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Une contribution aux inégalités stochastiques et applications

Soutenue le............................
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Publications

- Mekki, Slimane; Nieto, Juan J., Ouahab, Abdelghani Stochastic version of Henry type Gronwall's inequality. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **24**, No. 2, Article ID 2150013, 10 p. (2021).
- Mekki, Sliman; Blouhi, Tayeb; Nieto, Juan J.; Ouahab, Abdelghani Some existence results for systems of impulsive stochastic differential equations. *Ann. Math. Sil.* **35**, No. 2, 260-281 (2021).

Contents

Introduction

Stochastic differential equations are a natural choice to model the dynamic systems which are subject to random influences.

Random stochastic differential equations and integral stochastic inequalities theory are needed for the study of various classes of random equations. However, it is usually the case that the mathematical models or equations used to describe phenomena in the biological, physical, engineering, and systems sciences contain certain parameters or coefficients that have specific interpretations, but whose values are unknown. Therefore, it is more realistic to consider such equations as random operator equations, which, in fact, are much more difficult to handle mathematically than deterministic equations. Important contributions to the study of the mathematical aspects of such random equations have been undertaken in [11, 21, 24, 25, 45, 46, 48, 63, 65, 66, 71, 73, 74, 78] among others.

It is natural that a model contains unknown parameters. We consider the model as the parametric Itô stochastic differential equation

$$dX_t = f(\alpha, t, X_t)dt + \sigma(\beta, t, X_t)dW_t, \quad t \geq 0, \quad X_0 = \eta.$$

where $(W_t)_{t\geq 0}$ is a standard Wiener process, $f : I \times [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$, called the drift coefficient, and $\sigma : J \times [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}^+$, called the diffusion coefficient, are known functions except the unknown parameters α and β; I, J are subset of $\mathbb{R}$ and $\mathbb{E}(\eta) < \infty$. The drift coefficient is also called the trend coefficient or damping coefficient or translation coefficient. The diffusion coefficient is also called volatility coefficient. Under local Lipschitz and the linear growth conditions on the coefficients f and σ, there exists a unique strong solution of the above Itô SDE, called the diffusion process or simply a diffusion, which is a continuous strong Markov semimartingale.

This thesis is divided into four chapters. The first chapter is devoted to for some definitions and owners which are ports to the other explaining the stochastic calculus and the martingales with properties and giving a light to the following chapters. The first chapter sheds light on some definitions and owners for a Stochastic process, Martingale, Martingale inequality, Maximal martingale inequality and Burkholder Martingale inequality which are ports to the other chapters, explains the stochastic calculus and the martingales with given properties a light following chapters. In Chapter II, we introduced the notion of Brownian motion, stochastic integrals and derived Itô's formula and stochastic differential equation. These are fundamental notions in stochastic calculus and its applications.

The theory of Itô stochastic differential equations is one of the most convenient and useful areas of the theory of stochastic processes. However, until recently the range of investigations in this theory had been, in our view, unjustifiably restricted: only equations were studied which could, in analogy with the deterministic case,be called ordinary stochastic equations. The situation has began to change in the last $10 - 15$ years. The necessity of considering equations combining the features of partial differential equations and Itô equations has appeared both in the theory of stochastic processes and in related areas.
For the 3^{rd} and 4^{th} chapters, we find the results which are passed on the fractional inequalities and the fixed point theorem to show our rise results in the publications.

Chapter 1

Stochastic calculus

In this chapter, we will give definitions, set up notations that will be used in the rest of the thesis and give some basic results. We denote E a complete separable metric space, $\mathbf{C}(E)$ will denote the space of real-valued continuous functions on E, $\mathbf{C}_b(E)$ is subset of bounded functions in $\mathbf{C}(E)$, and $\mathfrak{B}(E)$ is ω-field on E. $\mathbb{R}^d$ will denote the d-dimensional Euclidean space and $L(m,d)$ will denote the space of $m \times d$ matrices with real entries. For $x \in \mathbb{R}^d$ and $A \in L(m,d)$, $|x|$ and $\|A\|$ will denote the Euclidean norms of x, A, respectively.
$(\Omega, \mathfrak{F}, \mathbb{P})$ will denote a generic probability space, and $\mathbf{B}(\Omega, \mathfrak{F})$ will denote the class of real-valued bounded $\mathfrak{F}$ measurable functions. When $\Omega = E$ and $\mathfrak{F} = \mathbb{B}(E)$, e will write $\mathbb{B}(E)$ for real-valued bounded Borel measurable functions. It is well known and easy to prove that for a complete separable metric space E,

$$\sigma(\mathbf{C}_b) = \mathfrak{B}(E)$$

For an integer $d \geq 1$, let $\mathbf{C}_d = \mathbf{C}([0,+\infty), \mathbb{R}^d)$ with the ucc topology, (i.e. ucc is the uniform convergence on compact subsets of $[0, +infty)$).With this topology, $\mathbf{C}_d$ is itself a complete separable metric space. We will denote a generic element in $\mathbf{C}_d$ by ζ. Denoting the coordinate mappings on $\mathbf{C}_d$ by

$$\beta_t(\zeta) = \zeta(t), \quad \zeta \in \mathbf{C}_d \quad and \quad t \in [0,+\infty)$$

it can be shown that

$$\mathfrak{B}(\mathbf{C}_d) = \sigma\{\beta_t, \quad 0 \leq t < +\infty\}$$

A function f from $[0,+\infty)$ to $\mathbb{R}^d$ is said to be r.c.l.l. (right continuous with left limits) if f is right continuous everywhere ($f(t) = \liminf\limits_{u \longrightarrow t} f(u)$ for all

$0 \leq t < +\infty$) and such that the left limit $f(t-) = \lim_{u \longrightarrow t} \sup f(u)$ exists for all $0 \leq t < +\infty$. We define $f(0-) = 0$ and for $t \geq 0$, $\Delta f(t) = (t) - (t-)$. For an integer $d \geq 1$, let $\mathbb{D}_d = \mathbb{D}([0, +\infty), \mathbb{R}^d)$ be the space of all r.c.l.l.functions g from $[0, +\infty)$ to $\mathbb{R}^d$ with the topology of uniform convergence on compact subsets, abbreviated it as ucc. Thus f_n converges to f in ucc topology if

$$\sup_{t \leq T} |f_n(t) - f(t)| \longrightarrow 0, \quad \forall T < \infty.$$

1.1 Stochastic process

In this section, we will discuss martingales indexed by integers (mostly positive integers) and obtain basic inequalities on martingales and other results which are the basis of most of the developments in later chapters on stochastic integration.We will begin with a discussion on conditional expectation and then on filtration $\mathfrak{F}_.$ two notions central to martingales.
For all information of this chapter we can see Rao *et al* [71].

1.1.1 Generated σ-algebras

Let a set S and for a collection Λ of subsets of a set S, we write $\sigma_S(\Lambda)$ for the σ-algebra of subsets of S generated by Λ, that is, the smallest σ-algebra of subsets of S containing Λ. Similarly, we write $\delta_S(\Lambda)$ for the algebra of subsets of S generated by Λ. For a topological space S we write $\mathfrak{B}_S$ for the Borel σ-algebra of subsets of S, that is, the σ-algebra of subsets of S generated by the collection of the open sets in S. In particular, $\mathfrak{B}_{\mathbb{R}}$ is the smallest σ-algebra containing all the open sets in the space of the real numbers $\mathbb{R}$ with the usual topology, and $\mathfrak{B}_{\overline{\mathbb{R}}}$ is the smallest σ-algebra containing all the open sets in the space of the extended real numbers $\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$ with the topology of the two-point compactification of $\overline{\mathbb{R}}$, that 87is, $\mathfrak{B}_{\mathbb{R}}$ is the smallest σ-algebra of subsets of $\overline{\mathbb{R}}$ containing $\mathfrak{B}_{\mathbb{R}}$, $\{-\infty\}$, and $\{\infty\}$. For a subset A of S and the collection Λ of subsets of a set S, we write $\Lambda \cap A$ for the collection $\{E \cap A : E \in \Lambda\}$ of subsets of S.
The following theorems concerning generation of a σ-algebra will be used frequently.

Theorem 1.1.1. *Let f be a mapping of a set S into a set T. Then for an* *arbitrary collection* Λ *of subsets of* T*, we have*

$$\sigma(f^{-1}(\Lambda)) = f^{-1}(\sigma(\Lambda))$$

Corollary 1.1.2. *Let* S *be a set and* Λ *be a collection of subsets of a set* T*. Then*

$$S \times \sigma(\Lambda) = \sigma(S \times \Lambda).$$

Lemma 1.1.3. *Let* $(\Lambda_i)_{\in I}$ *be an arbitrary collection of subsets of a set* S_i *for* $i = 1, 2, 3, \ldots, k$ *where* $k \in \mathbb{N}^*$ *Then*

$$\sigma(\Lambda_1) \times \sigma(\Lambda_2) \times \ldots \sigma(\Lambda_k) \subset \sigma(\Lambda_1 \times \Lambda_2 \times \ldots \Lambda_k).$$

Theorem 1.1.4. *Let* S_i*,* $i = 1, 2, \ldots, k$ *be a topological spaces satisfying the second axiom of countability, let* $\mathfrak{D}_{S_i}$ *be the collection of the open sets in* S_i*, and let* $\mathfrak{B}_{S_i} = \sigma(\mathfrak{D}_{S_i})$*, that is, the Borel 08*σ*-algebra of subsets of* S_i*, for* $i = 1, 2, \ldots, k$*. Then for the Borel* σ*-algebra* $\mathfrak{B}_{S_1 \times S_2 \times \ldots \times S_k}$ *of 09subsets of* $S_1 \times S_2 \times \ldots \times S_k$ *in its product topology we have*

$$\mathfrak{B}_{S_1 \times S_2 \times \ldots \times S_k} = \sigma(\mathfrak{B}_{S_1} \times \mathfrak{B}_{S_2} \times \ldots \times \mathfrak{B}_{S_k})$$

Theorem 1.1.5. *Let* Λ *be a collection of subsets of a set* S *and let* $A \subset S$*. Then*

$$\sigma_A(\Lambda \cap A) = \sigma(\Lambda) \cap A.$$

Definition 1.1.1. *A collection* Λ *of sets is called a* π*-class if it is closed under finite intersection, that is,* $A, B \in \Lambda \Longrightarrow A \cap B \in \Lambda$*. A collection* $\aleph$ *of subsets of a set* S *is called a d-class of subsets of* S *if:*

1. $S \in \aleph$.

2. $A, B \in \aleph, A \subset B \Longrightarrow B - A \in \aleph$.

3. $A_n \in \aleph, n \in \mathbb{N}, A_n \uparrow \Longrightarrow \lim\limits_{n \longrightarrow \infty} A_n \in \aleph$.

For a collection Λ *of subsets of a set* S*, the d-class of subsets of* S *generated by* Λ*, that is, the smallest d-class of subsets of* S *containing* Λ *is denoted by* $d(\Lambda)$

Theorem 1.1.6. *Let Λ be a π-class of subsets of a set S. Then $d(\Lambda) = \sigma(\Lambda)$. For all resilttes of this section we see ([24]).*

Definition 1.1.2. *A measurable space $(\Omega, \mathfrak{F})$ is called a standard measurable space if it is Borel isomorphic to one of the following measurable spaces: $(\langle 1; n\rangle, \mathfrak{B}(\langle 1, n\rangle))$, $(N, \mathfrak{B}(N))$ or $(M, \mathfrak{B}(M))$, where $\langle 1, n\rangle = \{1, 2, \ldots, n\}$ with the discrete topology, $\mathbb{N} = \{l, 2, \ldots\}$ with the discrete topology and $M = \{0, l\}^N = \{\omega = (\omega_1, \omega_1, \ldots); \quad \omega_i = 0 \quad or \quad 1\}$ with the product topology.*
It is well known that a Polish space (a complete separable metric space) with the topological σ-field is a standard measurable space and every measurable subset of a standard measurable space with the induced σ-field is a standard measurable space ([24, 59]).

Theorem 1.1.7. *Let $(\Omega, \mathfrak{F})$ be a standard measurable space and $\mathbb{P}$ be a probability on $(\Omega, \mathfrak{F})$. Let $\mathfrak{G}$ be a sub σ-field of $\mathfrak{G}$. Then a regular conditional probability $(p(\omega, B))$ given $\mathfrak{G}$ exists uniquely.*

1.1.2 Continuous stochastic processes

Let $W^d = \mathbf{C}\left([0, +\infty), \mathbb{R}^d\right)$ be the set of all continuous functions.
We define a metric ρ on W^d by:

$$\rho(w_1, w_2) \quad = \quad \sum_{k=1}^{+\infty} 2^{-k}\left[\left(\sum_{0\leq t\leq n} |w_1(t) - w_2(t)|\right) \wedge 1\right]; \quad w_1, w_2 \in W^d \tag{1.1.1}$$

It is easy to see that W^d is complete and separable under this metric.
Let $\mathfrak{B}(W^d)$ be the topological σ-field. By a Borel cylinder set we mean a set $\mathfrak{H} \subset W^d$ of the fol1owing form

$$\mathfrak{H} = \{w = (w(t_1), w(t_2), \ldots, w(t_n)) \in K\}$$

for some sequence $0 \leq t_1 < t_2 < \ldots < t_n$ and $K \in \mathfrak{B}\left(\mathbb{R}^{nd}\right)$. The totality of Borel cylinder sets is denoted by $\mathfrak{C}$. Since the mapping

$$w \in W^d \longmapsto w = (w(t_1), w(t_2), \ldots, w(t_n)) \in \mathbb{R}^{nd}$$

is continuous, it is clear that

$$\mathfrak{C} \subset \mathfrak{B}\left(W^d\right), \quad and \quad \sigma\left(\mathfrak{C}\right) = \mathfrak{B}\left(W^d\right)$$

Remark 1.1.1. *Every probability on $\left(W^d, \mathfrak{B}(W^d)\right)$ is uniquely determined by its values on $\mathfrak{C}$.*

1.1.3 Progressively measurable processes

Definition 1.1.3. *A stochastic process $X = \{X_t : t \in \mathbb{R}^+\}$ on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ is said to be continuous, left-continuous, or right-continuous if $X(., \omega)$ is a continuous, left-continuous, or right-continuous function on $\mathbb{R}^+$ for every $\omega \in \Omega$. X is said to be a.s. continuous, a.s. left-continuous, or a.s. right-continuous if $X(\cdot, \omega)$ is continuous, left-continuous, or right-continuous on $\mathbb{R}^+$ for a.e. $\omega \in \Omega$.*
Note that by left-continuity of $X(., \omega)$ on $\mathbb{R}^+$ we mean left-continuity on $(0, \infty)$.

Theorem 1.1.8. *Let $X = \{X_t : t \in \mathbb{R}^+\}$ and $Y = \{Y_t : t \in \mathbb{R}^+\}$ be two a.s. left- (or a.s. right-) continuous processes on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$.*
If $X_t = Y_t$ a.e. on $(\Omega, \mathfrak{F}, \mathbb{P})$ for every $t \in \mathbb{R}^+$ then X and Y are equivalent processes.

Definition 1.1.4. *Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space. A system of sub-σ-algebras $\mathfrak{F}$, $\{\mathfrak{F}_t : t \in \mathbb{R}^+\}$, is called a filtration of the probability space if it is an increasing system in the sense that $\mathfrak{F}_s \subset \mathfrak{F}_t$ for $s, t \in \mathbb{R}^+, \quad s < t$.*
The quadruple $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t : t \in \mathbb{R}^+\}, \mathbb{P})$, or more briefly $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}, \mathbb{P})$, is called a filtered space. We call $\mathfrak{F}_\infty = \sigma\left(\bigcup_{t \in \mathbb{R}^+} \mathfrak{F}_t\right)$ the σ-algebra at infinity of the filtration. A stochastic process $X = \{X_t : t \in \mathbb{R}^+\}$ on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ for which a filtration $\{\mathfrak{F}_t : t \in \mathbb{R}^+\}$ is said to be $\{\mathfrak{F}_t\}$-adapted, or simply adapted, if X_t is an $\mathfrak{F}_t$-measurable mapping of Ω into $\mathbb{R}$ for every $t \in \mathbb{R}^+$.
Note that if $X = \{X_t : t \in \mathbb{R}^+\}$ is a stochastic process on a probability space $(\Omega, \mathfrak{F}, P)$ and if we let $\mathfrak{F}_t^X = \sigma\{X_s : s \in [0, t]\}$, then $\{\mathfrak{F}_t^X, t \in \mathbb{R}\}$ is a filtration of stochastic process. We call $\{\mathfrak{F}_t^X t \in \mathbb{R}\}$ the filtration generated by X.

Then, we get the notations for an integer $d \geq 1$, $\mathbb{R}^d$ denotes the d-dimensional Euclidean space, and $\mathfrak{B}(\mathbb{R}^d)$ will 64denote the Borel σ-fieldon $\mathbb{R}^d$. Further, $\mathcal{C}(\mathbb{R}^d)$ and $\mathcal{C}_b(\mathbb{R}^d)$ will denote the 65classes of continuous functions and bounded continuous functions on $\mathbb{R}^d$, respectively. When $d = 1$, we will write $\mathbb{R}$ in place of $\mathbb{R}^1$. $\mathbb{Q}$ will denote the set of rational numbers in $\mathbb{R}$.
$(\Omega, \mathfrak{F}, \mathbb{P})$ will denote a generic probability space, and $\mathfrak{B}(\Omega, \mathfrak{F})$ will denote the class of real-valued bounded $\mathfrak{F}$ measurable functions.
For a collection $A \subset \mathfrak{F}$, $\sigma(A)$ will denote the smallest σ-field which contains A and for a collection $G \subset \mathfrak{B}(\Omega, \mathfrak{F})$; $\sigma(G)$ will likewise denote the smallest

σ-field with respect to which each function in G is measurable. An $\mathbb{R}^d$-valued random variable X, on a probability space $(\Omega, \mathfrak{F}, \mathbf{P})$, is function from $(\Omega, \mathfrak{F})$ to $(\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d))$. For such an X and a function f, $\mathcal{C}^b(\mathbb{R}^d)$, $\mathbb{E}[f(X)]$ (or $\mathbb{E}_{\mathbb{P}}[f(X)]$ if there are more than one probability measure under consideration) will denote the integral

$$\mathbb{E}_{\mathbb{P}}\left[f(X)\right] = \int_{\Omega} f(X(w))d\mathbb{P}(w).$$

For any measure μ on $(\Omega, \mathfrak{F})$ and for $1 \leq p < \infty$, we will denote by $L^p(\mu)$ the space $L^p(\Omega, \mathfrak{F}, \mu)$ of real-valued $\mathfrak{F}$ measurable functions equipped with the norm

$$\|f\|_p = \left(\int_{\Omega} |f|^p d\mu\right)^{\frac{1}{p}}$$

It is well known that $L^p(\mu)$ is a Banach space under the norm $\|f\|_p$. For more details and discussions as well as proofs of statements quoted in this chapter, see Billingsley [4], Breiman [5], Ethier and Kurtz [18].

1.2 Martingale

In this section, we will fix a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ and a filtration $(\mathfrak{F}_t)_{t \geq 0}$. We will only be considering $\mathbb{R}$-valued processes in this section.

Definition 1.2.1. *M is said to be a $\mathfrak{F}_t$-martingale if M is $(\mathfrak{F}_t)$ adapted and integrable for all $t \geq 0$ and for all $0 \leq s \leq t$ one has*

$$\mathbb{E}\left(M_t | \mathfrak{F}_s\right) = M_s.$$

Definition 1.2.2. *M is said to be a $\mathfrak{F}_t$-submartingale if M is $(\mathfrak{F}_t)$ adapted and integrable for all $t \geq 0$ and for all $0 \leq s \leq t$ one has*

$$\mathbb{E}\left(M_t | \mathfrak{F}_s\right) \geq M_s.$$

Remark 1.2.1. *If M is a martingale and ϕ is a convex function on $\mathbb{R}$, then Jensen's inequality implies that the process $X = (X_t)$ defined by $X_t = \phi(M_t)$ is a submartingale provided X_t is integrable for all $t \geq 0$. If M is a submartingale and ϕ is an increasing convex function then X is also a submartingale provided X_t is integrable for each n. In particular, if M is a martingale or a positive submartingale with $\mathbb{E}\left[M_t^2\right] < +\infty$ for all $t \geq 0$, then Y defined by $Y_t = M_t^2$ is a submartingale.*

Definition 1.2.3. *M is said to be a $\mathfrak{F}_t$-supermartingale if M is $(\mathfrak{F}_t)$ adapted and integrable for all $t \geq 0$ and for all $0 \leq s \leq t$ one has*

$$\mathbb{E}\left(M_t | \mathfrak{F}_s\right) \leq M_s.$$

Example 1.2.1. *We say that a process $(Z_t)_{t\geq 0}$ with values in $\mathbb{R}^d$, $d \in \mathbb{N}^*$ has independent increments with respect to the filtration $\mathfrak{F}_t$ if Z is adapted and if, for every $0 \leq s < t$, $Z_t - Z_s$ is independent of $\mathfrak{F}_s$ (for instance, a Brownian motion has independent increments with respect to its canonical filtration). If Z is a real-valued process having independent increments with respect to $(\mathfrak{F}_t)$, then*

(i) *if $Z_t \in L^1$ for every $t \geq 0$, then $Y_t = Z_t - \mathbb{E}(Z_t)$ is a martingale.*

(ii) *if $Z_t \in L^2$ for every $t \geq 0$, then $V_t = Y_t^2 - \mathbb{E}(Y_t^2)$ is a martingale.*

(iii) *if, for some $\theta \in \mathbb{R}$, we have $\mathbb{E}(e^{\theta Z_t}) < \infty$ for every $t \geq 0$, then*

$$X_t = \frac{e^{\theta Z_t}}{\mathbb{E}(e^{\theta Z_t})}.$$

is a martingale.

Theorem 1.2.1. *Let $(M^m)_{m\geq 0}$ be a sequence of martingales on probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. Suppose that*

$$M_t^m \longrightarrow M_t \quad if \quad m \longrightarrow +\infty \quad in \quad L^1(\mathbb{P}) \quad \forall t \geq 0$$

Then M is also a $(\mathfrak{F}_t)$-martingale .

Theorem 1.2.2. *Let X be a submartingale. Let $A = (A_n)$ be defined by $A_0 = 0$ and for $n \geq 1$*

$$A_n = \sum_{k=1}^{n} \left[\mathbb{E}\left(X_k - X_{k-1}\right) | \mathfrak{F}_{k-1}\right]$$

Then A is an increasing process (i.e. $A_n \leq A_{n+1} for n \geq 0$) such that $A_0 = 0$, A_n is $\mathfrak{F}_{n-1}$ measurable for each n and $M = (M_n)$ defined by

$$M_n = X_n - A_n$$

is a martingale. Further, if $B = (B_n)$ is a process such that $B_0 = 0$, B_n is $\mathfrak{F}_{n-1}$ measurable for each n and $N = (N_n)$ defined by $N_n = X_n - B_n$ is a martingale, then

$$\mathbb{P}\left(A_n = B_n, \forall n \geq 1\right) = 1$$

Remark 1.2.2. *The uniqueness in the result above depends strongly on the assumption that B_n is $\mathfrak{F}_{n-1}$- measurable. The process A is called the compensator of the submartingale X. et $M = (M_n)$ be a martingale. The sequence $\mathcal{D}$ defined by $\mathcal{D}_n = M_n - M_{n-1}$, for $n \leq 1$ and $\mathcal{D}_0 = M_0$ clearly satisfies*

$$\mathbb{E}\left[\mathcal{D}_n | \mathfrak{F}_{n-1}\right] = 0, \quad \forall n \geq 1 \tag{1.2.1}$$

An adapted sequence $(\mathcal{D}_n)_{n\geq 0}$ satisfying (1.3.1) is called a martingale difference sequence.

