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Solutions positives pour les équations différentielles d'ordre fractionnaire

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A la mémoire de ma mére
A mon pére : Ali qui a si attendu ma réussite.
A mon frére et mes Soeurs, neveus et niéces et toute ma famille.

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"Soit A un succès dans la vie.
Alors A = x + y + z,
où x = travailler, y = s'amuser,
z = se taire."
Albert Einstein.

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Abstract

In this thesis, we are interested in two different axes :
The first axis, having the R_K function and inspired by the work of Agarwal et al. we introduce a new generalized Riemann-Liouville fractional derivation operator, and we obtain generating relations involving an extended and generalized hypergeometric Gauss function.
In the second axis, we are interested in studying some fractional differential equations with integral boundary conditions, our objective is to present some results of existence and uniqueness of solutions and positive solutions for these fractional differential equations. The approach followed consists in bringing back the search for the existence under suitable conditions by means of the Green function by applying different methods in Banach spaces.

Mots clés : Fractional differential equation, boundary value problem, fractional derivative, Green's function, Integral boundary conditions, Positive solution, Fixed point theorem, Generalized extended incomplete Gamma function, Generalized extended beta function, extended Riemann-Liouville fractional derivative, Mellin transform, Extended Gauss hypergeometric function, Integral representation.
AMS (MOS)Subject Classification : 26A33, 33B15, 33B20, 33C20, 33C65 ; 34B18

Résumé

Dans cette thèse, nous nous intéressons à deux axes différents :
Le premier axe, ayant la fonction R_K et inspiré par les travaux d'Agarwal *et al.*, nous introduisons un nouvel opérateur de dérivation fractionnaire de Riemann-Liouville généralisé, et nous obtenons des relations génératrices impliquant une fonction hypergéométrique de Gauss étendue et généralisée.
Dans le second axe, nous nous intéressons à étudier certaines équations différentielles d'ordre fractionnaire avec des conditions aux limites intégrales. Notre objectif est de présenter quelques résultats d'existence et d'unicité de solutions et l'existance de solutions positives pour ces équations différentielles d'ordre fractionnaire. La démarche suivie consiste à ramener la recherche de l'existence sous des conditions convenables moyennant la fonction de Green en appliquant différentes méthodes dans les espaces de Banach.
Ces méthode sont basée sur des célèbres théorèmes du point fixe tels que le théorème de point fixe le théorème de Guo-Krasnoselskii.

Mots clés : équation différentielle fractionnaire, problème à valeurs limites, dérivée fractionnaire, fonction de Green, conditions aux limites intégrales, solution positive, théorèmes du point fixe, fonction Gamma incomplète étendue généralisée, fonction bêta étendue généralisée, dérivée fractionnaire de Riemann-Liouville étendue, transformée de Mellin, Fonction hypergéométrique de Gauss étendue, représentation intégrale.
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Publications

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Introduction

Fractional calculus is a generalization of the classical calculus, It is a field of mathematics study, that grows out of the traditional definitions of the calculus integral and derivative operators. It has been used successfully in various fields of science and engineering as rheology, viscoelasticity, electrochemistry, electromagnetism, and so forth. Many different books and monographs are devoted to the development of fractional calculus. See for instance [25, 35, 39, 40, 43, 51, 53, 59]. The interest of the study of fractional order differential equations lies in the fact that there are more degrees of freedom in the fractional-order models. Furthermore, fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. Recent results on fractional differential equations can be seen in [26, 28, 33, 41].

The concepts of fractional differentiation and integration are usually associated with the name of Liouville. However, the creators of differential and integral calculus had already considered derivatives not only of integer order, but of fractional order too. We learn that fractional derivatives were the subjects of Leibniz's study. Euler also took an interest in fractional derivatives Liouville, Abel, Riemann, Letnikov, Weyl, Hadamard and many other well-known mathematicians of the past and present influenced the development of fractional integra-differentiation, which has now become a significant topic in mathematical analysis.

Mathematical modelling of real-life problems usually results in fractional differential equations and various other problems thus linking to their extensions and generalizations in one or more variables, in addition most physical phenomena of fluid dynamics, quantum mechanics, electricity ecological systems and many other models are controlled within their domain of validity by fractional differential equations , therefore it becomes increasingly important to be used to all traditional and recently developed methods for solving his equations and the implementations of these methods, Note that nowadays a lot of attention is directed to the development of extensions of fractional differential operators, readers can find it in [4, 11, 13, 38, 37, 46, 48, 50, 55, 66].

Boundary value problem are problems associated into partial differential equations of elliptic type, they consist to finding a function that satisfies a given partial differential equation and particular boundary conditions, this kind of problem is considerably difficult to solve this is due to the requirement that the solutions must be valid in the large domain.

Boundary value problems with positive solutions describe many phenomena in the applied sciences such as chemical reactors neutron, transport, population, biology (infectious diseases), economics and other systems can be reduced to nonlinear differential problems with integral boundary conditions, we need to discuss the existence of positive solutions with certain desired qualitative properties, thus it is clear that investigating nonlinear problems through abstract cones is an important branch of nonlinear analysis and this thesis is devoted to a systematic research of this aspect for more details see [10, 19, 57].

Nowadays, the fixed point theory has much applications in nonlinear analysis and lots other branch of modern mathematics, is one of the most powerful and efficient tools of modern mathematics considered such a core of nonlinear analysis. This theory has been a productive area of research for many mathematicians the origins of the theory which date from the latter part of the nineteenth century are based on the use of sequential approximations to prove the existence and uniqueness of solutions in particular of differential equations, this method is associated with many famous mathematicians.

Metric fixed point theory is an important mathematical discipline because of its applications in different areas such as variational and linear inequalities, optimization theory, boundary value problems in particular when we deal with the solvability of a certain functional differential equations like integral equations, matrix equations etc we formulate the problem in terms of finding a fixed point of a certain mapping.

The study of nonlinear boundary value problems ($BVPs$) is an important field of research. Indeed, their importance is due to the fact that boundary value problems model a large number of phenomena, whether in physics, technological sciences, chemistry, biology, engineering, economics, or applied mathematics. In practice, only positive solutions can be useful because they correspond to measurable parameters such as temperature, density.....; parameters that are used in different laws of physics. The resolution of differential equations or even boundary problems associated with differential equations, is a very large field of investigation.

In 1955 Krasnoselskii proved a fixed point theorem motivated by an observation that the inversion of a perturbed differential operator can give the sum of compact and contraction operators, his theorem in fact combines both Banach's contraction principle and Schauder's fixed point theorem and becôme useful for establishing existence theorems for equations of perturbed operators since then a large number of papers have been published contributing to generalizations or modifications of krasnoselskii's fixed point theorem and their applications. In recent years the krasnoselskii fixed point theorem for cone and its many generalizations have been successfully applied to establish the existence of positive (even multiple) solutions in the study of boundary value problems of various types.

The main impetus for seeking new cone fixed point theorems is to apply them to obtain better criteria for the existence of solutions, one of the reasons why the close relationship between the Krasnoselskii's theorem and the Brouwer-Schauder theorem has been overlooked is that the former is usually stated in the setting of a cone embedded in a Banach space with a given norm. In this setting, the norm functional plays a couple of important roles : in defining the region of points we are interested in, and in stating the properties of the images under the given map. When attempting to extend Krasnoselskii's theorem, one naturally focuses on finding similar functionals to replace the norm while still preserving these roles. On the other hand, the Brouwer-Schauder theorem is more topological in nature, being free from the concept of a metric.

This thesis is organized as follows :

The first chapter will be devoted to the basic elements of fractional calculus, a reminder of the Riemann-Liouville approach to the generalization of the notions of derivation will then be considered, as well as the different fixed point theorems used in this work, especially the one of Guo-Krasnoselskii's.

The objective of The second chapter entitled "Generalized Extended Riemann-Liouville type fractional derivative operator" is the development of new extensions of the incomplete gamma function, the beta function by means of extended Bessel-Riemann function :

1.

$$\gamma_\mu(\alpha, x; q; \lambda; p) = \sqrt{\frac{2p}{\pi}} \int_0^x t^{\alpha-\frac{3}{2}} e^{-t} R_K\left(\frac{p}{t}, -\mu - \frac{1}{2}, q, \lambda\right) dt \tag{1}$$

$$\Gamma_\mu(\alpha, x; q; \lambda; p) = \sqrt{\frac{2p}{\pi}} \int_x^\infty t^{\alpha-\frac{3}{2}} e^{-t} R_K\left(\frac{p}{t}, -\mu - \frac{1}{2}, q, \lambda\right) dt \tag{2}$$

where $\Re_e(x) > 0$, $0 < q \leq 1$, $-1 \leq \lambda \leq 1$ and $\Re_e(p) > 0$.

2.

$$B_{\mu}(x,y;q;\lambda;p;m)=\sqrt{\frac{2p}{\pi}}\int_{0}^{1}t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}}R_{K}\left(\frac{p}{t^{m}(1-t)^{m}},-\mu-\frac{1}{2},q,\lambda\right)dt, \tag{3}$$

where $x,y\in\mathbb{C}$, $0<q\leq 1$, $-1\leq\lambda\leq 1$, $m>0$ and $\Re_{e}(p)>0$.

some of their properties and related integral tranforms will be investigated. We introduce the extended hypergeometric functions and confluent hypergeometric functions by mean of extended Euler beta function, and establish some interesting properties. The extended Appell and Lauricella hypergeometric function are given. A new definition of the extended generalized Riemann Liouville fractional derivative operator is given and known properties are investigated.

The third chapter entitled "Fractional boundary value problems with integral boundary conditions" concerns the existence of solutions for the following fractional boundary value problem :

$$D^{\delta}u(t)+f(t,u(t))=0,\ 0<t<1,\ 1<\delta\leq 2 \tag{4}$$

subject to the integral boundary conditions :

$$u(0)=0,\int_{0}^{1}u(s)ds=u(1) \tag{5}$$

Where D^{δ} is the standard Riemann-Liouville derivation of order δ.
using the Green's function , $f(t,u)$ continuous function on $[0,1]\times\mathbb{R}$, we show that the problem $(4)-(5)$ has at least one solution according to the theorem used and the hypothesis imposed, The results of this chapter are based on the Banach fixed point theorem, Schaefer fixed point theorem and Nonlinear alternative of Leray-Schauder type, finally we will give examples to see the utility of our result.
this article is submitted since the year 2016 not yet published .

In the last chapter entitled "Positive solutions for fractional boundary value problems with integral boundary conditions and parameter dependence" we focus our attention to study the existence of positive solutions of the following fractional boundary value problem :

$$D^{\delta}u(t)+f(t,u(t))=0,\quad 0<t<1,\ 1<\delta\leq 2, \tag{6}$$

$$u(0) = 0, \ u(1) = \lambda \int_0^1 h(r)u(r)dr. \tag{7}$$

Where D^δ is the Riemann-Liouville fractional derivative and f is a given function. The boundary conditions (7) can be thought as a mechanism putted at the end point of an oscillator. which is characterized by the weighted function h and the parameter λ, that controls its displacement according to the feedback from devices measuring the displacements along different parts of the oscillator.

Using the principle of compression and expansion of the norm of a cone of a Banach space (Guo-Krasnosel'skii fixed point theorem), we prove an existence result for each case where the function f is sublinear and the case where f is superlinear. Finally, we will give an example to see the usefulness of our result.

Chapitre 1

Fundamentals

Contents

This chapter is introductory, it aims to present some notations and basic facts on Analysis as functional spaces, fractional calculus, and fixed points theorem.

1.1 Definitions and notations

Let E be the set of all functions defined and continuous on I a compact of $\mathbb{R}$, i.e :

$$E = \{f : I \to \mathbb{R} \text{ } \textit{is continuous}\}$$

E is a Banach space endowed with the norm :

$$\|f\|_{\infty} = \lim_{t \in I} |f(t)|$$

Definition 1.1. Let T be a mapping from a metric space (X, d) to another metric space (Y, ρ). Then, T is said to be

(i) Continuous at $x_0 \in X$ if for given $\epsilon > 0$, there exists $\delta = \delta(\epsilon, x_0) > 0$ such that

$$\rho(Tx, Tx_0) < \epsilon \quad whenever \quad d(x, x_0) < \delta \quad for\ all\ x \in X.$$

In general, T is said to be continuous at $x_0 \in X$ if for any sequence $\{x_n\}$ in X such that $x_n \to x_0$ implies $Tx_n \to Tx_0$ in Y.

(ii) Uniformly continuous on X if for given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$\rho(Tx, Ty) < \epsilon \quad whenever \quad d(x, y) < \delta \quad for\ all\ x, y \in X$$

Definition 1.2. [35]
A subset A of a real Banach space E is called uniformly bounded in E if and only if :

$$\exists M > 0,\ \forall f \in A,\ \|f\|_\infty \leq M.$$

Definition 1.3. [35]
A subset A of E is equicontinuous on I if :

$$\forall \epsilon > 0,\ \exists \delta_\epsilon > 0,\ \forall x, y \in I,\ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon \qquad for\ all\ f \in A$$

Let us define the compactness and relative compactness of a given operator (see [15])

Definition 1.4. Let E be a Banach space and $[a, b] = \Omega \subset E$. The operator $T : \Omega \to E$ is called completely continuous if A is continuous and if for any bounded subset C of Ω, $T(C)$ is relatively compact on E.

One way to prove that a subset $C \subset E$ is relatively compact is to show if C is both continuous and uniformly bounded in E

Definition 1.5. Let F be a Banach space. The mapping $f : E \to F$ is called compact if :

(i) f is continuous in E.

(ii) $f(E)$ is relatively compact in F.

Definition 1.6. The mapping $f : \Omega \to F$ is said to be completely continuous if and only if :

1. f is continuous.
2. $\overline{f(B)}$ is compact for any bounded set B of Ω.

Remark 1.1. If f is linear, then f is completely continuous $\Leftrightarrow f$ compact.

Let $C \subset R^n$, $C(K)$ denotes the space of all continuous function on K

Theorem 1.1. (Arzela-Ascoli)
Let $K \subset \mathbb{R}^n$ be a compact set. A subset $F \subset C(K)$ is relatively compact if and only if it is uniformly bounded and equicontinuous.

Definition 1.7. [36]
Let (M, d) be a metric space. A mapping $T : M \to M$ is said to be lipschitzian if there is a constant $k \geq 0$ such that for all $x, y \in M$

$$d(T(x), T(y)) \leq kd(x, y) \tag{1.1}$$

The smallest number k for which (1.1) holds is called the Lipschitz constant of T.

The contraction mapping principle as it first appeared in Banach's 1922 thesis.

Definition 1.8. [36]
A lipschitzian mapping $T : M \to M$ with Lipschitz constant $k < 1$ is said to be a contraction mapping.

Definition 1.9. [31] (Normal Cones)
Let E be a real Banach space.

1. A nonempty convex closed set $P \subset E$ is called a cone if it satisfies the following conditions :
   (a) $x \in P,\ \lambda \geq 0$ implies $\lambda x \in P$.
   (b) $x \in P,\ -x \in P$ implies $x = \theta$, where θ denotes the zero element E.
2. A cone is called solid if it contains some interior points, i.e $\mathring{P} \neq \emptyset$.
3. A cone P is said to be generating if $E = P - P$.
   i.e., every element $x \in E$ can be represented in the form $x = u - v$, where $u, v \in P$.

4. Every cone P in E define a partial ordering in E defined by $x \preceq y$ if $y-x \in P$ If $x \leq y$ and $x \neq y$, we write $x \prec y$, if the cone P is solid and $y-x \in \mathring{P}$, we write $x \ll y$

Definition 1.10. [31]
A cone P in E is said to be normal if there exists a positive constant δ such that :

$$\|x+y\| \geq \delta \quad \forall x, y \in P, \quad \|x\| = 1, \ \|y\| = 1$$

1.2 Special functions

Many important functions in applied sciences are defined via improper integrals or series (or infinite products). The general name of these important functions are called special functions.
The simple interpretation of the gamma function is simply the generalization of the factorial for all real numbers.

1.2.1 Gamma function

Let $p \in \mathbb{C}$

$$\Gamma(p) = \int_0^\infty e^{-x} x^{p-1} dx \tag{1.2}$$

Theorem 1.2. [42]
The gamma function is convergent for $\Re_e(p) > 0$ (the right half of the complex plane)

The gamma function is viewed as most important eulerian function used in fractional calculus, it appears in almost every fractional integral and derivative definitions. we cite some basic but fundamental properties of Γ, that we will use later :

Proposition 1.1. [42]

1. The gamma function $\Gamma(p)$ is continuous for $\Re_e(p) > 0$.
2. $\Gamma(p+1) = p\Gamma(p)$ Recurrence relation
3. $\Gamma(p+n) = (p+n-1)....(p+1)p\Gamma(p)$

1.2.2 Beta function

The beta function is defined by :

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$$

where $\Re_e(p) > 0$ and $\Re_e(q) > 0$.
we denote the beta function by B in the following chapters.
Some properties of beta function can be summarized as follows :

Proposition 1.2. [42]

1. For every $\Re_e(p) > 0$ and $\Re_e(q) > 1$, we have :

$$B(p,q) = B(q,p)$$

2. For every $\Re_e(p) > 0$ and $\Re_e(q) > 1$, the beta function B satisfies the property :

$$B(p,q) = \frac{q-1}{p+q-1}B(p,q-1)$$

3. For every $\Re_e(p) > 0$ and $\Re_e(q) > 0$, it is valid the identity :

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

4. For every $\Re_e(p) > 0$, and for an integer number n, it can be proved :

$$B(p,n) = B(n,p) = \frac{1.2.3......(n-1)}{p(p+1)......(p+n)}$$

and so :

$$B(p,1) = \frac{1}{p}$$

For any given integers m, n we obtain :

$$B(m,n) = \frac{(n-1)!(m-1)!}{(m+n-1)!}$$

We refer to books [16, 42] for more reading.
In 1994, Chaudhry and Zubair [22] introduced the following extension gamma function.

Definition 1.11. The extended gamma function is defined by :

$$\Gamma_p(x) := \int_0^\infty t^{x-1} exp(-t - pt^{-1})dt \tag{1.3}$$

$$(\Re_e(x) > 0,\ \Re_e(p) > 0)$$

In 1997, Chaudhry et al. [9] presented the following extension of Euler's beta function.

Definition 1.12. The extended beta function is defined by :

$$B_p(x) := \int_0^1 t^{x-1}(1-t)^{y-1} exp(-\frac{p}{t(1-t)})dt \tag{1.4}$$

$$(\Re_e(p) > 0,\ \Re_e(x) > 0,\ \Re_e(y) > 0)$$

Obviously we see that $\Gamma_0(x) = \Gamma(x)$ and $B_0(x, y) = B(x, y)$

1.2.3 Hypergeometric Functions and their extended versions

In this part, we give definitions and some properties of the hypergeometric functions, (see [47] for details).
The second order linear differential equation

$$z(1-z)\frac{d^2y}{dz^2} + [c - (a+b+1)z]\frac{dy}{dz} - aby = 0$$

where a, b and c are complex parameters, is called hypergeometric equation. The solutions (as series expansion) of the hypergeometric equation are valid in the neighborhood of $z = 0$, 1 or ∞. Thus, if c is not an integer, the general solution of differential equation is valid in a neighborhood of the origin and can be given by :

$$y = A_2\ F_1(a, b; c; z) + Bz^{1-c}{}_2F_1(a - c + 1, b - c + 1; 2 - c; z)$$

where A and B are arbitrary constants, and

$$\begin{aligned} {}_2F_1(a, b; c; z) &= 1 + \frac{ab}{c.1}z + \frac{a(a+1)b(b+1)}{c(c+1).1.2}z^2 + ... \\ &= \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c_n)} \frac{z^n}{n!} \end{aligned}$$

$$(c \neq 0, -1, -2, ...)$$

and $(\lambda)_v$ denotes the Pochhammer symbol defined by

$$(\lambda)_0 \equiv 1 \ and \ (\lambda)_v := \frac{\Gamma(\lambda + v)}{\Gamma(\lambda)}$$

Hence

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

is called Gauss hypergeometric function. This serie is convergent for $|z| < 1$ where $\Re_e(c) > \Re_e(b) > 0$ and $|z| = 1$ where $\Re_e(c - a - b) > 0$.

The Gauss hypergeometric function can be given by Euler's integral representation as follows :

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - zt)^{-a} dt$$

$$(|z| < 1; \ \Re_e(c) > \Re_e(b) > 0)$$

Replacing $z = \frac{z}{b}$ and by letting $|b| \to \infty$ in Gauss's hypergeometric equation, we have

$$z\frac{d^2y}{dz^2} + (c - z)\frac{dy}{dz} - ay = 0$$

This equation has a regular singularity at $z = 0$, The simplest solution of the equation is

$$\begin{aligned} \phi(a; c; z) &= 1 + \frac{a}{c.1} z + \frac{a(a+1)}{c(c+1).1.2} z^2 + ... \\ &= \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!} \end{aligned}$$

$$(c \neq 0, -1, -2, ...)$$

Hence, we get

$$\phi(a; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!}$$

which is called confluent hypergeometric function.

