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This thesis is dedicated to . . . ✍

To the memory of my mother

To that great person who always prayed and wished to see me crowned with a doctorate until death stole her from us before her wish was fulfilled. To the secret of my struggle and diligence, to my beloved mother, may God have mercy on her, who worked hard for my success by her love, her constant support, her sacrifices that she never stopped to give for me and her unconditional help. She did not stop for encouraging me and praying for me. Her prayer and blessing have been a great help to me in completing my studies.

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SUMMARY

This thesis is devoted to the nonparametric modelization of a real response variable conditioned by a functional covariate (valued in infinite dimensional space (semi metric space/ Hilbert space)). We study the asymptotic properties of some functional parameters such as the conditional density function, the conditional mode, the regression operator, the conditional cumulative distribution and the quantile regression function for complete and incomplete data in different situations.

In the first part, we focus on the ergodic process forecasting via a functional kernel estimation for incomplete data. The randomly censored density prediction is treated as a preliminary study of the conditional mode function, when we establish the explicit expressions of the almost sure consistency rates and the asymptotic normality of the built estimators. To illustrate the effectiveness of our method, we propose a simulation study. Moreover, we evaluate the regression function expectation in a functional single index framework by adapting the nonparametric methodology in two directions: ergodic property and missing at random data (MAR).

Secondly, we deal with the central limit theorem by using single index approach. More precisely, we examine some conditional problems including the estimation of the conditional density and the conditional distribution functions with applications to the conditional mode and the conditional quantile, respectively, of a scalar response variable given an explanatory variable which is assumed to be of functional feature in the sense that is supposed to take its values in Hilbert space. On the one hand, we prove the asymptotic normality of the conditional density estimator from which we derive the central limit theorem of the conditional mode function in the independent data case. On the other hand, we contribute to the functional nonparametric research by investigating the quantile regression estimation as an useful alternative to the regression model with randomly right-censored data when the sample is considered as an α-mixing sequence. In order to illustrate the validity and the performance of the suggested models in this part, we introduce examples on simulated data.

Keywords : Nonparametric estimation, Functional data, Regression function, Conditional cumulative distribution, Conditional density function, Functional single index model, Censored data, Missing at random data, Ergodic processes, α-mixing sequence, Convergence rate, Asymptotic normality.

RÉSUMÉ

Cette thèse est consacrée à la modélisation non paramétrique d'une variable de réponse réelle conditionnée par une covariable fonctionnelle (à valeurs dans un espace de dimension infini (espace semi-métrique/ espace de Hilbert)). Nous étudions les propriétés asymptotiques de certains paramètres fonctionnels tels que la fonction de densité conditionnelle, le mode conditionnel, la fonction de régression, la fonction de répartition conditionnelle et le quantile conditionnel pour des données complètes et incomplètes dans des différentes situations.

Dans la première partie, nous nous intéressons à la prévision du processus ergodique via l'estimation à noyau fonctionnel pour des données incomplètes. La prédiction de la densité censurée aléatoirement est traitée comme étude préliminaire de la fonction du mode conditionnel, où nous établissons les expressions explicites des vitesses de convergence presque-sûre et de la normalité asymptotique des estimateurs construits. Pour illustrer l'efficacité de notre méthode, nous proposons une étude de simulation. De plus, nous évaluons l'estimation de la fonction de régression dans un cadre à indice fonctionnel simple en adaptant la méthodologie non paramétrique dans deux directions : propriété ergodique et données manquantes au hasard (MAR).

Deuxièmement, nous traitons le théorème central limite en utilisant une approche à indice unique. Plus précisément, nous examinons quelques problèmes conditionnels, notamment, l'estimation de la fonction densité conditionnelle et la fonction de distribution conditionnelle avec des applications au quantile conditionnel et au mode conditionnel, respectivement, d'une variable de réponse scalaire conditionnée par une variable explicative qui est supposée d'être de caractéristique fonctionnelle au sens qu'il prend ses valeurs dans un espace de Hilbert. D'une part, nous prouvons la normalité asymptotique de l'estimateur de la densité conditionnelle à partir duquel nous dérivons le théorème central limite de la fonction du mode conditionnelle dans le cas des données indépendantes. D'autre part, nous contribuons à la recherche non paramétrique fonctionnelle en étudiant l'estimation du quantile comme alternative utile au modèle de régression avec des données censurées à droite aléatoirement lorsque l'échantillon est considéré comme une séquence α-mélange. Afin d'illustrer la validité et la performance des modèles proposés dans cette partie, nous introduisons des exemples sur des données simulées.

Mots clés : Estimation non paramétrique, Données fonctionnelles, Fonction de régression, Fonction de répartition conditionnelle, Fonction de densité conditionnelle, Indice fonctionnel simple, Données censurées, Données manquantes au hasard, Processus ergodique, Séquence α-mélange, Vitesse de convergence, Normalité asymptotique.

Achieved Works

Publications

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Chapter 1

General Introduction

1.1 Bibliographic note on the functional nonparametric statistics

The functional data analysis is a typical issue of current statistical research which has encountered a strong infatuation recently. The many books and the recognized journals devoted to this type of data (Statistica Sinica (2004), Computational Statistics and Data Analysis (2006), Computational Statistics (2007), Journal of Multivariate Analysis (2008)), in addition to the possibility of application in several areas (such as meteorology, quantitative chemistry, biometrics, econometrics or medical imaging) are all testimonies of this success which is essentially due to the progress in the informatics world. consequently, the use of functional data become customary in statistical problems on both the theoretical and practical sides. Indeed, this field facilitates the data collection on thinner and thinner discretization grid of mathematical objets such as curves, surfaces...due to the great advancement in the domain of measuring devices and their treatment efficiency as well as the informatics systems improvement in terms of storage capacities that allowing larger data to be recorded. Contradictory to the standard multivariate methods that are often not adapted to treat this sort of data, this new axis of infinite-dimensional techniques is more powerful because it permits to properly examine these data with conserving the functional feature.
This branch of modern statistics was popularized during the last two decades, particularly with the monographs of Ramsay and Silverman ((1997) [52]-(2002) [53]-(2005) [54]) and Bosq (2000) [8] for the theoretical aspects. We must cite the celebre groundbreaking monograph of Ferraty and Vieu (2006) [33] for nonparametric statistical modelling with functional variables and Ferraty and Romain (2011) [27] for recent developments. In the same context, we refer to Manteiga and Vieu (2007) [45], Ferraty (2010) [21], Horvath and Kokoszka (2012)[35] as well as Cuevas (2014)[11].

The modelization of statistical models adapted to such kind of infinite dimensional data has attracted an increasing interest in the statistical literature. Ferraty *et al.* (2006) [26] investigated a kernel estimate for a real response variable given an explicatory variable valued in a semi metric space whenever an independent observations were considered. Precisely, this article dealt with some conditional nonparametric problems including the estimation of the conditional distribution and the conditional density functions with an application to the conditional quantile and the conditional mode in addition to a chemiometrical functional data application. The almost complete rate of convergence of each estimator was stated. Based on the local linear approach, an application to Ozone pollution prediction via the conditional quantile estimation when the data were in the form of curves was presented by Cardo *et al.* (2004) [9]. In (2003), Dabo Niang in collaboration with Rhomari [13] established a nonparametric treatment of the regression function when the covariate was assumed to be of functional feature. By using a kernel smoothing method, they constructed a consistent estimator of the regression function and demonstrated its convergence rate almost surely and in norm L^1. Moreover, they determined the upper bound of each estimation. Few years later, exactly in (2009) [14], the same researchers introduced similar results by considering doubly functional model (i.e. both the response variable and the covariate were of functional kind). This latter estimator was treated uniformly by Ferraty *et al.* [23] in a contribution dating in 2011. The authors studied the almost complete consistency of the kernel estimate and they specified the explicit expression of the convergence speed terms. Laksaci *et al.* (2009) [41] were interested on the robust modelization of the conditional distribution as a preliminary study of the conditional quantile whenever a functional sample was considered. The asymptotic results of this contribution are: Rates of almost complete consistencies and the asymptotic normality of the kernel estimators. A smooth kernel prediction of the conditional mode in the i.i.d case was proved by Dabo Niang and Laksaci (2007) [12], Ezzahrioui and Ould-Saïd (2008a) [18]. The work of Dabo Niang and Laksaci was devoted to the study of the consistency in L^1 norm of the nonparametric estimate when the random covariate was assumed to be of functional type in the sense that was supposed to take its values in some abstract semi metric space.The main purpose of the other paper is to evaluate functionally the asymptotic normality of the suggested estimator in addition to the presentation of an application to confidence intervals and a simulation study. In the uniform aspect, Ferraty *et al.* [22] introduced an important contribution in (2010) focused on the functional kernel prediction via certain conditional characteristics, where they studied the almost complete consistency rates of the nonparametric estimators.

The fact of assuming that the treated data are always independent is not realistic, for this, many authors focused their studies on the dependent case. The functional almost complete convergence of the conditional mode estimate via the estimator of the conditional density function by using a kernel smoothing method, and the asymptotic behavior of the kernel estimator of the regression function whenever the response variable was in a Banach space and the explanatory variable valued in a semi-metric space were obtained by Ferraty *et al.* (2005) [25] and Ferraty *et al.* (2012) [24], respectively, under α and β-mixing hypotheses. Ezzahrioui and Ould-Saïd (2010) [20] established the consistency as well as the asymptotic normality of the conditional mode estimate under α-mixing condition. Always in the functional space, the regression function estimation was studied by Ferraty and Vieu (2004) [32], Masry (2005) [46] when the data were assumed to be sampled from a strong mixing process. By using the kernel approach, Ferraty *et al.* (2005) [29], Ezzahrioui and Ould-Saïd (2008b) [19], Delsol (2009) [17] dealt with the nonparametric functional time series forecasting based on conditional expectation estimation when the sample was considered as an α-mixing sequence.

In the ergodic case, it worth telling that the ergodic theory finds its roots in the branch of statistical physic examined in the second half of the 19^{th} century. It was firstly determined by Boltzmann (1871) [7] for the purposes of his kinetic theory of gases. The ergodicity hypothesis is also applied in signal treatment in order to evaluate the evolution of the random signal. The value of this condition is assured in the study of Markov chains, the stationary processes and for digital learning. The first significant result for ergodic theory in mathematics is the famous Poincaré's recurrence theorem (1980). The development of the theory was undoubtedly only in 1931 with the ergodic theorems of Birkhoff and Neumann. For further discussion on ergodic theory results, we refer the reader to Krengel (1985) [38] or Peskir (2000) [50].
The context of ergodic observations has motivated a number of papers in the literature. Ould-Saïd (1997) [48] proposed some interesting results on the kernel conditional density function estimate, where he investigated the uniform almost sure consistency of the conditional mode via the conditional density convergence under ergodic hypotheses. A nonparametric estimate of a regression function and its derivatives formulated by the kernel method was proposed by Delecroix and Rosa (1996) [16], where the researchers demonstrated almost surely the strong uniform convergence of the studied estimator. In the same setting, we can mention also Rosa (1992) [55].
Recall that the ergodicity condition is weaker than any other kind of dependence (α-mixing, β-mixing, ϕ-mixing...). It covers several cases that do not satisfy the usual mixture structures. For example, there are some processes where the α-mixing condition doesn't hold such as the autoregressive process of order 1 (AR(1)). Further, it allows to avoid the widely used strong mixing condition and its variants to measure the dependency and the very involved probabilistic calculations that it implies.
The combination of the functional and ergodic data in nonparametric treatment is a recent axis in statistics. This problem was established firstly by Laib and Louani (2010) [39], who proposed the first functional ergodic version of the kernel smoothing estimation. This last paper is based on the prediction via the classical regression function illustrated by examples. The researchers proved the convergence rate in probability, in addition to the asymptotic normality of the built estimator. Rates of pointwise and uniform consistencies of the same estimator were investigated almost surely by Laib in collaboration with Louani (2011) [40]. Inspired y the groundbreaking contributions of Laib and Louani ((2010)-(2011)), the nonparametric estimate when the data were sampled from an ergodic

process was the subject of a limited number of works in functional statistics.
We cite among them Chaouch and Khardani (2015) [10], Ling *et al.* (2016) [43] for the case of incomplete and ergodic observations. The contribution of Chaouch and Khardani dealt with the smooth kernel nonparametric modelization of the conditional quantile function when the covariates were in the form of curves. Under random censorship, the explicit expression of the almost sure pointwise consistency rate as well as the asymptotic normality of the constructed estimator were established in this paper. In order to prove the effectiveness of the presented results, some applications including the confidence bands, a simulation study in addition to a real data analysis of the electricity peak demand prediction were also introduced. The principal idea of the other work is the nonparametric treatment via the conditional mode estimation whenever missing at random responses are considered. The publishers investigated some asymptotic properties of the kernel estimator when the explicatory variable take its values in a semi metric space.

1.2 Incomplete data

The survival analysis area is becoming increasingly popular because of its importance in several applied sciences as medicine, epidemiology, industry, finance, biometry, economy, sociology...This axis is interested on the term "lifetime", that is a positive variable designates the time elapsed until the realization of the event of interest. This variable can be the lifetime of a patient after treatment, the duration of unemployment, the time between two successive breakdowns of a device, the lifetime of a company..., and all which is linked with measuring the moment of arrival of the events relating to (breakdown, death, failure...).
For such sort of studies and for various reasons, the variable of interest is not always completely observed for certain individuals. We are talking then about incomplete data which equates to loss of information.
Distinguishing three types of these data: censoring data, truncated data and missing data.

In this manuscript, we deal only with censoring and missing models. A brief discussion of each of these kinds, in addition to a general historical overview on the topic are introduced below.

1.2.1 Censored data

In the censorship case, the lifetime T is only known for a part of the sample individuals. Data for which survival time is unknown are said to be censored. There are many applied scenarios that can generate this mechanism, as the end of study while some patients still alive (excluded-alive), patient's withdrawal from study (for example stopping or changing treatment due of side effects or treatment ineffectiveness), loss of follow-up (interruption of individual monitoring before the event of interest occurrence)...
This kind of data may be generally classified into three big categories: right censoring, left censoring and interval censoring, presented as follows

Right censoring

Right censoring is the most frequent example of incomplete data in survival analysis, and it was widely described in the literature. A survival time is said to be censored on the right if the individual did not experienced the event of interest at his last observation. In this case, the lifetime T may not be directly observable. Instead, we observe only censored lifetimes of items under study (C). It is worth defining this situation by the couple $(Y,\, \delta)$, with

$$Y = T \wedge C = \min(T,\, C).$$

and

$$\delta = \begin{cases} 1 & \text{if } T \leq C \\ 0 & \text{if } T > C, \end{cases}$$

where, δ is named the indicator of censure: a binary variable represents the nature of the observed duration.
The true survival time T is observed only if it is less than C. In this case, the data are not censored and $\delta = 1$. If $\delta = 0$, the data are said to be right censored.
We obtain the following types:

• Type 1: fixed censoring

The survival time cannot be observed beyond a fixed maximum duration (identical for all individuals). This type of censorship therefore comes from stopping the collection of information on a time fixed a priori (C). For an individual i, the lifetime T_i is observed only when $T_i \leq C$. In this context, we introduce the following expression:

$$Y_i = \min(T_i,\, C).$$

The fixed censoring is used for example in the industrial field, when we observe the service life of a component electronic over a time interval $[0,\, C]$.

• Type 2: censorship waiting

The experimenter fixes a priori the number of events to be observed (denoted r). Therefore, the end date of the experiment becomes random, while the number of events being non-random. This model is often used in reliability studies.
Given an n-sample $T_1 \cdots, T_n$ of ordered variables, where we obtain the statistics of order $T_{(1)}, T_{(2)}, \cdots, T_{(n)}$. Let r be a fixed positive number, such that $(1 \leq r \leq n)$. The censorship date is then $T_{(r)}$, where we only observe $T_1 \leq T_2 \leq \cdots \leq T_r$.
Otherwise, the observations in this case depend on the couple (Y_i, δ_i) as follows

$$Y_i = T_i \wedge T_{(r)} \qquad \text{and} \qquad \delta_i = \mathbf{1}_{\{T_i \leq T_{(r)}\}}.$$

• Type 3: random censoring

Consider (T_i) a sequence of i.i.d random variables, and suppose that they form a strictly stationary sequence of lifetimes. Let C_i be a sequence of i.i.d. censoring random variables. The censorship of type (3) is achieved if the observed random variables are not T_i, but rather $(Y_i,\, \delta_i)$, with $Y_i = \min(T_i, C_i)$ and $\delta_i = \mathbf{1}_{\{T_i \leq C_i\}}$ is the censorship indicator which informs us if the duration actually observed corresponds to a true survival time T_i (when $\delta_i = 1$), or to a random censoring if δ_i takes the value zero, namely

- If $\delta_i = 1 \Rightarrow Y_i = T_i$,
- if $\delta_i = 0 \Rightarrow Y_i = C_i$.

Left censoring

Left censoring is much rare. An observation is said to be censored on the left if we only know that the individual has already experienced the event of interest before entering the study, but the exact time of this experience is unknown. The information available on the lifetime really observed can be summarized in the couple (Y, δ), with

$$Y = \max(T,\, C),$$

and

$$\delta = \begin{cases} 1 & \text{if } T > C \Rightarrow Y = T \text{ (the lifetime is observed)} \\ 0 & \text{if } T \leq C \Rightarrow Y = C \text{ (left censoring).} \end{cases}$$

Remark 1.2.1. *Right and left censoring types can be combined in the same sample. This can happen in the case of double (or mixed) censorship. Let C_r and C_l be right and left censorship variables, respectively, with assuming that $C_r < C_l$. The triplet $(Y,\, \delta_r,\, \delta_l)$ is observable, where*
$\delta_r = \mathbf{1}_{\{T \leq C_r\}},\ \delta_l = \mathbf{1}_{\{T \leq C_l\}}$,
and
- *if* $T \leq C_r \Rightarrow Y = C_r$,
- *if* $C_r < T \leq C_l \Rightarrow Y = T$,
- *if* $C_l < T \Rightarrow Y = C_l$.

Interval censoring

In interval censorship, the exact time of the event of interest occurrence cannot be observable. Instead, we know only the lower and the upper bounds between which the event takes place. This type of censorship can appear in medical experiments, for example in clinical trials where patients are checked periodically, if a disease arises, the only information available is that it produced between two medical visits. The industry is another area to apply interval censoring, where there are periodic inspections of machines.

In this thesis and exactly in Chapters 3 and 6, we focus only on the third type of right censoring (random censorship). For such data and for the censoring in general, the cause of censoring must be independent of the event of interest. This condition is very useful and essential for classical survival analysis models, due to its ability to ensure the identifiability of the studied models. In other words, if T_i and C_i are independent, then the law of T is identifiable from the law of the observations.

1.2.2 Missing data

Now, we turn to another particularity of the data incompleteness so-called missing data. It occurred when there is no observation of a variable for a given individual. In statistical analysis, such data are as important as the observed data, what requires solutions to manage the study successfully in the presence of the miss. We distinguish in the missing models two big categories. The first is ignorable missing mechanism (the probability of observing a missing data element is independent of the value of that data element) and it includes both of **Missing Completely At Random** and **Missing At Random** kinds, the other group is named non-ignorable missing data (contradictory to the previous family, the probability of observing a missing data element is dependent of the value of that data element) and it contains the type of **Missing Not At Random** data. These mechanisms are suggested as follows:

Missing Completely At Random

In this missing data type, the probability that an observation is missing is not related to any other variable. More precisely, Missing Completely At Random (MCAR) occurred when the missing value is independent of both observed and missing data.

Missing At Random

In the aspect of Missing At Random (MAR), missing data are explained from other observed variables. In other words, this mechanism means that the missing observations in the data are independent of the missing variables themselves, but dependent on other observed variables.

Missing Not At Random

If the missing value depend on the missing values of the variable itself, the mechanism is then called Missing Not At Random (MNAR).

In this manuscript (Chapter 4), we are interested in the Random At Messing data type which is central in the missing models.

Historically, many authors were interested on the nonparametric modelization for incomplete data. In the censored case, Ould-Saïd and Cai (2005) [49] established consistency properties of the conditional mode estimate. More exactely, they investigated uniformly the convergence rate of the kernel estimator. In (2010), Khardani *et al.* [37] estimated the same function almost surely. The randomly censored mode prediction was treated by proving the consistency rate of the studied estimator and presenting its asymptotic normality. In the same setting of incomplete observations, Ferraty *et al.* (2013) [31] examined the mean estimate for functional data with responses missing at random. The researcher dealt with two estimators, the first one is based on the average of the predicted values and the other was a functional adaptation of the Horvitz-Thompson estimator. More recently, Chaouch in collaboration with Khardani [10] suggested an important work dated in (2015) on the ergodic process forecasting via a kernel estimation in a functional framework. In the presence of random censorship, the pointwise convergence of the conditional quantile function was proved. One refers also to Ling *et al.* (2016) [43] for data random at missing. The publishers evaluated nonparametrically the conditional mode estimate through the conditional density estimator whenever the functional explicatory variables were considered. Under general ergodic assumption, the almost sure convergence (with rate) and the asymptotic normality in addition to an illustrative simulation study were introduced in this contribution.

1.3 Originality and modernization of the single index approach

1.3.1 The single index approach

The models that incorporate simultaneously parametric and nonparametric treatments and permit to extract a compromise between the two mechanisms are called in the literature semi-parametric models or single index approaches. This sort of modeling was intensively discussed in various situations throughout the recent statistical literature. It progressively occupied an important place in different scientific fields, particulary econometrics, due to its flexibility in dealing with the high dimensions. Indeed, this alternative approach introduce a parameter θ, which consists in bring to the covariates a dimension in smaller than dimension of the space variable and therefore qualifies the single index approach to be an efficient tool for treating the curse of dimensionality problem and also for increasing the explanatory power of each variable. As a simple example, the classical regression model $r(x) = \mathbb{E}(Y|X = x)$ will reformulate by using the semi-parametric modelization as the following way

$$r_\theta(x) = \mathbb{E}_\theta(Y|X = x) = \mathbb{E}(Y| < X, \theta >=< x, \theta >).$$

In the multivariate case, the topic of semi-parametric models has a long history in the literature. For example, Ichimura (1993) [36] introduced a contribution dealt with two semi-parametric estimators, where the author studied the consistency as well as the asymptotic normality of the proposed estimates. In the same year, another work focused on this type of models was suggested by Härdle *et al.* (1993) [34] for the optimal smoothing. The estimators effectiveness in the single index model was the subject of some contributions. We quote among them Newey and Stoker (1993) [47] and Delecroix *et al.* (2003) [15]. The first paper included effectiveness properties for weighted average derivative estimators and the construction of estimators with elevated efficiency, while the main purpose of other work is to formulate an asymptotically effective estimator. The publishers in this last one investigated a nonparametric conditional density estimate and studied some asymptotic properties.

1.3.2 The semi-parametric models in functional statistics

The functional single index framework has aroused growing interest, we cite some interesting works in this area. This trend was first initiated by Ferraty *et al.* (2003) [28]. The contribution focused on the kernel-type modelization of the regression function, when the observations were linked with a single index structure. The authors determined the full expression of the almost complete consistency rate of the constructed estimator when the data were in the form of curves. The same results for time series forecasting were investigated by Ait-Saïdi *et al.* (2005) [1] in the dependent situation. In (2008), Ait-Saïdi *et al.* [2] highlighted the problem of unknown functional index estimation, where they used the cross-validation method. The topic of functional derivative estimation for this type of models was examined by Ferraty *et al.* (2011) [27]. Always by using the single index approach, Attaoui *et al.* (2011) [4] treated nonparametrically the kernel conditional density prediction for a scalar response variable given an explicatory variable valued in an Hilbert space whenever independent observations were considered. Precisely, they studied the pointwise and the uniform almost complete consistency rate of the suggested estimator. An application to the conditional mode including the convergence rate of the estimator was also stated. Few years later and exactly in (2014), the dependent version of this paper was proposed by Ling [42], where he studied the same results under α-mixing assumption. The asymptotic normality of this last estimator was established by Ling *et al.* [44] in 2012. In the same context, one can refer to Attaoui and Ling (2016) [6] for the conditional cumulative distribution estimate, and Rabhi *et al.* (2017) [51] for the uniform convergence of the conditional quantile function.