Definition 1.2.4. *A sequence of random variables $V = (V_n)$ is said to be predictable if for all $n \geq 1$, V_n is $\mathfrak{F}_{n-1}$ measurable and V_0 is $\mathfrak{F}_0$ measurable. The compensator A appearing in the Doob decomposition of a submartingale M is predictable.*

Definition 1.2.5. *Let $M = (M_n)$ be a martingale and $U = (U_n)$ be a predictable sequence of random variables. The process $K = (K_n)$ defined by $K_0 = 0$ and for $n \leq 1$.*

$$K_n = \sum_{k=1}^{n} U_k (M_k - M_{k-1}). \tag{1.2.2}$$

is called the martingale transform of M by the sequence U.

The following theorem gives conditions under which the transformed sequence is a martingale.

Theorem 1.2.3. *Suppose $U = (U_n)$ is a sequence of random variables and $M = (M_n)$ is a martingale such that*

$$\mathbb{E}\left[U_n M_n\right] < \infty \quad \forall n \geq 1.$$

Then the martingale transform K defined by (1.3.2) is a martingale.

The proof given above essentially also yields the following:

Theorem 1.2.4. *Suppose $U = (U_n)$ is a sequence of random variables and $M = (M_n)$ is a submartingale such that $U_n \geq 0$ and*

$$\mathbb{E}\left[M_{n-1}U_n\right] < \infty \quad \mathbb{E}\left[M_n U_n\right] < \infty \quad \forall n \geq 1$$

Then the transform K defined by (1.3.2) is a submartingale.

Remark 1.2.3. *A process M such that $-M$ is a submartingale is called a supermartingale and the process which is both a sub and a supermartingale is a martingale. In other words, a martingale is an adapted family of integrable random variables such that*

$$\int_A M_s d\mathbb{P} = \int_A M_t d\mathbb{P}.$$

for every pair s, t with $s < t$ and $A \in \mathfrak{F}_s$.
A sub(super)martingale such that all the variables M_t are integrable is called an integrable sub(super)martingale.

Properties 1.2.5. *Let B be a standard linear BM; then the following processes are martingales with respect to $\sigma(B_s, s \leq t)$:*

i) B_t *it self.*

ii) $B_t^2 - t$.

iii) $M_t^\alpha = exp(\alpha B_t - \frac{\alpha^2}{2}t), \quad for \quad \alpha \in \mathbb{R}$.

1.2.1 Stopping Times

Definition 1.2.6. *A stopping time with respect to a filtration $(\mathfrak{F}_t)$ is a mapping τ from Ω into $[0, +\infty]$ such that for all $t < \infty$,*

$$\{\tau \leq t\} \in \mathfrak{F}_t.$$

Proposition 1.2.6. *If the filtration under consideration is fixed, we will only refer to it as a stopping time.*

1. *for a stopping time,* $\{\tau < t\} = \bigcup_n \{\tau < t - \frac{1}{n}\}$.

2. *For stopping times τ and ω, it is easy to see that $\tau \wedge \omega$ and $\tau \vee \omega$ are stopping times.*

3. *In particular, $\tau \wedge \omega$ and $\tau \vee \omega$ are stopping times for any $t \geq 0$.*

Remark 1.2.4. *Martingales and stopping times together are very important tools in the theory of stochastic process in general and stochastic calculus in particular.*

Proposition 1.2.7. *For the discrete case; let $(X_n), n = 0, 1, 2, ...$ be a submartingale with respect to a discrete filtration $(\mathfrak{F}_n)$ and $H_n, n = 1, 2, ...$ a positive bounded process such that $H_n \in \mathfrak{F}_{n-l}$ for $n > 1$; the process Y defined by*

$$Y_0 = X_0, Y_n = Y_{n-1} + H_n(X_n - X_{n-1}).$$

is a (sub)martingale. In particular, if T is a stopping time, the stopped process X^T is a (sub)martingale.

Proposition 1.2.8. *Let S and T are two bounded stopping times and $S < T$, i.e. there is a constant L such that for every ω,*

$$S(\omega) \leq T(\omega) \leq L < \infty$$

then

$$X_S \leq \mathbb{E}\left[X_T | \mathfrak{F}_S\right] \quad a.s.$$

with equality in case X is a martingale. Moreover, an adapted and integrable process X is a martingale if and only if

$$\mathbb{E}[X_S] = \mathbb{E}[X_T]$$

for any such pair of stopping times.

Theorem 1.2.9. *Let $M = (M_t)_{t \geq 0}$ be a submartingale and τ be a stopping time. Then the process $N = (N_t)_{t \geq 0}$ defined by*

$$N_t = M_{t \wedge \tau}$$

is a submartingale. Further, if M is a martingale then so is N.

The following version of this result is also useful.

Theorem 1.2.10. *Let $M = (M_t)_{t\geq 0}$ be a submartingale and τ, σ be stopping times such that $\sigma \leq \tau$. Let $R = (R_t)_{t\geq 0}$, $S = (S_t)_{t\geq 0}$ be defined as follows: $R_0 = S_0 = 0$ and for $t > 0$*

$$\begin{aligned} S_t &= M_n - M_{t\wedge\tau} \\ R_t &= M_{t\wedge\tau} - M_{\tau\wedge\sigma}. \end{aligned}$$

Then R, S are submartingales. Further, if M is a martingale then so are S, R.

Corollary 1.2.11. *Let $M = (M_t)_{t\geq 0}$ be a submartingale and τ, σ be stopping times such that $\sigma \leq \tau$. Then for all $t > 0$*

$$\mathbb{E}\left[M_{t\wedge\sigma}\right] \leq \mathbb{E}\left[M_{t\wedge\tau}\right].$$

However, we do have the following result.

Theorem 1.2.12. *Let $M = (M_t)_{t\geq 0}$ be an adapted process such that $\mathbb{E}[\sup\limits_{t\geq 0}|M_t|] < \infty$ for all $n \geq 0$. Then M is a martingale if and only if for all bounded stopping times τ,*

$$\mathbb{E}\left[M_\tau\right] = \mathbb{E}\left[M_0\right]. \tag{1.2.3}$$

Definition 1.2.7. The process X is continuous if all its paths $t \rightsquigarrow X_t(w)$ are continuous. The process X is cadlag if all its paths are right-continuous with left-hand limits. If X is cadlag, we define the left-continuous process X_{t^-} as

$$X_{t^-} := \begin{cases} X_0, & t = 0 \\ \lim_{s\uparrow t} X_s, & t > 0 \end{cases}$$

and the jumps as

$$\Delta X_t := X_t - X_{t^-}, \quad t \geq 0.$$

For $T > 0$ we define the system of sets

$$\mathcal{G}_T = \{A \times \{0\} : \quad A \in \mathcal{F}\} \cup \{A \times (s,t] : \quad 0 \leq s \leq t \leq T \quad A \in \mathcal{F}_s\}$$

and the predictable σ-algebra $\mathcal{P}_T = \sigma(\mathcal{G}_T)$. An $\mathcal{F}$-valued process $X = (X_t)_{t\in[0,T]}$ is called predictable if it is $\mathcal{P}_T$ -measurable.

Definition 1.2.8. For a stochastic process $X = (X_t)_{t\geq 0}$ and a stopping time T, the stopped process $X^T := (X_t^T)_{t\geq 0}$ is defined by

$$X_t^T := X_{T\wedge t} = \begin{cases} X_t, & \text{if} \quad t \leq T, \\ X_T, & \text{if} \quad t > T \end{cases}$$

Definition 1.2.9. An adapted, right-continuous stochastic process $X = (X_t)_{t\geq 0}$ is called a local martingale, if there exists a sequence of stopping times (T_n) with $T_n \to \infty$ P-a.s, such that the stopped process

$$X^{T_n} I_{T_n>0} = (X_{T_n \wedge t} I_{T_n>0})_{t\geq 0}$$

is a (uniformly integrable) martingale with respect to $(\mathcal{F}_t)$.

Expectations

Let X be a real (or complex) random variable defined on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. Two random variables X and Y are identified if

$$\mathbb{P}[m; X(\omega) \neq Y(\omega)] = 0.$$

X is called integrable if

$$\int_\Omega |X(\omega)| \mathbb{P}(d\omega) < \infty.$$

More generally, if

$$\int_\Omega |X(\omega)|^p \mathbb{P}(d\omega) < \infty \quad p > 0,$$

it is called p-th integrable. Let $p \geq 1$. The totality of p-th integrable random variables, denoted by $\mathfrak{L}_p(\Omega, \mathfrak{F}, \mathbb{P})$ or simply by $\mathfrak{L}_p(\Omega)$ or $\mathfrak{L}_p(\mathbb{P})$, forms a Banach space with the norm

$$\|X\|_p = \left(\int_\Omega |X(\omega)|^p \mathbb{P}(d\omega) \right)^{1/p}.$$

$\mathfrak{L}_\infty(\Omega, \mathfrak{F}, \mathbb{P})$ is the Banach space of essentially bounded random variables with the norm

$$\|X\|_p = ess.\sup |X(\omega)|.$$

Conditional expectations

For an integrable random variable X, $\mathbb{E}(X) = \displaystyle\int_\Omega X(\omega)\mathbb{P}(d\omega)$ is called the expectation of X. For a square-integrable random variable X,

$$V(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \mathbb{E}\left((X - \mathbb{E}(X))^2\right),$$

is called the variance of X.
A family $(\mathfrak{F}_\alpha)_{\alpha\in\Lambda}$ of sub σ-fields of $\mathfrak{F}$ is called mutually independent if for every distinct choice of $\alpha_1, \alpha_2, \ldots, \alpha_k \in \Lambda$ and $A_i \in \mathfrak{F}_{\alpha_i}, \quad i = 1, 2, \ldots, k$,

$$\mathbb{P}\left(A_1 \cap A_2 \cap \ldots \cap A_k\right) = \mathbb{P}(A_1).\mathbb{P}(A_2)\ldots\mathbb{P}(A_k)$$

A family of random variables $(X_\alpha)_{\alpha\in\Lambda}$ is called mutually independent if $(\sigma[X_\alpha])_{\alpha\in\Lambda}$ is mutually independent. A family of random variables $(X_\alpha)_{\alpha\in\Lambda}$ is called independent of a σ-field , $\mathfrak{G} \in \mathfrak{F}$, if $\sigma\left[X_\alpha; \alpha \in \Lambda\right]$ and $\mathfrak{G}$ are mutually independent.
Let X be an integrable random variable and $\mathfrak{G} \in \mathfrak{F}$ be a sub σ-field of $\mathfrak{F}$. Then

$$\mu(B) = \mathbb{E}(X : B) := \int_B X(\omega)\mathbb{P}(d\omega), \quad B \in \mathfrak{G}.$$

defines a σ-additive set function on $\mathfrak{G}$ with finite total variation and is clearly absolutely continuous with respect to $\lambda = \mathbb{P}|_{\mathfrak{G}}$. The Radon-Nikodym derivative $d\mu/d\lambda(\omega)$ is denoted by $\mathbb{E}\left(X|\mathfrak{G}\right)(\omega)$; thus $\mathbb{E}\left(X|\mathfrak{G}\right)$ is the unique (up to identification) $\mathfrak{G}$-measurable integrable random variable Y such that $\mathbb{E}(Y : B) = \mathbb{E}(X : B)$ for every $B \in \mathfrak{G}$.

Regular conditional probabilities

Definition 1.2.10. $\mathbb{E}\left(X|\mathfrak{G}\right)(\omega)$ *is called the conditional expectation of* X *given* $\mathfrak{G}$.
The following properties of conditional expectations are easily proved.
(X, Y, X_n, *below are integrable real random variables anda*, b*are real numbers.*)

(1) $\mathbb{E}(aX + bY|\mathfrak{G}) = a\mathbb{E}(X|\mathfrak{G}) + b\mathbb{E}(Y|\mathfrak{G}) \quad a.s.$

(2) *if* $X \geq 0 \quad a.s$ *then* $\mathbb{E}(X|\mathfrak{G}) \quad a.s.$

(3) $\mathbb{E}(1|\mathfrak{G}) = 1 \quad a.s.$

(4) *If* X *is* $\mathfrak{G}$*-measurable, then* $\mathbb{E}(X|\mathfrak{G}) = X \quad a.s.$*, more generally, if* XY *is integrable and* X *is* $\mathfrak{G}$*-measurable*

$$\mathbb{E}\left(XY|\mathfrak{G}\right) = X\mathbb{E}\left(Y|\mathfrak{G}\right) \quad a.s.$$

(5) *if* Θ *is sub* σ*-field of* $\mathfrak{G}$*, then*

$$\mathbb{E}\left(\mathbb{E}(X|\mathfrak{G})|\Theta)\right) = \mathbb{E}\left(X|\Theta\right) \quad a.s.$$

(6) *if* $X_n \longrightarrow X$ *in* $\mathfrak{L}_1(\Omega)$*, then* $\mathbb{E}(X_n|\mathfrak{G}) \longrightarrow \mathbb{E}(X|\mathfrak{G})$ *in* $\mathfrak{L}_1(\Omega)$*.*

(7) *(Jones's inequality) if* $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ *is convex and* $\varphi(X)$ *integrable then*

$$\varphi\left(\mathbb{E}(X|\mathfrak{G})\right) \leq \mathbb{E}\left(\varphi(X)|\mathfrak{G}\right) \quad a.s.$$

In particular, $|\mathbb{E}(X|\mathbb{G})| \leq \mathbb{E}(|X||\mathfrak{G})$ *and if* X *is a square-integrable,* $|\mathbb{E}(X|\mathbb{G})|^2 \leq \mathbb{E}(|X|^2|\mathfrak{G})$*.*

(8) X *is independent of* $\mathfrak{G}$ *if and only if for every Borel measurable function* f *such that* $f(X)$ *is integrable,* $\mathbb{E}\left(f(X)|\mathfrak{G}\right) = \mathbb{E}\left(f(X)\right) \quad a.s.$

Definition 1.2.11. *Let* $(\Omega, \mathfrak{F}, \mathbb{P})$ *be a probability space and* $\mathfrak{G}$ *be a* σ*-field of* $\mathfrak{F}$*. A system* $(p(\omega, B))_{\omega\in\Omega, B\in\mathfrak{F}}$ *is called a regular conditional probability given* $\mathfrak{G}$ *if it satisfies the following conditions:*

1) *for fixed* $(\omega, B) \longmapsto p(\omega, B)$ *is a probability in* $(\Omega, \mathfrak{F})$*;*

2) *for fixed* $B \in \mathfrak{F}$*,* $\omega \longmapsto p(\omega, B)$ *is* $\mathfrak{G}$*-measurable;*

3) *for* $B \in \mathfrak{F}$ *and* $A \in \mathfrak{G}$*,*

$$\mathbb{P}(A \cap B) = \int_A p(\omega, B)\mathbb{P}(d\omega),$$

1.3 Martingale inequality

In this section, we study the regularity properties of martingales and supermartingales paths. We, first, establish continuous-time analogues of classical inequalities in the framework of discrete time.

1.3.1 Doobs Maximal Inequality

We will now derive an inequality for martingales known as Doob's maximal inequality. It plays a major role in stochastic calculus as we will see later.

Theorem 1.3.1. *Let M be a martingale or a positive submartingale. Then, for $\lambda > 0$, $n > 1$ one has*

$$\mathbb{P}(\max_{1\leq k\leq n}|M_k| > \lambda) \leq \frac{1}{\lambda}\mathbb{E}[|M_n|\mathbb{k}1_{\{\max\limits_{1\leq k\leq n}|M_k| > \lambda\}}]$$

Further, for $1 < p < +\infty$, there exists a universal constant C_p depending only on p such that

$$\mathbb{E}[(\max_{1\leq k\leq n}|M_k|)^p] \leq C_p\mathbb{E}(|M_n|)^p)$$

Where $C_p = (\frac{p}{p-1})^p$

1.3.2 Martingale Convergence Theorem

Martingale convergence theorem is one of the main results on martingales.We begin this section with an upcrossings inequality (a key step in its proof). Let $(a_n)_{1\leq n\leq m}$ be a sequence of real numbers and $\alpha < \beta$ be real numbers. Let s_k, t_k be defined (inductively) as follows: $s_0 = 0$, $t_0 = 0$, and for $k = 1, 2, ..., m$

$$s_k = \inf\{n > t_{k-1} : a_n < \alpha\}, \quad t_k = \inf\{n > s_k : a_n < \beta\}$$

. Recall our convention-infimum of an empty set is taken to be ∞. It is easy to see that if $t_k = j < \infty$, then

$$0 \leq s_0 < t_0 < ... < s_k < t_k < \infty. \tag{1.3.1}$$

and for $i \leq j$

$$a_{s_i} \leq \alpha, \quad a_{t_i} \geq \beta. \tag{1.3.2}$$

We define

$$U_k(\{a_j : 1 \leq j \leq k\}, \alpha, \beta) = \max\{i : t_i \leq k\}$$

$U_k(\{a_j : 1 \leq j \leq k\}, \alpha, \beta)$ is the number of upcrossings of the interval (α, β) by the sequence $U_k(\{a_j : 1 \leq j \leq k\}, \alpha, \beta)$. Eqs.(1.5.1) and (1.5.2) together also imply $t_i > (2i - 1)$ and also writing $c_j = \max(\alpha, a_j)$ that

$$\sum_{j=1}^{k}(c_{t_j\wedge k} - c_{s_j\wedge k}) \geq (\beta - \alpha)U_k(\{a_j\}, \alpha, \beta) \tag{1.3.3}$$

This inequality follows because each completed upcrossings contributes at least $\beta - \alpha$ to the sum, one term could be non-negative and rest of the terms are zero.

Lemma 1.3.2. *For a sequence $(a_n)_{n\geq 1}$ of real numbers,*

$$\liminf_{n\longrightarrow\infty} \quad a_n = \limsup_{n\longrightarrow\infty} \quad a_n \tag{1.3.4}$$

if and only if

$$\lim_{k\longrightarrow\infty} U_k(\{a_j : 1 \leq j \leq k\}, \alpha, \beta) < \infty, \quad \forall \alpha \leq \beta \quad \alpha, \beta \quad rationals. \tag{1.3.5}$$

It follows that if (1.5.5) holds, then $\lim\limits_{n\longrightarrow\infty} a_n$ exists in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. The next result gives an estimate on expected value of

$$U_k(\{X_j : 1 \leq j \leq k\}, \alpha, \beta)$$

for a submartingale X.

Theorem 1.3.3. *(Doob's submartingale inequality). Let X be a submartingale. Then for $\alpha < \beta$*

$$\mathbb{E}[U_k(\{a_j : 1 \leq j \leq k\}, \alpha, \beta)] \leq \frac{\mathbb{E}(|X_k| + |\alpha|)}{\beta - \alpha}. \tag{1.3.6}$$

We recall the notion of uniform integrability of a class of random variables and related results.
A collection $\{Z_\alpha : \alpha \in \Delta\}$ of random variables is said to be uniformly integrable if

$$\lim_{n\longrightarrow\infty} (\sup_{\alpha\in\Delta} \mathbb{E}[|Z_\alpha| 1_{\{|Z_\alpha| \geq n\}}]) = 0. \tag{1.3.7}$$

Lemma 1.3.4. *Let $X \in L^1(P)$ and $(X_n)_{n\geq 1}$ be a sequence of element in $L^1(P)$.*

1. *$(X_n)_{n\geq 1}$ converges in $L^1(P)$ to X if and only if it converges to X in probability and (X_n) is uniformly integrable.*

2. *Let $\{G_\alpha : \alpha \in \Delta\}$ be a collection of sub-σ-fields of $\mathfrak{F}$. Then $\{\mathbb{E}[Z|G_\alpha] : \alpha \in \Delta\}$ is uniformly integrable.*

Theorem 1.3.5. *(Martingale Convergence Theorem). Let $(X_n)_{n\geq 0}$ be a sub-martingale such that*

$$\sup_n \quad \mathbb{E}(|X_n|) = K_1 < \infty. \tag{1.3.8}$$

Then the sequence of random variables $(X_n)_{n\geq 0}$ converges a.e. to a random variable ξ with $\mathbb{E}[|\xi|] < \infty$. Further if $(X_n)_{n\geq 0}$ is uniformly integrable, then X_n converges to ξ in $L^1(\mathbb{P})$. If $(X_n)_{n\geq 0}$ is a martingale or a positive sub-martingale and if for some p, $1 < p < +\infty$

$$\sup_n \quad \mathbb{E}(|X_n|^p) = K_p < \infty. \tag{1.3.9}$$

then X_n converges to ξ in $L^p(\mathbb{P})$.

Theorem 1.3.6. *Let $(\mathfrak{F}_n)$ be an increasing family of sub-σ-fields of $\mathfrak{F}$, and $\mathfrak{F}_\infty = \sigma(\bigcup_{n=1}^{\infty}\mathfrak{F}_n)$. Let $Z \in L^1(\mathbb{P})$ and for $1 \leq n < \infty$, where*

$$Z_n = \mathbb{E}[Z/\mathfrak{F}_n].$$

Then

$$Z_n \longrightarrow Z^* = \mathbb{E}[Z|F_\infty] \in L^1(\mathbb{P}) \quad \textit{and} \quad \textit{a.s.}$$

The previous result has an analogue when the ω-fields are decreasing. Usually one introduces a reverse martingale (martingale indexed by negative integers) to prove this result.

Theorem 1.3.7. *Let $(\mathfrak{G}_n)$ be a decreasing family of sub-σ-fields of $\mathfrak{G}$, i.e. $\mathfrak{G}_m \supseteq \mathfrak{G}_{m+1}$ $\forall m \geq 1$. Let $\mathfrak{G}_\infty = \bigcap_{n=1}^{\infty}\mathfrak{G}_n)$. Let $Y_0 \in L^1(\mathbb{P})$, and for $1 \leq n < \infty$,*
and let

$$Y_n = \mathbb{E}[Y_0|\mathfrak{G}_n].$$

Then

$$Y_n \longrightarrow Y \in L^1(\mathbb{P}) \quad \textit{and} \quad \textit{a.s.}$$

1.3.3 Stochastic Integrators

Let us fix an r.c.l.l. (F_t) adapted stochastic process X. Recall, $\mathbb{S}$ consists of the class of processes f of the form

$$f(s) \quad = \quad a_0.1_{\{0\}}(s) + \sum_{k=0}^{n} a_{k+1}.1_{(a_k, a_{k+1}]}(s). \tag{1.3.10}$$

where $0 = s_0 < s_1 < s_2 < \ldots < s_{n+1} < \infty$ a_j is bounded $\mathfrak{F}_{s_{j-1}}$ measurable random variable, $1 \leq j \leq (n+1)$, and a_0 is bounded $\mathfrak{F}_0$ measurable. For simple process $f \in \mathbb{S}$ given by (1.1.1), let $J_X(f)$ be the r.c.l.l. process defined by

$$J_X(f)(t) \quad = \quad a_0 X_0(t) + \sum_{k=0}^{n} a_{k+1}(X_{s_{k+1}\wedge t} - X_{s_k \wedge t}). \tag{1.3.11}$$

One needs to verify that J_X is unambiguously defined on $\mathbb{S}$. That is, if a given f has two representations of type (1.1.1), then the corresponding expressions in (1.1.2) agree, as well as linearity of $J_X(f)$ for $f \in \mathbb{S}$ can be verified using elementary algebra. By definition, for $f \in \mathbb{S}$, $J_X(f)$ is an r.c.l.l.adapted process. In analogy with the Itô's integral with respect to Brownian motion discussed in the earlier chapter, we wish to explore if we can extend J_X to the smallest bp-closed class of integrands that contain $\mathbb{S}$. Each $f \in \mathbb{S}$ can be viewed as a real-valued function on $\tilde{\Omega} = [0, +\infty) \times \Omega$. Since $\mathcal{P}$ is the σ-field generated by $\mathbb{S}$, by theorem 2.66, the smallest class of functions that contains $\mathbb{S}$ and is closed under *bp*-convergence is $\mathbb{B}(\tilde{\Omega}, \mathcal{P})$. When the space, filtration and the probability measures are clear from the context, we will write the class of adapted r.c.l.l. processes $\mathbb{R}^0(\Omega, (\mathfrak{F}_t), \mathbb{P})$ simply as $\mathbb{R}^0$.

