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Intitulée

Stabilité Exponentielle de quelques problèmes d'évolution avec retard

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Dédicace

I dedicate this work to my father Mohammed Benali;May ALLAH the Most Merciful grant him his blessing in his vast Heaven,

To my mother Fatiha ; May ALLAH the Almighty protect here and keep here safe and healthy,

To my wife ,

To my brothers and sisters ,

and to my friends .

Dédicace

Je dédie ce modeste travail à mon pére Mohammed Benali qu'ALLAH le Tout Miséricordieu lui accorde sa bénédiction dans son vaste Paradis,
à ma mère Fatiha qu'ALLAH le Tout Puissant la protége et la préserve saine et sauve,
à ma femme ,
à mes frères et soeurs,
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Resumé

Dans cette thèse on a considère un système thermoelastique avec retard, ce retard met en cause l'existence de la solution et l'effet de stabilisation de conduction de la chaleur,pour surmonter ce problème on a ajouté a l'équation du système où figure le retard un amortissement de Kelvin-Voigt. Apres un apercus sur la notion de la thermoelasticite classique, lineaire et non lineaire,on a d'abord prouvé l'existance de la solution du système en utilisant la théorie du semigroupe, puis on a prouvé sous des assumptions appropriées la stabilité exponentielle du système en introduisant une fonctionalle de Lyapunov convenable. Dans le dernier chapitre, on a considéré l'équation des ondes en présence de termes dissipatifs: un terme dissipatif viscoélastique et un terme dissipatif non local de type fractionnaire. On obtient des résultats de stabilité forte, uniforme et polynomiale.

Mots Clés: Systeme Thermoelsatique, retard, amortissement de Kelvin-voigt,stabilité exponentielle.

Abstract

In this Thesis we considere a thermoelastic systeme whith delay,the presence of this delay destroys the well-posedness and the stabilizing effect of the heat conduction.To avoid this problem, we add to the system, at the delayed equation, a Kelvin-Voigt damping.After giving a short summary on classical thermoelasticte, we prove at first the well-posedness of the system by the semigroup theory.then , under appropriate assumptions, we prove the exponential stability of the system by introducing a suitable Lyapunov functional.

In the last chapter we considere a wave equation which present a viscoelastic dissipatif term and a non local dissipatif term of fractionnaire type.We obtain a strong, uniform and polynomial stability result.

Keywords: Thermoelastic system, delay, Kelvin-voigt damping, well posedness, exponential stability.

Contents

Notations

$\mathbb{R}$: The set of real numbers.
$\mathbb{R}_+$: The set of non negative real numbers.
$\mathbb{R}^*$: The set of non zero real numbers.
$\mathbb{C}$: The set of complex numbers.
i: The imaginary unit.
L^p: The Lebesgue space.
H^m: The sobolev space.
C^0: The space of continuous function.
C^1: the space of continuously differentiable functions.
$L(X, Y)$: The space of bounded linear operators from X into Y .
$|.|$: The modulus.
$\|.\|$: The norm.
inf: The infimum.
sup: The supremum.
$\Re$: The real part.
$\Im$: The imaginary part.
∂: The partial derivative.
∂_t: The partial derivative with respect of t.
$\partial_{tt} f$: The second partial derivative of f with respect of t.
$\partial_t^{\alpha,\eta}$: Fractional Derivative.
$D(A)$: Domain of A.
$R(A)$: The range of A.
$ker(A)$: The kernel of A.
A^*: The adjoint operator of A.
$\rho(A)$: The resolvent set of A.
$\sigma(A)$: The spectrum of A.
$\sigma_p(A)$: The punctual spectrum of A.

Introduction

1 Time delay

Systems with delays abound in the world. One reason is that nature is full of transparent delays. Another reason is that time-delay systems are often used to model a large class of engineering systems, where propagation and transmission of information or material are involved. The presence of delays (especially, long delays) makes system analysis and control design much more complex. In this section, an example of time-delay system is discussed.

What is a delay?

Time delay is the property of a physical system by which the response to an applied force (action) is delayed in its effect. Whenever material, information or energy is physically transmitted from one place to another, there is a delay associated with the transmission. The value of the delay is determined by the distance and the transmission speed. Some delays are short, some are very long. The presence of long delays makes system analysis and control design much more complex. What is worse is that some delays are too long to perceive and the system is misperceived as one without delays.

Time delays abound in the world. They appear in various systems such as biological, ecological, economic, social, engineering systems etc. For example, over-exposure to radiation increases the risk of cancer, but the onset of cancer typically follows exposure to radiation by many years. In economics, the central bank in a country often attempts to influence the economy by adjusting interest rates; the effect of a change in interest rates takes months to be translated into an impact on the economy. In politics, politicians need some time to make decisions and they will have to wait for some time before they find out if the decisions are correct or not. When reversing a car around a corner, the driver has to wait for the steering to take effect. In engineering, on which this section focuses, there are a lot of systems with delays.

A general tendency in responding to some errors in a system is to react immediately to the errors and to react more if the errors are not lessened or eliminated in time as expected. However, for a system with time delays, only after the inherent delays will the errors start to change. Hence, it is very important to properly understand the existence of delays and not to over-react. Otherwise, the system is very likely to overshoot or even become unstable. When dealing with time-delay systems, "patience is a virtue." For a given delay element with a delay $h \geq 0$, the output y(t) corresponding to the input $u(t)$ is

$$y(t) = u(t - h).$$

Hence, the transfer function of a delay element is given by e^{-sh}.

An example of time-delay system

A simple example of a time-delay system from everyday life is the shower, as depicted in Figure 1.

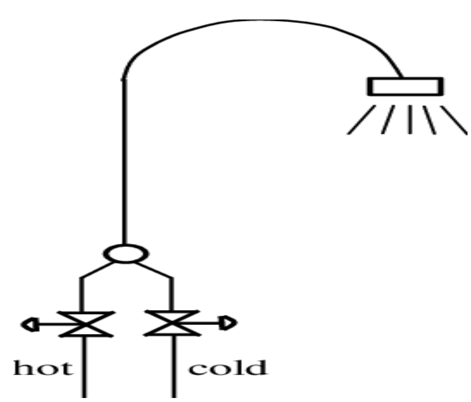

Figure 1: Sketch of a shower system.

Most people have experienced the difficulty in adjusting the water temperature: it gets too cold or too warm. The actual temperature often overshoots the desired and, sometimes, it takes a while to get the temperature right. This is because it takes time for the increased (or decreased) hot/cold water to flow from the tap to the shower head (or the human body). This time is a delay, which depends on the water pressure and the length of the pipe. The change of the faucet position is almost immediate, however, the change of the water temperature has to wait until the delay has elapsed. If the faucet position is constantly adjusted according to the currently perceived temperature, then it is very likely that the temperature will fluctuate. Assume that the water is an incompressible fluid and stationary flow. According to the Poiseuille law, the flow rate of water is

$$F = \frac{\pi R^4}{8\mu l}\Delta p,$$

where $\mu = 0.01$ is the kinematic viscosity of water, R is the radius of the pipe, l is the length of the pipe and Δp is the pressure difference between the two ends of the pipe. The time delay h can then be found as

$$h = \frac{\pi R^2 l}{F} = \frac{8\mu}{\Delta p}\left(\frac{l}{R}\right).$$

2 Evolution equations in Thermoelasticity

The equations of thermoelasticity describe the elastic and the thermal behavior of elastic, heat conductive media, in particular the reciprocal actions between elastic stresses and temperature differences. We consider the classical thermoelastic system where the elastic part is the usual second-order one in the space variable. The equations are a coupling of the equations of elasticity and of the heat equation and thus build a hyperbolic-parabolic system.

Indeed, both hyperbolic and parabolic effects are encountered. This section discusses the mathematical questions arising in the study of initial value problems and of initial boundary value problems to these equations, both for linear and for nonlinear systems. Classical boundary conditions of the Dirichlet type (rigidly clamped, constant temperature) or the Neumann type (traction free, insulated) are considered, as well as the linearized equations together with contact boundary conditions.

It is known both for hyperbolic and for parabolic nonlinear equations and systems that global smooth solutions in general might not exist but that a blow-up may occur. The criteria according to which global solutions still exist are different for hyperbolic and for parabolic equations. Hence the question arises whether the behavior will be dominated by the hyperbolic or by the parabolic part. The answer will depend on the number of space dimensions. This also applies to the question of asymptotic behavior of solutions to the linearized system, where the behavior will also depend on the space dimension, or to be more precise, one dimension on one side and two or three dimensions on the other side.

The methods used for obtaining global existence theorems for small data consist of proving suitable a priori estimates, where one often exploits the decay of solutions to the linearized equations. This requires a precise analysis of the asymptotic behavior of such solutions as time tends to infinity, which will finally allow us to describe the asymptotic behavior of solutions to the nonlinear systems as well.

We are mainly interested in proving the well-posedness in the class of smooth solutions and in describing the asymptotic behavior of the solutions as time tends to infinity. This will be possible in the linear case and also in the nonlinear case provided the nonlinearities and the data satisfy certain conditions. Otherwise, a blow-up in finite time has to be expected as examples will show; then weak solutions must be considered.

In one space dimension the picture is almost complete. Bounded or unbounded intervals representing the reference configuration can be dealt with for all the classical boundary conditions. The asymptotic behavior is known, the decay rates are known to be optimal (in the case of absence of forces and heat supplies). For small data global smooth solutions to the nonlinear system will exist; large data lead to a blow-up.

In two or three space dimensions generic nonlinear cases are understood, although there are unsolved problems. Local well-posedness is known in most cases, but concerning global solutions or blow-up for nonlinear situations, only the Cauchy problem and the bounded radially symmetric case have been investigated. This corresponds to the fact that the dynamics in the linear case is complicated apart from these situations, as we shall describe in detail.

Although the system to be considered is a rather special one, it should be stressed that the methods employed are rather general and have been or can be used for purely hyperbolic or parabolic problems.

The aim of this section is to present a state of the art in the treatment of initial value problems and of initial boundary value problems both in linear and nonlinear thermoelasticity. Although well-posedness in the linear theory has been studied for years, the description of the general dynamical system with its asymptotic behavior as time tends to infinity and, in particular, the study of nonlinear systems only started in the late sixties and the early eighties, respectively, and led to very interesting features.

Derivation of the equations

In this section we give a short summary of the derivation of the nonlinear resp. linearized equations that describe the thermoelastic behavior of a body $\mathcal{B}$.

Let the body $\mathcal{B}$ be represented by the undistorted reference configuration Ω, Ω being a domain in $\mathbb{R}^n$, $n = 1, 2$ or 3.

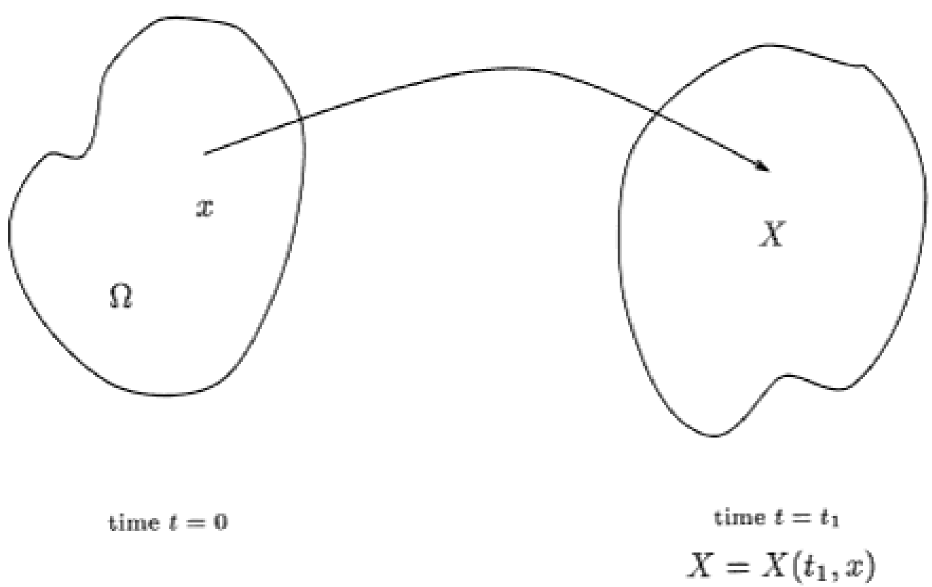

Figure 2: displacement $U(t_1, x) = X(t_1, x) - x$

A deformation goes along with a change of temperature $T = T(t, x)$ and vice versa. The balance of linear momentum in local form reads:

$$\rho X_{tt} - \nabla' \tilde{S} = \rho b \tag{1}$$

where ρ is the material density in Ω, $\tilde{S}$ is the Piola-Kirchhoff stress tensor, and b is the specific external body force, while $'$ denotes the transposition.

This system of equations essentially describes the elastic part; actually, if $\tilde{S}$ does not depend on the temperature, it may represent the (hyperbolic) partial differential equations in pure elasticity. The main differential equation for the temperature arises from the local form of the first law of thermodynamics, the balance of energy:

$$\epsilon_t - tr\{\tilde{S}F_t\} + \nabla' q = r \tag{2}$$

where ϵ is the internal energy, q is the heat flux, r is the external heat supply, tr denotes the trace and F is the deformation gradient,

$$F^{ij} = \frac{\partial}{\partial x_j} X_i,$$

or, in terms of the displacement vector,

$$F = 1 + \nabla U$$

By η we denote the entropy and by

$$\psi = \epsilon - T\eta$$

the Helmholtz free energy.

The constitutive assumptions defining an elastic medium in thermoelasticity are that $\tilde{S}, p, \psi$ and η are functions of the present values of F, T and ∇T (and x). It is always assumed that these functions are smooth and that

$$detF \neq 0,\ T > 0$$

The local form of the second law of thermodynamics reads

$$\eta_t \geq -\nabla'(q/T) + r/T$$

which, combined with (2), yields the local dissipation inequality

$$\psi_t + \eta T_t - tr\{\tilde{S}F_t\} + (q\nabla T)/T \leq 0 \tag{3}$$

The second law of thermodynamics implies the following restrictions on the response functions $\tilde{S}, \eta, \psi, q$.

Lemma 2.1 *A necessary and sufficient condition that the local dissipation inequality* (3) *is* *always satisfied, is that the following three statements hold:*

- *(i) The response functions $\tilde{S}, \eta$ and ψ are independent of the temperature gradient ∇T:*

$$\tilde{S} = \tilde{S}(F,T), \;\; \psi = \psi(F,T), \;\; \eta = \eta(F,T)$$

- *(ii) ψ determines $\tilde{S}$ through the stress relation*

$$\tilde{S}(F,T) = \frac{\partial \psi}{\partial F}(F,T)$$

and η through the entropy relation

$$\eta(F,T) = -\frac{\partial \psi}{\partial T}(F,T)$$

- *(iii) q obeys the heat conduction inequality*

$$q(F,T,\nabla T)\nabla T \leq 0$$

Using these relations we rewrite (2) as

$$\psi_t + T_t\eta + T\eta_t - tr\{\tilde{S}F_t\} + \nabla' q = r$$

or

$$tr\{\tilde{S}F_t\} - \eta T_t + T_t\eta + T\eta_t - tr\{\tilde{S}F_t\} + \nabla' q = r$$

which implies

$$T\eta_t + \nabla' q = r$$

or

$$T\left\{-\frac{\partial^2\psi}{\partial T^2}T_t - \frac{\partial^2\psi}{\partial F \partial T}F_t\right\} + \nabla' q = r \tag{4}$$

The equation (1) is mainly a hyperbolic system for X; the equation (4) is mainly a parabolic equation for T.

Instead of X the variable $U = (X - x)$ is often used, and instead of T the temperature difference is often expressed as $\theta = T - T_0$, where T_0 is a constant reference temperature. We shall write

$$\psi(F,T) = \psi(\nabla U, \theta)$$

with the same symbol ψ, analogously for the other response functions.

The problem of finding U and θ will become well-posed if additionally initial conditions

$$U(t=0) = U^0,\ \ U_t(t=0) = U^1,\ \ \theta(t=0) = \theta^0, \tag{5}$$

and, if $\Omega \neq \mathbb{R}^n$, boundary conditions, are prescribed, for example “rigidly clamped, constant temperature”,

$$U = 0,\ \ \theta = 0,\ \text{ on } \partial\Omega,$$

or “traction free, insulated”,

$$\tilde{S}\nu = 0,\ \ \nu^{'} q = 0, \tag{6}$$

or other combinations of the boundary conditions for U and θ. Here $\nu - \nu(x)$ denotes the exterior normal in $x \in \partial\Omega$, $\partial\Omega$ being the boundary of Ω.

The investigation of the linearized equations will play an important role. The linearized equations arise from (1), (4) by assuming that

$$|\nabla U|,\ |\nabla U_t|,\ |\theta|,\ |\theta_t|,\ |\nabla\theta|$$

are small.

Using Taylor expansions $\left(\text{for example } \dfrac{\partial^2\psi}{\partial F \partial T}(\nabla U, \theta, x) = \dfrac{\partial^2\psi}{\partial F \partial T}(0,0,x) + \mathcal{O}(|\nabla \mathcal{U}| + |\theta|)\right)$ we arive at ($T_0 = 1$ without loss of generality)

$$\rho U_{tt} - \mathcal{D}^{'}\mathcal{S}\mathcal{D}\mathcal{U} + \mathcal{D}^{'}{-}\theta = \rho, \tag{7}$$

$$\delta\theta_t - \nabla^{'} K \nabla\theta + \Gamma^{'}\mathcal{D}\mathcal{U} = , \tag{8}$$

where $\rho - \rho(x)$ can be regarded as a symmetric density matrix, $S - S(x)$ is an $M \times M$ symmetric, positive definite matrix containing the elastic moduli ($M = 6$ in $\mathbb{R}^3$), $\Gamma = \Gamma(x)$ is a vector with coefficients determining the so-called stress-temperature tensor, $\delta = \delta(x)$ is the specific heat and $K = K(x)$ is the heat conductivity tensor. All functions are assumed to be smooth. $\mathcal{D}$ is an abbreviation for a generalized gradient,

$$\mathcal{D} = \begin{pmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_2 & 0 \\ 0 & 0 & \partial_3 \\ 0 & \partial_3 & \partial_2 \\ \partial_3 & 0 & \partial_1 \\ \partial_2 & \partial_1 & 0 \end{pmatrix}, \text{ in } \mathbb{R}^3, \quad \mathcal{D} = \begin{pmatrix} \partial_1 & 0 \\ 0 & \partial_2 \\ \partial_2 & \partial_1 \end{pmatrix} \text{ in } \mathbb{R}^2, \quad \mathcal{D} = \partial_1 \text{ in } \mathbb{R}^1,$$

In this way the general (linear) non-homogeneous, anisotropic case is described. The linear counterpart of the boundary conditions (6) reads

$$\mathcal{N}'(S\mathcal{D}U - \Gamma\theta) = 0, \;\; \nu' K \nabla\theta = 0,$$

where $\mathcal{N}$ arises from the normal vector ν in the same way as $\mathcal{D}$ arises from the gradient vector ∇.

