 = J_\sigma(x_2),\ t = \tau_\sigma, \end{cases} \tag{2}$$

where $x_1 \in \mathbb{R}^n,\ x_2 \in \mathbb{R}^m$, and $x := col(x_1, x_2)$. The function $f_1,\ f_2$ and h are continuous, locally Lipschitz in x uniformly in t, and f_1 is continuously differentiable in both arguments.

We establish sufficient conditions for the practical exponential stability of a class of nonlinear nonautonomous impulsive systems. In the spirit of a result of [13] and [26], we develop the exponential practical stability with more general assumptions for cascades impulsive systems.

Chapter 1

Preliminaries

1.1 Introduction

As is well known, the field of differential equations is an old subject. which remains topical and useful to engineers, scientists and mathematicians. The study of differential equations began with the birth of calculus, which dates of the 1660s. Part of Newton's motivation in the development of calculus was to solve the problems that could be approached by differential equations. After more than 300 years of history, differential equations represent a huge field of knowledge and a wide range of applications in many disciplines. These are very important problems in the theory and applications of differentials equations, hence the interest of mentioning a few, for example, in applied sciences practical problems concerning mechanics, economics, control theory, physical sciences are associated with the nonlinear differential equations.

At the end of the 19th century, three types of stability were established for motion in continuous dynamical systems: Lyapunov stability, Zhukovskij stability, and Poincaré stability. Among them, Lyapunov stability, Poincaré stability are the most known. One can determine the properties of stability and boundedness of a system or solutions of a differential equation, directly using the definitions of stability and boundness. As is well known, in general it is impossible to find the solution of all differential equations, except numerically. Therefore, due to the absence of the analytical expression of solutions for differentials equations it is very important to obtain information on the behavior of the stability and boundness of these solutions. So far, the most effective tool for the study of the stability and boundedness of solutions of a given nonlinear system is provided by Lyapunov's theory, which is Lyapunov's second (or direct) method. It is also important to mention that this theory has become an important part of mathematics and theoretical mechanics in the twentieth century.

In this introductory chapter, we briefly present Lyapunov's theory as well as the main results concerning the stability of the solutions of differential equations using Lyapunov functions.

1.2 Notions on stability

Definition 1.2.1. *Consider the nonautonomous system*

$$\dot{x} = f(t, x), \tag{1.1}$$

where $f : \mathbb{R}_+ \times D \to \mathbb{R}^n$ *continuous in* t *and locally lipschitz in* x, *where* $D \subset \mathbb{R}^n$.
We say that a *is an equilibrium point for the system (1.1) if :*

$$f(t, a) = 0, \qquad \forall t \geq 0.$$

An equilibrium point at the origin could be a translation of a nonzero equilibrium point or, more generally, a translation of a nonzero solution of the system.

Definition 1.2.2. *(Stability) An equilibrium point* 0 *of (1.1) is stable if:*

$$\forall \varepsilon > 0, \exists \delta(\varepsilon, t_0) > 0 : \|x(t_0)\| < \delta \Longrightarrow \|x(t)\| < \varepsilon, \forall t \geq t_0 \geq 0.$$

Obviously: an equilibrium point x *is unstable if it is not stable.*

In other words, the stability in the sense of Lyapunov of the origin of the system means that for all $t \geq 0$, the solution $x(t)$ remains in the neighborhood of the origin if $x(t_0)$ is in the neighborhood of the origin. So a small perturbation of the initial condition $x(t_0)$ around the origin gives a solution that remains close to the origin.

The stability of a system does not imply the convergence of solutions towards the origin, it is therefore insufficient for the study of the behavior of solutions.

Definition 1.2.3. *An equilibrium point* 0 *of (1.1) is asymptotically stable* (AS) *if :*

- *i) it is stable.*
- *ii) there is* $\delta_1 = \delta_1(t_0) > 0$:

$$\|x(t_0)\| < \delta_1 \Longrightarrow \lim_{t\to\infty} x(t) = 0.$$

Definition 1.2.4. *(Uniform boundedness) the solutions of the system (1.1) are said:*

- *i) Uniformly bounded if:* $\exists c > 0$ *such that*

$$\forall a \in]0, c[\,, \exists \beta(a) > 0 : \|x(t_0)\| < a \Longrightarrow \|x(t)\| < \beta, \forall t \geq t_0.$$

- *ii) Globally uniformly bounded if the previous property is true for* c *quite large.*

Definition 1.2.5. *An equilibrium point is*

- *i) Uniformly stable* $(U.S)$ *if:*

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 : \|x(t_0)\| < \delta \Longrightarrow \|x(t)\| < \varepsilon, \forall t \geq t_0 \geq 0.$$

- *ii) Globally uniformly stable* $(G.U.S)$ *if it is uniformly stable and solutions of the system solutions are globally uniformly bounded.*

Definition 1.2.6. *An equilibrium point* 0 *is:*

- *Exponentially stable* (ES) *if there exist positive constants* c, k *and* λ *such that:*

$$\|x(t)\| \leq k\,\|x(t_0)\|\, e^{-\lambda(t-t_0)}, \forall\, \|x(t_0)\| \leq c.$$

  the constant k *is called the rate or also the speed of convergence.*

- *Globally exponentially stable* $(G.E.S)$ *if the previous condition is checked* $\forall\ x(t_0) \in \mathbb{R}^n$.

Remark 1.2.7. *It is important to note that the exponential stability property of the system necessarily results in the asymptotic stability of the system.*

1.2.1 Lyapunov Theory

Definition 1.2.8. *Consider the system (1.1). Let $D \subset \mathbb{R}^n$ be a neighborhood of 0 and $V : \mathbb{R}_+ \times D \to \mathbb{R}$ a continuously differentiable function on D. The function V is said to be positive semi definite if:*

1. $V(t, 0) = 0, \forall t \in \mathbb{R}_+$.

2. $V(t, x) \geq 0$ *if* $\forall x \in D - \{0\}$.

The use of positive definite functions is a highly effective technique for analyzing the stability of a system governed by an ordinary differential equation. But there is no method to determine these kinds of functions.

Definition 1.2.9. *Let $D \subset \mathbb{R}^n$ a neighborhood of 0 and $V : \mathbb{R}_+ \times D \to \mathbb{R}$ a continuously differentiable function on D. The function V is said to be positive definite if:*

1. $V(t, 0) = 0, \forall t \in \mathbb{R}_+$.

2. *There is a definite positive function $W_1(x)$ such as:*

$$W_1(x) \leq V(t, x), \forall x \in D.$$

Definition 1.2.10. *Let $D \subset \mathbb{R}^n$ and $V : \mathbb{R}_+ \times D \to \mathbb{R}$ a continuously differentiable function on D. The function V is said to be decreasing if there is a positive semi definite function $W_2(x)$ such that:*

$$V(t, x) \quad \leq W_2(x), \quad \forall x \in D.$$

Definition 1.2.11. *(Lyapunov function) We consider the system (1.1).*

,

1) *It is said that V is a Lyapunov function in the broad sense in 0 if it checks the following two properties:*

i. *V is positive definite.*

ii. $\dot{V}(t,x) \leq 0$ *for all* $x \in D$.

2) *It is said that* V *is a strict Lyapunov function in* 0, *if it checks the following two properties:*

i. V *is positive definite.*

ii. $\dot{V}(t,x) < 0$ *for all* $x \in D - \{0\}$.

The use of these functions provides criteria for concluding the stability or asymptotic stability of a equilibrium point without the integration of the equation considered. Here are two important theorems of the stability of a equilibrium point. These results date from the 19 century and are due to Lyapunov.

Theorem 1.2.12. *Let* $x = 0$ *be an equilibrium point for (1.1) and* D *be a domain containing* $x = 0$ *. Let* $V : \mathbb{R}_+ \times D \to \mathbb{R}$ *be a continuously differentialble function sush as:*

$$W_1(x) \leq V(t,x) \leq W_2(x);$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq 0;$$

for all $t \geq 0$ *and* $x \in D$, *where* $W_1(x)$ *and* $W_2(x)$ *are continues positive definite functions. Then* $x = 0$ *uniformly stable.*

Theorem 1.2.13. *Let* $x = 0$ *be an equilibrium point for (1.1) and* D *be a domain containing* $x = 0$. *Let* $V : \mathbb{R}_+ \times D \to \mathbb{R}$ *be a continuously differentiable function on* D *such that:*

$$W_1(x) \leq V(t,x) \leq W_2(x);$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W_3(x);$$

for all $t \geq 0$ *and* $x \in D$, *where* $W_1(x)$, $W_2(x)$ *and* $W_3(x)$ *are continues positive definite functions. Then* $x = 0$ *is uniformly asymptotically stable. If* $D = \mathbb{R}^n$ *then* $x = 0$ *is globally uniformly asymptotically stable.*

1.3 Stability of perturbed systems

Physical systems are often complex and their representations with precision and a simple structure model, remain a great concern to guarantee the fidelity of the model towards the real process. Indeed many dynamics are described by disturbed systems of form:

$$\dot{x} = f(t,x) + g(t,x), \tag{1.2}$$

where $f : \mathbb{R}_+ \times D \to \mathbb{R}^n$, and $g : \mathbb{R}_+ \times D \to \mathbb{R}^n$ are continuous functions in relation to t, and locally Lipschitzian in x on D where $D \subset \mathbb{R}^n$ a domain such as $0 \in D$.

Such a system is seen as the sum of the nominal system $\dot{x} = f(t,x)$ and the perturbation term $g(t,x)$. The latter term represents the modeling errors of the real system given the presence of uncertainties. Generally, we do not know this function but we have some information about it, like for example the upper terminal of $\|g(t,x)\|$. One of the classic problems that can arise for this class of systems is the problem of stability. Assuming that the nominal system is uniformly exponentially stable (U.E.S), will the disturbed system behave the same or not?. The idea of solving this problem is to use the Lyapunov function of the nominal system as the Lyapunov function depending on the perturbed system. This approach is used to study the local stability of the $\dot{x} = F(t,x)$ system during linearization. The limitations of this method lie in the fact that the perturbation term considered in linearization verifies the hypothesis: $\|g(t,x)\| < L\|x\|^2, \forall \|x(t)\| \leq r$, this is not the case in general. However, the exponential stability of the perturbed system can be ensured by imposing on the function $g(t,x)$ a definite condition so that the term destabilizing, $g(t,x)$, which is assumed to be a destabilizing term, does not affect the stability of the nominal system. For example, it is sufficient to assume the following condition:

$$\|g(t,x)\| < L\|x\|^2, \ \forall t \geq 0,$$

L a positive constant. Note that functions that tend towards 0 and that are locally Lipschitzian in x, uniformly in t for all $t \geq 0$ check the latest inequality.

1.3.1 Perturbation of type A

Consider the system(1.2) such that $g(t,0) = 0$. It is assumed that $x = 0$ is exponentially stable equilibrium point for the nominal system and let $V(t,x)$ be a Lyapunov function satisfying

$$
\begin{aligned}
c_1 \|x_1\|^2 &\le V_1(t,x_1) \le c_2 \|x_1\|^2 && (1.3)\\
\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t,x_1) &\le -c_3 V_1(t,x_1) && (1.4)\\
\left\| \frac{\partial V_1}{\partial x_1} \right\| &\le c_4 \|x_1\| && (1.5)
\end{aligned}
$$

for all $(t,x) \in \mathbb{R}_+ \times D$ and c_1, c_2, c_3 and c_4 positive constants. It is assumed that the perturbation term $g(t,x)$ verifies:

$$
\|g(t,x)\| < L\|x\|, \ \forall t \ge 0, \forall x \in D, \tag{1.6}
$$

where L is a positive constant.

Lemma 1.3.1. *Let $x = 0$ an exponentially stable equilibrium point of the nominal system. Let $V(t,x)$ be a Lyapunov function which verifies hypotheses (1.3) on $\mathbb{R}_+ \times D$. It is assumed that the perturbation term $g(t,x)$ verifies (1.6). Then the origin is an exponentially stable equilibrium point for the system (1.2) Moreover, if the assumptions are verified globally,then the origin is globally exponentially stable.*

1.3.2 Perturbation of type B

If it is not known that $g(t,x) = 0$, the origin $x = 0$ may not be a point of equilibrium for the disturbed system.

Lemma 1.3.2. *Let $x = 0$ be an exponentially stable equilibrium point for the nominal system $x = f(t,x)$. Let $V(t,x)$ a Lyapunov function of the nominal system satisfies 1.3 on $\mathbb{R}^+ \times D$ where*

$$
D = \{x \in \mathbb{R}^n, \|x\| < r\}.
$$

Suppose the perturbation term verifies:

$$
\|g(t,x)\| \le \delta < \frac{c_3}{c_4}\sqrt{\frac{c_1}{c_2}}\theta r,
$$

for all $t \ge 0$, all $x \in D$ and $0 < \theta < 1$. Then for all $\|x(t_0)\| < \sqrt{\frac{c_1}{c_2}}r$, solution of the perturbed system verifies:

$$
\begin{aligned}
\|x(t)\| &\le k\|x(t_0)\| e^{-\gamma(t-t_0)}, \quad \forall t_0 \le t \le t_0 + T \quad \textit{and}\\
\|x(t)\| &\le b, \quad \forall t \ge t_0 + T,
\end{aligned}
$$

for T finite, where

$$k = \sqrt{\frac{c_1}{c_2}}, \gamma = \frac{(1-\theta)c_3}{2c_2}, \; b = \frac{c_4}{c_3}\sqrt{\frac{c_2}{c_1}}\frac{\delta}{\theta}.$$

Lemma 1.3.3. *(Comparison Lemma) [13]If W verifies*

$$\dot{W}(t) \leq -a(t)W(t) + b(t),$$

where a and b are continuous functions then

$$W(t) \leq e^{\sigma(t)}\left[W_0 + \int_{t_0}^{t} e^{-\sigma(t)}b(s)ds\right] \quad \text{and } \sigma(t) = -\int_{t_0}^{t} a(s)ds.$$

1.4 Exponential stability of nonautonomous non-linear differential equations

Sufficient conditions for exponential stability of a class of nonlinear autonomous, differential equations are given in this part by using the Lyapunov method.

Theorem 1.4.1. *Let $x = 0$ the point of equilibrium of the system (1.1) and $D \subset \mathbb{R}^n$ a domain that contains $x = 0$. Let*

$V : [0; +\infty(\times D \to \mathbb{R}$ a continuously differentiable function such that

$$k_1 \|x\|^a \leq V(t,x) \leq k_2 \|x\|^a \tag{1.7}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -k_3 \|x\|^a, \tag{1.8}$$

for $t \geq 0$ and $\forall x \in D$ and k_1, k_2, k_3 positive constants, then $x = 0$ is exponentially stable. If these assumptions are checked globally then $x = 0$ is globally exponentially stable.

