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Evolution equations with singular coefficients

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Abstract

We study the Cauchy problem for certain classes of evolution equations with singular coefficients and/or data in the framework of the concept of very weak solutions. This concept allows to consider equations with highly singular coefficients and/or data for which the classical theory fail. In particular, it is possible to deal with equations involving the Dirac delta function and its powers as coefficients and/or data. Our study deals with three important questions: The well-posedness of the considered Cauchy problems and the study, either analytically or numerically, of the phenomenon of propagation of coefficients/data singularities. The essential methods for our existence and uniqueness results are based on energy estimates and techniques from the classical analysis of differential equations. In order to describe the behaviour of the very weak solutions near the singularities of the coefficients/data, a detailed phase space analysis is carried out. The approach is based on a decomposition into different zones where different techniques of asymptotic analysis are used.

ملخص

في هذه الأطروحة ندرس مسألة كوشي لفئات معينة من معادلات التطور التي تتضمن معاملات أو معطيات إبتدائية غير منتظمة. نهتم بمسألتين : الأولى، دراسة وجود ووحدانية الحلول للمسائل المقترحة. ثانياً: الدراسة سواء التحليلية أو الرقمية لسلوك الحلول بالقرب من النقاط الغير منتظمة للمعاملات والمعطيات الإبتدائية. نقود هذه الدراسة في إطار المفهوم الذي تم تعريفه حديثًا، والذي يسمى الحلول الضعيفة جدًا. هذا المفهوم يسمح لنا بالنظر في معادلات ذات معاملات أو معطيات غير منتظمة التي لا يمكن طرحها في الإطار الكلاسيكي. على وجه الخصوص، من الممكن التعامل مع المعادلات التي تتضمن دالة ديراك وقواها كمعامِلات أو معطيات.

Résumé

Dans cette thèse, nous étudions le problème de Cauchy pour quelques classes d'équations d'évolutions avec des coefficients ou des données initiales non-régulières. On traite deux questions : Premièrement, l'existence et l'unicité de solutions pour les problèmes considérés. Deuxièmement, l'étude du comportement des solutions au voisinage des singularités des coefficients et des données initiales. On utilise le cadre du concept des “solutions très faibles”. Ce concept permet de considérer des équations avec des coefficients ou des données très singulières pour lesquelles la théorie classique échoue. En particulier, il est possible de traiter des équations faisant intervenir la fonction de Dirac et ses puissances comme coefficients ou données initiales.

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Contents

Chapter 1

Introduction

1.1 Background

Partial differential equations (PDEs) are at the heart of many physical models. Famous and often celebrated examples include the Schrödinger equation in quantum mechanics, the heat equation, the Dirac equation and the Klein-Gordon equation in relativistic quantum mechanics, the Maxwell equations in electrodynamics, and the Einstein equations in general relativity.

The theory of partial differential equations with smooth (or otherwise sufficiently regular) coefficients is a well-established subject in analysis. The aim of the present thesis is to contribute to generalisations of the theory. It addresses the study of certain classes of evolution equations involving singular objects (coefficients and/or data), which naturally emerge in various physical contexts, and whose investigation has led to the development of deep tools in various mathematical fields. We use the framework of the recently introduced solution concept, called "very weak solutions".

What are the reasons to get into this concept of solutions? Nearly 70 years ago, Laurent Schwartz introduced the theory of distributions [58] which provided a convenient setting that proved to be powerful in handling a variety of ill-defined mathematical techniques such as the problem of differentiation in the presence of irregular objects (non-differentiable functions or measures) in mathematical models. Irregular functions such as the Heaviside step function $H(x)$ and the Dirac delta function $\delta(x)$ are identified to linear functionals called "distributions".

The theory was very successful and is still widely used, but suffers from the main drawback that it allows only linear operations. In other words, distributions cannot be multiplied, in view of the celebrated work of Laurent Schwartz [59] from around 1954 in which he proved the impossibility of the generalisation to distributions, of the usual pointwise product of continuous functions (except for very special cases). For example it is not meaningful to square the Dirac delta function. However, in physics we are often faced with situations where the need to involve irregular objects as coefficients or data into the mathematical models is natural. Examples range from classical mechanics over electrodynamics to general relativity. For instance, in continuum mechanics, the Heaviside function provides an accurate description of the behaviour of certain material parameters (such as the heat conductivity) across a boundary surface between two regions made up of different compounds. Distributions appear also as coefficients in PDEs in many branches of physics, for example in theory of fluids, in the microscopic scale, masses behave like distributions [7]. Moreover, the charge density of an electron within the framework of classical electrodynamics has to be a delta distribution. Thereby, we inevitably have to deal with ill-defined multiplications, either as interaction between distributional coefficients, for instance in hydrodynamics as products like $H(x)\delta(x)$ of shock waves and their derivatives, or when studying PDEs with irregular data where the solution is expected to be as singular as the initial data and thus the equation may involves interaction between coefficient singularities with singularities in the solution, hence a product of distributions.

Physicists have long ago developed heuristic multiplication of distributions, often adapted to the physical problem under consideration. Some of these methods can be justified and explained mathematically by invoking distribution theory, but some go beyond the theory of distributions. This provide motivation for mathematicians to give a meaning and study the multiplication of distributions in order to explain these heuristic computations in physics. Many attempts have been made, see [26, 27, 71]. In order to give a neat solution to the problems that Schwartz theory of distributions is concerned with, Colombeau [13, 14] introduced differential algebras of generalized functions $\mathcal{G}(\mathbb{R}^n)$ containing the space of distributions $\mathcal{D}'(\mathbb{R}^n)$ as a linear subspace and the algebra of C^∞-functions as a subalgebra. Namely, Let Ω be an open subset of $\mathbb{R}^n$ and set

$$\mathcal{E}(\Omega) := \left\{(u_\varepsilon)_{\varepsilon\in]0,1]} | u_\varepsilon \in C^\infty(\Omega)\right\},$$

where $(u_\varepsilon)_\varepsilon$ are nets of smooth functions indexed by a parameter $\varepsilon \in]0,1]$. The Colombeau algebra of generalized functions on Ω is defined as the quotient space

$$\mathcal{G}(\Omega) := \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega),$$

where

$$\mathcal{E}_M(\Omega) := \{(u_\varepsilon)_\varepsilon \in \mathcal{E}(\Omega) | \forall K \Subset \Omega, \forall \alpha \in \mathbb{N}_0^n, \exists N \in \mathbb{N}_0 : \\ \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = \mathcal{O}(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\},$$

and

$$\mathcal{N}(\Omega) := \{(u_\varepsilon)_\varepsilon \in \mathcal{E}(\Omega) | \forall K \Subset \Omega, \forall \alpha \in \mathbb{N}_0^n, \forall m \in \mathbb{N}_0 : \\ \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = \mathcal{O}(\varepsilon^{m}) \text{ as } \varepsilon \to 0\}.$$

$\mathcal{E}_M(\Omega)$ and $\mathcal{N}(\Omega)$ are called the sets of moderate and negligible nets respectively. Convolution with Friedrichs mollifiers yields an embedding of both smooth functions $C^\infty(\Omega)$ and distributions $\mathcal{D}'(\Omega)$ into this algebra extending in particular multiplication of smooth functions. For more details, we refer the reader to [48]. These algebras permit then to bypass the Schwartz impossibility result, and thus constitute a framework in which singular problems arising from nonlinear operations on distributions can be dealt with a mathematically rigorous basis. In particular, the theory provides solution concept for PDEs with coefficients and data whose regularity is below the required regularity in the classical theory.

This approach has a serious drawback. On the one hand, the multiplication in these algebras is only consistent with the multiplication of smooth functions, and hence, in general not consistent with the algebra structure of continuous or measurable functions. This is in particular problematic when applying this concept to well-posedness issues of singular partial differential equations, where the natural spaces are usually of lower regularity than C^∞. On the other hand, when studying existence and uniqueness of solutions to PDEs involving singular objects in the setting of Colombeau, one needs to construct an algebra, which is not obvious. There are many examples when it is not easy to construct. For instance, L^2 is not an algebra under multiplication, and it is not clear how would one modify to make into algebra. So, the Colombeau theory in this case would not be applicable.

To overcome these issues, in [33] Ruzhansky and Garetto introduced the concept of very weak solutions by defining a different concept of moderateness and negligibility based on natural norms associated to the problem under consideration. For instance, for hyperbolic partial differential equations it seems natural to consider solutions of finite energy and the modification in the approach would be to call a family of solutions moderate if the energy satisfies a polynomial bound with respect to the regularisation parameter, while negligible nets are such that the energy is smaller than any power of the regularisation parameter. We refer the reader to Chapters 2, 3 and 4 for more details about the concept. In order to show a wide applicability, the concept was later on applied in a series of papers for different situations, either for physical models or abstract mathematical problems, we cite [4, 46, 47, 53, 54, 55, 57] where the authors consider the situation when the coefficients depends only on time and we cite the recent paper [30] where the author starts to study very weak solutions for the wave equation with spatial variable coefficients. A classical question that arises naturally when studying PDEs with singular coefficients and initial data is to study either analytically or numerically the influence of the singularities of the coefficients and the Cauchy data on the "solution". In [17, 21], the authors considered a wave equation with coefficients having jumps in the propagation speed and studied the propagation of singularities in the setting of Colombeau algebras.

This thesis is a contribution to the theory, we use the concept of very weak solutions to study the well-posedness of singular PDEs and the propagation of singularities, for different physical models. The advantage in using this framework is twofold. First, the concept of very weak solutions depends heavily on the equation under consideration and is consistent with the classical theory, in contrast with the framework of Colombeau algebras, where the consistency with classical solutions maybe lost in the case of non-smooth functions. Second, working within this framework allow to consider equations with highly singular coefficients, distributional or not distributional, for which distributional solution concepts fail. In particular, it is possible to deal with equations incorporating the Dirac delta function and its powers. The concept of very weak solutions can be considered as a relaxed version of the concept of solutions in the framework of Colombeau algebras. it constitutes a natural framework in which PDEs with coefficients and data of low regularity can be rigorously studied. The concept is also proving to be a valuable tool and easy for application.

Because of the non-local nature of the fractional derivatives, the fractional generalisation of partial differential equations have been found to be very accurate to model real-world problems, see for instance [11, 45, 51], while in the near past, they were thought to be the subject of the pure mathematics. Consequently, considerable attention has been given to the solution of fractional partial differential equations of physical interest. As mentioned above, the main contribution in the present thesis is the study in the very weak sense of some evolution equations with singular coefficients and data. In addition, we consider equations that involve the fractional Laplacian.

1.2 Main results and outline

We briefly summarize the main results of the thesis:

- We provide results of existence and uniqueness in the very weak sense for three types of equations, the fractional Klein-Gordon equation with singular mass term (Theorems 2.2.3, 2.2.4), the fractional Schrödinger equation with singular potential (Theorems 3.2.4, 3.2.5) and the heat equation with strongly singular potentials (Theorems 4.2.4, 4.2.5, 4.3.3, 4.3.4). Furthermore, we prove, in the three situations, the consistency of the very weak solutions with the classical ones (Theorems 2.3.1, 3.3.1, 4.2.6, 4.3.5). These results were published in [1, 2, 3]

- We conduct numerical experiments to study the behaviour of the very weak solutions to the given Cauchy problems near the singularities of the coefficients and/or the Cauchy data.

- We investigate the phenomenon of propagation of singularities and regularity of very weak solutions to a particular Cauchy problem, and give mathematical justification (Theorem 5.4.3) to the interesting numerically observed phenomenon pointed out in the work of Munoz et al [46]. These results were presented in [62].

The outline of the remaining parts of the thesis is as follows:

Section 1.3 introduces the notions of moderateness of families of regularised functions and negligiblity that will be used to define the notion of very weak solution. We briefly review the definition of the fractional Laplacian and the fractional Sobolev spaces.

In chapter 2 we consider the Cauchy problem for a space-fractional wave equation with a singular mass term depending on the position and study existence and uniqueness of a very weak solution. The uniqueness is proved in some appropriate sense. Moreover, we show the consistency of the very weak solution with classical solutions when they exist. Proofs of main theorems are based on energy estimates and techniques from the classical theory. In order to study the behaviour of the very weak solution near the singularities of the coefficient, some numerical experiments are conducted where the appearance of a wall effect for the singular masses of the strength of δ^2 is observed.

Chapter 3 is devoted to the study of the Cauchy problem for the space-fractional Schrödinger equation with singular potential. Delta like or even higher-order singularities are allowed. By using similar arguments as in chapter 2 we establish existence and uniqueness results. Numerical simulations are done, and a particles accumulating effect is observed in the singular cases. From the mathematical point of view a "splitting of the strong singularity" phenomena is also observed.

In chapter 4 we study existence and uniqueness of very weak solutions to a heat equation with strongly singular potentials. The consistency with classical solutions is also discussed. The cases of positive and negative potentials are studied. Numerical simulations are done: one suggests so-called "laser heating and cooling"effects depending on a sign of the potential.

In chapter 5 we consider a singular wave equation with distributional and more singular non-distributional coefficients and develop tools and techniques for the phase-space analysis of such problems. In particular we provide a detailed analysis for the interaction of singularities of solutions with strong singularities of the coefficient in a model problem and we show that the interesting phenomenon observed in [46] really occurs.

We conclude this manuscript in chapter 6, by reference to further developments in the field and some open problems.

In chapter 7, we collect several basic tools, which are essential for the results of this thesis. They are well-known and, only if necessary and possible, we sketch the main ideas of the proofs.

We should remind the reader that each chapter is independent with its own notations, however, in chapter 7 we collect some common symbols and notations used through the thesis.

1.3 The notion of Very weak solutions

In this section we introduce the basic concepts on the notion of very weak solutions for singular problems. The basic idea is as follows. Instead of considering the singular problem itself, one consider a family (net) of regularised problems depending on a regularising parameter. These regularised problems result from approximation of irregular objects in the considered problem by convolution with suitable mollifiers and satisfying a polynomial bound (**moderateness**) with respect to the regularisation parameter. Treating these regularised problems in a classical way leads to a net of solutions, which if **moderate** is called **very weak solution**. Let us make this more precise. We just mention that the definition of a very weak solution depend on the equation under consideration. We introduce here the notion of moderateness that we will need to define the notion of very weak solution and we introduce also the notion of negligibility used to define in which sense we understand the uniqueness and we refer the reader to chapters 2, 3 and 4 for more details about the definition of a very weak solution.

Definition 1 (**Friedrichs mollifier**). *A function $\psi \in C_0^\infty(\mathbb{R}^d)$ is said to be a* Friedrichs *mollifier if and only if it satisfies the following properties:*

1. *(Positivity) ψ is non-negative,*
2. *(Unit mass) $\int_{\mathbb{R}^d} \psi(x)\mathrm{d}x = 1$.*

Example 1.3.1. *A standard example of such function is given by:*

$$\psi(x) = \begin{cases} \alpha e^{-\frac{1}{1-|x^2|}} & |x| < 1, \\ 0 & |x| \geq 1, \end{cases}$$

where α is a normalizing factor, i.e. one that makes $\int_{\mathbb{R}^d} \psi(x)\mathrm{d}x = 1$.

Assume now ψ as described a Friedrichs mollifier.

Definition 2 (**Mollifying net**). *For $\varepsilon \in (0,1]$, $k \in \mathbb{Z}_+^*$ and $x \in \mathbb{R}^d$, a net of functions $(\psi_\varepsilon)_{\varepsilon\in(0,1]}$ is called a mollifying net if*

$$\psi_\varepsilon(x) = \omega(\varepsilon)^{-k}\psi\left(x/\omega(\varepsilon)\right),$$

where $\omega(\varepsilon)$ is a positive function converging to 0 as $\varepsilon \to 0$.

Now, let f be a given function (distribution). Regularising f by convolution with a mollifying net $(\psi_\varepsilon)_{\varepsilon\in(0,1]}$ as defined in definition 2, we get a net of smooth functions, namely

$$(f_\varepsilon)_{\varepsilon\in(0,1]} = (f * \psi_\varepsilon)_{\varepsilon\in(0,1]}. \tag{1.3.1}$$

Definition 3 (**Moderateness**). *Let X and Y be Banach spaces endowed with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively.*

1. *A net of functions $(f_\varepsilon)_{\varepsilon\in(0,1]}$ from X is said to be X-moderate, if there exist $N \in \mathbb{N}_0$ and $c > 0$ such that*

$$\|f_\varepsilon\|_X \leq c\omega(\varepsilon)^{-N}. \tag{1.3.2}$$

2. *For $T > 0$. A net of functions $(u_\varepsilon(\cdot,\cdot))_\varepsilon$ from $C([0,T],X)$ is said to be $C([0,T],X)$-moderate, if there exist $N \in \mathbb{N}_0$ and $c > 0$ such that*

$$\sup_{t\in[0,T]} \|u_\varepsilon(t,\cdot)\|_X \leq c\omega(\varepsilon)^{-N}. \tag{1.3.3}$$

3. *For $T > 0$. A net of functions $(u_\varepsilon(\cdot,\cdot))_\varepsilon$ from $C([0,T],X) \cap C^1([0,T],Y)$ is said to be $C([0,T],X) \cap C^1([0,T],Y)$-moderate, if there exist $N \in \mathbb{N}_0$ and $c > 0$ such that*

$$\sup_{t\in[0,T]} \|u_\varepsilon(t,\cdot)\|_X + \sup_{t\in[0,T]} \|\partial_t u_\varepsilon(t,\cdot)\|_Y \leq c\omega(\varepsilon)^{-N}. \tag{1.3.4}$$

For the second and the third definition of moderateness we will shortly write C-moderate and C^1-moderate.

Remark 1.3.2. *We mention that in the very weak solution concept, derivatives are not required to be moderate, as it is mostly the case when working in the framework of Colombeau algebras [48]. We also mention that the notion of moderateness can be defined by other ways, the most general one is to use families of seminorms as it was used in [33, 30].*

Remark 1.3.3. *We note that by regularising a distribution $T \in \mathcal{E}'(\mathbb{R}^d)$ by convolution with a mollifying net $(\psi_\varepsilon)_\varepsilon$ as defined in definition 2, we get a net of moderate functions. Indeed, by the structure theorems of distributions (Theorem 7.2.5 and Theorem 7.2.6), we know that every compactly supported distribution can be represented by a finite sum of (distributional) derivatives of continuous functions. Precisely, for $T \in \mathcal{E}'(\mathbb{R}^d)$ we can find $n \in \mathbb{N}$ and functions $f_\alpha \in C(\mathbb{R}^d)$ such that, $T = \sum_{|\alpha|\leq n} \partial^\alpha f_\alpha$. The convolution of T with a mollifying net $(\psi_\varepsilon)_\varepsilon$ gives*

$$T*\psi_\varepsilon = \sum_{|\alpha|\leq n} \partial^\alpha f_\alpha * \psi_\varepsilon = \sum_{|\alpha|\leq n} f_\alpha * \partial^\alpha \psi_\varepsilon = \sum_{|\alpha|\leq n} \omega(\varepsilon)^{-|\alpha|} f_\alpha * \big(\omega(\varepsilon)^{-1} \partial^\alpha \psi(x/\omega(\varepsilon))\big).$$

Using an appropriate norm, we see that the net $(T_\varepsilon)_\varepsilon = (T * \psi_\varepsilon)_\varepsilon$ is moderate.

Example 1.3.4. *Let* $(\psi_\varepsilon)_\varepsilon$ *be a mollifying net such that* $\psi_\varepsilon(x) = \varepsilon^{-1}\psi(\varepsilon^{-1}x)$ *and* ψ *is a Friedrichs mollifier. Then, we have*

(1) For $f(x) = \delta_0(x)$*, we have* $f_\varepsilon(x) = \varepsilon^{-1}\psi(\varepsilon^{-1}x) \le C\varepsilon^{-1}$.

(2) For $f(x) = \delta_0^2(x)$*, we obtain* $f_\varepsilon(x) = \varepsilon^{-2}\psi^2(\varepsilon^{-1}x) \le C\varepsilon^{-2}$.

In the next definition we introduce the notion of **negligibility** used to define what we mean by uniqueness of a very weak solution.

Definition 4 (**Negligibility**). *Let* X *be a Banach space with the norm* $\|\cdot\|$. *A net of functions* $(f_\varepsilon)_{\varepsilon\in(0,1]}$ *from* X *is said to be* X*-negligible, if there exists* $C_K > 0$ *such that*

$$\|f_\varepsilon\|_X \le C_k \varepsilon^k \text{ for all } k > 0. \tag{1.3.5}$$

Remark 1.3.5. *The notion of negligibility can be defined in a more general way. Indeed, in the estimate* (1.3.5), ε *can be replaced by a positive function* $\omega(\varepsilon)$ *converging to* 0 *as* $\varepsilon \to 0$.

Remark 1.3.6. *Roughly speaking, the uniqueness of a very weak solution to a considered Cauchy problem will be understood in the sense that negligible changes in the regularisation of the equation coefficients and initial data lead to negligible changes in the corresponding very weak solutions. Since it depend on the problem under consideration, we refer then the reader to definitions 8, 10, 12 and 14 for more details.*

1.4 Fractional Laplacian and fractional Sobolev spaces

The fractional Laplacian can be defined in several equivalent ways. We give the simplest way to define it, as a Fourier multiplier, that we use in this thesis.

Definition 5 (**Fractional Laplacian**). *For* $s > 0$, $(-\Delta)^s$ *denotes the fractional Laplacian defined in terms of the Fourier transform by* $(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s}(\mathcal{F}u))$ *for all* $\xi \in \mathbb{R}^d$.

In other words, the fractional Laplacian $(-\Delta)^s$ is the pseudo-differential operator with symbol $|\xi|^{2s}$.

The fractional Sobolev spaces are related to the fractional Laplacian. Precisely,

Definition 6 (**Fractional Sobolev spaces**). *For* $s > 0$,

$$\begin{aligned} H^s(\mathbb{R}^d) &= \{u \in L^2(\mathbb{R}^d) : (-\Delta)^{\frac{s}{2}} u \in L^2(\mathbb{R}^d)\} \\ &= \{u \in L^2(\mathbb{R}^d) : (1+|\xi|^2)^{\frac{s}{2}} \hat{u} \in L^2(\mathbb{R}^d)\}. \end{aligned}$$

We refer the reader to [22, 29, 38] for more details and alternative definitions of the fractional Laplacian and the fractional Sobolev spaces.

Chapter 2

Fractional Klein-Gordon equation with singular mass term

2.1 Introduction

In this chapter we consider the Cauchy problem

$$\left\{ \begin{array}{l} u_{tt}(t,x) + (-\Delta)^{\alpha} u(t,x) + m(x)u(t,x) = 0, \quad (t,x) \in (0,T] \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \;\; x \in \mathbb{R}^d, \end{array} \right. \tag{2.1.1}$$

where the spatially dependent coefficient m indicates the mass and the differential operator $(-\Delta)^{\alpha}$ stands for the fractional Laplacian. When $\alpha = 1$ and the mass is constant, the equation in (2.1.1) reduces to the well known Klein-Gordon equation which plays a very important role in modelling many problems in classical and quantum mechanics, solitons and condensed matter physics.

The study of analytical and numerical solutions of the space and/or time-fractional Klein-Gordon equation has been investigated considerably in the last years by many authors, we cite for instance [60, 61, 70, 72] to mention only few of many recent publications. We also cite [6, 18, 68] where the authors consider the case when the mass term depends on the position and we refer to [31, 32] where the fractional Laplacian is introduced.

In this chapter, we study the well-posedness in the very weak sense of the Cauchy problem (2.1.1), where we allow the spatially dependent coefficient m to be singular. We have in mind the Dirac delta function and its powers. The uniqueness is proved in an appropriate sense. Moreover, we prove the consistency of the very weak solution with the classical ones when they exist. The mass coefficient is spatially dependent, some numerical experiments are conducted in order to study the behaviour of the very weak solution near the singularities of the coefficient.

2.2 Very weak well-posedness

2.2.1 Notation

Throughout this chapter we use the following specific notation:

$$\|u(t,\cdot)\| := \|u(t,\cdot)\|_{H^\alpha} + \|\partial_t u(t,\cdot)\|_{L^2}.$$

2.2.2 Key lemma

In the regular situation, i.e. in the case when the coefficient m is a regular function we have the following lemma.

Lemma 2.2.1. *Let $m \in L^\infty(\mathbb{R}^d)$ and $m \geq 0$. Suppose that $u_0 \in H^\alpha(\mathbb{R}^d)$ and $u_1 \in L^2(\mathbb{R}^d)$. Then, there is a unique solution $u \in C([0,T]; H^\alpha(\mathbb{R}^d)) \cap C^1([0,T]; L^2(\mathbb{R}^d))$ to (2.1.1), and it satisfies the estimate*

$$\|u(t,\cdot)\|^2 \lesssim (1 + \|m\|_{L^\infty}) \left[\|u_1\|^2_{L^2} + \|u_0\|^2_{H^\alpha}\right]. \tag{2.2.1}$$

Proof. Multiplying the equation (2.1.1) on both sides by u_t and integrating, we get

$$\begin{aligned} Re(\langle \partial_t^2 u(t,\cdot), \partial_t u(t,\cdot)\rangle_{L^2} + \langle (-\Delta)^\alpha u(t,\cdot), \partial_t u(t,\cdot)\rangle_{L^2} \\ + \langle m(\cdot) u(t,\cdot), \partial_t u(t,\cdot)\rangle_{L^2}) = 0, \end{aligned} \tag{2.2.2}$$

where $\langle\cdot,\cdot\rangle_{L^2}$ is the inner product of $L^2(\mathbb{R}^d)$.
Easy calculations show that

$$Re\langle \partial_t^2 u(t,\cdot), \partial_t u(t,\cdot)\rangle_{L^2} = \frac{1}{2}\partial_t \langle \partial_t u(t,\cdot), \partial_t u(t,\cdot)\rangle_{L^2},$$

$$Re\langle (-\Delta)^\alpha u(t,\cdot), \partial_t u(t,\cdot)\rangle_{L^2} = \frac{1}{2}\partial_t \langle (-\Delta)^{\frac{\alpha}{2}} u(t,\cdot), (-\Delta)^{\frac{\alpha}{2}} u(t,\cdot)\rangle_{L^2},$$

and

$$Re\langle m(\cdot) u(t,\cdot), \partial_t u(t,\cdot)\rangle_{L^2} = \frac{1}{2}\partial_t \langle m^{\frac{1}{2}}(\cdot) u(t,\cdot), m^{\frac{1}{2}}(\cdot) u(t,\cdot)\rangle_{L^2}.$$

Let us denote by

$$E(t) := \|\partial_t u(t,\cdot)\|^2_{L^2} + \|(-\Delta)^{\frac{\alpha}{2}} u(t,\cdot)\|^2_{L^2} + \|m^{\frac{1}{2}}(\cdot) u(t,\cdot)\|^2_{L^2},$$

the energy functional of the system (2.1.1). From (2.2.2) it follows that $\partial_t E(t) = 0$, and thus $E(t) = E(0)$. By taking in consideration that $\|m^{\frac{1}{2}} u_0\|^2_{L^2}$ can be estimated by $\|m^{\frac{1}{2}} u_0\|^2_{L^2} \leq \|m\|_{L^\infty} \|u_0\|^2_{L^2}$, it follows that

$$\|\partial_t u(t,\cdot)\|^2_{L^2} \lesssim \left(\|u_1\|^2_{L^2} + \|(-\Delta)^{\frac{\alpha}{2}} u_0\|^2_{L^2} + \|m\|_{L^\infty}\|u_0\|^2_{L^2}\right), \tag{2.2.3}$$

$$\|(-\Delta)^{\frac{\alpha}{2}} u(t,\cdot)\|^2_{L^2} \lesssim \left(\|u_1\|^2_{L^2} + \|(-\Delta)^{\frac{\alpha}{2}} u_0\|^2_{L^2} + \|m\|_{L^\infty}\|u_0\|^2_{L^2}\right), \tag{2.2.4}$$

and

$$\|m^{\frac{1}{2}}(\cdot)u(t,\cdot)\|_{L^2}^2 \lesssim \left(\|u_1\|_{L^2}^2 + \|(-\Delta)^{\frac{\alpha}{2}}u_0\|_{L^2}^2 + \|m\|_{L^\infty}\|u_0\|_{L^2}^2\right). \tag{2.2.5}$$

Hence, the desired estimates for $\partial_t u(t,\cdot)$ and $(-\Delta)^{\frac{\alpha}{2}}u(t,\cdot)$ are proved. Let us now estimate u. Applying the Fourier transform to (2.1.1), the problem can be rewritten as a second order ordinary differential equation

$$\hat{u}_{tt}(t,\xi) + |\xi|^{2\alpha}\hat{u}(t,\xi) = \hat{f}(t,\xi), \tag{2.2.6}$$

with the initial conditions $\hat{u}(0,\xi) = \hat{u}_0(\xi)$ and $\hat{u}_t(0,\xi) = \hat{u}_1(\xi)$. Here $\hat{f}$, $\hat{u}$, denote the Fourier transform of f and u in the spatial variable and

$$f(t,x) := -m(x)u(t,x).$$

We note that in (2.2.6), we see $\hat{f}$ as a source term.

By solving first the homogeneous equation and by application of Duhamel's principle (Theorem 7.2.3), we get the following representation of the solution

$$\hat{u}(t,\xi) = \cos(t|\xi|^\alpha)\hat{u}_0(\xi) + \frac{\sin(t|\xi|^\alpha)}{|\xi|^\alpha}\hat{u}_1(\xi) + \int_0^t \frac{\sin((t-s)|\xi|^\alpha)}{|\xi|^\alpha}\hat{f}(s,\xi)ds. \tag{2.2.7}$$

Taking the L^2 norm in (2.2.7) and using the following estimates:

1) $|\cos(t|\xi|^\alpha)| \leq 1$, for $t \in [0,T]$ and $\xi \in \mathbb{R}^d$,

2) $|\sin(t|\xi|^\alpha)| \leq 1$, for large frequencies and $t \in [0,T]$ and,

3) $|\sin(t|\xi|^\alpha)| \leq t|\xi|^\alpha \leq T|\xi|^\alpha$, for small frequencies and $t \in [0,T]$,

we get that

$$\|\hat{u}(t,\cdot)\|_{L^2}^2 \lesssim \|\hat{u}_0\|_{L^2}^2 + \|\hat{u}_1\|_{L^2}^2 + \int_0^t \|\hat{f}(s,\cdot)\|_{L^2}^2 ds.$$

By Parseval-Plancherel formula (see Theorem 7.2.8) we arrive at

$$\|u(t,\cdot)\|_{L^2}^2 \lesssim \|u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2 + \int_0^T \|m(\cdot)u(s,\cdot)\|_{L^2}^2 ds.$$

Using the estimate (2.2.5) and taking in consideration that the last term in the above estimate can be estimated by $\|m(\cdot)u(t,\cdot)\|_{L^2} \leq \|m\|_{L^\infty}^{\frac{1}{2}}\|m^{\frac{1}{2}}u(t,\cdot)\|_{L^2}$, we get

$$\|u(t,\cdot)\|_{L^2}^2 \lesssim (1 + \|m\|_{L^\infty})\left[\|u_0\|_{H^\alpha}^2 + \|u_1\|_{L^2}^2\right]. \tag{2.2.8}$$

The estimate (2.2.1) follows by summing the estimates (2.2.3), (2.2.4) and (2.2.8), ending the proof. $\square$

2.2.3 Existence of a very weak solution

Here, we consider an irregular case when the mass term m of the equation (2.1.1) has strong singularities, namely, δ-function or "δ^2-function" like behaviours. In what follows, we will understand a multiplication of distributions in the sense of the Colombeau algebra [48].

Now we introduce a notion of the very weak solution to the Cauchy problem (2.1.1) and prove the existence result. We start by regularising the coefficient m by convolution with a suitable mollifying net $(\psi_\varepsilon)_{\varepsilon\in(0,1]}$ as defined in definition 2, generating families of smooth functions $(m_\varepsilon)_\varepsilon$, namely,

$$m_\varepsilon(x) = m * \psi_\varepsilon(x),$$

where $\psi_\varepsilon(x) = \varepsilon^{-d}\psi(x/\varepsilon)$ and $\varepsilon \in (0,1]$. Here we consider $\omega(\varepsilon) = \varepsilon$.

Assumption 2.2.2. *Using the definition of moderateness as in definition 3 ,we assume that the regularisation $(m_\varepsilon)_\varepsilon$ of the coefficient m is L^∞-moderate. That is, there exist $N_0 \in \mathbb{N}_0$ and $c > 0$ such that*

$$\|m_\varepsilon\|_{L^\infty} \leq c\varepsilon^{-N_0}, \tag{2.2.9}$$

for all $\varepsilon \in (0,1]$.