The confluent hypergeometric function can be given by an integral representation as follows :

$$\phi(a; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \int_0^1 t^{a-1}(1 - t)^{c-a-1} exp(zt) dt$$

$$(\Re_e(c) > \Re_e(a) > 0)$$

A generalized form of the hypergeometric function is

$$ {}_pF_q(\alpha_1, ..., \alpha_p; \gamma_1, ..., \gamma_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n.....(\alpha_p)_n}{(\gamma_1)_n.....(\gamma_q)_n} \frac{z^n}{n!} \tag{1.5}$$

$$(p, q = 0, 1, ...)$$

Setting $p = 2$, $q = 1$ in (1.5), we get the Gauss hypergeometric function,

$$F(\alpha_1, \alpha_2; \gamma_1; z) :=_2 F_1(\alpha_1, \alpha_2; \gamma_1; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n}{(\gamma_1)_n} \frac{z^n}{n!}$$

Setting $p = q = 1$ in (1.5) , we get confluent hypergeometric function

$$\phi(\alpha_1; \gamma_1; z) =_1 F_1(\alpha_1; \gamma_1; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n}{(\gamma_1)_n} \frac{z^n}{n!}$$

In 2004, Chaudhry et al. [20] used extended beta function $B_p(x, y)$ to extend the hypergeometric function (and confluent hypergeometric function) as follows :

$$F_p(a, b; c; z) = \sum_{n=0}^{\infty} \frac{B_p(b + n, c - b)}{B(b, c - b)} (a)_n \frac{z^n}{n!}$$

$$(p \geq 0;\ \Re_e(c) > \Re_e(b) > 0)$$

$$\phi_p(b; c; z) = \sum_{n=0}^{\infty} \frac{B_p(b + n, c - b)}{B(b, c - b)} \frac{z^n}{n!}$$

$$(p \geq 0;\ \Re_e(c) > \Re_e(b) > 0)$$

and gave the Euler type integral representations :

$$F_p(a, b; c; z) = \frac{1}{B(b, c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - zt)^{-a} exp\left[-\frac{p}{t(1 - t)}\right] dt$$

$$(p > 0;\ p = 0 \ and \ |arg(1 - z)| < \pi < p,\ \Re_e(c) > \Re_e(b) > 0)$$

and

$$\phi_p(b; c; z) = \frac{exp(z)}{B(b, c - b)} \int_0^1 t^{c-b-1}(1 - t)^{b-1} exp\left[-zt - \frac{p}{t(1 - t)}\right] dt$$

$$(p > 0;\ p = 0 \ and \ \Re_e(c) > \Re_e(b) > 0)$$

They called these functions by extended Gauss hypergeometric function (EGHF) and extended confluent hypergeometric function respectively, since
$F_0(a,b;c;z) =_2 F_1(a,b;c;z)$ and $\phi_0(b;c;z) =_1 F_1(b;c;z)$

1.2.4 Extended Appell's Functions

Let us define the extensions of the Appell's functions $F_1(a,b,c;d;x,y;p)$ and $F_1(a,b,c;d,c;x,y;p)$, and extended Lauricella's hypergeometric function $F^3_{D,p}(a,b,c,d;e;x,y,z)$ by

$$F_1(a,b,c;d;x,y;p) := \sum_{n,m=0}^{\infty} \frac{B_p(a+m+n,d-a)}{B(a,d-a)}(b)_n(c)_m \frac{x^n}{n!}\frac{y^m}{m!}$$

$$(max\{|x|,|y|\} < 1)$$

$$F_2(a,b,c;d,e;x,y;p) := \sum_{n,m=0}^{\infty} \frac{B_p(b+n,d-b)B_p(c+m,e-c)}{B(b,d-b)B(c;e-c)} \frac{x^n}{n!}\frac{y^m}{m!}$$

$$(|x|+|y| < 1)$$

and

$$F^3_{D,p}(a,b,c,d;e;x,y;z) := \sum_{m,n,r=0}^{\infty} \frac{B_p(a+m+n+r,e-a)(b)_m(c)_n(d)_r}{B(a,e-a)} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^r}{r!}$$

$$(\sqrt{|x|}+\sqrt{|y|}+|z| < 1)$$

respectively. Notice that the case $p = 0$ give the original functions. Now we proceed by obtaining the integral representations of the functions $F_1(a,b,c;d;x,y;p)$ and $F_2(a,b,c;d,e;x,y;p)$

Theorem 1.3. [47]
We have the following integral representation :

$$\begin{aligned} F_1(a,b,c;d;x,y;p) &= \frac{\Gamma(d)}{\Gamma(a)\Gamma(d-a)} \int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c} \\ &\times \ exp\left[-\frac{p}{t(1-t)}\right] dt \end{aligned}$$

$$(p > 0;\ p = 0 \quad and \quad |arg(1-x)| < \pi,\ |arg(1-y)| < \pi;\ \Re_e(d) > \Re_e(a) > 0)$$

$$(\Re_e(b) > 0,\ \Re_e(c) > 0)$$

Theorem 1.4. [47]
We have the following integral representation :

$$F_2(a,b,c;d,e;x,y;p) = \frac{1}{B(b,d-b)B(c,e-c)}.\int_0^1\int_0^1 \frac{t^{b-1}(1-t)^{d-b-1}s^{c-1}(1-s)^{e-c-1}}{(1-xt-ys)^a} \times exp\left[-\frac{p}{t(1-t)}-\frac{p}{s(1-s)}\right]dtds$$

$$(p>0; p=0 \; and \; |x|+|y|<1)$$

$$\Re_e(d)>\Re_e(b),\ \Re_e(e)>\Re_e(c)>0,\ \Re_e(a)>0$$

1.3 Fractional calculus

Fractional calculus concerns integrals and derivatives of (real or multivalued) functions at the non-integer order integrals and derivatives. These are called fractional derivatives and fractional integrals, which can be of real or complex orders and may include integer orders. During the last century, fractional differential equations have been proved to be powerful tools in the modelling of many phenomena in various fields of engineering, biology, physics and economics. We refer for more details to monographs of Kilbas et al. [35], ...
In this section, we shall give some basic formulas and techniques which are necessary to better understand the rest of this thesis. The Riemann-Liouville approach will be explored by means of Euler-gamma and beta functions connected with this function.

1.3.1 Fractional integral and fractional derivative

The fractional derivatives are defined using fractional integrals. There are several known forms of the fractional integrals, the three most commonly used fractional derivatives are Riemanne-Liouville, Caputo, and GrünwaldeLetnikov. [See . e.g [35]]
In this section we give the definitions of the Riemann-Liouville fractional integrals and fractional derivatives on a finite interval of the real line and present some of their properties in space of summable and continuous functions.
For every $\delta>0$ and a given local integrable function f.

Definition 1.13. [35, 42]
Let $\Omega = [a, b](-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The Riemann-Liouville integrals

$$(I_{a^+}^{\delta} f)(x) := \frac{1}{\Gamma(\delta)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\delta}} \quad (x > a; \Re_e(\delta) > 0) \tag{1.6}$$

and

$$(I_{b^-}^{\delta} f)(x) := \frac{1}{\Gamma(\delta)} \int_x^b \frac{f(t)dt}{(t-x)^{1-\delta}} \quad (x < b; \Re_e(\delta) > 0) \tag{1.7}$$

respectively. Here $\Gamma(\delta)$ is the Gamma function . These integrals are called the left-sided and the right-sided fractional integrals. When $\delta = n \in \mathbb{N}$, the definitions 1.6 and 1.7 coincide with the nth integral of the form

$$\begin{aligned} (I_{a^+}^{n} f)(x) &= \int_a^x dt_1 \int_a^{t_1} dt_2 ...... \int_a^{t_{n-1}} f(t_n) dt_n \\ &= \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) dt \quad (n \in \mathbb{N}) \end{aligned}$$

and

$$\begin{aligned} (I_{b^-}^{n} f)(x) &= \int_x^b dt_1 \int_{t_1}^{b} dt_2 ...... \int_{t_{n-1}}^{b} f(t_n) dt_n \\ &= \frac{1}{(n-1)!} \int_x^b (t-x)^{n-1} f(t) dt \quad (n \in \mathbb{N}) \end{aligned}$$

For particular values of the a and b, the following cases are known :

(i) Riemann : $a = 0, \quad b = +\infty$

(ii) Liouville : $a = -\infty, \quad b = 0$

Theorem 1.5. [35, 42]
We have

$${}_aI_t^{\delta}[C_1 f(t) + C_2 g(t)] = C_1 \, {}_aI_t^{\delta} f(t) + C_2 \, {}_aI_t^{\delta} g(t)$$

where C_1 and C_2 are constants and f, g are two arbitrary functions.

Definition 1.14. [35, 42] (Fractional derivative of order δ)
For every δ, Let $n = [\delta]$. The Riemann-Liouville derivative of order δ can be defined

as :

$$
\begin{aligned}
{}_aD_t^\delta f(t) &= \left(\frac{d}{dt}\right)^n {}_aI_t^{n-\delta} f(t) &(1.8)\\
&= \frac{1}{\Gamma(n-\delta)}\left(\frac{d}{dx}\right)^n \int_a^t (t-u)^{n-\delta-1} f(u)du &(1.9)
\end{aligned}
$$

Theorem 1.6. [35, 42]

The following integro-derivation rules are valid :

$$\int_a^b \phi(x)_a I_x^\delta \psi(x)dx = \int_a^b \psi(x)_x I_b^\delta f(x)dx \tag{1.10}$$

$$\int_a^b f(x)_a D_x^\delta g(x)dx = \int_a^b g(x)_x D_b^\delta f(x)dx \tag{1.11}$$

Also, notice that ${}_aI_x^\delta {}_aD_x^\delta f(x) = f(x)$ for $0 < \delta < 1$.

Theorem 1.7. [35, 42]

The following integration and derivation rules are valid :

(a) ${}_aI_t^{\delta+1}[Df(t)] =_a I_t^\delta f(t) - \frac{(t-a)^\delta}{\Gamma(\delta+1)} f(a)$

(b) ${}_aI_t^\delta[{}_aD_t^\delta f(t)] = f(t) - \sum_{k=1}^n {}_aD_t^{\delta-k} f(t)|_{t=a} \frac{(t-a)^{\delta-k}}{\Gamma(\delta-k+1)}$

(c) $D[{}_aI_t^\delta f(t)] =_a I_t^\delta[Df(t)] + \frac{(t-a)^{\delta-1}}{\Gamma(\delta)} f(a)$

(d) ${}_aI_t^\delta f(t) =_a I_t^{\delta+p}[D^p f(t)] + \sum_{k=0}^{p-1} \frac{D^k f(a)(t-a)^{\delta+k}}{\Gamma(\delta+k+1)}$, where $\Re_e(p)$ is positive.

(e) $D^p[{}_aI_t^\delta f(t)] =_a I_t^\delta[D^p f(t)] + \sum_{k=0}^{p-1} \frac{D^k f(a)(t-a)^{\delta+k}}{\Gamma(\delta+k+1)}$, where $\Re(p)_e$ is positive.

Theorem 1.8. [35, 42]

1. ${}_aI_t^\delta {}_aI_t^\beta f(t) =_a I_t^{\delta+\beta} f(t)$
2. ${}_aD_t^\delta[{}_aI_t^\beta f(t)] =_a D_t^{\delta-\beta} f(t)$
3. ${}_aI_t^\delta[{}_aD_t^\beta f(t)] =_a I_t^{\delta-\beta} f(t) - \sum_{k=1}^m \frac{(t-a)^{\delta-k}}{\Gamma(\delta+1-k)} f(t)|_{t=a}$
   where : $m = [\beta] + 1$
4. ${}_aD_t^\delta[{}_aD_t^\beta f(t)] =_a D_t^{\delta+\beta} f(t) - \sum_{k=1}^m {}_aD_t^{\beta-k} f(t)|_{t=a} \frac{(t-a)^{-\delta-k}}{\Gamma(1-\delta-k)}$

Example 1.1. To solve the following FDE with initial value :

$$D^{\frac{1}{2}} y(t) = y(t),$$

$$D^{-\frac{1}{2}} y(0) = -2\sqrt{\pi}$$

We transforme it in first order differential equation.

Using theorem 1.8 (4), we obtain :

$$D^{\frac{1}{2}}[D^{\frac{1}{2}}y(t)] = y'(t) - D^{\frac{1}{2}}y(0)\frac{t^{-\frac{1}{2}-1}}{\Gamma(1-\frac{1}{2}-1)} = D^{\frac{1}{2}}y(t) = y(t)$$

$$y'(t) - t^{-\frac{3}{2}} = y(t)$$

Theorem 1.9. [35, 42]

If the function $f(t)$ possess continuous derivative, then for $\delta > 0$, $n = [\delta] + 1$:

$$_aI_t^\delta f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(t-a)^{k-\delta}}{\Gamma(k+1-\delta)} + \frac{1}{\Gamma(n-\delta)} \int_a^t (t-y)^{n-\delta-1} f^{(n)}(y)dy$$

Theorem 1.10. [35, 42]

We denote $_0I_t^\delta$ with I^δ, for $p \in N$, $\delta > 0$. It can be proved that :

(a) $I^\delta[t^p f(t)] = \sum_{k=0}^{p} \binom{-\delta}{k} \frac{d^k}{dt^k} t^p I^{\delta+k} f(t)$

(b) $D^\delta[t^p f(t)] = \sum_{k=0}^{p} \binom{\delta}{k} \frac{d^k}{dt^k} t^p D^{\delta-k} f(t)$

Example 1.2.

Let calculate the Riemann-Liouville fractional derivative of the function $f(t) = t^\beta$ for $\delta > 0$, $n-1 < \delta < n$, $\beta > n-1$.

For instance, we can write :

$$I = D^\delta t^\beta = \frac{1}{\Gamma(n-\delta)} \frac{d^n}{dt^n} \int_0^t u^\beta (t-u)^{n-\delta-1} du$$

and we take :

$$u = vt, \quad du = tdv$$

It follows :

$$\begin{aligned} I &= \frac{1}{\Gamma(n-\delta)} \frac{d^n}{dt^n} \int_0^t (vt)^\beta [(1-v)t]^{n-\delta-1} tdv \\ &= \frac{1}{\Gamma(n-\delta)} \frac{d^n}{dt^n} \int_0^t (1-v)^{n-\delta-1} v^\beta t^{n-\delta+\beta} dv \\ &= \frac{1}{\Gamma(n-\delta)} \int_0^t (1-v)^{n-\delta-1} v^\beta \frac{d^n}{dt^n} t^{n-\delta+\beta} dv \end{aligned}$$

but

$$\frac{d^n}{dt^n}t^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-n+1)}t^{\lambda-n}$$

Recall that :

$$B(p,q) = \int_0^1 v^{p-1}(1-v)^{q-1}dv$$

so that it results :

$$I = \frac{1}{\Gamma(n-\delta)}\frac{\Gamma(n-\delta+\beta+1)}{\Gamma(-\delta+\beta+1)}t^{-\delta+\beta}\int_0^1 (1-v)^{n-\delta-1}v^\beta dv$$

$$\int_0^1 (1-v)^{n-\delta-1}v^\beta dv = B(n-\delta,\beta+1) = \frac{\Gamma(n-\delta)\Gamma(\beta+1)}{\Gamma(n-\delta+\beta+1)}$$

$$D^\delta t^\beta = I = \frac{\Gamma(\beta+1)}{\Gamma(-\delta+\beta+1)}t^{\beta-\delta}$$

Example 1.3. We can also find the Riemann-Liouville fractional integral and fractional derivative of

$$f(t) = (t-a)^\beta$$

For the fractional integral we apply the Riemann-Liouville definition :

$$I = {}_a I_t^\delta f(t) = \frac{1}{\Gamma(\delta)}\int_a^t (t-u)^{\delta-1}(u-a)^\beta du$$

The following change of variables

$$\frac{u-a}{t-a} = v$$

$$du = (t-a)dv$$

allows us to calculate :

$$I = \frac{(t-a)^{\delta+\beta}}{\Gamma(\delta)}\int_0^1 (1-v)^{\delta-1}v^\beta dv = \frac{(t-a)^{\delta+\beta}}{\Gamma(\delta)}B(\delta,\beta+1)$$

$$I = \frac{\Gamma(\beta+1)}{\Gamma(\delta+\beta+1)}(t-a)^{\delta+\beta}$$

For the fractional derivative we apply the Riemann-Liouville definition :

$$Df = {}_a D_t^\delta (t-a)^\beta = \frac{d^n}{dt^n}{}_a I^{n-\delta}(t-a)^\beta$$

and finally :

$$Df = \frac{\Gamma(\beta+1)}{\Gamma(\beta+n-\delta+1)} \frac{d^n}{dt^n}(t-a)^{\beta+n-\delta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\delta+1)}(t-a)^{\beta-\delta}$$

1.4 Basic Fixed Point Theorems

In this section, we present some allied theorems to fixed point theory. Firstly, The contraction mapping principle proved independently by Banach and Cacciopoli (see [75]) is the fundamental fixed point theorem. This theorem has a wide spectrum of applications and is a natural choice in approximating solutions to nonlinear problems.

Theorem 1.11. [62] (Banach contraction mapping principle)
Let (X, d) be a complete metric space, and $T : \Omega \to \Omega$ a contraction mapping with constant $0 < k < 1$ Then there exists a unique fixed point x of T in Ω, i.e, $Tx = x$

The next fixed point theorem is an extension to Brouwer's fixed point theorem to infinite-dimensional spaces. The need for such an extension arose because existence of solutions to nonlinear equations, especially nonlinear integral and differential equations can be formulated as fixed point problems in function-spaces.

Theorem 1.12. [62] (Schauder fixed point theorem)
If K is a compact convex subset of a normed linear space, then every continuous function f mapping K into itself has a fixed point.

For applications the following generalization proves to be useful

Theorem 1.13. [75] (Generalization of Schauder's fixed point theorem)
Let Ω be a closed convex set in a Banach space X and assume that $T : \Omega \longrightarrow \Omega$ is a continuous mapping such that $T(\Omega)$ is a relatively compact set of Ω. Then T has a fixed point in Ω.

Theorem 1.14. [75] (Schaefer fixed point theorem)
Let X be a Banach space and let $F : X \to X$ be a completely continuous mapping. Then either

(i) The equation $x = \lambda Fx$ has a solution for $\lambda = 1$, or

(ii) The set $\{x \in X : x = \lambda Fx\}$ is unbounded for some $\lambda \in (0, 1)$

Theorem 1.15. [75] (Nonlinear alternative of Leray-Schauder)
Let $\mathcal{C}$ be a nonempty convex subset of X. Let U be a nonempty open subset of $\mathcal{C}$ with $0 \in U$ and $F : \bar{U} \to \mathcal{C}$ be a compact and continuous operator. Then either

(i) F has fixed point, or

(ii) There exist $y \in \partial U$ and $\lambda^* \in [0, 1]$ with $y = \lambda^* F(y)$

Theorem 1.16. [31] (Krasnoselskii Fixed Point theorem)
Let E be a Banach space, and $P \subset E$ a cone, Ω_1, Ω_2 be two bounded open sets in E such that $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$. Let $T : P \cap (\overline{\Omega_2} \backslash \Omega_1) \to P$ be completely continuous. Suppose that one of the two conditions :

(i) $\|Tu\| \geq \|u\|,\ u \in P \cap \partial\Omega_1 \quad and \quad \|Tu\| \leq \|u\|,\ u \in P \cap \partial\Omega_2,$

(ii) $\|Tu\| \leq \|u\|,\ u \in P \cap \partial\Omega_1 \quad and \quad \|Tu\| \geq |u\|,\ u \in P \cap \partial\Omega_2$

is satisfied. Then A has at least one fixed point in $P \cap (\overline{\Omega_2} \backslash \Omega_1)$.

Chapitre 2

Generalized Extended Riemann-Liouville type fractional derivative operator

Contents

2.1 Introduction

Many problems in applied mathematics, statitics, engineering and many other fields of physics, biology are being solved by incomplete gamma functions. Most generaly, special functions became powerful tools to treat all these areas. We recall gamma and Euler's beta functions :

$$\gamma(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t} dt, \quad \text{for } \Re_e(\alpha) > 0, \tag{2.1}$$

$$\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} e^{-t} dt, \tag{2.2}$$

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \quad \text{for } \Re_e(x) > 0 \text{ and } \Re_e(y) > 0. \tag{2.3}$$

Their general properties can be found in [67].

Nevertheless, these functions give rise to some difficulties mostly in the neighborhood of 0. In order to overcome these problems, Chaudhry et al. [22, 24], using an exponential regulazing term, Chaudhry *et al.* extended the incomplete gamma function as follow [22]

$$\gamma(\alpha, x; p) = \int_0^x t^{\alpha-1} e^{-t-\frac{p}{t}} dt, \quad \text{for } p = 0, \Re_e(\alpha) > 0, \tag{2.4}$$

$$\Gamma(\alpha, x; p) = \int_x^\infty t^{\alpha-1} e^{-t-\frac{p}{t}} dt. \tag{2.5}$$

They proved the following recurrence formula

$$\gamma(\alpha, x; p) + \Gamma(\alpha, x; p) = 2p^{\alpha/2} K_\alpha(2\sqrt{p}),$$

where $K_\alpha(z)$ is the Macdonald function, known also as modified bessel function of third kind, defined for any $\Re_e(z) > 0$ by

$$K_\alpha(z) = \frac{(z/2)^\alpha}{2} \int_0^\infty t^{-\alpha-1} e^{-t-z^2/4t} dt.$$

Later, a new extension of Euler's beta function is given by Chaudhry *et al.* [21] as follow

$$\beta(x, y, p) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{\frac{-p}{t(1-t)}} dt. \tag{2.6}$$

These extensions are useful and provide new connections with error and Whittaker functions, for $p = 0$ they will be reduced to the known gamma and beta functions. Instead of using the exponential function, Chaudhry and Zubair [24] proposed generalized extensions of (2.4), (2.5) in the following form

$$\gamma_\mu(\alpha, x; p) = \sqrt{\frac{2p}{\pi}} \int_0^x t^{\alpha-\frac{3}{2}} e^{-t} K_{\mu+\frac{1}{2}}\left(\frac{p}{t}\right) dt, \tag{2.7}$$

$$\Gamma_\mu(\alpha, x; p) = \sqrt{\frac{2p}{\pi}} \int_x^\infty t^{\alpha-\frac{3}{2}} e^{-t} K_{\mu+\frac{1}{2}}\left(\frac{p}{t}\right) dt. \tag{2.8}$$

Many authors later developed more extensions of Euler's gamma and beta functions and hypergeometric functions based on the paper of Chaudhry and Zubair by considering exponential function and certain modified special functions (see for details [48].....). Recently, and inspired by their construction Agarwal *et al.* [4] developed an extension of the Euler's beta function as follow :

$$B_\mu(x, y; p; m) = \sqrt{\frac{2p}{\pi}} \int_0^1 t^{x-\frac{3}{2}} (1-t)^{y-\frac{3}{2}} K_{\mu+\frac{1}{2}}\left(\frac{p}{t^m(1-t)^m}\right) dt \tag{2.9}$$

where $x, y \in \mathbb{C}$, $m > 0$ and $\Re_e(p) > 0$.

Inspired by the work of Agarwal *et al.* [4] we introduce a generalized incomplete Riemann-Liouville fractional integral operators, and we obtain some generating relations involving generalized hypergeometric and confluent hypergeometric functions.

In this chapter, we introduce a new generalized incomplete gamma and Euler's beta functions by replacing in (2.7), (2.8) and (2.9) the Macdonald function $K_\alpha(z)$ by it's extended one developed by Boudjelkha [14], namely the function

$$R_K(z, \alpha, q, \lambda) = \frac{(z/2)^\alpha}{2} \int_0^\infty t^{-\alpha-1} \frac{e^{-qt-z^2/4t}}{1-\lambda e^{-t}} dt, \tag{2.10}$$

where $|\arg z^2| < \pi/2$, $0 < q \leq 1$ and $-1 \leq \lambda \leq 1$.

Motivations by the kernel R_K.

In his article [14] Boujelkha studied the extended Riemann Bessel functions, the integer representation and the series development of the Riemann zeta function and the Bessel functions. According to the complete representation we distinguish three types of bessel functions : K, I and J (see for more explanations [69]), defined by :

$$K_\alpha(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\alpha \int_0^{+\infty} t^{-\alpha-1} e^{\frac{-t-z^2}{4t}} dt, \quad |argz^2| < \frac{\pi}{2}$$

of this function and the Lerch zeta function [60], it seems natural to define (3) types of Riemann Bessel functions, one of them is our R_K function

$$R_K(z,\alpha,q,\lambda) = \frac{(z/2)^\alpha}{2}\int_0^\infty t^{-\alpha-1}\frac{e^{-qt-z^2/4t}}{1-\lambda e^{-t}}dt, \tag{2.11}$$

or $|\arg z^2| < \pi/2$, $0 < q \leq 1$ et $-1 \leq \lambda \leq 1$.

Remark 2.1. When $\lambda = 0$ et $q = 1$, $R_K(z,\alpha,q,\lambda)$ is reduced to $K_\alpha(z)$.

Proposition 2.1. [14]

1. The function $R_K(z,-\alpha,q,\lambda)$ can be expressed in terms of $K_\alpha(z)$ as follows :

$$R_K(z,-\alpha,q,\lambda) = \sum_{n=0}^\infty \lambda^n \frac{K_\alpha(z\sqrt{q+n})}{(q+n)^{\alpha/2}},\ \Re_e(z^2) > 0,\ 0 < q \leq 1,\ -1 \leq \lambda \leq 1, \tag{2.12}$$

2. The behavior of the function $R_K(z,-\alpha,q,\lambda)$ for small values of z is described by the following asymptotic formulas :

$$R_K(z,-\alpha,q,\lambda) \sim \begin{cases} \frac{1}{2}\frac{\Gamma(-z)}{(z/2)^{-\alpha}}(1-\lambda)^{-1},\ z \to 0, -1 < \lambda < 1, & \Re_e(\alpha) < 0, \\ \frac{1}{2}\frac{\Gamma(z)}{(z/2)^{\alpha}}\Phi(\lambda,\alpha,q),\ z \to 0,\ -1 \leq \lambda \leq 1, & \Re_e(\alpha) > 1, \end{cases} \tag{2.13}$$

   where $\Phi(\lambda,\alpha,q)$ is the Lerch function [65].