In the strongly mixing setting, some asymptotic properties of the conditional density and the conditional mode functions when the sample was assumed to be of functional feature, and the uniform consistency with rate of the conditional density in addition to its asymptotic normality with functional covariates were examined in the same year by Attaoui (2014a) [3] and Attaoui (2014b) [4] , respectively, when the data were based always on the single index modeling.

1.4 Plan of the thesis

This manuscript is organized in six chapters, described successively as follows :
Naturally, we open our work with an introductory chapter of the different thematics covered in our research trend. we start with a bibliographic study of the problems related to the functional nonparametric statistics. Next, we propose a brief presentation of incomplete data notations, specially censored data and missing data with their deferent types, in addition to a general historical overview on these sorts of modeling. This part is followed by a bibliographic context on the single index approach in both multivariate and functional cases. Finally, our introduction ends with the exposition of the obtained asymptotic results.

Chapter 2 represents a short mathematical background that briefly includes some basic definitions, concepts and inequalities needed throughout this manuscript.

In the third chapter, we treat nonparametrically the conditional mode function via the conditional density estimator of a randomly censored scalar response given a functional covariate when the data are sampled from a stationary and ergodic process. The principal obtained results are the establishment of the almost sure consistency rates and the central limit theorems of the build estimators by the kernel-type method. In order to prove the effectiveness of the presented results, some applications including the confidence bands in addition to a simulation study are also introduced.

The following chapter is devoted to the convergence in probability as well as the asymptotic normality of the regression function operator under ergodic assumption. Here, we deal with the functional single index model whenever missing at random responses are considered. As an application, the asymptotic $(1-\zeta)$ confidence interval of the regression operator is suggested for $0 < \zeta < 1$.

In the chapter five, we are interested on the nonparametric kernel estimation of a randomly scalar response given a functional Hilbertian explanatory variable, when the data are sampled from an i.i.d process with a single-index relationship. The asymptotic normality of the conditional density estimator is investigated from which one derives the central limit theorem of the conditional mode function. These results lead to construct prediction intervals. At the end of this chapter, we clarify our methodology with examples on simulated data where the objective is the comparative study between the functional single index setting (FSIM) and the non-parametric functional data analysis case (NPFDA).

The last chapter focus on the dependent data. In particular, we examine the conditional cumulative distribution and the conditional quantile estimators when the data are of functional feature and linked with a single-index structure. In the presence of the right censorship, the asymptotic normality of the constructed estimators is treated under some mild conditions. These asymptotic properties are illustrated through a simulation study to prove the validity and finite sample performance of the considered estimators.

The thesis closes with a general conclusion as well as some open questions and presumable future prolongations of our works in order to ameliorate and extend the obtained results to short and long terms.

1.5 Brief presentation of the results

We give hereafter a short overview of the results obtained throughout our thesis.

1.5.1 Results : Ergodic case for censored data

In this part, we consider the case where the data are simultaneously functional and ergodic. Under random censorship, we establish the almost sure convergence of the conditional density and the conditional mode functions, in addition to the central limit theorem of both of these nonparametric estimates.

Proposition 1.5.1. *Under assumptions of concentration of the probability measure of the functional variable, as well as other conditions of regularities and techniques, one gets*

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}} |\widehat{\varphi_n}(t|z) - \varphi(t|z)| = O_{a.s.}(h_K^{\alpha_1} + h_H^{\alpha_2}) + O_{a.s.}\left(\sqrt{\frac{\log n}{nh_H\phi(h_K)}}\right).$$

Theorem 1.5.1. *In view of some hypotheses detailed in Chapter 3, it yields*

$$|\widehat{\theta}(z) - \theta(z)| = O_{a.s.}\big(h_K^{\alpha_1} + h_H^{\alpha_2}\big) + O_{a.s.}\left(\sqrt{\frac{\log n}{nh_H\phi(h_K)}}\right).$$

Theorem 1.5.2. *Under certain assumptions relating to the estimator, we have for the asymptotic normality*

$$\sqrt{nh_H\phi(h_K)}(\widehat{\varphi}_n(t|z) - \varphi(t|z)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2(z,t)),$$

where

$$\sigma^2(z,t) = \frac{M_2}{M_1^2}\frac{\varphi(t|z)}{\bar{G}(t)f_1(z)}\int_{\mathbb{R}}(H'(v))^2\,dv,$$

with $M_j = K^j(1) - \int_0^1 (K^j)'\varsigma_0(u)\,du \quad for \quad j = 1, 2.$

Noting that " $\xrightarrow{\mathcal{D}}$ " symbolyzes the convergence in distribution.

Theorem 1.5.3. *For all $z \in E$, we obtain, as n goes to infinity*

$$\sqrt{\frac{nh_H^3\phi(h_K)}{\varrho^2(z,\theta(z))}}(\widehat{\theta}(z) - \theta(z)) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1),$$

where

$$\varrho^2(z,\theta(z)) = \frac{M_2}{M_1^2}\frac{\varphi(\theta(z)|z)}{\bar{G}(t)f_1(z)(\varphi^{(2)}(\theta(z)|z))^2}\int_{\mathbb{R}}(H^{(2)}(v))^2\,dv.$$

The proofs of these results and the detail of the imposed hypotheses will be given in the third chapter of this thesis.

1.5.2 Results : Ergodic case for missing data

In this part, we suppose that the data are linked with the semi-parametric modeling. We investigate some asymptotic properties of the estimator $\hat{r}_n(\theta,x)$ for the regression operator based on the functional stationary ergodic data with MAR. More precisely, Theorem 1.5.4 shows the convergence in probability of the estimator. The asymptotic distribution of this kernel estimator is presented in Theorem 1.5.5.

Theorem 1.5.4. *Using the general ergodicity condition and the assumptions of functional concentration, we write*

a)

$$\left(\frac{n\phi_\theta(h)}{\log\log(n)}\right)^{\frac{1}{2}}(\hat{r}_n(\theta,x) - C_n(\theta,x)) \xrightarrow{\mathbb{P}} 0,$$

where $\xrightarrow{\mathbb{P}}$ *means the convergence in probability.*

b)

$$\left(\frac{n\phi_\theta(h)}{\log\log(n)}\right)^{\frac{1}{2}}(\hat{r}_n(\theta,x) - r(\theta,x)) \xrightarrow{\mathbb{P}} 0.$$

Theorem 1.5.5. *For any* $x \in \mathcal{H}$*, we have*

a)

$$\sqrt{n\phi_\theta(h)}(\hat{r}_n(\theta,x) - C_n(\theta,x)) \xrightarrow{D} \mathcal{N}(0,\sigma^2(\theta,x)),$$

where $\xrightarrow{D}$ *means the convergence in distribution and* $\sigma^2(\theta,x) = \dfrac{M_2}{M_1^2}\dfrac{V(\theta,x)}{p(\theta,x)f_1(\theta,x)}$*,*

with $M_j = K^j(1) - \int_0^1 (K^j)'(u)\tau_0(u)du$ *for* $j = 1,2$.

b)

$$\sqrt{n\phi_\theta(h)}(\hat{r}_n(\theta,x) - r(\theta,x)) \xrightarrow{D} \mathcal{N}(0,\sigma^2(\theta,x)).$$

The required assumptions and the demonstrations of the above theorems will be introduced in detail in Chapter 4.

1.5.3 Results : Independent case for complete data

In the next part, we will explicit the asymptotic normality results of the conditional density and conditional mode estimators when the observations are independent and linked with single index structure.

Theorem 1.5.6. *Under some classical hypotheses in functional nonparametric estimation and other technical conditions, one obtains*

$$\sqrt{\frac{nh_H\phi_{\theta,x}(h_K)}{\sigma^2(\theta,y,x)}}\left(\widehat{f}(\theta,y,x)-f(\theta,y,x)\right)\xrightarrow{\mathcal{D}}\mathcal{N}(0,1),\quad \textit{as } n\to\infty,$$

where $\sigma^2(\theta,y,x)=\dfrac{\alpha_2(\theta,x)f(\theta,y,x)}{(\alpha_1(\theta,x))^2}\displaystyle\int H^2(t)dt,$

with $\alpha_l(\theta,x)=K^l(1)-\displaystyle\int_0^1 (K^l)^{'}(u)\beta_{\theta,x}(u)du,\ l=1,2.$

" $\xrightarrow{\mathcal{D}}$ *" means the convergence in distribution.*

Theorem 1.5.7. *For all* $x\in\mathcal{H}$, *we have as* $n\to\infty$

$$\sqrt{\frac{nh_H^3\phi_{\theta,x}(h_K)}{\nu^2(\theta,M_\theta(x),x)}}(\widehat{M}_\theta(x)-M_\theta(x))\xrightarrow{\mathcal{D}}\mathcal{N}(0,1),\quad \textit{as }\ n\to\infty,$$

where

$$\nu^2(\theta,M_\theta(x),x)=\frac{\alpha_2(\theta,x)f(\theta,M_\theta(x),x)}{\left(\alpha_1(\theta,x)f^{(2)}(\theta,M_\theta(x),x)\right)^2}\int(H^{'}(t))^2dt.$$

The demonstrations and the necessary conditions to obtain these two results will be detailed in Chapter 5 of our manuscript.

1.5.4 Results : Dependent case for incomplete data

Always in the functional single index framework, we assume now that the observations are censored. If the conditional distribution function and the conditional quantile satisfy the nonparametric models given in Chapter 6, then, under general technical assumptions in NPFDA and other slightly restrictive conditions on the mixing coefficient, one obtains the following results on the asymptotic normality of these kernel estimators, and we leave the assumptions and the details of the proofs for the sixth chapter.

Theorem 1.5.8. *Using some regularity condition, it yields*

$$\left(\frac{n\phi_{\theta,x}(h_K)}{\sigma^2(\theta,t,x)}\right)^{1/2}\left(\widehat{F}(\theta,t,x)-F(\theta,t,x)\right)\xrightarrow{\mathcal{D}}\mathcal{N}(0,1),$$

where $\sigma^2(\theta,t,x)=\dfrac{a_2(\theta,x)}{(a_1(\theta,x))^2}F(\theta,t,x)\left(\dfrac{1}{\bar{G}(t)}-F(\theta,t,x)\right),$

with $a_l(\theta,x) = K^l(1) - \int_0^1 (K^l)'(u)\xi_h^{\theta,x}(u)\,du$, *for* $l = 1, 2$,

and " $\xrightarrow{\mathcal{D}}$ " *means the convergence in distribution.*

Theorem 1.5.9. *By some basic hypotheses, and if* γ *is the unique order of the quantile such that* $\gamma = F(\theta, \zeta_\theta(\gamma,x), x) = \widehat{F}(\theta, \widehat{\zeta}_\theta(\gamma,x), x)$, *one gets*

$$\left(\frac{n\phi_{\theta,x}(h_K)}{\Sigma^2(\theta,\zeta_\theta(\gamma,x),x)}\right)^{1/2}\left(\widehat{\zeta}_\theta(\gamma,x) - \zeta_\theta(\gamma,x)\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1),$$

where $\Sigma(\theta,\zeta_\theta(\gamma,x),x) = \dfrac{\sigma(\theta,\zeta_\theta(\gamma,x),x)}{f(\theta,\zeta_\theta(\gamma,x),x)}$.

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Chapter 2

Necessary Tools and Definitions

In order to facilitate the reader's examination of this thesis, we propose this short mathematical background involving briefly some required definitions, concepts and inequalities that are essential to derive the main results of this manuscript.

2.1 Definitions

Small Ball Probabilities

Firstly, we will present a powerful quantity so-called small ball probability (or concentration property), which plays a key role in the functional nonparametric problems. This is evident from its ability to propose an alternative to the curse of dimensionality problem well-known for the nonparametricians and from the strong relationship between this measure and the asymptotic results for functional framework, where it directly effects on the rates of convergence. Indeed, the small ball probability controls the concentration of the probability measure of the functional variable on a small ball, and it is constructed as reported in Ferraty and Vieu (2006) [55] as

$$\phi_x(h) = \mathbb{P}(X \in B(x, h)),$$

where X is a random variable taking its values in a semi metric space (E, d), x is a fixed element of E, and $B(x, h) = \{x' \in E,\ d(x, x') \le h\}$ is the ball centered at x and of radius h (h is a real positive number).

The kernel-types are defined as follows:

Definition 2.1.1. *(Ferraty and Vieu (2006) [55])*

i) *A function K from $\mathbb{R}$ into $\mathbb{R}^+$ such that $\int K = 1$ with compact support $[-1, 1]$ and such that $\forall u \in (0, 1)$, $K(u) > 0$ is called **a kernel of type** 0.*

ii) *A function K from $\mathbb{R}$ into $\mathbb{R}^+$ such that $\int K = 1$ is called **a kernel of type** I if there exist two real constants $0 < C_1 < C_2 < \infty$, such that*

$$C_1 \mathbf{1}_{[0,1]} \le K \le C_2 \mathbf{1}_{[0,1]}.$$

iii) *A function K from $\mathbb{R}$ into $\mathbb{R}^+$ such that $\int K = 1$ is called* ***a kernel of type** II if its support is $[0, 1]$ and if its derivative K' exists on $[0, 1]$ and satisfies for two real constants $-\infty < C_2 < C_1 < 0$:*

$$C_2 \leq K' \leq C_1.$$

Definition 2.1.2. ***"Martingale differences"*** *(Laib and Louani (2011) [70]) A sequence of random variables $(W_n)_{n\geq 0}$ is said to be a sequence of martingale differences with respect to the sequence of σ-fields $(\mathcal{F}_n)_{n\geq 0}$ if*

(i) $(W_n)_{n\geq 0}$ *is* $(\mathcal{F}_n)_{n\geq 0}$*-measurable,*

(ii) $\forall n \geq 0,\ \mathbb{E}(W_n|\mathcal{F}_{n-1}) = 0$ *almost surely.*

Noting that this last definition is related to the exponential inequality for ergodic case, where this latter will be introduced later (at the end of this chapter).

2.2 Convergence notions

Let's $(X_n)_{n\in\mathbb{N}}$ be a sequence of real random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, while $(u_n)_{n\in\mathbb{N}}$ is a deterministic sequence of positive real numbers.

Definition 2.2.1. ***"Almost complete convergence"****(Ferraty and Vieu (2006) [55]) One says that $(X_n)_{n\in\mathbb{N}}$ converges almost completely (a.co.) to some real random variable (r.r.v.) X, if and only if:*

$$\forall\varepsilon > 0, \quad \sum_{n\in\mathbb{N}} \mathbb{P}(|X_n - X| > \varepsilon) < \infty,$$

and the almost complete convergence of $(X_n)_{n\in\mathbb{N}}$ to X is denoted by

$$\lim_{n\to\infty} X_n = X,\ a.co. \quad or \quad X_n \xrightarrow[n\to\infty]{a.co.} X.$$

Definition 2.2.2. ***"Rate of almost complete convergence"****(Ferraty and Vieu (2006) [55]) One says that the rate of almost complete convergence of $(X_n)_{n\in\mathbb{N}}$ to X is of order u_n if and only if*

$$\exists\varepsilon_0 > 0, \quad \sum_{n\in\mathbb{N}} \mathbb{P}(|X_n - X| > \varepsilon_0 u_n) < \infty,$$

and we write

$$X_n - X = O_{a.co.}(u_n).$$

Definition 2.2.3. ***"Almost sure convergence"****(Ferraty and Vieu (2006) [55]) We say that $(X_n)_{n\in\mathbb{N}}$ converge almost surely (a.s.) to some r.r.v. X if:*

$$\mathbb{P}\Big(\lim_{n\to\infty} X_n = X\Big) = 1.$$

This is denoted

$$X_n \xrightarrow[n\to\infty]{a.s.} X.$$

Definition 2.2.4. ***"Convergence in probability"***(*Ferraty and Vieu (2006) [55]) We say that* $(X_n)_{n\in\mathbb{N}}$ *converge in probability to* X *if:*

$$\forall \varepsilon > 0, \quad \lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0.$$

This is written

$$\lim_{n\to\infty} X_n = X, \ \mathbb{P}.$$

Remark 2.2.1.

- *If* $X_n \xrightarrow[n\to\infty]{a.s.} X$, *then* $X_n \xrightarrow[n\to\infty]{p.} X$.
- *If* $X_n \xrightarrow[n\to\infty]{a.co.} X$, *then* $X_n \xrightarrow[n\to\infty]{a.s.} X$ *and* $X_n \xrightarrow[n\to\infty]{p.} X$.

2.3 Dependency notions

For several phenomena of the real world, observations in the past and present may have big impact on observations in the near future, but rather weak impact on observations in the far future. Mathematically, such sort of phenomena is expressed by using random sequences that satisfy strong mixing conditions. Indeed, There are various ways to modelize the dependence structure via the different types of mixing that are defined according to coefficients, noted: α, β, ρ, ψ and ϕ. For more discussions on the mixing hypothesis, we direct the reader to Bosq (1998) [10], Rio (2000) [98]. In this thesis, we deal specifically with the alpha-mixing (or strong mixing) sequence, which is the most general and the weakest among all the previously mentioned mixing processes and is therefore least restraining. In this setting, some definitions that can be found in Ferraty and Vieu (2006) [55] are suggested as follows:

Definition 2.3.1. *Let* $(\Delta_n)_{n\in\mathbb{Z}}$ *be a sequence of random variables defined on some probabilistic space* $(\Omega, \mathcal{A}, \mathbb{P})$ *and taking values in some space* $(\Omega', \mathcal{A}')$. *For* $-\infty \leq j \leq k \leq +\infty$, *one denotes by* $\mathcal{A}_j^k$ *the* σ*-algebra generated by the random variables* $(\Delta_s,\ j \leq s \leq k)$.
The strong mixing coefficients are identified in the following expression:

$$\alpha(n) = \sup_{\{k\in\mathbb{Z},\, A\in\mathcal{A}_{-\infty}^k,\, B\in\mathcal{A}_{n+k}^{+\infty}\}} |\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)|.$$

Definition 2.3.2. *The sequence* $(\Delta_n)_{n\in\mathbb{Z}}$ *is said to be* α*-mixing (or strongly mixing), if*

$$\lim_{n\to\infty} \alpha(n) = 0.$$

Definition 2.3.3. *One says that the sequence* $(\Delta_n)_{n\in\mathbb{Z}}$ *is arithmetically (or equivalently* *algebraically)* α*-mixing with rate* $a > 0$ *if*

$$\exists C > 0,\ \alpha(n) \leq Cn^{-a}.$$

Remark 2.3.1. *The following scheme summarizes the implications between the different types of dependency*

$$\psi - mixing \Longrightarrow \phi - mixing \Longrightarrow \begin{Bmatrix} \rho - mixing \\ \beta - mixing \end{Bmatrix} \Longrightarrow \alpha - mixing.$$

2.4 Some useful inequalities

Always for the aim of simplicity, we will recall some inequalities adapted to the framework of this thesis. Let's now introduce two powerful inequalities for mixing sequences of real random variables, which are stated bellow.

Proposition 2.4.1. ***"Davydov-Rio's inequality"****(Ferraty and Vieu (2006) [55])* $(W_n)_{n\in\mathbb{Z}}$ *will be a stationary sequence of real random variables assumed to be* α*-mixing. Let us, for some* $k \in \mathbb{Z}$*, consider a real variable* W *(resp.* W'*) which is* $\mathcal{A}_{-\infty}^{k}$*-measurable (resp.* $\mathcal{A}_{n+k}^{+\infty}$*-measurable).*

i) *If* W *and* W' *are bounded, then:*

$$\exists C,\ 0 < C < +\infty,\ Cov(W, W') \leq C\alpha(n).$$

ii) *If, for some positive numbers* p, q, r *such that* $p^{-1} + q^{-1} + r^{-1} = 1$*, we have* $\mathbb{E}W^p < \infty$ *and* $\mathbb{E}W'^q < \infty$*, then:*

$$\exists C,\ 0 < C < +\infty,\ Cov(W, W') \leq C(\mathbb{E}W^p)^{\frac{1}{p}}(\mathbb{E}W'^q)^{\frac{1}{q}}\alpha(n)^{\frac{1}{r}}.$$

Lemma 2.4.1. *(Volkonskii and Rozanov (1959) [106]) Let* $W_1, ..., W_L$ *be strongly mixing random variables measurable with respect to the* σ*-algebras* $\mathcal{F}_{i_1}^{j_1}, ..., \mathcal{F}_{i_L}^{j_L}$ *respectively with* $1 \leq i_1 < j_1 < i_2 < ... < j_L \leq n,\ i_{l+1} - j_l \geq V \geq 1$ *and* $|W_j| \leq 1$ *for* $j = 1, ..., L$*. Then,*

$$\left| E\left(\prod_{j=1}^{L} W_j\right) - \prod_{j=1}^{L} E(W_j) \right| \leq 16(L-1)\alpha(V),$$

where $\alpha(V)$ *is the strongly mixing coefficient.*

The exponential inequality for partial sums of unbounded martingale differences is necessary in the studies focused on the ergodic processes forecasting via functional estimation, that is used to prove the asymptotic results of the constructed estimates. This inequality is given in the following lemma:

Lemma 2.4.2. ***"Exponential inequality"*** *(Laib and Louani (2011) [70]) Let* $(W_n)_{n\geq 1}$ *be a sequence of real martingale differences with respect to the sequence of* σ*-fields* $(\mathcal{F}_n = \sigma(W_1, \cdots, W_n))_{n\geq 1}$*, where* $\sigma(W_1, \cdots, W_n)$ *is the* σ*-field generated by the random variables* $W_1, \cdots, W_n$*. Set* $S_n = \sum_{i=1}^{n} W_i$*. For any* $p \geq 2$ *and any* $n \geq 1$*, assume that there exist some nonnegative constants* c *and* d_n *such that*

$$\mathbb{E}(W_n^p|\mathcal{F}_{n-1}) \leq C^{p-2}p!d_n^2, \quad \text{almost surely.}$$

Then, for any $\varepsilon > 0$*, we have*

$$\mathbb{P}(|S_n| > \varepsilon) \leq 2\exp\left\{ -\frac{\varepsilon^2}{2(D_n + C\varepsilon)} \right\},$$

where $D_n = \sum_{i=1}^{n} d_i^2$*.*

Chapter 3

Some asymptotic properties of conditional density function for functional data under random censorship

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Some asymptotic properties of conditional density function for functional data under random censorship

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Abstract: In this work, we investigate the asymptotic properties of a nonparametric mode of a conditional density when the real response variable is censored and the explanatory variable is valued in a semi-metric space under ergodic data. First of all, we establish asymptotic properties for a conditional density estimator from which we derive a central limit theorem (CLT) of the conditional mode estimator. Simulation study is also presented to illustrate the validity and finite sample performance of the considered estimator.

Keywords: Asymptotic normality, Censored data, Conditional mode, Ergodic processes, Functional data, Strong consistency.

2010 Mathematics Subject Classification: 62G20, 62N01, 62P30, 62M10.

3.1 Introduction

Survival analysis methods have been used in a number of applied fields (medicine, biology, epidemiology, engineering, econometrics, finance, social sciences, demography...). The analysis of failure time data usually means addressing one of three problems: the estimation of survival functions, the comparison of treatments or survival functions, and the assessment of covariate effects or the dependence of failure time on explanatory variables. There are many reasons that make it difficult to get complete data in studies involving survival times. A study is often finished before the death of all patients, and we may keep only the information that some patients are still alive at the end of the study, not observing when they really die. In the presence of censored data, the time to event is unknown, and all we know is that the survival time has occurred before, between or after certain time points, this obviates the need for inference methods for censored data. When the failure time is observed completely, there are numerous methods to make non parametric inference on its conditional distribution. For instance Nadaraya (1964) [15] and Watson [18] proposed a nonparametric estimator to estimate the conditional expectation as a locally weighted average using a kernel function. Beran (1981) [1] extended the Kaplan-Meier estimator and proposed a method for non-parametric estimation (generalized Kaplan-Meier) of the conditional survival function for right-censored data.