Definition 7 (Very weak solution)**.** *Let $(u_0, u_1) \in H^\alpha(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. Then the net $(u_\varepsilon)_\varepsilon \in C([0,T]; H^\alpha(\mathbb{R}^d)) \cap C^1([0,T]; L^2(\mathbb{R}^d))$ is a very weak solution to the Cauchy problem (2.1.1) if there exists an L^∞-moderate regularisation $(m_\varepsilon)_\varepsilon$ of the coefficient m such that $(u_\varepsilon)_\varepsilon$ solves the regularised problem*

$$\left\{\begin{array}{l} \partial_t^2 u_\varepsilon(t,x) + (-\Delta)^\alpha u_\varepsilon(t,x) + m_\varepsilon(x)u_\varepsilon(t,x) = 0, \quad (t,x) \in (0,T] \times \mathbb{R}^d, \\ u_\varepsilon(0,x) = u_0(x), \partial_t u_\varepsilon(0,x) = u_1(x), \;\; x \in \mathbb{R}^d, \end{array}\right. \tag{2.2.10}$$

for all $\varepsilon \in (0,1]$, and is C^1-moderate.

Remark 2.2.1. *C^1-moderate is understood in the sense of definition 3. That is, there exist $N \in \mathbb{N}_0$ and $c > 0$ such that*

$$\sup_{t\in[0,T]} \|u_\varepsilon(t,\cdot)\| \leq c\varepsilon^{-N}.$$

Theorem 2.2.3 (Existence)**.** *Assume that the regularisation $(m_\varepsilon)_\varepsilon$ of the coefficient m satisfies the moderateness condition (2.2.9). Then the Cauchy problem (2.1.1) has a very weak solution.*

Proof. Since u_0 and u_1 are smooth enough, using the moderateness assumption (2.2.9) and the energy estimate (2.2.1), we arrive at

$$\|u_\varepsilon\| \leq C\varepsilon^{-N_0/2},$$

where N_0 is from (2.2.9), which means that $(u_\varepsilon)_\varepsilon$ is C^1-moderate. $\square$

2.2.4 Uniqueness of the very weak solution

The uniqueness of the very weak solution is proved in the sense of the following definition.

Definition 8 (Uniqueness)**.** *We say that the Cauchy problem (2.1.1) has a unique very weak solution, if for all families of regularisations $(m_\varepsilon)_\varepsilon$ and $(\tilde{m}_\varepsilon)_\varepsilon$, of the coefficient m, such that the net $(m_\varepsilon - \tilde{m}_\varepsilon)_\varepsilon$ is L^∞-negligible, it follows that the net $(u_\varepsilon(t,\cdot) - \tilde{u}_\varepsilon(t,\cdot))_\varepsilon$ is L^2-negligible for all $t \in [0,T]$, where $(u_\varepsilon)_\varepsilon$ and $(\tilde{u}_\varepsilon)_\varepsilon$ are the families of solutions corresponding to $(m_\varepsilon)_\varepsilon$ and $(\tilde{m}_\varepsilon)_\varepsilon$, respectively.*

Theorem 2.2.4 (Uniqueness)**.** *Let $T > 0$. Assume that $m \geq 0$ in the sense that its regularisations as functions are non-negative. Suppose that $(u_0, u_1) \in H^\alpha(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. Then, the very weak solution to the Cauchy problem (2.1.1) is unique.*

Proof. Let $(u_\varepsilon)_\varepsilon$ and $(\tilde{u}_\varepsilon)_\varepsilon$ be very weak solutions to the Cauchy problem (2.1.1) corresponding to the coefficients $(m_\varepsilon)_\varepsilon$ and $(\tilde{m}_\varepsilon)_\varepsilon$ and assume that $\|m_\varepsilon - \tilde{m}_\varepsilon\|_{L^\infty} \leq C_k\varepsilon^k$ for all $k > 0$. Let us denote by

$$U_\varepsilon(t,x) := u_\varepsilon(t,x) - \tilde{u}_\varepsilon(t,x),$$

then, U satisfies the equation

$$\begin{cases} \partial_t^2 U_\varepsilon(t,x) + (-\Delta)^\alpha U_\varepsilon(t,x) + m_\varepsilon(x)U_\varepsilon(t,x) = f_\varepsilon(t,x), \\ U(0,x) = 0, \quad \partial_t U_\varepsilon(0,x) = 0, \end{cases} \tag{2.2.11}$$

with $f_\varepsilon(t,x) = (\tilde{m}_\varepsilon(x) - m_\varepsilon(x))\tilde{u}_\varepsilon(t,x)$. Using Duhamel's principle (Theorem 7.2.3), U_ε is given by

$$U_\varepsilon(x,t) = \int_0^t V_\varepsilon(x,t;s)ds,$$

where $V_\varepsilon(x,t;s)$ solves the problem

$$\begin{cases} \partial_t^2 V_\varepsilon(x,t;s) + (-\Delta)^\alpha V_\varepsilon(x,t;s) + m_\varepsilon(x)V_\varepsilon(x,t;s) = 0, \;\; (t,x) \in (s,T] \times \mathbb{R}^d, \\ V_\varepsilon(x,s;s) = 0, \;\; \partial_t V_\varepsilon(x,s;s) = f_\varepsilon(s,x). \end{cases}$$

Taking U_ε in L^2-norm and using (2.2.1) to get estimate for V_ε, we arrive at

$$\begin{aligned} \|U_\varepsilon(\cdot,t)\|_{L^2} &\leq C\left(1 + \|m_\varepsilon\|_{L^\infty}\right)^{\frac{1}{2}} \int_0^T \|f_\varepsilon(s,\cdot)\|_{L^2} ds \\ &\leq C\left(1 + \|m_\varepsilon\|_{L^\infty}\right)^{\frac{1}{2}} \|\tilde{m}_\varepsilon - m_\varepsilon\|_{L^\infty} \int_0^T \|\tilde{u}_\varepsilon(s,\cdot)\|_{L^2} ds. \end{aligned}$$

We have that $\|m_\varepsilon - \tilde{m}_\varepsilon\|_{L^\infty} \leq C_k\varepsilon^k$ for all $k > 0$, the net $(m_\varepsilon)_\varepsilon$ is moderate by assumption and $(\tilde{u}_\varepsilon)_\varepsilon$ is moderate as a very weak solution to the Cauchy problem (2.1.1). Then, for all $N > 0$, we obtain

$$\|U_\varepsilon(\cdot,t)\|_{L^2} = \|u_\varepsilon(t,\cdot) - \tilde{u}_\varepsilon(t,\cdot)\|_{L^2} \lesssim \varepsilon^N.$$

Thus, the very weak solution is unique. $\square$

2.3 Consistency with classical theory

We want to prove that in the case when a classical solution exists for the Cauchy problem (2.1.1) as in Lemma 2.2.1, the very weak solution recaptures the classical one.

Theorem 2.3.1 (Consistency). *Let $(u_0, u_1) \in H^\alpha(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. Assume that m is a non-negative continuous function and, let us consider the Cauchy problem*

$$\begin{cases} u_{tt}(t,x) + (-\Delta)^\alpha u(t,x) + m(x)u(t,x) = 0, \quad (t,x) \in (0,T] \times \mathbb{R}^d, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), \;\; x \in \mathbb{R}^d. \end{cases} \tag{2.3.1}$$

Let $(u_\varepsilon)_\varepsilon$ be a very weak solution of (2.3.1). Then for any regularising family $m_\varepsilon = m * \psi_\varepsilon$, for any $\psi \in C_0^\infty$, $\psi \geq 0$, $\int \psi = 1$, such that

$$\|m_\varepsilon - m\|_{L^\infty} \to 0 \quad as \quad \varepsilon \to 0,$$

the net $(u_\varepsilon)_\varepsilon$ converges to the classical solution of the Cauchy problem (2.3.1) in L^2 as $\varepsilon \to 0$.

Proof. The classical solution satifies

$$\begin{cases} u_{tt}(t,x) + (-\Delta)^\alpha u(t,x) + m(x)u(t,x) = 0, \quad (t,x) \in (0,T] \times \mathbb{R}^d, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), \;\; x \in \mathbb{R}^d. \end{cases}$$

For the very weak solution, there is a representation $(u_\varepsilon)_\varepsilon$ such that

$$\begin{cases} \partial_t^2 u_\varepsilon(t,x) + (-\Delta)^\alpha u_\varepsilon(t,x) + m_\varepsilon(x)u_\varepsilon(t,x) = 0, \quad (t,x) \in (0,T] \times \mathbb{R}^d, \\ u_\varepsilon(0,x) = u_0(x), \partial_t u_\varepsilon(0,x) = u_1(x), \;\; x \in \mathbb{R}^d. \end{cases}$$

Taking the difference of the above equations, we get

$$\begin{cases} \partial_t^2 (u - u_\varepsilon)(t,x) + (-\Delta)^\alpha (u - u_\varepsilon)(t,x) + m_\varepsilon(x)(u - u_\varepsilon)(t,x) = \eta_\varepsilon(t,x), \\ (u - u_\varepsilon)(0,x) = 0, \quad \partial_t (u - u_\varepsilon)(0,x) = 0, \;\; x \in \mathbb{R}^d, \end{cases}$$

where $\eta_\varepsilon(t,x) = (m(x) - m_\varepsilon(x))u(t,x)$. Let us denote by

$$W_\varepsilon(t,x) := (u - u_\varepsilon)(t,x).$$

Once again, using Duhamel's principle (Theorem 7.2.3), W_ε is given by

$$W_\varepsilon(x,t) = \int_0^t V_\varepsilon(x,t;s)ds,$$

where $V_\varepsilon(x,t;s)$ solves the problem

$$\begin{cases} \partial_t^2 V_\varepsilon(x,t;s) + (-\Delta)^\alpha V_\varepsilon(x,t;s) + m_\varepsilon(x)V_\varepsilon(x,t;s) = 0, \quad (t,x) \in (s,T] \times \mathbb{R}^d, \\ V_\varepsilon(x,s;s) = 0, \quad \partial_t V_\varepsilon(x,s;s) = \eta_\varepsilon(s,x). \end{cases}$$

We have that $\|m - m_\varepsilon\|_{L^\infty} \to 0$ as $\varepsilon \to 0$. Taking the L^2-norm for W_ε and using the energy estimate (2.2.1), we get

$$\begin{aligned}\|W_\varepsilon(\cdot, t)\|_{L^2} &\leq \int_0^T \|V_\varepsilon(\cdot, t; s)\|_{L^2} ds \\ &\leq C\left(1 + \|m_\varepsilon\|_{L^\infty}\right)^{1/2} \|m - m_\varepsilon\|_{L^\infty}^{1/2} \int_0^T \|u(s, \cdot)\|_{L^2} ds.\end{aligned}$$

Since $\|m_\varepsilon\|_{L^\infty} \leq C$ it follows that $(u_\varepsilon)_\varepsilon$ converges to u in L^2 as $\varepsilon \to 0$. $\square$

Remark 2.3.1. *We proved in theorem 2.3.1 that the very weak solution converges in L^2 to the classical solution, provided that $\|m_\varepsilon - m\|_{L^\infty} \to 0$ as $\varepsilon \to 0$. This is in particular true when $m \in C_0(\mathbb{R}^d)$.*

2.4 Propagation of singularities: Numerical experiments

In this Section, we do some numerical experiments. We note that in the case when the mass m depends only on the parameter t, the simulations were done in [4]. Let us analyse our problem by regularising a distributional mass term $m(x)$ by a parameter ε. We define $m_\varepsilon(x) := (m * \varphi_\varepsilon)(x)$, as the convolution with the mollifier $\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi(x/\varepsilon)$, where

$$\varphi(x) = \begin{cases} c\exp\left(\frac{1}{x^2-1}\right), & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

with $c \simeq 2.2523$ to have $\int\limits_{-\infty}^{\infty} \varphi(x)dx = 1$. Then, instead of (2.1.1) we consider the regularised problem

$$\partial_t^2 u_\varepsilon(t,x) - \partial_x^2 u_\varepsilon(t,x) + m_\varepsilon(x)u_\varepsilon(t,x) = 0,\ (t,x) \in (0,T] \times \mathbb{R}, \qquad (2.4.1)$$

with the initial data $u_\varepsilon(0,x) = u_0(x)$ and $\partial_t u_\varepsilon(0,x) = u_1(x)$, for all $x \in \mathbb{R}$. Here, we put

$$u_0(x) = \begin{cases} \exp\left(\frac{1}{(x-50)^2-0.25}\right), & |x-50| < 0.5, \\ 0, & |x-50| \geq 0.5, \end{cases}$$

and $u_1(x) \equiv 0$. Note that $\operatorname{supp} u_0 \subset [49.5, 50.5]$.

For m we consider the following cases, with δ denoting the standard Dirac's delta-distribution:

Case 1: $m(x) = 0$ with $m_\varepsilon(x) = 0$;

Case 2: $m(x) = \delta(x-40)$ with $m_\varepsilon(x) = \varphi_\varepsilon(x-40)$;

Case 3: $m(x) = \delta(x-40) \times \delta(x-40)$. Here, we understand $m_\varepsilon(x)$ as $m_\varepsilon(x) = (\varphi_\varepsilon(x-40))^2$.

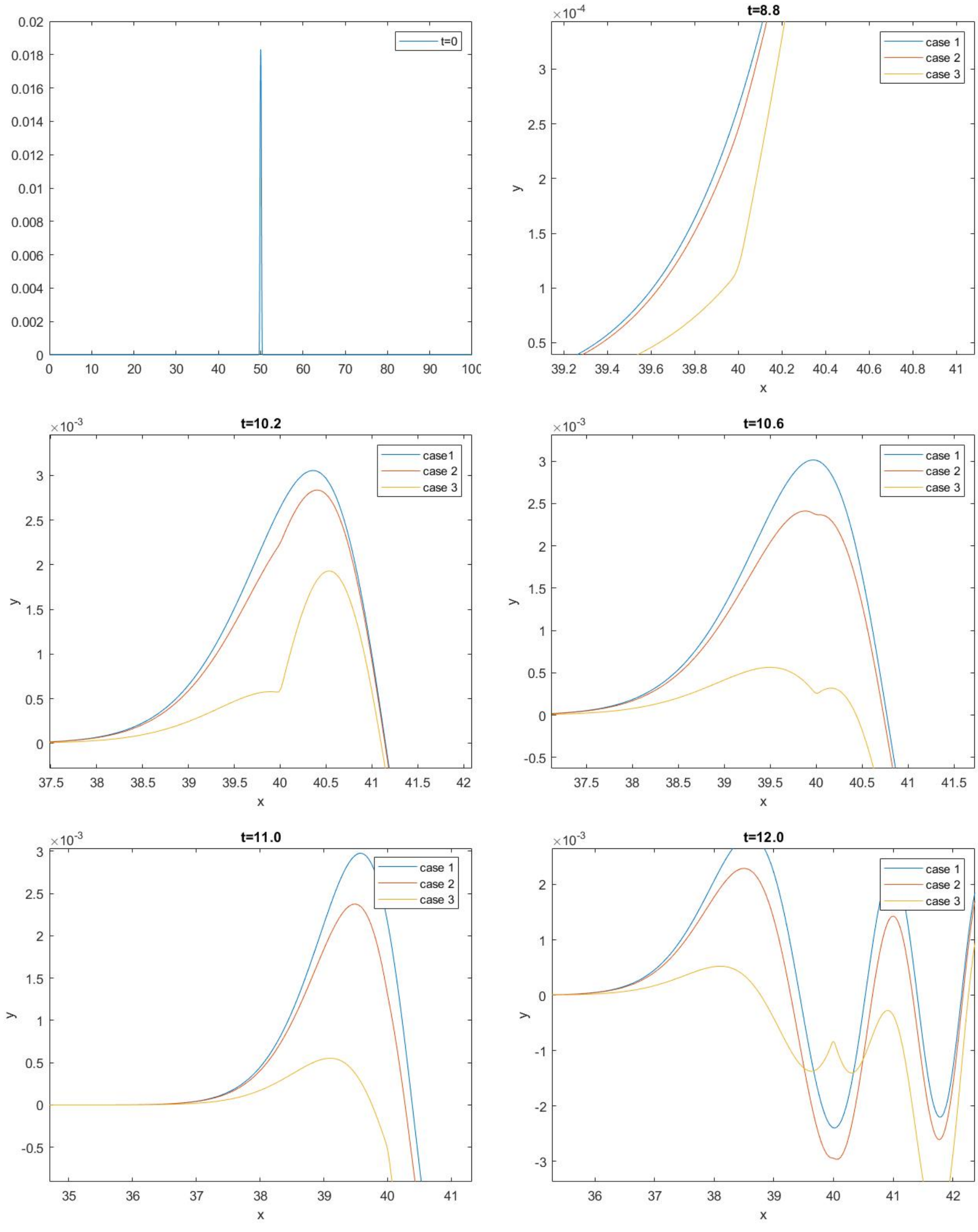

Figure 2.1: In these plots, we analyse behaviours of the solutions of the equation (2.4.1) in the cases of different mass terms. In the upper-left plot, the graphic of the initial function u_0 is given. In the further plots, we compare the replacement function u at $t = 8.8, 10.2, 10.6, 11.0, 12.0$ for $\varepsilon = 0.05$ in the three cases of the mass term, which are described below.

In Figure 2.1, we analyse behaviours of the solutions to the equation (2.4.1) with the initial function u_0 (given in the upper-left plot) in the cases of different mass terms. The further plots of Figure 2.1 are comparing the replacement function u at $t = 8.8, 10.2, 10.6, 11.0, 12.0$ for $\varepsilon = 0.05$ in the following three cases: Case 1 is corresponding to the mass term m is equal to zero; Case 2 is corresponding to the case when the mass term m is like a δ-function; Case 3 is corresponding to the mass term m is like a square of the δ-function.

By analysing Figure 2.1, we see that a delta-function mass term affects less on the behaviour of the solution of (2.4.1) compared to the square delta-function like mass term by reflecting some waves in the opposite direction. In the upper-right plot and in the lower plots of Figure 2.1, we observe that the replacement function u is almost fully reflected in the square delta-function like mass term case. At $t = 8.8$ we see that the yellow coloured wave is starting to settle and, from $t = 10.2$ is moving in opposite direction. We call the last phenomena, a "wall effect".

Remark 2.4.1. *All numerical computations are made in C++ by using the sweep method. In above numerical simulations, we use the Matlab R2018b. For all simulations we take* $\Delta t = 0.2$, $\Delta x = 0.01$.

2.5 Conclusion

The numerical simulations conducted in this chapter show that a delta-function mass term affects less on the behaviour of the waves compared to the square of the delta-function case, the latter causing a so-called "wall effect".

Chapter 3

Fractional Schrödinger equation with singular potentials of higher order

3.1 Introduction

In this chapter we study the fractional Schrödinger equation with distributional potentials. Namely, the following Cauchy problem

$$\begin{cases} iu_t(t,x) + (-\Delta)^s u(t,x) + p(x)u(t,x) = 0, \quad (t,x) \in (0,T) \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{3.1.1}$$

is a subject to our investigation. Here p is assumed to be non-negative, and $s > 0$. We consider the fractional Laplacian as a spatial operator instead of the classical one and prove that the problem has a very weak solution.

While the study of the fractional Schrödinger equation is mathematically challenging, from the physical point of view it is a natural extension of the standard Schrödinger equation when the Brownian trajectories in Feynman path integrals are replaced by Levy flights. The fractional Schrödinger equation is introduced by Laskin in quantum mechanics [39], [40]. More recently, it is proposed as a model in optics by Longhi [43] and applied to laser implementation. For more general overview about the fractional Schrödinger equation and its related topics in physics, one can see [36, 41]. In recent years, it has attracted a lot of interest by many authors, for instance in [12, 42, 52].

On the other hand, our intention to consider singular potentials is also natural from a physical point of view. It can describe a particle which is free to move in two regions of space with a barrier between the two regions. For example, an electron can move almost freely in a conducting material, but if two conducting surfaces are put close together, the interface between them acts as a barrier for the electron. The fractional Schrödinger equation with singular potentials has been also investigated by many authors, we cite for instance [5, 35, 49, 50] and the references mentioned there.

3.2 Very weak well-posedness

3.2.1 Notation

The following notation is proper to this chapter :

For $k \in \mathbb{Z}_+$, we denote by $\|\cdot\|_k$ the norm defined by

$$\|u(t,\cdot)\|_k := \sum_{l=0}^{k} \|\partial_t^l u(t,\cdot)\|_{L^2} + \|(-\Delta)^{\frac{s}{2}} u(t,\cdot)\|_{L^2},$$

and simply denote it by $\|u(t,\cdot)\|$, when $k = 0$.

3.2.2 Some useful lemmas

Let us fix $T > 0$. For a positive s, we consider the initial problem for the space-fractional Schrödinger equation

$$\begin{cases} iu_t(t,x) + (-\Delta)^s u(t,x) + p(x)u(t,x) = 0, & (t,x) \in (0,T) \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{3.2.1}$$

where the potential p is non-negative and can be singular.

We start by stating the following results dealing with the case of regular enough coefficient p.

Lemma 3.2.1. *Let $s > 0$. Suppose that $p \in L^\infty(\mathbb{R}^d)$ be non-negative and assume that $u_0 \in H^s(\mathbb{R}^d)$. Then the estimate*

$$\|u(t,\cdot)\|_{H^s} \lesssim (1 + \|p\|_{L^\infty}) \|u_0\|_{H^s}, \tag{3.2.2}$$

holds for the unique solution $u \in C([0,T]; H^s)$ to the Cauchy problem (3.2.1).

Proof. We multiply the equation in (3.2.1) by u_t and by integrating, we get

$$\begin{aligned} Re(\langle i\partial_t u(t,\cdot), \partial_t u(t,\cdot)\rangle_{L^2} + \langle(-\Delta)^s u(t,\cdot), \partial_t u(t,\cdot)\rangle_{L^2} \\ + \langle p(\cdot)u(t,\cdot), \partial_t u(t,\cdot)\rangle_{L^2}) = 0. \end{aligned} \tag{3.2.3}$$

It is easy to see that

$$Re\langle i\partial_t u(t,\cdot), \partial_t u(t,\cdot)\rangle_{L^2} = 0,$$

$$Re\langle p(\cdot)u(t,\cdot), \partial_t u(t,\cdot)\rangle_{L^2} = \frac{1}{2}\partial_t \|p^{\frac{1}{2}}(\cdot)u(t,\cdot)\|_{L^2}^2,$$

and

$$Re\langle(-\Delta)^s u(t,\cdot), \partial_t u(t,\cdot)\rangle_{L^2} = \frac{1}{2}\partial_t \|(-\Delta)^{\frac{s}{2}} u(t,\cdot)\|_{L^2}^2.$$

The last equality is a consequence of the fact that $(-\Delta)^s$ is a self-adjoint operator. Let us denote by

$$E(t) := \|(-\Delta)^{\frac{s}{2}} u(t,\cdot)\|_{L^2}^2 + \|p^{\frac{1}{2}}(\cdot)u(t,\cdot)\|_{L^2}^2.$$

It follows from (3.2.3) that $\partial_t E(t) = 0$ and thus

$$E(t) = E(0).$$

Therefore

$$\|p^{\frac{1}{2}} u(t,\cdot)\|_{L^2}^2 \lesssim \|(-\Delta)^{\frac{s}{2}} u_0\|_{L^2}^2 + \|p\|_{L^\infty}\|u_0\|_{L^2}^2 \tag{3.2.4}$$

and

$$\|(-\Delta)^{\frac{s}{2}} u(t,\cdot)\|_{L^2}^2 \lesssim \|(-\Delta)^{\frac{s}{2}} u_0\|_{L^2}^2 + \|p\|_{L^\infty}\|u_0\|_{L^2}^2,$$

where we use that $\|p^{\frac{1}{2}}\, u_0\|_{L^2}^2$ can be estimated by

$$\|p^{\frac{1}{2}}\, u_0\|_{L^2}^2 \leq \|p\,\|_{L^\infty}\|u_0\|_{L^2}^2.$$

Moreover, it follows that

$$\|(-\Delta)^{\frac{s}{2}} u(t,\cdot)\|_{L^2} \lesssim \left(1 + \|p\|_{L^\infty}^{\frac{1}{2}}\right)\|u_0\|_{H^s}. \tag{3.2.5}$$

Let us estimate u. After application of the Fourier transformation in (3.2.1), we get the auxiliary Cauchy problem

$$i\hat{u}_t(t,\xi) + |\xi|^{2s}\hat{u}(t,\xi) = \hat{f}(t,\xi); \quad \hat{u}(0,\xi) = \hat{u}_0(\xi), \tag{3.2.6}$$

where $\hat{u}$, $\hat{f}$ denote the Fourier transforms of u and f with respect to the spatial variable x and $f(t,x) := -p(x)u(t,x)$. Using Duhamel's principle (Theorem 7.2.2), we get the following representation of the solution to the Cauchy problem (3.2.6)

$$\hat{u}(t,\xi) = \hat{u}_0(\xi)\exp(-i|\xi|^{2s}t) + \int_0^t \exp\big(-i|\xi|^{2s}(t-s)\big)\hat{f}(s,\xi)ds. \tag{3.2.7}$$

Taking the L^2 norm in (3.2.7) and using the fact that $\exp(-i|\xi|^{2s}t)$ is a unitary operator, we get the estimate

$$\|\hat{u}(t,\cdot)\|_{L^2} \lesssim \|\hat{u}_0\|_{L^2} + \int_0^T \|\hat{f}(s,\cdot)\|_{L^2}ds. \tag{3.2.8}$$

Using the Plancherel-Parseval formula (Theorem 7.2.8), the estimate (3.2.4) and the fact that $\|f(t,\cdot)\|_{L^2} = \|p(\cdot)u(t,\cdot)\|_{L^2}$ can be estimated by

$$\|p(\cdot)u(t,\cdot)\|_{L^2} \leq \|p\|_{L^\infty}^{\frac{1}{2}}\|p^{\frac{1}{2}}u(t,\cdot)\|_{L^2},$$

we arrive at

$$\|u(t,\cdot)\|_{L^2} \lesssim \left(1 + \|p\|_{L^\infty}^{\frac{1}{2}}\right)^2\|u_0\|_{H^s}. \tag{3.2.9}$$

By summing (3.2.5) and (3.2.9), we get our estimate. Thus, the lemma is proved. $\square$

Remark 3.2.1. *Requiring further regularity on the initial data u_0, one can prove that the estimate*

$$\|u(t,\cdot)\|_k \lesssim (1+\|p\|_{L^\infty})\,\|u_0\|_{H^{s(1+2k)}},$$

holds for all $k \geq 0$. For this, we use the estimate (3.2.9) and proceed by induction on $k \geq 1$, on the property that, if $v_k := \partial_t^k u$, where u is the solution to the Cauchy problem (3.2.1), solves the equation

$$i\partial_t v_k(t,x) + (-\Delta)^s v_k(t,x) + p(x)v_k(t,x) = 0,$$

with initial data $v_k(0,x)$, then $v_{k+1} = \partial_t v_k$ solves the same equation with initial data

$$v_{k+1}(0,x) = -i(-\Delta)^s v_k(0,x) - ip(x)v_k(0,x).$$

In order to prove uniqueness and consistency of the very weak solution, we will need the following lemma.

Lemma 3.2.2. *Let $u_0 \in H^s(\mathbb{R}^d)$ and assume that $p \in L^\infty(\mathbb{R}^d)$ is non-negative. Then, the energy conservation*

$$\|u(t,\cdot)\|_{L^2} = \|u_0\|_{L^2}, \tag{3.2.10}$$

holds for all $t \in [0,T]$, for the unique solution $u \in C([0,T]\,;H^s)$ to the Cauchy problem (3.2.1).

Proof. We first multiply the equation in (3.2.1) by $-i$, we obtain

$$u_t(t,x) - i(-\Delta)^s u(t,x) - ip(x)u(t,x) = 0.$$

Multiplying the last equation by u, integrating over $\mathbb{R}^d$ and taking the real part, we get

$$Re\,(\langle u_t(t,\cdot),u(t,\cdot)\rangle_{L^2} - i\langle(-\Delta)^s u(t,\cdot),u(t,\cdot)\rangle_{L^2} - i\langle p(\cdot)u(t,\cdot),u(t,\cdot)\rangle_{L^2}) = 0.$$

Using similar arguments as in Lemma 3.2.1, it is easy to see that

$$Re\langle u_t(t,\cdot),u(t,\cdot)\rangle_{L^2} = \frac{1}{2}\partial_t\|u(t,\cdot)\|_{L^2}^2$$

and that

$$Re\,(-i\langle(-\Delta)^s u(t,\cdot),u(t,\cdot)\rangle_{L^2}) = Re\,(-i\langle p(\cdot)u(t,\cdot),u(t,\cdot)\rangle_{L^2}) = 0.$$

Thus, we have the energy conservation law, i.e. $\|u(t,\cdot)\|_{L^2}$ is constant for all $t \in [0,T]$ and the statement is proved. $\square$

3.2.3 Existence of a very weak solution

In what follows, we consider the case when the potential p is strongly singular, we have in mind δ or δ^2-functions. We want to prove the existence of a very weak solution to the Cauchy problem (3.2.1). We first regularise the coefficient p and the data u_0 by convolution with a suitable mollifying net $(\psi_\varepsilon)_{\varepsilon\in(0,1]}$ (see definition 2) and obtain families of smooth functions $(p_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$, namely

$$p_\varepsilon(x) = p * \psi_\varepsilon(x) \quad \textit{and} \quad u_{0,\varepsilon}(x) = u_0 * \psi_\varepsilon(x),$$

where

$$\psi_\varepsilon(x) = \omega(\varepsilon)^{-d}\psi(x/\omega(\varepsilon)), \quad \varepsilon \in (0,1]\,,$$

and $\omega(\varepsilon) = \varepsilon$. The above regularisation works when p is at least a distribution.

For more generality, we will make assumptions on the regularisations $(p_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$, instead of making them on p and u_0.

Assumption 3.2.3. *We assume that the regularisations $(p_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$ of the coefficient p and the Cauchy data u_0 are L^∞-moderate and H^s-moderate respectively. That is, there exist $N, N_0 \in \mathbb{N}_0$ such that*

$$\|p_\varepsilon\|_{L^\infty} \leq C\varepsilon^{-N} \tag{3.2.11}$$

and

$$\|u_{0,\varepsilon}\|_{H^s} \leq C_0\varepsilon^{-N_0}, \tag{3.2.12}$$

for all $\varepsilon \in (0,1]$.

Now, let us introduce the notion of a very weak solution to the Cauchy problem (3.2.1).

Definition 9 (Very weak solution)**.** *The net $(u_\varepsilon)_\varepsilon \in C([0,T]\,; H^s)$ is said to be a very weak solution of order s to the Cauchy problem (3.2.1) if there exist an L^∞-moderate regularisation of the coefficient p and H^s-moderate regularisation of u_0 such that $(u_\varepsilon)_\varepsilon$ solves the regularised problem*

$$\left\{ \begin{array}{l} i\partial_t u_\varepsilon(t,x) + (-\Delta)^s u_\varepsilon(t,x) + p_\varepsilon(x)u_\varepsilon(t,x) = 0, \quad (t,x) \in [0,T]\times\mathbb{R}^d, \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x), \end{array} \right. \tag{3.2.13}$$

for all $\varepsilon \in (0,1]$, and is C-moderate.

Remark 3.2.2. *We recall that according to definition 3, by C-moderate, we mean that there exist $N \in \mathbb{N}_0$ and $c > 0$ such that*

$$\sup_{t\in[0,T]} \|u_\varepsilon(t,\cdot)\|_{H^s} \leq c\varepsilon^{-N}.$$

We can now state the following theorem, the proof of which follows immediately from the definition. In what follows we understand $p \geq 0$ as its regularisations p_ε satisfy $p_\varepsilon \geq 0$ for all $\varepsilon \in (0,1]$. This is clearly the case when p is a distribution.

Theorem 3.2.4 (Existence)**.** *Let $p \geq 0$ and $s > 0$. Assume that the regularisations of the coefficient p and the Cauchy data u_0 satisfy the assumptions (3.2.11) and (3.2.12). Then the Cauchy problem (3.2.1) has a very weak solution.*

Proof. The coefficient p and the data u_0 are moderate by assumptions. To prove that a very weak solution exists, we need to prove that the net $(u_\varepsilon)_\varepsilon$, solution to the family of regularised Cauchy problems

$$\begin{cases} i\partial_t u_\varepsilon(t,x) + (-\Delta)^s u_\varepsilon(t,x) + p_\varepsilon(x) u_\varepsilon(t,x) = 0, & (t,x) \in [0,T] \times \mathbb{R}^d, \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x), \end{cases}$$

is C-moderate. Indeed, using the assumptions (3.2.11), (3.2.12) and the energy estimate (3.2.2), we arrive at

$$\|u(t,\cdot)\|_{H^s} \lesssim \varepsilon^{-N_0-N},$$

For all $t \in [0,T]$. The net $(u_\varepsilon)_\varepsilon$ is then C-moderate and the existence of a very weak solution is proved. □

3.2.4 Uniqueness of the very weak solution

We prove the uniqueness of a very weak solution to the Cauchy problem (3.2.1) in the sense of the following definition.