3. As for the asymptotic behavior of this function z to infinity it is given by :

$$R_K(z,-\alpha,q,\lambda) \sim \sqrt{\frac{\pi}{2z}}\frac{e^{-z\sqrt{q}}}{q^{\alpha/2+1/4}},\ \text{as } z \to \infty,\ |\arg z| < \frac{\pi}{4},\ -1 \leq \lambda \leq 1. \tag{2.14}$$

4. In particular when $q = 1$, we have

$$R_K(z,-\alpha,1,\lambda) \sim \sqrt{\frac{\pi}{2z}}e^{-z},\ \text{as } z \to \infty,\ |\arg z| < \frac{\pi}{4}, \tag{2.15}$$

   which is the same asymptotic formula as for K_α.

5. Differential equations satisfied by R_K

$$\frac{\partial^2 R_K}{\partial z^2} + z^{-1}\frac{\partial R_K}{\partial z} - \left(1 + \frac{\alpha^2}{z^2}\right) R_K = \lambda \frac{\partial R_K}{\partial \lambda}$$

6. The recurrence relations : Consider the function$R_K(z, -\alpha, \lambda) := R_K(z, -\alpha, 1, \lambda)$ then

$$R_k(z, -\alpha+1, \lambda) - R_K(z, -\alpha-1, \lambda) + \frac{2\alpha}{z} R_K(z, -\alpha, \lambda) = \lambda \frac{\partial}{\partial \lambda} R_K(z, -\alpha-1, \lambda)$$

2.2 Extended generalized gamma and Euler's beta functions

In this section, we define a new extended incomplete gamma and Euler's beta functions based on the extension of Bessel function (2.11) and we give some properties.

2.2.1 Extended generalized gamma function

Definition 2.1. The extended generalized incomplete gamma functions are given by [1]

$$\gamma_\mu(\alpha, x; q; \lambda; p) = \sqrt{\frac{2p}{\pi}} \int_0^x t^{\alpha-\frac{3}{2}} e^{-t} R_K\left(\frac{p}{t}, -\mu - \frac{1}{2}, q, \lambda\right) dt \qquad (2.16)$$

$$\Gamma_\mu(\alpha, x; q; \lambda; p) = \sqrt{\frac{2p}{\pi}} \int_x^\infty t^{\alpha-\frac{3}{2}} e^{-t} R_K\left(\frac{p}{t}, -\mu - \frac{1}{2}, q, \lambda\right) dt \qquad (2.17)$$

where $\Re_e(x) > 0$, $0 < q \leq 1$, $-1 \leq \lambda \leq 1$ and $\Re_e(p) > 0$.

Remark 2.2. When $\lambda = 0$ and $q = 1$, (2.16) and (2.17) will be reduced respectively to the extended incomplete gamma functions (2.7) and (2.8) defined by Chaudhry and Zubair [23, 24].

Proposition 2.2 (Decomposition theorem)**.**

$$\begin{aligned}\Gamma_\mu(\alpha,x;q;\lambda;p)+\gamma_\mu(\alpha,x;q;\lambda;p) &= \frac{\Gamma(\alpha+\mu)}{\sqrt{\pi}}\left(\frac{p}{2}\right)^{-\mu}\Phi_{1-\frac{\alpha+\mu}{2},\frac{1}{2}-\frac{\alpha+\mu}{2}}\left(\lambda,\mu+\frac{1}{2},q,\frac{p^2}{16}\right)\\ &+ \frac{\Gamma\left(-\frac{\alpha+\mu}{2}\right)}{2\sqrt{\pi}}\left(\frac{p}{2}\right)^{\alpha}\Phi_{\frac{1}{2},\frac{\alpha+\mu+2}{2}}\left(\lambda,\frac{\mu-\alpha+1}{2},q,\frac{p^2}{16}\right)\\ &- \frac{\Gamma\left(-\frac{\alpha+\mu+1}{2}\right)}{2\sqrt{\pi}}\left(\frac{p}{2}\right)^{\alpha+1}\Phi_{\frac{3}{2},\frac{\alpha+\mu+3}{2}}\left(\lambda,\frac{\mu-\alpha}{2},q,\frac{p^2}{16}\right)\end{aligned} \tag{2.18}$$

with $\Re_e(p)>0$, $-\infty<\alpha<\infty$ and

$$\begin{aligned}\Phi_{b_1,b_2}(\lambda,s,q,\xi) &= \int_0^\infty \frac{t^{s-1}e^{-qt}}{1-\lambda e^{-t}}\,{}_0F_2\left(\begin{matrix}\cdot \\ & ;-\xi\backslash t\\ b_1,b_2\end{matrix}\right)dt\\ &= \int_0^\infty \frac{t^{s-1}e^{-(q-1)t}}{e^t-\lambda}\,{}_0F_2\left(\begin{matrix}\cdot \\ & ;-\xi\backslash t\\ b_1,b_2\end{matrix}\right)dt,\end{aligned} \tag{2.19}$$

$(s\in\mathbb{C},\ \Re_e(\xi)>0,\ b_1,b_2\in\mathbb{C}\setminus\mathbb{Z}_0^-)$.

Proof We have

$$\begin{aligned}\Gamma_\mu(\alpha,x;q;\lambda;p)+\gamma_\mu(\alpha,x;q;\lambda;p) &= \sqrt{\frac{2p}{\pi}}\int_0^\infty t^{\alpha-\frac{3}{2}}e^{-t}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)dt\\ &= \frac{1}{\sqrt{\pi}}\left(\frac{p}{2}\right)^{-\mu}\int_0^\infty t^{\alpha+\mu-1}e^{-t}\\ &\times \left(\int_0^\infty \tau^{\mu-\frac{1}{2}}\frac{e^{-q\tau-\frac{p^2}{4t^2\tau}}}{1-\lambda e^{-\tau}}d\tau\right)dt\\ &= \frac{1}{\sqrt{\pi}}\left(\frac{p}{2}\right)^{-\mu}\int_0^\infty \tau^{\mu-\frac{1}{2}}\frac{e^{-q\tau}}{1-\lambda e^{-\tau}}\\ &= \left(\int_0^\infty t^{\alpha+\mu-1}e^{-t}e^{-\frac{p^2}{4t^2\tau}}dt\right)d\tau\end{aligned} \tag{2.20}$$

By using the integral [54, pp. 31, formula 6] we obtain

$$\begin{aligned}\int_0^\infty t^{\alpha+\mu-1}e^{-t}e^{-\frac{p^2}{4t^2\tau}}dt &= \Gamma(\alpha+\mu)\,{}_0F_2\left(\begin{matrix}. \\ & ;-\frac{p^2}{16\tau} \\ 1-\frac{\alpha+\mu}{2},\frac{1}{2}-\frac{\alpha+\mu}{2}\end{matrix}\right)\\ &+ \frac{\Gamma\left(-\frac{\alpha+\mu}{2}\right)}{2}\left(\frac{p^2}{4\tau}\right)^{\frac{\alpha+\mu}{2}}\,{}_0F_2\left(\begin{matrix}. \\ & ;-\frac{p^2}{16\tau} \\ \frac{1}{2},\frac{\alpha+\mu+2}{2}\end{matrix}\right)\\ &- \frac{\Gamma\left(-\frac{\alpha+\mu+1}{2}\right)}{2}\left(\frac{p^2}{4\tau}\right)^{\frac{\alpha+\mu+1}{2}}\,{}_0F_2\left(\begin{matrix}. & ;-\frac{p^2}{16\tau} \\ \frac{3}{2},\frac{\alpha+\mu+3}{2}\end{matrix}\right). \qquad (2.21)\end{aligned}$$

Finaly, substituting (2.21) in (2.20) and by using the notation (2.19) we get the desired result.

Proposition 2.3 (Recurrence relation)**.**

$$\begin{aligned}\Gamma_\mu(\alpha+1,x;q;\lambda;p) &= (\alpha+\mu)\Gamma_\mu(\alpha,x;q;\lambda;p)+p\Gamma_{\mu-1}(\alpha-1,x;q;\lambda;p)\\ &+ \sqrt{\frac{2p}{\pi}}x^{\alpha-\frac{1}{2}}e^{-x}R_K\left(\frac{p}{x},-\mu-\frac{1}{2},q,\lambda\right) \qquad (2.22)\end{aligned}$$

$$(\Re_e(p)>0,\ -\infty<\alpha<\infty).$$

Proof We have

$$\begin{aligned}\frac{d}{dt}\left[R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)\right] &= \frac{d}{dt}\left[\frac{\left(\frac{p}{2t}\right)^{-\mu-\frac{1}{2}}}{2}\int_0^\infty \tau^{\mu-\frac{1}{2}}\frac{e^{-q\tau-\frac{p^2}{4t^2\tau}}}{1-\lambda e^{-\tau}}d\tau\right]\\ &= \frac{\mu+\frac{1}{2}}{t}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)+\frac{p}{t^2}R_K\left(\frac{p}{t},-\mu+\frac{1}{2},q,\lambda\right) \qquad (2.23)\end{aligned}$$

Differentiating $t^{\alpha-\frac{1}{2}}e^{-t}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)$ with respect to t and using (2.23),we get

$$\begin{aligned}&\frac{d}{dt}\left[t^{\alpha-\frac{1}{2}}e^{-t}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)\right]=(\alpha+\mu)t^{\alpha-\frac{3}{2}}e^{-t}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)\\ &+p\,t^{\alpha-\frac{5}{2}}e^{-t}R_K\left(\frac{p}{t},-\mu+\frac{1}{2},q,\lambda\right)-t^{\alpha-\frac{1}{2}}e^{-t}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right). \qquad (2.24)\end{aligned}$$

Multiplying both sides in (2.24) by $\sqrt{\frac{2p}{\pi}}$ and integrating from x to ∞ and using (2.17), we get

$$\begin{aligned}0-\sqrt{\frac{2p}{\pi}}x^{\alpha-\frac{1}{2}}e^{-x}R_K\left(\frac{p}{x},-\mu-\frac{1}{2},q,\lambda\right) &= (\alpha+\mu)\Gamma_\mu(\alpha,x;q;\lambda;p)+p\\ &\times\Gamma_{\mu-1}(\alpha-1,x;q;\lambda;p)\Gamma_\mu(\alpha+1,x;q;\lambda;p).\end{aligned}$$

which can be also writen as

$$\begin{aligned}\Gamma_\mu(\alpha+1,x;q;\lambda;p) &= (\alpha+\mu)\Gamma_\mu(\alpha,x;q;\lambda;p)+p\Gamma_{\mu-1}(\alpha-1,x;q;\lambda;p)\\ &+ \sqrt{\frac{2p}{\pi}}x^{\alpha-\frac{1}{2}}e^{-x}R_K\left(\frac{p}{x},-\mu-\frac{1}{2},q,\lambda\right).\end{aligned}$$

Proposition 2.4. The following formula holds

$$\Gamma_{\mu-1}(\alpha,x;1;\lambda;p)-\Gamma_{\mu+1}(\alpha,x;1;\lambda;p)+\frac{2\mu+1}{p}\Gamma_\mu(\alpha+1,x;1;\lambda;p)=\lambda\frac{\partial}{\partial\lambda}\Gamma_{\mu+1}(\alpha,x;1;\lambda;p), \tag{2.25}$$

$$(\Re_e(p)>0,\ -\infty<\alpha<\infty).$$

Proof

We use (2.17) for $q=1$ and the following relation [14, (22)]

$$R_K(z,-\alpha+1,1,\lambda)-R_K(z,-\alpha-1,1,\lambda)+\frac{2\alpha}{z}R_K(z,-\alpha,1,\lambda)=\lambda\frac{\partial}{\partial\lambda}R_K(z,-\alpha-1,1,\lambda). \tag{2.26}$$

Proposition 2.5 (Parametric differentiation)**.**

$$\frac{\partial}{\partial p}\left(\Gamma_\mu(\alpha,x;q;\lambda;p)\right)=-\frac{1}{p}\left[\mu\Gamma_\mu(\alpha,x;q;\lambda;p)+p\Gamma_{\mu-1}(\alpha-1,x;q;\lambda;p)\right]. \tag{2.27}$$

Proof

$$\begin{aligned}\frac{\partial}{\partial p}\left(\Gamma_\mu(\alpha,x;q;\lambda;p)\right) &= \frac{1}{2p}\sqrt{\frac{2p}{\pi}}\int_x^\infty t^{\alpha-\frac{3}{2}}e^{-t}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)dt\\ &+\sqrt{\frac{2p}{\pi}}\int_x^\infty t^{\alpha-\frac{3}{2}}e^{-t}\frac{\partial}{\partial p}\left(R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)\right)dt\end{aligned} \tag{2.28}$$

We have

$$\begin{aligned}\frac{\partial}{\partial p}\left(R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)\right) &= -\frac{\mu+\frac{1}{2}}{p}\frac{(p/2t)^{-\mu-\frac{1}{2}}}{2}\int_0^\infty \tau^{\mu-\frac{1}{2}}\frac{e^{-q\tau-\frac{p^2}{4t^2\tau}}}{1-\lambda e^{-\tau}}d\tau\\ &= \frac{1}{t}\frac{(p/2t)^{-\mu+\frac{1}{2}}}{2}\int_0^\infty \tau^{\mu-\frac{3}{2}}\frac{e^{-q\tau-\frac{p^2}{4t^2\tau}}}{1-\lambda e^{-\tau}}d\tau\\ &= -\frac{\mu+\frac{1}{2}}{p}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)-\frac{1}{t}R_K\left(\frac{p}{t},-\mu+\frac{1}{2},q,\lambda\right)\end{aligned} \tag{2.29}$$

Finaly, by Substituting (2.29) in (2.28) we get the desired result.

2.2.2 Extended generalized beta function

Definition 2.2. The extended generalized beta function is given by

$$B_\mu(x,y;q;\lambda;p;m)=\sqrt{\frac{2p}{\pi}}\int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}}R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt, \tag{2.30}$$

where $x,y\in\mathbb{C}$, $0<q\leq 1$, $-1\leq\lambda\leq 1$, $m>0$ and $\Re_e(p)>0$.

Remark 2.3. Taking $\lambda=0$ and $q=1$, the equation (2.30) reduces to the extended Euler beta function (2.9) defined by Agarwal *et al.* [4].

Proposition 2.6 (Functional relations)**.**

1. The following formula holds :

$$B_\mu(x,y;q;\lambda;p;m)=B_\mu(x+1,y;q;\lambda;p;m)+B_\mu(x,y+1;q;\lambda;p;m). \tag{2.31}$$

2. Let $n\in\mathbb{N}$. Then the following summation formula holds :

$$B_\mu(x,y;q;\lambda;p;m)=\sum_{k=0}^{n}B_\mu(x+k,y+n-k;q;\lambda;p;m). \tag{2.32}$$

Proof

1. The right-hand side of (2.31) reads

$$\sqrt{\frac{2p}{\pi}}\int_0^1\left\{t^{x-\frac{1}{2}}(1-t)^{y-\frac{3}{2}}+t^{x-\frac{3}{2}}(1-t)^{y-\frac{1}{2}}\right\}R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt,$$

which, after simplification, yields

$$\sqrt{\frac{2p}{\pi}}\int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}}R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt,$$

which is equal to the left-hand side of (2.31).

2. The case $n=0$ of (2.32) holds trivially. The case $n=1$ of (2.32) is just the relation (2.31). For the other cases we can easily proceed by induction on n.

Proposition 2.7. The following formula holds

$$B_\mu(x,1-y;q;\lambda;p;m)=\sum_{n=0}^{\infty}\frac{(y)_n}{n!}B_\mu(x+n,1;q;\lambda;p;m). \tag{2.33}$$

Proof We have

$$B_\mu(x,1-y;q;\lambda;p;m)=\sqrt{\frac{2p}{\pi}}\int_0^1 t^{x-\frac{3}{2}}(1-t)^{-y-\frac{1}{2}}R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt. \tag{2.34}$$

By substituting the formula

$$(1-t)^{-y}=\sum_{n=0}^{\infty}(y)_n\frac{t^n}{n!},\quad (|t|<1,\ \ y\in\mathbb{C}), \tag{2.35}$$

in the right-hand of (2.34) and after interchanging the order of integral and summation we get (2.33).

Proposition 2.8. The following formula holds

$$B_\mu(x,y;q;\lambda;p;m)=\sum_{n=0}^{\infty}B_\mu(x+n,y+1;q;\lambda;p;m). \tag{2.36}$$

Proof By substituting the following formula

$$(1-t)^{y-1}=(1-t)^y\sum_{n=0}^{\infty}t^n,\quad (|t|<1), \tag{2.37}$$

in the right-hand of (2.30) and similarly as in the proof of proposition 2.7 we get the desired result.

2.3 Extended Gauss hypergeometric and confluent hypergeometric functions

2.3.1 Extended Gauss hypergeometric and confluent hypergeometric functions

We use the extended beta function (2.30) to extend the hypergeometric and confluent hypergeometric functions, respectively, as follows :

Definition 2.3. The extended Gauss hypergeometric function $F_\mu(a, b; c; z; q; \lambda; p; m)$ and the confluent hypergeometric function $\Phi_\mu(b; c; z; q; \lambda; p; m)$ are respectively defined by

$$F_\mu(a, b; c; z; q; \lambda; p; m) = \sum_{n=0}^{\infty} (a)_n \frac{B_\mu(b+n, c-b; q; \lambda; p; m)}{B(b, c-b)} \frac{z^n}{n!} \tag{2.38}$$

$$(|z| < 1,\ \Re_e(c) > \Re_e(b) > 0,\ 0 < q \leq 1,\ -1 \leq \lambda \leq 1,\ m > 0,\ \Re_e(p) > 0).$$

$$\Phi_\mu(b; c; z; q; \lambda; p; m) = \sum_{n=0}^{\infty} \frac{B_\mu(b+n, c-b; q; \lambda; p; m)}{B(b, c-b)} \frac{z^n}{n!} \tag{2.39}$$

$$(z \in \mathbb{C}, \Re_e(c) > \Re_e(b) > 0, -1 \leq \lambda \leq 1, m > 0, \Re_e(p) > 0).$$

Remark 2.4. For $\lambda = 0$ and $q = 1$, (2.38) reduces to the extended hypergeometric Gauss function established by Agarwal *et al.* [4].

Proposition 2.9 (Integral representation)**.**

1. The following integral representation for the extended Gauss hypergeometric function $F_\mu(a, b; c; z; q; \lambda; p; m)$ is valid

$$\begin{aligned} F_\mu(a, b; c; z; q; \lambda; p; m) &= \sqrt{\frac{2p}{\pi}} \frac{1}{B(b, c-b)} \int_0^1 t^{b-\frac{3}{2}} (1-t)^{c-b-\frac{3}{2}} (1-zt)^{-a} \\ &\times R_K\left(\frac{p}{t^m(1-t)^m}, -\mu - \frac{1}{2}, q, \lambda\right) dt \end{aligned} \tag{2.40}$$

$$(\arg(1-z) < \pi, \Re_e(c) > \Re_e(b) > 0, 0 < q \leq 1, -1 \leq \lambda \leq 1, m > 0, \Re_e(p) > 0).$$

2. The following integral representation for the extended confluent hypergeome-

tric function $\Phi_\mu(b;c;z;q;\lambda;p;m)$ is valid :

$$\begin{aligned}\Phi_\mu(b;c;z;q;\lambda;p;m) &= \sqrt{\frac{2p}{\pi}}\frac{1}{B(b,c-b)}\int_0^1 t^{b-\frac{3}{2}}(1-t)^{c-b-\frac{3}{2}}e^{zt} \\ &\times R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt, \qquad (2.41)\end{aligned}$$

$$(\Re_e(c)>\Re_e(b)>0, 0<q\leq 1, -1\leq\lambda\leq 1, m>0, \Re_e(p)>0).$$

Proof

1. By using (2.30) and the generalized binomial expansion

$$(1-zt)^{-a}=\sum_{n=0}^{\infty}(a)_n\frac{(zt)^n}{n!},\quad (|zt|<1), \qquad (2.42)$$

   we get the required result.

2. Similarly as in the proof of 1.

Proposition 2.10 (Differential formula)**.**

1. For $n\in\mathbb{N}$,

$$\frac{d^n}{dz^n}\{F_\mu(a,b;c;z;q;\lambda;p;m)\}=\frac{(a)_n(b)_n}{(c)_n}F_\mu(a+n,b+n;c+n;z;q;\lambda;p;m), \qquad (2.43)$$

$$(|z|<1,\ \Re_e(c)>\Re_e(b)>0,\ 0<q\leq 1,\ -1\leq\lambda\leq 1,\ m>0,\ \Re_e(p)>0).$$

2. For $n\in\mathbb{N}$,

$$\frac{d^n}{dz^n}\{\Phi_\mu(b;c;z;q;\lambda;p;m)\}=\frac{(b)_n}{(c)_n}\Phi_\mu(b+n;c+n;z;q;\lambda;p;m), \qquad (2.44)$$

$$(z\in\mathbb{C},\ \Re_e(c)>\Re_e(b)>0,\ 0<q\leq 1,\ -1\leq\lambda\leq 1, m>0, \Re_e(p)>0).$$

Proof

1. For $n=1$, we have

$$\begin{aligned}\frac{d}{dz}\{F_\mu(a,b;c;z;q;\lambda;p;m)\} &= \sum_{n=1}^{\infty}(a)_n\frac{B_\mu(b+n,c-b;q;\lambda;p;m)}{B(b,c-b)} \\ &\times \frac{z^{n-1}}{(n-1)!}\end{aligned}$$

$$= \sum_{n=0}^{\infty}(a)_{n+1}\frac{B_{\mu}(b+n+1,c-b;q;\lambda;p;m)}{B(b,c-b)}$$
$$= \times\frac{z^n}{n!} \qquad (2.45)$$

Using the identities $B(b,c-b)=\frac{c}{b}B(b+1,c-b)$ and $(a)_{n+1}=a(a+1)_n$ in (2.45), we get

$$\begin{aligned}\frac{d}{dz}\{F_{\mu}(a,b;c;z;q;\lambda;p;m)\} &= \frac{ab}{c}\sum_{n=0}^{\infty}(a+1)_n\frac{B_{\mu}(b+n+1,c-b;q;\lambda;p;m)}{B(b+1,c-b)}\frac{z^n}{n!}\\ &= \frac{ab}{c}F_{\mu}(a+1,b+1;c+1;z;q;\lambda;p;m), \qquad (2.46)\end{aligned}$$

and hence

$$\frac{d}{dz}\{F_{\mu}(a,b;c;z;q;\lambda;p;m)\}=\frac{ab}{c}F_{\mu}(a+1,b+1;c+1;z;q;\lambda;p;m). \qquad (2.47)$$

Repeated application of (2.47) leads to the formula (2.43).

2. Similarly as in the proof of 1.

Proposition 2.11 (Transformation formula)**.**

1. For $\arg(1-z)<\pi$, we have

$$F_{\mu}(a,b;c;z;q;\lambda;p;m)=(1-z)^{-a}F_{\mu}(a,c-b;c;\frac{z}{z-1};q;\lambda;p;m), \qquad (2.48)$$

$$(\Re_e(c)>\Re_e(b)>0,\ 0<q\leq 1,\ -1\leq\lambda\leq 1,\ m>0,\ \Re_e(p)>0).$$

2.

$$\Phi_{\mu}(b;c;z;q;\lambda;p;m)=e^z\Phi_{\mu}(c-b;c;-z;q;\lambda;p;m), \qquad (2.49)$$

$$(z\in\mathbb{C},\ \Re_e(c)>\Re_e(b)>0,\ 0<q\leq 1,\ -1\leq\lambda\leq 1,\ m>0,\Re_e(p)>0).$$

Proof

Replacing t by $1-t$ in the integral representations (2.40) and (2.41).

2.3.2 Generating functions involving the extended generalized Gauss hypergeometric function

We establish in this section some generating functions for the generalized Gauss hypergeometric functions. The results derived here can be compared to a number of (known and new) results in the theory of generating functions.