Definition 1.3.1. *An r.c.l.l. adapted process X is said to be a stochastic integrator if the mapping J_X from $\mathbb{S}$ to $\mathbb{R}^0(\Omega, (\mathfrak{F}_t), \mathbb{P})$) has an extension $J_X : \mathbb{B}(\tilde{\Omega}, \mathcal{P}) \longrightarrow \mathbb{R}^0(\Omega, (\mathfrak{F}_t), \mathbb{P})$ satisfying the following continuity property:*

$$f^n \longrightarrow f \quad a.s.pb \quad \Longrightarrow \quad J_X(f^n) \longrightarrow J_X(f) \quad a.s.ucp. \tag{1.3.12}$$

It should be noted that for a given r.c.l.l. process X, J_X may not be continuous on $\mathbb{S}$.

1.3.4 Square Integrable Martingales

The main aim of this section is to show that square integrable martingales are stochastic integrators.
The treatment is essentially classical, as in Kunita Watanabe [73], but with an exception. The role of $\langle M, M\rangle$ the predictable quadratic variation in the Kunita Watanabe treatment is here played by the quadratic variation $[M, M]$.
Recall that $\mathbb{M}^2$ denotes the class of r.c.l.l. martingales M such that $\mathbb{E}[M_t^2] < \infty$ for all $t < \infty$ with $M_0 = 0$.

$$\mathbb{E}\left(|M_n|^2\right) < \infty \quad \forall n \geq 0. \tag{1.3.13}$$

are called square integrable martingales, and they play a special role in the theory of stochastic integration as we will see later. Let us note that for $p = 2$, the constant C_p appearing in (1.7.2) equals 4. Thus for a square integrable martingale M, we have

$$\mathbb{E}(\max_{1\leq k\leq n} |M_k|^2) \leq 4\mathbb{E}(|M_n|^2) \tag{1.3.14}$$

As seen earlier, $X_n = M_n^2$ is a submartingale and the compensator of X namely the predictable increasing process A such that X_n. A_n is a martingale, is given by $A_0 = 0$ and for $n \geq 1$,

$$A_n = \sum_{j=1}^{n} \mathbb{E}\left[(X_j - X_{j-1})/\mathfrak{F}_{j-1}\right]$$

The compensator A is denoted as $\langle M, M\rangle$. Using

$$\begin{aligned} \mathbb{E}\left[(M_j - M_{j-1})^2|\mathfrak{F}_{j-1}\right] &= \mathbb{E}\left[(M_j^2 - 2M_jM_{j-1} + M_{j-1}^2)|\mathfrak{F}_{j-1}\right] && (1.3.15)\\ &= \mathbb{E}\left[\left(M_j^2 - M_{j-1}^2\right)|\mathfrak{F}_{j-1}\right] && (1.3.16)\end{aligned}$$

it follows that the compensator can be described as

$$\langle M, M\rangle_n = \sum_{j=1}^{n} \mathbb{E}\left[(X_j - X_{j-1})\,|\mathfrak{F}_{j-1}\right] \tag{1.3.17}$$

Thus $\langle M, M\rangle_n$ is the unique predictable increasing process with $\langle M, M\rangle_0 = 0$ such that $M_n^2 - \langle M, M\rangle_n$ is a martingale. Let us also define another increasing process $[M,M]$ associated with a martingale $M : [M,M]_0 = 0$ and

$$[M,M]_n = \sum_{j=1}^{n} (M_j - M_{j-1})^2 \tag{1.3.18}$$

The process $[M,M]$ is called the quadratic variation of M, and the process $\langle M,M\rangle$ is called the predictable quadratic variation of M. It can be easily checked that

$$M_n^2 - [M,M]_n = M_0^2 + 2\sum_{j=1}^{n} M_{j-1}(M_j - M_{j-1}) \tag{1.3.19}$$

and hence using theorem 1.15 it follows that $M_n^2 - [M,M]_n$ is also a martingale. If $M_0 = 0$, then it follows that

$$\mathbb{E}\left[M_n^2\right] = \mathbb{E}\left[\langle M,M\rangle_n\right] = \mathbb{E}\left[[M,M]_n\right] \tag{1.3.20}$$

We have already seen that if U is a bounded predictable sequence, then the transform Y defined by (1.5.2) is itself a martingale. The next result includes an estimate on the $L^2(P)$ norm of Z_n.

Proposition 1.3.8. *Let $M = (M_t)_{t\geq 0}$ be a supermartingale with right-continuous. Then, for every $t > 0$ and every $\alpha > 0$,*

$$\alpha\mathbb{P}(\sup_{0\leq s<t} |M_s| > \alpha) \leq \mathbb{E}\left[|M_0|\right] + 2\mathbb{E}\left[|M_t|\right]$$

1.3.5 Doobs inequality in L^p

Proposition 1.3.9. *Let $M = (M_t)_{t\geq 0}$ be a rmartingale with right-continuous and $\mathbb{E}[|M_t|^p] < \infty$, for $p > 1$. Then, for every $t > 0$, we have:*

$$\mathbb{E}\left(\sup_{0\leq s<t} |M_s|\right) \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[|M_t|^p\right]$$

The next result is known as the Burkholder-Davis-Gundy inequalities. It was first proved for discrete martingales and $p > 0$ by Burkholder [16] in 1966. In 1968, Millar [50] extended the result to continuous martingales. In 1970, Davis [20] extended the result for discrete martingales to $p = 1$. The extension to $p > 0$ was obtained independently by Burkholder and Gundy [17] in 1970 and Novikov [56] in 1971.

Theorem 1.3.10. *[66] For each* $p > 0$ *there exist constants* $c_p, C_p \in (0, \infty)$*, such that for any progressive process* x *with the property that for some* $t \in [0, \infty)$*,*$\int_0^t X_s^2 ds < \infty$ *a.s., we have*

$$c_p \mathbb{E} \left(\int_0^t X_s^2 ds \right)^{\frac{p}{2}} \leq \mathbb{E} \left(\sup_{s \in [0,t]} \int_0^t X_s^2 dW(s) \right)^p \leq C_p \mathbb{E} \left(\int_0^t X_s^2 ds \right)^{\frac{p}{2}}. \tag{1.3.21}$$

1.3.6 Burkholder-Davis-Gundy Martingale inequality

For an L^2-bounded continuous martingale M vanishing at zero, the norme $\|M_\infty^*\|_2$ and $\|\langle M, M\rangle_\infty^{1/2}\|_2$ are equivalent. We now use the Ito formula to generalize this to other L^P-norms. We recall that if M is a continuous local martingale, we write

$$M_t^* = \sup_{s \leq t} |M_s|$$

.

Theorem 1.3.11. $\forall p \in]0, +\infty[$*, there exist two constants* α_p *and* β_p *such that, for all continuous local martingales* M *vanishing at zero,*

$$\alpha_p \mathbb{E}\left[\langle M, M\rangle_\infty^{p/2}\right] \leq \mathbb{E}\left[(M_\infty^*)^p\right] \leq \beta_p \mathbb{E}\left[\langle M, M\rangle_\infty^{p/2}\right]. \tag{1.3.22}$$

It is customary to say that the constants α_p and β_p are "universal" because they can be taken the same for all local martingales on any probability space whatsoever. If we call $\mathbb{H}^P$ the space of continuous local martingales such that M_∞^* is in L^p, the paste theorem gives us two equivalent norms on this space. For $p \geq 1$, the elements of $\mathbb{H}^P$ are true martingales and, for $p > 1$, the spaces $\mathbb{H}^P$ are the spaces of continuous martingales bounded in L^P; this is not true for $p = 1$, the space $\mathbb{H}^1$ is smaller than the space of continuous L^1-bounded martingales and even of uniformly integrable martingales.
Let us also observe that, by stopping, the theorem has the obvious, but nonetheless important

Corollary 1.3.12. *For any stopping time* T

$$\alpha_p \mathbb{E}\left[\langle M, M\rangle_T^{p/2}\right] \leq \mathbb{E}\left[(M_T^*)^p\right] \leq \beta_p \mathbb{E}\left[\langle M, M\rangle_T^{p/2}\right]. \tag{1.3.23}$$

More generally, for any bounded predictable process H

$$\begin{aligned}\alpha_p \mathbb{E}\left[\left(\int_0^T d\langle M,M\rangle_s\right)^{p/2}\right] &\leq \mathbb{E}\left[|\int_0^T H_s dM_s|^p\right] \\ &\leq \beta_p \mathbb{E}[(\int_0^T d\langle M,M\rangle_s)^{p/2}]\end{aligned}$$

Proposition 1.3.13. *For $p \geq 2$, there exists a constant λ_p such that for any continuous local martingale M such that $M_0 = 0$,*

$$\mathbb{E}\left[(M_\infty^*)^p\right] \leq \lambda_p \mathbb{E}\left[\langle M,M\rangle_\infty^{p/2}\right]. \tag{1.3.24}$$

Proposition 1.3.14. *For $p \geq 4$, there exists a constant γ_p such that,*

$$\gamma_p \mathbb{E}\left[\langle M,M\rangle_\infty^{p/2}\right] \leq \mathbb{E}\left[(M_\infty^*)^p\right]. \tag{1.3.25}$$

1.3.7 Domination relation

Definition 1.3.2. *A positive, adapted right-continuous process X is dominated by an increasing process A, if*

$$\mathbb{E}\left(X_\tau/\mathfrak{F}_0\right) \leq \mathbb{E}\left(A_\tau/\mathfrak{F}_0\right).$$

for any bounded stopping time τ.

Lemma 1.3.15. *If X is dominated by A and A is continuous, for $x, y \in]0,+\infty[$,*

$$\mathbb{P}[X_\infty^* > x; A_\infty \leq Y] \leq \frac{1}{x}\mathbb{E}\left(A_\infty \wedge Y\right).$$

where $X^* = \sup_s X_s$.

Proposition 1.3.16. *Under the hypothesis of Lemma (4.6),for any $\alpha \in]0,1[$,*

$$\mathbb{E}\left[(X_\infty^*)^\alpha\right] \leq \left(\frac{2-\alpha}{1-\alpha}\right)\mathbb{E}\left[A_\infty^\alpha\right].$$

To finish the proof of Theorem (4.1), it is now enough to use the above result with $X = (M^*)^2$ and $A = C_2\langle M,M\rangle$ for the right-hand side inequality, $X = \langle M,M\rangle^2$ and $A = C_4(M^*)^4$ for the left-hand side inequality. The necessary domination relations follow from Propositions (4.3) and (4.4), by stopping as in Corollary (4.2).

Definition 1.3.3. *Let ψ be a positive real function defined on $]0,\alpha]$, such that $\lim_{x\longrightarrow+\infty}\psi(x)=0$ and β a real number >1. An ordered pair (X,Y) of positive random variables is said to satisfy the "good λ inequality $I(\psi,\beta)$" if*

$$\mathbb{P}[X\leq\beta\lambda;Y<\xi\lambda]\leq\psi(\xi)P(X\geq\lambda)$$

for every $\lambda>0$ and $\xi\in]0,\alpha]$. We will write $(X,Y)\in I(\psi,\beta)$.

Let F be a moderate function, that is, an increasing, continuous function vanishing at 0 and such that

$$\sup_{x>0}F(\alpha x)/F(x)=\eta<\infty,\quad\forall\alpha>1.$$

Lemma 1.3.17. *There is a constant c depending only on ψ,β and η such that if $(X,Y)\in I(\psi,\beta)$, then*

$$\mathbb{E}\left[F(X)\right]\leq c\mathbb{E}\left[F(Y)\right].$$

The foregoing lemma may be put to use to prove Theorem (4.1). We will presently use it for a result on BM. We consider the canonical BM with the probability measures $P_x,\quad x\in\mathbb{R}$, and translation operators $\pi_t,\quad t\geq 0$. We denote by $(\mathfrak{F}_t)$ the Brownian filtration. Then, we have the theorem

Theorem 1.3.18. *Let $X_t,t\geq 0$, be an $\mathfrak{F}_t$-adapted. continuous, increasing process such that*

1. $\lim_{n\longrightarrow+\infty}\sup_{x,\alpha}\mathbb{P}_x\left[X_{\alpha^2}>n\alpha\right]=0.$

2. *there is a constant a such that for every s and t*

$$X_{t+s}-X_s\leq aX_t\circ\pi_s.$$

Then, there exists a constant c_F such that for any stopping time τ,

$$\mathbb{E}_0\left[F\left(X_\tau\right)\right]\leq c_F\mathbb{E}_0\left[F(\tau^{1/2})\right].$$

We may likewise obtain the reverse inequality.

Theorem 1.3.19. *If $X_t,t\geq 0$, is an $\mathfrak{F}_t$-adapted. continuous, increasing process such that*

1. $\lim_{n \longrightarrow +\infty} \sup_{x,\alpha} \mathbb{P}_x \left[X_{\alpha^2} < n\alpha \right] = 0$

2. *there is a constant* a *such that for every* s *and* t

$$X_{t-s} \circ \pi_s \leq a X_t.$$

Then, there exists a constant C_F *such that for any stopping time* τ,

$$\mathbb{E}_0 \left[F(\tau^{1/2}) \right] \leq C_F \mathbb{E}_0 \left[F(X_\tau) \right].$$

Chapter 2

Stochastic integral

In this chapter we consider processes X that are integrators: i.e.

$$J_X(f)(t) = \int_0^t f dX$$

can be defined for a suitable class of integrands f and the integral has some natural continuity properties. We will call such a process a stochastic integrator. In this chapter, we will prove basic properties of the stochastic integral $\int_0^t f dX$ for a stochastic integrator X.
We need to consider the corresponding right continuous filtration

$$(\mathfrak{F}_.^+) = \{(\mathfrak{F}_t^+); t \geq 0\}$$

where

$$(\mathfrak{F}_t^+) = \bigcap_{s<t} \mathfrak{F}_s.$$

2.1 Brownian Motion

In 1828 Robert Brown, observed that pollent grains suspended in water perform an unending chaotic motion .L.Bachelier (1900) derived the law governing the position W_t at time t of single grain performing a one -dimensional Brownain motion starting at $a \in \mathbb{R}^+$ at time $t = 0$,

$$\mathbb{P}_a\{W_t \in dx\} = p(t, a, x)dx, \tag{2.1.1}$$

where

$$p(t,a,x)=\frac{1}{\sqrt{2\pi t}}e^{-(x-a)^2/2t}$$

is the fundamental solution of the heat equation

$$\frac{\partial u}{\partial t}=\frac{1}{2}\frac{\partial^2 u}{\partial a^2}.$$

Bachelier (1900) also pointed out the Markovian nature of the Brownain path and used it to establish the of maximum displacement

$$\mathbb{P}_a\{\max_{s\leq t} W_t\leq b\}=\frac{2}{\sqrt{2\pi t}}\int_0^b e^{\frac{-x^2}{2t}}dx,\quad t>0,b\geq 0.$$

Einstein (1905) also derived (2.1.1) from statistical machanis considerations and applied it to the determination of molecular diameters .Bachelier was unable to obtain a clear picture of the Brownain motion ,and has ideas were unappreciated at the time .This is not surprising ,because the precise mathematical definition of the Brownain motion involves a measure on the path space ,and even after the ideas of Borel, Lebesgue,and Daniell appeared ,N.Winer (1923) only constructed a Daniell integral on the path space which later was revealed to be the Lebesgue integral against a measure the so - called Wiener measure.

The simplest model describing movement of a particle subject to hists by much smaller particles is the following .Let η_k,$k=1,2,...$ be independent identically distributed random variables which $E\eta_k=0$ and $E\eta_k^2=1$.Fix an integer n, and at times $1/n,2/n,\ldots$ let our particle experience instant displacements by $\eta_1 n^{\frac{-1}{2}},\eta_1 n^{\frac{-1}{2}},....$ At moment zero let our particle be at zero. If

$$S_k=\eta_1+\eta_2+\ldots+\eta_k,$$

then at moment k/n our particle will be at the point $S_k/\sqrt{n}$ and will stay there during the time interval $[k/n,(k+1)/n)$ .Since real Brownain motion has continues paths ,we replace our piecewise constant trajectory by continuous piecewise linear one preserving its positions at times k/n .Thus we come to the process,

$$\xi_t^n=S_{[nt]}/\sqrt{n}+(nt-[nt])\eta_{[nt]+}/\sqrt{n}. \tag{2.1.2}$$

This process gives a very rough caricature of Brownain motion .Clearly ,to get a better model we have to let $n\to\infty$.By the way , precisely this necessity dictates the intervals of time between collisions to be $1/n$ and the displacements due to collisions to be $\eta_k/\sqrt{n}$,since then ξ_t^n is asymptotically normal with parameters $(0, 1)$

Definition 2.1.1. The standard Brownian motion is a stochastic process $(W_t)_{t\in\mathbb{R}^+}$ such that

(i) $W_0 = 0$ almost surely.

(ii) The sample trajectories $t \to W_t$ are continuous, with probability 1.

(iii) For any finite sequence of times $t_0 < t_1 < \ldots < t_n$, the increments

$$W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$$

are independent.

(iv) For any given times $0 \le s \le t$, $W_t - W_s$ has the Gaussian distribution $\mathcal{N}(0, t-s)$ with mean zero and variance $t - s$.

Theorem 2.1.1. ***(Bachelier)****. For every $t \in (0, 1]$ we have $\max_{s\le t} W_s \sim |W_t|$, which is to say that for every $x \ge 0$*

$$\mathbb{P}(\max_{s\le t} W_s \le x) = \frac{2}{\sqrt{2\pi t}} \int_0^x e^{-y^2/2t} dy.$$

The continuity question of Brownian motion can be answered by using another famous theorem of Kolmogorov:

Theorem 2.1.2. *Suppose that the process ξ_t^n satisfies the following condition: For all $T > 0$ there exist positive constants α, β, N such that*

$$\mathbb{E}|\xi_t^n - \xi_s^n|^\alpha \le N|t-s|^{1+\beta} \quad \forall s, t \in [0, 1].$$

Then there exists a continuous version of ξ_t^n.

2.1.1 Functions of Bounded Variation

Definition 2.1.2. *The variation of a function $f : [0, T] \longrightarrow \mathbb{R}$ is defined to be*

$$\limsup_{\Delta t\to 0} \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|,$$

where $t = (t_0, t_1, \ldots, t_n)$ *is a partition of* $[0, T]$*, i.e.* $0 = t_0 < t_1 < \ldots < t_n = T$*,and where*

$$\Delta t = \max_{i=0,1,\ldots,n-1} |t_{i+1} - t_i|,$$

is bounded variation of f *on* $[0, T]$ *is defined to be* $V(f, [0, T])$*.*

Example 2.1.1. *If* f *is constant on* $[0, T]$ *then* f *is of bounded variation on* $[0, T]$*.*
Consider the constant function $f(x) = c$ *on* $[0, T]$*. Notice that*

$$\limsup_{\Delta t \to 0} \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|,$$

is zero for every partition of $[0, T]$*. Thus* $V(f; [0, T])$ *is zero.*

Another example of a function of bounded variation is a monotone function on $[0, T]$.

Theorem 2.1.3. *If* f *is increasing on* $[0, T]$*, then* f *is of bounded variation on* $[0, T]$ *and* $V(f, [0, T]) = f(T) - f(0)$*.*

Similarly, if f is decreasing on $[0, T]$ then $V(f, [0, T]) = f(0) - f(T)$. For the next example we first recall a theorem involving rational and irrational numbers.

Theorem 2.1.4. *Let* $f : [a, b] \to \mathbb{R}$ *be a function and let* $c \in (a, b)$*. If* f *is of bounded variation on* $[a, c]$ *and* $[c, b]$*, then* f *is of bounded variation on* $[a, b]$ *and* $V(f, [a, b]) = V(f, [a, c]) + V(f, [c, b])$

A proof of Theorem 2.1.4 is provided in Gordon's text [**?**].

Theorem 2.1.5. *The variation of the paths of* $W(t)$ *is infinite a.s.*

Lemma 2.1.6. *Let* $s \leq t$*.*

$$cov(W_s, W_t) = t \wedge s.$$

Lemma 2.1.7. *Let* W *Winer process is a d-dimensional and two constants* $\lambda > 0$ *and* $r > 0$*.*

1. *(self-similar) the process* $X_t = \frac{1}{\sqrt{\lambda}} W_{\lambda t}$ *is a Winer process.*

2. *the process* $Y_t = W_{t+s} - W_s$ *is a Winer process.*

Lemma 2.1.8. *For every* $0 \leq t_1 < t_2 < t_3 \ldots < t_k \leq 1$ *the vectors* $(\xi^n_{t_1}, \xi^n_{t_2} \ldots \xi^n_{t_k})$ *are asymptotically normal with parameters* $(0, t_i \wedge t_j)$.

Theorem 2.1.9. *(Quadratic Variation) Let* $0 = t_{0n} \leq t_{1n} \leq \ldots \leq t_{k_n n} = 1$ *be a sequence of partitions of* $[0,1]$ *such that* $\max_i(t_{i+1,n} - t_{i,n}) \to 0$ *a.s* $n \to \infty$ *Also let* $0 \leq t \leq 1$*. Then, in probability as n to* ∞

$$\sum_{s \leq t_{i,n} \leq t_{i+1,n} \leq t} (W_{t_{i+1,n}} - W_{t_{i,n}})^2 \to t - s. \tag{2.1.3}$$

Definition 2.1.3. *(Riemann-Stieltjes Integrals) Let* $V, f : \mathbb{R}^+ \to \mathbb{R}$ *and* $0 = t_0 < t_1 < t_2 < \ldots < t_n = T$ *is a partition of* $[0,T]$.

1. *A sum of the form*

$$\sum_{i=1}^{n} f(s_i^n)(V(t_i^n) - V(t_{i-1}^n)).$$

   is called a Riemann-Stieltjes sum of f *with respect to* V.

2. *A function* f *is Riemann-Stieltjes Integrable with respect to* V *on* $[a,b]$ *such that*

$$\int_0^t f(s)dV = \lim_{n\to\infty} \sum_{i=1}^{n} f(s_i^n)(V(t_i^n) - V(t_{i-1}^n)).$$

2.1.2 Itô Stochastic Integrals

Assume $W(t)$ is an infinite sequence of independent standard Brownian motions, defined on $(\Omega, \mathcal{F}, \mathbb{P})$ hat is, $W(t) = (W^1(t), W^2(t), \ldots)^T$. An $\mathbb{R}$-valued random variable is an $\mathcal{F}$-measurable function $x(t) : \Omega \to \mathbb{R}$ and the collection of random variables

$$S = \{x(t,\omega) : \Omega \to \mathbb{R} |\ t \in J\}$$

is called a stochastic process. Generally, we just write $x(t)$ instead of $x(t,\omega)$.

Definition 2.1.4. *An $\mathcal{F}$-adapted process X on $[0,T]\times\Omega$ is elementary processes if for a partition $\phi = \{t = 0 < t_1 < \ldots < t_n = T\}$ and $(\mathcal{F}_{t_i})$-measurable random variables $(X_{t_i})_{i<n}$, X_t satisfies*

$$X_t(\omega) = \sum_{i=0}^{n-1} X_i(\omega)\chi_{[t_i,t_{i+1})}(t), \quad for \;\; 0 \le t \le T, \quad \omega \in \Omega.$$

The Itô integral of the simple process X is defined as

$$\int_0^T X(s)dW^l(s) = \sum_{i=0}^{n-1} X_l(t_i)(W^l(t_{i+1}) - W^l(t_i)), \tag{2.1.4}$$

whenever $X_{t_i} \in L^2(\mathcal{F}_{t_i})$ for all $i \le n$.

The following result is one of the elementary properties of square-integrable stochastic processes [**?**, 45].