Definition 10 (Uniqueness)**.** *We say that the Cauchy problem (3.2.1) has a unique very weak solution, if for all families of regularisations $(p_\varepsilon)_\varepsilon$, $(\tilde{p}_\varepsilon)_\varepsilon$, $(u_{0,\varepsilon})_\varepsilon$ and $(\tilde{u}_{0,\varepsilon})_\varepsilon$ of the coefficient p and the Cauchy data u_0, such that the nets $(p_\varepsilon - \tilde{p}_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})_\varepsilon$ are L^∞-negligible and H^s-negligible, it follows that the net $(u_\varepsilon(t,\cdot) - \tilde{u}_\varepsilon(t,\cdot))_\varepsilon$ is L^2-negligible for all $t \in [0,T]$, where $(u_\varepsilon)_\varepsilon$ and $(\tilde{u}_\varepsilon)_\varepsilon$ are the families of solutions to the corresponding regularised Cauchy problems.*

Theorem 3.2.5 (Uniqueness)**.** *Let $p \geq 0$ and assume that p and u_0 satisfy the assumptions (3.2.11) and (3.2.12). Then, the Cauchy problem (3.2.1) has a unique very weak solution.*

Proof. Let $(p_\varepsilon)_\varepsilon$, $(\tilde{p}_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$, $(\tilde{u}_{0,\varepsilon})_\varepsilon$, regularisations of p and u_0, satisfying

$$\|p_\varepsilon - \tilde{p}_\varepsilon\|_{L^\infty} \leq C_k \varepsilon^k \quad \text{for all} \quad k > 0$$

and

$$\|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} \leq C_l \varepsilon^l \quad \text{for all} \quad l > 0,$$

and let us denote by $U_\varepsilon(t,x) := u_\varepsilon(t,x) - \tilde{u}_\varepsilon(t,x)$, where $(u_\varepsilon)_\varepsilon$ and $(\tilde{u}_\varepsilon)_\varepsilon$ are the families of solutions to the regularised Cauchy problems, corresponding to the families $(p_\varepsilon, u_\varepsilon)_\varepsilon$ and $(\tilde{p}_\varepsilon, \tilde{u}_{0,\varepsilon})_\varepsilon$. Then, U_ε solves the Cauchy problem

$$\begin{cases} i\partial_t U_\varepsilon(t,x) + (-\Delta)^s U_\varepsilon(t,x) + p_\varepsilon(x) U_\varepsilon(t,x) = f_\varepsilon(t,x), & (t,x) \in (0,T) \times \mathbb{R}^d, \\ U_\varepsilon(0,x) = (u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})(x), \end{cases} \tag{3.2.14}$$

where

$$f_\varepsilon(t,x) = (\tilde{p}_\varepsilon(x) - p_\varepsilon(x))\, \tilde{u}_\varepsilon(t,x).$$

Let $(V_\varepsilon)_\varepsilon$ and $(W_\varepsilon)_\varepsilon$, the families of solutions to the auxiliary Cauchy problems

$$\begin{cases} i\partial_t V_\varepsilon(x,t;s) + (-\Delta)^s V_\varepsilon(x,t;s) + p_\varepsilon(x) V_\varepsilon(x,t;s) = 0, \; (t,x) \in (s,T] \times \mathbb{R}^d, \\ V_\varepsilon(x,s;s) = f_\varepsilon(s,x), \end{cases}$$

and

$$\begin{cases} i\partial_t W_\varepsilon(t,x) + (-\Delta)^s W_\varepsilon(t,x) + p_\varepsilon(x) W_\varepsilon(t,x) = 0, & (t,x) \in (0,T] \times \mathbb{R}^d, \\ W_\varepsilon(0,x) = (u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})(x). \end{cases}$$

Using Duhamel's principle (see, Theorem 7.2.2), U_ε is given by

$$U_\varepsilon(t,x) = W_\varepsilon(t,x) + \int_0^t V_\varepsilon(x,t;s)\,ds. \tag{3.2.15}$$

Taking the L^2 norm in (3.2.15) and using (3.2.10) to estimate V_ε and W_ε, we get

$$\begin{aligned} \|U_\varepsilon(t,\cdot)\|_{L^2} &\leq \|W_\varepsilon(t,\cdot)\|_{L^2} + \int_0^T \|V_\varepsilon(\cdot,t;s)\|_{L^2} ds \\ &\lesssim \|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} + \int_0^T \|f_\varepsilon(s,\cdot)\|_{L^2} ds \\ &\lesssim \|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} + \|\tilde{p}_\varepsilon - p_\varepsilon\|_{L^\infty} \int_0^T \|\tilde{u}_\varepsilon(s,\cdot)\|_{L^2} ds. \end{aligned}$$

From the one hand, we have that $\|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} \leq C_l \varepsilon^l$, for all $l > 0$. On the other hand, $(u_\varepsilon)_\varepsilon$ as a very weak solution to the Cauchy problem (3.2.1) is moderate and $\|\tilde{p}_\varepsilon - p_\varepsilon\|_{L^\infty} \leq C_k \varepsilon^k$ for all $k > 0$. Therefore,

$$\|U_\varepsilon(t,\cdot)\|_{L^2} = \|u_\varepsilon(t,\cdot) - \tilde{u}_\varepsilon(t,\cdot)\|_{L^2} \lesssim \varepsilon^N,$$

for all $N > 0$, which means that the very weak solution is unique. □

3.3 Consistency with classical theory

Now we give the consistency result, which means that the very weak solution to the Cauchy problem (3.2.1) converges in an appropriate norm, to the classical solution, when the latter exists.

Theorem 3.3.1 (Consistency). *Let $p \in L^{\infty}(\mathbb{R}^d)$ be non-negative. Assume that $u_0 \in H^s(\mathbb{R}^d)$ for $s > 0$, and let us consider the Cauchy problem*

$$\begin{cases} iu_t(t,x) + (-\Delta)^s u(t,x) + p(x)u(t,x) = 0, & (t,x) \in (0,T) \times \mathbb{R}^d, \\ u(0,x) = u_0(x). \end{cases} \tag{3.3.1}$$

Let $(u_\varepsilon)_\varepsilon$ be a very weak solution of (3.3.1). Then for any regularising families of the coefficient p such that $\|p_\varepsilon - p\|_{L^\infty} \to 0$ as $\varepsilon \to 0$ and the Cauchy data u_0, the net $(u_\varepsilon)_\varepsilon$ converges in L^2 as $\varepsilon \to 0$ to the unique classical solution of the Cauchy problem (3.3.1).

Proof. Let u be the classical solution to

$$\begin{cases} iu_t(t,x) + (-\Delta)^s u(t,x) + p(x)u(t,x) = 0, & (t,x) \in (0,T) \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases}$$

and let $(u_\varepsilon)_\varepsilon$ its very weak solution. It satisfies

$$\begin{cases} i\partial u_\varepsilon(t,x) + (-\Delta)^s u_\varepsilon(t,x) + p_\varepsilon(x)u_\varepsilon(t,x) = 0, & (t,x) \in (0,T) \times \mathbb{R}^d, \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x). \end{cases}$$

Let us denote by $W_\varepsilon(t,x) := u(t,x) - u_\varepsilon(t,x)$. It solves

$$\begin{cases} i\partial W_\varepsilon(t,x) + (-\Delta)^s W_\varepsilon(t,x) + p_\varepsilon(x)W_\varepsilon(t,x) = \eta_\varepsilon(t,x), & (t,x) \in (0,T) \times \mathbb{R}^d, \\ W_\varepsilon(0,x) = (u_0 - u_{0,\varepsilon})(x), \end{cases}$$

where $\eta_\varepsilon(t,x) := (p_\varepsilon(x) - p(x))u(t,x)$. Using Duhamel's principle (Theorem 7.2.2) and similar arguments as in Theorem (3.2.5), we get the estimate

$$\begin{aligned} \|W_\varepsilon(t,\cdot)\|_{L^2} &\lesssim \|u_0 - u_{0,\varepsilon}\|_{L^2} + \int_0^T \|\eta_\varepsilon(s,\cdot)\|_{L^2} ds \\ &\lesssim \|u_0 - u_{0,\varepsilon}\|_{L^2} + \|p_\varepsilon - p\|_{L^\infty} \int_0^T \|u(s,\cdot)\|_{L^2} ds. \end{aligned}$$

When $\varepsilon \to 0$, the right hand side of the last inequality tends to 0, since $\|p_\varepsilon - p\|_{L^\infty} \to 0$ by assumption and $\|u_0 - u_{0,\varepsilon}\|_{L^2} \to 0$. The latter is a consequence of the fact that C^∞ is dense in L^2. Hence, the very weak solution converges to the classical one in L^2 as $\varepsilon \to 0$. ☐

3.4 Propagation of singularities: Numerical experiments

In this Section, we do some numerical experiments. Let us analyse our problem by regularising a distributional potential $p(x)$ by a parameter ε. We define $p_\varepsilon(x) := (p * \varphi_\varepsilon)(x)$, as the convolution with the mollifier $\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi(x/\varepsilon)$, where

$$\varphi(x) = \begin{cases} c\exp\left(\frac{1}{x^2-1}\right), & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

with $c \simeq 2.2523$ to have $\int\limits_{-\infty}^{\infty} \varphi(x)dx = 1$. Then, instead of (3.2.1) we consider the regularised Cauchy problem for the 1D Schrödinger equation

$$i\partial_t u_\varepsilon(t,x) - \partial_x^2 u_\varepsilon(t,x) + p_\varepsilon(x)u_\varepsilon(t,x) = 0, \; (t,x) \in (0,T) \times \mathbb{R}, \tag{3.4.1}$$

with the initial data $u_\varepsilon(0,x) = u_0(x)$, for all $x \in \mathbb{R}$. Here, we put

$$u_0(x) = \begin{cases} \exp\left(\frac{1}{(x-5)^2-0.25}\right), & |x-5| < 0.5, \\ 0, & |x-5| \geq 0.5. \end{cases}$$

Note that $\operatorname{supp} u_0 \subset [4.5, 5.5]$.

Here, we consider the following cases when the potential is a regular function:

1) $p(x) = 0$,
2) $p(x) = 1$,
3) $p(x) = (x-5)^2$;

and when potential is a singular function:

1) $p(x) = \frac{1}{30}\delta(x-3)$ with $p_\varepsilon(x) = \frac{1}{30}\varphi_\varepsilon(x-3)$,
2) $p(x) = \frac{1}{30}\delta^2(x-3)$ in the sense $p_\varepsilon(x) = \frac{1}{30}\varphi_\varepsilon^2(x-3)$,

where δ denotes the standard Dirac's delta-distribution.

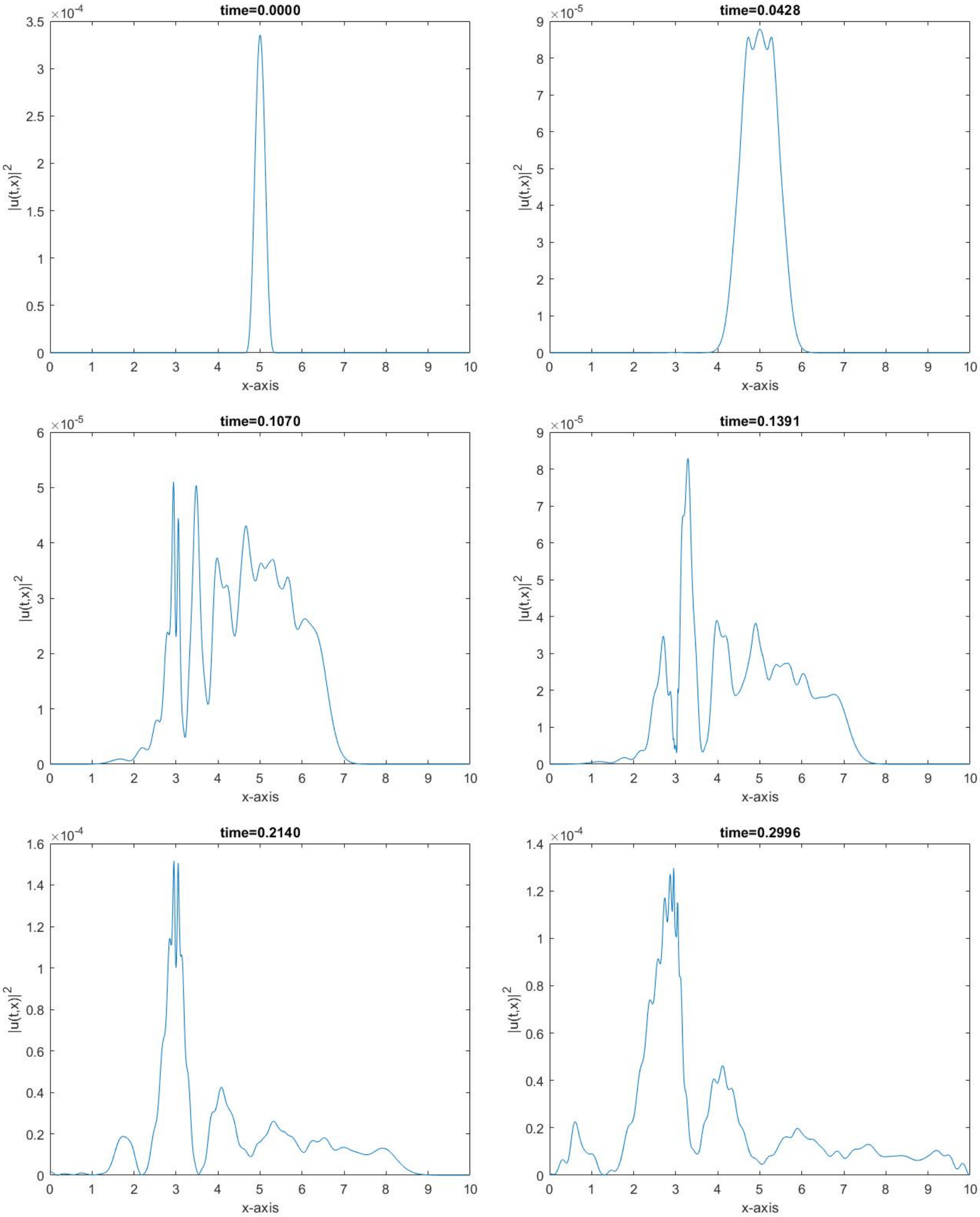

Figure 3.1: In these plots, we analyse behaviour of the solution of the Schrödinger equation (3.4.1) with a δ-like potential. In the top left plot, the graphic of the position density of particles at the initial time is given. In the further plots, we draw the position density function $|u|^2$ at $t = 0.0428, 0.1070, 0.1391, 0.2140, 0.2996$ for $\varepsilon = 0.05$. Here, a δ-like function with the support at point 3 is considered.

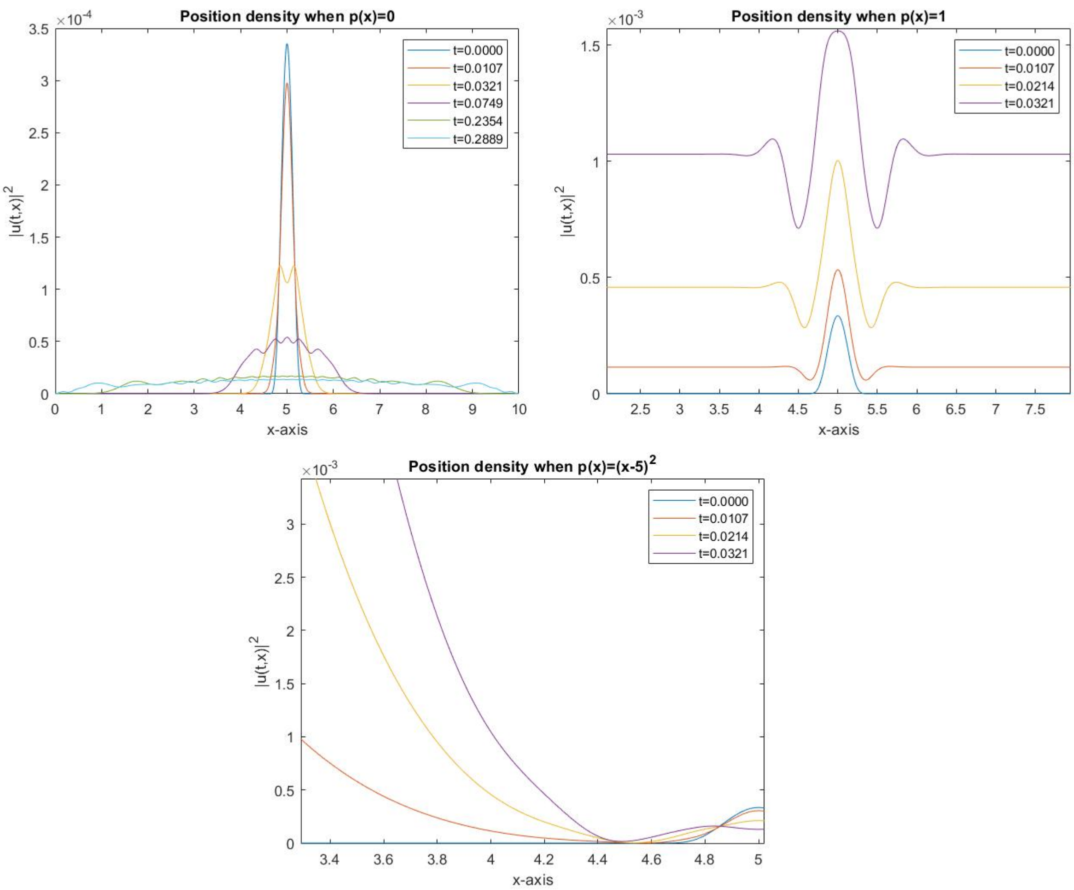

Figure 3.2: In these plots, we analyse the time evolution of the position density $|u|^2$ for different regular potentials. Here, the cases of the potentials with $p(x) = 0$, $p(x) = 1$, and $p(x) = (x-5)^2$ are considered.

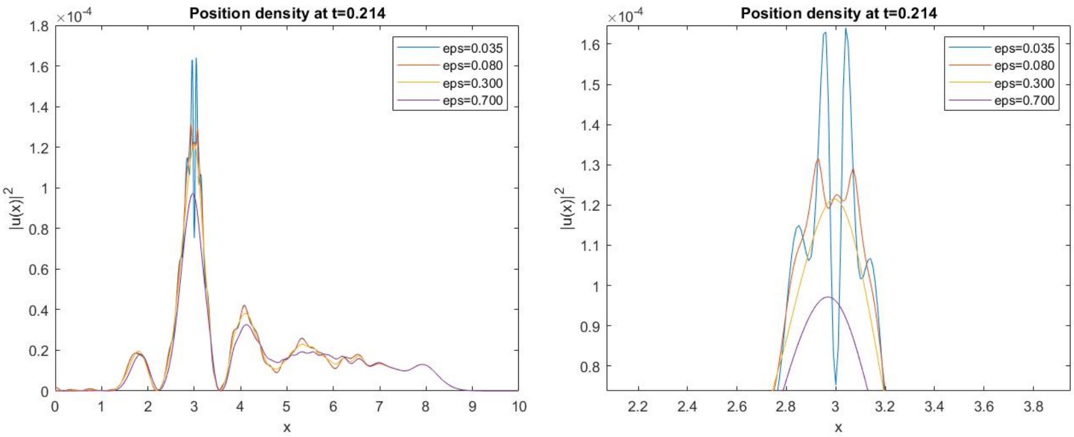

Figure 3.3: In these plots, we analyse behaviour of the solution of the Schrödinger equation (3.4.1) with a δ-like potential for different values of the parameter ε. Here, we compare the position density function of particles $|u|^2$ at $t = 0.214$ for $\varepsilon = 0.035, 0.080, 0.300, 0.800$. Here, the case of the potential with a δ-like function behaviour with the support at point 3 is considered.

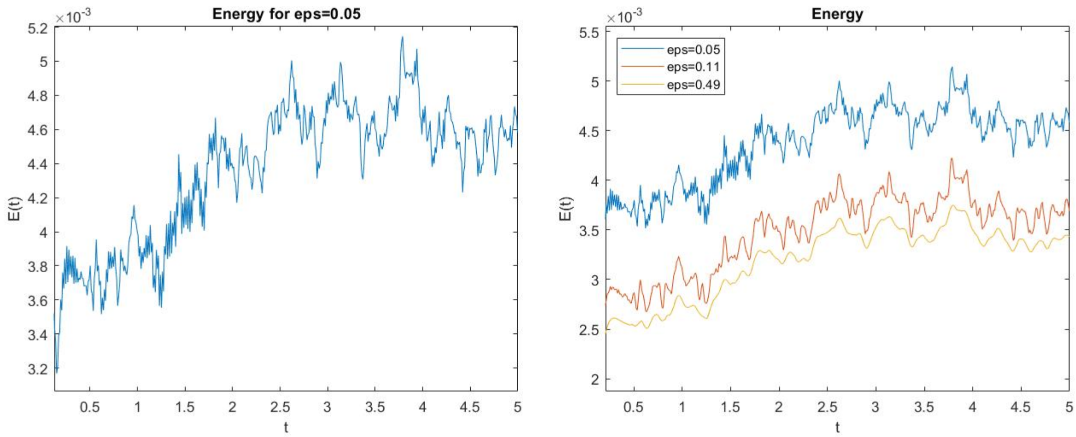

Figure 3.4: In these plots, we compare the energy function $E(t)$ of the Schrödinger equation (3.4.1) corresponding to the δ-potential case for $\varepsilon = 0.05, 0.11, 0.49$.

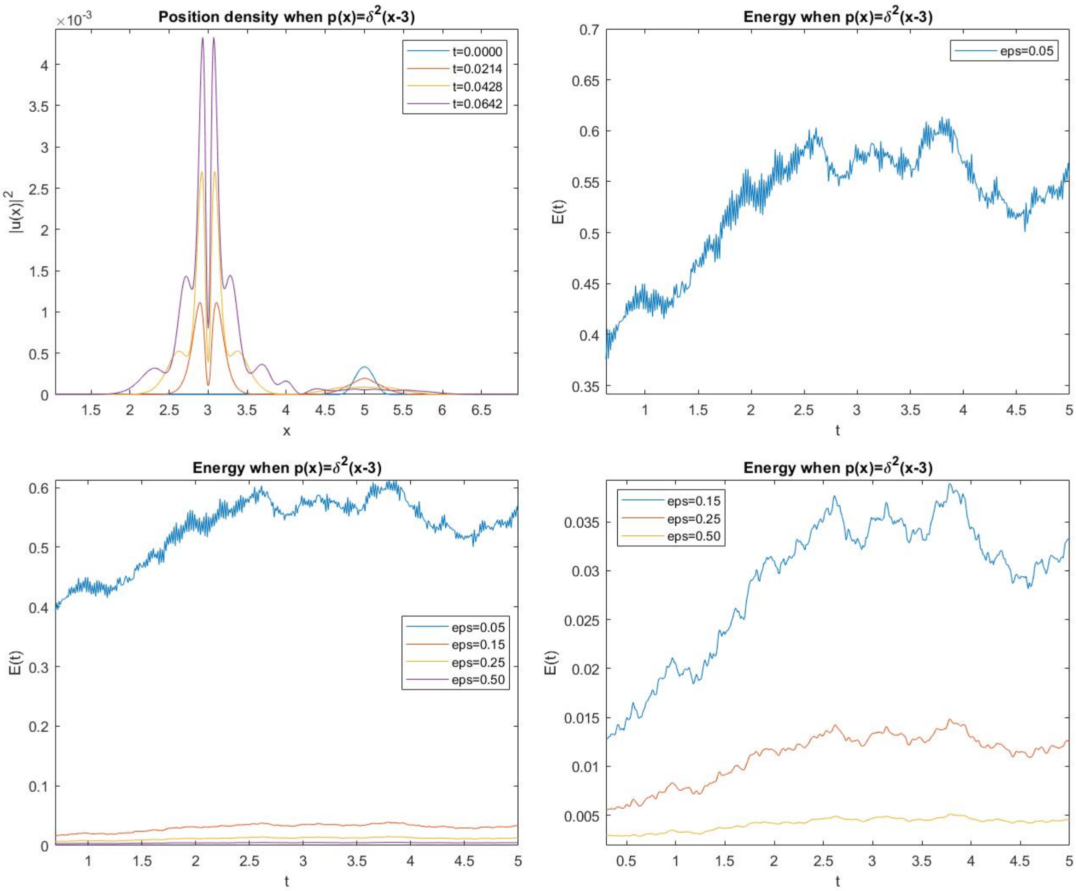

Figure 3.5: In these plots, we analyse the solution of the Schrödinger equation (3.4.1) with a δ^2-like potential. In the top left plot, we study the position density function $|u(t,x)|^2$ at $t = 0.0000, 0.0214, 0.0428, 0.0642$ for $\varepsilon = 0.05$. In further plots, we compare the energy function $E(t)$ of the Schrödinger equation (3.4.1) corresponding to the δ^2-potential case for $\varepsilon = 0.05, 0.15, 0.25, 0.50$. In the right-bottom plot, we compare the energy function for $\varepsilon = 0.15, 0.25, 0.50$.

In Figure 3.1, we analyse behaviour of the solution with a δ-like potential. In the top left plot, the graphic of the position density of particles at the initial time is given. In the further plots, we draw the position density function $|u|^2$ at $t = 0.0428, 0.1070,\ 0.1391, 0.2140, 0.2996$ for $\varepsilon = 0.05$. Here, a δ-like function with the support at point 3 is considered. We observe a delta-function potential causing an accumulation of particles in the place of the support of the singularity.

In Figure 3.2, we analyse the time evolution of the position density for different regular potentials. Here, the cases of the potentials with $p(x) = 0, p(x) = 1$, and $p(x) = (x-5)^2$ are considered.

In Figure 3.3, we analyse behaviour of the solution of the Schrödinger equation (3.4.1) with a δ-like potential for different values of the parameter ε. Here, we compare the position density function of particles $|u|^2$ at $t = 0.214$ for $\varepsilon = 0.035, 0.080, 0.300, 0.800$. Here, the case of the potential with a δ-like function behaviour with the support at point 3 is considered. Here, we can see that the numerical simulations of the regularised equation (3.4.1) are stable under the changing of the values of the parameter ε.

In Figure 3.4, we compare the energy function

$$E(t) = \|\nabla u(t,\cdot)\|_{L^2}^2 + \|p^{\frac{1}{2}}(\cdot)u(t,\cdot)\|_{L^2}^2. \tag{3.4.2}$$

of the Schrödinger equation (3.4.1) corresponding to the δ-potential case for different values of the parameter ε. Simulations show that $E(t) \approx E(0)$ for $t > 0$.

In Figure 3.5, we analyse the solution of the Schrödinger equation (3.4.1) with a δ^2-like potential. In the left plot, we study the position density function $|u(t,x)|^2$ at $t = 0.0000, 0.0214, 0.0428, 0.0642$ for $\varepsilon = 0.05$. In the right plot, we compare the energy function $E(t)$ of the Schrödinger equation (3.4.1) corresponding to the δ^2-potential case for $\varepsilon = 0.05, 0.15, 0.25,\ 0.50$.

Remark 3.4.1. *By analysing these cases, from Figures 3.4 and 3.5 we see that the energy function $E(t)$ given by* (3.4.2) *satisfies $E(t) \approx E(0)$ for $t > 0$. Moreover, it is observed that $E(t)$ depends on ε by confirming the theory, that is, $E(t) = E_\varepsilon(t)$. From the bottom plots of Figure 3.5 we observe that the energy $E(t)$ of the Schrödinger equation* (3.4.1) *with a δ^2-like potential corresponding to the case $\varepsilon = 0.5$ is increased around* 200 *times as ε is decreased* 10 *times by justifying the theoretical part.*

Remark 3.4.2. *From the behaviours of the density function $|u(t,x)|^2$ of the Schrödinger equation* (3.4.1) *corresponding to the cases of δ-like and δ^2-like potentials, namely, from the left plot of Figure 3.3 and the upper–left plot of Figure 3.5 we observe a "splitting of the strong singularity" effect. Explanation of this phenomena is still an open question from the theoretical point of view.*

Remark 3.4.3. *A second order in time and in space Crank-Nicolson scheme is used for the numerical analysis of the equation* (3.4.1)*. All numerical computations are made in C++ by using the sweep method. In above numerical simulations, we use the Matlab R2018b. For all simulations we take $\Delta t = 0.0107$, $\Delta x = 0.01$.*

3.5 Conclusion

The theoretical and numerical analysis conducted in this chapter showed that numerical methods work well in situations where a rigorous mathematical formulation of the problem is difficult in the framework of the classical theory of distributions. The ideology of very weak solutions eliminates this difficulty in the case of the terms with multiplication of distributions. In particular, in the case of the Schrödinger equation, we see that a delta-function potential is causing an effect of accumulating particles in the place of the support of the singularity.

Chapter 4

The heat equation with strongly singular potentials

4.1 Introduction

After the pioneering works due to Baras and Goldstein [8, 9], the heat equation with inverse-square potential in bounded and unbounded domains has attracted considerable attention during the last decades, we cite [34, 44, 64] to name only few. In this chapter we consider the heat equation with irregular potentials, in particular, with a δ-function and with a behaviour like "multiplication" of δ-functions and study its very weak well-posedness. That is, we study the Cauchy problem

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) + q(x)u(t,x) = 0, \quad (t,x) \in (0,T) \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \;\; x \in \mathbb{R}^d, \end{cases}$$

where the coefficient q indicates the potential. We consider first the case of positive potential, and due to the interesting numerically observed behaviours, we consider also the case of negative potential.

To start with, let us fix some notations used through this chapter.

4.1.1 Notation

We define

$$\|u(t,\cdot)\|_k := \|\nabla u(t,\cdot)\|_{L^2} + \sum_{l=0}^{k} \|\partial_t^l u(t,\cdot)\|_{L^2},$$

for all $k \in \mathbb{Z}_+$. In the case when $k = 0$, we simply use $\|u(t,\cdot)\|$ instead of $\|u(t,\cdot)\|_0$.

4.2 Part I: Non-negative potential

In this section we consider the case when the potential q is non-negative. That is, for fixed $T > 0$, in the domain $\Omega := (0, T) \times \mathbb{R}^d$ we consider the heat equation

$$\partial_t u(t,x) - \Delta u(t,x) + q(x)u(t,x) = 0, \ (t,x) \in \Omega, \tag{4.2.1}$$

with the Cauchy data $u(0,x) = u_0(x)$, where the potential q is assumed to be non-negative and singular.

4.2.1 Some useful lemmas

In the case when the potential is a regular function, we have the following lemma.

