N° d'ordre :

RÉPUBLIQUE ALGÉRIENNE DÉMOCRATIQUE & POPULAIRE

MINISTÈRE DE L'ENSEIGNEMENT SUPÉRIEUR & DE LA RECHERCHE SCIENTIFIQUE

UNIVERSITÉ DJILLALI LIABES
FACULTÉ DES SCIENCES EXACTES
SIDI BEL-ABBÈS

THÈSE DE DOCTORAT EN SCIENCES

Présentée par

Hanene ZERKOUK

Spécialité : MATHÉMATIQUES

Option : ÉQUATIONS AUX DÉRIVÉES PARTIELLES

Intitulée

Étude qualitative de quelque équation d'évolution du second ordre.

Soutenue le 17/04/ 2022 devant le jury composé de :

Président : Ali HAKEM Professeur à l'Université de Sidi Bel-Abbés.

Examinateurs :

Selma BAGHLI-BENDIMERAD Professeur à l'université de Sidi Bel-Abbés.

Nadia MEZOUAR MCA à l'Université de Mascara.

Mounir BAHLIL MCA à l'Université de Mascara.

Abderrahmane BENIANI MCA à l'Université d'Aïn-Temouchent.

Directeur de thèse : Abbès BENAISSA Professeur à l'Université de Sidi Bel-Abbés.

Dedication

I dedicate this thesis to:

- The memory of my father-in-law ami Mahmoud.

- My father Salah and my mother Rezkia.

- My mother-in-law Rabiaa.

-My husband Samir and my sweethearts, Ahmed and Adem.

My brothers and sisters, my brothers-in-law and sisters-in-law and their families and especially to the last one in the family: little Karim.

-My cousins: Meriem, Meriem, Nawel, Hadjira, Zineb.....

-All my family.

-My friends: Nabila, Amina, Mokhtara...
-My colleagues at the Higher School of Economics in Oran.

Acknowledgements

I would like to thank my supervisor Mr Abbes BENAISSA Professor at the Djillali Liabes University of Sidi Belabes, who knew with his knowledge, his seriousness and his kindness, to guide me in my work, his useful suggestions allowed me to complete my work.

I also thank Pr. Ali HAKEM of the University of Djillali Liabes of Sidi Belabes for having done me the honour of chairing the jury.

Pr Selma BAGHLI-BENDIMERAD of the University of Djillali Liabes of Sidi Belabes, Ms Nadia MEZOUAR of the University of Mascara, Mr Mounir BAHLIL of the University of Mascara and Mr Abderrahmane BENIANI of the University of Ain-Temouchent are warmly thank for having agreed to examine my work and to be a member of the jury.

Finally, I thank any person who has contributed, in any way, to the realisation of this work.

Contents

Introduction

Many physical phenomena in nature can be described by partial differential equations and the control of such equations is a quite recent and very active field of investigation. The aim of this dissertation is to survey several issues related to the study of the Lamé system under fractional controls.

The problem of well-posedness and stability for elasticity systems in general, and the wave equation in particular, has attracted considerable attention in recent years, where diverse types of dissipative mechanisms have been introduced and several stability and boundedness results have been obtained. The main problem concerning the stability of solutions is to determine and estimate the best decay rate for solutions.

Real progress has been realized during the last three decades, Let us recall here some known results addressing problems of existence, uniqueness and asymptotic behavior of solutions.

In particular, in the works of Haraux and Chentouf **[37]**, **[20]**, considering the problem of observability, exact controllability and stability of general elasticity systems with variable coeffcients depending on both time and space variables in bounded domains, the results hold under linear or nonlinear, global or local feedbacks, and they generalize and improve, in some cases, the decay rate obtained by Alabau and Komornik **[41]**.

This thesis focuses on fractional calculus which has been applied successfully in various areas to modify many existing models of physical processes such as heat conduction, diffusion, viscoelasticity, wave propagation, electronics etc. Caputo and Mainardi [10] have established the relation between fractional derivative and theory of viscoelasticity. The feedback under consideration here is of fractional type and is described by the following fractional derivative:

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\, ds, \qquad \eta \geq 0. \tag{1}$$

The order of the derivative is between 0 and 1. In addition to being nonlocal, fractional derivative involves singular and non-integrable kernels ($t^\alpha, 0 < \alpha < 1$). It has been shown (see **[49]**) that, as ∂_t, the fractional derivative ∂_t^α forces the system to become dissipative and the solution to approach the equilibrium state.

Furthermore, This thesis intended also to state the well-posedness result for the wave equation using the theory of semigroups. Linear semigroup theory received considerable attention in the 1930s as a new approach in the study of linear partial differential equations. Note that the linear semigroup theory has been later developed as an independent theory, with applications in some other fields, such as ergodic theory, the theory of Markov processes, etc.

Outline: This dissertation is split into three chapters.

CHAPTER 1: Preliminaries

In this chapter, we present some well known results, definitions, properties and theorems that are used throughout the dissertation. Firstly, we recall some basic knowledge on linear operators and semigroups without proofs, including some theorems on strong, exponential and polynomial stability of C_0-semigroups. Next, we display a brief historical introduction to fractional derivatives and we define the fractional derivative operator in the sense of Caputo. After that, we introduce some preliminary facts on the Bessel functions and lastly, we define two different types of geometric conditions.

CHAPTER 2: On the Stability of a DegenerateWave Equation Under Fractional Feedbacks Acting on the Degenerate Boundary

This Chapter is devoted to the study of boundary stabilization of fractional type for degenerate wave equation of the form

$$(P) \qquad \begin{cases} u_{tt}(x,t) - (x^\gamma u_x(x,t))_x = 0 & \text{in } (0,1)\times(0,+\infty), \\ (x^\gamma u_x)(0,t) = \varrho \partial_t^{\alpha,\eta} u(0,t) & \text{in } (0,+\infty), \\ u(1,t) = 0 & \text{in } (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } (0,1), \end{cases}$$

where $\gamma \in [0,1)$ and $\varrho > 0$. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α, $(0 < \alpha \le 1)$, with respect to the time variable (see **[21]**). It is defined as follows

$$\partial_t^{\alpha,\eta} u(t) = \begin{cases} u_t(t) & \text{for } \alpha = 1, \eta \ge 0, \\ \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{du}{ds}(s)\, ds, & \text{for } 0 < \alpha < 1, \eta \ge 0. \end{cases}$$

We will explain the influence of the relation between the degenerate coefficient and the fractional feedback on decay estimates.

CHAPTER 3: Global Existence and Asymptotic Behavior of the Solutions to a Class of Nonlinear Second Order ODE With Delay Term

This Chapter is devoted to the study of following scalar nonlinear second order ODE with delay term of the type

$$u'' + c_1|u'(t)|^{\alpha}u'(t) + c_2|u'(t-\tau)|^{\alpha}u'(t-\tau) + c_3|u|^{\beta}u = 0$$

where α, β, c_1, c_2 et τ are positive constants.

We prove the global existence of its solutions in energy spaces by means of the energy method under a condition between the weight of the delay term in the feedback and the weight of the term without delay. Furthermore, we study the asymptotic behavior of solutions using multiplier method and general weighted integral inequalities.

Chapter 1

Preliminaries

1.1 Reminder on Real and complex analysis

Definition 1.1.1 *A function f defined on a real interval I is said to be absolutely continuous if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite sequence of sub-intervals $[a_n, b_n]_{n \leq N}$ of I with disjoint interiors we have*

$$\sum_{n=0}^{N}(b_n - a_n) \leq \delta \Rightarrow \sum_{n=0}^{N} |F(b_n) - F(a_n)| < \varepsilon$$

Proposition 1.1.1 ([57]) *If $I = [a, b]$ then we have the equivalnces*

1. *f is absolutely continuous on I*
2. *$\exists g \in L^1(I)$ such that*
$$\int_a^x g(t)dt = f(x) - f(a), \forall x \in I$$
3. *f is derivable almost everywhere, and its derivative f' satisfies*
$$\int_a^x f'(t)dt = f(x) - f(a), \text{ almost everywhere}$$

Definition 1.1.2 *A function $f : \mathbb{R} \to \mathbb{R}$ is said to bo locally absolutely continuous if f is absolutely continuous on every $[a, b] \subset \mathbb{R}$.*

Theorem 1.1.1 (Rouche's theorem, stronger version [32]) *Let f and g be holomorphic functions in a domain G. If $K \subset G$ is a bounded region with continuous boundary ∂K and*

$$|f(z) - g(z)| < |g(z)| \qquad \forall z \in \partial K$$

then f and g have the same number of roots (counting multiplicity) in K.

Definition 1.1.3 *For a complex number z with $\Re(z) > 0$ we set*

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt$$

The function $z \to \Gamma(z)$ is said the gamma function.

Proposition 1.1.2 ([28]) *For $\Re(z) > 0$, we have*

1. $$\Gamma(z) = 2\int_0^{+\infty} x^{2z-1} e^{-x^2} dx$$

2. $$\Gamma(z) = \int_0^{1} (-ln(x))^{z-1} dx$$

3. $$\Gamma(z+1) = z\Gamma(z)$$

4. $$\Gamma(n+1) = n!, \forall n \in \mathbb{N}$$

Definition 1.1.4 *Let Ω be an open subset of $\mathbb{R}^n$. For $p \in [1, +\infty[$ and $m \in \mathbb{N}$, the set*

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) | D^\alpha u \in L^p(\Omega), \forall \alpha \in \mathbb{N}^n, |\alpha| \leq m\}$$

is called the (m,p)-Sobolev space on Ω.

$W^{m,p}$ is a Banach space when equipped with the norm

$$\| u \|_{W^{m,p}} = (\sum_{|\alpha| \leq m} \| D^\alpha u \|_{L^p}^p)^{\frac{1}{p}}$$

Proposition 1.1.3 ([14]) *The space $W^{1,1}(]0,1[)$ is equal to the set of absolutely continuous fonctions on $]0,1[$.*

$$W^{1,1}(]0,1[) = \{f :]0,1[\to \mathbb{R} | f \textit{ is absolutely continuous}\}$$

Definition 1.1.5 *For $p = 2$, the Sobolev space $W^{m,2}(\Omega)$ is denoted $H^m(\Omega)$*

$$W^{m,2}(\Omega) = H^m(\Omega)$$

$H^m(\Omega)$ is a Hilbert space, where the scalar product is given by

$$< u, v >_{H^m(\Omega)} = \sum_{|\alpha| \leq m} < D^\alpha u, D^\alpha v >_{L^2(\Omega)}$$

Theorem 1.1.2 (Rellich-Kondrachov theorem [14]) *If Ω is a bounded open set of $\mathbb{R}^n$ with a C^1-boundary, then any bounded sequence in $H^1(\Omega)$ admits a convergent subsequence in $L^2(\Omega)$.*

This theorem stats that any bounded set of $H^1(\Omega)$ is compact in $L^2(\Omega)$. We said that the canonical injection $i : H^1(\Omega) \to L^2(\Omega)$ is compact.

1.2 Linear Operators

The proofs of the following results can be found in [26] and [14].

Definition 1.2.1 *Let X and Y be two Banach spaces. A linear mapping:*

$$\mathcal{A} : D(\mathcal{A})(\subset X) \to Y$$

is called a linear operator. The $D(\mathcal{A}) \subset X$ is called the domain of $\mathcal{A}$ and $\mathcal{R}(\mathcal{A}) \subset Y$ is called the range of $\mathcal{A}$:

$$\mathcal{R}(\mathcal{A}) = \{\mathcal{A}x \mid x \in D(\mathcal{A})\}.$$

$\mathcal{A}$ is said to be one-to-one (or injective) if $\mathcal{A}x = 0$ if and only if $x = 0$; $\mathcal{A}$ is said to be onto (or surjective) if $\mathcal{R}(\mathcal{A}) = Y$; $\mathcal{A}$ is said to be densely defined if

$$\overline{D(\mathcal{A})} = X$$

.

Definition 1.2.2 *A linear operator $\mathcal{A}$ is said to be closed if for any $(x_n)_{n \geq 1} \subset D(\mathcal{A})$ such that*

$$x_n \to x, \;\; \mathcal{A}x_n \to y, \;\; \text{as } n \to \infty,$$

then $x \in D(\mathcal{A})$ and $\mathcal{A}x = y$. $\mathcal{A}$ is said to be bounded if $D(\mathcal{A}) = X$ and $\mathcal{A}$ maps a bounded set of X into a bounded set of Y. A linear operator is bounded if and only if it is continuous, that is,

$$x_n \to x_0 \in X \Longrightarrow \mathcal{A}x_n \to \mathcal{A}x_0 \in Y$$

for any $(x_n)_{n \geq 1} \subset X$.

Obviously, any operator which has bounded inverse must be closed. All the bounded operators from X to Y are denoted by $\mathcal{L}(X,Y)$. In particular, when $X=Y$, $\mathcal{L}(X,Y)$ is abbreviated as $\mathcal{L}(X)$.

Theorem 1.2.1 *Let X and Y be Banach spaces. Then $\mathcal{L}(X,Y)$ is a Banach space with the norm*

$$\|\mathcal{A}\| = \sup\left\{\|\mathcal{A}x\| \mid x\in X,\ \|x\|=1\right\}.$$

Definition 1.2.3 *Let X be a Banach space. If $Y=\mathbb{R}$ or $Y=\mathbb{C}$, then the operator in $\mathcal{L}(X,Y)$ is called a linear functional on X. A bounded functional is also denoted by f.*

By Theorem 1.2.1, all linear bounded functionals on X consist of a Banach space which is called the dual of the space X, denoted by X^*.

A bounded operator is called compact operator if $\mathcal{A}$ maps any bounded set into a relatively compact set which is a compact set but not necessarily closed. For a closed operator $\mathcal{A}$, we can define the graph space $[D(\mathcal{A})]$ where the norm is defined by

$$\|x\|_{[D(\mathcal{A})]} = \|x\| + \|\mathcal{A}x\|, \quad \forall x\in D(\mathcal{A}). \tag{1.1}$$

Let X and Y be two normed spaces. If there exists a one-to-one linear operator $\mathcal{A}$ mapping X into Y having the property $\|\mathcal{A}x\|_Y = \|x\|_X$ for every $x\in X$ and $y\in Y$, then we call $\mathcal{A}$ an isometric isomorphism between X and Y, and we say that X and Y are isometrically isomorphic.

Definition 1.2.4 *An operator sequence $\{\mathcal{A}_n\}\subset\mathcal{L}(X,Y)$ is said to be convergent to an operator $\mathcal{A}\in\mathcal{L}(X,Y)$ in terms of the operator norm, if*

$$\|\mathcal{A}_n-\mathcal{A}\| \ \to 0 \text{ as } n\to\infty.$$

$(\mathcal{A}_n)_n$ is said to be strongly convergent to $\mathcal{A}\in\mathcal{L}(X,Y)$, if for all $x\in X$,

$$\mathcal{A}_n x \ \to Ax \text{ as } n\to\infty.$$

$(\mathcal{A}_n)_n$ is said to be weak convergent to $\mathcal{A}\in\mathcal{L}(X,Y)$, if for all $f\in Y^*$,*

$$f(\mathcal{A}_n x) \ \to f(Ax) \text{ as } n\to\infty.$$

which is generally denoted by

$$\langle \mathcal{A}_n x, f\rangle \ \to \langle \mathcal{A}x, f\rangle \ \text{as}\ n \to \infty.$$

where $\langle ., . \rangle$ *stands for the duality product between* X *and* X^**, that is,* $< x, f >$ *simply means that*

$$\langle x, f\rangle \ = \langle x, f\rangle_{X,X^*} = f(x).$$

Theorem 1.2.2 [Banach inverse theorem]
Let X *and* Y *be two Banach spaces. If a linear operator* $\mathcal{A} : X \to Y$ *defined on the whole space* X *is an invertible and onto mapping, then* $\mathcal{A}^{-1} \in \mathcal{L}(Y, X)$*.*

Theorem 1.2.3 [Open mapping theorem]
Let X *and* Y *be Banach spaces and let* $\mathcal{A}$ *be a bounded operator from* X *to* Y*. If* $\mathcal{R}(\mathcal{A}) = Y$*, then* $\mathcal{A}$ *maps an open set of* X *into an open set of* Y*.*

Theorem 1.2.4 [Closed graph theorem]
Suppose that $\mathcal{A}$ *is a closed operator in a Banach space* X*. Then* $\mathcal{A}$ *must be bounded provided* $D(\mathcal{A}) = X$*.*

Theorem 1.2.5 [Uniform convergence theorem]
Let X *and* Y *be Banach spaces. Suppose that* $\{T_n\} \subset \mathcal{L}(X, Y)$*. If*

$$\sup_n \{\|T_n x\|\} < \infty, \ \forall x \in X,$$

then

$$\sup_n \{\|T_n\|\} < \infty.$$

Let $\mathcal{A}$ *be a linear operator in a Banach space.* $\mathcal{A}$ *is said to be densely defined in* X *if* $D(\mathcal{A})$ *is dense in* X*. For a densely defined operator* $\mathcal{A}$*, there exists a unique operator* $\mathcal{A}^*$ *defined in* X^**, which is called the adjoint operator of* $\mathcal{A}$ *satisfying*

$$\langle \mathcal{A}x, y\rangle = \langle x, \mathcal{A}^* y\rangle, \ \forall x \in D(\mathcal{A}), \ y \in D(\mathcal{A}^*),$$

where

$$D(\mathcal{A}^*) = \{f \in X^* \mid \exists z \in X^* \ \text{such that} \langle \mathcal{A}x, f\rangle = \langle x, z\rangle, \ \forall x \in D(\mathcal{A})\}.$$

When X is a Hilbert space, we consider $X^* = X$ attributed to the following Riesz representation theorem.

Theorem 1.2.6 [Riesz representation theorem]
Suppose that H is a Hilbert space. Then $f \in H^*$ if and only if there is an $x \in H$ such that

$$f(y) = \langle y, x \rangle, \ \forall x, y \in H$$

Definition 1.2.5 *A closed linear operator $\mathcal{A} : X \to Y$ between two Banach spaces is said to be of Fredholm if*

1. $D(\mathcal{A})$ *is dense in* X.
2. $\dim \ker \mathcal{A} < +\infty$
3. $\dim$ *coker* $\mathcal{A} := \dim(Y/ran\mathcal{A}) < +\infty$

In this case, the index of Fredholm of $\mathcal{A}$ is

$$ind\ \mathcal{A} = \dim \ker \mathcal{A} - \dim\ coker\ \mathcal{A}$$

Theorem 1.2.7 (Fredholm's alternative) *If $\mathcal{A} : X \to X$ is a compact linear operator on a Banach space, then, exactly one of the following assertions holds*

- $I - \mathcal{A}$ *is surjective (and hence it is bijective).*
- $\dim \ker I - \mathcal{A} > 0$.

Theorem 1.2.8 [Lax Milgram theorem]
Let $a(x, y)$ be a sesquilinear form, that is, it is linear in x and conjugate linear in y, and satisfies

- *there is an $M > 0$ such that $|a(x, y)| \leq M\|x\|\|y\|$ for all $x, y \in H$;*
- *there is a $\delta > 0$ such that for any $x \in H$, $|a(x, x)| \geq \delta\|x\|^2$.*

Then there exists a unique $\mathcal{A} \in \mathcal{L}(H)$ which is bounded invertible and satisfies

$$a(x, y) = \langle x, Ay \rangle, \ \forall x, y \in H.$$

Definition 1.2.6 *A linear operator in a Hilbert space is said to be symmetric if*

$$\mathcal{A}^* = \mathcal{A} \text{ on } D(\mathcal{A}) \text{ and } D(\mathcal{A}^*) \supseteq D(\mathcal{A})$$

A symmetric operator is said to be self-adjoint, if $\mathcal{A}^* = \mathcal{A}$.

For bounded operators, the symmetric and self-adjoint are the same. But for unbounded operators, they are different.

Definition 1.2.7 *A linear operator $\mathcal{B}$ in a Hilbert space H is said to be A-bounded if*

- $D(\mathcal{B}) \supset D(\mathcal{A})$, *and*
- *there are $a, b > 0$ such that*

$$\|\mathcal{B}x\| \leq a\|\mathcal{A}x\| + b\|x\|, \ \forall x \in D(\mathcal{A}).$$

Theorem 1.2.9 [Kato-Rellich theorem]
Let $\mathcal{A}$ be a self-adjoint operator in a Hilbert space H and $\mathcal{B}$ be symmetric and A-bounded, such that

$$\|\mathcal{B}x\| \leq a\|\mathcal{A}x\| + b\|x\|, \ \forall x \in D(\mathcal{A}), \quad 0 < a < 1, b > 0$$

then $\mathcal{A} + \mathcal{B}$ is self-adjoint in $D(\mathcal{A})$. In particular, when $\mathcal{B}$ is bounded, $\mathcal{A} + \mathcal{B}$ is self-adjoint.

Definition 1.2.8 *Let $\mathcal{A} \in \mathcal{L}(H)$ be a self-adjoint operator in a Hilbert space H. $\mathcal{A}$ is said to be positive if*

$$\langle \mathcal{A}x, x\rangle \geq 0, \ \forall x \in H. \tag{1.2}$$

A positive operator is denoted by $\mathcal{A} \geq 0$; $\mathcal{A}$ is said to be positive definite if the equality in 1.2 holds true only if $x = 0$, which is denoted by $\mathcal{A} > 0$; A positive operator $\mathcal{A}$ is said to be strictly positive if there exists an $m > 0$ such that

$$\langle \mathcal{A}x, x\rangle \geq m\|x\|^2, \ \forall x \in D(\mathcal{A}). \tag{1.3}$$

1.3 The spectrum of linear operators

The proofs of the following results can be found in [26] and [14].

Definition 1.3.1 *Suppose that X is a Banach space and $\mathcal{A} : D(\mathcal{A})(\subset X) \to X$ is a linear operator. The resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$ is an open set in the complex plane, which is defined by*

$$\rho(\mathcal{A}) = \left\{\lambda \in \mathbb{C} \mid (\lambda - \mathcal{A})^{-1} \in \mathcal{L}(X)\right\}.$$

When $\lambda \in \rho(\mathcal{A})$, the operator $R(\lambda, \mathcal{A}) = (\lambda - \mathcal{A})^{-1}$ is called the resolvent of $\mathcal{A}$. If one of resolvents is compact, then any of the resolvents must be compact. This comes from the following resolvent formula:

$$(\lambda - \mathcal{A})^{-1} - (\mu - \mathcal{A})^{-1} = (\mu - \lambda)(\lambda - \mathcal{A})^{-1}(\mu - \mathcal{A})^{-1}, \qquad \forall \lambda, \mu \in \rho(\mathcal{A})$$

The spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ is the supplement set of the resolvent set in the complex plane, that is,

$$\sigma(\mathcal{A}) = \mathbb{C} \backslash \rho(\mathcal{A}).$$

Generally, the spectrum $\sigma(\mathcal{A})$ is decomposed into three parts:

$$\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) \cup \sigma_c(\mathcal{A}) \cup \sigma_r(\mathcal{A})$$

where

- *the point spectrum*

$$\sigma_p(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid \exists x \in X \backslash \{0\} \text{ so that } \mathcal{A}x = \lambda x\};$$

- *the continuous spectrum*

$$\sigma_c(\mathcal{A}) = \left\{\lambda \in \mathbb{C} \mid (\lambda - \mathcal{A}) \text{ is invertible and } \overline{\mathcal{R}(\lambda - \mathcal{A})} = X\right\};$$

- *the residual spectrum*

$$\sigma_r(\mathcal{A}) = \left\{\lambda \in \mathbb{C} \mid (\lambda - \mathcal{A}) \text{ is invertible and } \overline{\mathcal{R}(\lambda - \mathcal{A})} \neq X\right\};$$

When $\lambda \in \sigma_p(\mathcal{A})$, any nonzero vector x satisfying $\mathcal{A}x = \lambda x$ is said to be an eigenvector (it is also called eigenfunction if the space is a function space) of $\mathcal{A}$. For a matrix in $\mathbb{C}^n$, the spectrum is just the set of eigenvalues.

1.4 Semigroups of linear operators

The proofs of the following results can be found in [26].

Definition 1.4.1 *Semigroup theory is aiming to solve the following linear differential equation in Banach space X:*

$$\begin{cases} \dot{u}(t) = \mathcal{A}u(t), \ t > 0, \\ u(0) = x \in X, \end{cases} \tag{1.4}$$

where $\mathcal{A} : D(\mathcal{A})(\subset X) \to X$ is a linear operator.

Equation (1.4) is said to be well-posed (for bounded A) If:

- *for any initial value $x \in D(\mathcal{A}) = X$, there exists a solution $u(x,t)$ to (1.4) which is differentiable for $t > 0$, continuous at $t = 0$ and $u(x,t)$ satisfies (1.4) for $t > 0$,*

- *$u(x,t)$ depends continuously on the initial condition x, that is:*

$$x \to 0 \text{ implies } u(x,t) \to 0 \text{ for each } t > 0;$$

- *$u(x,t)$ is unique for each $x \in D(\mathcal{A}) = X$.*

We can then define an operator T by

$$T(t)x = u(x,t) \text{ for each } t \geq 0.$$

From the existence and uniqueness of the solution $u(x,t)$, we know that $T(t)$, $t \geq 0$ is well defined on X.

Definition 1.4.2 *Let X be a Banach space and $T(t) : X \to X$ be a family of linear bounded operators, for $t \geq 0$, $T(t)$ is called a semigroup of linear bounded operators, or simply a semigroup, on X if*

- $T(0) = I$;
- $T(t+s) = T(t)T(s), \ \forall t \geq 0, \ \forall s \geq 0$

A semigroup $T(t)$ is called uniformly continuous if

$$\lim_{t \to 0} \|T(t) - I\| = 0,$$

and is called strongly continuous, (or C_0-semigroup for short), if

$$\lim_{t \to 0} T(t)x - x = 0, \ \forall x \in X$$

Definition 1.4.3 *Let $T(t)$ be a C_0-semigroup on a Banach space X. The operator $\mathcal{A}$ defined as*

$$\begin{cases} \mathcal{A}x = \lim\limits_{t \longrightarrow 0} \dfrac{T(t)x - x}{t}, & \forall x \in D(\mathcal{A}), \\ D(\mathcal{A}) = \left\{ x \in X \mid \lim\limits_{t \longrightarrow 0} \dfrac{T(t)x - x}{t} \quad \text{exists} \right\} \end{cases}$$

is called the infinitesimal generator of the C_0-semigroup $T(t)$.

Theorem 1.4.1 *Let X be a Banach space. For any bounded linear operator $\mathcal{A}$ on X,*

$$T(t) = e^{\mathcal{A}t}$$

is a uniformly continuous semigroup and $\mathcal{A}$ is the infinitesimal generator of $T(t)$ and we have $D(\mathcal{A}) = X$.