Consider nonlinear autonomous system

$$\dot{x} = f(t, x(t)); \; x(t_0) = x_0, \quad t \geq t_0 \geq 0, \tag{1.9}$$

where $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ nonlinear function such that $f(t, 0) = 0$ for all $t \in \mathbb{R}^+$.
It is assumed that the conditions imposed on the system (1.9) such as the existence of the solutions are guaranteed.

Definition 1.4.2. *Function $V(t,x)$ definite on $W=\mathbb{R}^{+}\times D$, where is D an open of $\mathbb{R}^{n}$ and $0\in D$ is called function L- Lyapunov for (1.9) if $V(t,x)$ is continuously differentiable in $t\in\mathbb{R}^{+}$; $x\in D$ and there are positive constants λ_1, λ_2, λ_3, k, p, q, r, δ such that:*

$$\lambda_1 \left\|x\right\|^{p} \leq V(t,x) \leq \lambda_2 \left\|x\right\|^{q}, \ \forall (t,x)\in W, \tag{1.10}$$

$$D_f V(t,x) \leq -\lambda_3 \left\|x\right\|^{r} + Ke^{-\delta t}, \ \forall t\geq 0, \ x\in D-\{0\}. \tag{1.11}$$

1.5 Criteria for exponential stability of nonautonomous nonlinear differential equations

1.5.1 Practical stability

Practical stability concerns the fact that an arbitrarily small neighborhood of the origin (at place of origin itself) is stable. In this case, the objective is reduced to the study of the stability of a whole ball centered at the origin.

The definitions of practical stability are very varied and present more or less important differences in the literature, but the idea remains the same to make stability flexible and to facilitate its applications. The notion of practical stability was developed by Lasalle and Lefschetz (1961). It is defined for a disturbed system, with initial conditions. The time interval over which stability is studied is infinite. In the next part some definitions and criteria of practical stability will be given.
Consider the system

$$\begin{cases} \dot{x} = f(t,x(t)) \\ x(t_0) = x_0. \end{cases} \tag{1.12}$$

Let $r\geq 0$ and $B_r=\{x\in\mathbb{R}^n / \left\|x\right\|\leq r\}$. Note by B^r the following set

$$B^r = \{x\in\mathbb{R}^n / \left\|x\right\|\geq r\}.$$

In what follows we are interested in some particular notions of stability. The results that we will develop in the following chapters are based on these concepts.

We will give definition of uniform stability of B_r, and then the uniform asymptotic stability.

Definition 1.5.1. *(Uniform stability of B_r)*

1. *B_r is uniformly stable if for all $\varepsilon > r$, $\exists\delta = \delta(\varepsilon) > 0$ such that* $(t_0 \geq 0)$,

$$\|x(t_0)\| \leq \delta \Longrightarrow \|x(t)\| \leq \varepsilon,\ \forall t \geq t_0.$$

2. *B_r is globally uniformly stable if it is uniformly stable and system solutions (1.12) are globally uniformly bounded.*

Definition 1.5.2. *If there is a function α of the class $\mathcal{K}$, a positive γ constant and a positive r constant such that for any initial condition x_0, the solution verifies*

$$\|x(t)\| \leq \gamma(\|x_0\|) + r,\ \forall t \geq t_0.$$

Then the system (1.12) is globally uniformly practically stable.

Definition 1.5.3. *If there is a function β of class $\mathcal{KL}$ and a constant positive r such as for any initial condition x_0, the solution verifies*

$$\|x(t)\| \leq \beta(\|x_0\|, t - t_0) + r,\ \forall t \geq t_0.$$

So the system (1.12) is globally uniformly practically asymptotically stable.

Definition 1.5.4. *B_r is uniformly exponentially stable if there exists $\gamma > 0$ and $k > 0$ such as for everything $t_0 \in \mathbb{R}_+$ and $x_0 \in \mathbb{R}^n$*

$$\|x(t)\| \leq r + k\|x_0\| e^{-\gamma(t-t_0)}, \forall t \geq t_0.$$

Uniformly exponentially stable, if there exists $r > 0$ such that B_r is globally uniformly exponentially stable.

Definition 1.5.5. *The system (1.12) is globally uniformly exponentially convergent to $B_r ifandonlyif \exists\alpha > 0$ such as $\forall t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$, $c(\ x_0) \geq 0$ if $x(t)$ is a solution of (1.12) such that $x(t_0) = x_0$ then:*

$$\|x(t)\| \leq r + c(x_0)e^{-\alpha(t-t_0)}, \forall t \geq t_0. \tag{1.13}$$

Lemma 1.5.6. *Consider the system $\dot{x} = f(t,x)$. Let*

$$V : [0, +\infty[\times \mathbb{R}^n \to \mathbb{R},$$

a continuously differentiable function such as:

$$\begin{aligned} c_1 \|x\|^p &\leq V(t,x) \leq c_2 \|x\|^q, \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) &\leq -c_3 \|x\|^r + k, \end{aligned}$$

for all $t \geq 0$ et $x \in \mathbb{R}^n$,where c_1, c_2 , c_3, k, p, q and r are positive constants; then V is bounded.

Proof. There are three possible behaviors of $\dot{V}$:

1) If $\dot{V} \leq 0$

V is positive definite and decreasing so V is bounded.

2) If $\dot{V} \geq 0$

$$\begin{aligned} 0 &\leq \dot{V} \leq -c_3 \|x\|^r + k \\ &\leq -\frac{c_3}{c_2^{\frac{r}{q}}} V^{\frac{r}{q}} + k. \end{aligned}$$

By taking

$$\mu = -\frac{c_3}{c_2^{\frac{r}{q}}},$$

we obtain

$$-\mu V^{\frac{r}{q}} + k \geq 0,$$

which implies

$$V \leq \left(\frac{k}{\mu}\right)^{\frac{q}{r}} < \infty.$$

So V is bounded.

3) If $\dot{V}$ is oscillatory:

There exists $(t_n)_{n \geq 0}$, $t_n \geq 0$ and $\lim_{n \to +\infty} t_n = +\infty$ such that $\dot{V}(t_n) = 0$.

without loss of generality we assume that

for $t \in [t_n; t_{n+1}] : \dot{V}(t) \geq 0$ which implies than $V(t) \leq \left(\frac{k}{\mu}\right)^{\frac{q}{r}}$;

for $t \in [t_{n+1}; t_{n+2}] : \dot{V}(t) \leq 0$ implies that $V(t) \leq V(t_{n+1}) \leq \left(\frac{k}{\mu}\right)^{\frac{q}{r}}$. ☐

Remark 1.5.7. *Inequality* $c_1 \|x\|^p \leq V(t,x) \leq c_2 \|x\|^q$ *leads :*

$$\begin{cases} \text{for } p > q : \|x\| \leq \left(\frac{c_2}{c_1}\right)^{\frac{1}{p-q}}, \\ \text{for } p < q : \|x\| \geq \left(\frac{c_1}{c_2}\right)^{\frac{1}{p-q}}. \end{cases}$$

We have also the following theorem :

Theorem 1.5.8. *We consider the system* $\dot{x} = f(t,x)$*. Let* $V : [0,+\infty[\times \mathbb{R}^n \to \mathbb{R}$ *be a continuously differentiable function such that:*

$$\begin{aligned} c_1 \|x\|^p &\leq V(t,x) \leq c_2 \|x\|^q, \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) &\leq -c_3 p(\|x\|^r - k), \end{aligned}$$

for all $t \geq 0$ *and* $x \in W$*, such that*

$$W = \begin{cases} B_{r_1} & \text{if } p > q \text{ such that } r_1 = \left(\frac{c_2}{c_1}\right)^{\frac{1}{p-q}}, \\ B^{r_2} & \text{if } p < q \text{ such that } r_2 = \left(\frac{c_1}{c_2}\right)^{\frac{1}{p-q}}, \end{cases}$$

where c_1, c_2 , c_3, k, p, q *and* r *are positive constants and let*

$$\rho = \begin{cases} \left(\frac{c_2}{c_1}\right)^{\frac{1}{p-q}} & \text{if } p > q, \\ \sqrt[p]{\frac{kc_2}{c_1}} & \text{if } p = q = r, \\ \sqrt[p]{\frac{kc_2}{c_1 \eta^{r-q}}} & \text{if } p < q \leq r, \\ \sqrt[p]{\frac{\max(c_2\ (k)^{\frac{r}{q}}, V(t_0,x_0)}{c_1}} & \text{if } p < q \text{ and } q > r. \end{cases} \tag{3.12}$$

Then (1.12) is exponentially convergent to B_ρ *where:*

$$c(x_0) = \begin{cases} 0 & \text{if } V(x_0) \leq k, \\ \sqrt[p]{\frac{(V(x_0)-k)}{c_1}} & \text{if } V(x_0) > k, \end{cases} \tag{3.13}$$

we also have that (1.12) is exponentially stable in B_ρ*.*

Chapter 2

Results on the practical stability of cascade impulsive systems

2.1 Introduction

Impulsive systems (systems of impulsive differential equations) model real world processes that under go abrupt changes (impulses) in the state at a sequence of discrete times. These abrupt changes in system's states inspire the impulsive control mechanism. Mechanical systems with impacts are not a main focus of control community, predator-prey systems can not attract serious attention of control engineers. Unfortunately, mathematicians only know the above few kinds of real examples that fall into the scope of impulsive differential equations.

Mathematical foundation for the concept of impulsive control called impulsive differential equations, or differential equations with impulse effects, or differential equations with discontinuous righthand sides have a long history that can be traced back to the beginning of modern control theory. The mathematical investigations of the impulsive ordinary differential equations mark their beginning with the work of Mil'man and Myshkis 1960. In it some general concepts are given about the systems with impulse effect and the first results on stability of such systems solutions are obtained. The theory of impulsive differential equations and its applications to impulsive control problems has been an active research area since 1990s. for many years, the study of impulsive control problems had been restricted to only a few kinds of special problems such as mechanical systems with impacts and the optimal control of spacecraft. Another fact contributed to the slow development of impulsive control is that the early research activities were reported as Russian literature and therefore was not well-known to the English community.

Even after the publication of many English books on impulsive differential equations since 1982, the control community still saw nothing exciting about these mathematical tools because the well-known plants that can be studied by these mathematical tools seem to be too limited. Impulsive dynamical systems have become increasingly popular during the past decades because they provide a natural framework for mathematical modeling of many real-world phenomena. Applications of impulsive dynamical systems can be found in a variety of fields such as aeronautics, ecology, economics, epidemiology, finance, medicine and robotics, just to name a few.

An impulsive dynamical system normally consists of three elements: a continuous system of differential equations, which governs the motion of the dynamical system between impulsive and resetting events; a discrete system of difference equations, which governs the way the system states are instantaneously changed when a setting event occurs; and a criterion for determining when the states of the system are to be reset. The solutions of impulsive dynamical systems are in general discontinuous, which often renders some of the standard analysis and control design methods ineffective. Nonetheless, significant progress has been made in theory and applications of impulsive dynamical systems in the past few decades,especially when the underlying continuous portions are described by ordinary differential equations. The latter are often referred to as impulsive (ordinary) differential equations.

In recent years the qualitative and qualitative theory of such equations has been extensively studied. A number of results on existence, uniqueness, continuability, stability, boundedness, oscillations, asymptotic properties, etc. were published. Scientists have been aware of the fact that many applicable problems are pointless unless the dependence on previous states is being taken into account. But until Volterra's work [65], a bigger part of the obtained results refers to several specific properties of a narrow type of equations. This work marks the beginning of the development of the functional differential equations theory. Impulsive functional differential equations are a natural generalization of impulsive ordinary differential equations (without delay) and of functional differential equations (without impulses). At the present time the qualitative theory of such equations undergoes rapid development.

Many results on the stability of their solutions are obtained. It is natural to ask whether we can find a systematic account of recent developments in the stability theory for impulsive functional differential equations. Our aim is to present the main results on stability theory for impulsive functional differential equations by means of the second method of Lyapunov and provide a unified general structure applicable to study the dynamics of mathematical models based on such equations.

Recently, many interesting results in stability analysis of impulsive systems have been reported in the literature by using Lyapunov function approach. However, sometimes it seems to be more difficult to appropriate Lyapunov functions for stability analysis. Many real-world processes exhibit continuous-time evolution with intermittent bursts of comparatively fast dynamics. In mathematical models of such processes, these bursts of activity are sometimes intrinsic to the dynamics.

For example, the Hodgkin-Huxley model [19] is a nonlinear ordinary differential equation that describes the propagation of action potentials of neurons; here, the bursts of activity correspond to the action potentials and are an intrinsic feature of the model. In the Hodgkin-Huxley model, these bursts arise from slow-fast dynamics in the continuous-time model, but in other neuronal models such as integrate-and-fire, the bursts are introduced synthetically using a logic rule. In other situations, these bursts of activity or impulses enter into the model in the form of a control that is designed to (ideally) force or constrain the dynamics in a desired way. The applications of this idea are quite diverse, including control theory, multi-agent systems, epidemiology, population dynamics, medicine and robotics. The mathematical formalism in which these ideas take concrete form is impulsive dynamical systems.

Also as an example, one of the first mathematical models which incorporate interaction between two species (predator-prey, or herbivore-plant, or parasitoid-host) was proposed by Alfred Lotka [45] and Vito Volterra [65]. The classical predator-prey model is based on the following system of two differential equations:

$$\begin{cases} \dot{H}(t) = H(t)\,(r_1 - bP(t)) \\ \dot{P}(t) = P(t)\,(-r_2 + cH(t)) \end{cases}$$

where $H(t)$ and $P(t)$ represent the population densities of prey and predator at time t, respectively; $t \geq 0$; $r_1 > 0$ is the intrinsic growth rate of the prey; $r_2 > 0$ is the death rate of the predator or consumer; b and c are the interaction constants. More concrete, the constant b is the per-capita rate of the predator predation and the constant c is the product of the predation per-capita rate and the rate of converting the prey into the predator.

However, in the study of the dynamic relationship between species, the effect of some impulsive factors has been ignored, which exists widely in the real world. For example, the birth of many species is an annual birth pulse or harvesting. Moreover, the human beings have been harvesting or stocking species at some time, then the species is affected by another impulsive type. Also, impulsive reduction of the population density of a given species is possible after its partial destruction by catching or poisoning with chemicals used at some transitory slots in fishing or agriculture. Such factors have a great impact on the population growth. If we incorporate these impulsive factors into the model of population interaction, the model must be governed by impulsive functional differential system.

For example, if at the moment $t = t_k$ the population density of the predator is changed, then we can assume that

$$\Delta P(t_k) = P(t_k + 0) - P(t_k - 0) = g_k P(t_k),$$

where $P(t_k - 0) = P(t_k)$ and $P(t_k + 0)$ are the population densities of the predator before and after impulsive perturbation, respectively, and $g_k \in \mathbb{R}$ are constants which characterize the magnitude of the impulsive effect at the moment t_k. If $g_k > 0$, then the population density increases and if $g_k < 0$, then the population density decreases at the moment t_k.