Lemma 2.1.10. *(Itô Isometry for Elementary Processes) Let $(X_l)_{l\in\mathbb{N}}$ be a sequences of elementary processes. Assume that*

$$\int_0^T \mathbb{E}|X(s)|^2 ds < \infty,$$

where $|X| = (\sum_{l=1}^{\infty} X_l^2)$. Then

$$\mathbb{E}\Big(\sum_{l=1}^{\infty}\int_0^T X_l(s)dW^l(s)\Big)^2 = \mathbb{E}\Big(\sum_{l=1}^{\infty}\int_0^T X_l^2(s)ds\Big). \tag{2.1.5}$$

Proof. Set $\Delta W_i^l = W_{t_{i+1}}^l - W_{t_i}^l$. Let $M^k = \sum_{l=1}^k \int_0^T X_l(s)dW^l(s)$, observe that if $k = 1$ we have

$$\begin{aligned}
\mathbb{E}\Big(\int_0^T X_1(s)dW(s)\Big)^2 \;\; &= \;\; \mathbb{E}\Big(\sum_{i=0}^{n-1} X_1(t_i)\Delta W_i^1\Big)^2 \\
&= \;\; \sum_{i=0}^{n-1} E\Big(X_1^2(t_i)(W_{t_{i+1}}^1 - W_{t_i}^1)^2\Big) \\
&\quad\; +2\sum_{i<j}\mathbb{E}\Big(X_1(t_i)X_1(t_j)(W_{t_{i+1}}^1 - W_{t_i}^1)(W_{t_{j+1}}^1 - W_{t_j}^1)\Big).
\end{aligned}$$

For $i < j$, since $t_i < t_j$ and $(W_{t_{j+1}} - W_{t_j})$ is independent from

$$X_i(t_i)X_j(t_j)(W_{t_{j+1}} - W_{t_j}).$$

Hence

$$\begin{aligned}
\mathbb{E}\Big(X_1(t_i)X_1(t_j)(W^1_{t_{i+1}} - W^1_{t_i})(W^1_{t_{j+1}} - W^1_{t_j})\Big) &= \mathbb{E}\Big(X_1(t_i)X_1(t_j)(W^1_{t_{i+1}} - W^1_{t_i}\Big) \\
&\quad \times\mathbb{E}\Big(W^1_{t_{j+1}} - W^1_{t_j})\Big) \\
&= 0.
\end{aligned}$$

This implies that

$$\begin{aligned}
\mathbb{E}\Big(\int_0^T X_1(s)dW^1(s)\Big)^2 &= \sum_{i=0}^{n-1}\mathbb{E}\Big(X_1^2(t_i)(W^1_{t_{i+1}} - W^1_{t_i})^2\Big) \\
&= \sum_{i=0}^{n-1}\mathbb{E}\Big(X_1^2(t_i)\Big)\mathbb{E}\Big(W^1_{t_{i+1}} - W^1_{t_i})\Big)^2 \\
&= \sum_{i=0}^{n-1}\mathbb{E}\Big(X_1^2(t_i)\Big)(t_{i+1} - t_i) \\
&= \mathbb{E}\Big(\int_0^T X_1^2(s)ds\Big).
\end{aligned}$$

Therefore

$$\mathbb{E}\Big(\int_0^T X_1(s)dW^1(s)\Big)^2 = \Big(\mathbb{E}\int_0^T X_1^2(s)ds\Big). \tag{2.1.6}$$

When $k = 1$ the result is true, For $k = 2$ we use the estimate (2.1.6), to get

$$
\begin{aligned}
M^2 = \mathbb{E}\Big(\sum_{l=1}^{2}\int_0^T X_l(s)dW^l(s)\Big)^2 &= \mathbb{E}\Big(\sum_{l=1}^{2}\sum_{i=0}^{n-1} X_l(t_i)\Delta W_i^l\Big)^2 \\
&= \mathbb{E}\Big(\sum_{i=0}^{n-1} X_1(t_i)\Delta W_i^1\Big)^2 + \mathbb{E}\Big(\sum_{i=0}^{n-1} X_2(t_i)\Delta W_i^2\Big)^2 \\
&+ 2\mathbb{E}\Big(\sum_{i=0}^{n-1} X_1(t_i)\Delta W_i^1\Big)\mathbb{E}\Big(\sum_{i=0}^{n-1} X_2(t_i)\Delta W_i^2\Big) \\
&= \sum_{i=0}^{n-1} E\Big(X_1^2(t_i)\Big)(t_{i+1}-t_i) + \sum_{i=0}^{n-1}\mathbb{E}\Big(X_2^2(t_i)\Big)(t_{i+1}-t_i) \\
&= \mathbb{E}\int_0^T X_1^2(s)ds + \mathbb{E}\int_0^T X_2^2(s)ds \\
&= \mathbb{E}\Big(\sum_{l=0}^{2}\int_0^T X_l^2(s)\Big)ds.
\end{aligned}
$$

Suppose that

$$
\mathbb{E}\Big(\sum_{l=1}^{k}\int_0^T X_l(s)dW^l(s)\Big)^2 = \sum_{l=1}^{k}\mathbb{E}\Big(\int_0^T X_l^2(s)ds\Big), \tag{2.1.7}
$$

is true for fixed $k \in \mathbb{N}$. We show that

$$
\mathbb{E}\Big(\sum_{l=1}^{k+1}\int_0^T X_l(s)dW^l(s)\Big)^2 = \sum_{l=1}^{k+1}\int_0^T \mathbb{E}(X_l^2(s))ds.
$$

Since $(X_l)_{l=1}^{k+1}$ is a set of elementary stochastic process, then

$$\begin{aligned}
M^{k+1} = \mathbb{E}\Big(\sum_{l=1}^{k+1}\int_0^T X_l(s)dW^l(s)\Big)^2 &= \mathbb{E}\Big(\sum_{l=1}^{k+1}\sum_{i=0}^{n-1} X_l(t_i)\Delta W_i^l\Big)^2 \\
&= \mathbb{E}\Big(\sum_{l=1}^{k}\sum_{i=0}^{n-1} X_l(t_i)\Delta W_i^l\Big)^2 \\
&+ \mathbb{E}\Big(\sum_{i=0}^{n-1} X_{k+1}(t_i)\Delta W_i^{k+1}\Big)^2 \\
&+ 2\Big(\mathbb{E}\sum_{i=0}^{n-1} X_{k+1}(t_i)\Delta W_i^{k+1}\sum_{l=1}^{k+1}\sum_{i=0}^{n-1} X_l(t_i)\Delta W_i^l\Big).
\end{aligned}$$

From (2.1.7) and using the fact that $(W^l)_{l=1}^{k+1}$ is a set of independent standard Brownian motions, we have

$$\begin{aligned}
\mathbb{E}\Big(\sum_{l=1}^{k+1}\int_0^T X_l(s)dW^l(s)\Big)^2 &= \mathbb{E}\sum_{l=1}^{k}\int_0^T X_l^2(s)ds + \mathbb{E}\int_0^T X_{k+1}^2(s)ds \\
&= \mathbb{E}\Big(\sum_{l=1}^{k+1}\int_0^T X_l^2(s)ds\Big).
\end{aligned}$$

Hence the formula is true for $k+1$. This implies that for every $k \in \mathbb{N}$ we have

$$M^k = \mathbb{E}\Big(\sum_{l=1}^{k}\int_0^T X_l(s)dW^l(s)\Big)^2 = \mathbb{E}\Big(\sum_{l=1}^{k}\int_0^T X_l^2(s)ds\Big).$$

From Lemma 2.1, proved in [?], M^k is a convergent in $L^2(\mathcal{F}_t)$, then

$$\lim_{k\to\infty} M^k = \lim_{k\to\infty}\mathbb{E}\Big(\sum_{l=1}^{k}\int_0^T X_l(s)dW^l(s)\Big)^2 = \mathbb{E}\Big(\sum_{l=1}^{\infty}\int_0^T X_l^2(s)ds\Big).$$

Hence

$$\mathbb{E}\Big(\sum_{l=1}^{\infty}\int_0^T X_l(s)dW^l(s)\Big)^2 = \mathbb{E}\Big(\sum_{l=1}^{\infty}\int_0^T X_l^2(s)ds\Big).$$

□

Remark 2.1.1. *For an square integrable stochastic process X on $[0,T]$, its Itô integral is defined by*

$$\int_0^T X(s)dW(s) = \lim_{n\to\infty} \int_0^T X_n(s)dW(s)$$

taking the limit in L^2, with X_n is defined in definition 2.1.4. Then the Itô isometry holds for all Itô-integrable X.

Theorem 2.1.11. *[?] Let X, $Y \in S$, $a, b \in \mathbb{R}$. Then :*

- *(linearity) for all $t$at once with probability one*

$$\mathbb{E}\int_0^t \Big(aX(s)+bY(s)dW(s)\Big) = a\mathbb{E}\int_0^t X(s)dW(s)+b\mathbb{E}\int_0^t Y(s)dW(s).$$

- $\mathbb{E}(\int_0^t X(s)dW(s)) = 0$;

- *the process $\int_0^t X(s)dW(s)$ is a martingale relative to $\mathcal{F}_t$;*

- *Doob inequality holds:*

$$\mathbb{E}\sup_t(\int_0^t X(s)dW(s))^2 \le 4\mathbb{E}\sup_t(\int_0^t X^2(s)ds;$$

- *if $A \in \mathcal{F}$, $T \in [0,\infty]$, and $X_t(w) = Y_t(w)$ for all $w \in A$ and $t \in [0,T)$, then*

$$I_A \int_0^t X(s)dW(s) = I_A \int_0^t Y(s)dW(s)$$

  for all $t \in [0,T]$ at once with probability one.

Theorem 2.1.12. *if $f \in S$ and f is continuous ,then, for any sequence Π_i of $0 = t_{n,0} < t_{n,1} < t_{n,2} < \ldots < t_{n,n} = T$ is a partition of $[0,T]$ with $|\Pi_i| \to 0$,*

$$\mathbb{P}\Big(\int_0^T f(s)dW(s) = \lim_{n\to\infty} \sum_{k=0}^{m_n-1} X_l(t_i)(W(t_{n,k+1}) - W(t_{n,k}))\Big) = 1, \quad (2.1.8)$$

Remark 2.1.2. *For an square integrable stochastic process X on $[0,T]$, its Itô integral is defined by*

$$\int_0^T X(s)dW(s) = \lim_{n\to\infty}\int_0^T X_n(s)dW(s),$$

taking the limit in L^2, with X_n is defined in definition 2.1.4. Then the Itô isometry holds for all Itô-integrable X.

The next result is known as the Burholder-Davis-Gundy inequalities. It was first proved for discrete martingales and $p>0$ by Burkholder [16]. In 1968, Millar [50] extended the result to continuous martingales. In 1970, Davis [20] extended the result for discrete martingales to $p=1$. The extension to $p>0$ was obtained independently by Burkholder and Gundy [17] in 1970 and Novikov [56] in 1971.

Theorem 2.1.13. *[66] For each $p>0$ there exist constants $c_p, C_p \in (0,\infty)$, such that for any progressive process x with the property that for some $t\in [0,\infty)$, $\int_0^t X_s^2 ds < \infty$,$\mathbb{P}$.a.s we have*

$$c_p\mathbb{E}\big(\int_0^t X_s^2 ds\big)^{\frac{p}{2}} \le \mathbb{E}\big(\sup_{s\in[0,t]}\int_0^t X_s^2 dW(s)\big)^p \le C_p\mathbb{E}\big(\int_0^t X_s^2 ds\big)^{\frac{p}{2}}. \tag{2.1.9}$$

Example 2.1.2. *Mean and mean square of a stochastic integral Let $I(f) = \int_0^1 W(s)dW(s)$. Then, by the theorem 2.1.11 of It ô integrals,*

$$\mathbb{E}(I(f)) = 0 \quad \text{and} \quad \mathbb{E}(|I(f)|^2) = \int_0^1 \mathbb{E}|W(s)|^2 ds = \frac{1}{2}.$$

Lemma 2.1.14. *(Stochastic Product Rule) Let $X, Y\in S$. Then*

$$X_tY_t = X_0Y_0 + \int_0^t Y_s dX_s + \int_0^t X(s)dY_s + < X, Y >_t, \tag{2.1.10}$$

for all $t\ge 0$.

2.1.3 Itô formula

Definition 2.1.5. *Let W_t be 1-dimensional Brownian motion on $(\Omega,\mathcal{F},\mathbb{P})$ (i.e.1-dimensional) Itô process (or stochastic integral) is a stochastic process X_t on $(\Omega,\mathcal{F},\mathbb{P})$ of the form*

$$X_t = X_0 + \int_0^t u(s,w)ds + \int_0^t v(s,w)dW(s), \tag{2.1.11}$$

where $u(t,w), v(t,w)$ is $\mathcal{F}_t$-adapted.

$$\mathbb{P}\Big(\int_0^t v^2(s,w)ds < \infty \quad \text{for all} \quad t \geq 0\Big) = 1,$$

and

$$\mathbb{P}\Big(\int_0^t u(s,w)ds < \infty \quad \text{for all} \quad t \geq 0\Big) = 1.$$

If X_t is an Itô process of the form (2.1.11) is sometimes written in the shorter differential form

$$dX_t = udt + vdW(t).$$

2.1.4 The Multi-dimensional Itô Formula

Itô formula Let $X = (X^1, X^2, \ldots, X^d)$ be an $\mathbb{R}^d$-valued process with continuously differentiable paths and consider the process $Y_t = f(X_t)$, where $f \in C^2(\mathbb{R}^d)$. Let us write

$$D_j f = \frac{\partial f}{\partial X_j} \quad \text{and} \quad D_{i,j} f = \frac{\partial^2 f}{\partial X_i \partial X_j}.$$

The process Y has continuously differentiable paths with

$$\frac{d}{dt} f(X_t) = \sum_{j=1}^{d} D_j f(X_t(w)) \frac{d}{dt} X_t^j(w).$$

Fixing $w \in \Omega$ and integrating yields

$$f(X_t(w)) - f(X_0(w)) = \sum_{j=1}^{d} \int_0^t D_j f(X_s(w)) \frac{d}{ds} X_s^j(w),$$

where this integral is to be interpreted pathwise. Written as

$$f(X_t(w)) - f(X_0(w)) = \sum_{j=1}^{d} \int_0^t D_j f(X_s(w)) dX_s^j(w). \tag{2.1.12}$$

This equation remains true if X is a continuous, bounded variation process. The situation becomes more complicated if the process X is a continuous semi martingale and hence no longer has paths which are of bounded variation on finite intervals in general. Then a new term appears on the right hand side of (2.1.12) (Itô's formula). We will give a very explicit derivation which shows clearly where the new term comes from.

Theorem 2.1.15. *Let $G \subset \mathbb{R}^d$ be an open set, $X = (X^1, X^2, \ldots, X^d)$ a continuous semi martingale with values in G and $f \in C^2(G)$. Then*

$$f(X_t(w)) - f(X_0(w)) = \sum_{j=1}^{d} \int_0^t D_j f(X_s(w)) dX_s^j(w)$$

$$+\frac{1}{2} \sum_{i,j=1}^{d} \int_0^t D_{i,j} f(X_s(w)) d\langle X_s^i(w), X_s^j(w)\rangle \quad \mathbb{P}.a.s, \tag{2.1.13}$$

for each $t \geq 0$.

Theorem 2.1.16. *Let $G \subset \mathbb{R}^d$ be an open set, $X = (X^1, X^2, \ldots, X^d)$ a continuous semi martingale with values in G and $f \in C^{1,2}([0,T] \times G)$. Then*

$$f(t, X_t(w)) - f(0, X_0(w)) = \int_0^t \frac{\partial f(s, X_s(w))}{\partial s} ds + \sum_{j=1}^{d} \int_0^t D_j f(s, X_s(w)) dX_s^j(w)$$

$$+\frac{1}{2} \sum_{i,j=1}^{d} \int_0^t D_{i,j} f(s, X_s(w)) d\langle X_s^i(w), X_s^j(w)\rangle, \ \mathbb{P}.a.s \tag{2.1.14}$$

for each $t \geq 0$.

Let us write down the special case where $X \in H_{SP}$ is a scalar semi martingale ($d = 1$):

Theorem 2.1.17. *Let $G \subset \mathbb{R}$ be an open set, $X = (X^1)$ (i.e $d = 1$) a continuous semi martingale with values in G and $f \in C^2(G)$. Then*

$$f(X_t(w)) - f(X_0(w)) = \int_0^t \frac{\partial f(X_s(w))}{\partial X_s} dX_s(w)$$

$$+\frac{1}{2} \int_0^t \frac{\partial^2 f(X_s(w))}{\partial X_s^2} d\langle X_s(w), X_s(w)\rangle, \mathbb{P}.a.s \tag{2.1.15}$$

for each $t \geq 0$.

Theorem 2.1.18. *Let $G \subset \mathbb{R}$ be an open set, X a continuous semi martingale with values in G and $f \in C^{1,2}([0,T] \times G)$. Then*

$$f(t, X_t(w)) - f(0, X_0(w)) = \int_0^t \frac{\partial f(s, X_s(w))}{\partial s} ds + \int_0^t \frac{\partial f(X_s(w))}{\partial X_s} dX_s(w)$$

$$+\frac{1}{2}\int_0^t \frac{\partial^2 f(X_s(w))}{\partial X_s^2} d\langle X_s(w), X_s(w)\rangle \ \mathbb{P}.a.s. \tag{2.1.16}$$

for each $t \geq 0$.

As a first consequence of Itô's formula we show that the family S of continuous semi martingales is not only a real algebra but is in fact closed under the application of twice continuously differentiable functions:

Theorem 2.1.19. *Let* G *be an open subset of* $\mathbb{R}^d$, $X \in S^d$. *Let* S^d *denote the family of all* $\mathbb{R}^d$*-valued with values in* G, $f \in C^2(G)$. *For each* $i = 1, \ldots, d$ *let* $X^i = M^i + A^i$ *be the semi martingale decomposition of* X^i, *especially* $A^i = u_{X^i}$ *.Then* $Z = f(X)$ *is again a continuous semi martingale and its local martingale part* M_t *and compensator* $u_Z(t)$ *are given by,*

$$M = Z_0 + \sum_{j=1}^{d} \int_0^t D_j f(X_s(w)) dM_s^i(w).$$

and

$$u_Z = \sum_{j=1}^{d} \int_0^t D_j f(X_s(w)) dA_s^i(w) + \frac{1}{2}\sum_{i,j=1}^{d} \int_0^t D_{i,j} f(X_s(w)) d\langle X_s^i(w), X_s^j(w)\rangle.$$

2.2 Stochastic Differential Equation

One of the most important applications of Itô stochastic integral is in the construction of stochastic differential equations. These are important for a number of reasons. We are going to consider stochastic differential equations of the type

$$dX_t = \sigma(t, X_t)dW_t + b(t, X_t)dt \tag{2.2.1}$$

Equation (3.5.1) is to be interpreted as an integral equation:

$$X_t = X_0 + \int_0^t \sigma(s, X_s)dW_s + \int_0^t b(s, X_s)ds \tag{2.2.2}$$

Here W is an $\mathbb{R}^d$-valued Brownian motion, X_0 is an $\mathbb{R}^d$-valued $\mathfrak{F}_0$ measurable random variable, $\sigma : [0, +\infty) \longrightarrow L(m, d)$ and $b : [0, +\infty) \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ are given functions, and one is seeking a process X such that (2.3.2) is true. The solution X to the SDE (2.3.1), when it exists, is called a diffusion process with diffusion cœfficient $\sigma\sigma^*$. and drift cœfficient b. We shall impose the following conditions on σ, b:

$$\begin{cases} \sigma : [0,+\infty) \longrightarrow L(m,d), & \text{is a continuous function} \\ b : [0,+\infty) \longrightarrow \mathbb{R}^m, & \text{is a continuous function} \end{cases} \tag{2.2.3}$$

$\forall T < \infty\ \exists C_T < \infty$ such that for all $t \in [0,+\infty)$, $x, \overline{x} \in \mathbb{R}^d$

$$\begin{cases} \|\sigma(t,x) - \sigma(t,\overline{x})\| \leq C_T|x-\overline{x}|, \\ \|b(t,x) - b(t,\overline{x})\| \leq C_T|x-\overline{x}|, & . \end{cases} \tag{2.2.4}$$

Since $t \longmapsto \sigma(t,0)$ and $t \longmapsto b(t,0)$ are continuous and hence bounded on $[0,T]$ for every $T < \infty$, using the Lipschitz conditions (2.3.4), we can conclude that for each $T < \infty$, $K_T < \infty$ such that

$$\begin{cases} \|\sigma(t,x)\| \leq K_T(1+|x|), \\ \|b(t,x)\| \leq K_T(1+|x|) \end{cases} \tag{2.2.5}$$

We will need the following lemma, known as Gronwalls lemma, for proving uniqueness of solution to (2.3.2) under the Lipschitz conditions.

Lemma 2.2.1. *(Gronwall Inequality) Let $f(t)$ be a bounded measurable function on $[0,T]$ satisfying, for some $0 \leq a < \infty$, $0 \leq b < \infty$,*

$$f(t) \quad \leq \quad a + b\int_0^t f(s)ds; \quad 0 \leq t \leq T. \tag{2.2.6}$$

Then

$$f(t) \quad \leq \quad ae^{bt} \tag{2.2.7}$$

Proof. Let

$$h(t) = e^{-bt}\int_0^t f(s)ds$$

Then by definition, h is absolutely continuous and

$$h^{'}(t) = e^{-bt}f(t) - be^{-bt}\int_0^t f(s)ds$$

where almost everywhere refers to the Lebesgue measure on $\mathbb{R}$. Using (2.3.6), it follows that

$$h^{'}(t) \leq ae^{-bt} \qquad a.e$$

Hence (using $h(0) = 0$ and that h is absolutely continuous) $h(t) \leq \frac{a}{b}(1-e^{-bt})$ from which we get

$$\int_0^t f(s)ds \leq \frac{a}{b}(e^{bt} - 1)$$

The conclusion $f(t) \leq ae^{bt}$ follows immediately from (2.3.6). $\square$

So now let $(\mathfrak{F}_t)_{t\geq 0}$ be a filtration on $(\Omega, \mathfrak{F}, P)$ and W be a d-dimensional Brownian motion adapted to $(\mathfrak{F}_t)_{t\geq 0}$ and such that $(W_t, \mathfrak{F}_t)_{t\geq 0}$ is a Wiener martingale. Without loss of generality, let us assume that $(\Omega, \mathfrak{F}, P)$ is complete and that $\mathfrak{F}_0$ contains all P null sets in $\mathfrak{F}$. Let $\mathbb{K}_m$ denote the class of $\mathbb{R}^m$-valued continuous $(\mathfrak{F}_t)_{t\geq 0}$ adapted process Z such that $\mathbb{E}[\int_0^T |Z(s)|^2 ds < \infty \quad \forall T < \infty$. For $Y \in \mathbb{K}_m$ let

$$\xi(t) \quad = \quad Y_0 + \int_0^t \omega(s, Y_s)dW_s + \int_0^t b(s, Y_s)ds. \tag{2.2.8}$$

Note that in view of the growth condition (2.3.5) the Itos integral above is defined. Using the growth estimate $\mathbb{E}[\sup_{0\leq t\leq T} |\int_0^t f dW|^2] \leq 4\mathbb{E}[\int_0^t \|f_s\|^2 ds]$ we see that

$$\begin{aligned}
\mathbb{E}[\sup_{0\leq t\leq T} |\xi_t|^2] &\leq 3\mathbb{E}[|Y_0|^2] + 4\mathbb{E}[\int_0^t \|\sigma 95(s, Y_s)\|^2 ds] \\
&+ \mathbb{E}[(\int_0^t \|b(s, Y_s)\|^2 ds)] \\
&\leq 3\mathbb{E}[|Y_0|^2] + 3K_T^2(4+T)\int_0^t (1 + \mathbb{E}[|Y_s|^2])ds
\end{aligned}$$

and hence $\xi \in \mathbb{K}_m$. Let us define a mapping Γ from $\mathbb{K}_m$ into itself as follows: $\Gamma(Y) = \xi$ where ξ is defined by (2.3.8). Thus solving the SDE (2.3.2) amounts to finding a fixed point Z of the functional Γ with $Z_0 = X_0$, where X_0 is prespecified. We are going to prove that given X_0, there exists a unique solution (or a unique fixed point of Γ) with the given initial condition. The following lemma is an important step in that direction.