Theorem 1.4.2 *Let $T(t)$ be a C_0-semigroup on a Banach space X, then the following holds*

- *There exists constants $M > 1$ and $\omega \geq 0$ such that*

$$\|T(t)\| \leq Me^{\omega t}, \ \forall t \geq 0$$

- *Suppose that $\mathcal{A}$ is the generator of $T(t)$. Then*

$$\{\lambda \in \mathbb{C} \mid Re(\lambda) > \omega\} \subset \rho(\mathcal{A}).$$

- *In addition, if $Re(\lambda) > \omega$, then*

$$\mathcal{R}(\lambda; \mathcal{A})x = (\lambda - \mathcal{A})^{-1}x = \int_0^\infty e^{-\lambda t} T(t)x dt, \ \forall x \in X.$$

- *$T(t)$ is strongly continuous on X. i.e. for any $x \in X$, the map $t \to T(t)x$ is continuous.*

Theorem 1.4.3 *Let $\mathcal{A}$ be the generator of a C_0-semigroup $T(t)$ on a Banach space X. we have the following*

- *$D(\mathcal{A})$ is dense in X*
- *$\mathcal{A}$ is a closed operator.*
- *For any $n \geq 1$, $D(\mathcal{A}^n)$ is dense in X. The set*

$$D = \bigcap_{n=1}^{\infty} D(\mathcal{A}^n)$$

  is also dense in X and is invariant under $T(t)$. i.e. for $x \in D$, $T(t)x \in D$ for $t \geq 0$. Moreover, if we define

$$D^\infty = \{x \in X | t \to T(t)x \in C^\infty\}.$$

  then we have $D = D^\infty$

Theorem 1.4.4 *Let $T(t)$ and $S(t)$ be C_0-semigroups, and let $\mathcal{A}$ and $\mathcal{B}$ be their infinitesimal generators, respectively. Then*

$$\mathcal{A} = \mathcal{B} \Rightarrow T(t) = S(t) \ \ \forall t \geq 0.$$

Definition 1.4.4 *Let $T(t)$ be a C_0-semigroup on a Banach space X and let $M \geq 1$ and $\omega \geq 0$.*

If $\|T(t)\| \leq M$ for $t \geq 0$ then $T(t)$ is called uniformly bounded.

Moreover, if we have $M = 1$, then $T(t)$ is called a contraction.

Theorem 1.4.5 [Hille-Yosida]
Let X be a Banach space and let $\mathcal{A}$ be a linear (not necessirely bounded) operator in X. Then, $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions $T(t)$ on X, if and only if

- *$\mathcal{A}$ is closed and $D(\mathcal{A})$ is dense in X*
- *There exist positive constants M and ω verifying the property: for all $\lambda \in \rho(\mathcal{A})$, $\Re\lambda > \omega$, the following holds*
$$\|\mathcal{R}(\lambda, \mathcal{A})^n\| \leq \frac{M}{(\Re\lambda - \omega)^n}, \quad n = 1, 2, ...$$

Corollary 1.4.1 *Let X be a Banach space and let $\mathcal{A}$ be a linear (not necessirely bounded) operator in X. Then, $\mathcal{A}$ is the infinitesimal generator of the C_0-semigroup of contractions $T(t)$ on X, if and only if the following holds.*

- *$\mathcal{A}$ is closed and $D(\mathcal{A})$ is dense in X*
- *For any $\lambda > 0$, $\lambda \in \rho(\mathcal{A})$ and*
$$\|\mathcal{R}(\lambda, \mathcal{A})\| \leq \frac{1}{\lambda}$$
.

Definition 1.4.5 *Let X be a Banach space and let $F(X)$ be the duality set. A linear operator $\mathcal{A}$ in X is said to be dissipative if for every $x \in D(\mathcal{A})$ there is an $x^* \in F(X)$ such that*

$$Re\langle \mathcal{A}x, x^* \rangle \leq 0$$

Corollary 1.4.2 *Let $\mathcal{A}$ be a linear operator in a Banach space X.Then $\mathcal{A}$ is dissipative if and only if*

$$\|x\| \leq \|x - h\mathcal{A}x\|, \text{ for each } h > 0 \text{ and all } x \in D(\mathcal{A}).$$

Definition 1.4.6 *A linear operator $\mathcal{A}$ in a Banach space X is called m-dissipative if $\mathcal{A}$ is dissipative and $\mathcal{R}(\lambda - \mathcal{A}) = X$, for some $\lambda > 0$.*

Remark 1.4.1 *In a Hilbert space H, the dissipativity of $\mathcal{A}$ simply means that*

$$Re\langle \mathcal{A}x, x\rangle \leq 0, \ \forall x \in D(\mathcal{A}).$$

Theorem 1.4.6 [L$\ddot{u}$mer-Phillips]
Let $\mathcal{A}$ be a linear operator in a Banach space X. Then $\mathcal{A}$ generates a C_0-semigroup of contractions on X if and only if

- $\overline{D(\mathcal{A})} = X$.
- *$\mathcal{A}$ is dissipative.*

Corollary 1.4.3 *Let $\mathcal{A}$ be a linear operator in a Banach space X. Then $\mathcal{A}$ generates a C_0-semigroup of contractions on X if and only if*

- *$\mathcal{A}$ is densely defined and closed.*
- *Both $\mathcal{A}$ and $\mathcal{A}^*$ are dissipative.*

Remark 1.4.2 *When X is reflexive, the condition $\overline{D(\mathcal{A})} = X$ can be removed in the Lümer-Phillips theorem.*

1.5 Stability of C_0-semigroups.

The proofs of the following results can be found in [26].

Definition 1.5.1 *Let $T(t)$ be a C_0-semigroup on a Banach space X.*

- *$T(t)$ is said to be exponentially stable, if there exist two positive constants M, $\omega > 0$ such that*
$$\|T(t)\| \leq Me^{-\omega t}, \ \forall t \geq 0.$$
- *$T(t)$ is said to be strongly or asymptotically stable, if*
$$\lim_{t\to+\infty} \|T(t)x\| = 0 \ \forall x \in X.$$
- *$T(t)$ is said to be weakly stable, if*
$$\langle T(t)x, y\rangle \to 0 \ \text{as} \ t \to \infty, \ \forall x \in X, \ y \in X^*.$$

- *$T(t)$ is said to be polynomially stable if there exist two positive constants C and α such that*
$$\|T(t)\| \leq Ct^{-\alpha} \quad \forall t > 0,\ \forall x \in X.$$

Theorem 1.5.1 [Spectral mapping theorem]
Let $T(t)$ be a C_0-semigroup on a Banach space X and $\mathcal{A}$ be its infinitesimal generator. Then

$$e^{t\sigma_p(\mathcal{A})} \subset \sigma_p(T(t)) \subset e^{t\sigma(\mathcal{A})} \cup \{0\}\,.$$

More precisely, if $\lambda \in \sigma_p(\mathcal{A})$. then $e^{\lambda t} \in \sigma_p(T(t))$, and if $e^{\lambda t} \in \sigma_p(T(t))$ then there exists an integer k such that $\lambda_k = \lambda + 2\pi ik/t \in \sigma_p(\mathcal{A})$.

Theorem 1.5.2 *Let $T(t)$ be a C_0-semigroup on a Banach space with generator $\mathcal{A}$. Then*

$$e^{t\sigma(\mathcal{A})} \subset \sigma(T(t)).$$

Proposition 1.5.1 *Let $X = H$ be a Hilbert space. Suppose that $T(t)$ is a weakly stable C_0-semigroup on H. i.e. $\langle T(t)x, y\rangle \to 0$ as $t \to \infty$ for all $x, y \in H$. If its infinitesimal generator $\mathcal{A}$ has compact resolvent, then $T(t)$ is asymptotically stable. i.e. $\|T(t)z\| \to 0$ as $t \to \infty$ for all $z \in H$.*

Theorem 1.5.3 *Let $T(t)$ be a uniformly bounded C_0-semigroup on a Banach space X and let $\mathcal{A}$ be its generator. Then*

- *If $T(t)$ is asymptotically stable then $\sigma(\mathcal{A}) \cap i\mathbb{R} \subset \sigma_c(\mathcal{A})$.*

- *If $\sigma(\mathcal{A}) \cap i\mathbb{R} \subset \sigma_c(\mathcal{A})$ and $\sigma_c(\mathcal{A})$ is countable, then $T(t)$ is asymptotically stable.*

- *If $\mathcal{R}(\lambda, \mathcal{A})$ is compact, then $T(t)$ is asymptotically stable if and only if $Re\lambda < 0$ for all $\lambda \in \sigma(\mathcal{A})$.*

Corollary 1.5.1 *Let $T(t)$ be a C_0-semigroup on a Banach space X and $\mathcal{A}$ be its generator. Suppose that $\sigma(\mathcal{A}) \cap i\mathbb{R} \subset \sigma_c(\mathcal{A})$ and $\sigma_c(\mathcal{A})$ is countable, then $T(t)$ is weakly stable if and only if $T(t)$ is asymptotically stable.*

Theorem 1.5.4 *Let $\mathcal{A}$ be the infinitesimal generator of a C_0-semigroup $T(t)$ on a Banach space X. If for some $p \geq 1$*

$$\int_0^\infty \|T(t)x\|^p dt < \infty, \quad \textit{for every } x \in X,$$

then $T(t)$ is exponentially stable.

Remark 1.5.1 *We say that $T(t)$ is exponentially asymptotically stable if for every $x \in X$, there exist $M_x, \omega_x > 0$ depending on x such that*

$$\|T(t)x\| \leq M_x e^{-\omega_x t}.$$

Theorem 1.5.4 shows that a linear C_0-semigroup is exponentially asymptotically stable if and only if it is exponentially stable.

Theorem 1.5.5 *Let $T(t)$ be a C_0-semigroup with infinitesimal generator $\mathcal{A}$. The following statements are equivalent.*

- *$T(t)$ is exponentially stable, i.e.*

  $$\|T(t)x\| \leq M e^{-\omega t}.$$

  for $M \geq 1$ and $\omega > 0$.
- $\lim_{t\to\infty} \|T(t)\| = 0.$
- *There exists a $t_0 > 0$ such that*

  $$\|T(t_0)\| < 1.$$

We assume, that $X = H$ is a Hilbert space with the inner product $\langle .,. \rangle$ and the induced norm $\|.\|$. Recall that if $\mathcal{A}$ generates a C_0-semigroup $T(t)$ on H with $\|T(t)\| \leq Me^{wt}$, then for all λ with $Re\lambda > \omega$,

$$\mathcal{R}(\lambda, \mathcal{A})x = \int_0^\infty e^{-\lambda t} T(t)x\, dt$$

Theorem 1.5.6 *Let $T(t)$ be a C_0-semigroup on a Hilbert space H with generator $\mathcal{A}$. Then $T(t)$ is exponentially stable if and only if*

$$\{\lambda \mid Re\lambda \geq 0\} \subset \sigma(\mathcal{A})$$

and

$$|\mathcal{R}(\lambda, \mathcal{A})\| \leq M$$

for all λ with $Re\lambda \geq 0$ and some constant $M > 0$.

Theorem 1.5.7 [Huang-Pruss]

Assume that $\mathcal{A}$ is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on H. $S(t)$ is uniformly stable if and only if

1. $i\mathbb{R} \subset \rho(\mathcal{A})$.

2. $\sup\limits_{\beta\in\mathbb{R}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(H)} < +\infty$.

Theorem 1.5.8 [Batty , A.Borichev and Y.Tomilov, Z. Liu and B. Rao.]
Assume that $\mathcal{A}$ is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on H. If $i\mathbb{R} \subset \rho(\mathcal{A})$, then for a fixed $l > 0$ the following conditions are equivalent

1. $\lim\limits_{|\lambda|\to+\infty} \sup \frac{1}{\lambda^l} \|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(H)} < +\infty$.

2. $\|S(t)U_0\|_H \leq \frac{C}{t^{l-1}} \|U_0\|_{D(\mathcal{A})}\ \ \forall t > 0,\ U_0 \in D(\mathcal{A}),$ *for some* $C > 0$.

1.6 Bessel functions

The proofs of the following results can be found in [28] and [59].

The second order differential equation given as

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0$$

is known as Bessel′s differential equation which is often encountered when solving boundary value problems, especially when working in cylindrical or spherical coordinates. The constant ν, determines the order of the Bessel functions found in the solution to Bessel′s differential equation and can take any real numbered value. For cylindrical problems the order of the Bessel function is an integer value ($\nu = n$) while for spherical problems the order is of half integer value ($\nu = n + 1/2$).

Since Bessel′s differential equation is a second-order equation, there must be two linearly independent solutions. Typically the general solution is given as:

$$y = AJ_\nu(x) + BY_\nu(x)$$

where A and B are arbitrary constants and the special functions $J_\nu(x)$ and $Y_\nu(x)$ are:

- Bessel functions of the first kind, $J_\nu(x)$, which are finite at $x = 0$ for all real values of ν

- Bessel functions of the second kind, $Y_\nu(x)$, (also known as Weber or Neumann functions) which are singular at $x = 0$.

The Bessel function of the first kind of order ν can be determined using an infinite power series expansion as follows:

$$\begin{aligned} J_\nu(x) &= \sum_{\kappa=0}^{\infty} \frac{(-1)^\kappa (x/2)^{\nu+2\kappa}}{\kappa!\Gamma(\nu+\kappa+1)} \\ &= \frac{1}{\Gamma(1+\nu)} \left(\frac{x}{2}\right)^\nu \left\{1 - \frac{(x/2)^2}{1(1+\nu)} \left(1 - \frac{(x/2)^2}{2(2+\nu)} \left(1 - \frac{(x/2)^2}{3(3+\nu)} (1 - ...)\right)\right)\right\} \end{aligned}$$

or by noting that $\Gamma(\nu+\kappa+1) = (\nu+\kappa)!$, we can write

$$J_\nu(x) = \sum_{\kappa=0}^{\infty} \frac{(-1)^\kappa (x/2)^{\nu+2\kappa}}{\kappa!(\nu+\kappa)!}$$

The Bessel function of the second kind, $Y_\nu(x)$ is sometimes referred to as a Weber function or a Neumann function (which can be denoted as $N_\nu(x)$). It is related to the Bessel function of the first kind as follows:

$$Y_\nu(x) = \frac{J_\nu(x) cos(\nu\pi) - J_{-\nu}(x)}{sin(\nu\pi)}$$

where we take the limit $\nu \to n$ for integer values of ν.

For integer order ν, J_ν, $J_{-\nu}$ are not linearly independent:

$$\begin{aligned} J_{-\nu}(x) &= (-1)^\nu J_\nu(x) \\ Y_\nu(x) &= (-1)^\nu Y_\nu(x) \end{aligned}$$

in which case Y_ν is needed to provide the second linearly independent solution of Bessel's equation. In contrast, for non-integer orders, J_ν and $J_{-\nu}$ are linearly independent and Y_ν is redundant.

The Bessel function of the second kind of order ν can be expressed in terms of the Bessel function of the first kind as follows:

$$\begin{aligned} Y_\nu(x) &= \frac{2}{\pi} J_\nu(x) \left(\ln\frac{x}{2} + \gamma\right) - \frac{1}{\pi} \sum_{\kappa=0}^{\nu-1} \frac{(\nu-\kappa-1)!}{\kappa!} \left(\frac{x}{2}\right)^{2\kappa-\nu} + \\ &+ \frac{1}{\pi} \sum_{\kappa=0}^{\infty} \frac{(-1)^{\kappa-1} \left[\left(1 + \frac{1}{2} + ... + \frac{1}{\kappa}\right) + \left(1 + \frac{1}{2} + ... + \frac{1}{\kappa+\nu}\right)\right]}{\kappa!(\kappa+\nu)!} \left(\frac{x}{2}\right)^{2\kappa+\nu} \end{aligned}$$

Recurrence formulas satisfied by the Bessel functions

Bessel functions of higher order be expressed by Bessel functions of lower orders for all real values of ν.

$$J_{\nu+1}(x) = \frac{2\nu}{x}J_\nu(x) - J_{\nu-1}(x), \qquad Y_{\nu+1}(x) = \frac{2\nu}{x}Y_\nu(x) - Y_{\nu-1}(x)$$

$$J'_{\nu+1}(x) = \frac{1}{2}\left[J_{\nu-1}(x) - J_{\nu+1}(x)\right], \qquad Y'_{\nu+1}(x) = \frac{1}{2}\left[Y_{\nu-1}(x) - Y_{\nu+1}(x)\right]$$

$$J'_\nu(x) = J_{\nu-1}(x) - \frac{\nu}{x}J_\nu(x), \qquad Y'_\nu(x) = Y_{\nu-1}(x) - \frac{\nu}{x}Y_\nu(x)$$

$$J'_\nu(x) = \frac{\nu}{x}J_\nu(x) - J_{\nu+1}(x), \qquad Y'_\nu(x) = \frac{\nu}{x}Y_\nu(x) - Y_{\nu+1}(x)$$

$$\frac{d}{dx}\left[x^\nu J_\nu(x)\right] = x^\nu J_{\nu-1}(x), \qquad \frac{d}{dx}\left[x^\nu Y_\nu(x)\right] = x^\nu Y_{\nu-1}(x)$$

$$\frac{d}{dx}\left[x^{-\nu} J_\nu(x)\right] = -x^{-\nu} J_{\nu+1}(x), \qquad \frac{d}{dx}\left[x^{-\nu} Y_\nu(x)\right] = -x^{-\nu} Y_{\nu+1}(x)$$

Polynomial approximations of The Bessel functions:

For $x \geq 2$, we can use the following approximation based upon asymptotic expansions:

$$J_n(x) = \left(\frac{2}{\pi x}\right)^{1/2}\left[P_n(x)cosu - Q_n(x)sinu\right]$$

where $u = x - (2n+1)\frac{\pi}{4}$ and the polynomials $P_n(x)$ and $Q_n(x)$ are given by

$$P_n(x) = 1 - \frac{(4n^2-1^2)(4n^2-3^2)}{2.1(8x)^2}\left(1 - \frac{(4n^2-5^2)(4n^2-7^2)}{4.3(8x)^2}\left(1 - \frac{(4n^2-9^2)(4n^2-11^2)}{6.5(8x)^2}(1 - ...)\right)\right)$$

and

$$Q_n(x) = \frac{(4n^2-1^2)}{1!(8x)}\left(1 - \frac{(4n^2-3^2)(4n^2-5^2)}{3.2(8x)^2}\left(1 - \frac{(4n^2-7^2)(4n^2-9^2)}{5.4(8x)^2}(1 - ...)\right)\right)$$

The general form of these terms can be written as

$$P_n(x) = \frac{(4n^2 - (4k-3)^2)(4n^2 - (4k-1)^2)}{2k(2k-1)(8x)^2}, \quad k = 1, 2, 3...$$

$$Q_n(x) = \frac{(4n^2 - (4k-1)^2)(4n^2 - (4k+1)^2)}{2k(2k+1)(8x)^2}, \quad k = 1, 2, 3...$$

- **Asymptotic approximation of Bessel Functions (large values of x):**

$$Y_0(x) = \left(\frac{2}{\pi x}\right)^{1/2} [P_0(x) sin(x - \pi/4) + Q_0(x) cos(x - \pi/4)]$$

$$Y_1(x) = \left(\frac{2}{\pi x}\right)^{1/2} [P_1(x) sin(x - 3\pi/4) + Q_1(x) cos(x - 3\pi/4)]$$

1.7 Fractional derivatives

Some history of fractional calculus:

In a letter dated **September 30th, 1695** L'Hospital wrote to Leibniz asking him about the meaning of $d^n y/dx^n$ if $n = 1/2$, that is "what if n is fractional?". Leibniz's response:"An apparent paradox, from which one day useful consequences will be drawn."
In **1819** S. F. Lacroix was the first to mention in some two pages a derivative of arbitrary order. Thus for $y = x^a$, $a \in \mathbb{R}_+$, he showed that

$$\frac{d^{1/2}y}{dx^{1/2}} = \frac{\Gamma(a+1)}{\Gamma(1+1/2)} x^{a-1/2}.$$

In particular he had $(d/dx)^{1/2} x = 2\sqrt{x/\pi}$.
In **1822** J. B. J. Fourier derived an integral representation for f(x),

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} f(\alpha) d\alpha \int_{\mathbb{R}} \cos p(x - \alpha) dp,$$

obtained (formally) the derivative version

$$\frac{d^\nu}{dx^\nu} f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} f(\alpha) d\alpha \int_{\mathbb{R}} p^\nu \cos[p(x-\alpha) + \frac{\nu\pi}{2}] dp$$

where "the number v will be regarded as any quantity whatever, positive or negative".
In **1823** Abel resolved the integral equation arising from the brachistochrone problem, namely

$$\frac{1}{\Gamma(\alpha)} \int_0^x \frac{g(u)}{(x-u)^{1-\alpha}} du = f(x), \qquad 0 < \alpha < 1$$

with the solution

$$g(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{f(u)}{(x-u)^\alpha} du$$

Abel never solved the problem by fractional calculus but, in **1832** Liouville did solve this integral equation.

Perhaps the first serious attempt to give a logical definition of a fractional derivative is due to Liouville; he published nine papers on the subject between **1832** and **1837**, the last in the field in **1855**. They grew out of Liouville's early work on electromagnetism. There is further work of George Peacock (**1833**), D. F. Gregory (**1841**), Augustus de Morgan (**1842**), P. Kelland (**1846**), William Center (**1848**). Especially basic is Riemann's student paper of **1847**.

After the participation of Riemann and the work of Cayley in **1880**, among the mathematicians spearheading research in the broad area of fractional calculus until **1941** were S.F. Lacroix, J.B.J. Fourier, N.H. Abel, J. Liouville, A. De Morgan, B. Riemann, Hj. Holmgren, K. Griinwald, A.V. Letnikov, N.Ya. Sonine, J. Hadamard, G.H. Hardy, H. Weyl, M. Riesz, H.T. Davis, A. Marchaud, J.E. Littlewood, E.L. Post, E.R. Love, B.Sz.-Nagy, A. Erdelyi and H. Kober.

Fractional calculus has developed especially intensively since **1974** when the first international conference in the field took place.It was organized by Bertram Ross.

Samko et al in their encyclopedic volume state and we cite: "We pay tribute to investigators of recent decades by citing the names of mathematicians who have made a valuable scientific contribution to fractional calculus development from **1941** until the present [**1990**]. These are M.A. Al- Bassam, L.S. Bosanquet, P.L. Butzer, M.M. Dzherbashyan, A. Erdelyi, T.M. Flett, Ch. Fox, S.G. Gindikin, S.L. Kalla, LA. Kipriyanov, H. Kober, P.I. Lizorkin, E.R. Love, A.C. McBride, M. Mikolas, S.M. Nikol'skii, K. Nishimoto, LI. Ogievetskii, R.O. O'Neil, T.J. Osier, S. Owa, B. Ross, M. Saigo, I.N. Sneddon, H.M. Srivastava, A.F. Timan, U. Westphal, A. Zygmund and others". To this list must of course be added the names of the authors of Samko et al and many other mathematicians, particularly those of the younger generation. Books especially devoted to fractional calculus include K.B. Oldham and J. Spanier, S.G. Samko, A.A. Kilbas and O.I. Marichev, V.S. Kiryakova [91], K.S. Miller and B. Ross, B. Rubin. Books containing a chapter or sections dealing with certain aspects of fractional calculus include H.T. Davis, A. Zygmund, M.M.Dzherbashyan, I.N. Sneddon, P.L. Butzer and R.J. Nessel, P.L. Butzer and W. Trebels, G.O. Okikiolu, S. Fenyo and H.W. Stolle, H.M. Srivastava and H.L. Manocha, R. Gorenfio and S. Vessella.

Definitions of fractional integrals and derivatives of different kind

Let f be a real function of a real variable. All the following definitions are formal.

Definition 1.7.1 *The left Riemann-Liouville fractional integral of order $\alpha > 0$ starting from* *a has the following form*

$$({}_aI^\alpha f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt.$$

Definition 1.7.2 *The right Riemann-Liouville fractional integral of order $\alpha > 0$ ending at $b > a$ is defined by*

$$(I_b^\alpha f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (x-t)^{\alpha-1} f(t)dt.$$

Definition 1.7.3 *The left Riemann-Liouville fractional derivative of order $\alpha > 0$ starting at a is given below*

$$({}_aD^\alpha f)(x) = (\frac{d}{dx})^n({}_aI^{n-\alpha} f)(x), \ \ n = [\alpha] + 1.$$

Definition 1.7.4 *The right Riemann-Liouville fractional derivative of order $\alpha > 0$ ending at b becomes*

$$(D_b^\alpha f)(x) = (-\frac{d}{dx})^n(I_b^{n-\alpha} f)(x).$$

Definition 1.7.5 *The left Caputo fractional derivative of order $\alpha > 0$ starting from a has the following form*

$$({}_aD^\alpha f)(x) = (aI^{n-\alpha} f^{(n)})(x), \ \ n = [\alpha] + 1.$$

Definition 1.7.6 *The right Caputo fractional derivative of order $\alpha > 0$ ending at b becomes*

$$(D_b^\alpha f)(x) = (I_b^{n-\alpha}(-1)^n f^{(n)})(x).$$

The Hadamard type fractional integrals and derivatives were introduced in [15] as:

Definition 1.7.7 *The left Hadamard fractional integral of order $\alpha > 0$ starting from a has the following form*

$$({}_aI^\alpha f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (\ln x - \ln t)^{\alpha-1} f(t)dt$$

Definition 1.7.8 *The right Hadamard fractional integral of order $\alpha > 0$ ending at $b > a$ is defined by*

$$(I_b^\alpha f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (\ln t - \ln x)^{\alpha-1} f(t)dt$$

Definition 1.7.9 *The left Hadamard fractional derivative of order $\alpha > 0$ starting at a is given below*

$$({}_aD^\alpha f)(x) = (x\frac{d}{dx})^n(aI^{n-\alpha} f)(x), \ \ n = [\alpha] + 1.$$

Definition 1.7.10 *The right Hadamard fractional derivative of order $\alpha > 0$ ending at b is*

$$(D_b^\alpha f)(x) = (-x\frac{d}{dx})^n (I_b^{n-\alpha} f)(x).$$

Definition 1.7.11 *The fractional derivative of order α, $0 < \alpha < 1$, in sense of Caputo, is defined by*

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{df}{ds}(s) ds.$$

Definition 1.7.12 *The fractional integral of order α, $0 < \alpha < 1$, in sense of Riemann-Liouville, is defined by*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds.$$

Remark 1.7.1 *From the above definitions, clearly*

$$D^\alpha f = I^{1-\alpha} Df, \;\; 0 < \alpha < 1.$$

Lemma 1.7.1

$$I^\alpha D^\alpha f(t) = f(t) - f(0), \;\; 0 < \alpha < 1.$$

Lemma 1.7.2 *If*

$$D^\beta f(0) = 0.$$

then

$$D^\alpha D^\beta f = D^{\alpha+\beta} f, \;\; 0 < \alpha < 1, \;\; 0 < \beta < 1.$$

Now, we give the definitions of the generalized Caputo's fractional derivative and the generalized fractional integral. These exponentially modified fractional integro-differential operators were first proposed in [54].