Lemma 4.2.1. *Let $u_0 \in H^1(\mathbb{R}^d)$ and suppose that $q \in L^\infty(\mathbb{R}^d)$ is non-negative. Then, there is a unique solution $u \in C^1([0,T]\,; L^2) \cap C([0,T]\,; H^1)$ to (4.2.1) and it satisfies the energy estimate*

$$\|u(t,\cdot)\| \lesssim (1 + \|q\|_{L^\infty}) \|u_0\|_{H^1}. \tag{4.2.2}$$

Proof. By multiplying the equation (4.2.1) by u_t and integrating with respect to x, we obtain

$$Re\left(\langle u_t(t,\cdot), u_t(t,\cdot)\rangle_{L^2} + \langle -\Delta u(t,\cdot), u_t(t,\cdot)\rangle_{L^2} + \langle q(\cdot)u(t,\cdot), u_t(t,\cdot)\rangle_{L^2}\right) = 0. \tag{4.2.3}$$

One observes

$$Re\langle u_t(t,\cdot), u_t(t,\cdot)\rangle_{L^2} = \langle u_t(t,\cdot), u_t(t,\cdot)\rangle_{L^2} = \|u_t(t,\cdot)\|^2_{L^2}.$$

Also, we see that

$$Re\langle -\Delta u(t,\cdot), u_t(t,\cdot)\rangle_{L^2} = \frac{1}{2}\partial_t\langle \nabla u(t,\cdot), \nabla u(t,\cdot)\rangle_{L^2} = \frac{1}{2}\partial_t\|\nabla u(t,\cdot)\|^2_{L^2}$$

and

$$Re\langle q(\cdot)u(t,\cdot), u_t(t,\cdot)\rangle_{L^2} = \frac{1}{2}\partial_t\langle q^{\frac{1}{2}}(\cdot)u(t,\cdot), q^{\frac{1}{2}}(\cdot)u(t,\cdot)\rangle_{L^2} = \frac{1}{2}\partial_t\|q^{\frac{1}{2}}(\cdot)u(t,\cdot)\|^2_{L^2}.$$

It follows from (4.2.3) that

$$\partial_t\left[\|\nabla u(t,\cdot)\|^2_{L^2} + \|q^{\frac{1}{2}}(\cdot)u(t,\cdot)\|^2_{L^2}\right] = -2\|u_t(t,\cdot)\|^2_{L^2}. \tag{4.2.4}$$

Let us denote by

$$E(t) := \|\nabla u(t,\cdot)\|^2_{L^2} + \|q^{\frac{1}{2}}(\cdot)u(t,\cdot)\|^2_{L^2},$$

the energy functional. It follows from (4.2.4) that $E'(t) \leq 0$, and thus

$$E(t) \leq E(0).$$

By taking into account that $\|q^{\frac{1}{2}}(\cdot)u_0(\cdot)\|_{L^2}^2$ can be estimated by

$$\|q^{\frac{1}{2}}(\cdot)u_0(\cdot)\|_{L^2}^2 \leq \|q(\cdot)\|_{L^\infty}\|u_0(\cdot)\|_{L^2}^2,$$

we get

$$\|\nabla u(t,\cdot)\|_{L^2}^2 + \|q^{\frac{1}{2}}(\cdot)u(t,\cdot)\|_{L^2}^2 \leq \|\nabla u_0\|_{L^2}^2 + \|q(\cdot)\|_{L^\infty}\|u_0\|_{L^2}^2.$$

Thus, we have

$$\|q^{\frac{1}{2}}(\cdot)u(t,\cdot)\|_{L^2}^2 \leq \|\nabla u_0\|_{L^2}^2 + \|q(\cdot)\|_{L^\infty}\|u_0\|_{L^2}^2 \tag{4.2.5}$$

and

$$\|\nabla u(t,\cdot)\|_{L^2}^2 \leq \|\nabla u_0\|_{L^2}^2 + \|q(\cdot)\|_{L^\infty}\|u_0\|_{L^2}^2,$$

and consequently, one can be seen that

$$\|\nabla u(t,\cdot)\|_{L^2} \leq \left(1 + \|q\|_{L^\infty}^{\frac{1}{2}}\right)^2 \|u_0\|_{H^1}. \tag{4.2.6}$$

To obtain the estimate for u, we rewrite the equation (4.2.1) as follows

$$u_t(t,x) - \Delta u(t,x) = -q(x)u(t,x), \quad (t,x) \in (0,T) \times \mathbb{R}^d. \tag{4.2.7}$$

Here, considering $-q(x)u(t,x)$ as a source term, we denote it by $f(t,x) := -q(x)u(t,x)$. By using Duhamel's principle (Theorem 7.2.2), we represent the solution to (4.2.7) in the form

$$u(t,x) = \phi_t * u_0(x) + \int_0^t \phi_{t-s} * f_s(x)ds, \tag{4.2.8}$$

where $f_s = f(s,\cdot)$ and $\phi_t = \phi(t,\cdot)$. Here, ϕ is the fundamental solution (heat kernel) to the heat equation, and it satisfies

$$\|\phi(t,\cdot)\|_{L^1} = 1.$$

Now, taking the L^2-norm in (4.2.8) and using Young's inequality (see Theorem 7.2.7), we arrive at

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2} &\leq \|\phi_t\|_{L^1}\|u_0\|_{L^2} + \int_0^T \|\phi_{t-s}\|_{L^1}\|f_s\|_{L^2}ds \\
&\leq \|u_0\|_{L^2} + \int_0^T \|f_s\|_{L^2}ds \\
&\leq \|u_0\|_{L^2} + \int_0^T \|q(\cdot)u(s,\cdot)\|_{L^2}ds.
\end{aligned}$$

We estimate the term $\|q(\cdot)u(s,\cdot)\|_{L^2}$ as

$$\|q(\cdot)u(s,\cdot)\|_{L^2} \leq \|q\|_{L^\infty}^{\frac{1}{2}}\|q^{\frac{1}{2}}u(s,\cdot)\|_{L^2},$$

and using the estimate (4.2.5), one observes

$$\|u(t,\cdot)\|_{L^2} \lesssim \left(1+\|q\|_{L^\infty}^{\frac{1}{2}}\right)^2 \|u_0\|_{H^1}. \tag{4.2.9}$$

Summing the estimates proved above, we conclude (4.2.2).

□

Remark 4.2.1. *We can also prove that the estimate*

$$\|\partial_t^k u(t,\cdot)\|_{L^2} \lesssim (1+\|q\|_{L^\infty})\,\|u_0\|_{H^{2k+1}},$$

is valid for all $k \geq 0$, by requiring higher regularity on u_0. To do so, we denote by $v_0 := u$ and its derivatives by $v_k := \partial_t^k u$, where u is the solution of the Cauchy problem (4.2.1). Using (4.2.9) and the property that if v_k solves the equation

$$\partial_t v_k(t,x) - \Delta v_k(t,x) + q(x)v_k(t,x) = 0,$$

with the initial data $v_k(0,x)$, then $v_{k+1} = \partial_t v_k$ solves the same equation with the initial data

$$v_{k+1}(0,x) = \Delta v_k(0,x) - q(x)v_k(0,x),$$

we get our estimate for $\partial_t^k u$ for all $k \geq 0$.

To prove the uniqueness and consistency of the very weak solution, we will also need the following lemma.

Lemma 4.2.2. *Let $u_0 \in H^1(\mathbb{R}^d)$ and assume that $q \in L^\infty(\mathbb{R}^d)$ is non-negative. Then, the estimate*

$$\|u(t,\cdot)\|_{L^2} \lesssim \|u_0\|_{L^2}, \tag{4.2.10}$$

holds for the unique solution $u \in C^1([0,T]\,;L^2) \cap C([0,T]\,;H^1)$ of the Cauchy problem (4.2.1).

Proof. Again, by multiplying the equation (4.2.1) by u and integrating over $\mathbb{R}^d$ in x, we derive

$$Re\left(\langle u_t(t,\cdot),u(t,\cdot)\rangle_{L^2} + \langle -\Delta u(t,\cdot),u(t,\cdot)\rangle_{L^2} + \langle q(\cdot)u(t,\cdot),u(t,\cdot)\rangle_{L^2}\right) = 0.$$

Using the similar arguments as in Lemma 4.2.1, we obtain

$$\partial_t\|u(t,\cdot)\|_{L^2}^2 = -\|\nabla u(t,\cdot)\|_{L^2}^2 - \|q^{\frac{1}{2}}(\cdot)u(t,\cdot)\|_{L^2}^2 \leq 0. \tag{4.2.11}$$

This ends the proof of the lemma. □

4.2.2 Existence of very weak solutions

In this subsection we deal with the existence of very weak solutions. We first fix a notation. By writing $q \geq 0$, we mean that all regularisations q_ε in our calculus are non-negative functions.

To show that the Cauchy problem (4.2.1) has a very weak solution, we start by regularising the coefficient q and the initial data u_0 using a suitable mollifying net $(\psi_\varepsilon)_{\varepsilon\in(0,1]}$, generating families of smooth functions $(q_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$. Namely,

$$q_\varepsilon(x) = q * \psi_\varepsilon(x), \qquad u_{0,\varepsilon}(x) = u_0 * \psi_\varepsilon(x),$$

where

$$\psi_\varepsilon(x) = \omega(\varepsilon)^{-d}\psi(x/\omega(\varepsilon)), \quad \varepsilon \in (0,1]\,,$$

and as in definition 2, $\omega(\varepsilon)$ is a positive function converging to 0 as $\varepsilon \to 0$ to be chosen later.

Assumption 4.2.3. *We assume that the regularisations of the coefficient q and the Cauchy data u_0 are L^∞-moderate and H^1-moderate respectively. Namely, there exist $N, N_0 \in \mathbb{N}_0$, such that*

$$\|q_\varepsilon\|_{L^\infty} \leq C\omega(\varepsilon)^{-N}, \tag{4.2.12}$$

and

$$\|u_{0,\varepsilon}\|_{H^1} \leq C_0\omega(\varepsilon)^{-N_0}, \tag{4.2.13}$$

for all $\varepsilon \in (0,1]$.

Definition 11 (Very weak solution)**.** *Let $q \geq 0$. The net $(u_\varepsilon)_\varepsilon$ is said to be a very weak solution to the Cauchy problem (4.2.1), if there exist an L^∞-moderate regularisation $(q_\varepsilon)_\varepsilon$ of the coefficient q and H^1-moderate regularisation $(u_{0,\varepsilon})_\varepsilon$ of the initial function u_0, such that $(u_\varepsilon)_\varepsilon$ solves the regularised equation*

$$\partial_t u_\varepsilon(t,x) - \Delta u_\varepsilon(t,x) + q_\varepsilon(x)u_\varepsilon(t,x) = 0, \quad (t,x) \in (0,T)\times\mathbb{R}^d, \tag{4.2.14}$$

with the Cauchy data $u_\varepsilon(0,x) = u_{0,\varepsilon}(x)$, for all $\varepsilon \in (0,1]$, and is C-moderate.

With this setup the existence of a very weak solution becomes straightforward. But we will also analyse its properties later on.

Theorem 4.2.4 (Existence)**.** *Let $q \geq 0$. Assume that the regularisations of the coefficient q and the Cauchy data u_0 satisfy the assumptions (4.2.12) and (4.2.13). Then the Cauchy problem (4.2.1) has a very weak solution.*

Proof. Using the moderateness assumptions (4.2.13), (4.2.12), and the energy estimate (4.2.2), we arrive at

$$\begin{aligned}\|u_\varepsilon(t,\cdot)\| &\lesssim \omega(\varepsilon)^{-N} \times \omega(\varepsilon)^{-N_0}\\ &\lesssim \omega(\varepsilon)^{-N-N_0},\end{aligned}$$

for all $t \in [0,T]$, concluding that $(u_\varepsilon)_\varepsilon$ is C-moderate. $\square$

4.2.3 Uniqueness result

In this subsection we discuss uniqueness of the very weak solution to the Cauchy problem (4.2.1). We prove it in the sense of the following definition.

Definition 12 (Uniqueness). *We say that the very weak solution to the Cauchy problem (4.2.1) is unique, if for all families $(q_\varepsilon)_\varepsilon$, $(\tilde{q}_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$, $(\tilde{u}_{0,\varepsilon})_\varepsilon$, regularisations of the coefficient q and u_0, such that the nets $(q_\varepsilon - \tilde{q}_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})_\varepsilon$ are L^∞-negligible and L^2-negligible respectively, we have that the net $(u_\varepsilon(t,\cdot) - \tilde{u}_\varepsilon(t,\cdot))_\varepsilon$ is L^2-negligible for all $t \in [0,T]$, where $(u_\varepsilon)_\varepsilon$ and $(\tilde{u}_\varepsilon)_\varepsilon$ solve, respectively, the families of the Cauchy problems*

$$\begin{cases} \partial_t u_\varepsilon(t,x) - \Delta u_\varepsilon(t,x) + q_\varepsilon(x) u_\varepsilon(t,x) = 0, & (t,x) \in [0,T] \times \mathbb{R}^d, \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x), \end{cases}$$

and

$$\begin{cases} \partial_t \tilde{u}_\varepsilon(t,x) - \Delta \tilde{u}_\varepsilon(t,x) + \tilde{q}_\varepsilon(x) \tilde{u}_\varepsilon(t,x) = 0, & (t,x) \in [0,T] \times \mathbb{R}^d, \\ \tilde{u}_\varepsilon(0,x) = \tilde{u}_{0,\varepsilon}(x). \end{cases}$$

We have the following theorem.

Theorem 4.2.5 (Uniqueness). *Let $T > 0$. Assume that $q \geq 0$ and u_0 satisfy the moderateness assumptions (4.2.12) and (4.2.13), respectively. Then, the very weak solution to the Cauchy problem (4.2.1) is unique.*

Proof. Let $(q_\varepsilon)_\varepsilon$, $(\tilde{q}_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$, $(\tilde{u}_{0,\varepsilon})_\varepsilon$, regularisations of the coefficient q and the data u_0, satisfying

$$\|q_\varepsilon - \tilde{q}_\varepsilon\|_{L^\infty} \leq C_k \varepsilon^k, \quad \text{for all} \quad k > 0,$$

and

$$\|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} \leq C_l \varepsilon^l, \quad \text{for all} \quad l > 0.$$

Then, $(u_\varepsilon)_\varepsilon$ and $(\tilde{u}_\varepsilon)_\varepsilon$, the solutions to the related Cauchy problems, satisfy

$$\begin{cases} \partial_t (u_\varepsilon - \tilde{u}_\varepsilon)(t,x) - \Delta (u_\varepsilon - \tilde{u}_\varepsilon)(t,x) + q_\varepsilon(x)(u_\varepsilon - \tilde{u}_\varepsilon)(t,x) = f_\varepsilon(t,x), \\ (u_\varepsilon - \tilde{u}_\varepsilon)(0,x) = (u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})(x), \end{cases} \tag{4.2.15}$$

with

$$f_\varepsilon(t,x) = (\tilde{q}_\varepsilon(x) - q_\varepsilon(x)) \tilde{u}_\varepsilon(t,x).$$

Let us denote by $U_\varepsilon(t,x) := u_\varepsilon(t,x) - \tilde{u}_\varepsilon(t,x)$ the solution to the equation (4.2.15). Using Duhamel's principle (Theorem 7.2.2), U_ε is given by

$$U_\varepsilon(t,x) = W_\varepsilon(t,x) + \int_0^t V_\varepsilon(x,t;s) ds,$$

where $W_\varepsilon(t,x)$ is the solution to the problem

$$\begin{cases} \partial_t W_\varepsilon(t,x) - \Delta W_\varepsilon(t,x) + q_\varepsilon(x) W_\varepsilon(t,x) = 0, & (t,x) \in (0,T] \times \mathbb{R}^d, \\ W_\varepsilon(0,x) = (u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})(x), \end{cases}$$

and $V_\varepsilon(x,t;s)$ solves

$$\begin{cases} \partial_t V_\varepsilon(x,t;s) - \Delta V_\varepsilon(x,t;s) + q_\varepsilon(x)V_\varepsilon(x,t;s) = 0, & (t,x) \in (s,T] \times \mathbb{R}^d \\ V_\varepsilon(x,s;s) = f_\varepsilon(s,x). \end{cases}$$

Taking U_ε in L^2-norm and using (4.2.10) to estimate V_ε and W_ε, we arrive at

$$\begin{aligned} \|U_\varepsilon(t,\cdot)\|_{L^2} &\leq \|W_\varepsilon(t,\cdot)\|_{L^2} + \int_0^T \|V_\varepsilon(\cdot,t;s)\|_{L^2} ds \\ &\lesssim \|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} + \int_0^T \|f_\varepsilon(s,\cdot)\|_{L^2} ds \\ &\lesssim \|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} + \|\tilde{q}_\varepsilon - q_\varepsilon\|_{L^\infty} \int_0^T \|\tilde{u}_\varepsilon(s,\cdot)\|_{L^2} ds. \end{aligned}$$

The family $(\tilde{u}_\varepsilon)_\varepsilon$ is a very weak solution to the Cauchy problem (4.2.1), it is then moderate, i.e. there exists $N_0 \in \mathbb{N}_0$ such that

$$\|\tilde{u}_\varepsilon(s,\cdot)\|_{L^2} \leq c\omega^{-N_0}(\varepsilon).$$

On the other hand, we have that $\|q_\varepsilon - \tilde{q}_\varepsilon\|_{L^\infty} \leq C_k\varepsilon^k$, for all $k > 0$, and $\|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} \leq C_l\varepsilon^l$, for all $l > 0$. Thus, we obtain that

$$\|U_\varepsilon(t,\cdot)\|_{L^2} := \|u_\varepsilon(t,\cdot) - \tilde{u}_\varepsilon(t,\cdot)\|_{L^2} \lesssim \varepsilon^N,$$

for all $N > 0$, showing the uniqueness of the very weak solution. Here we choose $\omega(\varepsilon) = \varepsilon$. □

4.2.4 Consistency with the classical case

Now we show that if the classical solution of the Cauchy problem (4.2.1) given by Lemma 4.2.1 exists then the very weak solution recaptures it.

Theorem 4.2.6 (Consistency)**.** *Let $u_0 \in H^1(\mathbb{R}^d)$. Assume that $q \in L^\infty(\mathbb{R}^d)$ is non-negative and consider the Cauchy problem*

$$\begin{cases} u_t(t,x) - \Delta u(t,x) + q(x)u(t,x) = 0, & (t,x) \in [0,T] \times \mathbb{R}^d, \\ u(0,x) = u_0(x). \end{cases} \tag{4.2.16}$$

Let $(u_\varepsilon)_\varepsilon$ be a very weak solution of (4.2.16). Then, for any regularising families $(q_\varepsilon)_\varepsilon$ such that $\|q_\varepsilon - q\|_{L^\infty} \to 0$ as $\varepsilon \to 0$ and $(u_{0,\varepsilon})_\varepsilon$, the net $(u_\varepsilon)_\varepsilon$ converges in L^2 as $\varepsilon \to 0$ to the classical solution of the Cauchy problem (4.2.16).

Proof. Consider the classical solution u to

$$\begin{cases} u_t(t,x) - \Delta u(t,x) + q(x)u(t,x) = 0, & (t,x) \in [0,T] \times \mathbb{R}^d, \\ u(0,x) = u_0(x). \end{cases}$$

Note that for the very weak solution there is a representation $(u_\varepsilon)_\varepsilon$ such that

$$\begin{cases} \partial_t u_\varepsilon(t,x) - \Delta u_\varepsilon(t,x) + q_\varepsilon(x) u_\varepsilon(t,x) = 0, & (t,x) \in [0,T] \times \mathbb{R}^d, \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x). \end{cases}$$

Taking the difference, we get

$$\begin{cases} \partial_t (u - u_\varepsilon)(t,x) - \Delta (u - u_\varepsilon)(t,x) + q_\varepsilon(x)(u - u_\varepsilon)(t,x) = \eta_\varepsilon(t,x), \\ (u - u_\varepsilon)(0,x) = (u_0 - u_{0,\varepsilon})(x), \end{cases} \tag{4.2.17}$$

where

$$\eta_\varepsilon(t,x) = (q_\varepsilon(x) - q(x)) u(t,x).$$

Let us denote $U_\varepsilon(t,x) := (u - u_\varepsilon)(t,x)$ and let $W_\varepsilon(t,x)$ be the solution to the auxiliary homogeneous problem

$$\begin{cases} \partial_t W_\varepsilon(t,x) - \Delta W_\varepsilon(t,x) + q_\varepsilon(x) W_\varepsilon(t,x) = 0, \\ W_\varepsilon(0,x) = (u_0 - u_{0,\varepsilon})(x). \end{cases}$$

Then, by Duhamel's principle (Theorem 7.2.2), the solution to (4.2.17) is given by

$$U_\varepsilon(t,x) = W_\varepsilon(t,x) + \int_0^t V_\varepsilon(x,t;s)\,ds, \tag{4.2.18}$$

where $V_\varepsilon(x,t;s)$ is the solution to the problem

$$\begin{cases} \partial_t V_\varepsilon(x,t;s) - \Delta V_\varepsilon(x,t;s) + q_\varepsilon(x) V_\varepsilon(x,t;s) = 0, & (t,x) \in (s,T] \times \mathbb{R}^d, \\ V_\varepsilon(x,s;s) = \eta_\varepsilon(t,x). \end{cases}$$

As in Theorem 4.2.5, taking the L^2-norm in (4.2.18) and using (4.2.10) to estimate V_ε and W_ε, we get

$$\begin{aligned} \|U_\varepsilon(t,\cdot)\|_{L^2} &\leq \|W_\varepsilon(t,\cdot)\|_{L^2} + \int_0^T \|V_\varepsilon(\cdot,t;s)\|_{L^2} ds \\ &\lesssim \|u_0 - u_{0,\varepsilon}\|_{L^2} + \int_0^T \|\eta_\varepsilon(s,\cdot)\|_{L^2} ds \\ &\lesssim \|u_0 - u_{0,\varepsilon}\|_{L^2} + \|q_\varepsilon - q\|_{L^\infty} \int_0^T \|u(s,\cdot)\|_{L^2} ds, \end{aligned}$$

and taking into account that

$$\|q_\varepsilon - q\|_{L^\infty} \to 0 \quad \text{as} \quad \varepsilon \to 0$$

and

$$\|u_{0,\varepsilon} - u_0\|_{L^2} \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

consequently, it implies that u_ε converges to u in L^2 as $\varepsilon \to 0$. $\square$

4.3 Part II: Negative potential

In this part we aim to study the case when the potential is negative and to show that the problem is still well-posed. Namely, we consider the Cauchy problem for the heat equation

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) - q(x)u(t,x) = 0, & (t,x) \in (0,T) \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{4.3.1}$$

where q is non-negative.

4.3.1 Useful lemma

In the classical case, we have the following energy estimates for the solution of the problem (4.3.1).

Lemma 4.3.1. *Let $u_0 \in L^2(\mathbb{R}^d)$ and suppose that $q \in L^\infty(\mathbb{R}^d)$ is non-negative. Then, there is a unique solution $u \in C([0,T]\,; L^2)$ to (4.3.1) and it satisfies the estimate*

$$\|u(t,\cdot)\|_{L^2} \lesssim \exp\left(t\|q\|_{L^\infty}\right)\|u_0\|_{L^2}, \tag{4.3.2}$$

for all $t \in [0,T]$.

Proof. Multiplying the equation in (4.3.1) by u, integrating with respect to x, and taking the real part, we obtain

$$Re\left(\langle u_t(t,\cdot), u(t,\cdot)\rangle_{L^2} + \langle -\Delta u(t,\cdot), u(t,\cdot)\rangle_{L^2} - \langle q(\cdot)u(t,\cdot), u(t,\cdot)\rangle_{L^2}\right) = 0,$$

for all $t \in [0,T]$. Using similar arguments as in Lemma 4.2.1 and noting that the term $\|q(\cdot)u(t,\cdot)\|_{L^2}$ can be estimated by $\|q\|_{L^\infty}\|u(t,\cdot)\|_{L^2}$, we get

$$\partial_t\|u(t,\cdot)\|_{L^2} \lesssim \|q\|_{L^\infty}\|u(t,\cdot)\|_{L^2},$$

for all $t \in [0,T]$. The desired estimate follows by the application of Gronwall's inequality (see theorem 7.2.4). $\square$

4.3.2 Existence of very weak solutions

Let now assume that the potential q and the initial data u_0 are singular. Consider the Cauchy problem for the heat equation

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) - q(x)u(t,x) = 0, & (t,x) \in (0,T) \times \mathbb{R}^d, \\ u(0,x) = u_0(x). \end{cases} \tag{4.3.3}$$

In order to prove the existence of a very weak solution to (4.3.3), we proceed as in the case of the positive potential. We start by regularising the equation in (4.3.3). In other words, using

$$\psi_\varepsilon(x) = \omega(\varepsilon)^{-d}\psi(x/\omega(\varepsilon)), \quad \varepsilon \in (0,1]\,,$$

where $(\psi_\varepsilon)_\varepsilon$ is a mollyfing net, we regularise q and u_0 obtaining the nets $(q_\varepsilon)_\varepsilon = (q * \psi_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon = (u_0 * \psi_\varepsilon)_\varepsilon$. For this, we can assume that q and u_0 are distributions.

Assumption 4.3.2. *We assume that the nets $(q_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$ are respectively, L^∞-moderate and L^2-moderate. That is, there exist $N_0, N_1 \in \mathbb{N}_0$ such that*

$$\|q_\varepsilon\|_{L^\infty} \leq C_0 \omega(\varepsilon)^{-N_0}, \tag{4.3.4}$$

and

$$\|u_{0,\varepsilon}\|_{L^2} \leq C_1 \omega(\varepsilon)^{-N_1}. \tag{4.3.5}$$

Let us now give the definition of a very weak solution adapted to the problem (4.3.3).

Definition 13 (Very weak solution)**.** *Let q be non-negative. The net $(u_\varepsilon)_\varepsilon$ is said to be a very weak solution to the problem (4.3.3), if there exist an L^∞-moderate regularisation $(q_\varepsilon)_\varepsilon$ of the coefficient q and an L^2-moderate regularisation $(u_{0,\varepsilon})_\varepsilon$ of u_0 such that $(u_\varepsilon)_\varepsilon$ solves the regularised problem*

$$\left\{\begin{array}{l} \partial_t u_\varepsilon(t,x) - \Delta u_\varepsilon(t,x) - q_\varepsilon(x) u_\varepsilon(t,x) = 0, \quad (t,x) \in (0,T) \times \mathbb{R}^d, \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x), \end{array}\right. \tag{4.3.6}$$

for all $\varepsilon \in (0,1]$, and is C-moderate.

Remark 4.3.1. *In this section, by C-moderate we mean $C([0,T], L^2)$-moderate (see definition 3).*

Theorem 4.3.3 (Existence)**.** *Let $q \geq 0$. Assume that the nets $(q_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$ satisfy the assumptions (4.3.4) and (4.3.5), respectively. Then the problem (4.3.3) has a very weak solution.*

Proof. The nets $(q_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$ are moderate by the assumption. To prove that a very weak solution to the Cauchy problem (4.3.3) exists, we need to show that the net $(u_\varepsilon)_\varepsilon$, a solution to the regularised problem (4.3.6), is C-moderate. Indeed, using the assumptions (4.3.4), (4.3.5) and the estimate (4.3.2), we get

$$\|u(t,\cdot)\|_{L^2} \lesssim \exp\left(t\omega(\varepsilon)^{-N_0}\right)\omega(\varepsilon)^{-N_1},$$

for all $t \in [0,T]$. Choosing the function ω of logarithmic type, that is

$$\omega(\varepsilon) = \left(\log \varepsilon^{-N_0}\right)^{-\frac{1}{N_0}},$$

we obtain that

$$\begin{aligned} \|u(t,\cdot)\|_{L^2} &\lesssim \varepsilon^{-tN_0} \times \left(\log \varepsilon^{-N_0}\right)^{\frac{N_1}{N_0}} \\ &\lesssim \varepsilon^{-TN_0} \times \varepsilon^{-N_1}, \end{aligned}$$

where the fact that $t \in [0,T]$ and that $\log \varepsilon^{-N_0}$ can be estimated by ε^{-N_0} are used. Then the net $(u_\varepsilon)_\varepsilon$ is C-moderate, implying the existence of a very weak solution. $\square$

4.3.3 Uniqueness result

Here, we prove the uniqueness of the very weak solution to the heat equation with a non-positive potential (4.3.3) in the spirit of Definition 12, adapted to our problem.

Definition 14 (Uniqueness). *Let the regularisations $(q_\varepsilon)_\varepsilon$ and $(\tilde{q}_\varepsilon)_\varepsilon$ of q and the regularisations $(u_{0,\varepsilon})_\varepsilon$ and $(\tilde{u}_{0,\varepsilon})_\varepsilon$ of u_0 satisfy Assumption 4.3.2. Then we say that the very weak solution to the heat equation (4.3.3) is unique, if for all families $(q_\varepsilon)_\varepsilon$, $(\tilde{q}_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$, $(\tilde{u}_{0,\varepsilon})_\varepsilon$, such that the nets $(q_\varepsilon - \tilde{q}_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})_\varepsilon$ are L^∞-negligible and L^2-negligible respectively, we have that the net $(u_\varepsilon(t,\cdot) - \tilde{u}_\varepsilon(t,\cdot))_\varepsilon$ is L^2-negligible for all $t \in [0,T]$, where $(u_\varepsilon)_\varepsilon$ and $(\tilde{u}_\varepsilon)_\varepsilon$ solve, respectively, the families of the Cauchy problems*

$$\left\{\begin{array}{l} \partial_t u_\varepsilon(t,x) - \Delta u_\varepsilon(t,x) - q_\varepsilon(x) u_\varepsilon(t,x) = 0, \quad (t,x) \in (0,T) \times \mathbb{R}^d, \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x), \end{array}\right.$$

and

$$\left\{\begin{array}{l} \partial_t \tilde{u}_\varepsilon(t,x) - \Delta \tilde{u}_\varepsilon(t,x) - \tilde{q}_\varepsilon(x) \tilde{u}_\varepsilon(t,x) = 0, \quad (t,x) \in (0,T) \times \mathbb{R}^d, \\ \tilde{u}_\varepsilon(0,x) = \tilde{u}_{0,\varepsilon}(x). \end{array}\right.$$

Theorem 4.3.4 (Uniqueness). *Let $T > 0$. Assume that the nets $(q_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$ satisfy the assumptions (4.3.4) and (4.3.5), respectively. Then, the very weak solution to the Cauchy problem (4.3.3) is unique.*

Proof. Let us consider $(q_\varepsilon)_\varepsilon$, $(\tilde{q}_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$, $(\tilde{u}_{0,\varepsilon})_\varepsilon$, regularisations of the q and u_0, satisfying

$$\|q_\varepsilon - \tilde{q}_\varepsilon\|_{L^\infty} \leq C_k \varepsilon^k \quad \text{for all} \quad k > 0$$

and

$$\|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} \leq C_l \varepsilon^l \quad \text{for all} \quad l > 0.$$

Then, $(u_\varepsilon)_\varepsilon$ and $(\tilde{u}_\varepsilon)_\varepsilon$, the solutions to the related Cauchy problems, satisfy

$$\left\{\begin{array}{l} \partial_t (u_\varepsilon - \tilde{u}_\varepsilon)(t,x) - \Delta (u_\varepsilon - \tilde{u}_\varepsilon)(t,x) - q_\varepsilon(x)(u_\varepsilon - \tilde{u}_\varepsilon)(t,x) = f_\varepsilon(t,x), \\ (u_\varepsilon - \tilde{u}_\varepsilon)(0,x) = (u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})(x), \end{array}\right. \tag{4.3.7}$$

with

$$f_\varepsilon(t,x) = (q_\varepsilon(x) - \tilde{q}_\varepsilon(x)) \tilde{u}_\varepsilon(t,x).$$

Let us denote by $U_\varepsilon(t,x) := u_\varepsilon(t,x) - \tilde{u}_\varepsilon(t,x)$ the solution to the equation (4.3.7). Arguing as in Theorem 4.2.5 and using the estimate (4.3.2), we arrive at

$$\|U_\varepsilon(t,\cdot)\|_{L^2} \lesssim \exp\left(t\|q_\varepsilon\|_{L^\infty}\right)\|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} + \|q_\varepsilon - \tilde{q}_\varepsilon\|_{L^\infty} \int_0^T \exp\left(s\|q_\varepsilon\|_{L^\infty}\right)\|\tilde{u}_\varepsilon(s,\cdot)\|_{L^2} ds.$$

On the one hand, the net $(q_\varepsilon)_\varepsilon$ is moderate by the assumption and $(\tilde{u}_\varepsilon)_\varepsilon$ is moderate as a very weak solution. From the other hand, we have that

$$\|q_\varepsilon - \tilde{q}_\varepsilon\|_{L^\infty} \leq C_k \varepsilon^k \quad \text{for all} \quad k > 0,$$

and

$$\|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} \leq C_l \varepsilon^l \quad \text{for all} \quad l > 0.$$

By choosing $\omega(\varepsilon) = \left(\log \varepsilon^{-N_0}\right)^{-\frac{1}{N_0}}$ for q_ε in (4.3.4), it follows that

$$\|U_\varepsilon(t,\cdot)\|_{L^2} = \|u_\varepsilon(t,\cdot) - \tilde{u}_\varepsilon(t,\cdot)\|_{L^2} \lesssim \varepsilon^N,$$

for all $N > 0$, ending the proof. □

4.3.4 Consistency with the classical case

We conclude this section by showing that if the coefficient and the Cauchy data are regular then the very weak solution coincides with the classical one, given by Lemma 4.3.1.

Theorem 4.3.5 (Consistency)**.** *Let $u_0 \in L^2(\mathbb{R}^d)$. Assume that $q \in L^\infty(\mathbb{R}^d)$ is non-negative and consider the Cauchy problem for the heat equation*

$$\left\{ \begin{array}{l} u_t(t,x) - \Delta u(t,x) - q(x)u(t,x) = 0, \quad (t,x) \in (0,T) \times \mathbb{R}^d, \\ u(0,x) = u_0(x). \end{array} \right. \tag{4.3.8}$$

Let $(u_\varepsilon)_\varepsilon$ be a very weak solution of the heat equation (4.3.8). Then, for any regularising families $(q_\varepsilon)_\varepsilon$ such that $\|q_\varepsilon - q\|_{L^\infty} \to 0$ as $\varepsilon \to 0$ and $(u_{0,\varepsilon})_\varepsilon$, the net $(u_\varepsilon)_\varepsilon$ converges in L^2 as $\varepsilon \to 0$ to the classical solution of the Cauchy problem (4.3.8).