Theorem 2.1. Let $\Re_e(\beta) > 0$ and $\Re_e(\gamma) > \Re_e(\alpha) > -\frac{1}{2}$. Then we have

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} F_\mu(\beta+n, \alpha+\frac{1}{2}; \gamma+1; z; q; p; \lambda; m) t^n \\ = \quad &(1-t)^{-\beta} \times F_\mu\left(\beta, \alpha+\frac{1}{2}; \gamma+1; \frac{z}{1-t}; q; p; \lambda; m\right) \end{aligned} \tag{2.50}$$

where $|z| < min\{1, |1-t|\}$.

Proof we start by recalling the elementary identity

$$[(1-z)-t]^{-\beta} \sum_{n=0}^{\infty} \frac{(\beta)_n t^n}{n!} \frac{t}{(1-z)^n} = (1-t)^{-\beta}\left(1-\frac{z}{1-t}\right)^{-\beta}, \quad \text{for } |t| < |1-z|.$$

Multiplying both sides of the above equality by $z^{\alpha-1}$ and applying the extended Riemann-Liouville fractional derivative operator $D^{\alpha-\gamma;\mu;q;p;\lambda;m}$ on both sides, we find

$$D^{\alpha-\gamma;\mu;q;p;\lambda;m}\left\{\sum_{n=0}^{\infty} \frac{(\beta)_n t^n}{n!} z^{\alpha-1}(1-z)^{-\beta-n}\right\} = D^{\alpha-\gamma;\mu;q;p;\lambda;m}\{(1-t)^{-\beta} z^{\alpha-1}(1-\frac{z}{1-t})^{-\beta}\}.$$

Uniform convergence of the involved series makes it possible to exchange the summation and fractional operator to give

$$\sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} D^{\alpha-\gamma;\mu;q;p;\lambda;m}\{z^{\alpha-1}(1-z)^{-\beta-n}\} t^n = (1-t)^{-\beta} D^{\alpha-\gamma;\mu;q;p;\lambda;m}\{z^{\alpha-1}(1-\frac{z}{1-t})^{-\beta}\}.$$

The result follows by theorem 2.5 to both sides of the last identity.

Theorem 2.2. Let $\Re_e(\beta) > 0$, $\Re_e(\tau) > 0$ and $\Re_e(\gamma) > \Re_e(\alpha) > -\frac{1}{2}$. Then we have

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} F_\mu(\beta-n, \alpha+\frac{1}{2}; \gamma+1; z; q; p; \lambda; m) t^n \\ = \quad &(1-t)^{-\beta} \times F_{1,\mu}(\alpha+\frac{1}{2}, \tau, \beta; \gamma+1; z; \frac{-zt}{1-t}; q; p; \lambda; m) \end{aligned}$$

where $|z| < 1$, $|t| < |1-z|$ and $|z||t| < |1-t|$.

Proof Considering the following identity

$$[1-(1-z)t]^{-\beta} = (1-t)^{-\beta}(1+\frac{zt}{1-t})^{-\beta},$$

and expanding its left-hand side as a power series, we get

$$\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}(1-z)^n t^n = (1-t)^{-\beta}(1-\frac{-zt}{1-t})^{-\beta}, \quad \text{for } |t| < |1-z|.$$

Multiplying both sides by $z^{\alpha-1}(1-z)^{-\tau}$ and applying the definition of the extended Riemann-Liouville fractional derivative operator $D_z^{\alpha-\gamma;\mu;q;p;\lambda;m}$ on both sides, we find

$$\begin{aligned} & D_z^{\alpha-\gamma;\mu;q;p;\lambda;m}\left\{\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}z^{\alpha-1}(1-z)^{-\tau}(1-z)^n t^n\right\} \\ = \ & D_z^{\alpha-\gamma;\mu;q;p;\lambda;m}\left\{(1-t)^{-\beta}z^{\alpha-1}(1-z)^{-\tau}(1-\frac{-zt}{1-t})^{-\beta}\right\}. \end{aligned}$$

The given condition are found to allow us to exchange the order of the summation and fractional derivative to yield

$$\begin{aligned} & \sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}D^{\alpha-\gamma;\mu;q;p;\lambda;m}\{z^{\alpha-1}(1-z)^{-\tau+n}\}t^n \\ = \ & (1-t)^{-\beta}\times D^{\alpha-\gamma;\mu;q;p;\lambda;m}\left\{z^{\alpha-1}(1-z)^{-\tau}(1-\frac{z}{1-t})^{-\beta}\right\}. \end{aligned}$$

Finally the result follows by theorems 2.5 and 2.6.

Theorem 2.3. Let $\Re_e(\xi) > \Re_e(\upsilon) > -\frac{1}{2}$, $\Re_e(\gamma) > \Re_e(\alpha) > -\frac{1}{2}$ and $\Re_e(\beta) > 0$. Then we have

$$\begin{aligned} & \sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}F_\mu(\beta+n,\alpha+\frac{1}{2};\gamma+1;z;q;\lambda;p;m)F_\mu(-n,\upsilon+\frac{1}{2};\xi+1;u;q;\lambda;p;m)t^n \\ = \ & (1-t)^{-\beta}F_{2,\mu}\left(\beta,\alpha+\frac{1}{2},\upsilon+\frac{1}{2};\gamma+1,\xi+1;\frac{z}{1-t},\frac{-ut}{1-t};q;\lambda;p;m\right). \end{aligned}$$

where $|z| < 1$, $|\frac{1-u}{1-z}t| < 1$ and $|\frac{z}{1-t}| + |\frac{ut}{1-t}| < 1$.

Proof

Replacing t by $(1-u)t$ in (2.50) and multiplying both sides of the resulting identity by $u^{\upsilon-1}$ gives

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}F_\mu(\beta+n,\alpha+\frac{1}{2};\gamma+1;z;q;\lambda;p;m)u^{\upsilon-1}(1-u)^n t^n\\
=\ & u^{\upsilon-1}[1-(1-u)t]^{-\beta}F_\mu\left(\beta,\alpha+\frac{1}{2};\gamma+1;\frac{z}{1-(1-u)t};q;\lambda;p;m\right).
\end{aligned}$$

where $\Re_e(\beta)>0$ and $\Re_e(\gamma)>\Re_e(\alpha)>-\frac{1}{2}$.

Applying the fractional derivative $D^{\upsilon-\xi,\mu;q;\lambda;p;m}$ to both sides of resulting identity and changing the order of the summation and the fractional derivative yields

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}F_\mu(\beta+n,\alpha+\frac{1}{2};\gamma+1;z;q;\lambda;p;m)D^{\upsilon-\xi,\mu;q;\lambda;p;m}\{u^{\upsilon-1}(1-u)^n\}t^n\\
=\ & D^{\upsilon-\xi,\mu;q;\lambda;p;m}\left\{u^{\upsilon-1}[1-(1-u)t]^{-\beta}F_\mu\left(\beta,\alpha+\frac{1}{2};\gamma+1;\frac{z}{1-(1-u)t};q;\lambda;p;m\right)\right\},
\end{aligned}$$

where $|(1-u)t|<1$, $|ut|<|1-t|$.

The last identity can be written as follows :

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}F_\mu(\beta+n,\alpha+\frac{1}{2};\gamma+1;z;q;\lambda;p;m)D^{\upsilon-\xi,\mu;q;\lambda;p;m}\{u^{\upsilon-1}(1-u)^n\}t^n\\
=\ & (1-t)^{-\beta}\\
\times\ & D^{\upsilon-\xi,\mu;q;\lambda;p;m}\left\{u^{\upsilon-1}\left[1-\frac{-ut}{1-t}\right]^{-\beta}F_\mu\left(\beta+n,\alpha+\frac{1}{2};\gamma+1;\frac{\frac{z}{1-t}}{1-\frac{-ut}{1-t}};q;\lambda;p;m\right)\right\}
\end{aligned}$$

Finally the use of theorems 2.5 and 2.8 in the resulting identity is seen to give the desired result.

2.4 Extended Appell and Lauricella hypergeometric functions

Definition 2.4. The extended Appell hypergeometric functions $F_{1,\mu}$, $F_{2,\mu}$ and the Lauricella hypergeometric function $F_{D,\mu}^3$ are, respectively, defined by

$$\begin{aligned}
F_{1,\mu}(a,b,c;d;x,y;q;\lambda;p;m) &= \sum_{n,k=0}^{\infty}(b)_n(c)_k\frac{B_\mu(a+n+k,d-a;q;\lambda;p;m)}{B(a,d-a)}\\
&\times\ \frac{x^n}{n!}\frac{y^k}{k!} \qquad (2.51)
\end{aligned}$$

$$(|x| < 1,\ |y| < 1,\ \Re_e(d) > \Re_e(a) > 0,\ 0 < q \leq 1)$$

$$(-1 \leq \lambda \leq 1,\ m > 0,\ \Re_e(p) > 0).$$

$$\begin{aligned} F_{2,\mu}(a, b, c; d, e; x, y; q; \lambda; p; m) &= \sum_{n,k=0}^{\infty} (a)_{n+k} \frac{B_\mu(b+n, d-b; q; \lambda; p; m)}{B(b, d-b)} \\ &\times \frac{B_\mu(c+k, e-c; q; \lambda; p; m)}{B(c, e-c)} \frac{x^n}{n!} \frac{y^k}{k!} \end{aligned} \tag{2.52}$$

$$(|x| + |y| < 1,\ \Re_e(d) > \Re_e(b) > 0,\ \Re_e(e) > \Re_e(c) > 0)$$

$$(0 < q \leq 1,\ -1 \leq \lambda \leq 1,\ m > 0,\ \Re_e(p) > 0)$$

$$\begin{aligned} &F^3_{D,\mu}(a, b, c, d; e; x, y, z; q; \lambda; p; m) \\ &= \sum_{n,k,r=0}^{\infty} (b)_n (c)_k (d)_r \frac{B_\mu(a+n+k+r, e-a; q; \lambda; p; m)}{B(a, e-a)} \frac{x^n}{n!} \frac{y^k}{k!} \frac{z^r}{r!} \end{aligned} \tag{2.53}$$

$$(|x| < 1,\ |y| < 1,\ |z| < 1,\ \Re_e(e) > \Re_e(a) > 0,\ 0 < q \leq 1)$$

$$(-1 \leq \lambda \leq 1,\ m > 0,\ \Re_e(p) > 0).$$

Proposition 2.12 (Integral representation)**.**
The following integral representations for the extended Appell hypergeometric functions $F_{1,\mu}$, $F_{2,\mu}$ and the Lauricella hypergeometric function $F^3_{D,\mu}$ are, respectively, valid

$$\begin{aligned} F_{1,\mu}(a, b, c; d; x, y; q; \lambda; p; m) &= \sqrt{\frac{2p}{\pi}} \frac{1}{B(a, d-a)} \int_0^1 t^{a-\frac{3}{2}} (1-t)^{d-a-\frac{3}{2}} (1-xt)^{-b} \\ &\times (1-yt)^{-c} R_K\left(\frac{p}{t^m (1-t)^m}, -\mu - \frac{1}{2}, q, \lambda\right) dt \end{aligned} \tag{2.54}$$

$$\begin{aligned} F_{2,\mu}(a, b, c; d; x, y; q; \lambda; p; m) &= \frac{2p}{\pi} \frac{1}{B(b, d-b) B(c, e-c)} \int_0^1 \int_0^1 t^{b-\frac{3}{2}} (1-t)^{d-b-\frac{3}{2}} \\ &\times w^{b-\frac{3}{2}} (1-w)^{e-c-\frac{3}{2}} (1-xt-yw)^{-a} \\ &\times R_K\left(\frac{p}{t^m (1-t)^m}, -\mu - \frac{1}{2}, q, \lambda\right) \\ &\times R_K\left(\frac{p}{w^m (1-w)^m}, -\mu - \frac{1}{2}, q, \lambda\right) dt dw \end{aligned} \tag{2.55}$$

$$\begin{aligned} F^3_{D,\mu}(a,b,c,d;e;x,y,z;q;\lambda;p;m) &= \sqrt{\frac{2p}{\pi}}\frac{1}{B(a,e-a)}\int_0^1 t^{a-\frac{3}{2}}(1-t)^{e-a-\frac{3}{2}} \\ &\times (1-xt)^{-b}(1-yt)^{-c}(1-zt)^{-d} \\ &\times R_K\left(\frac{p}{t^m(1-t)^m}, -\mu-\frac{1}{2}, q, \lambda\right) dt \qquad (2.56) \end{aligned}$$

Proofs are very similar to those of theorems (2.13), (2.15) and (2.16) in [4].

2.5 Contribution to fractional calculus

Let's begin by a litte recall of the classical Riemmann-Liouville fractional derivative operator of order δ defined by

$$D_z^\delta f(z) := \frac{1}{\Gamma(-\delta)}\int_0^z (z-t)^{-\delta-1} f(t)dt,$$

where $\Re_e(\delta) < 0$. It coincides with the fractional integral of order $-\delta$. In the case $m-1 < \Re_e(\delta) < m$, $m \in \mathbb{N}$, it is customary to write

$$D_z^\delta f(z) := \frac{d^m}{dz^m} D_z^{\delta-m} f(z) = \frac{d^m}{dz^m}\left\{\frac{1}{\Gamma(m-\delta)}\int_0^z (z-t)^{m-\delta-1} f(t)dt\right\}.$$

We start by a new extended generalized Riemann-Liouville fractional derivative operator [1].

Definition 2.5. The extended generalized Riemann-Liouville fractional derivative is defined as

$$D_z^{\delta,\mu;p;q;\lambda;m} f(z) := \frac{1}{\Gamma(-\delta)}\sqrt{\frac{2p}{\pi}}\int_0^z (z-t)^{-\delta-1} f(t) R_K\left(\frac{pz^{2m}}{t^m(z-t)^m}, -\mu-\frac{1}{2}, q, \lambda\right) dt, \qquad (2.57)$$

where $\Re_e(\delta) < 0$, $\Re_e(p) > 0$, $\Re_e(m) > 0$, $\Re_e(\mu) > 0$ and $0 < q \leq 1$, $-1 \leq \lambda \leq 1$.

For $n-1<\Re_e(\delta)<n$, $n\in\mathbb{N}$ we write

$$\begin{aligned} D_z^{\delta,\mu;p;q;\lambda;m}f(z):=\frac{d^n}{dz^n}D_z^{\delta-n,\mu;p;q;\lambda;m}f(z) &= \frac{d^n}{dz^n}\Bigg\{\frac{1}{\Gamma(n-\delta)}\sqrt{\frac{2p}{\pi}}\int_0^z(z-t)^{n-\delta-1}f(t)\\ &\times\ R_K\left(\frac{pz^{2m}}{t^m(z-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt\Bigg\} \end{aligned} \tag{2.58}$$

Remark 2.5. If we take $m=0$, $\mu=0$, $\lambda=1$, and $p\to 0$, then the above extended generalized Riemann-Liouville fractional derivative operator reduces to the classical Riemann-Liouville fractional derivative operator

In order to calculate the extended generalized fractional derivatives for some functions, We begin by two results involving extended generalized Riemann-Liouville fractional derivative operator of some elementary functions which will be useful in the sequel.

Lemma 2.1. Let $-m-1<\Re_e(\delta)<-m$ for some positif integer m and $\beta>-\frac{3}{2}$. Then we have

$$D_z^{\delta,\mu;p;q;\lambda;m}\{z^\beta\}=\frac{z^{\beta-\delta}}{\Gamma(-\delta)}B_\mu(\beta+\frac{3}{2},-\delta+\frac{1}{2};p;q;\lambda;m). \tag{2.59}$$

Proof Using definition 2.5, and a local setting $t=zu$ we obtain

$$\begin{aligned} D_z^{\delta,\mu;p;q;\lambda;m}\{z^\beta\} &= \frac{1}{\Gamma(-\delta)}\sqrt{\frac{2p}{\pi}}\int_0^z(z-t)^{-\delta-1}t^\beta R_K\left(\frac{pz^{2m}}{t^m(z-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt\\ &= \frac{z^{\beta-\delta}}{\Gamma(-\delta)}\sqrt{\frac{2p}{\pi}}\int_0^z(1-u)^{(-\delta+\frac{1}{2})-\frac{3}{2}}u^{(\beta+\frac{3}{2})-\frac{3}{2}}\\ &\times\ R_K\left(\frac{p}{u^m(1-u)^m},-\mu-\frac{1}{2},q,\lambda\right)dt\\ &= \frac{z^{\beta-\delta}}{\Gamma(-\delta)}B_\mu(\beta+\frac{3}{2},-\delta+\frac{1}{2};p;q;\lambda;m). \end{aligned}$$

More generally, we give the extended generalized Riemann-Liouville fractional derivative of an analytic function f at the origin.

Lemma 2.2. Let $-m-1<\Re_e(\delta)<-m$ for some positif integer m. If a function f is analytic at the origin then we have

$$D_z^{\delta,\mu;p;q;\lambda;m}\{f(z)\}=\sum_{n=0}^{\infty}a_nD_z^{\delta,\mu;p;q;\lambda;m}\{z^n\}.$$

Proof

Since f is analytic at the origin, its Maclaurin expansion is given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$ (for $|z| < \rho$ with $\rho \in \mathbb{R}^+$ is the convergence radius). Substitute entire power series in definition 2.5, we obtain

$$D_z^{\delta,\mu;p;q;\lambda;m}\{f(z)\} = \frac{1}{\Gamma(-\delta)}\sqrt{\frac{2p}{\pi}}\int_0^z (z-t)^{-\delta-1} R_K\left(\frac{pz^{2m}}{t^m(z-t)^m}, -\mu-\frac{1}{2}; q; \lambda\right)\sum_{n=0}^{\infty} a_n t^n dt.$$

By virtue of the uniform continuity on the convergence disk, we can do integration term by term in the equation above, so we obtain yet :

$$\begin{aligned} D_z^{\delta,\mu;p;q;\lambda;m}\{f(z)\} &= \sum_{n=0}^{\infty} a_n \left(\frac{1}{\Gamma(-\delta)}\sqrt{\frac{2p}{\pi}}\int_0^z (z-t)^{-\delta-1}\right) \\ &\times \; R_K\left(\frac{pz^{2m}}{t^m(z-t)^m}, -\mu-\frac{1}{2}; q; \lambda\right) t^n dt \\ &= \sum_{n=0}^{\infty} a_n D_z^{\delta,\mu;p;q;\lambda;m}\{z^n\}. \end{aligned}$$

Corollary 2.1.

$$D_z^{\delta,\mu;p;q;\lambda;m}\{(1-z)^{-\alpha}\} = \frac{z^{-\delta}}{\Gamma(-\delta)} B(\frac{3}{2}, -\delta+\frac{1}{2}) F_\mu(\alpha, \frac{3}{2}, -\delta+2; z; q; \lambda; p; m)$$

Where $\Re_e(\alpha) > 0$ and $\Re_e(\delta) < 0$.

Proof

Using binomial theorem for $(1-z)^{-\alpha}$ and lemma 2.1 we obtain :

$$\begin{aligned} D_z^{\delta,\mu;p;q;\lambda;m}\{(1-z)^{-\alpha}\} &= D_z^{\delta,\mu;p;q;\lambda;m}\left\{\sum_{n=0}^{\infty}(\alpha)_n \frac{z^n}{n!}\right\} = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!} D_z^{\delta,\mu;p;q;\lambda;m}\{z^n\} \\ &= \frac{z^{-\delta}}{\Gamma(-\delta)}\sum_{n=0}^{\infty}(\alpha)_n B_\mu\left(n+\frac{3}{2}, -\delta+\frac{1}{2}; p, q; \lambda; m\right)\frac{z^n}{n!}. \end{aligned}$$

Whence the result.

Combining the previous lemmas we obtain again a generalized extended derivative of the product of analytic with power function.

Theorem 2.4. Let $m-1 \leq \Re_e(\beta) < m$ for some $m \in \mathbb{N}$. Suppose that a function $f(z)$ is analytic at the origin with its Maclaurin expansion given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$,

$(|z| < \rho)$ for some $\rho \in \mathbb{R}^+$. Then we have

$$\begin{aligned} D_z^{\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}f(z)\} &= \sum_{n=0}^{\infty} a_n D_z^{\delta,\mu;p;q;\lambda;m}\{z^{\beta+n-1}\} \\ &= \frac{z^{\beta-\delta-1}}{\Gamma(-\delta)} \\ &\times \sum_{n=0}^{\infty} a_n B_\mu(\beta+n+\frac{1}{2}, -\delta+\frac{1}{2}; p; q; \lambda; m) z^n. \end{aligned}$$

Proof

Since the function $z^{\beta-1}f(z)$ can be rewritten as a serie expansion, by definition 2.5, we get

$$\begin{aligned} D_z^{\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}f(z)\} &= \sum_{n=0}^{\infty} a_n D_z^{\delta,\mu;p;q;\lambda;m}\{z^{\beta+n-1}\} \\ &= \frac{z^{\beta-\delta-1}}{\Gamma(-\delta)} \sum_{n=0}^{\infty} a_n B_\mu\left(\beta+n+\frac{1}{2}, -\delta+\frac{1}{2}; p; q; \lambda; m\right) z^n \end{aligned} \tag{2.60}$$

Theorem 2.5. For $\Re_e(\delta) > \Re_e(\beta) > -\frac{1}{2}$, we have

$$D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-z)^{-\alpha}\} = \frac{z^{\delta-1}}{\Gamma(\delta-\beta)} B(\beta+\frac{1}{2}, \delta-\beta+\frac{1}{2}) F_\mu(\alpha, \beta+\frac{1}{2}; \delta+1; z; q; \lambda; p; m). \tag{2.61}$$

Proof

The result is easily recovered by taking $f(z) = (1-z)^{-\alpha}$, so we have

$$\begin{aligned} D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-z)^{-\alpha}\} &= D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}\sum_{k=0}^{\infty}(\alpha)_k \frac{z^k}{k!}\} \\ &= \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta+k-1}\} \\ &= \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} \frac{B_\mu(\beta+k+\frac{1}{2}, \delta-\beta+\frac{1}{2}; p; q; \lambda; m)}{\Gamma(\delta-\beta)} z^{\delta+k-1}. \end{aligned}$$

By using the expression (2.38) we can get

$$D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-z)^{-\alpha}\} = \frac{z^{\delta-1}}{\Gamma(\delta-\beta)} B(\beta+\frac{1}{2}, \delta-\beta+\frac{1}{2}) F_\mu(\alpha, \beta+\frac{1}{2}; \delta+1; z; q; \lambda; p; m).$$

The following makes straight connection between F_μ and $F_{1,\mu}$

Theorem 2.6. For $\Re_e(\delta) > \Re_e(\beta) > -\frac{1}{2}$, $\Re_e(\delta) > 0$, $\Re_e(\gamma) > 0$, $|az| < 1$ and $|bz| < 1$. Then the following generating relation holds true

$$\begin{aligned} D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-az)^{-\alpha}(1-bz)^{-\gamma}\} &= \frac{z^{\delta-1}}{\Gamma(\delta-\beta)} B\left(\beta+\frac{1}{2}, \delta-\beta+\frac{1}{2}\right) \\ &\times F_{1,\mu}\left(\beta+\frac{1}{2}, \alpha, \gamma; \delta+1; az, bz; q; \lambda; p; m\right) \end{aligned} \tag{2.62}$$

Proof

By the binomial theorem once again for $(1-az)^{-\alpha}$ and $(1-bz)^{-\gamma}$ and applying lemma 2.1 and lemma 2.2, we obtain

$$\begin{aligned} & D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-az)^{-\alpha}(1-bz)^{-\gamma}\} \\ = \; & D_z^{\beta-\delta,\mu;p;q;\lambda;m}\left\{z^{\beta-1}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}(\alpha)_k(\gamma)_r\frac{(az)^k}{k!}\frac{(bz)^r}{r!}\right\} \\ = \; & \sum_{k,r=0}^{\infty}(\alpha)_k(\gamma)_r D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta+k+r-1}\}\frac{a^k}{k!}\frac{b^r}{r!} \\ = \; & z^{\delta-1}\sum_{k,r=0}^{\infty}(\alpha)_k(\gamma)_r\frac{B_\mu(\beta+k+r+\frac{1}{2}, \delta-\beta+\frac{1}{2}; p; q; \lambda; m)}{\Gamma(\delta-\beta)}\frac{(az)^k}{k!}\frac{(bz)^r}{r!}. \end{aligned}$$

By using (2.51) we can get

$$\begin{aligned} D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-az)^{-\alpha}(1-bz)^{-\gamma}\} &= \frac{z^{\delta-1}}{\Gamma(\delta-\beta)} B\left(\beta+\frac{1}{2}, \delta-\beta+\frac{1}{2}\right) \\ &\times F_{1,\mu}\left(\beta+\frac{1}{2}, \alpha, \gamma; \delta+1; az, bz; q; \lambda; p; m\right) \end{aligned}$$

Theorem 2.7. For $\Re_e(\delta) > \Re_e(\beta) > -\frac{1}{2}$, $\Re_e(\alpha) > 0$, $\Re_e(\gamma) > 0$, $\Re_e(\tau) > 0$, $|az| < 1$, $|bz| < 1$ and $|cz| < 1$.