Thus the following impulsive functional differential system is obtained :

$$\begin{cases} \dot{H}(t) = H(t)\left(r_1 - a\int\limits_{-\tau_1}^{0} H(t+s)d\mu_1(s) - bP(t)\right) \\ \dot{P}(t) = P(t)\left(-r_2 + cH(t) - d\int\limits_{-\tau_2}^{0} P(t+s)d\mu_2(s)\right) \end{cases}$$

where $\tau_i \geq 0, \mu_i\ [-\tau_i, 0] \to \mathbb{R}$ is non-decreasing on$[-\tau_i, 0]$, $i = 1, 2$; $a; d$ are the intra-species competition coefficients. and t_k are fixed moments of time, $0 < t_1 < t_2 < .....$, $\lim\limits_{k\to\infty} t_k = \infty$. Which denote in mathematical ecology system a model of the dynamics of a predator-prey system, which is subject to impulsive effects at certain moments of time.

By means of such models, it is possible to take into account the possible environmental changes or other exterior effects due to which the population density of the predator is changed momentary.

We are interested in this these to the cascade systems which appear in various contexts related to control and automatic. In addition to the fact that this structure is commonly found in physical systems, it often reduces the complexity of the analysis considerably. Cascade interconnected subsystems are interconnected unilaterally, i.e. the output of one subsystem is the input of another subsystem, this structure has been widely studied. Numerous results were developed for the study of the behavior of cascade systems: Coron, Kokotovi'c, Ortega, Praly, Sepulcher, Sontag, Sussmann, Teel, Vidyasagar.
The main advantage of this structure is that each subsystem can be studied separately, and a stability property can be deduced for the entire cascade. The interest of the control community in this structure is not recent. It has its roots in [56], which presents some theoretical results for the analysis of system stability.

2.2 Some reminders on the practical stability of impulsive equations

In this section, we develop notation and introduce some basic properties of impulsive dynamical systems.

Let $t_0 \in \mathbb{R}$, and Ω a domain of $\mathbb{R}^n$ which contains the origin with $\|x\| = \sqrt{\sum\limits_{k=1}^{n} x_k^2}$ the norm for all $x \in \mathbb{R}$. We consider the following system of impulsive differential equations:

$$\begin{cases} \dot{x} = f(t,x), t \neq t_k; \\ \Delta x(t) = I_k(x), t = t_k; \end{cases} \tag{2.1}$$

with the following conditions:

(H_1) $0 < t_1 < t_2 < ... < t_k < ...$ and $\lim\limits_{k \to \infty} t_k = \infty$.
(H_2) $f :]t_{k-1}, t_k] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous on $]t_{k-1}, t_k] \times \mathbb{R}^n$ and for all $x \in \mathbb{R}^n$, $k = 1, 2, ...$ we have

$$\lim_{(t,y) \to (t_k^+, x)} f(t,y) = f(t_k^+, x).$$

(H_3) $f(t,x)$ is Lipschitzian in x.
(H_4) $I_k : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and for all $\rho > 0$ there is $\rho_1 \in (0, \rho)$ such that:

$$x \in S(\rho_1)\ , I_k(x) \in S(\rho) \text{ and } S(\rho) = \{x \in \mathbb{R}^n ; \|x\| < \rho\}.$$

Definition 2.2.1. *Let $V : \mathbb{R}_+ \times R^n \to \mathbb{R}_+ . V$ is said class V_0 if:*
i)V is continuous on $]t_{k-1}, t_k] \times \mathbb{R}^n$ and for all $x \in R^n$, $k = 1, 2, ...$, such that

$$_{(t,y) \to (t_k^+, x)} V(t,y) = V(t_k^+, x) \quad \textit{exist.}$$

ii)V is locally Lipschitzian in x for $]t_{k-1}, t_k]$.

Definition 2.2.2. *for $(t,x) \in]t_{k-1}, t_k] \times \mathbb{R}^n$, we define*

$$D^+V(t,x) = \lim_{h \longrightarrow 0^+} \sup \frac{1}{h} \left[V(t+h, x + hf(t,x)) - V(t,x) \right].$$

Theorem 2.2.3. *Let $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ and $V \in V_0$. Suppose that*

$$\begin{cases} D^+V(t,x) \leq g(t, V(t,x)), t \neq t_k, \\ V(t, x + I_k(x)) \leq \psi_k(V(t,x)), k = 1, 2, ... \end{cases}$$

where $g : \mathbb{R}_+^2 \to \mathbb{R}$ checks the condition (H_2) and $\psi_k : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing. Let $r(t) = r(t, t_{0,} u_0)$ the maximum solution of the differential equation

$$\begin{cases} u' = g(t,u), t \neq t_k, u(t_0^+) = u_0 \geq 0, \\ u(t_k^+) = \psi_k(u(t_k)), k = 1, 2, ... \end{cases}$$

that exists on $[t_0, \infty[$. *then* $V(t_0^+, x_0) \leq u_0$ *implies that*

$$V(t, x(t)) \leq r(t), t \geq t_0,$$

where $x(t) = x(t, t_0, x_0)$ *is solution of (2.1) existing on* $[t_0, \infty[$.

Corollaire 2.2.4. *If we assume in the theorem 2.2.3, the following hypotheses:*

$$\begin{aligned} a) g(t,u) &= 0, \psi_k(u) = u \text{ for all } k; \\ b) g(t,u) &= 0, \psi_k(u) = d_k u, d_k \geq 0 \text{ for all } k; \\ c) g(t,u) &= -\alpha u, \alpha > 0, \psi_k(u) = d_k u, d_k \geq 0 \text{ for all } k; \\ d) g(t,u) &= \lambda'(t) u, \psi_k(u) = d_k u, d_k \geq 0 \text{ for all } k \text{ and } \lambda \in C^1 [\mathbb{R}_+, \mathbb{R}_+]; \end{aligned}$$

Then the following conclusions are verify respectively:

$$\begin{aligned} &a) V(t, x(t)) \text{ is strictly increasing in } t \text{ and } V(t, x(t)) \leq V(t_0^+, x_0), t \geq t_0. \\ &b) V(t, x(t)) \leq V(t_0^+, x_0) \prod_{t_0 < t_k < t} d_k, t \geq t_0. \\ &c) V(t, x(t)) \leq \left[V(t_0^+, x_0) \prod_{t_0 < t_k < t} d_k \right] e^{-\alpha(t - t_0)}, t \geq t_0. \\ &d) V(t, x(t)) \leq \left[V(t_0^+, x_0) \prod_{t_0 < t_k < t} d_k \right] e^{(\lambda(t) - \lambda(t_0))}, t \geq t_0. \end{aligned}$$

The following theorem gives sufficient conditions of practical stability of the impulsive system (2.1) satisfying the conditions $(H_1) - (H_4)$.

Theorem 2.2.5. *Assume that the hypotheses (i) - (v) hold*

(i) λ, $A \in R$ *are given such that* $0 < \lambda < A$;

ii) $V : \mathbb{R}_+ \times \mathbb{R}^n \to R_+$ *and* $V \in V_0$ *verified (c) (corollaire 2.2.4);*

iii) it exists $\rho = \rho(A)$ *such that* $x \in S(A)$; $I_k \in S(\rho)$ *for all* k;

iv) for $(t, x) \in \mathbb{R}_+ \times S(\rho)$, $b(\|x\|) \leq V(t, x) \leq a(\|x\|), a, b \in K$; *such that* K *is strictly increasing, where* K *is strictly increasing* $\{a \in C(\mathbb{R}_+, \mathbb{R}_+), a(0) = 0\}$

v) $a(\lambda) < b(\lambda)$ *holds.*

Then, the practical stability properties of (2.2.3) imply the corresponding practical stability properties of the system (2.1).

2.3 Assumptions and main results

In the following section we have obtained sufficient conditions to guarantee the practical stability of system (1).

We suppose that the following hypotheses are verified :

H1) There is a function V_1 continuously differentiable and constants c_1, c_2, c_3, c_4 such that:

$$\begin{cases} c_1 \|x_1\|^2 \leq V_1(t, x_1) \leq c_2 \|x_1\|^2, \\ \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t, x_1) \leq -c_3 V_1(t, x_1) + r_1, \ t \neq t_k \\ \left\| \frac{\partial V_1}{\partial x_1} \right\| \leq c_4 \|x_1\|, \\ V_1(t, x_1 + I_k(x_1)) \leq (\psi_1)_k (V_1(t, x_1)), t = t_k, k = 1, 2, ... \end{cases} \tag{2.2}$$

H2) There is a function V_2 continuously differentiable and constants b_1, b_2, b_3 such that

$$\begin{cases} b_1 \|x_2\|^2 \leq V_2(t, x_2) \leq b_2 \|x_2\|^2, \\ \frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2} f_2(t, x_2) \leq -b_3 V_2(t, x_2) + r_2; \quad t \neq \tau_\sigma \\ V_2(t, x_1 + J_\sigma(x_2)) \leq (\psi_2)_\sigma (V_2(t, x_2)), t = \tau_\sigma, \sigma = 1, 2, ... \end{cases}$$

such that

$$(\psi_1)_k (V_1(t, x_1)) = (d_1)_k V_1(t, x_1), \quad t = t_k, \quad k = 1, 2, ...$$

and

$$(\psi_2)_\sigma (V_2(t, x_2)) = (d_2)_\sigma V_2(t, x_2), \quad t = \tau_\sigma, \quad \sigma = 1, 2, ...$$

Theorem 2.3.1. *Suppose that assumptions (H_1) and (H_2) hold, and that the interconnection term is limited, i.e ,*

$$\exists M > 0 : \|h(t, x)\| \leq M \text{ for all } (t, x).$$

Then the system (1) is practically stable.

Proof. Let $W(t, x_1, x_2) = V_1(t, x_1) + \delta V_2(t, x_2)$ where δ is a positive constant that will be specified later. The derivative of W along the trajectories of the system (1) is

$$
\begin{aligned}
\dot{W} &= \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1}\frac{\partial x_1}{\partial t} + \delta(\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2}\frac{\partial x_2}{\partial t}) \\
&= \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t,x_1) + \frac{\partial V_1}{\partial x_1} h(t,x)x_2 + \delta(\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2} f_2(t,x_2)) \\
&\leq -c_3 V_1(t,x_1) + c_4 M \|x_1\| \|x_2\| + \delta(-b_3 V_2(t,x_2)) + r \\
&\leq -c_3 V_1(t,x_1) + \frac{c_4 M}{2}(\frac{\|x_1\|^2}{\varepsilon} + \varepsilon \|x_2\|^2) - \delta b_3 V_2(t,x_2) + r \\
&\leq -\big(c_3 - \frac{c_4 M}{2\varepsilon c_1}\big) V_1(t,x_1) - \big(b_3 - \frac{c_4 M \varepsilon}{2 b_1 \alpha}\big)\delta V_2(t,x_2) + r. \qquad (2.3)
\end{aligned}
$$

where $r = r_1 + r_2$.

By taking

$$\varepsilon = \frac{c_4 M}{c_1 c_3},$$

we get

$$\frac{c_4 M}{2\varepsilon c_1} = \frac{c_3}{2},$$

and

$$\frac{c_4 M \varepsilon}{2 b_1 \delta} = \frac{(c_4 M)^2}{2 c_1 c_3 b_1 \delta}.$$

Thus,

$$\dot{W} \leq -\frac{c_3}{2} V_1(t,x_1) + (-b_3 + \frac{(c_4 M)^2}{2 c_1 c_3 b_1 \delta})\delta V_2(t,x_2) + r.$$

It is convenient to choose δ such that

$$-b_3 + \frac{(c_4 M)^2}{2 c_1 c_3 b_1 \delta} < 0.$$

Therefore

$$\delta > \frac{(c_4 M)^2}{2 c_1 c_3 b_1 b_3}.$$

Fix

$$\delta = \frac{(c_4 M)^2}{c_1 c_3 b_1 b_3},$$

we have

$$-b_3 + \frac{(c_4 M)^2}{2 c_1 c_3 b_1 \delta} = -\frac{b_3}{2}.$$

Consequently

$$\dot{W} \leq -\frac{c_3}{2} V_1(t,x_1) - \delta \frac{b_3}{2} V_2(t,x_2) + r.$$

Let $\gamma = \min\left(\frac{c_3}{2}, \frac{b_3}{2}\right)$. Hence
$$\dot{W} \leq -\gamma W(t) + r.$$
We also have
$$V_1(t, x_1 + I_k(x_1)) \leq \psi_{1k}(V_1(t, x_1)), t = t_k, k = 1, 2, ...$$
and
$$V_2(t, x_2 + J_\sigma(x_2)) \leq \psi_{2\sigma}(V_2(t, x_2)), t = \tau_\sigma, \sigma = 1, 2, ...$$
It follows that
$$\begin{aligned} W(t, x_1 x_1 + I_k(x_1), x_2 + J_\sigma(x_2)) &\leq V_1(t, x_1 + I_k(x_1)) + \alpha V_2(t, x_2 + J_\sigma(x_2)) \\ &\leq (\psi_1)_k(V_1(t, x_1)) + \alpha(\psi_2)_\sigma(V_2(t, x_2)) \\ &= (d_1)_k V_1(t, x_1) + \alpha\,(d_2)_\sigma V_2(t, x_2) \\ &\leq \gamma_{k,\sigma} W, \end{aligned}$$
such that $\gamma_{k,\sigma=} \max((d_1)_k, (d_2)_\sigma)$. By Theorems (2.2.3) and (2.2.5) system (1) is practically stable. □

2.3.1 Example 1

Consider the system
$$\begin{cases} \dot{x}_1 = -\left(1 + e^t\right) x_1 + \dfrac{1}{1+t^2} x_2, t \neq t_k \\ \Delta x_1 = I_k(x_1) = \alpha_k x_1, t = t_k, \alpha_k \in R, k = 1, 2... \\ \dot{x}_2 = -x_2 + e^{-x_2^2}\ t \neq \tau_\sigma \\ \ \Delta x_2 = J_\sigma(x_2) = \beta_\sigma x_2,\ \ t = \tau_\sigma, \beta_\sigma \in \mathbb{R}, \sigma = 1, 2.... \end{cases} \tag{2.4}$$
In this case,
$$\begin{aligned} f_1(t, x_1) &= -\left(1 + e^t\right) x_1, \\ f_2(t, x_2) &= -x_2 + e^{-x_2^2}, \\ h(t, x) &= \frac{1}{1+t^2}. \end{aligned}$$
Set $V_1(t, x_1) = x_1^2$ and $V_2(t, x_2) = x_2^2$.
Verification of assumptions:

We have

$$\begin{aligned}
\|x_1\|^2 &\leq V_1(t,x_1) \leq \|x_1\|^2, \\
\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t,x_1) &= -2x_1^2 \leq -2V_1, \\
\left\| \frac{\partial V_1}{\partial x_1} \right\| &\leq 2\|x_1\|, \\
V_1(t, x_1 + I_k(x_1)) &= V_1(t, x_1 + \alpha_k x_1) \leq \psi_{1k}(V_1(t,x_1)) = (1+\alpha_k)^2 x_1^2, t = t_k, k = 1, 2, \ldots.
\end{aligned}$$

With $c_1 = 1$, $c_2 = 1$, $c_3 = 2, c_4 = 2$ and $\psi_{1k}(V_1(t,x_1)) = (1+\alpha_k)^2 V_1$.
We have also

$$\begin{aligned}
\|x_2\|^2 &\leq V_2(t,x_2) \leq \|x_2\|^2 \\
\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2} f_2(t,x_1) &= -2x^2 \leq -2V_2 \\
V_2(t, x_2 + J_\sigma(x_2)) &= V_2(t, x_2 + \beta_\sigma x_2) = (1+\beta_\sigma)^2 x_2^2 \leq \psi_{2\sigma}(V_2(t,x_2)), t = \tau_\sigma, \sigma = 1, 2, \ldots
\end{aligned}$$

With $b_1 = 1$, $b_2 = 1$, $b_3 = 2$,and $\psi_{2\sigma}(V_2(t,x_2)) = (1+\beta_\sigma)^2 V_2$.
We have also $\|h(t,x)\| \leq 1 = M$.
Therefore, we can apply Theorem 3.2.1 to prove that system 2.4 is practically stable.