Proof. Let us denote the classical solution and the very weak one by u and $(u_\varepsilon)_\varepsilon$, respectively. It is clear, that they satisfy

$$\left\{ \begin{array}{l} u_t(t,x) - \Delta u(t,x) - q(x)u(t,x) = 0, \;\; (t,x) \in (0,T) \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{array} \right.$$

and

$$\left\{ \begin{array}{l} \partial_t u_\varepsilon(t,x) - \Delta u_\varepsilon(t,x) - q_\varepsilon(x)u_\varepsilon(t,x) = 0, \;\; (t,x) \in (0,T) \times \mathbb{R}^d, \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x), \end{array} \right.$$

respectively. Let us denote by $V_\varepsilon(t,x) := (u_\varepsilon - u)(t,x)$. Using the estimate (4.3.2) and the same arguments as in the positive potential case, we show that

$$\|V_\varepsilon(t,\cdot)\|_{L^2} \lesssim \exp\left(t\|q_\varepsilon\|_{L^\infty}\right)\|u_{0,\varepsilon} - u_0\|_{L^2} + \|q_\varepsilon - q\|_{L^\infty} \int_0^T \exp\left(s\|q_\varepsilon\|_{L^\infty}\right)\|u(s,\cdot)\|_{L^2} ds.$$

By taking into account that

$$\|q_\varepsilon - q\|_{L^\infty} \to 0 \quad \text{as} \quad \varepsilon \to 0$$

and

$$\|u_{0,\varepsilon} - u_0\|_{L^2} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

From the other hand, due to the facts that q_ε is bounded as a regularisation of an essentially bounded function and $\|u(s,\cdot)\|_{L^2}$ is bounded as well as u is a classical solution, we conclude that $(u_\varepsilon)_\varepsilon$ converges to u in L^2 as $\varepsilon \to 0$. □

4.4 Propagation of singularities: Numerical experiments

In this Section, we do some numerical experiments. Let us analyse our problem by regularising a distributional potential $q(x)$ by a parameter ε. We define $q_\varepsilon(x) := (q * \varphi_\varepsilon)(x)$, as the convolution with the mollifier $\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi(x/\varepsilon)$, where

$$\varphi(x) = \begin{cases} c\exp\left(\frac{1}{x^2-1}\right), & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

with $c \simeq 2.2523$ to have $\int\limits_{-\infty}^{\infty} \varphi(x)dx = 1$. Then, instead of (4.2.1) we consider the regularised problem

$$\partial_t u_\varepsilon(t,x) - \partial_x^2 u_\varepsilon(t,x) + q_\varepsilon(x)u_\varepsilon(t,x) = 0,\ (t,x) \in [0,T] \times \mathbb{R}, \tag{4.4.1}$$

with the initial data $u_\varepsilon(0,x) = u_0(x)$, for all $x \in \mathbb{R}$. Here, we put

$$u_0(x) = \begin{cases} \exp\left(\frac{1}{(x-50)^2-0.25}\right), & |x-50| < 0.5, \\ 0, & |x-50| \geq 0.5. \end{cases} \tag{4.4.2}$$

Note that $\operatorname{supp} u_0 \subset [49.5, 50.5]$.

In the non-negative potential case, for q we consider the following cases, with δ denoting the standard Dirac's delta-distribution:

Case 1: $q(x) = 0$ with $q_\varepsilon(x) = 0$;

Case 2: $q(x) = \delta(x-40)$ with $q_\varepsilon(x) = \varphi_\varepsilon(x-40)$;

Case 3: $q(x) = \delta(x-40) \times \delta(x-40)$. Here, we understand $q_\varepsilon(x)$ as follows $q_\varepsilon(x) = (\varphi_\varepsilon(x-40))^2$.

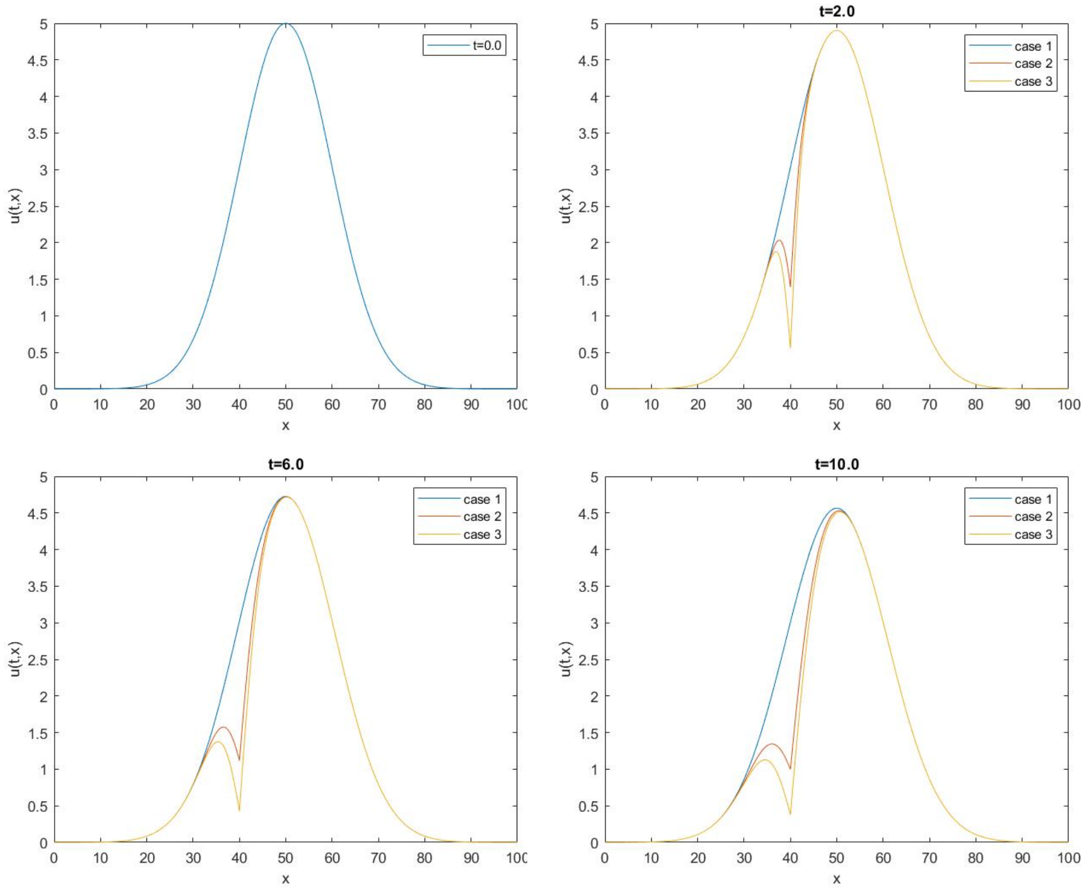

Figure 4.1: In these plots, we analyse behaviour of the temperature in three different cases. In the top left plot, the graphic of the initial function is given. In the further plots, we compare the temperature function u which is the solution of (4.4.1) at $t = 2, 6, 10$ for $\varepsilon = 0.2$ in three cases. Case 1 is corresponding to the potential q equal to zero. Case 2 is corresponding to the case when the potential q is a δ-function with the support at point 40. Case 3 is corresponding to a δ^2-like function potential with the support at point 40.

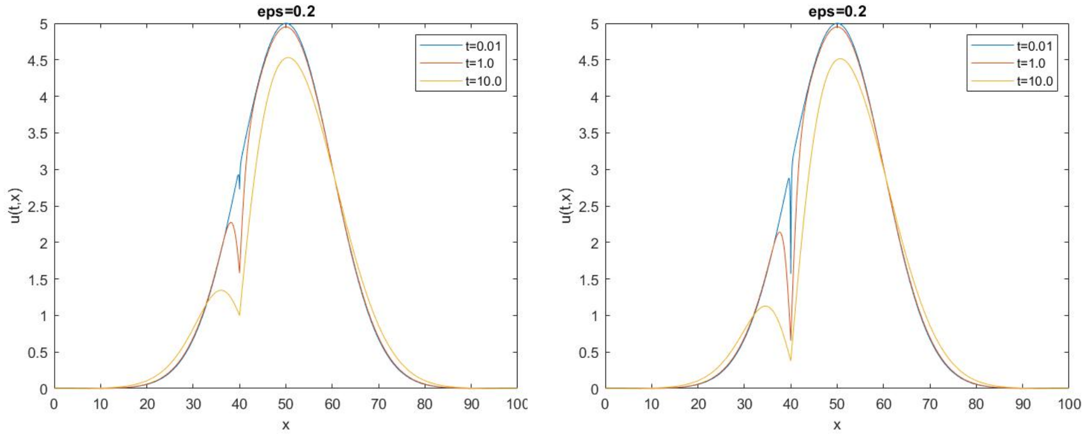

Figure 4.2: In these plots, we compare the temperature function u at $t = 0.01, 1.0, 10.0$ for $\varepsilon = 0.2$ in the second and third cases: when the potential is δ-like and δ^2-like functions with the support at point 40, respectively. The left picture is corresponding to the second case. The right picture is corresponding to the third case.

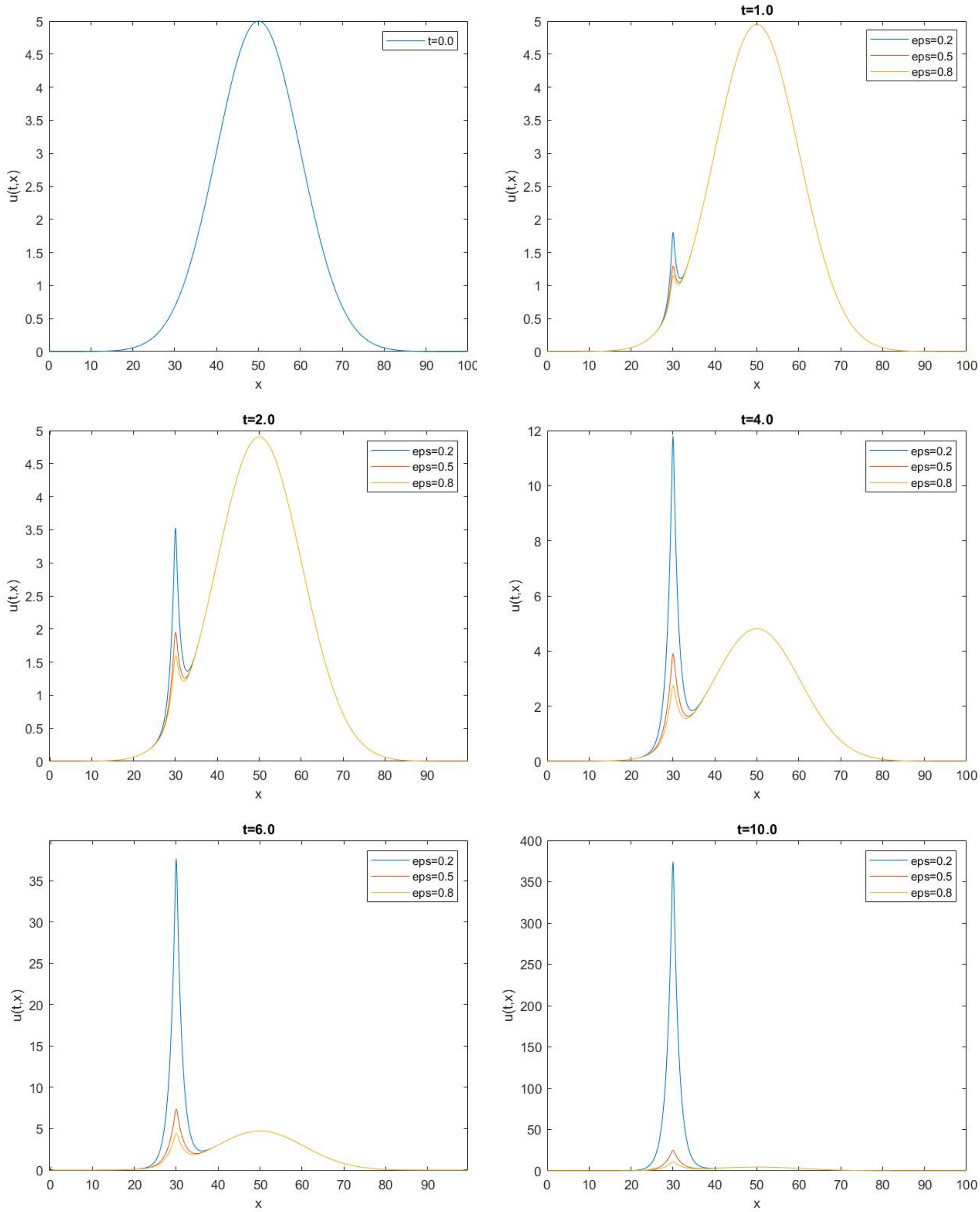

Figure 4.3: In these plots, we analyse behaviour of the solution of the heat equation (4.4.3) with the negative potential. In the top left plot, the graphic of the temperature distribution at the initial time. In the further plots, we compare the temperature function u at $t = 1, 2, 4, 6, 10$ for $\varepsilon = 0.8, 0.5, 0.2$. Here, the case of the potential with a δ-like function behaviour with the support at point 30 is considered.

In Figure 4.1, we study behaviour of the temperature function u which is the solution of (4.4.1) at $t = 2, 6, 10$ for $\varepsilon = 0.2$ in three cases: the first case is corresponding to the potential q equal to zero; the second case is corresponding to the case when the potential q is a δ-function with the support at point 40; the third case is corresponding to a δ^2-like function potential with the support at point 40. By comparing these cases, we observe that in the second and in the third cases, the place of the support of the δ-function is cooling down faster rather than the zero-potential case. This phenomena can be described as a "point cooling" or "laser cooling" effect.

In Figure 4.2, we compare the temperature function u at $t = 0.01, 1.0, 10.0$ for $\varepsilon = 0.2$ in the second and third cases: when the potential is δ-like and δ^2-like functions with the supports at point 40, respectively. The left picture is corresponding to the second case. The right picture is corresponding to the third case.

In Figures 4.1 and 4.2, we analyse the equation (4.4.1) with positive potentials. Now, in Figure 4.3, we study the following equation with negative potentials:

$$\partial_t u_\varepsilon(t,x) - \partial_x^2 u_\varepsilon(t,x) - q_\varepsilon(x) u_\varepsilon(t,x) = 0, \ (t,x) \in [0,T] \times \mathbb{R}, \tag{4.4.3}$$

with the same initial data u_0 as in (4.4.2). In these plots, we compare the temperature function u at $t = 1, 2, 4, 6, 10$ for $\varepsilon = 0.8, 0.5, 0.2$ corresponding to the potential with a δ-like function with the support at point 30.

Remark 4.4.1. *Numerical simulations justify the theory developed in Section 4.3. Moreover, we observe that for the negative δ-potential case, the place of the support of the δ-function is heating up. This phenomena can be described as a "point heating" or "laser heating" effect. Also, one observes that our numerical calculations prove the behaviour of the solution related to the parameter ε.*

Remark 4.4.2. *All numerical computations are made in C++ by using the sweep method. In above numerical simulations, we use the Matlab R2018b. For all simulations we take $\Delta t = 0.2$, $\Delta x = 0.01$.*

4.4.1 Conclusion

The numerical analysis conducted in this chapter showed that a delta-function potential helps to loose/increase energy in a less time, the latter causing a so-called "laser cooling/heating" effect in the positive/negative potential cases.

Chapter 5

Propagation of singularities for very weak solutions to a wave equation with singular dissipation

5.1 Introduction

A starting point of this thesis was a recent paper, where Munoz, Ruzhansky and Tokmagambetov [46] investigated a particular wave model with singular dissipation arising from acoustic problems. They considered the Cauchy problem

$$u_{tt} - \Delta u + \frac{b'(t)}{b(t)} u_t = 0, \qquad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x),$$

where b is a piecewise continuous and positive function allowing in particular for jumps and in consequence a non-distributional singular coefficient in this Cauchy problem. Based on results from [65] and [66], they considered the notion of very weak solutions for this singular problem and showed that this problem is well-posed in this very weak sense and that the very weak solution is independent of the choice of the regularising family. Moreover, they numerically observed in one space dimension a very interesting phenomenon, namely the appearance of a new wave after the singular time travelling in the opposite direction to the main one.

The aim in this chapter is twofold. On the one hand we consider this model and carry out a detailed phase space analysis for families of regularised problems in order to describe the behaviour of the very weak solution in the vicinity of the singular time. This will allow us to show that the numerically observed partial reflection of wave packets at the singular time is really appearing and to calculate the partial reflection indices in terms of the jump of the coefficient. On the other hand this is a model study to develop tools and techniques to treat more general singular hyperbolic problems within the framework of very weak solutions and to provide a symbolic calculus framework for analysing singularities of such solutions.

5.2 Our model problem and general strategy

We consider the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + \frac{b'(t)}{b(t)} u_t = 0, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), \end{cases} \tag{5.2.1}$$

where b is a piecewise smooth and piecewise continuous function. We are interested in solutions close to a singularity of the coefficient and hence, without loss of generality, we assume that b has exactly one jump at $t = 1$. In particular, we require that the limits

$$b(1_{\pm 0}) = \lim_{t \to 1 \pm 0} b(t) \tag{5.2.2}$$

exist for the function itself and also its derivatives. Thus, we ask for b to satisfy the following two assumptions:

(H1) There exists a strictly positive number b_0 such that $b(t) \geq b_0 > 0$.

(H2) $b \in C_b^\infty(-\infty, 1] \cap C_b^\infty[1, +\infty)$, having a jump at $t = 1$.

In contrast to [46] we do *not* require b to be monotonically increasing. Thus, we will not make use of any sign properties of the coefficient later.

5.2.1 Notation

Throughout this chapter we will use the following special conventions and symbols:

- We denote the height of the jump of b at $t = 1$ by

  $$h = b(1_{+0}) - b(1_{-0})$$

  and denote $H = \frac{b(1_{-0})}{b(1_{+1})}$.

- We denote by $\Phi \in C_0(\mathbb{R})$ a fixed non-negative, continuous and symmetric function, such that

  $$\Phi(-t) = \Phi(t) \qquad \text{and} \qquad \operatorname{supp} \Phi = [-K', K'] \tag{5.2.3}$$

  holds. We further assume that Φ is differentiable outside the origin and that

  $$\Phi^2(t) \lesssim \begin{cases} \Phi'(t), & t < 0, \\ -\Phi'(t), & t > 0, \end{cases} \tag{5.2.4}$$

  holds. This function will play an important role in the definition of zones and symbol classes and will be referred to as the shape function.

- We denote by $\psi \in C_0^\infty(\mathbb{R})$ a fixed non-negative and symmetric mollifier such that
$$\psi(-t) = \psi(t), \qquad \operatorname{supp}\psi = [-K, K], \qquad \text{and} \qquad \int \psi(t)\,\mathrm{d}t = 1 \tag{5.2.5}$$
with $0 < K \le K'$ describing the size of its support. We further require that derivatives of ψ be bounded by powers of the shape function Φ, i.e.
$$|\partial_t^k \psi(t)| \lesssim \Phi^k(t) \tag{5.2.6}$$
for any number $k \in \mathbb{N}$.

- The identity matrix will be denoted by I. Furthermore for any matrix A we denote by $\|A\|$ its Euclidean matrix norm.

5.2.2 Regularisation of the problem

In order to consider very weak solutions of our model problem, we solve families of regularised problems using the regularisations
$$b_\varepsilon(t) = b * \psi_\varepsilon(t) \qquad \text{and} \qquad b'_\varepsilon(t) = b' * \psi_\varepsilon(t) = b * \psi'_\varepsilon(t) \tag{5.2.7}$$
in terms of the mollifier $\psi_\varepsilon(t) = \varepsilon^{-1}\psi(\varepsilon^{-1}t)$ and with $\varepsilon \in (0, 1]$. This gives rise to the family of Cauchy problems
$$\begin{cases} u_{tt} - \Delta u + \frac{b'_\varepsilon(t)}{b_\varepsilon(t)} u_t = 0, \\ u(0, x) = u_0(x),\ u_t(0, x) = u_1(x), \end{cases} \tag{5.2.8}$$
parameterised by $\varepsilon \in (0, 1]$. Our approach is based on a detailed phase space analysis for this family of problems treating ε as an additional variable of the extended phase space. For this, we will introduce two zones and apply a diagonalisation based technique to extract leading order terms in each of them. For details on the diagonalisation procedure and its use in a related singular context we refer to [23] or [67], and for a broader discussion of the techniques used see [56].

As the coefficients of (5.2.8) depend on t only, we apply a partial Fourier transform with respect to the spatial variables and, thus, reduce consideration to the ordinary differential equation
$$\begin{cases} \hat{u}_{tt} + |\xi|^2 \hat{u} + \frac{b'_\varepsilon(t)}{b_\varepsilon(t)} \hat{u}_t = 0, \\ \hat{u}(0, \xi) = \hat{u}_0(\xi),\ \hat{u}_t(0, \xi) = \hat{u}_1(\xi), \end{cases} \tag{5.2.9}$$
parameterised by both $\varepsilon \in (0, 1]$ and $\xi \in \mathbb{R}^n$. We construct its solutions for $t \in [0, 2]$ and investigate the limiting behaviour of solutions as $\varepsilon \to 0$. To write the equation in system form, we introduce the micro-energy
$$U(t, \xi, \varepsilon) = \begin{pmatrix} |\xi|\,\hat{u} \\ \mathrm{D}_t \hat{u} \end{pmatrix}, \tag{5.2.10}$$

where $\mathrm{D}_t = -\mathrm{i}\partial_t$ denotes the Fourier derivative. Then (5.2.9) can be rewritten as

$$\mathrm{D}_t U(t,\xi,\varepsilon) = \left[\begin{pmatrix} 0 & |\xi| \\ |\xi| & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{i}\mathfrak{d}_\varepsilon(t) \end{pmatrix}\right] U(t,\xi,\varepsilon), \tag{5.2.11}$$

where we used the notation $\mathfrak{d}_\varepsilon(t) = \frac{b'_\varepsilon(t)}{b_\varepsilon(t)}$ for the net of dissipation coefficients. Denoting the coefficient matrices arising in this system by

$$A(\xi) = \begin{pmatrix} 0 & |\xi| \\ |\xi| & 0 \end{pmatrix} \quad \text{and} \quad B(t,\varepsilon) = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{i}\mathfrak{d}_\varepsilon(t) \end{pmatrix}, \tag{5.2.12}$$

we see that depending on the values $|\xi|$, ε and t either the matrix $A(\xi)$ or the matrix $B(t,\varepsilon)$ is dominant. If $A(\xi)$ is dominant, we apply a standard hyperbolic approach and diagonalise the system. If $B(t,\varepsilon)$ is dominant, we use a transformation of variables to reduce consideration to a model equation describing the behaviour close to the singularity.

5.2.3 Zones

To make use of different leading terms, we use the following definition of zones. For a zone constant N to be fixed later we define the *hyperbolic zone*

$$Z_{\mathrm{hyp}}(N) = \left\{(t,\xi,\varepsilon) \in [0,2]\times\mathbb{R}^n\times(0,1] \mid |\xi| \geq N\big(\Phi_\varepsilon(t-1)+1\big)\right\}, \tag{5.2.13}$$

where $\Phi_\varepsilon(t) = \varepsilon^{-1}\Phi\left(\varepsilon^{-1}t\right)$ is defined in terms of the function Φ from Section 5.2.1. The *singular zone*

$$Z_{\mathrm{sing}}(N) = \left\{(t,\xi,\varepsilon) \in [0,2]\times\mathbb{R}^n\times(0,1] \mid N < |\xi| \leq N\big(\Phi_\varepsilon(t-1)+1\big)\right\} \tag{5.2.14}$$

is used to investigate the vicinity of the jump of the coefficient, while the remaining *bounded frequencies*

$$Z_{\mathrm{bd}}(N) = \{(t,\xi,\varepsilon) \in [0,2]\times\mathbb{R}^n\times(0,1] \mid |\xi| \leq N\} \tag{5.2.15}$$

will be dealt with later by a simple argument. The common boundary of the hyperbolic and the singular zone will be denoted by $(t_{\xi_i}(\varepsilon))_{i=1,2}$ and is defined implicitly by the equation

$$|\xi| = N\big(\Phi_\varepsilon(t_{\xi_i}-1)+1\big) \tag{5.2.16}$$

for ξ satisfying $N < |\xi| \leq N(\varepsilon^{-1}\Phi(0)+1)$ and with the convention that t_{ξ_1} is the solution branch for $t<1$ and t_{ξ_2} when $t>1$. The zones are depicted in Figure 5.1 for fixed $\varepsilon>0$.

The singular zone $Z_{\mathrm{sing}}(N)$ is better understood in the variables $\Lambda = \varepsilon|\xi|$ and $\tau = \varepsilon^{-1}(t-1)$. Then the definition of the singular zone can be rewritten as

$$Z_{\mathrm{sing}}(N) = \{(\tau,\Lambda,\varepsilon) \mid N\varepsilon < \Lambda \leq N\Phi(\tau)+N\varepsilon\} \tag{5.2.17}$$

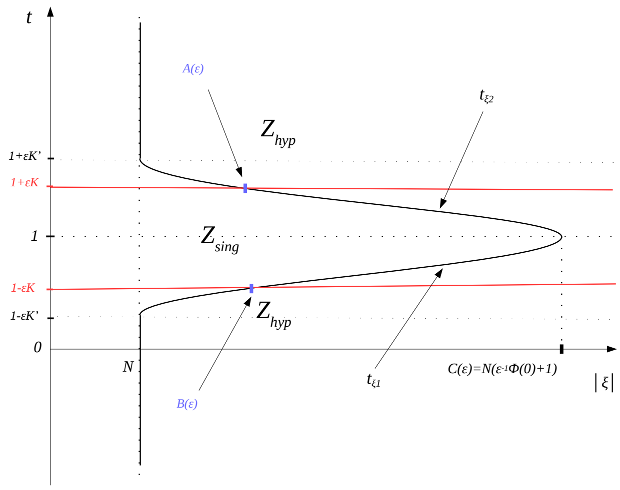

Figure 5.1: Zones in coordinates (t, ξ) for fixed $\varepsilon > 0$

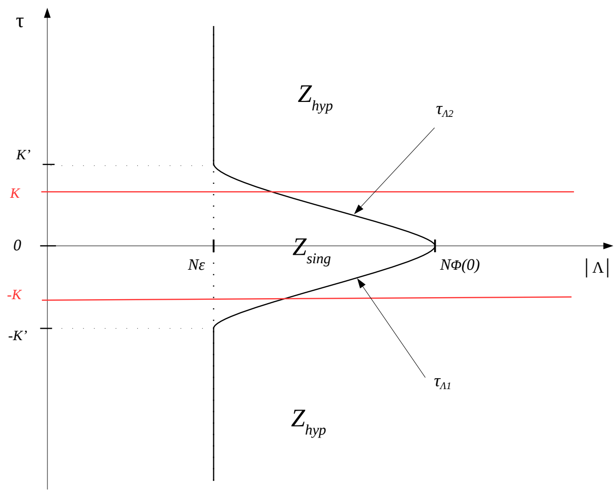

Figure 5.2: Zones in coordinates (τ, Λ) again for a fixed $\varepsilon > 0$

and stabilises as $\varepsilon \to 0$. We will use these singular variables when discussing the solutions of the regularised problem in the singular zone. For convenience, the zone is depicted in Figure 5.2 using these variables. We will also use a notation for the zone-boundary and denote it by $\tau_{\Lambda_1}(\varepsilon)$ and $\tau_{\Lambda_2}(\varepsilon)$.

Our strategy is as follows. Within the hyperbolic zone we will apply a diagonalisation procedure taking care of the ε-dependence of the transformation matrices and all appearing symbols in an appropriate way. This allows to construct the fundamental solution of the parameter-dependent family (5.2.8) within the hyperbolic zone and to investigate its limiting behaviour as $\varepsilon \to 0$. Within the singular zone, we transform the problem to the singular variables and construct its fundamental solution as power series in Λ with τ, ε-dependent coefficients and again study the limiting behaviour of this solution as $\varepsilon \to 0$.

Remark 5.2.1. *We note that in coordinates (t, ξ) the point $C(\varepsilon)$ tends to ∞*

when ε tends to 0 *and that* $t_{\min}$ *and* $t_{\max}$ *depend on* ε *and tend to* 1_{-0} *and* 1_{+0} *when* $\varepsilon \to 0$*, respectively.*

Remark 5.2.2. *The interval* $[1-\varepsilon K, 1+\varepsilon K]$ *is the support of* $\psi_\varepsilon(t-1)$*. The lines* $t = 1-\varepsilon K$ *and* $t = 1+\varepsilon K$ *divide the hyperbolic zone into two parts, one with* $|t-1| > \varepsilon K$ *and the other one with* $|t-1| < \varepsilon K$*. The last one is of minor interest since the points* $A(\varepsilon) = N\varepsilon^{-1}\Phi(K) + N$ *and* $B(\varepsilon)$ *tend to infinity when* ε *tends to* 0*.*

5.2.4 Regular faces of the zones.

The hyperbolic zone $Z_{\text{hyp}}(N)$ and the zone of bounded frequencies $Z_{\text{bd}}(N)$ have a boundary on which $\varepsilon \to 0$. This will be of importance later when relating our representation of very weak solutions to the standard theory for smooth coefficients for $t \neq 1$. We refer to the two parts $\{(t,\xi,0) \mid |\xi| > N,\ t \neq 1\}$ as the *regular face* of $Z_{\text{hyp}}(N)$ and the set $\{(t,\xi,0) \mid |\xi| \leq N\}$ as the *regular face* of $Z_{\text{bd}}(N)$. The singular zone does not have a regular face.

5.3 Representation of solutions

5.3.1 Some useful lemmas

The nets $\mathfrak{d}_\varepsilon(t) = b'_\varepsilon(t)/b_\varepsilon(t)$, $b_\varepsilon(t)$ and its derivatives $b_\varepsilon^{(k)}(t)$ defined in terms of (5.2.7) satisfy the following inequalities.

Lemma 5.3.1. *The estimates*

$$|\partial_t^k b_\varepsilon(t)| \leq C_{1,k}\big(\Phi_\varepsilon(t-1)+1\big)^k \text{ and } |\partial_t^k \mathfrak{d}_\varepsilon(t)| \leq C_{2,k}\big(\Phi_\varepsilon(t-1)+1\big)^{k+1} \tag{5.3.1}$$

hold for all $k \geq 0$ *and all* $t \in [0,2]$ *and* $\varepsilon \in (0,1]$*.*

Proof. The second estimate will follow from the first one, so we only concentrate on the first. For $k = 0$ we have by the Assumptions **(H1)** and **(H2)** and the positivity of the mollifier ψ

$$0 < b_0 \leq b_\varepsilon(t) = \int_{-\infty}^{\infty} b(t-\varepsilon s)\psi(s)\mathrm{d}s \leq \max_{s\in[-K,1+K]} b(s). \tag{5.3.2}$$

For $k = 1$ we apply integration by parts. As

$$\begin{aligned}
b'_\varepsilon(t) &= \varepsilon^{-2}\int_{-\infty}^{\infty} b(s)\psi'(\varepsilon^{-1}(t-s))\mathrm{d}s \\
&= \varepsilon^{-2}\left[\int_{-\infty}^{1} b(s)\psi'(\varepsilon^{-1}(t-s))\mathrm{d}s + \int_{1}^{\infty} b(s)\psi'(\varepsilon^{-1}(t-s))\mathrm{d}s\right] \\
&= \varepsilon^{-1}\psi(\varepsilon^{-1}(t-1))\big[b(1_{-0}) - b(1_{+0})\big] \\
&\quad - \varepsilon^{-1}\left[\int_{-\infty}^{1} b'(s)\psi(\varepsilon^{-1}(t-s))\mathrm{d}s + \int_{1}^{\infty} b'(s)\psi(\varepsilon^{-1}(t-s))\mathrm{d}s\right]
\end{aligned} \tag{5.3.3}$$

we obtain

$$|b_\varepsilon'(t)| \leq |h|\psi_\varepsilon(t-1) + \sup_{s\in[-K,1)\cup(1,1+K]} |b'(s)| \leq C_1\left(\Phi_\varepsilon(t-1)+1\right) \tag{5.3.4}$$

using the bound $\psi \lesssim \Phi$. For higher k we need to apply several steps of integration by parts. For $k \geq 2$ we obtain by induction

$$\begin{aligned}\partial_t^k b_\varepsilon(t) = &\sum_{\ell=0}^{k-1}(-1)^{k-\ell-1}\left[b^{(k-\ell-1)}(1_{-0}) - b^{(k-\ell-1)}(1_{+0})\right]\partial_t^\ell\psi_\varepsilon(t-1)\\ &+ (-1)^k\left[\int_{-\infty}^1 b^{(k)}(s)\psi_\varepsilon(t-s)\mathrm{d}s + \int_1^\infty b^{(k)}(s)\psi_\varepsilon(t-s)\mathrm{d}s\right].\end{aligned} \tag{5.3.5}$$

Again, the remaining integrals can be estimated by uniform bounds on the derivatives of b outside the singularity, and the statement follows from

$$\begin{aligned}|\partial_t^k b_\varepsilon(t)| &\leq \sum_{\ell=0}^{k-1}|b^{(k-\ell-1)}(1_{-0}) - b^{(k-\ell-1)}(1_{+0})|\,|\partial_t^\ell\psi_\varepsilon(t)|\\ &\quad + \sup_{s\in[-K,1)\cup(1,1+K]}|b^{(k)}(s)|\\ &\leq C_k\big(\Phi_\varepsilon(t)+1\big)^k\end{aligned} \tag{5.3.6}$$

by the estimate $|\partial_t^k\psi(t)| \leq C\Phi^k(t)$. Finally, the estimate for derivatives of $\mathfrak{d}_\varepsilon(t)$ follow from applying the quotient rule and using the uniform lower bound $b_0 \leq b_\varepsilon(t)$ when estimating the denominator. ☐

Lemma 5.3.2. *The estimates*

$$|\partial_t^k b_\varepsilon(t) - \partial_t^k b(t)| \lesssim \varepsilon \tag{5.3.7}$$

hold for all $k \geq 0$ uniformly in $\varepsilon \in (0,1]$ and t satisfying $|t-1| > \varepsilon K$.