The elastic moduli $C_{ijkl}, \; i,j,k,l = 1, \cdots, n$, which are given in general by:

$$C_{ijkl} = \frac{\partial^2 \psi(0,0,x)}{\partial(\partial_j U_i)\partial(\partial_k U_l)}$$

satisfy in the linear case

$$C_{ijkl} = C_{klij} = C_{jikl} = C_{ijkl}(x)$$

The assumption of positive definiteness of $(C_{ijkl})_{ijkl}$ in the sense

$$\exists k_0 > 0, \forall \xi_{ij} = \xi_{ji} \in \mathbb{R}, \forall x \in \Omega : \;\; \xi_{ij} C_{ijkl} \xi_{kl} \geq k_0 \sum_{j,k=1}^{n} |\xi_{jk}|^2,$$

where and throughout this section, the Einstein summation convention (i.e. repeated indices indicate summation) is used unless there is a statement to the contrary, implies that the matrix $S = S(x)$ is uniformly positive definite since

$$S = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} \\ \cdot & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} \\ \cdot & \cdot & C_{3333} & C_{3323} & C_{3331} & C_{3312} \\ \cdot & \cdot & \cdot & C_{2323} & C_{2331} & C_{2312} \\ \cdot & \cdot & \cdot & \cdot & C_{3131} & C_{3112} \\ \cdot & \cdot & \cdot & \cdot & \cdot & C_{1212} \end{pmatrix} \text{ in } \mathbb{R}^3,$$

$$S = \begin{pmatrix} C_{1111} & C_{1122} & C_{1112} \\ \cdot & C_{2222} & C_{2212} \\ \cdot & \cdot & C_{1212} \end{pmatrix} \text{ in } \mathbb{R}^2, \quad S = C_{1111} \text{ in } \mathbb{R}^1,$$

In the simplest case of a homogeneous medium which is isotropic we have

$$S = \begin{pmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \cdot & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \cdot & \cdot & 2\mu+\lambda & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \mu & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \mu & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \mu \end{pmatrix} \text{ in } \mathbb{R}^3,$$

$$S = \begin{pmatrix} 2\mu+\lambda & \lambda & 0 \\ \cdot & 2\mu+\lambda & 0 \\ \cdot & \cdot & \mu \end{pmatrix} \text{ in } \mathbb{R}^2, \quad S = \tau > 0 \text{ in } \mathbb{R}^1,$$

and the equations reduce in two or three space dimensions to

$$U_{tt} - ((2\mu+\lambda)\nabla\nabla' - \mu\nabla\times\nabla\times)U + \gamma\nabla\theta = b, \tag{9}$$

$$\delta\theta_t - \kappa\Delta\theta + \gamma\nabla' U_t = r, \tag{10}$$

where the density $\rho = 1$ without loss of generality and $\mu, \lambda, \gamma, \delta$ and κ, are constants; μ, λ are the Lamé moduli,

$$\mu > 0, \quad 2\mu + n\lambda > 0,$$

moreover

$$\delta, \kappa > 0, \quad \gamma \neq 0.$$

Notice that in two space dimensions the rotation of a scalar field f in $\mathbb{R}^2$ is defined to be the vector field

$$\nabla \times f = (\partial_2 f, -\partial_1 f)^{'}$$

and the rotation of a vector field F in $\mathbb{R}^2$ is defined to be the scalar

$$\nabla \times f = \partial_1 F_2 - \partial_2 F_1$$

In particular the formula

$$\Delta = \nabla \nabla^{'} - \nabla \times \nabla$$

holds in $\mathbb{R}^2$ and in $\mathbb{R}^2$.

In one space dimension the basic equations for the homogeneous (and necessarily isotropic) case are:

$$U_{tt} - \tau U_{xx} + \gamma \theta_x = b, \tag{11}$$

$$\delta \theta_t - \kappa \theta_{xx} + \gamma U_{tx} = r, \tag{12}$$

and we shall often write in this case u instead of U.

Chapter I

Preliminaries

1 Sobolev spaces

We denote by Ω an open domain in $\mathbb{R}^n, n \geq 1$, with a smooth boundary $\Gamma = \partial\Omega$. In general, some regularity of Ω will be assumed. We will suppose that either Ω is Lipschitz, a.e., the boundary Γ is locally the graph of a Lipschitz function, or

$$\Omega \text{ is of class } C^r, r \geq 1,$$

a.e., the boundary Γ is a manifold of dimension $n \geq 1$ of class C^r. In both cases we assume that Ω is totally on one side of Γ. These definition mean that locally the domain Ω is below the graph of some function ψ, the boundary Γ is represented by the graph of ψ and its regularity is determined by that of the function ψ. Moreover, it is necessary to note that a domain with a continuous boundary is never on both sides of its boundary at any point of this boundary and that a Lipschitz boundary has almost everywhere a unit normal vector ν.

We will also use the following multi-index notation for partial differential derivatives of a function:

$$\partial_i^k u = \frac{\partial^k u}{\partial x_i^k} \text{ for all } k \in \mathbb{N} \text{ and } i = 1, ..., n,$$

$$D^\alpha u = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n} u = \frac{\partial^{\alpha_1 + \dots + \alpha_n} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}},$$

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n, |\alpha| = \alpha_1 + \dots + \alpha_n.$$

We denote by $C(D)$ (respectively $C^k(D), k \in \mathbb{N}$ or $k = +\infty$) the space of real continuous functions on D (respectively the space of k times continuously differentiable functions on D), where D plays the role of Ω or its closure $\overline{\Omega}$. The space of real C^∞ functions on Ω with a compact support in Ω is denoted by $C_0^\infty(\Omega)$ or $\mathcal{D}(\Omega)$ as in the distributions theory of Schwartz.The distributions space on Ω is denoted by $\mathcal{D}'(\Omega)$, a.e., the space of continuous linear form over $\mathcal{D}(\Omega)$.

Definition 1.1 *We define $L^p(\Omega)$ as:*
If $p \in [1,+\infty[$,

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \Big/ \int_\Omega |f(x)|^p dx < \infty \right\}$$

We define on $L^p(\Omega)$ the norm :

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}}$$

If $p = +\infty$,

$$L^\infty(\Omega) = \{ f : \Omega \longrightarrow \mathbb{R} \; / \exists c \in \mathbb{R} \quad |f(x)| \leqslant c \quad a.e \; in \quad \Omega \}.$$

We define in $L^\infty(\Omega)$ the norm :

$$||f||_{L^\infty(\Omega)} = \inf \{ c \in \mathbb{R} : |f| \leqslant c \quad a.e \; in \;\; \Omega \}.$$

For $1 \leq p \leq \infty$, we call $L^p(\Omega)$ the space of measurable functions f on Ω such that

$$\begin{aligned} \|f\|_{L^p(\Omega)} &= \left(\int_\Omega |f(x)|^p dx \right)^{1/p} < +\infty \quad \text{for} \quad p < +\infty \\ \|f\|_{L^\infty(\Omega)} &= \sup_\Omega |f(x)| < +\infty \qquad\qquad \text{for} \quad p = +\infty \end{aligned}$$

The space $L^p(\Omega)$ equipped with the norm $f \longrightarrow \|f\|_{L^p}$ is a Banach space: it is reflexive and separable for $1 < p < \infty$ (its dual is $L^{\frac{p}{p-1}}(\Omega)$), separable but not reflexive for $p = 1$ (its dual is $L^\infty(\Omega)$), and not separable, not reflexive for $p = \infty$ (its dual contains strictly $L^1(\Omega)$). In particular the space $L^2(\Omega)$ is a Hilbert space equipped with the scalar product defined by

$$(f, g)_{L^2(\Omega)} = \int_\Omega f(x) g(x) dx.$$

We denote by $L^p_{loc}(\Omega)$ the space of functions which are L^p on any bounded sub-domain of Ω.

Similar space can be defined on any open set other than Ω, in particular, on the cylinder set $\Omega \times]a, b[$ or on the set $\Gamma \times]a, b[$, where $a, b \in \mathbb{R}$ and $a < b$. Let X the space of Hilbert and $]a, b[$ an open interval in $\mathbb{R}^n$. we refer to the measure of Lebesgue as dt in $]a, b[$ and as $\|.\|_X$ The norm in X.

Definition 1.2 - *We call $L^p(a, b; X)$, $\quad 1 \leqslant p \leqslant +\infty$, the space*

$$L^p(a,b;X) = \{f :]a,b[\longrightarrow X \quad mesurable\}$$

We provide the space $L^p(a,b;X)$ of the norm

$$\|f\|_{L^p(a,b;X)} = \left(\int_a^b \|f(t)\|_X^p\right)^{\frac{1}{p}}$$

- We define $L^\infty(a,b;X)$, as being the espace of functions defined from $]a,b[\to X$, measurable et bounded presume all over in $]a,b[$, provided of the norm

$$\|f\|_{L^\infty(a,b;X)} = \sup_{t\in[a,b]} ess\|f(t)\|_X$$

- $L^p(a,b;X) \quad (1\leqslant p\leqslant +\infty)$ is space of Banach for the norm defined above.

a- *The space $L^2(a,b;X)$ is space of Hilbert for the inner product*

$$(f,g)_{L^2(a,b;X)} = \int_a^b (f(t),g(t))_x dt$$

or $(.,.)_X$ is the inner product in X.

b- *For $\quad 1\leqslant p\leqslant +\infty$, $L^p(a,b;X)$ is an space separable.*

The injection of $L^p(a,b;X)$ in $D'(]a,b[;X)$ is strongly and weakly sequentializing continuous a.e: If : $f_j \to f$ strongly (resp weakly) in $L^p(a,b;X)$, the f_j tends strongly (resp weakly) towards f in the meaning of $D'(]a,b[;X)$.

$$\|f\|_{L^p(a,b;X)} = \left(\int_a^b \|f(x)\|_X^p\, dt\right)^{1/p} < +\infty \quad \text{for} \quad p < +\infty$$

and for the norm

$$\|f\|_{L^\infty(a,b;X)} = \sup_{t\in(a,b)} \|f(x)\|_X < +\infty \quad \text{for} \quad p = +\infty$$

Similarly, for a Banach space $X, k \in \mathbb{N}$ and $-\infty < a < b < +\infty$, we denote by $C([a,b];X)$ (respectively $C^k([a,b];X)$) the space of continuous functions (respectively the space of k times continuously differentiable functions) f from $[a,b]$ into X, which are Banach spaces, respectively, for the norms

$$\|f\|_{C(a,b;U)} = \sup_{t\in(a,b)} \|f(x)\|_X, \quad \|f\|_{C^k(a,b;X)} = \sum_{i=0}^{k} \left\| \frac{\partial^i f}{\partial t^i} \right\|_{\mathcal{C}(a,b;X)}$$

Definition of Sobolev Spaces

Now, we will introduce the Sobolev spaces: The Sobolev space $W^{k,p}(\Omega)$ is defined to be the subset of L^p such that function f and its weak derivatives up to some order k have a finite L^p norm, for given $p \geq 1$.

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega); D^\alpha f \in L^p(\Omega). \ \forall\alpha; |\alpha| \leq k\} \ ,$$

With this definition, the Sobolev spaces admit a natural norm,

$$f \longrightarrow \|f\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\leq m} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p} \ , \text{ for } p < +\infty$$

and

$$f \longrightarrow \|f\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha|\leq m} \|D^\alpha f\|_{L^\infty(\Omega)} \ , \text{ for } p = +\infty$$

Space $W^{k,p}(\Omega)$ equipped with the norm $\| \ . \ \|_{W^{k,p}}$ is a Banach space. Moreover is a reflexive space for $1 < p < \infty$ and a separable space for $1 \leq p < \infty$. Sobolev spaces with $p = 2$ are especially important because of their connection with Fourier series and because they form a Hilbert space. A special notation has arisen to cover this case:

$$W^{k,2}(\Omega) = H^k(\Omega)$$

the H^k inner product is defined in terms of the L^2 inner product:

$$(f, g)_{H^k(\Omega)} = \sum_{|\alpha|\leq k} (D^\alpha f, D^\alpha g)_{L^2(\Omega)} \ .$$

The space $H^m(\Omega)$ and $W^{k,p}(\Omega)$ contain $C^\infty(\overline{\Omega})$ and $C^m(\overline{\Omega})$. The closure of $\mathcal{D}(\Omega)$ for the $H^m(\Omega)$ norm (respectively $W^{m,p}(\Omega)$ norm) is denoted by $H^m_0(\Omega)$ (respectively $W^{k,p}_0(\Omega)$).

Now, we introduce a space of functions with values in a space X (a separable Hilbert space).

The space $L^2(a,b;X)$ is a Hilbert space for the inner product

$$(f,g)_{L^2(a,b;X)} = \int_a^b (f(t),g(t))_X \, dt$$

We note that $L^\infty(a,b;X) = (L^1(a,b;X))'$.

Now, we define the Sobolev spaces with values in a Hilbert space X

For $k \in \mathbb{N}$, $p \in [1,\infty]$, we set:

$$W^{k,p}(a,b;X) = \left\{ v \in L^p(a,b;X); \frac{\partial v}{\partial x_i} \in L^p(a,b;X). \ \ \forall i \leq k \right\} \ ,$$

The Sobolev space $W^{k,p}(a,b;X)$ is a Banach space with the norm

$$\begin{aligned} \|f\|_{W^{k,p}(a,b;X)} &= \left(\sum_{i=0}^{k} \left\| \frac{\partial f}{\partial x_i} \right\|_{L^p(a,b;X)}^p \right)^{1/p}, \text{ for } p < +\infty \\ \|f\|_{W^{k,\infty}(a,b;X)} &= \sum_{i=0}^{k} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^\infty(a,b;X)} , \qquad \text{for } p = +\infty \end{aligned}$$

The spaces $W^{k,2}(a,b;X)$ form a Hilbert space and it is noted $H^k(0,T;X)$. The $H^k(0,T;X)$ inner product is defined by:

$$(u,v)_{H^k(a,b;X)} = \sum_{i=0}^{k} \int_a^b \left(\frac{\partial u}{\partial x^i}, \frac{\partial v}{\partial x^i} \right)_X dt \ .$$

Theorem 1.3 *Let $1 \leq p \leq n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$$

where p^* is given by $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}$ (where $p = n, \ p^* = \infty$). Moreover there exists a constant $C = C(p,n)$ such that

$$\|u\|_{L^{p^*}} \leq C \|\nabla u\|_{L^p(\mathbb{R}^n)} \forall u \in W^{1,p}(\mathbb{R}^n).$$

Corolary 1.4 *Let $1 \leq p < n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \quad \forall q \in [p,p^*]$$

with continuous imbedding.

For the case $p = n$, we have

$$W^{1,n}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \quad \forall q \in [n, +\infty[$$

Theorem 1.5 *Let $p > n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$$

with continuous imbedding.

Corolary 1.6 *Let Ω a bounded domain in $\mathbb{R}^n$ of C^1 class with $\Gamma = \partial\Omega$ and $1 \leq p \leq \infty$. We have*

$$\begin{aligned}
&if \quad 1 \leq p < \infty, \text{ then } W^{1,p}(\Omega) \subset L^{p^*}(\Omega) \text{ where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}.\\
&if \quad p = n, \text{ then } W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [p, +\infty[.\\
&if \quad p > n, \text{ then } W^{1,p}(\Omega) \subset L^\infty(\Omega)
\end{aligned}$$

with continuous imbedding.
Moreover, if $p > n$, we have: $\forall u \in W^{1,p}(\Omega)$,

$$|u(x) - u(y)| \leq C|x - y|^\alpha \|u\|_{W^{1,p}(\Omega)} \ a.e\ x, y \in \Omega$$

with $\alpha = 1 - \dfrac{n}{p} > 0$ and C is a constant which depend on p, n and Ω. In particular $W^{1,p}(\Omega) \subset C(\overline{\Omega})$.

Corolary 1.7 *Let Ω a bounded domain in $\mathbb{R}^n$ of C^1 class with $\Gamma = \partial\Omega$ and $1 \leq p \leq \infty$. We have*

$$\begin{aligned}
&if \quad p < n, \text{ then } W^{1,p}(\Omega) \ \subset L^q(\Omega) \forall q \in [1, p^*[\text{ where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}.\\
&if \quad p = n, \text{ then } W^{1,p}(\Omega) \ \subset L^q(\Omega), \forall q \in [p, +\infty[.\\
&if \quad p > n, \text{ then } W^{1,p}(\Omega) \subset C(\overline{\Omega})
\end{aligned}$$

with compact imbedding.

Remark 1.8 *We remark in particular that*

$$W^{1,p}(\Omega) \subset L^q(\Omega)$$

with compact imbedding for $1 \leq p \leq \infty$ and for $p \leq q < p^*$.

Corolary 1.9

$$\begin{aligned}
&if \quad \frac{1}{p}-\frac{m}{n}>0,\ then\ W^{m,p}(\mathbb{R}^n)\subset L^q(\mathbb{R}^n)\ where\ \frac{1}{q}=\frac{1}{p}-\frac{m}{n}.\\
&if \quad \frac{1}{p}-\frac{m}{n}=0,\ then\ W^{m,p}(\mathbb{R}^n)\subset L^q(\mathbb{R}^n), \forall q\in[p,+\infty[.\\
&if \quad \frac{1}{p}-\frac{m}{n}<0,\ then\ W^{m,p}(\mathbb{R}^n)\subset L^\infty(\mathbb{R}^n)
\end{aligned}$$

with continuous imbedding.

Some inequalities.

Formula of Green: Let $u, v \in W(a,b,V,V')$ with a, b finished. Then we have the formula of the Green:

$$\int_a^b \left\langle \frac{du}{dt}(x), v(t) \right\rangle_{V\times V'} dt + \int_a^b \left\langle \frac{dv}{dt}(x), u(t) \right\rangle_{V\times V'} dt = (u(b),v(b)) - (u(a),v(a)).$$

Proposition 1.10 *For $u \ \in W(a,b,V,V')$ et $v\in V$, we have:*

$$\left\langle \frac{du}{dt}(.), v \right\rangle_{V\times V'} = \frac{d}{dt}(u(.),v),\ in \quad D'(]a,b[).$$

Young Inequality : For all $a, b\in \mathbb{R}$, $(or\ \mathbb{C})$ and for all $p, q \in [1,+\infty[$ with $\frac{1}{q}+\frac{1}{p}=1$, we have :

$$|ab|\leqslant \frac{1}{p}|a|^p+\frac{1}{q}|b|^q.$$

Hölder Inequality : Let $1<p,\ q<+\infty$, with $\frac{1}{p}+\frac{1}{q}=1$. Let f the function de $L^p(\Omega)$ et g one function de $L^q(\Omega)$. Then Hölder l'inequality writes:

$$\|fg\|_{L^1(\Omega)} = \|f\|_{L^p(\Omega)}\ \|g\|_{L^q(\Omega)}.$$

a.e

$$\begin{cases}
\int_\Omega |f(x)g(x)|\,dx \leqslant \left(\int_\Omega |f(x)^p|\,dx\right)^{\frac{1}{p}} \left(\int_\Omega |g(x)^q|\,dx\right)^{\frac{1}{q}}, & \text{if} \quad p,q\in[1,+\infty[,\\[2ex]
\int_\Omega |f(x)g(x)|dx \leqslant \|g\|_{L^\infty}\int_\Omega |f(x)|\,dx, \quad \text{if} \quad p=1, \quad \text{and} \quad q=+\infty.
\end{cases}$$

Green Formula: Let Ω an open bounded of frontiers regulars $\partial\Omega$ and $v(x)$ the normal exteriors the point x. Let u a function of $H^2(\Omega)$ and v a function de $H^1(\Omega)$. then the Green formula write :

$$\begin{aligned}\int_\Omega (\Delta u) v dx &= \int_{\partial\Omega} \frac{\partial u}{\partial n} v ds - \int_\Omega \nabla u \ \nabla v dx \\ \int_\Omega (u \triangle v - v \triangle u) dx &= \int_{\partial\Omega} \left(u \frac{\partial u}{\partial n} - v \frac{\partial u}{\partial n} \right)\end{aligned}$$

2 Weak convergence

Let $(E; \|.\|_E)$ a Banach space and E' its dual space, a.e., the Banach space of all continuous linear forms on E endowed with the norm $\|.\|'_E$ defined by

$$\|f\|_{E'} =: \sup_{x \neq 0} \frac{|\langle f, x\rangle|}{\|x\|}$$

; where $\langle f, x\rangle$; denotes the action of f on$x$, i.e.$\langle f, x\rangle := f(x)$. In the same way, we can define the dual space of E' that we denote by E''. (The Banach space E'' is also called the bi-dual space of E.) An element x of E can be seen as a continuous linear form on E' by setting $x(f) := \langle x, f\rangle$, which means that $E \subset E''$:

Definition 2.1 *The Banach space E is said to be reflexive if $E = E''$.*

Definition 2.2 *The Banach space E is said to be separable if there exists a countable subset D of E which is dense in E, a.e. $\overline{D} = E$.*

Theorem 2.3 *(Riesz). If $(H; \langle .,.\rangle)$ is a Hilbert space, $\langle .,.\rangle$ being a scalar product on H, then $H' = H$ in the following sense: to each $f \in H'$ there corresponds a unique $x \in H$ such that $f = \langle x, .\rangle$ and $\|f\|'_H = \|x\|_H$*

Remark : From this theorem we deduce that $H'' = H$. This means that a Hilbert space is reflexive.