Chapter 3

Stochastic Henry inequalies

It is well known that Gronwall inequality play a crucial role in study of differential equations, Volterra integral equations and stochastic differential equations (see [9, 34, 58, 63, 64]), and is allowing to estimate bounds which help prove uniqueness results, global existence of solutions and stability results.

Fractional calculus and differential equations of fractional order have recently proved to be valuable tools in the modeling of many physical phenomena [27, 47, 49, 52]. There has also been a significant theoretical development in fractional differential equations in recent years; see the monographs of Abbas *et al* [3], Kilbas *et al* [40], Podlubny [61], Samko *et al* [69].

The Gronwall's lemma has been already generalized in framework of fractional differential equations.

In 1981, Henry [36], established the following nonlinear integral inequality:
If

$$v(t) \leq w(t) + a \int_0^t \frac{v(s)}{(t-s)^{\gamma}} ds,$$

for some function w and constants $a > 0$ and $0 < \gamma < 1$, then there exists a constant $K = K(\gamma)$ such that

$$v(t) \leq w(t) + Ka \int_0^t \frac{w(s)}{(t-s)^{\gamma}} ds, \quad \text{for every} \quad t \in [0, b].$$

The above inequality was generalized by Ye *el al* [77]. More precisely, Ye *el* *al*, studied the following fractional integral inequality:

Let $v : [0, b) \to [0, \infty)$ be a real function and $w(\cdot)$ is a nonnegative, locally integrable function on $[0, b)$(some $b \leq \infty$) and $a(t)$ is a nonnegative, nondecreasing continuous function defined on $0 \leq t < b$, $a(t) \leq M$ (constant), and suppose $v(t)$ is nonnegative and locally integrable on $0 \leq t < b$. Assume $\gamma > 0$ such that

$$v(t) \leq w(t) + a(t) \int_0^t \frac{v(s)}{(t-s)^{1-\gamma}} ds,$$

then

$$v(t) \leq w(t) + \int_0^t \phi(s) w(s) ds, \quad \text{for every } t \in [0, b),$$

where

$$\phi(s) = \sum_{n=1}^{\infty} \frac{(a(t)\Gamma(\gamma))^n}{\Gamma(n\gamma)} (t-s)^{n\gamma - 1}.$$

Based on above inequalities many researchers developed many useful and new integral inequalities, mainly inspired by their applications in various branches of fractional differential equations (see [1, 2, 8, 10, 22, 23, 36, 72, 77] and the references therein).

In 2013 Scheutzow [71], by martingale inequality due to Burkholder [15] used to prove the stochastic linear version of Granwall lemma. Some extensions and discrete version of Scheutzow stochastic Gronwall inequality was proved by Makasu [45, 46] and Kruse and Scheutzow [41].

However, in certain situations, such as some classes of fractional stochastic differential equations or fractional integral stochastic equations, the right hands has nonlinear growth, it is desirable to find the stochastic version of Henry type inequalities, in order to get some estimates. We use some mathematical analysis techniques combined with Young's and Hölder inequalities to obtain the explicit estimates.

3.1 Gronwall stochastic inequality

In the first step, we proposed this lemma for to obtain the demonstration of Gronwall lemma

Lemma 3.1.1. *Let Λ be a topological space which also has a partial order which is sequentially closed in $\Lambda \times \Lambda$. Suppose that a map $\Psi : \Lambda \longrightarrow \Lambda$*

preserves the order relation and has an attractive fixed point v. Then for all $x \in \Lambda$ we have

$$x \leq \Psi(x) \Longrightarrow x \leq v$$

Proof. Assume $x \leq \Psi(x)$. Since Ψ preserves the order relation we get $x \leq \Psi^n(x)$ by induction. Since v is an attractive fixed point we have $v = \lim_{n \longrightarrow +\infty} \Psi^n(x)$. Since the order relation is sequentially closed, we conclude $x \leq v$ as required. □

Lemma 3.1.2. *(Gronwall Lemma) Assume that the continuous functions $f, g : [0, T] \longrightarrow [0, +\infty)$ and $K > 0$ satisfy*

$$f(t) \leq K + \int_0^t f(s)g(s)ds$$

$\forall t \in [0, T]$. *Then the usual Gronwall inequality is*

$$f(t) \quad \leq \quad K \exp(\int_0^t g(s)ds) \tag{3.1.1}$$

Proof. The hypothesis is

$$\frac{f(s)}{K + \int_0^s f(\tau)g(\tau)d\tau} \leq 1$$

Multiply this by $g(s)$ to get

$$\frac{d}{ds}(K + \int_0^t f(s)g(s)ds) \leq g(s)$$

By hypothesis, the left side is $\geq f(t)$. We now show how to derive the usual Gronwall inequality from the abstract Gronwall inequality. For $v : [0, T] \longrightarrow [0, +\infty)$ define $\Psi(v)$ by

$$\Psi(v)(t) \quad = \quad K + \int_0^t v(s)g(s)ds. \tag{3.1.2}$$

In this notation, the hypothesis of Gronwall's inequality is $f \leq \Psi(f)$ where $v \leq \omega$ means $v(t) \leq \omega(t)$ for all $t \in [0, T]$. Since $g(t) \geq 0$ we have

$$v \leq \omega \Longrightarrow \Psi(v) \leq \Psi(\omega)$$

Hence iterating the hypothesis of Gronwalls inequality gives

$$f \leq \Psi(f)$$

Now change the dummy variable in (2) from s to s1 and apply the inequality $f(s_1) \leq \Psi(f(s_1))$ to obtain

$$\Psi^2(f)(t) = K + \int_0^t Kg(s)ds + \int_0^t \int_0^{s_1} g(s_1)g(s_2)ds_2 ds_1$$

More generally, by induction we have

$$\Psi^n(f)(t) = K\sum_{i=0}^{n-1} G_i(t) + E_i(t)$$

where

$$G_i(t) = \int_0^t \int_0^{s_1} \dots \int_0^{s_{i-1}} g(s_1)\dots g(s_i)ds_i ds_{i-1}\dots ds_1$$

Withe $G_0(t) = 1$ and

$$E_n(t) = \int_0^t \int_0^{s_1} \dots \int_0^{s_{n-1}} g(s_1)\dots g(s_n)f(s_n)ds_n ds_{i-1}\dots ds_1$$

Now $G_i(t)$ is an integral over the i-simplex $0 \leq s_i \leq \dots \leq_1 \leq t$ and the integrand is symmetric under a permutation of the variables. Hence

$$\begin{aligned} G_i(t) &= \frac{1}{i!}\int_0^t \int_0^t \dots \int_0^t g(s_1)\dots g(s_i)ds_i ds_{i-1}\dots ds_1 \\ &= \frac{1}{i!}(\int_0^t g(s)ds)^i \end{aligned}$$

Also $|E_n(t)|$ is bounded by an n^{th} power times the area $\frac{1}{n!}$ of the n-simplex. Hence the term $E_n(t)$ converges uniformly to zero and the series limits to the series for the exponential function. The above argument shows $\Psi.$ has an attractive fixed point so we can also prove the Gronwall inequality by solving $v = \Psi(v)$; the solution is

$$v(t) = K\exp[\int_0^t g(s)ds]$$

We use this approach to prove a more general form of Gronwall's inequality where the constant K is replaced by a continuous function $K : [0,T] \longrightarrow [0,+\infty)$. Namely, assume that

$$f(t) \quad \le \quad K(t) + \int_0^t f(s)g(s)ds \tag{3.1.3}$$

$\forall t \in [0,T]$. We prove that

$$f(t) \quad \le \quad K(t) + \int_0^t K(s)g(s)\exp(\int_s^t g(\tau)d\tau)ds(4) \tag{3.1.4}$$

The abstract Gronwall inequality applies much as before so to prove (4) we show that the solution of

$$v(t) \quad = \quad K(t) + \int_0^t v(s)g(s)ds(5) \tag{3.1.5}$$

is

$$v(t) \quad = \quad K(t) + \int_0^t K(s)g(s)\exp(\int_s^t g(\tau)d\tau)ds(4) \tag{3.1.6}$$

Equation (5) implies $\dot{v} = \dot{K} + Kv$. By variation of constants we seek a solution in the form

$$v(t) = C(t)\exp(\int_0^t g(\tau)d\tau)$$

so

$$C(t) = K(t) + \int_0^t K(s)g(s)\exp(-\int_0^s g(\tau)d\tau)ds$$

so

$$v(t) = C(t)\exp(\int_0^t g(\tau)d\tau)ds + \int_0^t K(s)\exp(\int_s^t g(\tau)d\tau)ds$$

Equation (5) requires $v(0) = K(0)$ so

$$C(0) = K(0).$$

Integration by parts gives

$$\begin{aligned} \int_0^t \dot{K}(t)\exp(\int_s^t g(\tau)d\tau)ds \quad &= \quad K(t) - K(0)\exp(\int_0^t g(s)ds) \\ &+ \quad \int_0^t K(t)\exp(\int_s^t g(\tau)d\tau)ds \end{aligned}$$

Combining the last three displayed equations give (6).
For the proof of (4), Define

$$R(t) := \int_0^t g(s)f(s)ds$$

Then the derivative R0 satisfies

$$R'(s) - g(s)R(s) = g(s)(f(sa' - R(s)) \leq g(s)K(s)$$

Hence

$$\frac{d}{ds}R(s)\exp(\int_s^t g(\tau)d\tau) \leq g(s)K(s)\exp(\int_s^t g(\tau)d\tau)$$

so integrating gives

$$R(t) = \int_0^t g(s)K(s)\exp(\int_s^t g(\tau)d\tau)ds$$

Now add $K(t)$ to both sides and use the hypothesis $f(t) \leq K(t) + R(t)$. If $K(t)$ is a constant, the right hand side of (4) reduces to the right hand side of (1). This follows on taking $K(t)$ constant in the fixed point equation $v = K + \int gv$, but here's a direct proof. $\square$

3.2 Henry type inequality

We wish to establish an integral inequality which can be used in a fractional differential equation. The proof is based on an iteration argument.

Theorem 3.2.1. *Let $\alpha > 0$ and $h(t)$ is a nonnegative function locally integrable on $t \in [0, T]$, (some $T \leq +\infty$) and $g(t)$ is a nonnegative, nondecreasing continuous function defined on $[0, T]$, and suppose $f(t)$ is nonnegative and locally integrable on $[0, T]$ with*

$$f(t) \leq h(t) + g(t)\int_0^t (t-s)^{\alpha-1} f(s)ds.$$

Then

$$f(t) \leq h(t) + \int_0^t [\sum_0^{+\infty} \frac{(g(t)\gamma(\alpha))^n}{\gamma(n\alpha)}(t-s)^{n\alpha-1}h(s)]ds, \quad t \in [0, T]$$

Proof. Let Φ operator be defined by

$$\Phi\xi(t) = g(t)\int_0^t (t-s)^{\alpha-1}\xi(s)ds, \quad t \geq 0$$

for locally integrable functions ξ. Then

$$f(t) \leq h(t) + \Phi f(t)$$

implies

$$f(t) \leq \sum_{k=0}^{n-1} \Phi^k h(t) + \Phi^n f(t)$$

Let us prove that

$$\Phi^n f(t) \quad \leq \quad \int_0^t \frac{(g(t)\gamma(\alpha))^n}{\gamma(n\alpha)}(t-s)^{n\alpha-1} f(s)ds \tag{3.2.1}$$

and

$$\Phi^n f(t) \longrightarrow 0 \quad as \quad n \longrightarrow +\infty$$

for all $t \in [0, T]$ We know this relation (3.1.13) is true for $n = 1$. Assume that it is true for some $n = k$.
If $n = k + 1$, then the induction hypothesis implies

$$\Phi^{k+1} f(t) = \Phi(\Phi^k f(t)) \leq g(t)\int_0^t (t-s)^{\alpha-1}[\int_0^s \frac{(g(s)\gamma(\alpha))^k}{\gamma(k\alpha)}(s_\tau)^{k\alpha-1} f(\tau)d\tau]ds$$

Since $g(t)$ is nondecreasing, it follows that

$$\Phi^{k+1} f(t) \leq (g(t))^{k+1}\int_0^t (t-s)^{\alpha-1}[\int_0^s \frac{(\gamma(\alpha))^k}{\gamma(k\alpha)}(s_\tau)^{k\alpha-1} f(\tau)d\tau]ds$$

By interchanging the order of integration, we have

$$\begin{aligned}
\Phi^{k+1} f(t) \quad &\leq \quad (g(t))^{k+1}\int_0^t [\int_\tau^t \frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)}(t-s)^{\alpha-1}(s-\tau)^{k\alpha-1} f(\tau)d\tau]ds \\
&= \quad \int_0^t \frac{(g(t)\Gamma(\alpha))^{k+1}}{\Gamma((k+1)\alpha)}(t-s)^{(k+0)\alpha-1} f(s)ds
\end{aligned}$$

where the integral, with the help of the substitution $s = t + \mu(t - t)$ and the definition of the beta function $\beta(.,.)$ we opened

$$\begin{aligned}\int_{\tau}^{t}(t-s)^{\alpha-1}(s-\tau)^{k\alpha-1}ds &= (t-\tau)^{(k+1)\alpha-1}\int_{0}^{1}(1-\mu)^{\alpha-1}\mu^{k\alpha-1}d\mu \\ &= (t-\tau)^{(k+1)\alpha-1}\beta(k\alpha,\alpha) \\ &= \frac{\Gamma(\alpha)\Gamma(k\alpha)}{\Gamma((k+1)\alpha}(t-\tau)^{(k+1)\alpha-1}\end{aligned}$$

The relation (3.1.13) is proved.
Since

$$\Phi^n f(T) \leq \int_0^t \frac{(g(s)\Gamma(\alpha))^n}{\Gamma(k\alpha)}(t-s)^{n\alpha-1}f(s)ds \longrightarrow 0, \quad as \quad n \longrightarrow +\infty \quad for \quad t \in [0,T]$$

. We conclude that

$$f(t) \leq \sum_{k=0}^{+\infty}\Phi^k h(t)$$

$\square$

Remark 3.2.1.

- *In theorem (3.1.1)the author use that g is bounded function, from above the pounded of g is not messier*

- *For $\alpha = 1$, $g \equiv b$ (constante) and h integrable function we obtain the classical Gronwall lemma.*

3.3 Stochastic fractional inequalities

In all throughout this section, the random process considered are real valued. We denote

$$X^*(t) = \sup_{0\leq s\leq t} X_s$$

the ruining supremely of a process $X = (X_t)_{t\geq 0}$ and every integral considered in this note are well defined.

For the first result, we apply the martingale inequality to prove a stochastic version of Henry lemma.

Theorem 3.3.1. *Let $0 < \alpha < 1$, $b \geq 0$, $q \in (1, \infty)$, X and H be non-negative, adapted processes with continuous paths, and let M be a continuous local martingale starting at zero. Assume that X satisfies the following stochastic inequality:*

$$X_t \leq b \int_0^t (t-s)^{\alpha-1} X_s ds + M_t + H_t$$

then, there exists $A_{r,q} > 0$ such that

$$\mathbb{E}(X^*(t)) \leq 2^{1-\frac{1}{q}} A_{r,q} E_\alpha(b\Gamma(\alpha)) \left(\left(\mathbb{E}\langle M \rangle_t^{\frac{rq}{2}} \right)^{\frac{1}{rq}} + (\mathbb{E}(H^*(t))^q)^{\frac{1}{q}} \right). \quad (3.3.1)$$

where $\langle M \rangle$ is the quadratic variation of M_t, $A_{r,q} = \max\left(C_{r,q}^{\frac{1}{q}}, 1 \right)$ and $E_\alpha(\cdot)$ is Mittag-Leffler function.

Proof. From Ye [77], inequality, we get

$$\begin{aligned} X_t &\leq M_t + H_t + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(b\Gamma(\alpha))^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} \right] (M_s + H_s) ds \\ &\leq \left(1 + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(b\Gamma(\alpha))^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} \right] ds \right) (|M^*(t)| + H^*(t)). \end{aligned}$$

Hence

$$X^*(t) \leq \left(1 + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(b\Gamma(\alpha))^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} \right] ds \right) (|M^*(t)| + H^*(t)).$$

Therefore

$$\mathbb{E}X^*(t) \leq \left(1 + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(b\Gamma(\alpha))^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} \right] ds \right) \mathbb{E}(|M^*(t)| + H^*(t)).$$

By Hölder inequality we obtain

$$\begin{aligned} \mathbb{E}X^*(t) &\leq \left(1 + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(b\Gamma(\alpha))^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} \right] ds \right) (\mathbb{E}(|M^*(t)| + H^*(t))^q)^{\frac{1}{q}} \\ &\leq 2^{1-\frac{1}{q}} \left(1 + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(b\Gamma(\alpha))^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} \right] ds \right) \\ &\quad \times \left((\mathbb{E}(|M^*(t)|^q)^{\frac{1}{q}} + (\mathbb{E}(H^*(t))^q)^{\frac{1}{q}} \right). \end{aligned}$$

Since M_t is continuus local martingale and $M(0) = 0$. Then from there exists $C_{r,q}$ constant positive numbre for $r \in (1, \infty)$ such that

$$\mathbb{E}(M^*(t))^q \leq C_{r,q} \left(\mathbb{E}(\langle M \rangle_t^{\frac{rq}{2}}) \right)^{\frac{1}{r}}.$$

Finaly, we get

$$\begin{aligned}
\mathbb{E}X^*(t) &\leq 2^{1-\frac{1}{q}} \left(1 + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(b\Gamma(\alpha))^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} \right] ds \right) \left(C_{r,q}^{\frac{1}{q}} \left(\mathbb{E}\langle M \rangle_t^{\frac{rq}{2}} \right)^{\frac{1}{rq}} + (\mathbb{E}(H^*(t))^q)^{\frac{1}{q}} \right) \\
&\leq 2^{1-\frac{1}{q}} \left(1 + \sum_{n=1}^{\infty} \frac{(b\Gamma(\alpha))^n}{\Gamma(n\alpha+1)} \right) \left(C_{r,q}^{\frac{1}{q}} \left(\mathbb{E}\langle M \rangle_t^{\frac{rq}{2}} \right)^{\frac{1}{rq}} + (\mathbb{E}(H^*(t))^q)^{\frac{1}{q}} \right).
\end{aligned}$$

This completes the proof. □

Theorem 3.3.2. *Let $0 < \alpha < 1$, q, $\theta \in (1, \infty)$, $\frac{1}{\theta} + \frac{1}{q} = 1$, X and H be non-negative, adapted processes with continuous paths, let M be a continuous local martingale starting at zero and f is non-negative and progressively measurable. Assume that X satisfies the following stochastic inequality:*

$$X_t \leq f(t) \int_0^t (t-s)^{\alpha-1} X_s ds + M_t + H_t$$

then, there exists $A_{r,q} > 0$ such that

$$\mathbb{E}(X^*(t))^p \leq 2^{-\frac{1}{\theta}-\frac{1}{q}} A_{r,q} \left(\mathbb{E}(E_\alpha(f(t)\Gamma(\alpha)))^\theta \right)^{\frac{1}{\theta}} \left(\left(\mathbb{E}\langle M \rangle_t^{\frac{rq}{2}} \right)^{\frac{1}{rq}} + (\mathbb{E}(H^*(t))^q)^{\frac{1}{q}} \right).$$

If f is deterministic function, then

$$\mathbb{E}(X^*(t))^p \leq 2^{1-\frac{1}{q}} A_{r,q} E_\alpha(f(t)\Gamma(\alpha)) \left(\left(\mathbb{E}\langle M \rangle_t^{\frac{rq}{2}} \right)^{\frac{1}{rq}} + (\mathbb{E}(H^*(t))^q)^{\frac{1}{q}} \right)$$

where $\langle M \rangle$ is the quadratic variation of M_t and $A_{r,q} = \max\left(C_{r,q}^{\frac{1}{q}}, 1 \right)$.

Proof. Applying the fractional Gronwall inequality proved in [77], we get

$$\begin{aligned} X_t &\leq M_t + H_t + \int_0^t \sum_{n=1}^{\infty} \frac{(f(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}(M_s+H_s)ds \\ &\leq \left(1+\int_0^t \sum_{n=1}^{\infty} \frac{(f(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}ds\right)(M^*(t)+H^*(t)) \end{aligned}$$

then

$$\mathbb{E}X^*(t) \leq \mathbb{E}\left[\left(1+\int_0^t \sum_{n=1}^{\infty} \frac{(f(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}ds\right)(M^*(t)+H^*(t))\right].$$

By Hölder and continuous martingale inequality, we have

$$\begin{aligned} \mathbb{E}(X^*(t))^p &\leq \left(\mathbb{E}\left(1+\int_0^t \sum_{n=1}^{\infty} \frac{(f(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}ds\right)^{\theta}\right)^{\frac{1}{\theta}} (\mathbb{E}(M^*(t)+H^*(t))^q)^{\frac{1}{q}} \\ &\leq 2^{1-\frac{1}{q}}\left(\mathbb{E}\left(1+\int_0^t \sum_{n=1}^{\infty} \frac{(f(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}ds\right)^{\theta}\right)^{\frac{1}{\theta}} \\ &\quad \times \left(C_{r,q}^{\frac{1}{q}}\left(\mathbb{E}\langle M\rangle_t^{\frac{rq}{2}}\right)^{\frac{1}{rq}} + (\mathbb{E}(H^*(t))^q)^{\frac{1}{q}}\right) \\ &\leq 2^{-\frac{1}{\theta}-\frac{1}{q}}\left(1+\mathbb{E}\left(\int_0^t \sum_{n=1}^{\infty} \frac{(f(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}ds\right)^{\theta}\right)^{\frac{1}{\theta}} \\ &\quad \times \left(C_{r,q}^{\frac{1}{q}}\left(\mathbb{E}\langle M\rangle_t^{\frac{rq}{2}}\right)^{\frac{1}{rq}} + (\mathbb{E}(H^*(t))^q)^{\frac{1}{q}}\right) \\ &\leq 2^{-\frac{1}{\theta}-\frac{1}{q}}\left(1+\mathbb{E}\left(\sum_{n=1}^{\infty} \frac{(f(t)\Gamma(\alpha))^n}{\Gamma(n\alpha+1)}\right)^{\theta}\right)^{\frac{1}{\theta}} \\ &\quad \times \left(C_{r,q}^{\frac{1}{q}}\left(\mathbb{E}\langle M\rangle_t^{\frac{rq}{2}}\right)^{\frac{1}{rq}} + (\mathbb{E}(H^*(t))^q)^{\frac{1}{q}}\right). \end{aligned}$$

□

Remark 3.3.1. *If f is deterministic can be easily obtained the conclusion of theorem 3.3.1.*

Theorem 3.3.3. *Let* $0 < \alpha < 1$, q, $\gamma \in (1, \infty)$, $\frac{1}{q} + \frac{1}{\gamma} = 1$, $\alpha\gamma \leq 1$, *X and H be non-negative, adapted processes with continuous paths, let M be a continuous local martingale starting at zero and ψ is non-negative and progressively measurable. Assume that X satisfies the following stochastic inequality:*

$$X_t \leq \int_0^t (t-s)^{\alpha-1}\psi_s X_s ds + M_t + H_t$$

then, there exists $A_{r,q} > 0$ *such that*

$$\mathbb{E}X^*(t) \leq 2^{1-\frac{1}{q}} A_{r,q} E_\alpha(b\Gamma(\alpha)) \left(\left(\mathbb{E}\langle M\rangle_t^{\frac{rq}{2}}\right)^{\frac{1}{rq}} + (\mathbb{E}(H^*(t))^q)^{\frac{1}{q}} \right). \tag{3.3.2}$$

Proof. By Hölder inequality, we get

$$\begin{aligned} X_t &\leq \left(\int_0^t (t-s)^{\gamma\alpha-\gamma} ds\right)^{\frac{1}{\gamma}} \left(\int_0^t \psi_s^q X_s^q ds\right)^{\frac{1}{q}} + |M_t| + H_t \\ &\leq \frac{\gamma t^{\alpha-1+\frac{1}{\gamma}}}{\alpha\gamma-\gamma+1} \left(\int_0^t \psi_s^q X_s^q ds\right)^{\frac{1}{q}} + |M_t| + H_t. \end{aligned}$$

It follows that from [76],

$$\begin{aligned} X^*(t) &\leq \frac{\gamma t^{\alpha-1+\frac{1}{\gamma}}}{(\alpha\gamma-\gamma+1)(1-(1-e(t))^{\frac{1}{q}})} \left(\int_0^t \psi_s^q (H_s + |M_s|)^q e(s) ds\right)^{\frac{1}{q}} + |M_t| + H_t \\ &\leq \left(1 + \frac{\gamma t^{\alpha-1+\frac{1}{\gamma}}}{(\alpha\gamma-\gamma+1)(1-(1-e(t))^{\frac{1}{q}})} \left(\int_0^t \psi_s^q e(s) ds\right)^{\frac{1}{q}}\right)(M^*(t) + H^*(t)) \end{aligned}$$

where

$$e(t) = \exp\left(-\int_0^t \frac{s^{q\alpha-q+\frac{q}{\gamma}}}{(\alpha\gamma-\gamma+1)} \psi_s^q ds\right), \quad t \in \mathbb{R}_+.$$

Then, by Hölder inequality,

$$\begin{aligned} \mathbb{E}X^*(t) &\leq \mathbb{E}\left[\left(1 + \frac{\gamma t^{\alpha-1+\frac{1}{\gamma}}}{(\alpha\gamma-\gamma+1)(1-(1-e(t))^{\frac{1}{q}})} \left(\int_0^t \psi_s^q e(s) ds\right)^{\frac{1}{q}}\right)(M^*(t) + H^*(t))\right] \\ &\leq \left(\mathbb{E}\left(1 + \frac{\gamma t^{\alpha-1+\frac{1}{\gamma}}}{(\alpha\gamma-\gamma+1)(1-(1-e(t))^{\frac{1}{q}})} \left(\int_0^t \psi_s^q e(s) ds\right)^{\frac{1}{q}}\right)^q\right)^{\frac{1}{q}} (\mathbb{E}(M^*(t) + H^*(t))^\gamma)^{\frac{1}{\gamma}}, \end{aligned}$$

which implies that

$$\begin{aligned}\mathbb{E}X^*(t) \;\; &\leq \;\; 2^{-\frac{1}{\gamma}-\frac{1}{q}}\left(1+\frac{\gamma^q t^{q\alpha-q+\frac{q}{\gamma}}}{(\alpha\gamma-\gamma+1)^q}\mathbb{E}\left(\frac{\int_0^t \psi_s^q e(s)ds}{(1-(1-e(t)))}\right)\right)^{\frac{1}{q}} \\ &\quad \times\left(C_{r,\gamma}^{\frac{1}{\gamma}}\left(\mathbb{E}\langle M\rangle_t^{\frac{r\gamma}{2}}\right)^{\frac{1}{r\gamma}}+(\mathbb{E}(H^*(t))^\gamma)^{\frac{1}{\gamma}}\right).\end{aligned}$$

□

Next, we present the stochastic version of Henry inequality without quadratic term in the estimate part. The our result based on the following proposition [71].

