N° d'ordre :

REPUBLIQUE ALGERIENNE DEMOCRATIQUE & POPULAIRE

MINISTERE DE L'ENSEIGNEMENT SUPERIEUR & DE LA RECHERCHE SCIENTIFIQUE

UNIVERSITE DJILLALI LIABES
FACULTE DES SCIENCES EXACTES
SIDI BEL ABBES

THESE DE DOCTORAT EN SCIENCES

Présentée par *Keddar Naima.*

Spécialité : **Mathématiques**

Option : **Equations aux dérivées partielles**

Intitulée

Multiple solutions for weighted nonlinear elliptic equations and systems with critical exponents

Soutenue le 07/03/2022

Devant le jury composé de :

Président : Mr Hakem Ali	**Prof. Univ. Sidi Bel Abbes**
Examinateurs : Mr Amroun Noureddine	**Prof. Univ Sidi Bel Abbes**
Mr Messirdi Sofiane	**MCA univ Oran**
Mme Benmanssour Safia	**MCA ESM Tlemcen**
Directeur de thèse :Mme Matallah Atika	**MCA ESM Tlemcen**
Co-Directeur de thèse :Mr Benaissa Abbes	**Prof. Univ. Sidi Bel Abbes**

Année universitaire 2021/2022.

Dedication

I perfect this work at height of my parents and my husband for their help, motivation and encouragement.

To my sister Farah, my brothers.

To my children Rayane and Walid.

Acknowledgement

First of all, I would like to send my warmest and sincere thanks to Madame **Atika Matallah**, I thank her for being the researcher who gave me the opportunity to work in this direction, I thank her as the colleague and the supervisor to the total commitment; her advices and indications were always successful.

All my gratitude goes to Professor **Abbes Benaissa**, to have contributed to the enrichment of this thesis and to take part in my jury. I present to professor **Ali Hakem**, the expression of my sincere thanks, for the honor he makes me for chairing the jury of this thesis. I pray **Mr Amroun Nouredine**, to find the expression of all my gratitude, for the honor he makes me by accepting to be part of the jury that will examines this thesis. I extend to Madame **Safia Benmansour**, the expression of my sincere thanks and my whole gratitude, to have agreed to assess and examine my work and to take part in the jury. I thank warmly, **Mr. Messirdi Sofiane**, to have agreed to participate in the jury and examine this manuscript.

In the end, I would like to thank all people who helped me in the completion of this thesis.

Contents

Introduction

This thesis is concerned with the following class of elliptic equations

$$-\left(a\left\|u\right\|_{\alpha,\mu}^{p}+b\right)\left(\operatorname{div}\left(\frac{\left|\nabla u\right|^{p-2}\nabla u}{\left|x\right|^{p\alpha}}\right)+\mu\frac{\left|u\right|^{p-2}u}{\left|x\right|^{p(\alpha+1)}}\right)=\frac{\left|u\right|^{p^{*}-2}u}{\left|x\right|^{p^{*}\beta}}+\lambda f\left(x\right)\quad\text{in }\Omega\qquad(1)$$

where $\Omega\subseteq\mathbb{R}^{N}$ $(N\geq3)$ containing 0 in its interior, $1<p<N$, a, $b\geq0$, $a+b>0$, $0\leq\alpha<\left(N-p\right)/p$, $\alpha\leq\beta<\alpha+1$, $-\infty<\mu<\overline{\mu}:=\left[\left(N-\left(\alpha+1\right)p\right)/p\right]^{p}$, λ is a parameter, $p^{*}=pN/\left[N-p\left(1+\alpha-\beta\right)\right]$ is the critical Caffarelli-Kohn-Nirenberg exponent and $f\in W^{*}/\left\{0\right\}$. Here, $W_{\alpha,\mu}^{1,p}\left(\Omega\right)$ denotes the completion of $C_{0}^{\infty}(\Omega)$ with respect to the norm

$$\left\|u\right\|_{\alpha,\mu}^{p}:=\int\limits_{\Omega}\left(\frac{\left|\nabla u\right|^{p}}{\left|x\right|^{p\alpha}}-\mu\frac{\left|u\right|^{p}}{\left|x\right|^{p\alpha+p}}\right)dx,$$

and W^{*} is the dual space of $W_{\alpha,\mu}^{1,p}\left(\Omega\right)$. For $(\alpha,\mu)=(0,0)$ we shall work with the space $W^{1,p}\left(\Omega\right)$ endowed with the norme

$$\left\|u\right\|^{p}:=\int\limits_{\Omega}\left|\nabla u\right|^{p}dx.$$

This problem is related to the following well known Caffarelli-Kohn-Nirenberg inequality [17]:

$$\left(\int_{\Omega}\left|x\right|^{-p^{*}\beta}\left|u\right|^{p^{*}}dx\right)^{1/p^{*}}\leq C_{\alpha,\beta}\left(\int_{\Omega}\left|x\right|^{-p\alpha}\left|\nabla u\right|^{p}dx\right)^{1/p}\quad\text{for all }u\in C_{0}^{\infty}\left(\Omega\right),\qquad(2)$$

for some positive constant $C_{\alpha,\beta}$. For sharp constants and extremal functions associated to (2), see [19, 35, 49]. If $\beta = \alpha + 1$ in (2), then $p^* = p$, $C_{\alpha,\beta} = 1/\overline{\mu}$, and we have the following weighted Hardy inequality [4, 6, 1]:

$$\int_\Omega \frac{|u|^p}{|x|^{p\alpha+p}}dx \leq \frac{1}{\overline{\mu}} \int_\Omega \frac{|\nabla u|^p}{|x|^{p\alpha}}dx, \text{ for all } u \in C_0^\infty(\Omega). \tag{3}$$

If $\alpha = \beta = 0$ in (2), then $p^* = pN/(N-p)$ we obtain the following Sobolev inequality

$$\left(\int_\Omega |u|^{p^*} dx\right)^{1/p^*} \leq C_{0,0} \left(\int_\Omega |\nabla u|^p dx\right)^{1/p} \text{ for all } u \in C_0^\infty(\Omega), \tag{4}$$

for some positive constant $C_{0,0}$.

If $a \neq 0$, the problem (1) is called nonlocal because of the presence of the nonlocal term $a\|u\|_{\alpha,\mu}^p$, which implies that problem (1) is no longer a pointwise identity. This phenomenon causes some mathematical difficulties, which make the study of problem (1) interesting. It is called also non degenerate if $b > 0$ and $a \geq 0$, while it is named degenerate if $b = 0$ and $a > 0$.

Such class of nonlocal elliptic problem like (1) without singular weights ($\alpha = \beta = \mu = 0$) is related to the original Kirchhoff's equation, which arises in nonlinear vibrations, namely

$$\begin{cases} u_{tt} - M\left(\int |\nabla u|^2 dx\right)\Delta u = g(x,t) & \text{in } \Omega \times (0,T) \\ u = 0 & \text{in } \partial\Omega \times (0,T) \\ u(0,x) = u_0,\ u_t(0,x) = u_1, & \end{cases}$$

which was first introduced by Kirchhoff as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the strings produced by transverse vibrations.

Problems which involve nonlocal operator have been widely studied due to their numerous and relevant applications in various fields of sciences. In particular, Kirchhoff-type problems proved to be valuable tools for modeling several physical and biological phenomena and many works have been made to ensure the existence of solutions for such problems; we quote in particular the article of Lions [42]. Since this famous paper, very fruitful development has given rise to many works on this advantageous axis and in most of them, the used approach relies on topological methods. However, just few improvements were held concerning the multiplicity of solutions. At this regard, variational approach was solicited instead of topological methods to solve this kind of problems and also to prove the existence of multiple solutions; we refer interested readers to the works [3], [11], [43] and [46].

In the last few years, great attention has been paid to the study of elliptic problems involving critical nonlinearities. This problems create many difficulties in applying variational methods. It is worth mentioning that the semilinear Laplace equation of elliptic type involving critical exponent of Sobolev was investigated in the crucial paper of Brézis and Nirenberg [16]. After that, many researchers dedicated to the study of several kinds of elliptic equations with critical exponent of Sobolev or Caffarelli-Kohn-Nirenberg in bounded domain or in the whole space. For $p = 2$ and $a = \alpha = \beta = \mu = 0$, Tarantello [50] treated the problem (1) in a bounded domain of $\mathbb{R}^N$ and proved the existence of at least two solutions by using Nehari manifold methods. The first work on the Kirchhoff-type problem with critical Sobolev exponent is Alves, Corrêa and Figueiredo in [3]. Naimen in [46] showed a Brézis-Nirenberg type result for Kirchhoff problem in bounded domain. In [29], Figueiredo and al. considered Kirchhoff elliptic equations with critical exponent of Caffarelli-Kohn-Nirenberg.

Recently, Benaissa and al. in [30] discussed the problem

$$-\left(a\int_{\mathbb{R}^N}|\nabla u|^p\,dx+b\right)\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)=|u|^{p^*-2}u+\lambda f(x)\quad\text{in }\mathbb{R}^N,$$

here $p^*=\dfrac{pN}{N-p}$ is the critical Sobolev exponent. For a particular dimension $N=3p/2$, they proved the existence of two solutions.

Note that the problem (1) without Kirchhoff terms ($a=0$) comes from the consideration of starting waves in anisotropic Schrödinger equations (see [1]). It was also introduced as models for several physical phenomena related to equilibrium of anisotropic media that possibly are some where perfect insulator or perfect conductors [1]. This class of equations has been investigated in a series of works see [9], [14], [49], [13], [12], [39] and the references therein.

This thesis is presented as follows.

Chapter 1 of preliminaries is devoted to the basic definitions, results and useful inequalities which we use frequently in the proof of our results in this thesis .

In **Chapter 2**, we firstly consider the case where Ω is a bounded domain in $\mathbb{R}^N$ ($N\geq 3$) containing 0 in its interior and $(a,b)=(0,1)$ in (1). So, we study the following nonhomogenous singular elliptic equation with the critical Caffarelli-Kohn-Niremberg exponent

$$\begin{cases}-div\left(\dfrac{|\nabla u|^{p-2}\nabla u}{|x|^{p\alpha}}\right)-\mu\dfrac{|u|^{p-2}u}{|x|^{p(\alpha+1)}}=\dfrac{|u|^{p^*-2}u}{|x|^{p^*\beta}}+f(x)\text{ in }\Omega,\\ u=0\qquad\qquad\text{on }\partial\Omega.\end{cases}\tag{5}$$

The purpose of this chapter is to investigate the existence of a ground state solution for the problem (5) by a "smallness" condition on f. By using the Nehari manifold we proved our result. On the other hand, when $(\alpha, \mu) = (0,0)$ we proved to the existence of a second solution of problem (5).

Chapter 3, is devoted to the case where $\Omega = \mathbb{R}^N$, $a \neq 0$ and $(\alpha, \beta, \mu) = (0,0,0)$ in (1). So, we are concerned with the existence, multiplicity infinity and the non existence of solutions for the following Kirchhoff-type problem

$$-\left(a\int_{\mathbb{R}^N} |\nabla u|^p \, dx + b\right) \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = |u|^{p^*-2}u + \lambda f(x) \text{ in } \mathbb{R}^N \tag{6}$$

where $p^* = pN/[N-p]$ is the critical Sobolev exponent, $f \in W^* \setminus \{0\}$. Here, W^* is the dual space of $W^{1,p}(\mathbb{R}^N)$. Note that if $a = \lambda = 0$, $b = 1$ and $1 < p < N$, the equation (6) reduces to the following problem

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = |u|^{p^*-2}u, \quad \text{in } \mathbb{R}^N \tag{7}$$

Sciunzi in [2] provided that if u is a positive solution of (7) then $u(x) = v_{\varepsilon,x_0}(x)$ where

$$v_{\varepsilon,x_0}(x) := \left[\frac{\varepsilon^{\frac{1}{p-1}} N^{\frac{1}{p}} \left(\frac{N-p}{p-1}\right)^{\frac{p-1}{p}}}{\varepsilon^{\frac{p}{p-1}} + |x-x_0|^{\frac{p}{p-1}}}\right]^{\frac{N-p}{p}} \qquad \varepsilon > 0,\ x_0 \in \mathbb{R}^N. \tag{8}$$

Consequently, u a minimizer for

$$S := \inf_{u \in W^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|^p}{\left(\int_{\mathbb{R}^N} |u|^{p^*} \, dx\right)^{p/p^*}},$$

and satisfies

$$\|v_{\varepsilon,x_0}\|^p = \int_{\mathbb{R}^N} |v_{\varepsilon,x_0}|^{p^*} \, dx = S^{\frac{p^*}{p^*-p}}, \tag{9}$$

We make the following assumptions to prove the following results:

$(\mathcal{H}_0)$ $p^* > 2p$, $a \geq 0$, $b \geq 0$ and $a + b > 0$.

$(\mathcal{H}_1)$ $p^* = 2p$, $a \geq 0$ and $b > 0$.

$(\mathcal{H}_2)$ $p^* = 2p$, $0 \leq a < S^{-2}$ and $b > 0$

$(\mathcal{H}_3)$ $p^* > 2p$, $a > 0$ and $b > 0$.

$(\mathcal{H}_4)$ $p^* = 2p$, $a > S^{-2}$ and $b = 0$.

$(\mathcal{H}_5)$ $p^* = 2p$, $a \geq S^{-2}$ and $b > 0$.

$(\mathcal{H}_6)$ $p^* < 2p$, $a > 0$ and $b > b^*$.

$(\mathcal{H}_7)$ $p^* = 2p$, $a > 0$ and $b = 0$.

$(\mathcal{H}_8)$ $p^* < 2p$, $a > 0$ and $b = b^*$ where

$$b^* = \frac{2p - p^*}{p} \left(\frac{p}{p^* - p} a \right)^{-\frac{p^* - p}{2p - p^*}} S^{-\frac{p^*}{2p - p^*}},$$

and we define the energy functional

$$I(u) = \frac{a}{2p} \|u\|^{2p} + \frac{b}{p} \|u\|^p - \frac{1}{p^*} \int_{\mathbb{R}^N} |u|^{p^*} dx - \lambda \int_{\mathbb{R}^N} f(x)\, u dx,$$

then we obtain the following results.

Theorem 0.1 *Suppose that $f \in W^* \backslash \{0\}$ and assume $(\mathcal{H}_0)$ or $(\mathcal{H}_1)$ holds. Then there exists a constants $\lambda_- > 0$ such that for any $\lambda \in (0, \lambda_-)$ problem (6) has a solution u_- with negative energy.*

Theorem 0.2 *Suppose that $f \in W^* \backslash \{0\}$ such that $\int_{\mathbb{R}^N} f(x)\, v_{\varepsilon,x_0} dx \neq 0$. Assume $(\mathcal{H}_0)$ or $(\mathcal{H}_2)$ holds. Then there exists a constant $\lambda_+ \in (0, \lambda_-]$ such that for any $\lambda \in (0, \lambda_+)$ problem (6) has a second solution u_+ with positive energy.*

Theorem 0.3 *Let $\lambda = 0$, $a > 0$, $b \geq 0$, $1 < p < N$. For v_{ε,x_0} given by (8) the following conclusions hold:*

(1) *If* $p^* = 2p$*, then under the hypothesis* $(\mathcal{H}_2)$ *, the problem* (6) *has infinitely many nonnegative solutions and these solutions are*

$$\left(\frac{b}{1-S^2a}\right)^{\frac{1}{p^*-p}} v_{\varepsilon,x_0} \quad \textit{for all } \varepsilon > 0,$$

and under the hypothesis $(\mathcal{H}_7)$*, the problem* (6) *has infinitely many positive solutions* $\delta^{\frac{1}{p}} v_{\varepsilon,x_0}$ *(for any* $\delta > 0$*) if and only if* $a = S^{-2}$*.*

(2) *If* $p^* \neq 2p$*,* $b = 0$ *and* $a > 0$*, then problem* (6) *has infinitely many nonnegative solutions and these solutions*

$$\left(aS^{\frac{p^*}{p^*-p}}\right)^{-\frac{1}{2p-p^*}} v_{\varepsilon,x_0} \quad \textit{for all } \varepsilon > 0.$$

(3) *If* $(\mathcal{H}_3)$ *satisfied, then there exists* $\delta_2 > S^{-1}\left(\frac{ap}{p^*-p}S^2\right)^{\frac{p^*-p}{p^*-2p}}$ *such that* $\delta_2^{\frac{1}{p^*-p}} v_{\varepsilon,x_0}$ *are solutions of problem* (6)*, for all* $\varepsilon > 0$*.*

(4) *If* $(\mathcal{H}_8)$ *satisfied, then problem* (6) *has infinitely many nonnegative solutions and these solutions are*

$$S^{-\frac{1}{p^*-p}}\left(\frac{p^*-p}{pa}S^{-2}\right)^{\frac{1}{2p-p^*}} v_{\varepsilon,x_0} \quad \textit{for all } \varepsilon > 0.$$

(5) *If* $(\mathcal{H}_6)$ *satisfied, then there exist* $\delta_3 \in \left(0,\ S^{-1}\left(\frac{p^*-p}{pa}S^{-2}\right)^{\frac{p^*-p}{2p-p^*}}\right)$ *and* $\delta_4 \in \left(S^{-1}\left(\frac{p^*-p}{pa}S^{-2}\right)^{\frac{p^*-p}{2p-p^*}},\ +\infty\right)$ *such that* $\delta_3^{\frac{1}{p^*-p}} v_{\varepsilon,x_0}$ *and* $\delta_4^{\frac{1}{p^*-p}} v_{\varepsilon,x_0}$ *solutions of problem* (6) *for all* $\varepsilon > 0$*.*

Theorem 0.4 *Assume that one of the hypotheses* $(\mathcal{H}_i)$ *holds for* $4 \leq i \leq 6$*. Then problem* (6) *has no non-trivial solution for* $\lambda = 0$*.*

In **Chapter 4** we generalise some results of chapter 3 in the following p-Kirchhoff-type systems

$$\begin{cases} -\left(a_1 + b_1 \|u\|^p\right) \left[\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)\right] = \dfrac{2q}{q+q'}|u|^{q-2}u \ |v|^{q'} + \lambda_1 f\left(x\right), \\ -\left(a_2 + b_2 \|v\|^p\right) \left[\operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right)\right] = \dfrac{2q'}{q+q'}|u|^{q} |v|^{q'-2} v + \lambda_2 g\left(x\right), \\ (u,v) \in W^{1,p}\left(\mathbb{R}^N\right) \times W^{1,p}\left(\mathbb{R}^N\right) \end{cases} \tag{10}$$

where $1 < p < N,\ a_1, a_2 \geq 0,\ b_1, b_2 > 0,\ q, q' > 1,\ q + q' = p^*,\ p^* = pN/\left[N - p\right]$ $\lambda_1, \lambda_2 \geq 0$ and $f, g \in W^* \backslash \{0\}$, W^* is the dual space of $W^{1,p}(\mathbb{R}^N)$.

In this chapter we establish the existence of solutions with negative and positive energy, infinity results and non existence of solution for the Kirchhoff-type systems involving the critical Sobolev exponent.

Note that the problem (10) has infinitely many nonnegative solutions for $\lambda_1 = \lambda_2 = b_1 = b_2 = 0$, $a_1 = a_2 = 1$ and $1 < p < N$. These solutions are

$$\begin{cases} u_\varepsilon = \left(\dfrac{2}{p^*}\right)^{\frac{1}{(p-p^*)}} (q)^{\frac{p-q'}{p(p-p^*)}} \left(q'\right)^{\frac{q'}{p(p-p^*)}} v_{\varepsilon,x_0} \\ v_\varepsilon = \left(\dfrac{2}{p^*}\right)^{\frac{1}{(p-p^*)}} \left(q'\right)^{\frac{p-q}{p(p-p^*)}} q^{\frac{q}{p(p-p^*)}} v_{\varepsilon,x_0} \end{cases} \text{for all } \varepsilon > 0. \tag{11}$$

Let the constant

$$S_{q,q'} := \inf_{\substack{(u,v)\in W^{1,p}\left(\mathbb{R}^N\right)\times W^{1,p}\left(\mathbb{R}^N\right) \\ (u,v)\neq(0,0)}} \frac{\|u\|^p + \|v\|^p}{\left(\int_{\mathbb{R}^N} |u|^q |v|^{q'} dx\right)^{p/p^*}}$$

which is positive. Let $a = \max\left(a_1, a_2\right) \geq 0$, $b = \max\left(b_1, b_2\right) \geq 0$. The following assumptions are used in this chapter:

(H_1) : $p^* \geq 2p$, $b \geq 0$ and $a > 0$.

(H_2) : $p^* > 2p$, $b > 0$ and $a = 0$.

(H_3) $p^* = 2p$, $a_2 = a_2 = 0$ and $b_1, b_2 > S_{q,q'}^{-2}$.

(H_4) $p^* = 2p$, $b_1, b_2 \geq S_{q,q'}^{-2}$ and $a_1, a_2 > 0$.

(H_5) $p^* > 2p$, $a > 0$ and $b > \frac{p^*-p}{p}\left(2\frac{2p-p^*}{pa}\right)^{\frac{2p-p^*}{p^*-p}} 2^{\frac{p}{p^*-p}} \left(S_{q,q'}\right)^{-\frac{p^*}{p^*-p}}$.

(H_6) $p^* \geq 2p$, $a_1 = a_2 = 0$, b_1, $b_2 > 0$.

(H_7) $p^* \geq 2p$, $a_1 = 0, a_2 \neq 0$, b_1, $b_2 > 0$.

(H_8) $p^* \geq 2p$, $a_1 \neq 0$, $a_2 \neq 0$, b_1, $b_2 > 0$

We define the energy functional

$$\begin{aligned} I(u,v) &= \frac{1}{2p}\left(b_1\|u\|^{2p} + b_2\|v\|^{2p}\right) + \frac{1}{p}\left(a_1\|u\|^p + a_2\|v\|^p\right) \\ &\quad -\frac{2}{p^*}\int_{\mathbb{R}^N}|u|^q|v|^{q'}dx - \int_{\mathbb{R}^N}\lambda_1 f(x)\,u + \lambda_2 g(x)\,v dx, \end{aligned}$$

and we present our results:

Case 1: $(\lambda_1, \lambda_2) = (0, 0)$.

Result 1: If one of assumptions (H_3), (H_4) or (H_5) is satisfied then problem (10) has no non-trivial solution.

Result 2: If one of assumptions (H_6), (H_7) or (H_8) is satisfied and if $a_1 = a_2 = 1$, $b_1 = b_2 = 0$, then Problem (10) has infinitely many nonnegative solutions $(u'_\varepsilon, v'_\varepsilon) = (\theta_1 u_\varepsilon, \theta_2 v_\varepsilon)$ for any $\theta_1, \theta_2 > 0$.

Case 2: $(\lambda_1, \lambda_2) \neq (0, 0)$.

Result 3: If (H_1) or (H_2) is satisfied then system (10) has a solution (u_1, v_1) with negative energy for some conditions in (λ_1, λ_2).

Result 4: If $a_1 = a_2 = 1$, $b_1 = b_2 = 0$ and $\int_{\mathbb{R}^N} f(x)\,u_\varepsilon dx \neq 0$ or $\int_{\mathbb{R}^N} g(x)\,v_\varepsilon dx \neq 0$. Then problem (10) has a second solution (u_2, v_2) with positive energy.

Chapter 1

Preliminaries

In this chapter, we briefly recall the basic definitions and some important results which we need in the proof of our results in the following chapters (see [30], [6], [14], [54], [27]).

1.1 Palais-Smale condition

Let K a Banach space, $J \in C^1(K, \mathbb{R})$ (K^* the dual of K).

Definition 1.1 *A function J is called Frechet differentiable at $u \in K$ if there exists a bounded linear application $J'(u) \in K^*$ such that*

$$\left[\frac{|J(u+v) - J(u) - \langle J'(u), v\rangle|}{\|v\|_K}\right] \to 0, \text{ when } \|v\|_K \to 0$$

A function J that is Frechet differentiable for any point of K is said to be C^1 if the function $J'$is continuous.

Definition 1.2 *We call that $u \in K$ is a critical point of J if $J'(u) = 0$, otherwise u is called a regular point.*

Let $c \in \mathbb{R}$; we say that c is a critical value of J if there exists a critical point u in K such that $J(u) = c$, otherwise c is called regular.

Definition 1.3 *We call a sequence $(u_n) \in K$ is a Palais-Smale sequence on K if $J(u_n) \to c$ and $\|J'(u_n)\|_{K^*} \to 0$ as $n \to +\infty$.*

Definition 1.4 *Let $c \in \mathbb{R}$, We say that J satisfies the Palais-Smale condition at level c we also note $(PS)_c$ for short, if for any sequence $(u_n) \in K$ such that*

$$\begin{cases} J(u_n) \to c & \text{in } \mathbb{R} \\ J'(u_n) \to 0 & \text{in } K^* \end{cases}$$

contains a convergent subsequence in K.

Let us observe that if $J \in C^1(K, \mathbb{R})$ satisfies the Palais-Smale condition, any point of accumulation $\overline{u}$ of a Palais-Smale sequence (u_n), is a critical point of J. We have implicitly $J'(\overline{u}) = 0,\ J(\overline{u}) = c$.

1.2 Mountain Pass Theorem and Ekeland's variational principle

A powerful tool for proving the existence of a critical point of a functional, is given by the following theorem.

Theorem 1.5 *[38] Let (K, d) be a complete metric space, and $J : K \to \mathbb{R}$. a lower semicontinuous functional, not identically equal to $+\infty$, which is bounded from below*

($c = \inf_K J >$ -∞), Then, for all $\varepsilon > 0$; there exists $\gamma_\varepsilon \in K$ such that

$$\begin{aligned} c &< J(\gamma_\varepsilon) < c + \varepsilon, \\ J(\gamma) - J(\gamma_\varepsilon) + \varepsilon d(\gamma, \gamma_\varepsilon) &> 0 \ \forall \gamma \in K \ , \ \textit{such that } \gamma \neq \gamma_\varepsilon \end{aligned}$$

Corollary 1.6 *[38] If K is a Banach space and $J \in C^1(K,\mathbb{R})$ is bounded from below, then there exists a minimizing sequence (u_n) for J in K such that*

$$J(u_n) \to \inf_K J, \ \ J'(u_n) \to 0 \ \textit{in} \ \ K^* \ \textit{as} \ n \to +\infty.$$

Theorem 1.7 [6] *Let $J \in C^1(K,\mathbb{R})$ satisfying the Palais-Smale condition.*

Assume that

1) $J(0) = 0$,

2) *There exists two numbers σ and ρ such that $J(u) \geq \sigma$ for every $u \in K$ with $\|u\|_K = \rho$.*

3) *There exists $v \in K$ such that $J(v) < \sigma$ and $\|v\|_K \geq \rho$.*

Define

$$\Gamma := \{\gamma \in C(0,1), \gamma(0) = 0, \ \gamma(1) = v\},$$

then

$$c := \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} J(u) \geq \sigma$$

is a critical value.

1.3 The Sobolev spaces

Definition 1.8 *Let* $\Omega \subseteq \mathbb{R}^N$*. We define the Sobolev space* $W^{1,p}(\Omega)$ *by*

$$\left\{ u \in L^p(\Omega) / \exists\ f_1, f_2, ...f_N \text{ such that } \int_\Omega u \frac{\partial \varphi}{\partial x_i} = -\int_\Omega f_i \varphi\ \forall \varphi \in C_0^\infty(\Omega);\ \forall\ i = 1...N \right\},$$

and $u \in W^{1,p}(\Omega)$ *by*

$$\frac{\partial u}{\partial x_i} = f_i \quad , \ \nabla u = \left(\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_N} \right).$$

Definition 1.9 *Let* $\Omega \subseteq \mathbb{R}^N$*. We define* $W_0^{1,p}(\Omega)$ *by the completion of* $C_0^\infty(\Omega)$ *in* $W^{1,p}(\Omega)$.

Remark 1.10 *We have* $W_0^{1,p}(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N)$.

Theorem 1.11 *[16] Let* $u \in W^{1,p}(\Omega)$*, then* $u \in W_0^{1,p}(\Omega)$ *if and only if* $u = 0$ *on* $\partial\Omega$.

Definition 1.12 *Let* $\Omega \subseteq \mathbb{R}^N$*. We define for* $p \geq 1$

$$\|u\|_{W^{1,p}(\Omega)}^p := \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)}^p$$

when $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$*, there exists a constant* $S > 0$ *such that*

$$S := \inf_{u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|^p}{\left(\int_{\mathbb{R}^N} |u|^{p^*} dx \right)^{p/p^*}}$$

Theorem 1.13 *[47] Let* $\Omega \subseteq \mathbb{R}^N$*. If* u *is a positive solution of*

$$-\operatorname{div}(|\nabla u|^{p-2} \nabla u) = |u|^{p^*-2} u, \quad \text{in } \Omega$$

then $u(x) = v_{\varepsilon, x_0}(x)$ *where*

$$v_{\varepsilon, x_0}(x) := \left[\frac{\varepsilon^{\frac{1}{p-1}} N^{\frac{1}{p}} \left(\frac{N-p}{p-1} \right)^{\frac{p-1}{p}}}{\varepsilon^{\frac{p}{p-1}} + |x - x_0|^{\frac{p}{p-1}}} \right]^{\frac{N-p}{p}} \qquad \varepsilon > 0,\ x_0 \in \Omega \tag{1.1}$$

Consequently, u a minimizer for

$$S := \inf_{u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|^p}{\left(\int_{\mathbb{R}^N} |u|^{p^*} \, dx \right)^{p/p^*}}$$

and satisfies

$$\|v_{\varepsilon,x_0}\|^p = \int_{\Omega} |v_{\varepsilon,x_0}|^{p^*} \, dx = S^{\frac{p^*}{p^*-p}}, \tag{1.2}$$

Theorem 1.14 *Assume $q, q' > 1,\ q + q' \leq p^*$, we define the constant*

$$S_{q,q'} := \inf_{\substack{(u,v) \in W^{1,p}(\mathbb{R}^N) \times W^{1,p}(\mathbb{R}^N) \\ (u,v) \neq (0,0)}} \frac{\|u\|^p + \|v\|^p}{\left(\int_{\mathbb{R}^N} |u|^q \, |v|^{q'} \, dx \right)^{p/p^*}}$$

which is positive, then

$$S_{q,q'} = \left[\left(\frac{q}{q'} \right)^{\frac{q'}{q+q'}} + \left(\frac{q'}{q} \right)^{\frac{q}{q+q'}} \right] S_0$$

1.4 Needed inequalities and Sobolev's embedding

Theorem 1.15 *(Sobolev-Gagliardo-Nirenberg)*

Let $1 \leq p < N$ and $\Omega \subseteq \mathbb{R}^N$, Sobolev embedding gives

$$W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$$

where $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{N}$. Moreover there exists a constant $C = C(p, N)$ such that

$$\|u\|_{L^{p^*}(\Omega)} \leq C \, \|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W^{1,p}(\Omega).$$

Corollary 1.16 *Let $1 \leq p < N$, then*

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \forall q \in [p, p^*]$$

with continuous embedding.