Proposition 2.4 *If E is reflexive and if F is a closed vector subspace of E, then F is reflexive.*

Corolary 2.5 *The following two assertions are equivalent: (i) E is reflexive; (ii) E' is reflexive.*

Weak and strong convergence

Definition 2.6 *(Weak convergence in E). Let $x \in E$ and let $\{x_n\} \subset E$. We say that $\{x_n\}$ weakly converges to x in E, and we write $x_n \rightharpoonup x$ in E, if $\langle f, x_n \rangle \to \langle f, x \rangle$, for all $f \in E'$.*

Definition 2.7 *(weak convergence in E'). Let $f \in E'$ and let $\{f_n\} \subset E'$. We say that $\{f_n\}$ weakly converges to f in E', and we write $f_n \rightharpoonup f$ in E', if $\langle f_n, x \rangle \to \langle f, x \rangle$, for all $x \in E''$.*

Definition 2.8 *(strong convergence). Let $x \in E$(resp. $f \in E'$) and let $\{x_n\} \subset E$ (resp $\{f_n\} \subset E'$). We say that $\{x_n\}$ (resp. $\{f_n\}$) strongly converges to x (resp. f), and we write $x_n \to x$ in E (resp. $f_n \to f$ in E'), if*

$$\lim_n \|x_n - x\|_E = 0; \ \ (\textit{resp.} \ \lim_n \|f_n - f\|'_E = 0)$$

Proposition 2.9 *Let $x \in E$, let $\{x_n\} \subset E$, let $f \in E'$ and let $\{f_n\} \subset E'$.*

i. If $x_n \to x$ in E then $x_n \rightharpoonup x$ in E.

ii. If $x_n \rightharpoonup x$ in E then $\{x_n\}$ is bounded.

iii. If $x_n \rightharpoonup x$ in E then $\liminf\limits_{n\to\infty} \|x_n\|_E \geq \|x\|_E$

iv. If $f_n \to f$ in E' then $f_n \rightharpoonup f$ inE' (and so $f_n \overset{*}{\rightharpoonup} f$ in E').

v. If $f_n \rightharpoonup f$ in E' then $\{f_n\}$ is bounded.

vi. If $f_n \rightharpoonup f$ in E' then then $\liminf\limits_{n\to\infty} \|f_n\|'_E \geq \|f\|'_E$.

Proposition 2.10 *(finite dimension). If $\dim E < \infty$ then strong, weak and weak star convergence are equivalent.*

Bounded and unbounded linear operators

Let $(E, \|.\|_E)$ and $(F, \|.\|_F)$ be two Banach spaces over $\mathbb{C}$, and H will always denote a Hilbert space equipped with the scalar product $< .,. >_H$ and the corresponding norm $\|.\|_H$. A linear operator $T : E \longrightarrow F$ is a transformation which maps linearly E in F, that is

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v), \ \ \forall u, v \in E \text{ and } \alpha, \beta \in \mathbb{C}.$$

Definition 2.11 *A linear operator $T : E \longrightarrow F$ is said to be bounded if there exists $c \geq 0$ such that*

$$\|Tu\|_F \leq c\|u\|_E \ \ \forall u \in E.$$

The set of all bounded linear operators from E into F is denoted by $\mathcal{L}(E, F)$. Moreover, the set of all bounded linear operators from E into E is denoted by $\mathcal{L}(E)$.

Definition 2.12 *A bounded operator $T \in \mathcal{L}(E, F)$ is said to be compact if for each sequence $(x_n)_{n\in\mathbb{N}} \in E$ with $\|x_n\|_E = 1$ for each $n \in \mathbb{N}$, the sequence $(Tx_n)_{n\in\mathbb{N}}$ has a subsequence which converges in F.*
The set of all compact operators from E into F is denoted by $\mathcal{K}(E, F)$. For simplicity one writes $\mathcal{K}(E) = \mathcal{K}(E, F)$.

Definition 2.13 *Let $T \in \mathcal{L}(E, F)$ we define*
• *Range of T by*

$$\mathcal{R}(T) = \{Tu : u \in E\} \subset F.$$

• *Kernel of T by*

$$ker(T) = \{u \in E : Tu = 0\} \subset E.$$

Theorem 2.14 (Fredholm alternative)
If $T \in \mathcal{K}(E)$, then
• *$ker(I - T)$ is finite dimension, (I is the identity operator on E) .*
• *$\mathcal{R}(I - T)$ is closed.*
• *$ker(I - T) = 0 \Leftrightarrow \mathcal{R}(I - T) = E$.*

Definition 2.15 *An unbounded linear operator T from E into F is a pair $(T, D(T))$, consisting of a subspace $D(T) \subset E$ (called the domain of T) and a linear transformation.*

$$T : D(T) \subset E \mapsto F.$$

In the case when $E = F$ then we say $(T, D(T))$ is an unbounded linear operator on E. If $D(T) = E$ then $T \in \mathcal{L}(E, F)$.

Definition 2.16 *Let $T : D(T) \subset E \mapsto F$ be an unbounded linear operator.*
• *The range of T is defined by*

$$\mathcal{R}(T) = \{Tu : u \in D(T)\} \subset F.$$

- *The Kernel of* T *is defined by*

$$ker(T) = \{u \in D(T) : Tu = 0\} \subset E.$$

- *The graph of* T *is defined by*

$$G(T) = \{(u, Tu) : u \in D(T)\} \subset E \times F.$$

Definition 2.17 *A map* T *is said to be closed if* $G(T)$ *is closed in* $E \times F$. *The closedness of an unbounded linear operator* T *can be characterize as following if* $u_n \in D(T)$ *such that* $u_n \to u$ *in* E *and* $Tu_n \to v$ *in* F, *then* $u \in D(T)$ *and* $Tu = v$.

Definition 2.18 *Let* $T : D(T) \subset E \mapsto F$ *be a closed unbounded linear operator.*

- *The resolvent set of* T *is defined by*

$$\rho(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is bijective from } D(T) \text{ onto } F\}.$$

- *The resolvent of* T *is defined by*

$$\mathcal{R}(\lambda, T) = \{(\lambda I - T)^{-1} : \lambda \in \rho(T)\}$$

- *The spectrum set of* T *is the complement of the resolvent set in* $\mathbb{C}$, *denoted by*

$$\sigma(T) = \mathbb{C}/\rho(T)$$

Definition 2.19 *Let* $T : D(T) \subset E \mapsto F$ *be a closed unbounded linear operator. we can split the spectrum* $\sigma(T)$ *of* T *into three disjoint sets, given by*

- *The punctual spectrum of* T *is define by*

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : ker(\lambda I - T) \neq \{0\}\}$$

in this case λ *is called an eigenvalue of* T.

- *The continuous spectrum of* T *is define by:*

$$\sigma_c(T) = \{\lambda \in \mathbb{C} \,:\, ker(\lambda I - T) = 0,\ \overline{\mathcal{R}(\lambda I - T)} = F \text{ and } (\lambda I - T)^{-1} \text{ is not bounded } \}.$$

- *The residual spectrum of T is define by*

$$\sigma_r(T) = \{\lambda \in \mathbb{C} \,:\, ker(\lambda I - T) = 0, \ and \ \mathcal{R}(\lambda I - T) \ \ is \ not \ dense \ in \ F\}$$

Definition 2.20 *Let $T : D(T) \subset E \longrightarrow F$ be a closed unbounded linear operator and let λ be an eigevalue of A. non-zero element $e \in E$ is called a generalized eigenvector of T associated with the eigenvalue value λ, if there exists $n \in \mathbb{N}^*$ such that*

$$(\lambda I - T)^n e = 0 \quad and \quad (\lambda I - T)^{n-1} e \neq 0.$$

if $n = 1$, then e is called an eigenvector.

Definition 2.21 *Let $T : D(T) \subset E \longrightarrow F$ be a closed unbounded linear operator. We say that T has compact resolvent, if there exist $\lambda_0 \in \rho(T)$ such that $(\lambda_0 I - T)^{-1}$ is compact.*

Theorem 2.22 *Let $(T, D(T))$ be a closed unbounded linear operator on H then the space $(D(T), \|.\|_{D(T)})$ where $\|u\|_{D(T)} = \|Tu\|_H + \|u\|_H \quad \forall u \in D(T)$ is Banach space .*

Theorem 2.23 *Let $(T, D(T))$ be a closed unbounded linear operator on H then, $\rho(T)$ is an open set of $\mathbb{C}$.*

3 Linear operators

Definition 3.1 *Let X and Y be two Banach spaces. A linear mapping: $\mathcal{A} : D(\mathcal{A})(\subset X) \to Y$ is called a linear operator. The $D(\mathcal{A}) \subset X$ is called the domain of $\mathcal{A}$ and $\mathcal{R}(\mathcal{A}) \subset Y$ is called the range of $\mathcal{A}$:*

$$\mathcal{R}(\mathcal{A}) = \{\mathcal{A}x \mid x \in D(\mathcal{A})\}.$$

$\mathcal{A}$ is said to be invertible (or injective) if $\mathcal{A}x = 0$ if and only if $x = 0$; $\mathcal{A}$ is said to be onto (or surjective) if $\mathcal{R}(\mathcal{A}) = Y$; $\mathcal{A}$ is said to be densely defined if $\overline{D(\mathcal{A})} = X$.

Definition 3.2 *A linear operator $\mathcal{A}$ is said to be closed if for any $x_n \in D(\mathcal{A}),\ n \geq 1$,*

$$x_n \to x, \ \ \mathcal{A}x_n \to y, \ \ as \ \ n \to \infty,$$

it must have $x \in D(\mathcal{A})$ and $\mathcal{A}x = y$. $\mathcal{A}$ is said to be bounded if $D(\mathcal{A}) = X$ and $\mathcal{A}$ maps a bounded set of X into a bounded set of Y. A linear operator is bounded if and only if it is *continuous, that is,*

$$x_n \to x_0 \in X \Longrightarrow \mathcal{A}x_n \to \mathcal{A}x_0 \in Y$$

for any $x_n \subset X$.

Obviously, any operator which has bounded inverse must be closed. All the bounded operators from X to Y are denoted by $\mathcal{L}(X, Y)$. In particular, when $X = Y$, $\mathcal{L}(X, Y)$ is abbreviated as $\mathcal{L}(X)$.

Theorem 3.3 *Let X and Y be Banach spaces. Then $\mathcal{L}(X, Y)$ is a Banach space with the norm*

$$\|\mathcal{A}\| = \sup\left\{\|\mathcal{A}x\| \mid x \in X,\ \|x\| = 1\right\}.$$

Definition 3.4 *Let X be a Banach space. If $Y = \mathbb{R}$ or $Y = \mathbb{C}$, then the operator in $\mathcal{L}(X, Y)$ is called a linear functional on X. A bounded functional is also denoted by f.*

By Theorem 3.3, all linear bounded functionals on X consist of a Banach space which is called the dual of the space X, denoted by X^*.

A bounded operator is called compact operator if $\mathcal{A}$ maps any bounded set into a relatively compact set which is a compact set but not necessarily closed. For a closed operator $\mathcal{A}$, we can define the graph space $[D(\mathcal{A})]$ where the norm is defined by

$$\|x\|_{[D(\mathcal{A})]} = \|x\| + \|\mathcal{A}x\|, \quad \forall x \in D(\mathcal{A}). \tag{I.1}$$

Theorem 3.5 [Open mapping theorem]
Let X and Y be Banach spaces and let $\mathcal{A}$ be a bounded operator from X to Y. If $\mathcal{R}(\mathcal{A}) = Y$, then $\mathcal{A}$ maps an open set of X into an open set of Y.

Theorem 3.6 [Closed graph theorem]
Suppose that $\mathcal{A}$ is a closed operator in a Banach space X. Then $\mathcal{A}$ must be bounded provided $D(\mathcal{A}) = X$.

Theorem 3.7 [Lax Milgram theorem]
Let $a(x, y)$ be a bilinear form, that is, it is linear in x and conjugate linear in y, and satisfies

- *there is an* $M > 0$ *such that* $|a(x, y)| \leq M\|x\|\|y\|$ *for all* $x, y \in H$*;*

- *there is a* $\delta > 0$ *such that for any* $x \in H$*,* $|a(x, x)| \geq \delta\|x\|^2$*.*

Then there exists a unique $\mathcal{A} \in \mathcal{L}(H)$ *which is bounded invertible and satisfies*

$$a(x, y) = \langle x, Ay \rangle, \ \forall x, y \in H.$$

Definition 3.8 *A linear operator in a Hilbert space is said to be symmetric if*

$$\mathcal{A}^* = \mathcal{A} \text{ on } D(\mathcal{A}) \text{ and } D(\mathcal{A}^*) \supseteq D(\mathcal{A})$$

A symmetric operator is said to be self-adjoint, if $\mathcal{A}^* = \mathcal{A}$*.*

For bounded operators, the symmetric and self-adjoint are the same. But for unbounded operators, they are different.

Definition 3.9 *A linear operator* $\mathcal{B}$ *in a Hilbert space* H *is said to be A-bounded if*

- $D(\mathcal{B}) \supset D(\mathcal{A})$*, and*

- *there are* $a, b > 0$ *such that*

$$\|\mathcal{B}x\| \leq a\|\mathcal{A}x\| + b\|x\|, \ \forall x \in D(\mathcal{A}).$$

Definition 3.10 *Let* $\mathcal{A} \in \mathcal{L}(H)$ *be a self-adjoint operator in a Hilbert space* H*.* $\mathcal{A}$ *is said to be positive if*

$$\langle \mathcal{A}x, x \rangle \geq 0, \ \forall x \in H. \tag{I.2}$$

A positive operator is denoted by $\mathcal{A} \geq 0$*;* $\mathcal{A}$ *is said to be positive definite if the equality in I.2 holds true only if* $x = 0$*, which is denoted by* $\mathcal{A} > 0$*; A positive operator* $\mathcal{A}$ *is said to be strictly positive if there exists an* $m > 0$ *such that*

$$\langle \mathcal{A}x, x \rangle \geq m\|x\|^2, \ \forall x \in D(\mathcal{A}). \tag{I.3}$$

4 The spectrum of linear operators

Definition 4.1 *Suppose that X is a Banach space and $\mathcal{A} : D(\mathcal{A})(\subset X) \to X$ is a linear operator. The resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$ is an open set in the complex plane, which is defined by*

$$\rho(\mathcal{A}) = \left\{ \lambda \in \mathbb{C} \mid (\lambda - \mathcal{A})^{-1} \in \mathcal{L}(X) \right\}.$$

when $\lambda \in \rho(\mathcal{A})$, the operator $R(\lambda, \mathcal{A}) = (\lambda - \mathcal{A})^{-1}$ is called the resolvent of $\mathcal{A}$. If one of resolvents is compact, then any of the resolvents must be compact. This comes from the following resolvent formula:

$$(\lambda - \mathcal{A})^{-1} - (\mu - \mathcal{A})^{-1} = (\mu - \lambda)(\lambda - \mathcal{A})^{-1}(\mu - \mathcal{A})^{-1}, \ \forall \lambda, \mu \in \rho(\mathcal{A}).$$

The spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ is the supplement set of the resolvent set in the complex plane, that is,

$$\sigma(\mathcal{A}) = \mathbb{C} \setminus \rho(\mathcal{A}).$$

Generally, the spectrum $\sigma(\mathcal{A})$ is decomposed into three parts:

$$\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) \cup \sigma_c(\mathcal{A}) \cup \sigma_r(\mathcal{A})$$

where

- *the point spectrum*

$$\sigma_p(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid \textit{there exists a } 0 \neq x \in X \textit{ so that } \mathcal{A}x = \lambda x\};$$

- *the continuous spectrum*

$$\sigma_c(\mathcal{A}) = \left\{ \lambda \in \mathbb{C} \mid (\lambda - \mathcal{A}) \textit{ is invertible and } \overline{\mathcal{R}(\lambda - \mathcal{A})} = X \right\};$$

- *the residual spectrum*

$$\sigma_r(\mathcal{A}) = \left\{ \lambda \in \mathbb{C} \mid (\lambda - \mathcal{A}) \textit{ is invertible and } \overline{\mathcal{R}(\lambda - \mathcal{A})} \neq X \right\};$$

When $\lambda \in \sigma_p(\mathcal{A})$, any nonzero vector x satisfying $\mathcal{A}x = \lambda x$ is said to be an eigenvector *(it is also called eigenfunction if the space is the function space) of $\mathcal{A}$. For a matrix in $\mathbb{C}^n$, the spectrum is just the set of eigenvalues.*

5 Semigroups of linear operators

Definition 5.1 *Semigroup theory is aiming to solve the following linear differential equation in Banach space X:*

$$\begin{cases} \dot{u}(t) = \mathcal{A}u(t), \ t > 0, \\ u(0) = x \in X, \end{cases} \tag{I.4}$$

where $\mathcal{A} : D(\mathcal{A})(\subset X) \to X$ is a linear operator.

Eq (I.4) is said to be well-posed (for bounded A) If:

- *for any initial value $x \in D(\mathcal{A}) = X$, there exists a solution $u(x,t)$ to (I.4) which is differentiable for $t > 0$, continuous at $t = 0$ and $u(x,t)$ satisfies (I.4) for $t > 0$,*
- *$u(x,t)$ depends continuously on the initial condition x, that is:*

$$x \to 0 \ \textit{ implies } \ u(x,t) \to 0 \ \textit{ for each } \ t > 0;$$

- *$u(x,t)$ is unique for each $x \in D(\mathcal{A}) = X$.*

We can then define an operator $T(t)$ by $T(t)x = u(x,t)$ for each $t \geq 0$. From the existence and uniqueness of the solution $u(x,t)$, we know that $T(t), t \geq 0$ is well defined on X.

Definition 5.2 *Let X be a Banach space and $T(t) : X \to X$ be a family of linear bounded operators, for $t \geq 0$, $T(t)$ is called a semigroup of linear bounded operators, or simply a semigroup, on X if*

- *$T(0) = I$;*
- *$T(t+s) = T(t)T(s), \ \forall t \geq 0, \ s \geq 0$*

A semigroup $T(t)$ is called uniformly continuous if

$$\lim_{t \to 0} \|T(t) - I\| = 0,$$

and is called strongly continuous, (or C_0-semigroup for short), if

$$\lim_{t\to 0} T(t)x - x = 0,\ \forall x \in X$$

Definition 5.3 *Let $T(t)$ be a C_0-semigroup on a Banach space X. The operator $\mathcal{A}$ is defined as*

$$\begin{cases} \mathcal{A}x = \lim\limits_{t\longrightarrow 0} \dfrac{T(t)x - x}{t}, & \forall x \in D(\mathcal{A}), \\ D(\mathcal{A}) = \left\{ x \in X \mid \lim\limits_{t\longrightarrow 0} \dfrac{T(t)x - x}{t} \quad \textit{exists} \right\} \end{cases}$$

is called the infinitesimal generator of the C_0-semigroup $T(t)$.