Proposition 3.3.4. *For all $0<p<1$ and every continues local martingale $M(t),\ t\geq 0$ staring at $M(0)=0$, we have*

$$\mathbb{E}\left(\sup_{t\geq 0} M^p(t)\right)\leq c_p\mathbb{E}\left(\left(-\inf_{t\geq 0} M(t)\right)^p\right),$$

where $c_p=(4\wedge\frac{1}{p})\frac{\pi p}{\sin(\pi p)}$.

Theorem 3.3.5. *Let $0<\alpha<1,\ q,\ \gamma,\ \nu,\ \theta\in(1,\infty),\ \frac{1}{q}+\frac{1}{\gamma}=\frac{1}{\theta}+\frac{\nu}{\gamma}=1,\ \alpha\gamma\leq 1,\ \nu<q,\ X$ and H be non-negative, adapted processes with continuous paths, let M be a continuous local martingale starting at zero and ψ is non-negative and progressively measurable. Assume that X satisfies the following stochastic inequality:*

$$X_t^q\leq\int_0^t (t-s)^{\alpha-1}\psi_s X_s ds+M_t+H_t$$

then, there exists $c_q>0$ such that

$$\mathbb{E}X^*(t)\leq(1+c_q)^{\frac{1}{\nu}}\mathbb{E}\left(H^*(t)\right)^{\frac{1}{q}}+\frac{\gamma^{\frac{\gamma-1}{q}}t^{\frac{\gamma\alpha-\gamma+1}{q}}}{(\alpha\gamma-\gamma+1)^{\frac{\gamma}{q}}}\left(\mathbb{E}\left(e_q(t,0)\right)^{\frac{1}{q}}+c_q^{\frac{1}{\nu}}\left(\mathbb{E}\left(e_q(t,0)\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}}\right).$$

Proof. By Hölder and Young inequalities, we get

$$\begin{aligned}X_t^q \;\; &\leq \;\; \left(\int_0^t (t-s)^{\gamma\alpha-\gamma}ds\right)^{\frac{1}{\gamma}}\left(\int_0^t \psi_s^q X_s^q ds\right)^{\frac{1}{q}}+|M_t|+H_t \\ &\leq \;\; \frac{\gamma t^{\alpha-1+\frac{1}{\gamma}}}{\alpha\gamma-\gamma+1}\left(\int_0^t \psi_s^q X_s^q ds\right)^{\frac{1}{q}}+M_t+H_t \\ &\leq \;\; \frac{\gamma^\gamma t^{\gamma\alpha-\gamma+1}}{\gamma(\alpha\gamma-\gamma+1)^\gamma}+\frac{1}{q}\int_0^t \psi_s^q X_s^q ds+M_t+H_t.\end{aligned}$$

Then by classical Gronwall's lemma, we obtain

$$(X^*(t))^q \leq \frac{\gamma^\gamma t^{\gamma\alpha-\gamma+1}}{\gamma(\alpha\gamma-\gamma+1)^\gamma}+H_t+M_t+\frac{1}{q}\int_0^t \psi_s^q e_q(t,s)\left(M_s+H_s+\frac{\gamma^\gamma s^{\gamma\alpha-\gamma+1}}{\gamma(\alpha\gamma-\gamma+1)^\gamma}\right)ds,$$

where

$$e_q(t,s)=\exp\left(\int_s^t \frac{\psi_r^q}{q}dr\right),\ e_q(0,s)=\exp\left(-\int_0^s \frac{\psi_r^q}{q}dr\right) \quad t\geq s\geq 0.$$

Therefore

$$(X^*(t))^q \leq e_q(t,0)\left(\int_0^t e_q(0,s)dM_s+\frac{\gamma^\gamma t^{\gamma\alpha-\gamma+1}}{\gamma(\alpha\gamma-\gamma+1)^\gamma}+H^*(t)\right).$$

This implies that

$$R(t)+\frac{\gamma^\gamma t^{\gamma\alpha-\gamma+1}}{\gamma(\alpha\gamma-\gamma+1)^\gamma}+H^*(t)\geq 0 \Longrightarrow \frac{\gamma^\gamma t^{\gamma\alpha-\gamma+1}}{\gamma(\alpha\gamma-\gamma+1)^\gamma}+H^*(t)\geq -R^*(t).$$

Hence

$$\begin{aligned} X^*(t) &\leq (e_q(t,0))^{\frac{1}{q}}\left(R(t)+\frac{\gamma^\gamma t^{\gamma\alpha-\gamma+1}}{\gamma(\alpha\gamma-\gamma+1)^\gamma}+H^*(t)\right)^{\frac{1}{q}} \\ &\leq (e_q(t,0))^{\frac{1}{q}}(R(t)+H^*(t))^{\frac{1}{q}}+\frac{\gamma^{\frac{\gamma-1}{q}}t^{\frac{\gamma\alpha-\gamma+1}{q}}(e_q(t,0))^{\frac{1}{q}}}{(\alpha\gamma-\gamma+1)^{\frac{\gamma}{q}}}. \end{aligned}$$

By Hölder inequality, we have

$$\begin{aligned} \mathbb{E}X^*(t) &\leq \mathbb{E}\left(e_q(t,0)\right)^{\frac{1}{q}}(R(t)+H^*(t))^{\frac{1}{q}}+\frac{\gamma^{\frac{\gamma-1}{q}}t^{\frac{\gamma\alpha-\gamma+1}{q}}\mathbb{E}\left(e_q(t,0)\right)^{\frac{1}{q}}}{(\alpha\gamma-\gamma+1)^{\frac{\gamma}{q}}} \\ &\leq \left(\mathbb{E}\left(e_q(t,0)\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}}\left(\mathbb{E}\left(R(t)+H^*(t)\right)^{\frac{\nu}{q}}\right)^{\frac{1}{\nu}}+\frac{\gamma^{\frac{\gamma-1}{q}}t^{\frac{\gamma\alpha-\gamma+1}{q}}\mathbb{E}\left(e_q(t,0)\right)^{\frac{1}{q}}}{(\alpha\gamma-\gamma+1)^{\frac{\gamma}{q}}} \\ &\leq \left(\mathbb{E}\left(e_q(t,0)\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}}\left(\mathbb{E}\left(R(t)\right)^{\frac{\nu}{q}}+\mathbb{E}\left(H^*(t)\right)^{\frac{\nu}{q}}\right)^{\frac{1}{\nu}}+\frac{\gamma^{\frac{\gamma-1}{q}}t^{\frac{\gamma\alpha-\gamma+1}{q}}\mathbb{E}\left(e_q(t,0)\right)^{\frac{1}{q}}}{(\alpha\gamma-\gamma+1)^{\frac{\gamma}{q}}}. \end{aligned}$$

From proposition 3.3.4, we obtain

$$
\begin{aligned}
\mathbb{E}X^*(t) \quad &\leq \quad \left(\mathbb{E}\left(e_q(t,0)\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}} \left(c_p \mathbb{E}\left(\frac{\gamma^\gamma t^{\gamma\alpha-\gamma+1}}{\gamma(\alpha\gamma-\gamma+1)^\gamma} + H^*(t)\right)^{\frac{\nu}{q}} + \mathbb{E}\left(H^*(t)\right)^{\frac{\nu}{q}}\right)^{\frac{1}{\nu}} \\
&+ \quad \frac{\gamma^{\frac{\gamma-1}{q}} t^{\frac{\gamma\alpha-\gamma+1}{q}} \mathbb{E}\left(e_q(t,0)\right)^{\frac{1}{q}}}{(\alpha\gamma-\gamma+1)^{\frac{\gamma}{q}}} \\
&\leq \quad \left(\mathbb{E}\left(e_q(t,0)\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}} \left(\frac{c_q \gamma^{\frac{\nu(\gamma-1)}{q}} t^{\frac{\nu(\gamma\alpha-\gamma+1)}{q}}}{(\alpha\gamma-\gamma+1)^{\frac{\nu\gamma}{q}}} + (1+c_q)\mathbb{E}\left(H^*(t)\right)^{\frac{\nu}{q}}\right)^{\frac{1}{\nu}} \\
&+ \quad \frac{\gamma^{\frac{\gamma-1}{q}} t^{\frac{\gamma\alpha-\gamma+1}{q}} \mathbb{E}\left(e_q(t,0)\right)^{\frac{1}{q}}}{(\alpha\gamma-\gamma+1)^{\frac{\gamma}{q}}}.
\end{aligned}
$$

In conclusion, the theorem is proved. $\square$

Chapter 4

Systems of Impulsive Stochastic Differential Equations

Differential equations with impulses were considered for the first time by Milman and Myshkis [51] and then followed by a period of active research which culminated with the monograph by Halanay and Wexler [33]. Many phenomena and evolution processes in physics, chemical technology, population dynamics, and natural sciences may change state abruptly or be subject to short-term perturbations. These perturbations may be seen as impulses. Impulsive problems arise also in various applications in communications, mechanics (jump discontinuities in velocity), electrical engineering, medicine and biology. A comprehensive introduction to the basic theory is well developed in the monographs by Benchohra et al [6], Graef *et al* [29], Laskshmikantham *et al.* [4], Samoilenko and Perestyuk [70].

Random differential and integral equations play an important role in characterizing many social, physical, biological and engineering problems; see for instance the monograph of Da Prato and Zabczyk [21], Gard [25], Gikhman and Skorokhod [28], Sobzyk [73] and Tsokos and Padgett [74]. For example, a stochastic model for drug distribution in a biological system was described by Tsokos and Padgett [74] to a closed system with a simplified heat, one organ or capillary bed, and re-circulation of blood with a constant rate of flow, where the heart is considered as a mixing chamber of constant volume. For the basic theory concerning stochastic differential equations see the monographs of Bharucha-Reid [11], Tsokos and Padgett [74], Sobczyk [73] and Da Prato and Zabczyk [21].

By using classical fixed point theory, in [18, 26, 32, 38, 42–44, 68], the authors studied the existence and asymptotic stability and exponential stability for impulsive stochastic differential equations.

In [5], the authors studied the following system of impulsive random semilinear differential equations without Brownian motion,

$$\begin{cases} x'(t,\omega) = A_1(\omega)x(t,\omega) + f_1(t, x(t,\omega), y(t,\omega), \omega), \quad t \in J = [0,b] \\ y'(t,\omega) = A_2(\omega)y(t,\omega) + f_2(t, x(t,\omega), y(t,\omega), \omega), \quad t \in J = [0,b] \\ x(t_k^+,\omega) - x(t_k^-,\omega) = I_k(x(t_k^-,\omega), y(t_k^-,\omega)) \quad k = 1, 2, \dots, m \\ y(t_k^+,\omega) - y(t_k^-,\omega) = \overline{I}_k(x(t_k^-,\omega), y(t_k^-,\omega)) \quad k = 1, 2, \dots, m \\ x(\omega, 0) = \varphi_1(\omega), \quad \omega \in \Omega \\ y(\omega, 0) = \varphi_2(\omega), \quad \omega \in \Omega, \end{cases} \tag{4.0.1}$$

where X is a Banach space and $A_i : \Omega \times X \to X, i = 1, 2$ are random operators. They obtained the existence and uniqueness of solutions using fixed point theory in vector Banach spaces.

Recently in [14], the authors used the idea of fixed point theory in generalized Banach spaces to prove the existence of mild solutions of impulsive coupled systems of stochastic differential equations with fractional Brownian motion.

In this paper, we are interested in the questions of existence and uniqueness of solutions of the following system of problems:

$$\begin{cases} dx(t) &= \sum\limits_{l=1}^{\infty} f_l^1(t, x(t), y(t))dW^l(t) + g^1(t, x(t), y(t))dt, \ t \in J, t \neq t_k \\ dy(t) &= \sum\limits_{l=1}^{\infty} f_l^2(t, x(t), y(t))dW^l(t) + g^2(t, x(t), y(t))dt, \ t \in J, t \neq t_k \\ \Delta x(t) &= I_k(x(t_k)), \quad \Delta y(t) = \overline{I}_k(y(t_k)) \quad t = t_k \quad k = 1, 2, \dots, m \\ x(0) &= x_0 \in \mathbb{R}, \quad y(0) = y_0 \in \mathbb{R}, \end{cases} \tag{4.0.2}$$

where $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T, \ \ J := [0,T]$. $f^1, f^2, g^1, g^2 : J \times \mathbb{R}^2 \to \mathbb{R}$ are Carathéodory functions, and W^l is an infinite sequence of independent standard Brownian motions, $l = 1, 2, \dots$ and $I_k, \overline{I}_k \in C(\mathbb{R}, \mathbb{R})$ $(k = 1, \dots, m)$, and $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-), \Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$. The notations $y(t_k^+) = \lim\limits_{h \to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim\limits_{h \to 0^+} y(t_k - h)$ stand for the right and the left limits of the function y at $t = t_k$, respectively. Set

$$\begin{cases} f^i(\cdot, x, y) = (f_1^i(\cdot, x, y), f_2^i(\cdot, x, y), \ldots), \\ \|f^i(\cdot, x, y)\| = \Big(\sum_{l=1}^{\infty} (f_l^i)^2(\cdot, x, y)\Big)^{\frac{1}{2}} \end{cases} \tag{4.0.3}$$

where $i = 1, 2,\ f^i(\cdot, x, y) \in l^2$ for all $x \in \mathbb{R}$.

In recent years, in the absence of random effect and stochastic analysis many authors studied the existence of solutions for systems of differential and difference equations with and without impulses by using the vector version of the fixed point theorem (see [7, 13, 31, 37, 39, 53–55, 62], the monograph of Graef *et al* [29], and the references therein).

This paper is organized as follows. In Sections 2, we introduce all the background material used in this paper such as stochastic calculus and some properties of generalized Banach space. In Section 4, we state some results for fixed point theorems in generalized Banach spaces. Finally, an application of Schaefer's and and Perov fixed point theorems in generalized Banach spaces are used to prove the existence of solutions to problem (4.0.2).

4.1 Generalized metric and Banach spaces

In this section we define vector metric spaces and generalized Banach spaces and prove some properties. If, $x, y \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for all $i = 1, \ldots, n$. Also $|x| = (|x_1|, \ldots, |x_n|)$ and $\max(x, y) = \max(\max(x_1, y_1), \ldots, \max(x_n, y_n))$. If $c \in \mathbb{R}$, 36 then $x \leq c$ means $x_i \leq c$ for each $i = 1, \ldots, n$. For $x \in \mathbb{R}^n$, $(x)_i = x_i,\ i = 1, \ldots, n$.

Definition 4.1.1. *Let X be a nonempty set. By a vector-valued metric on X we mean a map $d : X \times X \to \mathbb{R}^n$ with he following properties:*

(i) $d(u, v) \geq 0$ for all $u, v \in X$; if $d(u, v) = 0$ if and only if $u = v$.

(ii) $d(u, v) = d(v, u)$ for all $u, v \in X$.

(iii) $d(u, v) \leq d(u, w) + d(w, v)$ for all $u, v, w \in X$.

Note that for any $i \in \{1, \ldots, n\}$ $\ (d(u, v))_i = d_i(u, v)$ is a metric space in X.

We call the pair (X, d) a generalized metric space. For $r = (r_1, r_2, \ldots, r_n) \in \mathbb{R}^n_+$, we will denote by

$$B(x_0, r) = \{x \in X : \; d(x_0, x) < r\}$$

the open ball centrad in x_0 with radius r and

$$\overline{B(x_0, r)} = \{x \in X : d(x_0, x) \leq r\}$$

the closed ball centered in x_0 with radius r.

Definition 4.1.2. *Let E be a vector space on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. By a vector-valued norm on E we mean a map $\| \cdot \| : \; E \to \mathbb{R}^n_+$ with the following properties:*

(i) $\|x\| \geq 0$ *for all* $x \in E$ *; if* $\|x\| = 0$ *then* $x = 0$

(ii) $\|\lambda x\| = |\lambda| \|x\|$ *for all* $x \in E$ *and* $\lambda \in \mathbb{K}$

(iii) $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in E$.

The pair $(E, \| \cdot \|)$ is called a generalized normed space. If the generalized metric generated by $\| \cdot \|$ (i.e $d(x, y) = \|x - y\|$) is complete then the space $(E, \| \cdot \|)$ is called a generalized Banach space, where

$$\|x - y\| = \begin{pmatrix} \|x - y\|_1 \\ \cdots \\ \|x - y\|_n \end{pmatrix}.$$

Notice that $\| \cdot \|$ is a generalized Banach space on E if and only if $\| \cdot \|_i, \; i = 1, \ldots, n$ are norms on E.

Remark 4.1.1. *In generalized metric space in the sense of Perov's, the notations of convergence sequence, Cauchy sequence, completeness, open subset and closed subset are similar to those for usual metric spaces.*

Definition 4.1.3. *A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc.*

Lemma 4.1.1. *[67] Let M be a square matrix of nonnegative numbers. The following assertions are equivalent:*

(i) M is convergent towards zero;

(ii) the matrix $I - M$ is non-singular and

$$(I - M)^{-1} = I + M + M^2 + \ldots + M^k + \ldots;$$

(iii) $\|\lambda\| < 1$ for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$

(iv) $(I - M)$ is non-singular and $(I - M)^{-1}$ has nonnegative elements.

4.2 Fixed point theory

From the theorem of contraction principle extended an contractive maps on Banach spaces endowed with vector-valued metric space by Perov in 1964 [60], Perov and Kibenko [59] and Precup [59]. In this section is to present the version of Schaefer's fixed point theorem and nonlinear alternative of Leary-Schauder type in generalized Banach spaces.

Theorem 4.2.1. *[60]Let (X, d) be a complete generalized metric space and let $N : X \longrightarrow X$ be such that*

$$d(N(x), N(y)) \leq Md(x, y)$$

for all $x, y \in X$ and some square matrix M of nonnegative numbers. If the matrix M is convergent to zero, that is $M^k \longrightarrow 0$ as $k \longrightarrow \infty$, then N has a unique fixed point $x_* \in X$

$$d(N^k(x_0), x_*) \leq M^k(I - M)^{-1}d(N(x_0), x_0)$$

for every $x_0 \in X$ and $k \geq 1$.

Theorem 4.2.2. *[30, 75] Let E be a generalized Banach space, $C \subset E$ be a nonempty closed convex subset of E and $N : C \to C$ be a continuous operator such that $N(C)$ is relatively compact. Then N has at least fixed point in C.*

As a consequence of Schauder fixed point theorem we present the version of Schaefer's fixed point theorem in generalized Banach space.

Theorem 4.2.3. *[30] Let $(E, \|\cdot\|)$ be a generalized Banach space and $N : E \to E$ is a continuous compact mapping. Moreover assume that the set*

$$\mathcal{A} = \{x \in E : x = \lambda N(x) \quad \text{for some } \lambda \in (0, 1)\}$$

is bounded. Then N has a fixed point.

4.3 Existence and Uniqueness Results

Let $J_k = (t_k, t_{k+1}],\ k = 1, 2, \ldots, m$. In order to define a solution for Problem (4.0.2), consider the pace of piece-wise continuous functions

$$\begin{array}{rcl} PC & = & \{x:\ \Omega\times J \longrightarrow \mathbb{R},\ x \in C(J_k, \mathbb{R}),\ k = 1, \ldots, m \text{ such that} \\ & & x(t_k^+, \cdot) \text{ and } x(t_k^-, \cdot) \text{ exist with } x(t_k^-, \cdot) = x(t_k, \cdot)\}, \end{array}$$

Endowed with the norm

$$\|x\|_{PC}^2 = \sup_{t\in J} \mathbb{E}|x(t,\cdot)|^2$$

PC is a Banach space with norm $\|\cdot\|_{PC}$.