Proof. Let $|t-1| > \varepsilon K$. For $k=0$ we have

$$\begin{aligned}b_\varepsilon(t) - b(t) &= \int_{-\infty}^\infty b(s)\psi_\varepsilon(t-s)\mathrm{d}s - b(t)\\ &= \int_{[t-\varepsilon K, t+\varepsilon K]}\left(b(s)-b(t)\right)\psi_\varepsilon(t-s)\mathrm{d}s\end{aligned} \tag{5.3.8}$$

seeing that $\int\psi(s)\mathrm{d}s = 1$ and that $\operatorname{supp}\psi_\varepsilon = [-\varepsilon K, \varepsilon K]$. Hence,

$$|b_\varepsilon(t) - b(t)| \leq \int_{[t-\varepsilon K, t+\varepsilon K]}|b(s)-b(t)|\psi_\varepsilon(t-s)\mathrm{d}s. \tag{5.3.9}$$

As the range of integration does not contain 1 we can use the differentiability of b to estimate

$$|b(s)-b(t)| \leq |s-t|\sup_{\theta\in[t-\varepsilon K, t+\varepsilon K]}|b'(\theta)| = M|s-t| \tag{5.3.10}$$

for $s \in [t - \varepsilon K, t + \varepsilon K]$. Therefore

$$|b_\varepsilon(t) - b(t)| \le \varepsilon MK. \tag{5.3.11}$$

For $k \ge 1$ the argumentation is similar using

$$\partial_t^k b_\varepsilon(t) - \partial_t^k b(t) = \int_{[t-\varepsilon K, t+\varepsilon K]} \left(\partial_s^k b(s) - \partial_t^k b(t)\right) \psi_\varepsilon(t-s) \mathrm{d}s \tag{5.3.12}$$

together with the corresponding bound on the derivatives of b on the interval of integration. □

These two technical lemmas are the model behaviours for our symbol classes and the key estimates for the boundary behaviour at regular faces of the zones.

5.3.2 Treatment in the hyperbolic zone

Symbol classes and their properties

For the treatment within the hyperbolic zone, symbol classes and their basic calculus properties are used.

Definition 15 (Symbol classes)**.** *Let $N > 0$ be fixed and Φ as in Section 5.2.1.*

(i) *We say that a function*

$$a \in C^\infty([0,2] \times \mathbb{R}^n \times (0,1]) \tag{5.3.13}$$

belongs to the hyperbolic symbol class $\mathcal{S}_{N,\Phi}\{m_1, m_2\}$ if it satisfies the estimates

$$|\partial_t^k \partial_\xi^\alpha a(t,\xi,\varepsilon)| \le C_{k,\alpha} \left(\Phi_\varepsilon(t-1) + 1\right)^{m_2+k} |\xi|^{m_1 - |\alpha|} \tag{5.3.14}$$

uniformly within $Z_{\mathrm{hyp}}(N)$ for all non-negative integers $k \in \mathbb{N}_0$ and all multi-indices $\alpha \in \mathbb{N}_0^n$ together with the existence of the limits

$$a(t,\xi,0) = \lim_{\varepsilon \to 0} a(t,\xi,\varepsilon), \qquad t \ne 1, \tag{5.3.15}$$

at the regular face of the zone satisfying the estimates

$$|\xi|^{|\alpha| - m_1} |\partial_t^k \partial_\xi^\alpha a(t,\xi,0)| \le C'_{k,\alpha}, \tag{5.3.16}$$

$$|\xi|^{|\alpha| - m_1} |\partial_t^k \partial_\xi^\alpha \left(a(t,\xi,\varepsilon) - a(t,\xi,0)\right)| \le C''_{k,\alpha} \varepsilon \tag{5.3.17}$$

with the latter one uniformly on $|t-1| \ge \varepsilon K$.

(ii) *We say that a matrix-valued function A belongs to $\mathcal{S}_{N,\Phi}\{m_1, m_2\}$ if all its entries belongs to the scalar-valued symbol class $\mathcal{S}_{N,\Phi}\{m_1, m_2\}$.*

Example 5.3.1. *Due to Lemma 5.3.1, we know that the regularising families b_ε and $\mathfrak{d}_\varepsilon$ satisfy*

$$(b_\varepsilon) \in \mathcal{S}_{N,\Phi}\{0,0\} \qquad \text{and} \qquad (\mathfrak{d}_\varepsilon) \in \mathcal{S}_{N,\Phi}\{0,1\} \tag{5.3.18}$$

for any zone constant $N > 0$. Similarly $|\xi|$ is a symbol from $\mathcal{S}_{N,\Phi}\{1,0\}$ for any admissible Φ and $N > 0$.

Remark 5.3.2. *The boundary behaviour of symbols given by* (5.3.16) *corresponds to a characterisation of symbol classes defined on the regular face $\{(t,\xi,0) \mid |\xi| \geq N\}$ of $Z_{\mathrm{hyp}}(N)$ with symbol estimates uniform with respect to t.*

Increasing the zone constant N makes the hyperbolic zone $Z_{\mathrm{hyp}}(N)$ smaller and thus the symbol class $\mathcal{S}_{N,\Phi}\{m_1,m_2\}$ larger. We will make use of this fact later by choosing N sufficiently large in order to guarantee the smallness of some terms. We will omit the indices N and Φ to simplify notation.

Proposition 5.3.3 (Properties of symbol classes)**.** *For any fixed $N > 0$ and admissible Φ the following statements hold:*

(1) *$\mathcal{S}\{m_1,m_2\}$ is a vector space.*

(2) *$\mathcal{S}\{m_1,m_2\} \subset \mathcal{S}\{m_1+\ell_1, m_2-\ell_2\}$ for all $\ell_1 \geq \ell_2 \geq 0$.*

(3) *If $f \in \mathcal{S}\{m_1,m_2\}$ and $g \in \mathcal{S}\{m_1',m_2'\}$ then $f \cdot g \in \mathcal{S}\{m_1+m_1', m_2+m_2'\}$.*

(4) *If $f \in \mathcal{S}\{m_1,m_2\}$ then $\partial_t^k f \in \mathcal{S}\{m_1, m_2+k\}$ and $\partial_\xi^\alpha f \in \mathcal{S}\{m_1-|\alpha|, m_2\}$.*

(5) *If $f \in \mathcal{S}\{m_1,0\}$ satisfies $|f(t,\xi,\varepsilon)| > c|\xi|^{m_1}$ for a positive constant c, then one has $1/f \in \mathcal{S}\{-m_1,0\}$.*

Proof. Properties (1) and (4) follow immediately from the definition of the symbol classes. For (3) we apply the product rule for derivatives to derive the symbol estimate (5.3.16). The boundary behaviour (5.3.17) follows by writing the product of two symbols $f \in \mathcal{S}\{m_1,m_2\}$ and $g \in \mathcal{S}\{m_1',m_2'\}$ as

$$\begin{aligned} f(t,\xi,\varepsilon)g(t,\xi,\varepsilon) - f(t,\xi,0)g(t,\xi,0) &= \big(f(t,\xi,\varepsilon) - f(t,\xi,0)\big)g(t,\xi,\varepsilon) \\ &\quad + f(t,\xi,0)\big(g(t,\xi,\varepsilon) - g(t,\xi,0)\big), \end{aligned} \tag{5.3.19}$$

applying the product rule for derivatives

$$\begin{aligned} &\partial_t^k\partial_\xi^\alpha\big(f(t,\xi,\varepsilon)g(t,\xi,\varepsilon) - f(t,\xi,0)g(t,\xi,0)\big) = \\ &\qquad \sum_{\substack{\ell\leq k\\ \beta\leq\alpha}} \binom{k}{\ell}\binom{\alpha}{\beta}\big(\partial_t^{k-\ell}\partial_\xi^{\alpha-\beta}\big(f(t,\xi,\varepsilon) - f(t,\xi,0)\big)\big)\big(\partial_t^\ell\partial_\alpha^\beta g(t,\xi,\varepsilon)\big) \\ &\qquad + \binom{k}{\ell}\binom{\alpha}{\beta}\big(\partial_t^{k-\ell}\partial_\xi^{\alpha-\beta} f(t,\xi,0)\big)\big(\partial_t^\ell\partial_\alpha^\beta\big(g(t,\xi,\varepsilon) - g(t,\xi,0)\big)\big), \end{aligned} \tag{5.3.20}$$

and estimating each resulted differences on the right by (5.3.17) and all the remaining factors by (5.3.16). To prove (2) we use the definition of the hyperbolic zone $Z_{\mathrm{hyp}}(N)$ in the form

$$\big(\Phi_\varepsilon(t-1)+1\big)^{-\ell_2}|\xi|^{\ell_1} \geq N^{\ell_1}\big(\Phi_\varepsilon(t-1)+1\big)^{\ell_1-\ell_2} \geq N^{\ell_1} \tag{5.3.21}$$

and conclude that symbol estimates from $\mathcal{S}\{m_1, m_2\}$ imply symbol estimates from $\mathcal{S}\{m_1+\ell_1, m_2-\ell_2\}$. It remains to prove (5). Here we use Faà di Bruno's formula (see Proposition 7.2.12) and write

$$\partial_t^k \partial_\xi^\alpha \frac{1}{f(t,\xi,\varepsilon)} = \sum_{\ell=1}^{k+|\alpha|} \sum_{\substack{j_1+\cdots+j_\ell=k \\ |\alpha_1|+\cdots+|\alpha_\ell|=|\alpha|}} C_{k,\alpha,j_i,\alpha_i} \frac{\partial_t^{j_1}\partial_\xi^{\alpha_1} f(t,\xi,\varepsilon)\cdots\partial_t^{j_\ell}\partial_\xi^{\alpha_\ell} f(t,\xi,\varepsilon)}{f^{\ell+1}(t,\xi,\varepsilon)} \tag{5.3.22}$$

where $C_{k,\alpha,j_i,\alpha_i}$ are constants depending on the order of the derivatives. Each term in the last sum can be estimated in the following way. As $f \in S\{m_1, 0\}$ property (4) implies for $i = 1, \ldots, \ell$ that

$$|\partial_t^{j_i}\partial_\xi^{\alpha_i} f| \leq C_{j_i,\alpha_i}\big(\Phi_\varepsilon(t-1)+1\big)^{j_i}|\xi|^{m_1+|\alpha_i|}. \tag{5.3.23}$$

Therefore,

$$|\partial_t^{j_1}\partial_\xi^{\alpha_1} f\cdots\partial_t^{j_\ell}\partial_\xi^{\alpha_\ell} f| \lesssim \big(\Phi_\varepsilon(t-1)+1\big)^{j_1+\cdots+j_\ell}|\xi|^{\ell m_1+|\alpha_1|+\cdots+|\alpha_\ell|} \tag{5.3.24}$$

and using the condition $|f(t,\xi,\varepsilon)| > c|\xi|^{m_1}$ we obtain

$$\begin{aligned} \left|\frac{\partial_t^{j_1}\partial_\xi^{\alpha_1} f\cdots\partial_t^{j_\ell}\partial_\xi^{\alpha_\ell} f}{f^{\ell+1}}\right| &\lesssim \frac{\big(\Phi_\varepsilon(t-1)+1\big)^{j_1+\cdots+j_\ell}|\xi|^{\ell m_1+|\alpha_1|+\cdots+|\alpha_\ell|}}{|\xi|^{m_1(\ell+1)}} \\ &\lesssim \frac{\big(\Phi_\varepsilon(t-1)+1\big)^{k}|\xi|^{\ell m_1+|\alpha|}}{|\xi|^{m_1(\ell+1)}} \\ &\lesssim \big(\Phi_\varepsilon(t-1)+1\big)^{k}|\xi|^{-m_1+|\alpha|}. \end{aligned} \tag{5.3.25}$$

Summing all these terms yields the desired estimate. The boundary estimate follows similarly. $\square$

These symbol classes and in particular the embeddings

$$\mathcal{S}\{-1,2\} \hookrightarrow \mathcal{S}\{0,1\} \hookrightarrow \mathcal{S}\{1,0\} \tag{5.3.26}$$

will be of importance for the treatment within the hyperbolic zone. The gain of decay in $|\xi|$ will be paid for by a loss of point-wise control in the t-variable near the singularity. What we gain, are integrability properties and improved limits at the regular face.

Proposition 5.3.4. *Within the hyperbolic zone $Z_{\text{hyp}}(N)$,*

(1) *symbols from $\mathcal{S}\{0,0\}$ are uniformly bounded;*

(2) *symbols from $\mathcal{S}\{0,1\}$ are uniformly integrable with respect to t;*

(3) *symbols $a \in \mathcal{S}\{-1,2\}$ satisfy*

$$\int_0^t |a(\theta,\xi,\varepsilon)|\mathrm{d}\theta \le C|\xi|^{-1}\big(\Phi_\varepsilon(t-1)+1\big) \tag{5.3.27}$$

for all $0 < t \le t_{\xi_1}$, and

$$\int_t^2 |a(\theta,\xi,\varepsilon)|\mathrm{d}\theta \le C|\xi|^{-1}\big(\Phi_\varepsilon(t-1)+1\big) \tag{5.3.28}$$

for all $t_{\xi_2} \le t \le 2$.

Proof. Statement (1) is obvious from the definition of the symbol class. Next we prove (2). If $f \in \mathcal{S}\{0,1\}$ then it satisfies the point-wise estimate

$$|f(t,\xi,\varepsilon)| \le C\big(\Phi_\varepsilon(t-1)+1\big), \tag{5.3.29}$$

and therefore after integrating over $t \in [0, t_{\xi_1}(\varepsilon)]$ (or similarly over $t \in [t_{\xi_2}(\varepsilon), 2]$) one has

$$\begin{aligned}\int_0^{t_{\xi_1}} |f(t,\xi,\varepsilon)\mathrm{d}s|\mathrm{d}t &\le C\int_0^{t_{\xi_1}} \Phi_\varepsilon(t-1)\mathrm{d}t + C\int_0^{t_{\xi_1}} \mathrm{d}t \\ &= C\int_{-\varepsilon^{-1}}^{\varepsilon^{-1}(t_{\xi_1}-1)} \Phi(\tau)\mathrm{d}\tau + C \\ &\le C\left[1+\int_{-\infty}^{0}\Phi(\tau)\mathrm{d}\right]\end{aligned} \tag{5.3.30}$$

for any fixed $\varepsilon \in (0,1]$ and $\xi \in \mathbb{R}^n$. It remains to prove (3). If $a \in S\{-1,2\}$ then it satisfies the point-wise estimate

$$|a(t,\xi,\varepsilon)| \le C|\xi|^{-1}\big(\Phi_\varepsilon(t-1)+1\big)^2 \tag{5.3.31}$$

and the only new term needing to be treated is the one arising from the square of the shape function. This can be estimated by means of (5.2.4) for $t<1$ as

$$\begin{aligned}\int_0^t \Phi_\varepsilon^2(\theta)\mathrm{d}\theta = \varepsilon^{-1}\int_{-\varepsilon^{-1}}^{\varepsilon^{-1}(t-1)} \Phi^2(\tau)\mathrm{d}\tau \le C\varepsilon^{-1}\int_{-\varepsilon^{-1}}^{\varepsilon^{-1}(t-1)} \Phi'(\tau)\mathrm{d}\tau \\ \le C\varepsilon^{-1}\Phi(\varepsilon^{-1}(t-1)) - C\varepsilon^{-1}\Phi(-\varepsilon^{-1}) \le C\Phi_\varepsilon(t-1)\end{aligned} \tag{5.3.32}$$

and similarly for the case $t > 1$. $\square$

Transformations

Within the hyperbolic zone, we apply transformations to our system in order to extract precise information about the behaviour of its fundamental solution. Recall that (5.2.11) is of the form $\mathrm{D}_t U = (A+B)U$ with $A \in \mathcal{S}\{1,0\}$ and $B \in \mathcal{S}\{0,1\}$. Using the diagonaliser of the principal part A

$$M = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{with inverse} \quad M^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \tag{5.3.33}$$

the matrix A can be written as

$$A(\xi) = MD(\xi)M^{-1} \tag{5.3.34}$$

with $D(\xi) = \operatorname{diag}(|\xi|, -|\xi|)$. Hence, setting $V(t,\xi,\varepsilon) = M^{-1}U(t,\xi,\varepsilon)$ system (5.2.11) can be rewritten as

$$\mathrm{D}_t V(t,\xi,\varepsilon) = \big(D(\xi) + R(t,\varepsilon)\big)V(t,\xi,\varepsilon) \tag{5.3.35}$$

with a remainder given by

$$R(t,\varepsilon) = M^{-1}B(t,\varepsilon)M = \frac{\mathrm{i}}{2}\mathfrak{d}_\varepsilon(t)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathcal{S}\{0,1\}. \tag{5.3.36}$$

Our aim is to further improve the remainder within the hyperbolic hierarchy (5.3.26). This allows to extract more detailed information on the propagation of singularities close to the singularity later. For this we follow [56] to construct transformation matrices $N_k(t,\xi,\varepsilon)$, transforming the system (5.3.35) into a new system with an updated diagonal part and an improved remainder. The construction is done in such a way, that the operator identity

$$\big(\mathrm{D}_t - D(\xi) - R(t,\varepsilon)\big)N_k(t,\xi,\varepsilon) = N_k(t,\xi,\varepsilon)\left(D_t - D_k(t,\xi,\varepsilon) - R_k(t,\xi,\varepsilon)\right) \tag{5.3.37}$$

holds for $k \geq 1$ and

(1) the matrix-valued symbols $D_k(t,\xi,\varepsilon)$ are given by

$$D_k(t,\xi,\varepsilon) = D(\xi) + F^{(0)}(t,\xi,\varepsilon) + \cdots + F^{(k-1)}(t,\xi,\varepsilon) \tag{5.3.38}$$

with diagonal $F^{(j)}(t,\xi,\varepsilon) \in S\{-j, j+1\}$;

(2) the transformation matrices $N_k(t,\xi,\varepsilon)$ are of the form

$$N_k(t,\xi,\varepsilon) = \mathrm{I} + N^{(1)}(t,\xi,\varepsilon) + \cdots + N^{(k)}(t,\xi,\varepsilon) \tag{5.3.39}$$

with $N^{(j)}(t,\xi,\varepsilon) \in S\{-j, j\}$;

(3) the remainder satisfies $R_k(t,\xi,\varepsilon) \in S\{-k, k+1\}$.

We give the construction for $k = 1$ in full detail. In this case (5.3.37) simplifies modulo $S\{-1, 2\}$ to the commutator equation

$$\big[D(\xi), N^{(1)}(t, \xi, \varepsilon)\big] = F^{(0)}(t, \xi, \varepsilon) - R(t, \varepsilon). \tag{5.3.40}$$

As the diagonal part of the commutator vanishes, we set

$$F^{(0)}(t, \varepsilon) = \operatorname{diag} R(t, \varepsilon) = \frac{\mathrm{i}}{2}\mathfrak{d}_\varepsilon(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in S\{0, 1\}, \tag{5.3.41}$$

and determine the off-diagonal entries of

$$N^{(1)}(t, \xi, \varepsilon) = \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} \tag{5.3.42}$$

from (5.3.40) as $n_{12} = -\frac{\mathrm{i}}{4|\xi|}\mathfrak{d}_\varepsilon(t)$ and $n_{21} = \frac{\mathrm{i}}{4|\xi|}\mathfrak{d}_\varepsilon(t)$. The diagonal entries are chosen to be zero, and hence

$$N^{(1)}(t, \xi, \varepsilon) = \frac{\mathrm{i}}{4|\xi|}\mathfrak{d}_\varepsilon(t) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in S\{-1, 1\}. \tag{5.3.43}$$

The transformation matrix $N_1(t, \xi, \varepsilon) = \mathrm{I} + N^{(1)}(t, \xi, \varepsilon)$ is invertible, provided that the zone constant is chosen large enough.

Proposition 5.3.5. *Assume Hypotheses* **(H1)** *and* **(H2)** *hold. Then, there exists a matrix $N^{(1)}(t, \xi, \varepsilon) \in S\{-1, 1\}$ and a diagonal matrix $F^{(0)}(t, \varepsilon) \in S\{0, 1\}$, such that the identity (5.3.37) is satisfied with a remainder $R_1(t, \xi, \varepsilon) \in S\{-1, 2\}$. Moreover, we can find a zone constant N, such that the transformation matrix $N_1(t, \xi, \varepsilon) = I + N^{(1)}(t, \xi, \varepsilon)$ is invertible in $Z_{\mathrm{hyp}}(N)$ and $N_1^{-1}(t, \xi, \varepsilon) \in S\{0, 0\}$.*

Proof. It remains to show the invertibility of $N_1(t, \xi, \varepsilon)$. Indeed, by (5.3.43) it follows that

$$\det N_1 = 1 - \frac{\mathfrak{d}_\varepsilon^2(t)}{16|\xi|^2} \tag{5.3.44}$$

and by Lemma 5.3.1 one has

$$\frac{\mathfrak{d}_\varepsilon^2(t)}{16|\xi|^2} \le \frac{\big(c_1\Phi_\varepsilon(t-1) + c_2\big)^2}{16|\xi|^2}. \tag{5.3.45}$$

Hence, by choosing the zone constant N large enough such that

$$c_1\psi_\varepsilon(t-1) + c_2 \le N\psi_\varepsilon(t-1) + N, \tag{5.3.46}$$

the invertibility follows. By the calculus rules of Proposition 5.3.3, we also conclude $N_1^{-1} \in S\{0, 0\}$.

The matrices $N^{(1)}(t, \xi, \varepsilon) \in S\{-1, 1\}$ and $F^{(0)}(t, \varepsilon) \in S\{0, 1\}$ are already constructed in such a way that (5.3.37) holds with the remainder

$$\begin{aligned} R_1(t, \xi, \varepsilon) = N_1^{-1}(t, \xi, \varepsilon)\big(&R(t, \varepsilon)N^{(1)}(t, \xi, \varepsilon) - \mathrm{D}_t N^{(1)}(t, \xi, \varepsilon) \\ &- N^{(1)}(t, \xi, \varepsilon)F^{(0)}(t, \varepsilon)\big) \in S\{-1, 2\} \end{aligned} \tag{5.3.47}$$

and the statement is proved. □

Remark 5.3.3. *Taking the limits $\varepsilon \to 0$ at the regular faces of $Z_{\mathrm{hyp}}(N)$ the diagonalisation procedure yields in particular the transformations needed to construct representations of solutions in the case of smooth coefficients. In particular the limit*

$$N^{(1)}(t,\xi,0) = \lim_{\varepsilon\to 0} N^{(1)}(t,\xi,\varepsilon) \tag{5.3.48}$$

exists for $t \neq 1$ and $|\xi| > N$ and satisfies

$$\|N_1(t,\xi,\varepsilon) - N_1(t,\xi,0)\| = \|N^{(1)}(t,\xi,\varepsilon) - N^{(1)}(t,\xi,0)\| \le C\varepsilon|\xi|^{-1}, \tag{5.3.49}$$

and as the inverse $N_1^{-1}(t,\xi,\varepsilon)$ can be written as a Neumann series, we know that $N_1^{-1}(t,\xi,\varepsilon) - \mathrm{I} \in \mathcal{S}\{-1,1\}$ and consequently

$$\|N_1^{-1}(t,\xi,\varepsilon) - N_1(t,\xi,0)^{-1}\| \le C\varepsilon|\xi|^{-1}. \tag{5.3.50}$$

Similarly the limit $R_1(t,\xi,0) = \lim_{\varepsilon\to 0} R_1(t,\xi,\varepsilon)$ satisfies

$$\|R_1(t,\xi,\varepsilon) - R_1(t,\xi,0)\| \le C\varepsilon|\xi|^{-1}. \tag{5.3.51}$$

This enables us to relate the construction of fundamental solutions for the regularised family to the fundamental solution of the original problem outside the singularity.

Fundamental solution to the diagonalised system

We now fix the zone constant N large enough to guarantee that $N_1(t,\xi,\varepsilon)$ be uniformly invertible within the hyperbolic zone $Z_{\mathrm{hyp}}(N)$. Then for V solving (5.3.35), the transformed function

$$V_1(t,\xi,\varepsilon) = N_1^{-1}(t,\xi,\varepsilon)V(t,\xi,\varepsilon) \tag{5.3.52}$$

satisfies due to (5.3.37)

$$\mathrm{D}_t V_1(t,\xi) = \left(D(\xi) + F^{(0)}(t,\xi,\varepsilon) + R_1(t,\xi,\varepsilon)\right) V_1(t,\xi) \tag{5.3.53}$$

with the diagonal matrix $F^{(0)} \in S\{0,1\}$ given by (5.3.41) and the remainder $R_1(t,\xi,\varepsilon) \in S\{-1,2\}$ specified by (5.3.47). We construct its fundamental solution.

Theorem 5.3.6. *Assume the Hypotheses* **(H1)** *and* **(H2)** *hold. Then the fundamental solution $\mathcal{E}_1(t,s,\xi,\varepsilon)$ to the transformed system (5.3.53) can be represented by*

$$\mathcal{E}_1(t,s,\xi,\varepsilon) = \sqrt{\frac{b_\varepsilon(s)}{b_\varepsilon(t)}}\mathcal{E}_0(t,s,\xi)\mathcal{Q}(t,s,\xi,\varepsilon) \tag{5.3.54}$$

for $[s,t]\times\{(\xi,\varepsilon)\} \subset Z_{\mathrm{hyp}}(N)$, where

(1) *the factor $\sqrt{\frac{b_\varepsilon(s)}{b_\varepsilon(t)}}$ describes the main influence of the dissipation term;*

(2) *the matrix $\mathcal{E}_0(t,s,\xi)$ is the fundamental solution of the hyperbolic principal part $\mathrm{D}_t - D(\xi)$ given by*

$$\mathcal{E}_0(t,s,\xi) = \begin{pmatrix} \exp(\mathrm{i}(t-s)|\xi|) & 0 \\ 0 & \exp(-\mathrm{i}(t-s)|\xi|) \end{pmatrix}; \tag{5.3.55}$$

(3) *the matrix $\mathcal{Q}(t,s,\xi,\varepsilon)$ is uniformly bounded*

$$\|\mathcal{Q}(t,s,\xi,\varepsilon)\| \le \exp\bigg(\int_s^t \|R_1(\tau,\xi,\varepsilon)\|\mathrm{d}\tau\bigg), \tag{5.3.56}$$

uniformly invertible within the hyperbolic zone due to

$$|\det \mathcal{Q}(t,s,\xi,\varepsilon)| \ge \exp\bigg(\int_s^t \|R_1(\tau,\xi,\varepsilon)\|\mathrm{d}\tau\bigg), \tag{5.3.57}$$

and has the precise behaviour for large $|\xi|$ determined by the identity matrix

$$\|\mathcal{Q}(t,s,\xi,\varepsilon) - \mathrm{I}\| \le \int_s^t \|R_1(t_1,\xi,\varepsilon)\| \exp\bigg(\int_s^{t_1} \|R_1(\tau,\xi,\varepsilon)\|\mathrm{d}\tau\bigg)\mathrm{d}t_1. \tag{5.3.58}$$

Proof. We consider first $\mathrm{D}_t - D(\xi) - F^0$, i.e. the main diagonal part of the transformed system (5.3.53). Its fundamental solution is given by

$$\mathcal{E}_0(t,s,\xi)\exp\bigg(-\frac{1}{2}\int_s^t \mathfrak{d}_\varepsilon(\tau)\mathrm{d}\tau\bigg) = \sqrt{\frac{b_\varepsilon(s)}{b_\varepsilon(t)}}\mathcal{E}_0(t,s,\xi,\varepsilon), \tag{5.3.59}$$

where $\mathcal{E}_0(t,s,\xi)$ is the fundamental solution to $\mathrm{D}_t - D(\xi)$ given by (5.3.55). For the fundamental solution to the system (5.3.53) we use an ansatz in the form

$$\mathcal{E}_1(t,s,\xi) = \sqrt{\frac{b_\varepsilon(s)}{b_\varepsilon(t)}}\mathcal{E}_0(t,s,\xi,\varepsilon)\mathcal{Q}(t,s,\xi,\varepsilon) \tag{5.3.60}$$

for a still to be determined matrix $\mathcal{Q}(t,s,\xi,\varepsilon)$. A simple calculation shows that $\mathcal{Q}(t,s,\xi,\varepsilon)$ must solve

$$\mathrm{D}_t\mathcal{Q}(t,s,\xi,\varepsilon) = \mathcal{R}(t,s,\xi,\varepsilon)\mathcal{Q}(t,s,\xi,\varepsilon), \qquad \mathcal{Q}(s,s,\xi,\varepsilon) = \mathrm{I}, \tag{5.3.61}$$

with coefficient matrix

$$\mathcal{R}(t,s,\xi,\varepsilon) = \mathcal{E}_0(s,t,\xi)R_1(t,\xi,\varepsilon)\mathcal{E}_0(t,s,\xi) \tag{5.3.62}$$

determined by the remainder R_1 and the fundamental solution of the hyperbolic principal part $\mathcal{E}_0$. The solution $\mathcal{Q}$ can thus be represented in terms of the Peano–Baker series (see Theorem 7.2.9)

$$\mathcal{Q}(t,s,\xi,\varepsilon) = \mathrm{I} + \sum_{k=1}^{\infty} \mathrm{i}^k \int_s^t \mathcal{R}(t_1,s,\xi,\varepsilon)\cdots\int_s^{t_{k-1}} \mathcal{R}(t_k,s,\xi,\varepsilon)\mathrm{d}t_k\ldots\mathrm{d}t_1, \tag{5.3.63}$$

and it remains to provide estimates based on this series representation. As $\mathcal{E}_0$ is unitary, we obtain from the symbol estimate of the remainder R_1

$$\begin{aligned}\|\mathcal{R}(t,s,\xi,\varepsilon)\| = \|R_1(t,\xi,\varepsilon)\| &\leq C|\xi|^{-1}\left(\Phi_\varepsilon(t-1)+1\right)^2 \\ &\leq \frac{C}{N}\left(\Phi_\varepsilon(t-1)+1\right),\end{aligned} \tag{5.3.64}$$

and thus it follows that

$$\|\mathcal{Q}(t,s,\xi,\varepsilon)\| \leq \exp\left(\int_s^t \|R_1(\tau,\xi,\varepsilon)\|\mathrm{d}\tau\right) \leq \exp(C/N). \tag{5.3.65}$$

Together with

$$\det \mathcal{Q}(t,s,\xi,\varepsilon) = \exp\left(\int_s^t \operatorname{trace} R_1(\tau,\xi,\varepsilon)\mathrm{d}\tau\right) \tag{5.3.66}$$

the uniform invertibility of $\mathcal{Q}$ follows. Furthermore, by using (5.2.4) we obtain

$$\|\mathcal{Q}(t,s,\xi,\varepsilon) - \mathrm{I}\| \leq C|\xi|^{-1}\big(\Phi_\varepsilon(t-1)+1\big)\exp(C/N), \tag{5.3.67}$$

and the main contribution of $\mathcal{Q}$ for large $|\xi|$ is given by the identity matrix. $\square$

Fundamental solution to the original system

After obtaining the fundamental solution to the transformed system (5.3.53), we go back to the original problem (5.2.11) and obtain in the hyperbolic zone the representation

$$\mathcal{E}_{\mathrm{hyp}}(t,s,\xi,\varepsilon) = \sqrt{\frac{b_\varepsilon(s)}{b_\varepsilon(t)}} M N_1(t,\xi,\varepsilon)\mathcal{E}_0(t,s,\xi)\mathcal{Q}(t,s,\xi,\varepsilon)N_1^{-1}(s,\xi,\varepsilon)M^{-1} \tag{5.3.68}$$

for the fundamental solution. We will briefly discuss its limiting behaviour as $\varepsilon \to 0$ for fixed $s < t < 1$ or $1 < s < t$. As $\mathcal{E}_0(t,s,\xi)$ is independent of ε and the transformation matrix $N_1(t,\xi,\varepsilon)$ is already estimated by (5.3.49), this boils down to considering $\mathcal{Q}(t,s,\xi,\varepsilon)$.