Results regarding the estimation of the conditional models from right censored data can be found for instance in Dabrowska (1992) [4], where author gave the nonparametric regression with censored survival time data. In Li and Doss [13] an approach to nonparametric regression for life history data using local linear fitting was given. Dehgham and Duchesne (2016)[6] established the estimation of the conditional survival function of a failure time given a time-varying covariate with interval-censored observations. Many works in the statistical literature deal with nonparametric estimation when the variable of interest is either complete or singly censored. However, in reliability and survival time studies, one can encounter a more complicated random censorship situation. An example of such a model, given in Patilea and Rolin (2006) [17], is to consider a reliability system consisting of three components with two components in series and one component in parallel with the series system, the authors defined the product-limit estimators of the survival function with twice censored data.

On the other hand, the problem of nonparametric conditional models for censored data where the observations can be censored from either left or right are very limited in the literature. This gap can partially be explained by the difficulties arising in the estimation of the conditional distribution and/or density function with two-sided censored data. The problem of estimating the (unconditional) distribution function for data that may be censored from above and below has been considered by several authors.

Despite the regression function is of interest, other statistics such as quantile and mode regression might be important from a theoretical and a practical point of view. Quantile and/or mode regression is a common way to describe the dependence structure between a response variable T and some covariate Z. Unlike the regression function that relies only on the central tendency of the data, the conditional quantile function allows the analyst to estimate the functional dependence between variables for all portions of the conditional distribution of the response variable.

Mode regression is a common way to describe the dependence structure between a response variable T and some covariate Z. Unlike the regression function (which is defined as the conditional mean) that relies only on the central tendency of the data, the conditional mode function allows the analysts to estimate the functional dependence between variables for all portions of the conditional distribution of the response variable. On the other hand, compared with the standard approach based on functional conditional mean prediction that is sensitive to outliers, functional condition mode prediction could be seen as a reasonable alternative to conditional mean because of its robustness. Moreover, quantiles are well known for their robustness to heavy-tailed error distributions and outliers which allow to consider them as a useful alternative to the regression function see Chaouch and Khardani (2015) [2]. Conditional model are used in finance and/or insurance to model the risks of extreme values. The regression quantile function provide a well description of the data, specifically the conditional median function (see Chaudhuri *et al.* (1997) [3]). Estimation of the conditional mode of a scalar response given a functional covariate has attracted the attention of many researchers.

In the censored case, Ould-Saïs and Cai (2005) [16] stated the uniform strong consistency with rates of the kernel estimator of the conditional mode function, in this context, we refer to Ling *et al.* (2016) [14] for the estimation of conditional mode for functional stationary ergodic data with missing at random. The ergodic theory has appeared in statistical mechanics, notably in Maxwell's and Gibbs's theories. It is necessary to make a sort of logical transition between the average behavior of the set of dynamic systems and the temporal average of the behaviors of a single dynamic system. It is derived from an ingenious hypothesis used for a long time without justifying it, and in various forms. In the context of the ergodic functional case with censored observations the literature is very restricted.

So, in the present work, we investigate the asymptotic properties of the conditional mode function of a randomly censored scalar response given a functional covariate when the data are sampled from a stationary and ergodic process. In practice, this study has great importance, because, it permits us to construct a prediction method based on the conditional mode estimator. Here, we consider a model in which the response variable is censored but not the covariate. Besides the infinite dimensional character of the data, we avoid here the widely used strong mixing condition and its variants to measure the dependency and the very involved probabilistic calculations that it implies. Therefore, we consider in our setting the ergodic property to allow the maximum possible generality with regard to the dependence setting. Further motivations to consider ergodic data are discussed in Laib and Louani (2010 [11] -2011 [12]) where details defining the ergodic property of processes are also given.

The layout of the paper is as follows. In the next section, our model is described. Section 3.3 is dedicated to fixing notations and hypotheses. We state our main result of strong consistency rate, the asymptotic normality as well as an application to confidence bands, where the technical proofs are given with some auxiliary results in Section 3.4. A simulation study is also presented in Section 3.5 to illustrate the validity of the kernel estimator. Lastly, our contribution ends with a general conclusion proposed in Section

3.6.

Consider a random pair (Z, T) which is valued in $E \times \mathbb{R}$, where E is some semi-metric abstract space equipped with semi-metric $d(\cdot,\cdot)$, and T takes values in $\mathbb{R}$. Let $(Z_i, T_i)_{1\le i\le n}$ be the statistical sample of pairs which are identically distributed as (Z, T) and supposed to be stationary and ergodic. Henceforward, Z is called functional random variable *f.r.v.* For $z \in E$, we denote by $\varphi(\cdot|z)$ the conditional density function of T given $Z = z$ and we assume that $\varphi(\cdot|z)$ has an unique conditional mode $\theta(z)$ defined as

$$\theta(z) = \arg\sup_{t\in\mathcal{S}_{\mathbb{R}}} \varphi(t|z), \tag{3.1}$$

where $\mathcal{S}_{\mathbb{R}}$ is a fixed compact subset of $\mathbb{R}$.

3.2 The model

Consider a randomly censored model given by two nonnegative sequences of random variables $T_1, \cdots, T_n$ (survival times) and $C_1, \cdots, C_n$ (i.i.d censoring r.v) with the distribution functions F and G, respectively. In pratice, particularly, in medical applications, it is not possible to observe the lifetimes T of all patients under study in the presence of censoring. We only observe the triples (X_i, δ_i, Z_i), where $X_i = \min\{T_i, C_i\}$ and $\delta_i = \mathbf{1}_{\{T_i\le C_i\}}$, $1 \le i \le n$ with $\mathbf{1}_A$ denotes the indicator function of the set A, where both of T_i and C_i are expected to exhibit some kind of dependence which ensures the identifiability of the model.

In biomedical case studies, it is assumed that C_i and (Z_i, T_i) are independent, this condition is plausible whenever the censoring is independent of the patient's modality.

In this kind of model, it is well known that the empirical distribution is not a consistent estimator for the distribution function G. Therefore, Kaplan and Meier (1958) [9] proposed a consistent estimator, for the survival function $\bar{G}(\cdot) = 1 - G(\cdot)$ which is constructed by

$$\bar{G}_n(t) = \begin{cases} \prod_{i=1}^{n} \left(1 - \dfrac{1-\delta_{(i)}}{n-i+1}\right)^{\mathbf{1}_{\{X_{(i)}\le t\}}}, & \text{if } t < X_{(n)}, \\ 0, & \text{Otherwise}, \end{cases}$$

where $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ are the order statistics of $(X_i)_{1\le i\le n}$ and $\delta_{(i)}$ is concomitant with $X_{(i)}$.

Because of the relation between the conditional mode and the conditional density given in statement (3.1), an estimator of $\theta(z)$ follows straightforwardly from an estimator of $\varphi(t|z)$. Now, we represent the kernel estimator of the conditional density function in the case of complete data, set

$$\varphi_n(t|z) = \frac{h_H^{-1}\sum_{i=1}^{n} K(h_K^{-1}d(z,Z_i))H'(h_H^{-1}(t-T_i))}{\sum_{i=1}^{n} K(h_K^{-1}d(z,Z_i))}, \tag{3.2}$$

where, K is a probability density function (so-called kernel function), $h_K = h_{n,K}$(resp. $h_H = h_{n,H}$) is a sequence of positive real numbers (so-called bandwidth) which goes to zero as n tends to infinity, $H'(\cdot)$ is the first derivative of a given distribution function $H(\cdot)$. An analogous estimator to equation (3.2) was already given in Ferraty and Vieu (2006) [7] in the general setting.

Firstly, we must know that our kernel type estimator of the conditional density $\varphi(t|z)$ adapted for censored samples is based on "a pseudo-estimator" of $\varphi(t|z)$ that is defined as

$$\widetilde{\varphi}_n(t|z) = \frac{\sum_{i=1}^{n} \delta_i \bar{G}^{-1}(X_i)K(h_K^{-1}d(z,Z_i))H'(h_H^{-1}(t-X_i))}{h_H\sum_{i=1}^{n} K(h_K^{-1}d(z,Z_i))} = \frac{\widetilde{\varphi}_n(z,t)}{\psi_n(z)},$$

where

$$\widetilde{\varphi}_n(z,t) = \frac{1}{nh_H\mathbb{E}(\Delta_1(z))}\sum_{i=1}^{n} \delta_i \bar{G}^{-1}(X_i)H'(h_H^{-1}(t-X_i))\Delta_i(z),$$

and

$$\psi_n(z) = \frac{1}{n\mathbb{E}(\Delta_1(z))}\sum_{i=1}^{n}\Delta_i(z), \quad \text{with} \quad \Delta_i(z) = K(d(z,Z_i)/h_K).$$

In fact, this pseudo-estimator is not efficient since $\bar{G}(\cdot)$ is unknown in practice. So, we should replace $\bar{G}(\cdot)$ by its Kaplan and Meier's estimator $\bar{G}_n(\cdot)$ previously defined.

Therefore, feasible estimator of the conditional density function $\varphi(t|z)$ is denoted by

$$\widehat{\varphi}_n(t|z) = \frac{\sum_{i=1}^{n} \delta_i \bar{G}_n^{-1}(X_i)K(h_K^{-1}d(z,Z_i))H'(h_H^{-1}(t-X_i))}{h_H\sum_{i=1}^{n} K(h_K^{-1}d(z,Z_i))} = \frac{\widehat{\varphi}_n(z,t)}{\psi_n(z)}, \tag{3.3}$$

where

$$\widehat{\varphi}_n(z,t) = \frac{1}{nh_H\mathbb{E}(\Delta_1(z))}\sum_{i=1}^{n} \delta_i \bar{G}_n^{-1}(X_i)H'(h_H^{-1}(t-X_i))\Delta_i(z).$$

Then, a natural kernel estimator of $\theta(z)$ which maximizes the kernel estimator $\widehat{\varphi}_n(\cdot|z)$

of $\varphi(\cdot|z)$ is given by

$$\widehat{\theta}(z) = \arg\sup_{t \in \mathcal{S}_{\mathbb{R}}} \widehat{\varphi}_n(t|z). \tag{3.4}$$

3.3 Notations and hypotheses

To formulate our assumptions, some additional notations are required. For $i = 1, \ldots, n$, we represent $\mathcal{F}_i$ as the σ-field generated by $((Z_1, T_1), \ldots, (Z_i, T_i))$ and $\mathcal{G}_i$ the one generated by $((Z_1, T_1), \ldots, (Z_i, T_i), Z_{i+1})$. Let $\mathcal{N}_z$ be a fixed neighborhood of z, and let $\mathcal{B}(z, h)$ the ball of center z and radius h, denote $D_i(z) = d(z, Z_i)$ a nonnegative random variable such that its cumulative distribution function is determined by $F_z(u) = \mathbb{P}(D_i(z) \le u) = \mathbb{P}(Z_i \in \mathcal{B}(z, u))$. Furthermore, we define $F_z^{\mathcal{F}_{i-1}}(u) = \mathbb{P}(D_i(z) \le u|\mathcal{F}_{i-1}) = \mathbb{P}(Z_i \in \mathcal{B}(z, u)|\mathcal{F}_{i-1})$ the conditional distribution function given the σ-field $\mathcal{F}_{i-1}$ of $(D_i(z))_{i\ge1}$.

Our nonparametric model will be quite general in the sense that we will just need the following hypotheses:

(H0) For $z \in E$, there exists a sequence of nonnegative random functions $(f_{i,1})_{i\ge1}$ almost surely bounded by a sequence of deterministic quantities $(b_i(z))_{i\ge1}$ accordingly, a sequence of random functions $(g_{i,z})_{i\ge1}$, a deterministic nonnegative bounded function f_1 and a nonnegative real function ϕ tending to zero, as its argument tends to 0, such that if $n \to \infty$ and $h \to 0$

- **(a)** $F_z(h) = \phi(h)f_1(z) + o(\phi(h))$.
- **(b)** For any $i \in \mathbb{N}$, $F_z^{\mathcal{F}_{i-1}}(h) = \phi(h)f_{i,1}(z) + g_{i,z}(h)$ with $g_{i,z}(h) = o_{a.s.}(\phi(h))$ as $\frac{g_{i,z}(h)}{\phi(h)}$ almost surely bounded and $n^{-1}\sum_{i=1}^n g_{i,z}^j(h) = o_{a.s.}(\phi^j(h))$ for $j = 1, 2$.
- **(c)** $n^{-1}\sum_{i=1}^n f_{i,1}^j(z) \to f_1^j(z)$, almost surely, for $j = 1, 2$.
- **(d)** There exists a nondecreasing bounded function ς_0 such that, uniformly in $s \in [0, 1]$,
  $\phi(hs)/\phi(h) = \varsigma_0(s) + o(1)$, and, for $j \ge 1$, $\int_0^1 (K^j(t))'\varsigma_0(t)\,dt < \infty$.
- **(e)** $n^{-1}\sum_{i=1}^n b_i(z) \longrightarrow D(z) < \infty$.

(H1) The conditional density function $\varphi(t|z)$ satisfies

- **(a)** $\displaystyle\int_{\mathbb{R}} |t|\varphi(t|z)\,dt < \infty$, for all $z \in E$.
- **(b)** The Hölder condition, that is
  $\forall (t_1, t_2) \in \mathcal{S}_{\mathbb{R}}^2$, $\forall (z_1, z_2) \in \mathcal{N}_x^2$, for some $\alpha_1 > 0$ and $\alpha_2 > 0$

$$|\varphi^{z_1}(t_1) - \varphi^{z_2}(t_2)| \le C_z(d(z_1, z_2))^{\alpha_1} + |t_1 - t_2|^{\alpha_2}),$$

  with C_z is a positive constant depending on z.

(H2) $\varphi(\cdot|z)$ is twice continuously differentiable in a neighbourhood of $\theta(z)$ with

$$\begin{cases} \varphi^{(1)}(\theta(z)|z) = 0, \\ |\varphi^{(2)}(\theta(z)|z)| \neq 0. \end{cases}$$

(H3) The cumulative kernel H is derivable such that

$$\begin{cases} \exists C < \infty, \forall (v_1, v_2) \in \mathbb{R}^2, \quad |H'(v_1) - H'(v_2)| \leq C|v_1 - v_2|, \\ \int |v|^{\alpha_2} H'(v)\,dv < \infty, \quad \text{and} \quad \int H'(v)\,dv = 1. \end{cases}$$

(H4) For any $m \geq 1$, $\mathbb{E}[(H'(h_H^{-1}(t - T_i)))^m | \mathcal{G}_{i-1}] = \mathbb{E}[(H'(h_H^{-1}(t - T_i)))^m | Z_i]$.

(H5) For any $z' \in E$ and $m \geq 2$, $\sup_{t \in \mathcal{S}_\mathbb{R}} |g_m(z', t)| = \sup_{t \in \mathcal{S}_\mathbb{R}} |\mathbb{E}[H^m(h_H^{-1}(t - T_1))|Z_1 = x']| < \infty$ and $g_m(z', t)$ is continuous in $\mathcal{N}_z$ uniformly in t:

$$\sup_{t \in \mathcal{S}_\mathbb{R}} \sup_{z' \in B(z,h)} |g_m(z', t) - g_m(z, t)| = o(1).$$

(H6) K is a differentiable positive bounded function supported on $[0, 1]$ of class $\mathcal{C}^1(0, 1)$: $\exists C', C''$, $-\infty < C' < K'(t) < C'' < 0$ for $0 < t < 1$, $|\int_0^1 (K^j)'(t)\,dt| < \infty$ for $j = 1, 2$.

(H7) The bandwidth h_K and h_H, satisfying $\lim_{n\to\infty} h_K = 0$, $\lim_{n\to\infty} h_H = 0$ and $\dfrac{\log n}{n h_H \phi(h_K)} \underset{n\to\infty}{\longrightarrow} 0$.

(H8) $(C_n)_{n\geq 1}$ and $(Z_n, T_n)_{n\geq 1}$ are independent.

Remark 3.3.1. *Our assumptions are very standard for this kind of models. Assumption (H0) plays an important role in our methodology, it is devoted to the ergodicity of functional data. (H1) is a regularity condition which characterizes the functional space of our model and is needed to evaluate the bias terms of our asymptotic results, while hypotheses (H3) and (H7) are technical conditions and are similar to those given in Ferraty and Vieu (2006) [7]. As for (H6), it is classical in nonparametric estimation.*

3.4 Main resullts

3.4.1 Pointwise almost sure rate of convergence

We establish in Proposition 3.4.1 the rates of convergence of the kernel density estimator $\widehat{\varphi}_n(t|z)$. An immediate consequence is the almost sure convergence with a rate of the kernel mode estimator, as stated in Theorem 3.4.1.

Proposition 3.4.1. *Suppose that assumptions (H6)-(H7) and (H8) hold true, we get*

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}} |\widehat{\varphi_n}(t|z) - \varphi(t|z)| = O_{a.s.}(h_K^{\alpha_1} + h_H^{\alpha_2}) + O_{a.s.}\left(\sqrt{\frac{\log n}{nh_H\phi(h_K)}}\right).$$

Proof. First of all, denote

$$\bar{\widetilde{\varphi}}_n(z,t) = \frac{1}{nh_H\mathbb{E}(\Delta_1(z))}\sum_{i=1}^{n}\mathbb{E}[\delta_i\bar{G}^{-1}(X_i)H'(h_H^{-1}(t-X_i))\Delta_i(z)|\mathcal{F}_{i-1}],$$

and

$$\bar{\psi}_n(z) = \frac{1}{n\mathbb{E}(\Delta_1(z))}\sum_{i=1}^{n}\mathbb{E}[\Delta_i(z)|\mathcal{F}_{i-1}],$$

the conditional bias which is given by

$$B_n(z,t) = \frac{\bar{\widetilde{\varphi}}_n(z,t)}{\bar{\psi}_n(z)} - \varphi(t|z). \tag{3.5}$$

In addition to quantities:

$$R_n(z,t) = -B_n(z,t)(\psi_n(z) - \bar{\psi}_n(z)),$$

and

$$Q_n(z,t) = (\widetilde{\varphi}_n(z,t) - \bar{\widetilde{\varphi}}_n(z,t)) - \varphi(t|z)(\psi_n(z) - \bar{\psi}_n(z)).$$

Lets's now introduce the following decomposition which is important to prove Proposition 3.4.1. For all $z \in E$, we state

$$\widehat{\varphi}_n(t|z) - \varphi(t|z) = \widehat{\varphi}_n(t|z) - \widetilde{\varphi}_n(t|z) + \widetilde{\varphi}_n(t|z) - \varphi(t|z). \tag{3.6}$$

The proof of this proposition is a direct consequence of the following intermediate results. It suffices to combine Lemmas 3.4.1, 3.4.2 and decomposition (3.6).

Lemma 3.4.1. *Using (H6)-(H7) and (H8), we can show that*

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}} |\widehat{\varphi}_n(t|z) - \widetilde{\varphi}(t|z)| = O_{a.s.}\left(\sqrt{\frac{\log\log n}{n}}\right).$$

Proof. By following the same steps as for the proof of Lemma 5.2 Khardani *et al.* (2010) [10], we can also prove our Lemma.

Lemma 3.4.2. *Because of the conditions (H6)-(H7) and (H8), we have as $n \to \infty$*

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}} |\widetilde{\varphi}_n(t|z) - \varphi(t|z)| = O_{a.s.}(h_K^{\alpha_1} + h_H^{\alpha_2}) + O_{a.s.}\left(\sqrt{\frac{\log n}{nh_H\phi(h_K)}}\right).$$

Proof. Since $\widetilde{\varphi}_n(t|z) - \varphi(t|z) = B_n(z,t) + \dfrac{R_n(z,t) + Q_n(z,t)}{\psi_n(z)}$, then the proof can be achieved by combining Lemmas 3.4.3-3.4.4 and 3.4.6 proposed below.

Lemma 3.4.3. *Suppose that assumptions (H0)-(H6) and (H7) hold true. Then, for any $z \in E$, set*

(i) $\psi_n(z) - \bar{\psi}_n(z) = O_{a.s.}\left(\sqrt{\log n / n\phi(h_K)}\right).$

(ii) $\lim_{n\to\infty} \psi_n(z) = \lim_{n\to\infty} \bar{\psi}_n(z) = 1, \quad a.s.$

Proof. The proof of this Lemma is the same of Lemma 3 and Lemma 5 in Laib and Louani (2011) [12].

Lemma 3.4.4. *Under the hypotheses (H3)-(H6) and (H7) together with (H8), we have as n goes to infinity*

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}} |B_n(z,t)| = O_{a.s.}(h_K^{\alpha_1} + h_H^{\alpha_2}), \tag{3.7}$$

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}} |R_n(z,t)| = O_{a.s.}\left((h_K^{\alpha_1} + h_H^{\alpha_2})\left(\frac{\log n}{n\phi(h_K)}\right)^{1/2}\right). \tag{3.8}$$

Proof. In the beginning, we rewrite the statement (3.5)

$$B_n(z,t) = \frac{\bar{\widetilde{\varphi}}_n(z,t) - \bar{\psi}_n(z)\varphi(t|z)}{\bar{\psi}_n(z)}.$$

If $(H4)$ is verified, and in addition if $\mathbf{1}_{\{T_i \le C_i\}}\chi(X_i) = \mathbf{1}_{\{T_i \le C_i\}}\chi(T_i)$, we obtain

$$\begin{aligned}
\bar{\widetilde{\varphi}}_n(z,t) &= \frac{1}{nh_H\mathbb{E}(\Delta_1(z))}\sum_{i=1}^n \mathbb{E}\{\Delta_i(z)\mathbb{E}[\delta_i\bar{G}^{-1}(X_i)H'(h_H^{-1}(t - X_i))|\mathcal{G}_{i-1}, T_i]|\mathcal{F}_{i-1}\} \\
&= \frac{1}{nh_H\mathbb{E}(\Delta_1(z))}\sum_{i=1}^n \mathbb{E}\{\Delta_i(z)\mathbb{E}[\delta_i\bar{G}^{-1}(X_i)H'(h_H^{-1}(t - X_i))|Z_i, T_i]|\mathcal{F}_{i-1}\} \\
&= \frac{1}{nh_H\mathbb{E}(\Delta_1(z))}\sum_{i=1}^n \mathbb{E}\{\bar{G}^{-1}(T_i)H'(h_H^{-1}(t - T_i))\Delta_i(x)\mathbb{E}[\mathbf{1}_{\{T_i\le C_i\}}|Z_i, T_i]|\mathcal{F}_{i-1}\} \\
&= \frac{1}{nh_H\mathbb{E}(\Delta_1(z))}\sum_{i=1}^n \mathbb{E}\{\Delta_i(z)H'(h_H^{-1}(t - T_i))|\mathcal{F}_{i-1}\}.
\end{aligned}$$

Furthermore, simple calculations by using always a double conditioning with respect to $\mathcal{G}_{i-1}$ leads to

$$\bar{\widetilde{\varphi}}_n(z,t) - \bar{\psi}_n(z)\varphi(t|z) = \frac{1}{nh_H\mathbb{E}(\Delta_1(z))}\sum_{i=1}^n \mathbb{E}\left\{\Delta_i(z)\left[\mathbb{E}\left(H'\left(\frac{(t-T_i)}{h_H}\right)|Z_i\right) - h_H\varphi(t|z)\right]\Bigg|\mathcal{F}_{i-1}\right\}.$$

In view of conditions (H1) and (H3), it follows that

$$|\mathbb{E}(H'(h_H^{-1}(t-T_i)|Z_i) - h_H\varphi(t|z)| \le C_z h_H \int_{\mathbb{R}} H'(u)(h_K^{\alpha_1} + |u|^{\alpha_2} h_H^{\alpha_2})\, du. \tag{3.9}$$

Hence, we get

$$\begin{aligned}
\bar{\widetilde{\varphi}}_n(z,t) - \bar{\psi}_n(z)\varphi(t|z) &= O_{a.s.}(h_K^{\alpha_1} + h_H^{\alpha_2}) \times \frac{1}{n\mathbb{E}(\Delta_1(z))} \sum_{i=1}^{n} \mathbb{E}\{\Delta_i(z)|\mathcal{F}_{i-1}\}. \\
&= O_{a.s.}(h_K^{\alpha_1} + h_H^{\alpha_2}) \times \bar{\psi}_n(z).
\end{aligned}$$

As a last step, we combine the above result with Lemma 3.4.3(ii) to obtain the following:

$$\frac{\bar{\widetilde{\varphi}}_n(z,t) - \bar{\psi}_n(z)\varphi(t|z)}{\bar{\psi}_n(z)} = O_{a.s.}(h_K^{\alpha_1} + h_H^{\alpha_2}).$$

Now, the second part of Lemma 3.4.4 will be easily deduced from the definition of $R_n(z,t)$, together with Lemma 3.4.3 and equation (3.7).