Theorem 1.17 *[30] Let $n \geq 1$ and $1 \leq p < \infty$. We have*

$$\text{if } \frac{1}{p} - \frac{n}{N} > 0, \text{ then } W^{n,p}\left(\mathbb{R}^N\right) \hookrightarrow L^q\left(\mathbb{R}^N\right) \text{ where } \frac{1}{q} = \frac{1}{p} - \frac{n}{N},$$

Corollary 1.18 *If $\frac{1}{p} - \frac{n}{N} = 0,$ then $W^{n,p}\left(\mathbb{R}^N\right) \hookrightarrow L^q\left(\mathbb{R}^N\right), \ \forall q \in [p, +\infty[\,,$*

if $\frac{1}{p} - \frac{n}{N} < 0,$ then $W^{n,p}\left(\mathbb{R}^N\right) \hookrightarrow L^\infty\left(\mathbb{R}^N\right),$

with continuous embedding.

1.4.1 Some integral inequalities

We will give here some important integral inequalities. These inequalities play an important role in applied mathematics and also, they are very useful in our next chapters.

Theorem 1.19 *[30] Let q and q' such that $1 < q,\ q' < \infty$ and $\frac{1}{q} + \frac{1}{q'} = 1$. If $f \in L^q$ and $g \in L^{q'}$, then*

$$fg \in L^1(\Omega) \ \text{ and } \int |fg|\, dx \leq \left(\int |f|^q\, dx\right)^{\frac{1}{q}} \left(\int |g|^{q'}\, dx\right)^{\frac{1}{q'}}.$$

Lemma 1.20 *[30] Let $0 \leq m \leq 1$. Then*

$$\|u\|_{L^r(\Omega)} \leq \|u\|_{L^t(\Omega)}^m \|u\|_{L^q(\Omega)}^{1-m},$$

valid for $u \in L^q(\Omega)$ with $1 \leq t \leq r \leq q,\ \frac{1}{r} = \frac{m}{t} + \frac{1-m}{q}$

Lemma 1.21 (Brézis-Lieb Lemma) *[16]Let (u_n) be a sequence in $W^{1,p}(\Omega)$, if (u_n) is bounded in $W^{1,p}(\Omega)$ and $u_n \to u$ a. e. in Ω, then*

$$\lim_{n\to\infty} \left(\|u_n\|^p - \|u_n - u\|^p\right) = \|u\|^p$$

Theorem 1.22 *[17] (Caffarelli-Kohn-Nirenberg inequality)*

Let $\Omega \subseteq \mathbb{R}^N$, $1 < p < N$, $0 \leq \alpha < (N-p)/p$, $\alpha \leq \beta < \alpha + 1$

$$\left(\int_{\Omega} \frac{|u|^{p^*}}{|x|^{p^*\beta}} dx \right)^{1/p^*} \leq C_{\alpha,\beta} \left(\int_{\Omega} \frac{|\nabla u|^p}{|x|^{p\alpha}} dx \right)^{1/p} \text{ for all } u \in C_0^{\infty}(\Omega), \tag{1.3}$$

where $\Omega \subseteq \mathbb{R}^N$, *for some positive constant* $C_{\alpha,\beta}$.

If $\beta = \alpha + 1$ *in* (1.3), *then* $p^* = p$, $C_{\alpha,\beta} = \dfrac{1}{\mu} = \left[\dfrac{p}{N - (\alpha+1)p} \right]^p$ *and we have the following weighted Hardy inequality*

$$\int_{\Omega} \frac{|u|^p}{|x|^{p(\alpha+1)}} dx \leq \frac{1}{\mu} \int_{\Omega} \frac{|\nabla u|^p}{|x|^{p\alpha}} dx, \text{ for all } u \in C_0^{\infty}(\Omega).$$

Chapter 2

Nonlinear elliptic equations with critical Caffarelli-Kohn-Nirenberg exponent in bounded domain

2.1 Introduction

In this chapter we are interested to study the existence of solution to the nonhomogeneous problem

$$\begin{cases} -div(\frac{|\nabla u|^{p-2}}{|x|^{p\alpha}}\nabla u) - \mu\frac{|u|^{p-2}}{|x|^{p(\alpha+1)}}u = \frac{|u|^{p^*-2}}{|x|^{p^*\beta}}u + f(x) \text{ in } \Omega, \\ u = 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ containing 0 in its interior,

$$1 < p < N, 0 \leq \alpha < (N-p)/p, \alpha \leq \beta < \alpha+1, -\infty < \mu < \overline{\mu} := [(N-(\alpha+1)p)/p]^p,$$

λ is a parameter, $p^* = pN/[N-p(1+\alpha-\beta)]$ is the critical Caffarelli-Kohn-Nirenberg exponent, $f \in W^{*}/\{0\}$. Here, $W_{\alpha,\mu}^{1,p}(\Omega)$ denotes the completion of $C_{0}^{\infty}(\Omega)$ with respect to the norm $\|.\|_{\alpha,\mu}$ and W^{*} is the dual space of $W_{\alpha,\mu}^{1,p}(\Omega)$.

To state our result, let set for $u \in W_{\alpha,\mu}^{1,p}\left(\mathbb{R}^{N}\right)$and $f \in W_{\alpha,\mu}^{*}$(the dual of $W_{\alpha,\mu}^{1,p}(\Omega)$)

$$\|u\|_{p^*}^{p^*} := \int_{\Omega} \frac{|u|^{p^*}}{|x|^{p^*\beta}} dx,$$

To start this section, we need to introduce the following notation:

$$I_f(u) := \int_{\Omega} fu \; dx,$$

$$\gamma_f := \inf_{\|u\|^{p^*}=1} \left\{ (p^* - p) \left[\frac{1}{p^* - 1} \|u\|_{\alpha,\mu}^{p} \right]^{\frac{p^*-1}{p^*-p}} - I_f(u) \right\}.$$

We define for $0 \leq \mu < \overline{\mu}$

$$S_\mu := \inf_{u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|^p}{\left(\int_{\mathbb{R}^N} |u|^{p^*} dx \right)^{p/p^*}}$$

and

$$S_0 := \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\|u\|_0^p}{\left(\int_{\mathbb{R}^N} |u|^{p^*} dx \right)^{p/p^*}}$$

where $W^{1,p}(\Omega) = W_{0,0}^{1,p}(\Omega)$

From [36], S_μ is independent of any $\Omega \subset \mathbb{R}^N$ in the sense that $S_\mu(\Omega) = S_\mu\left(\mathbb{R}^N\right) = S_\mu$. In addition, the constant S_μ is achieved by a family of functions

$$V_\varepsilon(x) := \varepsilon^{(p-N)/p} \tilde{u}_{p,\mu}\left(\frac{x}{\varepsilon}\right), \varepsilon > 0,$$

where $\tilde{u}_{p,\mu}(x) = \tilde{u}_{p,\mu}(|x|)$ is the unique radial solution for the problem

$$\begin{cases} -div\left(\frac{|\nabla u|^{p-2}}{|x|^{p\alpha}} \nabla u\right) - \mu \frac{|u|^{p-1} u}{|x|^{p(\alpha+1)}} = \frac{|u|^{p^*-2}}{|x|^{p^*\beta}} u & \text{in } \mathbb{R}^N \setminus \{0\} \\ u \longrightarrow 0 & \text{as } |x| \longrightarrow \infty. \end{cases}$$

In the other hand, from [30] S_0 is independent of any $\Omega \subset \mathbb{R}^N$ and it is achieved by a family of functions

$$U_\varepsilon(x) := \left[\varepsilon\,(N)\left(\frac{N-p}{p-1}\right)^{p-1}\right]^{\frac{N-p}{p(p)}} \left(\varepsilon + |x|^{\frac{p}{p-1}}\right)^{\frac{p-N}{p}}, \ \varepsilon > 0,$$

Moreover the functions U_ε solve the equation

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right) = \dfrac{|u|^{p^*-2}}{|x|^{p^*\beta}}u & \text{in } \mathbb{R}^N \backslash \{0\} \\ u \longrightarrow 0 & \text{as } |x| \longrightarrow \infty. \end{cases}$$

and define

$$D := \left\{g \in W^*,\ g \neq 0;\ \gamma_g > 0\right\}.$$

Note that $D \neq \varnothing$ and if $f \in L^p(\Omega)$ then

$$\int_\Omega |f|^p\,dx < (p^* - p)^p \left[\frac{1}{(p^*-1)}\right]^{\frac{p(p^*-1)}{p^*-p}} S_\mu^{p^*/(p^*-p)},$$

which implies that $f \in D$.

Set $\delta > 0$ small enough such that $B\,(0,\delta) \subset \Omega$, $\varphi \in C_0^\infty(\Omega)$ such that for

$$0 \leq \varphi\,(x) \leq 1, \varphi\,(x) = \begin{cases} 0 & \text{if } |x| \geq 2\delta \\ 1 & \text{if } |x| \leq \delta \end{cases};\ \text{and } |\nabla\varphi\,(x)| \leq C.$$

Put $u_\varepsilon = \varphi\,(x)\,U_\varepsilon(x)$.

By [30] we have the following estimates.

Lemma 2.1 *Assume that* $2 \leq p < N$ *and* $\varepsilon > 0$ *small enough. By taking*

$$v_\varepsilon = \frac{u_\varepsilon}{\|u_\varepsilon\|_{p^*}},$$

so that $\|u_\varepsilon\|_{p^*}^{p^*}=1$, *we have the following estimates:*

$$(1)\quad \|v_\varepsilon\|_0^p = S_0 + O\left(\varepsilon^{\frac{N-p}{p}}\right),$$

$$(2)\quad \int_\Omega |\nabla v_\varepsilon|^\alpha\, dx = O\left(\varepsilon^{\frac{\alpha(N-p)}{p^2}}\right) \text{ for } \alpha = 1...p-1,$$

$$(3)\quad \int_\Omega \frac{v_\varepsilon^{p^*-1}}{|x|^{p^*\beta}}\, dx = O\left(\varepsilon^{\frac{(p-1)(N-p)}{p^2}}\right),$$

$$(4)\quad \int_\Omega \frac{v_\varepsilon}{|x|^{p^*\beta}}\, dx = O\left(\varepsilon^{\frac{N-p}{p^2}}\right).$$

2.2 Nehari manifold

First we give some preliminaries about the so called Nehari manifold.

Since $f \in W_{\alpha,\mu}^{*}(\Omega)$ then the Euler-Lagrange functional I_1 associated to the problem (2.1) is given by

$$I_1(u) = \frac{1}{p}\|u\|_{\alpha,\mu}^p - \frac{1}{p^*}\|u\|_{p^*}^{p^*} - I_f(u) \quad \text{for all } u \in W_{\alpha,\mu}^{1,p}(\Omega),$$

it's clear that $I_1 \in C^1(W_{\alpha,\mu}^{1,p}(\Omega), \mathbb{R})$ and satisfies

$$\langle I'_1(u), v\rangle = (\int_\Omega \frac{|\nabla u|^{p-2}}{|x|^{p\alpha}}\nabla u \nabla v - \mu\frac{|u|^{p-2}}{|x|^{p(\alpha+1)}}uv - \frac{|u|^{p^*-2}}{|x|^{p^*\beta}}uv - fv)dx,$$

for all $u,\ v \in W_{\alpha,\mu}^{1,p}(\Omega)$.

Hence, weak solution of (2.1) are critical points of the functional I_1.

We denote the Nehari manifold by

$$\mathcal{N} = \left\{u \in W_{\alpha,\mu}^{1,p}(\Omega)\ /\{0\}, \langle I'_1(u), u\rangle = 0\right\}.$$

It is easy to see that $u \in \mathcal{N}$ if and only if

$$J(u) = \|u\|_{\alpha,\mu}^p - \|u\|_{p^*}^{p^*} - I_f(u) = 0.$$

Lemma 2.2 *The function I_1 is coercive and bounded from below in $\mathcal{N}$.*

Proof. Let $u \in \mathcal{N}$, by Holder and Young inequalities we have

$$\begin{aligned} I_1(u) &= \frac{1}{p}\|u\|_{\alpha,\mu}^p - \frac{1}{p^*}\|u\|_{p^*}^{p^*} - I_f(u) \\ &\geq \frac{1}{p}\|u\|_{\alpha,\mu}^p - \frac{1}{p^*}\|u\|_{p^*}^{p^*} - \|u\|_{\alpha,\mu}^p + \|u\|_{p^*}^{p^*} \\ &\geq -\left(\frac{p-1}{p}\right)\|u\|_{\alpha,\mu}^p + \left(\frac{p^*-1}{p^*}\right)S_\mu^{p^*/p}\|u\|_{p^*}^{p^*} \end{aligned}$$

Let $\rho = \|u\|_{\alpha,\mu}^p$ and

$$h(\rho) = -\left(\frac{p-1}{p}\right)\rho^p + \left(\frac{p^*-1}{p^*}\right)S_\mu^{p^*/p}\rho^{p^*}.$$

Direct calculations show that h is convex and achieves its minimum at

$$\rho_0 = \left[\frac{p-1}{p^*-1}S_\mu^{p^*/p}\right]^{\frac{1}{p^*-p}}$$

So

$$I_1(u) \geq h(\rho_0) = -\frac{(p-1)(p^*-p)}{pp^*}[\frac{p-1}{p^*-1}S_\mu^{p^*/p}]^{\frac{p}{p^*-p}}$$

Then $I_1$is coercive and bounded from below in $\mathcal{N}$. ■

The Nehari manifold $\mathcal{N}$ is closely linked to the behavior of the application

$$\Phi_u(t) : t \longrightarrow I_1(tu),$$

which for $t > 0$ is defined by

$$\Phi_u(t) = \frac{t^p}{p}\|u\|_{\alpha,\mu}^p - \frac{t^{p^*}}{p^*}\|u\|_{p^*}^{p^*} - tI_f(u).$$

Lemma 2.3 *Let $u \in W_{\alpha,\mu}^{1,p}(\Omega)$, then $tu \in \mathcal{N}$ if and only if $\Phi_u'(t) = 0$.*

Proof. We have

$$\begin{aligned}\Phi_u'(t) &= \langle I\,'_1\,(tu)\,,u\,\rangle \\ &= \frac{1}{t}\langle I\,'_1\,(tu)\,,tu\,\rangle.\end{aligned}$$

Then the conclusion holds ■

The elements in $\mathcal{N}$ correspond to stationary points of the maps Φ_u.

We note that

$$\Phi_u'(t) = t^{p-1}\left\|u\right\|_{\alpha,\mu}^{p} - t^{\,p^*-1}\left\|u\right\|_{p^*}^{p^*} - I_f\left(u\right).$$

and

$$\Phi"_u(t) = (p-1)\,t^{p-2}\left\|u\right\|_{\alpha,\mu}^{p} - (p^*-1)\,t^{\,p^*-2}\left\|u\right\|_{p^*}^{p^*} \quad .$$

By Lemma 2.3 we have $u \in \mathcal{N}$ if and only if $\Phi_u'(1) = 0$.Hence

$$\Phi"_u(1) = (p-1)\left\|u\right\|_{\alpha,\mu}^{p} - (p^*-1)\left\|u\right\|_{p^*}^{p^*} \quad .$$

Then it is natural to split $\mathcal{N}$ into three subsets corresponding to local minima, local maxima, and point of inflexion, i.e,

$$\begin{aligned}\mathcal{N}^+ &= \left\{u \in \mathcal{N} : \Phi"_u(1) > 0\right\}, \\ \mathcal{N}^- &= \left\{u \in \mathcal{N} : \Phi"_u(1) < 0\right\},\end{aligned}$$

and

$$\mathcal{N}^0 = \left\{u \in \mathcal{N} : \Phi"_u(1) = 0\right\}.$$

First, we prove that $\Phi"_u(1) \neq 0$ for all $u \in \mathcal{N}/\left\{0\right\}$.

Lemma 2.4 *Assume that* $f \in D$. *Then* $\mathcal{N}^0 = \varnothing$.

Proof. Suppose that $\mathcal{N}^0 \neq \varnothing$. For $u \in \mathcal{N}^0$, we have

$$\begin{aligned}(p-1)\|u\|_{\alpha,\mu}^{p} &= (p^*-1)\|u\|_{p^*}^{p^*} \\ (p-1)\, I_f(u) &= (p^*-p)\|u\|_{p^*}^{p^*}\end{aligned}$$

and

$$(p^*-1)\, I_f(u) = (p^*-p)\|u\|_{\alpha,\mu}^{p}.$$

Using the definition of S_μ we get

$$\begin{aligned}\|u\|_{p^*}^{p^*} &= (p-1)\|u\|_{\alpha,\mu}^{p} \; / \, (p^*-1) \\ &\geq \left[\left[\frac{(p-1)}{(p^*-1)} S_\mu\right]^{p^*/(p^*-p)}\right].\end{aligned}$$

Thus

$$\frac{\|u\|_{\alpha,\mu}^{p}}{\|u\|_{p^*}^{p^*}} = \frac{p^*-1}{p-1}.$$

Therefore,

$$\begin{aligned}0 &= \frac{p^*-p}{p^*-1}\|u\|_{\alpha,\mu}^{p} - I_f(u) \\ &= \|u\|_{p^*}^{p^*}\left[\frac{p^*-p}{p^*-1}\frac{\|u\|_{\alpha,\mu}^{p}}{\|u\|_{p^*}^{p^*}} - \frac{I_f(u)}{\|u\|_{p^*}^{p^*}}\right] \\ &\geq \|u\|_{p^*}^{p^*}\left[(p^*-p)\left[\frac{\|u\|_{\alpha,\mu}^{p}}{(p^*-1)\|u\|_{p^*}^{p^*}}\right]^{(p^*-1)\;/(p^*-p)} - \frac{I_f(u)}{\|u\|_{p^*}^{p^*}}\right] \\ &> 0.\end{aligned}$$

Which is impossible. ■

Define for all $u \in W_{\alpha,\mu}^{1,p}(\Omega) / \{0\}$

$$t_u^{\max} := \left[\|u\|_{\alpha,\mu}^{p}(p-1)/(p^*-1)\|u\|_{p^*}^{p^*}\right]^{\frac{1}{p^*-p}}.$$

Lemma 2.5 *Assume that $f \in D$. Then for any $u \in W^{1,p}_{\alpha,\mu}/\{0\}$, there exists a unique positive value t_u^+ such that*

$$t_u^+ > t_u^{\max},\ t_u^+ u \in \mathcal{N}^- \text{ and } I_1\left(t_u^+ u\right) = \max_{t \geq t_u^{\max}} I_1(tu).$$

Moreover, if $I_f(u) > 0$, then there exists a unique positive value t_u^- such that

$$0 < t_u^- < t_u^{\max},\ t_u^- u \in \mathcal{N}^+ \text{ and } I_1\left(t_u^- u\right) = \inf_{0 \leq t \leq t_u^{\max}} I_1(tu).$$

Proof. Set

$$\Psi_u(t) = t^{p-1} \|u\|_{\alpha,\mu}^p - t^{p^*-1} \|u\|_{p^*}^{p^*}$$

for $u \in W^{1,p}_{\alpha,\mu}/\{0\}$, then

$$\Phi_u'(t) = \Psi_u(t) - I_f(u)$$

Easy computations show that Ψ_u is concave and achieves its maximum at $t_u^{\max}$, also

$$\Psi_u(t_u^{\max}) = (p^* - p)\left(\frac{\|u\|_{\alpha,\mu}^p}{p^*-1}\right)^{(p^*-1)/(p^*-p)} \left(\frac{p-1}{\|u\|_{p^*}^{p^*}}\right)^{(p-1)/(p^*-p)}.$$

Then we can get easily the conclusion of our Lemma. ■

By the previous lemma we know that $\mathcal{N}^+$ and $\mathcal{N}^-$ are not empty, so we can define

$$\theta^+ := \inf_{u \in \mathcal{N}^+} I_1(u) \text{ and } \theta^- := \inf_{u \in \mathcal{N}^-} I_1(u).$$

Lemma 2.6 *Assume that $f \in D$. Then for any $u \in \mathcal{N}^{\pm}$, there exist $\varepsilon > 0$ and a differentiable function $\zeta = \zeta(v)$, $v \in W^{1,p}_{\alpha,\mu}(\Omega)$, $\|v\|_{\alpha,\mu} < \varepsilon$, such that $\xi(0) = 1$, $\zeta(v)(u - v) \in \mathcal{N}^{\pm}$ and*

$$(\zeta'(0), v) = \frac{\int_\Omega \left[p\left(\frac{|\nabla u|^{p-2} \nabla u\ \nabla v}{|x|^{p\alpha}} - \mu \frac{u^{p-2} uv}{|x|^{p(\alpha+1)}}\right) - p^* \frac{|u|^{p^*-2} uv}{|x|^{p^*\beta}} - fv\right] dx}{(p-1)\|u\|_{\alpha,\mu}^p - (p^*-1)\|u\|_{p^*}^{p^*}}.$$

Proof. Define $\varphi : \mathbb{R} \times W^{1,p}_{\alpha,\mu}(\Omega) \longrightarrow \mathbb{R}$ such that

$$\varphi(\zeta, v) = \zeta^{p-1} \|u - v\|^p_{\alpha,\mu} - \zeta^{p^*-1} \|u - v\|^{p^*}_{p^*} - \int_\Omega f(u - v)\ dx.$$

As $u \in \mathcal{N}$ and $\mathcal{N}^0 = \varnothing$, we have

$$\varphi(1, 0) = 0,\ \frac{\partial \varphi}{\partial \zeta}(1, 0) = (p-1)\|u\|^p_{\alpha,\mu} - (p^* - 1)\|u\|^{p^*}_{p^*} \neq 0.$$

Then by the implicit function Theorem, we get our result. ■

Lemma 2.7 *Let $f \in D$, then there exist $\theta_0^+ < 0$ and $\theta_0^- > 0$ such that $\theta^+ \leq \theta_0^+$ and $\theta^- \geqslant \theta_0^-$.*

Proof. Let $v \in W^{1,p}_{\alpha,\mu}(\Omega)$ be the unique solution of the following problem

$$\begin{cases} -div(\dfrac{|\nabla u|^{p-2} \nabla u}{|x|^{p\alpha}}) - \mu \dfrac{|u|^{p-2} u}{|x|^{p(\alpha+1)}} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, as $f \not\equiv 0$ we have $I_f(v) = \|v\|^p_{\alpha,\mu} > 0$ and $\|v\|^p_{\alpha,\mu} = \|f\|^p_-$ where $\|.\|_- = \|.\|_{W^*_\mu}$. Moreover from Lemma 2.5, there exists $t_v^- > 0$ such that $t_v^- v \in \mathcal{N}^+$. This implies that

$$\begin{aligned} \theta^+ &\leq I_1\left(t_v^- v\right) \\ &= \frac{(1-p)\left(t_v^-\right)^p}{p} \|v\|^p_{\alpha,\mu} + \frac{1-p^*}{p^*}\left(t_v^-\right)^{p^*} \|v\|^{p^*}_{p^*} \\ &\leq \frac{(1-p)\left(t_v^-\right)^p}{p} \|v\|^p_{\alpha,\mu} \\ &\leq \frac{(1-p)}{p}\left(t_v^-\right)^p \|f\|^p_- . \end{aligned}$$

We deduce that $\theta^+ \leq \theta_0 < 0$ where $\theta_0 = \frac{(1-p)}{p}\left(t_v^-\right)^p \|f\|^p_-$.

On the other hand , there exists. $t_v^+ > 0$ such that $t_v^+ v \in \mathcal{N}^-$which yields

$$\begin{aligned}
\theta^{-} &\geq I_1\left(t_v^{+} v\right) \\
&= \left(t_v^{+}\right)^p \|v\|_\mu^p - \frac{p^*-1}{p-1}\left(t_v^{+}\right)^{p^*} \|v\|_{p^*}^{p^*} \\
&\geq \left(t_v^{+}\right)^p \ [\frac{(p-1)}{(p^*-1)} S_\mu]^{(p^*/p^*-p)}.
\end{aligned}$$

Therefore, $\theta^{-} \geq \theta_0^{-} > 0$ where

$$\theta_0^{-} = \left(t_v^{+}\right)^p [\frac{(p-1)}{(p^*-1)} S_\mu]^{(p^*/p^*-p)}.$$

The proof is complete. ■

Lemma 2.8 *Assume that $f \in D$. Then, there exists a minimizing sequence (u_n) such that*

$$I_1\left(u_n\right) \longrightarrow \theta^{+} \text{ and } I_1'\left(u_n\right) \rightarrow 0 \text{ in } W^*\left(\Omega\right).$$

Proof. It is easy to prove that $I_1$is bounded in $\mathcal{N}^{+}$, then by applying Ekeland's variational principle, there exists a minimizing sequence $(u_n) \subset \mathcal{N}^{+}$ satisfying

$$\theta^{+} \leq I_1\left(u_n\right) \leq \theta^{+} + \frac{1}{n} \text{ and } I_1\left(u\right) \geq I_1\left(u_n\right) - \frac{1}{n}\left\|u - u_n\right\|_{\alpha,\mu} \text{ for all } u \in \mathcal{N}^{+}.$$

From the preceding lemma we have $\theta^{+} \leq \theta_0$. So that

$$\left(\frac{1}{p} - \frac{1}{p^*}\right)\|u\|_{\alpha,\mu}^p < \left(\frac{1}{p} - \frac{1}{p^*}\right) \frac{(1-p)}{p}\left(t_v^{-}\right)^p \|f\|_{-}^p + \frac{p^*-1}{p^*}\|f\|_{-}^{p-1}\left\|u_n\right\|_{\alpha,\mu},$$

and

$$\frac{p^*(p-1)}{p}\left(t_v^{-}\right)^p \|f\|_{-}^p \leq I_f\left(u_n\right) \leq \|f\|_{-}^{p-1}\left\|u_n\right\|_{\alpha,\mu},$$

for n large, this implies that $C_1 \leq \left\|u_n\right\|_{\alpha,\mu} \leq C_2$ with

$$C_1 = \frac{p^*(p-1)}{p\left(p^*-1\right)}\left(t_v^{-}\right)^p \|f\|$$

and

$$C_2 = \frac{p\,(p^* - 1)}{(p-1)(p^* - p)} \left\| f \right\|_- .$$

Now, we show that $I_1'(u_n) \to 0$ in $W_{\alpha,\mu}^*$, For that, fix n such that $\left\| I_1'(u_n) \right\|_- \neq 0$.

Then by Lemma 2.6 there exist $\varepsilon > 0$ and a function $\zeta_n : B_\varepsilon \longrightarrow \mathbb{R}$ such that

$$w_n = \zeta_n(v_n)(u_n - v_n) \in \mathcal{N}^+$$

with

$$v_n = \delta \frac{I_1'(u_n)}{\left\| I_1'(u_n) \right\|_-} \text{ and } 0 < \delta < \varepsilon.$$

Let $A_n = \left\| w_n - u_n \right\|_{\alpha,\mu}$, by the Taylor expansion of I_1, we obtain

$$\begin{aligned}
-\frac{1}{n} A_n &\leq I_1(w_n) - I_1(u_n) \\
&\leq \left\langle I_1'(u_n),\ w_n - u_n \right\rangle + \circ(A_n) \\
&= (\zeta_n(v_n) - 1) \left\langle I_1'(u_n),\ u_n \right\rangle - \delta \zeta_n(v_n) \left\langle I_1'(u_n),\ \frac{I_1'(u_n)}{\left\| I_1'(u_n) \right\|_-} \right\rangle + \\
&\quad \circ(A_n).
\end{aligned}$$

Then

$$\zeta_n(v_n) \left\| I_1'(u_n) \right\|_- \leq \frac{\zeta_n(v_n) - 1}{\delta} \left\langle I_1'(u_n), u_n \right\rangle + \frac{A_n}{n\delta} + \frac{\circ(A_n)}{\delta}. \tag{2.2}$$

We have

$$\lim_{\delta \to 0} \zeta_n(v_n) = 1, \lim_{\delta \to 0} \frac{\left| \zeta_n(v_n) - 1 \right|}{\delta} = \lim_{\delta \to 0} \frac{\left| \zeta_n(v_n) - \zeta_n(0) \right|}{\delta} \leq \left\| \zeta_n'(0) \right\|_-,$$

and

$$\begin{aligned}
\lim_{\delta \to 0} \frac{A_n}{n\delta} &= \lim_{\delta \to 0} \frac{1}{n\delta} \left\| (\zeta_n(v_n) - 1) u_n - \zeta_n(v_n) v_n \right\|_\mu \\
&\leq \frac{1}{n} \left(\left\| \zeta_n'(0) \right\|_- \left\| u_n \right\|_{\alpha,\mu} + 1 \right).
\end{aligned}$$

Taking $\delta \to 0$ in (2.2) and since (u_n) is a bounded sequence we get

$$\|I_1'(u_n)\|_{\alpha,\mu} \leq \frac{C_3}{n}\left(\|\zeta_n'(0)\|_- + 1\right),$$

for a suitable constant $C_3 > 0$. Now, we must show that $\|\zeta_n'(0)\|_-$ is uniformly bounded in n.

From the boundedness of (u_n) we have by Lemma 2.6

$$\langle \zeta_n'(0), v\rangle \leq \frac{C_4 \|v\|_{\alpha,\mu}}{\left|(p-1)\|u_n\|_{\alpha,\mu}^p - (p^*-1)\|u_n\|_{p^*}^{p^*}\right|},$$

for all $v \in W_{\alpha,\mu}^{1,p}(\Omega)$ and some constant $C_4 > 0$. We only need to show that for any sequence $(u_n) \subset \mathcal{N}^+$

$$\left|(p-1)\|u_n\|_{\alpha,\mu}^p - (p^*-1)\|u_n\|_{p^*}^{p^*}\right| > C_5,$$

for some constant $C_5 > 0$.