Theorem 5.4 *Let X be a Banach space. For any bounded linear operator $\mathcal{A}$ on X, $T(t) = e^{\mathcal{A}t}$ is a uniformly continuous semigroup and $\mathcal{A}$ is the infinitesimal generator of $T(t)$ with $D(\mathcal{A}) = X$.*

Theorem 5.5 *Let $T(t)$ be a C_0-semigroup on a Banach space X, then the following holds*

- *There exists constants $M > 1$ and $\omega \geq 0$ such that*

$$\|T(t)\| \leq Me^{\omega t},\ \forall t \geq 0$$

- *Suppose that $\mathcal{A}$ is the generator of $T(t)$. Then*

$$\{\lambda \in \mathbb{C} \mid Re(\lambda) > \omega\} \subset \rho(\mathcal{A}).$$

- *In addition, if $Re(\lambda) > \omega$, then*

$$\mathcal{R}(\lambda, \mathcal{A})x = (\lambda - \mathcal{A})^{-1}x = \int_0^\infty e^{-\lambda t} T(t)x dt,\ \forall x \in X.$$

- *$T(t)$ is strongly continuous on X. i.e. for any $x \in X$, the map $t \to T(t)x$ is continuous.*

Theorem 5.6 *Let $\mathcal{A}$ be the generator of a C_0-semigroup $T(t)$ on a Banach space X. we have the following*

- *$D(\mathcal{A})$ is dense in X*

- *$\mathcal{A}$ is a closed operator.*

- *For any* $n \geq 1$, $D(\mathcal{A}^n)$ *is dense in* X. *The set* $D = \cap_{n=1}^{\infty} D(\mathcal{A}^n)$ *is also dense in* X *and is invariant under* $T(t)$. *i.e. for* $x \in D$, $T(t)x \in D$ *for* $t \geq 0$. *Moreover, if we define* $D^\infty = \{x \in X | t \to T(t)x \in C^\infty\}$. *then we have* $D = D^\infty$

Theorem 5.7 [Hille-Yosida]
Let X *be a Banach space and let* $\mathcal{A}$ *be a linear (not necessirely bounded) operator in* X. *Then,* $\mathcal{A}$ *is the infinitesimal generator of a* C_0*-semigroup of contractions* $T(t)$ *on* X, *if and only if*

- $\mathcal{A}$ *is closed and* $D(\mathcal{A})$ *is dense in* X
- *There exist positive constants* M *and* ω *verifying the property: for all* $\lambda > \omega$, $\lambda \in \rho(\mathcal{A})$, *the following holds*
$$\|\mathcal{R}(\lambda, \mathcal{A})^n\| \leq \frac{M}{(\lambda - \omega)^n}, \ \ n = 1, 2, ...$$

Definition 5.8 *Let* $T(t)$ *be a* C_0*-semigroup on a Banach space* X *and let* $M \geq 1$ *and* $\omega \geq 0$.

If $\omega = 0$, *then we have* $\|T(t)\| \leq M$ *for* $t \geq 0$ *and* $T(t)$ *is called uniformly bounded.*

Moreover, if we have $M = 1$, *then* $T(t)$ *is called a contraction.*

Corolary 5.9 *Let* X *be a Banach space and let* $\mathcal{A}$ *be a linear (not necessirely bounded) operator in* X. *Then,* $\mathcal{A}$ *is the infinitesimal generator of the* C_0*-semigroup of contractions* $T(t)$ *on* X, *if and only if the following holds.*

- $\mathcal{A}$ *is closed and* $D(\mathcal{A})$ *is dense in* X
- *For any* $\lambda > 0$, $\lambda \in \rho(\mathcal{A})$ *and*
$$\|\mathcal{R}(\lambda, \mathcal{A})\| \leq \frac{1}{\lambda}$$
.

Definition 5.10 *Let* X *be a Banach space and let* $F(x)$ *be the duality set. A linear operator* $\mathcal{A}$ *in* X *is said to be dissipative if for every* $x \in D(\mathcal{A})$ *there is an* $x^* \in F(x)$ *such that*

$$Re\langle \mathcal{A}x, x^* \rangle \leq 0$$

Definition 5.11 *A linear operator* $\mathcal{A}$ *in a Banach space* X *is called m-dissipative if* $\mathcal{A}$ *is dissipative and* $\mathcal{R}(\lambda - \mathcal{A}) = X$, *for some* $\lambda > 0$.

Remark 5.12 *In a Hilbert space H, the dissipativity of $\mathcal{A}$ simply means that*

$$Re\langle \mathcal{A}x, x\rangle \leq 0,\ \forall x \in D(\mathcal{A}).$$

Theorem 5.13 **[L*ü*mer-Phillips]**
Let $\mathcal{A}$ be a linear operator in a Banach space X. Then $\mathcal{A}$ generates a C_0-semigroup of contractions on X if and only if

- $\overline{D(\mathcal{A})} = X$.
- $\mathcal{A}$ *is dissipative.*

Remark 5.14 *When X is reflexive, the condition $\overline{D(\mathcal{A})} = X$ can be removed in the Lümer-Phillips theorem.*

6 Fractional Derivative control

In this part, we introduce the necessary elements for the good understanding of this manuscript. It includes a brief reminder of the basic elements of the theory of fractional computation as well as some examples of applications of this theory in this scientific field.

The concept of fractional computation is a generalization of ordinary derivation and integration to an arbitrary order. Derivatives of non-integer order are now widely applied in many domains, for example in economics, electronics, mechanics, biology, probability and viscoelasticity.

A particular interest for fractional derivation is related to the mechanical modeling of gums and rubbers. In short, all kinds of materials that preserve the memory of previous deformations in particular viscoelastic. Indeed, the fractional derivation is introduced naturally.

There exists a many mathematical definitions of fractional order integration and derivation. These definitions do not always lead to identical results but are equivalent for a wide large of functions. We introduce the fractional integration operator as well as the two most definitions of fractional derivatives, used, namely that Riemann-Liouville and Caputo, by giving the most important properties of the notations.

Fractional systems appear in different fields of research. however, the progressive interest in their applications in the basic and applied sciences. It can be noted that for most of the domains presented (automatic, physics, mechanics of continuous media). The fractional operators are used to take into account memory effects. We can mention the works that reroute various applications of fractional computation.

In physics, on of the most remarkable applications of fractional computation in physics was in the context of classical mechanics. Riewe, has shown that the Lagragien of the notion of temporal derivatives of fractional orders leads to an equation of motion with friction forces and nonconservative are essential in macroscopic variational processing such as friction. This result are remarkable because friction forces and non conservative forces are essential in the usual macroscopic variational processing and therefore in the most advances methods classical mechanics.

Riewe, has generalized the usual Lagrangian variation which depends on the fractional derivatives in order to deal with the usual non-conservative forces. On the another hand, serval approaches have been developed to generalize the principle of least action and the Euler-Lagrange equation to the case of fractional derivative.

The definition of the fractional order derivation is based on that of a fractional order integration, a fractional order derivation takes on a global character in contrast to an integral derivation. It turns out that the derivative of a fractional order integration, a fractional order derivation takes on a global character in contrast to an integral derivation.

It turns out that the derivative of a fractional order of a function requires the knowledge of $f(t)$ over the entire interval $]a, b[$, where in the whole case only the local knowledge of f around around t is necessary. This property allows to interpret fractional order systems as long memory systems, the whole systems being then interpretable as systems with short memory. Now, we give the definition of the fractional derivatives in the sense of Riemann-Liouville as well as some essential properties.

Definition 6.1 *The fractional integral of order $\alpha > 0$, in sense Rieamann-Liouville is given by*

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \quad t > a$$

Definition 6.2 *The fractional derivative of order $\alpha > 0$, in sens of Rieamann-Liouville of* *a function f defined on the interval $[a,b]$ is given by*

$$D^{\alpha}_{RL,a}(t) = D^n I^{n-\alpha}_{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s) ds, \quad n = [\alpha]+1, t > a$$

In particular, if $\alpha = 0$, then

$$D^{0}_{RL,a} f(t) = I^0_a f(t) = f(t)$$

If $\alpha = n \in \mathbb{N}$, then

$$D^{0}_{RL,a} f(t) = f^{(n)}(t).$$

moreover, if $0 < \alpha < 1$, then $n = 1$, then

$$D^{\alpha}_{RL,a} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{-\alpha} f(s) ds, \quad t > a$$

Example 6.1 *Let $\alpha > 0, \quad \gamma > -1$ and $f(t) = (t-a)^{\gamma}$, then*

$$I^{\alpha}_a f(t) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} (t-a)^{\gamma+\alpha},$$

$$D^{\alpha}_{RL,a} f(t) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha+1)} (t-a)^{\gamma-\alpha}$$

In particular, if $\gamma = 0$ and $\alpha > 0$, then $D^{\alpha}_{RL,a}(C) = C\dfrac{(t-\alpha)^{-\alpha}}{\Gamma(1-\alpha)}$

The derivatives of Riemann-Liouville have certain disadvantages when attempting to model real world phenomena. The problems studied require a definition of the fractional derivatives allowing the use of the physically interpretable initial conditions introducing $y(0), y\prime(0)$, etc. There shortcomings led to an alternatives that satisfies these demands in the last sixties. It was introduced by Caputo.In fact, Caputo and Minardi used this definition in their work on viscoelasticity.

Now, we give the definition of the fractional derivatives in the sense of Caputo as well as some essential properties.

Definition 6.3 *The fractional derivative of order $\alpha > 0$, in sense of Caputo, defined on the* *interval* $[a,b]$, *is given by*

$$D^{\alpha}_{C,a}f(t) = D^{\alpha}_{RL,a}\left(f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^k\right).$$

where

$$n = \begin{cases} [\alpha]+1 & if \quad \alpha \notin \mathbb{N}, \\ \alpha & if \quad \alpha \in \mathbb{N}^*, \end{cases}$$

In particular, where $0 < \alpha < 1$, *the relation (6.3) take the form*

$$\begin{aligned} D^{\alpha}_{C,a}f(t) &= D^{\alpha}_{C,a}([f(t) - f(a)]) \\ &= I_a^{1-\alpha}f'(t) \\ &= \frac{1}{\Gamma(1-\alpha)}\int_a^t (t-a)^{-\alpha}f'(s)ds \end{aligned}$$

If $\alpha \in \mathbb{N}$, *then* $f^{(n)}(t) \quad D^{\alpha}_{C,a}f(t) = f^n(t)$ *coincides i.e*

$$D^{\alpha}_{Ca}f(t) = f^n(t).$$

Example 6.2 *Let* $\alpha > 0$ *and* $f(t)(t-a)^{\gamma}$ *where* $\gamma > -1$. *then*

$$D^{\alpha}_{Ca}f(t) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha+1)}(t-a)^{\gamma-\alpha}.$$

In particular, if $\gamma = 0$ *and* $\alpha > 0$, *then* $D^{\alpha}_{Ca}C = 0$.

7 Caputo's fractional derivative

There are various ways of defining the fractional derivative, but we will focus primarily on the Caputo fractional derivative defined by Podlubny [**?**] (chapter 2.4) who gave few formal definitions and theorems.

The approach suggested by Caputo is very useful for the formulation and solution of applied problems and their transparency. It allows the formulation of initial conditions for initial-value problems for fractional-order differential equations in a form involving the limit values of integer-order derivatives at the lower terminal (initial time) $t = a$, such as $y^{'}(a), y^{''}(a)$ etc.

The definition of the fractional derivative of the Reimann-Liouville type played an important role in the development of the theory of fractional derivatives and integrals and for its applications in pure mathematics (solution of integer-order differential equations, definitions of new function classes, summation of series, etc.). We define it by

$$_{a}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s)\, ds, \quad (n-1 \leq \alpha < n)$$

or

$$_{a}D_t^{\alpha} f(t) = \frac{d^n}{dt^n} \left({}_{a}D_t^{-(n-\alpha)} f(t) \right), \quad (n-1 \leq \alpha < n)$$

Moreover, we see that for $\alpha = n \geq 1$ and $t > a$

$$_{a}D_t^{\alpha} f(t) = \frac{d^n}{dt^n} \left({}_{a}D_t^{0} f(t) \right) = \frac{d^n f(t)}{dt^n} = f^n(t)$$

which means that for $t > a$ the Riemann-Liouville fractional derivative of order $\alpha = n > 1$ coincides with the conventional derivative of order n.

However, there have appeared a number of works, especially in the theory of viscoelasticity and in solid mechanics, where fractional derivatives are used for a better description of material properties. Mathematical modeling naturally leads to differential equations of fractional order, and to the necessity of the formulation of initial conditions to such equations. This means that the Riemann-Liouville is not the best definition to take when solving some problems, their solutions are practically useless because there is no known physical interpretation for such types of initial conditions, it is better to use a different definition, such as the Caputo definition which makes initial conditions for differential equations nicer.

Caputo's definition can be written as

$$_{a}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-n)} \int_a^t \frac{f^{(n)}(s)\, ds}{(t-s)^{\alpha+1-n}}, \quad (n-1 < \alpha < n).$$

Under natural conditions on the function $f(t)$, for $\alpha \to n$ the Caputo derivative becomes a conventional n^{th} derivative of the function $f(t)$. Indeed, let us assume that $0 \leq n-1 < \alpha < n$ and that the function $f(t)$ has $n+1$ continuous bounded derivatives in $[a, t]$ for every $t > a$, then

$$
\begin{aligned}
\lim_{\alpha\to n} {}_a^C D_t^\alpha f(t) &= \lim_{\alpha\to n}\left(\frac{f^{(n)}(a)(t-a)^{n-\alpha}}{\Gamma(n-\alpha+1)}+\frac{1}{\Gamma(n-\alpha+1)}\int_a^t (t-s)^{n-\alpha} f^{(n+1)}(s)\,ds\right)\\
&= f^{(n)}(a)+\int_a^t f^{(n+1)}(s)\,ds\\
&= f^n(t) \quad n=1,2,...
\end{aligned}
$$

The main advantage of Caputo's approach is that the initial conditions for fractional differential equation with Caputo derivatives take on the same form as for integer-order differential equations, i.e. contain the limit values of integer-order derivatives of unknown functions at the lower terminal $t = a$.

Definition 7.1 *The fractional derivative of order α, $0 < \alpha < 1$, in sense of Caputo, is defined by*

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{df}{ds}(s)ds.$$

Definition 7.2 *The fractional integral of order α, $0 < \alpha < 1$, in sense Riemann-Liouville, is defined by*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds.$$

Remark 7.1 *From the above definitions, clearly*

$$D^\alpha f = I^{\alpha-1} Df, \;\; 0 < \alpha < 1.$$

Now, we give the definitions of the generalized Caputo's fractional derivative and the generalized fractional integral.

Definition 7.3 *The generalized Caputo's fractional derivative is given by*

$$D^{\alpha,\eta} f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)}\frac{df}{ds}(s)\,ds, \;\; 0<\alpha<1, \;\; \eta \geq 0.$$

Definition 7.4 *The generalized fractional integral in sense Riemann-Liouville, is given by*

$$I^{\alpha,\eta} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} e^{-\eta(t-s)} f(s)\,ds, \;\; 0<\alpha<1, \eta \geq 0.$$

Remark 7.2 *We have*

$$D^{\alpha,\eta} f = I^{1-\alpha,\eta} Df, \;\; 0<\alpha<1, \eta\geq 0.$$

Chapter II

Well-posedness and exponential stability of a thermoelastic system with internal delay

1 Introduction

Let us consider the following thermoelastic system with delay

$$\begin{cases} u_{tt}(x,t) - \alpha u_{xx}(x,t-\tau) + \gamma\theta_x(x,t) = 0, & \text{in } (0,\ell)\times(0,\infty), \\ \theta_t(x,t) - \kappa\theta_{xx}(x,t) + \gamma u_{xt}(x,t) = 0, & \text{in } (0,\ell)\times(0,\infty), \\ u(0,t) = u(\ell,t) = \theta_x(0,t) = \theta_x(\ell,t) = 0, & t \geq 0 \end{cases} \tag{II.1}$$

where α, γ, κ and ℓ are some positive constants. The functions $u = u(x,t)$ and $\theta = \theta(x,t)$ describe respectively the displacement and the temperature difference, with $x \in (0,\ell)$ and $t \geq 0$. Moreover, $\tau > 0$ is the time delay. Racke proved in [32] that, under some initial and boundary conditions, the system (II.1) is not well posed and unstable even if τ is relatively small. However, it is well known that, in the absence of delay, the damping through the heat conduction is strong enough to produce an exponential stable system (see for example [18, 30, 33]), and specially, [30] and [19] where various types of boundary conditions are associated to the one dimensional thermoelastic systems.

In recent years, the PDEs with time delays effects become an active area of research. In fact, time delays so often arise in many applications since, most physical phenomena not only depend on the present state but also on some past occurrences, see for instance [34] and references therein, but as for the classical thermoelastic system, an arbitrary small delay may destroy the well-posedness of the problem or may destroy the stability, see also [7, 14, 15, 28].

In order to solve the problem, additional conditions or control terms have been used, we refer to [2, 3, 14, 16, 27, 31], see also [23] and references therein . In this paper we add to the delayed equation, a Kelvin-Voigt damping of the form $-\beta u_{xxt}(x,t)$ for some real positive number β, which eventually depends on α, γ, κ and τ. Then our system takes the form

$$\begin{cases} u_{tt}(x,t) - \alpha u_{xx}(x,t-\tau) - \beta u_{xxt}(x,t) + \gamma\theta_x(x,t) = 0, & \text{in } \Omega \times (0,\infty), \\ \theta_t(x,t) - \kappa\theta_{xx}(x,t) + \gamma u_{xt}(x,t) = 0, & \text{in } \Omega \times (0,\infty), \\ u(0,t) = u(\ell,t) = 0, & \text{in } (0,\infty), \\ \theta_x(0,t) = \theta_x(\ell,t) = 0, & \text{in } (0,\infty), \\ u_x(x,t-\tau) = f_0(x,t-\tau), & \text{in } \Omega \times (0,\tau), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \theta(x,0) = \theta_0(x), & \text{in } \Omega \end{cases} \tag{II.2}$$

where the initial data $(u_0, u_1, f_0, \theta_0)$ belongs to a suitable space and with $\Omega = (0,\ell)$. We meanly investigate well-posedness and exponential stability of such initial-boundary value problem.

This idea arises from [4] where the authors added a Kelvin-Voigt damping term to the abstract equation. More precisely, they considered the following system

$$\begin{cases} u''(t) + aBB^*u'(t) + BB^*u(t-\tau) = 0, & \text{in } (0,\infty), \\ u(0) = u_0, \quad u'(0) = u_1, \\ B^*u(t-\tau) = f_0(t-\tau), & \text{in } (0,\tau), \end{cases} \tag{II.3}$$

where a "prime" denotes a one-dimensional derivative with respect to "t" and where $B : \mathcal{D}(B) \subset H_1 \to H$ is a linear unbounded operator from a Hilbert space H_1 to a Hilbert space H, such that B^*, the adjoint of B, satisfies some properties of coercivity and compact embedding. They obtained an exponential decay result under the assumption $\tau \le a$.

In [26], the authors dropped the time delay in the harmonic term of the elastic equation in (II.1) and added a delay term of the form $\int_{\tau_1}^{\tau_2} \mu(s)\theta_{xx}(x,t-s\tau)ds$ in the heat equation, where τ_1 and τ_2 are non-negative constants such that $\tau_1 < \tau_2$ and $\mu : [\tau_1,\tau_2] \to \mathbb{R}$ is a bounded function. They proved an exponential decay result under the condition $\int_{\tau_1}^{\tau_2} |\mu(s)|ds < \kappa$.

We define the energy of a solution of problem (II.2) as

$$E(t) := \frac{1}{2}\int_\Omega \left(u_t^2(x,t) + \alpha u_x^2(x,t) + \theta^2(x,t)\right) dx + \xi \int_\Omega \int_0^1 u_x^2(x,t-\tau\rho)d\rho dx$$

where $\xi > 0$ is a parameter fixed later on.

We first formulate the problem (II.2) into an appropriate Hilbert space, and then we study the well-posedness of the system using semigroup theory.Then we prove, using Lyapunov's method, a result of exponential stability of system (II.2).