Lemma 3.4.5. *Assume that (H0)-(H4) and (H6)-(H8) are satisfied. Then, for any $z \in E$, set*

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}} |\widetilde{\varphi}_n(z,t) - \bar{\widetilde{\varphi}}_n(z,t)| = O_{a.s.}\left(\left(\frac{\log n}{n h_H \phi(h_K)}\right)^{1/2}\right).$$

Proof. To prove our result we need the decomposition below

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}} |\widetilde{\varphi}_n(z,t) - \bar{\widetilde{\varphi}}_n(z,t)| \le \mathcal{J}_{1,n} + \mathcal{J}_{2,n} + \mathcal{J}_{3,n},$$

where

$$\mathcal{J}_{1,n} = \max_{1\le k\le \gamma_n} \sup_{t\in\mathcal{B}_k} |\widetilde{\varphi}_n(z,t) - \widetilde{\varphi}_n(z,t_k)|, \quad \mathcal{J}_{2,n} = \max_{1\le k\le \gamma_n} |\widetilde{\varphi}_n(z,t_k) - \bar{\widetilde{\varphi}}_n(z,t_k)|,$$

$$\mathcal{J}_{3,n} = \max_{1\le k\le \gamma_n} \sup_{t\in\mathcal{B}_k} |\widetilde{\varphi}_n(z,t_k) - \bar{\widetilde{\varphi}}_n(z,t)|.$$

Indeed, $\mathcal{S}_{\mathbb{R}}$ may be written as: $\mathcal{S}_{\mathbb{R}} \subset \cup_{k=1}^{\gamma_n} \mathcal{B}_k = \cup_{k=1}^{\gamma_n} \mathcal{B}_k(t_k, \Re_n)$, with $t_k (1 \le k \le \gamma_n)$ are the balls centers. Let's now study our three terms.

On the one hand, by a standard analytical argument and by using hypothesis (H3) and the result of Lemma 3.4.3, we can evaluate the first term in the following way

$$\begin{aligned}\mathcal{J}_{1,n} &\leq \frac{1}{nh_H\mathbb{E}(\Delta_1(z))}\max_{1\leq k\leq \gamma_n}\sup_{t\in\mathcal{B}_k}\sum_{i=1}^{n}\left|\delta_i\bar{G}^{-1}(X_i)[H'(h_H^{-1}(t-X_i))-H'(h_H^{-1}(t_k-X_i))]\Delta_i(z)\right|\\ &\leq \frac{C}{nh_H\mathbb{E}(\Delta_1(z))}\max_{1\leq k\leq \gamma_n}\sup_{t\in\mathcal{B}_k}\frac{|t-t_k|}{h_H}\sum_{i=1}^{n}\delta_i\bar{G}^{-1}(X_i)\Delta_i(z)\\ &\leq \frac{\gamma_n}{nh_H^2\mathbb{E}(\Delta_1(z))}\sum_{i=1}^{n}\delta_i\bar{G}^{-1}(X_i)\Delta_i(z),\end{aligned}$$

more precisely, by the fact that $\lim_{n\to\infty} n^{\vartheta}h_H^2=\infty$, we obtain

$$\mathcal{J}_{1,n}\longrightarrow 0 \qquad \text{a.s.} \quad \text{as } n\to\infty.$$

As the first and the third terms can be treated in the same manner, so $\mathcal{J}_{3,n}$ is also negligible almost surely

$$\mathcal{J}_{3,n}\longrightarrow 0 \;\; a.s. \;\; \text{as} \;\; n\to\infty.$$

On the other hand, to examine the rest term, we start by showing that

$$\widetilde{\varphi}_n(z,t_k)-\bar{\widetilde{\varphi}}_n(z,t_k)=\frac{1}{nh_H\mathbb{E}(\Delta_1(z))}\sum_{i=1}^{n}\Psi_{i,n}(z,t_k),$$

where

$\Psi_{i,n}(z,t_k)=\delta_i\bar{G}^{-1}(X_i)H'(h_H^{-1}(t_k-X_i))\Delta_i(z)-\mathbb{E}(\delta_i\bar{G}^{-1}(X_i)H'(h_H^{-1}(t_k-X_i))\Delta_i(z)|\mathcal{F}_{i-1})$,

represents a triangular array of stationary martingale differences with respect to the σ-field $\mathcal{F}_{i-1}$. Based on the proof of Lemma 5 in Laib and Louani (2011) [12] and the assumptions (H0)-(H4) and (H5), the quantity $\mathbb{E}(\Psi_{i,n}^p(z,t_k)|\mathcal{F}_{i-1})$ can be developed as

$$|\mathbb{E}(\Psi_{i,n}^p(z,t)|\mathcal{F}_{i-1})|=p!C^{p-2}[C_2\phi(h_K)f_{i,1}(z)+O_{a.s}(g_{i,z}(h_K))]\leq p!C^{p-2}\phi(h_K)[Mb_i(z)+1],$$

where $C=2\max(1,a_1^2)$ and $M=(C_2C)^2$.

Choosing $D_n=\sum_{i=1}^n d_i^2$ with $d_i^2=\phi(h_K)[Mb_i(z)+1]$. By using hypotheses (H0)(b) and (H0)(e), it yields $n^{-1}D_n=\phi(h_K)[MD(z)+o_{a.s}(1)]$ as $n\to\infty$.

Thus, we apply the exponential inequality given in Lemma 1 in Laib and Louani (2011) [12] with taking $D_n=O_{a.s}(n\phi(h_K))$, $S_n=\sum_{i=1}^n\Psi_{i,n}(z,t)$, and for any $\epsilon_0>0$ and C_1 is a positive constant, the following calculations is valid

$$\begin{aligned}
\mathbb{P}\left(|\mathcal{J}_{2,n}| > \epsilon_0\sqrt{\frac{\log n}{nh_H\phi(h_K)}}\right) &\leq \mathbb{P}\left(\max_{k\in 1\cdots\gamma_n} |\widetilde{\varphi}_n(z,t_k) - \bar{\widetilde{\varphi}}_n(z,t_k)| > \epsilon_0\sqrt{\frac{\log n}{nh_H\phi(h_K)}}\right) \\
&\leq \max_{k\in 1\cdots\gamma_n} \mathbb{P}\left(|\sum_{i=1}^{n}\Psi_{i,n}(z,t_k)| > nh_H\mathbb{E}(\Delta_1(z))\epsilon_0\sqrt{\frac{\log n}{nh_H\phi(h_K)}}\right) \\
&\leq 2\gamma_n \exp\left(\frac{-\left(nh_H\epsilon_0\mathbb{E}(\Delta_1(z))\right)^2\frac{\log n}{nh_H\phi(h_K)}}{2D_n + 2Cnh_H\mathbb{E}(\Delta_1(x))\epsilon_0\sqrt{\frac{\log n}{nh_H\phi(h_K)}}}\right) \\
&\leq 2\gamma_n \exp\{-C_1\epsilon_0^2\log n\} \\
&\leq \frac{2}{n^{C_1\epsilon_0^2}}.
\end{aligned}$$

Lastly, to achieve the proof we need only to take ϵ_0 large enough and to use the Borel-Cantelli Lemma.

Lemma 3.4.6. *By the same hypotheses of Lemma 3.4.5, it yields*

$$\sup_{t\in\mathcal{S}_\mathbb{R}} |Q_n(z,t)| = O_{a.s.}\left(\sqrt{\frac{\log n}{nh_H\phi(h_K}}\right).$$

Proof. Lemmas 3.4.3 and 3.4.5 lead directly to the proof.

Finally, the proof of Proposition 3.4.1 is completed.

Theorem 3.4.1. *Again by (H6)-(H7) and (H8) in conjunction with (H2), we obtain*

$$|\widehat{\theta}(z) - \theta(z)| = O_{a.s.}\left(h_K^{\alpha_1} + h_H^{\alpha_2}\right) + O_{a.s.}\left(\sqrt{\frac{\log n}{nh_H\phi(h_K)}}\right).$$

Proof. The proof of Theorem 3.4.1 can be completed by the following lemma.

Lemma 3.4.7. *Under the assumptions of Proposition 3.4.1, we obtain*

$$\lim_{n\to\infty} |\widehat{\theta}(z) - \theta(z)| = 0, \quad a.s.$$

Proof. By the continuity of the function $f(t|x)$, it follows that

$$\forall\epsilon > 0,\ \exists\zeta(\epsilon) > 0, \quad |\varphi(t|z) - \varphi(\theta(z)|z)| \leq \zeta(\epsilon) \Rightarrow |t - \theta(z)| \leq \epsilon.$$

This allowing us to write

$$\forall\epsilon > 0,\ \exists\zeta(\epsilon) > 0, \quad \mathbb{P}\left(|\widehat{\theta}(z) - \theta(z)| > \epsilon\right) \leq \mathbb{P}\left(|\varphi(\widehat{\theta}(z)|z) - \varphi(\theta(z)|z)| > \zeta(\epsilon)\right). \quad (3.10)$$

Next, by simple algebra, we also have

$$|\varphi(\widehat{\theta}(z)|z) - \varphi(\theta(z)|z)| \leq 2 \sup_{t\in\mathcal{S}_{\mathbb{R}}} |\widehat{\varphi}_n(t|z) - \varphi(t|z)|. \tag{3.11}$$

Lastly, the convergence of $\widehat{\theta}(z)$ to $\theta(z)$ almost surely will be easily deduced from the latter together with (3.10) and Proposition 3.4.1.

• The proof of Theorem 3.4.1 is based on the Taylor expansion of order two of $\varphi(\widehat{\theta}(z)|z)$ at the point $\theta(z)$, on the use of the first part of (H2). Let

$$\varphi(\widehat{\theta}(z)|z) - \varphi(\theta(z)|z) = \frac{1}{2}\varphi^{(2)}(\theta^*(z)|z)(\widehat{\theta}(z) - \theta(z))^2,$$

where $\min(\theta(z), \widehat{\theta}(z)) < \theta^*(z) < \max(\theta(z), \widehat{\theta}(z))$.
Consequently, by considering the last equality with the statement (3.11), we derive

$$|(\widehat{\theta}(z) - \theta(z))|^2 \leq \frac{1}{\varphi^{(2)}(\theta^*(z)|z)} \sup_{t\in\mathcal{S}_{\mathbb{R}}} |\widehat{\varphi}_n(t|z) - \varphi(t|z)|.$$

Now, because of $\varphi^{(2)}(\theta^*(z)|z) \longrightarrow \varphi^{(2)}(\theta(z)|z)$, and on the use of the second part of (H2), we directly obtain

$$|(\widehat{\theta}(z) - \theta(z))|^2 = O_{a.s.}\Big(\sup_{t\in\mathcal{S}_{\mathbb{R}}} |\widehat{\varphi}_n(t|z) - \varphi(t|z)|\Big).$$

Thus, Proposition 3.4.1 allow us to get the claimed result.

3.4.2 Asymptotique normality

The aim of this section is to establish the asymptotic normality which induces a confidence interval of the conditional mode estimator. For this purpose, we shall list some basic conditions

(A0) The smoothing parameter h_H satisfies: $nh_H^3\phi(h_K) \longrightarrow 0$, as $n \to \infty$.

(A1) The distribution function of the censored random variable G has a bounded first derivative $G^{(1)}$.

(A2) The *cdf* $\varphi(t|z)$ verifies the Hölder condition, $\forall(t_1, t_2) \in \mathcal{S}_{\mathbb{R}}^2$, $\forall j = 1, 2,$ for some $\alpha_0 > 0$,

$$|\varphi^{(j)}(t_1|z) - \varphi^{(j)}(t_2|z)| \leq C(|t_1 - t_2|^{\alpha_0}).$$

(A3) The kernel H is twice differentiable such that

$$\int |t|^{\alpha_0}(H^{(j)}(v))^2\,dv < \infty, \text{ for } j = 1, 2, \quad \text{and} \quad \int (H'(v))^2\,dv < \infty.$$

Theorem 3.4.2. *Using the conditions (H0)-(H6)-(H7) and (A1)-(A3), it results*

$$\sqrt{nh_H\phi(h_K)}(\widehat{\varphi}_n(t|z) - \varphi(t|z)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2(z,t)),$$

where

$$\sigma^2(z,t) = \frac{M_2}{M_1^2}\frac{\varphi(t|z)}{\bar{G}(t)f_1(z)}\int_{\mathbb{R}}(H'(v))^2\,dv,$$

with $M_j = K^j(1) - \int_0^1 (K^j)'\varsigma_0(u)\,du$ *for* $j = 1, 2.$

Noting that " $\xrightarrow{\mathcal{D}}$ " symbolyzes the convergence in distribution.

Proof. Initially, we suggest the following decomposition:

$$\begin{aligned}\widehat{\varphi}_n(t|z) - \varphi(t|z) &= [\widehat{\varphi}_n(t|z) - \widetilde{\varphi}_n(t|z)] + [\widetilde{\varphi}_n(t|z) - \bar{\widetilde{\varphi}}_n(t|z)] + [\bar{\widetilde{\varphi}}_n(t|z) - \varphi(t|z)] \\ &= \mathcal{U}_{1,n} + \mathcal{U}_{2,n} + \mathcal{U}_{3,n}.\end{aligned}$$

According to the Lemma 3.4.1, the term $\mathcal{U}_{1,n}$ converges almost surely to zero when n goes to infinity, where

$$\mathcal{U}_{1,n} = O_{a.s.}\left(\sqrt{\frac{\log\log n}{n}}\right). \tag{3.12}$$

Moreover, it is simple to show that $\mathcal{U}_{3,n}$ is also negligible, where we readily get

$$\mathcal{U}_{3,n} = \bar{\widetilde{\varphi}}_n(t|z) - \varphi(t|z) = B_n(z,t).$$

Therefore, from Lemma 3.4.4, we obtain

$$\mathcal{U}_{3,n} = O_{a.s.}(h_K^{\alpha_1} + h_H^{\alpha_2}). \tag{3.13}$$

Now, it suffices to prove the asymptotic normality of $\mathcal{U}_{2,n} = \dfrac{Q_n(z,t) + R_n(z,t))}{\psi_n(z)}$, where $R_n(z,t)$ is negligible as $n \to \infty$, and $\psi_n(z)$ converges almost surely towards 1, where

$$R_n(z,t) = -B_n(z,t)(\psi_n(z) - \bar{\psi}_n(z)),$$

with

$$B_n(z,t) = \frac{\bar{\widetilde{\varphi}}_n(z,t)}{\bar{\psi}_n(z)} - \varphi(t|z).$$

Thus, the asymptotic normality will be proved by the term $Q_n(z,t) = [\widetilde{\varphi}_n(z,t) - \bar{\widetilde{\varphi}}_n(z,t)] - \varphi(t|z)(\psi_n(z) - \bar{\psi}_n(z))$ which is treated by the Lemmas 3.4.8 and 3.4.9 below.

Lemma 3.4.8. *Assume that conditions (H0)(a),(H0)(b) and (H0)(d) as well as (H6) are satisfied. Then, For any real numbers* $1 \le j \le 2+\delta$ *and* $1 \le k \le 2+\delta$ *with* $\delta > 0$, *as* $n \to \infty$, *one has*

(i) $\frac{1}{\phi(h_K)}\mathbb{E}\Big[\Delta_i^j(z)|\mathcal{F}_{i-1}\Big] = M_j f_{i,1}(z) + O_{a.s.}\left(\frac{g_{i,z(h_K)}}{\phi(h_K)}\right).$

(ii) $\frac{1}{\phi(h_K)}\mathbb{E}\Big[\Delta_i^j(z)\Big] = M_j f_1(z) + o(1).$

(iii) $\frac{1}{\phi^k(h_K)}(\mathbb{E}(\Delta_1(z)))^k = M_1^k f_1^k(z) + o(1).$

Proof. The proof is given in Lemma 1 by Laib and Louani (2010) [11].

Lemma 3.4.9. *By the same hypotheses of Theorem 3.4.2, one writes as* $n \to \infty$

$$\sqrt{nh_H\phi(h_K)}Q_n(z,t) \xrightarrow{\mathcal{D}} \mathcal{N}(0,\sigma^2(z,t)).$$

Recall that $\sigma^2(z,t)$ *is defined in Theorem 3.4.2.*

Proof. Easily, we get

$$\sqrt{nh_H\phi(h_K)}Q_n(z,t) = \sum_{i=1}^n \mu_{ni}, \tag{3.14}$$

where

$$\mu_{ni} = \Xi_{ni} - \mathbb{E}[\Xi_{ni}|\mathcal{F}_{i-1}],$$

with

$$\Xi_{ni} = \left(\frac{\phi(h_K)}{nh_H}\right)^{1/2}\left(\frac{\delta_i}{\bar{G}(X_i)}H'(h_H^{-1}(t-X_i) - h_H\varphi(t|z)\right)\frac{\Delta_i(z)}{\mathbb{E}(\Delta_1(z))}.$$

Obviously, based on the central limit theorem for discrete-time arrays of real-valued martingales (see Hall and Heyde (1980) [8]), the asymptotic normality of $Q_n(z,t)$ can be obtained if we demonstrate this two statements:

I. $\sum_{i=1}^n \mathbb{E}[\mu_{ni}^2|\mathcal{F}_{i-1}] \xrightarrow{\mathbb{P}} \sigma^2(z,t).$

II. $n\mathbb{E}[\mu^2\mathbf{1}_{[|\mu_{ni}|>\epsilon]}] = o(1)$ holds for any $\epsilon > 0$ (Linderberg condition).

• **Proof of the first part(I.)**:

Firstly, let us consider

$$\left|\sum_{i=1}^n \mathbb{E}[\Xi_{ni}^2|\mathcal{F}_{i-1}] - \sum_{i=1}^n \mathbb{E}[\mu_{ni}^2|\mathcal{F}_{i-1}]\right| \leq \sum_{i=1}^n(\mathbb{E}[\Xi_{ni}|\mathcal{F}_{i-1}])^2.$$

Applying Lemma 3.4.8 together with inequality (3.9), it yields

$$\begin{aligned}|\mathbb{E}[\Xi_{ni}|\mathcal{F}_{i-1}]| &= \frac{1}{\mathbb{E}(\Delta_1(z))}\left(\frac{\phi(h_K)}{nh_H}\right)^{1/2}\left|\mathbb{E}\left[\Delta_i(z)\left(\frac{\delta_i}{\bar{G}(X_i)}H'(h_H^{-1}(t-X_i)) - h_H\varphi(t|z)\right)|\mathcal{F}_{i-1}\right]\right| \\ &\leq C(h_K^{\alpha_1}+h_H^{\alpha_2})\left(\frac{\phi(h_K)h_H}{n}\right)^{1/2}\left(\frac{f_{i,1}(z)}{f_1(z)} + O_{a.s}\left(\frac{g_{i,z}(h_K)}{\phi(h_K)}\right)\right).\end{aligned}$$

Subsequently, by (H0)(b)-(c), it follows that

$$\sum_{i=1}^{n}(\mathbb{E}[\Xi_{ni}|\mathcal{F}_{i-1}])^2 = O_{a.s.}\Big(h_H\phi(h_K)(h_K^{\alpha_1}+h_H^{\alpha_2})^2\Big).$$

So, we just need to prove the following

$$\lim_{n\to\infty}\sum_{i=1}^{n}\mathbb{E}[\Xi_{ni}^2|\mathcal{F}_{i-1}] \overset{\mathbb{P}}{\longrightarrow} \sigma^2(z,t), \tag{3.15}$$

for this, let using (H4) to get

$$\begin{aligned}
\sum_{i=1}^{n}\mathbb{E}[\Xi_{ni}^2|\mathcal{F}_{i-1}] &= \frac{\phi(h_K)}{nh_H(\mathbb{E}(\Delta_1(z)))^2}\sum_{i=1}^{n}\mathbb{E}\Bigg\{\Delta_i^2(z)\Bigg(\frac{\delta_i}{\bar{G}(X_i)}H'(h_H^{-1}(t-X_i)) - h_H\varphi(t|z)\Bigg)^2|\mathcal{F}_{i-1}\Bigg\}\\
&= \frac{\phi(h_K)}{nh_H(\mathbb{E}(\Delta_1(z)))^2}\sum_{i=1}^{n}\mathbb{E}\Bigg\{\Delta_i^2(z)\mathbb{E}\Bigg[\Bigg(\frac{\delta_i}{\bar{G}(X_i)}H'(h_H^{-1}(t-X_i))\\
&- h_H\varphi(t|z)\Bigg)^2|Z_i\Bigg]|\mathcal{F}_{i-1}\Bigg\}.
\end{aligned}$$

Moreover, set

$$\begin{aligned}
\mathbb{E}\Bigg[\Bigg(\frac{\delta_i}{\bar{G}(X_i)}H'(h_H^{-1}(t-X_i)) - h_H\varphi(t|z)\Bigg)^2|Z_i\Bigg] &= Var\Bigg[\frac{\delta_i}{\bar{G}(X_i)}H'(h_H^{-1}(t-X_i))|Z_i\Bigg]\\
&+\Bigg[\mathbb{E}\Bigg(\frac{\delta_i}{\bar{G}(X_i)}H'(h_H^{-1}(t-X_i))|Z_i\Bigg) - h_H\varphi(t|z)\Bigg]^2\\
&= \Gamma_{1,n}+\Gamma_{2,n}.
\end{aligned}$$

It should be noted that the second term is negligible: $\Gamma_{2,n}\longrightarrow 0, \quad \text{as}\, n\to\infty$, where we used inequality (3.9) and assumptions (H1), (H3) in order to get our result.