Assume by contradiction that there exists $(u_n) \subset \mathcal{N}^+$ such that

$$\lim_{n\to\infty}\left[(p-1)\|u_n\|_{\alpha,\mu}^p - (p^*-1)\|u_n\|_{p^*}^{p^*}\right] = 0.$$

Then as $\|u_n\|_\mu \geq C_1 > 0$, we get

$$\frac{\|u_n\|_{p^*}^{p^*}}{\|u_n\|_{\alpha,\mu}^p} = \frac{(p-1)}{p^*-1} + \circ_n(1) \text{ and } (p-1)I_f(u_n) = (p^*-p)\|u_n\|_{p^*}^{p^*} + \circ_n(1),$$

where $\circ_n(1) \to 0$ as $n \to \infty$. But this is impossible since, as in the proof of Lemma 2.4 we have

$$\begin{aligned}
\circ_n(1) &= (p-1)\|u_n\|_{\alpha,\mu}^p - (p^*-1)\|u_n\|_{p^*}^{p^*} \\
&= (p^*-p)\|u_n\|_{p^*}^{p^*} - (p-1)I_f(u_n) \\
&= \|u_n\|_{p^*}\left[(p^*-p)\left(\frac{\|u_n\|_{\alpha,\mu}^p}{(p^*-1)\|u_n\|_{p^*}^{p^*}}\right)^{(p^*-1)/(p^*-p)} - \frac{I_f(u_n)}{\|u_n\|_{p^*}}\right] \\
&> 0.
\end{aligned}$$

At this point we conclude that $I_1'(u_n) \to 0$ in $W_\mu^*(\Omega)$. ■

2.3 Existence of ground state solution

By previous results about Nehari manifold and precedent preliminary results we prove the existence of a ground state solution of problem (2.1).

Theorem 2.9 *Let* $-\infty < \alpha < (N-p)/p$, $\alpha \le \beta < \alpha + 1$ *and* $-\infty \le \mu < \bar{\mu}$. *Assume that* $f \in D$, *then problem* (2.1) *has a ground state solution* u.

Proof. First, we prove that I_1 can achieve a local minimum on $\mathcal{N}^+$.

According to the proof of lemma 2.8, there exists a minimizing sequence $(u_n) \subset \mathcal{N}^+$ such that $C_1 \le \|u_n\|_{\alpha,\mu} \le C_2$. Up to a subsequence if necessary, we have

$$\begin{aligned} u_n &\rightharpoonup u_1 \text{ in } W_{\alpha,\mu}^{1,p}(\Omega) \\ u_n &\rightharpoonup u_1 \text{ in } L^{p^*}(\Omega, |x|^{-p^*\beta}) \\ u_n &\to u_1 \text{ a.e in } \Omega. \end{aligned}$$

For some $u_1 \in W_{\alpha,\mu}^{1,p}(\Omega)$. As $\theta^+ < 0$ then $u_1 \not\equiv 0$. Suppose otherwise, so $\|u_1\|_{\alpha,\mu} < \underline{\lim}_{n\to\infty} \|u_n\|_{\alpha,\mu}$, which implies that

$$\begin{aligned} \theta^+ &\le I_1(u_1) \\ &= \left(\frac{1}{p} - \frac{1}{p^*}\right) \|u_1\|_{\alpha,\mu}^p - \left(1 - \frac{1}{p^*}\right) I_f(u_1) \\ &< \underline{\lim}_{n\to\to\infty} \left(\frac{p^*-p}{p^*p} \|u_n\|_{\alpha,\mu}^p - \frac{p^*-1}{p^*} I_f(u_n)\right) \\ &= \theta^+ \end{aligned}$$

This is a contradiction, which leads to conclude that $u_n \to u_1$ in $W^{1,p}_{\alpha,\mu}(\Omega)$ and $I_1(u_1) = \theta^+$.

Moreover, we have $u_1 \in \mathcal{N}^+$. In fact, if $u_1 \in \mathcal{N}^-$ then by Lemma 2.5, $t^+_{u_1} = 1$ and there exists unique $t^-_{u_1} > 0$ such that $t^-_{u_1} u_1 \in \mathcal{N}^+$.

Since

$$\left.\frac{dI_1(tu_1)}{dt}\right|_{t=t^-_{u_1}} = 0, \quad \left.\frac{d^2 I_1(tu_1)}{dt}\right|_{t=t^-_{u_1}} > 0,$$

there exists $t^-_{u_1} < t^0_{u_1} < t^+_{u_1}$ such that $I_1\left(t^-_{u_1} u_1\right) < I_1\left(t^0_{u_1} u_1\right) \leq I_1\left(t^+_{u_1} u_1\right) = I_1(u_1)$, which is a contradiction.

Hence $u_1 \in \mathcal{N}^+$ and

$$\theta^+ = \inf_{u \in \mathcal{N}^+} I_1\ (u) = \inf_{u \in \mathcal{N}} I_1\ (u).$$

By the Lagrange multiplier rule, there exists $\lambda \in \mathbb{R}$ such that

$$\Phi'_{u_1}(1) = I\ '_1(u_1) = \lambda \Phi\text{''}(1),$$

with implies that

$$0 = \langle I\ '_1(u_1), u_1 \rangle = \lambda \langle J\ '(u_1), u_1 \rangle,$$

we have $\langle J\ '(u_1), u_1 \rangle \neq 0$, so $\lambda = 0$ and $I\ '_1(u_1) = 0$.

Thus u_1 is a ground state solution of problem (2.1) . ■

2.4 Existence of the second solution

In the following, we prove that problem (2.1) has a second solution u_2.

Theorem 2.10 *Suppose that* $2 \leq p < N$, $\mu = 0$, $\alpha = 0$, $p^* = pN/(N - p\beta)$ *and* $f(x) \geq a_0 > 0$ *in a small neighborhood of* 0 *and satisfies* $\gamma_f > 0$. *Then, problem* (2.1) *has a second solution.*

Lemma 2.11 *Let* $1 < p < N$, $\mu = 0, \alpha = 0$ *and* $f \not\equiv 0$ *satisfies* $\gamma_f > 0$. *Then* $I_1(u)$ *verifies the Palais-Smale condition at level* c *for all* $c < \theta^+ + \frac{1}{N}(S_0)^{\frac{N}{p}}$.

Proof. Assume that (u_n) is a sequence in $W_0^{1,p}(\Omega)$ satisfying as $n \to \infty$

$$I_1(u_n) \to c < \frac{1}{N}(S_0)^{\frac{N}{p}} \text{ and } I_1'(u_n) \to 0 \text{ in } W_0^*(\Omega). \tag{2.3}$$

By Lemma 2.8, we know that (u_n) is bounded in $W_0^{1,p}(\Omega)$. Then, there exist a subsequence (still denoted by (u_n)) and u_2 in $W_0^{1,p}(\Omega)$ such that $u_2 \not\equiv 0$ and

$$\begin{aligned} u_n &\rightharpoonup u_2 \text{ in } W_0^{1,p}(\Omega), \\ u_n &\rightharpoonup u_2 \text{ in } L_{p^*}\left(\Omega, |x|^{-p^*\beta}\right), \\ u_n &\to u_2 \text{ a.e.in } \Omega. \end{aligned}$$

Denote $v_n = u_n - u_2$, then

$$\begin{aligned} v_n &\rightharpoonup 0 \text{ in } W_0^{1,p}(\Omega), \\ v_n &\rightharpoonup 0 \text{ in } L_{p^*}\left(\Omega, |x|^{-p^*\beta}\right), \\ v_n &\to 0 \; a.e.in \; \Omega. \end{aligned}$$

By the Brézis - Lieb Lemma [16] we have

$$\|u_n\|_0^p = \|v_n\|_0^p + \|u_2\|_0^p + \circ_n(1),$$

and

$$\|u_n\|_{p_*}^{p_*} = \|v_n\|_{p_*}^{p_*} + \|u_2\|_{p_*}^{p_*} + \circ_n(1).$$

Then, from (2.3) we deduce that

$$c + \circ_n(1) = I_1(u_2) + \frac{1}{p}\|v_n\|_0^p - \frac{1}{p^*}\|v_n\|_{p^*}^{p^*}$$

and

$$\|v_n\|_0^p - \|v_n\|_{p^*}^{p^*} = \circ_n(1).$$

Using the fact that $v_n \rightharpoonup 0$ in $W_0^{1,p}(\Omega)$, we can assume that

$$\|v_n\|_0^p \to l \text{ and } \|v_n\|_{p^*}^{p^*} \to l \geq 0.$$

So, by the Sobolev-Hardy inequality, we get $l \geq S_0 l^{p/p^*}$.

Now, assume that $l \neq 0$, then

$$l \geq (S_0)^{p^*/(p^*-p)}$$

and we obtain

$$c = I_1(u_2) + \left(\frac{1}{p} - \frac{1}{p^*}\right) l \geq I_1(u_2) + \frac{1}{N}(S_0)^{\frac{N}{p}}.$$

As $I_1(u_2) \geq \theta^+$, we get a contradiction. So again $u_n \to u$ in $W_0^{1,p}(\Omega)$ strongly. ■

In order, to prove Theorem 2.10, we need the following key lemma.

Lemma 2.12 *Suppose that* $2 \leq p < N$, $\mu = 0, \alpha = 0$, $f(x) \geq a_0 > 0$ *in a small neighborhood of* 0 *and satisfies* $\gamma_f > 0$. *Then*

$$\theta^- < \theta^+ + \frac{1}{N}(S_0)^{\frac{N}{p}}.$$

Proof. *Set*

$$\mathcal{M}_1 = \{0\} \cup \left\{u \in W_0^{1,p}(\Omega) : \|u\|_0 < t^+_{u\|u\|_0^{-1}}\right\} \text{ and } \mathcal{M}_2 = \left\{u \in W_0^{1,p}(\Omega) : \|u\|_0 > t^+_{u\|u\|_0^{-1}}\right\}.$$

We have $W_0^{1,p}(\Omega)\backslash\mathcal{N}^- = \mathcal{M}_1 \cup \mathcal{M}_2$, $\mathcal{N}^+ \subset \mathcal{M}_1$, $u_1 \in \mathcal{M}_1$ *and* $u_1 + Tv_\varepsilon \in \mathcal{M}_2$ *for some real* $T > 0$. *Let*

$$\Gamma = \left\{h : [0,1] \to W_0^{1,p}(\Omega) \text{ continuous, } h(0) = u_1,\ h(1) = u_1 + Tv_\varepsilon\right\},$$

and

$$\tilde{h}(t) = u_1 + tTv_\varepsilon \quad \text{with } t \in [0,1].$$

It is obvious that $\tilde{h}$ *belongs to* Γ *and the range of any* $h \in \Gamma$ *intersects* $\mathcal{N}^-$. *Then*

$$\theta^- \le \inf_{h\in\Gamma} \max_{t\in[0,1]} I_1(h(t)).$$

Now, we show that

$$\sup_{t\ge 0} I_1(u_1 + tv_\varepsilon) < \theta^+ + \frac{1}{(N)}(S_0)^{\frac{N}{p}}.$$

To this purpose, we define $g(t) := I_1(u_1 + tv_\varepsilon)$, *then*

$$g(0) = I_1(u_1) < \theta^+ + \frac{1}{N}(S_0)^{\frac{N}{p}},$$

and by the continuity of g *there exists* $t_0 > 0$ *small enough such that*

$$g(t) < \theta^+ + \frac{1}{N}(S_0)^{\frac{N}{p}},$$

for all $t \in (0,\ t_0)$. *On the other hand, it is easy to see that* $g(t) \to -\infty$ *as* $t \to +\infty$, *that is, there exists* $t_1 > 0$ *large enough such that*

$$g(t) < \theta^+ + \frac{1}{N}(S_0)^{\frac{N}{p}},$$

for all $t \ge t_1$. *So we only need to show that*

$$\sup_{t_0\le t\le t_1} g(t) < \theta^+ + \frac{1}{N}(S_0)^{\frac{N}{p}}.$$

Let ε *be sufficiently small such that* $f(x) \geq a_0 > 0$ *in* $B(0,\varepsilon)$. *Then, we get from Lemma* 2.1

$$\begin{aligned}
\sup_{t_0\leq t\leq t_1} I_1(tv_\varepsilon) &\leq \sup_{t\geq 0}\left(\frac{1}{p}\|tv_\varepsilon\|_0^p - \frac{1}{p^*}\|tv_\varepsilon\|_{p^*}^{p^*}\right) - t_0\int_\Omega f v_\varepsilon dx \\
&\leq \sup_{t\geq 0}\left(\frac{1}{p}\|tv_\varepsilon\|_0^p - \frac{1}{p^*}\|tv_\varepsilon\|_{p^*}^{p^*}\right) - t_0 a_0\int_\Omega v_\varepsilon dx \\
&\leq \frac{1}{N}(S_0)^{\frac{N}{p}} + O\left(\varepsilon^{\frac{N-p}{p}}\right) - O(\varepsilon^{\frac{N-p}{p^2}}).
\end{aligned}$$

For the second one, we can assume that the first solution u_1 *is smooth and* $\nabla u_1 \in L_\infty(\Omega)$. *Thus we have*

$$\begin{aligned}
\sup_{t_0\leq t\leq t_1} g(t) &= \sup_{t_0\leq t\leq t_1} I_1(u_1+tv_\varepsilon) \\
&\leq I_1(u_1) + \sup_{t\geq 0} I_1(tv_\varepsilon) + C_1\int_\Omega\left(|\nabla u_1|^{p-1}|\nabla v_\varepsilon| + |\nabla v_\varepsilon|^{p-1}|\nabla u_1|\right)dx + \\
&\quad \int_\Omega\left(|u_1|^{p^*-1}v_\varepsilon + |v_\varepsilon|^{p^*-1}u_1\right)dx \\
&\leq \theta^+ + \frac{1}{N}(S_0)^{\frac{N}{p}} + O\left(\varepsilon^{\frac{N-p}{p}}\right) - O(\varepsilon^{\frac{N-p}{p^2}}) + O\left(\varepsilon^{\frac{N-p}{p^2}}\right) + O\left(\varepsilon^{\frac{(N-p)(p-1)}{p^2}}\right)
\end{aligned}$$

From

$$\frac{N-p}{p} > \frac{N-p}{p^2} > \frac{(N-p)(p-1)}{p^2}$$

we have

$$O\left(\varepsilon^{\frac{N-p}{p}}\right) - O(\varepsilon^{\frac{N-p}{p^2}}) + O\left(\varepsilon^{\frac{N-p}{p^2}}\right) + O\left(\varepsilon^{\frac{(N-p)(p-1)}{p^2}}\right) = O\left(\varepsilon^{\frac{(N-p)(p-1)}{p^2}}\right) + O(\varepsilon^{\frac{N-p}{p}}).$$

Since

$$\frac{(N-p)(p-1)}{p^2} + \frac{N-p}{p} > 0,$$

then

$$\sup_{t_0 \le t \le t_1} I_1 \left(u_1 + tv_\varepsilon\right) < \theta^+ + \frac{1}{N} \left(S_0\right)^{\frac{N}{p}},$$

for ε small enough.

The proof is now complete. ■

Chapter 3

Elliptic p-Kirchhoff type equations with critical Sobolev exponent in $\mathbb{R}^N$

3.1 Introduction

In this chapter we are concerned with the following regular $p-$Kirchhoff type problem in $\mathbb{R}^N$ with critical Sobolev exponent.

$$-\left(a\int_{\mathbb{R}^N}|\nabla u|^p\,dx+b\right)\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)=|u|^{p^*-2}u+\lambda f\left(x\right)\quad\text{in }\mathbb{R}^N \tag{3.1}$$

where $1<p<N$, $a,\ b\geq 0$, $a+b>0$, λ is a parameter, $p^*=pN/\left[N-p\right]$, $f\in W^*\backslash\left\{0\right\}$. Here, W^* is the dual space of $W^{1,p}(\mathbb{R}^N)$ and $W^{1,p}(\mathbb{R}^N)$ denotes the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \left(\int\limits_{\mathbb{R}^N} |\nabla u|^p \, dx \right)^{1/p}.$$

Note that if $a = \lambda = 0$, $b = 1$ and $1 < p < N$, (3.1) reduces to the following problem

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = |u|^{p^*-2}u, \quad \text{in } \mathbb{R}^N \tag{3.2}$$

Sciunzi in [47] provided that if u is a positive solution of (3.2) then $u(x) = v_{\varepsilon,x_0}(x)$ where

$$v_{\varepsilon,x_0}(x) := \left[\frac{\varepsilon^{\frac{1}{p-1}} N^{\frac{1}{p}} \left(\frac{N-p}{p-1}\right)^{\frac{p-1}{p}}}{\varepsilon^{\frac{p}{p-1}} + |x - x_0|^{\frac{p}{p-1}}} \right]^{\frac{N-p}{p}} \qquad \varepsilon > 0, \, x_0 \in \mathbb{R}^N \tag{3.3}$$

Consequently, u is a minimizer for

$$S := \inf_{u \in W^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|^p}{\left(\int_{\mathbb{R}^N} |u|^{p^*} \, dx \right)^{p/p^*}}$$

and satisfies

$$\|v_{\varepsilon,x_0}\|^p = \int_{\mathbb{R}^N} |v_{\varepsilon,x_0}|^{p^*} \, dx = S^{\frac{p^*}{p^*-p}}, \tag{3.4}$$

Definition 3.1 *We say that $u \in W^{1,p}(\mathbb{R}^N)\setminus\{0\}$ is a weak solution of equation* (3.1) *if*

$$(a\|u\|^p + b) \int\limits_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u dx - \int\limits_{\mathbb{R}^N} \left(|u|^{p^*-2}u + \lambda f(x)\right) v dx = 0$$

for any $v \in W^{1,p}(\mathbb{R}^N)$.

Next, we define the energy functional

$$I_2(u) = \frac{a}{2p}\|u\|^{2p} + \frac{b}{p}\|u\|^p - \frac{1}{p^*}\int\limits_{\mathbb{R}^N} |u|^{p^*} dx - \lambda \int\limits_{\mathbb{R}^N} f(x) u dx,$$

associated to problem (3.1), for all $u \in W^{1,p}(\mathbb{R}^N)$.

Notice that the functional I_2 is well defined in $W^{1,p}(\mathbb{R}^N)$, belongs to $C^1\left(W^{1,p}(\mathbb{R}^N),\ \mathbb{R}\right)$ and a critical point of I_2 is a weak solution of problem (3.1).

First, we make the following assumptions:

$(\mathcal{H}_0)$ $p^* > 2p$, $a \geq 0$, $b \geq 0$ and $a + b > 0$,

$(\mathcal{H}_1)$ $p^* = 2p$, $a > 0$ and $b > 0$,

$(\mathcal{H}_2)$ $p^* = 2p$, $0 < a < S^{-2}$ and $b > 0$.

When $\lambda > 0$, we have the following results.

3.2 Palais Smale condition

Lemma 3.2 *Suppose that $f \in W^* \backslash \{0\}$ and assume that $(\mathcal{H}_0)$ or $(\mathcal{H}_1)$ holds. Let $c \in \mathbb{R}$ and $(u_n) \subset W^{1,p}(\mathbb{R}^N)$ be a $(PS)_c$ sequence for I_2, then*

$$u_n \rightharpoonup u \text{ in } W^{1,p}(\mathbb{R}^N)$$

for some $u \in W^{1,p}(\mathbb{R}^N)$ with $I_2'(u) = 0$.

Proof. We have

$$I_2(u_n) \to c \text{ and } I_2'(u_n) \to 0,$$

that is

$$c + o_n(1) = I_2(u_n) \text{ and } o_n(1)\|v\| = \langle I_2'(u_n), v\rangle,$$

for any $v \in W^{1,p}(\mathbb{R}^N)$, where $o_n(1)$ denotes any quantity that tends to zero as $n \to \infty$.

Then as $n \to \infty$, it follows that

$$c + o_n(1) - \tfrac{1}{p^*} o_n(1) \|u_n\| = I_2(u_n) - \tfrac{1}{p^*} \langle I_2'(u_n), u_n \rangle$$
$$= a\tfrac{p^*-2p}{2pp^*} \|u_n\|^{2p} + b\tfrac{p^*-p}{pp^*} \|u_n\|^p - \lambda \tfrac{p^*-1}{p^*} \int_{\mathbb{R}^N} f(x) u_n dx,$$
$$\geq a\tfrac{p^*-2p}{2pp^*} \|u_n\|^{2p} + b\tfrac{p^*-p}{pp^*} \|u_n\|^p - \lambda \tfrac{p^*-1}{p^*} \|f\|_{W^*} \|u_n\|,$$

that is, (u_n) is bounded in $W^{1,p}(\mathbb{R}^N)$ if $(\mathcal{H}_0)$ or $(\mathcal{H}_1)$ holds. Up to a subsequence if necessary, there exists a function $u \in W^{1,p}(\mathbb{R}^N)$ such that

$$u_n \rightharpoonup u \text{ in } W^{1,p}(\mathbb{R}^N) \text{ and in } L^{p^*}\left(\mathbb{R}^N, |x|^{-p^*}\right), \; u_n \to u \text{ a. e. in } \mathbb{R}^N,$$

and

$$\int_{\mathbb{R}^N} f(x) u_n dx \to \int_{\mathbb{R}^N} f(x) u dx.$$

Then

$$\langle I_2'(u_n), v \rangle = 0 \text{ for all } v \in C_0^\infty\left(\mathbb{R}^N\right),$$

thus $I_2'(u) = 0$. This completes the proof. ■

Before giving the local Palais Smale condition, we need the following lemma which is a key step to obtain a solution with positive energy (Mountain Pass type solution).

Lemma 3.3 *Let $a, b \geq 0$, $a + b > 0$ and $\sigma \geq 1$. For $y \geq 0$ we consider the function $\Psi : \mathbb{R}^+ \to \mathbb{R}^*$, given by*

$$\Psi(y) = S^{-1} y^\sigma - aSy - b.$$

Then

(1) *If $\sigma = 1$, $0 \leq a < S^{-2}$ and $b > 0$ then the equation $\Psi(y) = 0$ has a unique positive solution*

$$y_1 = \frac{b}{(S^{-2} - a) S}$$

and $\Psi(y) \geq 0$ *for all* $y \geq y_1$.

(2) *If* $\sigma > 1$ *then the equation* $\Psi(y) = 0$ *has a unique positive solution* $y_2 > \left(\frac{a}{\sigma}S^2\right)^{\frac{1}{\sigma-1}}$ *and* $\Psi(y) \geq 0$ *for all* $y \geq y_2$.

(3) *If* $\sigma < 1$. *Let* $\tilde{y} = \left(\frac{\sigma}{a}S^{-2}\right)^{\frac{1}{1-\sigma}}$, *then we have:*

i) Ψ *has no zero point for* $\Psi(\tilde{y}) < 0$.

ii) Ψ *has unique zero point for* $\Psi(\tilde{y}) = 0$, *Consequently, for*

$$b = S^{-1}(1-\sigma)\left(\frac{\sigma}{a}S^{-2}\right)^{\frac{\sigma}{1-\sigma}}$$

iii) Ψ *has two different zero points for* $\Psi(\tilde{y}) > 0$, *with*

$$0 < y_3 < \tilde{y} < y_4.$$

Proof. (1) For $\sigma = 1$, $0 \leq a < S^{-2}$ and $b > 0$, we have

$$\Psi(y) = S\left(S^{-2} - a\right)y - b$$

that is, the equation $\Psi(y) = 0$ has a unique positive solution

$$y_1 = \frac{b}{(S^{-2} - a)S}$$

and $\Psi(y) \geq 0$ for all $y \geq y_1$.

(2) For $\sigma > 1$ we have $\Psi'(y) = \sigma S^{-1}y^{\sigma-1} - aS$ and

$$\Psi''(y) = \sigma(\sigma-1)S^{-1}y^{\sigma-2} > 0, \quad \forall y > 0.$$

Then $\Psi'\left(\left(\frac{a}{\sigma}S^2\right)^{\frac{1}{\sigma-1}}\right) = 0$, $\Psi'(y) < 0$ for $y < \left(\frac{a}{\sigma}S^2\right)^{\frac{1}{\sigma-1}}$ and $\Psi'(y) > 0$ for $y > \left(\frac{a}{\sigma}S^2\right)^{\frac{1}{\sigma-1}}$. Hence Ψ is a concave function and

$$\min_{y\geq 0}\Psi(y) = \Psi\left(\left(\frac{a}{\sigma}S^2\right)^{\frac{1}{\sigma-1}}\right) = -(\sigma-1)S^{-1}\left(\frac{a}{\sigma}S^2\right)^{\frac{\sigma}{\sigma-1}} < 0. \tag{3.5}$$

Moreover, we have $\Psi\left(\left(\frac{a}{\sigma}S^2\right)^{\frac{1}{\sigma-1}}\right) < 0$ and $\lim_{y\to+\infty}\Psi(y) = +\infty$, thus from (3.5) and the concavity of Ψ we can conclude that the equation $\Psi(y) = 0$ has a unique positive solution $y_2 > \left(\frac{a}{\sigma}S^2\right)^{\frac{1}{\sigma-1}}$ and $\Psi(y) \geq 0$ for all $y \geq y_2$.

(3) For $\sigma < 1$. Let $\Psi'(y) = 0$, one has

$$\tilde{y} = \left(\frac{\sigma}{a}S^{-2}\right)^{\frac{1}{1-\sigma}},$$

and when $0 < y < \tilde{y}$, Ψ is increasing, while $y > \tilde{y}$, Ψ is decreasing. Moreover, from $\Psi(0) = -b < 0$, we obtain that

i) Ψ has no zero point for $\Psi(\tilde{y}) < 0$.

ii) Ψ has unique zero point for $\Psi(\tilde{y}) = 0$, Consequently, for

$$b = S^{-1}(1-\sigma)\left(\frac{\sigma}{a}S^{-2}\right)^{\frac{\sigma}{1-\sigma}}$$

iii) Ψ has two different zero points for $\Psi(\tilde{y}) > 0$. ■

Next, for $i \in \{1, 2\}$ we put

$$C_i = a\left(\frac{1}{2p} - \frac{1}{p^*}\right)(Sy_i)^2 + b\left(\frac{1}{p} - \frac{1}{p^*}\right)Sy_i$$

and

$$C^* = \begin{cases} C_1 & \text{if } p^* = 2p,\ 0 \leq a < S^{-2},\ b > 0, \\ C_2 & \text{if } p^* > 2p,\ a \geq 0,\ b \geq 0,\ a + b > 0. \end{cases} \tag{3.6}$$

Now, we prove an important lemma which ensures the local compactness of the Palais Smale sequence for I_2.

Lemma 3.4 *Suppose that $f \in W^* \backslash \{0\}$ and $(\mathcal{H}_0)$ or $(\mathcal{H}_2)$ holds. Let $(u_n) \subset W^{1,p}(\mathbb{R}^N)$ be a Palais Smale sequence for I_2 for some $c \in \mathbb{R}$. Then*

$$\text{either } u_n \to u \quad \text{or } c \geq I_2(u) + C^*.$$

Proof. By the proof of Lemma 3.2 we have (u_n) is a bounded sequence in $W^{1,p}(\mathbb{R}^N)$ and $u_n \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N)$ for some $u \in W^{1,p}(\mathbb{R}^N)$ with $I_2'(u) = 0$. Furthermore, if we write $v_n = u_n - u$, we derive

$$\begin{cases} v_n \rightharpoonup 0 \text{ in } W^{1,p}(\mathbb{R}^N) \text{ and in } L^{p^*}\left(\mathbb{R}^N, |x|^{-p^*}\right) \\ v_n \to 0 \text{ a. e. in } \mathbb{R}^N \\ \int_{\mathbb{R}^N} f(x)\, v_n dx \to 0. \end{cases} \tag{3.7}$$

On the one hand, by using Brézis-Lieb's Lemma [16], one has

$$\begin{cases} \|u\|^p = \|v_n\|^p + \|u\|^p + o_n(1), \\ \int_{\mathbb{R}^N} \frac{|u_n|^{p^*}}{|x|^{p^*}} dx = \int_{\mathbb{R}^N} \frac{|v_n|^{p^*}}{|x|^{p^*}} dx + \int_{\mathbb{R}^N} |u|^{p^*} dx + o_n(1). \end{cases} \tag{3.8}$$

As $\langle I_2'(u), u\rangle = 0$ we obtain by (3.7) and (3.8) that

$$o_n(1) = \langle I_2'(u_n), u_n\rangle = \|v_n\|^p - \int_{\mathbb{R}^N} \frac{|v_n|^{p^*}}{|x|^{p^*}} dx \tag{3.9}$$

and

$$\begin{aligned} c + o_n(1) &= I_2(u_n) - \frac{1}{p^*}\langle I_2'(u_n), u_n\rangle \\ &= a\left(\frac{1}{2p} - \frac{1}{p^*}\right)(\|v_n\|^p + \|u\|^p)^2 + b\left(\frac{1}{p} - \frac{1}{p^*}\right)(\|v_n\|^p + \|u\|^p) \\ &\quad + \lambda\left(\frac{1}{p^*} - 1\right)\int_{\mathbb{R}^N} f(x)\, v_n dx - \lambda\left(\frac{1}{p^*} - 1\right)\int_{\mathbb{R}^N} f(x)\, u dx \\ &\geq a\left(\frac{1}{2p} - \frac{1}{p^*}\right)\|v_n\|^{2p} + b\left(\frac{1}{p} - \frac{1}{p^*}\right)\|v_n\|^p + I_2(u) - \frac{1}{p^*}\langle I_2'(u), u\rangle. \end{aligned}$$

Consequently,

$$c + o_n(1) \geq I_2(u) + \left(\frac{a}{2p} - \frac{a}{p^*}\right)\|v_n\|^{2p} + \left(\frac{b}{p} - \frac{b}{p^*}\right)\|v_n\|^p. \tag{3.10}$$

Assume that $\lim_{n\to\infty} \|v_n\| = l > 0$, then by (3.9) and the Caffarelli-Kohn-Nirenberg inequality we obtain

$$l^p \geq S\left(bl^p + al^{2p}\right)^{\frac{p}{p^*}},$$

this implies that

$$S^{-\frac{p^*}{p}} l^{p^*-p} - al^p - b \geq 0. \tag{3.11}$$

Let $y = S^{-1}l^p$ and $\sigma = \frac{p^*-p}{p}$, then by (**??**) we get

$$S^{-1}y^{\sigma} - aSy - b \geq 0.$$

It is clear that $\sigma \geq 1$, thanks to $p^* \geq 2p$. So, from the definition of Ψ we get $\Psi(y) \geq 0$.

We will discuss two cases:

Case 1. $p^* = 2p$, $0 \leq a < S^{-2}$ and $b > 0$. According to Lemma 3.3, we have $\Psi(y) \geq 0$ if $y \geq y_1$ with

$$y_1 = \frac{b}{(S^{-2} - a)\, S},$$

which implies that $l^p \geq Sy_1$.