We propose in the following to state several generalizations of Theorem3.2.1. To do this, let's establish the following lemma:

Lemma 2.3.2. *We consider the system $\dot{x} = f(t,x)$. Let*

$$V : [0, +\infty[\times \mathbb{R}^n \to \mathbb{R}, \tag{2.5}$$

a continuously differentiable function such that

$$\begin{aligned}
c_1 \|x\|^p &\leq V(t,x) \leq c_2 \|x\|^q, \\
\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) &\leq -c_3 \|x\|^r + r,
\end{aligned}$$

for all $t \geq 0$ and $x \in \mathbb{R}^n$, where c_1, c_2, c_3, p, q and r are positive constants.
Then V is bounded.

Theorem 2.3.3. *Consider the system (1), and suppose that* *i) The hypotheses (H_1) and (H_2) verified.*

ii) There is a constant ε such that

$$\|h(t,x)\| \leq \varepsilon \|x\| + L, \textit{for all}\,(t,x) \in \mathbb{R}^+ \times D,$$

where D an open of $\mathbb{R}^n$. Then the system (1)is practically stable.

Proof. We have

$$\begin{aligned}\dot{V}_1 &= \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t,x_1) + \frac{\partial V_1}{\partial x_1} h(t,x) x_2 \\ &\leq -c_3 V_1(t,x_1) + c_4 \|x_1\| \|x_2\| (\varepsilon \|x\| + L) + r_1.\end{aligned}$$

The existence of λ such that $\|x_2\| \leq \lambda$, and the use of triangular inequality

$$\|x\| \leq \|x_1\| + \|x_2\|,$$

give

$$\begin{aligned}\dot{V}_1 &\leq -c_3 c_1 \|x_1\|^2 + c_4 \lambda \varepsilon \|x_1\|^2 + c_4 \lambda (\varepsilon \lambda + L) \|x_1\| + r_1 \\ &\leq -\mu_1 \|x_1\|^2 + c_4 \lambda (\varepsilon \lambda + L) \|x_1\| + r_1,\end{aligned}$$

with $\mu_1 = c_3 c_1 - c_4 \lambda \varepsilon$. Therefore, just choose ε such that $c_4 \varepsilon \lambda < c_3 c_1$.
Let $0 < \theta < 1$, we can rewrite the previous inequality as follows

$$\begin{aligned}\dot{V}_1 &\leq -\mu_1 \|x_1\|^2 + \mu_1 \theta \|x_1\|^2 - \mu_1 \theta \|x_1\|^2 + c_4 \lambda (\varepsilon \lambda + L) \|x_1\| + r_1 \\ &\leq -\mu_1 (1-\theta) \|x_1\|^2 - \mu_1 \theta \|x_1\|^2 + c_4 \lambda (\varepsilon \lambda + L) \|x_1\| + r_1.\end{aligned}$$

So

$$-\mu_1 \theta \|x_1\|^2 + c_4 \lambda (\varepsilon \lambda + L) \|x_1\| < 0,$$

just take

$$\|x_1\| \geq \frac{c_4 \lambda (\varepsilon \lambda + L)}{\mu_1 \theta}.$$

Hence,

$$\dot{V}_1 \leq -\mu_1 (1-\theta) \|x_1\|^2.$$

Therefore

$$\dot{V}_1 \leq -\frac{\mu_1 (1-\theta)}{c_1} V_1 + r_1, \quad \forall \|x_1\| \geq \frac{c_4 \lambda (\varepsilon \lambda + L)}{\mu_1 \theta}.$$

Now, consider

$$W(t,x_1,x_2) = V_1(t,x_1) + V_2(t,x_2). \tag{2.6}$$

The derivative of W along system trajectories of (1) is equal to

$$\dot{W} = \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t,x_1) + \frac{\partial V_1}{\partial x_1} h(t,x)x_2 + \frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2}\frac{\partial x_2}{\partial t}. \tag{2.7}$$

From conditions (i) and (ii) we get

$$\dot{W} \leq -\frac{\mu_1(1-\theta)}{c_1} V_1 - b_3 V_2 + r.$$

where $r = r_1 + r_2$.
Thus

$$\dot{W} \leq -\mu W,$$

with $\mu = \min\{\frac{\mu_1(1-\theta)}{c_1}, b_3\}$.
We also have
$V_1(t, x_1 + I_k(x_1)) \leq (\psi_1)_k(V_1(t,x_1)), t = t_k, k = 1,2,...$
and
$V_2(t, x_2 + J_\sigma(x_2)) \leq (\psi_2)_\sigma(V_2(t,x_2)), t = \tau_\sigma, \sigma = 1,2,...$
Then

$$\begin{aligned} W(t, x_1 + I_k(x_1), x_2 + J_\sigma(x_2)) &\leq V_1(t, x_1 + I_k(x_1)) + V_2(t, x_2 + J_\sigma(x_2)) \\ &\leq (\psi_1)_k(V_1(t,x_1)) + (\psi_2)_\sigma(V_2(t,x_2)) \\ &= (d_1)_k V_1(t,x_1) + (d_2)_\sigma V_2(t,x_2) \\ &\leq \gamma_{k,\sigma} W, \end{aligned}$$

such that $\gamma_{k,\sigma=}\max((d_1)_k, (d_2)_\sigma)$. By Theorems2.2.3 and 2.2.5 the system (1) is practically stable. □

We are now able to present our second generalization of Theorem3.2.1. We begin by stating the following assumptions.

H₃) There is a function V_1 continuously differentiable and constants c_1, c_2, c_3, c_4, r, p

and q such that :

$$\begin{aligned}
c_1 \|x_1\|^p &\leq V_1(t,x_1) \leq c_2 \|x_1\|^q \\
\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t,x_1) &\leq -c_3 \|x_1\|^r + \rho_1, t \neq t_k \\
\left\| \frac{\partial V_1}{\partial x_1} \right\| &\leq c_4 \|x_1\|, \\
V_1(t, x_1 + I_k(x_1)) &\leq \psi_{1k}(V_1(t,x_1)), t = t_k, k = 1, 2, ...
\end{aligned}$$

$\mathbf{H_4}$) There is a function V_2 continuously differentiable and constants b_1, b_2, b_3, r, p and q such that :

$$\begin{aligned}
b_1 \|x_2\|^p &\leq V_2(t,x_2) \leq b_2 \|x_2\|^q \\
\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2} f_2(t,x_2) &\leq -b_3 \|x_2\|^r + \rho_2, t \neq t_\sigma \\
V_2(t, x_2 + J_\sigma(x_2)) &\leq \psi_{2\sigma}(V_2(t,x_2)), t = \tau_\sigma \sigma = 1, 2, ...
\end{aligned}$$

Lemma 2.3.4. *Let V be a positive definite and continuously differentiable function defined such that*

$$\dot{V}(t) \leq -\alpha V(t) + \beta \sqrt[s]{V(t)} + k,$$

where α, β, k are positives constants, and $s > 1$. Then V is bounded.

Proof. Take

$$f(V) = -\alpha V + \beta \sqrt[s]{V} + k.$$

There are three possibilities for the behavior of $\dot{V}(t)$.
Case 1) If $\dot{V}(t) \leq 0$, since V is a positive definite function, then V is a decreasing function. Hence, V is necessarily bounded.
Case 2) If $\dot{V}(t) \geq 0$, in this case $f(V) \geq 0$ and $f'(V) = \dfrac{-s\alpha V^{1-\frac{1}{s}} + \beta}{sV^{1-\frac{1}{s}}}$.
It is easy to see that

$$f'(\overline{V}) = 0 \text{ and } f(\overline{V}) = \frac{s-1}{\alpha^{\frac{1}{s-1}}} \left(\frac{\beta}{s}\right)^{\frac{s}{s-1}} + k > 0,$$

where $\overline{V} = (\dfrac{\beta}{s\alpha})^{\frac{s}{s-1}}$ and $f'(V) < 0$ for $V(t) > \overline{V}$ and $\lim\limits_{V \to +\infty} f(V) = -\infty$. Thus, there exists $\xi > \overline{V}$ such that $f(\xi) = 0$. Consequently

$$f(V) > 0 \quad \text{for all} \quad V(t) < \xi.$$

Hence, V is bounded.

Case 3) If $\dot{V}$ is oscillatory.

There exists the sequence $(t_n)_{n\geq 0}$ such that $t_n \geq 0$, and $\lim\limits_{n\to+\infty} t_n = +\infty$ with $\dot{V}(t_n) = 0$, $\forall n$. Without loss of generality, we suppose that on $[t_n; t_{n+1}] : \dot{V}(t) \geq 0$, from case 2 there exists finite constant $\xi_n > 0$ such that $V(t) \leq \xi_n$ for all $t \in [t_{n+1}; t_{n+2}]$.

If $t \in [t_{n+1}; t_{n+2}] : \dot{V}(t) \leq 0$ and $V(t) \leq V(t_{n+1}) \leq \xi_n$ so $V(t) \leq \xi_n$ for all $t \in [t_n; t_{n+2}]$, consequently , $V(t) \leq \sup_{n\geq 0}\ \xi_n$, for all $t \geq t_0$. □

Theorem 2.3.5. *We consider the system (1) and we suppose that*

i) The assumptions (H_3) *and* (H_4) *are verified.*

ii) There is a positive constant M *such that*

$$\|h(t,x)\| \leq M, \ for\ all\ \ (t,x) \in \mathbb{R}^+ \times D,$$

where D *is an open of* $\mathbb{R}^n$*. Then the system (1) is practically stable.*

Proof. Since V is positive and decreasing, V is bounded

Consider W defined by (3.8), and its derivative along the trajectories of the system (1) satisfaisant (3.9). By Conditions (H_3) and (H_4) we have

$$\begin{aligned} \dot{W} &\leq -c_3 \|x_1\|^r + c_4 M \|x_1\| \|x_2\| - b_3 \|x_2\|^r + \rho \\ &\leq -\mu_1 V_1^{\frac{r}{q}} - \mu_2 V_2^{\frac{r}{q}}, \end{aligned}$$

where$\rho = \rho_1 + \rho_2$ $\mu_1 = \dfrac{\beta_1}{c_2^{\frac{r}{q}}}$ and $\mu_2 = \dfrac{b_3}{b_2^{\frac{r}{q}}}$.

Note that

$$\begin{aligned} \dot{W} &\leq -\mu_1 V_1 - \mu_2 V_2 + \mu_1(V_1 - V_1^{\frac{r}{q}}) \ + \mu_2(V_2 - V_2^{\frac{r}{q}}) \ + \rho. \\ &\leq -\mu W + \mu_1(V_1 - V_1^{\frac{r}{q}}) \ + \mu_2(V_2 - V_2^{\frac{r}{q}}) + \rho, \end{aligned}$$

where $\mu = \min\{\mu_1, \mu_2\}$. As V_1 and V_2 are bounded

$$\mu_1(V_1 - V_1^{\frac{r}{q}}) \ + \mu_2(V_2 - V_2^{\frac{r}{q}}) \leq 0.$$

Hence

$$\dot{W} \leq -\mu W + \rho,$$

We also have

$$V_1(t, x_1 + I_k(x_1)) \leq \psi_{1k}(V_1(t, x_1)), \quad t = t_k, \quad k = 1, 2, ...$$

and

$$V_2(t, x_2 + J_\sigma(x_2)) \leq \psi_{2\sigma}(V_2(t, x_2)), \quad t = \tau_\sigma, \quad \sigma = 1, 2, ...$$

Then

$$\begin{aligned} W(t, x_1 + I_k(x_1), x_2 + J_\sigma(x_2)) &\leq V_1(t, x_1 + I_k(x_1)) + \alpha V_2(t, x_2 + J_\sigma(x_2)) \\ &\leq (\psi_1)_k(V_1(t, x_1)) + \alpha(\psi_2)_\sigma(V_2(t, x_2)) \\ &= (d_1)_k V_1(t, x_1) + \alpha (d_2)_\sigma V_2(t, x_2) \\ &\leq \gamma_{k,\sigma} W, \end{aligned}$$

such that $\gamma_{k,\sigma=} \max\{(d_1)_k, (d_2)_\sigma\}$.