Definition 1.7.13 *The generalized Caputo's fractional derivative is given by*

$$D^{\alpha,\eta} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{df}{ds}(s)\, ds, \;\; 0 < \alpha < 1, \eta \geq 0.$$

Remark 1.7.2 *The operators D^α and $D^{\alpha,\eta}$ differ just by their kernels.*

Definition 1.7.14 *The generalized fractional integral is given by*

$$I^{\alpha,\eta} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\eta(t-s)} f(s)\, ds, \;\; 0 < \alpha < 1, \eta \geq 0.$$

Remark 1.7.3 *We have*

$$D^{\alpha,\eta} f = I^{1-\alpha,\eta} Df, \;\; 0 < \alpha < 1, \eta \geq 0.$$

Chapter 2

On the stability of a degenerate wave equation under fractional feedbacks acting on the degenerate boundary

2.1 Introduction

In this chapter, we are concerned with the boundary stabilization of fractional type for degenerate wave equation of the form

$$(P) \qquad \begin{cases} u_{tt}(x,t) = (x^\gamma u_x(x,t))_x 0 & \text{in } (0,1)\times(0,+\infty), \\ (x^\gamma u_x)(0,t) = \varrho \partial_t^{\alpha,\eta} u(0,t) & \text{in } (0,+\infty), \\ u(1,t) = 0 & \text{in } (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } (0,1), \end{cases}$$

where $\gamma \in [0,1)$ and $\varrho > 0$. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α, $(0 < \alpha \leq 1)$, with respect to the time variable (see **[21]**). It is defined as follows

$$\partial_t^{\alpha,\eta} u(t) = \begin{cases} u_t(t) & \text{for } \alpha = 1, \eta \geq 0, \\ \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{du}{ds}(s)\, ds, & \text{for } 0 < \alpha < 1, \eta \geq 0. \end{cases}$$

The degenerate wave equation (P) (i.e $\gamma \neq 0$) can describe the vibration problem of an elastic string. In a neighborhood of an endpoint $x = 0$ of this string, the elastic is sufficiently small or the linear density is large enough. Indeed a mathematical model that describes transverse vibration of an elastic string is given by

$$u_{tt}(x,t) - \left(\frac{T(x)}{\rho(x)}u_x(x,t)\right)_x + \text{lower terms} = 0,$$

where T is the tension of a string and ρ is the density of the string. The elasticity of the string can stretch proportionally to a variation in tension. Hence, the wave equation becomes degenerate when $T(x) \to 0$ as $x \to 0$ or $\rho(x) \to +\infty$ as $x \to 0$.

In **[55]**, G. Propst and J. Pruss consider the following model for the evolution of sound in a compressible fluid with viscoelastic surface, that is

$$\begin{cases} p_{tt}(x,t) - \Delta p(x,t) = 0, \quad t \in \mathbb{R},\ x \in \Omega \\ \dfrac{\partial p}{\partial n}(x,t) + a * p_t(x,t) = 0, \quad t \in \mathbb{R},\ x \in \partial\Omega, \end{cases}$$

where $p(x,t) \in \mathbb{R}$ denote acoustic pressure, $\Omega \subset \mathbb{R}^3$ is a domain with smooth boundary and $n(x)$ is the outer normal to $\partial\Omega$ at x. The convolution is $a * v(t,.) = \int_{-\infty}^{t} a(t-s)v(s,.)\,ds$, a is a given real-valued function on $[0,\infty)$. Physically, the boundary condition models the interaction of a viscoelastic boundary material with memory and the incident waves. It is mentioned in **[55]** that these boundary conditions model well the reflexion of sound at surfaces of materials that are of interest in ingineering practice.

It has been shown (see **[50]**, **[48]** and **[24]**) that, as ∂_t, the fractional derivative ∂_t^α forces the system to become dissipative and the solution to converge to the equilibrium state. Therefore, when applied on the boundary, we can consider them as controllers which help to reduce the vibrations.

Moreover, fractional derivatives can improve performance not achievable before using controls of integer-order type and provide an excellent instrument for the description of memory and hereditary properties of various materials and processes (see **[45]**). This is the main advantage of fractional derivatives in comparison with classical integer-order models, in which such effects are in fact neglected.

The bibliography of works concerning the stabilization of nondegenerate wave equation with different types of dampings is truly long (see e.g. **[22]**, **[23]** and **[20]** and the references therein). In **[23]**, for $a(x) = a_1 x + a_0$: D'Andrea-Novel, F. Boustany and B. Rao have established aymptotics stabilization with the following boundary damping

$$\begin{cases} (au_x)(0,t) = 0, \\ (au_x)(1,t) = -ku(1,t) - u_t(1,t),\ k > 0. \end{cases}$$

In **[20]**, B. Chentouf, C.Z. Xu and G. Sallet considered the following modelization of a flexible torque arm controlled by two feedbacks depending only on the boundary velocities:

$$\begin{cases} u_{tt}(x,t) - (a(x)u_x)_x + \alpha u_t(x,t) + \beta u(x,t) = 0,\ 0 < x < 1,\ t > 0, \\ (a(x)u_x)(0) = k_1 u_t(0,t), t > 0, \\ (a(x)u_x)(1) = -k_2 u_t(1,t), t > 0, \end{cases}$$

where

$$\begin{cases} \alpha \geq 0, \beta > 0, k_1, k_2 \geq 0, k_1 + k_2 \neq 0, \\ a \in W^{1,\infty}(0,1), a(x) \geq a_0 \text{ for all } x \in [0,1]. \end{cases}$$

They proved the exponential decay of the solutions.

On the contrary, when the coefficient $a(x)$ is degenerate very little is known in the literature, even though many problems that are relevant for applications are described by hyperbolic equations degenerating at the boundary of the space domain (see **[33]**).

The controllability problems for degenerate/singular heat equations have been studied by many authors in the last decade (see, for instance, **[12]**, **[18]**, **[4]**, **[17]**, **[16]**, **[30]** and **[31]**).

New Carleman estimates and moment methods (based on spectral analysis) have been used to derive observability inequalities for the corresponding dual problems. Recently, there are more and more authors who studied the exact controllability of the degenerate wave equations, see related studies **[33]**, **[63]**, **[8]**. In **[33]**, for any $0 < \gamma < 1$, the null controllability of the following degenerate wave equation was considered:

$$(PC) \qquad \begin{cases} u_{tt}(x,t) - (x^{\gamma} u_x(x,t))_x = 0 & \text{on } (0,1) \times (0,T), \\ u(0,t) = \theta(t), u(1,t) = 0 & \text{on } (0,T), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } (0,1), \end{cases}$$

where $\theta(t)$ is the control variable and it acts on the degenerate boundary. He proved that the degeneracy affect the exact controllability and an explicit expression for the controllability time depending on the parameter γ is given.

Very recently, A. Benaissa and C. Aichi **[10]** studied the degenerate wave equation of the type

$$u_{tt}(x,t) = (a(x)u_x(x,t))_x \quad \text{in } (0,1) \times (0,+\infty), \tag{2.1}$$

where the coefficient a is a positive function on $]0,1]$ but vanishes at zero. The degeneracy of (2.1) at $x = 0$ is measured by the parameter μ_a defined by

$$\mu_a = \sup_{0<x\leq 1} \frac{x|a'(x)|}{a(x)} \tag{2.2}$$

and the initial conditions are

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \tag{2.3}$$

followed by the boundary conditions

$$(P1) \qquad \begin{cases} \begin{cases} u(0,t) = 0 & \text{if } 0 \leq \mu_a < 1 \\ (au_x)(0,t) = 0 & \text{if } 1 \leq \mu_a < 2 \end{cases} & \text{in } (0,+\infty), \\ (au_x)(1,t) + \varrho \partial_t^{\alpha,\eta} u(1,t) + \beta u(1,t) = 0 & \text{in } (0,+\infty). \end{cases}$$

They obtained optimal polynomial stability of the solutions. Moreover, the degeneracy does not affect the decay rates of the energy.

Here we want to focus on the following remarks:

- System (P) under study is different from one studied on [**10**]. Indeed, the control is located at $x = 0$.
- The explicit representation of the resolvent gives us a sharp polynomial decay rate, however in [**10**], stabilization is done under the frequency domain method based on multiplier techniques (see [**41**]). Unfortunately, this method does not seem to be applicable in the case of damping acting at $x = 0$.

In this work, we explain the influence of the relation between the degenerate coefficient and the fractional feedback on decay estimates. To our best knowledge, this is the first attempt to study the global decaying solutions for a degenerate wave equation under a control acting on the degenerate boundary.

This chapter is organized as follows. In sections 2 and 3, we give preliminary results and we reformulate the system (P) into an augmented system by coupling the degenerate wave equation with a suitable diffusion equation and we show the well-posedness of our problem by semigroup theory. In section 4, we prove lack of exponential stability by spectral analysis and by using Bessel functions. In the last section, we prove an optimal decay rate. The proof heavily relies on Bessel equations and Borichev-Tomilov Theorem.

2.2 Preliminary results

Now, we introduce, as in [**19**], the following weighted Sobolev spaces:

$$H^1_{0,\gamma}(0,1) = \left\{u \text{ is locally absolutely continuous in } (0,1] : x^{\gamma/2}u_x \in L^2(0,1)/\ u(1) = 0\right\},$$

$$H^1_{\gamma}(0,1) = \left\{u \text{ is locally absolutely continuous in } (0,1] : x^{\gamma/2}u_x \in L^2(0,1)\right\}.$$

We remark that $H^1_\gamma(0,1)$ is a Hilbert space with the scalar product

$$(u,v)_{H^1_\gamma(0,1)} = \int_0^1 (u\overline{v} + x^\gamma u_x(x)\overline{v_x(x)})\,dx, \quad \forall u,v \in H^1_\gamma(0,1).$$

Let us also set

$$|u|_{H^1_{0,\gamma}(0,1)} = \left(\int_0^1 x^\gamma |u_x(x)|^2\,dx\right)^{1/2} \quad \forall u \in H^1_\gamma(0,1).$$

Actually, $|\cdot|_{H^1_{0,\gamma}(0,1)}$ is an equivalent norm on the closed subspace $H^1_{0,\gamma}(0,1)$ to the norm of $H^1_\gamma(0,1)$. This fact is a simple consequence of the following version of Poincaré's inequality.

Proposition 2.2.1 *There is a positive constant $C_* = C(\gamma)$ such that*

$$\|u\|^2_{L^2(\Omega)} \le C_* |u|^2_{H^1_{0,\gamma}(0,1)} \quad \forall u \in H^1_{0,\gamma}(0,1). \tag{2.4}$$

Proof. Let $u \in H^1_{0,\gamma}(0,1)$. For any $x \in]0,1]$ we have that

$$|u(x)| = \left| \int_x^1 u_x(s)\, ds \right| \le |u|_{H^1_{0,\gamma}(0,1)} \left\{ \int_0^1 \frac{1}{s^\gamma}\, ds \right\}^{1/2}.$$

Therefore

$$\int_0^1 |u(x)|^2\, dx \le \frac{1}{1-\gamma} |u|^2_{H^1_{0,\gamma}(0,1)}.$$

Next, we define

$$H^2_\gamma(0,1) = \{u \in H^1_\gamma(0,1) :\ x^\gamma u_x \in H^1(0,1)\},$$

where $H^1(0,1)$ denotes the classical Sobolev space.

Remark 2.2.1 *Notice that if $u \in H^2_\gamma(0,1), \gamma \in [1,2)$, we have $(x^\gamma u_x)(0) \equiv 0$ since $1/x^\gamma$ is not integrable over neighbourhoods of 0. Hence the problem is not well-posed in terms of the semigroups in the Hilbert space.*

The elements of $H^1_{0,\gamma}(0,1)$ satisfy the following property.

Proposition 2.2.2 *For every $u \in H^1_{0,\gamma}(0,1)$, u is absolutely continuous in $[0,1]$ and we have*

$$(*) \qquad \|u\|_{L^\infty(0,1)} \le \frac{1}{\sqrt{1-\gamma}} \|u\|_{H^1_{0,\gamma}(0,1)}.$$

Proof. As

$$u'(x) = \frac{1}{x^{\gamma/2}} x^{\gamma/2} u'(x) \qquad \forall x \in]0,1].$$

then

$$\begin{aligned} \int_0^1 |u'(x)|\, dx &\le \left(\int_0^1 \frac{1}{x^\gamma}\, dx \right)^{1/2} \|u\|_{H^1_{0,\gamma}(0,1)} \\ &= \frac{1}{\sqrt{1-\gamma}} \|u(x)\|_{H^1_{0,\gamma}(0,1)} \end{aligned}$$

u' is summable over $(0,1)$. So u is absolutely continuous in $[0,1]$. Moreover

$$|u(x)| = \left| \int_x^1 u'(s)\, ds \right| \le \int_0^1 |u'(x)|\, dx \le \frac{1}{\sqrt{1-\gamma}} \|u\|_{H^1_{0,\gamma}(0,1)}.$$

$(*)$ is thus proved.

2.2.1 Augmented model

In this section, we reformulate (P) into an augmented system. For that, we need the following proposition.

Proposition 2.2.3 (see [49]) *Let μ be the function:*

$$\mu(\xi) = |\xi|^{(2\alpha-1)/2}, \quad -\infty < \xi < +\infty, \ 0 < \alpha < 1. \tag{2.5}$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t \phi(\xi, t) + (\xi^2 + \eta)\phi(\xi, t) - U(t)\mu(\xi) = 0, \quad -\infty < \xi < +\infty, \eta \geq 0, t > 0, \tag{2.6}$$

$$\phi(\xi, 0) = 0, \tag{2.7}$$

$$O(t) = (\pi)^{-1} \sin(\alpha\pi) \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi, t)\, d\xi, \tag{2.8}$$

where $U \in C^0([0, +\infty))$, is given by

$$O = I^{1-\alpha,\eta} U, \tag{2.9}$$

where

$$[I^{\alpha,\eta} f](t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{-\eta(t-\tau)} f(\tau)\, d\tau.$$

Lemma 2.2.1 (see [3]) *If $\lambda \in D_\eta = \mathbb{C} \backslash] - \infty, -\eta]$ then*

$$\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2}\, d\xi = \frac{\pi}{\sin \alpha\pi} (\lambda + \eta)^{\alpha - 1}.$$

Using now Proposition 2.2.3 and relation (2.9), system (P) may be recast into the following augmented system

$$(P') \qquad \begin{cases} u_{tt}(x, t) = (x^\gamma u_x(x, t))_x, \\ \phi_t(\xi, t) + (\xi^2 + \eta)\phi(\xi, t) = u_t(0, t)\mu(\xi), & -\infty < \xi < +\infty, \quad t > 0, \\ (x^\gamma u_x)(0, t) = \zeta \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi, t)\, d\xi, \\ u(1, t) = 0, \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), \phi(\xi, 0) = 0, \end{cases}$$

where $\zeta = \varrho(\pi)^{-1} \sin(\alpha\pi)$.

2.3 Well-posedness

In this section, we are interested in showing that system (P') is well posed in the sens of semigroups. We introduce the following Hilbert space

$$\mathcal{H} = H^1_{0,\gamma}(0,1) \times L^2(0,1) \times L^2(-\infty,+\infty)$$

equipped with the inner product

$$\left\langle \begin{pmatrix} u \\ v \\ \phi \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\phi} \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_0^1 x^\gamma u_x \overline{\tilde{u}}_x dx + \int_0^1 v\overline{\tilde{v}} dx + \zeta \int_{-\infty}^{+\infty} \phi \overline{\tilde{\phi}}\, d\xi.$$

If we put $U = (u, u_t, \phi)^T$, it is clear that (P') can be written in the following form

$$U_t = \mathcal{A}U, \quad U(0) = U_0, \tag{2.10}$$

where $U_0 = (u_0, u_1, 0)^T$ and $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ \phi \end{pmatrix} = \begin{pmatrix} v \\ (x^\gamma u_x)_x \\ -(\xi^2+\eta)\phi + v(0)\mu(\xi) \end{pmatrix} \tag{2.11}$$

with domain

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (u,v,\phi) \text{ in } \mathcal{H} : u \in H^2_\gamma(0,1) \cap H^1_{0,\gamma}(0,1), v \in H^1_{0,\gamma}(0,1), \\ -(\xi^2+\eta)\phi + v(0)\mu(\xi) \in L^2(-\infty,+\infty), \\ (x^\gamma u_x)(0) = \zeta \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi, \\ |\xi|\phi \in L^2(-\infty,+\infty) \end{array} \right\}. \tag{2.12}$$

Our main result is giving by the following theorem.

Theorem 2.3.1 *The operator $\mathcal{A}$ defined by (2.11) and (2.12), generates a C_0-semigroup of contractions $e^{t\mathcal{A}}$ in the Hilbert space $\mathcal{H}$.*

Proof.
To prove this result, we shall use the Lumer-Phillips' Theorem. Sine we have for for every $U = (u,v,\phi) \in D(\mathcal{A})$

$$\Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\zeta \int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi)|^2\, d\xi. \tag{2.13}$$

then $\mathcal{A}$ is dissipative.

Let $\lambda > 0$. we prove that the operator $(\lambda I - \mathcal{A})$ is a surjection. In other words, we shall demonstrate that given any triplet $F = (f_1, f_2, f_3) \in \mathcal{H}$, there is an other triplet $U = (u, v, \phi)$ in $D(\mathcal{A})$ such that

$$(\lambda I - \mathcal{A})U = F. \tag{2.14}$$

Equation (2.14) is equivalent to

$$\begin{cases} \lambda u - v = f_1, \\ \lambda v - (x^\gamma u_x)_x = f_2, \\ \lambda\phi + (\xi^2 + \eta)\phi - v(0)\mu(\xi) = f_3. \end{cases} \tag{2.15}$$

Suppose u is found with the appropriate regularity. Then, $(2.15)_1$ and $(2.15)_3$ yield

$$v = \lambda u - f_1 \in H^1_{0,\gamma}(0,1), \tag{2.16}$$

$$\phi = \frac{f_3(\xi) + \mu(\xi)v(0)}{\xi^2 + \eta + \lambda}. \tag{2.17}$$

By using $(2.15)_2$ and (2.16) it can easily be shown that u satisfies

$$\lambda^2 u - (x^\gamma u_x)_x = f_2 + \lambda f_1. \tag{2.18}$$

Solving equation (2.18) is equivalent to finding $u \in H^2_\gamma(0,1) \cap H^1_{0,\gamma}(0,1)$ such that

$$\int_0^1 (\lambda^2 u\overline{w} - (x^\gamma u_x)_x\overline{w})\,dx = \int_0^1 (f_2 + \lambda f_1)\overline{w}\,dx, \tag{2.19}$$

for all $w \in H^1_{0,\gamma}(0,1)$. By using (2.19), the boundary condition $(2.12)_3$ and (2.17) the function u satisfies the following equation

$$\begin{aligned} &\int_0^1 (\lambda^2 u\overline{w} + (x^\gamma u_x)\overline{w}_x)\,dx + \tilde{\zeta} v(0)\overline{w}(0) \\ &\qquad = \int_0^1 (f_2 + \lambda f_1)\overline{w}\,dx - \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + \lambda} f_3(\xi)\,d\xi\,\overline{w}(0), \end{aligned} \tag{2.20}$$

where $\tilde{\zeta} = \zeta \displaystyle\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2 + \eta + \lambda}\,d\xi$. Using again (2.16), we deduce that

$$v(0) = \lambda u(0) - f_1(0). \tag{2.21}$$

Inserting (2.21) into (2.20), we get

$$\begin{cases} \displaystyle\int_0^1 (\lambda^2 u\overline{w} + x^\gamma u_x\overline{w}_x)\,dx + \lambda\tilde{\zeta} u(0)\overline{w}(0) \\ \displaystyle = \int_0^1 (f_2 + \lambda f_1)\overline{w}\,dx - \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + \lambda} f_3(\xi)\,d\xi\,\overline{w}(0) + \tilde{\zeta} f_1(0)\overline{w}(0). \end{cases} \tag{2.22}$$

Problem (2.22) is of the form

$$\mathcal{B}(u,w) = \mathcal{L}(w), \tag{2.23}$$

where the sesquilinear[1] form $\mathcal{B} : [H^1_{0,\gamma}(0,1) \times H^1_{0,\gamma}(0,1)] \rightarrow \mathbb{C}$ and the antilinear[2] form $\mathcal{L} : H^1_{0,\gamma}(0,1) \rightarrow \mathbb{C}$ are defined by

$$\mathcal{B}(u,w) = \int_0^1 (\lambda^2 u\overline{w} + x^\gamma u_x \overline{w}_x)\, dx + \lambda\tilde{\zeta} u(0)\, \overline{w}(0),$$

$$\mathcal{L}(w) = \int_0^1 (f_2 + \lambda f_1)\overline{w}\, dx - \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + \lambda} f_3(\xi)\, d\xi \overline{w}(0) + \tilde{\zeta} f_1(0)\overline{w}(0).$$

It is clear that $\mathcal{B}$ is a continuous and coercive form on $H^1_{0,\gamma}(0,1) \times H^1_{0,\gamma}(0,1)$ and $\mathcal{L}$ is a continuous form on $H^1_{0,\gamma}(0,1)$. Hence, by means of the Lax-Milgram Lemma, system (2.23) has a unique solution $u \in H^1_{0,\gamma}(0,1)$. In particular, setting $w \in \mathcal{D}(0,1)$ in (2.23), we get

$$\lambda^2 u - (x^\gamma u_x)_x = f_2 + \lambda f_1 \text{ in } \mathcal{D}'(0,1). \tag{2.24}$$

As $f_2 + \lambda f_1 \in L^2(0,1)$, using (2.24), we deduce that

$$\lambda^2 u - (x^\gamma u_x)_x = f_2 + \lambda f_1 \text{ in } L^2(0,1). \tag{2.25}$$

Due to the fact that $u \in H^1_{0,\gamma}(0,1)$ we get $(x^\gamma u_x)_x \in L^2(0,1)$, and we deduce that

$$u \in H^2_\gamma(0,1) \cap H^1_{0,\gamma}(0,1).$$

Multiplying the conjugate of the equality (2.25) by $w \in H^1_{0,\gamma}(0,1)$, integrating by parts on $(0,1)$, and comparing with (2.23) we get

$$\begin{aligned} &-(x^\gamma u_x)(0)\overline{w}(0) + \varrho\lambda(\lambda+\eta)^{\alpha-1} u(0)\overline{w}(0) \\ &+\zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_3(\xi)\, d\xi \overline{w}(0) \ - \varrho(\lambda+\eta)^{\alpha-1} f_1(0)\overline{w}(0) = 0. \end{aligned}$$

Consequently, defining $v = \lambda u - f_1$ and ϕ by (2.17), we deduce that

$$-(x^\gamma u_x)(0) + \zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi = 0.$$

---

[1]An application $\mathcal{B} : V \times V \to \mathbb{C}$ where V is a $\mathbb{C}$ vector space is said to be sesquilinear if it is linear for one variable and antilinear for the other.

[2]An application $\mathcal{L} : V \to \mathbb{C}$ where V is a $\mathbb{C}$ vector space is said to be antilinear if $L(\lambda x) = \overline{\lambda}\mathbb{L}(x)$ for all $x \in V$ and $\lambda \in \mathbb{C}$.

In order to complete the existence of $U \in D(\mathcal{A})$, we need to prove ϕ and $|\xi|\phi \in L^2(-\infty, \infty)$. From (2.17), we get

$$\int_{\mathbb{R}} |\phi(\xi)|^2\, d\xi \leq 3 \int_{\mathbb{R}} \frac{|f_3(\xi)|^2}{(\xi^2+\eta+\lambda)^2}\, d\xi + 3(\lambda^2|u(0)|^2 + |f_1(0)|^2) \int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{(\xi^2+\eta+\lambda)^2}\, d\xi.$$

Using Proposition 2.2.3, it easy to see that

$$\int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{(\xi^2+\eta+\lambda)^2}\, d\xi = (1-\alpha)\frac{\pi}{\sin \alpha\pi}(\lambda+\eta)^{\alpha-2}.$$

On the other hand, using the fact that $f_3 \in L^2(\mathbb{R})$, we obtain

$$\int_{\mathbb{R}} \frac{|f_3(\xi)|^2}{(\xi^2+\eta+\lambda)^2}\, d\xi \leq \frac{1}{(\eta+\lambda)^2} \int_{\mathbb{R}} |f_3(\xi)|^2\, d\xi < +\infty.$$

It follows that $\phi \in L^2(\mathbb{R})$. Next, using (2.17), we get

$$\int_{\mathbb{R}} |\xi\phi(\xi)|^2\, d\xi \leq 3 \int_{\mathbb{R}} \frac{|\xi|^2|f_3(\xi)|^2}{(\xi^2+\eta+\lambda)^2}\, d\xi + 3(\lambda^2|u(0)|^2 + |f_1(0)|^2) \int_{\mathbb{R}} \frac{|\xi|^{2\alpha+1}}{(\xi^2+\eta+\lambda)^2}\, d\xi.$$

Using again Proposition 2.2.3, it easy to see that

$$\int_{\mathbb{R}} \frac{|\xi|^{2\alpha+1}}{(\xi^2+\eta+\lambda)^2}\, d\xi = \alpha\frac{\pi}{\sin \alpha\pi}(\lambda+\eta)^{\alpha-1}.$$

Now, using the fact that $f_3 \in L^2(\mathbb{R})$, we obtain

$$\int_{\mathbb{R}} \frac{|\xi|^2|f_3(\xi)|^2}{(\xi^2+\eta+\lambda)^2}\, d\xi \leq \frac{1}{(\eta+\lambda)} \int_{\mathbb{R}} |f_3(\xi)|^2\, d\xi < +\infty.$$

It follows that $|\xi|\phi \in L^2(\mathbb{R})$. Finally, since $\phi \in L^2(\mathbb{R})$, we get

$$-(\xi^2+\eta)\phi + v(0)\mu(\xi) = \lambda\phi(\xi) - f_3(\xi) \in L^2(\mathbb{R}).$$

Then $U \in D(\mathcal{A})$ and Therefore, the operator $\lambda I - \mathcal{A}$ is surjective for any $\lambda > 0$.

$\blacksquare$

As a consequence of Theorem 2.3.1, the system (P') is well-posed in the energy space $\mathcal{H}$ and we have the following proposition.