Then we have

$$\begin{aligned} & D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-az)^{-\alpha}(1-bz)^{-\gamma}(1-cz)^{-\tau}\} \\ = \; & \frac{z^{\delta-1}}{\Gamma(\delta-\beta)} B\left(\beta+\frac{1}{2}, \delta-\beta+\frac{1}{2}\right) F_{D,\mu}^{3}\left(\beta+\frac{1}{2}, \alpha, \gamma, \tau; \delta+1; az, bz; q; \lambda; p; m\right) \end{aligned} \tag{2.63}$$

Proof

As in the proof of theorem 2.6, using the binomial theorem for $(1-az)^{-\alpha}$, $(1-bz)^{-\gamma}$, $(1-cz)^{-\tau}$ and applying lemmas 2.1- 2.2 , one can easily prove theorem.

Theorem 2.8. For $\Re_e(\delta) > \Re_e(\beta) > -\frac{1}{2}$, $\Re_e(\alpha) > 0$, $\Re_e(\gamma) > \Re_e(\tau) > 0$, $\frac{x}{1-z} < 1$ and $|x| + |z| < 1$. Then we have

$$\begin{aligned} & D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-z)^{-\alpha}F_\mu(\alpha,\gamma;\tau;\frac{x}{1-z};q;\lambda;p;m)\} \\ = \; & z^{\delta-1}\frac{B(\beta+\frac{1}{2},\delta-\beta+\frac{1}{2})}{\Gamma(\delta-\beta)}F_{2,\mu}(\alpha,\gamma,\beta+\frac{1}{2},\tau;\delta+1;x,z;q;\lambda;p;m). \end{aligned}$$

Proof

First, By the binomial theorem (2.42) and according to definition 2.3 we expand $z^{\beta-1}(1-z)^{-\alpha}F_\mu(\alpha,\gamma;\tau;\frac{x}{1-z};q;\lambda;p;m)$ to get

$$\begin{aligned} & D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-z)^{-\alpha}F_\mu(\alpha,\gamma;\tau;\frac{x}{1-z};q;\lambda;p;m)\} \\ = \; & D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-z)^{-\alpha}\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\frac{B_\mu(\gamma+n,\tau-\gamma;q;\lambda;p;m)}{B(\gamma,\tau-\gamma)}(\frac{x}{1-z})^n \\ = \; & \sum_{n=0}^{\infty}(\alpha)_n\frac{B_\mu(\gamma+n,\tau-\gamma;q;\lambda;p;m)}{B(\gamma,\tau-\gamma)}D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-z)^{-\alpha-n}\}\frac{x^n}{n!}. \end{aligned}$$

In order to exhib $F_{2,\mu}$, we apply theorem 2.5 for $D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-z)^{-\alpha-n}\}$ and substitute the extended hypergeometric function F_μ by its series representation, we obtain

$$\begin{aligned} & D_z^{\beta-\delta,\mu;p;q;\lambda;m}\left\{z^{\beta-1}(1-z)^{-\alpha}F_\mu\left(\alpha,\gamma;\tau;\frac{x}{1-z};q;\lambda;p;m\right)\right\} \\ = \; & \frac{z^{\delta-1}}{\Gamma(\delta-\beta)}B\left(\beta+\frac{1}{2},\delta-\beta+\frac{1}{2}\right)F_{2,\mu}\left(\alpha,\gamma,\beta+\frac{1}{2},\tau;\delta+1;x,z;q;\lambda;p;m\right). \end{aligned}$$

This completes the proof

2.6 Integral transformations

2.6.1 Transformation of R_K function.

Proposition 2.13 (Laplace transformation)**.**

Let

$$H(\tau) = \begin{cases} 1 & \tau > 0 \\ 0 & \tau < 0 \end{cases}$$

The unit Heaviside function and $\mathcal{L}$ the Laplace transform operator. Then

$$\mathcal{L}\left\{t^{\alpha-\frac{3}{2}} R_K\left(\frac{p}{t}, -\mu - \frac{1}{2}, q, \lambda\right) H(t-x); s\right\} = \sqrt{\frac{\pi}{2p}} s^{-\alpha} \Gamma_\mu(\alpha, sx; q; \lambda; sp), \tag{2.64}$$

$$\mathcal{L}\left\{t^{\alpha-\frac{3}{2}} R_K\left(\frac{p}{t}, -\mu - \frac{1}{2}, q, \lambda\right) H(t-x)H(t); s\right\} = \sqrt{\frac{\pi}{2p}} s^{-\alpha} \gamma_\mu(\alpha, sx; q; \lambda; sp), \tag{2.65}$$

$(x > 0,\ \Re_e(p) > 0,\ -\infty < \alpha < \infty)$.

Proof We have

$$\begin{aligned} \mathcal{L}\left\{t^{\alpha-\frac{3}{2}} R_K\left(\frac{p}{t}, -\mu - \frac{1}{2}, q, \lambda\right) H(t-x); s\right\} &= \int_0^\infty t^{\alpha-\frac{3}{2}} R_K\left(\frac{p}{t}, -\mu - \frac{1}{2}, q, \lambda\right) \\ &\times\ e^{-st} H(t-x) dt \\ &= \int_x^\infty t^{\alpha-\frac{3}{2}} R_K\left(\frac{p}{t}, -\mu - \frac{1}{2}, q, \lambda\right) e^{-st} dt. \end{aligned}$$

let's replace $t = \frac{\tau}{s}$, $dt = \frac{d\tau}{s}$, on a

$$\begin{aligned} \int_x^\infty t^{\alpha-\frac{3}{2}} R_K\left(\frac{p}{t}, -\mu - \frac{1}{2}, q, \lambda\right) e^{-st} dt &= s^{-\alpha+\frac{1}{2}} \int_{sx}^\infty \tau^{\alpha-\frac{3}{2}} e^{-\tau} R_K\left(\frac{sp}{\tau}, -\mu - \frac{1}{2}, q, \lambda\right) dt \\ &= \sqrt{\frac{\pi}{2p}} s^{-\alpha} \Gamma_\mu(\alpha, sx; q; \lambda; sp). \end{aligned}$$

The proof of (2.65) is left as it follows from the above.

Lemma 2.3. Let $\mathcal{M}$ be the Mellin transform operator. Then

$$\mathcal{M}\{R_K(z, -\alpha, q, \lambda), z \to s\} = 2^{s-2} \Gamma\left(\frac{s-\alpha}{2}\right) \Gamma\left(\frac{s+\alpha}{2}\right) \Phi\left(\lambda, \frac{s+\alpha}{2}, q\right), \tag{2.66}$$

where $\Phi\left(\lambda, \frac{s+\alpha}{2}, q\right)$ stands for the Lerch function [29, pp.1039] and $0 < q \leq 1$, or $-1 \leq \lambda < 1$, $\Re_e(s) > |\Re_e(\alpha)|$ or $\lambda = 1$, $\Re_e(s) > \max(\Re_e(\alpha), 2 - \Re_e(\alpha))$.

Proof

$$\begin{aligned}
\mathcal{M}\{R_K(z,-\alpha,q,\lambda), z \to s\} &= \int_0^\infty z^{s-1} R_K(z,-\alpha,q,\lambda) dz \\
&= 2^{\alpha-1} \int_0^\infty z^{s-\alpha-1} \left(\int_0^\infty t^{\alpha-1} \frac{e^{-qt-z^2/4t}}{1-\lambda e^{-t}} dt \right) dz \\
&= 2^{\alpha-1} \int_0^\infty t^{\alpha-1} \frac{e^{-qt}}{1-\lambda e^{-t}} \left(\int_0^\infty z^{s-\alpha-1} e^{-z^2/4t} dz \right) dt \\
&= 2^{s-2} \Gamma\left(\frac{s-\alpha}{2}\right) \int_0^\infty t^{\frac{s+\alpha}{2}-1} \frac{e^{-qt}}{1-\lambda e^{-t}} dt \\
&= 2^{s-2} \Gamma\left(\frac{s-\alpha}{2}\right) \Gamma\left(\frac{s+\alpha}{2}\right) \Phi\left(\lambda, \frac{s+\alpha}{2}, q\right)
\end{aligned}$$

2.6.2 Eulerian functions

Proposition 2.14 (Mellin transform)**.**

The following expression holds true

$$\begin{aligned}
\mathcal{M}\{B_\mu(x,y;q;\lambda;p;m), p \to s\} &= \frac{2^{s-1}}{\sqrt{\pi}} B\left(x+ms+\frac{m-1}{2}, y+ms+\frac{m-1}{2}\right) \\
&\times \Gamma\left(\frac{s-\mu}{2}\right) \Gamma\left(\frac{s+\mu+1}{2}\right) \Phi\left(\lambda, \frac{s+\mu+1}{2}, q\right)
\end{aligned}$$

where $x, y \in \mathbb{C}$, $m > 0$ et $0 < q \leq 1$, où $-1 \leq \lambda < 1$,

$$\Re_e(s) > \max\left\{\Re_e(\mu), -1-\Re_e(\mu), -\frac{1}{2}+\frac{1}{2m}-\frac{\Re_e(x)}{m}, -\frac{1}{2}+\frac{1}{2m}-\frac{\Re_e(y)}{m}\right\}$$

où

$$\lambda = 1,\ \Re_e(s) > \max\left\{\Re_e(\mu), 1-\Re_e(\mu), -\frac{1}{2}+\frac{1}{2m}-\frac{\Re_e(x)}{m}, -\frac{1}{2}+\frac{1}{2m}-\frac{\Re_e(y)}{m}\right\}$$

Proof

$$\begin{aligned}
\mathcal{M}\{B_\mu(x,y;q;\lambda;p;m), p \to s\} &= \int_0^\infty p^{s-1} B_\mu(x,y;q;\lambda;p;m) dp \\
&= \int_0^\infty p^{s-1} \sqrt{\frac{2p}{\pi}} \bigg(\int_0^1 t^{x-\frac{3}{2}} (1-t)^{y-\frac{3}{2}} \\
&\times R_K\left(\frac{p}{t^m(1-t)^m}, -\mu-\frac{1}{2}, q, \lambda\right) dt \bigg) dp
\end{aligned}$$

$$\begin{aligned}
&= \sqrt{\frac{2}{\pi}}\int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}}\left(\int_0^\infty p^{s+\frac{1}{2}-1}\right.\\
&\times \left.R_K\left(\frac{p}{t^m(1-t)^m}, -\mu-\frac{1}{2}, q, \lambda\right)dp\right)dt\\
&= \sqrt{\frac{2}{\pi}}\int_0^1 t^{x+m(s+\frac{1}{2})-\frac{3}{2}}(1-t)^{y+m(s+\frac{1}{2})-\frac{3}{2}}dt\\
&\times \int_0^\infty u^{s+\frac{1}{2}-1}R_K\left(u, -\mu-\frac{1}{2}, q, \lambda\right)du\\
&= \sqrt{\frac{2}{\pi}}B\left(x+ms+\frac{m-1}{2}, y+ms+\frac{m-1}{2}\right)\\
&\times \int_0^\infty u^{s+\frac{1}{2}-1}R_K\left(u, -\mu-\frac{1}{2}, q, \lambda\right)du
\end{aligned}$$

Finally, by using lemma 2.3 we get the desired result.

2.6.3 Generalized fractional derivative of Riemann-Liouville

Proposition 2.15 (Mellin transform).
The following expression holds true For $\Re_e(s) > \max\left\{\Re_e(\mu), -1-\Re_e(\mu), \frac{1}{2}+\frac{\beta}{m}, \frac{\delta}{m}+\frac{1}{2}\right\}$

$$\begin{aligned}
\mathcal{M}\{D_z^{\delta,\mu,p;q;\lambda;m}z^\beta, p\to s\} &= 2^{s-1}z^{\beta-\delta}\frac{1}{\sqrt{\pi}}B\left(\beta+m(s-\frac{1}{2}), -\delta+m(s-\frac{1}{2})\right)\\
&\times \Gamma\left(\frac{s-\mu}{2}\right)\Gamma\left(\frac{s+\mu+1}{2}\right)\Phi\left(\lambda, \frac{s+\mu+1}{2}, q\right)
\end{aligned}$$

Where Φ stands to Lerch function.

Proof

We can prove this result by applying Mellin transform and using lemma 2.1

$$\begin{aligned}
\mathcal{M}\{D_z^{\delta,\mu,p;q;\lambda;m}z^\beta, p\to s\} &= \frac{1}{\Gamma(-\delta)}\int_0^\infty p^{s-1}z^{\beta-\delta}B_\mu\left(\beta+\frac{3}{2}, -\delta+\frac{1}{2}; p; q; \lambda; m\right)dp\\
&= \frac{z^{\beta-\delta}}{\Gamma(-\delta)}\int_0^\infty p^{s-1}B_\mu\left(\beta+\frac{3}{2}, -\delta+\frac{1}{2}; p; q; \lambda; m\right)dp
\end{aligned}$$

Since the last intergral is the Mellin transform of $B_\mu(\beta+\frac{3}{2}, -\delta+\frac{1}{2}; p; q; \lambda; m)$, we conclude the result by virtue of proposition 2.14.

Proposition 2.16.

$$\begin{aligned}\mathcal{M}\{D_z^{\delta,\mu,p;q;\lambda;m}(1-z)^{-\beta}, p\to s\} &= 2^{s-1}z^{-\delta}\frac{1}{\sqrt{\pi}}B\left(m\left(s+\frac{1}{2}\right)+1, -\delta+m\left(s+\frac{1}{2}\right)\right)\\ &\times \Gamma\left(\frac{s-\mu}{2}\right)\Gamma\left(\frac{s+\mu+1}{2}\right)\Phi\left(\lambda,\frac{s+\mu+1}{2},q\right)\\ &\times {}_2F_1\left(\beta, m\left(s+\frac{1}{2}\right)+1; -\delta+m(2s+1)+1; z\right)\end{aligned}$$

where

$$\Re_e(s) > \max\left\{Re(\mu), -1-\Re_e(\mu), \frac{1}{2}+\frac{\beta}{m}, \frac{\delta}{m}+\frac{1}{2}\right\}$$

Proof

The result can be proved using the binomial theorem for $(1-z)^{-\alpha}$ and the Mellin transform of the general term. Indeed ;

$$\begin{aligned}\mathcal{M}\{D_z^{\delta,\mu;p;q;\lambda;m}\{(1-z)^{-\alpha}\}, p\to s\} &= \mathcal{M}\{D_z^{\delta,\mu;p;q;\lambda;m}\left\{\sum_{n=0}^{\infty}(\alpha)_n\frac{z^n}{n!}\right\}, p\to s\}\\ &= \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\mathcal{M}\{D_z^{\delta,\mu;p;q;\lambda;m}z^n, p\to s\}\\ &= \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}2^{s-1}z^{n-\delta}\frac{1}{\sqrt{\pi}}\\ &\times B\left(n+m\left(s+\frac{1}{2}\right)+1, -\delta+m\left(s+\frac{1}{2}\right)\right)\\ &\times \Gamma\left(\frac{s-\mu}{2}\right)\Gamma\left(\frac{s+\mu+1}{2}\right)\Phi\left(\lambda,\frac{s+\mu+1}{2},q\right)\\ &= 2^{s-1}z^{-\delta}\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{s-\mu}{2}\right)\Gamma\left(\frac{s+\mu+1}{2}\right)\\ &\times \Phi\left(\lambda,\frac{s+\mu+1}{2},q\right)\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\\ &\times B\left(n+m\left(s+\frac{1}{2}\right)+1, -\delta+m\left(s+\frac{1}{2}\right)\right)z^n\\ &= 2^{s-1}z^{-\delta}\frac{1}{\sqrt{\pi}}B\left(m\left(s+\frac{1}{2}\right)+1, -\delta+m\left(s+\frac{1}{2}\right)\right)\\ &\times \Gamma\left(\frac{s-\mu}{2}\right)\Gamma\left(\frac{s+\mu+1}{2}\right)\Phi\left(\lambda,\frac{s+\mu+1}{2},q\right)\\ &\times {}_2F_1(\beta, m\left(s+\frac{1}{2}\right)+1; -\delta+m(2s+1)+1; z).\end{aligned}$$

Chapitre 3

Fractional boundary value problems with integral boundary conditions

Contents

3.1 Introduction

In recent years, boundary value problems (**BVP**) for nonlinear fractional differential equations have been addressed by many researchers. Fractional derivatives are an excellent tool for memory and hereditary properties of various processes.

Integral boundary conditions appear in various real-world phenomenas such as thermal conditions, chemical engineering, blood flow problems and population dynamics. For more details we mention following works ([19], [27], and [56]) and references therein.

3.2 Problem's position

Recently, the theory of existence and uniqueness of solutions of linear and nonlinear fractional differential equations has attracted the attention of many authors, see for example ([7], [30], [34], and [68])

In [68], Taige and Xie use a monotonous iterative method to prove the existence and uniqueness of the solution of fractional differential equation involving the Riemann-Liouville differential operators with integral boundary conditions defined by :

$$\begin{cases} D^{\delta}x(t) = f(t,x), & t \in J = [0,T],\ T \geq 0 \\ x(0) = \lambda \int_0^T x(s)ds + d, & d \in R. \end{cases}$$

where $0 < \delta < 1$, λ is 1 or -1 and $f \in [J \times \mathbb{R},\ \mathbb{R}]$. D^{δ} stands to the Riemann-Liouville fractional derivative of order δ.

Ahmed et al. in [7] studies a boundary value problem of nonlinear fractional differential equations of order $q \in [1,2]$ with three-point integral boundary conditions. Some new existence and uniqueness results are obtained by using standard fixed point theorems and Leray-Schauder degree theory. The problem is presented as follows :

$$\begin{cases} {}^{c}D^{\delta}x(t) = f(t,x(t)), & 0 < t < 1,\ 1 < \delta \leq 2 \\ x(0) = 0,\ x(1) = \alpha \int_0^{\eta} x(s)ds & 0 < \eta < 1. \end{cases}$$

where ${}^{c}D^{\delta}$ denotes the Caputo fractional derivative of order δ, $f : [0,1] \times X \to X$ is continuous, and $\alpha \in \mathbb{R}$ is such that $\alpha \neq \frac{2}{\eta^2}$.

In 2011, Khan et al. [34] study existence and uniqueness of solutions to nonlinear fractional differential equations with integral boundary conditions in ordered Banach space. They employ the nonlinear alternative of Leray-Schauder and Banach fixed point theorem :

$$\begin{cases} {}^{c}D^{\delta}_{0+}u(t) = f(t,u(t),{}^{c}D^{\sigma}_{0+}u(t)),\ t \in [0,T] & , \\ \alpha u(0) - \beta u'(0) = \int_0^T g(s,u)ds,\ \gamma u(1) + \delta u'(1) = \int_0^T h(s,u)ds & . \end{cases}$$

where $0 < \sigma < 1$, $1 < \delta < 2$, $\alpha, \delta > 0$, β, $\gamma \geq 0$, and the nonlinearity depends on the fractional derivative of an unknown function.

Later, Ahmed et *al*, in [6] investigated results of existence and uniqueness of the boundary value problem of nonlinear fractional differential equations of order $\delta\,(1 < \delta \leq 2)$ with non-separated integral boundary conditions, this results are obtained by using some standard fixed point theorems and Leray-Schauder degree theory, they consider the following problem :

$$\begin{cases} {}^{c}D^{\delta}x(t) = f(t, u(t), t \in [0,T],\ T > 0,\ 1 < \delta \leq 2 \\ x(0) - \lambda_1 x(T) = \mu_1 \int_0^T g(s, x(s))ds, \\ x'(0) - \lambda_2 x'(T) = \mu_2 \int_0^T h(s, x(s))ds, \end{cases}$$

where $f, g, h : [0,T) \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and $\lambda_1,\ \lambda_2,\ \mu_1,\ \mu_2 \in \mathbb{R}$ with $\lambda_1 \neq 1$, $\lambda_2 \neq 1$.

In 2011, Ahmed and Nieto [5] studied the following boundary value problem of Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary condition :

$$\begin{cases} D^{\delta}u(t) = f(t, u(t), (\phi u)(t), (\psi u)(t)),\ t \in [0,T],\ \delta \in (1,2] \\ D^{\delta-2}u(0^+) = 0, \\ D^{\delta-1}u(0^+) = vI^{\delta-1}u(\eta),\ 0 < \eta < T,\ v\ is\ a\ constant \end{cases}$$

where $f : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, and :

$$(\phi x)(t) = \int_0^t \gamma(t,s)x(s)ds, \quad (\psi x)(t) = \int_0^t \Theta(t,s)x(s)ds,$$

and γ et Θ continuous on $[0,T] \times [0,T]$.
They obtain some new existence results by applying standard fixed point theorems.

In his work [45] Ntouyas discuss the existence and uniqueness of solution for a boundary value problem of nonlinear fractional differential equations and inclusions of order $\delta \in (0,1]$ with fractional integral boundary conditions of the following problem :

$$\begin{cases} {}^cD^\delta x(t) = f(t, x(t)),\ 0 < t < 1,\ 0 < \delta \leq 1, \\ x(0) = \alpha I^\delta x(\eta),\ 0 < \eta < 1. \end{cases}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $\alpha \in \mathbb{R}$ is such that $\alpha \neq \frac{\Gamma(p+1)}{\eta^p}$ and I^δ, $0 < \delta < 1$ is the Riemann-Liouville fractional integral of order δ.

Using Banach contraction principle and Leray-Schauder nonlinear alternative, Lakoud and Khaldi in [30] establish sufficient conditions for the existence and uniqueness of solutions for boundary value problems for fractional differential equations with integral condition :

$$\begin{cases} {}^cD^\delta_{0+} u(t) = f(t, u(t), D^\delta_{0+} u(t))\ 0 < t < 1, \\ u(0) = 0,\ u'(1) = I^\alpha_{0+} u(1) \end{cases}$$

where $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function. $1 < \delta < 2$ and $0 < \alpha < 1$.

In this chapter, we investigate the existence of solutions to the following fractional differential equation see [3] :

$$D^\delta u(t) + f(t, u(t)) = 0,\ 0 < t < 1,\ 1 < \delta \leq 2 \tag{3.1}$$

subject to the following integral boundary conditions :

$$u(0) = 0,\ \int_0^1 u(s)ds = u(1) \tag{3.2}$$

where D^δ stands to the Riemann-Liouville fractional derivative and f is a continuous function.

3.3 Green's function

In order to find an integral representation of the boundary value problem, we need the following lemmas :

Lemma 3.1. ([52, 58])
Let $\delta > 0$, then the fractional differential equation

$$D^{\delta}u(t) = 0$$

has a unique solution given by

$$u(t) = c_1 t^{\delta-1} + c_2 t^{\delta-2} + ... + c_n t^{\delta-n}, c_i \in \mathbb{R}$$

where $i = 1, ..., n$ et

$$n = \begin{cases} [\delta] + 1, & si\ n \in \{0, 1, 2, ...\}; \\ \delta & si\ n \notin \{0, 1, 2, ...\}. \end{cases} \tag{3.3}$$

From lemma 3.1, we deduce the following result.