Case 2. $p^* > 2p$, $a \geq 0$, $b \geq 0$ and $a + b > 0$. In this case, it follows from lemma 3.3 that $\Psi(y) \geq 0$ if $y \geq y_2$ with

$$y_2 > \left(\frac{ap}{p^* - p} S^2 \right)^{\frac{p}{p^*-2p}},$$

which implies that $l^p \geq Sy_2$. Then by (3.10), one has

$$\begin{aligned}
c &\geq I_2(u) + \left(\frac{a}{2p} - \frac{a}{p^*} \right) l^{2p} + \left(\frac{b}{p} - \frac{b}{p^*} \right) l^p \\
&\geq I_2(u) + \begin{cases} b\dfrac{p^* - p}{pp^*} Sy_1 & \text{if } p^* = 2p, 0 \leq a < S^{-2} \text{ and } b > 0, \\ a\dfrac{p^* - 2p}{2pp^*} (Sy_2)^2 + b\dfrac{p^* - p}{pp^*} Sy_2 & \text{if } p^* > 2p,\ a, b \geq 0 \text{ and } a + b > 0\ . \end{cases} \\
&= I_2(u) + C^*.
\end{aligned}$$

The proof of Lemma 3.4 is completed. ∎

3.3 Existence of solution with negative energy

Theorem 3.5 *Suppose that $f \in W^{*}\backslash\{0\}$ and assume that $(\mathcal{H}_0)$ or $(\mathcal{H}_1)$ holds. Then there exists a constants $\lambda_{-} > 0$ such that for any $\lambda \in (0,\ \lambda_{-})$ problem (3.1) has a solution u_{-} with negative energy.*

Remark 3.6 *If $p^{*} < 2p$, $a \geq 0$, $b \geq 0$ and $a + b > 0$ or $p^{*} = 2p$, $a = S^{-2}$ and $b > 0$ or $p^{*} = 2p$, $a > S^{-2}$ and $b \geq 0$, then for any $\lambda > 0$, we can easily show the existence of one solution which is a ground state solution.*

We give here the proof of our Theorem 3.5 by using Ekeland's variational principle.

Proof. Let $u \in W^{1,p}(\mathbb{R}^N)\backslash\{0\}$, $b > 0$, $a \geq 0$ and $p^{*} \geq 2p$. By Hölder and Caffarelli-Kohn-Nirenberg inequalities we have

$$\begin{aligned} I_2(u) &= \frac{a}{2p}\|u\|^{2p} + \frac{b}{p}\|u\|^{p} - \frac{1}{p^{*}}\int_{\mathbb{R}^N} |u|^{p^{*}} dx - \lambda \int_{\mathbb{R}^N} f(x)\, u dx \\ &\geq \frac{b}{p}\|u\|^{p} + \frac{a}{2p}\|u\|^{2p} - \frac{S^{-p^{*}/p}}{p^{*}}\|u\|^{p^{*}} - \lambda \|f\|_{W^{*}}\|u\|. \end{aligned}$$

Now we divide the proof in two cases.

Firstly, assume that $b > 0$ and $a \geq 0$. If $(\mathcal{H}_0)$ or $(\mathcal{H}_1)$ holds, we get

$$I_2(u) \geq \frac{b}{p}\|u\|^{p} - \frac{S^{-p^{*}/p}}{p^{*}}\|u\|^{p^{*}} - \left(\frac{b}{2}\right)^{\frac{-1}{p}} \lambda \|f\|_{W^{*}} \left(\frac{b}{2}\right)^{\frac{1}{p}} \|u\|,$$

it follows from the inequality $XY \leq \dfrac{X^{q}}{q} + \dfrac{Y^{q'}}{q'}$ for any $X, Y \geq 0$ and $q, q' > 0$ with $\dfrac{1}{q} + \dfrac{1}{q'} = 1$, that

$$\begin{aligned} I_2(u) &\geq \frac{b}{p}\|u\|^{p} - \frac{S^{-p^{*}/p}}{p^{*}}\|u\|^{p^{*}} - \frac{p-1}{p}\left(\left(\frac{b}{2}\right)^{\frac{-1}{p}} \lambda \|f\|_{W^{*}}\right)^{\frac{p}{p-1}} - \frac{1}{p}\left(\left(\frac{b}{2}\right)^{\frac{1}{p}} \|u\|\right)^{p} \\ &\geq \frac{b}{2p}\|u\|^{p} - \frac{S^{-p^{*}/p}}{p^{*}}\|u\|^{p^{*}} - \frac{p-1}{p}\left(\left(\frac{b}{2}\right)^{\frac{-1}{p}} \lambda \|f\|_{W^{*}}\right)^{\frac{p}{p-1}}. \end{aligned}$$

For $\rho \geq 0$ we consider the function $h_1 : \mathbb{R}^+ \to \mathbb{R}^*$, given by

$$h_1(\rho) = \frac{b}{2p}\rho^p - \frac{S^{-p^*/p}}{p^*}\rho^{p^*},$$

direct calculation shows that

$$\max_{\rho\geq 0} h_1(\rho) = h_1(\rho_1) = \frac{p^*-p}{pp^*} S^{\frac{p^*}{p^*-p}} \left(\frac{b}{2}\right)^{\frac{p^*}{p^*-p}} \text{ with } \rho_1 = \left[\frac{b}{2}S^{p^*/p}\right]^{\frac{1}{p^*-p}}$$

and $h_1(\rho) \geq 0 \ \forall \rho \in B_{\rho_1}(0)$.

Consequently,

$$I_2(u)|_{B\rho_1(0)} \geq -\frac{p-1}{p}\left(\left(\frac{b}{2}\right)^{\frac{-1}{p}} \lambda \|f\|_{W^*}\right)^{\frac{p}{p-1}}. \tag{3.12}$$

Moreover, for $\|u\| = \rho_1$ we have

$$\begin{aligned} I_2(u) &\geq h_1(\rho_1) - \frac{p-1}{p}\left(\left(\frac{b}{2}\right)^{\frac{-1}{p}} \lambda \|f\|_{W^*}\right)^{\frac{p}{p-1}}, \\ &\geq \frac{1}{p}h_1(\rho_1) + \frac{p-1}{p}h_1(\rho_1) - \frac{p-1}{p}\left(\left(\frac{b}{2}\right)^{\frac{-1}{p}} \lambda \|f\|_{W^*}\right)^{\frac{p}{p-1}}, \\ &\geq \frac{1}{p}h_1(\rho_1) \\ &= : \delta_1 \end{aligned}$$

for all $\lambda \in (0, \ \lambda_1)$ with

$$\lambda_1 = \left(\frac{p^*-p}{pp^*}S^{\frac{p^*}{p^*-p}}\right)^{\frac{p-1}{p}} \|f\|_{W^*}^{-1} \left(\frac{b}{2}\right)^{\frac{p^*-1}{p^*-p}}.$$

We turn to the case where $a > 0$ and $b \geq 0$. If $(\mathcal{H}_0)$ holds we obtain

$$\begin{aligned} I_2(u) &\geq \frac{a}{2p}\|u\|^{2p} - \frac{S^{-p^*/p}}{p^*}\|u\|^{p^*} - \left(\left(\frac{a}{2}\right)^{\frac{-1}{2p}} \lambda \|f\|_{W^*}\right)\left(\left(\frac{a}{2}\right)^{\frac{1}{2p}} \|u\|\right) \\ &\geq \frac{a}{2p}\|u\|^{2p} - \frac{S^{-p^*/p}}{p^*}\|u\|^{p^*} - \frac{2p-1}{2p}\left(\left(\frac{a}{2}\right)^{\frac{-1}{2p}} \lambda \|f\|_{W^*}\right)^{\frac{2p}{2p-1}} - \frac{1}{2p}\left(\left(\frac{a}{2}\right)^{\frac{1}{2p}} \|u\|\right)^{2p} \\ &\geq \frac{a}{4p}\|u\|^{2p} - \frac{S^{-p^*/p}}{p^*}\rho^{p^*} - \frac{2p-1}{2p}\left(\left(\frac{a}{2}\right)^{\frac{-1}{2p}} \lambda \|f\|_{W^*}\right)^{\frac{2p}{2p-1}}. \end{aligned}$$

Now, we consider the function $h_2 : \mathbb{R}^+ \to \mathbb{R}^*$, given by

$$h_2(\rho) = \frac{a}{4p}\rho^{2p} - \frac{S^{-p^*/p}}{p^*}\rho^{p^*}$$

then

$$\max_{\rho\geq 0} h_2(\rho) = h_2(\rho_2) = \left(\frac{1}{2p} - \frac{1}{p^*}\right) S^{-p^*/p}\left[\frac{a}{2}S^{p^*/p}\right]^{\frac{p^*}{p^*-2p}} \text{ with } \rho_2 = \left[\frac{a}{2}S^{p^*/p}\right]^{\frac{1}{p^*-2p}}$$

and $h_2(\rho) \geq 0\ \forall \rho \in B_{\rho_2}(0)$.

Consequently,

$$I_2(u)|_{B_{\rho_2}(0)} \geq -\frac{2p-1}{2p}\left(\left(\frac{a}{2}\right)^{\frac{-1}{2p}} \lambda \|f\|_{W^*}\right)^{\frac{2p}{2p-1}}.$$

Moreover, for $\|u\| = \rho_2$ we have

$$\begin{aligned} I_2(u) &\geq h_2(\rho_2) - \frac{2p-1}{2p}\left(\left(\frac{a}{2}\right)^{\frac{-1}{2p}} \lambda \|f\|_{W^*}\right)^{\frac{2p}{2p-1}}, \\ &\geq \frac{2p-1}{2p}h_2(\rho_2) + \frac{1}{2p}h_2(\rho_2) - \frac{2p-1}{2p}\left(\left(\frac{a}{2}\right)^{\frac{-1}{2p}} \lambda \|f\|_{W^*}\right)^{\frac{2p}{2p-1}}, \\ &\geq \frac{1}{2p}h_2(\rho_2) \\ &= :\delta_2 \end{aligned}$$

for all $\lambda \in (0,\ \lambda_2)$ with

$$\lambda_2 = \left(\frac{p^*-2p}{2pp^*}S^{\frac{2p^*}{p^*-2p}}\right)^{\frac{2p-1}{2p}}\left(\frac{a}{2}\right)^{\frac{p^*-1}{p^*-2p}}\|f\|_{W^*}^{-1}.$$

Choosing $\delta_*,\ \rho_*$ and λ_- such that

$$(\delta_*,\ \rho_*,\ \lambda_-) = \begin{cases} (\delta_1,\ \rho_1,\ \lambda_1) & \text{if } (\mathcal{H}_0) \text{ satisfies} \\ (\delta_2,\ \rho_2,\ \lambda_2) & \text{if } (\mathcal{H}_1) \text{ satisfies.} \end{cases} \tag{3.13}$$

Then, for all $\lambda \in (0,\ \lambda_-)$ we have

$$I_2(u)|_{\partial B_{\rho_*}(0)} \geq \delta_* \tag{3.14}$$

and

$$I_2\,(u)|_{B_{\rho_*}(0)} \geq -C_\lambda \tag{3.15}$$

with

$$C_\lambda := \begin{cases} \dfrac{p-1}{p}\left(\left(\dfrac{b}{2}\right)^{\frac{-1}{p}}\lambda\,\|f\|_{W^*}\right)^{\frac{p}{p-1}} & \text{if } (\mathcal{H}_0) \text{ satisfies} \\ \dfrac{2p-1}{2p}\left(\left(\dfrac{a}{2}\right)^{\frac{-1}{2p}}\lambda\,\|f\|_{W^*}\right)^{\frac{2p}{2p-1}} & \text{if } (\mathcal{H}_1) \text{ satisfies.} \end{cases} \tag{3.16}$$

Now, we define

$$c_- = \inf\left\{I_2\,(u)\,,\ u \in \overline{B}_{\rho_*}\,(0)\right\}. \tag{3.17}$$

As $f \in W^*\backslash\,\{0\}$ we can choose $\varphi \in W^{1,p}(\mathbb{R}^N)$ such that $\displaystyle\int_{\mathbb{R}^N} f\,(x)\,\varphi dx > 0$. Then, for a fixed $\lambda \in (0,\ \lambda_-)$, there exists $t_0 > 0$ such that $\|t_0\varphi\| < \rho_*$ and

$$c_- \leq I_2(t_0\varphi) < 0 \ \text{ for } t \in (0,\ t_0)\,.$$

Hence, $c_- < I_2(0) = 0$. Using Ekeland's variational principle, for the complete metric space $\overline{B}_{\rho_*}\,(0)$ with respect to the norm of $W^{1,p}(\mathbb{R}^N)$, we obtain the result that there exists a Palais Smale sequence $u_n \in \overline{B}_{\rho_*}\,(0)$ at level c_-. From Lemma 3.2 there exists $u_- \in \overline{B}_{\rho_*}\,(0)$ such that $u_n \rightharpoonup u_-$ in $W^{1,p}(\mathbb{R}^N)$ and $I_2'\,(u_-) = 0$.

Now, we shall show that $u_n \to u_-$ in $W^{1,p}$. Suppose otherwise, then $\|u_-\| < \liminf_{n\to+\infty} \|u_n\|$, which implies that

$$\begin{aligned}
c_- &\le I_2(u_-) \\
&= I_2(u_-) - \frac{1}{p^*}\langle I_2'(u_-), u_-\rangle \\
&= a\frac{p^*-2p}{2pp^*}\|u_-\|^{2p} + b\frac{p^*-p}{pp^*}\|u_-\|^p - \lambda\frac{p^*-1}{p^*}\int_{\mathbb{R}^N} f(x)\,u_- dx \\
&< \liminf_{n\to+\infty}\left[a\frac{p^*-2p}{2pp^*}\|u_n\|^{2p} + b\frac{p^*-p}{pp^*}\|u_n\|^p - \lambda\frac{p^*-1}{p^*}\int_{\mathbb{R}^N} f(x)\,u_n dx\right] \\
&= \liminf_{n\to+\infty}\left[I_2(u_n) - \frac{1}{p^*}\langle I_2'(u_n), u_n\rangle\right] \\
&= c_-,
\end{aligned}$$

which is a contradiction. We conclude that $u_n \to u_-$ strongly in $W^{1,p}(\mathbb{R}^N)$. Therefore, $I_2'(u_-) = 0$ and $I_2(u_-) = c_- < 0 = I_2(0)$. Hence u_- is a nonzero solution of (3.1) with negative energy. ■

3.4 Existence of solution with positive energy

Theorem 3.7 *Suppose that $f \in W^*\setminus\{0\}$ such that $\int_{\mathbb{R}^N} f(x)\,v_{\varepsilon,x_0}dx \neq 0$. Assume that $(\mathcal{H}_0)$ or $(\mathcal{H}_2)$ holds. Then there exists a constant $\lambda_+ \in (0,\ \lambda_-]$ such that for any $\lambda \in (0,\ \lambda_+)$ problem (3.1) has a second solution u_+ with positive energy.*

Notice that assumption $\int_{\mathbb{R}^N} f(x)\,v_{\varepsilon,x_0}dx \neq 0$ certainly holds if $f \in W^*\setminus\{0\}$ does not change sign. Also we have $f \in L^{\frac{p^*}{p^*-1}}(\mathbb{R}^N)$ since $f \in W^*\setminus\{0\}$ and $u_-, u_+ \ge 0$ for $f \ge 0$. Furthermore, in Remark 3.6 [32], the authors mentioned that it difficult to obtain the second solution in the case $p < N < p^*$, $a > 0$ and $b > 0$. For special dimension $N = 3p/2$, this case is studied in [11].

Now, we prove the existence of a Mountain Pass type solution and we give the proof of Theorems 3.7 with the help of Theorem 3.5. Here we need the following lemma.

Lemma 3.8 *Assume that all conditions in Theorem* 3.7 *are fulfilled. Then there exists* $z_\varepsilon \in W^{1,p}(\mathbb{R}^N)$ *and* $\lambda^* > 0$ *such that*

$$\sup_{t\geq 0} I_2(tz_\varepsilon) < c_- + C^* \quad \forall\, \lambda \in (0, \lambda^*)$$

where c_-, C^* *are given in* (3.17) *and* (3.6) *respectively.*

Proof. Since $\int_{\mathbb{R}^N} f(x)\, v_{\varepsilon,x_0}(x)\, dx \neq 0$ there exists $z_\varepsilon = \pm v_{\varepsilon,x_0}$ satisfies

$$\int_{\mathbb{R}^N} f(x)\, z_\varepsilon(x)\, dx > 0.$$

Given any $\lambda > 0$ and fixed $t > 0$, then from (3.4) we have

$$\begin{aligned} I_2(tz_\varepsilon) &= \frac{a}{2p} t^{2p} \|z_\varepsilon\|^{2p} + \frac{b}{p} t^p \|z_\varepsilon\|^p - \frac{t^{p^*}}{p^*} \int_{\mathbb{R}^N} |z_\varepsilon|^{p^*} dx - \lambda t \int_{\mathbb{R}^N} f(x)\, z_\varepsilon dx \\ &= \frac{a}{2p} t^{2p} S^{\frac{2p^*}{p^*-p}} + \frac{b}{p} t^p S^{\frac{p^*}{p^*-p}} - \frac{t^{p^*}}{p^*} S^{\frac{p^*}{p^*-p}} - \lambda t \int_{\mathbb{R}^N} f(x)\, z_\varepsilon dx. \end{aligned}$$

Define $g, h :]0, +\infty[\to \mathbb{R}$ by $g(t) = I_2(tz_\varepsilon)$ and

$$h(t) = \frac{a}{2p} t^{2p} S^{\frac{2p^*}{p^*-p}} + \frac{b}{p} t^p S^{\frac{p^*}{p^*-p}} - \frac{t^{p^*}}{p^*} S^{\frac{p^*}{p^*-p}}.$$

Then

$$h'(t) = -t^{p-1} S^{\frac{p^*}{p^*-p}} \left(t^{p^*-p} - a S^{\frac{p^*}{p^*-p}} t^p - b \right),$$

it follows from $h'(t) = 0$ that

$$aS^{\frac{p^*}{p^*-p}}t^p + b - t^{p^*-p} = 0. \tag{3.18}$$

So

$$t^{p^*} = aS^{\frac{p^*}{p^*-p}}t^{2p} + bt^p. \tag{3.19}$$

Let $y = S^{\frac{p}{p^*-p}}t^p$, $\sigma = \frac{p^*-p}{p}$ and

$$y_* := \begin{cases} y_1 \text{ if } (\mathcal{H}_2) \text{ holds} \\ y_2 \text{ if } (\mathcal{H}_1) \text{ holds.} \end{cases}$$

Then by (3.18) and the definition of Ψ we get

$$\Psi(y) = S^{-1}y^\sigma - aSy - b = 0, \tag{3.20}$$

which implies from the proof of Lemma 3.2 that $\Psi(y_*) = 0$, $\Psi(y) < 0$ for all $y \in]0, y_*[$ and $\Psi(y) > 0$ for all $y \in]y_*, +\infty[$. Therefore, $h'(t_*) = 0$, $h'(t) > 0$ for all $t \in]0, t_*[$ and $h'(t) < 0$ for all $t \in]t_*, +\infty[$ where

$$t_* := \begin{cases} S^{\frac{-1}{p^*-p}}y_1^{\frac{1}{p}} & \text{if } (\mathcal{H}_2) \text{ holds} \\ S^{\frac{-1}{p^*-p}}y_2^{\frac{1}{p}} & \text{if } (\mathcal{H}_0) \text{ holds.} \end{cases}$$

Moreover, since $h(0) = 0$ and $\lim_{t\to+\infty} h(t) = -\infty$ if $(\mathcal{H}_0)$ or $(\mathcal{H}_2)$ holds, then h attains its maximum at t_*.

So, from (3.19) we deduce that

$$\begin{aligned}
\max_{t\geq 0} h(t) &= h(t_*) \\
&= \frac{a}{2p} t_*^{2p} S^{\frac{2p^*}{p^*-p}} + \frac{b}{p} t_*^p S^{\frac{p^*}{p^*-p}} - \frac{t_*^{p^*}}{p^*} S^{\frac{p^*}{p^*-p}} \\
&= \frac{a}{2p} t_*^{2p} S^{\frac{2p^*}{p^*-p}} + \frac{b}{p} t_*^p S^{\frac{p^*}{p^*-p}} - \left(\frac{a}{p^*} t_*^{2p} S^{\frac{2p^*}{p^*-p}} + \frac{b}{p^*} t_*^p S^{\frac{p^*}{p^*-p}} \right) \\
&= a\left(\frac{1}{2p} - \frac{1}{p^*}\right) t_\varepsilon^{2p} S^{\frac{2p^*}{p^*-p}} + b\left(\frac{1}{p} - \frac{1}{p^*}\right) t_\varepsilon^p S^{\frac{p^*}{p^*-p}} \\
&= a\left(\frac{1}{2p} - \frac{1}{p^*}\right) S^2 y_*^2 + b\left(\frac{1}{p} - \frac{1}{p^*}\right) S y_* \\
&= C^*.
\end{aligned}$$

We know from the proof of Theorem 2.5 that $c_- \geq -C_\lambda$ for all $\lambda \in (0,\ \lambda_*)$. So, we can choose $\lambda_3 \leq \lambda_-$ such that for any $\lambda \in (0,\ \lambda_3)$ we have $C^* - c_- \geq C^* - C_\lambda > 0$. Hence $C^* - c_- > 0$ for all $\lambda \in (0,\ \lambda_3)$.

Now, we consider the function $g(t) := I_2(tz_\varepsilon)$, $t \geq 0$. Then

$$g(t) = h(t) - \lambda t \int_{\mathbb{R}^N} f(x)\, z_\varepsilon dx.$$

So, for all $\lambda \in (0, \lambda_3)$ we have

$$g(0) = 0 < C^* - C_\lambda$$

Hence, by the continuity of $g(t)$, there exists $t_1 > 0$ small enough such that

$$g(t) < C^* - C_\lambda \ \forall t \in (0,\ t_1).$$

We know also that $\lim_{t\to+\infty} g(t) = -\infty$ if $(\mathcal{H}_0)$ or $(\mathcal{H}_2)$ holds. Then for $t_2 > 0$ sufficiently large, one has

$$g(t) < C^* - C_\lambda \ \forall t \in (t_2,\ +\infty).$$

On the other hand, as $\int_{\mathbb{R}^N} f(x)\, z_\varepsilon dx > 0$ we can deduce from the above estimate on $h(t)$ that for all $t \in [t_1, t_2]$

$$g(t) < C^* - \lambda t_1 \int_{\mathbb{R}^N} f(x)\, z_\varepsilon dx.$$

Set

$$\lambda_4 = \begin{cases} \left(\dfrac{p}{p-1} t_1 \displaystyle\int_{\mathbb{R}^N} f(x)\, z_\varepsilon dx\right)^{p-1} \dfrac{b}{2} \|f\|_{W^*}^{-p} & \text{if } (\mathcal{H}_2) \text{ or } (\mathcal{H}_0) \text{ with } b > 0 \text{ holds} \\ \left(\dfrac{2p}{2p-1} t_1 \displaystyle\int_{\mathbb{R}^N} f(x)\, z_\varepsilon dx\right)^{2p-1} \dfrac{a}{2} \|f\|_{W^*}^{-2p} & \text{if } (\mathcal{H}_0) \text{ with } a > 0 \text{ holds} \end{cases}$$

Then for any $\lambda \in (0,\ \lambda_4)$ one has

$$-\lambda t_1 \int_{\mathbb{R}^N} f(x)\, z_\varepsilon dx < -C_\lambda$$

Taking $\lambda_+ = \min\{\lambda_-,\ \lambda_3,\ \lambda_4\}$ then $c_- \geq -C_\lambda$ and we deduce that

$$\sup_{t \geq 0} I_2(t z_\varepsilon) < C^* + c_-, \text{ for all } \lambda \in (0,\ \lambda_+).$$

This concludes the proof of Lemma 3.8. ■

Now we can prove Theorem 3.7.

Proof. Note that $I_2(0) = 0$ and from (3.14) we have $I_2(u)|_{\partial B_{\rho_*}(0)} \geq \delta_* > 0$ for all $\lambda \in (0,\ \lambda_-)$ where ρ_*, δ_* are defined in (3.13). We know also that $\lim_{t \to \infty} I_2(t z_\varepsilon) = -\infty$ if $(\mathcal{H}_0)$ or $(\mathcal{H}_2)$ holds, then $I_2(T z_\varepsilon) < 0$ for T large enough, hence I_2 satisfies the geometry conditions of the Mountain Pass Theorem [6]. Then, there exists a Palais Smale sequence (u_n) at level c_+, such that

$$I_2(u_n) \to c_+ \text{ and } I_2'(u_n) \to 0 \text{ as } n \to +\infty$$

with

$$0 < c_+ = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_2(\gamma(t)) \leq \sup_{t \geq 0} I_2(t T z_\varepsilon) < C^* + c_-, \text{ for all } \lambda \in (0, \lambda_+),$$

where

$$\Gamma = \left\{ \gamma \in C\left([0,1]\,,\ W^{1,p}(\mathbb{R}^N)\right),\ \gamma\left(0\right) = 0,\ \gamma\left(1\right) = Tz_{\varepsilon} \right\}.$$

Using Lemma 3.1 we have that (u_n) has a subsequence, still denoted by (u_n), such that $u_n \rightharpoonup u_+$ in $W^{1,p}(\mathbb{R}^N)$ as $n \to +\infty$. Hence, from Lemma 3.4 if $u_n \nrightarrow u_+$ in $W^{1,p}(\mathbb{R}^N)$ as $n \to +\infty$, it holds

$$c_+ \geq I_2\left(u_+\right) + C^* \geq c_- + C^*$$

which is a contradiction with Lemma 3.4. Hence, $I_2'\left(u_+\right) = 0$ and

$$I_2\left(u_+\right) = c_+ > 0.$$

So, as $c_+ > 0 = I_2\left(0\right)$ we can conclude that u_+ is a nonzero solution of (3.1) with positive energy. This completes the proof of Theorem 3.7. ■

3.5 Infinitely solutions

we use the following assumptions:

$(\mathcal{H}_2)$ $p^* = 2p$, $0 < a < S^{-2}$ and $b > 0$,

$(\mathcal{H}_3)$ $p^* > 2p$, $a > 0$ and $b > 0$,

$(\mathcal{H}_6)$ $p^* < 2p$, $a > 0$ and $b > b^*$,

$(\mathcal{H}_7)$ $p^* = 2p$, $a > 0$ and $b = 0$,

$(\mathcal{H}_8)$ $p^* < 2p$, $a > 0$ and $b = b^*$ where

$$b^* = \frac{2p - p^*}{p}\left(\frac{p}{p^* - p}a\right)^{-\frac{p^*-p}{2p-p^*}} S^{-\frac{p^*}{2p-p^*}}.$$

Theorem 3.9 *Let* $\lambda = 0,\ a > 0,\ b \geq 0,\ 1 < p < N$. *For* v_{ε,x_0} *given by* (3.3) *the following conclusions hold:*

(1) *If* $p^* = 2p$, *then under the hypothesis* $(\mathcal{H}_2)$, *the problem* (3.1) *has infinitely many nonnegative solutions and these solutions are*

$$\left(\frac{b}{1 - S^2 a}\right)^{\frac{1}{p^*-p}} v_{\varepsilon,x_0} \quad \text{for all } \varepsilon > 0,$$

(2) *Under the hypothesis* $(\mathcal{H}_7)$, *the problem* (3.1) *has infinitely many positive solutions* $\delta^{\frac{1}{p}} v_{\varepsilon,x_0}$ *(for any* $\delta > 0$*) if and only if* $a = S^{-2}$.

(3) *If* $p^* \neq 2p$, $b = 0$ *and* $a > 0$, *then problem* (3.1) *has infinitely many nonnegative solutions and these solutions*

$$\left(aS^{\frac{p^*}{p^*-p}}\right)^{-\frac{1}{2p-p^*}} v_{\varepsilon,x_0} \quad \text{for all } \varepsilon > 0.$$

(4) *If* $(\mathcal{H}_3)$ *satisfied, then there exists* $\delta_2 > S^{-1}\left(\frac{ap}{p^*-p} S^2\right)^{\frac{p^*-p}{p^*-2p}}$ *such that* $\delta_2^{\frac{1}{p^*-p}} v_{\varepsilon,x_0}$ *are solutions of problem* (3.1), *for all* $\varepsilon > 0$.

(5) *If* $(\mathcal{H}_8)$ *satisfied, then problem* (3.1) *has infinitely many nonnegative solutions and these solutions are*

$$S^{-\frac{1}{p^*-p}} \left(\frac{p^* - p}{pa} S^{-2}\right)^{\frac{1}{2p-p^*}} v_{\varepsilon,x_0} \quad \text{for all } \varepsilon > 0.$$

(6) *If* $(\mathcal{H}_6)$ *satisfied, then there exist* $\delta_3 \in \left(0,\ S^{-1}\left(\frac{p^*-p}{pa} S^{-2}\right)^{\frac{p^*-p}{2p-p^*}}\right)$ *and* $\delta_4 \in \left(S^{-1}\left(\frac{p^*-p}{pa} S^{-2}\right)^{\frac{p^*-p}{2p-p^*}},\ +\infty\right)$ *such that* $\delta_3^{\frac{1}{p^*-p}} v_{\varepsilon,x_0}$ *and* $\delta_4^{\frac{1}{p^*-p}} v_{\varepsilon,x_0}$ *solutions of problem* (3.1) *for all* $\varepsilon > 0$.

Proof. We give the proof of Theorem 3.9.

For any $\delta > 0$ and v_{ε,x_0} in (3.4) define $V_{\varepsilon,\delta} = \delta^{\frac{1}{p^*-p}} v_{\varepsilon,x_0}$. Using the solutions v_{ε,x_0} of problem (3.1), then $V_{\varepsilon,\delta}$ weakly solves the following equation:

$$-\delta \operatorname{div}\left(|\nabla V_{\varepsilon,\delta}|^{p-2}\nabla V_{\varepsilon,\delta}\right) = |V_{\varepsilon,\delta}|^{p^*-2}V_{\varepsilon,\delta}$$

Moreover, according to (3.4), one has

$$\begin{aligned}\delta &= a\left\|V_{\varepsilon,\delta}\right\|^p + b \\ &= a\delta^{\frac{p}{p^*-p}}\left\|v_{\varepsilon,x_0}\right\|^p + b \\ &= aS^{\frac{p^*}{p^*-p}}\delta^{\frac{p}{p^*-p}} + b.\end{aligned}$$

Therefore, the positive solution of problem (3.1) is corresponding to the solution of the following equation about $\delta > 0$

$$\delta - aS^{\frac{p^*}{p^*-p}}\delta^{\frac{p}{p^*-p}} - b = 0 \tag{3.21}$$

1) For $p^* = 2p$, equation (3.21) is equal to

$$\delta\left(1 - aS^2\right) - b = 0$$

i) If $b > 0$ and $0 \leq a < S^{-2}$, we have that

$$\delta_0 = \frac{b}{1 - S^2 a}$$

is a solution of equation (3.21). Hence, $V_{\varepsilon,\delta_0} = \delta_0^{\frac{1}{p^*-p}} v_{\varepsilon,x_0}$ satisfies the following equation in the weak sense:

$$-\left(a\left\|u\right\|^p + b\right)\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = |u|^{p^*-2}u.$$

ii) If $b = 0$ and $a > 0$ equation (3.21) is equal to

$$\delta\left(1 - aS^2\right) = 0.$$

Obviously, for $\delta > 0$, this equality is true if and only if $1 - aS^2 = 0$. Thus, when $p^* = 2p$, problem (3.1) has infinity many positive solutions $V_{\varepsilon,\delta} = \delta^{\frac{1}{p}} v_{\varepsilon,x_0}$ (for any $\delta > 0$) if and only if $a = S^{-2}$.