Proposition 2.3.1 *For $(u_0, u_1, 0) \in \mathcal{H}$, the problem (P') admits a unique weak solution*

$$(u, u_t, \phi) \in C^0(\mathbb{R}_+, \mathcal{H}).$$

and for $(u_0, u_1, 0) \in D(\mathcal{A})$, the problem (P') admits a unique strong solution

$$(u, u_t, \phi) \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

Moreover, from the density of $D(\mathcal{A})$ in $\mathcal{H}$, the energy of $(u(t), \phi(t))$ at time $t \geq 0$ given by

$$E(t) = \frac{1}{2}\int_0^1 (|u_t|^2 + x^\gamma |u_x|^2)dx + \frac{\zeta}{2}\int_{-\infty}^{+\infty} |\phi(\xi,t)|^2 \, d\xi \tag{2.26}$$

decays as follows

$$E'(t) = -\zeta \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi(\xi,t)|^2 \, d\xi \leq 0. \tag{2.27}$$

Proof of Proposition 2.3.1. Noting that the regularity of the solution of the problem (P') is consequence of the semigroup properties. We have just to prove (2.27).

Multiplying the first equation in (P') by $\overline{u}_t$, integrating over $(0,1)$ and using integration by parts, we get

$$\int_0^1 u_{tt}(x,t)\overline{u}_t dx - \int_0^1 (x^\gamma u_x(x,t))_x \overline{u}_t dx = 0.$$

then

$$\frac{d}{dt}\left(\frac{1}{2}\int_0^1 |u_t(x,t)|^2 dx\right) + \frac{1}{2}\frac{d}{dt}\int_0^1 x^\gamma |u_x(x,t)|^2 \, dx - \Re\Big[(x^\gamma u_x)(x,t)\overline{u}_t\Big]_0^1 = 0.$$

then

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left(|u_t(x,t)|^2 + x^\gamma |u_x(x,t)|^2\right) dx + \zeta \Re \overline{u}_t(0,t)\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,t)\, d\xi = 0. \tag{2.28}$$

Multiplying the second equation in (P') by $\zeta\overline{\phi}$ and integrating over $(-\infty, +\infty)$, to obtain:

$$\zeta \int_{-\infty}^{+\infty} \phi_t(\xi,t)\overline{\phi}(\xi,t)d\xi + \zeta\int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi,t)|^2 d\xi - \zeta u_t(0,t)\int_{-\infty}^{+\infty} \mu(\xi)\overline{\phi}(\xi,t)d\xi = 0.$$

Hence

$$\frac{\zeta}{2}\frac{d}{dt}\int_{-\infty}^{+\infty} |\phi(\xi,t)|^2 d\xi + \zeta\int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi,t)|^2 d\xi - \zeta \Re u_t(0,t)\int_{-\infty}^{+\infty} \mu(\xi)\overline{\phi}(\xi,t)d\xi = 0. \tag{2.29}$$

Consequently, it is resulted from (2.26), (2.28) and (2.29) that

$$E'(t) = -\zeta \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi(\xi,t)|^2 \, d\xi \leq 0.$$

This completes the proof of the Proposition.

Remark 2.3.1 *In the case $\alpha = 1$, we take $\varrho u_t(0,t)$ instead of $\varrho \partial_t^{\alpha,\eta} u(0,t)$. We do not need to introduce an augmented system. In this case the operator $\mathcal{A}$ takes the form*

$$\tilde{\mathcal{A}} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ (x^\gamma u_x)_x \end{pmatrix} \tag{2.30}$$

with domain

$$D(\tilde{\mathcal{A}}) = \left\{ \begin{array}{l} (u,v) \ in \ \tilde{\mathcal{H}} : u \in H^2_\gamma(0,1) \cap H^1_{0,\gamma}(0,1), v \in H^1_{0,\gamma}(0,1), \\ (x^\gamma u_x)(0) = \varrho v(0), \end{array} \right\}, \tag{2.31}$$

where

$$\tilde{\mathcal{H}} = H^1_{0,\gamma}(0,1) \times L^2(0,1)$$

with inner product

$$\left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_0^1 x^\gamma u_x \overline{\tilde{u}}_x dx + \int_0^1 v \overline{\tilde{v}} dx.$$

The well-posedness result follows exactly as in the case $0 < \alpha < 1$. Moreover, the energy function is defined as

$$\tilde{E}(t) = \frac{1}{2} \int_0^1 (|u_t|^2 + x^\gamma |u_x|^2) dx \tag{2.32}$$

and decays as follows

$$\tilde{E}'(t) = -\varrho |u_t(0,t)|^2 \leq 0.$$

■

2.4 Spectral analysis and lack of uniform stability

This section will be devoted to the study of the lack of exponential decay of solutions associated with the system (2.10). To do this, we shall use the following well-known result from semigroup theory.

Theorem 2.4.1 ([56]-[40]) *Let $S(t)$ be a C_0-semigroup of contractions on Hilbert space $\mathcal{X}$ with generator $\mathcal{A}$. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{ i\beta : \beta \in \mathbb{R} \} \equiv i\mathbb{R} \tag{2.33}$$

and

$$\overline{\lim_{|\beta| \to \infty}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{X})} < \infty. \tag{2.34}$$

Our main result is the following.

Theorem 2.4.2 *The semigroup generated by the operator $\mathcal{A}$ is not exponentially stable if $\eta = 0$ or $\alpha \neq 2\nu_\gamma = 2\frac{1-\gamma}{2-\gamma}$.*

Proof. We will examine two cases.
•**Case 1** $\eta = 0$ and $\alpha \neq 1$: We shall show that $i\lambda = 0$ is not in the resolvent set of the operator $\mathcal{A}$. Indeed, noting that $F = (\sin(x-1), 0, 0)^T \in \mathcal{H}$, and assume that there exists $U = (u, v, \phi)^T \in D(\mathcal{A})$ such that $-\mathcal{A}U = F$. It follows

$$\begin{cases} -v = \sin(x-1), \\ -(x^\gamma u_x)_x = 0, \\ \xi^2\phi = v(0)\mu(\xi). \end{cases}$$

We see that $\phi(\xi) = |\xi|^{\frac{2\alpha-5}{2}} \sin 1$. But, then $\phi \notin L^2(-\infty, +\infty)$, since $\alpha \in]0, 1[$. So we get $(u, v, \phi)^T \notin D(\mathcal{A})$. Then the operator $-\mathcal{A}$ is not invertible.
• **Case 2** $\eta \neq 0$ and $\alpha \neq 2\nu_\gamma$:

We aim to show that an infinite number of eigenvalues of $\mathcal{A}$ approach the imaginary axis which prevents the system (P) from being exponentially stable. Indeed we first compute the characteristic equation that gives the eigenvalues of $\mathcal{A}$. Let λ be an eigenvalue of $\mathcal{A}$ with associated eigenvector $U = (u, v, \phi)^T$. Then $\mathcal{A}U = \lambda U$ is equivalent to

$$\begin{cases} \lambda u = v, \\ \lambda v = (x^\gamma u_x)_x, \\ \lambda\phi + (\xi^2 + \eta)\phi = v(0)\mu(\xi) \end{cases} \tag{2.35}$$

with boundary conditions

$$\begin{cases} u(1) = 0, \\ (x^\gamma u_x)(0) = \zeta \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi. \end{cases} \tag{2.36}$$

Inserting $(2.35)_1$ into $(2.35)_2$ and $(2.35)_3$, we get

$$\begin{cases} \lambda^2 u = (x^\gamma u_x)_x, \\ (\lambda + \xi^2 + \eta)\phi = \lambda u(0)\mu(\xi). \end{cases} \tag{2.37}$$

From the condition $(2.36)_2$, $(2.37)_2$ and Lemma 2.2.1, we obtain that

$$(x^\gamma u_x)(0) = \varrho\lambda(\lambda + \eta)^{\alpha-1} u(0). \tag{2.38}$$

Finally, we get the following problem

$$\begin{cases} \lambda^2 u = (x^\gamma u_x)_x, \\ u(1) = 0, \\ (x^\gamma u_x)(0) = \varrho\lambda(\lambda + \eta)^{\alpha-1} u(0). \end{cases} \tag{2.39}$$

It is well-known that Bessel functions play an important role in this type of problem. Assume that u is a solution of $(2.39)_1$ associated to eigenvalue $-\lambda^2$, then one easily checks that the function

$$u(x) = x^{\frac{1-\gamma}{2}} \Psi \left(\frac{2}{2-\gamma} i\lambda x^{\frac{2-\gamma}{2}} \right)$$

is a solution of the following problem:

$$y^2 \Psi''(y) + y\Psi'(y) + \left(y^2 - (\frac{\gamma-1}{2-\gamma})^2 \right) \Psi(y) = 0. \tag{2.40}$$

We have

$$u(x) = c_+ \tilde{\Phi}_+ + c_- \tilde{\Phi}_-, \tag{2.41}$$

where $\tilde{\Phi}_+$ and $\tilde{\Phi}_-$ are defined by

$$\tilde{\Phi}_+(x) := x^{\frac{1-\gamma}{2}} J_{\nu_\gamma} \left(\frac{2}{2-\gamma} i\lambda x^{\frac{2-\gamma}{2}} \right)$$

and

$$\tilde{\Phi}_-(x) := x^{\frac{1-\gamma}{2}} J_{-\nu_\gamma} \left(\frac{2}{2-\gamma} i\lambda x^{\frac{2-\gamma}{2}} \right),$$

where

$$J_\nu(y) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\nu+1)} \left(\frac{y}{2}\right)^{2m+\nu} = \sum_{m=0}^{\infty} c^+_{\nu,m} y^{2m+\nu}, \tag{2.42}$$

$$J_{-\nu}(y) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m-\nu+1)} \left(\frac{y}{2}\right)^{2m-\nu} = \sum_{m=0}^{\infty} c^-_{\nu,m} y^{2m-\nu}, \tag{2.43}$$

$$\nu_\gamma = \frac{1-\gamma}{2-\gamma}.$$

J_{ν_γ} and $J_{-\nu_\gamma}$ are Bessel functions of the first kind of order ν_γ and $-\nu_\gamma$. As $\nu_\gamma \notin \mathbb{N}$, so J_{ν_γ} and $J_{-\nu_\gamma}$ are linearly independent and therefore the pair $(J_{\nu_\gamma}, J_{-\nu_\gamma})$ (classical result) forms a fundamental system of solutions (2.40).

Then, using the series expansion of J_{ν_γ} and $J_{-\nu_\gamma}$, one obtains

$$\tilde{\Phi}_+(x) = \sum_{m=0}^{\infty} \tilde{c}^+_{\nu_\gamma,m} x^{1-\gamma+(2-\gamma)m}, \quad \tilde{\Phi}_-(x) = \sum_{m=0}^{\infty} \tilde{c}^-_{\nu_\gamma,m} x^{(2-\gamma)m}$$

with

$$\tilde{c}^+_{\nu_\gamma,m} = c^+_{\nu_\gamma,m} \left(\frac{2}{2-\gamma} i\lambda \right)^{2m+\nu_\gamma}, \quad \tilde{c}^-_{\nu_\gamma,m} = c^-_{\nu_\gamma,m} \left(\frac{2}{2-\gamma} i\lambda \right)^{2m-\nu_\gamma}.$$

Next one easily verifies that $\Phi_+, \Phi_- \in H^1_{0,\gamma}(0,1)$: indeed,

$$\begin{array}{ll} \tilde{\Phi}_+(x) \sim_0 \tilde{c}^+_{\nu_\gamma,0} x^{1-\gamma}, & x^{\gamma/2}\tilde{\Phi}'_+(x) \sim_0 (1-\gamma)\tilde{c}^+_{\nu_\gamma,0} x^{-\gamma/2}, \\ \tilde{\Phi}_-(x) \sim_0 \tilde{c}^-_{\nu_\gamma,0}, & x^{\gamma/2}\tilde{\Phi}'_-(x) \sim_0 (2-\gamma)\tilde{c}^-_{\nu_\gamma,0} x^{1-\gamma/2}, \end{array}$$

where we have used the following relation

$$xJ'_\nu(x) = \nu J_\nu(x) - xJ_{\nu+1}(x). \tag{2.44}$$

Hence, given c_+ and c_-, $u(x) = c_+\tilde{\Phi}_+(x) + c_-\tilde{\Phi}_-(x) \in H^1_{0,\gamma}(0,1)$ with the following boundary condition

$$\begin{cases} (x^\gamma u_x)(0) = \varrho\lambda(\lambda+\eta)^{\alpha-1}u(0), \\ u(1) = 0. \end{cases}$$

Then

$$M(\lambda)C(\lambda) = \begin{pmatrix} (1-\gamma)\tilde{c}^+_{\nu_\gamma,0} & -\varrho\lambda(\lambda+\eta)^{\alpha-1}\tilde{c}^-_{\nu_\gamma,0} \\ J_{\nu_\gamma}\left(\frac{2}{2-\gamma}i\lambda\right) & J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}i\lambda\right) \end{pmatrix} \begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.45}$$

Hence, a non-trivial solution u exists if and only if the determinant of $M(\lambda)$ vanishes. Set $f(\lambda) = detM(\lambda)$ thus the characteristic equation is $f(\lambda) = 0$.

Our purpose is to prove, thanks to Rouché's Theorem ([32]), that there is a subsequence of eigenvalues for which their real part tends to 0.

In the sequel, since $\mathcal{A}$ is dissipative, we study the asymptotic behavior of the large eigenvalues λ of $\mathcal{A}$ in the strip $-\alpha_0 \le \Re(\lambda) \le 0$, for some $\alpha_0 > 0$ large enough and for such λ, we remark that Φ_+, Φ_- remain bounded.

Lemma 2.4.1 *There exists $N \in \mathbb{N}$ such that*

$$\{\lambda_k\}_{k\in\mathbf{Z}^*, |k|\ge N} \subset \sigma(\mathcal{A}), \tag{2.46}$$

where
- *If $\gamma = 0$ and $\alpha = 1$, then*

$$\lambda_k = \left\{ \begin{array}{ll} \ln\sqrt{\dfrac{\varrho-1}{\varrho+1}} + ik\pi & \text{if } \varrho > 1 \\ \ln\sqrt{\dfrac{1-\varrho}{\varrho+1}} + i\left(k+\dfrac{1}{2}\right)\pi & \text{if } \varrho < 1 \end{array} \right\}, \quad k \in \mathbf{Z}.$$

- *If $0 < \gamma < 1$ and $\alpha = 1$, then*

$$\lambda_k = -\frac{2-\gamma}{2} i\left(k + \frac{\nu_\gamma}{2} + \frac{3}{4}\right)\pi + \frac{\beta}{k^{1-2\nu_\gamma}} + o\left(\frac{1}{k^{1-2\nu_\gamma}}\right), k \ge N, \beta \in \mathbb{R}, \beta < 0.$$

$$\lambda_k = \overline{\lambda_{-k}} \text{ if } k \le -N,$$

where

$$\beta = -\frac{1-\gamma}{\varrho}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\frac{\sin\nu_\gamma\pi}{\pi^{1-2\nu_\gamma}}.$$

- *If* $\alpha = 2\nu_\gamma$, *then*

$$\lambda_k = -i\frac{2-\gamma}{4}\left(2k\pi+\theta-\frac{\pi}{2}\right) - \frac{2-\gamma}{4}\ln\frac{1+\tilde{A}}{\sqrt{1+\tilde{A}^2+2\tilde{A}\cos 2\nu_\gamma\pi}} + O\left(\frac{1}{k}\right), \quad k\in\mathbf{Z},$$

$$\lambda_k = \overline{\lambda_{-k}} \text{ if } k \leq -N,$$

where

$$\tilde{A} = \frac{1-\gamma}{\varrho}\left(\frac{2}{2-\gamma}\right)^{2\nu_\gamma}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}$$

and θ *is such that*

$$\begin{cases} \cos\theta &= \dfrac{(1+\tilde{A})\cos\nu_\gamma\pi}{\sqrt{1+\tilde{A}^2+2\tilde{A}\cos 2\nu_\gamma\pi}}, \\ \sin\theta &= \dfrac{(1-\tilde{A})\sin\nu_\gamma\pi}{\sqrt{1+\tilde{A}^2+2\tilde{A}\cos 2\nu_\gamma\pi}}. \end{cases}$$

- *If* $\alpha > 2\nu_\gamma$, *then*

$$\lambda_k = -\frac{2-\gamma}{2}i\left(k+\frac{\nu_\gamma}{2}+\frac{3}{4}\right)\pi + \frac{\tilde{\alpha}}{k^{\alpha-2\nu_\gamma}} + \frac{\beta}{k^{\alpha-2\nu_\gamma}} + o\left(\frac{1}{k^{\alpha-2\nu_\gamma}}\right), k\geq N, \tilde{\alpha}\in i\mathbb{R}, \beta\in\mathbb{R}, \beta<0,$$

$$\lambda_k = \overline{\lambda_{-k}} \text{ if } k \leq -N,$$

where

$$\beta = -\frac{1-\gamma}{\varrho}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\left(\frac{2-\gamma}{2}\right)^{1-\alpha}\frac{\cos(1-\alpha)\frac{\pi}{2}\sin\nu_\gamma\pi}{\pi^{\alpha-2\nu_\gamma}}.$$

- *If* $\alpha < 2\nu_\gamma$, *then*

$$\lambda_k = -\frac{2-\gamma}{2}i\left(k-\frac{\nu_\gamma}{2}+\frac{3}{4}\right)\pi + \frac{\tilde{\alpha}}{k^{2\nu_\gamma-\alpha}} + \frac{\beta}{k^{2\nu_\gamma-\alpha}} + o\left(\frac{1}{k^{2\nu_\gamma-\alpha}}\right), k\geq N, \tilde{\alpha}\in i\mathbb{R}, \beta\in\mathbb{R}, \beta<0,$$

$$\lambda_k = \overline{\lambda_{-k}} \text{ if } k \leq -N,$$

where

$$\beta = -\frac{\varrho}{1-\gamma}\frac{c^-_{\nu_\gamma,0}}{c^+_{\nu_\gamma,0}}\left(\frac{2-\gamma}{2}\right)^{1+\alpha}\frac{\cos(1-\alpha)\frac{\pi}{2}\sin\nu_\gamma\pi}{\pi^{\alpha-2\nu_\gamma}}.$$

Moreover for all $|k|\geq N$, *the eigenvalues* λ_k *are simple.*

The proof of Lemma 2.4.1 will be given in Appendix A.

Now, setting $\tilde{U}_k = (\lambda_k^0 - \mathcal{A})U_k$, where U_k is a normalized eigenfunction associated to λ_k. We then have

$$\begin{aligned}\|(\lambda_k^0 - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} = \sup_{U\in\mathcal{H},U\neq 0} \frac{\|(\lambda_k^0 - \mathcal{A})^{-1}U\|_{\mathcal{H}}}{\|U\|_{\mathcal{H}}} &\geq \frac{\|(\lambda_k^0 - \mathcal{A})^{-1}\tilde{U}_k\|_{\mathcal{H}}}{\|\tilde{U}_k\|_{\mathcal{H}}} \\ &\geq \frac{\|U_k\|_{\mathcal{H}}}{\|(\lambda_k^0 - \mathcal{A})U_k\|_{\mathcal{H}}}.\end{aligned}$$

Hence, by Lemma 2.4.1, we deduce that

$$\|(\lambda_k^0 - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \geq c \begin{cases} |k|^{\alpha-2\nu_\gamma} & \text{if } \alpha > 2\nu_\gamma, \\ |k|^{2\nu_\gamma-\alpha} & \text{if } \alpha < 2\nu_\gamma. \end{cases}$$

Thus, (2.34) is not satisfied for $\alpha \neq 2\nu_\gamma$. So that, the semigroup $e^{t\mathcal{A}}$ is not exponentially stable. Thus the proof is complete.

■

2.5 Optimality of energy decay when $\eta \neq 0$

By Lemma 2.4.1, the spectrum of $\mathcal{A}$ is at the left of the imaginary axis, but approaches this axis for $\alpha \neq 2\nu_\gamma$. Hence, the decay of the energy depends on the asymptotic behavior of the real part of these eigenvalues.

Unfortunately, we were not able to prove this decay rate by frequency domain method based on multiplier method as the problem (P) is degenerate and the control is acting on the degenerate boundary.

To state and prove our stability results, we need some results from semigroup theory.

Theorem 2.5.1 ([6]) *Let $\mathcal{A}$ be the generator of a uniformly bounded C_0-semigroup $\{S(t)\}_{t\geq 0}$ on a Hilbert space $\mathcal{X}$. If:*

(i) $\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.

(ii) The intersection of the spectrum $\sigma(\mathcal{A})$ with $i\mathbb{R}$ is at most a countable set,

then the semigroup $\{S(t)\}_{t\geq 0}$ is asymptotically stable, i.e, $\|S(t)z\|_{\mathcal{X}} \to 0$ as $t \to \infty$ for any $z \in \mathcal{X}$.

Theorem 2.5.2 ([13]) *Let $S(t)$ be a bounded C_0-semigroup on a Hilbert space $\mathcal{X}$ with generator $\mathcal{A}$. If*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \text{ and } \overline{\lim_{|\beta|\to\infty}} \frac{1}{\beta^l}\|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{X})} < \infty,$$

for some l, then there exist c such that

$$\|e^{\mathcal{A}t}U_0\|^2 \leq \frac{c}{t^{\frac{2}{l}}}\|U_0\|^2_{D(\mathcal{A})}.$$

First, we use Theorem 2.5.1 to show the strong stability of the C_0-semigroup $e^{t\mathcal{A}}$ associated to the system (P'). Our main result is the following Theorem.

Theorem 2.5.3 *The C_0-semigroup $e^{t\mathcal{A}}$ is strongly stable in $\mathcal{H}$; i.e, for all $U_0 \in \mathcal{H}$, the solution of (2.10) satisfies*

$$\lim_{t\to\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0.$$

For the proof of Theorem 3.2.1, we need the following two lemmas.

Lemma 2.5.1 *$\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.*

Proof.
We make a distinction between $i\lambda = 0$ and $i\lambda \neq 0$.
Step 1. Solving for $\mathcal{A}U = 0$ leads to the system

$$\begin{cases} v = 0, \\ (x^\gamma u_x)_x = 0, \\ (\xi^2 + \eta)\phi = v(0)\mu(\xi) \end{cases} \tag{2.47}$$

together with the conditions (2.36).

Then $v = 0, \phi = 0, (x^\gamma u_x)(0) = 0$ and

$$(x^\gamma u_x)(x) = c,$$

where c is a constant. As $(x^\gamma u_x)(0) = 0$, we have $(x^\gamma u_x)(x) = 0$. Hence

$$u_x(x) = 0 \text{ for } x \in (0,1).$$

As $u(1) = 0$, then $u = 0$. We have $U = 0$. Hence, $i\lambda = 0$ is not an eigenvalue of $\mathcal{A}$.

Step 2. Let $\lambda \in \mathbb{R} - \{0\}$. We prove that $i\lambda$ is not an eigenvalue of $\mathcal{A}$. Let $U = (u, v, \phi)^T$ with $\|U\|_{\mathcal{H}} = 1$, be such that

$$\mathcal{A}U = i\lambda U. \tag{2.48}$$

Using the definition of $\mathcal{A}$ it follows that $\mathcal{A}U = i\lambda U$ if and only if

$$\begin{cases} i\lambda u = v, \\ i\lambda v = (x^\gamma u_x)_x, \\ i\lambda\phi + (\xi^2 + \eta)\phi = v(0)\mu(\xi) \end{cases} \tag{2.49}$$

together with the conditions (2.36). Using (2.13) and (2.48), we find

$$\phi \equiv 0, \tag{2.50}$$

then, using the third equation in (2.49), we deduce that

$$v(0) = 0. \tag{2.51}$$

Therefore, from $(2.49)_1$ and $(2.36)_2$, we get

$$u(0) = 0 \quad \text{and} \quad (x^\gamma u_x)(0) = 0. \tag{2.52}$$

Thus, by eliminating v, the system (2.49) implies that

$$\begin{cases} \lambda^2 u + (x^\gamma u_x)_x = 0 \text{ on } (0,1), \\ u(0) = u(1) = 0, \\ (x^\gamma u_x)(0) = 0. \end{cases} \tag{2.53}$$

The solution of the equation (2.53) is given by

$$u(x) = C_1\Phi_+(x) + C_2\Phi_-(x),$$

where Φ_+ and Φ_- are defined by

$$\Phi_+(x) = x^{\frac{1-\gamma}{2}} J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right), \quad \Phi_-(x) = x^{\frac{1-\gamma}{2}} J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right). \tag{2.54}$$

From boundary conditions $(2.53)_2$ and $(2.53)_3$, we deduce that

$$\begin{aligned} &C_2 \tilde{\tilde{c}}^-_{\nu_\gamma,0} = 0, \\ &C_1 J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda\right) + C_2 J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda\right) = 0, \\ &C_1(1-\gamma)\tilde{\tilde{c}}^+_{\nu_\gamma,0} = 0, \end{aligned}$$

where

$$\tilde{\tilde{c}}^-_{\nu_\gamma,0} = c^-_{\nu_\gamma,0}\left(\frac{2}{2-\gamma}\lambda\right)^{-\nu_\gamma}, \tilde{\tilde{c}}^+_{\nu_\gamma,0} = c^+_{\nu_\gamma,0}\left(\frac{2}{2-\gamma}\lambda\right)^{\nu_\gamma}.$$

Hence

$$u \equiv 0.$$

Therefore $U = 0$, which contradicts $\|U\|_{\mathcal{H}} = 1$. This completes the proof of Lemma 2.5.1. ■

Lemma 2.5.2
If $\lambda \neq 0$, the operator $i\lambda I - \mathcal{A}$ is surjective.
If $\lambda = 0$ and $\eta \neq 0$, the operator $i\lambda I - \mathcal{A}$ is surjective.

Proof.
Case 1: $\lambda \neq 0$. Let $F = (f_1, f_2, f_3)^T \in \mathcal{H}$ be given, and let $U = (u, v, \phi)^T \in D(\mathcal{A})$ be such that

$$(i\lambda I - \mathcal{A})U = F. \tag{2.55}$$

Equivalently, we have

$$\begin{cases} i\lambda u - v = f_1, \\ i\lambda v - (x^\gamma u_x)_x = f_2, \\ i\lambda\phi + (\xi^2 + \eta)\phi - \mu(\xi)v(0) = f_3 \end{cases} \tag{2.56}$$

together with the conditions (2.36).