Now, all what is left to be study is $\Gamma_{1,n}$, thus we state

$$\Gamma_{1,n} = \underbrace{\mathbb{E}\Bigg[\frac{\delta_i}{\bar{G}^2(X_i)}\Bigg(H'\Bigg(\frac{t-X_i}{h_H}\Bigg)\Bigg)^2\Bigg|Z_i\Bigg]}_{\Lambda_1} - \underbrace{\Bigg[\mathbb{E}\Bigg(\frac{\delta_i}{\bar{G}(X_i)}H'\Bigg(\frac{t-X_i}{h_H}\Bigg)\Bigg|Z_i\Bigg)\Bigg]^2}_{\Lambda_2}. \tag{3.16}$$

- Concerning Λ_1, by simple calculations, we obtain

$$\begin{aligned}
\Lambda_1 &= \mathbb{E}\Big[\mathbb{E}\Big(\frac{\delta_i}{\bar{G}^2(X_i)}H'^2\Big(\frac{t-X_i}{h_H}\Big)\Big|Z_i,T_i\Big)\Big] \\
&= \mathbb{E}\Big(\frac{1}{\bar{G}(T_i)}H'^2\Big(\frac{t-T_i}{h_H}\Big)\Big|Z_i\Big) \\
&= \int_{\mathbb{R}}\frac{1}{\bar{G}(\omega)}H'^2\Big(\frac{t-\omega}{h_H}\Big)f(\omega|Z_i)\,d\omega \\
&= \int_{\mathbb{R}}\frac{1}{\bar{G}(t-vh_H)}H'^2(v)dF(t-vh_H|Z_i).
\end{aligned}$$

Writing a Taylor expansion of order one of the function $\bar{G}^{-1}(\cdot)$ around zero leads to the existence of some t^* between t and $(t-vh_H)$ such that

$$\begin{aligned}
\Lambda_1 &= \int_{\mathbb{R}}\frac{1}{\bar{G}(t)}(H'(v))^2dF(t-vh_H|Z_i)+\frac{h_H^2}{\bar{G}^2(t)}\int_{\mathbb{R}}v(H'(v))^2\bar{G}^{(1)}(t^*)\varphi(t-vh_H|Z_i)\,dv+o(1) \\
&= \lambda_1+\lambda_2.
\end{aligned}$$

If the hypotheses (H1),(A3) are verified, one has

$$\begin{aligned}
\lambda_1 &= h_H\int_{\mathbb{R}}\frac{1}{\bar{G}(t)}(H'(v))^2\varphi(t-vh_H|Z_i)\,dv \\
&\leq \frac{h_H}{\bar{G}(t)}\int_{\mathbb{R}}(H'(v))^2(\varphi(t-vh_H|Z_i)-\varphi(t|z))\,dv \\
&\quad +\frac{h_H}{\bar{G}(t)}\int_{\mathbb{R}}(H'(v))^2\varphi(t|z)\,dv \\
&\leq \frac{h_H}{\bar{G}(t)}\Big(C_z\int_{\mathbb{R}}(H'(v))^2(h_K^{\alpha_1}+|v|^{\alpha_2}h_H^{\alpha_2})\,dv+\varphi(t|z)\int_{\mathbb{R}}(H'(v))^2\,dv\Big) \\
&= O\Big(h_K^{\alpha_1}+h_H^{\alpha_2}\Big)+\frac{h_H}{\bar{G}(t)}\varphi(t|z)\int_{\mathbb{R}}(H'(v))^2\,dv.
\end{aligned}$$

On the other hand, by (A1), one can write

$$\lambda_2\leq h_H^2(\sup_{v\in\mathbb{R}}|\bar{G}^{(1)}(v)|/\bar{G}^2(t))\int_{\mathbb{R}}v\varphi(t-vh_H|Z_i)\,dv.$$

This means that as $n\to\infty$, $\lambda_2=O(h_H^2)$.

- For the second term of (3.16), it suffices to evaluate its square root

$$\begin{aligned}\Lambda_2' &= \mathbb{E}\left(\frac{\delta_i}{\bar{G}(X_i)} H'\left(\frac{t-X_i}{h_H}\right)\Big| Z_i\right)\\ &= \mathbb{E}\left(H'\left(\frac{t-T_i}{h_H}\right)\Big| Z_i\right)\\ &= \int_{\mathbb{R}} H'\left(\frac{t-\omega}{h_H}\right) f(\omega|Z_i)\, d\omega.\end{aligned}$$

By changing variables, we arrive at
$\Lambda_2' = h_H \int_{\mathbb{R}} H'(v)(\varphi(t - vh_H|Z_i) - \varphi(t|z))\, dv + h_H\varphi(t|z) \int_{\mathbb{R}} H'(v)\, dv.$
So, under (H1) and (H3) we would have

$$\Lambda_2' = O\left(h_K^{\alpha_1} + h_H^{\alpha_2}\right) + h_H\varphi(t|z),$$

which permit us to conclude that Λ_2 is negligible. By Lemma 3.4.8, all of the above results leads to

$$\begin{aligned}\frac{\phi(h_K)}{nh_H(\mathbb{E}(\Delta_1(z)))^2}\sum_{i=1}^n \mathbb{E}\{\Delta_i^2(z)\Gamma_{1,n}|\mathcal{F}_{i-1}\} &= \frac{h_H}{\bar{G}(t)}\varphi(t|z)\int_{\mathbb{R}}(H'(v))^2\, dv\\ &\times\frac{\phi(h_K)}{nh_H(\mathbb{E}(\Delta_1(z)))^2}\sum_{i=1}^n \mathbb{E}(\Delta_i^2(z)|\mathcal{F}_{i-1}),\\ &\longrightarrow \frac{M_2}{M_1^2}\frac{\varphi(t|z)}{\bar{G}(t)f_1(z)}\int_{\mathbb{R}}(H'(v))^2\, dv.\end{aligned}$$

Lastly, we could establish that

$$\sum_{i=1}^n \mathbb{E}[\Xi_{ni}^2|\mathcal{F}_{i-1}] = \frac{M_2}{M_1^2}\frac{\varphi(t|z)}{\bar{G}(t)f_1(z)}\int_{\mathbb{R}}(H'(v))^2\, dv = \sigma^2(z,t),$$

which is enough to confirm part (I).

- **Proof of the second part(II.)**:

The definition of μ_{ni} allows us to write: $n\mathbb{E}[\mu_{ni}^2 \mathbf{1}_{[|\mu_{ni}|>\epsilon]}] \leq 4n\mathbb{E}[\Xi_{ni}^2 \mathbf{1}_{[|\Xi_{ni}|>\epsilon/2]}]$.

Denote: $A > 1$ and $B > 1$ such that $1/A + 1/A = 1$. According to Hölder and Markov inequalities, we have for any $\epsilon > 0$

$$\mathbb{E}[\Xi_{ni}^2 \mathbf{1}_{[|\Xi_{ni}|>\epsilon/2]}] \leq \frac{\mathbb{E}|\Xi_{ni}|^{2A}}{(\epsilon/2)^{2A/B}}.$$

Choosing C_0 a positive constant and $2A = 2 + \delta$ for all $\delta > 0$, it follows that

$$
\begin{aligned}
4n\mathbb{E}[\Xi_{ni}^2\mathbf{1}_{[|\Xi_{ni}|>\epsilon/2]}] &\leq C_0\left(\frac{\phi(h_K)}{nh_H}\right)^{(2+\delta)/2}\frac{n}{(\mathbb{E}(\Delta_1(z)))^{2+\delta}} \\
&\quad\times\mathbb{E}\left(\left[\Delta_i(z)\left|\frac{\delta_i}{\bar{G}(X_i)}H'(h_H^{-1}(t-X_i)-h_H\varphi(t|z)\right|\right]^{2+\delta}\right) \\
&\leq C_0\left(\frac{\phi(h_K)}{nh_H}\right)^{(2+\delta)/2}\frac{n}{(\mathbb{E}(\Delta_1(z)))^{2+\delta}}\mathbb{E}((\Delta_i(z))^{2+\delta} \\
&\quad\times\mathbb{E}\left[\left|H'(h_H^{-1}(t-T_i)-h_H\varphi(t|z)\right|^{2+\delta}\middle|Z_i\right].
\end{aligned}
$$

Meanwhile,

$$
\begin{aligned}
\mathbb{E}\left[\left|H'(h_H^{-1}(t-T_i)-h_H\varphi(t|z)\right|^{2+\delta}\middle|Z_i\right] &= \int_{\mathbb{R}}\left(H'\left(\frac{t-\omega}{h_H}\right)-h_H\varphi(t|z)\right)^{2+\delta}\varphi(\omega|Z_i)\,d\omega \\
&\leq C\int_{\mathbb{R}}H'^{2+\delta}\left(\frac{t-\omega}{h_H}\right)\varphi(\omega|Z_i)\,d\omega+h_H^{2+\delta}\varphi^{2+\delta}(t|z) \\
&= Ch_H\int_{\mathbb{R}}H'^{2+\delta}(v)\varphi(t-vh_H|Z_i)\,dv+h_H^{2+\delta}\varphi^{2+\delta}(t|z) \\
&= h_H\left[\int_{\mathbb{R}}H'^{2+\delta}(v)\varphi(t-vh_H|Z_i)\,dv+h_H^{1+\delta}\varphi^{2+\delta}(t|z)\right],
\end{aligned}
$$

which implies that

$$
\begin{aligned}
4n\mathbb{E}[\Xi_{ni}^2\mathbf{1}_{[|\Xi_{ni}|>\epsilon/2]}] &\leq C_0\left(\frac{\phi(h_K)}{nh_H}\right)^{(2+\delta)/2}\frac{nh_H}{\mathbb{E}(\Delta_1(z)))^{2+\delta}} \\
&\quad\times\mathbb{E}\left((\Delta_i(z))^{2+\delta}\left[\int_{\mathbb{R}}\left(H'^{2+\delta}(v)\varphi(t-vh_H|z)\,dv+h_H^{1+\delta}\varphi^{2+\delta}(t|z)\right]\right) \\
&\leq C_0\left(\frac{\phi(h_K)}{nh_H}\right)^{(2+\delta)/2}\frac{nh_H\mathbb{E}[(\Delta_i(z))^{2+\delta}]}{(\mathbb{E}(\Delta_1(z)))^{2+\delta}}.
\end{aligned}
$$

Making use of Lemma 3.4.8, then

$$
\begin{aligned}
4n\mathbb{E}[\Xi_{ni}^2\mathbf{1}_{[|\Xi_{ni}|>\epsilon/2]}] &\leq C_0(nh_H\phi(h_K))^{-\delta/2}\frac{M_{2+\delta}f_1(z)+o(1)}{M_1^{2+\delta}f_1^{2+\delta}(z)+o(1)} \\
&= O((nh_H\phi(h_K))^{-\delta/2}).
\end{aligned}
$$

Ultimately, the proof of the second part is completed. Thus, Lemma 3.4.9 is proved.

From that, the Theorem 3.4.2 is valid by combining equations (3.12), (3.13) and Lemma 3.4.9.

Theorem 3.4.3. *If the hypotheses (A0)-(A1)-(A3) as well as (H0)-(H2)-(H6) are satisfied, then we have*

$$\sqrt{\frac{nh_H^3\phi(h_K)}{\varrho^2(z,\theta(z))}}(\widehat{\theta}(z)-\theta(z)) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1),$$

where

$$\varrho^2(z,\theta(z)) = \frac{M_2}{M_1^2}\frac{\varphi(\theta(z)|z)}{\bar{G}(t)f_1(z)(\varphi^{(2)}(\theta(z)|z))^2}\int_{\mathbb{R}}(H^{(2)}(v))^2\,dv.$$

Proof. By the first order Taylor expansion of $\widehat{\varphi}_n^{(1)}(\cdot|z)$ in the neighborhood of $\widehat{\theta}(z)$, and since $\widehat{\varphi}_n^{(1)}(\widehat{\theta}(z)|z) = 0$, one has

$$\sqrt{nh^3\phi(h_K)}|\widehat{\theta}(z)-\theta(z)| = \frac{-\sqrt{nh^3\phi(h_K)}\widehat{\varphi}_n^{(1)}(\theta(z)|z)}{\widehat{\varphi}_n^{(2)}(\theta^*(z)|z)},$$

where $\theta^*(z)$ is between $\theta(z)$ and $\widehat{\theta}(z)$.

In the verity, the proof of the statement below is analogous to that of Theorem 3.4.2. Let's

$$-\sqrt{nh^3\phi(h_K)}\widehat{\varphi}_n^{(1)}(\theta(z)|z) \xrightarrow{\mathcal{D}} \mathcal{N}(0,\varrho_1^2(z,\theta(z)),$$

with $\varrho_1^2(z,\theta(z)) = \dfrac{M_2}{M_1^2}\dfrac{\varphi(\theta(z)|z)}{\bar{G}(t)f_1(z)}\displaystyle\int_{\mathbb{R}}(H^{(2)}(v))^2\,dv.$

Then, proceeding as in Ferraty and Vieu (2006) [7], where $\widehat{\varphi}_n^{(2)}(\theta(z)|z) \longrightarrow \varphi^{(2)}(\theta(z)|z)$ as $n \to \infty$, and the fact that $\theta^*(z)$ is lying between $\theta(z)$ and $\widehat{\theta}(z)$, which gives

$$\widehat{\varphi}_n^{(2)}(\theta^*(z)|z) \longrightarrow \varphi^{(2)}(\theta(z)|z), \quad \text{as}\, n \to \infty.$$

3.4.3 Application and Confidence bands

Observe that, both the asymptotic variance $\sigma^2(z,t)$ and $\varrho^2(z,\theta(z))$ are not useful in practice since some of its related quantities $(\varphi(\cdot|z), \varphi^{(2)}(\cdot|z), \theta(z), \bar{G}(\cdot), M_j$ for $j=1,2)$ and functions $(\phi(h_K), f_1(z))$ are unknown. To overcome this difficulty and to make it usable, we have to estimate it.

Hence, $\varphi(\cdot|z)$, $\varphi^{(2)}(\cdot|z)$, $\theta(z)$ and $\bar{G}(\cdot)$ must be changed respectively by the conditional density estimators $\widehat{\varphi}_n(\cdot|z)$ and $\widehat{\varphi}_n^{(2)}(\cdot|z)$, the conditional mode estimator $\widehat{\theta}(z)$ and the Kaplan-Meier's estimator $\bar{G}_n(\cdot)$. Furthermore, under the conditions (H0)-(a) and (H0)-(d), $\varsigma_0(\cdot)$ can be estimated by

$$\varsigma_n(\cdot) = \frac{F_{z,n}(uh)}{F_{z,n}(h)},$$

where

$$F_{z,n}(u)=\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{\{d(z,Z_i)\le u\}}.$$

Finally, since ς_0 is replaced with ς_n, so we can directly estimate M_1 and M_2 by $M_{1,n}$ and $M_{2,n}$,respectively.

Now, we can simply obtain a confidence interval in practice since all quantities are known. For this purpose, let us introduce the following corollaries.

Corollary 3.4.1. *By the same assumptions of Theorem 3.4.2, one gets*

$$\sqrt{\frac{nh_H F_{z,n}(h_K)}{\widehat{\sigma}^2(z,t)}}(\widehat{\varphi}_n(t|z)-\varphi(t|z))\xrightarrow{\mathcal{D}}\mathcal{N}(0,1), \tag{3.17}$$

where

$$\widehat{\sigma}^2(z,t)=\frac{M_{2,n}}{M_{1,n}^2}\frac{\widehat{\varphi}_n(t|z)}{\bar{G}_n(t)}\int_{\mathbb{R}}(H'(v))^2\,dv.$$

Corollary 3.4.2. *By the same assumptions of Theorem 3.4.3, one gets*

$$\sqrt{nh_H^3 F_{z,n}(h_K)}(\widehat{\theta}(z)-\theta(z))\xrightarrow{\mathcal{D}}\mathcal{N}(0,\widehat{\varrho}^2(z,\widehat{\theta}(z))), \tag{3.18}$$

where

$$\widehat{\varrho}^2(z,\widehat{\theta}(z))=\frac{M_{2,n}}{M_{1,n}^2}\frac{\widehat{\varphi}_n(\widehat{\theta}(z)|z)}{\bar{G}_n(t)(\widehat{\varphi}_n^{(2)}(\widehat{\theta}(z)|z))^2}\int_{\mathbb{R}}(H^{(2)}(v))^2\,dv.$$

Proof. Note that

$$\sqrt{\frac{nh_H^3 F_{z,n}(h_K)}{\widehat{\varrho}^2(z,\widehat{\theta}(x))}}(\widehat{\theta}(z)-\theta(z)) = \frac{M_{1,n}}{M_1}\frac{\sqrt{M_2}}{\sqrt{M_{2,n}}}\frac{[\widehat{\varphi}_n^{(2)}(\widehat{\theta}(z)|z)]}{[\varphi^{(2)}(\theta(z)|z)]}\sqrt{\frac{F_{z,n}(h_K)\bar{G}_n(t)\varphi(\theta(z)|z)}{\phi(h_K)\bar{G}(t)\widehat{\varphi}_n(\widehat{\theta}(z)|z)f_1(z)}}$$
$$\times\sqrt{\frac{nh_H^3\phi(h_K)}{\varrho^2(z,\theta(z))}}(\widehat{\theta}(z)-\theta(z)).$$

By Theorem 3.4.3, it follows that

$$\sqrt{\frac{nh_H^3\phi(h_K)}{\varrho^2(z,\theta(z))}}(\widehat{\theta}(z)-\theta(z))\xrightarrow{\mathcal{D}}\mathcal{N}(0,1).$$

Making use of results given by Laib and Louani (2010) [11], we obtain $M_{1,n}\xrightarrow{\mathbb{P}} M_1$, $M_{2,n}\xrightarrow{\mathbb{P}} M_2$, $F_{z,n}(h_K)/\phi(h_K)f_1(z)\xrightarrow{\mathbb{P}} 1\,\text{as}\,n\to\infty$. On the other hand, we have $\bar{G}_n\longrightarrow\bar{G}$, according to Deheuvels and Einmahl (2000) [5]. In addition, we have $\widehat{\varphi}_n^{(2)}(\widehat{\theta}(z)|z)\longrightarrow\varphi^{(2)}(\theta(z)|z)$.

Finally, in conjunction with Lemma 3.4.7 and Proposition 3.4.1, one writes

$$\frac{M_{1,n}}{M_1}\frac{\sqrt{M_2}}{\sqrt{M_{2,n}}}\frac{[\widehat{\varphi}_n^{(2)}(\widehat{\theta}(z)|z)]}{[\varphi^{(2)}(\theta(z)|z)]}\sqrt{\frac{F_{z,n}(h_K)\bar{G}_n(t)\varphi(\theta(z)|z)}{\phi(h_K)\bar{G}(t)\widehat{\varphi}_n(\widehat{\theta}(z)|z)f_1(z)}} \xrightarrow{\mathbb{P}} 1, \quad \text{as}\, n \to \infty.$$

This yields the proof.

- From Corollaries 3.4.1 and 3.4.2, it is possible to construct confidence bands. Exactly, we can obtain for each fixed $\eta \in (0,1)$ approximate $(1-\eta)\%$ confidence intervals for the conditional density and conditional mode, namely

$$\left[\widehat{\varphi}_n(t|z) - \frac{I_{\eta/2}\widehat{\sigma}(z,t)}{\sqrt{nh_H F_{z,n}(h_K)}}, \widehat{\varphi}_n(t|z) + \frac{I_{\eta/2}\widehat{\sigma}(z,t)}{\sqrt{nh_H F_{z,n}(h_K)}}\right],$$

and

$$\left[\widehat{\theta}(z) - \frac{I_{\eta/2}\widehat{\varrho}(z,\widehat{\theta}(z))}{\sqrt{nh_H^3 F_{z,n}(h_K)}}, \widehat{\theta}(z) + \frac{I_{\eta/2}\widehat{\varrho}(z,\widehat{\theta}(z))}{\sqrt{nh_H^3 F_{z,n}(h_K)}}\right],$$

where $I_{\eta/2}$ denotes the $\eta/2$ quantile of the standard normal distribution.

3.5 Simulation study

This section is proposed to illustrate our study for the conditional mode and to evaluate the effectiveness of the suggested estimator (i.e. in the censored nonparametric functional data analysis case) (CNPFDA) (3.3) in comparison with the one for complete data (NPFDA) (3.2).

First of all, let's note that all the routines for functional data used in this application (developed in R/S-Plus software) are available on the web site: https://www.math.univ-toulouse.fr/staph/npfda/

Now, we start by introducing The following stationary ergodic process defined on $[0, \pi/3]$, where the covariates are curves

$$Z_i(t) = -1 - \cos(2W_i(t-\pi/3)), \quad i = 1, \cdots, 200;\ t \in [0, \pi/3], \tag{3.19}$$

where W_i is generated by the model constructed as: $W_i = \dfrac{1}{\sqrt{2}}W_{i-1} + \zeta_i$, with ζ_i are i.i.d uniformly distributed on $(0,1)$ and W_i is also simulated independently by $W_0 \sim \mathcal{U}(0,1)$. For more clarification, some of these curves (200 samples) are simulated, and the corresponding graph is presented in Figure 3.1 below.

The scalar response variable is defined by the following regression relation $T_i = r(Z_i) + \epsilon_i$, where $r(Z_i) = \left(\int_0^1 Z_i'(t)\,dt\right)^2$ and $\epsilon \sim \mathcal{N}(0, 0.075)$. Then, n i.i.d random variables C_i, $i = 1, \ldots, n$ are drawn from an exponential distribution $\varepsilon(1.5)$.

Recall that the calculations of our estimator (for the incomplete data) are linked to the observed triplets $(Z_i, X_i, \delta_i)_{i=1,\ldots,n}$, where $X_i = \min(T_i, C_i)$ and $\delta_i = \mathbf{1}_{\{T_i \le C_i\}}$ denotes the censorship indicator.

Concerning the other parameters of our study: The regularity of the curves Z_i leads directly to choose the semi metric in E

$$d(z_i, z_j) = \sqrt{\int_0^{\pi/3} (z_i'(t) - z_j'(t))^2 \, dt} \quad z_i, z_j \in E.$$

For the kernels $K(\cdot)$ and $H(\cdot)$ were chosen to be of quadratic type as

$$K(u) = \frac{3}{2}(1-u^2)\mathbf{1}_{(0,1)}(u), \;\; H(u) = \int_{-\infty}^{u} \frac{3}{4}(1-y^2)\mathbf{1}_{(-1,1)}(y)\, dy,$$

respectively.

Then, the smoothing parameter $h_H \sim h_K =: h$ is obtained by the cross-validation method on the k-nearest neighbours (Ferraty and Vieu (2006) [7]).

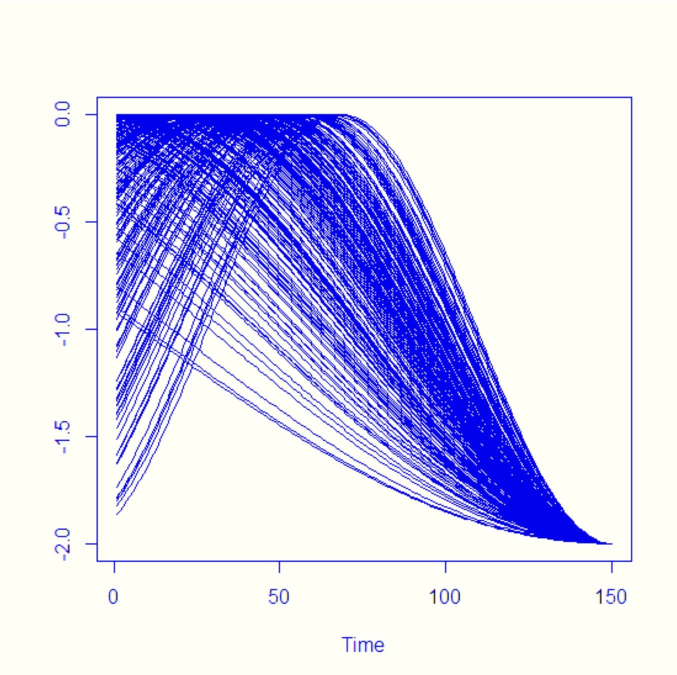

Figure 3.1: A sample of curves $\{Z_i(t), t \in [0, \pi/3]\}_{i=1,\cdots,200}$

In our experience, we consider a sample of 200 observations distributed on two parts A and B : The first one is a learning subsample $(Z_i, X_i)_{i\in A}$ with size$(A) = 150$, and the other is a testing subsample $(Z_j, X_j)_{j\in B}$ with size$(B) = 50$. We also compute the estimators $\widetilde{X}_j = \widetilde{\theta}(Z_j)$ and $\widehat{X}_j = \widehat{\theta}(Z_j)$ $j = \{151, \ldots, 200\}$ for complete data and censored data, respectively through the learning sample. To evaluate the performance of both estimators (3.2) and (3.3), we propose the following mean square errors(MSE):

✠ **Under the complete data case:** $NPFDA.MSE = \frac{1}{50}\sum_{j=151}^{200}(X_j - \widetilde{X}_j)^2.$

✠ **Under the censored data case:** $CNPFDA.MSE = \frac{1}{50}\sum_{j=151}^{200}(X_j - \widehat{X}_j)^2.$

In order to simplifying the obtained results, Figure 3.2 and Figure 3.3 plot the predicted values as functions of the true ones for the MSE under the complete data and censored data cases, respectively.