From Theorems 2.2.3 and 2.2.5 the system (1) is practically stable. □

2.3.2 Example 2

Consider the system

$$\begin{cases} \dot{x}_1 = -\frac{1}{4}x_1^{\frac{3}{2}} + \frac{x_1^{\frac{3}{2}}}{1 + x_1^2}e^{-t^2} + \frac{1}{1+t^2}x_2, t \neq t_k \\ \Delta x_1 = I_k(x_1) = \alpha_k x_1, t = t_k, \alpha_k \in \mathbb{R}, k = 1, 2... \\ \dot{x}_2 = -x_2^{\frac{3}{2}} + 2x_2^{\frac{3}{2}}e^{-x_2^2} \ t \neq \tau_\sigma \\ \Delta x_2 = J_\sigma(x_2) = \beta_\sigma x_2, \ t = \tau_\sigma, \beta_\sigma \in \mathbb{R}, \sigma = 1, 2.... \end{cases} \tag{2.8}$$

In this case,

$$\begin{aligned} f_1(t, x_1) &= -\frac{1}{4}x_1^{\frac{3}{2}} + \frac{x_1^{\frac{3}{2}}}{1 + x_1^2}e^{-t^2}, \\ f_2(t, x_2) &= -x_2^{\frac{3}{2}} + \frac{1}{2}x_2^{\frac{3}{2}}e^{-x_2^2}, \\ h(t, x) &= \frac{1}{1+t^2}. \end{aligned}$$

Set $V_1(t,x_1) = x_1^{\frac{5}{4}}$ and $V_2(t,x_2) = x_2^{\frac{5}{4}}$.
Verification of assumption H_3). We have

$$\begin{aligned}
\|x_1\|^{\frac{5}{4}} &\leq V_1(t,x_1) \leq \|x_1\|^{\frac{3}{2}}, \\
\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t,x_1) &\leq \frac{5}{4} x_1^{\frac{1}{4}} \left(-\frac{1}{4} x_1^{\frac{3}{2}} + \frac{x_1^{\frac{3}{2}}}{1+x_1^2} e^{-x_1^2}\right) \\
&\leq -15 x_1^{\frac{7}{4}} \\
&\leq -15 \|x_1\|^{\frac{7}{4}}. \\
V_1(t, x_1 + I_k(x_1)) &= V_1(t, x_1 + \alpha_k x_1) \leq \psi_{1k}(V_1(t,x_1)) = (1+\alpha_k)^2 x_1^2, t = t_k, k = 1, 2, .
\end{aligned}$$

With $p = \frac{5}{4}$, $q = \frac{3}{2}$, $r = \frac{7}{4}$, $c_1 = 1$, $c_2 = 1$, $c_3 = 15$, $c_4 = \frac{5}{4}$.
Verification of assumption H_4).We have

$$\begin{aligned}
\|x_2\|^{\frac{5}{4}} &\leq V_2(t,x_2) \leq \|x_2\|^{\frac{3}{2}} \\
\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2} f_2(t,x_1) &\leq \frac{5}{4} x_2^{\frac{1}{4}} \left(-x_2^{\frac{3}{2}} + \frac{1}{2} x_2^{\frac{3}{2}} e^{-x_2^2}\right) \\
&\leq -\frac{5}{8} x_2^{\frac{7}{4}} \\
&\leq -\frac{5}{8} \|x_2\|^{\frac{7}{4}}, \\
V_2(t, x_2 + J_\sigma(x_2)) &= V_2(t, x_2 + \beta_\sigma x_2) = (1+\beta_\sigma)^2 x_2^2 \leq \psi_{2\sigma}(V_2(t,x_2)), t = \tau
\end{aligned}$$

with $p = \frac{5}{4}$, $q = \frac{3}{2}$, $r = \frac{7}{4}$, $b_1 = 1$, $b_2 = 1$, $b_3 = \frac{5}{8}$.
Therefore, we can apply Theorem3.3.3 to prove that system (2.8) is practically stable.

Theorem 2.3.6. *Assume that the hypotheses* (H_3) *and* (H_4) *are checked, and that there are positive constants* L, ε *such as*

$$\|h(t,x)\| \leq \varepsilon \|x\| + L,$$

for all $(t,x) \in \mathbb{R}^+ \times D$, *where* D *is open from* R^n. *Then system (1) is practically stable.*

Proof. As V is positive and decreasing then V is bounded.
We consider W defined by (3.8) and its derivative along the trajectories of the system (1) given by (3.9), then using conditions (i) and (ii) we get

$$\begin{aligned}
\dot{W} &\leq -c_3 \|x_1\|^r + c_4(\varepsilon \|x\| + L) \|x_1\| \|x_2\| - b_3 \|x_2\|^r \\
&\leq -\beta_1 \|x_1\|^r - b_3 \|x_2\|^r \\
&\leq -\mu_1 V_1^{\frac{r}{q}} - \mu_2 V_2^{\frac{r}{q}},
\end{aligned}$$

with $\mu_1 = \dfrac{\beta_1}{c_2^{\frac{r}{q}}}$, $\mu_2 = \dfrac{b_3}{b_2^{\frac{r}{q}}}$.
Note that

$$\dot{W} \leq -\mu_1 V_1^{\frac{r}{q}} + \mu_1 V_1 - \mu_1 V_1 - \mu_2 V_2^{\frac{r}{q}} + \mu_2 V_2 - \alpha\mu_2 V_2.$$

Then

$$\dot{W} \leq -\mu W + \mu_1(V_1 - V_1^{\frac{r}{q}}) + \mu_2(V_2 - V_2^{\frac{r}{q}}) + K,$$

with $\mu = \min\{\mu_1, \mu_2\}$. The limits of V_1 and V_2 leads to the existence of $\rho > 0$ such that:

$$\mu_1(V_1 - V_1^{\frac{r}{q}}) + \mu_2(V_2 - V_2^{\frac{r}{q}}) + K \leq \rho.$$

So

$$\dot{W} \leq -\mu W + \rho. \tag{2.9}$$

We also have

$$V_1(t, x_1 + I_k(x_1)) \leq \psi_{1k}(V_1(t, x_1)), t = t_k, k = 1, 2, ...$$

and

$$V_2(t, x_2 + J_\sigma(x_2)) \leq \psi_{2\sigma}(V_2(t, x_2)), t = \tau_\sigma, \sigma = 1, 2, ...$$

Then

$$\begin{aligned} W(t, x_1 x_1 + I_k(x_1), x_2 + J_\sigma(x_2)) &\leq V_1(t, x_1 + I_k(x_1)) + V_2(t, x_2 + J_\sigma(x_2)) \\ &\leq (\psi_1)_k(V_1(t, x_1)) + (\psi_2)_\sigma(V_2(t, x_2)) \\ &= (d_1)_k V_1(t, x_1) + (d_2)_\sigma V_2(t, x_2) \\ &\leq \gamma_{k,\sigma} W, \end{aligned}$$

such as $\gamma_{k,\sigma=} \max((d_1)_k, (d_2)_\sigma)$. By Theorems 2.2.3 and 2.2.5 the system (1) is practically stable. □

Chapter 3

Results on the exponential practical stability of cascade impulsive systems

3.1 Introduction

Since analytical solutions of nonlinear differential equations cannot usually be obtained, the second Lyapunov method is of major importance in the determination of the stability of nonlinear systems. This method will be used in the paper of Mohsen Dlala and Mohamed Ali Hammami to investigate several criteria for perturbed impulsive systems in the case where the zero solution is an equilibrium point, they provide some sufficient conditions for the uniform exponential stability of perturbed impulsive systems by using the Lyapunov second method. Practical stability is also investigated for a class of perturbed impulsive systems.

In this section,first we introduce the results of Mohsen Dlala and Mohamed Ali Hammami concerning global practical exponential stability of impulsive perturbed systems. Let us consider the impulsive system

$$\begin{cases} \dot{x} = f(t,x), t \neq t_k \\ \Delta x = I_k(x), t = t_k, \quad k = 1, 2, ..., \\ x(t_0) = J_\sigma(x_2), \end{cases} \tag{3.1}$$

where

1. $t_1 = 0 < t_1 < t_2 < ... < t_k < ...$ and $\lim_{k \to \infty} t_k = \infty$.

2. $f : R_+ \times \ B_r \to \mathbb{R}^n$ is piecewise continuous in t with discontinuities of the first kind at $t = t_k$ and is left continuous at $t = t_k$ and locally Lipschitz with respect to x on the sets $]t_{k-1}, t_{k-1}] \ \times B_r$.

3. The fonction $I_k : \mathbb{B}_r \to \mathbb{R}^n$ is locally Lipschitz for $k = 1, 2, ...$and $I_k(0) = 0$.

4. $\Delta \ x(t_k) = x(t_k^+) - x(t_k^-)$, where

$$x(t_k^+) = \lim_{t \longrightarrow 0^+} x(t+h) x(t_k^-) = \lim_{t \longrightarrow 0^+} x(t-h),$$

and $x(t_k^-) = x(t_k^+)$ which implies that the solution of (3.1) is left continuous at t_k.

The perturbed impulsive system has the form:

$$\begin{cases} \dot{x} = f(t,x) + g(t,x), t \neq t_k \\ \Delta x = I_k(x) + J_k(x), t = t_k, \quad k = 1, 2, ..., \\ x(t_0) = J_\sigma(x_2), \end{cases} \tag{3.2}$$

where

1. The function g is continuous and locally Lipschitz with respect to x on the sets $]t_{k-1}, tk] \times B_r$ for $k = 1, 2, ...$, and for each k and $y \in B_r$, the $\lim_{(t,y) \to (t_k, x)} g(t, y)$ exists,

   and $J_k \ : \mathbb{B}_r \to \mathbb{R}^n$ is locally Lipschitz for $k = 1, 2, ...$

2. We denote
   (i) by K the class of continuous functions $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ such that α is strictly increasing and $\alpha(0) = 0$;
   (ii) by $C[\mathbb{R}^+]$ the set of functions $\psi : \mathbb{R}^+ \to \mathbb{R}$ which are continuous;
   (iii) by $PC^1[\mathbb{R}^+, \mathbb{R}]$ the set of functions $\psi : \mathbb{R}^+ \to \mathbb{R}$ which are piecewise continuous differentiable.

We introduce the following assumption:

(A_1) Suppose that the nominal system (2) is uniformly exponentially stable with a Lyapunov function $V \in \nu_0$ satisfying:

(i)$c_1\|x\|^2 \leq V(t,x) \leq c_2\|x\|^2$;

(ii) $|V(t,x) - V(t,y)| \leq L\|x-y\|$;

(iii) $D^+_{(2.1)}V(t,x) \leq -c_3\|x\|^2$ for $t \neq t_k$;

(iv) $V(t_k^+, x + I_k x) \leq V(t_k, x)$ for $t = t_k$,

where c_1, c_2, c_3, and L are positive constants. The perturbation term $g(t,x)$ and impulse effects satisfy the following condition:

(A_2) $\|g(t,x)\| \leq \gamma(t)\|x\|^2$, for all $t \in \mathbb{R}_+$ and $x \in B_r$, where $\gamma : \mathbb{R} \longrightarrow \mathbb{R}$ is a nonnegative continuous function on$[0, +\infty[$.

(A_3) For any $t \in \mathbb{R}_+$ and $x \in B_r$, we have

$$\|J_k(x)\| \leq \epsilon_k \|x\|^2,$$

where $\epsilon_k \geq 0$ and $\Sigma_k \epsilon_k < +\infty$.

(A_4) $\|g(t,x)\| \leq \gamma(t)\|x\|$ for all $t \in \mathbb{R}_+$ and all $x \in B_r$, where $\gamma : \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function on $[0, +\infty[$.

(A_5) For any $t \in \mathbb{R}_+$ and $x \in B_r$, we have

$$\|J_k(x)\| \leq \mu_k,$$

where $\mu_k \geq 0$.

(A_6) There exists a positive constant $\overline{\gamma}$ such that for any $t \in \mathbb{R}_+$ and $x \in B_r$, we have

$$\|g(t,x)\| \leq \overline{\gamma}.$$

Note that the convergence of the series $\Sigma_k \epsilon_k < +\infty$ is equivalent to the convergence of the infinite product$\prod_k(\epsilon_k + 1)$.

Theorem 3.1.1. *Let $(A_1), (A_3)$ hold. In addition, assume that the upper bound of the perturbed term satisfies Then the solution $x = 0$ of perturbed system (3.2) is an exponentially stable equilibrium point. Moreover, if all assumptions hold globally, then the origin is globally exponentially stable.*

Theorem 3.1.2. *Let assumptions $(A_1), (A_3)$ hold. In addition, assume that the upper bound of the perturbed term is integrable on $[0, +\infty[$. Then the solution $x = 0$ of perturbed system (3.2) is globally uniformly exponentially stable.*

Theorem 3.1.3. *Let assumptions* $(A_1), (A_4)$*, and* (A_5) *hold. In addition, assume that the series* $\Sigma_k \sqrt{\mu_k}$ *converges and the function* $\gamma(t)$ *in* (A_5) *is bounded or integrable on* $[0, +\infty[$*, then perturbed system (3.1) is practically exponentially stable. Moreover, if all assumptions hold globally, then the perturbed system is globally practically exponentially stable.*

Theorem 3.1.4. *Let assumptions* $(A_1), (A_3)$*, and* (A_4) *hold. In addition, assume that the function* $\gamma(t)$ *in* (A_5) *is bounded or integrable on* $[0, +\infty[$*. Then perturbed system (3.2) is practically exponentially stable. Moreover, if all assumptions hold globally, then the perturbed system is globally practically exponentially stable.*

Theorem 3.1.5. *Let assumptions* $(A_1), (A_5)$*, and* (A_6) *hold. In addition, assume that the series* $\Sigma_k \sqrt{\mu_k}$ *converges. Then perturbed system (3.2) is practically exponentially stable. Moreover, if all assumptions hold globally, then the perturbed system is globally practically exponentially stable.*

In the next paragraph, we will present another generalization of Theorem 3.2.1.

Let $t_0 \in \mathbb{R}$, and Ω a domain of $\mathbb{R}^n$ which contains the origin with $\|x\| = \sqrt{\sum_{k=1}^{n} x_k^2}$ for all $x \in \mathbb{R}^n$. We consider the following system

$$\begin{cases} \dot{x} = f(t, x), t \neq t_k, \\ \Delta x(t) = I_k(x), t = t_k, \end{cases} \tag{3.3}$$

with the following conditions:

1. $t_1 = 0 < t_1 < t_2 < ... < t_k < ...$ and $\lim_{k \to \infty} t_k = \infty$;
2. $f : (t_{k-1}, t_k] \times \mathbb{R}^n \to \mathbb{R}^n$ is piecewise continuous in t with discontinuities of the first kind at t_k and is left continuous at t_k and locally Lipschitz with respect to x on the sets $(t_{k-1}, t_k] \times B_r$ for $k = 1, 2, ..., r > 0$, where $B_r = \{x \in \mathbb{R}^n, \|x\| \leq r\}$;
3. $I_k : B_r \to \mathbb{R}^n$ is locally Lipschitz for $k = 1, 2, ...$ and $I_k(0) = 0$;
4. $x(t_k) = x(t_k^+) - x(t_k^-)$, where $x(t_k^+) = \lim_{h \to 0+} x(t + h)$, $x(t_k^-) = \lim_{h \to 0+} x(t - h)$, and $x(t_k^-) = x(t_k)$, which implies that the solution of (3.3) is left continuous at t_k.

Definition 3.1.6. *The equilibrium point $x = 0$ of system (3.3) is said to be :*
(i) uniformly exponentially stable if there exist positive constants $c, k,$ and λ such that

$$\|x(t, t_0, x_0)\| \leq k \|x(t_0)\| e^{-\lambda(t-t_0)}, \ \forall x(t_0) \leq c,$$

(ii) globally uniformly exponentially stable if (i) holds for all $x(t_0) \in \mathbb{R}^n$.