Lemma 3.2. ([52, 58])
Let $\delta > 0$ then :

$$I^{\delta}(D^{\delta}u(t)) = u(t) + c_1 t^{\delta-1} + c_2 t^{\delta-2} + ... + c_n t^{\delta-n}, c_i \in \mathbb{R}, \quad i = 1, ..., n,$$

where n is given in (3.3).

We start by solving an auxiliary problem to get an expression for the Green's function associated to the boundary value problem (3.1) – (3.2)

Lemma 3.3. Let $1 < \delta \leq 2$, Assume that $\sigma \in C[0, 1]$. A function $u \in C[0, 1]$ is a solution to the boundary value problem :

$$D^{\delta}u(t) + \sigma(t) = 0,\ 0 < t < 1,\ 1 < \delta \leq 2 \tag{3.4}$$

$$u(0) = 0, \int_0^1 u(s)ds = u(1) \tag{3.5}$$

if and only if it satisfies the integral equation

$$u(t) = \int_0^1 G_{\delta}(t, s)\sigma(s)ds$$

where $G_\delta(t,s)$ is given by

$$G_\delta(t,s)=\begin{cases}\dfrac{t^{\delta-1}(1-s)^{\delta-1}(s+\delta-1)+(1-\delta)(t-s)^{\delta-1}}{(\delta-1)\Gamma(\delta)}; & 0\le s\le t\le 1\\ \dfrac{t^{\delta-1}(1-s)^{\delta-1}(s+\delta-1)}{(\delta-1)\Gamma(\delta)}. & 0\le t\le s\le 1\end{cases}$$

Proof From lemma 3.2, the boundary value problem (3.4) – (3.5) is equivalent to the integral equation :

$$u(t)=-\int_0^t\frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\sigma(s)ds+c_1t^{\delta-1}+c_2t^{\delta-2}$$

Condition $u(0)=0$ implies necessarily that $c_2=0$.
Since $\int_0^1 u(s)ds=u(1)$, we deduce that

$$c_1=\int_0^1 u(s)ds+\int_0^1\frac{(1-s)^{\delta-1}}{\Gamma(\delta)}\sigma(s)ds$$

Finally, we have the following expression

$$u(t)=-\int_0^1\frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\sigma(s)ds+t^{\delta-1}\int_0^1 u(s)ds+t^{\delta-1}\int_0^1\frac{(1-s)^{\delta-1}}{\Gamma(\delta)}\sigma(s)ds \qquad (3.6)$$

From the previous equality, we deduce that

$$\begin{aligned}\int_0^1 u(s)ds &= -\int_0^1\int_0^t\frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\sigma(s)dsdt+\int_0^1\int_0^1 t^{\delta-1}u(s)dsdt\\ &+ \int_0^1\int_0^1 t^{\delta-1}\frac{(1-s)^{\delta-1}}{\Gamma(\delta)}\sigma(s)dsdt\\ &= -\int_0^1\frac{(1-s)^{\delta}}{\delta\Gamma(\delta)}\sigma(s)ds+\frac{1}{\delta}\int_0^1 u(s)ds+\int_0^1\frac{(1-s)^{\delta-1}}{\delta\Gamma(\delta)}\sigma(s)ds\end{aligned}$$

Then, we have

$$\int_0^1 u(s)ds=-\int_0^1\frac{(1-s)^{\delta}}{(\delta-1)\Gamma(\delta)}\sigma(s)ds+\int_0^1\frac{(1-s)^{\delta-1}}{(\delta-1)\Gamma(\delta)}\sigma(s)ds$$

By replacing this value in equation (3.6), we arrive to the following expression for function u :

$$\begin{aligned} u(t) &= -\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\sigma(s)ds + t^{\delta-1}\int_0^1 \frac{(1-s)^{\delta-1}(s+\delta-1)}{(\delta-1)\Gamma(\delta)}\sigma(s)ds \\ &= \int_0^t \frac{(1-\delta)(t-s)^{\delta-1} + t^{\delta-1}(1-s)^{\delta-1}(s+\delta-1)}{(\delta-1)\Gamma(\delta)}\sigma(s)ds \\ &+ \int_t^1 \frac{t^{\delta-1}(s+\delta-1)}{(\delta-1)\Gamma(\delta)}\sigma(s)ds \\ &= \int_0^1 G_\delta(t,s)\sigma(s)ds. \end{aligned}$$

This completes the proof.

Lemma 3.4. Fix $1 < \delta \leq 2$. Let $G_\delta(t,s)$ be the Green's function related to problem (3.4) – (3.5). Then the following inequalities hold :

$$t^{\delta-1}G_\delta(t,s) \leq \delta G_\delta(1,s), \quad for\ all\ t,s \in (0,1)$$

Proof Assume in a first moment that $0 < t \leq s < 1$. In such a case :

$$h(t,s) = \frac{G_\delta(t,s)}{G_\delta(1,s)} = \frac{t^{\delta-1}(s+\delta-1)}{s}, \quad for\ all\ 0 < t \leq s < 1.$$

Now, it is immediate to verify the following inequalities

$$t^{\delta-1} < t^{\delta-1}\left(1 + \frac{\delta-1}{s}\right) = h(t,s) \leq \delta t^{\delta-1} < \delta,\ for\ all\ 0 < t \leq s < 1$$

on the other hand, if $0 < s \leq t < 1$ we have

$$h(t,s) = \frac{t^{\delta-1}(1-s)^{\delta-1}(s+\delta-1) - (\delta-1)(t-s)^{\delta-1}}{s(1-s)^{\delta-1}},\ for\ all\ 0 < s \leq t < 1.$$

and since $s \geq ts$ we deduce that

$$h(t,s) \geq \frac{t^{\delta-1}(1-s)^{\delta-1[(s+\delta-1)-(\delta-1)]}}{s} = t^{\delta-1}$$

As in the previous case, it is not difficult to verify that $h(t,s) \leq \delta$ whenever $0 < s \leq t < 1$. As corollary of the previous result, we deduce the following :

Corollary 3.1. Let G_δ be the Green's function related to problem (3.4) – (3.5). Then for all $1 < \delta \leq 2$, the following inequalities hold :

$$0 < G_\delta(t,s) < \frac{\delta}{(\delta-1)\Gamma(\delta)}, \ for \ all \ t,s \in (0,1).$$

Remark 3.1. The function $t \to \int_0^1 |G_\delta(t,s)|ds$ is continuous on $[0,1]$, then it is bounded.

Let

$$M^* := \sup_{0\leq t\leq 1} \int_0^1 |G_\delta(t,s)|ds.$$

3.4 Existence of solution

Let $\mathcal{C} = C([0,1],\mathbb{R})$ be the Banach space of all continuous functions from $[0,1]$ to $\mathbb{R}$ endowed with the usual maximum norm $\|.\|$.
Assume that $f(t,u)$ is continuous on $[0,1]\times\mathbb{R}$ and define the operator $T : \mathcal{C} \to \mathcal{C}$ by

$$(Tu)(t) = \int_0^1 G_\delta(t,s)f(s,u(s))ds, \qquad t \in [0,1] \tag{3.7}$$

Where G_δ is the Green's function related to problem $(3.1) - (3.2)$. It is obvious that, if u is a fixed point of the operator T given by (3.7), then it coincides with the solution of problem $(3.1) - (3.2)$.
Since f is assumed to be continuous, we denote $f^* := \sup\limits_{0\leq t\leq 1} |f(t,0)|$.

Theorem 3.1. Assume that $f(t,u)$ is continuous on $[0,1]\times\mathbb{R}$ and satisfies a Lipschitz condition

$$|f(t,u) - f(t,v)| \leq k|u-v|, \qquad \forall t \in [0,1], \qquad u,v \in \mathbb{R}$$

Then the boundary value problem $(3.1) - (3.2)$ admits a unique solution provided that

$$kM^* < 1$$

Proof Define a closed ball as $B_\rho = \{u \in \mathcal{C} : \|u\| \leq \rho\}$, where $\rho > 0$ satisfies

$$\rho \geq \frac{M^* f^*}{1-kM^*}$$

Let $u \in B_\rho$, then for all $t \in [0,1]$ we have

$$\begin{aligned} |(Tu)(t)| &\leq \int_0^1 |G_\delta(t,s)||f(s,u(s))|ds \\ &\leq \int_0^1 |G_\delta(t,s)||f(s,u(s)) - f(s,0)|ds + \int_0^1 |G_\delta(t,s)||f(s,0)|ds \\ &\leq \int_0^1 |G_\delta(t,s)|k|u(s)|ds + \int_0^1 |G_\delta(t,s)||f(s,0)|ds \\ &\leq (k\rho + f^*)M^* \\ &\leq \rho \end{aligned}$$

due to the choice of ρ. It follows that $T(B_p) \subset B_p$
Now, for $u, v \in \mathcal{C}$ and for each $t \in [0,1]$, we have

$$\begin{aligned} |(Tu)(t) - (Tv)(t)| &\leq \int_0^1 |G_\delta(t,s)||f(s,u(s)) - f(s,v(s))|ds \\ &\leq k\int_0^1 |G_\delta(t,s)||u(s) - v(s)|ds \\ &\leq KM^*\|u - v\| \end{aligned}$$

Wich implies that the operator T is a contraction.

By theorem 3.1, the operator T admits a unique fixed points in B_ρ, which leads to a unique solution to problem $(3.1) - (3.2)$ since we have a unique fixed point of T, no matter how large is ρ.

The following lemma is needed to establish our next main result.

Lemma 3.5. Assume that $f(t,u)$ is continuous on $[0,1]\times$ $\mathbb{R}$. Then the operator $T : \mathcal{C} \rightarrow \mathcal{C}$ defined by (3.7) is completely continuous.

Proof In view of the continuity of the functions G_δ and f the operator $T : \mathcal{C} \rightarrow \mathcal{C}$ is continuous .
Let Ω $\subset \mathcal{C}$ be bounded, that is, there exists a positive constant $M > 0$ such that $\|u\| \leq M$ for all $u \in \Omega$. Define

$$L := \sup_{0 \leq t \leq 1, 0 \leq u \leq M} |f(t,u)| + 1$$

Then, for all $u \in \Omega$, we have

$$|(Tu)(t)| \leq L \int_0^1 |G_\delta(t,s)|ds \leq LM^*, \qquad t \in [0,1]$$

Hence, $T(\Omega)$ is bounded in $\mathcal{C}$.

For each $u \in \Omega$, and $t_1, t_2 \in [0,1]$, with $t_1 < t_2$, we have

$$\begin{aligned}
|(Tu((t_2) - (Tu)(t_1)| &= |\int_0^1 G_\delta(t_2,s)f(s,u(s))ds - \int_0^1 G_\delta(t_1,s)f(s,u(s))ds| \\
&= |\int_0^{t_1} \frac{[t_2^{\delta-1} - t_1^{\delta-1}](1-s)^{\delta-1}(s+\delta-1)}{(\delta-1)\Gamma(\delta)} f(s,u(s))ds \\
&+ \int_0^{t_1} \frac{(1-\delta)[(t_2-s)^{\delta-1} - (t_1-s)^{\delta-1}]}{(\delta-1)\Gamma(\delta)} f(s,u(s))ds \\
&+ \int_{t_1}^{t_2} \frac{[t_2^{\delta-1} - t_1^{\delta-1}](1-s)^{\delta-1}(s+\delta-1)}{(\delta-1)\Gamma(\delta)} f(s,u(s))ds \\
&+ \int_{t_1}^{t_2} \frac{(1-\delta)(t_2-s)^{\delta-1}}{(\delta-1)\Gamma(\delta)} f(s,u(s))ds \\
&+ \int_{t_2}^{1} \frac{[t_2^{\delta-1} - t_1^{\delta-1}](1-s)^{\delta-1}(s+\delta-1)}{(\delta-1)\Gamma(\delta)} f(s,u(s))ds \\
&\leq \frac{L|t_2^{\delta-1} - t_1^{\delta-1}|}{(\delta-1)\Gamma(\delta)} \int_0^{t_1} (1-s)^{\delta-1}(s+\delta-1)ds \\
&+ \frac{L|1-\delta|}{(\delta-1)\Gamma(\delta)} \int_0^{t_1} |(t_2-s)^{\delta-1} - (t_1-s)^{\delta-1}|ds \\
&+ \frac{L|t_2^{\delta-1} - t_1^{\delta-1}|}{(\delta-1)\Gamma(\delta)} \int_{t_1}^{t_2} (1-s)^{\delta-1}(s+\delta-1)ds \\
&+ \frac{L|1-\delta|}{(\delta-1)\Gamma(\delta)} \int_{t_1}^{t_2} (t_2-s)^{\delta-1}ds \\
&+ \frac{L|t_2^{\delta-1} - t_1^{\delta-1}|}{(\delta-1)\Gamma(\delta)} \int_{t_2}^{1} (1-s)^{\delta-1}(s+\delta-1)ds
\end{aligned}$$

$$\begin{aligned}
|(Tu)(t_2) - (Tu)(t_1)| &= \frac{L\mathcal{B}}{(\delta-1)\Gamma(\delta)} |t_2^{\delta-1} - t_1^{\delta-1}| \\
&+ \frac{L}{\Gamma(\delta)} \int_0^{t_1} |(t_2-s)^{\delta-1} - (t_1-s)^{\delta-1}|ds \\
&+ \frac{L}{\Gamma(\delta)} \int_{t_1}^{t_2} (t_2-s)^{\delta-1}ds
\end{aligned}$$

Where

$$\mathcal{B} = \int_0^1 (1-s)^{\delta-1}(s+\delta-1)ds = \frac{\delta}{\delta+1} < \infty$$

Note that

$$\int_{t_1}^{t_2} (t_2-s)^{\delta-1}ds = \frac{(t_2-t_1)^\delta}{\delta}$$

and, since $t_1 < t_2$, then $(t_1-s)^{\delta-1} < (t_2-s)^{\delta-1}$, so that

$$\begin{aligned}\int_0^{t_1} |(t_2-s)^{\delta-1}-(t_1-s)^{\delta-1}|ds &= \int_0^{t_1} [(t_2-s)^{\delta-1}-(t_1-s)^{\delta-1}]ds \\ &= (-1)\frac{(t_2-t_1)^\delta}{\delta} + \frac{t_2^\delta}{\delta} - \frac{t_1^\delta}{\delta}\end{aligned}$$

In consequence, for $u \in \Omega$, and $t_1, t_2 \in [0,1]$, with $t_1 < t_2$

$$\begin{aligned}|(Tu)(t_2)-(Tu)(t_1)| &= \frac{L\mathcal{B}}{(\delta-1)\Gamma(\delta)}|t_2^{\delta-1}-t_1^{\delta-1}| + \frac{L}{\delta\Gamma(\delta)}(t_2^\delta - t_1^\delta - (t_2-t_1)^\delta) \\ &+ \frac{L}{\delta\Gamma(\delta)}(t_2-t_1)^\delta \qquad (3.8)\end{aligned}$$

Using that the mappings $t \to t^\delta$ are uniformly continuous on $[0,1]$, we deduce that the right-hand side of the inequality in (3.8) tend to 0 as $|t_1-t_2| \to 0$ and, therefore, the set $T(\Omega)$ is equicontinuous in $\mathcal{C}$. Now, from the Arzelá-Áscoli theorem , we conclude that the set $\overline{T(\Omega)}$ is compact. Hence, the operator $T : \mathcal{C} \to \mathcal{C}$ is completely continuous.

Theorem 3.2. Suppose that $f(t,u)$ is continuous on $[0,1]\times\mathbb{R}$, Suppose also that there exists a constant c such that $0 \leq cM^* < 1$ and a constant $d > 0$ such that :
$|f(t,u)| \leq c|u| + d$. Then the problem (3.1) – (3.2) admits at least one solution.

Proof From lemma 3.5, the operator $T : \mathcal{C} \to \mathcal{C}$ is continuous and completely continuous. It remains to show that the set ε (see theorem 3.2 is bounded.
Let u be any element of ε. Then, for $0 < \mu < 1$ and $t \in [0,1]$, the following chain of inequalities holds :

$$\begin{aligned}|u(t)| &= \mu|(Tu)(t)| \\ &\leq \int_0^1 |G_\delta(t,s)||f(s,u(s))|ds \\ &\leq \int_0^1 |G_\delta(t,s)|[c|u(s)|+d]ds \\ &\leq [c\|u\|+d]\int_0^1 |G_\delta(t,s)|ds\end{aligned}$$

Which, furthermore, implies

$$\|u\| \leq \frac{M^*d}{1 - cM^*} < \infty$$

As a consequence of theorem 3.2, the operator T has at least a fixed point, which is a solution to problem $(3.1) - (3.2)$.

Theorem 3.3. Suppose that $f(t,u)$ is continuous $[0,1]\times\mathbb{R}$, Suppose also that there exist a nondecreasing $\psi : [0,\infty) \to [0,\infty)$ and a function $p \in \mathcal{C}([0,1],\mathbb{R}^+)$ such that

$$|f(t,u)| \leq p(t)\psi(|u|), \qquad \forall (t,u) \in [0,1] \times \mathbb{R} \tag{3.9}$$

Then the boundary value problem $(3.1) - (3.2)$ admits at least one solution provided that there exists a positive constant r such that :

$$r > \|p\|M^*\psi(r). \tag{3.10}$$

Proof From lemma 3.5, the operator $T : \mathcal{C} \to \mathcal{C}$ is completely continuous (it is proved that it is continuous and compact).
Choose $r > 0$ such that the condition (3.10) is satisfied, and define the ball $B_r \subset \mathcal{C}$ as :

$$B_r = \{u \in C : \|u\| < r\}$$

The set B_r is open and contains 0.
Now, assume that $u = \mu Tu$ holds for some $\mu \in [0,1]$ and all $t \in [0,1]$, then :

$$\begin{aligned} |u(t)| &= \mu|(Tu)(t)| \\ &\leq \int_0^1 |G_\delta(t,s)||f(s,u(s))|ds \\ &\leq \int_0^1 |G_\delta(t,s)|p(s)|\psi(|u(s)|)ds \\ &\leq \|p\|\psi(\|u\|)\int_0^1 |G_\delta(t,s)|ds. \end{aligned}$$

Which, furthermore, implies that :

$$\|u\| \leq \|p\|M^*\psi(\|u\|).$$

By (3.10), we conclude that there is no $u \in \partial B_r$ such that $u = \mu T u$ for some $\mu \in [0,1]$. Consequently, by theorem 3.3, we deduce that T has a fixed point $u \in B_r$, which gives rise to a solution of the boundary value problem $(3.1)-(3.2)$. This completes the proof.

3.5 Examples

Example 3.1. Consider the boundary value problem with integral condition :

$$D^{3/2}u(t) + \frac{e^{-t}}{1+3e^t}u(t) = 0, \qquad 0 < t < 1, \tag{3.11}$$

$$u(0) = 0, \int_0^1 u(s)ds = u(1) \tag{3.12}$$

Here :

$$f(t,u) = \frac{e^{-t}}{1+3e^t}u, \qquad t \in [0,1], u \in \mathbb{R}$$

Let $u_1, u_2 \in \mathbb{R}$ and $t \in [0,1]$. Then :

$$\begin{aligned} |f(t,u_1) - t(t,u_2)| &= \frac{e^{-t}}{1+3e^t}|u_1 - u_2| \\ &\leq \frac{1}{4}|u_1 - u_2| \end{aligned}$$

The condition of theorem 3.1 are satisfied since

$$kM^* \leq \frac{1}{4}\frac{3}{\Gamma(\frac{3}{2})} = \frac{3}{2\sqrt{\pi}} \approx 0.85 < 1$$

Hence, the boundary value problem $(3.11) - (3.12)$ has a unique solution.

Example 3.2. Consider the following boundary value problem :

$$D^{3/2}u(t) + \frac{e^{-t}}{1+3e^t}\sqrt{|u(t)|}, \qquad 0 < t < 1, \tag{3.13}$$

$$u(0) = 0, \int_0^1 u(s)ds = u(1) \tag{3.14}$$

Here

$$f(t,u) = \frac{e^{-t}}{1+3e^t}\sqrt{|u|}, \qquad t \in [0,1], \qquad u \in \mathbb{R},$$

Which is not Lipschitzien so theorem 3.1 can not be applied. On the other hand, we have

$$|f(t,u)| \leq c|u| + d$$

where

$$c = d = \frac{1}{4}$$

The condition of theorem 3.2 are satisfied since

$$cM^* \leq \frac{1}{4}\frac{3}{\Gamma(3/2)} = \frac{3}{2\sqrt{\Pi}} \approx 0.85 < 1$$

Hence, the boundary value problem (3.13) − (3.14) has at least one solution.

Example 3.3. Consider the boundary value problem :

$$D^{3/2}u(t) + e^{-t}u^2(t) = 0, \qquad 0 < t < 1, \tag{3.15}$$

$$u(0) = 0, \int_0^1 u(s)ds = u(1) \tag{3.16}$$

Here

$$f(t,u) = e^{-t}u^2, t \in [0,1], u \in \mathbb{R}$$

which is neither Lipschitzien nor sublinear. Then, theorem 3.1 and 3.2 can not be applied to this situation.

On the other hand, we can choose $r = \frac{1}{4}$, thus, the condition (3.10) in theorem 3.3 is satisfied since in this case we have :

$$\frac{r}{\|p\|M^*\psi(r)} = \frac{r}{\|p\|M^*r^2} > \frac{\frac{1}{4}}{\frac{3}{\Gamma(3/2)}(\frac{1}{4}^2)} \approx 1.17 > 1.$$

Hence, the boundary value problem (3.15) − (3.16) has at least one solution.

Chapitre 4

Positive solutions for fractional boundary value problems with integral boundary conditions and parameter dependence

Contents

4.1 Introduction

As in public health, epidemic diseases and dynamic of populations, many fields of engineering and sciences focus their interest on existence of positive solutions. We mention works [17, 18, 19, 44] for more reading.

One of the most common approaches to prove the existence of positive solutions for nonlinear problems is based on the Krasnoselskii's fixed point Theorem in cone, where the presumed solution is located in a cone of a normalized linear space.

Since then, a large number of papers have been published, bringing generalizations or modifications to the Krasnoselskii's fixed point Theorem and its applications see e.g [8, 12, 32, 49, 70].

4.2 Problem's position

The existence of positive solutions of ordinary and partial differential equations has been widely studied, less those of fractional differential equations.

In 2012, X.Zhao et al. [74] studied existence (and nonexistence) of positive solutions for the following fractional boundary value problem :

$$\begin{cases} D_{0^+}^{\delta} u(t) + \lambda h(t) f(u(t)) = 0, \ t \in (0,1) \\ u(0) = u'(0) = 0, \ u(1) = \int_0^1 g(s)u(s)ds \end{cases}$$

where δ is a real number with $2 < \delta \leq 3$, λ is a positive parameter, we obtain the existence and nonexistence results of positive solutions when the nonlinear term satisfies different requirements of superlinearity, sublinearity and the parameter λ lies to some intervals.

By means of the monotone iteration method and some inequalities associated with Green function, Sun and Zhao [63] obtain the existence of a positive solution and established an iteration sequence to approximate the solution of the fractional differential equation with integral boundary conditions :

$$\begin{cases} D^{\delta} u(t) + q(t) f(t, u(t)) = 0, \quad 0 < t < 1 \\ u(0) = u'(0) = 0, \ u(1) = \lambda \int_0^1 g(s)u(s)ds \end{cases}$$

with $2 < \delta \leq 3$, $f, \ q, \ g : [0,1] \times [0,\infty] \to [0,\infty]$ is continuous and $f(t,0) \neq 0$ on $[0,1]$.