Definition 4.3.1. *An $\mathbb{R}-$ valued stochastic process $u = (x, y) \in PC\times PC$ is said to be a solution of* (4.0.2) *with respect to the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, *if:*

1) $u(t)$ is $\mathcal{F}_t$-adapted for all $t \in J_k = (t_k, t_{k+1}]\quad k = 1, 2, \ldots, m$

2) $u(t)$ is right continuous and has limit on the left;

3) $u(t)$ satisfies that

$$\left\{\begin{array}{rcl} x(t) & = & x_0 + \displaystyle\sum_{l=1}^{\infty}\int_0^t f_l^1(s, x(s), y(s))dW^l(s) \\ & & +\int_0^t g^1(s, x(s), y(s))ds + \displaystyle\sum_{0\le t_k\le t} I_k(x(t_k)), \quad t\in J \\ \\ y(t) & = & y_0 + \displaystyle\sum_{l=1}^{\infty}\int_0^t f_l^2(s, x(s), y(s))dW^l(s) \\ & & +\int_0^t g^2(s, x(s), y(s))ds + \displaystyle\sum_{0\le t_k\le t} \overline{I}_k(y(t_k)),\ t\in J. \end{array}\right.$$

Let us introduce the following hypothesis:

(H_1) There exist nonnegative numbers a_i and $b_i,\ i = 1, 2$ such that for all x, y, $\overline{x}$, $\overline{y} \in \mathbb{R}, t \in J$ we have

$$\mathbb{E}(|f^i(t, x, y) - f^i(t, \overline{x}, \overline{y}|^2) \le a_i\mathbb{E}(|x-\overline{x}|^2) + b_i\mathbb{E}(|y-\overline{y}|^2).$$

(H_2) There exist positive constants α_i and β_i, $i = 1, 2$ such that for all x, y, $\overline{x}$, $\overline{y} \in \mathbb{R}, t \in J$ we have

$$\mathbb{E}(|g^i(t,x,y) - g^i(t,\overline{x},\overline{y})|^2) \leq \alpha_i \mathbb{E}(|x-\overline{x}|)^2 + \beta_i \mathbb{E}(|y-\overline{y}|^2).$$

(H_3) There exist constants $d_k \geq 0$ and $\overline{d}_k \geq 0$, $k = 1, \dots, m$ such that for all x, y, $\overline{x}$, $\overline{y} \in \mathbb{R}$

$$\mathbb{E}(|I_k(x)-I_k(\overline{x})|^2) \leq d_k \mathbb{E}(|x-\overline{x}|)^2, \;\; \mathbb{E}(|\overline{I}_k(y)-\overline{I}_k(\overline{y})|^2) \leq \overline{d}_k \mathbb{E}(|y-\overline{y}|^2).$$

Our first main result in this section is based on Perov's fixed point theorem.

Theorem 4.3.1. *Assume that* $(H_1) - (H_3)$ *are satisfied and the matrix*

$$M = \sqrt{3}\begin{pmatrix} \sqrt{C_2 a_1 + \alpha_1 T + l_1} & \sqrt{C_2 b_1 + \beta_1 T} \\ \sqrt{C_2 a_2 + \alpha_2 T} & \sqrt{C_2 b_2 + \beta_2 T + l_2} \end{pmatrix}, \;\; l_1 = \sum_{k=1}^{m} d_k, \;\; l_2 = \sum_{k=1}^{m} \overline{d}_k,$$

where $C_2 \geq 0$ *is defined in Theorem 2.1.13. If* M *converges to zero. Then the problem* (4.0.2) *has unique solution.*

Proof. Let $X = PC \times PC$. Consider the operator $N : X \to X$ defined by

$$N(x,y) = (N_1(x,y), N_2(x,y)), \;\; (x,y) \in PC \times PC$$

where

$$N_1(x(t),y(t)) = x_0 + \sum_{l=1}^{\infty} \int_0^t f_l^1(s,x(s),y(s))dW^l(s) + \int_0^t g^1(s,x(s),y(s))ds + \sum_{0<t_k\leq t} I_k(x(t_k))$$

and

$$N_2(x(t),y(t)) = y_0 + \sum_{l=1}^{\infty} \int_0^t f_l^2(s,x(s),y(s))dW^l(s) + \int_0^t g^2(s,x(s),y(s))ds + \sum_{0<t_k\leq t} \overline{I}_k(y(t_k)).$$

Fixed points of operator N are solutions of problem (4.0.2).

We shall use Theorem **??**to prove that N has a fixed point. Indeed, let $(x,y), (\overline{x},\overline{y}) \in X$. Then we ave for each $t \in J$

$$\begin{aligned} |N_1(x(t),y(t)) - N_1(\overline{x}(t),\overline{y}(t))|^2 &\leq 3\left|\sum_{l=1}^{\infty}\int_0^t (f_l^1(s,x(s),y(s)) - f_l^1(s,\overline{x}(s),\overline{y}(s)))dW^l(s)\right|^2 \\ &\quad +3\left|\int_0^t (g^1(s,x(s),y(s) - g^1(s,\overline{x}(s),\overline{y}(s))ds\right|^2 \\ &\quad +3\sum_{k=1}^{m} |I_k(x(t_k)) - I_k(\overline{x}(t_k))|^2. \end{aligned}$$

By theorem 2.1.13, we get

$$\begin{aligned}\mathbb{E}|N_1(x(t),y(t))-N_1(\overline{x}(t),\overline{y}(t))|^2 &\leq 3C_2\int_0^t \mathbb{E}|f^1(s,x(s),y(s))-f^1(s,\overline{x}(s),\overline{y}(s))|^2ds\\ &+3t\int_0^t \mathbb{E}|g^1(s,x(s),y(s))-g^1(s,\overline{x}(s),\overline{y}(s))|^2ds\\ &+3\sum_{k=1}^m \mathbb{E}|I_k(x(t_k))-I_k(\overline{x}(t_k))|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\sup_{t\in J}\mathbb{E}|N_1(x(t),y(t))-N_1(\overline{x}(t),\overline{y}(t))|^2 &\leq 3(C_2a_1+\alpha_1T+l_1)\|x-\overline{x}\|^2_{PC}\\ &+3(C_2b_1+\beta_1T)\|y-\overline{y}\|^2_{PC}.\end{aligned}$$

Similarly we have

$$\begin{aligned}\|N_2(x,y)-N_2(\overline{x},\overline{y})\|^2_{PC} &\leq 3(C_2a_2+\alpha_2T)\|x-\overline{x}\|^2_{PC}\\ &+(C_2b_2+\beta_2T+l_2)\|y-\overline{y}\|^2_{PC}.\end{aligned}$$

Hence

$$\begin{aligned}\|N(x,y)-N(\overline{x},\overline{y})\|_X &= \begin{pmatrix}\|N_1((x,y)-N_1(\overline{x},\overline{y})\|_{PC}\\ \|N_2(x,y)-N_2(\overline{x},\overline{y})\|_{PC}\end{pmatrix}\\ &\leq \sqrt{3}\begin{pmatrix}\sqrt{C_2a_1+\alpha_1T+l_1} & \sqrt{C_2b_1+\beta_1T}\\ \sqrt{C_2a_2+\alpha_2T} & \sqrt{C_2b_2+\beta_2T+l_2}\end{pmatrix}\begin{pmatrix}\|x-\overline{x}\|_{PC}\\ \|y-\overline{y}\|_{PC}\end{pmatrix}.\end{aligned}$$

Therefore

$$\|N(x,y)-N(\overline{x},\overline{y})\|_X \leq M\begin{pmatrix}\|x-\overline{x}\|_{PC}\\ \|y-\overline{y}\|_{PC}\end{pmatrix},\ \text{for all}, (x,y),(\overline{x},\overline{y})\in X.$$

From Perov's fixed point theorem, the mapping N has a unique fixed $(x,y)\in PC\times PC$ which is unique solution of problem (4.0.2). □

In this section we present the existence result under nonlinearities f^i and g^i, $i=1,2$ satisfying a Nagumo type growth conditions:

(H_4) There exist a function $p_i\in L^1(J,\mathbb{R}^+)$ and a continuous nondecreasing function $\psi_i:[0,\infty)\to[0,\infty)$for each $i=1,2$ such that for all $x,y\in\mathbb{R}$

$$\mathbb{E}(\|f^1(t,x,y)\|^2)\leq p_1(t)\psi_1(\mathbb{E}(|x|^2+|y|^2)),\ \mathbb{E}(\|f^2(t,x,y)\|)^2\leq p_2(t)\psi_2(\mathbb{E}(|x|^2+|y|^2))$$

with
$$\int_0^T m_1(s)ds < \int_{v_1}^{\infty} \frac{ds}{\psi_1(s)+\psi_2(s)}$$
where $m_1(t) = \max\{4C_2p_1(t),\ 4Tp_2(t)\}$,$v_1 = 4\mathbb{E}|x_0|^2 + 4\sum_{k=1}^m c_k$

(H_5) There exist a function $p_i \in L^1(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi_i : [0,\infty) \to [0,\infty)$ for each $i = 3,4$ such that for all $x, y, \in \mathbb{R}$ we have
$$E(|g^i(t,x,y)|^2) \leq p_i(t)\psi_3(E(|x|^2+|y|^2)),$$
with
$$\int_0^T m_2(s)ds < \int_{v_2}^{\infty} \frac{ds}{\psi_3(s)+\psi_4(s)}$$
where
$$m_2(t) = \max\{4C_2p_3(t), 4Tp_4(t)\},\ v_2 = 4\mathbb{E}|y_0|^2 + \sum_{k=1}^m \widetilde{c}_k.$$

(H_6) There exist positive constants $c_k, \widetilde{c}_k, k = 1,\ldots,m$, such that
$$\mathbb{E}(|I_k(x)|)^2 \leq c_k, \quad \mathbb{E}(|\overline{I}_k(y)|)^2 \leq \widetilde{c}_k \quad \text{for all } (x,y) \in \mathbb{R}^2.$$

Theorem 4.3.2. *Assume that* $(H_4) - (H_6)$ *hold. Then* (4.0.2) *has at least one solution on* J.

Proof. Clearly, the fixe point of N are solutions to (4.0.2), where N is defined in Theorem 4.3.1. In order to apply Lemma **??**, we first show that N is completely continuous. The proof will be given in several steps.

- **Step 1.** $N = (N_1, N_2)$ is continuous.
  Let (x_n, y_n) be a sequence such that $(x_n, y_n) \to (x, y) \in PC \times PC$ as $n \to \infty$. Then
$$\begin{array}{c} |N_1(x_n(t), y_n(t)) - N_1(x(t), y(t))|^2 \\ \leq \\ 3\left|\sum_{l=1}^{\infty}\int_0^t (f_l^1(s, x_n(s), y_n(s)) - f_l^1(s, x(s), y(s)))dW^l(s)\right|^2 \\ +3\left|\int_0^t (g^1(s, x(s), y(s)) - g^1(s, x(s), y(s)))ds\right|^2 \\ +3\sum_{k=1}^m |I_k(x_n(t_k)) - I_k(x(t_k))|^2 . \end{array}$$

From Theorem 2.1.13, we obtain

$$\begin{aligned}
&\mathbb{E}|N_1(x_n(t),y_n(t)) - N_1(x(t),y(t))|^2\\
&\leq\\
&3C_2\int_0^t \mathbb{E}|f^1(s,x_n(s),y_n(s)) - f^1(s,x(s),y(s))|^2 ds\\
&+3t\int_0^t \mathbb{E}|g^1(s,x_n(s),y_n(s)) - g^1(s,x(s),y(s))|^2 ds\\
&+3\sum_{k=1}^m \mathbb{E}|I_k(x_n(t_k)) - I_k(x(t_k))|^2.
\end{aligned}$$

Since f^1, g^1 is an Carathéodory function and $I_k, \overline{I}_k$ are continuous functions. By Lebesgue dominated convergence theorem, we get

$$\begin{gathered}
\sup_{t\in J}\mathbb{E}|N_1(x_n(t),y_n(t)) - N_1(x(t),y(t))|^2\\
\leq\\
3C_2\mathbb{E}\|f^1(\cdot,x_n,y_n) - f^1(\cdot,x,y)\|_{L^2}^2\\
+3T\mathbb{E}\|g^1(\cdot,x_n,y_n) - g^1(\cdot,x,y)\|_{L^2}^2\\
+3\sum_{k=1}^m \mathbb{E}|I_k(x_n(t_k)) - I_k(x(t_k))|^2 \to 0 \text{ as } n\to\infty.
\end{gathered}$$

Similarly

$$\begin{gathered}
\sup_{t\in J}\mathbb{E}|N_2(x_n(t),y_n(t)) - N_1(x(t),y(t))|^2\\
\leq\\
3C_2\mathbb{E}\|f^2(\cdot,x_n,y_n) - f^2(\cdot,x,y)\|_{L^2}^2\\
+3T\mathbb{E}\|g^2(\cdot,x_n,y_n) - g^2(\cdot,x,y)\|_{L^2}\\
+3\sum_{k=1}^m \mathbb{E}|\overline{I}_k(y_n(t_k)) - \overline{I}_k(y(t_k))|^2 \to 0 \text{ as } n\to\infty.
\end{gathered}$$

Thus N is continuous.

- **Step 2.** N maps bounded sets into bounded sets in $PC\times PC$. Indeed, it is enough to show that for any $q>0$ there exists a positive constant l such that for each

$$(x,y)\in B_q = \{(x,y)\in PC\times PC : \|x\|_{PC}\leq q,\ \|y\|\leq q\},$$

we have

$$\|N(x,y)\|_{PC}\leq l = (l_1,l_2).$$

Then for each $t\in J$, we get

$$\begin{gathered}
|N_1(x(t),y(t))|^2\\
\leq\\
4|x_0|^2 + 4|\sum_{l=1}^\infty \int_0^t f_l^1(s,x(s),y(s))dW^l(s)|^2\\
+4|\int_0^t g^1(s,x(s),y(s))ds|^2 + 4|\sum_{k=1}^m I_k(x(t_k))|^2.
\end{gathered}$$

Using the inequality (1.3.21), so we get

$$\begin{aligned}\mathbb{E}|N_1(x(t),y(t))|^2 &\leq 4\mathbb{E}|x_0|^2+4C_2\|p_1\|_{L^1}\psi_1(2q)\\ &\quad +4T\|p_3\|_{L^1}\psi_2(2q)ds+4\sum_{k=1}^{m}c_k.\end{aligned}$$

Therefore

$$\|N_1(x,y)\|_{PC}\leq 4\mathbb{E}|x_0|^2+4C_2\|p_1\|_{L^1}\psi_1(2q)+4\|p_2\|_{L^1}\psi_2(2q)ds+4\sum_{k=1}^{m}c_k:=l_1.$$

Similarly, we have

$$\|N_2(x,y)\|_{PC}\leq 4\mathbb{E}|x_0|^2+4C_2\|p_3\|_{L^1}\psi_2(q)+4\|p_4\|_{L^1}\psi_4(q)ds+4\sum_{k=1}^{m}\widetilde{c}_k:=l_2.$$

- **Step 3.** N maps bounded sets into equicontinuous sets of $PC\times PC$. Let B_q be a bounded set in $PC\times PC$ as in Step 2. Let $r_1,r_2\in J, r_1<r_2$ and $u\in B_q$. Thus we have

$$\begin{aligned}|N_1(x(r_2),y(r_2))-N_1(x(r_1),y(r_1))|^2 &\leq 3\left|\sum_{l=1}^{\infty}\int_{r_1}^{r_2}f_l^1(s,x(s),y(s))dW^l(s)\right|^2\\ &\quad +3\left|\int_{r_1}^{r_2}g^1(s,x(s),y(s))ds\right|^2\\ &\quad +3\sum_{r_1\leq t_k\leq r_2}|I_k(x(t_k))|^2.\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{E}|N_1(x(r_2),y(r_2))-N_1(x(r_1),y(r_1))|^2 &\leq 3C_2\psi_1(q)\int_{r_1}^{r_2}p_1(s)ds+T\psi_2(q)\int_{r_1}^{r_2}p_2(s)ds\\ &\quad +3\sum_{r_1\leq t_k\leq t_2}c_k.\end{aligned}$$

The right-hand term tends to zero as $|r_2-r_1|\to 0$. As a consequence of Steps 1 to 3 together with the Arzelà-Ascoli, we conclude that N maps B_q into a precompact set in $PC\times PC$.

- **Step 4.** It remains to show that

$$\mathcal{A} = \{(x, y) \in PC \times PC : \ (x, y) = \lambda N(x, y), \lambda \in (0, 1)\}$$

is bounded.
Let $(x, y) \in \mathcal{A}$. Then $x = \lambda N_1(x, y)$ and $y = \lambda N_2(x, y)$ for some $0 < \lambda < 1$. Thus, for $t \in J$, we have

$$\begin{aligned}\mathbb{E}|x(t)|^2 \ &\leq \ 4\mathbb{E}|x_0|^2 + 4C_2 \int_0^t p_1(s)\psi_1(\mathbb{E}|x(s)|^2 + \mathbb{E}|y(s)|^2)ds \\ &\quad +4T \int_0^t p_2(s)\psi_2(\mathbb{E}|x(s)|^2 + \mathbb{E}|y(s)|^2)ds + 4\sum_{k=1}^m c_k.\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{E}|x(t)|^2 \ &\leq \ 4\mathbb{E}|x_0|^2 + 4C_2 \int_0^t p_1(s)\psi_1(\mathbb{E}|x(s)|^2 + \mathbb{E}|y(s)|^2)ds \\ &\quad +4T \int_0^t p_2(s)\psi_2(\mathbb{E}|x(s)|^2 + \mathbb{E}|y(s)|^2)ds + 4\sum_{k=1}^m c_k\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}|y(t)|^2 \ &\leq \ 4\mathbb{E}|x_0|^2 + 4C_2 \int_0^t p_3(s)\psi_2(\mathbb{E}|x(s)|^2 + E|y(s)|^2)ds \\ &\quad +4T \int_0^t p_4(s)\psi_3(\mathbb{E}|x(s)|^2 + \mathbb{E}|y(s)|^2)ds + 4\sum_{k=1}^m \widetilde{c}_k.\end{aligned}$$

Therefore

$$\mathbb{E}|x(t)|^2 + \mathbb{E}|y(t)|^2 \ \leq \ \gamma + \int_0^t p(s)\phi(\mathbb{E}|x(s)|^2 + \mathbb{E}|y(s)|^2)ds,$$

where

$$\gamma = 8\mathbb{E}|x_0|^2 + 4\sum_{k=1}^m (c_k + \widetilde{c}_k), \ p(t) = m_1(t) + m(t), \text{ and } \phi(t) = \sum_{i=1}^m \psi_i(t).$$

By the Gronwal inequality, we have

$$\mathbb{E}|x(t)|^2 + \mathbb{E}|y(t)|^2 \ \leq \ \Gamma^{-1}\left(\int_\gamma^T p(s)ds\right) := K, \text{ for each } t \in J,$$

where

$$\Gamma(z) = \int_{\gamma}^{z} \frac{du}{\phi(u)}.$$

Consequently

$$\|x\|_{PC} \le K \quad \text{and } \|y\|_{PC} \le K.$$

This shows that $\mathcal{E}$ is bounded. As a consequence of Theorem **??** we deduce that N has a fixed point (x, y) which is a solution to the problem (4.0.2).

□

The goal of the second result of this section is to apply Schauder's fixed point. For the study of this problem we first introduce the following hypotheses:

(H_7) There exist nonnegative numbers $\overline{a}_i$ and $\overline{b}_i, c_i,\ i = 1, 2$ such that for all $x, y \in \mathbb{R}$, we have

$$\mathbb{E}(|f_i(t, x, y)|^2) \le \overline{a}_i \mathbb{E}(|x|)^2 + \overline{b}_i \mathbb{E}(|y|)^2 + c_1.$$

(H_8) There exist positive constants $\overline{\alpha}_i$ and $\overline{\beta}_i, \lambda_i,\ i = 1, 2$ such that for all $x, y \in \mathbb{R}$, we have

$$\mathbb{E}(|g_i(t, x, y)|^2) \le \overline{\alpha}_i \mathbb{E}(|x|)^2 + \overline{\beta}_i \mathbb{E}(|y|)^2 + \lambda_1.$$

(H_9) There exist constants $d \ge 0, \overline{d} \ge 0$ and $e_i \ge 0,\ i = 1, 2$ and $k = 1, \dots, m$ such that

$$\sum_{k=1}^{m} \mathbb{E}|I_k(x)|^2 \le dE|x|^2 + e_1, \quad \sum_{k=1}^{m} \mathbb{E}|\overline{I}_k(x)|^2 \le \overline{d}\mathbb{E}|x|^2 + e_2, \quad \text{for all } x \in \mathbb{R}.$$

Theorem 4.3.3. *Assume $(H_7) - (H_9)$ hold and*

$$M_{a,b} = \sqrt{2} \begin{pmatrix} \sqrt{C_2\overline{a}_1 + \overline{\alpha}_1 T + d} & \sqrt{C_2\overline{b}_1 + \overline{\beta}_1 T} \\ \sqrt{C_2\overline{a}_2 + \overline{\alpha}_2 T} & \sqrt{C_2\overline{b}_2 + \overline{\beta}_2 T + \overline{d}} \end{pmatrix}$$

converges to zero, then problem (4.0.2) has at least one solution.

Proof. Let $X = PC \times PC$. Consider the operator $N = (N_1, N_2) : PC \times \times PC \longrightarrow PC \times \times PC$ defined for $x, y \in PC$ by

$$\begin{aligned} N_1(x(t), y(t)) &= x_0 + \sum_{l=1}^{\infty} \int_0^t f_l^1(s, x(s), y(s)) dW^l(s) + \int_0^t g^1(s, x(s), y(s)) ds \\ &\quad + \sum_{0<t_k<t} I_k(x(t_k)) \end{aligned}$$

and

$$\begin{aligned} N_2(x(t), y(t)) &= y_0 + \sum_{l=1}^{\infty} \int_0^t f_l^2(s, x(s), y(s)) dW^l(s) + \int_0^t g^2(s, x(s), y(s)) ds \\ &\quad + \sum_{0<t_k<t} \overline{I}_k(y(t_k)). \end{aligned}$$

Set

$$D = \{(x, y) \in PC \times PC : \|x\|_{PC} \leq R_1, \|y\|_{PC} \leq R_2\}.$$

Obviously, the set D is a bounded closed convex set in space $PC \times PC$. Clear that

$$\begin{aligned} |N_1(x(t), y(t))|^2 &\leq 4|x_0|^2 + 4|\textstyle\sum_{l=1}^{\infty} \int_0^t f_l^1(s, x(s), y(s)) dW^l(s)|^2 \\ &\quad + 4|\textstyle\int_0^t g^1(s, x(s), y(s)) ds|^2 + |4 \displaystyle\sum_{k=1}^{m} I_k(x(t_k))|^2. \end{aligned}$$

From the inequality (1.3.21), we get

$$\begin{aligned} \mathbb{E}|N_1(x(t), y(t))|^2 &\leq 4\mathbb{E}|x_0|^2 + 4C_2\overline{a}_1 \int_0^t \mathbb{E}|x(s)|^2 ds + 4\overline{b}_1 C_2 \int_0^t \mathbb{E}|y(s)|^2 ds + 4c_1 T \\ &\quad + 4T\overline{\alpha}_1 \int_0^t \mathbb{E}|x(s)|^2 ds + 4\overline{\beta}_1 T \int_0^t \mathbb{E}|y(s)|^2 ds \\ &\quad + 4\lambda_1 T + 4d\mathbb{E}|x|^2 + 4e_1, \end{aligned}$$

thus

$$\begin{aligned} \sup_{t\in J} \mathbb{E}|N_1(x(t), y(t))|^2 &\leq 4(C_2\overline{a}_1 + \overline{\alpha}_1 T + d)\|x\|_{PC} + 4(C_2\overline{b}_1 + \overline{\beta}_1 T)\|y\|_{PC} \\ &\quad + 4\mathbb{E}|x_0|^2 + 4e_1 + 4Tc_1 + 4T\lambda_1. \qquad (4.3.1) \end{aligned}$$

From (4.3.1) we obtain that

$$\|N_1(x,y))\|_{PC} \leq \widetilde{a}_1\|x\|_{PC} + \widetilde{b}_1\|y\|_{PC} + \widetilde{c}_1 \tag{4.3.2}$$

where

$$\widetilde{a}_1 = 2\sqrt{C_2\overline{a}_1 + \overline{\alpha}_1 T + d},\ \widetilde{b}_1 = 2\sqrt{C_2\overline{b}_1 + \overline{\beta}_1 T},\ \widetilde{c}_1 = 2\sqrt{\mathbb{E}|x_0|^2 + e_1 + Tc_1 + T\lambda_1}.$$

Similarly we have

$$\|N_2(x,y)\|_{PC} \leq \widetilde{a}_2\|x\|_{PC} + \widetilde{b}_2\|y\|_{PC} + \widetilde{c}_2 \tag{4.3.3}$$

where

$$\widetilde{a}_2 = 2\sqrt{C_2\overline{a}_2 + 4\overline{\alpha}_2 T},\ \ \widetilde{b}_2 = 2\sqrt{C_2\overline{b}_2 + \overline{\beta}_2 T + \overline{d}},\text{ and } \widetilde{c}_2 = 2\sqrt{\mathbb{E}|y_0|^2 + 4e_2 + Tc_2 + T\lambda_2}.$$