2) For $p^* \neq 2p$

i) If $b = 0$ and $a > 0$ it is easy to see that

$$\delta_1 = \left(aS^{\frac{p^*}{p^*-p}}\right)^{-\frac{p^*-p}{2p-p^*}}$$

is a solution of equation (3.21). Then, problem (3.1) has infinity many positive solutions $V_{\varepsilon,\delta_1} = \delta_1^{\frac{1}{p^*-p}} v_{\varepsilon,x_0}$.

3) Let $y = (S\delta)^{\frac{p}{p^*-p}}$, equation (3.21) is equal to

$$S^{-1}y^{\frac{p^*-p}{p}} - aSy - b = 0.$$

Now we consider the following equation:

$$\Psi(y) = S^{-1}y^{\frac{p^*-p}{p}} - aSy - b = 0.$$

i) For $p^* > 2p$, according to Lemma 2.3, we have that $\Psi(y) = 0$ has a unique positive solution $y_2 > \left(\frac{ap}{p^*-p}S^2\right)^{\frac{p}{p^*-2p}}$. Thus, problem (3.1) has infinity many positive solutions $V_{\varepsilon,\delta_2} = \delta_2^{\frac{1}{p^*-p}} v_{\varepsilon,x_0}$, with $\delta_2 = S^{-1}y_2^{\frac{p^*-p}{p}} > S^{-1}\left(\frac{ap}{p^*-p}S^2\right)^{\frac{p^*-p}{p^*-2p}}$.

ii) For $2p > p^*$, according to Lemma 2.3, we have:

For $b = \frac{2p-p^*}{p}\left(\frac{p}{p^*-p}a\right)^{-\frac{p^*-p}{2p-p^*}} S^{-\frac{p^*}{2p-p^*}}$ then $\Psi(y) = 0$ has a unique positive solution $\tilde{y} = \left(\frac{p^*-p}{pa}S^{-2}\right)^{\frac{p}{2p-p^*}}$. Thus, problem (3.1) has infinity many positive solutions $V_{\varepsilon,\tilde{\delta}} = \tilde{\delta}^{\frac{1}{p^*-p}} v_{\varepsilon,x_0}$, with $\tilde{\delta} = S^{-1}\tilde{y}^{\frac{p^*-p}{p}}$, for $b < \frac{2p-p^*}{p}\left(\frac{p}{p^*-p}a\right)^{-\frac{p^*-p}{2p-p^*}} S^{-\frac{p^*}{2p-p^*}}$, Ψ has two different zero points y_3 and y_4 with $0 < y_3 < \tilde{y} < y_4$. Consequently, problem (3.1) has infinitely many positive solutions $V_{\varepsilon,\delta_3} = \delta_3^{\frac{1}{p^*-p}} v_{\varepsilon,x_0}$ and $V_{\varepsilon,\delta_4} = \delta_4^{\frac{1}{p^*-p}} v_{\varepsilon,x_0}$ with $\delta_3 = S^{-1} y_3^{\frac{p^*-p}{p}} \in \left(0, S^{-1}\left(\frac{p^*-p}{pa}S^{-2}\right)^{\frac{p^*-p}{2p-p^*}}\right)$ and $\delta_4 = S^{-1} y_4^{\frac{p^*-p}{p}} \in \left(S^{-1}\left(\frac{p^*-p}{pa}S^{-2}\right)^{\frac{p^*-p}{2p-p^*}}, +\infty\right)$. ■

3.6 Non-existence Result

Now we make the following assumptions:

$(\mathcal{H}_4)$ $p^* = 2p$, $a > S^{-2}$ and $b = 0$,

$(\mathcal{H}_5)$ $p^* = 2p$, $a \geq S^{-2}$ and $b > 0$,

$(\mathcal{H}_6)$ $p^* < 2p$, $a > 0$ and $b > b^*$, where

$$b^* = \frac{2p - p^*}{p}\left(\frac{p}{p^* - p}a\right)^{-\frac{p^*-p}{2p-p^*}} S^{-\frac{p^*}{2p-p^*}}.$$

Theorem 3.10 *Assume that one of the hypotheses* $(\mathcal{H}_i)$ *holds for* $4 \leq i \leq 6$. *Then problem* (3.1) *has no non-trivial solution for* $\lambda = 0$.

Remark 3.11 *The authors in [34] proved the non existence solution only in the case* $p^* < 2p$, *while the case* $p^* = 2p$ *is considered in the preceeding theorem.*

From this point of view, Theorem 3.10 *could be viewed as some extension and completeness of related results in [34].*

Proof. Suppose that $(\mathcal{H}_4)$ is satisfied and that $u \in W^{1,p}(\mathbb{R}^N)\backslash\{0\}$ is a solution of the problem (3.1). Then

$$a\|u\|^{2p} = \int_{\mathbb{R}^N} |u|^{p^*} dx. \tag{3.22}$$

As $a > S^{-2}$ and $\int_{\mathbb{R}^N} |u|^{p^*} dx \leq S^{-2} \|u\|^{\,p^*}$, we have by (3.22)

$$\begin{aligned} S^{-2} \|u\|^{2p} &< a \|u\|^{2p} \\ &= \int_{\mathbb{R}^N} |u|^{p^*} dx \\ &\leq S^{-2} \|u\|^{2p}, \end{aligned}$$

which leads to a contradiction.

Suppose now that $(\mathcal{H}_5)$ is satisfied and that $u \in W^{1,p}(\mathbb{R}^N)\backslash\{0\}$ is a solution of (3.1).

Then

$$a \|u\|^{2p} + b \|u\|^{p} = \int_{\mathbb{R}^N} |u|^{p^*} dx.$$

From this last equality and because $a \geq S^{-2}$, $b > 0$ and the fact that

$$\int_{\mathbb{R}^N} |u|^{p^*} dx \leq S^{-2} \|u\|^{p^*}$$

we get

$$\begin{aligned} S^{-2} \|u\|^{2p} &< a \|u\|^{2p} + b \|u\|^{p} \\ &= \int_{\mathbb{R}^N} |u|^{p^*} dx \\ &\leq S^{-2} \|u\|^{2p}, \end{aligned}$$

which is a contradiction.

In the same way as above, we suppose that under the condition $(\mathcal{H}_6)$ we have the existence of a solution $u \in W^{1,p}(\mathbb{R}^N)\backslash\{0\}$, that is,

$$a \|u\|^{2p} + b \|u\|^{p} = \int_{\mathbb{R}^N} |u|^{p^*} dx,$$

and then we got

$$\begin{aligned}
\int_{\mathbb{R}^N} |u|^{p^*} dx &\leq S^{-\frac{p^*}{p}} \|u\|^{p^*} = \|u\|^{p^*-(2p-p^*)} S^{-\frac{p^*}{p}} \|u\|^{2p-p^*} \\
&= \left(\frac{p}{p^*-p}a\right)^{\frac{p^*-p}{p}} \|u\|^{2(p^*-p)} \left(\frac{p}{p^*-p}a\right)^{-\frac{p^*-p}{p}} S^{-\frac{p^*}{p}} \|u\|^{2p-p^*} \\
&\leq \frac{p^*-p}{p} \left(\left(\frac{p}{p^*-p}a\right)^{\frac{p^*-p}{p}} \|u\|^{2(p^*-p)} \right)^{\frac{p}{p^*-p}} \\
&\quad + \frac{2p-p^*}{p} \left(\left(\frac{p}{p^*-p}a\right)^{-\frac{p^*-p}{p}} S^{-\frac{p^*}{p}} \|u\|^{2p-p^*} \right)^{\frac{p}{2p-p^*}} \\
&\leq a\|u\|^{2p} + \frac{2p-p^*}{p} \left(\left(\frac{p}{p^*-p}a\right)^{-\frac{p^*-p}{p}} S^{-\frac{p^*}{p}} \right)^{\frac{p}{2p-p^*}} \|u\|^{p} \\
&= a\|u\|^{2p} + \frac{2p-p^*}{p} \left(\frac{p}{p^*-p}a\right)^{-\frac{p^*-p}{2p-p^*}} S^{-\frac{p^*}{2p-p^*}} \|u\|^{p} \\
&< a\|u\|^{2p} + b\|u\|^{p} \\
&= \int_{\mathbb{R}^N} |u|^{p^*} dx,
\end{aligned}$$

which lead to a contradiction. ■

Chapter 4

Elliptic p-Kirchhoff type systems with critical Sobolev exponent in $\mathbb{R}^N$

4.1 Introduction

In this chapter, we study the following Kirchhoff-type systems involving the critical Sobolev exponent

$$\begin{cases} -(a_1 + b_1 \|u\|^p) \left[\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)\right] = \dfrac{2q}{q+q'} |u|^{q-2} u \ \ |v|^{q'} + \lambda_1 f(x), \\ -(a_2 + b_2 \|v\|^p) \left[\operatorname{div}\left(|\nabla v|^{p-2} \nabla v\right)\right] = \dfrac{2q'}{q+q'} |u|^{q} |v|^{q'-2} v + \lambda_2 g(x), \quad \text{in } \mathbb{R}^N \\ (u,v) \in W^{1,p}\left(\mathbb{R}^N\right) \times W^{1,p}\left(\mathbb{R}^N\right) \end{cases} \tag{4.1}$$

where $1 < p < N$, $a_1,\ a_2 \geq 0,\ b_1,\ b_2 > 0,\ q,\ q' > 1,\ q + q' = p^*,\ p^* = pN/[N-p]$ is the critical Sobolev exponent, $\lambda_1,\ \lambda_2 > 0$ are a parameters, $f,\ g \in W^* \backslash \{0\}$ and

$$\|u,v\|^p := \int_{\mathbb{R}^N} (|\nabla u|^p + |\nabla v|^p)\, dx$$

is the norm in $W^{1,p}(\mathbb{R}^N) \times W^{1,p}(\mathbb{R}^N)$.

The problem (4.1) is related to the following well known Sobolev inequality [17]

$$\left(\int_{\mathbb{R}^N} |u|^{p^*}\, dx\right)^{1/p^*} \leq C\left(\int_{\mathbb{R}^N} |\nabla u|^p\, dx\right)^{1/p} \text{ for all } u \in C_0^\infty(\mathbb{R}^N), \tag{4.2}$$

for some positive constant C.

Sciunzi in [47] provided that if V_ε is a positive solution of the critical problem

$$-\left[\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)\right] = |u|^{p^*-2}u \qquad \text{in } \mathbb{R}^N \tag{4.3}$$

then, for any $\varepsilon > 0$ the extremal functions of (4.3) is $V_\varepsilon(x) = V_{\varepsilon,x_0}(x)$ where

$$V_{\varepsilon,x_0}(x) := \left[\frac{\varepsilon^{\frac{1}{p-1}} N^{\frac{1}{p}} \left(\frac{N-p}{p-1}\right)^{\frac{p-1}{p}}}{\varepsilon^{\frac{p}{p-1}} + |x-x_0|^{\frac{p}{p-1}}}\right]^{\frac{N-p}{p}} \qquad \varepsilon > 0,\ x_0 \in \mathbb{R}^N \tag{4.4}$$

is a minimizer for

$$S := \inf_{u \in W^{1,p}\setminus\{0\}} \frac{\|u\|^p}{\left(\int_{\mathbb{R}^N} |u|^{p^*}\, dx\right)^{p/p^*}}$$

and satisfies

$$\|V_\varepsilon\|^p = \|V_{\varepsilon,x_0}\|^p = \int_{\mathbb{R}^N} |V_{\varepsilon,x_0}|^{p^*}\, dx = S^{\frac{p^*}{p^*-p}}. \tag{4.5}$$

Note that if $a_1 = a_2 = 1$, $\lambda_1 = \lambda_2 = 0$ and $b_1 = b_2 = 0$, system (4.1) reduces to the following system:

$$\begin{cases} -\left[\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)\right] = \frac{2q}{q+q'}|u|^{q-2}u\ \ |v|^{q'}, & \\ & \text{in } \mathbb{R}^N \\ -\left[\operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right)\right] = \frac{2q'}{q+q'}|u|^{q}\,|v|^{q'-2}\,v, & \\ (u,v) \in W^{1,p}(\mathbb{R}^N) \times W^{1,p}(\mathbb{R}^N). & \end{cases} \tag{4.6}$$

Let the constant

$$S_{q,q'} := \inf_{\substack{(u,v)\in W^{1,p}\left(\mathbb{R}^N\right)\times W^{1,p}\left(\mathbb{R}^N\right) \\ (u,v)\neq(0,0)}} \frac{\|u\|^p + \|v\|^p}{\left(\int_{\mathbb{R}^N} |u|^q \, |v|^{q'} \, dx\right)^{p/p^*}}$$

which is positive.

Next we define the energy functional

$$\begin{aligned} I_3(u,v) &= \frac{1}{2p}\left(b_1 \|u\|^{2p} + b_2 \|v\|^{2p}\right) + \frac{1}{p}\left(a_1 \|u\|^p + a_2 \|v\|^p\right) \\ &\quad - \frac{2}{p^*} \int_{\mathbb{R}^N} |u|^q |v|^{q'} dx - \int_{\mathbb{R}^N} \lambda_1 f(x)\, u + \lambda_2 g(x)\, v dx, \end{aligned}$$

associated to problem (4.1), for all $(u,v) \in W^{1,p}\left(\mathbb{R}^N\right) \times W^{1,p}\left(\mathbb{R}^N\right)$

Notice that the functional I_3 is well defined in $W^{1,p}\left(\mathbb{R}^N\right)$ and belongs to $C^1\left(W^{1,p},\ \mathbb{R}\right)$ and that we have

$$\begin{aligned} \langle I_3'(u,v),(u,v)\rangle &= \left(b_1 \|u\|^{2p} + b_2 \|v\|^{2p}\right) + \left(a_1 \|u\|^p + a_2 \|v\|^p\right) \\ &\quad -2 \int_{\mathbb{R}^N} |u|^q |v|^{q'} dx - \int_{\mathbb{R}^N} \lambda_1 f(x)\, u + \lambda_2 g(x)\, v dx \end{aligned}$$

for all $(u,v) \in W^{1,p}\left(\mathbb{R}^N\right) \times W^{1,p}\left(\mathbb{R}^N\right)$. Hence a critical point of functional I_3 is a weak solution of problem (4.1).

4.2 Non-existence of solutions

First we introduced some assumptions which we need to prove non-existence of solution for problem (4.1)

(H_1) $p^* = 2p,\ a_2 = a_2 = 0,\ b_1,\ b_2 > S_{q,q'}^{-2}$.

(H_2) $p^* = 2p$, $b_1,\ b_2 \geq S_{q,q'}^{-2}$, $a_1,\ a_2 > 0$.

(H_3) $p^* > 2p$, $a > 0$, $b > \frac{p^*-p}{p}\left(2\frac{2p-p^*}{pa}\right)^{\frac{2p-p^*}{p^*-p}} 2^{\frac{p}{p^*-p}} \left(S_{q,q'}\right)^{-\frac{p^*}{p^*-p}}$.

Theorem 4.1 *Suppose that* $(\lambda_1, \lambda_2) = 0$ *and assume that* (H_1) *or* (H_2) *or* (H_3) *. Then the problem* (4.1) *has no non-trivial solution.*

Proof. Suppose that (H_1) is satisfied and $(u, v) \in W^{1,p}\backslash\{0\} \times W^{1,p}\backslash\{0\}$ is a solution of the problem (4.1). Then

$$b_1 \|u\|^{2p} + b_2 \|v\|^{2p} = 2 \int_{\mathbb{R}^N} |u|^q |v|^{q'} dx. \tag{4.7}$$

As $b_1, b_2 > S_{q,q'}^{-2}$, $x^2 + y^2 \geq \frac{1}{2}(x+y)^2$ and $\int_{\mathbb{R}^N} |u|^q |v|^{q'} dx \leq S_{q,q'}^{-2} (\|u\|^p + \|v\|^p)^2$, we have by (4.7)

$$\begin{aligned} S_{q,q'}^{-2} \|u\|^{2p} + S_{q,q'}^{-2} \|v\|^{2p} &< b_1 \|u\|^{2p} + b_2 \|v\|^{2p} \\ &= 2 \int_{\mathbb{R}^N} |u|^q |v|^{q'} dx \\ &\leq 2S_{q,q'}^{-2} (\|u\|^p + \|v\|^p)^2 \\ &\leq S_{q,q'}^{-2} \|u\|^{2p} + S_{q,q'}^{-2} \|v\|^{2p} \end{aligned}$$

which leads to a contradiction.

Suppose now that (H_2) is satisfied and that $(u, v) \in W^{1,p}\backslash\{0\} \times W^{1,p}\backslash\{0\}$ is a solution of (4.1). Then

$$\left(b_1 \|u\|^{2p} + b_2 \|v\|^{2p}\right) + \left(a_1 \|u\|^p + a_2 \|v\|^p\right) = p \int_{\mathbb{R}^N} |u|^q |v|^{q'} dx.$$

From this last equality and as $b_1, b_2 \geq S_{q,q'}^{-2}$, $a_1, a_2 > 0$ and the fact that

$\int_{\mathbb{R}^N} |u|^q |v|^{q'} dx \leq S_{q,q'}^{-2} (\|u\|^p + \|v\|^p)^2$, we get

$$S_{q,q'}^{-2}\|u\|^{2p}+S_{q,q'}^{-2}\|v\|^{2p}\leq\left(b_1\|u\|^{2p}+b_2\|v\|^{2p}\right)$$
$$<\left(b_1\|u\|^{2p}+b_2\|v\|^{2p}\right)+\left(a_1\|u\|^{p}+a_2\|v\|^{p}\right)$$
$$=2\int_{\mathbb{R}^N}|u|^q|v|^{q'}dx$$
$$\leq 2S_{q,q'}^{-2}\left(\|u\|^p+\|v\|^p\right)^2\leq S_{q,q'}^{-2}\|u\|^{2p}+S_{q,q'}^{-2}\|v\|^{2p}$$

is a contradiction.

In the same way as above, we suppose that under the condition (H_3) we have the existence of a solution $(u,v)\in W^{1,p}\backslash\{0\}\times W^{1,p}\backslash\{0\}$, that is,

$$\left(b_1\|u\|^{2p}+b_2\|v\|^{2p}\right)+\left(a_1\|u\|^{p}+a_2\|v\|^{p}\right)=2\int_{\mathbb{R}^N}|u|^q|v|^{q'}dx$$

and then we get

$$\int_{\mathbb{R}^N}|u|^q|v|^{q'}dx\leq S_{q,q'}^{-\frac{p^*}{p}}\left(\|u\|^p+\|v\|^p\right)^{\frac{p^*}{p}}$$
$$\leq\left(2\frac{2p-p^*}{pa}\right)^{\frac{2p-p^*}{p}}S_{q,q'}^{-\frac{p^*}{p}}\left(\|u\|^p+\|v\|^p\right)^{\frac{2p^*-2p}{p}}\left(\frac{pa}{2(2p-p^*)}\right)^{\frac{2p-p^*}{p}}\left(\|u\|^p+\|v\|^p\right)^{\frac{2p-p^*}{p}}$$
$$\leq\frac{p^*-p}{p}\left(\left(2\frac{2p-p^*}{pa}\right)^{\frac{2p-p^*}{p}}\left(S_{q,q'}\right)^{-\frac{p^*}{p}}\left(\|u\|^p+\|v\|^p\right)^{\frac{2p^*-2p}{p}}\right)^{\frac{p}{p^*-p}}$$
$$+\frac{2p-p^*}{p}\left(\left(\frac{pa}{2(2p-p^*)}\right)^{\frac{2p-p^*}{p}}\left(\|u\|^p+\|v\|^p\right)^{\frac{2p-p^*}{p}}\right)^{\frac{p}{2p-p^*}}$$
$$\leq\frac{p^*-p}{p}\left[\left(2\frac{2p-p^*}{pa}\right)^{\frac{2p-p^*}{p}}2\left(S_{q,q'}\right)^{-\frac{p^*}{p^*-p}}\left(\|u\|^{2p}+\|v\|^{2p}\right)^{\frac{2p^*-2p}{2p}}\right]^{\frac{p}{p^*-p}}$$
$$+\frac{2p-p^*}{p}\left(\left(\frac{pa}{2(2p-p^*)}\right)^{\frac{2p-p^*}{p}}\left(\|u\|^p+\|v\|^p\right)^{\frac{2p-p^*}{p}}\right)^{\frac{p}{2p-p^*}}$$
$$\leq\frac{1}{2}\frac{p^*-p}{p}\left(2\frac{2p-p^*}{pa}\right)^{\frac{2p-p^*}{p^*-p}}2^{\frac{p}{p^*-p}}\left(S_{q,q'}\right)^{-\frac{p^*}{p^*-p}}\left(\|u\|^{2p}+\|v\|^{2p}\right)$$
$$\leq\frac{1}{2}\frac{p^*-p}{p}\left(2\frac{2p-p^*}{pa}\right)^{\frac{2p-p^*}{p^*-p}}2^{\frac{p}{p^*-p}}\left(S_{q,q'}\right)^{-\frac{p^*}{p^*-p}}\left(\|u\|^{2p}+\|v\|^{2p}\right)$$
$$+\frac{1}{2}a\left(\|u\|^p+\|v\|^p\right)$$
$$<\frac{1}{2}b\left(\|u\|^{2p}+\|v\|^{2p}\right)+\frac{1}{2}a\left(\|u\|^p+\|v\|^p\right)$$
$$<\frac{1}{2}\left(b_1\|u\|^{2p}+b_2\|v\|^{2p}\right)+\frac{1}{2}\left(a_1\|u\|^{p}+a_2\|v\|^{p}\right)$$
$$=\int_{\mathbb{R}^N}|u|^q|v|^{q'}dx,$$

which lead to a contradiction. ∎

4.3 Infinity solutions

Now, we prove that the problem (4.1) has infinitely many nonnegative solutions, we present the following results.

Lemma 4.2 *Let* $a_1 = a_2 = 1,\ \lambda_1 = \lambda_2 = 0,\ b_1 = b_2 = 0,$ *and* $1 < p < N.$ *For* V_ε *given by* (4.4) *the following conclusions hold:*

If $p^* \geq 2p$ *then the problem* (4.1) *has infinitely many nonnegative solutions and these solutions are* $(u_\varepsilon, v_\varepsilon)$, *which give*

$$\begin{cases} u_\varepsilon = \left(\frac{2}{p^*}\right)^{\frac{1}{(p-p^*)}} (q)^{\frac{p-q'}{p(p-p^*)}} (q')^{\frac{q'}{p(p-p^*)}} V_\varepsilon \\ v_\varepsilon = \left(\frac{2}{p^*}\right)^{\frac{1}{(p-p^*)}} (q')^{\frac{p-q}{p(p-p^*)}} q^{\frac{q}{p(p-p^*)}} V_\varepsilon. \end{cases} \quad \text{for all } \varepsilon > 0. \tag{4.8}$$

Proof. Indeed by [6] , we know that

$$u_\varepsilon = kV_\varepsilon \text{ and } v_\varepsilon = lV_\varepsilon \tag{4.9}$$

$$V_\varepsilon = \frac{1}{k} u_\varepsilon \text{ and } V_\varepsilon = \frac{1}{l} v_\varepsilon \tag{4.10}$$

is a solution of he following problem

$$-\left[\operatorname{div}\left(|\nabla V_\varepsilon|^{p-2}\nabla V_\varepsilon\right)\right] = |V_\varepsilon|^{p^*-2} V_\varepsilon$$

then

$$\begin{cases} -\frac{1}{k^{p-1}}\left[\operatorname{div}\left(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\right)\right] = \frac{1}{k^{q-1} l^{q'}} |u_\varepsilon|^{q-2} u_\varepsilon \ \ |v_\varepsilon|^{q'} \\ -\frac{1}{l^{p-1}}\left[\operatorname{div}\left(|\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon\right)\right] = \frac{1}{l^{q'-1} k^{q}} |v_\varepsilon|^{q'-2} v_\varepsilon \ \ |u_\varepsilon|^{q} \end{cases}$$

$$\begin{cases} -\frac{l^{q'}}{k^{p-q}}\left[\operatorname{div}\left(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\right)\right] = |u_\varepsilon|^{q-2} u_\varepsilon \ \ |v_\varepsilon|^{q'} \\ -\frac{k^{q}}{l^{p-q'}}\left[\operatorname{div}\left(|\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon\right)_\varepsilon\right] = |v_\varepsilon|^{q'-2} v_\varepsilon \ \ |u_\varepsilon|^{q} \end{cases}$$

which implies

$$\begin{cases} \dfrac{k^{p-q}}{l^{\,q'}} = \dfrac{2q}{p^*} \\ \dfrac{l^{\,p-q'}}{k^{\,q}} = \dfrac{2q'}{p^*} \end{cases}$$

wiche implies that

$$\begin{cases} k = \left(\dfrac{2}{p^*}\right)^{\frac{1}{(p-p^*)}} (q)^{\frac{p-q'}{p(p-p^*)}} (q')^{\frac{q'}{p(p-p^*)}} \\ l = \left(\dfrac{2}{p^*}\right)^{\frac{1}{(p-p^*)}} (q')^{\frac{p-q}{p(p-p^*)}} q^{\frac{q}{p(p-p^*)}} \end{cases}$$

$$k = \left[\frac{q}{q'}\right]^{\frac{1}{p}} l$$

then

$$\begin{cases} u_\varepsilon = \left(\dfrac{2}{p^*}\right)^{\frac{1}{(p-p^*)}} (q)^{\frac{p-q'}{p(p-p^*)}} (q')^{\frac{q'}{p(p-p^*)}} V_\varepsilon \\ v_\varepsilon = \left(\dfrac{2}{p^*}\right)^{\frac{1}{(p-p^*)}} (q')^{\frac{p-q}{p(p-p^*)}} q^{\frac{q}{p(p-p^*)}} V_\varepsilon \end{cases}$$

solution of (4.6) . ■

Now, we introduce some assumptions :

$(H_6)\, p^* \geq 2p,\ a_1 = a_2 = 0,\ b_1,\ b_2 > 0.$

$(H_7)\, p^* \geq 2p,\ a_1 = 0, a_2 \neq 0,\ b_1,\ b_2 > 0.$

$(H_8)\, p^* \geq 2p,\ a_1 \neq 0,\ a_2 \neq 0,\ b_1,\ b_2 > 0$

Theorem 4.3 *Assume that* $\lambda_1 = \lambda_2 = 0$*. Suppose that* (H_6) *or* (H_7) *or* (H_8) *and* $(u_\varepsilon, v_\varepsilon)$ *is a positive solution of* (4.6)

then we have that the problem (4.1) *has infinitely many nonnegative solutions* $(u'_\varepsilon, v'_\varepsilon)$ *for any* $\theta_1, \theta_2 > 0$*, where*

$$\begin{cases} u'_\varepsilon = \theta_1 u_\varepsilon \\ v'_\varepsilon = \theta_2 v_\varepsilon \end{cases}$$

Proof. The proof of theorem is inspired by the idea in [43],

For any $\theta_1, \theta_2 > 0$ we define $(u'_\varepsilon, v'_\varepsilon) = (\theta_1 u_\varepsilon, \theta_2 v_\varepsilon)$ where $(u_\varepsilon, v_\varepsilon)$ is given in (4.8). Since $(u_\varepsilon, v_\varepsilon)$ is a solution of problem (4.6), then $(u'_\varepsilon, v'_\varepsilon)$ solves the following system:

$$\begin{cases} -\left[\operatorname{div}\left(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\right)\right] = \frac{2q}{q+q'}|u_\varepsilon|^{q-2}u_\varepsilon \ \ |v_\varepsilon|^{q'}, \\ -\left[\operatorname{div}\left(|\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon\right)\right] = \frac{2q'}{q+q'}|u_\varepsilon|^{q}\,|v_\varepsilon|^{q'-2}\,v_\varepsilon, \end{cases}$$

$$\begin{cases} -\left(\frac{1}{\theta_1}\right)^{p-q}\left(\frac{1}{\theta_2}\right)^{-q'}\left[\operatorname{div}\left(|\nabla u'_\varepsilon|^{p-2}\nabla u'_\varepsilon\right)\right] = \frac{2q}{q+q'}|u'_\varepsilon|^{q-2}u'_\varepsilon\,|v'_\varepsilon|^{q'}, \\ -\left(\frac{1}{\theta_1}\right)^{-q}\left(\frac{1}{\theta_2}\right)^{p-q'}\left[\operatorname{div}\left(|\nabla v'_\varepsilon|^{p-2}\nabla v'_\varepsilon\right)\right] = \frac{2q'}{q+q'}|u'_\varepsilon|^{q}\,|v'_\varepsilon|^{q'-2}\,v'_\varepsilon, \end{cases}$$

Moreover, according to (4.5) and (4.8), one has

$$\begin{cases} \left(\frac{1}{\theta_1}\right)^{p-q}\left(\frac{1}{\theta_2}\right)^{-q'} = a_1 + b_1\left\|u'_\varepsilon\right\|^p = a_1 + b_1\theta_1^p\left\|u_\varepsilon\right\|^p, \\ \left(\frac{1}{\theta_1}\right)^{-q}\left(\frac{1}{\theta_2}\right)^{p-q'} = a_2 + b_2\left\|v'_\varepsilon\right\|^p = a_2 + b_2\theta_2^p\left\|v_\varepsilon\right\|^p, \end{cases}$$

$$\begin{cases} \left(\frac{1}{\theta_1}\right)^{p-q}\left(\frac{1}{\theta_2}\right)^{-q'} = a_1 + b_1\theta_1^p\left(\dfrac{2}{p^*}\right)^{\frac{p}{(p-p^*)}}(q)^{\frac{p-q'}{p-p^*}}(q')^{\frac{q'}{p-p^*}}\left\|V_\varepsilon\right\|^p, \\ \left(\frac{1}{\theta_1}\right)^{-q}\left(\frac{1}{\theta_2}\right)^{p-q'} = a_2 + b_2\theta_2^p\left(\dfrac{2}{p^*}\right)^{\frac{p}{(p-p^*)}}(q')^{\frac{p-q}{p-p^*}}q^{\frac{q}{p-p^*}}\left\|V_\varepsilon\right\|^p, \end{cases}$$

$$\begin{cases} \left(\frac{1}{\theta_1}\right)^{p-q}\left(\frac{1}{\theta_2}\right)^{-q'} = a_1 + b_1\theta_1^p\left(\dfrac{2}{p^*}\right)^{\frac{p}{(p-p^*)}}(q)^{\frac{p-q'}{p-p^*}}(q')^{\frac{q'}{p-p^*}}S^{\frac{p^*}{p^*-p}}, \quad (1) \\ \left(\frac{1}{\theta_1}\right)^{-q}\left(\frac{1}{\theta_2}\right)^{p-q'} = a_2 + b_2\theta_2^p\left(\dfrac{2}{p^*}\right)^{\frac{p}{(p-p^*)}}(q')^{\frac{p-q}{p-p^*}}q^{\frac{q}{p-p^*}}S^{\frac{p^*}{p^*-p}}, \quad (2) \end{cases}$$

$$\begin{cases} b_1\theta_1^p\left(\dfrac{2}{p^*}\right)^{\frac{p}{(p-p^*)}}(q)^{\frac{p-q'}{p-p^*}}(q')^{\frac{q'}{p-p^*}}S^{\frac{p^*}{p^*-p}} - (\theta_1)^{q-p}\,\theta_2^{q'} + a_1 = 0 \\ b_2\theta_2^p\left(\dfrac{2}{p^*}\right)^{\frac{p}{(p-p^*)}}(q')^{\frac{p-q}{p-p^*}}q^{\frac{q}{p-p^*}}S^{\frac{p^*}{p^*-p}} - \theta_1^q\theta_2^{q'-p} + a_2 = 0 \quad (3) \end{cases} \tag{4.11}$$

We have

$$\begin{aligned} & b_1\left(\dfrac{2}{p^*}\right)^{\frac{p}{(p-p^*)}}(q)^{\frac{p-q'}{p-p^*}}(q')^{\frac{q'}{p-p^*}}S^{\frac{p^*}{p^*-p}}\theta_1^{2p} \\ & +a_1\theta_1^p - \left(a_2\theta_2^p + b_2\theta_2^{2p}\left(\dfrac{2}{p^*}\right)^{\frac{p}{(p-p^*)}}(q')^{\frac{p-q}{p-p^*}}q^{\frac{q}{p-p^*}}S^{\frac{p^*}{p^*-p}}\right) \\ = \ & 0, \end{aligned}$$

and we have

$$\begin{aligned}\Delta &= a_1^2 + 4b_1 \left(\frac{2}{p^*}\right)^{\frac{p}{(p-p^*)}} (q)^{\frac{p-q'}{p-p^*}} (q')^{\frac{q'}{p-p^*}} S^{\frac{p^*}{p^*-p}} [a_2\theta_2^p \\ &\quad + b_2\theta_2^{2p} \left(\frac{2}{p^*}\right)^{\frac{p}{(p-p^*)}} (q')^{\frac{p-q}{p-p^*}} q^{\frac{q}{p-p^*}} S^{\frac{p^*}{p^*-p}} \\ &> 0,\end{aligned} \tag{4.12}$$

we deduce that

$$\theta_1 = \frac{\left(-a_1 + \sqrt{\Delta}\right)^{\frac{1}{p}}}{(2b_1)^{\frac{1}{p}} \left(\frac{2}{p^*}\right)^{\frac{1}{(p-p^*)}} (q)^{\frac{p-q'}{(p-p^*)P}} (q')^{\frac{q'}{(p-p^*)p}} S^{\frac{p^*}{(p^*-p)p}}},$$

by (4.11) we have

$$b_2 \left(\frac{2}{p^*}\right)^{\frac{p}{(p-p^*)}} (q')^{\frac{p-q}{p-p^*}} q^{\frac{q}{p-p^*}} S^{\frac{p^*}{p^*-p}} \theta_2^p \tag{4.13}$$

$$-\frac{\left(-a_1 + \sqrt{\Delta}\right)^{\frac{q}{p}}}{(2b_1)^{\frac{q}{p}} \left(\frac{2}{p^*}\right)^{\frac{q}{(p-p^*)}} (q)^{\frac{(p-q')q}{(p-p^*)P}} (q')^{\frac{qq'}{(p-p^*)p}} S^{\frac{qp^*}{(p^*-p)p}}} \theta_2^{q'-p} + a_2 \tag{4.14}$$

$$= 0 \tag{4.15}$$

so, θ_2 is solution of (4.13).