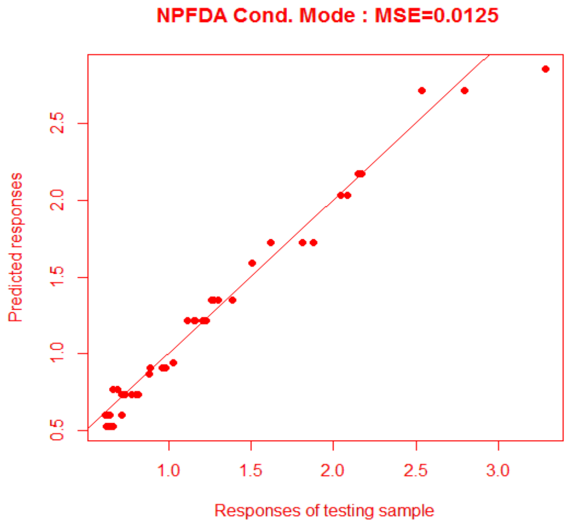

Figure 3.2: Prediction via the conditional mode for complete data case

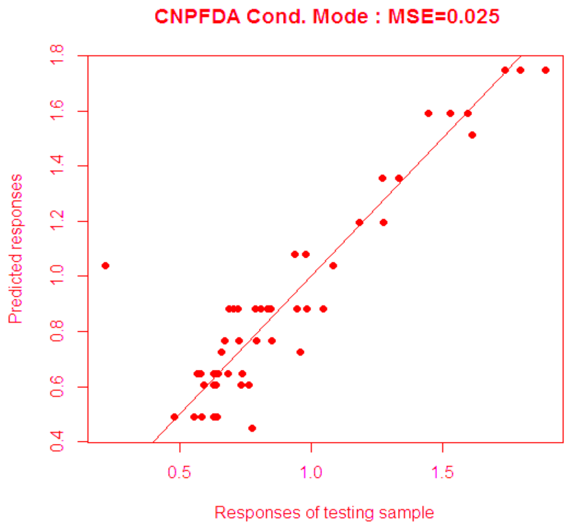

Figure 3.3: Prediction via the conditional mode for censored data case (CR$\sim$ 3%)

The performance of the conditional mode estimator $\widehat{\theta}(z)$ is evaluated on $N = 300$ replications using different sample sizes. The mean square error (MSE) is considered here, such that, for a fixed z. It can be observed that the proposed estimator performs well, especially when the sample size increases. This conclusion is confirmed by Table 3.1 which provides a numerical summary of the distribution of the MSE, with different Censored Rates (CR).

Table 3.1: MSE under the case of censored data

size(n)	CR%	MSE(for CNPFDA)	size(n)	CR%	MSE(for CNPFDA)
	6%	0.0443		6%	0.0401
200	17%	0.1005	300	17%	0.0941
	25%	0.1330		25%	0.1306
	50%	0.2712		50%	0.2487

3.6 Conclusion

This paper focused on nonparametric estimation of conditional mode for dependant stationary ergodic data under random censorship and defined as an argument of the maximum of the conditional density. The resulting estimator has been shown to be asymptotically normally distributed under some regularity conditions. The main implication is to obtain the confidence bands which have been given in Subsection 3.4.3. Of course, we use the plug-in rules to obtain an estimator of the asymptotic variance term.

Our prime aim was to improve the performance of this model for the conditional mode with censored response variable under the ergodic property. The simulations experiments in this paper show that our methodology can be easily implemented and work very well. It is well known that the kernel choice do not affect substantially the quality of the estimator.

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Chapter 4

Missing at random in nonparametric regression for functional stationary ergodic data in the functional index model

This chapter is the subject of an article published in ***MathLAB Journal***.

Missing at random in nonparametric regression for functional stationary ergodic data in the functional index model

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Abstract: The main objective of this paper is to estimate non-parametrically the estimator for the regression function operator when the observations are linked with a single-index model. The functional stationary ergodic data with missing at random (MAR) are considered. In particular, we construct the kernel type estimator of the regression operator, some asymptotic properties such as the convergence in probability as well as the asymptotic normality of the estimator are established under some mild conditions, respectively. As an application, the asymptotic $(1-\zeta)$ confidence interval of the regression operator is also presented for $0 < \zeta < 1$.

Keywords: Convergence in probability, Ergodic processes, Functional data analysis, Kernel estimator, Nonparametric estimation, Missing at random, Regression operator.

2010 Mathematics Subject Classification: 62G05, 62G99, 62M10.

4.1 Introduction

The focal point of this article is to study a nonparametric regression model in the case where the variable of interest Y (called response variable) is a scalar response variable and the explanatory variable X is of functional nature.
Let (X, Y) be $\mathcal{H} \times \mathbb{R}$-valued random elements, where $\mathcal{H}$ is a separable real Hilbert space with the norm $\|\cdot\|$ generated by an inner product $< \cdot, \cdot >$. Moreover, we consider $d_\theta(\cdot,\cdot)$ a semi-metric associated with the single index $\theta \in \mathcal{H}$ defined by $d_\theta(x_1, x_2) :=$

$| < x_1 - x_2, \theta > |$, for x_1 and x_2 in $\mathcal{H}$. Suppose now that $(X_i, Y_i)_{1 \leq i \leq n}$ is a sequence of stationary and ergodic functional samples.

Let us consider the following functional nonparametric regression model:

$$Y = r(\theta, X) + \varepsilon, \tag{4.1}$$

where $r(\theta, .)$ is an unknown smooth functional regression operator from $\mathcal{H}$ to $\mathbb{R}$, and ε is the random error with $\mathbb{E}(\varepsilon) = 0$ and $0 < Var(\varepsilon) < \infty$.

Compared with the classical nonparametric regression framework that the explanatory variable is a real or finite dimensional case, model (4.1) where the explanatory variables X are often curves or surfaces is widely applied in many fields such as in medicine, economics, environmetrics, chemometrics and others, The reason is that the data we observed or collected in these fields are exceptionally high-dimensional or even functional.

Let's not that the nonparametric regression for functional modelization was widely studied by Ferraty and Vieu ((2000) [12], (2002) [13], (2003) [14], (2004) [12]) and Ferraty *et al.* (2006) [9], and the references therein, in the case that the samples are observed completely.

However, in many practical works such as sampling survey, pharmaceutical tracing test and reliability test and so on, some pairs of observations may be incomplete, which is often called the case of missing data. Many examples of missing data and its statistical inferences for regression model can be found in statistical literature when explanatory variables are of finite dimensionality (Cheng (1994) [4], Little and Rubin (2002) [23], Nittner (2003) [24], Tsiatis (2006) [26], Liang *et al.* (2007) [22], Efromovich ((2011a) [7] , (2011b) [8]) and references therein for details. When explanatory variables are in the case of infinite dimensionality or it is of functional nature, only very few literature was reported to investigate the statistical properties of functional nonparametric regression model for missing data.

Recently, Ferraty *et al.* (2013) [11] first proposed to estimate the mean of a scalar response based on an i.i.d. functional sample in which explanatory variables are observed for every subject, while the response variables are missing at random by happenstance for some of them. It generalized the results in Cheng (1994) [4] to the case where the explanatory variables are of functional nature.

The single-index models are becoming increasingly popular because of their importance in several areas of science such as econometrics, biostatistics, medicine, financial econometric and so on. The single-index model, a special case of projection pursuit regression, has proven to be a very efficient way of coping with the high dimensional problem in nonparametric regression. Such kind of modelization is intensively studied in the multivariate case. We quote for example Härdle *et al.* (1993) [18], Hristache *et al.* (2001) [19]. Delecroix *et al.* (2003) [5] have studied the estimation for the single-index approach of the regression function and established some asymptotic properties. The first work in the fixed functional single-model was given by Ferraty *et al.* (2003) [10], where the authors obtained almost complete convergence (with the rate) of the regression function in the i.i.d. case. Their results have been extended to dependent case by Ait-Saïdi *et al.* (2005) [1]. Ait-Saïdi *et al.* (2008) [2] studied the case where the functional single-index is unknown. The authors have proposed for this parameter an estimator based on the cross-validation procedure.

The goal of this paper is to establish a nonparametric estimation on functional regression model (4.1). At first, an estimator of the regression operator in the functional single index model of a scalar response and the functional covariate which are assumed to be sampled from a stationary and ergodic process is constructed. Meanwhile, the response variables are MAR. Then, the asymptotic properties of the estimator are obtained under some mild conditions. To the best of our knowledge, the estimation of the nonparametric regression operator in the functional single index structure combining missing data and stationary ergodic processes has not been studied in the statistical literature.

4.2 The model and the estimates

4.2.1 The functional nonparametric framework

The estimators

In the complete data case, the kernel estimator $\widetilde{r}_n(\theta, x)$ of $r(\theta, x)$ is presented as follows:

$$\widetilde{r}_n(\theta, x) = \frac{\sum_{i=1}^{n} Y_i K\left(h^{-1}(< x - X_i, \theta >)\right)}{\sum_{i=1}^{n} K\left(h^{-1}(< x - X_i, \theta >)\right)}, \tag{4.2}$$

where K is a kernel function, $h = h_n$ is a sequence of positive real numbers.
Meanwhile, in incomplete case with missing at random for the response variable, we observe $(X_i, Y_i, \delta_i)_{1 \leq i \leq n}$ where X_i is observed completely, and $\delta_i = 1$ if Y_i is observed, and $\delta_i = 0$ otherwise. We define the Bernoulli random variable δ by

$$\mathbb{P}(\delta = 1| < \theta, X >=< \theta, x >, Y = y) = \mathbb{P}(\delta = 1| < \theta, X >=< \theta, x >) = p(\theta, x),$$

where $p(\theta, x)$ is a functional operator which is conditionally only on (θ, X).
Therefore, the estimator of $r(\theta, x)$ in the single index model with response MAR is given by

$$\widehat{r}_n(\theta, x) = \frac{\sum_{i=1}^{n} \delta_i Y_i K\left(h^{-1}(< x - X_i, \theta >)\right)}{\sum_{i=1}^{n} \delta_i K\left(h^{-1}(< x - X_i, \theta >)\right)} = \frac{\widehat{r}_{n,2}(\theta, x)}{\widehat{r}_{n,1}(\theta, x)}, \tag{4.3}$$

where

$$\widehat{r}_{n,j}(\theta, x) = \frac{1}{n\,\mathbb{E}(K_1(\theta, x))} \sum_{i=1}^{n} \delta_i Y_i^{j-1} K_i(\theta, x), \quad j = 1, 2, \tag{4.4}$$

with

$$K_i(\theta,x) = K(h^{-1}(< x - X_i, \theta >)).$$

Some notations and assumptions

For $1 \leq i \leq n$, let $\mathcal{F}_i$ and g_i be the σ-fields generated by $((< \theta, X_1 >, Y_1), \cdots, (< \theta, X_i >, Y_i))$ and $((< \theta, X_1 >, Y_1), \cdots, (< \theta, X_i >, Y_i), (< \theta, X_{i+1} >))$, respectively. Let, $B_\theta(x,h) = \{f \in \mathcal{H} : 0 < | < x - f, \theta > | < h\}$ the ball of center x and radius h. Define $F_{x,\theta}(u) = \mathbb{P}(< x - X_i, \theta > \leq u) = \mathbb{P}(X_i \in B_\theta(x,u))$ the distribution function and $F_{x,\theta}^{\mathcal{F}_{i-1}}(u) = \mathbb{P}(< x - X_i, \theta > \leq u | \mathcal{F}_{i-1}) = \mathbb{P}(X_i \in B_\theta(x,u)|\mathcal{F}_{i-1})$ the conditional distribution function given the σ-field $\mathcal{F}_{i-1}$.
Let

$$\overline{r}_{n,j}(\theta,x) = \frac{1}{n\,\mathbb{E}(K_1(\theta,x))} \sum_{i=1}^{n} \mathbb{E}(\delta_i Y_i^{j-1} K_i(\theta,x)/\mathcal{F}_{i-1}), \;\; j = 1,2, \tag{4.5}$$

$$C_n(\theta,x) = \frac{\overline{r}_{n,2}(\theta,x)}{\overline{r}_{n,1}(\theta,x)}, \tag{4.6}$$

and

$$B_n(\theta,x) = C_n(\theta,x) - r_n(\theta,x). \tag{4.7}$$

Then,

$$\widehat{r}_n(\theta,x) - C_n(\theta,x) = \frac{Q_n(\theta,x) + R_n(\theta,x)}{\widehat{r}_{n,1}(\theta,x)}, \tag{4.8}$$

where

$$Q_n(\theta,x) = (\widehat{r}_{n,2}(\theta,x) - \bar{r}_{n,2}(\theta,x)) - r(\theta,x)(\widehat{r}_{n,1}(\theta,x) - \bar{r}_{n,1}(\theta,x)), \tag{4.9}$$

and

$$R_n(\theta,x) = -B_n(\theta,x)(\widehat{r}_{n,1}(\theta,x) - \bar{r}_{n,1}(\theta,x)). \tag{4.10}$$

Our results are stated under some mild assumptions we gather below for easy references. Throughout the paper, when no confusion will be possible, we will denote by C, C_0 some positive generic constants whose values are allowed to change.

(A1) Assumptions on the kernel function K

K is a nonnegative bounded kernel function with support $[0,1]$, and the derivative K' exists on $[0,1]$ with $K'(t) < 0$ for all $t \in [0,1]$ and $|\int_0^1 (K^j)'(t)dt| < \infty$, for $j = 1,2$.

(A2) Assumptions on the stationary ergodic nature

For x$\in \mathcal{H}$, there exist a sequence of nonnegative bounded random functions $(f_{i,1})_{i\geq 1}$, a sequence of random functions $(g_{i,x,\theta})_{i\geq 1}$, a deterministic nonnegative bounded function f_1 and a nonnegative real function $\phi_\theta(\cdot)$ tending to zero, as its argument tends to 0, such that

(i) $F_{x,\theta}(h) = \phi_\theta(h) f_1(\theta,x) + o(\phi_\theta(h))$ as $h \to 0$.

(ii) For any i $\in \mathbb{N}$, $F_{x,\theta}^{\mathcal{F}_{i-1}}(h) = \phi_\theta(h) f_{i,1}(\theta, x) + g_{i,x,\theta}(h)$ with $g_{i,x,\theta} = o_{a.s}(\phi(h))$ as $h \to 0$, $\dfrac{g_{i,x,\theta}(h)}{\phi_\theta(h)}$ almost surely bounded and $n^{-1}\sum_{i=1}^{n} g_{i,x,\theta}^{j}(h) = o_{a.s}(\phi_\theta^{j}(h))\, as\ n \to \infty$, for $j = 1, 2$.

(iii) $n^{-1}\sum_{i=1}^{n} f_{i,1}^{j}(\theta, x) \to f_1^{j}(\theta, x)$ almost surely as $n \to \infty$, for $j = 1, 2$.

(iv) There exists a nondecreasing bounded function τ_0 such that, uniformly in $t \in [0, 1]$, $\dfrac{\phi_\theta(ht)}{\phi_\theta(h)} = \tau_0 + o(1)$, as $h \downarrow 0$, and $\int_0^1 (K^{j})' \tau_0(t) dt < \infty$ for $j \geq 1$.

(A3) Assumptions on the conditional moments

(i) The conditional mean of Y_i given the σ-field g_{i-1} depends only on (θ, X_i), i.e, for any $i \geq 1$, $\mathbb{E}(Y_i|\mathrm{g}_{i-1}) = \mathbb{E}(Y_i| <\theta, X_i>) = r(\theta, X_i)$, a.s.

(ii) For any $i \geq 1, \mathbb{E}[(Y_i - r(\theta, X_i))^2|\mathrm{g}_{i-1}] = \mathbb{E}[(Y_i - r(\theta, X_i))^2| <\theta, X_i>] = V(\theta, X_i)$, a.s.

(A4) Local smoothness and continuous conditions

(i) $\exists \beta > 0$ and a constant $C > 0$ such that $|r(u) - r(v)| \leq Cd(u, v)^{\beta}$ for all $(u, v) \in \mathcal{H} \times \mathcal{H}$.

(ii) $V(\cdot)$ and $p(\cdot)$ are continuous in a neighborhood of (θ, x) respectively, that is as $h \to 0$

$$\sup_{u:<x-u,\theta>\leq h} |V(u) - V(\theta, x)| = o(1),$$

$$\sup_{u:<x-u,\theta>\leq h} |p(u) - p(\theta, x)| = o(1).$$

(iii) $\exists \delta > 0$: $\mathbb{E}|Y_1|^{2+\delta} < \infty$, and let $\overline{W}_{2+\delta}(u) = \mathbb{E}\big(|\big(Y_1 - r(\theta, x)\big)|^{2+\delta}|X_1 = u\big)$ be continuous in a neighborhood of (θ, x) for $u \in \mathcal{H}$.

• Remarks on the assumptions

Similar to the discussions in Laib and Louani ((2010) [20], (2011) [21]), (A1), (A4)(i) are the quite usual conditions on the kernel function and regression operator for nonparametric functional data analysis. (A2) shows the ergodic nature of the data and the small ball techniques used in this paper. Assumption (A3) on condition moment shows the Markovian nature of the functional stationary ergodic data. (A4)(ii) and (A4)(iii) stand as local continuous conditions, which is necessary to establish the main results and make the results concise in this paper.

4.3 Asymptotic properties

In this section, we show some asymptotic properties of the estimator $\hat{r}_n(\theta, x)$ for the regression operator based on the functional stationary ergodic data with MAR in the single index model. More precisely, Theorem 4.3.1 shows the convergence in probability of the estimator. The asymptotic distribution of the estimator is presented in Theorem 4.3.2.

Theorem 4.3.1. *Under assumptions (A1)-(A4)(i),*

(a) If

$$\frac{n\phi_\theta(h)}{\log\log(n)} \to \infty, \quad \textit{as } n \to \infty, \tag{4.11}$$

for any $x \in \mathcal{H}$ *such that*$f_1(\theta, x) > 0$, *then we have*

$$\left(\frac{n\phi_\theta(h)}{\log\log(n)}\right)^{\frac{1}{2}} (\widehat{r}_n(\theta, x) - C_n(\theta, x)) \xrightarrow{\mathbb{P}} 0. \tag{4.12}$$

(b) In addition, if

$$\frac{n\phi_\theta(h)h^{2\beta}}{\log\log(n)} \to 0, \quad \textit{as } n \to \infty, \tag{4.13}$$

where β *is satisfied in (A4)(i), then we have*

$$\left(\frac{n\phi_\theta(h)}{\log\log(n)}\right)^{\frac{1}{2}} (\widehat{r}_n(\theta, x) - r(\theta, x)) \xrightarrow{\mathbb{P}} 0, \tag{4.14}$$

with $\xrightarrow{\mathbb{P}}$ *means the convergence in probability.*

Theorem 4.3.2. *Under assumptions (A1)-(A4),*

(a) If

$$n\phi_\theta(h) \to \infty, \quad \textit{as } n \to \infty, \tag{4.15}$$

for any $x \in \mathcal{H}$ *such that* $f_1(\theta, x) > 0$, *then we have*

$$\sqrt{n\phi_\theta(h)}(\widehat{r}_n(\theta, x) - C_n(\theta, x)) \xrightarrow{D} \mathcal{N}(0, \sigma^2(\theta, x)), \tag{4.16}$$

where $\xrightarrow{D}$ *means the convergence in distribution and* $\sigma^2(\theta, x) = \dfrac{M_2}{M_1^2}\dfrac{V(\theta, x)}{p(\theta, x)f_1(\theta, x)}$ *with* $M_j = K^j(1) - \int_0^1 (K^j)'(u)\tau_0(u)du$, *for* $j = 1, 2$.

(b) In addition, if

$$h^\beta \big(n\phi_\theta(h)\big)^{\frac{1}{2}} \to 0, \quad \textit{as } n \to \infty, \tag{4.17}$$

where β is specified in (A4)(i), then we have

$$\sqrt{n\phi_\theta(h)}(\widehat{r}_n(\theta,x)-r(\theta,x)) \xrightarrow{D} \mathcal{N}(0,\sigma^2(\theta,x)). \tag{4.18}$$

It is worth being noted that the results in our work extend the complete data in Laib and Louani ((2010) [20], (2011) [21]) to MAR case. On the other hand, as for the asymptotic normality, we also solve the second important open issue in MAR modeling proposed by Ferraty *et al.* (2013) [11]. In fact, the limiting variance in Theorem 4.3.2 contains the unknown function operator $f_1(\theta,\cdot)$, $V(\theta,\cdot)$, $p(\theta,\cdot)$ and unknown parameter M_j for $j=1,2$, respectively. Meanwhile, the normalization depends on the function $\phi_\theta(\theta,\cdot)$ which is also not identifiable explicitly. Therefore, we have to estimate them respectively so as to obtain asymptotic confidence interval of $r(\theta,x)$ in practice. First, the estimator of the conditional variance $V(\theta,x)$ can be defined as:

$$\begin{aligned}
V_n(\theta,x) &= \frac{\sum_{i=1}^{n}(\delta_i Y_i-\widehat{r}_n(\theta,x))^2 K\left(\frac{<x-X_i,\theta>}{h}\right)}{\sum_{i=1}^{n}\delta_i K\left(\frac{<x-X_i,\theta>}{h}\right)} \\
&= \frac{\sum_{i=1}^{n}\delta_i Y_i^2 K\left(\frac{<x-X_i,\theta>}{h}\right)}{\sum_{i=1}^{n}\delta_i K\left(\frac{<x-X_i,\theta>}{h}\right)} - \left(\widehat{r}_n(\theta,x)\right)^2 \\
&= \hat{\mathrm{g}}_n(\theta,x) - \left(\widehat{r}_n(\theta,x)\right)^2.
\end{aligned} \tag{4.19}$$

Second, by the assumptions (A2)(i) and (A2)(iv), the estimator of $\tau_0(\theta,x)$ is defined as

$$\tau_n(u)=\frac{F_{x,\theta,n}(uh)}{F_{x,\theta,n}(h)},$$

where

$$F_{x,\theta,n}(u)=\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{\{<x-X_i,\theta>\leq u\}}$$

can be used to estimate $\phi_\theta(h)$. Therefore, for a given kernel K, the estimator of M_1 and M_2, namely $M_{1,n}$ and $M_{2,n}$ respectively, is obtained by replacing τ_0 with τ_n in their respective expressions. Finally, the estimator of $p(\theta, x)$ is denoted by

$$P_n(\theta, x) = \frac{\displaystyle\sum_{i=1}^{n} \delta_i K\left(\frac{< x - X_i, \theta >}{h}\right)}{\displaystyle\sum_{i=1}^{n} K\left(\frac{< x - X_i, \theta >}{h}\right)}. \tag{4.20}$$

Then, the following Corollary is obtained immediately.

Corollary 4.3.1. *Under the conditions of Theorem 4.3.2, we have*

$$\frac{M_{1,n}}{\sqrt{M_{2,n}}}\sqrt{\frac{nF_{x,\theta,n}(h)P_n(\theta,x)}{V_n(\theta,x)}}\left(\widehat{r}_n(\theta,x) - r(\theta,x)\right) \xrightarrow{D} \mathcal{N}(0,1). \tag{4.21}$$

Thus, by (4.21), the asymptotic $(1-\zeta)$ confidence interval for the regression function operator $r(\theta, x)$ is given by

$$\widehat{r}_n(\theta,x) \pm \mu_{\frac{\zeta}{2}} \frac{\sqrt{M_{2,n}}}{M_{1,n}}\sqrt{\frac{V_n(\theta,x)}{nF_{x,\theta,n}(h)P_n(\theta,x)}},$$

where $\mu_{\frac{\zeta}{2}}$ is the upper $\frac{\zeta}{2}$ quantile of the Normal distribution $\mathcal{N}(0,1)$.