Inserting $(2.56)_1$ into $(2.56)_2$, we get

$$-\lambda^2 u - (x^\gamma u_x)_x = (f_2 + i\lambda f_1). \tag{2.57}$$

Solving system (2.57) is equivalent to finding $u \in H^2_\gamma \cap H^1_{0,\gamma}(0,1)$ such that

$$\int_0^1 (-\lambda^2 u\overline{w} - (x^\gamma u_x)_x \overline{w}\, dx = \int_0^1 (f_2 + i\lambda f_1)\overline{w}\, dx \tag{2.58}$$

for all $w \in H^1_{0,\gamma}(0,1)$. By using $(2.56)_3$ and $(2.56)_1$ the function u satisfies the following system

$$\begin{cases} \displaystyle\int_0^1 (-\lambda^2 u\overline{w} + x^\gamma u_x \overline{w}_x)\, dx + i\varrho\lambda(i\lambda + \eta)^{\alpha-1} u(0)\overline{w}(0) \\ \displaystyle = \int_0^1 (f_2 + i\lambda f_1)\overline{w}\, dx - \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + i\lambda} f_3(\xi)\, d\xi\ \overline{w}(0) + \varrho(i\lambda + \eta)^{\alpha-1} f_1(0)\overline{w}(0). \end{cases} \tag{2.59}$$

We can rewrite (2.59) as

$$\mathcal{B}(u, w) = l(w), \quad \forall w \in H^1_{0,\gamma}(0,1), \tag{2.60}$$

where

$$\mathcal{B}(u, w) = \mathcal{B}_1(u, w) + \mathcal{B}_2(u, w)$$

with

$$\begin{cases} \mathcal{B}_1(u, w) = \displaystyle\int_0^1 x^\gamma u_x \overline{w}_x\, dx + i\varrho\lambda(i\lambda + \eta)^{\alpha-1} u(0)\overline{w}(0), \\ \mathcal{B}_2(u, w) = -\displaystyle\int_0^1 \lambda^2 u\overline{w}\, dx, \end{cases} \tag{2.61}$$

and

$$l(w) = \int_0^1 (f_2 + i\lambda f_1)\overline{w}\, dx - \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + i\lambda} f_3(\xi)\, d\xi\ \overline{w}(0) \\ +\varrho(i\lambda + \eta)^{\alpha-1} f_1(0)\overline{w}(0).$$

Let $(H^1_{0,\gamma}(0,1))'$ be the dual space of $H^1_{0,\gamma}(0,1)$. Let us define the following operators

$$\begin{array}{ll} B : H^1_{0,\gamma}(0,1) \to (H^1_{0,\gamma}(0,1))' & B_i : H^1_{0,\gamma}(0,1) \to (H^1_{0,\gamma}(0,1))' \quad i \in \{1,2\} \\ \qquad u \mapsto Bu & \qquad u \mapsto B_i u \end{array} \tag{2.62}$$

such that

$$\begin{array}{l} (Bu)w = \mathcal{B}(u,w),\ \forall w \in H^1_{0,\gamma}(0,1), \\ (B_i u)w = \mathcal{B}_i(u,w),\ \forall w \in H^1_{0,\gamma}(0,1), i \in \{1,2\}. \end{array} \tag{2.63}$$

We need to prove that the operator B is an isomorphism. For this aim, we divide the proof into three steps:

Step 1. In this step, we want to prove that the operator B_1 is an isomorphism. For this aim, it is easy to see that $\mathcal{B}_1$ is sesquilinear, continuous form on $H^1_{0,\gamma}(0,1)$. Furthermore

$$\begin{array}{rcl} \Re\mathcal{B}_1(u,u) & = & \|x^{\gamma/2}u_x\|_2^2 + \varrho\lambda\Re\left(i(i\lambda+\eta)^{\alpha-1}\right)|u(0)|^2 \\ & \geq & \|x^{\gamma/2}u_x\|_2^2, \end{array}$$

where we have used the fact that

$$\varrho\lambda\Re\left(i(i\lambda+\eta)^{\alpha-1}\right) = \zeta\lambda^2 \int_{-\infty}^{+\infty} \frac{\mu(\xi)^2}{\lambda^2 + (\eta+\xi^2)^2}\, d\xi > 0.$$

Thus $\mathcal{B}_1$ is coercive. Then, from (2.62) and Lax-Milgram theorem, the operator B_1 is an isomorphism.

Step 2. In this step, we want to prove that the operator B_2 is compact. For this aim, from (2.61) and (2.63), we have

$$|\mathcal{B}_2(u,w)| \leq c\|u\|_{L^2(0,1)}\|w\|_{L^2(0,1)},$$

and consequently, using the compact embedding from $H^1_{0,\gamma}(0,1)$ to $L^2(0,1)$ (see **[4]**) we deduce that B_2 is a compact operator. Therefore, from the above steps, we obtain that the operator $B = B_1 + B_2$ is a Fredholm operator of index zero. Now, following Fredholm alternative, we still need to prove that the operator B is injective to obtain that the operator B is an isomorphism.

Step 3. Let $u \in ker(B)$, then

$$\mathcal{B}(u,w) = 0 \quad \forall w \in H^1_{0,\gamma}(0,1). \tag{2.64}$$

In particular for $w = u$, it follows that

$$\lambda^2\|u\|^2_{L^2(0,1)} - i\varrho\lambda(i\lambda+\eta)^{\alpha-1}|u(0)|^2 = \|x^{\gamma/2}u_x\|^2_{L^2(0,1)}.$$

Hence, we have

$$u(0) = 0. \tag{2.65}$$

From (2.64), we obtain

$$(x^{\gamma/2}u_x)(0) = 0 \tag{2.66}$$

and then

$$\begin{cases} -\lambda^2 u - (x^\gamma u_x)_x = 0, \\ u(0) = (x^{\gamma/2}u_x)(0) = 0, \\ u(1) = 0. \end{cases} \tag{2.67}$$

Then, according to Lemma 2.5.1, we deduce that $u = 0$ and consequently $Ker(B) = \{0\}$. Finally, from Step 3 and Fredholm alternative, we deduce that the operator B is isomorphism. It is easy to see that the operator l is a antilinear and continuous form on $H^1_{0,\gamma}(0,1)$. Consequently, (2.60) admits a unique solution $u \in H^1_{0,\gamma}(0,1)$. By using the classical elliptic regularity, we deduce that $U \in D(\mathcal{A})$ is a unique solution of (2.55). Hence $i\lambda - \mathcal{A}$ is surjective for all $\lambda \in \mathbb{R}^*$.

Case 2: $\lambda = 0$ **and** $\eta \neq 0$**.** Using Lax-Milgram Lemma, we obtain the result.

Taking account of Lemmas 2.5.1, 2.5.2 and from Theorem 2.5.1 The C_0-semigroup $e^{t\mathcal{A}}$ is strongly stable in $\mathcal{H}$.

■

Next, by an explicit representation of the resolvent of the generator on the imaginary axis and the use of Theorem 2.5.2, we prove an optimal decay rate. Our main result is the following.

Theorem 2.5.4 *If $\eta \neq 0$, then the global solution of the problem (P) has the following energy decay property*

$$E(t) = \|S_{\mathcal{A}}(t)U_0\|^2_{\mathcal{H}} \leq \begin{cases} \dfrac{c}{t^{\frac{2}{\alpha - 2\nu_\gamma}}}\|U_0\|^2_{D(\mathcal{A})} & \text{if } \alpha > 2\nu_\gamma, \\ \dfrac{c}{t^{\frac{2}{2\nu_\gamma - \alpha}}}\|U_0\|^2_{D(\mathcal{A})} & \text{if } \alpha < 2\nu_\gamma, \\ ce^{-\omega t}\|U_0\|^2_{\mathcal{H}} & \text{if } \alpha = 2\nu_\gamma. \end{cases}$$

Moreover, the rate of energy decay is optimal for general initial data in $D(\mathcal{A})$.

Proof.
Let us consider the resolvent equation

$$\begin{cases} i\lambda u - v = f_1, \\ i\lambda v - (x^\gamma u_x)_x = f_2, \\ i\lambda\phi + (\xi^2 + \eta)\phi - v(0)\mu(\xi) = f_3, \end{cases} \tag{2.68}$$

where $F = (f_1, f_2, f_3)^T \in \mathcal{H}$. From $(2.68)_1$ and $(2.68)_2$, we have

$$\lambda^2 u + (x^\gamma u_x)_x = -(f_2 + i\lambda f_1) \tag{2.69}$$

with

$$\begin{cases} (x^\gamma u_x)(0) = \zeta \displaystyle\int_{-\infty}^{\infty} \mu(\xi)\phi(\xi)\, d\xi, \\ u(1) = 0. \end{cases} \tag{2.70}$$

The substitution of ϕ given by $(2.68)_3$ into $(2.70)_1$ gives us

$$(x^\gamma u_x)(0) = \varrho(i\lambda + \eta)^{\alpha-1} v(0) + \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi) f_3(\xi)}{i\lambda + \xi^2 + \eta}\, d\xi. \tag{2.71}$$

Moreover, from $(2.68)_1$, we have

$$v(0) = i\lambda u(0) - f_1(0).$$

Then, the condition (2.71) become

$$(x^\gamma u_x)(0) - \varrho i\lambda(i\lambda + \eta)^{\alpha-1} u(0) = -\varrho(i\lambda + \eta)^{\alpha-1} f_1(0) + \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi) f_3(\xi)}{i\lambda + \xi^2 + \eta}\, d\xi. \tag{2.72}$$

Assume that Φ is a solution of (2.69), then one easily checks that the function Ψ defined by

$$\Phi(x) = x^{\frac{1-\gamma}{2}} \Psi\left(\frac{2}{2-\gamma} \lambda x^{\frac{2-\gamma}{2}}\right) \tag{2.73}$$

is solution of the following inhomogeneous Bessel equation:

$$\begin{aligned} &y^2 \Psi''(y) + y\Psi'(y) + \left(y^2 - \left(\frac{\gamma-1}{2-\gamma}\right)^2\right)\Psi(y) = \\ &-(\tfrac{2}{2-\gamma})^2 (\tfrac{2-\gamma}{2}\tfrac{1}{\lambda} y)^{\frac{3-\gamma}{2-\gamma}} \left(f_2\left((\tfrac{2-\gamma}{2}\tfrac{1}{\lambda} y)^{\frac{2}{2-\gamma}}\right) + i\lambda f_1\left((\tfrac{2-\gamma}{2}\tfrac{1}{\lambda} y)^{\frac{2}{2-\gamma}}\right)\right). \end{aligned} \tag{2.74}$$

The solution can be written as

$$\Psi(y) = AJ_{\nu_\gamma}(y) + BJ_{-\nu_\gamma}(y) - \frac{\pi}{2\sin\nu_\gamma\pi} \int_0^y \frac{f(s)}{s} \left(J_{\nu_\gamma}(s) J_{-\nu_\gamma}(y) - J_{\nu_\gamma}(y) J_{-\nu_\gamma}(s)\right) ds,$$

where

$$f(s) = -(\frac{2}{2-\gamma})^2 (\frac{2-\gamma}{2}\frac{1}{\lambda} s)^{\frac{3-\gamma}{2-\gamma}} \left(f_2\left((\frac{2-\gamma}{2}\frac{1}{\lambda} s)^{\frac{2}{2-\gamma}}\right) + i\lambda f_1\left((\frac{2-\gamma}{2}\frac{1}{\lambda} s)^{\frac{2}{2-\gamma}}\right)\right)$$

Thus,

$$u(x) = Ax^{\frac{1-\gamma}{2}} J_{\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right) + Bx^{\frac{1-\gamma}{2}} J_{-\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right)$$
$$+\frac{\pi}{2\sin\nu_\gamma\pi}\left(\frac{2}{2-\gamma}\right) x^{\frac{1-\gamma}{2}}\int_0^x s^{\frac{1-\gamma}{2}}(f_2(s)+i\lambda f_1(s))\left(J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda s^{\frac{2-\gamma}{2}}\right)J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right)\right.$$
$$\left.-J_{\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right)J_{-\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda s^{\frac{2-\gamma}{2}}\right)\right)\,ds.$$

Therefore,

$$\begin{aligned} u(x) &= A\Phi_+(x) + B\Phi_-(x) \\ &\quad + \frac{\pi}{2\sin\nu_\gamma\pi}\left(\frac{2}{2-\gamma}\right)\int_0^x (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi_-(x)-\Phi_+(x)\Phi_-(s))\,ds, \end{aligned} \tag{2.75}$$

where Φ_+ and Φ_- are defined by

$$\Phi_+(x) = x^{\frac{1-\gamma}{2}} J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right), \quad \Phi_-(x) = x^{\frac{1-\gamma}{2}} J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right). \tag{2.76}$$

Then

$$\begin{aligned} u_x(x) &= A\Phi'_+(x) + B\Phi'_-(x) \\ &\quad + \frac{\pi}{2\sin\nu_\gamma\pi}\left(\frac{2}{2-\gamma}\right)\int_0^x (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi'_-(x)-\Phi'_+(x)\Phi_-(s))\,ds. \end{aligned} \tag{2.77}$$

From (2.72), (2.77) and (2.75), we conclude that

$$(1-\gamma)\tilde{c}^{+}_{\nu_\gamma,0}A - \varrho i\lambda(i\lambda+\eta)^{\alpha-1}\tilde{c}^{-}_{\nu_\gamma,0}B = -\varrho(i\lambda+\eta)^{\alpha-1}f_1(0) + \zeta\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_3(\xi)}{i\lambda+\xi^2+\eta}\,d\xi \tag{2.78}$$

$$A\Phi_+(1) + B\Phi_-(1) = -\frac{\pi}{2\sin\nu_\gamma\pi}\left(\frac{2}{2-\gamma}\right)\int_0^1 (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi_-(1)-\Phi_+(1)\Phi_-(s))\,ds, \tag{2.79}$$

where

$$\tilde{c}^{+}_{\nu_\gamma,m} = c^{+}_{\nu_\gamma,m}\left(\frac{2}{2-\gamma}\lambda\right)^{2m+\nu_\gamma}, \quad \tilde{c}^{+}_{\nu_\gamma,m} = c^{-}_{\nu_\gamma,m}\left(\frac{2}{2-\gamma}\lambda\right)^{2m-\nu_\gamma}$$

and

$$\Phi_+(1) = J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda\right), \qquad \Phi_-(1) = J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda\right).$$

Using (2.78) and (2.79), a linear system in A and B is obtained

$$\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} C \\ \tilde{C} \end{pmatrix}, \tag{2.80}$$

where

$$\begin{aligned} r_{11} &= (1-\gamma)\tilde{c}^{+}_{\nu_\gamma,0}, \\ r_{12} &= -\varrho i\lambda(i\lambda+\eta)^{\alpha-1}\tilde{c}^{-}_{\nu_\gamma,0}, \\ r_{21} &= J_{\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda\right), \\ r_{22} &= J_{-\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda\right), \end{aligned}$$

$$\begin{aligned} C &= -\varrho(i\lambda+\eta)^{\alpha-1}f_1(0) + \zeta\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_3(\xi)}{i\lambda+\xi^2+\eta}\,d\xi, \\ \tilde{C} &= -\frac{\pi}{2\sin\nu_\gamma\pi}(\frac{2}{2-\gamma})\int_0^1 (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi_-(1)-\Phi_+(1)\Phi_-(s))\,ds. \end{aligned}$$

Let the determinant of the linear system given in (2.80) be denoted by D. Then

$$\begin{aligned} D &= (1-\gamma)\tilde{c}^{+}_{\nu_\gamma,0}J_{-\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda\right) + \varrho i\lambda(i\lambda+\eta)^{\alpha-1}\tilde{c}^{-}_{\nu_\gamma,0}J_{\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda\right) \\ &= (1-\gamma)c^{+}_{\nu_\gamma,0}\left(\tfrac{2}{2-\gamma}\right)^{\nu_\gamma}\lambda^{\nu_\gamma}\left[\left(\frac{2-\gamma}{\pi\lambda}\right)^{1/2}\cos\left(\tfrac{2}{2-\gamma}\lambda+\nu_\gamma\tfrac{\pi}{2}-\tfrac{\pi}{4}\right)+O(\frac{1}{\lambda^{3/2}})\right] \\ &\quad +\varrho i\lambda(i\lambda+\eta)^{\alpha-1}c^{-}_{\nu_\gamma,0}\left(\tfrac{2}{2-\gamma}\right)^{-\nu_\gamma}\lambda^{-\nu_\gamma}\left[\left(\frac{2-\gamma}{\pi\lambda}\right)^{1/2}\cos\left(\tfrac{2}{2-\gamma}\lambda-\nu_\gamma\tfrac{\pi}{2}-\tfrac{\pi}{4}\right)+O(\frac{1}{\lambda^{3/2}})\right] \\ &= (1-\gamma)c^{+}_{\nu_\gamma,0}\left(\tfrac{2}{2-\gamma}\right)^{\nu_\gamma}\left(\frac{2-\gamma}{\pi}\right)^{1/2}\lambda^{\nu_\gamma-\frac{1}{2}}\cos\left(\tfrac{2}{2-\gamma}\lambda+\nu_\gamma\tfrac{\pi}{2}-\tfrac{\pi}{4}\right) \\ &\quad +\varrho i^{\alpha}c^{-}_{\nu_\gamma,0}\left(\tfrac{2}{2-\gamma}\right)^{-\nu_\gamma}\left(\frac{2-\gamma}{\pi}\right)^{1/2}\lambda^{\alpha-\nu_\gamma-\frac{1}{2}}\cos\left(\tfrac{2}{2-\gamma}\lambda-\nu_\gamma\tfrac{\pi}{2}-\tfrac{\pi}{4}\right) \\ &\qquad O(\frac{1}{\lambda^{3/2-\nu_\gamma}})+O(\frac{1}{\lambda^{3/2+\nu_\gamma-\alpha}}). \end{aligned}$$

As $D\neq 0$ for all $\lambda\neq 0$, then A and B are uniquely determined by (2.80).

Now, we will prove that

$$|D| \geq \begin{cases} c|\lambda|^{\nu_\gamma-1/2} \text{ for large } \lambda \text{ if } \alpha\geq 2\nu_\gamma, \\ c|\lambda|^{\alpha-\nu_\gamma-1/2} \text{ for large } \lambda \text{ if } \alpha\leq 2\nu_\gamma. \end{cases} \tag{2.81}$$

Indeed, suppose (2.81) was wrong. We consider the case $\alpha\geq 2\nu_\gamma$. The case $\alpha\leq 2\nu_\gamma$ is similar. Then $\exists\lambda_n$ such that $|\lambda_n|\to\infty$ with

$$|D||\lambda_n|^{1/2-\nu_\gamma}\to 0 \text{ as } n\to\infty. \tag{2.82}$$

By $\Re D$,

$$|\lambda_n|^{1/2}J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda_n\right)\to 0 \text{ as } n\to+\infty.$$

By $\Im D$,

$$|\lambda_n|^{\alpha-2\nu_\gamma+1/2} J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda_n\right) \to 0 \text{ as } n \to +\infty.$$

By asymptotic behavior of J_{ν_γ} and $J_{-\nu_\gamma}$ (see formula (2.87)), we obtain

$$\begin{cases} \cos\left(\frac{2}{2-\gamma}\lambda_n + \nu_\gamma\frac{\pi}{2} - \frac{\pi}{4}\right) \to 0 \text{ as } n \to +\infty, \\ |\lambda_n|^{\alpha-2\nu_\gamma}\cos\left(\frac{2}{2-\gamma}\lambda_n - \nu_\gamma\frac{\pi}{2} - \frac{\pi}{4}\right) \to 0 \text{ as } n \to +\infty. \end{cases}$$

This is impossible. Indeed, $\exists k_n \in \mathbf{Z}$ with $|k_n| \to +\infty$ $n \to +\infty$ such that

$$\frac{2}{2-\gamma}\lambda_n + \nu_\gamma\frac{\pi}{2} - \frac{\pi}{4} = (k_n + \frac{1}{2})\pi + o(1).$$

Then

$$\left|\cos\left(\frac{2}{2-\gamma}\lambda_n - \nu_\gamma\frac{\pi}{2} - \frac{\pi}{4}\right)\right| \to \sin\nu_\gamma\pi \text{ as } n \to +\infty.$$

In the following Lemma, we will prove some technical inequalities which will be useful for showing the optimal polynomial decay of the solution.

Lemma 2.5.3
(I) *For all $\lambda \in \mathbb{R} - \{0\}$ large, we have*

$$\|\Phi_+\|_{L^2(0,1)}, \|\Phi_-\|_{L^2(0,1)} \le \frac{c}{\sqrt{|\lambda|}}. \tag{2.83}$$

(II)

$$\left\|x^{-\frac{1}{2}}J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right)\right\|_{L^2(0,1)}, \left\|x^{-\frac{1}{2}}J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right)\right\|_{L^2(0,1)} \le c\sqrt{|\lambda|}. \tag{2.84}$$

(III) *There exists a constant $C > 0$ such that, for all $f_1 \in H^1_{0,\gamma}(0,1), f_2 \in L^2(0,1)$ and $\lambda \in \mathbb{R} - \{0\}$,*

$$\left|\int_0^1 (f_2(s) + i\lambda f_1(s))(\Phi_+(s)\Phi_-(1) - \Phi_+(1)\Phi_-(s))\,ds\right| \le \frac{C\left(\|f_1\|_{H^1_{0,\gamma}(0,1)} + \|f_2\|_{L^2(0,1)}\right)}{|\lambda|}. \tag{2.85}$$

The proof of Lemma 2.5.3 will be given in Appendix B.

Now

$$A = \frac{1}{D}(Cr_{22} - \tilde{C}r_{12}),$$

$$B = \frac{1}{D}(-Cr_{21} + \tilde{C}r_{11}).$$

Considering only the dominant terms of λ, the following is obtained:

$$\begin{array}{rcl} |D||A| & \leq & c_1|\lambda|^{\alpha-\frac{3}{2}} + c_2|\lambda|^{\alpha-\nu_\gamma-1} \leq c_3|\lambda|^{\alpha-\nu_\gamma-1}, \\ |D||B| & \leq & c_1|\lambda|^{\alpha-\frac{3}{2}} + c_2|\lambda|^{\nu_\gamma-1} \leq \begin{cases} c_3|\lambda|^{\alpha-\nu_\gamma-1} \text{ if } \alpha > 2\nu_\gamma, \\ \tilde{c}_3|\lambda|^{\nu_\gamma-1} \text{ if } \alpha < 2\nu_\gamma, \end{cases} \end{array}$$

where we have used the fact that $f_1 \in H^1_{0,\gamma}(0,1)$ and

$$\left| \int_0^1 (f_2(s) + i\lambda f_1(s))(\Phi_+(s)\Phi_-(1) - \Phi_+(1)\Phi_-(s))\, ds \right| \leq \frac{1}{|\lambda|} \left(\|f_1\|_{H^1_{0,\gamma}(0,1)} + \|f_2\|_{L^2(0,1)} \right),$$

$$\left| \int_0^1 (f_2(s) + i\lambda f_1(s))(\Phi_+(s)\Phi_-'(1) - \Phi_+'(1)\Phi_-(s))\, ds \right| \leq \left(\|f_1\|_{H^1_{0,\gamma}(0,1)} + \|f_2\|_{L^2(0,1)} \right).$$

Then, we conclude that

$$|A| \leq \begin{cases} c|\lambda|^{\alpha-2\nu_\gamma-\frac{1}{2}} \text{ if } \text{ if } \alpha > 2\nu_\gamma, \\ c|\lambda|^{-1/2} \text{ if } \text{ if } \alpha < 2\nu_\gamma, \end{cases}$$

$$|B| \leq \begin{cases} c|\lambda|^{\alpha-2\nu_\gamma-\frac{1}{2}} \text{ if } \text{ if } \alpha > 2\nu_\gamma, \\ c|\lambda|^{2\nu_\gamma-\alpha-\frac{1}{2}} \text{ if } \text{ if } \alpha < 2\nu_\gamma. \end{cases}$$

Then

$$\|u\|_{L^2(0,1)} \leq \begin{cases} c|\lambda|^{\alpha-2\nu_\gamma-1} \left(\|f_1\|_{H^1_{0,\gamma}(0,1)} + \|f_2\|_{L^2(0,1)} \right) \text{ if } \alpha > 2\nu_\gamma, \\ c|\lambda|^{2\nu_\gamma-\alpha-1} \left(\|f_1\|_{H^1_{0,\gamma}(0,1)} + \|f_2\|_{L^2(0,1)} \right) \text{ if } \text{ if } \alpha < 2\nu_\gamma. \end{cases}$$

Using $(2.68)_1$ and (2.75), we get

$$\|v\|_{L^2(0,1)} \leq \begin{cases} c|\lambda|^{\alpha-2\nu_\gamma} \left(\|f_1\|_{H^1_{0,\gamma}(0,1)} + \|f_2\|_{L^2(0,1)} \right) \text{ if } \alpha > 2\nu_\gamma, \\ c|\lambda|^{2\nu_\gamma-\alpha} \left(\|f_1\|_{H^1_{0,\gamma}(0,1)} + \|f_2\|_{L^2(0,1)} \right) \text{ if } \text{ if } \alpha < 2\nu_\gamma. \end{cases}$$

From (2.76) and (2.44), we have

$$\begin{cases} x^{\gamma/2}\Phi_+'(x) &=& (\frac{1-\gamma}{2} + \frac{2\nu_\gamma}{2-\gamma})x^{-1/2}J_{\nu_\gamma}\left(\frac{2}{2-\gamma}x\right) - \lambda x^{\frac{1-\gamma}{2}}J_{1+\nu_\gamma}\left(\frac{2}{2-\gamma}x\right), \\ x^{\gamma/2}\Phi_-'(x) &=& (\frac{1-\gamma}{2} - \frac{2\nu_\gamma}{2-\gamma})x^{-1/2}J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}x\right) - \lambda x^{\frac{1-\gamma}{2}}J_{1-\nu_\gamma}\left(\frac{2}{2-\gamma}x\right). \end{cases}$$

Then from (2.77), we can get

$$\|x^{\gamma/2}u_x\|_{L^2(0,1)} \leq \begin{cases} c|\lambda|^{\alpha-2\nu_\gamma} \left(\|f_1\|_{H^1_{0,\gamma}(0,1)} + \|f_2\|_{L^2(0,1)} \right) \text{ if } \alpha > 2\nu_\gamma, \\ c|\lambda|^{2\nu_\gamma-\alpha} \left(\|f_1\|_{H^1_{0,\gamma}(0,1)} + \|f_2\|_{L^2(0,1)} \right) \text{ if } \alpha < 2\nu_\gamma. \end{cases}$$

Now, taking inner product of (2.68) with U in $\mathcal{H}$ and using (2.13) we get

$$|Re\langle \mathcal{A}U, U\rangle| \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

This implies that

$$\zeta \int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi)|^2\, d\xi \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{2.86}$$

Since $\eta > 0$, we have

$$\|\phi\|^2_{L^2(-\infty,\infty)} \leq \int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi)|^2\, d\xi \leq c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

Thus, we conclude that

$$\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \begin{cases} c|\lambda|^{\alpha-2\nu_\gamma} & \text{as } |\lambda| \to \infty \quad \text{if } \alpha > 2\nu_\gamma, \\ c|\lambda|^{2\nu_\gamma-\alpha} & \text{as } |\lambda| \to \infty \quad \text{if } \alpha < 2\nu_\gamma, \\ c & \text{as } |\lambda| \to \infty \quad \text{if } \alpha = 2\nu_\gamma. \end{cases}$$

The conclusion then follows by applying Theorem 2.5.2 for $\alpha \neq 2\nu_\gamma$ and Theorem 2.4.1 for $\alpha = 2\nu_\gamma$.