Definition 3.1.7. *System (3.3) is said to be :*

1. *uniformly exponentially practically stable (UEPS) with respect to B_ρ with the attraction region Ω if there exist constants $\rho > 0$ such that for all $t_0 \geq 0, x_0 \in \Omega$, there exists $k \geq 0$ such that*

$$x(t, t_0, x_0) \leq \rho + kx(t_0)e^{-\lambda(t-t_0)}.$$

2. *uniformly globally exponentially practically stable (UGEPS) if it is UEPS with $\mathbb{R}^n$ as the attraction region.*

Definition 3.1.8. *For $(t, x) \in (t_{k-1}, t_k] \times \mathbb{R}^n$ we define*

$$D^+V(t, x) = \lim_{h \longrightarrow 0^+} \sup \frac{1}{h} \left[V(t + h, x + hf(t, x)) - V(t, x)\right].$$

Lemma 3.1.9. ***(Comparison Lemma)**[22] Assume that*

1) $v \in PC^1(\mathbb{R}_+, \mathbb{R})$ and $v(t)$ is left continuous at t_k, $k = 1, 2, ...,$
2) For $k = 1, 2, ..., t \geq t_0$,

$$D^+v(t) \leq a(t)v(t) + b(t), \ v(t_k^+) \leq c_k v(t_k) + d_k,$$

where $a, b \in C(\mathbb{R}_+), c_k > 0$ and d_k are constants.
Then

$$v(t) \leq v(t_0) \left(\prod_{t_0<t_k<t} c_k \right) e^{\int_{t_0}^{t} a(s)ds} + \sum_{t_0<t_k<t} \left(\prod_{t_k<t_j<t} c_j \right) e^{\int_{t_k}^{t} a(s)ds} d_k + \int_{t_0}^{t} \left(\prod_{s<t_k<t} c_k \right) e^{\int_{s}^{t} a(u)du} b(s)ds.$$

3.2 Assumptions and main results

We are now able to present our generalization of Theorem (3.2.1) were we have obtained sufficient conditions to guarantee the practical stability of system (1).

We start by stating the hypotheses considered.

$H_1)$ There is a function V_1 continuously differentiable and constants c_1, c_2, c_3, c_4 and r_1 such that:

$$\begin{cases} c_1 \left\|x_1\right\|^2 \leq V_1(t, x_1) \leq c_2 \left\|x_1\right\|^2, \\ \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t, x_1) \leq -c_3 V_1(t, x_1) + r_1, \ t \neq t_k \\ \left\|\frac{\partial V_1}{\partial x_1}\right\| \leq c_4 \left\|x_1\right\|, \\ V_1(t, x_1 + I_k(x_1)) \leq (\psi_1)_k (V_1(t, x_1)), t = t_k, k = 1, 2, ... \end{cases} \tag{3.4}$$

$H_2)$ There is a function V_2 continuously differentiable and constants b_1, b_2, b_3 and r_2 such that

$$\begin{cases} b_1 \left\|x_2\right\|^2 \leq V_2(t, x_2) \leq b_2 \left\|x_2\right\|^2, \\ \frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2} f_2(t, x_2) \leq -b_3 V_2(t, x_2) + r_2; \quad t \neq \tau_\sigma \\ V_2(t, x_1 + J_\sigma(x_2)) \leq (\psi_2)_\sigma (V_2(t, x_2)), t = \tau_\sigma, \sigma = 1, 2, ... \end{cases}$$

such that

$$(\psi_1)_k (V_1(t, x_1)) = (1 + (d_1)_k) V_1(t, x_1), t = t_k, k = 1, 2, ...$$

and

$$(\psi_2)_\sigma (V_2(t, x_2)) = (1 + (d_2)_\sigma) V_2(t, x_2), t = \tau_\sigma, \sigma = 1, 2, ...$$

Theorem 3.2.1. *Suppose that assumptions (H_1) and (H_2) hold, and that the interconnection term is limited, i.e ,*

$$\exists M > 0 : \|h(t, x) \leq< M, \ \textit{for all} \ \ (t, x).$$

Then the system (1) is uniformly exponentially practically stable provided that

$$\sum_i \gamma_i < +\infty \quad \textit{where} \quad \gamma_i = \max\{(d_1)_k, (d_2)_\sigma\}. \tag{3.5}$$

Proof. Let $W(t, x_1, x_2) = V_1(t, x_1) + \delta V_2(t, x_2)$ where δ is a positive constant that will be specified later. The derivative of W along the trajectories of the system (1) is

$$\begin{aligned}\dot{W} &= \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1}\frac{\partial x_1}{\partial t} + \delta(\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2}\frac{\partial x_2}{\partial t}) \\ &= \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1}f_1(t,x_1) + \frac{\partial V_1}{\partial x_1}h(t,x)x_2 + \delta(\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2}f_2(t,x_2)) \\ &\leq -c_3V_1(t,x_1) + c_4M\left\|x_1\right\|\left\|x_2\right\| + \delta(-b_3V_2(t,x_2)) + r \\ &\leq -c_3V_1(t,x_1) + \frac{c_4M}{2}(\frac{\left\|x_1\right\|^2}{\varepsilon} + \varepsilon\left\|x_2\right\|^2) - \delta b_3V_2(t,x_2) + r \\ &\leq -\big(c_3 - \frac{c_4M}{2\varepsilon c_1}\big)V_1(t,x_1) - \big(b_3 - \frac{c_4M\varepsilon}{2b_1\alpha}\big)\delta V_2(t,x_2) + r, \end{aligned} \tag{3.6}$$

where $r = r_1 + r_2$. By taking

$$\varepsilon = \frac{c_4M}{c_1c_3},$$

we get

$$\frac{c_4M}{2\varepsilon c_1} = \frac{c_3}{2},$$

and

$$\frac{c_4M\varepsilon}{2b_1\delta} = \frac{(c_4M)^2}{2c_1c_3b_1\delta}.$$

Thus,

$$\dot{W} \leq -\frac{c_3}{2}V_1(t,x_1) + (-b_3 + \frac{(c_4M)^2}{2c_1c_3b_1\delta})\delta V_2(t,x_2) + r.$$

It is convenient to choose δ such that

$$-b_3 + \frac{(c_4M)^2}{2c_1c_3b_1\delta} < 0.$$

Therefore

$$\delta > \frac{(c_4M)^2}{2c_1c_3b_1b_3}.$$

By taking

$$\delta = \frac{(c_4M)^2}{c_1c_3b_1b_3},$$

we have

$$-b_3 + \frac{(c_4M)^2}{2c_1c_3b_1\delta} = -\frac{b_3}{2}.$$

Consequently

$$\dot{W} \leq -\frac{c_3}{2}V_1(t,x_1) - \delta\frac{b_3}{2}V_2(t,x_2) + r.$$

Let $\lambda = \min\left(\frac{c_3}{2}, \frac{b_3}{2}\right)$. Hence

$$\dot{W} \leq -\lambda W(t) + r.$$

We also have

$$V_1(t, x_1 + I_k(x_1)) \leq (1 + (d_1)_k) V_1(t, x_1), t = t_k, k = 1, 2, ...$$

and

$$V_2(t, x_2 + J_\sigma(x_2)) \leq (1 + (d_2)_\sigma) V_2(t, x_2), t = \tau_\sigma, \sigma = 1, 2, ...$$

It follows that

$$\begin{aligned} W(t, x_1 x_1 + I_k(x_1), x_2 + J_\sigma(x_2)) &\leq V_1(t, x_1 + I_k(x_1)) + \delta V_2(t, x_2 + J_\sigma(x_2)) \\ &= (1 + (d_1)_k) V_1(t, x_1) + \delta (1 + (d_2)_\sigma) V_2(t, x_2) \\ &\leq (1 + \gamma_i) W, \end{aligned}$$

such that from (3.5) we have $\sum_i \gamma_i < +\infty$.
By Comparison lemma we get

$$W \leq W(t_0) \left(\prod_{t_0 < t_i < t} (1 + \gamma_i) \right) e^{\int_{t_0}^{t} -\lambda ds} + \int_{t_0}^{t} \left(\prod_{s < t_i < t} (1 + \gamma_i) \right) e^{\int_{s}^{t} -\lambda du} r ds.$$

Since the convergence of the series $\sum_i \gamma_i$ is equivalent to the convergence of the infinite product $\prod_i (1 + \gamma_i)$, then, there exists positive constant μ such that $\prod_i (1 + \gamma_i) \leq \mu$.
Hence,

$$\begin{aligned} W &\leq W(t_0) \mu e^{-\lambda(t - t_0)} + \mu r \int_{t_0}^{t} e^{-\lambda(t-s)} ds \\ &= W(t_0) \mu e^{-\lambda(t - t_0)} + \frac{\mu r}{\lambda} e^{-\lambda t_0}. \end{aligned}$$

Thus, we obtain

$$\|x\| \leq \sqrt{\frac{(c_2 + \delta b_2)}{(c_1 + \delta b_1)}} \|x(t_0)\| \mu e^{\frac{-\lambda(t-t_0)}{2}} + \sqrt{\frac{\mu r}{\lambda (c_1 + \delta b_1)}} e^{-\frac{\lambda t_0}{2}}.$$

We conclude that (1) is uniformly exponentially practically stable. □

3.2.1 Example 1

Consider the system

$$\begin{cases} \dot{x}_1 = -\left(1 + e^t\right) x_1 + \dfrac{1}{1 + t^2} x_2, t \neq t_k \\ \Delta x_1 = I_k(x_1) = \alpha_k x_1, t = t_k, \alpha_k \in R, k = 1, 2... \\ \dot{x}_2 = -x_2 + e^{-x_2^2} \; t \neq \tau_\sigma \\ \Delta x_2 = J_\sigma(x_2) = \beta_\sigma x_2, \;\; t = \tau_\sigma, \beta_\sigma \in R, \sigma = 1, 2.... \end{cases} \tag{3.7}$$

In this case,

$$\begin{aligned} f_1(t,x_1) &= -\left(1+e^t\right)x_1, \\ f_2(t,x_2) &= -x_2+e^{-x_2^2}, \\ h(t,x) &= \frac{1}{1+t^2}. \end{aligned}$$

Set $V_1(t,x_1)=x_1^2$ and $V_2(t,x_2)=x_2^2$.
Verification of assumptions :
We have

$$\begin{aligned} \|x_1\|^2 &\leq V_1(t,x_1)\leq \|x_1\|^2, \\ \frac{\partial V_1}{\partial t}+\frac{\partial V_1}{\partial x_1}f_1(t,x_1) &= -2x_1^2\leq -2V_1, \\ \left\|\frac{\partial V_1}{\partial x_1}\right\| &\leq 2\|x_1\|, \\ V_1(t,x_1+I_k(x_1)) &= V_1(t,x_1+\alpha_k x_1)\leq \psi_{1k}(V_1(t,x_1))=(1+\alpha_k)^2 x_1^2, t=t_k, k=1,2,.... \end{aligned}$$

with $c_1=1$, $c_2=1$, $c_3=2, c_4=2$ and $\psi_{1k}(V_1(t,x_1))=(1+\alpha_k)^2 V_1$.
We have also

$$\begin{aligned} \|x_2\|^2 &\leq V_2(t,x_2)\leq \|x_2\|^2 \\ \frac{\partial V_2}{\partial t}+\frac{\partial V_2}{\partial x_2}f_2(t,x_1) &= -2x^2\leq -2V_2 \\ V_2(t,x_2+J_\sigma(x_2)) &= V_2(t,x_2+\beta_\sigma x_2)=(1+\beta_\sigma)^2 x_2^2\leq \psi_{2\sigma}(V_2(t,x_2)), t=\tau_\sigma, \sigma=1,2,... \end{aligned}$$

with

$$b_1=1, b_2=1, b_3=2, \text{and} \psi_{2\sigma}(V_2(t,x_2))=(1+\beta_\sigma)^2 V_2, \text{and} \quad \|h(t,x)\|\leq 1=M.$$

Therefore, we can apply Theorem 3.2.1 to prove that system (3.7) is uniformly exponentially practically stable.

Theorem 3.2.2. *Consider the system (1), and suppose that ii) there is a constant ε such that*

$$\|h(t,x)\|\leq \varepsilon\|x\|+L, \quad \text{for all}\,(t,x)\in\mathbb{R}^+\times D,$$

where D an open of $\mathbb{R}^n$.
Then, the system (1) is uniformly exponentially practically stable.

Proof. We have

$$\begin{aligned}\dot{V}_1 &= \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t, x_1) + \frac{\partial V_1}{\partial x_1} h(t, x) x_2 \\ &\leq -c_3 V_1(t, x_1) + c_4 \|x_1\| \|x_2\| (\varepsilon \|x\| + L) + r_1.\end{aligned}$$

The existence of λ such that $\|x_2\| \leq \lambda$, and the use of triangular inequality

$$\|x\| \leq \|x_1\| + \|x_2\|,$$

give

$$\begin{aligned}\dot{V}_1 &\leq -c_3 c_1 \|x_1\|^2 + c_4 \lambda \varepsilon \|x_1\|^2 + c_4 \lambda (\varepsilon \lambda + L) \|x_1\| + r_1 \\ &\leq -\mu_1 \|x_1\|^2 + c_4 \lambda (\varepsilon \lambda + L) \|x_1\| + r_1,\end{aligned}$$

with $\mu_1 = c_3 c_1 - c_4 \lambda \varepsilon$. Therefore, just choose ε such that $c_4 \varepsilon \lambda < c_3 c_1$.
Let $0 < \theta < 1$, we can rewrite the previous inequality as follows

$$\begin{aligned}\dot{V}_1 &\leq -\mu_1 \|x_1\|^2 + \mu_1 \theta \|x_1\|^2 - \mu_1 \theta \|x_1\|^2 + c_4 \lambda (\varepsilon \lambda + L) \|x_1\| + r_1 \\ &\leq -\mu_1 (1 - \theta) \|x_1\|^2 - \mu_1 \theta \|x_1\|^2 + c_4 \lambda (\varepsilon \lambda + L) \|x_1\| + r_1.\end{aligned}$$

So

$$-\mu_1 \theta \|x_1\|^2 + c_4 \lambda (\varepsilon \lambda + L) \|x_1\| < 0,$$

just take

$$\|x_1\| \geq \frac{c_4 \lambda (\varepsilon \lambda + L)}{\mu_1 \theta}.$$

Hence,

$$\dot{V}_1 \leq -\mu_1 (1 - \theta) \|x_1\|^2.$$

Therefore

$$\dot{V}_1 \leq -\frac{\mu_1 (1 - \theta)}{c_1} V_1 + r_1, \quad \forall \|x_1\| \geq \frac{c_4 \lambda (\varepsilon \lambda + L)}{\mu_1 \theta}.$$

Consider

$$W(t, x_1, x_2) = V_1(t, x_1) + V_2(t, x_2). \tag{3.8}$$

The derivative of W along system trajectories of (1) satisfied

$$\dot{W} = \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t, x_1) + \frac{\partial V_1}{\partial x_1} h(t, x) x_2 + \frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2} \frac{\partial x_2}{\partial t}. \tag{3.9}$$