4.4 Proofs of some lemmas and main results

In this section, we first present some lemmas and their proofs which are necessary to establish our main results.

Lemma 4.4.1. *Assume that assumptions (A1) and (A2)(i)(ii)(iv) hold true. For any real numbers $1 \leq j \leq 2+\delta$ and $1 \leq k \leq 2+\delta$ with $\delta > 0$, as $n \to \infty$, we have*

(i) $$\frac{1}{\phi_\theta(h)}\mathbb{E}\left[K_i^j(\theta,x)|\mathcal{F}_{i-1}\right] = M_j f_{i,1}(\theta,x) + O_{a.s}\left(\frac{g_{i,\theta,x}(h)}{\phi_\theta(h)}\right).$$

(ii) $$\frac{1}{\phi_\theta(h)}\mathbb{E}\left[K_i^j(\theta,x)\right] = M_j f_1(\theta,x) + o(1).$$

(iii) $$\frac{1}{\phi_\theta^k(h)}\left(\mathbb{E}(K_1^j(\theta,x))\right)^k = M_1^k f_1^k(\theta,x) + o(1).$$

Proof of Lemma 4.4.1. See the proof of Lemma 1 in Laib and Louani (2010) [20]. □

Lemma 4.4.2. *Under the assumptions (A1)-(A2) in addition to (A3), for any $x \in \mathcal{H}$ such that $f_1(\theta, x) > 0$, we have*

$$\widehat{r}_{n,1}(\theta,x) \xrightarrow{\mathbb{P}} p(\theta,x), \quad as\ n \to \infty. \tag{4.22}$$

Proof of Lemma 4.4.2. By (4.4), we have the decomposition as follows

$$\widehat{r}_{n,1}(\theta,x) = R_{n,1}(\theta,x) + \bar{r}_{n,1}(\theta,x), \tag{4.23}$$

where

$$R_{n,1}(\theta,x) = \frac{1}{n\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^{n}\big(\delta_i K_i(\theta,x) - \mathbb{E}\big[\delta_i K_i(\theta,x)|\mathcal{F}_{i-1}\big]\big),$$

and

$$\bar{r}_{n,1}(\theta,x) = \frac{1}{nE\big(K_1(\theta,x)\big)}\sum_{i=1}^{n}\mathbb{E}\big[\delta_i K_i(\theta,x)|\mathcal{F}_{i-1}\big].$$

First, we need to establish

$$\bar{r}_{n,1}(\theta,x) \xrightarrow{\mathbb{P}} p(\theta,x), \quad \text{as } n\to\infty. \tag{4.24}$$

By the properties of conditional expectation and the mechanism of MAR, combining the assumptions (A2)(ii)(iii), (A3) and the continuous property of $p(\theta,x)$ with Lemma 4.4.2, we have

$$\begin{aligned}
\bar{r}_{n,1}(\theta,x) &= \frac{1}{n\mathbb{E}\big(K_1(\theta,x)\big)}\sum_{i=1}^{n}\mathbb{E}\big[\mathbb{E}[(\delta_i K_i(\theta,x)|\mathcal{F}_{i-1}]|\mathrm{g}_{i-1}\big] \\
&= \frac{1}{n\mathbb{E}\big(K_1(\theta,x)\big)}\sum_{i=1}^{n}\mathbb{E}\big[p(\theta,x)+o(1)K_i(\theta,x)|\mathcal{F}_{i-1}\big] \\
&= \big(p(\theta,x)+o(1)\big)\frac{1}{n\mathbb{E}\big(K_1(\theta,x)\big)}\sum_{i=1}^{n}\mathbb{E}[K_i(\theta,x)|\mathcal{F}_{i-1}] \\
&= \big(p(\theta,x)+o(1)\big)\frac{1}{n\mathbb{E}\big(K_1(\theta,x)\big)}\sum_{i=1}^{n}\left(\phi_\theta(h)M_1 f_{i1}(\theta,x) + O_{a.s}\left(\frac{\mathrm{g}_{i,\theta,x}(h)}{\phi_\theta(h)}\right)\right) \\
&= \big(p(\theta,x)+o(1)\big)\frac{\phi_\theta(h)}{\mathbb{E}\big(K_1(\theta,x)\big)}\left(\frac{1}{n}\sum_{i=1}^{n}M_1 f_{i1}(\theta,x) + \frac{1}{n}\sum_{i=1}^{n}O_{a.s}\left(\frac{\mathrm{g}_{i,\theta,x}(h)}{\phi_\theta(h)}\right)\right) \\
&= \big(p(\theta,x)+o(1)\big)\frac{1}{M_1 f_1(\theta,x)+o(1)}\Big(M_1\big(f_1(\theta,x)+o(1)\big)+O_{a.s}(1)\Big)
\end{aligned}$$

$\to p(\theta,x) \quad a.s, \quad as\ n\to\infty.$

Second, we will prove that

$$R_{n,1}(\theta,x) \xrightarrow{\mathbb{P}} 0, \quad \text{as } n\to\infty. \tag{4.25}$$

On the one hand, we denote $\eta_{n.i}(\theta,x) = \delta_i K_i(\theta,x) - \mathbb{E}\big(\delta_i K_i(\theta,x)|\mathcal{F}_{i-1}\big)$.

Then,$(\eta_{n.i},\ 1\le i\le n)$ forms a triangular array of martingale differences with respect to the σ-field $\mathcal{F}_{i-1}$ and

$$R_{n,1}(\theta,x)=\frac{1}{n\mathbb{E}\big(K_1(\theta,x)\big)}\sum_{i=1}^{n}\eta_{n,i}(\theta,x).$$

On the other hand, by Burkholders inequality of martingale differences (Hall and Heyde (1980) [17]), we have, as $n\to\infty$

$$\begin{aligned}\mathbb{P}\big(|R_{n,1}(\theta,x)|>\varepsilon\big) &= \mathbb{P}\Bigg(\Bigg|\sum_{i=1}^{n}\eta_{n.i}(\theta,x)\Bigg|>\varepsilon n\mathbb{E}\big(K_1(\theta,x)\big)\Bigg)\\ &\leq C_0\frac{\mathbb{E}\eta_{n.i}^2(\theta,x)}{\varepsilon^2 n\big(\mathbb{E}(K_1(\theta,x)\big)}\\ &< C_0\frac{\mathbb{E}\big(\delta_1K_1^2(\theta,x)\big)}{\varepsilon^2 n\mathbb{E}(K_1^2(\theta,x))}\to 0,\end{aligned}$$

which means that (4.25) is correct. Finally, (4.22) follows from (4.23) to (4.25).

□

Lemma 4.4.3. *Under the assumptions (A1)-(A2), (A3)(i), (A4)(i) and the condition (4.15), for any $x\in\mathcal{H}$ such that $f_1(\theta,x)>0$, we have*

$$B_n(\theta,x)=O_{\mathbb{P}}(h^{\beta}),\tag{4.26}$$

and

$$\sqrt{n\phi_\theta(h)}R_n(\theta,x)\xrightarrow{\mathbb{P}}0,\quad \textit{as } n\to\infty.\tag{4.27}$$

Proof of Lemma 4.4.3. First, by (4.6) and (4.7), we have

$$B_n(\theta,x)=\frac{\bar{r}_{n,2}(\theta,x)-r(\theta,x)\bar{r}_{n,1}(\theta,x)}{\bar{r}_{n,1}}=\frac{\bar{B}_n(\theta,x)}{\bar{r}_{n,1}(\theta,x)}.$$

Then, by (4.24), we need to show that

$$\bar{B}_n(\theta,x)=\bar{r}_{n,2}(\theta,x)-r(\theta,x)\bar{r}_{n,1}(\theta,x)=O_{a.s}(h^{\beta}).\tag{4.28}$$

In fact, by the assumptions (A3)(i) and (A4)(i), similar to the proof of Lemma 4.4.2, it follows that

$$\begin{aligned}
|\overline{B}_n(\theta,x)| &= \left|\frac{1}{n\mathbb{E}\big(K_1(\theta,x)\big)}\sum_{i=1}^{n}\mathbb{E}\big[\big(Y_i-r(\theta,x)\big)\delta_i K_i(\theta,x)|\mathcal{F}_{i-1}\big]\right| \\
&= \left|\frac{1}{n\mathbb{E}\big(K_1(\theta,x)\big)}\sum_{i=1}^{n}\mathbb{E}\Big[\mathbb{E}\big[(Y_i-r(\theta,x))\delta_i K_i(\theta,x)|\mathrm{g}_{i-1}\big]|\mathcal{F}_{i-1}\Big]\right| \\
&= \left|\frac{1}{n\mathbb{E}\big(K_1(\theta,x)\big)}\sum_{i=1}^{n}\mathbb{E}\Big[\mathbb{E}\big[(Y_i-r(\theta,x))\delta_i K_i(\theta,x)| <\theta,X_i> \big]|\mathcal{F}_{i-1}\Big]\right| \\
&= \left|\frac{1}{n\mathbb{E}\big(K_1(\theta,x)\big)}\sum_{i=1}^{n}\mathbb{E}\big[(r(\theta,X_i)-r(\theta,x))p(X_i)K_i(\theta,x)| <\theta,X_i> \big]|\mathcal{F}_{i-1}\right| \\
&\le \sup_{u\in B_\theta(x,h)}|r(u)-r(\theta,x)|\left|\frac{1}{n\mathbb{E}\big(K_1(\theta,x)\big)}\sum_{i=1}^{n}\mathbb{E}\Big(p(\theta,X_i)K_i(\theta,x)|\mathcal{F}_{i-1}\Big)\right| \\
&= O_{a.s}(h^\beta).
\end{aligned}$$

Thus, (4.26) follows from (4.24) and (4.28).

Finally, in order to establish (4.27), observe that

$$\widehat{r}_{n,1}(\theta,x)-\bar{r}_{n,1}(\theta,x)=\frac{1}{n\mathbb{E}\big(K_1(\theta,x)\big)}\sum_{i=1}^{n}\eta_{n.i}(\theta,x)$$

is a summation of a martingale difference $\{\eta_{n.i}, 1\le i\le n\}$. Following the same steps as that in Laib and Louani (2010) [20], if we establish that

$$\sqrt{n\phi_\theta(h)}\big(\widehat{r}_{n,1}(\theta,x)-\bar{r}_{n,1}(\theta,x)\big)\xrightarrow{D}\mathcal{N}(0,\rho^2(\theta,x)), \tag{4.29}$$

where, $\rho(\theta,x)=\dfrac{M_2}{M_1^2}\dfrac{p(\theta,x)}{f_1(\theta,x)}$, then by (4.29), (4.26) and (4.10), (4.27) is follows.

In fact, the proof of (4.29) is similar to that of Lemma 4.4.4 which establishes the asymptotic normality of $Q_n(\theta,x)$. □

Lemma 4.4.4. *Under the assumptions (A1)-(A2) as well as (A3)(A4) and the condition (4.15), for any $x\in\mathcal{H}$ such that $f_1(\theta,x)>0$, we have*

$$\sqrt{n\phi_\theta(h)}Q_n(\theta,x)\xrightarrow{D}\mathcal{N}(0,\sigma_0^2(\theta,x)), \tag{4.30}$$

where

$$\sigma_0^2=\frac{M_2}{M_1^2}\frac{p(\theta,x)V(\theta,x)}{f_1(\theta,x)}.$$

Proof of Lemma 4.4.4. Let's denote

$$\zeta_{ni} = \Big(\frac{\phi_\theta(h)}{n}\Big)^{\frac{1}{2}}\delta_i(Y_i - r(\theta,x))\frac{K_i(\theta,x)}{\mathbb{E}\big(K_1(\theta,x)\big)},$$

and

$$\xi_{ni} = \zeta_{ni} - \mathbb{E}[\zeta_{ni}^2|\mathcal{F}_{i-1}].$$

It is easy to see that

$$\big(n\phi_\theta(h)\big)^2 Q_n(\theta,x) = \sum_{i=1}^n \xi_{ni}. \tag{4.31}$$

Thus the $\{\xi_{ni}, 1 \le i \le n\}$ forms a triangular array of stationary martingale differences with respect to the σ-field $\mathcal{F}_{i-1}$. We apply the central limit theorem for discrete-time arrays of real-valued martingales (Hall and Heyde (1980) [17]) to obtain the asymptotic normality of $Q_n(\theta,x)$. Therefore, we have to establish the following statements:

(a) $\sum_{i=1}^n \mathbb{E}[\xi_{ni}^2|\mathcal{F}_{i-1}] \xrightarrow{\mathbb{P}} \sigma_0^2(\theta,x)$;

(b) $n\mathbb{E}\big[\xi_{ni}^2 I_{[|\xi_{ni}|>\varepsilon]}\big] = o(1)$, for $\forall\varepsilon > 0$.

Proof of part (a). Observe that

$$\Big|\sum_{i=1}^n \mathbb{E}\big[\zeta_{ni}^2|\mathcal{F}_{i-1}\big] - \sum_{1}^n \mathbb{E}\big[\xi_{ni}^2|\mathcal{F}_{i-1}\big]\Big| \le \sum_{i=1}^n \big(\mathbb{E}\big[\zeta_{ni}|\mathcal{F}_{i-1}\big]\big)^2. \tag{4.32}$$

By (A4), the continuous condition of $p(\theta,x)$ and Lemma 4.4.1, we obtain that

$$\begin{aligned}
\big|\mathbb{E}[\zeta_{ni}|\mathcal{F}_{i-1}]\big| &= \frac{\big(\frac{\phi_\theta(h)}{n}\big)^{\frac{1}{2}}}{\mathbb{E}\big(K_1(\theta,x)\big)}\Big|\mathbb{E}\big[(r(\theta,X_i) - r(\theta,x))p(\theta,X_i)K_i(\theta,x)|\mathcal{F}_{i-1}\big]\Big| \\
&\le \frac{\big(\frac{\phi_\theta(h)}{n}\big)^{\frac{1}{2}}}{\mathbb{E}\big(K_1(\theta,x)\big)}\sup_{u\in B_\theta(x,h)}\big|r(u) - r(\theta,x)\big|\mathbb{E}\big(K_i(\theta,x)|\mathcal{F}_{i-1}\big)h^\beta\big(o(1) + p(\theta,x)\big) \\
&\le C\Big(\frac{\phi_\theta(h)}{n}\Big)^{\frac{1}{2}}\Big(\frac{f_{i1}(\theta,x)}{f_1(\theta,x)} + O_{a.s}\Big(\frac{\mathrm{g}_{i,\theta,x}(h)}{\phi_\theta(h)}\Big)\Big)h^\beta\big(o(1) + p(\theta,x)\big).
\end{aligned}$$

Thus, by (A2)(ii) and (A2)(iii), we have

$$\begin{aligned}
\sum_{i=1}^n \big(\mathbb{E}[\zeta_{ni}|\mathcal{F}_{i-1}]\big)^2 &\le O_{a.s}\big(h^{2\beta}\phi_\theta(h)\big)\Big(\frac{1}{f_1^2(\theta,x)}\frac{1}{n}\sum_{i=1}^n f_{i1}^2(\theta,x) + O_{a.s}(1)\Big)\big(o(1) + p(\theta,x)\big)^2 \\
&= O_{a.s}(h^{2\beta}\phi_\theta(h)).
\end{aligned}$$

Hence, the statement (a) follows if we show that

$$\sum_{i=1}^{n} \mathbb{E}[\zeta_{ni}^2 | F_{i-1}] \xrightarrow{\mathbb{P}} \sigma_0^2(\theta, x). \tag{4.33}$$

To establish (4.33), we have the decomposition as follows

$$\sum_{i=1}^{n} \mathbb{E}[\zeta_{ni}^2 | \mathcal{F}_{i-1}] = \frac{\phi_\theta(h)}{n\big(\mathbb{E}K_i(\theta,x)\big)^2} \sum_{i=1}^{n} \mathbb{E}\big[\big(Y_i - r(\theta,x)\big)^2 \delta_i K_i^2(\theta,x) | \mathcal{F}_{i-1}\big] = J_{1n} + J_{2n}, \tag{4.34}$$

where,

$$J_{1n} = \frac{\phi_\theta(h)}{n\big(EK_1(\theta,x)\big)^2} \sum_{i=1}^{n} \mathbb{E}\big[\big(Y_i - r(\theta,X_i)\big)^2 \delta_i K_i^2(\theta,x) | \mathcal{F}_{i-1}\big],$$

and

$$J_{2n} = \frac{\phi_\theta(h)}{n\big(\mathbb{E}K_1(\theta,x)\big)^2} \sum_{i=1}^{n} \mathbb{E}\big[\big(r(\theta,X_i) - r((\theta,x)\big)^2 \delta_i K_i^2(\theta,x) | \mathcal{F}_{i-1}\big].$$

Thus, by the properties of conditional expectation and on the use of assumption (A3)(ii), we obtain that

$$\begin{aligned}
J_{1n} &= \frac{\phi_\theta(h)}{n\big(\mathbb{E}K_1(\theta,x)\big)^2} \sum_{i=1}^{n} \mathbb{E}\Big[\mathbb{E}\big[\big(Y_i - r(\theta,X_i)\big)^2 \delta_i K_i^2(\theta,x) | \mathrm{g}_{i-1}\big] | \mathcal{F}_{i-1}\Big] \\
&= \frac{\phi_\theta(h)}{n\big(\mathbb{E}K_1(\theta,x)\big)^2} \sum_{i=1}^{n} \mathbb{E}\Big[K_i^2(\theta,x)\mathbb{E}\big[\big(Y_i - r(\theta,X_i)\big)^2 \delta_i | < \theta, X_i > \big] | \mathcal{F}_{i-1}\Big] \\
&= \frac{\phi_\theta(h)}{n\big(\mathbb{E}K_1(\theta,x)\big)^2} \sum_{i=1}^{n} \mathbb{E}\big[V(\theta,X_i)p(\theta,X_i)K_i^2(\theta,x) | \mathcal{F}_{i-1}\big].
\end{aligned}$$

Then, by (A2)(ii) and smoothness conditions (A4) as well as Lemma 4.4.1, we have that

$$\begin{aligned}
J_{1n} &= \frac{\phi_\theta(h)}{n\big(\mathbb{E}K_1(\theta,x)\big)^2} \sum_{i=1}^{n} \mathbb{E}\Big[\big(o(1) + V(\theta,x)\big)\big(o(1) + p(\theta,x)\big)K_i^2(\theta,x) | \mathcal{F}_{i-1}\Big] \\
&= \frac{\phi_\theta(h)}{n\big(\mathbb{E}K_1(\theta,x)\big)^2} \sum_{i=1}^{n} \big(o(1) + V(\theta,x)\big)\big(o(1) + p(\theta,x)\big)\left(M_2\phi_\theta(h)f_{i1}(\theta,x) + O_{a.s}\left(\frac{\mathrm{g}_{i,\theta,x}(h)}{\phi_\theta(h)}\right)\right) \\
&\to \frac{M_2 V(\theta,x)p(\theta,x)}{M_1^2 f_1(\theta,x)} = \sigma_0^2(\theta,x).
\end{aligned} \tag{4.35}$$

Similarly, by the assumptions (A2)(ii)(iii) and (A4)(i) together with Lemma 4.4.1

again, it follows that

$$\begin{aligned} J_{2n} &= O(h^{2\beta})\frac{\phi_\theta(h)}{n\big(\mathbb{E}K_1(\theta,x)\big)^2}\sum_{i=1}^{n}\mathbb{E}\big[\delta_i K_i^2(\theta,x)|\mathcal{F}_{i-1}\big] \\ &\le O(h^{2\beta})\left(\frac{M_2}{M_1^2}\frac{1}{f_1(\theta,x)}+o_{a.s}(1)\right)\longrightarrow 0, \quad \text{as } n\to\infty. \qquad (4.36) \end{aligned}$$

Finally, by (4.34)-(4.36), (4.33) is valid.

Proof of part (b).

The proof of this part is also similar to that in Laib and Louani (2010) [20]. In fact, by the definition of ξ_{ni}, we have $n\mathbb{E}\big[\xi_{ni}^2 I_{(|\xi_{ni}|>\varepsilon)}\big] \le 4n\mathbb{E}\big[\zeta_{ni}^2 I_{(|\zeta_{ni}|>\frac{\varepsilon}{2})}\big]$, where I_A is an indicator function of a set A. Let $a>1$ and $b>1$ such that $\frac{1}{a}+\frac{1}{b}=1$. By Hölder and Markov inequalities, one can write, for all $\varepsilon>0$,

$$\mathbb{E}\left[\zeta_{ni}^2 I_{\left(|\zeta_{ni}|>\frac{\varepsilon}{2}\right)}\right]\le \frac{\mathbb{E}|\zeta_{ni}|^{2a}}{\left(\frac{\varepsilon}{2}\right)^{\frac{2a}{b}}}. \qquad (4.37)$$

Taking C_0 a positive constant and $2a=2+\delta$ (with δ as in (A4)(iii)), by the local continuous condition, we can obtain

$$\begin{aligned} 4n\mathbb{E}\big[\zeta_{ni}^2 I_{(|\zeta_{ni}|>\frac{\varepsilon}{2})}\big] &\le C_0\left(\frac{\phi_\theta(h)}{n}\right)^{\frac{2+\delta}{2}}\frac{n}{\Big(\mathbb{E}\big(K_1(\theta,x)\big)\Big)^{2+\delta}}\mathbb{E}\Big(\big[|Y_i-r(\theta,x_i|\big)^2\delta_i K_i^2(\theta,x)\big]^{2+\delta}\Big) \\ &\le C_0\left(\frac{\phi_\theta(h)}{n}\right)^{\frac{2+\delta}{2}}\frac{n}{\Big(\mathbb{E}\big(K_1(\theta,x)\big)\Big)^{2+\delta}}\mathbb{E}\Big(\mathbb{E}\big[|Y_i-r(\theta,x_i)|^{2+\delta}\delta_i\big(K_i(\theta,x)\big)^{2+\delta}|<\theta,X_i>\big]\Big) \\ &\le C_0\left(\frac{\phi_\theta(h)}{n}\right)^{\frac{2+\delta}{2}}\frac{n}{\Big(\mathbb{E}\big(K_1(\theta,x)\big)\Big)^{2+\delta}}\mathbb{E}\big[\big(K_i(\theta,x)\big)^{2+\delta}p(\theta,X_i)\overline{W}_{2+\delta}(\theta,X_i)\big] \\ &= C_0\left(\frac{\phi_\theta(h)}{n}\right)^{\frac{2+\delta}{2}}\frac{n}{\Big(\mathbb{E}\big(K_1(\theta,x)\big)\Big)^{2+\delta}}\mathbb{E}\Big[\big(K_i(\theta,x)\big)^{2+\delta}\big(p(\theta,x)+o(1)\big)\big(\overline{W}_{2+\delta}(\theta,x)+o(1)\big)\Big] \\ &\le C_0\left(\frac{\phi_\theta(h)}{n}\right)^{\frac{2+\delta}{2}}\frac{n\mathbb{E}\big(K_1(\theta,x)\big)^{2+\delta}}{\big(E(K_1(\theta,x))\big)^{2+\delta}}\big(p(\theta,x)\overline{W}_{2+\delta}(\theta,x)+o(1)\big). \end{aligned}$$

Thus, by Lemma 4.4.1, it follows that

$$\begin{aligned} 4n\mathbb{E}\big[\zeta_{ni}^2 I_{(|\zeta_{ni}|>\frac{\varepsilon}{2})}\big] &\le C_0\big(n\phi_\theta(h)\big)^{-\frac{\delta}{2}}\frac{M_{2+\delta}f_1(\theta,x)+o(1)}{M_1^{2+\delta}f_1^{2+\delta}(\theta,x)+o(1)}\big(p(\theta,x)\overline{W}_{2+\delta}(\theta,x)+o(1)\big) \\ &= O\big(n\phi_\theta(h)\big)^{-\frac{\delta}{2}}. \end{aligned}$$

Finally, by (4.15), the proof of part (b) is completed. Then, (4.30) is valid.