Now (4.3.2), (4.3.3) can be put together as

$$\begin{aligned}
\|N(x,y)\|_X &= \begin{pmatrix} \|N_1(x,y)\|_{PC} \\ \|N_2(x,y)\|_{PC} \end{pmatrix} \\
&\leq 2\begin{pmatrix} \sqrt{C_2\overline{a}_1 + \overline{\alpha}_1 T + d} & \sqrt{C_2\overline{b}_1 + \overline{\beta}_1 T} \\ \sqrt{C_2\overline{a}_2 + \overline{\alpha}_2 T} & \sqrt{C_2\overline{b}_2 + \overline{\beta}_2 T + \overline{d}} \end{pmatrix} \begin{pmatrix} \|x\|_{PC} \\ \|y\|_{PC} \end{pmatrix} + \begin{pmatrix} \widetilde{c}_1 \\ \widetilde{c}_2 \end{pmatrix}.
\end{aligned}$$

Therefore

$$\|N(x,y)\|_X \leq M_{a,b}\begin{pmatrix} \|x\|_{PC} \\ \|y\|_{PC} \end{pmatrix} + \begin{pmatrix} \widetilde{c}_1 \\ \widetilde{c}_2 \end{pmatrix}.$$

Since $M_{a,b} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$, $N(x,y)$ converges to zero. Next, we look for two positive numbers R_1, R_2 such that if $\|x\|_{PC} \leq R_1$, $\|y\|_{PC} \leq R_2$, then $\|N_1(x,y)\|_{PC} \leq R_1$, $\|N_2(x,y)\|_{PC} \leq R_1$. To this end it is sufficient that

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \leq M_{a,b}\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} + \begin{pmatrix} \widetilde{c}_1 \\ \widetilde{c}_2 \end{pmatrix}$$

whence

$$(I - M_{a,b})\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \leq \begin{pmatrix} \widetilde{c}_1 \\ \widetilde{c}_2 \end{pmatrix}$$

that is

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \leq (I - M_{a,b})^{-1} \begin{pmatrix} \widetilde{c}_1 \\ \widetilde{c}_2 \end{pmatrix}.$$

We show that there exist $R_1, R_2 > 0$ such that

$$N(D) \subseteq D$$

where

$$D = \{(x, y) \in PC \times PC : \|x\|_{PC} \leq R_1, \|y\|_{PC} \leq R_2\}.$$

It clear that $N(D) \subset D$. Hence by Theorem 4.1.1,the operator N has at least one fixed point which is solution of (4.0.2). □

4.3.1 An example

In this section we consider the following example of stochastic differential equation:

$$\begin{cases} dx(t) &= \sum_{l=1}^{\infty}(a_{2l+1}\sin k^2 x + a_{2l}\cos l^2 y)dW^l(t) \\ & +d_1(t + x(t) + y(t))dt, \ t \in [0,1], t \neq \frac{1}{2} \\ dy(t) &= \sum_{l=1}^{\infty}(b_{2l+1}\sin k^2 x + b_{2l}\cos l^2 y)dW^l(t) \\ & +d_2(t + x(t) + y(t))dt, \ t \in [0,1], t \neq \frac{1}{2} \\ \Delta x(t) &= c_1\dfrac{x(t)}{1+|x(t)|}, \ \Delta y(t) = c_1\dfrac{y(t)}{1+|y(t)|}, \quad t = \dfrac{1}{2} \\ x(0) &= x_0, \ y(0) = y_0 \end{cases} \tag{4.3.4}$$

where $c_1, c_2 \in \mathbb{R}$, $(a_l)_{l\in\mathbb{N}}, (b_l)_{l\in\mathbb{N}} \in l^2$, $f_1, f_2 : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$f_1(t,x,y) = \sum_{k=1}^{\infty}(a_{2k+1}\sin k^2 x + a_{2k}\cos k^2 y), \ f_2(t,x,y) = \sum_{k=1}^{\infty}(b_{2k+1}\sin k^2 x + b_{2k}\cos k^2 y).$$

We deduce

$$\|f_1(t,x,y)\|^2 \leq 4\sum_{k=1}^{\infty} a_k^2 < \infty, \quad \|f_2(t,x,y)\|^2 \leq 4\sum_{k=1}^{\infty} b_k^2 < \infty.$$

Hence

$$\mathbb{E}|f_1(t,x,y)|^2 \leq 4\sum_{k=1}^{\infty} a_k^2 + \mathbb{E}(|x|^2+|y|^2), \quad \text{for all } x,y \in \mathbb{R}.$$

and

$$\mathbb{E}|f_2(t,x,y)|^2 \leq 4\sum_{k=1}^{\infty} b_k^2 + \mathbb{E}(|x|^2+|y|^2) \text{ for all } x,y \in \mathbb{R}.$$

Also we have

$$I_1(x) = c_1\frac{x(t)}{1+|x(t)|},\ I_2(y) = c_2\frac{y}{1+|y|} \Rightarrow \mathbb{E}|I_1(x)|^2 \leq c_1,\ \mathbb{E}|I_2(x)|^2 \leq c_2,$$

and

$$g^1(t,x,y) = d_1(t+x+y),\ g^2(t,x,y) = d_2(t+x+y),\ x,y \in \mathbb{R},\ t \in [0,1].$$

Hence

$$\mathbb{E}|g^1(t,x,y)|^2 \leq 3d_1^2(1+\mathbb{E}|x|^2+\mathbb{E}|y|^2),\ \mathbb{E}|g^2(t,x,y)|^2 \leq 3d_2^2(1+\mathbb{E}|x|^2+\mathbb{E}|y|^2).$$

Thus all the conditions of Theorem 4.3.2 hold, and then Problem (4.3.4) has at least one solution.

Conclusion and perspective

In this thesis, we have presented some results to the class of impulsive systems of stochastic differential equations with infinite Brownian motions. Sufficient conditions for the existence and uniqueness of solutions are established by mean of some fixed point theorems in vector Banach spaces and we use Young's and Hölder inequality combined with classical Gronwall's inequality to derive present a new version of the stochastic of Gronwall's inequalities with singular kernels. In most of these works sufficient conditions were considered to get the existence of solution by reducing the research to the search of the existence of fixed points by applying different fixed points argument. Existence results was given for some classes by Perov's and Schaefer's fixed point theorems in generalized Banach spaces.

Bibliography

[1] T. Abdeljawad and J. Alzabut, The q-fractional analogue for Gronwall-type inequality. *J. Funct. Spaces Appl.* 2013, Article ID 543839, 7 p. (2013).

[2] T. Abdeljawad, J. Alzabut and D. Baleanu, A generalized q-fractional Gronwall inequality and its applications to nonlinear delay q-fractional difference systems. *J. Inequal. Appl.* 2016, 240, 1-13.

[3] S. Abbas, M. Benchohra, J. R. Graef and J. Henderson, *Implicit fractional differential and integral equations. Existence and stability.* De Gruyter Series in Nonlinear Analysis and Applications 26. Berlin: De Gruyter, 2018.

[4] D.D. Bainov, V. Lakshmikantham and P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.

[5] A. Baliki, J.J. Nieto, A. Ouahab and M. Sinacer, Random semilinear system of differential equations with impulses, *Fixed Point Theory and Applications* 2017(1),27, 29 pp.

[6] M. Benchohra, J. Henderson, and S.K. Ntouyas, *Impulsive Differential Equations and Inclusions*, Hindawi Publishing Corporation, **2**, New York, 2006.

[7] H. Berrezoug, J. Henderson and A. Ouahab, Existence and uniqueness of solutions for a system of impulsive differential equations on the half-line, *Journal of Nonlinear Functional Analysis*, Vol. 2017 (2017), Article ID 38, pp. 1-16.

[8] M. Benchohra, J. Henderson, S. K. Ntouyas and A. Ouahab, Existence results for fractional order functional differential equations with infinite delay, *J. Math. Anal. Appl.* **338** (2008), 1340-1350.

[9] R. Bellman, *Stability Theory of Differential Equations*, McGraw-Hill, New York, 1953.

[10] M. Benchohra, J. Henderson, S. K. Ntouyas and A. Ouahab, Existence results for fractional functional differential inclusions with infinite delay and application to control theory, *Fract. Calc. Appl. Anal.* **11** (2008), 35-56.

[11] A. T. Bharucha-Reid, *Random Integral Equations*, Academic Press, New York, 1972.

[12] J. M. Bismut and M. Yor, An inequality for processes which satisfy Kolmogorov's continuity criterion. Application to continuous martingales, *J. Funct. Anal.* 51 (1983), 166-173.

[13] O. Bolojan-Nica, G. Infante and R. Precup, Existence results for systems with coupled nonlocal initial conditions, *Nonlinear Anal.* **4** (2014), 231–242.

[14] A. Boudaoui, T. Caraballo and A. Ouahab, Stochastic differential equations with non-instantaneous impulses driven by a fractional Brownian motion, *Discrete Contin. Dyn. Syst. Ser.* B 22 , no. 7, (2017), 2521–2541.

[15] D.L. Burkholder, *Explorations in Martingale Theory and Its Applications*, Lecture Notes in Math., vol. 1464, Springer, New York, Berlin, Heidelberg, 1991, pp. 66.

[16] D. L. Burkholder, Martingale transforms, *Ann. Math. Statist.* **37** (1966) 1494-1504.

[17] D. L. Burkholder and R. F. Gundy, Extrapolation and interpolation of quasi-linear operators on martingales, *Acta Math.*, 124 (1970), 249-304.

[18] T. Caraballo, C. Ogouyandjou, F. K. Allognissode and A. Fulbert Kuessi; Diop, Mamadou, Existence and exponential stability for neutral 34 stochastic integro-differential equations with impulses driven by a Rosenblatt process, *Discrete Contin. Dyn. Syst., Ser.* B , No. 2, (2020), 35 507-528.

[19] R. Cristescu, *Order Structures in Normed Vector Spaces*, Editura Ştiin ţifică şi Enciclopedică, Bucureşti, 1983 (in Romanian).

[20] B. Davis, On the integrability of the martingale square function, *Israel J. Math.* **8** (1970) 187-190.

[21] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, 1992.

[22] X.L. Ding, D. Cao-Labora and J. J. Nieto, A new generalized Gronwall inequality with a double singularity and its applications to fractional stochastic differential equations. *Stoch. Anal. Appl.* **37** (2019), no. 6, 1042–1056.

[23] R.A.C. Ferreira, A Discrete fractional Gronwall inequality. *Proc. Amer. Math. Soc.* (2012), 5, 1605-1612.

[24] D. L. Fisk, Quasi-martingales and stochastic integrals, *Tech. Rep. 1, Dept. Math. Michigan State Univ.*, 1963.

[25] T.C. Gard, *Introduction to Stochastic Differential Equations*, Marcel Dekker, New York, 1988.

[26] D. Gao and J. Li, Existence and mean-square exponential stability of mild solutions for impulsive stochastic partial differential equations 44 with noncompact semigroup, *J. Math. Anal. Appl.* 484(1), 2020, 123717.

[27] L. Gaul, P. Klein and S. Kempfle, Damping description involving fractional operators, *Mech. Systems Signal Processing* **5** (1991), 81-88.

[28] I.I. Gikhman and A. Skorokhod, *Stochastic Differential Equations*, Springer-Velag, 1972.

[29] J. R. Graef, J. Henderson and A. Ouahab, *Impulsive differential inclusions. A fixed point approach.* De Gruyter Series in Nonlinear 19 Analysis and Applications 20. Berlin: de Gruyter, 2013.

[30] J. R. Graef, J. Henderson, and A. Ouahab, *Topological Methods for Differential Equations and Inclusions*, Monographs and Research Notes in Mathematics Series Profile. Boca Raton, FL: CRC Press, 2019.

[31] J. R. Graef, H. Kadari, A. Ouahab and A. Oumansour, Existence results for systems of second-order impulsive differential equations. *23 Acta Math. Univ. Comenian. (N.S.)* **88** (2019), 51–66.

[32] Y. Guo, Q. Zhu and F. Wang, Stability analysis of impulsive stochastic functional differential equations, *Commun. Nonlinear Sci. 48 Numerical Simul.* 82, 2020, 105013.

[33] A. Halanay and D. Wexler, *Teoria Calitativa a sistemelor Impulsuri*, (in Romanian), Editura Academiei Republicii Socialiste România, Bucharest, 1968.

[34] X. Han and P. E. Kloeden, *Random ordinary differential equations and their numerical solution*. Springer, 2017.

[35] R. Hilfer, *Application of Fractional Calculus in Physics*, World Scientic, Singapore 1999.

[36] D. Henry, *Geometric Theory of Semilinear Parabolic Partial differential Equations*, Springer-Verlag, Berlin/New York, 1989.

[37] J. Henderson, A. Ouahab, and M. Slimani, Existence results of system of semilinear discrete equations, *Int. J. Difference Equ.* 50 **12**, (2017), 235–253.

[38] W. Hu and Q. Zhu, Stability analysis of impulsive stochastic delayed differential systems with unbounded delays, *Systems and Control 52 Letters* 136, 2020, 104606.

[39] H. Kadari, J.J. Nieto, A. Ouahab and A. Oumansour, Implicit first order impulsive differential systems. *Int. J. Difference Equ.* 54 **15**, (2020), 429–451.

[40] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, **204**, Elsevier Science B. V. Amsterdam, 2006.

[41] R. Kruse and M. Scheutzow, A discrete stochastic Gronwall lemma. *Math. Comput. Simulation* **143** (2018), 149–157.

[42] C. Li, J. Shi and J. Sun, Stability of impulsive stochastic differential delay systems and its application to impulsive stochastic neural 59 networks, *Nonlinear Anal.* **74**, (2011), 3099-3111.

[43] J. Liu, X. Liu and W.C. Xie, Existence and uniqueness results for impulsive hybrid stochastic delay systems, *Commun. Appl. Nonlinear 57 Anal.* **17** (2010) 37-54.

[44] M. Liu and K. Wang, On a stochastic logistic equation with impulsive perturbations, *Comput. Math. Appl.* **63** (2012), 871-886.

[45] C. Makasu, A stochastic Gronwall lemma revisited. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **22** (2019), no. 1, 1950007, 5 pp

[46] C. Makasu, Extension of a stochastic Gronwall lemma. *Bulletin Polish Acad. Sci. Math.*, in press.

[47] A.B. Malinowska and D.F.M. Torres, Towards a combined fractional mechanics and quantization, *Fract. Calc. Appl. Anal.* **15** (2012), 59 407-417.

[48] X. Mao, *Stochastic Differential Equations and Applications*, Horwood, Chichester, (1997).

[49] F. Metzler, W. Schick, H. G. Kilian and T. F. Nonnenmacher, Relaxation in filled polymers: A fractional calculus approach, *J. Chem. Phys.* **103** (1995), 7180-7186.

[50] P. Millar, Warwick Martingale integrals, *Trans. Amer. Math. Soc.* **133** (1968) 145-166.

[51] V.D. Milman and A.A. Myshkis, On the stability of motion in the presence of impulses, *Sib. Math. J.* (in Russian) **1** (1960) 233–237.

[52] S.P. Nasholm and S. Holm, On a fractional Zener elastic wave equation *Fract. Calc. Appl. Anal.*, **16** (2013), 26-50.

[53] O. Nica, Initial-value problems for first-order differential systems with general nonlocal conditions, *Electron. J. Differential Equations*, Vol. 2012 (2012), No. 74, pp. 1–15.

[54] O. Nica and R. Precup, On the nonlocal initial value problem for first order differential systems, *Stud. Univ. Babeş -Bolyai Math.* **56** (2011), No. 3, 125–137.

[55] J. J. Nieto, A. Ouahab and M.A. Slimani, Existence and boundedness of solutions for systems of difference equations with infinite delay, *Glas. Mat.*, 2018.

[56] A. A. Novikov, The moment inequalities for stochastic integrals. (Russian) *Teor. Verojatnost. i Primenen.*, **16** (1971) 548-551.

[57] S. Ogawa, On a Riemann definition of the stochastic integra1, (I), (II), Proc. Japan Acad.,46 (1970), 153-157, 158-161.

[58] B. G. Pachpatte, On some nonlinear generalizations of Gronwall's inequality. *Proc. Indian Acad. Sci. Sect. A* 84 (1976), no. 1, 1–9.

[59] K.P. Parthasarathy, Probability measures on metric spaces, Academic Press, New York, 1967.

[60] A.I. Perov, On the Cauchy problem for a system of ordinary differential equations, *Pviblizhen. Met. Reshen. Differ. Uvavn.,* **2**, (1964), 115-134. (in Russian).

[61] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[62] R. Precup and A. Viorel, Existence results for systems of nonlinear evolution equations, *Int. J. Pure Appl. Math. IJPAM* **47** 97 (2008), 199–206.

[63] Y. Qin, *Analytic Inequaliies and Applications in PDEs, Operator Theory, Adv. PDE.* Springer/Birkhäuser, Basel/Boston/Berlin, 2017.

[64] Y. Qin, *Integral and Discrete Inequalities and Their Applications.* Springer International Publishing AG, Birkhäuser, 2016.

[65] Rajeeva L. Karandikar. B. V. Rao, *Introduction to Stochastic Calculus*, Library of Congress Control, Springer Nature Singapore Pte Ltd. 2018.

[66] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, Springer. Third edition 1999.

[67] I.A.Rus, The theory of a metrical fixed point theorem: theoretical and applicative relevances, *Fixed Point Theory* **9** (2008), 541–559.

[68] R. Sakthivel and J. Luo, Asymptotic stability of nonlinear impulsive stochastic differential equations, *Statist. Probab. Lett.* **79** 86 (2009) 1219-1223.

[69] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[70] A.M. Samoilenko and N.A. Perestyuk, *Impulsive Differential Equations,* World Scientific, Singapore, 1995.

[71] M. Scheutzow, A stochastic Gronwall lemma, *Infin. Dimens. Anal. Quantum. Probab. Relat. Top.* 16 (2013), no. 2, 1350019, 4 pp.

[72] M. Seghier, A. Ouahab and J. Henderson, Random solutions to a system of fractional differential equations via the Hadamard fractional 79 derivative, *Eur. Phys. J. Spec. Top.* **226**, (2017), 3525-3549.

[73] H. Sobczyk, *Stochastic Differential Equations with Applications to Physics and Engineering*, Kluwer Academic Publishers, London, 1991.

[74] C. P. Tsokos and W.J. Padgett, *Random Integral Equations with Applications to Life Sciences and Engineering*, Academic Press, New York, 1974.

[75] A. Viorel, *Contributions to the Study of Nonlinear Evolution Equations,* Ph.D. thesis, Babeş-Bolyai University Cluj-Napoca, Department of Mathematics, 2011.

[76] D. Willet and J. S. W. Wong, On the discrete analogues of some generalizations of Gronwall's inequality, *Monatsh. Math.* **69**, (1965), 362-367.

[77] H. Ye, J. Gao and Y. Ding, A generalized Gronwall inequality and its application to a fractional differential equation. *J. Math. Anal. Appl.* **85**, 2007, 328, 1075-1081.

[78] J. Yeh. Martingales and stochastic analysis / James Yeh. p. cm. – (Series on multivariate analysis ; vol. 1) Includes bibliographical references and index. ISBN 981022477X

Index of Symbols

The most frequently used notations, symbols, and abbreviations are listed below.

$\langle x, y \rangle$ inner product of x and y.
$\alpha, \beta, \gamma, \delta$ members of the underlying scalar field.
$\mathbb{R}$ reel number.
$C([a, b])$ continuous functions on $[a, b]$. $\mathbb{R}^n$ space of n-tuples of reel numbers.
$f(A)$ image of set A under function J.
$d(x, y)$ distance (metric) from x to y.
$d(x, A)$ distance from the point x to the set A.
$d(A, C)$ distance between set B and set C.
$\overline{B(x_0, r)}$ the closed ball centered in x_0 with radius r.
$B(x_0, r)$ be the open ball centrad in x_0 with radius r.
∂B_n boundary of B_n unit sphere in $\mathbb{R}^n$.
$y(t_k^-)$ and $y(t_k^+)$ be the left and right limits of $y(t)$ at $t = t_k$.
l^2 square-summable sequences.
$C^i(J; X)$ be the space of functions $y : J \to X$,$i-$ differentiable in whose $i-$th derivative, $y^{(i)}$, is absolutely continuous.
$\sqrt{A}$ square root of the mapping A.
$\|A\|$ norm of the mapping to A.
$A^{'}$ derived set of A.
A^{-1} inverse function of the mapping A.
$A \geq 0$ mapping A greater than or equal to 0
$F_X(x)$ be the probability distribution function $\mathbb{R}$ to $[0, 1]$ of a random variable X.
$\mathcal{O}$ open sets of a topological space.
$\mathcal{F}$ be the collection of its subsets.
$\mathbb{P}$ is probability measure on $(\Omega, \mathcal{F})$.
$\mathbb{E}$ be the expectation mathematica on $(\Omega, \mathcal{F})$.

$(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a complete probability space.
$W(t)$ be a scalar Brownian motion defined on the probability space.
$Q_\Delta(M, M)$ be the quadratic variation process of M.
$\mathcal{N}(0, t)$ be the Gaussian distribution with mean zero and variance t.
$\mathcal{P}_{cl}(X)$ be the a multi-valued map is convex.
$s \wedge t$ the minimum of s and t. $(= \min(s, t))$.
$s \vee t$ the maximum of s and t $(= \max(s; t))$.
$\mathcal{F}_\infty$ the σ-algebra generated by $\cup_{t>0}\mathcal{F}_t$.
I_n the $n \times n$ identity matrix.
$B^{(H)}$ Fractional Brownian motion.

$\mathcal{P}_{cl}(X)$ multi-valued map a closed.
$\mathcal{P}_b(X)$ multi-valued map a bounded.
$\mathcal{P}_c(X)$ multi-valued map a convex.
$\mathcal{P}_{cp}(X)$ multi-valued map a compact.
$:=$ equal to by definition.
$I^\alpha_{a^+}f(x)$ The left fractional integrals of f of order α are defined for almost all $x \in (a, b)$
$I^\alpha_{b^-}f(x)$ The right-sided fractional integrals of f of order α are defined for almost all $x \in (a, b)$

$\mathcal{L}(X)$ be the set of all linear bounded operator from X
$C(J, E)$ be the Banach space of all continuous functions from J into E
$L^1(J, E)$ be the Banach space of all continuous functions from J into E
$\mathcal{L} \otimes \mathcal{B}$ product-measurable
$S_F(y)$ selection set of the multi-valued map F
$dimX$ dimension of the space X
$clA = \overline{A}$ closure of the set A

Résumé

Dans cette thèse nous trouvons quelque notion sur l'analyse stochastique, commencer par le calcule stochastique puis l'intégrale stochastique motionner dans les deux chapitres premier, et dans les deux chapitres suivantes, on trouve des résultats pour la méthode topologique d'un couple de système d'inclusions différentielles stochastique semi-linéaires avec mouvement Brownien fractionnel et quelque résultats d'existence de solutions d'un système infinie d'équations différentielles stochastiques.

Summary

In this thesis we find some notion on stochastic analysis, starting with the stochastic calculus and then the stochastic integral motion in the first two chapters, and in the next two chapters we find results for the topological method of a couple of semi-linear stochastic differential inclusions system with fractional Brownian motion and some results of existence of solutions of an infinite system of stochastic differential equations.

ملخص

نجد في هذه الأطروحة بعض التعريفات و الخواص حول التحليل العشوائي ، بدءًا من حساب التفاضل والتكامل العشوائي ثم الدافع المتكامل العشوائي في الفصلين الأولين ، والفصلين الأخيرين ، نجد نتائج للطريقة الطوبولوجية لمجموعة من التضمينات العشوائية شبه الخطية. مع الحركة البراونية الكسرية وبعض نتائج وجود حلول لنظام لانهائي من المعادلات التفاضلية العشوائية .