Lemma 5.3.7. *The limit*

$$\mathcal{Q}(t,s,\xi,0) = \lim_{\varepsilon\to 0}\mathcal{Q}(t,s,\xi,\varepsilon) \tag{5.3.69}$$

exists for fixed $s < t < 1$ or $1 < s < t$, is uniformly bounded and invertible, and satisfies the estimate

$$\|\mathcal{Q}(t,s,\xi,\varepsilon) - \mathcal{Q}(t,s,\xi,0)\| \lesssim \varepsilon|\xi|^{-1} \tag{5.3.70}$$

holds for all $[s,t]\times\{(\xi,\varepsilon)\}\subset Z_{hyp}$ with the condition that

$$\min\{|t-1|,|s-1|\} \geq \varepsilon K$$

.

Proof. We use (5.3.51) in combination with (5.3.63) and consider

$$\mathcal{Q}(t,s,\xi,0) = \mathrm{I} + \sum_{k=1}^{\infty} \mathrm{i}^k \int_s^t \mathcal{R}(t_1,s,\xi,0)\cdots \int_s^{t_{k-1}} \mathcal{R}(t_k,s,\xi,0)\mathrm{d}t_k \ldots \mathrm{d}t_1 \tag{5.3.71}$$

defined in terms of

$$\mathcal{R}(t,s,\xi,0) = \mathcal{E}_0(s,t,\xi)R_1(t,\xi,0)\mathcal{E}_0(t,s,\xi) \tag{5.3.72}$$

with

$$\|\mathcal{R}(t,s,\xi,0)\| = \|R_1(t,\xi,0)\| \le C|\xi|^{-1} \tag{5.3.73}$$

uniformly in $0 < s < t < 1$ or $1 < s < t < 2$ and $|\xi| > N$. It thus follows that $\mathcal{Q}(t,s,\xi,0)$ is uniformly bounded and uniformly invertible. To estimate the difference between $\mathcal{Q}(t,s,\xi,\varepsilon)$ and $\mathcal{Q}(t,s,\xi,0)$, we use a perturbation argument based on the estimate

$$\|\mathcal{R}(t,s,\xi,\varepsilon) - \mathcal{R}(t,s,\xi,0)\| = \|R_1(t,\xi,\varepsilon) - R_1(t,\xi,0)\| \le C\varepsilon|\xi|^{-1} \tag{5.3.74}$$

for $|t-1| \ge \varepsilon K$ following from (5.3.51). Differentiating

$$\mathcal{Q}(t,s,\xi,\varepsilon) = \mathcal{Q}(t,s,\xi,0)\Xi(t,s,\xi,\varepsilon) \tag{5.3.75}$$

yields for $\Xi(t,s,\xi,\varepsilon)$ the equation

$$\mathrm{D}_t\Xi(t,s,\xi,\varepsilon) = \mathcal{Q}(t,s,\xi,0)\big(\mathcal{R}(t,s,\xi,\varepsilon) - \mathcal{R}(t,s,\xi,0)\big)\mathcal{Q}(s,t,\xi,0)\Xi(t,s,\xi,\varepsilon) \tag{5.3.76}$$

with initial condition $\Xi(s,s,\xi,\varepsilon) = \mathrm{I}$ and coefficient matrix estimated by

$$\|\mathcal{Q}(t,s,\xi,0)\big(\mathcal{R}(t,s,\xi,\varepsilon) - \mathcal{R}(t,s,\xi,0)\big)\mathcal{Q}(s,t,\xi,0)\| \le C\varepsilon|\xi|^{-1}. \tag{5.3.77}$$

Therefore, using the representation of $\Xi(t,s,\xi,\varepsilon)$ in terms of the Peano–Baker series (see Corollary 7.2.11) we obtain the estimate

$$\|\Xi(t,s,\xi,\varepsilon)\| \le \exp\big(C\varepsilon|\xi|^{-1}|t-s|\big) = 1 + \mathcal{O}(\varepsilon|\xi|^{-1}) \tag{5.3.78}$$

uniform with respect to t and s for $|t-1|, |s-1| \ge \varepsilon K$ and thus the desired statement follows. □

Proposition 5.3.8. *The estimate*

$$\|\mathcal{E}_{\mathrm{hyp}}(t,s,\xi,\varepsilon) - \mathcal{E}_{\mathrm{hyp}}(t,s,\xi,0)\| \lesssim \varepsilon \tag{5.3.79}$$

holds for all $[s,t] \times \{(\xi,\varepsilon)\} \subset Z_{hyp}$ *with the condition that*

$$\min\{|t-1|, |s-1|\} \ge \varepsilon K$$

.

Proof. The proof follows directly from the representation (5.3.68) of $\mathcal{E}_{\mathrm{hyp}}(t,s,\xi,\varepsilon)$ combined with an analogous formula for the limit

$$\mathcal{E}_{\mathrm{hyp}}(t,s,\xi,0)=\lim_{\varepsilon\to 0}\mathcal{E}_{\mathrm{hyp}}(t,s,\xi,\varepsilon).$$

As all terms in (5.3.68) are uniformly bounded within the hyperbolic zone we obtain

$$\begin{aligned}\|\mathcal{E}_{\mathrm{hyp}}(t,s,\xi,\varepsilon)-\mathcal{E}_{\mathrm{hyp}}(t,s,\xi,0)\| \lesssim &\left|\sqrt{\frac{b_\varepsilon(s)}{b_\varepsilon(t)}}-\sqrt{\frac{b_\varepsilon(0)}{b_\varepsilon(0)}}\right| \\ &+\|N_1(t,\xi,\varepsilon)-N_1(t,\xi,0)\| \\ &+\|\mathcal{Q}(t,s,\xi,\varepsilon)-\mathcal{Q}(t,s,\xi,0)\| \\ &+\|N_1^{-1}(s,\xi,\varepsilon)-N_1^{-1}(s,\xi,0)\|\end{aligned}\tag{5.3.80}$$

Each of the last three differences appearing on the right hand side can be controlled by $\varepsilon|\xi|^{-1}$ by estimate (5.3.49) for $N_1(t,\xi,\varepsilon)$ and (5.3.50) for $N_1^{-1}(s,\xi,\varepsilon)$, and by estimate (5.3.70) for $\mathcal{Q}(t,s,\xi,\varepsilon)$. Furthermore, by Proposition 5.3.2 for $b_\varepsilon(s)$ and $b_\varepsilon(t)$, we know that the first difference is controlled by ε. The desired estimate for the fundamental solution follows. ☐

5.3.3 Treatment in the singular zone

Now we consider equation (5.2.9) within the singular zone. In order to describe its fundamental solution we use the substitution $\tau=\varepsilon^{-1}(t-1)$ and replace the parameter $|\xi|$ by $\Lambda=\varepsilon|\xi|$. Then the equation (5.2.9) can be rewritten as

$$\hat{u}_{\tau\tau}+\Lambda^2\hat{u}+\beta_\varepsilon(\tau)\hat{u}_\tau=0,\tag{5.3.81}$$

where

$$\beta_\varepsilon(\tau)=\varepsilon\mathfrak{d}_\varepsilon(1+\varepsilon\tau)=\frac{\int_{-\infty}^{\infty}b(1+\varepsilon(\tau-\theta))\psi'(\theta)\mathrm{d}\theta}{\int_{-\infty}^{\infty}b(1+\varepsilon(\tau-\theta))\psi(\theta)\mathrm{d}\theta}.\tag{5.3.82}$$

We recall here that in the new coordinates the singular zone is rewritten as

$$Z_{\mathrm{sing}}(N)=\{(\tau,\Lambda,\varepsilon)\mid \Lambda\le N\Phi(\tau)+N\varepsilon\}\tag{5.3.83}$$

so that the interval in which to solve our equation is given by $[\tau_{\Lambda_1}(\varepsilon),\tau_{\Lambda_2}(\varepsilon)]$ with implicitly defined endpoints through

$$\Lambda=N\Phi(\tau_\Lambda)+\varepsilon.\tag{5.3.84}$$

System form

Reformulating our equation as a system in

$$U(\tau,\Lambda,\varepsilon)=\begin{pmatrix}\Lambda\hat{u}\\ \partial_\tau\hat{u}\end{pmatrix}\tag{5.3.85}$$

yields

$$\partial_\tau U(\tau,\Lambda,\varepsilon) = \left[\begin{pmatrix} 0 & 0 \\ 0 & -\beta_\varepsilon(\tau) \end{pmatrix} + \begin{pmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{pmatrix} \right] U(\tau,\Lambda,\varepsilon), \tag{5.3.86}$$

where now the first matrix $A(\tau,\varepsilon) = \mathrm{diag}(0, -\beta_\varepsilon(\tau))$ is treated as the dominant part and the second matrix

$$\begin{pmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{pmatrix} = \Lambda \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \Lambda \mathrm{J} \tag{5.3.87}$$

plays the role of the remainder.

Proposition 5.3.9. *Under the assumptions* **(H1)** *and* **(H2)** *for the function* b

$$\beta_\varepsilon(\tau) = \beta_0(\tau) + \mathcal{O}(\varepsilon) \tag{5.3.88}$$

holds uniformly with respect to τ*, where* $\beta_0(\tau)$ *is given by*

$$\beta_0(\tau) = \frac{h\psi(\tau)}{h\int_{-K}^{\tau}\psi(\theta)\mathrm{d}\theta + b(1_{-0})} = \frac{h\psi(\tau)}{b(1_{+0}) - h\int_{\tau}^{K}\psi(\theta)\mathrm{d}\theta} \tag{5.3.89}$$

in terms of $h = b(1_{+0}) - b(1_{-0})$ *the jump of* b *at* $t = 1$.

Remark 5.3.4. *In* (5.3.89) *the numerator is the derivative of the denominator with respect to* τ*. We also see that* $\beta_0(\tau)$ *is compactly supported with* $\mathrm{supp}\,\beta_0 = \mathrm{supp}\,\psi = [-K, K]$.

Proof. The statement follows by considering both the numerator and the denominator of the representation (5.3.82) separately. First,

$$\begin{aligned} \int_{-\infty}^{\infty} b(1+\varepsilon(\tau-\theta))\psi'(\theta)\mathrm{d}\theta &= \int_{-\infty}^{\tau} b(1+\varepsilon(\tau-\theta))\psi'(\theta)\mathrm{d}\theta \\ &\quad + \int_{\tau}^{+\infty} b(1+\varepsilon(\tau-\theta))\psi'(\theta)\mathrm{d}\theta \\ &= h\psi(\tau) + \varepsilon\int_{-\infty}^{\tau} b'(1+\varepsilon(\tau-\theta))\psi(\theta)\mathrm{d}\theta \\ &\quad + \varepsilon\int_{\tau}^{+\infty} b'(1+\varepsilon(\tau-\theta))\psi(\theta)\mathrm{d}\theta \\ &= h\psi(\tau) + \mathcal{O}(\varepsilon) \end{aligned} \tag{5.3.90}$$

using integration by parts and the fact that b' is bounded on both $[0, 1]$ and $[1, 2]$.

Similarly, we obtain for the denominator

$$
\begin{aligned}
\Big| \int_{-\infty}^{\infty} & b(1+\varepsilon(\tau-\theta))\psi(\theta)\mathrm{d}\theta - \int_{-\infty}^{\tau} b(1_{+0})\psi(\theta)\mathrm{d}\theta - \int_{\tau}^{+\infty} b(1_{-0})\psi(\theta)\mathrm{d}\theta \Big| \\
&\leq \int_{-\infty}^{\tau} \big| b(1+\varepsilon(\tau-\theta)) - b(1_{+0}) \big| \psi(\theta)\mathrm{d}\theta \\
&\qquad + \int_{\tau}^{\infty} \big| b(1+\varepsilon(\tau-\theta)) - b(1_{-0})) \big| \psi(\theta)\mathrm{d}\theta \\
&\leq \int_{-K}^{\tau} C_1\varepsilon|\tau-\theta|\psi(\theta)\mathrm{d}\theta + \int_{\tau}^{K} C_2\varepsilon|\tau-\theta|\psi(\theta)\mathrm{d}\theta,
\end{aligned}
\tag{5.3.91}
$$

where for the last line we applied the mean value theorem to the function b fir $C_1 = \sup_{s\in[1,2]} |b'(s)|$ and $C_2 = \sup_{s\in[0,1]} |b'(s)|$. Hence

$$
\begin{aligned}
\int_{-\infty}^{\infty} b(1+\varepsilon(\tau-\theta))\psi(\theta)\mathrm{d}\theta = b(1_{+0}) & \int_{-\infty}^{\tau} \psi(\theta)\mathrm{d}\theta \\
& + b(1_{-0}) \int_{\tau}^{+\infty} \psi(\theta)\mathrm{d}\theta + \mathcal{O}(\varepsilon),
\end{aligned}
\tag{5.3.92}
$$

and therefore by combining (5.3.90) and (5.3.92) the desired statement follows. □

Construction of the fundamental solution in the singular zone

In the following we want to derive properties of the fundamental solution to (5.3.86). The strategy is again to use a perturbation argument to incorporate the remainder terms. Note, that in singular variables both τ and Λ stay bounded within $Z_{\mathrm{sing}}(N)$ and our main interest is in the characterisation of the solution when $\Lambda \to 0$ and $\varepsilon \to 0$.

Theorem 5.3.10. *The fundamental solution to the system (5.3.86) can be represented by*

$$
\mathcal{E}_{\mathrm{sing}}(\tau,\theta,\Lambda,\varepsilon) = \mathcal{F}(\tau,\theta,\varepsilon)\mathcal{G}(\tau,\theta,\Lambda,\varepsilon) \tag{5.3.93}
$$

for $[\theta,\tau]\times\{(\Lambda,\varepsilon)\} \subset Z_{\mathrm{sing}}(N)$ with $\theta<\tau$, where

(1) *$\mathcal{F}(\tau,\theta,\varepsilon)$ is the fundamental solution to the main part $\partial_\tau - \mathrm{diag}(0,-\beta_\varepsilon(\tau))$ given by*

$$
\mathcal{F}(\tau,\theta,\varepsilon) = \begin{pmatrix} 1 & 0 \\ 0 & \exp\left(-\int_\theta^\tau \beta_\varepsilon(\vartheta)\mathrm{d}\vartheta\right) \end{pmatrix} \tag{5.3.94}
$$

with

$$
\exp\left(\int_\theta^\tau \beta_\varepsilon(\theta)\mathrm{d}\theta\right) = \frac{h\Theta(\tau)+b(1_{-0})}{h\Theta(\theta)+b(1_{-0})}\left(1+\mathcal{O}(\varepsilon)\right) \tag{5.3.95}
$$

in terms of the smoothed Heaviside function

$$\Theta(\tau) = \int_{-\infty}^{\tau} \psi(\vartheta)\mathrm{d}\vartheta = 1 - \int_{\tau}^{\infty} \psi(\vartheta)\mathrm{d}\vartheta \tag{5.3.96}$$

and the height of the jump $h = b(1_{+0}) - b(1_{-0})$*; and*

(2) *the matrix* $\mathcal{G}(\tau, \theta, \Lambda, \varepsilon)$ *is given as a power series*

$$\mathcal{G}(\tau, \theta, \Lambda, \varepsilon) = \mathrm{I} + \sum_{k=1}^{\infty} \Lambda^k \mathcal{G}_k(\tau, \theta, \varepsilon) \tag{5.3.97}$$

with coefficients $\mathcal{G}_k$ *satisfying*

$$\|\mathcal{G}_k(\tau, \theta, \varepsilon)\| \le \frac{C^k |\tau - \theta|^k}{k!} \tag{5.3.98}$$

uniformly in k *and the occurring variables.*

Proof. The representation for $\mathcal{F}$ follows by integrating the main diagonal part in equation (5.3.86). Using the explicit form of $\beta_0(\tau)$ from (5.3.89) in combination with $\beta_\varepsilon(\tau) = \beta_0(\tau) + \mathcal{O}(\varepsilon)$, we obtain (5.3.95).

We make the ansatz

$$\mathcal{E}_{\mathrm{sing}}(\tau, \theta, \Lambda, \varepsilon) = \mathcal{F}(\tau, \theta, \varepsilon)\mathcal{G}(\tau, \theta, \Lambda, \varepsilon) \tag{5.3.99}$$

for the fundamental solution to system (5.3.86). Then by construction

$$\partial_\tau \mathcal{G}(\tau, \theta, \Lambda, \varepsilon) = \Lambda \widetilde{\mathcal{F}}(\tau, \theta, \varepsilon)\mathcal{G}(\tau, \theta, \Lambda, \varepsilon) \qquad \text{with} \qquad \mathcal{G}(\theta, \theta, \Lambda, \varepsilon) = \mathrm{I}, \tag{5.3.100}$$

where the coefficient matrix satisfies

$$\begin{aligned} \widetilde{\mathcal{F}}(\tau, \theta, \varepsilon) &= \mathcal{F}(\tau, \theta, \varepsilon)\mathrm{J}\mathcal{F}(\theta, \tau, \varepsilon) \\ &= \begin{pmatrix} 0 & \exp\left(-\int_\theta^\tau \beta_\varepsilon(\vartheta)\mathrm{d}\vartheta\right) \\ -\exp\left(-\int_\tau^\theta \beta_\varepsilon(\vartheta)\mathrm{d}\vartheta\right) & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \exp\left(-\int_\theta^\tau \beta_0(\vartheta)\mathrm{d}\vartheta\right) \\ -\exp\left(-\int_\tau^\theta \beta_0(\vartheta)\mathrm{d}\vartheta\right) & 0 \end{pmatrix} (1 + \mathcal{O}(\varepsilon)). \end{aligned} \tag{5.3.101}$$

In particular we obtain the uniform bound

$$\|\widetilde{\mathcal{F}}(\tau, \theta, \varepsilon)\| \le C \tag{5.3.102}$$

independent of τ, θ and ε. Writing the solution to (5.3.100) by the Peano–Baker series (Theorem 7.2.9), we have for (5.3.97) that

$$\mathcal{G}(\tau, \theta, \Lambda, \varepsilon) = \mathrm{I} + \sum_{k=1}^{\infty} \Lambda^k \int_\theta^\tau \widetilde{\mathcal{F}}(\tau_1, \theta, \varepsilon) \cdots \int_\theta^{\tau_{k-1}} \widetilde{\mathcal{F}}(\tau_k, \theta, \varepsilon)\mathrm{d}\tau_k \ldots \mathrm{d}\tau_1 \tag{5.3.103}$$

as a power series in Λ with coefficients

$$\mathcal{G}_k(\tau,\theta,\varepsilon) = \int_\theta^\tau \widetilde{\mathcal{F}}(\tau_1,\theta,\varepsilon)\int_\theta^{\tau_1}\cdots\int_\theta^{\tau_{k-1}}\widetilde{\mathcal{F}}(\tau_k,\theta,\varepsilon)\mathrm{d}\tau_k\ldots\mathrm{d}\tau_1. \qquad (5.3.104)$$

Combining this with (5.3.102) concludes the proof. □

Remark 5.3.5. *The asymptotic behaviour of the coefficient β_ε for $\varepsilon\to 0$ allows to simplify the formulas and to extract the asymptotic main contribution of the singular zone. Based on* (5.3.88) *and* (5.3.89) *we obtain*

$$\begin{aligned}\exp\left(-\int_{\tau_{\Lambda_1}(\varepsilon)}^{\tau_{\Lambda_2}(\varepsilon)}\beta_\varepsilon(\tau)\mathrm{d}\tau\right) &= \exp\left(-\int_{\tau_{\Lambda_1}}^{\tau_{\Lambda_2}}\beta_0(\tau)\mathrm{d}\tau\right)\big(1+\mathcal{O}(\varepsilon)\big)\\ &= \frac{b(1_{+0}) - h\int_{\tau_{\Lambda_1}}^K\psi(\theta)\mathrm{d}\theta}{b(1_{+0}) - h\int_{\tau_{\Lambda_2}}^K\psi(\theta)\mathrm{d}\theta}\big(1+\mathcal{O}(\varepsilon)\big)\\ &= \frac{b(1_{-0})}{b(1_{+0})}\big(1+\mathcal{O}(\varepsilon)\big)\end{aligned} \qquad (5.3.105)$$

with $\tau_{\Lambda_j} = \lim_{\varepsilon\to 0}\tau_{\Lambda_j}(\varepsilon)$ and for all Λ small enough to guarantee $\tau_{\Lambda_1}\le -K$ and $K\le\tau_{\Lambda_2}$. Thus for all these Λ we obtain

$$\mathcal{F}(\tau_{\Lambda_1},\tau_{\Lambda_2},\varepsilon) = \begin{pmatrix}1 & 0\\ 0 & \frac{b(1_{-0})}{b(1_{+0})}\end{pmatrix}\big(1+\mathcal{O}(\varepsilon)\big). \qquad (5.3.106)$$

From (5.3.97) *and* (5.3.98) *we conclude that $\mathcal{G}(\tau,\theta,\Lambda,\varepsilon) - \mathrm{I} = \mathcal{O}(\Lambda)$ holds uniform with respect to ε, τ and θ and hence*

$$\mathcal{E}_{\mathrm{sing}}(\tau_{\Lambda_2},\tau_{\Lambda_1},\Lambda,\varepsilon) = \begin{pmatrix}1 & 0\\ 0 & \frac{b(1_{-0})}{b(1_{+0})}\end{pmatrix}(1+\mathcal{O}(\varepsilon)+\mathcal{O}(\Lambda)) \qquad (5.3.107)$$

follows.

Limiting behaviour of the fundamental solution in the singular zone

We want to describe the behaviour of the fundamental solution $\mathcal{E}_{\mathrm{sing}}(\tau_{\Lambda_2},\tau_{\Lambda_1},\Lambda,\varepsilon)$ as $\varepsilon\to 0$ for fixed Λ. By (5.3.95) we already know that the limit

$$\mathcal{F}(\tau,\theta,0) = \lim_{\varepsilon\to 0}\mathcal{F}(\tau,\theta,\varepsilon) \qquad (5.3.108)$$

exists and satisfies

$$\|\mathcal{F}(\tau,\theta,\varepsilon) - \mathcal{F}(\tau,\theta,0)\| \le C\varepsilon \qquad (5.3.109)$$

uniform in $\tau_{\Lambda_1}\le\theta<\tau\le\tau_{\Lambda_2}$. In the next step we consider the limiting behaviour of the power series $\mathcal{G}$ and in particular its coefficients $\mathcal{G}_k$.

Lemma 5.3.11. *The limit*

$$\mathcal{G}_k(\tau,\theta,0) = \lim_{\varepsilon\to 0} \mathcal{G}_k(\tau,\theta,\varepsilon) \tag{5.3.110}$$

exists for all $\tau_{\Lambda_1} \leq \theta < \tau \leq \tau_{\Lambda_2}$ *and satisfies*

$$\|\mathcal{G}_k(\tau,\theta,\varepsilon) - \mathcal{G}_k(\tau,\theta,0)\| \leq C\varepsilon. \tag{5.3.111}$$

Furthermore,

$$\mathcal{G}_1(\tau,\theta,0) = \int_\theta^\tau \begin{pmatrix} 0 & \exp\left(\int_\theta^{\tau_1} \beta_0(\vartheta)\mathrm{d}\vartheta\right) \\ -\exp\left(\int_{\tau_1}^\theta \beta_0(\vartheta)\mathrm{d}\vartheta\right) & 0 \end{pmatrix} \mathrm{d}\tau_1. \tag{5.3.112}$$

Proof. The limiting behaviour of $\mathcal{F}$ implies that the limit

$$\widetilde{\mathcal{F}}(\tau,\theta,0) = \lim_{\varepsilon\to 0} \widetilde{\mathcal{F}}(\tau,\theta,\varepsilon) \tag{5.3.113}$$

exists and is uniformly bounded with respect to $\tau_{\Lambda_1} \leq \theta < \tau \leq \tau_{\Lambda_2}$ and therefore the functions

$$\mathcal{G}_k(\tau,\theta,0) = \int_\theta^\tau \widetilde{\mathcal{F}}(\tau_1,\theta,0) \int_\theta^{\tau_1} \cdots \int_\theta^{\tau_{k-1}} \widetilde{\mathcal{F}}(\tau_k,\theta,0)\mathrm{d}\tau_k \ldots \mathrm{d}\tau_1 \tag{5.3.114}$$

are good candidates to be considered for the limiting behaviour of $\mathcal{G}_k$. For $k=1$ the representation

$$\mathcal{G}_1(\tau,\theta,0) = \int_\theta^\tau \widetilde{\mathcal{F}}(\tau_1,\theta,0)\mathrm{d}\tau_1 \tag{5.3.115}$$

corresponds directly to (5.3.112) due to the formula (5.3.101) for $\widetilde{\mathcal{F}}(\tau,\theta,0)$. Hence, using the analogue to (5.3.109) we obtain

$$\|\mathcal{G}_1(\tau,\theta,\varepsilon) - \mathcal{G}_1(\tau,\theta,0)\| \leq \int_\theta^\tau \|\widetilde{\mathcal{F}}(\tau_1,\theta,\varepsilon) - \widetilde{\mathcal{F}}(\tau_1,\theta,0)\|\mathrm{d}\tau_1 \leq C|\tau-\theta|\varepsilon \tag{5.3.116}$$

and using $|\tau-\theta| \leq 2K'$ the first statement follows.

The estimate for $\mathcal{G}_k$ is obtained by telescoping the integral

$$\begin{aligned}
&\mathcal{G}_k(\tau,\theta,\varepsilon) - \mathcal{G}_k(\tau,\theta,0) = \\
&\quad \int_\theta^\tau \left(\widetilde{\mathcal{F}}(\tau_1,\theta,\varepsilon) - \widetilde{\mathcal{F}}(\tau_1,\theta,0)\right) \int_\theta^{\tau_1} \widetilde{\mathcal{F}}(\tau_1,\theta,0) \cdots \int_\theta^{\tau_{k-1}} \widetilde{\mathcal{F}}(\tau_k,\theta,0)\mathrm{d}\tau_k \ldots \mathrm{d}\tau_1 \\
&\quad + \int_\theta^\tau \left(\widetilde{\mathcal{F}}(\tau_1,\theta,\varepsilon) - \widetilde{\mathcal{F}}(\tau_1,\theta,0)\right) \int_\theta^{\tau_1} \left(\widetilde{\mathcal{F}}(\tau_1,\theta,\varepsilon) - \widetilde{\mathcal{F}}(\tau_1,\theta,0)\right) \\
&\qquad \times \int_\theta^{\tau_2} \widetilde{\mathcal{F}}(\tau_3,\theta,0) \cdots \int_\theta^{\tau_{k-1}} \widetilde{\mathcal{F}}(\tau_k,\theta,0)\mathrm{d}\tau_k \ldots \mathrm{d}\tau_1 \\
&\quad + \cdots,
\end{aligned} \tag{5.3.117}$$

each term containing one difference more up to having k differences as integrands. Note that this represents the difference $\mathcal{G}_k(\tau,\theta,\varepsilon) - \mathcal{G}_k(\tau,\theta,0)$ in terms of the differences $\widetilde{\mathcal{F}}(\tau_1,\theta,\varepsilon) - \widetilde{\mathcal{F}}(\tau_1,\theta,0)$ and the form $\mathcal{G}_{k-\ell}(\tau_\ell,\theta,0)$ already estimated in the previous induction step. Hence

$$\|\mathcal{G}_k(\tau,\theta,\varepsilon) - \mathcal{G}_k(\tau,\theta,0)\| \lesssim \sum_{\ell=1}^{k} \varepsilon^\ell \lesssim \varepsilon \tag{5.3.118}$$

and the lemma is proved. □

5.3.4 Bounded frequencies

We will give some remarks concerning estimates for the fundamental solution for $|\xi| \le N$. Here it suffices to consider the system (5.2.11) in original form and to observe that its coefficient matrices have norm estimates $\|A(\xi)\| \lesssim |\xi|$ and $\|B(t,\varepsilon)\| \lesssim 1 + \Phi_\varepsilon(t-1)$.

Representing its solution directly by the Peano–Baker series (Corollary 7.2.11) yields

$$\|\mathcal{E}(t,s,\xi,\varepsilon)\| \le \exp\left(C \int_s^t \big(|\xi| + 1 + \Phi_\varepsilon(\theta-1)\big)\mathrm{d}\theta \right) \le \tilde{C} \tag{5.3.119}$$

using that $\int_0^2 \Phi_\varepsilon(t-1)\mathrm{d}t$ is independent of ε and that both $|\xi|$ and s,t are bounded.

Remark 5.3.6. *Note that for dissipative problems the uniform boundedness of the fundamental solution follows already from the positivity of the coefficient of* (5.2.1) *in front of u_t. For more general wave models this statement needs a proof and the above reasoning seems viable for this case too.*

5.3.5 Combining the bits

We collect here the estimates obtained so far. As we are interested in the influence of the point singularity on the structure of the fundamental solution we consider $t_1, t_2 \in [0,2]$ with $t_1 < 1 < t_2$ and look at the fundamental solution to (5.2.11) for fixed ε chosen sufficiently small. This is given by

$$\begin{aligned}&\mathcal{E}(t_2,t_1,\xi,\varepsilon) = \\ &\quad \mathcal{E}_{\mathrm{hyp}}(t_2,t_{\xi_2}(\varepsilon),\xi,\varepsilon)T^{-1}(\varepsilon)\mathcal{E}_{\mathrm{sing}}(\tau_{\xi_2}(\varepsilon),\tau_{\xi_1}(\varepsilon),\varepsilon|\xi|,\varepsilon)T(\varepsilon)\mathcal{E}_{\mathrm{hyp}}(t_{\xi_1}(\varepsilon),t_1,\xi,\varepsilon)\end{aligned} \tag{5.3.120}$$

with $T(\varepsilon)$ the transformation matrix between the micro-energies used in the hyperbolic zone and in the singular zone, such as

$$T(\varepsilon) = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} = \varepsilon \mathrm{I}. \tag{5.3.121}$$

Note that both of these matrices cancel each other and can therefore be neglected. As ε tends to 0 we have $t_{\xi_1} \to 1_{-0}$ and $t_{\xi_2} \to 1_{+0}$. So using the estimates (5.3.79) and (5.3.107) we obtain for fixed ξ

$$\lim_{\varepsilon\to 0} \mathcal{E}(t_2, t_1, \xi, \varepsilon) = \mathcal{E}_{\mathrm{hyp}}(t_2, 1_{+0}, \xi, 0) \begin{pmatrix} 1 & 0 \\ 0 & H \end{pmatrix} \mathcal{E}_{\mathrm{hyp}}(1_{-0}, t_1, \xi, 0), \qquad (5.3.122)$$

where $H = \frac{b(1_{-0})}{b(1_{+0})}$ is given in terms of the jump of $\log b$ at $t = 1$.

5.4 Results

5.4.1 Existence of very weak solutions

Although in our model case the existence of very weak solutions was already established in [46], we will show how to obtain this from the properties of the fundamental solution just constructed.

Proposition 5.4.1. *For $\varepsilon \in (0, 1]$, $0 \le s < t \le 2$ and $|\xi| \ge N$ the fundamental solution to system* (5.2.11) *is uniformly bounded, i.e.*

$$\|\mathcal{E}(t, s, \xi, \varepsilon)\| \le C. \qquad (5.4.1)$$

Proof. If $[s, t] \times \{(\xi, \varepsilon)\} \subset Z_{\mathrm{hyp}}(N)$, the result follows directly from the construction in the hyperbolic zone. So it remains to consider only situations where the time interval intersects with the singular zone $Z_{\mathrm{sing}}(N)$.

We focus on the situation where $(s, \xi, \varepsilon) \in Z_{\mathrm{hyp}}(N)$ and $(t, \xi, \varepsilon) \in Z_{\mathrm{sing}}(N)$, i.e., $s < t_{\xi_1}(\varepsilon) < 1$ and $t_{\xi_1}(\varepsilon) < t < t_{\xi_2}(\varepsilon)$. Then the fundamental solution to system (5.2.11) is given by

$$\mathcal{E}(t, s, \xi, \varepsilon) = T^{-1}(\varepsilon)\mathcal{E}_{\mathrm{sing}}(\tau_{\xi_2}, \tau_{\xi_1}, \varepsilon|\xi|, \varepsilon)T(\varepsilon)\mathcal{E}_{\mathrm{hyp}}(t_{\xi_1}, t, \xi, \varepsilon). \qquad (5.4.2)$$

As the factors ε^{-1} and ε arising from $T^{\pm 1}(\varepsilon)$ cancel out, it suffices to show the uniform boundedness of $\mathcal{E}_{\mathrm{hyp}}(t, s, \xi, \varepsilon)$ for $s < t$ over the hyperbolic zone and that of $\mathcal{E}_{\mathrm{sing}}(\tau, \theta, \Lambda, \varepsilon)$ for $\theta < \tau$ over the singular zone (in singular variables).