In 2014, X.Zhang et al. in their paper [73] study the existence of positive solutions for the following nonlinear fractional differential equations with integral boundary conditions :

$$\begin{cases} D^{\delta}_{0+}u(t) + h(t)f(t, u(t)) = 0, \quad 0 < t < 1 \\ u(0) = u'(0) = u''(0) = 0, \\ u(1) = \lambda \int_0^{\eta} u(s)ds \end{cases}$$

according to sublinear and superlinear cases, where $3 < \delta \leq 4,\ 0 < \eta \leq 1, 0 \leq \frac{\lambda\eta^{\delta}}{\delta} < 1.$ The existence and multiplicity of positive solutions are obtained by means of the properties of the Green function, u_0-bounded function and the fixed point index theory under some conditions concerning the first eigenvalue with respect to the relevant linear operator.

By means of monotone iterative technique and some inequalities associated with the Green function, Zhang in [72] obtained the existence of nontrivial solutions or positive solutions, also iterative schemes to approximate the solution for the nonlinear boundary value problem :

$$\begin{cases} D^{\delta}_{0+}u(t) + f(t, u(t)) = 0, \quad t \in (0, 1) \\ u(0) = u'(0), \quad D^{\beta}_{0+}u(1) = \int_0^1 D^{\beta}_{0+}u(t)dA(t) \end{cases}$$

with $2 < \delta \leq 3,\ 0 < \beta \leq 1$ are real numbers and $\int_0^1 D^{\beta}_{0+}u(t)dA(t)$ denotes the Riemann-Stieltjes integral [See the definition in [64]].

Yang in [71] studied the existence and uniqueness of positive solutions to the following integral boundary value condition of fractional differential equation with a parameter :

$$\begin{cases} D^{\delta}_{0+}u(t) + \lambda f(t, u(t)) = 0, \quad 0 < t < 1 \\ u(0) = u'(0) = ... = u^{(n-2)}(0) = 0, \quad u(1) = \int_0^1 g(s)u(s)ds \end{cases}$$

with $2 < \delta \leq 3$, $\lambda > 0$ is a parameter, the method employed is a fixed point theorem of concave operators in partial ordering Banach spaces.

Again by the fixed point theorem and the properties of the theory of mixed monotone operator theory , Song and Bai [61] proved the existence and uniqueness of positive solutions of the following problem :

$$\begin{cases} D_{0+}^{\delta} u(t) + \lambda f(t, u(t), u(t)) = 0 \quad 0 < t < 1, n-1 < \delta \leq n \\ u^{(k)}(0) = 0,\ 0 \leq k \leq n-2, \quad u(1) = \int_0^1 u(s) dA(s) \end{cases}$$

with : $n-1 < \delta \leq n$, $\lambda > 0$, A is a function of bounded variations, $\int_0^1 u(s)dA(s)$ denotes the Riemann-Stieltjes integral of u with respect to A.

Motivated by all works cited below, we investigate the existence of positive solutions of the following fractional differential equation with integral boundary conditions.

$$D^{\delta} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1,\ 1 < \delta \leq 2, \tag{4.1}$$

$$u(0) = 0,\ u(1) = \lambda \int_0^1 h(r) u(r) dr. \tag{4.2}$$

where D^{δ} is the Riemann-Liouville fractional derivative and f is a given function. This chapter is an expansion of our published paper [2].

4.3 Green's function

We start by solving an auxiliary problem in order to get the expression for the Green's function of boundary value problem (4.1) – (4.2), the following auxiliary problem is :

$$D^{\delta} u(t) + \sigma(t) = 0, \quad 0 < t < 1,\ 1 < \delta \leq 2, \tag{4.3}$$

$$u(0) = 0,\ u(1) = \lambda \int_0^1 h(r) u(r) dr. \tag{4.4}$$

Lemma 4.1. Let $1 < \delta \leq 2$. Suppose that $1 - \lambda \int_0^1 h(r) r^{\delta-1} dr \neq 0$. A function $u \in C[0,1]$ is a solution of the linear boundary value problem (4.3)-(4.4) if and only if it satisfies the integral equation

$$u(t) = \int_0^1 G(t,s) \sigma(s) ds,$$

where $G(t,s)$ is the Green's function given by

$$G(t,s) = G_1(t,s) + G_2(t,s)$$

with

$$G_1(t,s) = \begin{cases} \dfrac{t^{\delta-1}(1-s)^{\delta-1} - (t-s)^{\delta-1}}{\Gamma(\delta)}, & 0 \le s \le t \le 1; \\ \dfrac{t^{\delta-1}(1-s)^{\delta-1}}{\Gamma(\delta)}, & 0 \le t \le s \le 1. \end{cases} \tag{4.5}$$

and

$$G_2(t,s) = \frac{\lambda t^{\delta-1}}{1 - \lambda \int_0^1 h(r) r^{\delta-1} dr} \int_0^1 h(r) G_1(r,s) dr \tag{4.6}$$

Proof By Lemma 2.2 we have that the u is a solution of the linear equation (4.3) if and only if it satisfies

$$u(t) = -\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} \sigma(s) ds + c_1 t^{\delta-1} + c_2 t^{\delta-2}.$$

Condition $u(0) = 0$ implies necessarily that $c_2 = 0$.
Since $u(1) = \lambda \int_0^1 h(r) u(r) dr$, we deduce that

$$c_1 = \int_0^1 \frac{(1-s)^{\delta-1}}{\Gamma(\delta)} \sigma(s) ds + \lambda c_1 \int_0^1 h(s) s^{\delta-1} ds - \frac{\lambda}{\Gamma(\delta)} \int_0^1 h(r) \int_0^r (r-s)^{\delta-1} \sigma(s) ds dr.$$

Now, since $1 - \lambda \int_0^1 h(r) r^{\delta-1} dr \neq 0$, we have

$$c_1 = \frac{1}{\Gamma(\delta)(1 - \lambda \int_0^1 h(r) r^{\delta-1} dr)} \left(\int_0^1 (1-s)^{\delta-1} \sigma(s) ds - \lambda \int_0^1 h(r) \int_0^r (r-s)^{\delta-1} \sigma(s) ds dr \right)$$

Finally, we have the expression

$$\begin{aligned} u(t) &= -\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} \sigma(s) ds \\ &+ \frac{t^{\delta-1}}{\Gamma(\delta)(1 - \lambda \int_0^1 h(r) r^{\delta-1} dr)} \int_0^1 (1-s)^{\delta-1} \sigma(s) ds \\ &- \frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1 - \lambda \int_0^1 h(r) r^{\delta-1} dr)} \int_0^1 h(r) \int_0^r (r-s)^{\delta-1} \sigma(s) ds dr \\ &= -\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} \sigma(s) ds \\ &+ \frac{t^{\delta-1}(1 - \lambda \int_0^1 h(r) r^{\delta-1} dr + \lambda \int_0^1 h(r) r^{\delta-1} dr)}{\Gamma(\delta)(1 - \lambda \int_0^1 h(r) r^{\delta-1} dr)} \int_0^1 (1-s)^{\delta-1} \sigma(s) ds \end{aligned}$$

$$
\begin{aligned}
- & \frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}dr)}\int_0^1 h(r)\cdot\int_0^r (r-s)^{\delta-1}\sigma(s)dsdr \\
= & -\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\sigma(s)ds+\frac{t^{\delta-1}}{\Gamma(\delta)}\int_0^1(1-s)^{\delta-1}\sigma(s)ds \\
& +\frac{\lambda t^{\delta-1}\int_0^1 h(r)r^{\delta-1}dr}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}dr)}\int_0^1(1-s)^{\delta-1}\sigma(s)ds \\
& -\frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}dr)}\int_0^1 h(r)\cdot\int_0^r (r-s)^{\delta-1}\sigma(s)dsdr \\
= & -\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\sigma(s)ds+\frac{t^{\delta-1}}{\Gamma(\delta)}\int_0^t(1-s)^{\delta-1}\sigma(s)ds+\frac{t^{\delta-1}}{\Gamma(\delta)}\int_t^1(1-s)^{\delta-1}\sigma(s)ds \\
+ & \frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}dr)}\int_0^1 h(r)r^{\delta-1}dr\cdot\int_0^1(1-s)^{\delta-1}\sigma(s)ds \\
- & \frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}dr)}\int_0^1 h(r)\cdot\int_0^r (r-s)^{\delta-1}\sigma(s)dsdr \\
= & -\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\sigma(s)ds+\frac{t^{\delta-1}}{\Gamma(\delta)}\int_0^t(1-s)^{\delta-1}\sigma(s)ds+\frac{t^{\delta-1}}{\Gamma(\delta)}\int_t^1(1-s)^{\delta-1}\sigma(s)ds \\
& +\frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}dr)}\int_0^1 h(r)\cdot\int_0^1 r^{\delta-1}(1-s)^{\delta-1}\sigma(s)dsdr \\
& -\frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}dr)}\int_0^1 h(r)\cdot\int_0^r (r-s)^{\delta-1}\sigma(s)dsdr \\
= & -\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\sigma(s)ds+\frac{t^{\delta-1}}{\Gamma(\delta)}\int_0^t(1-s)^{\delta-1}\sigma(s)ds+\frac{t^{\delta-1}}{\Gamma(\delta)}\int_t^1(1-s)^{\delta-1}\sigma(s)ds \\
& +\frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}dr)}\int_0^1 h(r)\cdot\int_0^r r^{\delta-1}(1-s)^{\delta-1}\sigma(s)dsdr \\
& -\frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}dr)}\int_0^1 h(r)\cdot\int_0^r (r-s)^{\delta-1}\sigma(s)dsdr \\
& +\frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}dr)}\int_0^1 h(r)\cdot\int_r^1 r^{\delta-1}(1-s)^{\delta-1}\sigma(s)dsdr \\
= & \int_0^1 G_1(t,s)\sigma(s)ds+\frac{\lambda t^{\delta-1}}{(1-\lambda\int_0^1 h(r)r^{\delta-1}dr)}\int_0^1 h(r)\cdot\int_0^1 G_1(r,s)\sigma(s)dsdr \\
= & \int_0^1 G_1(t,s)\sigma(s)ds+\int_0^1\frac{\lambda t^{\delta-1}}{(1-\lambda\int_0^1 h(r)r^{\delta-1}dr)}\int_0^1 h(r)G_1(r,s)dr\sigma(s)ds \\
= & \int_0^1 G_1(t,s)\sigma(s)ds+\int_0^1 G_2(t,s)\sigma(s)ds.
\end{aligned}
$$

□

As a direct consequence of the previous result, we deduce the following properties.

Lemma 4.2. The function $G_1(t,s)$ defined in Lemma 4.1 has the following properties :

1. $G_1(t,s)\in\mathcal{C}\ ([0,1]\times[0,1])$.

2. $G_1(t,s) > 0$ for $(t,s) \in (0,1)\times(0,1)$ and $G_1(0,s) = 0 = G_1(1,s)$ for $s \in [0,1]$.
3. $G_1(t,s) = G_1(1-s, 1-t),\ \forall t, s \in [0,1]$.

Proof

1.

$$G_1(t,s) = \begin{cases} \dfrac{t^{\delta-1}(1-s)^{\delta-1} - (t-s)^{\delta-1}}{\Gamma(\delta)}, & 0 \le s \le t \le 1; \\ \dfrac{t^{\delta-1}(1-s)^{\delta-1}}{\Gamma(\delta)}, & 0 \le t \le s \le 1. \end{cases}$$

is continuous (obvious)

2. We have : $(t - ts) > (t - s)$, $t \ge 0$ and $(1-s) > 0$ thus

$$G_1(t,s) = \begin{cases} \dfrac{t^{\delta-1}(1-s)^{\delta-1} - (t-s)^{\delta-1}}{\Gamma(\delta)} \ge 0, & 0 \le s \le t \le 1; \\ \dfrac{t^{\delta-1}(1-s)^{\delta-1}}{\Gamma(\delta)} \ge 0, & 0 \le t \le s \le 1. \end{cases}$$

on the other hand

$$\begin{cases} G_1(0,s) = 0 & 0 \le t \le s \le 1; \\ G_1(1,s) = 0 & 0 \le s \le t \le 1. \end{cases}$$

$\Rightarrow G_1(0,s) = G_1(1,s) = 0 \quad \forall s \in [0,1]$

3.

$$\begin{aligned} G_1(1-s,1-t) &= \begin{cases} \frac{(1-s)^{\delta-1}(1-1+t)^{\delta-1}-(1-s-1+t)^{\delta-1}}{\Gamma(\delta)} \ge 0, & 0 \le s \le t \le 1; \\ \frac{(1-s)^{\delta-1}(1-1+t)^{\delta-1}}{\Gamma(\delta)} \ge 0, & 0 \le t \le s \le 1. \end{cases} \\ &= \begin{cases} \frac{t^{\delta-1}(1-s)^{\delta-1}-(t-s)^{\delta-1}}{\Gamma(\delta)} \ge 0, & 0 \le s \le t \le 1; \\ \frac{t^{\delta-1}(1-s)^{\delta-1}}{\Gamma(\delta)} \ge 0, & 0 \le t \le s \le 1. \end{cases} \\ &= G_1(t,s) \end{aligned}$$

In next result, we deduce two inequalities that, as we will see, will be fundamental to ensure the existence of the solutions of the nonlinear problem (4.1)-(4.2).

Lemma 4.3. Fix $t_0 \in (0,1)$, then G_1 satisfies the following inequalities :

$$G_1(t,s) \leq \frac{s^{\delta-1}\,(1-s)^{\delta-1}}{\Gamma(\delta)}, \quad \forall t \in [0,1],\ s \in [0,1] \tag{4.7}$$

and

$$s^{\delta-1}\,(1-s)^{\delta-1}\,k(t,t_0) \leq G_1(t,s), \quad \forall t \in [0,1],\ s \in [t_0,1], \tag{4.8}$$

with

$$k(t,t_0) = \begin{cases} \frac{t^{\delta-1}}{\Gamma(\delta)} & \text{if} \quad 0 \leq t \leq t_0 < 1 \\ \\ \min\left\{\frac{t^{\delta-1}}{\Gamma(\delta)}, \frac{t^{\delta-1}(1-t_0)^{\delta-1}-(t-t_0)^{\delta-1}}{\Gamma(\delta)\,t_0^{\delta-1}\,(1-t_0)^{\delta-1}}\right\} & \text{if} \quad 0 < t_0 < t \leq 1 \end{cases}$$

Proof For $s > t$,we have

$$\frac{\partial G_1}{\partial t}(t,s) = \frac{\delta-1}{\Gamma(\delta)}(1-s)^{\delta-1}t^{\delta-2} > 0.$$

For $s < t$, since $1 < \delta \leq 2$, we have

$$\frac{\partial G_1}{\partial t}(t,s) = \frac{\delta-1}{\Gamma(\delta)}\left((1-s)^{\delta-1}t^{\delta-2} - (t-s)^{\delta-2}\right) \leq \frac{\delta-1}{\Gamma(\delta)}\left(t^{\delta-2} - (t-s)^{\delta-2}\right) < 0.$$

As a consequence, it is fulfilled that

$$G_1(t,s) \leq G_1(s,s) = \frac{s^{\delta-1}\,(1-s)^{\delta-1}}{\Gamma(\delta)} \quad \forall t,s \in [0,1]$$

and inequality (4.7) holds.

By using the third property on Lemma 4.2, we deduce that

$$\frac{\partial G_1}{\partial s}(t,s) > 0 \quad \text{for } 0 \leq s < t \leq 1$$

and

$$\frac{\partial G_1}{\partial s}(t,s) < 0 \quad \text{if } 0 \leq t < s \leq 1.$$

Now, we introduce the following function :

$$F_1(t,s) = \frac{G_1(t,s)}{s^{\delta-1}\,(1-s)^{\delta-1}}, \quad (t,s) \in [0,1] \times (0,1),$$

as a direct consequence of previous arguments, we deduce that

$$\frac{\partial F_1}{\partial t}(t,s) < 0 \quad \text{for } 0 \leq s < t \leq 1$$

and

$$\frac{\partial F_1}{\partial t}(t,s) > 0 \quad \text{if } 0 \leq t < s \leq 1.$$

As a consequence, we have that

$$\frac{G_1(t,s)}{s^{\delta-1}\,(1-s)^{\delta-1}} \leq \frac{G_1(s,s)}{s^{\delta-1}\,(1-s)^{\delta-1}} = \frac{1}{\Gamma(\delta)}.$$

By the other hand,

$$\frac{\partial F_1}{\partial s}(t,s) = \begin{cases} -\frac{t^{\delta-1}}{\Gamma(\delta-1)\,s^{\delta}} & 0 \leq t < s \leq 1, \\ \\ \frac{(\delta-1)\left(t\left(s^2-2st+t\right)(t-s)^{\delta}-(t-s)^2 t^{\delta}(1-s)^{\delta}\right)}{t\,\Gamma(\delta)(t-s)^2\,(1-s)^{\delta} s^{\delta}} & 0 \leq s < t \leq 1. \end{cases}$$

As a direct consequence, we deduce that

$$\frac{\partial F_1}{\partial s}(t,s) < 0 \quad \text{for } 0 \leq t < s \leq 1.$$

On the other hand, for the case $0 \leq s < t \leq 1$ we have that $\frac{\partial F_1}{\partial s}(t,s) > 0$ if and only if

$$h_1(t,s,\delta) := (1-s)^{\delta} t^{\delta-1}(t-s)^{2-\delta} < s^2 - 2\,s\,t + t =: h_2(t,s).$$

Now, since

$$\frac{\partial h_1}{\partial \delta}(t,s,\delta) = (1-s)^{\delta} t^{\delta-1}(t-s)^{2-\delta}\,\log\left(\frac{t-t\,s}{t-s}\right),$$

we have that h_1 is strictly increasing on the δ interval $[1,2]$ for any $0 \leq s < t \leq 1$ given.

Thus, since $h_2(t,s) - h_1(t,s,2) = (1-t)\,s^2 > 0$, we conclude that $\frac{\partial F_1}{\partial s}(t,s) > 0$ for all $0 < s < t < 1$.

So, for any $t_0 \in (0,1)$ fixed, we have that

$$\begin{aligned}
\frac{G_1(t,s)}{s^{\delta-1}\,(1-s)^{\delta-1}} &\geq \min\left\{\lim_{s\to 1^-}\frac{G_1(t,s)}{s\,(1-s)^{\delta-1}}, \frac{G_1(t,t_0)}{t_0^{\delta-1}\,(1-t_0)^{\delta-1}}\right\} \\
&= \min\left\{\frac{t^{\delta-1}}{\Gamma(\delta)}, \frac{G_1(t,t_0)}{t_0^{\delta-1}\,(1-t_0)^{\delta-1}}\right\} =: k(t,t_0), \quad \forall t\in[0,1],\ s\in[t_0,1],
\end{aligned}$$

and the result is concluded.

By virtue of this Lemma, we can give now the main result of this section.

Lemma 4.4. Let $t_0 \in (0,1)$ be fixed and h introduced at the boundary conditions (4.2). Denote by $A = \int_0^1 h(r)r^{\delta-1}dr$, $B = \int_0^1 h(r)dr$ and $C_0 = \int_{t_0}^1 k(r,t_0)\,h(r)dr$. Assume that $h \geq 0$ on $[0,1]$ and $1-\lambda\ A > 0$. Then, the Green's function $G(t,s)$ defined in Lemma 4.1 satisfies the inequalities

$$\frac{\lambda\, C_0\, t^{\delta-1}}{1-\lambda A}\, s^{\delta-1}(1-s)^{\delta-1} \leq G(t,s) \leq \frac{1}{\Gamma(\delta)}\left(1+\frac{\lambda B}{1-\lambda A}\right) s^{\delta-1}(1-s)^{\delta-1},\ \forall t,s\in[0,1]. \tag{4.9}$$

Proof From the definition of G, the inequality (4.7) and the fact that $1<\delta\leq 2$, we have the following inequalities for all $t,\ s\in[0,1]$:

$$\begin{aligned}
G(t,s) &\leq \frac{1}{\Gamma(\delta)}s^{\delta-1}(1-s)^{\delta-1} + \frac{\lambda t^{\delta-1}}{1-\lambda A}\int_0^1 \frac{1}{\Gamma(\delta)}s^{\delta-1}(1-s)^{\delta-1}h(r)dr \\
&\leq \frac{1}{\Gamma(\delta)}\left(1+\frac{\lambda B}{1-\lambda A}\right) s^{\delta-1}(1-s)^{\delta-1}.
\end{aligned}$$

On the other hand, by Lemma 4.2 (2) and (4.8), we have for all $t,\ s\in[0,1]$:

$$\begin{aligned}
G(t,s) &= G_1(t,s)+G_2(t,s) \\
&\geq \frac{\lambda t^{\delta-1}}{1-\lambda\int_0^1 h(r)r^{\delta-1}dr}\int_{t_0}^1 h(r)G_1(r,s)dr \\
&\geq \frac{\lambda t^{\delta-1}}{1-\lambda A}\,C_0\, s^{\delta-1}(1-s)^{\delta-1},
\end{aligned}$$

as we want to prove.

As a direct consequence, we deduce the following Corollary :

Corollary 4.1. If $h\geq 0$ on $[0,1]$ and $1-\lambda\, A>0$ then the Green's function $G(t,s)$

defined in Lemma 4.1 satisfies the inequalities

$$\frac{\lambda t}{1-\lambda A}\, C_0\, s^{\delta-1}(1-s)^{\delta-1} \leq t^{2-\delta} G(t,s) \leq \frac{1}{\Gamma(\delta)} \left(1+\frac{\lambda B}{1-\lambda A}\right) s^{\delta-1}(1-s)^{\delta-1},\ \forall t,s\in[0,1] \tag{4.10}$$

4.4 Existence of positive solutions

Now for any $u:(0,1]\to\mathbb{R}$, we define function $\overline{u}:[0,1]\to\mathbb{R}$ as follows :

$$\overline{u}(t)=\begin{cases} t^{2-\delta}u(t) & \text{if} \quad t\in(0,1],\\ \lim\limits_{t\to 0^+} t^{2-\delta}u(t) & \text{if} \quad t=0, \end{cases} \tag{4.11}$$

provided that such limit exists.

Consider the Banach space

$$E=C_\delta[0,1]:=\{\bar{u}:[0,1]\to\mathbb{R}, \text{is a continuous function in } [0,1]\}$$

endowed with the maximum norm $\|u\|=\max\limits_{0\leq t\leq 1}|\bar{u}(t)|$ and define the cone $P_0\subset E$ by

$$P_0=\{u\in E,\ \bar{u}(t)\geq t^{2-\delta}p(t,t_0)\|u\|,\ for\ all\ t\in[0,1]\},$$

where

$$p(t,t_0)=\Gamma(\delta)\,\frac{\lambda t^{\delta-1}}{1-\lambda A}\, C_0/\left(1+\frac{\lambda B}{1-\lambda A}\right),\ t\in[0,1],$$

with $t_0\in(0,1)$ fixed, and A, B and C_0 introduced in Lemma 4.4.
Notice that, provided that $h>0$ on $[0,1]$ and $1-\lambda A>0$, we deduce from (4.9) that $0<p(t,t_0)<1$ for all $t\in[0,1]$ and $t_0\in(0,1)$.
Now, we assume the following hypothesis on the nonlinear part of the equation :

(H_1) The function $f:[0,1]\times\mathbb{R}\to[0,\infty)$ is continuous.

So, we define the operator $T:P_0\to E$ by

$$(Tu)(t)=\int_0^1 G(t,s)f(s,\bar{u}(s))ds,\ \ t\in[0,1] \tag{4.12}$$

Lemma 4.5. $T:P_0\to P_0$ is completely continuous.