2 Well-posedness of the problem

We introduce, as in [4], the new variable

$$z(x,\rho,t) = u_x(x,t-\tau\rho), \qquad \text{in } \Omega\times(0,1)\times(0,\infty), \tag{II.4}$$

Clearly, $z(x,\rho,t)$ satisfies

$$\begin{aligned} \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) &= 0, \qquad x\in\Omega,\ \ \rho\in(0,1),\ \ t\in(0,+\infty), && \text{(II.5)}\\ z(x,0,t) &= u_x(x,t), \quad x\in\Omega,\ \ t\in(0,+\infty). && \text{(II.6)} \end{aligned}$$

Then, problem (II.2) takes the form

$$\begin{aligned} u_{tt}(x,t) - \alpha z_x(x,1,t) - \beta u_{xxt}(x,t) + \gamma\theta_x(x,t) = 0, &\qquad \text{in } \Omega\times(0,\infty), && \text{(II.7)}\\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, &\qquad \text{in } \Omega\times(0,1)\times(0,+\infty), && \text{(II.8)}\\ \theta_t(x,t) - \kappa\theta_{xx}(x,t) + \gamma u_{xt}(x,t) = 0, &\qquad \text{in } \Omega\times(0,\infty), && \text{(II.9)}\\ u(0,t) = u(\ell,t) = 0, &\qquad \text{in } (0,\infty), && \text{(II.10)}\\ \theta_x(0,t) = \theta_x(\ell,t) = 0, &\qquad \text{in } (0,\infty), && \text{(II.11)} \end{aligned}$$

$$\begin{aligned} z(x,0,t) = u_x(x,t), &\qquad \text{in } \Omega\times(0,\infty), && \text{(II.12)}\\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \theta(x,0) = \theta_0(x), &\qquad \text{in } \Omega, && \text{(II.13)}\\ z(x,\rho,0) = f_0(x,-\tau\rho), &\qquad \text{in } \Omega\times(0,1). && \text{(II.14)} \end{aligned}$$

Observe that it follows from (II.9)-(II.11) that $\int_\Omega \theta_t(x,t)dx = 0$ that is, $\int_\Omega \theta(x,t)dx$ is conservative all the time. Without loss of generality, we assume that $\int_\Omega \theta(x,t)dx = 0$. Otherwise, we can make the substitution $\tilde{\theta}(x,t) = \theta(x,t) - \frac{1}{\ell}\int_\Omega \theta_0(x)dx$, in fact (u,v,z,θ) and $(u,v,z,\tilde{\theta})$ satisfy the same system (III.2)-(III.4).

Let

$$\mathcal{H} = \left\{(f,g,p,h) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega \times (0,1)) \times L^2(\Omega) \mid \int_\Omega h(x)dx = 0\right\}.$$

Equipped with the following inner product: for any $U_k = (f_k, g_k, p_k, h_k) \in \mathcal{H},\ \ k = 1, 2,$

$$\langle U_1, U_2\rangle_{\mathcal{H}} = \int_\Omega \left(\alpha f_{1x}(x) f_{2x}(x) + g_1(x) g_2(x) + h_1(x) h_2(x)\right) dx + \xi \int_\Omega \int_0^1 p_1(x,\rho) p_2(x,\rho) d\rho dx,$$

$\mathcal{H}$ is a Hilbert space.

Define

$$U := (u, u_t, z, \theta)$$

then, problem (II.2) can be formulated as a first order system of the form

$$\begin{cases} U' = \mathcal{A}U, \\ U(0) = (u_0, u_1, f_0(., -.\tau), \theta_0) \end{cases} \tag{II.15}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ z \\ \theta \end{pmatrix} = \begin{pmatrix} v \\ (\alpha z(.,1) + \beta v_x)_x - \gamma\theta_x \\ -\frac{1}{\tau} z_\rho \\ -\gamma v_x + \kappa\theta_{xx} \end{pmatrix}$$

with domain

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{array}{c} U = (u, v, z, \theta) \in \mathcal{H} \cap [H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega; H^1(0,1)) \times H^2(\Omega)] \mid \\ z(.,0) = u_x \ \text{ and } \ (\alpha z(.,1) + \beta v_x) \in H^1(\Omega) \end{array} \right\}$$

in the Hilbert space $\mathcal{H}$.

To establish the existence of solution, we will prove that the operator $\mathcal{A}$ generates a $\mathcal{C}_0$-semigroup, and to do this, we will prove that $\mathcal{A} - mId$ generates a $\mathcal{C}_0$-semigroup (of contractions), for an appropriate real number m, function of ξ, α, β and τ. Then we apply the bounded perturbation theorem (Sect. III.1 of [17]). In fact, we begin by the following result

Lemma 2.1 *If $\xi > \frac{2\tau\alpha^2}{\beta}$, then there exists $m \in \mathbb{R}$ such that $\mathcal{A} - mId$ is dissipatif maximal.*

Proof 2.2 *Take $U = (u, v, z, h) \in \mathcal{D}(\mathcal{A})$.*

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= \alpha \int_\Omega v_x(x) u_x(x) dx + \int_\Omega \left((\alpha z(.,1) + \beta v_x)_x(x) - \gamma\theta_x\right) v(x) dx \quad &\text{(II.16)}\\ &+ \int_\Omega \left(-\gamma v_x + \kappa\theta_{xx}\right)(x)\theta(x) dx - \frac{\xi}{\tau}\int_\Omega \int_0^1 z_\rho(x,\rho) z(x,\rho) d\rho dx. \quad &\text{(II.17)}\end{aligned}$$

Integrating by parts, using boundary conditions of u, v and θ to get

$$\begin{aligned}&\int_\Omega \left((\alpha z(.,1) + \beta v_x)_x - \gamma\theta_x\right)(x) v(x) dx + \int_\Omega \left(-\gamma v_x + \kappa\theta_{xx}\right)(x)\theta(x) dx\\ &\qquad = -\alpha \int_\Omega z(x,1) v(x) dx - \beta \int_\Omega v_x^2(x) dx - \kappa \int_\Omega \theta_x^2(x) dx.\end{aligned}$$

Integrating by parts in ρ, we get

$$\int_\Omega \int_0^1 z_\rho(x,\rho) z(x,\rho) d\rho dx = \frac{1}{2}\int_\Omega \Big(z^2(x,1) - z^2(x,0)\Big) dx.$$

Then (II.17) become

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= \alpha \int_\Omega v_x(x) u_x(x) dx - \alpha \int_\Omega z(x,1) v_x(x) dx\\ &- \beta \int_\Omega v_x^2(x) dx - \kappa \int_\Omega \theta_x^2(x) dx - \frac{\xi}{2\tau}\int_\Omega \Big(z^2(x,1) - z^2(x,0)\Big) dx,\end{aligned}$$

from which follows, using the Young's inequality and that $z(x,0) = u_x(x)$,

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &\leq (\alpha\varepsilon - \beta)\int_\Omega v_x^2(x) dx + \left(\frac{\alpha}{2\varepsilon} - \frac{\xi}{2\tau}\right)\int_\Omega z^2(x,1) dx + \left(\frac{\alpha}{2\varepsilon} + \frac{\xi}{2\tau}\right)\int_\Omega u_x^2(x) dx\\ &- \kappa \int_\Omega \theta_x^2(x) dx.\end{aligned}$$

Choosing $\alpha\varepsilon = \frac{\beta}{2}$, or equivalently, $\varepsilon = \frac{\beta}{2\alpha}$, we get

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \leq -\frac{\beta}{2}\int_\Omega v_x^2(x) dx + \left(\frac{\alpha^2}{\beta} - \frac{\xi}{2\tau}\right)\int_\Omega z^2(x,1) dx + \left(\frac{\alpha^2}{\beta} + \frac{\xi}{2\tau}\right)\int_\Omega u_x^2(x) dx - \kappa\int_\Omega \theta_x^2(x) dx.$$

Then we choose $\xi > 0$ such that $\frac{\alpha^2}{\beta} - \frac{\xi}{2\tau} < 0$, that is, $\xi > \frac{2\tau\alpha^2}{\beta}$. Furthermore we take $m = \frac{\alpha^2}{\beta} + \frac{\xi}{2\tau} > \frac{2\alpha^2}{\beta}$, to get

$$\langle (\mathcal{A} - mId) U, U\rangle_{\mathcal{H}} \leq -\frac{\beta}{2}\int_\Omega v_x^2(x) dx + \left(\frac{\alpha^2}{\beta} - \frac{\xi}{2\tau}\right)\int_\Omega z^2(x,1) dx - \kappa \int_\Omega \theta_x^2(x) dx \leq 0$$

which means that the operator $\mathcal{A} - mId$ is dissipatif.

Now, we will prove the maximality of $\mathcal{A} - mId$. It suffices to show that $\lambda Id - \mathcal{A}$ is surjective for a fixed $\lambda > m$. Given $(f, g, p, h) \in \mathcal{H}$, we look for $U = (u, v, z, \theta) \in \mathcal{D}(\mathcal{A})$, solution of

$$(\lambda Id - \mathcal{A}) \begin{pmatrix} u \\ v \\ z \\ \theta \end{pmatrix} = \begin{pmatrix} f \\ g \\ p \\ h \end{pmatrix},$$

that is verifying

$$\begin{cases} \lambda u - v = f, \\ \lambda v - (\alpha z(.,1) + \beta v_x)_x + \gamma \theta_x = g, \\ \lambda z - \frac{1}{\tau} z_\rho = p, \\ \lambda \theta + \gamma v_x - \kappa \theta_{xx} = h. \end{cases} \tag{II.18}$$

Suppose that we have found u with the appropriate regularity. Then,

$$v = \lambda u - f. \tag{II.19}$$

To determine z, recall that $z(.,0) = u_x$, then, by $(II.18)_3$, we obtain

$$z(.,\rho) = e^{-\lambda\tau\rho} u_x + \tau e^{-\lambda\tau\rho} \int_0^\rho p(s) e^{\lambda\tau s} ds, \tag{II.20}$$

and, in particular

$$z(x,1) = e^{-\lambda\tau} u_x + z_0, \tag{II.21}$$

with $z_0 \in L^2(\Omega)$ defined by

$$z_0 = \tau e^{-\lambda\tau} \int_0^1 p(s) e^{\lambda\tau s} ds.$$

Now, Multiplying $(II.18)_2$ and $(II.18)_4$ respectively by $w \in H_0^1(\Omega)$ and $\varphi \in H^2(\Omega)$ such that $\varphi_x(0) = \varphi_x(\ell) = 0$, we obtain after some integrations by parts taking into account boundary conditions on v, θ and w,

$$\lambda \int_\Omega vw dx + \int_\Omega (\alpha z(.,1) + \beta v_x) w_x dx + \gamma \int_\Omega \theta_x w dx = \int_\Omega gw dx \tag{II.22}$$

and

$$\lambda \int_\Omega \theta\varphi dx - \gamma \int_\Omega v\varphi_x dx + \kappa \int_\Omega \theta_x \varphi_x dx = \int_\Omega h\varphi dx. \tag{II.23}$$

Substituting (II.19) and (II.21) into (II.22) and (II.23), we get

$$\lambda^2 \int_\Omega uw dx + \left(\alpha e^{-\lambda\tau} + \lambda\beta\right) \int_\Omega u_x w_x dx + \gamma \int_\Omega \theta_x w dx = \int_\Omega (g + \lambda f) w dx + \int_\Omega (f_x - \alpha z_0) w_x dx \tag{II.24}$$

and

$$\lambda \int_\Omega \theta\varphi dx - \lambda\gamma \int_\Omega u\varphi_x dx + \kappa \int_\Omega \theta_x \varphi_x dx = \int_\Omega (h - \gamma f)\varphi dx. \tag{II.25}$$

Summing (II.24), and (II.25) multiplied by $\frac{1}{\lambda}$, *we get*

$$b\left((u,\theta),(w,\varphi)\right) = F(w,\varphi) \tag{II.26}$$

with

$$b\left((u,\theta),(w,\varphi)\right) = \int_\Omega \left[\lambda^2 uw + \left(\alpha e^{-\lambda\tau} + \lambda\beta\right) u_x w_x\right] dx + \int_\Omega \left(\theta\varphi + \frac{\kappa}{\lambda}\theta_x\varphi_x\right) dx + \gamma \int_\Omega \left(\theta_x w - u\varphi_x\right) dx$$

and

$$F(w,\varphi) = \int_\Omega (g + \lambda f) w dx + \int_\Omega (f_x - \alpha z_0) w_x dx + \frac{1}{\lambda} \int_\Omega (h - \gamma f)\varphi dx.$$

The space

$$\mathcal{F} := \left\{(w,\varphi) \in H_0^1(\Omega) \times H^2(\Omega) \mid \theta_x(0) = \theta_x(\ell) = 0\right\},$$

equipped with the inner product

$$\langle (w_1,\varphi_1),(w_2,\varphi_2)\rangle_{\mathcal{F}} = \int_\Omega \left(w_1 w_2 + w_{1x} w_{2x} + \varphi_1\varphi_2 + \varphi_{1x}\varphi_{2x}\right) dx,$$

is a Hilbert space; the bilinear form b *on* $\mathcal{F} \times \mathcal{F}$ *and the linear form* F *on* $\mathcal{F}$ *are continuous. Moreover, for every* $(w,\varphi) \in \mathcal{F}$,

$$|b\left((w,\varphi),(w,\varphi)\right)| \geq c\|(w,\varphi)\|^2_{\mathcal{H}}$$

with $c := \min\left(\lambda^2, \left(\alpha e^{-\lambda\tau} + \lambda\beta\right), 1, \frac{\kappa}{\lambda}\right) > 0$.

By the Lax-Milgram lemma, equation (II.26) has a unique solution $(u,\theta) \in \mathcal{F}$. *Immediately, from (II.19), we have that* $v \in H_0^1(\Omega)$. *Now, if we consider* $(w,\varphi) \in \{0\} \times \mathcal{D}(\Omega)$ *in (II.26) we deduce that equation* $(II.18)_4$ *holds true. The function* z, *defined by (II.21), belongs to* $L^1(\Omega, H^1(0,1))$ *and satisfies* $(II.18)_3$ *and* $z(.,0) = u_x$. *The functions* $z(.,1)$ *and* v_x *belong to* $L^2(\Omega)$, *then we take* $(w,\varphi) \in \mathcal{D}(\Omega) \times \{0\}$ *in (II.26) to deduce that* $\alpha z(.,1) + \beta v_x$ *belongs to* $H_0^1(\Omega)$ *and that equation* $(II.18)_1$ *holds true.*

Let $\tilde{\theta} = \theta - \frac{1}{\ell}\int_\Omega \theta_0 dx$, *then we have that* $U = (u,v,z,\tilde{\theta})$ *belongs to* $\mathcal{D}(\mathcal{A})$, *and* $\mathcal{A}U =$

(f, g, p, h). *Thus,* $\lambda Id - \mathcal{A}$ *is surjective for every* $\lambda > 0$.

In conclusion the operator $\mathcal{A} - mId$ generates a $\mathcal{C}_0$-semigroup of contraction. By the bounded perturbation theorem (Sect. III.1 of [17]), we have

Lemma 2.3 *The operator* $\mathcal{A}$ *generates a* $\mathcal{C}_0$*-semigroup on* $\mathcal{H}$.

Finally, the well-posedness result follows from semigroup theory.

Theorem 2.4 *For any initial datum* $U_0 \in \mathcal{H}$ *there exists a unique solution* $U \in \mathcal{C}([0, +\infty), \mathcal{H})$ *of problem (II.15). Moreover, if* $U_0 \in \mathcal{D}(\mathcal{A})$, *then* $U \in \mathcal{C}([0, +\infty), \mathcal{D}(\mathcal{A})) \cap \mathcal{C}^1([0, +\infty), \mathcal{H})$.

3 Exponential stability

Based on Lyapunov method, we prove that the system (II.2) is exponentially stable for some $\beta > 0$. More precisely:

Theorem 3.1 *There exists* $\beta_0 > 0$ *such that for every* $\beta \geq \beta_0$, *the system (II.2) is exponentially stable.*

Proof 3.2 *We take as Lyapunov function*

$$V(t) := N_1 V_1(t) + \alpha N_2 V_2(t) + N_3 V_3(t) + N_4 V_4(t) + N_5 V_5(t) + N_6 V_6(t)$$

where

$$\begin{aligned}
V_1(t) &:= \frac{1}{2}\|u_t\|^2 = \frac{1}{2}\int_\Omega u_t^2 dx, \quad V_2(t) := \frac{1}{2}\|u_x\|^2 = \frac{1}{2}\int_\Omega u_x^2 dx, \quad V_3(t) := \frac{1}{2}\|\theta\|^2 = \frac{1}{2}\int_\Omega \theta^2 dx,\\
V_4(t) &:= \int_0^1 e^{-2\lambda\rho}\|z(., \rho, t)\|^2 d\rho = \int_0^1 e^{-2\lambda\rho}\int_\Omega z^2(x, \rho, t) dx d\rho,\\
V_5(t) &:= -\int_0^1 e^{-\lambda\rho} f(\rho)\langle z(., \rho, t), u_x\rangle d\rho = -\int_0^1 e^{-\lambda\rho} f(\rho)\int_\Omega z(x, \rho, t) u_x(x, t) dx d\rho \ \textit{ and}\\
V_6(t) &:= \langle u, u_t\rangle = \int_\Omega u u_t dx.
\end{aligned}$$

f *is a real function defined on* $[0, 1]$ *and that will be determined later. The constants* N_1, N_2, N_3, N_4, N_5 *and* N_6 *are positive numbers to be fixed later too.*

Denote by $\tilde{V}(t)$ *the energy defined by*

$$\tilde{V}(t) := N_1 V_1(t) + \alpha N_2 V_2(t) + N_3 V_3(t) + N_4 V_4(t).$$

It is clear that $\tilde{V}(t)$ is equivalent to $E(t)$. Then for a suitable choice of f we will prove that we can find $\{N_1, ..., N_6\}$ and $\beta > 0$ such that the following two assumptions are satisfied:

(A1) $V(t)$ is equivalent to $\tilde{V}(t)$,

(A2) $V'(t) \leq -n_0\tilde{V}(t)$, for some positive number n_0.