Hence, it remains to collect the already proved boundedness results. For $|\xi| \le N$ (i.e. for $\Lambda \le N\varepsilon$) the uniform bound was shown in (5.3.119). For $|\xi| > N$ and within the hyperbolic zone the boundedness follows from the representation (5.3.68) and the boundedness of each individual factor due to Theorem 5.3.6, while within the singular zone the representation in Theorem 5.3.10 gives a uniform bound on the fundamental solution based on the uniform boundedness of τ_{Λ_1} and τ_{Λ_2} with respect to both ε and Λ. $\square$

In combination with the bound $\varepsilon^{-1} + |\xi|$ for the coefficient matrix of (5.2.11) we conclude the bound

$$\|\mathrm{D}_t^k \mathcal{E}(t, s, \xi, \varepsilon)\| \le C_k \varepsilon^{-k} |\xi|^k \qquad (5.4.3)$$

uniform in $s < t$, $\varepsilon > 0$ and $\xi \in \mathbb{R}^n$.

Corollary 5.4.2. *Let the net $(u_\varepsilon)_{\varepsilon\in(0,1]}$ be a solution net to the Cauchy problem* (5.2.8) *for initial data $u_0 \in \mathrm{H}^1(\mathbb{R}^n)$ and $u_1 \in \mathrm{L}^2(\mathbb{R}^n)$. Then the estimate*

$$\|\partial_t^{1+k} u(t,\cdot)\|_{\mathrm{H}^{-k}(\mathbb{R}^n)} + \|\partial_t^k u(t,\cdot)\|_{\mathrm{H}^{1-k}(\mathbb{R}^n)} \le C_k \varepsilon^{-k} \tag{5.4.4}$$

holds.

Remark 5.4.1. *Note that the negative power of ε only appears for the solution at and after the singularity $t = 1$, and the estimates hold without ε when $t < 1$.*

5.4.2 Exceptional propagation of singularities

Now we want to prove the exceptional propagation of singularities already hinted by the numerical experiments from [46]. For this we consider the model problem in *one* space dimension and use specially prepared initial data in the form of wave packets

$$\begin{aligned} u_0(x) &= \mathrm{e}^{\mathrm{i}x\delta^{-1}\xi_0}\chi(x), \\ u_1(x) &= \partial_x u_0(x) = \mathrm{e}^{\mathrm{i}x\delta^{-1}\xi_0}\big(\mathrm{i}\xi_0\delta^{-1}\chi(x) + \chi'(x)\big) \end{aligned} \tag{5.4.5}$$

parameterised by a fixed frequency $\xi_0 \in \mathbb{R}\setminus\{0\}$ and for a smooth rapidly decaying function $\chi \in \mathcal{S}(\mathbb{R})$ with sufficiently small Fourier support around the origin. Applying a Fourier transform we see that

$$\begin{aligned} |\xi|\widehat{u_0}(\xi) \pm \mathrm{i}\widehat{u_1}(\xi) &= |\xi|\widehat{\chi}(\xi-\delta^{-1}\xi_0) \mp \xi\widehat{\chi}(\xi-\delta^{-1}\xi_0) \\ &= \begin{cases} 0, & \pm\xi > 0, \\ \pm 2\xi\widehat{\chi}(\xi-\delta^{-1}\xi_0), & \pm\xi < 0. \end{cases} \end{aligned} \tag{5.4.6}$$

Without loss of generality we can assume that $\xi_0 > 0$ and $\operatorname{supp}\widehat{\chi} \subset [-\xi_0/2, \xi_0/2]$. Hence, for such initial data the initial datum $U_0(\xi)$ to (5.2.11) satisfies

$$M^{-1}U_0(\xi,\varepsilon) = \sqrt{2}\begin{pmatrix} 0 \\ \xi\widehat{\chi}(\xi-\delta^{-1}\xi_0) \end{pmatrix} \tag{5.4.7}$$

for the diagonaliser M from (5.3.33). Let now $t < 1$. As $\mathcal{E}_0(t,s,\xi)$ is diagonal and $\mathcal{Q}(t,s,\xi,\varepsilon) - \mathrm{I}$ as well as $N_1(t,s,\xi,\varepsilon) - \mathrm{I}$ are both bounded by $|\xi|^{-1}$ uniformly in $\varepsilon > 0$ (small enough such that $(t,\xi,\varepsilon) \in Z_{\mathrm{hyp}}(N)$) and $s \in [0,t]$ we obtain that

$$V(t,\xi,\varepsilon) = \sqrt{\frac{b_\varepsilon(0)}{b_\varepsilon(t)}} N_1(t,\xi,\varepsilon)\mathcal{E}_0(t,0,\xi)\mathcal{Q}(t,0,\xi,\varepsilon)N_1^{-1}(0,\xi,\varepsilon)M^{-1}U_0(\xi,\varepsilon) \tag{5.4.8}$$

is given by

$$V(t,\xi,\varepsilon) = \sqrt{\frac{b(0)}{b(t)}}\sqrt{2}\begin{pmatrix} 0 \\ \mathrm{e}^{-\mathrm{i}t\xi}\xi\widehat{\chi}(\xi-\delta^{-1}\xi_0) \end{pmatrix} + \mathcal{O}(1), \qquad t < 1, \tag{5.4.9}$$

for fixed t and with a uniformly bounded remainder independent of the choice of δ. This corresponds to a wave traveling to the right plus remainder terms with smaller norm. Note that the first term behaves like δ^{-1} due to the support assumption made for $\widehat{\chi}$ and thus dominates the remainder term when choosing δ small enough.

In the following, we consider $t > 1$ and ask for the influence of the point singularity at time 1 on the behaviour of our net of solutions. If $\varepsilon > 0$ is small enough such that $(t, \xi, \varepsilon) \in Z_{\mathrm{hyp}}(N)$ the solution is represented by

$$\begin{aligned} V(t,\xi,\varepsilon) = \sqrt{\frac{b_\varepsilon(0) b_\varepsilon(t_{\xi_2})}{b_\varepsilon(t_{\xi_1}) b_\varepsilon(t)}} & N_1(t,\xi,\varepsilon)\mathcal{E}_0(t, t_{\xi_2}, \xi)\mathcal{Q}(t, t_{\xi_2}, \xi, \varepsilon) N_1^{-1}(t_{\xi_2}, \xi, \varepsilon) \\ & \times M^{-1} T^{-1}(\varepsilon)\mathcal{E}_{\mathrm{sing}}(\tau_{\xi_2}, \tau_{\xi_1}, \varepsilon|\xi|, \varepsilon) T(\varepsilon) M \\ & \times N_1(t_{\xi_1}, \xi, \varepsilon)\mathcal{E}_0(t_{\xi_1}, 0, \xi)\mathcal{Q}(t_{\xi_1}, 0, \xi, \varepsilon) N_1^{-1}(0, \xi, \varepsilon) M^{-1} U_0(\xi, \varepsilon). \end{aligned} \tag{5.4.10}$$

We again look at the main terms and estimates for remainders. In order to get the desired estimates we choose first the zone constant N large enough to control non-diagonal terms appearing in the transformation matrices and in $\mathcal{Q}$. This yields based on the symbol estimate for $N_1(t, \xi, \varepsilon) - \mathrm{I}$ and estimate (5.3.67) for $\mathcal{Q}(t, s, \xi, \varepsilon) - \mathrm{I}$

$$\begin{aligned} V(t,\xi,\varepsilon) = \sqrt{\frac{b(0)b(1_{+0})}{b(1_{-0})b(t)}} & \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{e}^{\mathrm{i}(t-1)\xi} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}(t-1)\xi} \end{pmatrix} \\ & \times \begin{pmatrix} H+1 & H-1 \\ H-1 & H+1 \end{pmatrix} \begin{pmatrix} 0 \\ \mathrm{e}^{-\mathrm{i}t\xi}\xi\widehat{\chi}(\xi - \delta^{-1}\xi_0) \end{pmatrix} \\ & + \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon|\xi|) + \mathcal{O}(1/N) \end{aligned} \tag{5.4.11}$$

using in an essential way that the $T(\varepsilon)$-terms cancel out, that $|t_{\xi_i}(\varepsilon) - 1| \le C\varepsilon$ combined with

$$\begin{gathered} \|N_1(t,\xi,\varepsilon) - \mathrm{I}\| + \|N_1^{-1}(0,\xi,\varepsilon) - \mathrm{I}\| \le C|\xi|^{-1}, \\ \|\mathcal{Q}(t, t_{\xi_2}, \xi, \varepsilon) - \mathrm{I}\| + \|\mathcal{Q}(t_{\xi_1}, 0, \xi, \varepsilon) - \mathrm{I}\| \le C/N, \\ \|N_1^{-1}(t, t_{\xi_2}, \xi, \varepsilon) - \mathrm{I}\| + \|N_1(t_{\xi_1}, \xi, \varepsilon) - \mathrm{I}\| \le C/N \end{gathered} \tag{5.4.12}$$

due to (5.3.43), (5.3.67) and (5.2.16) and that

$$\begin{aligned} M^{-1}\mathcal{E}_{\mathrm{sing}}(\tau_{\xi_2}, \tau_{\xi_1}, \varepsilon|\xi|, \varepsilon) M &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & H \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ &\qquad + \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon|\xi|) \\ &= \frac{1}{2}\begin{pmatrix} H+1 & H-1 \\ H-1 & H+1 \end{pmatrix} + \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon|\xi|) \end{aligned} \tag{5.4.13}$$

due to (5.3.107) with $H = \frac{b(1_{-0})}{b(1_{+0})} \in (0, 1]$. As for our net of initial data $|\xi| \sim \delta^{-1}$, the second remainder term is of order $\varepsilon\delta^{-1}$ and thus negligible for ε small enough and δ fixed.

To recover the solution $u(t,x)$, we have to multiply by the matrix M and apply the inverse Fourier transform. Thus we obtain the following theorem.

Theorem 5.4.3. *The very weak solution corresponding to the net of initial date* (5.4.5) *is described (up to terms small compared to the solution itself)*

- *by a wave travelling to the right for $t < 1$; and*
- *by two waves travelling to the left and to the right for all $t > 1$.*

Remark 5.4.2. *The partial reflection of rays at the singularity is characterised by the matrix*

$$\frac{1}{2}\begin{pmatrix} H+1 & H-1 \\ H-1 & H+1 \end{pmatrix} \tag{5.4.14}$$

in terms of the jump of $\log b$ *at $t = 1$.*

Thus, if the coefficient b has no jump and therefore $H = 1$ this matrix becomes the identity matrix and for $t > 1$ only one wave propagates to the right. Hence, no reflected wave occurs.

If b has a jump we can compare the amplitude of both travelling waves. For this we fix a sufficiently small $\delta > 0$ and write down the main terms of the travelling wave as

$$u(t,x) = \sqrt{\frac{b(0)}{b(t)}}\mathbf{u}(x-t), \qquad 0 < t < 1 \tag{5.4.15}$$

and

$$u(t,x) = \frac{H+1}{2\sqrt{H}}\sqrt{\frac{b(0)}{b(t)}}\mathbf{u}(x-t) + \frac{H-1}{2\sqrt{H}}\sqrt{\frac{b(0)}{b(t)}}\mathbf{u}(x-2+t), \qquad t > 1. \tag{5.4.16}$$

The first term corresponds to a wave continuing in the same direction but with amplitude multiplied by $\frac{H+1}{2\sqrt{H}}$, while the second term gives the reflected part with amplitude multiplied by $\frac{H-1}{2\sqrt{H}}$.

Remark 5.4.3. *The related wave model*

$$u_{tt} - \Delta u + \delta_1(t)u_t = 0 \tag{5.4.17}$$

with coefficient given by the Delta distribution supported in $t = 1$ appears almost as a special case of treatment here in this chapter. For the choice of $b(t) = 1/2$ for $0 \leq t < 1$ and $b(t) = 3/2$ for $1 < t \leq 2$ we obtain a closely related net of coefficients leading to $H = 1/3$ and a resulted transfer matrix at the singularity.

The true consideration of the above equation can be done on lines similar to the treatment provided here in the p. This would lead to the (related) transfer matrix

$$\frac{1}{2e}\begin{pmatrix} 1+e & 1-e \\ 1-e & 1+e \end{pmatrix} \tag{5.4.18}$$

.

Remark 5.4.4. *The arguments presented in this section for the case of one space dimension applies in a similar way to higher dimensions. The main reflected wave travels in the opposite direction to the main one, and lower order terms could propagate along cones emanating from the point of interaction of singularities.*

Remark 5.4.5. *In this chapter, the symbol classes used in the treatment were adapted to one point singularity at* $t = 1$*. This can clearly be extended to treat point singularities at a finite number of times.*

Chapter 6

Concluding remarks and perspectives

The theoretical and numerical analysis conducted in this thesis showed that numerical methods work well in situations where a rigorous mathematical formulation of the problem is difficult in the framework of the classical theory of distributions. The concept of very weak solutions eliminates this difficulty in the case of the terms with multiplication of distributions. In contrast with the framework of the Colombeau algebras where the consistency with classical solutions maybe lost, the concept of very weak solutions which depends heavily on the equation under consideration is consistent with classical theory. In addition, using the theory of very weak solutions, we can talk about the uniqueness of numerical solutions of differential equations with strongly singular coefficients in an appropriate sense.

In this short concluding chapter we give an overview on related questions arising in connection with the considerations of this thesis. The list is not complete in any sense, it should only give some hints of possible generalisations, applications and also parallel developments.

6.1 More physical and abstract examples

The study of singular PDEs in the framework of the concept of very weak solutions is a growing field. To show a wide applicability of the concept, it seems reasonable to apply it for more physical models. Throughout this thesis we treated linear equations. Still open is to consider the situation of nonlinear problems. It is also interesting to consider abstract problems. Recently, the authors in [15, 16] used the approach of very weak solutions for the Klein-Gordon and the Schrödinger equations for general classes of differential operators in the setting of graded Lie groups.

6.2 Microlocal analysis

The numerical simulations conducted in this thesis showed interesting behaviours of the very weak solutions for the considered equations near the singularities of the coefficients. This has to be justified mathematically. Using the same symbol classes as in chapter 5 one can treat other wave models with time-dependent coefficients having point singularities of suitable strength. This corresponds to the models proposed in [4] and will be considered in details in a forthcoming paper. A related problem are singular wave models with singularities depending on space and time. Here an adapted version of a full ε-dependent pseudo-differential calculus has to be used in order to describe the propagation of singularities for very weak solutions. However, this is a much harder problem and the description of a local scattering process of waves (and thus wave front sets) of very weak solutions at such singularities remains challenging.

6.3 Consistency

In this thesis we proved that the very weak solutions to our considered Cauchy problems recapture the classical solutions when they exist, provided that the regularisations of the equations coefficients approximate the coefficients in L^∞ (see Theorems 2.3.1, 3.3.1, 4.2.6 and 4.3.5). This is in particular true if we consider coefficients from the space $C_0(\mathbb{R}^d)$, but in general it is not true, since the space of C^∞ functions with compact support is not dense in L^∞. Recently in [15, 16], under suitable considerations on the equations coefficients and the Cauchy data, the authors proved the consistency results for more general cases, extending and improving the results obtained in this thesis.

Chapter 7

Appendices

7.1 Notation-Guide to the reader

7.1.1 Preliminaries

Unless indicated otherwise, the following notations are frequently used in this thesis:

$a \simeq b$	if a is similar or equal to b,
$f \lesssim g$	if there exists a positive constant C such that $f \leq Cg$,
$f = \mathcal{O}(g)$	where g is strictly positive, if there exists a positive constant C such that $\lvert f(x)\rvert \leq Cg(x)$,
$\lvert \cdot \rvert$	denotes the absolute value of a scalar expression,
$\lVert \cdot \rVert_{L^p}$	the norm in $L^p(\mathbb{R}^n)$ spaces where $\lVert u\rVert_{L^p}=\left(\int_{\mathbb{R}^n} \lvert u(x)\rvert^p \mathrm{d}x\right)^{\frac{1}{p}}$,
$\lVert \cdot \rVert_{L^\infty}$	the norm in $L^\infty(\mathbb{R}^n)$ space where $\lVert u\rVert_{L^\infty} = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} \lvert u(x)\rvert$,
$\lVert \cdot \rVert_{H^s}$	for $s > 0$, $\lVert u\rVert_{H^s} = \lVert u\rVert_{L^2} + \lVert(-\Delta)^{\frac{s}{2}} u\rVert_{L^2}$,
∂_x	denotes the partial derivative with respect to x,
D_t	denotes the Fourier derivative, i.e. $\mathrm{D}_t = -i\partial_t$,
∇	denotes the gradient $\nabla_x = (\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_n})$,
Δ	Laplace operator with respect to $x \in \mathbb{R}^n$, i.e. $\Delta_x = \partial_{x_1}^2 + \partial_{x_2}^2 + \cdots + \partial_{x_n}^2$,
$(-\Delta)^s$	for $s > 0$, denotes the fractional Laplacian defined in terms of the Fourier transform by $(-\Delta)^s u = \mathcal{F}^{-1}(\lvert\xi\rvert^{2s}(\mathcal{F}u))$ for all $\xi \in \mathbb{R}^d$.

7.1.2 Frequently used function spaces

We collect some of the function spaces occurring in this thesis together with a short definition:

$L^p(\mathbb{R}^n)$	Lebesgue spaces, $1 \leq p \leq \infty$,
$H^s(\mathbb{R}^n)$	Sobolev spaces based on $L^2(\mathbb{R}^n)$,
$C^k(\mathbb{R}^n)$	space of k-times continuously differentiable functions,
$C^\infty(\mathbb{R}^n)$	space of infinitely continuously differentiable functions,
$C_0(\mathbb{R}^n)$	space of continuous functions vanishing at infinity, that is $C_0(\mathbb{R}^n) = \{f \in C(\mathbb{R}^n), \forall \varepsilon > 0, \exists K \subset \mathbb{R}^n \text{ compact } : \lvert f\rvert < \varepsilon \text{ for } x \notin K\}$. Note that $C_0(\mathbb{R}^n)$ endowed with the norm $\lVert \cdot \rVert_{L^\infty}$ is a Banach space,
$C_0^\infty(\mathbb{R}^n)$	space of infinitely continuously differentiable functions with compact support,
$C^k([0,T];X)$	for $T > 0$ and X a Banach space, is the space of k-times continuously differentiable functions over $[0,T]$ with values in X,
$\mathcal{D}'(\mathbb{R}^n)$	space of distributions,
$\mathcal{E}'(\mathbb{R}^n)$	space of compactly supported distributions.

7.2 Basic tools

We collect in this appendix several basic tools, which are essential for the results of this thesis. They are well-known and, only if necessary and possible, we sketch the main ideas of the proof.

7.2.1 Duhamel's principle

Duhamel's principle plays a central role in proofs of existence and uniqueness of very weak solutions to our considered PDEs and consistency with classical solutions. For convenience of the reader we provide a proof of specific versions of this principle and refer the reader to [24, 25] for more details and applications.

Duhamel's principle in general case

Let us consider the following Cauchy problem

$$\begin{cases} Lu(t,x) = f(t,x), \quad (t,x) \in (0,\infty[\times \mathbb{R}^d, \\ u(0,x) = u_0(x), \\ u_t(0,x) = u_1(x), \;\; x \in \mathbb{R}^d, \end{cases} \tag{7.2.1}$$

where L is a second order linear partial differential operator.

Theorem 7.2.1 ([24]). *The solution to (7.2.1) is given by*

$$u(t,x) = E_0(t,0,x) * u_0(x) + E_1(t,0,x) * u_1(x) + \int_0^t E_1(t,s,x) * f(s,x)\mathrm{d}s,$$

where E_0 is the fundamental solution to

$$\begin{cases} Lu(t,x) = 0, \quad (t,x) \in (0,\infty[\times \mathbb{R}^d, \\ u(s,x) = \delta_0(s), \\ u_t(s,x) = 0, \;\; x \in \mathbb{R}^d, \end{cases}$$

and E_1 is the fundamental solution to the Cauchy problem

$$\begin{cases} Lu(t,x) = 0, \quad (t,x) \in (0,\infty[\times \mathbb{R}^d, \\ u(s,x) = 0, \\ u_t(s,x) = \delta_0(s), \;\; x \in \mathbb{R}^d, \end{cases}$$

where $s \in (0,\infty[$ and $\delta_0(s)$ is the Dirac delta function.

In particular, to prove our results, we use frequently the following versions of Duhamel's principle for which we give the proofs.

First order Duhamel's principle for evolution equations

Let us consider the following Cauchy problem for the first order inhomogeneous linear evolution equation

$$\begin{cases} u_t(t,x) - Lu(t,x) = f(t,x), \quad (t,x) \in (0,\infty[\times \mathbb{R}^d, \\ u(0,x) = u_0(x), \;\; x \in \mathbb{R}^d, \end{cases} \tag{7.2.2}$$

where L is a linear differential operator that involves no time derivatives.

Theorem 7.2.2. *The solution to the Cauchy problem (7.2.2) is given by*

$$u(t,x) = w(t,x) + \int_0^t v(t,x;s)\mathrm{d}s, \tag{7.2.3}$$

where $w(t,x)$ is the solution to the homogeneous problem

$$\begin{cases} w_t(t,x) - Lw(t,x) = 0, \quad (t,x) \in (0,\infty[\times \mathbb{R}^d, \\ w(0,x) = u_0(x), \;\; x \in \mathbb{R}^d, \end{cases} \tag{7.2.4}$$

and $v(t,x;s)$ solves the auxiliary Cauchy problem

$$\begin{cases} v_t(t,x;s) - Lv(t,x;s) = 0, \quad (t,x) \in (s,\infty[\times \mathbb{R}^d, \\ v(s,x;s) = f(s,x), \;\; x \in \mathbb{R}^d, \end{cases} \tag{7.2.5}$$

where s is a time-like parameter varying over $(0,\infty[$.

Proof. Applying the component ∂_t to u in (7.2.3) we get

$$\partial_t u(t,x) = \partial_t w(t,x) + v(t,x;t) + \int_0^t \partial_t v(t,x;s)\mathrm{d}s. \tag{7.2.6}$$

We note that $v(t,x;t) = f(t,x)$ by the initial condition in (7.2.5). For the spatial component L we simply have

$$Lu(t,x) = Lw(t,x) + \int_0^t Lv(t,x;s)\mathrm{d}s, \tag{7.2.7}$$

since L applies only to the variable x. Combining (7.2.6) and (7.2.7) and using that w and v are the solutions to (7.2.4) and (7.2.5) we arrive at

$$u_t(t,x) - Lu(t,x) = f(t,x).$$

Observing that $u(0,x) = u_0(x)$ from the initial condition in (7.2.4) completes the proof. □

Second order Duhamel's principle for evolution equations

Let us consider the Cauchy problem for the second order inhomogeneous linear evolution equation

$$\begin{cases} u_{tt}(t,x) - Lu(t,x) = f(t,x), & (t,x) \in (0,\infty[\times \mathbb{R}^d, \\ u(0,x) = u_0(x), \ u_t(0,x) = u_1(x), & x \in \mathbb{R}^d, \end{cases} \tag{7.2.8}$$

for some linear partial differential operator L over the space variable x.

Theorem 7.2.3. *The solution to the Cauchy problem (7.2.8) is given by*

$$u(t,x) = w(t,x) + \int_0^t v(t,x;s)\mathrm{d}s, \tag{7.2.9}$$

where $w(t,x)$ is the solution to the homogeneous problem

$$\begin{cases} w_{tt}(t,x) - Lw(t,x) = 0, & (t,x) \in (0,\infty[\times \mathbb{R}^d, \\ w(0,x) = u_0(x), \ w_t(0,x) = u_1(x), & x \in \mathbb{R}^d, \end{cases} \tag{7.2.10}$$

and $v(t,x;s)$ solves the auxiliary Cauchy problem

$$\begin{cases} v_{tt}(t,x;s) - Lv(t,x;s) = 0, & (t,x) \in (s,\infty[\times \mathbb{R}^d, \\ v(s,x;s) = 0, \ v_t(s,x;s) = f(s,x), & x \in \mathbb{R}^d, \end{cases} \tag{7.2.11}$$

where s is a time-like parameter varying over $(0,\infty[$.

Proof. As in Theorem 7.2.2, we apply the components of the operator $\partial_{tt} - L$ separately to u in (7.2.9). For the spatial component L we simply have

$$Lu(t,x) = Lw(t,x) + \int_0^t Lv(t,x;s)\mathrm{d}s. \tag{7.2.12}$$

For the temporal component we get

$$\partial_t u(t,x) = \partial_t w(t,x) + \int_0^t \partial_t v(t,x;s)\mathrm{d}s, \tag{7.2.13}$$

where we used the fact that $v(t,x;t) = 0$ by the imposed initial condition in (7.2.11). Differentiating (7.2.13) a second time and noting that $\partial_t v(t,x;t) = f(t,x)$, we arrive at

$$\partial_{tt} u(t,x) = \partial_{tt} w(t,x) + f(t,x) + \int_0^t \partial_{tt} v(t,x;s)\mathrm{d}s. \tag{7.2.14}$$

Combining (7.2.12) and (7.2.14) and using the fact that w and v solve (7.2.10) and (7.2.11) respectively, we arrive at

$$u_{tt}(t,x) - Lu(t,x) = f(t,x).$$

To conclude the proof, we just observe from (7.2.9) that $u(0,x) = w(0,x) = u_0(x)$ and from (7.2.13) that $u_t(0,x) = \partial_t w(0,x) = u_1(x)$. □

7.2.2 Gronwall's inequality

An important, useful tool for energy estimates is Gronwall's inequality. There are many variants of this inequality, we provide in this appendix the proof for a special form.

Theorem 7.2.4 (Gronwall's inequality - differential form). *Let $I = [t_0, t_1]$. Suppose a is a real-valued continuous function and suppose $u : I \to \mathbb{R}$ is in $C^1(I)$ and satisfies*

$$u'(t) \le a(t)u(t) \quad \text{for} \quad t \in I. \tag{7.2.15}$$

Then

$$u(t) \le u(t_0) \exp \int_{t_0}^t a(s)\mathrm{d}s \quad \text{for all} \quad t \in I. \tag{7.2.16}$$

Proof. Define the function

$$v(t) = \exp \int_{t_0}^t a(s)\mathrm{d}s, \quad t \in I.$$

We note that v satisfies $v'(t) = a(t)v(t)$, with $v(t_0) = 1$ and $v(t) > 0$ for all $t \in I$. By the quotient rule and using the estimate (7.2.15) we easily see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{u(t)}{v(t)} = \frac{v(t)\left(u'(t) - a(t)u(t)\right)}{v^2(t)} \le 0, \quad t \in I.$$

The derivative of the function $\frac{u(t)}{v(t)}$ is non-positive and thus the function is bounded above by its value at the initial point t_0 of the interval I, i.e.

$$\frac{u(t)}{v(t)} \le \frac{u(t_0)}{v(t_0)} = u(t_0).$$

The inequality (7.2.16) follows ending the proof. □

7.2.3 Structure theorems of distributions

We recall some theorems about the structure of distributions. The following one states that locally any distribution is represented as a derivative of a continuous function.

Theorem 7.2.5 (**Local structure of distributions on $\mathbb{R}^n$**). *The restriction of a distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ to a bounded open set $X \in \mathbb{R}^n$ is a derivative of finite order of a continuous function.*

Proof. We refer the reader to [28, Theorem 5.4.1]. □

In particular, when the distribution T has compact support we have,

Theorem 7.2.6 (**Global structure of compactly supported distributions**). *Let $T \in \mathcal{E}'(\mathbb{R}^n)$, then there is an integer $n \geq 0$ and a set of continuous functions f_α, $\alpha \leq n$, such that*

$$T = \sum_{|\alpha| \leq n} f_\alpha.$$

Proof. Please, see [28, Corollary 5.4.1]. □

7.2.4 Young's inequality

In real analysis, the following result is called Young's convolution inequality:

Theorem 7.2.7. *Let $f \in L^p(\mathbb{R}^n)$ and $f \in L^q(\mathbb{R}^n)$ such that*

$$1 \leq p \leq \infty, 1 \leq q \leq \infty \quad \text{and} \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \geq 0.$$

Then

$$f * g \in L^r(\mathbb{R}^n) \quad \text{and} \quad \|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Proof. There are various ways to prove this inequality. One of them is based on Riesz-Thorin interpolation theorem. See [10, Theorem IV.30.] or [63, Proposition 9.20]. □

7.2.5 Plancherel-Parseval identity

Here we review the important result in harmonic analysis, the Plancherel theorem (sometimes called the Parseval–Plancherel identity) which asserts that the integral of the square of the Fourier transform of a function is equal to the integral of the square of the function itself.

Definition 16 (**Fourier transform**). *Let $f : \mathbb{R}^n \to \mathbb{C}$ be an integrable function. The function denoted $\hat{f}$ and defined by*

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \mathrm{e}^{-ix\cdot\xi} f(x) \mathrm{d}x,$$

is called the Fourier transform of f.

Theorem 7.2.8 (**Plancherel theorem**). *Let $f \in L^2(\mathbb{R}^n)$. Then, $\hat{f} \in L^2(\mathbb{R}^n)$ and*

$$\|\hat{f}\|_{L^2} = \|f\|_{L^2}.$$

Proof. The proof is based on Schwarz inequality and Riesz's representation theorem. cf. [69, Plancherel's Theorem] □

7.2.6 The Peano-Baker formula

First order systems of ordinary differential equations

$$\frac{\mathrm{d}u}{\mathrm{d}t} = A(t)u, \quad u(0) = u_0 \in \mathbb{C}^n,$$

are solved in terms of the fundamental solution $\mathcal{E}(t,s)$ as $u(t) = \mathcal{E}(t,0)u_0$. The matrix function $\mathcal{E}(s,t)$ is the solution to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,s) = A(t)\mathcal{E}(t,s), \quad \mathcal{E}(s,s) = I \in \mathbb{C}^{n\times n}.$$

It is well known, that for a constant matrix $A(t) = A$, this fundamental solution can be expressed in terms of the exponential matrix,

$$\mathcal{E}(t,s) = \exp((t-s)A), \qquad \exp A = I + \sum_{k=1}^{\infty} \frac{1}{k!}A^k.$$

For variable coefficients this representation is not valid any more. We give here the representation used several times in chapter 5.

Theorem 7.2.9 (**Peano-Baker representation formula**). *Let $A(t) \in L^1_{loc}(\mathbb{R}, \mathbb{C}^{n\times n})$. Then the fundamental solution to the system $\dot{u}(t) = A(t)u(t)$ i.e. the matrix $\mathcal{E}(t,s)$ solution to*

$$\begin{cases} \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}(t,s) = A(t)\mathcal{E}(t,s), \\ \mathcal{E}(s,s) = I, \end{cases}$$

is given by the Peano-Baker formula

$$\mathcal{E}(t,s) = I + \sum_{k=1}^{\infty} \int_s^t A(t_1) \int_s^{t_1} A(t_2) \int_s^{t_2} \cdots \int_s^{t_{k-1}} A(t_k)\mathrm{d}t_k \ldots \mathrm{d}t_2\mathrm{d}t_1. \quad (7.2.17)$$

Proof. The proof follows by differentiating the series term by term. To prove the convergence of the series one uses the domination by the exponential series following from the following proposition. □

Proposition 7.2.10. *Assume $r \in L^1_{loc}(\mathbb{R})$. Then*

$$\left| \int_s^t r(t_1) \cdots \int_s^{t_k} r(t_{k-1})\mathrm{d}t_k \ldots \mathrm{d}t_1 \right| \le \frac{1}{k!} \left(\int_s^t r(\tau)\mathrm{d}\tau \right)^k, \qquad (7.2.18)$$

for all $k \in \mathbb{N}$.

Proof. The proof follows by induction over k. □

Corollary 7.2.11. *Let $A(t) \in L^1_{loc}(\mathbb{R}, \mathbb{C}^{n\times n})$. Then the fundamental solution to the system $\dot{u}(t) = A(t)u(t)$ satisfies the estimate*

$$\|\mathcal{E}(t,s)\| \leq \exp\left\{\int_s^t \|A(\tau)\| \mathrm{d}\tau\right\}.$$

Proof. The proof is a direct consequence of the representation formula (7.2.17) and the estimate (7.2.18). □

7.2.7 Faà di Bruno's formula

Faà di Bruno's formula [37] is an identity generalizing the chain rule to higher order derivatives. There are various forms of this formula, we give here a version that can be found in [19, 20].

Proposition 7.2.12. *For any given smooth functions $h : \mathbb{R} \to \mathbb{R}$ and $\omega : \mathbb{R}^n \to \mathbb{R}$ and a fixed multi-index $\beta \in \mathbb{N}^n$, we have*

$$D^\beta(h(\omega)) = \sum_{k=1}^{|\beta|} h^{(k)}(\omega) \sum_{\alpha(k,\beta)} C_{k,\alpha} \prod_{i=1}^{|\beta|} \sum_{|\gamma|=i, \gamma\neq 0} (D^\gamma \omega)^{\alpha_i},$$

where $\sum_{\alpha(k,\beta)}$ means summation over all $\alpha \in \mathbb{N}^n$, $\alpha \neq 0$, with $|\alpha| = k$ and $\sum_{i=1}^{|\beta|} i\alpha_i = |\beta|$. The coefficients $C_{k,\alpha} \in \mathbb{N}$ can be zero.

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