From conditions (i) and (ii) we get

$$\dot{W} \leq -\frac{\mu_1(1-\theta)}{c_1}V_1 - b_3V_2 + r.$$

where $r = r_1 + r_2$. Thus

$$\dot{W} \leq -\mu W + r,$$

with $\mu = \min\left\{\frac{\mu_1(1-\theta)}{c_1}, b_3\right\}$. We also have

$$V_1(t, x_1 + I_k(x_1)) \leq (1 + (d_1)_k)\, V_1(t, x_1), t = t_k, k = 1, 2, ...$$

and

$$V_2(t, x_2 + J_\sigma(x_2)) \leq (1 + (d_2)_\sigma)\, V_2(t, x_2), t = \tau_\sigma, \sigma = 1, 2, ...$$

It follows that

$$\begin{aligned} W(t, x_1x_1 + I_k(x_1), x_2 + J_\sigma(x_2)) &\leq V_1(t, x_1 + I_k(x_1)) + \delta V_2(t, x_2 + J_\sigma(x_2)) \\ &= (1 + (d_1)_k)\, V_1(t, x_1) + \delta\,(1 + (d_2)_\sigma)\, V_2(t, x_2) \\ &\leq (1 + \gamma_i)\, W, \end{aligned}$$

such that from (3.5) we have $\sum_i \gamma_i < +\infty$.
By Comparison lemma we get

$$W \leq W(t_0)\left(\prod_{t_0<t_i<t}(1+\gamma_i)\right)e^{\int_{t_0}^{t} -\mu ds} + \int_{t_0}^{t}\left(\prod_{s<t_i<t}(1+\gamma_i)\right)e^{\int_{s}^{t} -\mu du} r ds.$$

Note that the convergence of the series $\sum_i \gamma_i$ is equivalent to the convergence of the infinite product $\prod_i (1+\gamma_i)$, then, there exists positive constant μ such that $\prod_i (1+\gamma_i) \leq \beta$.
Hence

$$\begin{aligned} W &\leq W(t_0)\beta e^{-\mu(t-t_0)} + \beta r \int_{t_0}^{t} e^{-\mu(t-s)} ds \\ &= W(t_0)\beta e^{-\mu(t-t_0)} + \frac{\beta r}{\mu} e^{-\mu t_0}. \end{aligned}$$

Thus, we obtain

$$\|x\| \leq \sqrt{\frac{(c_2+\delta b_2)}{(c_1+\delta b_1)}}\,\|x(t_0)\|\,\beta e^{\frac{-\mu(t-t_0)}{2}} + \sqrt{\frac{\beta r}{\mu\,(c_1+\delta b_1)}}e^{-\frac{\mu t_0}{2}}.$$

We conclude that (1) is uniformly exponentially practically stable. □

3.2.2 Example 2

Consider the system

$$\begin{cases} \dot{x}_1 = -\left(1+e^t\right)x_1 + \dfrac{10^{-3}xe^{-x^2}}{1+t^2} + \dfrac{1}{5}, t \neq t_k \\ \Delta x_1 = I_k(x_1) = \alpha_k x_1, t = t_k, \alpha_k \in R, k = 1, 2... \\ \dot{x}_2 = -x_2 + e^{-x_2^2}\ t \neq \tau_\sigma \\ \ \Delta x_2 = J_\sigma(x_2) = \beta_\sigma x_2,\ t = \tau_\sigma, \beta_\sigma \in R, \sigma = 1, 2.... \end{cases} \tag{3.10}$$

We have $h(t,x) = \frac{10^{-3}xe^{-x^2}}{1+t^2} + \frac{1}{5}$. We have also $\|h(t,x)\| \leq \varepsilon \parallel x \parallel + M$ where $L = \dfrac{1}{5} < \dfrac{2c_3\eta^{r-2}}{c_4} = \dfrac{2}{9}$ and $\varepsilon = 10^{-3}e^{-2}$.
Therefore, we can apply Theorem 3.2.2 to prove that system (3.10) is uniformly exponentially practically stable.

3.3 Generalizations of the exponential practical stability of cascade impulsive systems

We propose in this part to state several generalizations of Theorem 3.2.1
Now we state our main results, to do this let's establish the following Lemma :

Lemma 3.3.1. *We consider the system $\dot{x} = f(t,x)$. Let*

$$V : [0, +\infty[\, \times \mathbb{R}^n \to \mathbb{R}, \tag{3.11}$$

a continuously differentiable function such that

$$\begin{aligned} c_1 \|x\|^p &\leq V(t,x) \leq c_2 \|x\|^q, \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) &\leq -c_3 \|x\|^r + r, \end{aligned}$$

for all $t \geq 0$ and $x \in \mathbb{R}^n$, where c_1, c_2, c_3, p, q and r are positive constants.
Then V is bounded.

We are now able to present our second generalization of Theorem 3.2.1. We begin by stating the following assumptions.

$\mathbf{H_3}$) There is a function V_1 continuously differentiable and constants c_1, c_2, c_3, c_4, r, p and q such that :

$$\begin{array}{rcl} c_1 \|x_1\|^p & \leq & V_1(t,x_1) \leq c_2 \|x_1\|^q \\ \dfrac{\partial V_1}{\partial t} + \dfrac{\partial V_1}{\partial x_1} f_1(t,x_1) & \leq & -c_3 \|x_1\|^r + \rho_1, t \neq t_k \\ \left\| \dfrac{\partial V_1}{\partial x_1} \right\| & \leq & c_4 \|x_1\|, \\ V_1(t, x_1 + I_k(x_1)) & \leq & \psi_{1k}(V_1(t,x_1)), t = t_k, k = 1,2,... \end{array}$$

$\mathbf{H_4}$) There is a function V_2 continuously differentiable and constants b_1, b_2, b_3, r, p and q such that:

$$\begin{array}{rcl} b_1 \|x_2\|^p & \leq & V_2(t,x_2) \leq b_2 \|x_2\|^q \\ \dfrac{\partial V_2}{\partial t} + \dfrac{\partial V_2}{\partial x_2} f_2(t,x_2) & \leq & -b_3 \|x_2\|^r + \rho_2, \ t \neq t_\sigma \\ V_2(t, x_2 + J_\sigma(x_2)) & \leq & \psi_{2\sigma}(V_2(t,x_2)), \ t = \tau_\sigma, \ \sigma = 1,2,... \end{array}$$

Lemma 3.3.2. *Let V be a positive definite and continuously differentiable function defined such that*

$$\dot{V}(t) \leq -\alpha V(t) + \beta \sqrt[s]{V(t)} + k,$$

where α, β, k are positives constants, and $s > 1$. Then V is bounded.

Proof. Take

$$f(V) = -\alpha V + \beta \sqrt[s]{V} + k.$$

There are three possibilities for the behavior of $\dot{V}(t)$.
Case 1) If $\dot{V}(t) \leq 0$,
since V is a positive definite function, then V is a decreasing function. Hence, V is necessarily bounded.
Case 2) If $\dot{V}(t) \geq 0$,
in this case $f(V) \geq 0$ and $f'(V) = \dfrac{-s\alpha V^{1-\frac{1}{s}} + \beta}{sV^{1-\frac{1}{s}}}$.
It is easy to see that

$$f'(\overline{V}) = 0 \text{ and } f(\overline{V}) = \frac{s-1}{\alpha^{\frac{1}{s-1}}} \left(\frac{\beta}{s} \right)^{\frac{s}{s-1}} + k > 0,$$

where $\overline{V} = (\dfrac{\beta}{s\alpha})^{\frac{s}{s-1}}$ and $f'(V) < 0$ for $V(t) > \overline{V}$ and $\lim_{V \to +\infty} f(V) = -\infty$. Thus, there exists $\xi > \overline{V}$ such that $f(\xi) = 0$. Consequently

$$f(V) > 0 \quad \text{for all} \quad V(t) < \xi.$$

Hence, V is bounded.
Case 3) If $\dot{V}$ is oscillatory.
there exists the sequence $(t_n)_{n \geq 0}$ such that $t_n \geq 0$, and $\lim_{n \to +\infty} t_n = +\infty$ *with* $\dot{V}(t_n) = 0$, $\forall n$. Without loss of generality, we suppose that on $[t_n; t_{n+1}] : \dot{V}(t) \geq 0$, from case 2 there exists finite constant $\xi_n > 0$ such that $V(t) \leq \xi_n$ for all $t \in [t_{n+1}; t_{n+2}]$.
If $t \in [t_{n+1}; t_{n+2}] : \dot{V}(t) \leq 0$ and $V(t) \leq V(t_{n+1}) \leq \xi_n$ so $V(t) \leq \xi_n$ for all $t \in [t_n; t_{n+2}]$, consequently , $V(t) \leq \sup_{n \geq 0} \ \xi_n$, for all $t \geq t_0$. □

Theorem 3.3.3. *Consider the system (1) and suppose that*
i) The assumptions (H_3) *and* (H_4) *are verified.*
ii) There is a positive constantM

$$\textit{such that} \, \|h(t,x)\| \leq M, \ \textit{for all} \ \ (t,x) \in \mathbb{R}^+ \times D,$$

where D *is an open of* $\mathbb{R}^n$. *Then the system (1) is uniformly exponentially practically stable.*

Proof. a) Boundedness of V_1:
case 1) If $p = q = r$

$$\begin{aligned} \dot{V}_1(t, x_1) &= \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t, x_1) + \frac{\partial V_1}{\partial x_1} g(t,x) x_2 \\ &\leq -c_3 \|x_1\|^r + \rho_1 + M \left\| \frac{\partial V_1}{\partial x_1} \right\| \|x_2\| \\ &\leq -c_3 \|x_1\|^r + \rho_1 + c_4 M \|x_1\| \|x_2\| . \end{aligned}$$

From assumption (H_4) we have that $\dot{x}_2 = f_2(t, x_2)$ is practically exponentially stable. Hence, by Theorem 3.5, there exists $\lambda > 0$ such that $\|x_2\| \leq \lambda$. Then

$$\dot{V}_1(t, x_1) \leq -\frac{c_3}{c_2} V_1(t, x_1) + \frac{c_4 M \lambda}{\sqrt[r]{c_1}} \sqrt[r]{V_1(t, x_1)} + \rho_1.$$

Take $f(V_1) = -\alpha V_1 + \beta \sqrt[r]{V_1} + \varrho_1$ with $\alpha = \dfrac{c_3}{c_2}$, and $\beta = \dfrac{c_4 M \alpha}{\sqrt[r]{c_1}}$. We conclude by Lemma 3.9 $(s = r)$ that V_1 is bounded.

case 2) If $p > q$ V_1 is bounded (see the proof of Theorem 3.4).
case 3) If $p < q$, we have

$$\|x_1\| > \left(\frac{c_2}{c_1}\right)^{\frac{1}{q-p}} = \eta,$$

and

$$\begin{aligned}
\dot{V}_1(t, x_1) &= \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t, x_1) + \frac{\partial V_1}{\partial x_1} g(t, x) x_2 \\
&\leq -c_3 \|x_1\|^r + \rho_1 + M \left\| \frac{\partial V_1}{\partial x_1} \right\| \|x_2\| \\
&\leq -c_3 \|x_1\|^r + \rho_1 + c_4 M \|x_1\| \|x_2\| .
\end{aligned}$$

Since for all $\xi > 0$, $\|x_1\| \|x_2\| \leq (\frac{\|x_1\|^2}{2\xi} + 2\xi \|x_2\|^2)$, we get

$$\dot{V}_1(t, x_1) \leq -c_3 \|x_1\|^r + \frac{c_4 M}{2\xi} \|x_1\|^2 + \frac{c_4 M \xi}{2} \|x_2\|^2 + \rho_1.$$

We discuss two cases.

1) *For* $r \geq q$ we have

$$\begin{aligned}
\dot{V}_1(t, x_1) &\leq -c_3 \eta^{r-q} \|x_1\|^q + \frac{c_4 M}{2\xi \eta^{p-2}} \|x_1\|^p + \frac{c_4 M \xi}{2} \lambda^2 + \rho_1 \\
&\leq -\frac{c_3 \eta^{r-q}}{c_2} V_1(t, x_1) + \frac{c_4 M}{2 c_1 \xi \eta^{p-2}} V_1(t, x_1) + \frac{c_4 M \xi}{2} \lambda^2 + \rho_1 \\
&\leq -(\frac{c_3 \eta^{r-q}}{c_2} - \frac{c_4 M}{2 c_1 \xi \eta^{p-2}}) V_1(t, x_1) + \frac{c_4 M \xi}{2} \lambda^2 + \rho_1.
\end{aligned}$$

We choose ξ such that

$$\frac{c_3 \eta^{r-q}}{c_2} - \frac{c_4 M}{2 c_1 \xi \eta^{p-2}} > 0.$$

It follows that

$$\dot{V}_1(t, x_1) \leq -\beta_1 V_1(t, x_1) + K_1,$$

where

$$\xi = \frac{c_2 c_4 M}{c_1 c_3 \eta^{r-q+p-2}}, \quad \beta_1 = \frac{c_3 \eta^{r-q}}{2 c_2}, \quad K_1 = \frac{c_4 M \xi}{2} \lambda^2 + \rho_1.$$

We conclude by Lemma 2.2, that V_1 is bounded.