The rest of the proof will be divided into three parts:

First part: it concerns the second assumption (A2). We start with the following lemma

***Lemma 3.3** Let $V(t)$ be defined as before. By choosing a function f satisfying*

$$-e^{-2\lambda\rho} = (e^{-\lambda\rho} f(\rho))', \quad \lambda > 0 \tag{II.27}$$

and by taking $N_3 = N_1$, $N_6\beta = N_2\alpha$ and $N_6\alpha = \frac{f(1)e^{-\lambda}}{\tau}N_5$ we have that for every positive real numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and ε_4,

$$\begin{aligned} V'(t) \quad \leq \quad & \left(-N_4\frac{e^{-2\lambda}}{\tau} + N_1\frac{\alpha\varepsilon_1}{2}\right)\|z(.,1,.)\|^2 \\ + \quad & \left(-2\frac{k_1}{\tau}N_4 + \frac{N_5}{2\tau}\left(\varepsilon_2 + \frac{\tau}{\varepsilon_3}\right)\right)V_4(t) \\ + \quad & \left(N_4\frac{1}{\tau} + \frac{N_5}{\tau}\left(\frac{\Gamma}{2\varepsilon_2} - \Lambda\right) + N_5\frac{\Psi\gamma\varepsilon_4 c_p}{2\alpha\tau}\right)\|u_x\|^2 \\ + \quad & \left(N_1\left(\frac{\alpha}{2\varepsilon_1} - \beta\right) + N_5\left(\frac{\varepsilon_3}{2}\Phi + \frac{\Psi c_p}{2\alpha\tau}\right)\right)\|u_{tx}\|^2 \\ + \quad & \left(-N_1\kappa + N_5\frac{\Psi\gamma}{2\alpha\tau\varepsilon_4}\right)\|\theta_x\|^2 \end{aligned}$$

where $c_p > 0$ is the Poincaré constant associated to Ω, (it can be taken equal to $\frac{\ell^2}{2}$) and

$$\Psi := f(1)e^{-\lambda}, \quad \Lambda := f(0) \quad \textit{and} \;\; \Phi := \int_0^1 f^2(\rho)d\rho.$$

Furthermore,

$$\Gamma := \int_0^1 e^{-2\lambda\rho}d\rho = \frac{1 - e^{-2\lambda}}{2\lambda}.$$

Notice that

$$\Lambda = \Psi + \Gamma.$$

***Proof 3.4** Computing the derivatives of V_1, V_2, V_3, V_4 and V_5 using integration by parts,* *boundary conditions and Youg's inequality, we have*

$$\begin{aligned}
V_1'(t) = \langle u_{tt}, u_t \rangle &= \langle (\alpha z(.,1,.) + \beta u_{tx})_x, u_t \rangle - \gamma \langle \theta_x, u_t \rangle \\
&= -\langle \alpha z(.,1,.) + \beta u_{tx}, u_{tx} \rangle - \gamma \langle \theta_x, u_t \rangle \\
&= -\langle \alpha z(.,1,.), u_{tx} \rangle - \beta \|u_{tx}\|^2 - \gamma \langle \theta_x, u_t \rangle \\
&\leq \left(\frac{\alpha}{2\varepsilon_1} - \beta \right) \|u_{tx}\|^2 + \frac{\alpha \varepsilon_1}{2} \|z(.,1,.)\|^2 - \gamma \langle \theta_x, u_t \rangle ,
\end{aligned}$$

$$V_2'(t) = \langle u_{xt}, u_x \rangle ,$$

$$\begin{aligned}
V_3'(t) = \langle \theta_t, \theta \rangle &= -\gamma \langle (u_{xt}, \theta \rangle + \kappa \langle \theta_{xx}, \theta \rangle \\
&= \gamma \langle u_t, \theta_x \rangle - \kappa \|\theta_x\|^2 .
\end{aligned}$$

The derivative of V_4 *is*

$$\begin{aligned}
V_4'(t) &= 2 \int_0^1 e^{-2\lambda\rho} \langle z(.,\rho,.), z_t(.,\rho,.) \rangle \, d\rho \\
&= -\frac{2}{\tau} \int_0^1 e^{-2\lambda\rho} \langle z(.,\rho,.), z_\rho(.,\rho,.) \rangle \, d\rho \\
&= -\frac{e^{-2\lambda}}{\tau} \|z(.,1,.)\|^2 + \frac{1}{\tau} \|u_x\|^2 - \frac{2\lambda}{\tau} \int_0^1 e^{-2\lambda\rho} \|z(.,\rho,.)\|^2 \, d\rho \\
&\leq -\frac{e^{-2\lambda}}{\tau} \|z(.,1,.)\|^2 + \frac{1}{\tau} \|u_x\|^2 - \frac{2\lambda}{\tau} V_4(t).
\end{aligned}$$

The derivative of V_5 *is calculated as follows*

$$\begin{aligned}
V_5'(t) &= \frac{1}{\tau} \int_0^1 e^{-\lambda\rho} f(\rho) \langle z_\rho(.,\rho,.), u_x \rangle \, d\rho - \int_0^1 e^{-\lambda\rho} f(\rho) \langle z(.,\rho,.), u_{xt} \rangle \, d\rho \\
&= \frac{1}{\tau} e^{-\lambda} f(1) \langle z(.,1,.), u_x \rangle - \frac{1}{\tau} f(0) \|u_x\|^2 \\
&+ \frac{1}{\tau} \int_0^1 (e^{-\lambda\rho} f(\rho))' \langle z(.,\rho,.), u_x \rangle \, d\rho - \int_0^1 e^{-\lambda\rho} f(\rho) \langle z(.,\rho,.), u_{xt} \rangle \, d\rho.
\end{aligned}$$

Replacing $e^{-\lambda} f(1)$ *by* Ψ, $f(0)$ *by* Λ *and* $(e^{-\lambda\rho} f(\rho))'$ *by* $-e^{-2\lambda\rho}$, *we obtain (using Young's inequality),*

$$V_5'(t) \leq \frac{1}{\tau} \left(\frac{\Gamma}{2\varepsilon_2} - \Lambda \right) \|u_x\|^2 + \frac{1}{2\tau} \left(\varepsilon_2 + \frac{\tau}{\varepsilon_3} \right) V_4(t) + \frac{\varepsilon_3}{2} \Phi \|u_{tx}\|^2 + \frac{\Psi}{\tau} \langle z(.,1,.), u_x \rangle .$$

Finally, the derivative of V_6 *is*

$$V_6'(t) \;=\; \|u_t\|^2 - \alpha \langle z(.,1,.), u_x\rangle - \beta \langle u_x, u_{xt}\rangle + \gamma \langle u, \theta_x\rangle .$$

To conclude, it suffices to sum up $N_1V_1'(t)$, $\alpha N_2V_2'(t)$, $N_3V_3'(t)$, $N_4V_4'(t)$, $N_5V_5'(t)$, *and* $N_6V_6'(t)$.

In view of Lemma 3.3, for the assumption (A2) to be satisfied, it suffices that

$$-N_4\frac{e^{-2\lambda}}{\tau} + N_1\frac{\alpha\varepsilon_1}{2} \;=\; 0, \tag{II.28}$$

$$n_1 := -2\frac{\lambda}{\tau}N_4 + \frac{N_5}{2\tau}\left(\varepsilon_2 + \frac{\tau}{\varepsilon_3}\right) \;<\; 0, \tag{II.29}$$

$$n_2 := N_4\frac{1}{\tau} + \frac{N_5}{\tau}\left(\frac{\Gamma}{2\varepsilon_2} - \Lambda\right) + N_5\frac{\Psi\gamma\varepsilon_4 c_p}{2\alpha\tau} \;<\; 0, \tag{II.30}$$

$$n_3 := N_1\left(\frac{\alpha}{2\varepsilon_1} - \beta\right) + N_5\left(\frac{\varepsilon_3}{2}\Phi + \frac{\Psi c_p}{\alpha\tau}\right) \;<\; 0, \tag{II.31}$$

$$n_4 := -N_1\kappa + N_5\frac{\Psi\gamma}{2\alpha\tau\varepsilon_4} \;<\; 0. \tag{II.32}$$

The first condition (II.28) is equivalent to

$$N_4 = aN_1$$

with $a := \frac{1}{2}\alpha\varepsilon_1\tau e^{2\lambda}$.

The second condition (II.29) means that there exists $0 < k < 1$ *such that*

$$N_5 = bN_4 = abN_1$$

with $b := \frac{4\lambda k}{\varepsilon_2 + \frac{\tau}{\varepsilon_3}}$.

Note that, we have then

$$N_6 = \frac{\Psi}{\alpha\tau}N_5 = \frac{ab\Psi}{\alpha\tau}N_1 \quad \textit{and} \quad N_2 = \frac{\beta}{\alpha}N_6 = \frac{ab\Psi\beta}{\alpha^2\tau}N_1.$$

Replacing N_5 *by* abN_1 *and* a *by* $\frac{1}{2}\alpha\varepsilon_1\tau e^{2\lambda}$ *in (II.31), then multiplying the inequality by* $\frac{\alpha}{\varepsilon_1}$, *we obtain*

$$\frac{1}{2}b\Psi\alpha e^{2\lambda}c_p + \frac{1}{4}\tau b\Phi\varepsilon_3\alpha^2 e^{2\lambda} < \frac{\alpha}{\varepsilon_1}\left(\beta - \frac{\alpha}{2\varepsilon_1}\right). \tag{II.33}$$

We take $\varepsilon_1 = \frac{\alpha}{\beta}$, *then (II.33) turns into*

$$b\Psi\alpha e^{2\lambda}c_p + \frac{1}{2}\tau b\Phi\varepsilon_3\alpha^2 e^{2\lambda} < \beta^2. \tag{II.34}$$

Return back to (II.32), replacing N_5 *by* abN_1 *to obtain*

$$\frac{ab\Psi\gamma}{2\alpha\tau\varepsilon_4} < \kappa. \tag{II.35}$$

Also inequality (II.30) becomes

$$\frac{b\Psi\gamma\varepsilon_4 c_p}{2\alpha} < \left(\left(\Lambda - \frac{\Gamma}{2\varepsilon_2}\right)b - 1\right). \tag{II.36}$$

Already, it is necessary that $\Lambda - \frac{\Gamma}{2\varepsilon_2} > 0$, *that is* $\Lambda > \frac{\Gamma}{2\varepsilon_2}$, *and* $\left(\Lambda - \frac{\Gamma}{2\varepsilon_2}\right)b - 1 > 0$, *that is,*

$$b = \frac{A}{\Lambda - \frac{\Gamma}{2\varepsilon_2}}, \quad A > 1, \tag{II.37}$$

hence (II.36) turns into

$$\frac{b\Psi\gamma\varepsilon_4 c_p}{2\alpha} < (A-1). \tag{II.38}$$

Combining (II.35) and (II.38) to obtain

$$\frac{ab\Psi\gamma}{2\alpha\tau\kappa} < \varepsilon_4 < \frac{2\alpha}{\gamma b\Psi c_p}(A-1). \tag{II.39}$$

Replacing a *by* $\frac{1}{2}\alpha\varepsilon_1\tau e^{2\lambda}$ *in (II.39) to get*

$$\frac{\alpha\gamma b\Psi}{4\beta\kappa}e^{2\lambda} < \varepsilon_4 < \frac{2\alpha}{\gamma b\Psi c_p}(A-1). \tag{II.40}$$

Now, going back with more detail on assumption (II.37). To do this, replacing b *by* $\frac{4\lambda k}{\varepsilon_2 + \frac{\tau}{\varepsilon_3}}$, *we obtain*

$$\frac{A}{\Lambda - \frac{\Gamma}{2\varepsilon_2}} = \frac{4\lambda k}{\varepsilon_2 + \frac{\tau}{\varepsilon_3}}, \tag{II.41}$$

or equivalently,

$$\frac{A}{4\lambda k}\varepsilon_2^2 - \left(\Lambda - \frac{A}{4\lambda k}\frac{\tau}{\varepsilon_3}\right)\varepsilon_2 + \frac{\Gamma}{2} = 0 \tag{II.42}$$

the discriminant of such equation in ε_2 is

$$\Delta := \left(\Lambda - \frac{A}{4\lambda k}\frac{\tau}{\varepsilon_3}\right)^2 - \frac{A\Gamma}{2\lambda k} \tag{II.43}$$

which must be at least zero. In the sequel, we choose it zero. On the other hand ε_2 is positive, then

$$\Lambda - \frac{A}{4\lambda k}\frac{\tau}{\varepsilon_3} = \sqrt{\frac{A\Gamma}{2\lambda k}} \tag{II.44}$$

or equivalently

$$\frac{A}{4\lambda k}\frac{\tau}{\varepsilon_3} = \Lambda - \sqrt{\frac{A\Gamma}{2\lambda k}}. \tag{II.45}$$

It is obvious that the left hand side of the last equation is positive, that is

$$\frac{A}{k} < 2\lambda\frac{\Lambda^2}{\Gamma}. \tag{II.46}$$

Moreover, since $A > 1$ and $0 < k < 1$ we have

$$1 < \frac{A}{k} < 2\lambda\frac{\Lambda^2}{\Gamma}. \tag{II.47}$$

Finally, note that

$$\varepsilon_2 = \sqrt{2\lambda\Gamma\frac{k}{A}}. \tag{II.48}$$

Second part: it concerns the equivalence between $V(t)$ and $\tilde{V}(t)$. Let $\varepsilon_5 > 0$ and $\varepsilon_6 > 0$, we have

$$|N_5V_5| \leq \frac{N_5}{2\varepsilon_5}V_4 + \frac{N_5\Phi\varepsilon_5}{2}\|u_x\|^2$$

and

$$|N_6V_6| \leq \frac{N_6\varepsilon_6}{2}c_p\|u_x\|^2 + \frac{N_6}{2\varepsilon_6}\|u_t\|^2.$$

For $V(t)$ to be equivalent to $\tilde{V}(t)$ it is sufficient that

$$\frac{N_6}{2\varepsilon_6} < \frac{N_1}{2}, \qquad \frac{N_5}{2\varepsilon_5} < N_4, \tag{II.49}$$

$$\frac{N_6\varepsilon_6}{2}c_p + \frac{N_5\Phi\varepsilon_5}{2} < \frac{N_2\alpha}{2}. \tag{II.50}$$

Using $N_6 = \frac{ab\Psi}{\alpha\tau} N_5$ *and* $N_5 = abN_1$ *in(II.49) we get*

$$\varepsilon_6 > \frac{ab\Psi}{\tau\alpha}, \qquad \varepsilon_5 > \frac{b}{2}.$$

We choose

$$\varepsilon_6 = 2\frac{ab\Psi}{\tau\alpha}, \qquad \varepsilon_5 = b.$$

Using again $N_6 = \frac{\Psi}{\alpha\tau} N_5$ *and* $N_2 = \frac{\beta\Psi}{\alpha^2\tau} N_5$*, inequality (II.50) become*

$$\frac{b\Psi^2}{\beta\tau} c_p + \Phi b < \frac{\beta\Psi}{\alpha\tau} \tag{II.51}$$

Third part: It is enough to examine the equations (II.47), (II.34), (II.40) and (II.51). We take $h := \frac{\Psi}{\Gamma}$*, then* $\Lambda = \Psi + \Gamma = (1+h)\Gamma = (1+h)\frac{2\lambda}{1-e^{-2\lambda}}$

First step. We begin by assumption (II.47) which can be translated into

$$1 < \frac{A}{k} < (1 - e^{-2\lambda})(1+h)^2 \tag{II.52}$$

We choose $h := e^{-2\lambda}$*. Then*

$$(1 - e^{-2\lambda})(1+h)^2 = (1-h)(1+h)^2 = 1 + h - h^2 - h^3 > 1$$

for λ *large enough. We choose* $A = 1 + h - h^2 - 2h^3 - 4h^4$ *and* $k = 1 - h^4$*. We have, for* λ *large enough,* $A > 1$*,* $0 < k < 1$ *and (II.52) is satisfied since*

$$1 < \frac{A}{k} = 1 + h - h^2 - 2h^3 + o(h^3) < (1-h)(1+h)^2 \tag{II.53}$$

Second step. Estimate of ε_2*,* b *and* ε_3 *according to* h *and* λ *for* λ *large enough. We have*

$$\frac{k}{A} = \frac{1 + o(h^2)}{1 + h - h^2 + o(h^2)} \tag{II.54}$$

$$= 1 - h + 2h^2 + o(h^2) \tag{II.55}$$

and

$$\Gamma = \frac{1}{2\lambda}(1-h)$$

then

$$\begin{aligned} 2\lambda\Gamma\frac{k}{A} &= (1-h)(1-h+2h^2+o(h^2)) \\ &= 1-2h+3h^2+o(h^2). \end{aligned}$$

Hence we obtain, using (II.48),

$$\begin{aligned} \varepsilon_2 &= (1-h+\frac{3}{2}h^2-\frac{1}{2}h^2+o(h^2)) \\ &= (1-h+h^2+o(h^2)). \end{aligned}$$

We evaluate b. First,

$$\begin{aligned} \frac{1}{2\varepsilon_2} &= \frac{1}{2(1-h+h^2+o(h^2))} \\ &= \frac{1}{2}\Big(1+h-h^2+(h-h^2)^2+o(h^2)\Big) \\ &= \frac{1}{2}\Big(1+h+o(h^2)\Big), \end{aligned}$$

then

$$1+h-\frac{1}{2\varepsilon_2}=\frac{1}{2}(1+h+o(h^2))$$

hence

$$\frac{1}{1+h-\frac{1}{2\varepsilon_2}} = 2(1-h+h^2+o(h^2)).$$

Finally, from(II.37) and using that $\Lambda=(1+h)\Gamma$ *we have*

$$b=\frac{A}{\Gamma(1+h-\frac{1}{2\varepsilon_2})} = 4\lambda\frac{(1+h-h^2+o(h^2))(1-h+h^2+o(h^2))}{1-h} \tag{II.56}$$

$$= 4\lambda(1+h+o(h^2)). \tag{II.57}$$

Now we evaluate ε_3 *:*

First, recall that

$$\frac{A}{2\lambda k}=\frac{1}{2\lambda}(1+h-h^2-2h^3-3h^4+o(h^4)) \tag{II.58}$$

then

$$\frac{A\Gamma}{2\lambda k} = \frac{1}{4\lambda^2}(1 - 2h^2 - h^3 - h^4 + o(h^4))$$

and

$$\sqrt{\frac{A\Gamma}{2\lambda k}} = \frac{1}{2\lambda}(1 - h^2 - \frac{1}{2}h^3 - h^4 + o(h^4)) \tag{II.59}$$

Using (II.45), (II.58), (II.59) and that $\Lambda = \frac{1}{2\lambda}(1-h^2)$ *we have*

$$\varepsilon_3 = \frac{\tau}{h^3}(1 - h + o(h)). \tag{II.60}$$

Third step. Interpretation of Inequality (II.34). First, we need to express Φ *according to* h *and* λ.

Since

$$\begin{aligned} f(\rho) &= e^{\lambda\rho}\left(h\Gamma + \int_\rho^1 e^{-2\lambda s} ds\right) \\ &= \frac{1}{2\lambda}e^{\lambda\rho}\left(e^{-2\lambda}(1 - e^{-2\lambda}) + (e^{-2\lambda\rho} - e^{-2\lambda})\right) \\ &= \frac{1}{2\lambda}e^{\lambda\rho}\left(e^{-2\lambda\rho} - e^{-4\lambda}\right) \end{aligned}$$

then,

$$\begin{aligned} f^2(\rho) &= \frac{1}{4\lambda^2}e^{2\lambda\rho}\left(e^{-4\lambda\rho} - 2e^{-4\lambda}e^{-2\lambda\rho} + e^{-8\lambda}\right) \\ &= \frac{1}{4\lambda^2}\left(e^{-2\lambda\rho} - 2e^{-4\lambda} + e^{-8\lambda}e^{2\lambda\rho}\right). \end{aligned}$$

Hence

$$\begin{aligned} \Phi &= \int_0^1 f^2(\rho)d\rho \\ &= \frac{1}{4\lambda^2}\left(\frac{1 - e^{-2\lambda}}{2\lambda} - 2e^{-4\lambda} + \frac{1}{2\lambda}\left(e^{-6\lambda} - e^{-8\lambda}\right)\right) \\ &= \frac{1}{8\lambda^3}(1 - h - 4\lambda h^2 + o(h^2)). \end{aligned}$$

Now, inequality (II.34) can be rewritten as:

$$2\alpha c_p(1 - h^2 + o(h^2)) + \frac{\tau^2\alpha^2}{4\lambda^2 h^4}(1 - h + o(h)) < \beta^2.$$

Then we take

$$2\left(\frac{c_p}{\alpha\tau^2}+\frac{e^{8\lambda}}{8\lambda^2}\right)<\left(\frac{\beta}{\alpha\tau}\right)^2 \tag{II.61}$$

with λ large enough.

Fourth step. Condition (II.40) and existence of ε_4. Inequality (II.40) can be rewritten as:

$$\frac{\alpha\gamma}{2\beta\kappa}(1-h^2+o(h^2))<\varepsilon_4<\frac{\alpha}{\gamma c_p}(1-h+o(h)).$$

Then we take

$$\frac{\gamma^2}{2\kappa}c_p<\beta(1-h+o(h)) \tag{II.62}$$

and ε_4 can be taken equal to $\frac{\alpha}{2}\left(\frac{\gamma}{2\beta\kappa}+\frac{1}{\gamma c_p}(1-h+o(h))\right)$, with λ large enough.