Besides, we prove that the decay rate is optimal. Indeed, the decay rate is consistent with the asymptotic expansion of eigenvalues.

■

2.6 Appendix

2.6.1 Appendix A. Proof of Lemma 2.4.1

This appendix is devoted to prove Lemma 2.4.1.

Proof.

• $\gamma = 0$ and $\alpha = 1$.

System (2.39) becomes

$$\begin{cases} \lambda^2 u - u_{xx} = 0, \\ u_x(0) = \varrho\lambda u(0), \\ u(1) = 0. \end{cases}$$

The solution u is given by

$$u = c_1 e^{\lambda x} + c_2 e^{-\lambda x}.$$

Thus, the boundary conditions give

$$e^{2\lambda} = \frac{\varrho - 1}{\varrho + 1}.$$

If $\varrho > 1$ and if we set $\lambda = x + iy$, then

$$e^{2x} = \frac{\varrho - 1}{\varrho + 1} \text{ and } e^{2iy} = 1.$$

Hence

$$x = \frac{1}{2} \ln \frac{\varrho - 1}{\varrho + 1} \text{ and } y = k\pi, \quad k \in \mathbf{Z}.$$

Then

$$\lambda = \frac{1}{2} \ln \frac{\varrho - 1}{\varrho + 1} + ik\pi, \quad k \in \mathbf{Z}.$$

Now if $\varrho < 1$, we have

$$e^{2x} = \frac{1 - \varrho}{\varrho + 1} \text{ and } e^{2iy} = -1.$$

Hence

$$x = \frac{1}{2} \ln \frac{1 - \varrho}{\varrho + 1} \text{ and } y = (k + \frac{1}{2})\pi, \quad k \in \mathbf{Z}.$$

Then

$$\lambda = \frac{1}{2} \ln \frac{1 - \varrho}{\varrho + 1} + i(k + \frac{1}{2})\pi, \quad k \in \mathbf{Z}.$$

• $0 < \gamma < 1$ and $\alpha = 1$.

Step 1. From (2.45), our aim is to solve the equation

$$f(\lambda) = (1-\gamma)\tilde{c}^{+}_{\nu_\gamma,0} J_{-\nu_\gamma}\left(\frac{2}{2-\gamma} i\lambda\right) + \varrho\lambda\tilde{c}^{-}_{\nu_\gamma,0} J_{\nu_\gamma}\left(\frac{2}{2-\gamma} i\lambda\right) = 0$$

We will use the following classical development (see **[44]** p. 122, (5.11.6)): for all $\delta > 0$, the following development holds when $|\arg z| < \pi - \delta$:

$$J_\nu(z) = \frac{\sqrt{2}}{\sqrt{\pi z}} \left[\cos\left(z - \nu\frac{\pi}{2} - \frac{\pi}{4}\right) - \frac{(\nu^2 - \frac{1}{4})}{2} \frac{\sin\left(z - \nu\frac{\pi}{2} - \frac{\pi}{4}\right)}{z} + O\left(\frac{1}{|z|^2}\right) \right]. \tag{2.87}$$

Then

$$f(\lambda) = \left(\frac{2}{\pi\tilde{z}}\right)^{1/2} \varrho\lambda^{1-\nu_\gamma} c^{-}_{\nu_\gamma,0} \left(\frac{2}{2-\gamma} i\right)^{-\nu_\gamma} \frac{e^{-i(\tilde{z} - \nu_\gamma\frac{\pi}{2} - \frac{\pi}{4})}}{2} \tilde{f}(\lambda), \tag{2.88}$$

where

$$\tilde{z} = \frac{2}{2-\gamma} i\lambda$$

and

$$\begin{aligned}
\tilde{f}(\lambda) &= (e^{2i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}+1)+\frac{1-\gamma}{\varrho}\left(\frac{2}{2-\gamma}i\right)^{2\nu_\gamma}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\frac{e^{2i(\tilde{z}-\frac{\pi}{4})}+e^{-i\nu_\gamma\pi}}{\lambda^{1-2\nu_\gamma}}+O\left(\frac{1}{\lambda}\right) \\
&= f_0(\lambda)+\frac{f_1(\lambda)}{\lambda^{1-2\nu_\gamma}}+O\left(\frac{1}{\lambda}\right),
\end{aligned} \tag{2.89}$$

where

$$f_0(\lambda)=e^{2i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}+1. \tag{2.90}$$

$$f_1(\lambda)=\frac{1-\gamma}{\varrho}\left(\frac{2}{2-\gamma}i\right)^{2\nu_\gamma}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}(e^{2i(\tilde{z}-\frac{\pi}{4})}+e^{-i\nu_\gamma\pi}). \tag{2.91}$$

Note that f_0 and f_1 remain bounded in the strip $-\alpha_0 \le \Re(\lambda) \le 0$.
Step 2. We look at the roots of f_0. From (2.90), f_0 has one family of roots that we denote λ_k^0.

$$f_0(\lambda)=0 \Leftrightarrow e^{2i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}+1=0$$

Hence

$$2i\left(\frac{2}{2-\gamma}i\lambda-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4}\right)=i(2k+1)\pi, \quad k\in\mathbf{Z},$$

i.e.,

$$\lambda_k^0=-\frac{2-\gamma}{2}i\left(k+\frac{\nu_\gamma}{2}+\frac{3}{4}\right)\pi, \quad k\in\mathbf{Z}.$$

We will now use Rouché's Theorem. Let $B_k(\lambda_k^0, r_k)$ be the ball of centrum λ_k^0 and radius $r_k=\dfrac{1}{k^{(1-2\nu_\gamma)/2}}$ and $\lambda\in\partial B_k$ (i.e $\lambda=\lambda_k^0+r_ke^{i\theta}, \theta\in[0,2\pi]$). Then, we successively have:

$$f_0(\lambda)=\frac{4}{2-\lambda}r_ke^{i\theta}+O(r_k^2).$$

It follows that there exists a positive constant c such that

$$\forall\lambda\in\partial B_k, |f_0(\lambda)|\ge cr_k=\frac{c}{k^{(1-2\nu_\gamma)/2}}.$$

Then we deduce from (2.91) that $|\tilde{f}(\lambda)-f_0(\lambda)|=O\left(\frac{1}{\lambda^{(1-2\nu_\gamma)}}\right)=O\left(\frac{1}{k^{(1-2\nu_\gamma)}}\right)$. It follows that, for k large enough

$$\forall\lambda\in\partial B_k, |\tilde{f}(\lambda)-f_0(\lambda)|<|f_0(\lambda)|,$$

Then $\tilde{f}$ and f_0 have the same number of zeros in B_k. Consequently, there exists a subsequence of roots of $\tilde{f}$ which tends to the roots λ_k^0 of f_0. Equivalently, it means that there exists $N\in\mathbb{N}$ and a subsequence $\{\lambda_k\}_{|k|\ge N}$ of roots of $f(\lambda)$, such that $\lambda_k=\lambda_k^0+o(1)$ which tends to the roots $-\dfrac{2-\gamma}{2}i\left(k+\dfrac{\nu_\gamma}{2}+\dfrac{3}{4}\right)\pi$ of f_0. Finally for $|k|\ge N, \lambda_k$ is simple since λ_k^0 is.

Step 3. From Step 2, we can write

$$\lambda_k = -\frac{2-\gamma}{2} i \left(k + \frac{\nu_\gamma}{2} + \frac{3}{4} \right) \pi + \varepsilon_k. \tag{2.92}$$

Using (2.92), we get

$$\begin{aligned} e^{2i\left(\frac{2}{2-\gamma} i\lambda_k - \nu_\gamma \frac{\pi}{2} - \frac{\pi}{4}\right)} &= -e^{-\frac{4}{2-\gamma}\varepsilon_k} \\ &= -1 + \tfrac{4}{2-\gamma}\varepsilon_k + O(\varepsilon_k^2). \end{aligned} \tag{2.93}$$

Substituting (2.93) into (2.90), using that $\tilde{f}(\lambda_k) = 0$, we get:

$$\tilde{f}(\lambda_k) = \frac{4}{2-\gamma}\varepsilon_k - \frac{1-\gamma}{\varrho} \left(\frac{2}{2-\gamma} i \right)^{2\nu_\gamma} \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}} \frac{2i \sin \nu_\gamma \pi}{(-\frac{2-\gamma}{2} ik\pi)^{1-2\nu_\gamma}} + o(\varepsilon_k) + o\left(\frac{1}{k^{1-2\nu_\gamma}} \right) = 0 \tag{2.94}$$

and hence

$$\varepsilon_k = -\frac{1-\gamma}{\varrho} \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}} \frac{\sin \nu_\gamma \pi}{(k\pi)^{1-2\nu_\gamma}} + o\left(\frac{1}{k^{1-2\nu_\gamma}} \right) \tag{2.95}$$

From (2.95) we have in that case $|k|^{1-\alpha} \Re \lambda_k \sim \beta$ with

$$\beta = -\frac{1-\gamma}{\varrho} \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}} \frac{\sin \nu_\gamma \pi}{(\pi)^{1-2\nu_\gamma}}.$$

- $\alpha = 2\nu_\gamma$.
  From (2.45), our aim is to solve the equation

$$f(\lambda) = (1-\gamma)\tilde{c}^+_{\nu_\gamma,0} J_{-\nu_\gamma}\left(\frac{2}{2-\gamma} i\lambda \right) + \varrho\lambda(\lambda+\eta)^{\alpha-1} \tilde{c}^-_{\nu_\gamma,0} J_{\nu_\gamma}\left(\frac{2}{2-\gamma} i\lambda \right) = 0$$

Then

$$f(\lambda) = \left(\frac{2}{\pi \tilde{z}} \right)^{1/2} \varrho \lambda^{1-\nu_\gamma} (\lambda+\eta)^{\alpha-1} c^-_{\nu_\gamma,0} \left(\frac{2}{2-\gamma} i \right)^{-\nu_\gamma} \frac{e^{-i(\tilde{z} - \nu_\gamma \frac{\pi}{2} - \frac{\pi}{4})}}{2} \tilde{f}(\lambda), \tag{2.96}$$

where

$$\begin{aligned} \tilde{f}(\lambda) &= (e^{2i(\tilde{z} - \nu_\gamma \frac{\pi}{2} - \frac{\pi}{4})} + 1) + \frac{1-\gamma}{\varrho} \left(\frac{2}{2-\gamma} i \right)^{2\nu_\gamma} \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}} (e^{2i(\tilde{z} - \frac{\pi}{4})} + e^{-i\nu_\gamma \pi}) + O\left(\frac{1}{\lambda} \right) \\ &= f_0(\lambda) + O\left(\tfrac{1}{\lambda}\right). \end{aligned} \tag{2.97}$$

We look at the roots of f_0. From (2.90), f_0 has one family of roots that we denote λ_k^0.

$$f_0(\lambda) = 0 \Leftrightarrow e^{2i\tilde{z}} = -i \frac{1+\tilde{A}}{e^{-i\nu_\gamma \pi} + \tilde{A} e^{i\nu_\gamma \pi}},$$

where
$$\tilde{A} = \frac{1-\gamma}{\varrho}\left(\frac{2}{2-\gamma}\right)^{2\nu_\gamma}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}.$$
Let us set $\lambda = x+iy$. Then, we have
$$\begin{cases} e^{-\frac{4}{2-\gamma}x} = \dfrac{1+\tilde{A}}{\sqrt{1+\tilde{A}^2+2\tilde{A}\cos 2\nu_\gamma\pi}}, \\ -\frac{4}{2-\gamma}y = 2k\pi - \dfrac{\pi}{2}+\theta, \quad k\in \mathbf{Z}, \end{cases}$$
where θ is such that
$$\begin{cases} \cos\theta &= \dfrac{(1+\tilde{A})\cos\nu_\gamma\pi}{\sqrt{1+\tilde{A}^2+2\tilde{A}\cos 2\nu_\gamma\pi}}, \\ \sin\theta &= \dfrac{(1-\tilde{A})\sin\nu_\gamma\pi}{\sqrt{1+\tilde{A}^2+2\tilde{A}\cos 2\nu_\gamma\pi}}. \end{cases}$$
Hence
$$\begin{cases} x = -\frac{2-\gamma}{4}\ln\dfrac{1+\tilde{A}}{\sqrt{1+\tilde{A}^2+2\tilde{A}\cos 2\nu_\gamma\pi}}, \\ y = -\frac{2-\gamma}{4}(2k\pi - \dfrac{\pi}{2}+\theta), \quad k\in \mathbf{Z}. \end{cases}$$
Now with the help of Rouché's Theorem, we conclude.

• $\alpha > 2\nu_\gamma$.

Step 1. From (2.45), we have
$$f(\lambda) = \left(\frac{2}{\pi\tilde{z}}\right)^{1/2}\varrho\lambda^{1-\nu_\gamma}(\lambda+\eta)^{\alpha-1}c^-_{\nu_\gamma,0}\left(\frac{2}{2-\gamma}i\right)^{-\nu_\gamma}\frac{e^{-i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}}{2}\tilde{f}(\lambda), \tag{2.98}$$
$$\begin{aligned} \tilde{f}(\lambda) &= (e^{2i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}+1) + \frac{1-\gamma}{\varrho}\left(\frac{2}{2-\gamma}i\right)^{2\nu_\gamma}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\frac{e^{2i(\tilde{z}-\frac{\pi}{4})}+e^{-i\nu_\gamma\pi}}{\lambda^{\alpha-2\nu_\gamma}} + O\left(\frac{1}{\lambda}\right) \\ &= f_0(\lambda) + \tfrac{f_1(\lambda)}{\lambda^{\alpha-2\nu_\gamma}} + O\left(\tfrac{1}{\lambda}\right), \end{aligned} \tag{2.99}$$
where
$$f_0(\lambda) = e^{2i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}+1. \tag{2.100}$$
$$f_1(\lambda) = \frac{1-\gamma}{\varrho}\left(\frac{2}{2-\gamma}i\right)^{2\nu_\gamma}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}(e^{2i(\tilde{z}-\frac{\pi}{4})}+e^{-i\nu_\gamma\pi}). \tag{2.101}$$
We look at the roots of f_0. From (2.100), f_0 has one family of roots that we denote λ^0_k.
$$f_0(\lambda) = 0 \Leftrightarrow e^{2i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}+1 = 0.$$

Hence

$$2i\left(\frac{2}{2-\gamma}i\lambda-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4}\right)=i(2k+1)\pi,\quad k\in\mathbf{Z},$$

i.e.,

$$\lambda_k^0=-\frac{2-\gamma}{2}i\left(k+\frac{\nu_\gamma}{2}+\frac{3}{4}\right)\pi,\quad k\in\mathbf{Z}.$$

Step 2. From Step 1, we can write

$$\lambda_k=-\frac{2-\gamma}{2}i\left(k+\frac{\nu_\gamma}{2}+\frac{3}{4}\right)\pi+\varepsilon_k. \tag{2.102}$$

Using (2.102), we get

$$\begin{aligned} e^{2i\left(\frac{2}{2-\gamma}i\lambda_k-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4}\right)} &= -e^{-\frac{4}{2-\gamma}\varepsilon_k}\\ &= -1+\tfrac{4}{2-\gamma}\varepsilon_k+O(\varepsilon_k^2). \end{aligned}\tag{2.103}$$

Substituting (2.103) into (2.99), using that $\tilde{f}(\lambda_k)=0$, we get:

$$\tilde{f}(\lambda_k)=\frac{4}{2-\gamma}\varepsilon_k-\frac{1-\gamma}{\varrho}\left(\frac{2}{2-\gamma}i\right)^{2\nu_\gamma}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\frac{2i\sin\nu_\gamma\pi}{(-\frac{2-\gamma}{2}ik\pi)^{\alpha-2\nu_\gamma}}+o(\varepsilon_k)+o\left(\frac{1}{k^{\alpha-2\nu_\gamma}}\right)=0 \tag{2.104}$$

and hence

$$\begin{aligned}\varepsilon_k &= -\frac{1-\gamma}{\varrho}\left(\frac{2-\gamma}{2}\right)^{1-\alpha}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\frac{\sin\nu_\gamma\pi}{(k\pi)^{\alpha-2\nu_\gamma}}(-i)^{1-\alpha}+o\left(\frac{1}{k^{\alpha-2\nu_\gamma}}\right)\\ &= \begin{cases}\dfrac{\gamma-1}{\varrho}\left(\dfrac{2-\gamma}{2}\right)^{1-\alpha}\dfrac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\dfrac{\sin\nu_\gamma\pi}{(k\pi)^{\alpha-2\nu_\gamma}}\left(\cos(1-\alpha)\dfrac{\pi}{2}-i\sin(1-\alpha)\dfrac{\pi}{2}\right)+o\left(\dfrac{1}{k^{\alpha-2\nu_\gamma}}\right)\\ \text{for } k\geq 0,\\ \dfrac{\gamma-1}{\varrho}\left(\dfrac{2-\gamma}{2}\right)^{1-\alpha}\dfrac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\dfrac{\sin\nu_\gamma\pi}{(-k\pi)^{\alpha-2\nu_\gamma}}\left(\cos(1-\alpha)\dfrac{\pi}{2}+i\sin(1-\alpha)\dfrac{\pi}{2}\right)+o\left(\dfrac{1}{k^{\alpha-2\nu_\gamma}}\right)\\ \text{for } k\leq 0.\end{cases}\end{aligned}\tag{2.105}$$

From (2.105) we have in that case $|k|^{\alpha-2\nu_\gamma}\Re\lambda_k\sim\beta$ with

$$\beta=-\frac{1-\gamma}{\varrho}\left(\frac{2-\gamma}{2}\right)^{1-\alpha}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\frac{1}{\pi^{\alpha-2\nu_\gamma}}\sin\nu_\gamma\pi\cos(1-\alpha)\frac{\pi}{2}.$$

• $\alpha<2\nu_\gamma$.
Step 1.

$$f(\lambda)=\left(\frac{2}{\pi\tilde{z}}\right)^{1/2}(1-\gamma)\lambda^{\nu_\gamma}c^+_{\nu_\gamma,0}\left(\frac{2}{2-\gamma}i\right)^{\nu_\gamma}\frac{e^{-i(\tilde{z}+\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}}{2}\tilde{f}(\lambda), \tag{2.106}$$

$$
\begin{aligned}
\tilde f(\lambda) &= (e^{2i(\tilde z+\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}+1)+\frac{\varrho}{1-\gamma}\left(\frac{2}{2-\gamma}i\right)^{-2\nu_\gamma}\frac{c^-_{\nu_\gamma,0}}{c^+_{\nu_\gamma,0}}\frac{e^{2i(\tilde z-\frac{\pi}{4})}+e^{i\nu_\gamma\pi}}{\lambda^{2\nu_\gamma-\alpha}}+O\left(\frac{1}{\lambda}\right)\\
&= f_0(\lambda)+\frac{f_1(\lambda)}{\lambda^{\alpha-2\nu_\gamma}}+O\left(\frac{1}{\lambda}\right),
\end{aligned}
\tag{2.107}
$$

where

$$f_0(\lambda)=e^{2i(\tilde z+\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}+1. \tag{2.108}$$

$$f_1(\lambda)=\frac{\varrho}{1-\gamma}\left(\frac{2}{2-\gamma}i\right)^{-2\nu_\gamma}\frac{c^-_{\nu_\gamma,0}}{c^+_{\nu_\gamma,0}}(e^{2i(\tilde z-\frac{\pi}{4})}+e^{i\nu_\gamma\pi}). \tag{2.109}$$

We look at the roots of f_0. From (2.108), f_0 has one family of roots that we denote λ_k^0.

$$2i\left(\frac{2}{2-\gamma}i\lambda+\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4}\right)=i(2k+1)\pi,\quad k\in\mathbf{Z},$$

i.e.,

$$\lambda_k^0=-\frac{2-\gamma}{2}i\left(k-\frac{\nu_\gamma}{2}+\frac{3}{4}\right)\pi,\quad k\in\mathbf{Z}.$$

Step 2. From Step 1, we can write

$$\lambda_k=-\frac{2-\gamma}{2}i\left(k-\frac{\nu_\gamma}{2}+\frac{3}{4}\right)\pi+\varepsilon_k. \tag{2.110}$$

Using (2.110), we get

$$
\begin{aligned}
e^{2i\left(\frac{2}{2-\gamma}i\lambda_k+\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4}\right)} &= -e^{-\frac{4}{2-\gamma}\varepsilon_k}\\
&= -1+\tfrac{4}{2-\gamma}\varepsilon_k+O(\varepsilon_k^2).
\end{aligned}
\tag{2.111}
$$

Substituting (2.111) into (2.107), using that $\tilde f(\lambda_k)=0$, we get:

$$\tilde f(\lambda_k)=\frac{4}{2-\gamma}\varepsilon_k+\frac{\varrho}{1-\gamma}\left(\frac{2}{2-\gamma}i\right)^{-2\nu_\gamma}\frac{c^-_{\nu_\gamma,0}}{c^+_{\nu_\gamma,0}}\frac{2i\sin\nu_\gamma\pi}{(-\frac{2-\gamma}{2}ik\pi)^{2\nu_\gamma-\alpha}}+o(\varepsilon_k)+o\left(\frac{1}{k^{\alpha-2\nu_\gamma}}\right)=0 \tag{2.112}$$

and hence

$$
\begin{aligned}
\varepsilon_k &= -\frac{\varrho}{1-\gamma}\left(\frac{2-\gamma}{2}\right)^{1+\alpha}\frac{c^-_{\nu_\gamma,0}}{c^+_{\nu_\gamma,0}}\frac{\sin\nu_\gamma\pi}{(k\pi)^{2\nu_\gamma-\alpha}}(-i)^{\alpha-1}+o\left(\frac{1}{k^{2\nu_\gamma-\alpha}}\right)\\
&= \begin{cases}
-\dfrac{\varrho}{1-\gamma}\left(\dfrac{2-\gamma}{2}\right)^{1+\alpha}\dfrac{c^-_{\nu_\gamma,0}}{c^+_{\nu_\gamma,0}}\dfrac{\sin\nu_\gamma\pi}{(k\pi)^{2\nu_\gamma-\alpha}}\left(\cos(1-\alpha)\dfrac{\pi}{2}+i\sin(1-\alpha)\dfrac{\pi}{2}\right)+o\left(\dfrac{1}{k^{2\nu_\gamma-\alpha}}\right)\\
\text{for } k\succeq 0,\\
-\dfrac{\varrho}{1-\gamma}\left(\dfrac{2-\gamma}{2}\right)^{1+\alpha}\dfrac{c^-_{\nu_\gamma,0}}{c^+_{\nu_\gamma,0}}\dfrac{\sin\nu_\gamma\pi}{(-k\pi)^{2\nu_\gamma-\alpha}}\left(\cos(1-\alpha)\dfrac{\pi}{2}-i\sin(1-\alpha)\dfrac{\pi}{2}\right)+o\left(\dfrac{1}{k^{2\nu_\gamma-\alpha}}\right)\\
\text{for } k\preceq 0.
\end{cases}
\end{aligned}
\tag{2.113}
$$

From (2.113) we have in that case $|k|^{2\nu_\gamma-\alpha}\Re\lambda_k \sim \beta$ with

$$\beta = -\frac{\varrho}{1-\gamma}\left(\frac{2-\gamma}{2}\right)^{1+\alpha}\frac{c^-_{\nu_\gamma,0}}{c^+_{\nu_\gamma,0}}\frac{1}{\pi^{2\nu_\gamma-\alpha}}\sin\nu_\gamma\pi\cos(1-\alpha)\frac{\pi}{2}.$$

2.6.2 Appendix B. Proof of Lemma 2.5.3

Suppose that $\lambda \neq 0$. It is enough to consider $\lambda > 0$. We will use the following results .