2) $For\ \ r < q$ we have

$$\dot{V}_1(t,x_1) \leq -c_3\|x_1\|^r + \rho_1 + c_4M\|x_1\|\|x_2\|$$
$$\leq -c_3\|x_1\|^r + \lambda c_4 M\|x_1\| + \rho_1$$
$$\leq -c_3\|x_1\|^r + \frac{\lambda c_4 M}{\eta^{r-1}}\|x_1\|^r + \rho_1$$
$$\leq -\beta_2\|x_1\|^r + \rho_1,$$

where $\beta_2 = c_3 - \dfrac{\lambda c_4 M}{\eta^{r-1}}$ and M is chosen such that $\beta_2 > 0$. Hence, by Theorem 3.4, V_1 is bounded.

b) Practical exponential stability of system (1):
Set $W(t,x_1,x_2) = V_1(t,x_1) + \alpha V_2(t,x_2)$ where α is a positive constant. The derivative of W along the trajectories of system (1) is

$$\begin{aligned}\dot{W}(t) &= \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1}f_1(t,x_1) + \frac{\partial V_1}{\partial x_1}g(t,x)x_2 + \alpha(\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2}f_2(t,x_2))\\ &\leq -c_3\|x_1\|^r + \rho_1 + c_4M\|x_1\|\|x_2\| + \alpha(-b_3\|x_2\|^r + \rho_2)\\ &\leq -\beta_3\|x_1\|^r + \rho_1 - \alpha b_3\|x_2\|^r + \rho_2\\ &\leq -\mu_1 V_1^{\frac{r}{q}}(t,x_1) + \rho_1 - \alpha\mu_2 V_2^{\frac{r}{q}}(t,x_2) + \rho_2,\end{aligned}$$

where $\beta_3 = min(\beta_1,\beta_2,c_3)$, $\mu_1 = \dfrac{\beta_3}{c_2^{\frac{r}{q}}}$ and $\mu_2 = \dfrac{b_3}{b_2^{\frac{r}{q}}}$.Remark that

$$\begin{aligned}\dot{W}(t) \leq &- \mu_1 V_1(t,x_1) - \alpha\mu_2 V_2(t,x_2) + \mu_1(V_1(t,x_1) - V_1^{\frac{r}{q}}(t,x_1))\\ &+ \alpha\mu_2(V_2(t,x_2) - V_2^{\frac{r}{q}}(t,x_2)) + \rho_1 + \rho_2.\end{aligned}$$

Let $\mu = \min(\mu_1,\mu_2)$, we obtain

$$\dot{W}(t) \leq -\mu W(t) + \mu_1(V_1(t,x_1) - V_1^{\frac{r}{q}}(t,x_1)) + \alpha\mu_2(V_2(t,x_2) - V_2^{\frac{r}{q}}(t,x_2)) + \rho_1 + \rho_2.$$

The boundedness of V_1 and V_2 implies that there exists $\rho_3 > 0$ such that

$$\mu_1(V_1(t,x_1) - V_1^{\frac{r}{q}}(t,x_1)) + \alpha\mu_2(V_2(t,x_2) - V_2^{\frac{r}{q}}(t,x_2)) \leq \rho_3,$$

thus,

$$\dot{W}(t) \leq -\mu W(t) + \rho,$$

where $\rho = \rho_1 + \rho_2 + \rho_3$ $\mu_1 = \dfrac{\beta_1}{c_2^{\frac{r}{q}}}$ and $\mu_2 = \dfrac{b_3}{b_2^{\frac{r}{q}}}$.
Note that

$$\begin{aligned}\dot{W} &\leq -\mu_1 V_1 - \mu_2 V_2 + \mu_1(V_1 - V_1^{\frac{r}{q}}) + \mu_2(V_2 - V_2^{\frac{r}{q}}) + \rho. \\ &\leq -\mu W + \mu_1(V_1 - V_1^{\frac{r}{q}}) + \mu_2(V_2 - V_2^{\frac{r}{q}}) + \rho,\end{aligned}$$

where $\mu = \min\{\mu_1, \mu_2\}$. As V_1 and V_2 are bounded

$$\mu_1(V_1 - V_1^{\frac{r}{q}}) + \mu_2(V_2 - V_2^{\frac{r}{q}}) \leq 0.$$

Hence

$$\dot{W} \leq -\mu W + \rho.$$

We also have

$$V_1(t, x_1 + I_k(x_1)) \leq \psi_{1k}(V_1(t, x_1)), t = t_k, \quad k = 1, 2, ...$$

and

$$V_2(t, x_2 + J_\sigma(x_2)) \leq \psi_{2\sigma}(V_2(t, x_2)), t = \tau_\sigma, \quad \sigma = 1, 2, ...$$

Then

$$\begin{aligned}W(t, x_1 + I_k(x_1), x_2 + J_\sigma(x_2)) &\leq V_1(t, x_1 + I_k(x_1)) + \alpha V_2(t, x_2 + J_\sigma(x_2)) \\ &\leq (\psi_1)_k(V_1(t, x_1)) + \alpha(\psi_2)_\sigma(V_2(t, x_2)) \\ &= (d_1)_k V_1(t, x_1) + \alpha (d_2)_\sigma V_2(t, x_2) \\ &\leq \gamma_{k,\sigma} W,\end{aligned}$$

such that $\gamma_{k,\sigma=} \max\{(d_1)_k, (d_2)_\sigma\}$.
By Comparison lemma we get

$$W \leq W(t_0) \left(\prod_{t_0 < t_i < t} (1 + \gamma_i) \right) e^{\int_{t_0}^{t} -\lambda ds} + \int_{t_0}^{t} \left(\prod_{s < t_i < t} (1 + \gamma_i) \right) e^{\int_{s}^{t} -\lambda du} r ds.$$

Note that the convergence of the series $\sum_i \gamma_i$ is equivalent to the convergence of the infinite product $\prod_i (1 + \gamma_i)$, then, there exist positive constant μ such that $\prod_i (1 + \gamma_i) \leq \mu$.
Hence

$$\begin{aligned}W &\leq W(t_0)\mu e^{-\lambda(t-t_0)} + \mu r \int_{t_0}^{t} e^{-\lambda(t-s)} ds \\ &= W(t_0)\mu e^{-\lambda(t-t_0)} + \frac{\mu r}{\lambda} e^{-\lambda t_0}.\end{aligned}$$

Thus, we obtain

$$\|x\| \leq \sqrt[p]{\frac{(c_2+\delta b_2)}{(c_1+\delta b_1)}}\,\|x(t_0)\|^{\frac{q}{p}}\,\mu e^{\frac{-\lambda(t-t_0)}{p}} + \sqrt[p]{\frac{\mu r}{\lambda\,(c_1+\delta b_1)}} e^{-\frac{\lambda t_0}{2}}.$$

We conclude that (1) is exponentially practically stable. □

3.3.1 Example

Consider the system

$$\begin{cases} \dot{x}_1 = -\frac{1}{4}x_1^{\frac{3}{2}} + \frac{x_1^{\frac{3}{2}}}{1+x_1^2}e^{-t^2} + \frac{1}{1+t^2}x_2, t \neq t_k \\ \Delta x_1 = I_k(x_1) = \alpha_k x_1, t = t_k, \alpha_k \in R, k = 1, 2... \\ \dot{x}_2 = -x_2^{\frac{3}{2}} + 2x_2^{\frac{3}{2}}e^{-x_2^2}\ t \neq \tau_\sigma \\ \ \Delta x_2 = J_\sigma(x_2) = \beta_\sigma x_2,\ t = \tau_\sigma, \beta_\sigma \in \mathbb{R}, \sigma = 1, 2.... \end{cases} \tag{3.12}$$

In this case,

$$\begin{aligned} f_1(t,x_1) &= -\frac{1}{4}x_1^{\frac{3}{2}} + \frac{x_1^{\frac{3}{2}}}{1+x_1^2}e^{-t^2}, \\ f_2(t,x_2) &= -x_2^{\frac{3}{2}} + \frac{1}{2}x_2^{\frac{3}{2}}e^{-x_2^2}, \\ h(t,x) &= \frac{1}{1+t^2}. \end{aligned}$$

Set $V_1(t,x_1) = x_1^{\frac{5}{4}}$ and $V_2(t,x_2) = x_2^{\frac{5}{4}}$.
Verification of assumption H_3). We have

$$\begin{aligned} \|x_1\|^{\frac{5}{4}} &\leq V_1(t,x_1) \leq \|x_1\|^{\frac{3}{2}}, \\ \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1}f_1(t,x_1) &\leq \frac{5}{4}x_1^{\frac{1}{4}}(-\frac{1}{4}x_1^{\frac{3}{2}} + \frac{x_1^{\frac{3}{2}}}{1+x_1^2}e^{-x_1^2}) \\ &\leq -15x_1^{\frac{7}{4}} \\ &\leq -15\,\|x_1\|^{\frac{7}{4}}. \\ V_1(t, x_1 + I_k(x_1)) &= V_1(t, x_1+\alpha_k x_1) \leq \psi_{1k}(V_1(t,x_1)) = (1+\alpha_k)^2\,x_1^2, t = t_k, k = 1, 2, . \end{aligned}$$

With $p = \frac{5}{4},\ \ q = \frac{3}{2}, r = \frac{7}{4}, c_1 = 1,\ c_2 = 1,\ c_3 = 15,\ c_4 = \frac{5}{4}.$

Verification of assumption $H4$). We have

$$\begin{array}{rcl} \|x_2\|^{\frac{5}{4}} & \leq & V_2(t,x_2) \leq \|x_2\|^{\frac{3}{2}} \\ \dfrac{\partial V_2}{\partial t} + \dfrac{\partial V_2}{\partial x_2} f_2(t,x_1) & \leq & \dfrac{5}{4} x_2^{\frac{1}{4}} (-x_2^{\frac{3}{2}} + \dfrac{1}{2} x_2^{\frac{3}{2}} e^{-x_2^2}) \\ & \leq & -\dfrac{5}{8} x_2^{\frac{7}{4}} \\ & \leq & -\dfrac{5}{8} \|x_2\|^{\frac{7}{4}}, \\ V_2(t, x_2 + J_\sigma(x_2)) & = & V_2(t, x_2 + \beta_\sigma x_2) = (1+\beta_\sigma)^2 x_2^2 \leq \psi_{2\sigma}(V_2(t,x_2)), t = \tau \end{array}$$

with $p = \frac{5}{4}$, $q = \frac{3}{2}$, $r = \frac{7}{4}$, $b_1 = 1$, $b_2 = 1$, $b_3 = \frac{5}{8}$.
Therefore, we can apply Theorem 3.3.3 to prove that system (3.12) is uniformly exponentially practically stable.

Theorem 3.3.4. *Assume that the hypotheses (H_3) and (H_4) are checked, and that there are positive constants L, ε such as*

$$\|h(t,x)\| \leq \varepsilon \|x\| + L, \quad \textit{for all} \ (t,x) \in R^+ \times D,$$

where D is open from $\mathbb{R}^n$. The system (1) is uniformly exponentially practically stable.

Proof. As V is positive and decreasing then V is bounded.
We consider W defined by (3.8) and its derivative along the trajectories of the system (1) given by (3.9), then using conditions (i) and (ii) we get

$$\begin{array}{rcl} \dot{W} & \leq & -c_3 \|x_1\|^r + c_4(\varepsilon \|x\| + L) \|x_1\| \|x_2\| - b_3 \|x_2\|^r \\ & \leq & -\beta_1 \|x_1\|^r - b_3 \|x_2\|^r \\ & \leq & -\mu_1 V_1^{\frac{r}{q}} - \mu_2 V_2^{\frac{r}{q}}, \end{array}$$

with $\mu_1 = \dfrac{\beta_1}{c_2^{\frac{r}{q}}}$, $\mu_2 = \dfrac{b_3}{b_2^{\frac{r}{q}}}$.
Note that

$$\dot{W} \leq -\mu_1 V_1^{\frac{r}{q}} + \mu_1 V_1 - \mu_1 V_1 - \mu_2 V_2^{\frac{r}{q}} + \mu_2 V_2 - \alpha \mu_2 V_2.$$

Then

$$\dot{W} \leq -\mu W + \mu_1 (V_1 - V_1^{\frac{r}{q}}) + \mu_2 (V_2 - V_2^{\frac{r}{q}}) + K,$$

with $\mu = \min\{\mu_1, \mu_2\}$. The limits of V_1 and V_2 leads to the existence of $\rho > 0$ such that:

$$\mu_1 (V_1 - V_1^{\frac{r}{q}}) + \mu_2 (V_2 - V_2^{\frac{r}{q}}) + K \leq \rho.$$

So

$$\dot{W} \leq -\mu W + \rho. \tag{3.13}$$

We also have

$$V_1(t, x_1 + I_k(x_1)) \leq \psi_{1k}(V_1(t, x_1)), t = t_k, k = 1, 2, ...$$

and

$$V_2(t, x_2 + J_\sigma(x_2)) \leq \psi_{2\sigma}(V_2(t, x_2)), t = \tau_\sigma, \sigma = 1, 2, ...$$

Then

$$\begin{aligned} W(t, x_1 x_1 + I_k(x_1), x_2 + J_\sigma(x_2)) &\leq V_1(t, x_1 + I_k(x_1)) + V_2(t, x_2 + J_\sigma(x_2)) \\ &\leq (\psi_1)_k(V_1(t, x_1)) + (\psi_2)_\sigma(V_2(t, x_2)) \\ &= (d_1)_k V_1(t, x_1) + (d_2)_\sigma V_2(t, x_2) \\ &\leq \gamma_{k,\sigma} W, \end{aligned}$$

such as $\gamma_{k,\sigma=} \max((d_1)_k, (d_2)_\sigma)$. By Comparison lemma we get

$$W \leq W(t_0) \left(\prod_{t_0 < t_i < t} (1 + \gamma_i) \right) e^{\int_{t_0}^{t} -\lambda ds} + \int_{t_0}^{t} \left(\prod_{s < t_i < t} (1 + \gamma_i) \right) e^{\int_{s}^{t} -\lambda du} r ds.$$

Hence

$$\begin{aligned} W &\leq W(t_0) \mu e^{-\lambda(t - t_0)} + \mu r \int_{t_0}^{t} e^{-\lambda(t-s)} ds \\ &= W(t_0) \mu e^{-\lambda(t - t_0)} + \frac{\mu r}{\lambda} e^{-\lambda t_0}. \end{aligned}$$

Thus, we obtain

$$\|x\| \leq \sqrt[p]{\frac{(c_2 + \delta b_2)}{(c_1 + \delta b_1)}} \|x(t_0)\|^{\frac{q}{p}} \mu e^{\frac{-\lambda(t - t_0)}{p}} + \sqrt[p]{\frac{\mu r}{\lambda (c_1 + \delta b_1)}} e^{-\frac{\lambda t_0}{2}},$$

We conclude that (1) is uniformly exponentially practically stable. □

CONCLUSION

In this thesis, as set out in the objectives of the research, we investigated the qualitative properties of solutions of certain classes of cascade impulsive differential systems. The Lyapunov second method was used (the use of Lyapunov functions) which remains one of the most effective method to discuss the concepts of stability.

Variants of the tool employed in these studies had been employed extensively by researchers to study the qualitative properties of solutions of these classes of differential systems. We give some sufficient conditions which guarantee practical stability and practical exponential stability of nonlinear time-varying cascade impulsive systems. In this way, we extend some existing results under more generalized assumptions.

Open Problems: The following are the open problems for further research:

- Better and easier ways to construct a Lyapunov function to impulsive cascade differential systems of higher orders.

- Study of the practical stability of impulsive cascade systems with delay.

- Due to high growth in the use of Lyapunov method, if possible, study practical stability of cascade systems of fractional-order with Caputo Derivatives.

- Find new criteria for stability of triangular higher order fractional differential equations.

The identified problems are subjects to be considered in the near future.

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ملخص

الهدف الرئيسي من أطروحتنا هو إجراء دراسة حول الاستقرار العملي والاستقرار العملي الأسي للأنظمة الاندفاعية المتتالية.

مساهمتنا هي تحسين وتعميم بعض الأعمال الموجودة ذات الصلة.

Abstract

The main objective of our thesis is to carry out a study on the practical stability and exponential practical stability of cascading impulsive systems.

Our contribution is the improvement and generalization of some relevant existing works.

Résumé

L'objectif principal de notre thèse est de réaliser une étude sur la stabilité pratique et la stabilité pratique exponentielle des systèmes impulsives en cascade.

Notre contribution est l'amélioration et la généralisation de certains travaux pertinents

existants.