Fifth step. Interpretation of assumption (II.51). It can be rewritten as:

$$2h\frac{c_p}{\tau\beta}(1-h^2+o(h^2))+\frac{1}{\lambda h}(1+h+o(h))<\frac{\beta}{\tau\alpha}.$$

It suffices to take

$$\left(2e^{-2\lambda}\frac{c_p}{\tau\beta}+\frac{1}{\lambda}(e^{2\lambda}+1+o(1))\right)<\frac{\beta}{\alpha\tau}. \tag{II.63}$$

with λ large enough.

Note that for λ large enough, $\beta=\alpha\tau e^{4\lambda}$ satisfies the three conditions (II.61, II.62) and (II.63). Moreover, there exists $\beta_0>0$ such that every $\beta>\beta_0$ satisfies the three conditions (II.61, II.62) and (II.63).

For every $\beta>\beta_0$ we have

$$\dot{V}(t)\le -n_0\tilde{V}(t) \tag{II.64}$$

where $n_0=min\{n_1,n_2,\frac{1}{c_p}n_3,\frac{1}{c_p}n_4\}$. Recall that $V(t)$, $\tilde{V}(t)$ and $E(t)$ are equivalent then, there exists $a_0>0, C>0$ such that

$$E(t)<Ce^{-a_0t}.$$

Comments

We can replace the Neumann conditions for θ

$$\theta_x(0,t) = \theta_x(\ell,t) = 0$$

by the Dirichlet conditions

$$\theta(0,t) = \theta(\ell,t) = 0,$$

we then obtain the same results.

Chapter III

On the stabilization of a wave equation with past history and fractional damping controls

1 Introduction

In this chapter, we are concerned with the well-posedness, smoothness and asymptotic behavior of the solution of the following wave equation

$$(P)\quad \begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \displaystyle\int_0^\infty g(s)u_{xx}(t-s)\,ds + \gamma\partial_t^{\alpha,\eta}u(x,t) = 0 & \text{in } (0,1)\times(0,+\infty), \\ u(0,t) = u(1,t) = 0 & \text{in } (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } (0,1), \end{cases}$$

where $g : \mathbb{R}_+ \to \mathbb{R}_+, \gamma > 0$. u_0 and u_1 are given initial data. The infinite integral term in (P) and $\gamma\partial_t^{\alpha,\eta}u$ represent, respectively, the past history (infinite memory) and the fractional damping. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α, $(0 < \alpha \leq 1)$, with respect to the time variable (see **[12]**). It is defined as follows

$$\partial_t^{\alpha,\eta}u(t) = \begin{cases} u_t(t) & \text{for } \alpha = 1, \eta \geq 0, \\ \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^t (t-s)^{-\alpha}e^{-\eta(t-s)}\frac{du}{ds}(s)\,ds, & \text{for } 0 < \alpha < 1, \eta \geq 0. \end{cases}$$

During the last few years, many people have been interested in the question of stability of wave equation with various kinds of (internal or boundary) controls. To focus on our motivation, let us mention here only some known results related to the stabilization with finite or infinite memory controls (for further results of stabilization, we refer the reader to the list of references of this paper, which is not exhaustive, and the references therein).

In the case $\gamma = 0$ and g satisfies

$$\exists \delta_1, \delta_2 > 0 : -\delta_1 g(s) \leq g'(s) \leq -\delta_2 g(s), \quad \forall s \in \mathbb{R}_+,$$

the authors of **[11]** proved that (P) is exponentially stable.

In the case $\gamma = 0$ and g satisfies

$$\exists \delta > 0, \exists p \in]1, 3/2[: g'(s) \leq -\delta_2 g^p(s), \quad \forall s \in \mathbb{R}_+,$$

it was proved in **[24]** that (P) is polynomially stable.

Very recently, in **[5]**, Ammari et al., studied the wave equation with internal fractional damping. The system considered is as follows:

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \gamma \partial_t^{\alpha,\eta} u(x,t) = 0 & \text{in } \Omega \times (0, +\infty), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{on } \Omega. \end{cases} \tag{III.1}$$

The authors proved that the energy decays polynomially as $t^{-2/(1-\alpha)}$. Our goals in this paper are: Investigating the effect of each control on the asymptotic behavior of the solutions of (P) and on the decay rate of its energy and giving an explicit and general characterization of the decay rate depending on the growth of g and α.

The chapter is organized as follows. In section 2, we give preliminary results and we reformulate the system (P) into an augmented system by coupling the viscoelastic wave equation with a suitable diffusion equation. Then, We convert the system into an evolution equation in an appropriate Hilbert space, and prove the well-posedness of our problem by semigroup theory. In section 3, we study asymptotic stability of above model and we establish an unform decay for $\eta \neq 0$ and polynomial energy decay for $\eta = 0$ for smooth solution.

2 Preliminary results

The integral term represents a history term with kernel g satisfying the following hypothesis:

$$\begin{cases} g : \mathbb{R}_+ \to \mathbb{R}_+ \text{ is a non-increasing differentiable function such that } \lim_{s\to 0^+} g(s) \\ \text{exists and there exists } c > 0 \text{ such that} \\ \qquad\qquad g'(s) \le -cg(s). \\ \text{Furthermore, we assume that } 1 - g_0 > 0, \text{ where } g_0 = \displaystyle\int_0^{+\infty} g(s)\, ds. \end{cases} \tag{H}$$

In this section we reformulate (P) into an augmented system. For that, we need the following proposition.

Proposition 2.1 (see [21]) *Let μ be the function:*

$$\mu(\xi) = |\xi|^{(2\alpha-1)/2}, \quad -\infty < \xi < +\infty,\ 0 < \alpha < 1. \tag{III.2}$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t \phi(\xi, t) + (\xi^2 + \eta)\phi(\xi, t) - U(t)\mu(\xi) = 0, \quad -\infty < \xi < +\infty, \eta \ge 0, t > 0, \tag{III.3}$$

$$\phi(\xi, 0) = 0, \tag{III.4}$$

$$O(t) = (\pi)^{-1} \sin(\alpha\pi) \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi, t)\, d\xi, \tag{III.5}$$

where $U \in C^0([0, +\infty))$, is given by

$$O = I^{1-\alpha,\eta} U, \tag{III.6}$$

where

$$[I^{\alpha,\eta} f](t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{-\eta(t-\tau)} f(\tau)\, d\tau.$$

We introduce, as in **[13]**, the new variables

$$\nu(x, t, s) = u(x, t) - u(x, t - s)$$

This functional satisfies

$$\begin{cases} \partial_t \nu + \partial_s \nu - u_t = 0 \text{ in }]0,1[\times \mathbb{R}_+ \times \mathbb{R}_+, \\ \nu(0,t,s) = \nu(1,t,s) = 0 \text{ in } \mathbb{R}_+ \times \mathbb{R}_+, \\ \nu(x,t,0) = 0 \text{ in }]0,1[\times \mathbb{R}_+ \end{cases} \tag{III.7}$$

In order to convert our problem to a system of first-order ordinary differential equations, we note the following:

$$\nu^0(x,s) = \nu(x,0,s)$$

$$U = (u, u_t, \nu, \phi)$$

and

$$U_0 = (u_0, u_1, \nu^0, 0)$$

Then (P) is equivalent to the following abstract system:

$$U_t = \mathcal{A}U, \quad U(0) = U_0, \tag{III.8}$$

where $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ \nu \\ \phi \end{pmatrix} = \begin{pmatrix} v \\ (1 - \int_0^\infty g(s)\,ds)u_{xx} + \int_0^\infty g(s)\nu_{xx}\,ds - \zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(x,\xi)d\xi \\ v - \partial_s \nu \\ -(\xi^2 + \eta)\phi + v(x)\mu(\xi) \end{pmatrix}. \tag{III.9}$$

We define the functional space of U as follows.

$$\mathcal{H} = H_0^1(0,1) \times L^2(0,1) \times H^* \times L^2((0,1) \times (0,\infty)), \tag{III.10}$$

where

$$H^* = \{f : \mathbb{R}_+ \to H_0^1(0,1), \int_0^1 \int_0^\infty g(s)|f_x(s)|^2\,ds\,dx < +\infty\} \tag{III.11}$$

The domain $D(\mathcal{A})$ of $\mathcal{A}$ is defined by

$$D(\mathcal{A}) = \{U \in \mathcal{H} \backslash \mathcal{A}U \in \mathcal{H}, \nu(x,t,0) = 0, |\xi|\phi \in L^2(-\infty,+\infty)\}.$$

$\mathcal{H}$ is a Hilbert spaces equipped with the inner product defined

$$\langle U,\tilde{U}\rangle_{\mathcal{H}} = \int_\Omega \left(v\overline{\tilde{v}} + (1-\int_0^\infty g(s)\,ds)u_x\overline{\tilde{u}_x}\right)dx + \int_0^1\int_0^{+\infty} g(s)\nu_x(s)\overline{\tilde{\nu}}_x(s)\,ds\,dx$$
$$+\zeta\int_0^1\int_{-\infty}^{+\infty}\phi\overline{\tilde{\phi}}\,d\xi dx,$$

Now, the domain $D(\mathcal{A})$ is dense in $\mathcal{H}$, and a simple computation implies that, for $U\in D(\mathcal{A})$,

$$\Re\langle \mathcal{A}U,U\rangle_{\mathcal{H}} = -\frac{1}{2}\Re\int_0^1 g(s)\int_0^{+\infty}\partial_s|\partial_x\nu|^2\,ds\,dx - \zeta\int_0^1\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi)|^2\,d\xi\,dx.$$

Integration by parts, using $(H1)$ and the boundary conditions in (III.7), yields

$$\Re\langle \mathcal{A}U,U\rangle_{\mathcal{H}} = \frac{1}{2}\int_0^1 g'(s)\int_0^{+\infty}|\partial_x\nu|^2\,ds\,dx - \zeta\int_0^1\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi)|^2\,d\xi\,dx \tag{III.12}$$

and then, because the kernel g is non-increasing,

$$\Re\langle \mathcal{A}U,U\rangle_{\mathcal{H}} \le 0.$$

This implies that $\mathcal{A}$ is a dissipative operator. Next, we prove that $\lambda Id-\mathcal{A}$ is surjective. Let $F=(f_1,f_2,f_3,f_3)\in\mathcal{H}$. We prove the existence of $U=(u,v,\phi)\in D(\mathcal{A})$ solution of the equation

$$(\lambda I-\mathcal{A})U = F. \tag{III.13}$$

Equation (III.13) is equivalent to

$$\begin{cases}\lambda u - v = f_1,\\ \lambda v - (1-\int_0^\infty g(s)\,ds)u_{xx} - \int_0^\infty g(s)\nu_{xx}\,ds + \zeta\int_{-\infty}^{+\infty}\mu(\xi)\phi(x,\xi)d\xi = f_2,\\ \lambda\nu - v + \partial_s\nu = f_3,\\ \lambda\phi + (\xi^2+\eta)\phi - v(x)\mu(\xi) = f_4.\end{cases} \tag{III.14}$$

The first equation of (III.14) gives

$$v = \lambda u - f_1 \in H_0^1(0,1) \tag{III.15}$$

The last equation of (III.14) gives

$$\phi = \frac{f_4(x,\xi)+\mu(\xi)v(x)}{\xi^2+\eta+\lambda} = \frac{f_4(\xi)}{\xi^2+\eta+\lambda} + \frac{\lambda u(x)\mu(\xi)}{\xi^2+\eta+\lambda} - \frac{f_1(x)\mu(\xi)}{\xi^2+\eta+\lambda}. \tag{III.16}$$

The third equation of (III.14) and (III.15) give

$$\partial_s \nu + \lambda \nu = \lambda u - f_1 + f_3$$

By integrating this differential equation and using the fact that $\nu(x,0) = 0$, we get

$$\nu = \frac{1}{\lambda}(1 - e^{-\lambda s})(\lambda u - f_1) + \int_0^s e^{\lambda(\tau - s)} f_3(\tau)\, d\tau. \tag{III.17}$$

Inserting the Equation (III.15) into $(III.14)_2$, we get

$$\lambda^2 u - (1 - \int_0^\infty g(s)\, ds) u_{xx} - \int_0^\infty g(s) \nu_{xx}\, ds + \zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(x,\xi) d\xi = \lambda f_1 + f_2 \tag{III.18}$$

Inserting Equations (III.16) and (III.17) into (III.18), we get

$$\begin{aligned} &(\lambda^2 + \gamma\lambda(\lambda+\eta)^{\alpha-1})u - \left(1 - \int_0^\infty g(s) e^{-\lambda s}\, ds\right) u_{xx} = \\ &\gamma(\lambda+\eta)^{\alpha-1} f_1 + \lambda f_1 + f_2 - \zeta \int_{-\infty}^{+\infty} \frac{f_4(x,\xi)\mu(\xi)}{\xi^2 + \eta + \lambda} d\xi \\ &-\frac{1}{\lambda} \int_0^\infty g(s)(1 - e^{-\lambda s})\, ds f_{1xx} + \int_0^\infty g(s) \int_0^s e^{\lambda(\tau - s)} f_{3xx}(\tau)\, d\tau\, ds \end{aligned} \tag{III.19}$$

multiplying them by $\overline{u}$ and integrating over $]0,1[$, we get

$$a(u,w) = L(w). \tag{III.20}$$

where the sesquilinear form $a : [H_0^1(0,1) \times H_0^1(0,1)] \to \mathbb{C}$ and the antilinear form $L : H_0^1(0,1) \to \mathbb{C}$ are defined by

$$a(u,w) = \int_0^1 ((\lambda^2 + \gamma\lambda(\lambda+\eta)^{\alpha-1}) u\overline{w} + \left(1 - \int_0^\infty g(s) e^{-\lambda s}\, ds\right) u_x \overline{w}_x)\, dx,$$

$$\begin{aligned} L(w) = &\int_0^1 (\gamma(\lambda+\eta)^{\alpha-1} f_1 + \lambda f_1 + f_2)\overline{w}\, dx - \zeta \int_0^1 \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_4(x,\xi)\, d\xi \overline{w}\, dx \\ &\frac{1}{\lambda}\int_0^\infty g(s)(1 - e^{-\lambda s})\, ds \int_0^1 f_{1x}\overline{w}_x\, dx - \int_0^\infty g(s) \int_0^s e^{\lambda(\tau-s)} \int_0^1 f_{3x}(\tau)\overline{w}_x\, dx\, d\tau\, ds \end{aligned}$$

It is easy to verify that a is continuous and coercive, and L is continuous. So applying the Lax-Milgram theorem, we deduce that for all $w \in H_0^1(0,1)$ problem (III.20) admits a unique solution $u \in H_0^1(0,1)$. Applying the classical elliptic regularity, it follows from (III.19) that $u \in H^2(0,1)$. Therefore, the operator $\lambda I - A$ is surjective for any $\lambda > 0$. Consequently, using Hille-Yosida theorem, we have the following results.

Theorem 2.2 (Existence and uniqueness)

(1) If $U_0 \in D(\mathcal{A})$, then system (III.8) has a unique strong solution

$$U \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

(2) If $U_0 \in \mathcal{H}$, then system (III.8) has a unique weak solution

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

□

3 Strong stability of the System

3.1 Strong stability of the System

In this subsection, we use a general criteria of Arendt-Batty in [**20**] to show the strong stability of the C_0-semigroup $e^{t\mathcal{A}}$ associated to the system (P) in the absence of the compactness of the resolvent of $\mathcal{A}$. Our main result is the following theorem:

Theorem 3.1 *Then, the C_0-semigroup $e^{t\mathcal{A}}$ is strongly stable in $\mathcal{H}$, i.e, for all $U_0 \in \mathcal{H}$, the solution of (III.8) satisfies*

$$\lim_{t \to +\infty} \|e^{t\mathcal{A}} U_0\|_{\mathcal{H}} = 0.$$

Lemma 3.2 *We have*

$$\sigma(\mathcal{A}) \cap \{i\lambda,\ \ \lambda \in \mathbb{R},\ \ \lambda \neq 0\} = \emptyset.$$

Lemma 3.3 *$\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.*

Let us first prove Lemma 3.3.

Proof.

From (III.9) we get that $(u, v, \nu, \phi)^T \in Ker(\mathcal{A}) \subset D(\mathcal{A})$ if and only if

$$\begin{cases} -v = 0, \\ -(1 - \int_0^\infty g(s)\, ds) u_{xx} - \int_0^\infty g(s) \nu_{xx}\, ds + \zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(x,\xi) d\xi = 0, \\ -v + \partial_s \nu = 0, \\ (\xi^2 + \eta)\phi - v(x)\mu(\xi) = 0. \end{cases} \tag{III.21}$$

This implies that $v = 0, \phi = 0$ and

$$\int_0^1 g'(s) \int_0^{+\infty} |\partial_x \nu|^2 \, ds \, dx = 0.$$

Due to hypothesis (H), it follows that

$$\int_0^1 g(s) \int_0^{+\infty} |\partial_x \nu|^2 \, ds \, dx = 0.$$

This implies that

$$\nu = 0.$$

Then, we have

$$u = cx + c'.$$

As $u(0) = u(1) = 0$, we deduce that $u = 0$. Thus $U = 0$. This concludes the proof of Lemma 3.3.

$\square$

Let us suppose that there is $\lambda \in \mathbb{R}, \lambda \neq 0$ and $U \neq 0$, such that $\mathcal{A}U = i\lambda U$. Then, we get

$$\begin{cases} i\lambda u - v = 0, \\ i\lambda v - (1 - \int_0^\infty g(s)\, ds) u_{xx} - \int_0^\infty g(s) \nu_{xx} \, ds + \zeta \int_{-\infty}^{+\infty} \mu(\xi) \phi(x, \xi) d\xi = 0, \\ i\lambda \nu - v + \partial_s \nu = 0, \\ i\lambda \phi + (\xi^2 + \eta)\phi - v(x)\mu(\xi) = 0. \end{cases} \tag{III.22}$$

Then, from (III.12) we have

$$\phi \equiv 0, \tag{III.23}$$

and

$$\partial_x \nu = 0. \tag{III.24}$$

From $(III.22)_4$ and $(III.22)_1$, we have

$$u = v = 0. \tag{III.25}$$

Hence, from $(III.22)_3$ we obtain

$$\nu = 0. \tag{III.26}$$

Then, we have

$$u = cx + c'.$$

As $u(0) = u(1) = 0$, we deduce that $u = 0$. Thus $U = 0$. This concludes the proof of Lemma 3.2.

Now, we prove Lemma 3.2.