$i)$ If $a_1 = a_2 = 0$, we have

$$\begin{cases} \theta_1 = \left(\frac{b_2 q'}{b_1 q}\right)^{\frac{1}{2p}} \theta_2 \\ \theta_2 = \left((b_2)^{\frac{2p-q}{2p}} (b_1)^{\frac{q}{2p}} \left(\frac{2}{p^*}\right)^{\frac{p}{(p-p^*)}} (q')^{\frac{2p(p-q)-q(p-p^*)}{2p(p-p^*)}} q^{\frac{q(p-p^*)}{2p(p-p^*)}} S^{\frac{p^*}{p^*-p}}\right)^{\frac{1}{p^*-2p}} \end{cases}$$

Hence we have

$$\begin{cases} u'_\varepsilon = \left(\frac{b_2 q'}{b_1 q}\right)^{\frac{1}{2p}} \theta_2 u_\varepsilon \\ v'_\varepsilon = \theta_2 v_\varepsilon \\ \theta_2 = \left((b_2)^{\frac{2p-q}{2p}} (b_1)^{\frac{q}{2p}} \left(\frac{2}{p^*}\right)^{\frac{p}{(p-p^*)}} (q')^{\frac{2p(p-q)-q(p-p^*)}{2p(p-p^*)}} q^{\frac{q(p-p^*)}{2p(p-p^*)}} S^{\frac{p^*}{p^*-p}}\right)^{\frac{1}{p^*-2p}} \end{cases}.$$

ii) If $a_1 = 0$ and $a_2 \neq 0$, we have by (4.11)

$$B\theta_2^p - A\left(a_2\theta_2^{\frac{(p^*-2p+q')p}{q}} + C\theta_2^{\frac{(p^*-p)2p}{q}}\right)^{\frac{q}{2p}} + a_2 = 0$$

where

$$A = \frac{\left(4b_1\left(\frac{2}{p^*}\right)^{\frac{p}{(p-p^*)}}(q)^{\frac{p-q'}{p-p^*}}(q')^{\frac{q'}{p-p^*}}S^{\frac{p^*}{p^*-p}}\right)^{\frac{q}{2p}}}{(2b_1)^{\frac{q}{p}}\left(\frac{2}{p^*}\right)^{\frac{q}{(p-p^*)}}(q)^{\frac{(p-q')q}{(p-p^*)P}}(q')^{\frac{qq'}{(p-p^*)p}}S^{\frac{qp^*}{(p^*-p)p}}}$$

$$\begin{aligned} B &= b_2\left(\frac{2}{p^*}\right)^{\frac{p}{(p-p^*)}}(q')^{\frac{p-q}{p-p^*}}q^{\frac{q}{p-p^*}}S^{\frac{p^*}{p^*-p}} \\ C &= b_2\left(\frac{2}{p^*}\right)^{\frac{p}{(p-p^*)}}(q')^{\frac{p-q}{p-p^*}}q^{\frac{q}{p-p^*}}S^{\frac{p^*}{p^*-p}}. \end{aligned}$$

We define

$$f(x) = Bx^p - A\left(a_2x^{\frac{(p^*-2p+q')p}{q}} + Cx^{\frac{(p^*-p)2p}{q}}\right)^{\frac{q}{2p}} + a_2$$

then

$$\begin{aligned} f'(x) &= pBx^{p-1} \\ &\quad -\tfrac{q}{2p}A\left(a_2\frac{(p^*-2p+q')p}{q}x^{\frac{(p^*-2p+q')p-q}{q}} + C\frac{(p^*-p)2p}{q}x^{\frac{(p^*-p)2p-q}{q}}\right) \\ &\quad \times\left(a_2x^{\frac{(p^*-2p+q')p}{q}} + Cx^{\frac{(p^*-p)2p}{q}}\right)^{\frac{q}{2p}-1} \\ &= x^{p-1}[pB \\ &\quad -\tfrac{q}{2p}A\left(a_2\frac{(p^*-2p+q')p}{q}x^{\frac{(q'-p)2p}{q}} + C\frac{(p^*-p)2p}{q}x^{\frac{(q'-p)2p+qp}{q}}\right)\left(a_2x^{\frac{(p^*-2p+q')p}{q}} + Cx^{\frac{(p^*-p)2p}{q}}\right)^{\frac{q}{2p}-1}] \end{aligned}$$

and $g(x) = pB$

$$-\tfrac{q}{2p}A\left(a_2\frac{(p^*-2p+q')p}{q}x^{\frac{(q'-p)2p}{q}} + C\frac{(p^*-p)2p}{q}x^{\frac{(q'-p)2p+qp}{q}}\right)\left(a_2x^{\frac{(p^*-2p+q')p}{q}} + Cx^{\frac{(p^*-p)2p}{q}}\right)^{\frac{q}{2p}-1}.$$

iii) If $a_1 \neq 0$ and $a_2 = 0$, we have by (4.12)

$$\Delta = a_1^2 + 4b_1b_2\left(\frac{2}{p^*}\right)^{\frac{2p}{(p-p^*)}}(q')^{\frac{p-q+q'}{p-p^*}}(q)^{\frac{p-q'+q}{p-p^*}}S^{\frac{2p^*}{p^*-p}}\theta_2^{2p}$$

then (4.13) implies

$$\begin{aligned} f(\theta_2) &= b_2\left(\frac{2}{p^*}\right)^{\frac{p}{(p-p^*)}}(q')^{\frac{p-q}{p-p^*}}q^{\frac{q}{p-p^*}}S^{\frac{p^*}{p^*-p}}\theta_2^p \\ &\quad -\frac{\left(-a_1+\sqrt{\Delta}\right)^{\frac{q}{p}}}{(2b_1)^{\frac{q}{p}}\left(\frac{2}{p^*}\right)^{\frac{q}{(p-p^*)}}(q)^{\frac{(p-q')q}{(p-p^*)P}}(q')^{\frac{qq'}{(p-p^*)p}}S^{\frac{qp^*}{(p^*-p)p}}}\theta_2^{q'-p} \\ &= 0 \end{aligned}$$

$$\begin{aligned} f(\theta_2) &= b_2\left(\frac{2}{p^*}\right)^{\frac{p}{(p-p^*)}}(q')^{\frac{p-q}{p-p^*}}q^{\frac{q}{p-p^*}}S^{\frac{p^*}{p^*-p}}\theta_2^p \\ &\quad -\left(\frac{-a_1+\sqrt{a_1^2+4b_1b_2\left(\frac{2}{p^*}\right)^{\frac{2p}{(p-p^*)}}(q')^{\frac{p-q+q'}{p-p^*}}(q)^{\frac{p-q'+q}{p-p^*}}S^{\frac{2p^*}{p^*-p}}\theta_2^{2p}}}{2b_1\left(\frac{2}{p^*}\right)^{\frac{p}{(p-p^*)}}(q)^{\frac{p-q'}{p-p^*}}(q')^{\frac{q'}{p-p^*}}S^{\frac{p^*}{p^*-p}}}\right)^{\frac{q}{p}}\theta_2^{q'-p} \\ &= 0 \end{aligned}$$

$$f(\theta_2) = b_2\left(\frac{2}{p^*}\right)^{\frac{p}{(p-p^*)}}(q')^{\frac{p-q}{p-p^*}}q^{\frac{q}{p-p^*}}S^{\frac{p^*}{p^*-p}}\theta_2^p$$
$$-\Bigg(\frac{-a_1}{2b_1\left(\frac{2}{p^*}\right)^{\frac{p}{(p-p^*)}}(q)^{\frac{p-q'}{p-p^*}}(q')^{\frac{q'}{p-p^*}}S^{\frac{p^*}{p^*-p}}}$$
$$+\sqrt{\frac{a_1^2+4b_1b_2\left(\frac{2}{p^*}\right)^{\frac{2p}{(p-p^*)}}(q')^{\frac{p-q+q'}{p-p^*}}(q)^{\frac{p-q'+q}{p-p^*}}S^{\frac{2p^*}{p^*-p}}\theta_2^{2p}}{(2b_1)^2\left(\frac{2}{p^*}\right)^{\frac{2p}{(p-p^*)}}(q)^{2\frac{p-q'}{p-p^*}}(q')^{\frac{2q'}{p-p^*}}S^{\frac{2p^*}{p^*-p}}}}\Bigg)^{\frac{q}{p}}\theta_2^{q'-p}$$
$$= 0.$$

Let

$$C_1 = \frac{a_1}{2b_1\left(\frac{2}{p^*}\right)^{\frac{p}{(p-p^*)}}(q)^{\frac{p-q'}{p-p^*}}(q')^{\frac{q'}{p-p^*}}S^{\frac{p^*}{p^*-p}}}$$

then

$$\left(b_2\left(\frac{2}{p^*}\right)^{\frac{p}{(p-p^*)}}(q')^{\frac{p-q}{p-p^*}}q^{\frac{q}{p-p^*}}S^{\frac{p^*}{p^*-p}}\right)\theta_2^p$$
$$-\left(\theta_2^{\frac{(q'-p)p+1}{q}}\sqrt{C_1^2+\frac{q'b_2}{qb_1}\theta_2^{2p}}-C_1\theta_2^{\frac{(q'-p)p+1}{q}}\right)^{\frac{q}{p}}\theta_2^p = 0$$

$$\left(b_2\left(\frac{2}{p^*}\right)^{\frac{p}{(p-p^*)}}(q')^{\frac{p-q}{p-p^*}}q^{\frac{q}{p-p^*}}S^{\frac{p^*}{p^*-p}}\right)\theta_2^p - \left(\sqrt{\theta_2^{\frac{2(q'-p)p+2}{q}}C_1^2+\frac{q'b_2}{qb_1}\theta_2^{\frac{2(q'-p)p+2+2pq}{q}}}-C_1\theta_2^{\frac{(q'-p)p+1}{q}}\right)^{\frac{q}{p}}\theta_2^p=0$$

$$\left(b_2\left(\frac{2}{p^*}\right)^{\frac{p}{(p-p^*)}}(q')^{\frac{p-q}{p-p^*}}q^{\frac{q}{p-p^*}}S^{\frac{p^*}{p^*-p}}\right)\theta_2^p - \left(\sqrt{\theta_2^{\frac{2(q'-p)p+2}{q}}C_1^2+\frac{q'b_2}{qb_1}\theta_2^{\frac{2(p^*-p)p+2}{q}}}-C_1\theta_2^{\frac{(q'-p)p+1}{q}}\right)^{\frac{q}{p}}\theta_2^p=0$$

$$\left(b_2\left(\frac{2}{p^*}\right)^{\frac{p}{(p-p^*)}}(q')^{\frac{p-q}{p-p^*}}q^{\frac{q}{p-p^*}}S^{\frac{p^*}{p^*-p}}\right)^{\frac{p}{q}}+C_1\theta_2^{\frac{(q'-p)p+1}{q}} = \sqrt{\theta_2^{\frac{2(q'-p)p+2}{q}}C_1^2+\frac{q'b_2}{qb_1}\theta_2^{\frac{2(p^*-p)p+2}{q}}}$$

then we have

$$\left(C_2+C_1\theta_2^{\frac{(q'-p)p+1}{q}}\right)^2=\theta_2^{\frac{2(q'-p)p+2}{q}}C_1^2+\frac{q'b_2}{qb_1}\theta_2^{\frac{2(p^*-p)p+2}{q}}$$

where $C_2=\left(b_2\left(\frac{2}{p^*}\right)^{\frac{p}{(p-p^*)}}(q')^{\frac{p-q}{p-p^*}}q^{\frac{q}{p-p^*}}S^{\frac{p^*}{p^*-p}}\right)^{\frac{p}{q}}$

we obtain that

$$C_2^2+C_1^2\theta_2^{\frac{2(q'-p)p+2}{q}}+2C_2C_1\theta_2^{\frac{(q'-p)p+1}{q}}=\theta_2^{\frac{2(q'-p)p+2}{q}}C_1^2+\frac{q'b_2}{qb_1}\theta_2^{\frac{2(p^*-p)p+2}{q}}.$$

Let

$$f(\theta_2)=\frac{q'b_2}{qb_1}\theta_2^{\frac{2(p^*-p)p+2}{q}}-2C_2C_1\theta_2^{\frac{(q'-p)p+1}{q}}-C_2^2=0$$

$$f'(\theta_2)=\frac{2(p^*-p)p+2}{q}\frac{q'b_2}{qb_1}\theta_2^{\frac{2(p^*-p)p+2}{q}-1}-2\frac{(q'-p)p+1}{q}C_2C_1\theta_2^{\frac{(q'-p)p+1}{q}-1}$$

if $q'>p$, we have

if $f'(\theta_2)=0$ we have $\theta_2^0=\left(2\frac{(q'-p)p+1}{2(p^*-p)p+2}\frac{qb_1}{q'b_2}C_2C_1\right)^{\frac{q}{[2p^*-p-q']p+1}}$

such that $f\left(\theta_2^0\right) < 0$, then there exist θ_2^1 such that $f\left(\theta_2^1\right) = 0$.

Hence we have

$$\begin{cases} u'_\varepsilon = \left(\dfrac{b_2 q'}{b_1 q}\right)^{\frac{1}{2p}} \theta_2^1 u_\varepsilon \\ v'_\varepsilon = \theta_2^1 v_\varepsilon. \end{cases}$$

where θ_2^1 is solution of $\frac{q' b_2}{q b_1} \theta_2^{\frac{2(p^*-p)p+2}{q}} - 2C_2C_1\theta_2^{\frac{(q'-p)p+1}{q}} - C_2^2 = 0$

and

$$\begin{aligned} C_1 &= \frac{a_1}{2b_1\left(\dfrac{2}{p^*}\right)^{\frac{p}{(p-p^*)}} (q)^{\frac{p-q'}{p-p^*}} (q')^{\frac{q'}{p-p^*}} S^{\frac{p^*}{p^*-p}}} \\ C_2 &= \left(b_2\left(\dfrac{2}{p^*}\right)^{\frac{p}{(p-p^*)}} (q')^{\frac{p-q}{p-p^*}} q^{\frac{q}{p-p^*}} S^{\frac{p^*}{p^*-p}}\right)^{\frac{p}{q}}. \end{aligned}$$

iv) If $a_1,\ a_2 \neq 0$, we have

$$B\theta_2^p - \frac{\left(-a_1 + \sqrt{\Delta}\right)^{\frac{q}{p}}}{A^{\frac{q}{p}}}\theta_2^{q'-p} + a_2 = 0$$

and $\Delta = a_1^2 + 2A\left(a_2\theta_2^p + B\theta_2^{2p}\right)$, where

$$A = 2b_1\left(\frac{2}{p^*}\right)^{\frac{p}{(p-p^*)}} (q)^{\frac{p-q'}{p-p^*}} (q')^{\frac{q'}{p-p^*}} S^{\frac{p^*}{p^*-p}}$$

$$B = b_2\left(\frac{2}{p^*}\right)^{\frac{p}{(p-p^*)}} (q')^{\frac{p-q}{p-p^*}} q^{\frac{q}{p-p^*}} S^{\frac{p^*}{p^*-p}}.$$

So

$$BA^{\frac{q}{p}}\theta_2^p + A^{\frac{q}{p}}a_2 = \left(-a_1 + \sqrt{a_1^2 + 2A\left(a_2\theta_2^p + B\theta_2^{2p}\right)}\right)^{\frac{q}{p}} \theta_2^{q'-p}$$

$$\begin{aligned} &\left(BA^{\frac{q}{p}}\theta_2^{2p-q'} + A^{\frac{q}{p}}\theta_2^{-(q'-p)}a_2\right)^{\frac{p}{q}} + a_1 = \sqrt{a_1^2 + 2A\left(a_2\theta_2^p + B\theta_2^{2p}\right)} \\ &\left(BA^{\frac{q}{p}}\theta_2^{2p-q'} + A^{\frac{q}{p}}\theta_2^{-(q'-p)}a_2\right)^{\frac{2p}{q}} + 2\left(BA^{\frac{q}{p}}\theta_2^{2p-q'} + A^{\frac{q}{p}}\theta_2^{-(q'-p)}a_2\right)^{\frac{p}{q}} a_1 = 2A\left(a_2\theta_2^p + B\theta_2^{2p}\right) \\ &A^2\left(\left[B\theta_2^p + a_2\right]\theta_2^{p-q'}\right)^{\frac{2p}{q}} + 2\left(\left[B\theta_2^p + a_2\right]\theta_2^{p-q'}\right)^{\frac{p}{q}} Aa_1 = 2A\left(a_2 + B\theta_2^p\right)\theta_2^p \\ &A\left(\left[B\theta_2^p + a_2\right]\theta_2^{p-q'}\right)^{\frac{2p}{q}} + 2\left(\left[B\theta_2^p + a_2\right]\theta_2^{p-q'}\right)^{\frac{p}{q}} a_1 = 2\left(a_2 + B\theta_2^p\right)\theta_2^p. \end{aligned} \tag{4.16}$$

Then there exist $\theta_2 > 0$ such that θ_2 is solution of (4.16). ■

4.4 Geometric conditions of the Mountain Pass Theorem

In first we verify that I_3 satisfies the geometric conditions of the Mountain Pass Theorem.

The following assumptions are used in this section :

(H_4) $p^* = 2p$, $a_2 = a_2 = 0$, $b_1, b_2 > S_{q,q'}^{-2}$.

(H_5) $p^* = 2p$, $b_1, b_2 \geq S_{q,q'}^{-2}$, $a_1, a_2 > 0$.

Lemma 4.4 *Let $f \in W^* \setminus \{0\}$, $a = \max(a_1, a_2)$, $b = \max(b_1, b_2) \geq 0$. Then there exist positive numbers δ_1, ρ_1 and λ_1^*, λ_2^*, $\lambda_3^* > 0$ such that*

$$I_3(u,v) \geq \delta_1 > 0, \quad \textit{with } \|u,v\| = \rho_1,$$

and

$$\begin{cases} \lambda_1 \leq \lambda_1^* & \textit{if } \lambda_1 \neq 0 \textit{ and } \lambda_2 = 0 \\ \lambda_2 \leq \lambda_2^* & \textit{if } \lambda_1 = 0 \textit{ and } \lambda_2 \neq 0 \\ \min(\lambda_1, \lambda_2) \leq \lambda_3^* & \textit{if } \lambda_1 \neq 0 \textit{ and } \lambda_2 \neq 0 \end{cases} \tag{4.17}$$

and

$$I_3(u,v) \geq \begin{cases} -\frac{p-1}{p}\left(\frac{a}{2}\right)^{\frac{-1}{p-1}}\left((\lambda_1 \|f\|_{W^*})^{\frac{p}{p-1}} + (\lambda_2 \|g\|_{W^*})^{\frac{p}{p-1}}\right) & \textit{if } (H_4) \\ -\frac{2p-1}{2p}\left(\frac{b}{4}\right)^{-\frac{1}{2p-1}}\left[(\lambda_1 \|f\|_{W^*})^{\frac{2p}{2p-1}} + (\lambda_2 \|g\|_{W^*})^{\frac{2p}{2p-1}}\right] & \textit{if } (H_5) \end{cases}$$

for all $(u,v) \in B_{\rho_1}(0,0)$.

Proof. Let $(u,v) \in W^{1,p}\backslash\{(0,0)\} \times W^{1,p}\backslash\{(0,0)\}$, $a = \max(a_1,a_2)$ and $b = \max(b_1,b_2) \geq 0$,

$$\begin{aligned} I_3(u,v) &= \frac{1}{2p}\left(b_1\|u\|^{2p} + b_2\|v\|^{2p}\right) + \frac{1}{p}\left(a_1\|u\|^p + a_2\|v\|^p\right) \\ &\quad -\frac{2}{p^*}\int_{\mathbb{R}^N}|u|^q|v|^{q'}dx - \int_{\mathbb{R}^N}\lambda_1 f(x)u + \lambda_2 g(x)v dx \\ &\geq \frac{b}{2p}\left(\|u\|^{2p} + \|v\|^{2p}\right) + \frac{a}{p}\left(\|u\|^p + \|v\|^p\right) \\ &\quad -\frac{2}{p^*}\int_{\mathbb{R}^N}|u|^q|v|^{q'}dx - \int_{\mathbb{R}^N}\lambda_1 f(x)u + \lambda_2 g(x)v dx \end{aligned}$$

by the elementary inequality

$$x^2 + y^2 \geq \frac{1}{2}(x+y)^2$$

we have that

$$\begin{aligned} I_3(u,v) &\geq \frac{b}{4p}\left(\|u\|^p + \|v\|^p\right)^2 + \frac{a}{p}\left(\|u\|^p + \|v\|^p\right) \\ &\quad -\frac{2}{p^*}\int_{\mathbb{R}^N}|u|^q|v|^{q'}dx - \int_{\mathbb{R}^N}\lambda_1 f(x)u + \lambda_2 g(x)v dx \\ &\geq \frac{b}{4p}\|u,v\|^{2p} + \frac{a}{p}\|u,v\|^p \\ &\quad -\frac{2}{p^*}\int_{\mathbb{R}^N}|u|^q|v|^{q'}dx - \int_{\mathbb{R}^N}\lambda_1 f(x)u + \lambda_2 g(x)v dx \end{aligned}$$

by the definition of $S_{q,q'}$, we have

$$\begin{aligned} I_3(u,v) &\geq \frac{b}{4p}\|u,v\|^{2p} + \frac{a}{p}\|u,v\|^p - \frac{2}{p^*}S_{q,q'}^{-p^*/p}\|u,v\|^{p^*} \\ &\quad -\lambda_1\|f\|_{W^*}\|u\| - \lambda_2\|g\|_{W^*}\|v\|. \end{aligned}$$

When $b \geq 0$, $a > 0$ and $p^* \geq 2p$, we have that

$$\begin{aligned} I_3(u,v) &\geq \frac{a}{p}\|u,v\|^p - \frac{2}{p^*}S_{q,q'}^{-p^*/p}\|u,v\|^{p^*} \\ &\quad -\left(\frac{a}{2}\right)^{\frac{-1}{p}}\lambda_1\|f\|_{W^*}\left(\frac{a}{2}\right)^{\frac{1}{p}}\|u\| - \left(\frac{a}{2}\right)^{\frac{-1}{p}}\lambda_2\|g\|_{W^*}\left(\frac{a}{2}\right)^{\frac{1}{p}}\|v\| \end{aligned}$$

by the elementary inequality

$$xy < \frac{x^p}{p} + \frac{y^q}{q}, \ x > 0, \ y > 0 \quad \text{such that } \frac{1}{p} + \frac{1}{q} = 1$$

we have that

$$\begin{aligned}
I_3(u,v) \ \geq \ & \frac{a}{p} \left\| (u,v) \right\|^p - \frac{p}{p^*} S_{q,q'}^{-p^*/p} \left\| (u,v) \right\|^{p^*} \\
& -\frac{p-1}{p} \left(\left(\frac{a}{2}\right)^{\frac{-1}{p}} \lambda_1 \left\| f \right\|_{W^*} \right)^{\frac{p}{p-1}} - \frac{1}{p} \left(\left(\frac{a}{2}\right)^{\frac{1}{p}} \left\| u \right\| \right)^p \\
& -\frac{p-1}{p} \left(\left(\frac{a}{2}\right)^{\frac{-1}{p}} \lambda_2 \left\| g \right\|_{W^*} \right)^{\frac{p}{p-1}} - \frac{1}{p} \left(\left(\frac{a}{2}\right)^{\frac{1}{p}} \left\| v \right\| \right)^p \\
\geq \ & \frac{a}{p} \left\| u,v \right\|^p - \frac{2}{p^*} S_{q,q'}^{-p^*/p} \left\| u,v \right\|^{p^*} - \frac{a}{2p} \left\| u \right\|^p \\
& -\frac{a}{2p} \left\| v \right\|^p - \frac{p-1}{p} \left(\left(\frac{a}{2}\right)^{\frac{-1}{p}} \lambda_1 \left\| f \right\|_{W^*} \right)^{\frac{p}{p-1}} - \frac{p-1}{p} \left(\left(\frac{a}{2}\right)^{\frac{-1}{p}} \lambda_2 \left\| g \right\|_{W^*} \right)^{\frac{p}{p-1}} \\
\geq \ & \frac{a}{2p} \left\| u,v \right\|^p - \frac{2}{p^*} S_{q,q'}^{-p^*/p} \left\| u,v \right\|^{p^*} \\
& -\frac{p-1}{p} \left(\left(\frac{a}{2}\right)^{\frac{-1}{p}} \lambda_1 \left\| f \right\|_{W^*} \right)^{\frac{p}{p-1}} - \frac{p-1}{p} \left(\left(\frac{a}{2}\right)^{\frac{-1}{p}} \lambda_2 \left\| g \right\|_{W^*} \right)^{\frac{p}{p-1}}.
\end{aligned}$$

Let $\rho = \left\| u,v \right\|$ we have that

$$\begin{aligned}
I_3(u,v) \ \geq \ & \frac{a}{2p} \rho^p - \frac{2}{p^*} S_{q,q'}^{-p^*/p} \rho^{p^*} \\
& -\frac{p-1}{p} \left(\left(\frac{a}{2}\right)^{\frac{-1}{p}} \lambda_1 \left\| f \right\|_{W^*} \right)^{\frac{p}{p-1}} - \frac{p-1}{p} \left(\left(\frac{a}{2}\right)^{\frac{-1}{p}} \lambda_2 \left\| g \right\|_{W^*} \right)^{\frac{p}{p-1}}.
\end{aligned}$$

Now we consider the function $h : \mathbb{R}^+ \to \mathbb{R}^*$, given by

$$h(\rho) = \frac{a}{2p} \rho^p - \frac{2}{p^*} S_{q,q'}^{-p^*/p} \rho^{p^*}$$

direct calculation shows that

$$h(\rho) \geq 0 \ \text{ for all } \rho \leq \rho_1 \text{ with } \rho_1 = \left(\frac{a}{2p} S_{q,q'}^{p^*/p} \right)^{\frac{1}{p^*-p}}$$

we immediately derive that

$$I_3(u,v)|_{B_{\rho_1}(0,0)} \geq -\frac{p-1}{p} \left(\frac{a}{2}\right)^{\frac{-1}{p-1}} \left[\left(\lambda_1 \left\| f \right\|_{W^*}\right)^{\frac{p}{p-1}} + \left(\lambda_2 \left\| g \right\|_{W^*}\right)^{\frac{p}{p-1}} \right].$$