Lemma 2.6.1 ([28]-[59]) *If $a \neq b$ are complex numbers and $\Re\vartheta > -1$, we have*

$$\begin{aligned}
(a^2-b^2)\int_0^x tJ_\vartheta(at)J_\vartheta(bt)\,dt &= x\left(J_\vartheta(ax)\frac{d}{dx}(J_\vartheta(bx)) - J_\vartheta(bx)\frac{d}{dx}(J_\vartheta(ax))\right),\\
2a^2\int_0^x t(J_\vartheta(at))^2\,dt &= (a^2x^2-\vartheta^2)\left(J_\vartheta(ax)\right)^2 + \left(x\frac{d}{dx}(J_\vartheta(ax))\right)^2,
\end{aligned} \tag{2.114}$$

$$\begin{aligned}
\frac{d}{dx}(x^\vartheta J_\vartheta(x)) &= x^\vartheta J_{\vartheta-1}(x),\\
\frac{d}{dx}(x^{-\vartheta} J_\vartheta(x)) &= -x^{-\vartheta} J_{\vartheta+1}(x).
\end{aligned} \tag{2.115}$$

Remark 2.6.1 *The equalities (2.114) are proposed as exercises in* **[28]** *but we can find the proof in* **[59]** *p. 134-135, formula 7 and 11.*

(I)

$$\|\Phi_+\|^2_{L^2(0,1)} = \int_0^1 x^{1-\gamma}\left(J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right)\right)^2 dx. \tag{2.116}$$

Let $z = \frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}$ in equation (2.116) and using $(2.114)_2$, we get

$$\begin{aligned}
\|\Phi_+\|^2_{L^2(0,1)} &= \frac{2-\gamma}{2\lambda^2}\int_0^r z\left(J_{\nu_\gamma}(z)\right)^2 dz \left(\text{ we set } r = \frac{2\lambda}{2-\gamma}\right)\\
&= \frac{1}{2-\gamma}\frac{1}{r^2}\left[\left(r^2-\nu_\gamma^2\right)\left(J_{\nu_\gamma}(r)\right)^2 + \left(rJ'_{\nu_\gamma}(r)\right)^2\right]\\
&= \frac{1}{2-\gamma}\frac{1}{r^2}\left[\left(rJ_{\nu_\gamma}(r)\right)^2 + \left(rJ_{\nu_\gamma+1}(r)\right)^2 - 2\nu_\gamma rJ_{\nu_\gamma}(r)\,J_{\nu_\gamma+1}(r)\right],
\end{aligned}$$

where we have used (2.44). Using the asymptotic formula (2.87) for large λ, we conclude that

$$\|\Phi_+\|_{L^2(0,1)} \le \frac{c}{\sqrt{\lambda}}.$$

Similarly, we prove that

$$\|\Phi_-\|_{L^2(0,1)} \le \frac{c}{\sqrt{\lambda}}.$$

(II)

$$\begin{aligned}\left\| x^{-\frac{1}{2}} J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right)\right\|^2_{L^2(0,1)} &= \int_0^1 x^{-1}\left(J_{\nu_\gamma}\left(rx^{\frac{2-\gamma}{2}}\right)\right)^2 dx\\ &= \frac{2}{2-\gamma}\int_0^r z^{-1}\left(J_{\nu_\gamma}(z)\right)^2 dz\end{aligned}$$

Now, using (2.44), we have

$$\begin{aligned} I &= \int_0^r z^{-1}\left(J_{\nu_\gamma}(z)\right)^2 dz = \frac{1}{\nu_\gamma}\int_0^r J_{\nu_\gamma}(z)(J'_{\nu_\gamma}(z)+J_{\nu_\gamma+1}(z))\,dz\\ &= \frac{1}{2\nu_\gamma}\left(J_{\nu_\gamma}(r)\right)^2 + \frac{1}{\nu_\gamma}\int_0^r J_{\nu_\gamma}(z)J_{\nu_\gamma+1}(z)\,dz\\ &\le \frac{1}{2\nu_\gamma}\left(J_{\nu_\gamma}(r)\right)^2 + \frac{1}{4}I + \frac{1}{\nu_\gamma^2}\int_0^r z(J_{\nu_\gamma+1}(z))^2\,dz.\end{aligned}$$

Using $(2.114)_2$ and (2.87), we obtain

$$\begin{aligned} I &\le c(J_{\nu_\gamma}(r))^2 + c'\int_0^r z(J_{\nu_\gamma+1}(z))^2\,dz.\\ &\le c\lambda.\end{aligned}$$

Hence

$$\left\| x^{-\frac{1}{2}} J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right)\right\|_{L^2(0,1)} \le c\sqrt{\lambda}.$$

Similarly, we prove that

$$\left\| x^{-\frac{1}{2}} J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right)\right\|_{L^2(0,1)} \le c\sqrt{\lambda}.$$

(III) Let $f_1 \in H^1_{0,\gamma}(0,1)$ and $f_2 \in L^2(0,1)$. First we estimate $I = \displaystyle\int_0^1 f_1(s)\Phi_+(s)\,ds$. After a change of variables $z = rs^{\frac{2-\gamma}{2}}$, using $(2.115)_2$ and integration by parts, we get

$$\begin{aligned} I &= \frac{1}{\lambda}\frac{1}{r^{\frac{1}{2-\gamma}}}\int_0^r f_1\left(\left(\frac{z}{r}\right)^{\frac{2}{2-\gamma}}\right) z^{\frac{1}{2-\gamma}} J_{\nu_\gamma}(z)\,dz\\ &= -\frac{1}{\lambda}\frac{1}{r^{\frac{1}{2-\gamma}}}\int_0^r f_1\left(\left(\frac{z}{r}\right)^{\frac{2}{2-\gamma}}\right)\frac{d}{dz}\left(z^{\frac{1}{2-\gamma}}J_{-\frac{1}{2-\gamma}}(z)\right)\,dz\\ &= \frac{1}{\lambda}\frac{1}{r^{\frac{1}{2-\gamma}}}\left[c^-_{\frac{1}{2-\gamma},0}f_1(0) + \int_0^r \frac{d}{dz}\left(f_1\left(\left(\frac{z}{r}\right)^{\frac{2}{2-\gamma}}\right)\right) z^{\frac{1}{2-\gamma}}J_{-\frac{1}{2-\gamma}}(z)\,dz\right]\\ &= \frac{1}{\lambda}\frac{1}{r^{\frac{1}{2-\gamma}}}c^-_{\frac{1}{2-\gamma},0}f_1(0) + \frac{1}{\lambda}\frac{2}{2-\gamma}\frac{1}{r^{\frac{3}{2-\gamma}}}\int_0^r f_1'\left(\left(\frac{z}{r}\right)^{\frac{2}{2-\gamma}}\right) z^{\frac{\gamma+1}{2-\gamma}}J_{-\frac{1}{2-\gamma}}(z)\,dz\\ &= \frac{1}{\lambda}\frac{1}{r^{\frac{1}{2-\gamma}}}c^-_{\frac{1}{2-\gamma},0}f_1(0) + \frac{1}{\lambda}\int_0^1 f_1'(s)s^{\frac{1}{2}}J_{-\frac{1}{2-\gamma}}\left(rs^{\frac{2-\gamma}{2}}\right)\,ds.\end{aligned}$$

Applying the Cauchy-Schwartz inequality, using the fact that

$$|f_1(0)| \leq \|f_1\|_{L^\infty(0,1)} \leq \frac{1}{\sqrt{1-\gamma}} \|f_1\|_{H^1_{\gamma,0}(0,1)}$$

and perform a change of variables, then by $(2.114)_2$, we have

$$\begin{aligned}
|I| &\leq \frac{1}{\lambda} \frac{1}{r^{\frac{1}{2-\gamma}}} c^-_{\frac{1}{2-\gamma},0} |f_1(0)| + \frac{1}{\lambda} \|s^{\gamma/2} f_1'\|_{L^2(0,1)} \left\| s^{\frac{1-\gamma}{2}} J_{-\frac{1}{2-\gamma}} \left(r s^{\frac{2-\gamma}{2}} \right) \right\|_{L^2(0,1)} \\
&\leq \frac{1}{\lambda} \frac{1}{r^{\frac{1}{2-\gamma}}} c^-_{\frac{1}{2-\gamma},0} |f_1(0)| + \frac{1}{\lambda} \left(\frac{2-\gamma}{2\lambda^2} \right)^{1/2} \|f_1\|_{H^1_{\gamma,0}(0,1)} \left(\int_0^r z (J_{-\frac{1}{2-\gamma}}(z))^2 \, dz \right)^{1/2} \\
&\leq c \frac{1}{|\lambda|^{\frac{3-\gamma}{2-\gamma}}} \|f_1\|_{H^1_{\gamma,0}(0,1)} + c' \frac{1}{|\lambda|^{\frac{3}{2}}} \|f_1\|_{H^1_{\gamma,0}(0,1)} \leq c \frac{1}{|\lambda|^{\frac{3}{2}}} \|f_1\|_{H^1_{\gamma,0}(0,1)}.
\end{aligned}$$

Hence, using (2.87), we deduce that

$$\left| i\lambda \int_0^1 f_1(s) \Phi_+(s) \Phi_-(1) \, ds \right| \leq \frac{c}{|\lambda|} \|f_1\|_{H^1_{\gamma,0}(0,1)}.$$

Also, we have

$$\left| \int_0^1 f_2(s) \Phi_+(s) \Phi_-(1) \, ds \right| \leq |\Phi_-(1)| \|f_2\|_{L^2(0,1)} \|\Phi_+(s)\|_{L^2(0,1)} \leq c \frac{1}{|\lambda|} \|f_2\|_{L^2(0,1)}.$$

In the same way, one can check that

$$\left| \int_0^1 f_1(s) \Phi_-(s) \, ds \right| \leq c \frac{1}{|\lambda|^{\frac{3}{2}}} \|f_1\|_{H^1_{\gamma,0}(0,1)}$$

and

$$\left| \int_0^1 f_2(s) \Phi_-(s) \Phi_+(1) \, ds \right| \leq c \frac{1}{|\lambda|} \|f_2\|_{L^2(0,1)}.$$

Consequently, we get (2.85). Thus, the proof of the Lemma 2.5.3 is complete.

■

Chapter 3

Global existence and asymptotic behavior of the solutions to a class of nonlinear second order ODE with delay term

3.1 Introduction

In this chapter we investigate the existence and decay properties of solutions to the scalar nonlinear second order ODE with delay term of the type

$$\begin{cases} u'' + c_1|u'(t)|^{\alpha}u'(t) + c_2|u'(t-\tau)|^{\alpha}u'(t-\tau) + c_3|u|^{\beta}u = 0 \\ u(0) = u_0, \\ u_t(0) = u_1, \\ u(\gamma) = u_0, \gamma \in]-\tau, 0[. \end{cases} \tag{3.1}$$

where $\alpha, \beta, c_1, c_2, c_3$ et τ are positive constants.

It is well known that if $c_2 = 0$, that is, in absence of delay, the energy of problem (P) is exponentially decaying to zero (see for instance **[37]**, **[34]**, **[36]** and **[51]**). On the contrary, if $c_1 = 0$, that is, there exits only the delay part in the internal, the system (P) becomes unstable (see, for instance **[25]**). In recent years, the PDEs with time delay effects have become an active area of research and arise in many pratical problems (see for example **[1]**, **[58]**). In **[25]**, the authors showed that a small delay in a boundary control could turn such well-behave hyperbolic system into a wild one and therefore, delay becomes a source of instability. To stabilize a hyperbolic system involving input delay terms, additional control terms will be necessary (see **[52]**, **[60]**, **[53]**). In **[52]** the authors examined the problem (P) and determined suitable relations between μ_1 and μ_2, for which the stability or alternatively instability takes place.

Our purpose in this chapter is to give a global solvability in Sobolev spaces and energy decay estimates of the solutions to the problem (P) for a nonlinear damping and a time varying delay term.

To prove decay estimates, we use a multiplier method. These arguments were introduced and developed by Haraux **[37]**, Komornik **[41]** and used by Liu and Zuazua **[46]**, Eller et al **[29]**.

3.2 Preliminaries and main results

First assume the following hypotheses:

$(H1)$ The positive constants c_1 and c_2 satisfy the next inequality

$$c_2 \leq c_1 \tag{3.2}$$

The following technical lemmas will play an important role in the sequel.

Lemma 3.2.1 ([37], [41]) *Let $\mathcal{E} : \mathbb{R}_+ \to \mathbb{R}_+$ be a non increasing function and assume that there are two constants $\sigma > -1$ and $\omega > 0$ such that*

$$\int_S^{+\infty} \mathcal{E}^{1+\sigma}(t)\, dt \leq \frac{1}{\omega}\mathcal{E}^{\sigma}(0)\mathcal{E}(S). \quad 0 \leq S < +\infty, \tag{3.3}$$

then we have

$$\mathcal{E}(t) = 0 \quad \forall t \geq \frac{\mathcal{E}(0)^{\sigma}}{\omega|\sigma|} \qquad \forall t \geq 0, \quad \text{if} \quad -1 < \sigma < 0, \tag{3.4}$$

$$\mathcal{E}(t) \leq \mathcal{E}(0)\left(\frac{1+\sigma}{1+\omega\sigma t}\right)^{\frac{1}{\sigma}} \quad \forall t \geq 0, \quad \textit{if} \quad \sigma > 0 \tag{3.5}$$

and

$$\mathcal{E}(t) \leq \mathcal{E}(0)e^{1-\omega t} \qquad \forall t \geq 0, \quad \textit{if} \quad \sigma = 0. \tag{3.6}$$

We introduce, as in **[52]**, the new variables

$$u'(t-\rho\tau) = Z(t,\rho) \tag{3.7}$$

Then, we have

$$\tau Z_t(t,\rho) + Z_\rho(t,\rho) = 0 \tag{3.8}$$

and

$$\begin{array}{rcl} -\tau u''(t-\rho\tau) & = & \frac{d}{d\rho}Z(t,\rho) = Z_\rho(t,\rho) \\ & \Rightarrow & -\frac{Z_\rho(t,\rho)}{\tau} = u''(t-\rho\tau) = Z_t(t,\rho) \\ & \Rightarrow & \tau Z_t(t,\rho) + Z_\rho(t,\rho) = 0 \end{array}$$

Therefore, problem (P) is equivalent to:

$$\begin{cases} u'' + c_1|u'(t)|^\alpha u'(t) + c_2|Z(t,1)|^\alpha Z(t,1) + c_3|u|^\beta u = 0 \\ \tau Z_t(t,\rho) + Z_p(t,\rho) = 0. \\ u(0) = u_0, \quad u_t(0) = u_1, \\ u(\gamma) = u_0, \gamma \in]-\tau, 0[, \\ Z(0,\rho) = f_0 \end{cases} \tag{3.9}$$

Let $\bar{\xi}$ be a positive constant such that

$$c_2 < \xi < c_1. \tag{3.10}$$

We define the energy associated to the solution of the problem (3.9) by:

$$E(t) = \frac{1}{2}|u'(t)|^2 + \frac{1}{\beta+2}|u|^{\beta+2} + \frac{\xi}{\alpha+2}\int_0^1 |Z(t,\rho)|^{\alpha+2}d\rho. \tag{3.11}$$

Lemma 3.2.2 *Let (u,z) be a solution of the problem (3.9). Then, the energy functional defined by (3.11) satisfies*

$$\begin{array}{rcl} E'(t) & \leq & -\left(c_1 - \frac{\xi}{(\alpha+2)\tau} - \frac{c_2}{(\alpha+2)}\right)|u'(t)|^{\alpha+2} - \left(-\frac{c_2}{\frac{\alpha+2}{\alpha+1}} + \frac{\xi}{(\alpha+2)\tau}\right)|Z(t,1)|^{\alpha+2} \\ & \leq & 0. \end{array} \tag{3.12}$$

Proof.
Multiplying the first equation in (3.9) by $u'(t)$, we get

$$\frac{1}{2}\frac{d}{dt}|u'(t)|^2 + c_1|u'(t)|^{\alpha+2} + c_2|Z(t,1)|^\alpha Z(t,1)u'(t) + \frac{1}{\beta+2}\frac{d}{dt}|u|^{\beta+2} = 0. \tag{3.13}$$

We multiply the second equation in (3.9) by $\xi|Z(t,\rho)|^\alpha Z(t,\rho)$, we obtain

$$\frac{\xi}{\alpha+2}\frac{d}{dt}|Z(t,\rho)|^{\alpha+2} + \frac{\xi}{(\alpha+2)\tau}\frac{d}{d\rho}|Z(t,\rho)|^{\alpha+2} = 0. \tag{3.14}$$

By integrating equation (3.14) over $(0,1)$ we get

$$\frac{\xi}{\alpha+2}\frac{d}{dt}\int_0^1 |Z(t,\rho)|^{\alpha+2}d\rho + \frac{\xi}{(\alpha+2)\tau}(|Z(t,1)|^{\alpha+2} - |Z(t,0)|^{\alpha+2}) = 0. \tag{3.15}$$

Summing (3.13) and (3.15) we get

$$\begin{aligned}&\tfrac{d}{dt}\left[\tfrac{1}{2}|u'(t)|^2+\tfrac{1}{\beta+2}|u|^{\beta+2}+\tfrac{\xi}{\alpha+2}\int_0^1|Z(t,\rho)|^{\alpha+2}d\rho\right]+c_1|u'(t)|^{\alpha+2}\\&+c_2|Z(t,1)|^{\alpha}Z(t,1)u'(t)+\tfrac{\xi}{(\alpha+2)\tau}(|Z(t,1)|^{\alpha+2}-|Z(t,0)|^{\alpha+2})=0.\end{aligned}$$

As

$$Z\,(t,0)=u'(t). \tag{3.16}$$

Hence

$$E'(t)=-\left(c_1-\frac{\xi}{(\alpha+2)\,\tau}\right)|u'(t)|^{\alpha+2}-c_2|Z(t,1)|^{\alpha}Z(t,1)u'(t)-\frac{\xi}{(\alpha+2)\,\tau}|Z(t,1)|^{\alpha+2}. \tag{3.17}$$

Using young's inequality

$$ab\le\frac{\varepsilon^p}{p}a^p+\frac{1}{q\varepsilon^q}b^q,\forall\varepsilon>0,\frac{1}{p}+\frac{1}{q}=1 \tag{3.18}$$

then

$$-c_2|Z(t,1)|^{\alpha}Z(t,1)u'(t)\le c_2|Z(t,1)|^{\alpha+1}|u'(t)|\le c_2\frac{\varepsilon^p}{p}|Z(t,1)|^{p(\alpha+1)}+c_2\frac{1}{q\varepsilon^q}|u'(t)|^q. \tag{3.19}$$

Hence

$$E'(t)\le-\left(c_1-\frac{\xi}{(\alpha+2)\,\tau}\right)|u'(t)|^{\alpha+2}+c_2\frac{\varepsilon^p}{p}|Z(t,1)|^{p(\alpha+1)}+c_2\frac{1}{q\varepsilon^q}|u'(t)|^q-\frac{\xi}{(\alpha+2)\,\tau}|Z(t,1)|^{\alpha+2}. \tag{3.20}$$

We Choose $q=\alpha+2$. Thus $p=(\alpha+2)/(\alpha+1)$. We obtain

$$E'(t)\le-\left(c_1-\frac{\xi}{(\alpha+2)\,\tau}-\frac{c_2}{(\alpha+2)\,\varepsilon^{(\alpha+2)}}\right)|u'(t)|^{\alpha+2}-\left(-c_2\frac{\varepsilon^{\frac{\alpha+2}{\alpha+1}}}{\frac{\alpha+2}{\alpha+1}}+\frac{\xi}{(\alpha+2)\,\tau}\right)|Z(t,1)|^{\alpha+2} \tag{3.21}$$

To get $E'(t)\le0$, it suffices

$$\begin{cases}\left(c_1-\frac{\xi}{(\alpha+2)\tau}-\frac{c_2}{(\alpha+2)\varepsilon^{(\alpha+2)}}\right)\ge0\\ \left(-c_2\frac{\varepsilon^{\frac{\alpha+2}{\alpha+1}}}{\frac{\alpha+2}{\alpha+1}}+\frac{\xi}{(\alpha+2)\tau}\right)\ge0\end{cases} \tag{3.22}$$

i.e

$$\begin{aligned}c_2\frac{\varepsilon^{\frac{\alpha+2}{\alpha+1}}}{\frac{\alpha+2}{\alpha+1}}&\le c_1-c_2\frac{1}{(\alpha+2)\,\varepsilon^{(\alpha+2)}}\\&\Rightarrow c_2\left(\frac{\varepsilon^{\frac{\alpha+2}{\alpha+1}}}{\frac{\alpha+2}{\alpha+1}}+\frac{1}{(\alpha+2)\,\varepsilon^{(\alpha+2)}}\right)\le c_1\end{aligned}$$

By Young's inequality

$$a^{\theta+1} = aa^{\theta} \le \frac{a^p}{p} + \frac{a^{q\theta}}{q} \tag{3.23}$$

for $p = \theta + 1$ and $q = \frac{\theta+1}{\theta}$we get

$$a^{\theta+1} = \frac{a^p}{p} + \frac{a^{q\theta}}{q} \tag{3.24}$$

so for

$$\theta = \frac{1}{\alpha + 1} \tag{3.25}$$

i.e for $b = \frac{1}{\varepsilon}$ and $a = \varepsilon$, wet get

$$\frac{1}{\varepsilon} = \varepsilon^{\frac{1}{\alpha+1}} \Rightarrow \varepsilon^{1+\frac{1}{\alpha+1}} = 1 \Rightarrow \varepsilon = 1. \tag{3.26}$$

Hence for $\varepsilon = 1$

$$c_2 \le c_1. \tag{3.27}$$

Our main result is the following.

Theorem 3.2.1 *Let $(u_0, u_1, f_0) \in \mathbb{R}^2 \times C^1(0,1)$ satisfy the compatibility condition*

$$f_0(0) = u_1.$$

Assume that $(H1)$ holds. Then problem (P) admits a unique

$$u \in C^1(\mathbb{R}^+), \qquad u' \in C^1(\mathbb{R}^+)$$

and we obtain the following decay property:

(i) If $\alpha \ge \dfrac{\beta}{\beta+2}$, then there exists a positive constant $C(E(0))$ depending continuously on $E(0)$ such that

$$E(t) \le \left(\frac{C(E(0))}{t}\right)^{-\frac{2}{\alpha}}.$$

(ii) If $\alpha < \dfrac{\beta}{\beta+2}$, then there exists a positive constant $C(E(0))$ depending continuously on $E(0)$ such that

$$E(t) \le \left(\frac{C(E(0))}{t}\right)^{-\frac{(\alpha+1)(\beta+2)}{\beta-\alpha}}.$$

Moreover, the rate of energy decay $t^{-2/\alpha}$ in part (i) is optimal.

3.2.1 Global existence

To show the existence of the solution for (P), we write the problem as first order system

$$\begin{cases} u' = v \\ v' = -c_1|v|^\alpha v - c_2|Z(t,1)|^\alpha Z(t,1) - c_3|u|^\beta u \\ Z_t(t,\rho) = -\dfrac{1}{\tau} Z_\rho(t,\rho) \end{cases} \tag{3.28}$$

This is the initial value problem for an ordinary differential equation, which admits a unique local solution $u \in C^2([0,T_{\max}))$. The energy in (3.28) is given by

$$E(t) = \frac{1}{2}|u'(t)|^2 + \frac{1}{\beta+2}|u|^{\beta+2} + \frac{\xi}{\alpha+2}\int_0^1 |Z(t,\rho)|^{\alpha+2} d\rho \tag{3.29}$$

From Lemma 3.2.2 for some constants M_1, M_2, we have

$$\forall t \in [0,T_{\max}), \quad |u'(t)| \le M_1, \ |u| \le M_2.$$

In particular $T_{\max} = +\infty, u$ is global, $u \in C^2([0,+\infty))$.

Now, we shall derive the decay estimate for the solutions in Theorem 3.2.1. For this we use the method of multipliers. We denote by c various positive constants which may be different at different occurrences.

3.2.2 Asymptotic behaviour

We multiply equation $(3.9)_1$ by $E^q u$, where q is positive constant, and integrate over (S,T) where S,T are positive constants

$$\begin{aligned} 0 &= \int_S^T E^q u \left(u'' + c_1|u'|^\alpha u' + c_2|z(t,1)|^\alpha z(t,1) + c_3|u|^\beta u\right) dt \\ &= [E^q u' u]_S^T - \int_S^T \left(qE'E^{q-1}\right) u' u \, dt - 2\int_S^T E^q u'^2 dt \\ &\quad + \int_S^T E^q \left(u'^2 + c_3|u|^{\beta+2}\right) dt + c_1 \int_S^T E^q |u'|^\alpha u u' \, dt \\ &\quad + c_2 \int_S^T E^q u |z(t,1)|^\alpha z(t,1) \, dt \end{aligned} \tag{3.30}$$

Similarly, we multiply the second equation of $(3.9)_2$ by $E^q e^{(-2\tau\rho)}(\alpha+2)\,|z|^{\alpha}\,z$, we have

$$\begin{aligned}
0 &= \int_S^T E^q \int_0^1 e^{-2\tau\rho}(\alpha+2)|z|^{\alpha} z\,(\tau z_t + z_\rho)\,d\rho dt \\
&= \left[E^q \int_0^1 \tau e^{-2\tau\rho}|z|^{\alpha+2}\,d\rho\right]_S^T - \tau \int_S^T \left(qE'E^{q-1}\right)\int_0^1 e^{-2\tau\rho}|z|^{\alpha+2}\,d\rho dt \\
&\quad + \int_S^T E^q \int_0^1 \left(\frac{\partial}{\partial\rho}\left(e^{-2\tau\rho}|z|^{\alpha+2}\right) + 2\tau e^{-2\tau\rho}|z|^{\alpha+2}\right) d\rho dt \\
&= \left[E^q \int_0^1 \tau e^{-2\tau\rho}|z|^{\alpha+2}\,d\rho\right]_S^T - \tau \int_S^T \left(qE'E^{q-1}\right)\int_0^1 e^{-2\tau\rho}|z|^{\alpha+2}\,d\rho dt \\
&\quad + \int_S^T E^q\left(e^{-2\tau}|z(t,1)|^{\alpha+2} - |z(t,0)|^{\alpha+2}\right) dt + 2\tau\int_S^T E^q e^{-2\tau\rho}|z|^{\alpha+2}\,d\rho dt
\end{aligned} \tag{3.31}$$

Summing (3.31) and (3.30), we obtain that

$$\begin{aligned}
A\int_S^T E^{q+1}dt \;\leq\; & -\left[E^q u'u\right]_S^T + \int_S^T \left(qE'E^{q-1}\right)u'u\,dt + 2\int_S^T E^q u'^2\,dt \\
& - c_1\int_S^T E^q u|u'|^{\alpha}u'\,dt - c_2\int_S^T E^q u|z(t,1)|^{\alpha} z(t,1)\,dt \\
& - \left[E^q \int_0^1 \tau e^{-2\tau\rho}|z|^{\alpha+2}\,d\rho\right]_S^T + \tau\int_S^T \left(qE'E^{q-1}\right)\int_0^1 e^{-2\tau\rho}|z|^{\alpha+2}\,d\rho dt \\
& - \int_S^T E^q\left(e^{-2\tau}|z(t,1)|^{\alpha+2} + |z(t,0)|^{\alpha+2}\right) dt
\end{aligned} \tag{3.32}$$

where $A = 2\min\left\{1, \tau e^{-2\tau}/\xi\right\}$. Since E is non-increasing, is abounded non-negative function on $\mathbb{R}^+$, using the Cauchy-Schwartz inequality, we find that

$$\begin{aligned}
-\left[E^q u'u\right]_S^T &= E^q(S)u'(S)u(S) - E^q(T)u'(T)u(T) \\
&\leq cE^{q+1/2+1/\beta+2}(S).
\end{aligned} \tag{3.33}$$

and

$$\begin{aligned}
\left|\int_S^T \left(qE'E^{q-1}\right)u'u\,dt\right| &\leq c\gamma\int_S^T q|E'|E^{q-\frac{1}{2}+\frac{1}{\beta+2}}dt \\
&\leq cE^{q+\frac{1}{2}+\frac{1}{\beta+2}}(S).
\end{aligned} \tag{3.34}$$

and

$$\begin{aligned}
& -\left[E^q\int_0^1 e^{-2\tau\rho}|z|^{\alpha+2}d\rho\right]_S^T \\
=\;& E^q(S)\int_0^1 e^{-2\tau\rho}|z(S,\rho)|^{\alpha+2}\,d\rho - E^q(T)\int_0^1 e^{-2\tau\rho}|z(T,\rho)|^{\alpha+2}\,d\rho \\
\leq\;& cE^{q+1}(S).
\end{aligned} \tag{3.35}$$

and

$$\begin{aligned}\int_S^T \left(qE'E^{q-1}\right)\int_0^1 e^{-2\tau\rho}|z|^{\alpha+2}\,d\rho dt &\leq c\int_S^T q\left(E'\right)E^q dt \\ &\leq cE^{q+1}(S).\end{aligned} \tag{3.36}$$

and

$$\begin{aligned}\int_S^T E^q e^{-2\tau}\left|z(t,1)\right|^{\alpha+2}\,dt &\leq c\int_S^T E^q\left(-E'\right)dt \\ &\leq cE^{q+1}(S).\end{aligned} \tag{3.37}$$

and

$$\begin{aligned}\int_S^T E^q|z(t,0)|^{\alpha+2}\,dt &= \int_S^T E^q|u'(t)|^{\alpha+2}dt \\ &\leq c\int_S^T E^q\left(-E'\right)dt \\ &\leq cE^{q+1}(S).\end{aligned} \tag{3.38}$$

Using estimates (3.33)-(3.38), we conclude from inequality (3.32) that

$$\begin{aligned}A\int_S^T E^{q+1}dt &\leq c\mu E^{q+1}(S)+c'E^{q+\frac{1}{2}+\frac{1}{\beta+2}}(S)+\int_S^T E^q u'^2 dt \\ &\quad +\mu_1\int_S^T E^q\left|u\right|\left|u'\right|^{\alpha+1}dt+\mu_2\int_S^T E^q\left|u\right|\left|z(t,1)\right|^{\alpha+1}dt \\ &\leq c\mu E^{q+1}(S)+c'E^{q+\frac{1}{2}+\frac{1}{\beta+2}}(S)+\int_S^T E^q(-E')^{\frac{2}{\alpha+2}}\,dt \\ &\quad +\mu_1\int_S^T E^{q+\frac{1}{\beta+2}}(-E')^{\frac{\alpha+1}{\alpha+2}}dt+\mu_2\int_S^T E^{q+\frac{1}{\beta+2}}(-E')^{\frac{\alpha+1}{\alpha+2}}\,dt\end{aligned} \tag{3.39}$$

Using young's inequality

$$(a\varepsilon)\left(\frac{1}{\varepsilon}\right)b\leq\frac{\varepsilon^p}{p}a^p+\frac{1}{q\varepsilon^q}b^q,\forall\varepsilon>0 \tag{3.40}$$

we deduce that

$$\int_S^T E^q(-E')^{\frac{2}{\alpha+2}}\,dt\leq\frac{\alpha}{\alpha+2}\varepsilon^{\frac{\alpha+2}{\alpha}}\int_S^T E^{q\frac{\alpha+2}{\alpha}}\,dt+\frac{2}{\alpha+2}\frac{1}{\varepsilon^{\frac{\alpha+2}{2}}}\int_S^T(-E')\,dt$$

$$\int_S^T E^{q+\frac{1}{\beta+2}}(-E')^{\frac{\alpha+1}{\alpha+2}}\,dt\leq\frac{1}{\alpha+2}\varepsilon^{\alpha+2}\int_S^T E^{(q+\frac{1}{\beta+2})(\alpha+2)}\,dt+\frac{\alpha+1}{\alpha+2}\frac{1}{\varepsilon^{\frac{\alpha+2}{\alpha+1}}}\int_S^T(-E')\,dt$$

We deduce that

$$\begin{aligned}&A\int_S^T E^{q+1}dt\leq c_1E^{q+1}(S)+c_2E^{q+\frac{1}{2}+\frac{1}{\beta+2}}(S)+c_3\frac{1}{\varepsilon^{\frac{\alpha+2}{2}}}E(S)+c_4\frac{1}{\varepsilon_1^{\frac{\alpha+2}{\alpha+1}}}E(S) \\ &+c_5\varepsilon^{\frac{\alpha+2}{\alpha}}\int_S^T E^{q\frac{\alpha+2}{\alpha}}\,dt+c_6\varepsilon_1^{\alpha+2}\int_S^T E^{(q+\frac{1}{\beta+2})(\alpha+2)}\,dt\end{aligned} \tag{3.41}$$

We distinguish five cases related to the parameters α and β.