We will prove that the operator $i\lambda I - \mathcal{A}$ is surjective for $\lambda \neq 0$. For this purpose, let $(f_1, f_2, f_3, f_4)^T \in \mathcal{H}$, we seek $U = (u, v, \nu, \phi)^T \in D(\mathcal{A})$ solution of solution of the following equation

$$(i\lambda - \mathcal{A})U = F. \tag{III.27}$$

Equivalently, we have the following system

$$\begin{cases} i\lambda u - v = f_1, \\ i\lambda v - (1 - \int_0^\infty g(s)\, ds) u_{xx} - \int_0^\infty g(s) \nu_{xx}\, ds + \zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(x,\xi) d\xi = f_2, \\ i\lambda \nu - v + \partial_s \nu = f_3, \\ i\lambda\phi + (\xi^2 + \eta)\phi - v(x)\mu(\xi) = f_4. \end{cases} \tag{III.28}$$

The function u satisfies the following equation

$$\begin{aligned} &(-\lambda^2 + i\gamma\lambda(i\lambda+\eta)^{\alpha-1})u - \left(1 - \int_0^\infty g(s) e^{-i\lambda s}\, ds\right) u_{xx} = \\ &\gamma(i\lambda+\eta)^{\alpha-1} f_1 + i\lambda f_1 + f_2 - \zeta \int_{-\infty}^{+\infty} \frac{f_4(x,\xi)\mu(\xi)}{\xi^2 + \eta + i\lambda} d\xi \\ &-\frac{1}{i\lambda} \int_0^\infty g(s)(1 - e^{-i\lambda s})\, ds f_{1xx} + \int_0^\infty g(s) \int_0^s e^{i\lambda(\tau - s)} f_{3xx}(\tau)\, d\tau\, ds \end{aligned} \tag{III.29}$$

Then

$$\begin{aligned} &\int_0^1 ((-\lambda^2 + i\gamma\lambda(i\lambda+\eta)^{\alpha-1}) u\overline{w} + \left(1 - \int_0^\infty g(s) e^{-i\lambda s}\, ds\right) u_x \overline{w}_x)\, dx \\ &= \int_0^1 (\gamma(i\lambda+\eta)^{\alpha-1} f_1 + i\lambda f_1 + f_2)\overline{w}\, dx - \zeta \int_0^1 \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + i\lambda} f_4(x,\xi)\, d\xi \overline{w}\, dx \\ &\frac{1}{i\lambda} \int_0^\infty g(s)(1 - e^{-i\lambda s})\, ds \int_0^1 f_{1x}\overline{w}_x\, dx - \int_0^\infty g(s) \int_0^s e^{i\lambda(\tau - s)} \int_0^1 f_{3x}(\tau)\overline{w}_x\, dx\, d\tau\, ds \end{aligned}$$

for all $w \in H_0^1(0,1)$. We can rewrite (III.29) as

$$\mathcal{B}(u, w) = l(w), \quad \forall w \in H_0^1(0,1), \tag{III.30}$$

where

$$\mathcal{B}(u, w) = \mathcal{B}_1(u, w) + \mathcal{B}_2(u, w)$$

with

$$(*)\qquad \begin{cases} \mathcal{B}_1(u,w) = \displaystyle\int_0^1 ((i\gamma\lambda(i\lambda+\eta)^{\alpha-1})u\overline{w} + \left(1 - \int_0^\infty g(s)e^{-i\lambda s}\,ds\right) u_x\overline{w}_x)\,dx, \\ \mathcal{B}_2(u,w) = -\displaystyle\int_0^1 \lambda^2 u\overline{w}\,dx, \end{cases}$$

and

$$l(w) = \int_0^1 (\gamma(i\lambda+\eta)^{\alpha-1}f_1 + i\lambda f_1 + f_2)\overline{w}\,dx - \zeta\int_0^1\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+i\lambda}f_4(x,\xi)\,d\xi\overline{w}\,dx$$
$$\frac{1}{i\lambda}\int_0^\infty g(s)(1-e^{-i\lambda s})\,ds\int_0^1 f_{1x}\overline{w}_x\,dx - \int_0^\infty g(s)\int_0^s e^{i\lambda(\tau-s)}\int_0^1 f_{3x}(\tau)\overline{w}_x\,dx\,d\tau\,ds.$$

Let $(H^{-1}(0,1))'$ be the dual space of $H_0^1(0,1)$. Let us define the following operators

$$(**)\qquad \begin{array}{cc} B : H_0^1(0,1) \to H^{-1}(0,1) & B_i : H_0^1(0,1) \to H^{-1}(0,1) \quad i\in\{1,2\} \\ u \mapsto Bu & u \mapsto B_i u \end{array}$$

such that

$$(***)\qquad \begin{array}{l} (Bu)w = \mathcal{B}(u,w),\ \forall w\in H_0^1(0,1), \\ (B_iu)w = \mathcal{B}_i(u,w),\ \forall w\in H_0^1(0,1), i\in\{1,2\}. \end{array}$$

We need to prove that the operator B is an isomorphism. For this aim, we divide the proof into three steps:

Step 1. In this step, we want to prove that the operator B_1 is an isomorphism. For this aim, it is easy to see that $\mathcal{B}_1$ is sesquilinear, continuous form on $H_0^1(0,1)$. Furthermore

$$\begin{array}{rcl} \Re\mathcal{B}_1(u,u) & = & \left(1 - \displaystyle\int_0^\infty g(s)\cos\lambda s\,ds\right)\|u_x\|_2^2 + \gamma\lambda\Re\left(i(i\lambda+\eta)^{\alpha-1}\right)\|u\|^2 \\ & \geq & \left(1 - \displaystyle\int_0^\infty g(s)\,ds\right)\|x^{\gamma/2}u_x\|_2^2, \end{array}$$

where we have used the fact that

$$\gamma\lambda\Re\left(i(i\lambda+\eta)^{\alpha-1}\right) = \zeta\lambda^2\int_{-\infty}^{+\infty}\frac{\mu(\xi)^2}{\lambda^2+(\eta+\xi^2)^2}\,d\xi > 0.$$

Thus $\mathcal{B}_1$ is coercive. Then, from $(**)$ and Lax-Milgram theorem, the operator B_1 is an isomorphism.

Step 2. In this step, we want to prove that the operator B_2 is compact. For this aim, from $(*)$ and $(***)$, we have

$$|\mathcal{B}_2(u,w)| \leq c\|u\|_{L^2(0,1)}\|w\|_{L^2(0,1)},$$

and consequently, using the compact embedding from $H_0^1(0,1)$ to $L^2(0,1)$ we deduce that

B_2 is a compact operator. Therefore, from the above steps, we obtain that the operator $B = B_1 + B_2$ is a Fredholm operator of index zero. Now, following Fredholm alternative, we still need to prove that the operator B is injective to obtain that the operator B is an isomorphism.

Step 3. Let $u \in ker(B)$, then

$$\mathcal{B}(u, w) = 0 \quad \forall w \in H_0^1(0,1). \tag{III.31}$$

In particular for $w = u$, it follows that

$$\lambda^2 \|u\|^2_{L^2(0,1)} + i\gamma\lambda(i\lambda + \eta)^{\alpha-1}\|u\|^2_{L^2(0,1)} = \left(1 - \int_0^\infty g(s)e^{-i\lambda s}\, ds\right)\|u_x\|^2_{L^2(0,1)}.$$

Hence, we have that $i\lambda$ is an eigenvalue of the operator $\mathcal{A}$. Then, according to Lemma 3.2, we deduce that $u = 0$ and consequently $Ker(B) = \{0\}$. Finally, from Step 3 and Fredholm alternative, we deduce that the operator B is isomorphism. It is easy to see that the operator l is a antilinear and continuous form on $H_0^1(0,1)$. Consequently, (III.30) admits a unique solution $u \in H_0^1(0,1)$. By using the classical elliptic regularity, we deduce that $U \in D(\mathcal{A})$ is a unique solution of (III.27). Hence $i\lambda - \mathcal{A}$ is surjective for all $\lambda \in \mathbb{R}^*$.

3.2 Exponential Stability (for $\eta \neq 0$)

In order to establish the exponential energy decay rate, we need the following theorem.

Theorem 3.4 ([29]) *Let $S(t)$ be a C_0-semigroup of contractions on Hilbert space with generator $\mathcal{A}$. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R}$$

and

$$\overline{\lim_{|\beta|\to\infty}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Our main result is the following.

Theorem 3.5 *The semigroup $S_{\mathcal{A}}(t)_{t\geq 0}$ is polynomially stable and there exists a positive constant ω such that*

$$E(t) = \|S_{\mathcal{A}}(t)U_0\|^2_{\mathcal{H}} \leq e^{-\omega t}\|U_0\|^2_{D(\mathcal{A})}.$$

Proof.

Given $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$, let $U = (u, v, \nu, \phi)^T \in D(\mathcal{A})$ be the solution of the resolvent equation $(i\lambda I - \mathcal{A})U = F$, for $\lambda \in \mathbb{R}$, i.e.,

$$\begin{cases} i\lambda u - v = f_1, \\ i\lambda v - (1 - \int_0^\infty g(s)\,ds)u_{xx} - \int_0^\infty g(s)\nu_{xx}\,ds + \zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(x,\xi)d\xi = f_2, \\ i\lambda\nu - v + \partial_s \nu = f_3, \\ i\lambda\phi + (\xi^2 + \eta)\phi - v(x)\mu(\xi) = f_4. \end{cases} \tag{III.32}$$

Taking the real part of the inner product of $(i\lambda I - \mathcal{A})U$ with U in $\mathcal{H}$ and using (**??**), we get

$$|Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}}| \leq \|U\|_{\mathcal{H}}\|G\|_{\mathcal{H}}. \tag{III.33}$$

This implies that

$$-\frac{1}{2}\int_0^1 g'(s)\int_0^{+\infty} |\partial_x \nu|^2\,ds\,dx + \zeta\int_0^1\int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi,x)|^2\,d\xi\,dx \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} \tag{III.34}$$

Using condition (H) into (III.34), we obtain

$$\int_0^1 g(s)\int_0^{+\infty} |\partial_x\nu|^2\,ds\,dx \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{III.35}$$

Multiplying $(III.32)_3$ by $\overline{u}$ in $L^2_g(\mathbb{R}_+, H^1_0)$, then using the fact that $\|u\|_g^2 = g_0\|u_x\|_2^2$, we get

$$\begin{aligned} g_0\|u_x\|_2^2 = \int_0^1\int_0^{+\infty} g(s)\nu_x\overline{u}_x\,ds\,dx + \frac{1}{i\lambda}\int_0^1\int_0^{+\infty} g(s)\nu_{sx}\overline{u}_x\,ds\,dx \\ -\frac{1}{i\lambda}\int_0^1\int_0^{+\infty} g(s)(f_3 - f_1)\overline{u}_{xx}\,ds\,dx \end{aligned} \tag{III.36}$$

Using by parts integration, condition (H) and the fact that $\nu(x,0) = 0$, we get

$$\frac{1}{i\lambda}\int_0^1\int_0^{+\infty} g(s)\nu_{sx}\overline{u}_x\,ds\,dx = -\frac{1}{i\lambda}\int_0^1\int_0^{+\infty} g'(s)\nu_x\overline{u}_x\,ds\,dx$$

Applying Holder's inequality in $L^2(0,1)$ and $L^2(0,+\infty)$, then using (III.34) and $\lim_{s\to 0}\sqrt{g(s)}$

exists, we obtain

$$\begin{aligned}\left|\frac{1}{\lambda}\int_0^1\int_0^{+\infty} g(s)\nu_{sx}\overline{u}_x\,ds\,dx\right| &\leq \frac{\lim_{s\to 0}\sqrt{g(s)}}{|\lambda|}\left(\int_0^1\int_0^{+\infty} -g'(s)|\nu_x|^2\,ds\,dx\right)^{1/2}\|u_x\|_2\\ &\leq \frac{C}{|\lambda|}(\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}})^{1/2}\|u_x\|_2.\end{aligned} \tag{III.37}$$

Using (III.35), we get

$$\begin{aligned}|\int_0^1\int_0^{+\infty} g(s)\nu_x\overline{u}_x\,ds\,dx| &\leq g_0^{1/2}\left(\int_0^1\int_0^{+\infty} g(s)|\nu_x|^2\,ds\,dx\right)^{1/2}\|u_x\|_2\\ &\leq g_0^{1/2}(\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}})^{1/2}\|u_x\|_2.\end{aligned} \tag{III.38}$$

$$\left|\frac{1}{\lambda}\int_0^1\int_0^{+\infty} g(s)(f_3-f_1)\overline{u}_{xx}\,ds\,dx\right| \leq \frac{1}{|\lambda|}(g_0^{1/2}+g_0)\|F\|_{\mathcal{H}}\|u_x\|_2 \tag{III.39}$$

Using (III.36), (III.37), (III.38) and (III.39), we deduce that

$$\|u_x\|_2^2 \leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{C}{|\lambda|^2}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{C}{|\lambda|^2}\|F\|_{\mathcal{H}}^2. \tag{III.40}$$

Now, multiplying $(III.32)_2$ by $\overline{u}$ in $L^2(0,1)$, we get

$$\begin{aligned}-\lambda^2\|u\|_2^2 + (1-\textstyle\int_0^\infty g(s)\,ds)\|u_x\|_2^2 + \int_0^1\int_0^\infty g(s)\nu_x\overline{u}_x\,ds\,dx + \zeta\int_0^1\overline{u}\int_{-\infty}^{+\infty}\mu(\xi)\phi(x,\xi)d\xi\,dx\\ = \textstyle\int_0^1(f_2+i\lambda f_1)\overline{u}\,dx\end{aligned}$$

$$\begin{aligned}&\left|\textstyle\int_0^1\overline{u}\int_{-\infty}^{+\infty}\mu(\xi)\phi(x,\xi)\,d\xi\,dx\right|\\ &\leq \left(\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}\,d\xi\right)^{\frac12}\|u\|_{L^2(0,1)}\left(\int_0^1\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(x,\xi)|^2\,dx\,d\xi\right)^{\frac12}\\ &\leq \frac{1}{\sqrt{\zeta}}\left(\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}\,d\xi\right)^{\frac12}\|u\|_{L^2(0,1)}(\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}})^{1/2}.\end{aligned}$$

Since $\eta > 0$, we have

$$\|\phi\|^2_{L^2((0,1)\times(-\infty,\infty))} \leq \frac{1}{\eta}\int_0^1\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi)|^2\,d\xi\,dx \leq c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

Thus, we conclude that

$$\|(i\lambda I-\mathcal{A})^{-1}\|_{\mathcal{H}} \leq C.$$

Applying Theorem 3.4, we obtain that

$$E(t) \leq e^{-\omega t}\|U_0\|^2_{D(\mathcal{A})}.$$

3.3 Polynomial Stability (for $\eta = 0$)

Lack of exponential stability

Theorem 3.6 *The semigroup generated by the operator $\mathcal{A}$ is not exponentially stable.*

Proof.
We shall show that $i\lambda = 0$ is not in the resolvent set of the operator $\mathcal{A}$. Indeed, noting that $(x \sin x\pi, 0, 0, 0)^T \in \mathcal{H}$, and denoting by $(u, v, \nu, \phi)^T$ the image of $(x \sin x\pi, 0, 0, 0)^T$ by $\mathcal{A}^{-1}$, we see that $\phi(x, \xi) = -|\xi|^{\frac{2\alpha-5}{2}} x \sin x\pi$. But, then $\phi \notin L^2((0,1) \times (-\infty, +\infty))$, since $\alpha \in]0, 1[$. So $(u, v, \nu, \phi)^T \notin D(\mathcal{A})$.

By theorem 3.6: 0 is a spectral point. Therefore it is convenient to have the following generalization of theorem 3.4 at hand:

Theorem 3.7 ([8]) *Let $S(t)$ be a bounded C_0-semigroup on a Hilbert space $\mathcal{H}$ with generator $\mathcal{A}$. Assume that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \{0\}$ and that there exist $\vartheta > 1$ and $\upsilon > 0$ such that*

$$\|(isI - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} = \begin{cases} O(|s|^{-\vartheta}), & s \to 0, \\ O(|s|^{\upsilon}), & |s| \to \infty. \end{cases}$$

Then there exist constants $C, t_0 > 0$ such that for all $t \geq t_0$ and $U_0 \in D(\mathcal{A}) \cap R(\mathcal{A})$ we have

$$\|e^{\mathcal{A}t} U_0\|^2 \leq C \frac{1}{t^{\frac{2}{\varsigma}}} \|U_0\|^2_{D(\mathcal{A}) \cap R(\mathcal{A})},$$

where $\varsigma = \max\{\vartheta, \upsilon\}$.

Our main result is the following.

Theorem 3.8 *The semigroup $S_{\mathcal{A}}(t)_{t \geq 0}$ is polynomially stable and*

$$E(t) = \|S_{\mathcal{A}}(t) U_0\|^2_{\mathcal{H}} \leq \frac{1}{t} \|U_0\|^2_{D(\mathcal{A}) \cap R(\mathcal{A})}.$$

Now, from $(III.32)_4$, we obtain

$$v(x)\mu(\xi) = (i\lambda + \xi^2 + \eta)\phi - f_4(x, \xi). \tag{III.41}$$

By multiplying $(III.41)$ by $(i\gamma + \xi^2 + \eta)^{-2}|\xi|$, we get

$$(i\lambda + \xi^2 + \eta)^{-2} v(x)\mu(\xi)|\xi| = (i\lambda + \xi^2 + \eta)^{-1}|\xi|\phi - (i\lambda + \xi^2 + \eta)^{-2}|\xi| f_4(x, \xi). \tag{III.42}$$

Hence, by taking absolute values of both sides of (III.42), integrating over the interval $]-\infty,+\infty[$ with respect to the variable ξ and applying Cauchy-Schwartz inequality, we obtain

$$\mathcal{S}|v(x)| \leq \sqrt{2}\mathcal{U}\left(\int_{-\infty}^{+\infty} \xi^2|\theta|^2\, d\xi\right)^{\frac{1}{2}} + 2\mathcal{V}\left(\int_{-\infty}^{+\infty} |f_4(x,\xi)|^2\, d\xi\right)^{\frac{1}{2}}, \tag{III.43}$$

where

$$\mathcal{S} = \left|\int_{-\infty}^{+\infty} (i\lambda + \xi^2 + \eta)^{-2}|\xi|\kappa(\xi)\, d\xi\right| = \frac{|1-2\alpha|}{4}\frac{\pi}{|\sin\frac{(2\alpha+3)}{4}\pi|}|i\lambda+\eta|^{\frac{(2\alpha-5)}{4}},$$

$$\mathcal{U} = \left(\int_{-\infty}^{+\infty} (|\lambda| + \xi^2 + \eta)^{-2}\, d\xi\right)^{\frac{1}{2}} = (\frac{\pi}{2})^{1/2}||\lambda|+\eta|^{-\frac{3}{4}},$$

$$\mathcal{V} = \left(\int_{-\infty}^{+\infty} (|\lambda| + \xi^2 + \eta)^{-4}|\xi|^2\, d\xi\right)^{\frac{1}{2}} = \left(\frac{\pi}{16}||\lambda|+\eta|^{-\frac{5}{2}}\right)^{1/2}.$$

Thus, by using the inequality $2PQ \leq P^2 + Q^2, P \geq 0, Q \geq 0$, again, we get

$$\mathcal{S}^2\int_0^1 |v(x)|^2\, dx \leq 2\mathcal{U}^2\left(\int_0^1\int_{-\infty}^{+\infty} (\xi^2+\eta)|\theta|^2\, d\xi\, dx\right) + 4\mathcal{V}^2\left(\int_0^1\int_{-\infty}^{+\infty} |f_4(x,\xi)|^2\, d\xi\, dx\right). \tag{III.44}$$

We deduce that For λ near 0, we have from (III.44)

$$\int_0^1 |v(x)|^2\, dx \leq c|\lambda|^{1-\alpha}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c|\lambda|^{-\alpha}\|F\|^2_{\mathcal{H}}. \tag{III.45}$$

$$\begin{aligned}
\|\phi\|^2 &\leq 2\int_0^1 |v(x)|^2\, dx\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{|i\lambda+\xi^2|^2}\, d\xi + 2\int_0^1\int_{-\infty}^{+\infty}\frac{|f_4(x,\xi)|^2}{|i\lambda+\xi^2|^2}\, d\xi\, dx \\
&\leq 4\int_0^1 |v(x)|^2\, dx\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{(|\lambda|+\xi^2)^2}\, d\xi + 4\int_0^1\int_{-\infty}^{+\infty}\frac{|f_4(x,\xi)|^2}{(|\lambda|+\xi^2)^2}\, d\xi\, dx \\
&\leq 4(1-\alpha)\frac{\pi}{\sin\alpha\pi}\|v\|^2_{L^2(0,1)}|\lambda|^{(\alpha-2)} + 4|\lambda|^{-2}\|f_4\|^2_{L^2((0,1)\times(-\infty,+\infty))}.
\end{aligned} \tag{III.46}$$

Then

$$\|\phi\|^2_{L^2((0,1)\times(-\infty,+\infty))} \leq c|\lambda|^{-1}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c|\lambda|^{-2}\|F\|^2_{\mathcal{H}}.$$

Finally, we deduce that

$$\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{H}} \leq \frac{C}{|\lambda|^2} \text{ as } \lambda \to 0.$$

Applying Theorem 3.7, we obtain that

$$E(t) \leq \frac{1}{t}\|U_0\|^2_{D(\mathcal{A})\cap R(\mathcal{A})}.$$

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