So, for $\|u,v\| = \rho_1$ we have

$$\begin{aligned} I_3(u,v) &\geq h(\rho_1) - \frac{p-1}{p}\left(\frac{a}{2}\right)^{\frac{-1}{p-1}} \left[(\lambda_1 \|f\|_{W^*})^{\frac{p}{p-1}} + (\lambda_2 \|g\|_{W^*})^{\frac{p}{p-1}}\right] \\ &\geq \frac{1}{p}h(\rho_1) + \frac{p-1}{p}h(\rho_1) - \frac{p-1}{p}\left(\frac{a}{2}\right)^{\frac{-1}{p-1}} \left[(\lambda_1 \|f\|_{W^*})^{\frac{p}{p-1}} + (\lambda_2 \|g\|_{W^*})^{\frac{p}{p-1}}\right] \\ &\geq \frac{1}{p}h(\rho_1) \end{aligned}$$

for

$$\begin{aligned} h(\rho_1) &\geq \left(\frac{a}{2}\right)^{\frac{-1}{p-1}} \left((\lambda_1 \|f\|_{W^*})^{\frac{p}{p-1}} + (\lambda_2 \|g\|_{W^*})^{\frac{p}{p-1}}\right) \\ &\geq \begin{cases} \left(\frac{a}{2}\right)^{\frac{-1}{p-1}} (\lambda_1 \|f\|_{W^*})^{\frac{p}{p-1}} & \text{if } \lambda_1 \neq 0 \text{ and } \lambda_2 = 0 \\ \left(\frac{a}{2}\right)^{\frac{-1}{p-1}} (\lambda_2 \|g\|_{W^*})^{\frac{p}{p-1}} & \text{if } \lambda_1 = 0 \text{ and } \lambda_2 \neq 0. \end{cases}. \end{aligned}$$

Finally, we obtain

$$I_3(u,v) \geq \frac{1}{p}h(\rho_1) = \frac{p^*-p}{pp^*} S_{q,q'}^{\frac{p}{p^*-p}} \left(\frac{a}{2p}\right)^{\frac{p^*}{p^*-p}},$$

for

$$0 < \lambda_1 \leq \left(\frac{1}{p}\right)^{\frac{p^*(p-1)}{p(p^*-p)}} \left(\frac{p^*-p}{p^*}\right)^{\frac{p-1}{p}} S_{q,q'}^{\frac{p-1}{p^*-p}} \left(\frac{a}{2}\right)^{\frac{p^*-1}{p^*-p}} \|f\|_{W^*}^{-1} \text{ and } \lambda_2 = 0$$

or

$$0 < \lambda_2 \leq \left(\frac{1}{p}\right)^{\frac{p^*(p-1)}{p(p^*-p)}} \left(\frac{p^*-p}{p^*}\right)^{\frac{p-1}{p}} S_{q,q'}^{\frac{p-1}{p^*-p}} \left(\frac{a}{2}\right)^{\frac{p^*-1}{p^*-p}} \|g\|_{W^*}^{-1} \text{ and } \lambda_1 = 0$$

and if $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ we have

$$\min(\lambda_1, \lambda_2) \leq \left(\frac{1}{p}\right)^{\frac{p^*(p-1)}{p(p^*-p)}} \left(\frac{p^*-p}{p^*}\right)^{\frac{p-1}{p}} \left(\frac{a}{2}\right)^{\frac{p^*-1}{p^*-p}} S_{q,q'}^{\frac{p-1}{p^*-p}} (\|f\|_{W^*}^{-1} + \|g\|_{W^*}^{-1}).$$

Then we can choose $\delta_1,\ \rho_1$and $\lambda_1^*,\ \lambda_2^*,\ \lambda_3^*$ such that

$$\delta_1 = \frac{p^*-p}{pp^*} S_{q,q'}^{\frac{p}{p^*-p}} \left(\frac{a}{2p}\right)^{\frac{p^*}{p^*-p}},\ \rho_1 = \left(\frac{a}{2p} S_{q,q'}^{p^*/p}\right)^{\frac{1}{p^*-p}},$$

$$\lambda_1^* = \left(\frac{1}{p}\right)^{\frac{p^*(p-1)}{p(p^*-p)}} \left(\frac{p^*-p}{p^*}\right)^{\frac{p-1}{p}} S_{q,q'}^{\frac{p-1}{p^*-p}} \left(\frac{a}{2}\right)^{\frac{p^*-1}{p^*-p}} \|f\|_{W^*}^{-1} \quad \text{if } \lambda_1 \neq 0 \text{ and } \lambda_2 = 0,$$

$$\lambda_2^* = \left(\frac{1}{p}\right)^{\frac{p^*(p-1)}{p(p^*-p)}} \left(\frac{p^*-p}{p^*}\right)^{\frac{p-1}{p}} S_{q,q'}^{\frac{p-1}{p^*-p}} \left(\frac{a}{2}\right)^{\frac{p^*-1}{p^*-p}} \|g\|_{W^*}^{-1} \text{ if } \lambda_1 = 0 \text{ and } \lambda_2 \neq 0,$$

and

$$\lambda_3^* = \left(\frac{1}{p}\right)^{\frac{p^*(p-1)}{p(p^*-p)}} \left(\frac{p^*-p}{p^*}\right)^{\frac{p-1}{p}} \left(\frac{a}{2}\right)^{\frac{p^*-1}{p^*-p}} S_{q,q'}^{\frac{p-1}{p^*-p}} (\|f\|_{W^*}^{-1} + \|g\|_{W^*}^{-1}) \text{ if } \lambda_1 \neq 0 \text{ and } \lambda_2 \neq 0$$

When $b > 0,\ a = 0$ and $p^* > 2p$, we have that

$$\begin{aligned} I_3(u,v) &\geq \tfrac{b}{4p} \|u,v\|^{2p} - \tfrac{2}{p^*} S_{q,q'}^{-p^*/p} \|u,v\|^{p^*} - (\lambda_1 \|f\|_{W^*}) \|u\| - (\lambda_2 \|g\|_{W^*}) \|v\| \\ &\geq \tfrac{b}{4p} \|(u,v)\|^{2p} - \tfrac{2}{p^*} S_{q,q'}^{-p^*/p} \|(u,v)\|^{p^*} - \left(\lambda_1 \left(\tfrac{b}{4}\right)^{-\frac{1}{2p}} \|f\|_{W^*}\right) \left(\tfrac{b}{4}\right)^{\frac{1}{2p}} \|u\| \\ &- \left(\lambda_2 \left(\tfrac{b}{4}\right)^{-\frac{1}{2p}} \|g\|_{W^*}\right) \left(\tfrac{b}{4}\right)^{\frac{1}{2p}} \|v\|. \end{aligned}$$

By the elementary inequality: $xy < \frac{x^p}{p} + \frac{y^q}{q},\ x > 0,\ y > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$ we have that

$$\begin{aligned} I_3(u,v) \quad &\geq \quad \frac{b}{4p} \|u,v\|^{2p} - \frac{2}{p^*} S_{q,q'}^{-p^*/p} \|u,v\|^{p^*} - \frac{2p-1}{2p} \left(\lambda_1 \left(\frac{b}{4}\right)^{-\frac{1}{2p}} \|f\|_{W^*}\right)^{\frac{2p}{2p-1}} - \frac{b}{8p} \|u\|^{2p} \\ &\quad -\frac{2p-1}{2p} \left(\lambda_2 \left(\frac{b}{4}\right)^{-\frac{1}{2p}} \|g\|_{W^*}\right)^{\frac{2p}{2p-1}} - \frac{b}{8p} \|v\|^{2p} \\ &\geq \quad \frac{b}{4p} \|(u,v)\|^{2p} - \frac{2}{p^*} S_{q,q'}^{-p^*/p} \|(u,v)\|^{p^*} - \frac{2p-1}{2p} \left(\lambda_1 \left(\frac{b}{4}\right)^{-\frac{1}{2p}} \|f\|_{W^*}\right)^{\frac{2p}{2p-1}} - \frac{b}{8p} \|u\|^{2p} \\ &\quad -\frac{2p-1}{2p} \left(\lambda_2 \left(\frac{b}{4}\right)^{-\frac{1}{2p}} \|g\|_{W^*}\right)^{\frac{2p}{2p-1}} - \frac{b}{8p} \|v\|^{2p} \\ &= \quad \frac{b}{8p} (\|u\|^{2p} + \|v\|^{2p} + 4 \|u\|^p \|v\|^p) - \frac{p}{p^*} S_{q,q'}^{-p^*/p} \|(u,v)\|^{p^*} \\ &\quad -\frac{2p-1}{2p} \left[\left(\lambda_1 \left(\frac{b}{4}\right)^{-\frac{1}{2p}} \|f\|_{W^*}\right)^{\frac{2p}{2p-1}} + \left(\lambda_2 \left(\frac{b}{4}\right)^{-\frac{1}{2p}} \|g\|_{W^*}\right)^{\frac{2p}{2p-1}}\right] \\ &> \quad \frac{b}{8p} (\|u\|^{2p} + \|v\|^{2p} + 2 \|u\|^p \|v\|^p) - \frac{2}{p^*} S_{q,q'}^{-p^*/p} \|(u,v)\|^{p^*} \\ &\quad -\frac{2p-1}{2p} \left[\left(\lambda_1 \left(\frac{b}{4}\right)^{-\frac{1}{2p}} \|f\|_{W^*}\right)^{\frac{2p}{2p-1}} + \left(\lambda_2 \left(\frac{b}{4}\right)^{-\frac{1}{2p}} \|g\|_{W^*}\right)^{\frac{2p}{2p-1}}\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{b}{8p}\|(u,v)\|^{2p} - \frac{2}{p^*}S_{q,q'}^{-p^*/p}\|(u,v)\|^{p^*} \\
&\quad -\frac{2p-1}{2p}\left(\frac{b}{4}\right)^{-\frac{1}{2p-1}}\left[(\lambda_1\|f\|_{W^*})^{\frac{2p}{2p-1}} + (\lambda_2\|g\|_{W^*})^{\frac{2p}{2p-1}}\right].
\end{aligned}
$$

Let $\rho = \|(u,v)\|$ we have that

$$
\begin{aligned}
I_3(u,v) &\geq \frac{b}{8p}\rho^{2p} - \frac{2}{p^*}S_{q,q'}^{-p^*/p}\rho^{p^*} \\
&\quad -\frac{2p-1}{2p}\left(\frac{b}{4}\right)^{-\frac{1}{2p-1}}\left[(\lambda_1\|f\|_{W^*})^{\frac{2p}{2p-1}} + (\lambda_2\|g\|_{W^*})^{\frac{2p}{2p-1}}\right].
\end{aligned}
$$

Now we consider the function $h : \mathbb{R}^+ \to \mathbb{R}^*$, given by

$$
h(\rho) = -\frac{2}{p^*}S_{q,q'}^{-p^*/p}\rho^{p^*} + \frac{b}{8p}\rho^{2p}
$$

and

$$
h'(\rho) = \rho^{2p-1}\left(-pS_{q,q'}^{-p^*/p}\rho^{p^*-2p} + \frac{b}{4}\right).
$$

Thus, $h'(\rho) = 0$ has a unique positive solution $\rho_1 = \left(\frac{b}{4p}S_{q,q'}^{p^*/p}\right)^{\frac{1}{p^*-2p}}$. Thus, direct calculation shows that

$$
h(\rho) \geq 0 \quad \text{for all } \rho \leq \rho_1,
$$

we immediately derive that

$$
I_3(u,v)|_{B_{\rho_1}(0,0)} \geq -\frac{2p-1}{2p}\left(\frac{b}{4}\right)^{-\frac{1}{2p-1}}\left[(\lambda_1\|f\|_{W^*})^{\frac{2p}{2p-1}} + (\lambda_2\|g\|_{W^*})^{\frac{2p}{2p-1}}\right].
$$

So, for $\|u,v\| = \rho_1$ we have

$$
\begin{aligned}
I_3(u,v) &\geq h(\rho_1) - \frac{2p-1}{2p}\left(\frac{b}{4}\right)^{-\frac{1}{2p-1}}\left[(\lambda_1\|f\|_{W^*})^{\frac{2p}{2p-1}} + (\lambda_2\|g\|_{W^*})^{\frac{2p}{2p-1}}\right] \\
&\geq \frac{1}{2p}h(\rho_1) + \frac{2p-1}{2p}h(\rho_1) - \frac{2p-1}{2p}\left(\frac{b}{4}\right)^{-\frac{1}{2p-1}}\left[(\lambda_1\|f\|_{W^*})^{\frac{2p}{2p-1}} + (\lambda_2\|g\|_{W^*})^{\frac{2p}{2p-1}}\right] \\
&\geq \frac{1}{2p}h(\rho_1)
\end{aligned}
$$

for

$$
\begin{aligned}
h(\rho_1) &\geq \left(\frac{b}{4}\right)^{-\frac{1}{2p-1}} \left[(\lambda_1 \|f\|_{W^*})^{\frac{2p}{2p-1}} + (\lambda_2 \|g\|_{W^*})^{\frac{2p}{2p-1}}\right] \\
&\geq \begin{cases} \left(\frac{b}{4}\right)^{\frac{-1}{2p-1}} (\lambda_1 \|f\|_{W^*})^{\frac{2p}{2p-1}} & \text{if } \lambda_1 \neq 0 \text{ and } \lambda_2 = 0 \\ \left(\frac{b}{4}\right)^{\frac{-1}{2p-1}} (\lambda_2 \|g\|_{W^*})^{\frac{2p}{2p-1}} & \text{if } \lambda_1 = 0 \text{ and } \lambda_2 \neq 0. \end{cases}.
\end{aligned}
$$

Finally, we obtain

$$
I_3(u,v) \geq \frac{p^* - 2p}{4pp^*} S_{q,q'}^{\frac{2p^*}{p^*-2p}} \left(\frac{b}{4p}\right)^{\frac{p^*}{p^*-2p}},
$$

for

$$
\begin{cases} \lambda_1 \leq \left(\frac{b}{4}\right)^{\frac{1}{2p}} \left[\frac{p^*-2p}{2p^*} S_{q,q'}^{\frac{2p^*}{p^*-2p}} \left(\frac{b}{4p}\right)^{\frac{p^*}{p^*-2p}}\right]^{\frac{2p}{2p-1}} \|f\|_{W^*}^{-1} & \text{if } \lambda_1 \neq 0 \text{ and } \lambda_2 = 0 \\ \lambda_2 \leq \left(\frac{b}{4}\right)^{\frac{1}{2p}} \left[\frac{p^*-2p}{2p^*} S_{q,q'}^{\frac{2p^*}{p^*-2p}} \left(\frac{b}{4p}\right)^{\frac{p^*}{p^*-2p}}\right]^{\frac{2p}{2p-1}} \|g\|_{W^*}^{-1} & \text{if } \lambda_1 = 0 \text{ and } \lambda_2 \neq 0, \end{cases}
$$

and if $\lambda_1 \neq 0,\ \lambda_2 \neq 0$,we have

$$
\min(\lambda_1, \lambda_2) \leq \left(\frac{b}{4}\right)^{\frac{1}{2p}} \left[\frac{p^* - 2p}{2p^*} S_{q,q'}^{\frac{2p^*}{p^*-2p}} \left(\frac{b}{4p}\right)^{\frac{p^*}{p^*-2p}}\right]^{\frac{2p}{2p-1}} (\|f\|_{W^*}^{-1} + \|g\|_{W^*}^{-1}).
$$

Then we can choose $\delta_1,\ \rho_1$and $\lambda_1^*, \lambda_2^*, \lambda_3^*$ are positives such that

$$
\begin{aligned}
\delta_1 &= \frac{p^* - 2p}{4pp^*} S_{q,q'}^{\frac{2p^*}{p^*-2p}} \left(\frac{b}{4p}\right)^{\frac{p^*}{p^*-2p}}, \\
\rho_1 &= \left(\frac{b}{4p} S_{q,q'}^{p^*/p}\right)^{\frac{1}{p^*-2p}}
\end{aligned}
$$

and

$$
\begin{cases} \lambda_1^* = \left(\frac{b}{4}\right)^{\frac{1}{2p}} \left[\frac{p^*-2p}{2p^*} S_{q,q'}^{\frac{2p^*}{p^*-2p}} \left(\frac{b}{4p}\right)^{\frac{p^*}{p^*-2p}}\right]^{\frac{2p}{2p-1}} \|f\|_{W^*}^{-1} & \text{if } \lambda_1 \neq 0 \text{ and } \lambda_2 = 0 \\ \lambda_2^* = \left(\frac{b}{4}\right)^{\frac{1}{2p}} \left[\frac{p^*-2p}{2p^*} S_{q,q'}^{\frac{2p^*}{p^*-2p}} \left(\frac{b}{4p}\right)^{\frac{p^*}{p^*-2p}}\right]^{\frac{2p}{2p-1}} \|g\|_{W^*}^{-1} & \text{if } \lambda_1 = 0 \text{ and } \lambda_2 \neq 0 \end{cases}
$$

and if $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$

$$\lambda_3^* = \left(\frac{b}{4}\right)^{\frac{1}{2p}} \left[\frac{p^* - 2p}{2p^*} S_{q,q'}^{\frac{2p^*}{p^*-2p}} \left(\frac{b}{4p}\right)^{\frac{p^*}{p^*-2p}}\right]^{\frac{2p}{2p-1}} \left(\|f\|_{W^*}^{-1} + \|g\|_{W^*}^{-1}\right)$$

This completes the proof of Lemma. ■

4.5 Palais Smale condition

Lemma 4.5 *Suppose that f , g $\in W^*\setminus\{0\}$ and assume that (H_4) or (H_5) holds. Let $c \in \mathbb{R}$ and $(u_n, v_n) \subset W^{1,p}\left(\mathbb{R}^N\right) \times W^{1,p}\left(\mathbb{R}^N\right)$ be a Palais Smale sequence for I_3, then*

$$(u_n, v_n) \rightharpoonup (u, v) \text{ in } W^{1,p}\left(\mathbb{R}^N\right) \times W^{1,p}\left(\mathbb{R}^N\right),$$

for some $(u, v) \in W^{1,p}\left(\mathbb{R}^N\right) \times W^{1,p}\left(\mathbb{R}^N\right)$ with $I_3'(u, v) = 0$.

Proof. Let $(u_n, v_n) \subset W^{1,p}\left(\mathbb{R}^N\right) \times W^{1,p}\left(\mathbb{R}^N\right)$ be a Palais Smale sequence for I_3 such that

$$I_3(u_n, v_n) \to c \in \mathbb{R}$$

and

$$I_3'(u_n, v_n) \to 0.$$

We have

$$\begin{aligned} c + o_n(1) &= I_3(u_n, v_n) \\ o_n(1) &= \langle I_3'(u_n, v_n), (u_n, v_n)\rangle, \end{aligned}$$

that is

$$c + o(\|u_n, v_n\|) = I_3(u_n, v_n) - \frac{1}{p^*}\langle I_3'(u_n, v_n), (u_n, v_n)\rangle$$

$$\begin{aligned}
c+o\left(\left\|u_n,v_n\right\|\right) &= \frac{1}{2p}\left(b_1\left\|u_n\right\|^{2p}+b_2\left\|v_n\right\|^{2p}\right)+\frac{1}{p}\left(a_1\left\|u_n\right\|^{p}+a_2\left\|v_n\right\|^{p}\right) \\
&\quad -\frac{2}{p^*}\int\limits_{\mathbb{R}^N}|u_n|^q|v_n|^{q'}dx-\int\limits_{\mathbb{R}^N}\lambda_1 f(x)\,u_n+\lambda_2 g(x)\,v_n dx \\
&\quad -\frac{1}{p^*}\left(b_1\left\|u_n\right\|^{2p}+b_2\left\|v_n\right\|^{2p}\right)-\frac{1}{p^*}\left(a_1\left\|u_n\right\|^{p}+a_2\left\|v_n\right\|^{p}\right) \\
&\quad +\frac{2}{p^*}\int\limits_{\mathbb{R}^N}|u_n|^q|v_n|^{q'}dx+\frac{1}{p^*}\int\limits_{\mathbb{R}^N}\lambda_1 f(x)\,u_n+\lambda_2 g(x)\,v_n dx \\
&= \frac{p^*-2p}{2pp^*}\left(b_1\left\|u_n\right\|^{2p}+b_2\left\|v_n\right\|^{2p}\right)+\frac{p^*-p}{pp^*}\left(a_1\left\|u_n\right\|^{p}+a_2\left\|v_n\right\|^{p}\right) \\
&\quad -\frac{p^*-1}{p^*}\int\limits_{\mathbb{R}^N} f(x)\,u_n\lambda_1+\lambda_2 g(x)\,v_n dx,
\end{aligned}$$

using $a=\max(a_1,a_2)$ and $b=\max(b_1,b_2)$, we have

$$\begin{aligned}
c+o\left(\left\|u_n,v_n\right\|\right) &\geq \frac{p^*-2p}{4pp^*}b\left\|(u_n,v_n)\right\|^{2p}+a\frac{p^*-p}{pp^*}\left\|(u_n,v_n)\right\|^{p} \\
&\quad -\frac{p^*-1}{p^*}\int\limits_{\mathbb{R}^N}\lambda_1 f(x)\,u_n+\lambda_2 g(x)\,v_n dx.
\end{aligned}$$

Then (u_n,v_n) is bounded in $W^{1,p}\left(\mathbb{R}^N\right)\times W^{1,p}\left(\mathbb{R}^N\right)$. Up to a subsequence if necessary, we obtain

$$\begin{aligned}
(u_n,v_n) &\rightharpoonup (u,v) \text{ in } W^{1,p}_{\alpha,\mu}\left(\mathbb{R}^N\right)\times W^{1,p}_{\alpha,\mu}\left(\mathbb{R}^N\right) \\
(u_n,v_n) &\rightharpoonup (u,v) \text{ in } L^{p^*}\left(\mathbb{R}^N\right), \\
(u_n,v_n) &\to (u,v) \text{ a. e. in } \mathbb{R}^N\times\mathbb{R}^N
\end{aligned}$$

and

$$\begin{aligned}
\int\limits_{\mathbb{R}^N} f(x)\,u_n dx &\to \int\limits_{\mathbb{R}^N} f(x)\,u dx \\
\int\limits_{\mathbb{R}^N} g(x)\,v_n dx &\to \int\limits_{\mathbb{R}^N} g(x)\,v dx.
\end{aligned}$$

Then

$$\left\langle I_3'(u_n,v_n),(\varphi,\psi)\right\rangle=0 \text{ for all } (\varphi,\psi)\in C_0^\infty\left(\mathbb{R}^N\right),$$

thus $I_3'(u,v)=0$. This completes the proof. ■

4.6 Existence of a critical point with negative energy

In this section we prove the existence of critical point with negative energy.

Theorem 4.6 *Suppose that* $f,g\in W^{*}\backslash\{0\}$ *and assume that* (H_4) *or* (H_5) *holds, then there exist constants* $\lambda_1^*,\lambda_2^*,\lambda_3^*>0$ *such that for any* λ_1,λ_2 *verifying* (4.17)*, system* (4.1) *has a solution* (u_1,v_1) *with negative energy.*

Proposition 4.7 *Let* $f,g\in W^{*}\backslash\{0\}$ *and* $p^*\geq 2p$*. For all* $\lambda_1,\ \lambda_2$*verifying* (4.17)*, there exists a nontrivial solution* (u_1,v_1) *of* (4.1) *with negative energy.*

Proof. First, by Lemma 4.4, we can define

$$c_1=\inf\left\{I_3\left(u,v\right),\ \ (u,v)\in\overline{B}_{\rho_1}\left(0,0\right)\right\} \tag{4.18}$$

.Now we claim that $-\infty<c_1<0$. As $f,g\in W^{*}\backslash\{0\}$ we can choose $\varphi_1,\varphi_1\in W^{1,p}\left(\mathbb{R}^N\right)\times W^{1,p}\left(\mathbb{R}^N\right)$ such that

$$\int\limits_{\mathbb{R}^N} f\left(x\right)\varphi_1 dx \text{ or } \int\limits_{\mathbb{R}^N} g\left(x\right)\varphi_2 dx>0.$$

Then, for a fixed λ_1 and λ_2 in (4.17), there exists $t_0>0$ such that $t_0\left\|\varphi_1,\varphi_2\right\|<\rho_1$ and $I_3\left(t_0\varphi_1,t_0\varphi_2\right)<0$ for $t\in\left]0,t_0\right[$.

Hence,

$$c_1<I_3(0,0)=0.$$

Using the Ekeland's variational principle, for the complete metric space $\overline{B}_{\rho_1}(0,0)$ with respect to the norm of $W^{1,p}(\mathbb{R}^N)$, we obtain the existence of a Palais-Smale sequence $(u_n,v_n)\in\overline{B}_{\rho_1}(0,0)$ at level c_1, and from Lemma 4.4 we have $(u_n,v_n)\rightharpoonup(u_1,v_1)$ in $W^{1,p}(\mathbb{R}^N)\times W^{1,p}(\mathbb{R}^N)$ for some (u_1,v_1) with $\|u_1,v_1\|<\rho_1$.

Now, we shall show that $(u_n,v_n)\to(u_1,v_1)$ in $W^{1,p}(\mathbb{R}^N)\times W^{1,p}(\mathbb{R}^N)$. Suppose otherwise, then $\|u_1,v_1\| < \underline{\lim}_{n\to+\infty}\|u_n,v_n\|$, which implies that

$$\begin{aligned}
c_1 &\le I_3(u_1,v_1)\\
&= I_3(u_1,v_1)-\frac{1}{p^*}\langle I_3'(u_1,v_1),(u_1,v_1)\rangle\\
&= \frac{p^*-2p}{2pp^*}\left(b_1\|u_1\|^{2p}+b_2\|v_1\|^{2p}\right)+\frac{p^*-p}{pp^*}\left(a_1\|u_1\|^p+a_2\|v_1\|^p\right)\\
&\quad -\frac{p^*-1}{p^*}\int_{\mathbb{R}^N} f(x)\,u_1\lambda_1+\lambda_2 g(x)\,v_1dx,\\
&< \underline{\lim}_{n\to+\infty}\left[\frac{p^*-2p}{2pp^*}\left(b_1\|u_n\|^{2p}+b_2\|v_n\|^{2p}\right)+\frac{p^*-p}{pp^*}\left(a_1\|u_n\|^p+a_2\|v_n\|^p\right)\right.\\
&\quad \left.-\frac{p^*-1}{p^*}\int_{\mathbb{R}^N} f(x)\,u_n\lambda_1+\lambda_2 g(x)\,v_ndx\right],\\
&= \underline{\lim}_{n\to+\infty}\left[I_3(u_n,v_n)-\frac{1}{p^*}\langle I_3'(u_n,v_n),(u_n,v_n)\rangle\right]\\
&= c_1.
\end{aligned}$$

This is a contradiction, we conclude that $(u_n,v_n)\to(u_1,v_1)$ strongly in $W^{1,p}(\mathbb{R}^N)\times W^{1,p}(\mathbb{R}^N)$. Therefore, $I_3'(u_1,v_1)=0$ and $I_3(u_1,v_1)=c_1<0$.

Thus (u_1,v_1) is a critical point of I_3 i.e. (u_1,v_1) is a weak solution of (4.1). As $I_3(0,0)=0$ and $I_3(u_1,v_1)<0$ then, $(u_1,v_1)\neq(0,0)$. Thus (u_1,v_1) is a nontrivial solution of (4.1) with negative energy. ■

Now assume that $a_1=a_2=1$ and $b_1=b_2=0$.

Let

$$C^* = \left(\frac{1}{p} - \frac{1}{p^*}\right)\left(\frac{1}{2} S_{q,q'}^{\frac{p^*}{p}}\right)^{\frac{p}{p^*-p}} . \tag{4.19}$$

Next, we prove an important lemma which ensures the local compactness of the Palais Smale sequence for I_3.

Lemma 4.8 *Suppose that* $f,\ g \in W^* \backslash \{0\}$*. Then if* $(u_n, v_n) \subset W^{1,p}\left(\mathbb{R}^N\right) \times W^{1,p}\left(\mathbb{R}^N\right)$ *is a Palais Smale sequence for* I_3 *for some* $c \in \mathbb{R}$*, then*

$$\textit{either } (u_n, v_n) \to (u, v) \quad \textit{or} \quad c \geq I_3(u, v) + C^*.$$

Proof. By the proof of Lemma 4.5 we have (u_n, v_n) is a bounded sequence in $W^{1,p}\left(\mathbb{R}^N\right) \times W^{1,p}\left(\mathbb{R}^N\right)$ and $(u_n, v_n) \rightharpoonup (u, v)$ in $W^{1,p}\left(\mathbb{R}^N\right) \times W^{1,p}\left(\mathbb{R}^N\right)$ for some $(u, v) \in W^{1,p}\left(\mathbb{R}^N\right) \times W^{1,p}\left(\mathbb{R}^N\right)$ with $I_3'(u, v) = 0$.