Proof Let us prove in first that $T(P_0) \subset P_0$. Notice from the definition of T and Corollary 4.1 that for $u \in P_0$, $Tu(t) \geq 0$ for all $t \in [0,1]$ and

$$\begin{aligned}
t^{2-\delta}(Tu)(t) &= \int_0^1 t^{2-\delta} G(t,s) f(s,\bar{u}(s))ds \\
&\geq \int_0^1 t^{2-\delta} \frac{\lambda t^{\delta-1}}{1-\lambda A} C_0\, s^{\delta-1}(1-s)^{\delta-1} f(s,\bar{u}(s))ds \\
&= t^{2-\delta} \frac{\Gamma(\delta)\, \frac{\lambda t^{\delta-1}}{1-\lambda A}\, C_0}{\left(1+\frac{\lambda B}{1-\lambda A}\right)} \int_0^1 \frac{\left(1+\frac{\lambda B}{1-\lambda A}\right)}{\Gamma(\delta)} s^{\delta-1}(1-s)^{\delta-1} f(s,\bar{u}(s))ds \\
&\geq t^{2-\delta} p(t,t_0) \int_0^1 \max_{0\leq t\leq 1} \left\{ t^{2-\delta} G(t,s) \right\} f(s,\bar{u}(s))ds \\
&\geq t^{2-\delta} p(t,t_0) \max_{0\leq t\leq 1} \left\{ \int_0^1 t^{2-\delta} G(t,s) f(s,\bar{u}(s))ds \right\} \\
&= t^{2-\delta} p(t,t_0) \|Tu\|.
\end{aligned}$$

Thus, $T(P_0) \subset P_0$.

In addition, since f is a continuous function it follows that T is a continuous operator.

Next, we show that T is uniformly bounded.

Let $D \subset P$ be a bounded set, i.e. there exists a constant $L > 0$ such that $\|u\| \leq L$, for all $u \in D$. Set

$$M = \max_{0\leq s\leq 1, 0\leq u\leq L} \{f(s,\bar{u}(s))\}.$$

Then, from Lemma 4.4, and for all $u \in D$, we have

$$\begin{aligned}
|t^{2-\delta} Tu(t)| &= \left| \int_0^1 t^{2-\delta} G(t,s) f(s,\bar{u}(s))ds \right| \\
&\leq \frac{M}{\Gamma(\delta)} \left(1+\frac{\lambda B}{1-\lambda A}\right) \int_0^1 s^{\delta-1}(1-s)^{\delta-1} ds \\
&= M \left(1+\frac{\lambda B}{1-\lambda A}\right) \frac{\Gamma(\delta)}{\Gamma(2\delta)}.
\end{aligned}$$

Hence, $T(D)$ is bounded.

Finally, we show that T is equicontinuous, as follows

For all $\epsilon > 0$ and for each $u \in P$, let t_1, $t_2 \in [0,1]$, be such that $t_1 < t_2$.

We have to prove that there is $\eta > 0$ valid for all $u \in D$, such that :

$$|t_2^{2-\delta} Tu(t_2) - t_1^{2-\delta} Tu(t_1)| < \epsilon, \ \textit{when}\ t_2 - t_1 < \eta.$$

One has

$$\begin{aligned}
|t_2^{2-\delta}Tu(t_2) - t_1^{2-\delta}Tu(t_1)| &= \left|\int_0^1 [t_2^{2-\delta}G(t_2,s)ds - t_1^{2-\delta}G(t_1,s)]f(s,\bar{u}(s))ds\right| \\
&\leq \int_0^1 |t_2^{2-\delta}G(t_2,s) - t_1^{2-\delta}G(t_1,s)|f(s,\bar{u}(s))ds \\
&\leq M\int_0^1 |t_2^{2-\delta}G(t_2,s) - t_1^{2-\delta}G(t_1,s)|ds
\end{aligned}$$

Then we have

$$\begin{aligned}
\int_0^1 |t_2^{2-\delta}G(t_2,s) - t_1^{\delta-2}G(t_1,s)|ds &\leq \int_0^1 |t_2^{2-\delta}G_1(t_2,s) - t_1^{\delta-2}G_1(t_1,s)|ds \\
&\quad + \int_0^1 |t_2^{2-\delta}G_2(t_2,s) - t_1^{\delta-2}G_2(t_1,s)|ds.
\end{aligned}$$

From the expression of G_1, we get

$$\begin{aligned}
\int_0^1 |t_2^{2-\delta}G_1(t_2,s) - t_1^{2-\delta}G_1(t_1,s)|ds &= \int_0^{t_1} |t_2^{2-\delta}G_1(t_2,s) - t_1^{2-\delta}G_1(t_1,s)|ds \\
&+ \int_{t_1}^{t_2} |t_2^{2-\delta}G_1(t_2,s) - t_1^{2-\delta}G_1(t_1,s)|ds \\
&+ \int_{t_2}^{1} |t_2^{2-\delta}G_1(t_2,s) - t_1^{2-\delta-2}G_1(t_1,s)|ds
\end{aligned}$$

$$\begin{aligned}
&= \int_0^{t_1} \left|\frac{t_2(1-s)^{\delta-1} - t_2^{2-\delta}(t_2-s)^{\delta-1}}{\Gamma(\delta)} - \frac{t_1(1-s)^{\delta-1} - t_1^{2-\delta}(t_1-s)^{\delta-1}}{\Gamma(\delta)}\right| ds \\
&+ \int_{t_1}^{t_2} \left|\frac{t_2(1-s)^{\delta-1} - t_2^{2-\delta}(t_2-s)^{\delta-1}}{\Gamma(\delta)} - \frac{t_1(1-s)^{\delta-1}}{\Gamma(\delta)}\right| ds \\
&+ \int_{t_2}^{1} \left|\frac{t_2(1-s)^{\delta-1}}{\Gamma(\delta)} - \frac{t_1(1-s)^{\delta-1}}{\Gamma(\delta)}\right| ds \\
&= \int_0^{t_1} \left|\frac{(t_2-t_1)(1-s)^{\delta-1} + t_1^{2-\delta}(t_1-s)^{\delta-1} - t_2^{2-\delta}(t_2-s)^{\delta-1}}{\Gamma(\delta)}\right| ds \\
&\quad + \int_{t_1}^{t_2} \left|\frac{(t_2-t_1)(1-s)^{\delta-1} - t_2^{2-\delta}(t_2-s)^{\delta-1}}{\Gamma(\delta)}\right| ds \\
&\quad + \int_{t_2}^{1} \left|\frac{(t_2-t_1)(1-s)^{\delta-1}}{\Gamma(\delta)}\right| ds
\end{aligned}$$

$$
\begin{aligned}
= & \left|\frac{t_2^{2-\delta}\left(t_2-t_1\right)^{\delta}+\left(t_1-t_2\right)\Big(\left(1-t_1\right)^{\delta}+t_1+t_2-1\Big)}{\Gamma(\delta+1)}\right| \\
& +\left|\frac{\left(t_2-t_1\right)\left(\left(1-t_1\right)^{\delta}-\left(1-t_2\right)^{\delta}\right)-t_2^{2-\delta}\left(t_2-t_1\right)^{\delta}}{\Gamma(\delta+1)}\right| \\
& +\frac{(t_2-t_1)(1-t_2^{\delta})}{\Gamma(\delta+1)}.
\end{aligned}
$$

So, we have that there is $\eta > 0$ valid for all $u \in D$, such that :

$$|t_2^{2-\delta}Tu(t_2) - t_1^{2-\delta}Tu(t_1)| < \epsilon, \ \textit{when } t_2 - t_1 < \eta.$$

Now, denote by $H(s) = \int_0^1 h(r)G_1(r,s)dr$ and $h^* = \max_{t\in[0,1]}\{h(t)\}$. Then, from the expression of $G_2(t,s)$ and the inequality (4.7), using that

$$\int_0^1 H(s)ds \leq h^* \int_0^1 \int_0^1 G_1(r,s)drds \leq h^* \int_0^1 \frac{s^{\delta-1}\left(1-s\right)^{\delta-1}}{\Gamma(\delta)}ds = h^* \frac{\Gamma(\delta)}{\Gamma(2\,\delta)},$$

we get

$$
\begin{aligned}
\int_0^1 |t_2^{2-\delta}G_2(t_2,s) - t_1^{2-\delta}G_2(t_1,s))|ds &= \int_0^1 \frac{\lambda(t_2-t_1)}{1-\lambda A}H(s)ds \\
&\leq \frac{\Gamma(\delta)}{\Gamma(2\,\delta)}\frac{\lambda\, h^*}{1-\lambda A}(t_2-t_1).
\end{aligned}
$$

Thus, we obtain that the set $T(D)$ is equicontinuous in E.

Now, we are in position to prove the existence of positive solutions of the nonlinear boundary value problem (BVP). For this we use the known Guo-Krasnoselskii fixed point Theorem.

Let introduce some notations,

$$f_0 = \lim_{\bar{u}\to 0^+}\left\{\min_{t\in[0,1]}\left\{\frac{f\left(t,\bar{u}\right)}{\bar{u}}\right\}\right\} \ \text{ and } f_\infty = \lim_{\bar{u}\to\infty}\left\{\min_{t\in[0,1]}\left\{\frac{f\left(t,\bar{u}\right)}{\bar{u}}\right\}\right\},$$

$$f^0 = \lim_{\bar{u}\to 0^+}\left\{\max_{t\in[0,1]}\left\{\frac{f\left(t,\bar{u}\right)}{\bar{u}}\right\}\right\} \ \text{ and } f^\infty = \lim_{\bar{u}\to\infty}\left\{\max_{t\in[0,1]}\left\{\frac{f\left(t,\bar{u}\right)}{\bar{u}}\right\}\right\}.$$

Theorem 4.1. Assume that $h \geq 0$ on $[0,1]$, $1-\lambda A > 0$ and (H_1) holds coupled with one of the following conditions

1. Sublinear case : $f_0 = \infty$ *and* $f^\infty = 0$.
2. Superlinear case : $f^0 = 0$ *and* $f_\infty = \infty$.

Then Problem (4.1)-(4.2) has at least one positive solution.

Proof Consider the first situation

1. Since $f_0 = \infty$, then there exists a constant $R_1 > 0$ such that $f(t, \bar{u}) \geq r_1 \bar{u}$ for all $0 < \bar{u} \leq R_1$ and $t \in [0, 1]$, where $r_1 > 0$ is defined as

$$r_1 := \frac{(1 - \lambda A)\,(1 - \lambda A + \lambda B)\,\Gamma(1 + 2\delta)}{\lambda^2\, C_0^2\, \delta\, \Gamma^3(\delta)} \tag{4.13}$$

Take $u \in P_0$ such that $\|u\| = R_1$. Then from expression (4.13), we get

$$\begin{aligned}
\|Tu\| &:= \max_{t\in[0,1]} \left\{ \int_0^1 t^{2-\delta}\, G(t,s) f(s, \bar{u}(s)) ds \right\} \\
&\geq r_1 \max_{t\in[0,1]} \left\{ \int_0^1 t^{2-\delta}\, G(t,s) \bar{u}(s) ds \right\} \\
&\geq r_1 \max_{t\in[0,1]} \left\{ \int_0^1 t^{2-\delta}\, G(t,s) s^{2-\delta} p(s, t_0) \|u\| ds \right\} \\
&\geq r_1\, \|u\| \max_{t\in[0,1]} \left\{ t \int_0^1 \frac{\lambda}{1 - \lambda A}\, C_0\, s\, (1 - s)^{\delta - 1} p(s, t_0)\, ds \right\} \\
&= r_1 \|u\| \, \frac{\lambda^2\, C_0^2\, \Gamma(\delta)}{(1 - \lambda A)\,(1 - \lambda A + \lambda B)} \int_0^1 s^\delta\, (1 - s)^{\delta - 1}\, ds \\
&= r_1 \|u\| \, \frac{\lambda^2\, C_0^2\, \delta\, \Gamma^3(\delta)}{(1 - \lambda A)\,(1 - \lambda A + \lambda B)\,\Gamma(1 + 2\delta)} \\
&= \|u\|.
\end{aligned}$$

By the other hand, since $f(t, \cdot)$ is a continuous function on $[0, \infty)$, we define a new function :

$$\hat{f}(t, \bar{u}) := \max_{y\in[0,\bar{u}]} \{f(t, y)\}.$$

Clearly $\hat{f}(t, \cdot)$ is nondecreasing on $[0, \infty)$. Moreover, since $f^\infty = 0$ it is obvious that

$$\lim_{\bar{u}\to\infty} \left\{ \max_{t\in[0,1]} \frac{\hat{f}\,(t, \bar{u})}{\bar{u}} \right\} = 0.$$

Choose now $r_2 > 0$ defined as the following constant :

$$r_2 = \frac{(1-\lambda A)\,\Gamma(2\delta)}{(1-\lambda A+\lambda B)\,\Gamma(\delta)}. \tag{4.14}$$

Therefore there exists a constant $R_2 > R_1 > 0$ such that $\hat{f}(t,\bar{u}) \le r_2\bar{u}$ for all $\bar{u} \ge R_2$ and $t \in [0,1]$.
Consider $u \in P_0$ such that $\|u\| = R_2$. Then from the definition of $\hat{f}$, inequality (4.14) and Lemma (4.3), we attain at the following inequalities :

$$\begin{aligned}
\|Tu\| &:= \max_{t\in[0,1]}\left\{\int_0^1 t^{2-\delta}\,G(t,s)f(s,\bar{u}(s))ds\right\}\\
&\le \max_{t\in[0,1]}\left\{\int_0^1 t^{2-\delta}\,G(t,s)\hat{f}(s,\|u\|)ds\right\}\\
&\le r_2\|u\|\max_{t\in[0,1]}\left\{\int_0^1 t^{2-\delta}\,G(t,s)ds\right\}\\
&\le r_2\|u\|\frac{(1-\lambda A+\lambda B)}{\Gamma(\delta)(1-\lambda A)}\int_0^1 s^{\delta-1}(1-s)^{\delta-1}ds\\
&= r_2\|u\|\frac{(1-\lambda A+\lambda B)\,\Gamma(\delta)}{(1-\lambda A)\,\Gamma(2\delta)}\\
&= \|u\|.
\end{aligned}$$

Thus, by the first part of Guo-Krasnoselskii fixed point Theorem, we conclude that the problem (4.1) – (4.2) has at least one positive solution u such that

$$R_1 \le \|u\| \le R_2.$$

2. Consider now the second case (ii)

Let $r_2 > 0$ be chosen as in equation (4.14). Since $f^0 = 0$, there exists a constant $\tau_1 > 0$ such that $f(t,\bar{u}) \le r_2\bar{u}$ for $0 \le \bar{u} \le \tau_1$ and $t \in [0,1]$.

Take $u \in P_0$ such that $\|u\| = \tau_1$. Then, arguing as in the previous case, we have

$$\begin{aligned}
\|Tu\| &:= \max_{t\in[0,1]}\left\{\int_0^1 t^{2-\delta}\,G(t,s)f(s,\bar{u}(s))ds\right\} \le r_2\|u\|\max_{t\in[0,1]}\left\{\int_0^1 t^{2-\delta}\,G(t,s)ds\right\}\\
&\le r_2\|u\|\frac{(1-\lambda A+\lambda B)\,\Gamma(\delta)}{(1-\lambda A)\,\Gamma(2\delta)} = \|u\|.
\end{aligned}$$

Now, by denoting the incomplete beta function as

$$B_z(a,b) := \int_0^z t^{a-1}(1-t)^{b-1}dt$$

For any fixed $t_1 \in (0,1)$. we define $r_3 > 0$ as follows :

$$r_3 = \frac{(1-\lambda A)\,(1-\lambda A+\lambda B)}{\lambda^2\, C_0^2\, \Gamma(\delta)} \left(\frac{\sqrt{\pi}\,\Gamma(\delta)}{\Gamma\left(\delta+\frac{1}{2}\right) 4^\delta} - B_{t_1}(\delta+1,\delta) \right)^{-1}. \qquad (4.15)$$

The fact that $f_\infty = \infty$ ensures that there exists a constant $\tau_2 > \tau_1 > 0$ such that $f(t,\bar{u}) \geq r_3 \bar{u}$ for all $\bar{u} \geq \tau_2$ and $t \in [0,1]$.

By the definition of $p(t,t_0)$ is clear that

$$p_1 := \min_{t\in[t_1,1]} \{t^{2-\delta}\, p(t,t_0)\} > 0.$$

Let now $u \in P_0$ be such that $\|u\| = \tau_2/p_1$. As consequence, since $u \in P_0$, the following inequality holds :

$$\bar{u}(t) \geq t^{2-\delta}\, p(t,t_0)\|u\| \geq p_1\,\|u\| = \tau_2 \qquad \text{for all } t \in [t_1,1].$$

So, condition (ii) gives us the following properties :

$$\begin{aligned}
\|Tu\| &:= \max_{t\in[0,1]} \left\{ \int_0^1 t^{2-\delta}\, G(t,s) f(s,\bar{u}(s)) ds \right\} \\
&\geq \max_{t\in[0,1]} \left\{ \int_{t_1}^1 t^{2-\delta}\, G(t,s) f(s,\bar{u}(s)) ds \right\} \\
&\geq r_3 \max_{t\in[0,1]} \left\{ \int_{t_1}^1 t^{2-\delta}\, G(t,s) \bar{u}(s) ds \right\} \\
&\geq r_3 \|u\| \max_{t\in[0,1]} \left\{ \int_{t_1}^1 t^{2-\delta}\, G(t,s) s^{2-\delta} p(s,t_0) ds \right\} \\
&\geq r_3\,\|u\| \max_{t\in[0,1]} \left\{ t \int_{t_1}^1 \frac{\lambda}{1-\lambda A}\, C_0\, s\,(1-s)^{\delta-1} p(s,t_0)\, ds \right\} \\
&= r_3 \|u\| \frac{\lambda^2\, C_0^2\, \Gamma(\delta)}{(1-\lambda A)\,(1-\lambda A+\lambda B)} \int_{t_1}^1 s^\delta\,(1-s)^{\delta-1}\, ds \\
&= r_3 \|u\| \frac{\lambda^2\, C_0^2\, \Gamma(\delta)}{(1-\lambda A)\,(1-\lambda A+\lambda B)} \left(\frac{\sqrt{\pi}\,\Gamma(\delta)}{\Gamma\left(\delta+\frac{1}{2}\right) 4^\delta} - B_{t_1}(\delta+1,\delta) \right) = \|u\|
\end{aligned}$$

Therefore, by the second part of Guo-Krasnoselskii fixed point Theorem, we conclude that the problem (4.1)-(4.2) has at least one positive solution.

4.5 Examples

Example 4.1. The problem

$$\begin{cases} D^{\frac{3}{2}}u(t) + f(t, u(t)) = 0 \\ u(0) = 0, \quad u(1) = \lambda \int_0^1 s^{\frac{1}{2}} u(s) ds, \end{cases} \tag{4.16}$$

with

$$f(t,x) = \begin{cases} t + \sqrt{x} \arctan\left(\frac{1}{x}\right) & \text{if} \quad x > 0,\ t \in [0,1] \\ t & \text{if} \quad x = 0,\ t \in [0,1], \end{cases}$$

has at least a positive solution for any $0 < \lambda < 2$.
Here, $\delta = \frac{3}{2}$ and $h(t) = t^{\frac{1}{2}}$. Then, $A = \int_0^1 t^{\frac{1}{2}} t^{\frac{3}{2}-1} dt = \frac{1}{2}$, and $1 - \lambda A > 0$ for any $\lambda < 2$.
It is clear that $f(t, u) : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous and,

$$\lim_{u \to 0^+} \left\{ \min_{t \in [0,1]} \left\{ \frac{f(t,u)}{u} \right\} \right\} = +\infty,$$

and

$$\lim_{u \to +\infty} \left\{ \max_{t \in [0,1]} \left\{ \frac{f(t,u)}{u} \right\} \right\} = 0.$$

Then by the first part of Theorem 4.1, the problem (4.16) has at least one positive solution.

Example 4.2. The problem

$$\begin{cases} D^{\frac{3}{2}}u(t) + u^{\beta}(t) + (t-1)u(t) = 0, \quad t \in (0,1) \\ u(0) = 0, \quad u(1) = \lambda \int_0^1 e^s u(s) ds, \end{cases} \tag{4.17}$$

has at least a positive solution for any $\beta > 1$ and $0 < \lambda < 0.796413$.
Here, $\delta = \frac{3}{2}$, $h(t) = e^t$ and $f(t, u) = u^{\beta} + (t - 1)\, u$.
A numerical calculation leads to $A = \int_0^1 e^t t^{\frac{1}{2}} dt \approx 1.25563$ and $1 - \lambda A > 0$ for any $\lambda \in (0, 0.796413)$.

$f(t,u):[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous and,

$$\lim_{u\to 0^+}\left\{\max_{t\in[0,1]}\left\{\frac{f(t,u)}{u}\right\}\right\}=0,$$

and

$$\lim_{u\to +\infty}\left\{\min_{t\in[0,1]}\left\{\frac{f(t,u)}{u}\right\}\right\}=+\infty.$$

Then by the second part of Theorem 4.1, the problem (4.17) has at least one positive solution.

Conclusion and Perspective

In this thesis, we were interested and positive solutions of fractional differential involving integral conditions as boundary conditions. The first part was devoted to a fractional differential equations with a kind of boundary conditions, while the second part was for a more general integral boundary conditions. We obtain very interesting results by using the fixed point theory Guo-Krasnosel'skii fixed point theorem.

No so far this area, we investigate some extensions of incomplete gamma function, beta function , Gaussian hypergeometrique function and confluent among other hypergeometric function by mean of a recent extension of the Bessel function. The results were very encouraging for us to establish an extension of the fractional derivative operator of Riemann-Liouville and its properties.

The perspective can be very large, we can discuss the influence of some parameters in integral condition for more general derivatives, we can also predict existence or positive solutions with a general scheme of Green function. Fractional derivative operators can be also an interesting axis since we able to improve some kernel or some special extended functions.

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الملخص:

في هذه الرسالة، نهتم بمحورين مختلفين:

المحور الأول: باستعمال الدالة ريمان بيسال معممة من عمل بوجلخة ومستوحى من أعمال أغاروال نقدم مشتق كسري معمم جديد باستخدام مقاربة ريمان ليوفيل ونحصل على العلاقات التوليدية التي تتضمن دالة هندسية غوسية موسعة ومعممة.

في المحور الثاني: نحن مهتمون بدراسة بعض المعادلات التفاضلية الكسرية باستخدام مقاربة ريمان ليوفيل بشروط حدية (تكامل) هدفنا هو تقديم بعض نتائج وجود ووحدانية الحل والحلول الإيجابية لهذه المعادلات التفاضلية الكسرية. يتمثل النهج المتبع في تحقيق الشروط مناسبة من خلال تطبيق طرق مختلفة في فضاءات باناخ تعتمد هذه الطرق على مبرهنات النقطة الصامدة مثل نظرية النقطة الصامدة لكرازونلسكي.

Résumé

Dans cette thèse, nous nous intéressons à deux axes différents :

Le premier axe, ayant la fonction R_K et inspiré par les travaux d'Agarwal {\it et al.} nous introduisons un nouvel opérateur de dérivation fractionnaire de Riemann-Liouville généralisé, et nous obtenons des relations génératrices impliquant une fonction hypergéométrique de Gauss étendue et généralisée.

Dans le second axe, nous intéressons à étudier certaines équations différentielles fractionnaire avec des conditions aux limites intégrales, nous présentons quelques résultats d'existence et unicité de solutions et solutions positives pour ces équations différentielles fractionnaire. Ces méthodes sont basées sur des célèbres théorèmes du point fixe tels que le théorème de point fixe le théorème de Guo-Krasnoselskii.

Abstract

In this thesis, we are interested in two different axes :

The first axis, having the function $R_K$and inspired by the work of Agarwal et al. we introduce a new generalized Riemann-Liouville fractional derivation operator, and we obtain generating relations involving an extended and generalized hypergeometric Gauss function.

In the second axis, we are interested in studying some fractional differential equations with integral boundary conditions, our objective is to present some results of existence and uniqueness of solutions and positive solutions for these fractional differential equations. The approach followed consists in bringing back the search for the existence under suitable conditions by means of the Green function by applying different methods in Banach spaces.