(i) If $\alpha \geq \dfrac{\beta}{\beta+2}$

We choose q such that

$$q\frac{\alpha+2}{\alpha} = q+1.$$

Thus, we find $q = \alpha/2$ and hence $(q + \frac{1}{\beta+2})(\alpha+2) = q+1+\gamma$ with

$$\gamma = \frac{(\alpha+2)(\alpha(\beta+2)-\beta)}{2(\beta+2)} > 0.$$

Set $\varepsilon_1 = \dfrac{\varepsilon_2}{E(0)^{\frac{\gamma}{\alpha+2}}}$. Choosing ε and ε_2 small enough, we deduce from (3.41) that

$$\begin{aligned}
\int_S^T E^{q+1}dt &\leq CE^{q+1}(S) + C'E(S) + C'''E^{q+\frac{1}{2}+\frac{1}{\beta+2}}(S) + C''''E(0)^{\frac{\gamma}{\alpha+1}}E(S) \\
&\leq \left(\frac{C' + CE(0)^q + C'''E(0)^{q-\frac{1}{2}+\frac{1}{\beta+2}} + C''''E(0)^{\frac{\gamma}{\alpha+1}}}{E(0)^q}\right) E(0)^q E(S)
\end{aligned}$$

where C, C', C'', C''' and C'''' are different positive constants independent of $E(0)$. Hence, we deduce from Lemma 3.2.1 that

$$E(t) \leq \left(\frac{1+q}{q}\right)^{1/q} \left(C' + CE(0)^q + C'''E(0)^{q-\frac{1}{2}+\frac{1}{\beta+2}} + C''''E(0)^{\frac{\gamma}{\alpha+1}}\right) t^{-1/q}.$$

(ii) If $\alpha < \dfrac{\beta}{\beta+2}$

We choose q such that

$$(q + \frac{1}{\beta+2})(\alpha+2) = q+1.$$

Thus, we find $q = \frac{(\beta-\alpha)}{(\alpha+1)(\beta+2)}$ and hence $q\frac{\alpha+2}{\alpha} = q+1+\gamma$ with

$$\gamma = \frac{(\alpha+2)(\beta-\alpha(\beta+2))}{\alpha(\alpha+1)(\beta+2)} > 0.$$

Set $\varepsilon = \dfrac{\varepsilon_2}{E(0)^{\frac{\alpha\gamma}{\alpha+2}}}$. Choosing ε_1 and ε_2 small enough, we deduce from (3.41) that

$$\begin{aligned}
\int_S^T E^{q+1}dt &\leq CE^{q+1}(S) + C'E(S) + C'''E^{q+\frac{1}{2}+\frac{1}{\beta+2}}(S) + C''''E(0)^{\frac{\alpha\gamma}{2}}E(S) \\
&\leq \left(\frac{C' + CE(0)^q + C'''E(0)^{q-\frac{1}{2}+\frac{1}{\beta+2}} + C''''E(0)^{\frac{\alpha\gamma}{2}}}{E(0)^q}\right) E(0)^q E(S)
\end{aligned}$$

where C, C', C'', C''' and C'''' are different positive constants independent of $E(0)$. Hence, we deduce from Lemma 3.2.1 that

$$E(t) \leq \left(\frac{1+q}{q}\right)^{1/q} \left(C' + CE(0)^q + C'''E(0)^{q-\frac{1}{2}+\frac{1}{\beta+2}} + C''''E(0)^{\frac{\alpha\gamma}{2}}\right) t^{-1/q}.$$

which completes the proof.

Conclusion and perspectives

The summary provided below considers some problems for future research works that arise from this dissertation.

The second chapter of this thesis, was devoted to study of the boundary stabilization of the degenerate wave system with dissipation law of fractional derivative type acting at a degenerate point. Using a spectral analysis we have proved a non-uniform stability. Using Arendt-Batty Theorem, we have proved the strong asymptotic stability. We obtain a sharp estimate for the rate of energy decay of classical solutions depending on parameters γ and α. Our approach is based on the asymptotic theory of C_0- semigroups and in particular on a result due to Borichev and Tomilov [**13**], which reduces the problem of estimating the rate of energy decay to finding a growth bound for the resolvent of the semigroup generator. In particular, we obtain uniform decay estimates for a weakly hyperbolic equation under a weak damping. This is a suprising effect. As for an interesting open problem, is to prove that the results obtained in this chapter hold for the case $\eta = 0$. The Borochev-Tomilov theorem do not work in this case.

Future works

1) It seems to be interesting to develop some multipliers method to treat the following problem (also in the case $a(x) = x^\gamma$)

$$(PR) \qquad \begin{cases} u_{tt}(x,t) - (a(x)u_x(x,t))_x = 0 & \text{in } (0,1)\times(0,+\infty), \\ (a(x)u_x)(0,t) = \varrho\partial_t^{\alpha,\eta}u(0,t) & \text{in } (0,+\infty), \\ u(1,t) = 0 & \text{in } (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } (0,1). \end{cases}$$

Here a is weakly degenerate at $x = 0$ in the sense that

$$\int_0^1 \frac{1}{a(s)}\, ds < +\infty.$$

Moreover, an explicit representation need to develop some tools similar to Bessel equations. This is an interesting problem.

2) More general problem is the following

$$\begin{cases} u_{tt}(x,t) - M(\|\sqrt{a(x)}u_x\|^2_{L^2(0,1)})(a(x)u_x)_x(x,t) = 0 & \text{in } (0,1)\times(0,+\infty) \\ M(\|\sqrt{a(x)}u_x\|^2_{L^2(0,1)})(au_x)(0,t) = \varrho\partial_t^{\alpha,\eta}u(0,t) & \text{in } (0,+\infty) \\ u(1,t) = 0 & \text{in } (0,+\infty) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } (0,1). \end{cases}$$

The problem of global existence and energy decay is open. It is clear that the energy decay rate depends on the order of degeneracy of M, a and the parameter α.

3) Another interesting problem is the following

$$\begin{cases} u_{tt}(x,t) - (a(x)u_x)_x(x,t) = 0 & \text{in } (0,1)\times(0,+\infty) \\ (au_x)(0,t) = h(u_t(0,t)) & \text{in } (0,+\infty) \\ u(1,t) = 0 & \text{in } (0,+\infty) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } (0,1). \end{cases}$$

at least for $a(x) = x^\gamma, 0 < \gamma < 1$ and $h(s) = |s|^{m-1}s$ $(m > 1)$. It is clear that the energy decay depends on the form of a and h.

4) It seems to be interesting to study stabilization, exact controllability and null controllability of solutions to the following hyperbolic-parabolic system

$$(P1) \qquad \begin{cases} y_t(x,t) - (x^{\gamma_1}y_x(x,t))_x = 0 & \text{in } (0,1)\times(0,+\infty), \\ z_{tt}(x,t) - (|x|^{\gamma_2}z_x(x,t))_x = 0 & \text{in } (-1,0)\times(0,+\infty), \\ (x^{\gamma_1}y_x)(0,t) = (|x|^{\gamma_2}z_x)(0,t) & \text{in } (0,+\infty), \\ y(0,t) = z(0,t) & \text{in } (0,+\infty), \\ y(1,t) = 0 & \text{in } (0,+\infty), \\ z(-1,t) = 0 & \text{in } (0,+\infty), \\ y(x,0) = y_0(x), & \text{on } (0,1), \\ z(x,0) = z_0(x), z_t(x,0) = z_1(x) & \text{on } (-1,0), \end{cases}$$

$$(P2) \qquad \begin{cases} y_t(x,t) - (x^{\gamma_1}y_x(x,t))_x = 0 & \text{in } (0,1)\times(0,+\infty), \\ z_{tt}(x,t) - (|x|^{\gamma_2}z_x(x,t))_x = 0 & \text{in } (-1,0)\times(0,+\infty), \\ (x^{\gamma_1}y_x)(0,t) = (|x|^{\gamma_2}z_x)(0,t) & \text{in } (0,+\infty), \\ y(0,t) = z(0,t) & \text{in } (0,+\infty), \\ y(1,t) = 0 & \text{in } (0,+\infty), \\ z(-1,t) = g(t) & \text{in } (0,+\infty), \\ y(x,0) = y_0(x), & \text{on } (0,1), \\ z(x,0) = z_0(x), z_t(x,0) = z_1(x) & \text{on } (-1,0), \end{cases}$$

and

$$(P3) \qquad \begin{cases} y_t(x,t) - (x^{\gamma_1} y_x(x,t))_x = 0 & \text{in } (0,1) \times (0,+\infty), \\ z_{tt}(x,t) - (|x|^{\gamma_2} z_x(x,t))_x = 0 & \text{in } (-1,0) \times (0,+\infty), \\ (x^{\gamma_1} y_x)(0,t) = (|x|^{\gamma_2} z_x)(0,t) & \text{in } (0,+\infty), \\ y(0,t) = z(0,t) & \text{in } (0,+\infty), \\ y(1,t) = g(t) & \text{in } (0,+\infty), \\ z(-1,t) = 0 & \text{in } (0,+\infty), \\ y(x,0) = y_0(x), & \text{on } (0,1), \\ z(x,0) = z_0(x), z_t(x,0) = z_1(x) & \text{on } (-1,0), \end{cases}$$

by means of the boundary control g.

Although, the last chapter has dealt with the multiplier technic which is widely used to control various systems and became nowadays, an indispensable tool for the study of all systems. whether they are finite or infinite, linear or nonlinear, time-invariant or time varying, continuous or discrete. Consequently, reproducing the result obtained in the chapter, with a time delay of fractional type and time-varying delay would be very interesting.

5) For the ordinary second order differential equation with delay, it is interesting to study the same problem, but replacing the nonlinear terms by a fractional derivatives

$$\begin{cases} u'' + c_1 \partial_t^{\alpha,\eta} u\,(t) + c_2 \partial_t^{\alpha,\eta} u\,(t-\tau) + c_3 |u|^{\beta} u = 0 \\ u(0) = u_0, \\ u_t(0) = u_1, \\ u(\gamma) = u_0, \gamma \in]-\tau, 0[. \end{cases} \tag{3.42}$$

Bibliography

Bibliography

[1] C. Abdallah, P. Dorato, J. Benitez-Read & R. Byrne, *Delayed Positive Feedback Can Stabilize Oscillatory System,* ACC, San Francisco, (1993), 3106-3107.

[2] M. Abdelli, A. Benaissa, Energy decay of solutions of a degenerate Kirchhoff equation with a weak nonlinear dissipation, Nonlinear Anal. **69** (2008). 1999-2008.

[3] Z. Achouri, N. Amroun, A. Benaissa, *The Euler-Bernoulli beam equation with boundary dissipation of fractional derivative type,* Mathematical Methods in the Applied Sciences **40**(2017)-11, 3887-3854.

[4] F. Alabau-Boussouira, P. Cannarsa, G. Fragnelli, *Carleman estimates for degenerate parabolic operators with applications to null controllability* , J. Evol. Equ. **6** (2006)-2, 161-204.

[5] N.D. Amroun, A. Benaissa, On exact solutions of a quasilinear wave equation with source term, Electron. J. Differential Equations **31** (2006) 231-238.

[6] W. Arendt, C. J. K. Batty, *Tauberian theorems and stability of one-parameter semigroups,* Trans. Amer. Math. Soc., **306** (1988)-2, 837-852.

[7] R. L. Bagley, P. J. Torvik, *On the appearance of the fractional derivative in the behavior of real material,* J. Appl. Mech. **51** (1983), 294-298.

[8] J. Bai, S. Chai, *Exact controllability for some degenerate wave equations,* Math. Methods Appl. Sci. **43** (2020)-12, 7292-7302.

[9] C. Bardos, G. Lebeau, & J. Rauch, *Sharp sufficient conditions for the observation, control and stabilisation of waves from the boundary,* SIAM J. Control Optim, **30** (1998)-5, 1024-1065, 92-2.

[10] A. Benaissa, C. Aichi, *Energy decay for a degenerate wave equation under fractional derivative controls,* Filomat **32** (2018)-17, 6045-6072.

[11] A. Benaissa, S. Mimouni, Energy decay of solutions of a wave equation of p-Laplacian type with a weakly nonlinear dissipation, J. Inequal. Pure Appl. Math. **7** (2006).

[12] U. Biccari, V. Hernández-Santamaría, J. Vancostenoble, *Existence and cost of boundary controls for a degenerate/singular parabolic equation,* arXiv:2001.11403.

[13] A. Borichev, Y. Tomilov, *Optimal polynomial decay of functions and operator semigroups,* Math. Ann. **347** (2010)-2, 455-478.

[14] H. Brézis, Analyse fonctionnelle. Théorie et Applications, Collection Mathématiques appliquées pour la maîtrise, deuxième tirage, Masson (1987).

[15] H. Brézis, Operateurs Maximaux Monotones et semi-groupes de contractions dans les espaces de Hilbert, Notas de Matemàtica (50), Universidade Federal do Rio de Janeiro and University of Rochester, North-Holland, Amsterdam, (1973).

[16] P. Cannarsa, P. Martinez, J. Vancostenoble, *Carleman estimates for a class of degenerate parabolic operators,* SIAM J. Control Optim. **47** (2008)-1, 1-19.

[17] P. Cannarsa, P. Martinez, J. Vancostenoble, *Null controllability of degenerate heat equations,* Adv. Differential Equations **10** (2005)-2, 153-190.

[18] P. Cannarsa, P. Martinez, J. Vancostenoble, *Sharp estimate of the cost of controllability for a degenerate parabolic equation with interior degeneracy,* Minimax Theory Appl. **6** (2021)-2, 251-280.

[19] P. Cannarsa, P. Martinez, J. Vancostenoble, *The cost of controlling weakly degenerate parabolic equations by boundary controls* , Math. Control Relat. Fields., **7** (2017)-2, 171-211.

[20] B. Chentouf, C. Z. Xu, G. Sallet, *On the stabilization of a vibrating equation,* Nonlinear Anal. **39** (2000)-5, 537-558.

[21] J. U. Choi, R. C. Maccamy, *Fractional order Volterra equations with applications to elasticity,* J. Math. Anal. Appl., **139** (1989), 448-464.

[22] F. Conrad, B. Rao, *Decay of solutions of the wave equation in a star-shaped dornain with nonlinear boundarv feedback,* Asymptotic Analysis, **7** (1993), 159-177.

[23] D'Andrea-Novel, F. Boustany, B. Rao, *Feedbacks stabilisation of a hybrid PDE-ODE system : Application to an overhead crane,* MCSS., **7** (1994), 1-22.

[24] S. Das, Functional Fractional Calculus for System Identification and Control Springer Science & Business Media, (2011).

[25] R. Datko, J. Lagnese & M.P. Polis, *An example on the effect of time delays in boundary feedback stabilization of wave equations,* SIAM J. Control Optim. **24** (1986), 152-156.

[26] R. Dautray, J.l. Lions *Mathematical Analysis and numerical methods for science and technology ,* Volume 5, Evolution Problems 1. Springer-Verlag. (1991).

[27] W. Desch, E. Fašangová, J. Milota & G. Propst, *Stabilization through viscoelastic boundary damping: a semigroup approach,* Semigroup Forum, **80** (2010)-3, 405-415.

[28] J. Dieudonné, Calcul infinitsimal, Collection Methodes, Herman, Paris, 1968.

[29] M. Eller, J. E. Lagnese & S. Nicaise, *Decay rates for solutions of a Maxwell system with nonlinear boundary damping,* Computational. Appl. Math., **21** (2002), 135-165.

[30] M. Fotouhi, L. Salimi, *Controllability Results for a Class of One Dimensional Degenerate/Singular Parabolic Equations,* Commun. Pure Appl. Anal., **12** (2013)-3, 1415-1430.

[31] M. Fotouhi, L. Salimi, *Null Controllability of Degenerate/Singular Parabolic Equations,*J. Dyn. Cont. Sys., **18** (2012)-4, 573-602.

[32] R. E. Green, S.G. Krantz, *Function theoru of one complex variable,* John Wiley and Sons, Inc, (1997).

[33] M. Gueye, *Exact boundary controllability of 1-D parabolic and hyperbolic degenerate equations,* SIAM J. Controle Optim., **52**(2014)-4,2037-2054.

[34] A. Haraux, Asymptotics for some nonlinear O.D.E of the second order, Nonlinear Anal. **10** (12) (1986), 1347-1355. **95** (2005) 297-321.

[35] A. Haraux, Sharp decay estimates of the solutions to a class of nonlinear second order ODEs, Anal. Appl. 9 (2011) 49-17.

[36] A. Haraux, Slow and fast decay of solutions to some second order evolution equations, J. Anal. Math. **95** (2005), 297-321.

[37] A. Haraux, *Two remarks on dissipative hyperbolic problems,* Research Notes in Mathematics, **122.** Pitman: Boston, MA, 1985; 161-179.

[38] A. Haraux, M.A. Jendoubi, Convergence of solutions of second-order gradientlike systems with analyc nonlinearities, J. Differential Equations **144** (1998)-2, 313-170.

[39] A. Haraux, M.A. Jendoubi, Decay estimates to equilibrium for some evolution equations with an analytic nonlinearity, Asymptot. Anal. **26** (1) (2001), 21-36.

[40] F. Huang, *Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces,* Ann. Differ. Equ., **1** (1985), 43-55.

[41] V. Komornik, Exact Controllability and Stabilization. The Multiplier Method, Masson-John Wiley, Paris, 1994.

[42] J. Lagnese, *Decay of solutions of wave equations in a bounded region with boundary dissipation,* J. Differential Equation, **83** (1983), 50(2) :163-182.

[43] I. Lasiecka, D. Tataru, *Uniform boundary stabilization of semilinear wave equations with nonlinear boundary dampin,* Diff. Inte. Equa., **6** (1993), 507-533.

[44] N. N. Lebedev, Special Functions and their Applications, Dover Publications, New York, (1972).

[45] J. Liang, Y. Q. Chen, B. M. Vinagre, I. Podlubny, *Fractional order boundary stabilization of a time-fractional wave equation,*
http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.471.3767&rep=rep1&type=pdf.

[46] W. J. Liu, E. Zuazua, *Decay rates for dissipative wave equations,* Ricerche di Matematica, **XLVIII** (1999), 61-75.

[47] Z. H. Luo, B. Z. Guo, O. Morgul, Stability and stabilization of infinite dimensional systems with applications, Communications and Control Engineering Series. Springer-Verlag London, Ltd., London, (1999).

[48] F. Mainardi, E. Bonetti, *The applications of real order derivatives in linear viscoelasticity,* Rheol. Acta., **26** (1988), 64-67.

[49] B. Mbodje, *Wave energy decay under fractional derivative controls,* IMA Journal of Mathematical Control and Information., **23** (2006), 237-257.

[50] B. Mbodje, G. Montseny, *Boundary fractional derivative control of the wave equation,* IEEE Transactions on Automatic Control., **40** (1995), 368-382.

[51] M. Nakao, *Decay of solutions of some nonlinear evolution equations,* J. Math. Anal. Appl., **60** (1977), 542-549.

[52] S. Nicaise, C. Pignotti, *Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks,* SIAM J. Control Optim. **45** (2006)-5, 1561-1585.

[53] S. Nicaise, C. Pignotti, *Stabilization of the wave equation with boundary or internal distributed delay,* Differ. Int. Equ. **21** (2008), 935-958.

[54] I. Podlubny, Fractional differential equations, Mathematics in Science and Engineering, **198** (1999), Academic Press.

[55] G. Propst, J. Prüss, *On wave equations with boundary dissipation of memory type,* J. Integral Equ. Appl. **8**, (1996)-1, 99-123.

[56] J. Prüss, *On the spectrum of C_0-semigroups,* Transactions of the American Mathematical Society, **284** (1984)-2, 847-857.

[57] W. Rudin, *Real and complex analysis,* Third edition, MacGraw-Hill book company, (1987) **284** (1984)-2, 847-857.

[58] I.H. Suh, Z. Bien, *Use of time delay action in the controller design,* IEEE Trans. Autom. Control **25** (1980) 600-603.

[59] G. N. Watson, A treatise on the theory of Bessel functions, second edition, Cambridge University Press, Cambridge, England, (1944).

[60] C.Q. Xu, S.P. Yung, L.K. Li, *Stabilization of the wave system with input delay in the boundary control,* ESAIM Control Optim. Calc. Var. **12** (2006) 770-785.

[61] P. Yao, *On the observability inequalities for exact controllability of wave equations with variable coeffcients,* SIAM J. Control Optim. **37** (1999)-5, 1568-1599.

[62] H.Zerkouk, C. Aichi, A. Benaissa. On The Stability Of Degenerate Wave Equation Under Fractional Feedbacks Acting On The Degenerate Boundary. Journal of Dynamical and control Systems. https://doi.org/10.1007/s10883-021-09578-7.

[63] M.M. Zhang, H. Gao *Null controllability of some degenerate wave equations,* Journal of Dynamical and Control Systems. J Syst Sci Complex. **30** (2017)-5, 1027-1041.

ملخص

العنوان : دراسة كيفية لبعض معادلات التطور من الدرجة الثانية.

في هذه الأطروحة، تم الإهتمام بدراسة إستقرار بعض مسائل التطور من الدرجة الثانية مع وجود تشتت. بالأخص، تم إعتبار بعض المعادلات التفاضلية من الدرجة الثانية. تحت بعض الفرضيات حول المعطيات الإبتدائية و الشروط الحدية، لتركزت الدراسة حول الوجود الكلي و السلوك المقارب للحلول، و تم التحصل على بعض النتائج حول خصائص الطاقة.

الكلمات المفتاحية :

معادلات الأمواج ضعيفة التشوه. التحكم الحدي الكسري. نظرية أنصاف الزمر. الإستقرار. دوال بيسيل. طريقة المضاعفات.

Abstract

Title : Qualitative study of some evolution equation of second order.

This thesis is devoted to the study of the stabilisation of some evolution problems of second order with the presence of dissipation. In particular, we consider some ordinary differential of second order. Under assumptions on initial data and boundary conditions, we focused our study on the global existence and asymptotic behavior of solutions where we obtained several results.

Keywords :

Weakly degenerate wave equation, Fractional boundary control, Semigroup theory, Stability, Bessel functions, Multiplier method.

Résumé

Titre : Etude qualitative de quelques équations d'évolution du second ordre.

Dans cette thèse, nous avons considéré quelques problèmes d'évolution du second ordre avec la présence des termes dissipatifs. En particulier, on considère quelques équations différentielles du second ordre. Sous quelques hypothèses sur les données initiales et aux bords, nous avons concentré notre étude sur l'existence globale et le comportement asymptotique des solutions où nous avons obtenu plusieurs résultats sur les propriétés de l'énergie.

Mots Clés :

Equation des ondes dégénérée, Contrôle frontière de type fractionnaire, Théorie des semi-groupes, Stabilité, Fonctions de Bessel, Méthode des multiplicateurs.

Laboratory of Analysis and Control of PDEs
Djillali Liabes University