Furthermore, if we write $w_n = u_n - u$ and $t_n = v_n - v$, we derive

$$\begin{aligned}
(u_n, v_n) &\rightharpoonup (0,0) \text{ in } W^{1,p}\left(\mathbb{R}^N\right) \times W^{1,p}\left(\mathbb{R}^N\right) \\
(u_n, v_n) &\rightharpoonup (0,0) \text{ in } L^{p^*}\left(\mathbb{R}^N\right), \\
(u_n, v_n) &\to (u, v) \text{ a. e. in } \mathbb{R}^N,
\end{aligned}$$

and

$$\int_{\mathbb{R}^N} f(x)\, w_n dx \to 0, \tag{4.20}$$

$$\int_{\mathbb{R}^N} g(x)\, t_n dx \to 0,$$

and by using Brézis-Lieb we have

$$\|u_n\|^p = \|w_n\|^p + \|u\|^p + o_n(1) \tag{4.21}$$

and

$$\|v_n\|^p = \|t_n\|^p + \|v\|^p + o_n(1),$$

and by a similar argument of [32] and Lemma 4.4 we have

$$\int_{\mathbb{R}^N} |u_n|^q |v_n|^{q'} dx = \int_{\mathbb{R}^N} |w_n|^q |t_n|^{q'} dx + \int_{\mathbb{R}^N} |u|^q |v|^{q'} dx + o_n(1).$$

Using together (4.20), (4.21) and [32]

$$I_3'(u_n, v_n) \to 0 \text{ as } n \to +\infty \tag{4.22}$$

and

$$I_3(u_n, v_n) \to c \text{ as } n \to +\infty .$$

Therefore,

$$c + o_n(1) = I_3(u_n, v_n) - \frac{1}{p^*}\langle I_3'(u_n, v_n), (u_n, v_n)\rangle,$$

so

$$\begin{aligned} c + o_n(1) &= \frac{1}{p}\left(\|u_n\|^p + \|v_n\|^p\right) - \frac{2}{p^*}\int_{\mathbb{R}^N} |u_n|^q |v_n|^{q'} \\ &\quad - \int_{\mathbb{R}^N} \lambda_1 f(x) u_n + \lambda_2 g(x) v_n dx \\ &\quad - \frac{1}{p^*}\left(\|u_n\|^p + \|v_n\|^p\right) + \frac{2}{p^*}\int_{\mathbb{R}^N} |u_n|^q |v_n|^{q'} \\ &\quad + \frac{1}{p^*}\int_{\mathbb{R}^N} \lambda_1 f(x) u_n + \lambda_2 g(x) v_n dx, \end{aligned}$$

this implique that

$$\begin{aligned} c+o_n(1) &= \frac{1}{p}\left(\|w_n\|^p+\|u\|^p+\|t_n\|^p+\|v\|^p\right) \\ &-\frac{1}{p^*}\left(\|w_n\|^p+\|u\|^p+\|t_n\|^p+\|v\|^p\right) \\ &-\int\limits_{\mathbb{R}^N}\lambda_1 f(x)\,u+\lambda_2 g(x)\,v dx \\ &+\frac{1}{p^*}\int\limits_{\mathbb{R}^N}\lambda_1 f(x)\,u+\lambda_2 g(x)\,v dx, \end{aligned}$$

and

$$\begin{aligned} c+o_n(1) &= \left(\frac{1}{p}-\frac{1}{p^*}\right)\left(\|w_n\|^p+\|t_n\|^p\right)+\frac{1}{p}\left(\|u\|^p+\|v\|^p\right) \\ &-\frac{2}{p^*}\int\limits_{\mathbb{R}^N}|u|^q|v|^{q'}dx-\int\limits_{\mathbb{R}^N}\lambda_1 f(x)\,u+\lambda_2 g(x)\,v dx \\ &-\frac{1}{p^*}\left(\|u\|^p+\|v\|^p\right)+\frac{1}{p^*}\int\limits_{\mathbb{R}^N}\lambda_1 f(x)\,u+\lambda_2 g(x)\,v dx \\ &+\frac{2}{p^*}\int\limits_{\mathbb{R}^N}|u|^q|v|^{q'}dx. \end{aligned}$$

We obtain

$$c+o_n(1)\geq\left(\frac{1}{p}-\frac{1}{p^*}\right)\left(\|w_n\|^p+\|t_n\|^p\right)+I_3'(u,v)-\frac{1}{p^*}\left\langle I_3'(u,v),(u,v)\right\rangle. \tag{4.23}$$

Consequently,

$$c+o_n(1)\geq I_3(u,v)+\left(\frac{1}{p}-\frac{1}{p^*}\right)\left(\|w_n\|^p+\|t_n\|^p\right),$$

using

$$c+o_n(1)\geq I_3(u,v)+\left(\frac{1}{p}-\frac{1}{p^*}\right)\|w_n,t_n\|^p. \tag{4.24}$$

By the definition of $S_{q,q'}$ and (4.23) we obtain

$$\int\limits_{\mathbb{R}^N}|w_n|^q\,|t_n|^{q'}dx+o_n(1)=\frac{1}{2}\left(\|w_n\|^p+\|t_n\|^p\right)+o_n(1)$$

then

$$\int_{\mathbb{R}^N} |w_n|^q |t_n|^{q'} dx + o_n(1) \leq S_{q,q'}^{-p^*/p} \|w_n, t_n\|^{p^*}. \tag{4.25}$$

On the other hand, (4.25)we have

$$\frac{1}{2}\left(\|w_n\|^p + \|t_n\|^p\right) + o_n(1) \leq S_{q,q'}^{-p^*/p} \|w_n, t_n\|^{p^*}. \tag{4.26}$$

So (4.26) becomes

$$\frac{1}{2}\|w_n, t_n\|^p + o_n(1) \leq S_{q,q'}^{-p^*/p} \|w_n, t_n\|^{p^*}. \tag{4.27}$$

Assume that $\|w_n, t_n\| \to l > 0$, then by (4.27) we obtain

$$\frac{1}{2} l^p \leq S_{q,q'}^{-p^*/p} l^{p^*},$$

this implies that

$$S_{q,q'}^{-\frac{p^*}{p}} l^{p^*-p} - \frac{1}{2} \geq 0$$

we obtain

$$l \geq \left(\frac{1}{2} S_{q,q'}^{\frac{p^*}{p}}\right)^{\frac{1}{p^*-p}}.$$

Using (4.24), consequently

$$\begin{aligned} c &\geq I_3(u,v) + \left(\frac{1}{p} - \frac{1}{p^*}\right) l^p \\ &\geq I_3(u,v) + \left(\frac{1}{p} - \frac{1}{p^*}\right)\left(\frac{1}{2} S_{q,q'}^{\frac{p^*}{p}}\right)^{\frac{p}{p^*-p}} \\ &\geq I_3(u,v) + C^*. \end{aligned}$$

■

4.7 Existence of a critical point with positive energy

Now, we proof the existence of a Mountain Pass type solution.

Lemma 4.9 *Suppose that* $f,\ g \in W^{*}\backslash\{0\}$ *such that* $\int_{\mathbb{R}^N} f(x)\, u_\varepsilon dx \neq 0$, $\int_{\mathbb{R}^N} g(x)\, v_\varepsilon dx \neq 0$ *and* $a_1 = a_2 = 1$ *and* $b_1 = b_2 = 0$. *Then there exists* $(u'_\varepsilon, v'_\varepsilon) \in W^{1,p}(\mathbb{R}^N) \times W^{1,p}(\mathbb{R}^N)$ *and* $\lambda_1^{**},\ \lambda_2^{**},\ \lambda_3^{**} > 0$ *such that*

$$\begin{cases} \lambda_1 \leq \lambda_1^{**} & \text{if } \lambda_1 \neq 0 \text{ and } \lambda_2 = 0 \\ \lambda_2 \leq \lambda_2^{**} & \text{if } \lambda_1 = 0 \text{ and } \lambda_2 \neq 0 \\ \min(\lambda_1, \lambda_2) \leq \lambda_3^{**} & \text{if } \lambda_1 \neq 0 \text{ and } \lambda_2 \neq 0 \end{cases} \tag{4.28}$$

and

$$\sup_{t \geq 0} I_3(tu'_\varepsilon, tv'_\varepsilon) < c_1 + C^* \quad \text{for all} \quad \lambda_1^{**},\ \lambda_2^{**},\ \lambda_3^{**} > 0$$

where c_1, C^* *are given in* (4.18) *and* (4.19) *respectively..*

Proof. Let

$$\begin{aligned} h(t) &= I_3(tu'_\varepsilon, tv'_\varepsilon) = \frac{t^p}{p}\left(\|u'_\varepsilon\|^p + \|v'_\varepsilon\|^p\right) \\ &\quad -\frac{2}{p^*} t^{p^*} \int_{\mathbb{R}^N} |u'_\varepsilon|^q |v'_\varepsilon|^{q'} dx - t \int_{\mathbb{R}^N} \lambda_1 f(x)\, u'_\varepsilon + \lambda_2 g(x)\, v'_\varepsilon dx.) \end{aligned}$$

and

$$\begin{aligned} p(t) &= \frac{t^p}{p}\left(\|u'_\varepsilon\|^p + \|v'_\varepsilon\|^p\right) - \frac{2}{p^*} t^{p^*} \int_{\mathbb{R}^N} |u'_\varepsilon|^q |v'_\varepsilon|^{q'} dx \\ p'(t) &= t^{p-1}\left(\|u'_\varepsilon\|^p + \|v'_\varepsilon\|^p\right) - 2t^{p^*-1} \int_{\mathbb{R}^N} |u'_\varepsilon|^q |v'_\varepsilon|^{q'} dx. \end{aligned}$$

Then there exists $t_\epsilon > 0$ such that $p'(t) = 0$, we have

$$t_\varepsilon = \left(\frac{\|u'_\varepsilon\|^p + \|v'_\varepsilon\|^p}{2 \displaystyle\int_{\mathbb{R}^N} |u'_\varepsilon|^q |v'_\varepsilon|^{q'} dx} \right)^{\frac{1}{p^*-p}} \tag{4.29}$$

the above estimate on $p(t)$ yields

$$\max_{t \geq 0} p(t) = p(t_\varepsilon) = \frac{t_\varepsilon^p}{p} \left(\|u'_\varepsilon\|^p + \|v'_\varepsilon\|^p \right) - \frac{2}{p^*} t_\varepsilon^{p^*} \int\limits_{\mathbb{R}^N} |u'_\varepsilon|^q |v'_\varepsilon|^{q'} dx \tag{4.30}$$

from $p'(t_\varepsilon) = 0$,we have

$$t_\varepsilon^{p^*} \int\limits_{\mathbb{R}^N} |u'_\varepsilon|^q |v'_\varepsilon|^{q'} dx = \frac{t_\varepsilon^p}{2} \left(\|u'_\varepsilon\|^p + \|v'_\varepsilon\|^p \right),$$

become (4.29) and (4.30)

$$t_\varepsilon = \left(\frac{\|u'_\varepsilon\|^p + \|v'_\varepsilon\|^p}{2 \displaystyle\int_{\mathbb{R}^N} |u'_\varepsilon|^q |v'_\varepsilon|^{q'} dx} \right)^{\frac{1}{p^*-p}} = 1$$

and

$$\begin{aligned} \max_{t \geq 0} p(t) &= p(t_\varepsilon) = t_\varepsilon^p \left(\|u'_\varepsilon\|^p + \|v'_\varepsilon\|^p \right) - \frac{t_\varepsilon^p}{p^*} \left(\|u'_\varepsilon\|^p + \|v'_\varepsilon\|^p \right) \\ &= \left(\frac{1}{p} - \frac{1}{p^*} \right) \left(\|u'_\varepsilon\|^p + \|v'_\varepsilon\|^p \right) \\ &= \left(\frac{1}{p} - \frac{1}{p^*} \right) 2^{\frac{p}{(p-p^*)}} S_{q,q'}^{\frac{p^*}{p^*-p}} \\ &= \left(\frac{1}{p} - \frac{1}{p^*} \right) \left(\frac{1}{2} S_{q,q'}^{\frac{p^*}{p}} \right)^{\frac{p}{p^*-p}} \\ &= C^*. \end{aligned}$$

By the above estimates, we deduce that $\sup\limits_{t \geq 0} p(t) = C^*$.

Choosing λ_3^* defined in (4.17) such that

$$C^* - \frac{p-1}{p} \left(\frac{1}{2} \right)^{\frac{-1}{p-1}} (\lambda_3^*)^{\frac{p}{p-1}} \left[\|f\|_{W^*}^{\frac{p}{p-1}} + \|g\|_{W^*}^{\frac{p}{p-1}} \right] > 0$$

then there exists $t_1 \in (0,1)$ such that

$$\begin{aligned}\sup_{0\le t\le t_1} I_3(tu_{\varepsilon 1}, tv_{\varepsilon 1}) &< C^* - \frac{p-1}{p}\left(\frac{1}{2}\right)^{\frac{-1}{p-1}} (\lambda_3^*)^{\frac{p}{p-1}} \left[\|f\|_{W^*}^{\frac{p}{p-1}} + \|g\|_{W^*}^{\frac{p}{p-1}}\right] \\ &< C^* - \frac{p-1}{p}\left(\frac{1}{2}\right)^{\frac{-1}{p-1}} \left[\|\lambda_1 f\|_{W^*}^{\frac{p}{p-1}} + \|\lambda_2 g\|_{W^*}^{\frac{p}{p-1}}\right]\end{aligned}$$

for all $\lambda_1,\ \lambda_2$ verifying (4.17). Moreover, since $f,\ g \neq 0$, we can choose $\varepsilon_1 > 0$ such that $\int_{\mathbb{R}^N} f(x)\, u_{\varepsilon_1} dx, \int_{\mathbb{R}^N} f(x)\, v_{\varepsilon_1} dx > 0$ then

$$-\frac{p-1}{p}\left(\frac{1}{2}\right)^{\frac{-1}{p-1}} \left[\|\lambda_1 f\|_{W^*}^{\frac{p}{p-1}} + \|\lambda_2 g\|_{W^*}^{\frac{p}{p-1}}\right] > -\lambda_1 t_1 \int_{\mathbb{R}^N} f(x)\, u_{\varepsilon_1} dx - \lambda_2 t_1 \int_{\mathbb{R}^N} g(x)\, v_{\varepsilon_1} dx$$

for each $\lambda_1,\ \lambda_2$ verifying (4.17).

Then, for any $\lambda_1,\ \lambda_2$ verifying (4.17), one has

$$\begin{aligned}\sup_{t\ge t_1} I_3(tu_{\varepsilon_1}, tv_{\varepsilon_1}) &< C^* - \lambda_1 t_1 \int_{\mathbb{R}^N} f(x)\, u_{\varepsilon_1} dx - \lambda_2 t_1 \int_{\mathbb{R}^N} g(x)\, v_{\varepsilon_1} dx \\ &< C^* - \frac{p-1}{p}\left(\frac{1}{2}\right)^{\frac{-1}{p-1}} \left[\|\lambda_1 f\|_{W^*}^{\frac{p}{p-1}} + \|\lambda_2 g\|_{W^*}^{\frac{p}{p-1}}\right].\end{aligned}$$

Using Lemma 4.5 we see that

$$c_1 \ge -\frac{p-1}{p}\left(\frac{1}{2}\right)^{\frac{-1}{p-1}} \left[(\lambda_1 \|f\|_{W^*})^{\frac{p}{p-1}} + (\lambda_2 \|g\|_{W^*})^{\frac{p}{p-1}}\right] .$$

Therefore, we have

$$\sup_{t\ge 0} I_3(tu_{\varepsilon 1}, tv_{\varepsilon 1}) < C^* + c_1.$$

Then we can choose

$$\begin{cases} \lambda_1^{**} < \left(\frac{p}{p-1}C^*\right)^{\frac{p-1}{p}} \left(\frac{1}{2}\right)^{\frac{1}{p}} \left[\|f\|_{W^*}^{\frac{p}{p-1}}\right]^{-\frac{p-1}{p}} & \text{if } \lambda_1 \neq 0 \text{ and } \lambda_2 = 0 \\ \lambda_2^{**} < \left(\frac{p}{p-1}C^*\right)^{\frac{p-1}{p}} \left(\frac{1}{2}\right)^{\frac{1}{p}} \left[\|g\|_{W^*}^{\frac{p}{p-1}}\right]^{-\frac{p-1}{p}} & \text{if } \lambda_1 = 0 \text{ and } \lambda_2 \neq 0 \\ \lambda_3^{**} < \left(\frac{p}{p-1}C^*\right)^{\frac{p-1}{p}} \left(\frac{1}{2}\right)^{\frac{1}{p}} \left[\|f\|_{W^*}^{\frac{p}{p-1}} + \|g\|_{W^*}^{\frac{p}{p-1}}\right]^{-\frac{p-1}{p}} & \text{if } \lambda_1 \neq 0 \text{ and } \lambda_2 \neq 0. \end{cases}$$

This concludes the proof of Lemma 4.9. ■

Theorem 4.10 *Suppose that* $f,\ g \in W^{*}\backslash\{0\}$ *such that* $\int_{\mathbb{R}^N} f(x)\, u_\varepsilon dx \neq 0$, $\int_{\mathbb{R}^N} g(x)\, v_\varepsilon dx \neq 0$. $a_1 = a_2 = 1$ *and* $b_1 = b_2 = 0$. *Then, there exists constants* $(\lambda_1^{**},\ \lambda_2^{**},\ \lambda_3^{**}) > 0$ *such that* $\lambda_1,\ \lambda_2$ *satisfying* (4.28), *such that the problem* (4.1) *has a nontrivial solution* (u_2, v_2) *with positive energy .*

Proof. Note that $I_3(0,0) = 0$ and by the fact that

$$\lim_{t\to\infty} I_3(tu'_\varepsilon, tv'_\varepsilon) = -\infty \ ,$$

then $I_3(Tu'_\varepsilon, Tv'_\varepsilon) < 0$ for T large enough, and by Lemma 4.7, we know that I_3 is satisfying the geometry conditions of the Mountain Pass theorem. Then, by the Mountain Pass theorem [6], there exists a Palais Smale sequence (u_n, v_n) at level c_2, such that

$$I_3(u_n, v_n) \to c_2 > 0 \text{ and } I'_3(u_n, v_n) \to 0 \text{ as } n \to +\infty$$

with

$$0 < c_2 = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_3(\gamma(t), \zeta(t)) < \sup_{t\geq 0} I_3(tu'_\varepsilon, tv'_\varepsilon) < \ C^* + c_1,$$

for all $\lambda_1,\ \lambda_2$ satisfying (4.28), where for T large enough

$$\Gamma = \left\{(\gamma,\zeta) \in C\left([0,1], W^{1,p}\left(\mathbb{R}^N\right)\right), (\gamma,\zeta)(0,0) = (0,0), (\gamma,\zeta)(1,1) = (Tu'_\varepsilon, Tv'_\varepsilon)\right\}.$$

Using Lemma 4.8 and Lemma 4.9 we have that (u_n, v_n) has a subsequence, still denoted by (u_2, v_2), such that $(u_n, v_n) \to (u_2, v_2)$ in $W^{1,p}\left(\mathbb{R}^N\right) \times W^{1,p}\left(\mathbb{R}^N\right)$ as $n \to +\infty$. Hence, it holds

$$I_3(u_2, v_2) = \lim_{n\to+\infty} I_3(u_2, v_2) = c_2 > 0,$$

which implies that $(u_2, v_2) \neq (0,0)$. Furthermore, from the continuity of I_3', we obtain that (u_2, v_2) is a nontrivial solution with energy positive that follows immediately from the preceding lemma. This completes the proof of theorem 4.10. ■

Chapter 5

Perspectives

1) The existence of the second solution to the following nonhomogeneous elliptic problem

$$\begin{cases} -div(\frac{|\nabla u|^{p-2}}{|x|^{p\alpha}}\nabla u) - \mu\frac{|u|^{p-2}}{|x|^{p(\alpha+1)}}u = \frac{|u|^{p^*-2}}{|x|^{p^*\beta}}u + f(x) \text{ in } \Omega, \\ u = 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ containing 0 in its interior, $1 < p < N$, $0 \leq \alpha < (N-p)/p$, $\alpha \leq \beta < \alpha + 1$, $-\infty < \mu < \overline{\mu} := [(N-(\alpha+1)p)/p]^p$, $p^* = pN/[N - p(1+\alpha-\beta)]$ is the critical Caffarelli-Kohn-Nirenberg exponent, and f is function different than 0.

2) The existence of the second solution to the following Kirchhoff-type systems involving the critical Sobolev exponent

$$\begin{cases} -(a_1 + b_1\|u\|^p)[\text{div}(|\nabla u|^{p-2}\nabla u)] = \frac{2q}{q+q'}|u|^{q-2}u\ |v|^{q'} + \lambda_1 f(x), \\ -(a_2 + b_2\|v\|^p)[\text{div}(|\nabla v|^{p-2}\nabla v)] = \frac{2q'}{q+q'}|u|^{q}|v|^{q'-2}v + \lambda_2 g(x), \quad \text{in } \mathbb{R}^N \\ (u,v) \in W^{1,p}(\mathbb{R}^N) \times W^{1,p}(\mathbb{R}^N), \end{cases} \tag{5.2}$$

where $1 < p < N$, $a_1,\ a_2 \geq 0,\ b_1,\ b_2 > 0,\ q,\ q' > 1,\ q + q' = p^*,\ p^* = pN/\left[N - p\right]$ is the critical Sobolev exponent, $\lambda_1,\ \lambda_2 > 0$ are parameters, $f,\ g \in W^* \backslash \left\{0\right\}.$

REFERENCES

1 B. Abdellaoui , V. Felli , I. Peral; Existence and nonexistence for quasilinear equations involving the p-Laplacian. Boll Unione Mat Ital Sez, B9 (2006) 445-484.

2 C. O. Alves, Multiple positive solutions for semilinear elliptic equations in $\mathbb{R}^N$ involving critical exponents, Electron. J. Differential Equations 13, 1–10 (1997).

3 C.O. Alves, F.J.S.A. Corrêa, G.M. Figueiredo, On a class of nonlocal elliptic problems with critical growth, Differential Equation and Applications 2, 409–417 (2010).

4 C.O. Alves, F J.S.A. Correa, T.F. Ma, Positive solutions for a quasilinear elliptic equation of Kirchhoff type, Comput. Math. Appl. 49, 85-93 (2005).

5 C.O. Alves, de Morais Filho, D.C., Souto, M.A.S.: On systems of elliptic equations involving subcritical or critical Sobolev exponents. Nonlinear Anal. 42, 771-787 (2000).

6 A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14, 349-381 (1973).

7 A. Arosio, A geometrical nonlinear correction to the Timoshenko beam equation, Nonlinear Anal., 47, 729–740 (2001).

8 A. Arosio, On the nonlinear Timoshenko-Kirchhoff beam equation, Chinese Ann. Math. Ser. B, 20, 495–506 (1999).

9 R. Assuncão, P. Carrião, O. Miyagaki, Subcritical perturbations of a singular quasilinear elliptic equation involving the critical Hardy–Sobolev exponent. Nonlinear Anal. 66, 1351–1364 (2007).

10 A. Azzollini; note on the elliptic Kirchhoff equation in $\mathbb{R}^N$ perturbed by a local nonlinearity, Commun. Contemp. Math. 17, 1–5 (2015).

11 A. Benaissa, A. Matallah; Nonhomogeneous Elliptic Kirchhoff Equations of the P -Laplacian Type, Ukr. Math.m J. 72, 203-210 (2020).

12 S. Benmansour, M. Bouchekif; Nonhomogeneous elliptic problems of kirchhoff type involving critical Sobolev exponents. J. Differential Equations. 63, 1-11 (2015).

13 S. Benmansour, A. Matallah; Multiple Solutions for Nonhomogeneous Elliptic Equations Involving Critical Caffarelli-Kohn-Nirenberg Exponent Mediterr. J. Math. (2016).

14 M. Bouchekif, A. Matallah; Multiple positive solutions for elliptic equations involving a concave term and critical Sobolev-Hardy exponent. Appl. Math. Lett. 22, 268-275 (2009).

16 H. Brézis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Comm. Pure Appl. Math. 36, 437–477 (1983).

17 L. Caffarelli, R. Kohn, L. Nirenberg, First order interpolation inequality with weights, Compos. Math. 53, 259–275 (1984).

18 D. M. Cao, G. B. Li and H. S. Zhou, Multiple solutions for nonhomogeneous elliptic equations involving critical Sobolev exponent, Proc. Roy. Soc. Edinburgh Sect. A 124, 1177–1191 (1994).

19 F. Catrina, Z. Wang, On the Caffarelli-Kohn -Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions, Comm. Pure Appl. Math. 54,229-257 (2001).

20 J. Chabrowski, On multiple solutions for the non-homogeneous p-Laplacian with a critical Sobolev exponent, Differential Integral Equations 8, 705–716 (1995).

21 G. Chen, Quasilinear elliptic equations with Hardy terms and Hardy-Sobolev critical exponents: nontrivial solutions. Bound. Value Probl. 2015, 171, 1-11 (2015).

22 C. Y. Chen, Y. C. Kuo, T. F. Wu, The Nehari manifold for a Kirchhoff type problem involving sign-changing weight functions, J. Differential Equations 250, 1876-1908 (2011).

23 S. J. Chena, L. Li, Multiple solutions for the nonhomogeneous Kirchhoff equation on $\mathbb{R}^N$, Nonlinear Analysis: Real World Applications 14, 1477-1486 (2013).

24 B. Cheng, New existence and multiplicity of nontrivial solutions for nonlocal elliptic Kirchhoff type problems, J. Math. Anal. Appl. 394, 488-495 (2012).

25 C. M. Chu, Multiplicity of positive solutions for Kirchhoff type problem involving critical exponent and sign-changing weight functions, Boundary Value Problems, 1-19 (2014).

26 K.S. Chou, C.W. Chu, On the best constant for a weighted Sobolev-Hardy Inequality, J. London Math. Soc. 2, 137-151 (1993).

R. Dautray, J. L. Lions, Physical origins and classical methods, Mathematical Analysis and Numerical Methods for Science and Technology, Springer, Berlin (1985).

27 I. Ekeland, On the variational principle, J. Math. Anal. Appl. 47, 324-353 (1974).

28 G. M. Figueiredo, Eyistence of a positive solution for a Kirchhoff problem type with critical growth via truncation argument, J. Math. Anal. Appl. 401, 706–713 (2013).

29 G.M. Figueiredo, G.G. dos Santos, Solutions for a Kirchhoff equation with critical Caffarelli–Kohn–Nirenberg growth and discontinuous nonlinearity. Z. Angew. Math. Phys. 69, 75 (2018).

30 N. Ghoussoub, C. Yuan; Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents. Trans. Amer. Math. Soc. 352, 5703-5743 (2000).

31 M. B. Guimaràes, R. D. S. Rodrigues, Existence of solutions for Kirchhoff equations involving p-Linear and p-superlinear therms and with critical growth; Electronic Journal of Differential Equations, 113, 1-14 (2016).

32 P. Han, The effect of the domain topology on the number of positive solutions of elliptic systems involving critical Sobolev exponents, Houston J. Math. 32 (2006) 1241–1257.

33 P.G. Han; Mutiple positive solutions of nonhomogeneous elliptic systems involving critical Sobolev exponents. Nonlinear Anal. 64, 869-886 (2006).

34 Y. He, G. B. Li and S. J. Peng, Concentrating bound states for Kirchhoff type problems in $\mathbb{R}^3$ involving critical Sobolev exponents, Adv. Nonlinear Stud. 14, 483–510 (2014).

35 T. Horiuchi, Best constant in weighted Sobolev inequality with weights being powers of distance from the origin, J.Inequal. Appl. 1 (1997) 275-292.

36 D.S. Kang; On the quasilinear elliptic problems with critical Sobolev-Hardy exponents and Hardy terms. Nonlinear Anal. 68, 1973-1985 (2008).

37 D. Kang, Positive solutions to the weighted critical quasilinear problems, Applied Mathematics and Computation 213, 432–439 (2009).

38 O. Kavian, Introduction a la théoriedes points critiques et applications aux problèmes elliptiques, Springer-Verlag, (1983).

39 N. Keddar, A. Boussettine and A. Benaissa: Ground state solution for Nonhomogeneous elliptic equations involving critical Caffarelli-Kohn-Nirenberg exponent. EJMAA 7(1), 341-350 (2019).

41 F. Li, Y. Li, J. Shi, Existence of positive solutions to Kirchhoff type problems with zero mass, J. Math. Anal. Appl. 410, 361-374(2014).

42 J.L. Lions, On some questions in boundary value problems of mathematical physics, in: Contemporary Developments in Continuum Mechanics and Partial Differential Equations in: North-Holland Math. Stud. vol. 30. North-Holland. Amsterdam. 284-346 (1978).

43 J. Liu, J. F. Liao and C. L. Tang, Positive solutions for Kirchhoff-type equations with critical exponent in $\mathbb{R}^N$, J. Math. Anal. Appl. 429, 1153-1172 (2015).

44 A. Matallah, S. Litimein and S. Messirdi; Existence of multiple solutions for a nonhomogeneous p-Laplacian elliptic equation with critical Sobolev-Hardy exponent. Bol. Soc. Paran. Mat 1-12 (2019).

45 T.F. Ma, J.E. Munoz Rivera, Positive solutions for a nonlinear elliptic transmission problem, Appl. Math. Lett. 16, 243-248 (2003).

46 D. Naimen, On The critical problem of Kirchhoff type elliptic equations in dimension four. J. Differ. Equ. 257, 1168–1193 (2014).

47 B. Sciunzi, Classification of positive $D^{1,p}(\mathbb{R}^N)$-solutions to the critical p-Laplace equation in $\mathbb{R}^N$, Adv. Math. 291, 12-23 (2016).

48 J. Sun, S. Liu, Nontrivial solutions of Kirchhoff type problems, Appl. Math. Lett. 25, 500–504 (2012).

49 G. Talenti, Best constant in Sobolev inequality, Ann. Mat. Pura Appl. 110, 353-372 (1976).

50 G. Tarantello, On nonhomogeneous elliptic equations involving critical Sobolev exponent, Ann. Inst. Henri Poincaré 9, 281-304 (1992).

51 S. Terracini, On positive entire solutions for a class of equations with singular coefficient and critical exponent, Adv. Differential Equations 1, 241-264 (1996).

52 M. Willem; Minimax theorems, progress in nonlinear differential equations and their applications, 24. Birkhauser Boston, Inc, Boston, MA, (1996).

53 X. Wu, Exitence of nontrivial solutions and high energy solutions for Schrödinger-Kirchhofftype equations in $\mathbb{R}^N$, Nonlinear Anal. Real World Appl. 12, 1278-1287 (2011).

54 B. Xuan, S. Su, Y. Yan, Existence results for Brézis-Nirenberg problems with Hardy potential and singular coefficients. Nonlinear Anal. 67, 2091-2106 (2007).

الملخص:
في هده الأطروحة درسنا بعض المعادلات شبه الخطية غيرالمتجانسة و الأنظمة من نوع كيرشوف التي تحتوي على الأس الحرج لسوبوليف او كافارلي ـ كون ـ نيرمبرج. لقد اظهرنا وجود حلول من خلال مبدا ايكلاند المتغير و نظرية ممر الجبل .
الكلمات المفتاحية: الطرق المتغيرة ، نظرية ممر الجبل، مبدا ايكلاند المتغير، الأس الحرج لسوبوليف ، الأس الحرج كافارلي ـ كون ـ نيرمبرج، مشاكل كيرشوف.

Résumé :
Dans cette thèse, nous avons considéré quelques équations et systèmes quasi linéaires elliptiques non homogènes de type Kirchhoff contenant l'exposant critique de Sobolev ou de Caffarelli-Kohn-Niremberg. , Nous avons montré l'existence des solutions par le principe variationel d'Ekeland et le Théorème de Pass Montagne.
Les mots clés : Méthodes variationnelles, Théorème de Pass Montagne, Principe variationel d'Ekeland, Exposant critique de Sobolev, Exposant critique de Caffarelli-Kohn-Niremberg , Problemes de Kirchhoff.

Abstract:
In this thesis we have considered some nonhomogeneous elliptic quasi-linear equations and systems of Kirchhoff type containing the critical exponent of Sobolev or of Caffarelli-Kohn-Niremberg. We have show the existence of solutions by Ekeland's variational principle and Mountain Pass Theorem.
Keywords: Variational methods, Mountain Pass Theorem, Ekeland Variational Principle, critical exponent of Sobolev, critical exponent of Caffarelli-Kohn-Niremberg, Kirchhoff problems.