□

Proof of Theorem 4.3.1. First, we present the proof of (4.12). By Lemma 4.4.4, it follows that $\big(n\phi_\theta(h)\big)^{\frac{1}{2}}Q_n(\theta,x) = O_{\mathbb{P}}(1)$, which leads to

$$\left(\frac{n\phi_\theta(h)}{\log\log n}\right)^{\frac{1}{2}} Q_n(\theta,x) = O_{\mathbb{P}}(1). \tag{4.38}$$

On the other hand, by Lemma 4.4.3, we have

$$\left(\frac{n\phi_\theta(h)}{\log\log n}\right)^{\frac{1}{2}} R_n(\theta,x) = O_{\mathbb{P}}(1). \tag{4.39}$$

Thus, by Lemma 4.4.2 and (4.8), (4.12) is valid.

Second, we give the proof of (4.14). Since

$$\widehat{r}_n(\theta,x) - r(\theta,x) = \widehat{r}_n(\theta,x) - C_n(\theta,x) + B_n(\theta,x). \tag{4.40}$$

Hence, by (4.40) together with (4.3), (4.4) and (4.15), (4.14) follows.

□

Proof of Theorem 4.3.2. On the one hand, (4.16) follows directly from (4.8), (4.22), (4.27), (4.30) and the Slulsky Theorem. On the other hand, by (4.16), (4.17), (4.26), (4.40) and the Slulsky Theorem again, (4.18) is also obtained.

□

Proof of Corollary 4.3.1. First, one can observe that

$$\frac{M_{1.n}}{\sqrt{M_{2.n}}}\sqrt{\frac{nP_n(\theta,x)F_{x,\theta,n}(h)}{V_n(\theta,x)}}\big(\widehat{r}_n(\theta,x) - r(\theta,x)\big)$$

$$= \frac{M_{1.n}}{M_1}\frac{\sqrt{M_2}}{\sqrt{M_{2.n}}}\sqrt{\frac{nF_{x,\theta,n}(h)P_n(\theta,x)V(\theta,x)}{p(\theta,x)V_n(\theta,x)n\phi_\theta(h)f_1(\theta,x)}} \times \frac{M_1}{\sqrt{M_2}}\sqrt{\frac{n\phi_\theta(h)f_1(\theta,x)p(\theta,x)}{V(\theta,x)}}\big(\widehat{r}_n(\theta,x) - r(\theta,x)\big).$$

By (4.18), we have

$$\frac{M_1}{\sqrt{M_2}}\sqrt{\frac{p(\theta,x)n\phi_\theta(h)f_1(\theta,x)}{V(\theta,x)}}\big(\widehat{r}_n(\theta,x) - r(\theta,x)\big) \xrightarrow{D} \mathcal{N}(0,1),\ as\, n\to\infty.$$

Therefore, we need to establish the following statement

$$\frac{M_{1.n}}{M_1}\frac{\sqrt{M_2}}{\sqrt{M_{2.n}}}\sqrt{\frac{nF_{x,\theta,n}(h)V(\theta,x)P_n(\theta,x)}{p(\theta,x)V_n(\theta,x)n\phi_\theta(h)f_1(\theta,x)}} \xrightarrow{\mathbb{P}} 1,\ \text{as}\, n\to\infty. \tag{4.41}$$

Similar to the proof of Corollary 1 in Laib and Louani (2010) [20], we have

$$M_{1.n} \xrightarrow{\mathbb{P}} M_1,\ M_{2.n} \xrightarrow{\mathbb{P}} M_2,\ \frac{F_{x,\theta,n}(h)}{\phi_\theta(h)f_1(\theta,x)} \xrightarrow{\mathbb{P}} 1,\ as\, n\to\infty. \tag{4.42}$$

In addition, by (4.11) and (4.14), it follows that

$$\hat{r}_n(\theta, x) \xrightarrow{\mathbb{P}} r(\theta, x), \; as\, n \to \infty. \tag{4.43}$$

On the other hand, by the same steps as in the proof of Theorem 4.3.1, we have

$$\hat{\text{g}}_n(\theta, x) \xrightarrow{\mathbb{P}} \mathbb{E}(Y^2| < \theta, X > = < \theta, x >), \; as\, n \to \infty. \tag{4.44}$$

Then, by (4.19), we obtain

$$V_n(\theta, x) \xrightarrow{\mathbb{P}} V(\theta, x), \; as\, n \to \infty. \tag{4.45}$$

Finally, by Proposition 2 in Laib and Louani (2010) [20], it follows that

$$P_n(\theta, x) \xrightarrow{\mathbb{P}} \mathbb{P}(\delta = 1| < \theta, X > = < \theta, x >) = p(\theta, x), \; as\, n \to \infty. \tag{4.46}$$

Hence, (4.41) follows from (4.42)-(4.46). □

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Chapter 5

Asymptotic normality of conditional density and conditional mode in the functional single index model

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Asymptotic normality of conditional density and conditional mode in the functional single index model

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Abstract: The main objective of this paper is to investigate the nonparametric estimation of the conditional density of a scalar response variable Y, given the explanatory variable X taking value in an Hilbert space when the sample of observations is considered as an independent random variables with identical distribution (i.i.d) and are linked with a single functional index structure. First of all, a kernel type estimator for the conditional density function (*cond-df*) is introduced. Afterwards, the asymptotic properties are stated for a conditional density estimator when the observations are linked with a single-index structure from which one derives a central limit theorem (CLT) of the conditional density estimator to show the asymptotic normality of the kernel estimate of this model. As an application, the conditional mode in functional single-index model is presented, and the asymptotic $(1-\xi)$ confidence interval of the conditional mode function is given for $0<\xi<1$. A simulation study is also presented to illustrate the validity and finite sample performance of the considered estimator. Finally, the estimation of the functional index via the pseudo-maximum likelihood method is discussed.

Keywords: Asymptotic normality, Conditional density, Functional single index model, Functional random variable, Nonparametric estimation.

JEL Classification: C13, C14, C15.

5.1 Introduction

The statistical analysis of functional variables has grown considerably over the last two decades. In fact, an important innovation in measuring devices has emerged, permitting to monitor several objects in a continuous way, such as stock market index, pollution, climatology, and satellite images, etc.

Thus, a new branch of statistics called functional statistics has been developed to treat observations as functional random elements.

As a conditional nonparametric model, the regression was one of the first predictive analysis tools. Quantile regression is the common way to describe the dependence structure between a response variable Y and some covariate X. Unlike the regression function (which is defined as the conditional mean) that relies only on the central tendency of the data, the conditional mode function allows the analyst to estimate the functional independence between variables for all portions of the conditional density of the response variable. However, compared with the standard approach based on functional conditional mean prediction that is sensitive to outliers, functional condition mode prediction could be seen as a reasonable alternative to the conditional mean because of its robustness, which allows to consider it as a useful alternative to the regression function.

The conditional model estimator has been widely used to estimate some characteristic features of the data set, such as the conditional mode, the conditional median, and the conditional quantiles. Many authors are interested in the estimation of the conditional mode of a scalar response given a functional covariate. Ferraty *et al.* (2006) [12] introduced a nonparametric study of the kernel-type estimation of some characteristics of the conditional cumulative distribution function and the conditional density and its derivatives. Some asymptotic properties were established with particular attention to the conditional mode and conditional quantiles. An application to a chemometrical data set coming from the food industry was also presented. The uniform strong consistency with rates and the asymptotic normality for the kernel conditional mode estimator were obtained by Ezzahrioui and Ould-Saïd (2008) [11] in the i.i.d case.

In the case of censoring data, Ould-Saïd and Cai (2005) [26] established the strong uniform convergence (with rate) of kernel conditional mode estimator for i.i.d random variables, while Ould-Saïd (2006) [25] constructed a kernel estimator of the conditional quantile and establish its strong uniform convergence rate. Next, Khardani *et al.* (2010) [19] obtained strong consistency with the rate and asymptotic normality of the conditional mode Khardani *et al.* (2011) [20] established strong consistency with the rate of the conditional mode for the censored dependent case, while Khardani *et al.* (2014) [21] presented asymptotic normality.

For infinite dimensional purpose, the study used the terminology *functional nonparametric*, where the term *functional* refers to the infinite dimensionality of the data, and where nonparametric refers to the infinite dimensionality of the model. Such *functional nonparametric* statistics is also called *doubly infinite dimensional* (see Ferraty and Vieu (2003) [14], for more details). Conditional density function estimation is one of the crucial problems in non-parametric statistics, see (De Gooijer and Zerom (2003) [9]). Ling and Xu (2012) [24] established the asymptotic normality of the conditional density estimator and the conditional mode estimator for the α-mixing dependence functional time series data. Ling *et al.* (2014) [23] investigated the pointwise almost complete consistency and the uniform almost complete convergence of the kernel estimation with a rate for the conditional density in the setting of the α-mixing functional data. Attaoui (2014b) [4] investigated the nonparametric estimation of the conditional density of a scalar response variable given a random variable taking values in separable Hilbert space. The author established under general conditions the uniform almost complete convergence rates and the asymptotic normality of the conditional density kernel estimator, when the variables satisfy the strong mixing dependency, based on the single-index structure.

The single index models have been used and studied in both statistical and econometric literature, and are very popular in the economics community as they address two important concerns. The first is the reduction of dimension, since this type of model makes it possible to solve the problem of the scourge of the dimension. The second is related to the interpretability of the index (parameter) introduced in these models. The statistical study of these models, in the context of vectorial explanatory random variables, was initiated by Härdle and Marron (1985) [59]. Hristache *et al.* (2001) [18] provided both new theoretical and bibliographic elements. Based on the regression function, Delecroix *et al.* (2003) [10] studied the estimation of the single-index and established some asymptotic properties. In the same setting, we can cite Härdle *et al.* (1993) [16]. Several authors have worked on simple functional index models, e.g. (Attaoui and Boudiaf (2014a) [3], Ait-Saïdi *et al.* (2008) [2], Belabbaci *et al* (2015) [7], Ferraty *et al.* (2003) [13]).

These models attracted the attention of many researchers, such as Ait-Saïdi *et al.* (2005) [1]. Bouchentouf *et al.* (2014) [8] established a nonparametric estimation of some characteristics of the conditional cumulative distribution function and the successive derivatives of the conditional density of a scalar response variable Y given a Hilbertian random variable X when the observations are linked with a single-index structure. Attaoui *et al.* (2011) [5] studied the functional single-index model via its conditional density kernel estimator, and established its pointwise and uniform almost complete convergence rates.

The main contribution of this work is to generalize the result of Ezzahrioui and Ould-Saïd (2008) [11], where a functional parameter θ is present in the model. The results can be used to construct prediction intervals, for instance regarding electricity when one wants to construct a maximum interval of demand (or need) for chemometrical data coming from the food industry.

This study established the asymptotic properties of the asymptotic normality for the estimators of conditional density function and conditional mode of a randomly scalar response, given a functional covariate when the data are sampled from an i.i.d process with a single-index structure.

The paper is organized as follows. The model and some basic assumptions are presented in Section 5.2. Section 5.3 shows the main results, and the proofs of some lemmas. In Section 5.4, an application of the conditional mode in functional single-index model is presented. The next section illustrates those asymptotic properties through some simulations. Finally, a general conclusion to this contribution is proposed in Section 5.6.

5.2 Model and some basic assumptions

Let $(X_i, Y_i)_{i=1,\ldots,n}$ be a sequence of independent functional samples, with the same distribution as (X, Y), where Y is a real-valued random variable and X be a functional random variable (*frv*), which takes its values in a separable real Hilbert space $\mathcal{H}$ with the norm $\|\cdot\|$ generated by an inner product $<\cdot,\cdot>$.

Moreover, we consider $d_\theta(\cdot,\cdot)$ a semi-metric associated with the single index $\theta \in \mathcal{H}$ defined by $d_\theta(x_1, x_2) := |<x_1 - x_2, \theta>|$, for x_1 and x_2 in $\mathcal{H}$.

For a fixed x in $\mathcal{H}$, let $F(\theta, y, x)$ be the conditional cumulative distribution function (*cond-cdf*) of Y given $<\theta, X>=<\theta, x>$, specifically:

$$\forall y \in \mathbb{R},\ F(\theta, y, x) = \mathbb{P}(Y \leq y| <X, \theta>=<x, \theta>).$$

Saying that, we are implicitly assuming the existence of a regular version of the conditional distribution and that it's absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, our aim is to build nonparametric estimates of several functions related with the conditional density of Y given $<X, \theta>=<x, \theta>$. Let

$$\forall y \in \mathbb{R},\ f_\theta(y|x) = f(y| <x, \theta>),$$

be the conditional density of Y given $<X, \theta>=<x, \theta>$, for $x \in \mathcal{H}$.

In the following, we denote by $f(\theta, \cdot, x)$ the conditional density of Y given $<x, \theta>$ and we define the kernel estimator $\widehat{f}(\theta, \cdot, x)$ of $f(\theta, \cdot, x)$ by:

$$\widehat{f}(\theta, y, x) = \frac{h_H^{-1} \sum_{i=1}^{n} K(h_K^{-1}(<x - X_i, \theta>)) H(h_H^{-1}(y - Y_i))}{\sum_{i=1}^{n} K(h_K^{-1}(<x - X_i, \theta>))}, \tag{5.1}$$

with the convention $0/0 = 0$, where K and H are kernel functions and $h_K := h_{n,K}$ (resp. $h_H = h_{n,H}$) is a sequence of bandwidths that decrease to zero as n goes to infinity.

Let, for any $x \in \mathcal{H},\ i = 1, \ldots, n$ and $y \in \mathbb{R}$:

$$K_i(\theta, x) := K(h_K^{-1}| <x - X_i, \theta>|),\ \ H_i(y) := H(h_H^{-1}(y - Y_i)).$$

We denote by $B_\theta(x, h_K) = \{\chi \in \mathcal{H}/0 < |<x - \chi, \theta>| < h_K\}$ the ball of center x and radius h. Let $\mathcal{N}_x$ be a fixed neighborhood of x in $\mathcal{H}$, $S_\mathbb{R}$ will be a fixed compact subset of $\mathbb{R}$. Now, we consider the following basic assumptions that are necessary in deriving the main result of this paper.

(H1) $\mathbb{P}(X \in B_\theta(x, h_K)) = \phi_{\theta,x}(h_K) > 0; \quad \phi_{\theta,x}(h_K) \longrightarrow 0 \ as\ h_K \to 0.$

(H2) The conditional density $f(\theta, y, x)$ satisfies the Hölder condition, that is:

$\forall (y_1, y_2) \in S_\mathbb{R} \times S_\mathbb{R}\ ,\ \forall (x_1, x_2) \in \mathcal{N}_x \times \mathcal{N}_x$

$$|f(\theta,y_1,x_1)-f(\theta,y_2,x_2)| \leq C_{\theta,x}(\|x_1-x_2\|^{b_1}+|y_1-y_2|^{b_2}),\ b_1>0,\ b_2>0.$$

(H3) The kernel H is a positive bounded function such that $\forall (t_1,t_2)\in\mathbb{R}^2$,

$$|H(t_1)-H(t_2)| \leq C|t_1-t_2|, \int H^2(t)dt < \infty \text{ and } \int |t|^{b_2} H(t)dt < \infty.$$

(H4) The kernel K is a positive bounded function supported on $[0,1]$ and is differentiable on $[0,1]$ with derivative such that: $\exists\, C_1,\ C_2,\ -\infty < C_1 < K'(t) < C_2 < 0$, for $0 < t < 1$.

(H5) There exists a function $\beta_{\theta,x}(\cdot)$ such that $\lim_{h_K\to+\infty} \frac{\phi_{\theta,x}(sh_K)}{\phi_{\theta,x}(h_K)} = \beta_{\theta,x}(s)$, for $\forall s\in[0,1]$.

(H6) The bandwidth h_K and h_H, small ball probability $\phi_{\theta,x}(h_K)$ satisfying

(i) $\lim_{n\to+\infty} h_K = 0$, $\lim_{n\to+\infty} h_H = 0$ and $\lim_{n\to+\infty} \frac{log\, n}{nh_H\phi_{\theta,x}(h_K)} = 0.$

(ii) $h_H^{b_2}\sqrt{nh_H\phi_{\theta,x}(h_K)} \longrightarrow 0, \quad$ as $n\to\infty$.

(iii) $nh_H^3\phi_{\theta,x}^3(h_K) \longrightarrow 0, \quad$ as $n\to\infty$.

5.3 Main results

In this section, the asymptotic normality of the estimator $\widehat{f}(\theta,\cdot,x)$ in the single functional index model is established.

Theorem 5.3.1. *Under the assumptions (H1)-(H6), we have for all $x\in\mathcal{H}$*

$$\sqrt{\frac{nh_H\phi_{\theta,x}(h_K)}{\sigma^2(\theta,y,x)}}\Big(\widehat{f}(\theta,y,x)-f(\theta,y,x)\Big)\xrightarrow{\mathcal{D}}\mathcal{N}(0,1), \quad as\ n\to\infty,$$

where $\sigma^2(\theta,y,x) = \frac{\alpha_2(\theta,x)f(\theta,y,x)}{(\alpha_1(\theta,x))^2}\int H^2(t)dt,$

with $\alpha_l(\theta,x) = K^l(1) - \int_0^1 (K^l)'(u)\beta_{\theta,x}(u)du,\ l=1,2,$

and " $\xrightarrow{\mathcal{D}}$ " *means the convergence in distribution.*

Proof. In order to establish the asymptotic normality of $\widehat{f}(\theta,y,x)$, we need further notations and definitions. First, we consider the following decomposition

$$\begin{aligned}
\widehat{f}(\theta,y,x)-f(\theta,y,x) &= \frac{\widehat{f}_N(\theta,y,x)}{\widehat{f}_D(\theta,x)}-\frac{\alpha_1(\theta,x)f(\theta,y,x)}{\alpha_1(\theta,x)}\\
&= \frac{1}{\widehat{f}_D(\theta,x)}\left(\widehat{f}_N(\theta,y,x)-\mathbb{E}\widehat{f}_N(\theta,y,x)\right)\\
&\quad -\frac{1}{\widehat{f}_D(\theta,x)}\left(\alpha_1(\theta,x)f(\theta,y,x)-\mathbb{E}\widehat{f}_N(\theta,y,x)\right)\\
&\quad +\frac{f(\theta,y,x)}{\widehat{f}_D(\theta,x)}\left(\alpha_1(\theta,x)-\mathbb{E}\widehat{f}_D(\theta,x)\right)\\
&\quad -\frac{f(\theta,y,x)}{\widehat{f}_D(\theta,x)}\left(\widehat{f}_D(\theta,x)-\mathbb{E}\widehat{f}_D(\theta,x)\right)\\
&= \frac{1}{\widehat{f}_D(\theta,x)}\left(A_n(\theta,y,x)+B_n(\theta,y,x)\right),
\end{aligned}$$

where $\widehat{f}_N(\theta,y,x)=\dfrac{\sum_{i=1}^n K_i(\theta,x)H_i(y)}{nh_H\mathbb{E}(K_1(\theta,x))}$, $\widehat{f}_D(\theta,x)=\dfrac{\sum_{i=1}^n K_i(\theta,x)}{n\mathbb{E}(K_1(\theta,x))}$ and

$$\begin{aligned}
A_n(\theta,y,x) &= \frac{1}{nh_H\mathbb{E}K_1(\theta,x)}\sum_{i=1}^n\left(H_i(y)-h_Hf(\theta,y,x)\right)K_i(\theta,x)-\mathbb{E}\left[\left(H_i(y)-h_Hf(\theta,y,x)\right)K_i(\theta,x)\right]\\
&= \frac{1}{nh_H\mathbb{E}K_1(\theta,x)}\sum_{i=1}^n N_i(\theta,y,x).
\end{aligned}$$

It follows that,

$$\begin{aligned}
nh_H\phi_{\theta,x}(h_K)Var(A_n(\theta,y,x)) &= \frac{\phi_{\theta,x}(h_K)}{h_H(\mathbb{E}K_1(\theta,x))^2}Var(N_1(\theta,y,x))\\
&= V_n(\theta,y,x),
\end{aligned}$$

and

$$B_n(\theta,y,x)=\alpha_1(\theta,x)f(\theta,y,x)-\mathbb{E}\widehat{f}_N(\theta,y,x)+f(\theta,y,x)(\alpha_1(\theta,x)-\mathbb{E}\widehat{f}_D(\theta,x)).$$

Then, the proof of Theorem 5.3.1 can be deduced from the following Lemmas. □

Lemma 5.3.1. *Assume that hypotheses (H1)-(H2)-(H3) in addition to (H4) hold true, then we get*

$$\sqrt{nh_H\phi_{\theta,x}(h_K)}A_n(\theta,y,x)\xrightarrow{\mathcal{D}}\mathcal{N}(0,\sigma^2(\theta,y,x)),$$

where $\sigma^2(\theta,y,x)$ is given in Theorem 5.3.1.

Proof.

$$\begin{aligned} V_n(\theta,y,x) &= \frac{\phi_{\theta,x}(h_K)}{h_H(\mathbb{E}K_1(\theta,x))^2}\mathbb{E}\Big[K_1^2(\theta,x)\left(H_1(y)-h_hf(\theta,y,x)\right)^2\Big] \\ &= \frac{\phi_{\theta,x}(h_K)}{h_H(\mathbb{E}K_1(\theta,x))^2}\mathbb{E}\Big[K_1^2(\theta,x)\mathbb{E}\Big((H_1(y) \\ &- h_Hf(\theta,y,x))^2| < \theta, X_1 > \Big)\Big]. \end{aligned} \tag{5.2}$$

Using the definition of conditional variance, we have

$$\mathbb{E}\Big[\left(H_1(y)-h_Hf(\theta,t,x)\right)^2 | < \theta, X_1 > \Big] = J_{1n}+J_{2n},$$

where

$$\begin{aligned} J_{1n} &= Var\left(H_1(y)| < \theta, X_1 >\right), \\ J_{2n} &= \Big[\mathbb{E}\left(H_1(y)| < \theta, X_1 >\right) - h_Hf(\theta,y,x)\Big]^2. \end{aligned}$$

- Concerning J_{1n}

$$J_{1n} = \mathbb{E}\left(H_1^2(y)| < \theta, X_1 >\right) - \mathbb{E}\left(H_1(y)| < \theta, X_1 >\right)^2 = J_1 + J_2.$$

As for J_1, by the property of conditional expectation and by changing variables, one gets

$$\begin{aligned} J_1 &= \mathbb{E}\left(H_1^2\left(\frac{y-Y_1}{h_H}\right) | < \theta, X_1 >\right) \\ &= \int_{\mathbb{R}} H_1^2\left(\frac{y-v}{h_H}\right) f(\theta,v,X_1)dv \\ &= \int_{\mathbb{R}} H_1^2(u)dF(\theta,y-uh_H,X_1). \end{aligned}$$

On the other hand, by applying (H2) and (H3), we have

$$\begin{aligned}
J_1 &= \int_{\mathbb{R}} H_1^2(u) dF(\theta, y - uh_H, X_1) \\
&= h_H \int_{\mathbb{R}} H_1^2(u) f(\theta, y - uh_H, X_1) du \\
&\leq h_H \int_{\mathbb{R}} H_1^2(u)(f(\theta, y - uh_H, X_1) - f(\theta, y, x)) du \\
&\quad + h_H \int_{\mathbb{R}} H_1^2(u) f(\theta, y, x) du \\
&\leq h_H \left(C_{x,\theta} \int_{\mathbb{R}} H^2(u) \left(h_K^{b_1} + |v|^{b_2} h_H^{b_2}\right) du + f(\theta, y, x) \int_{\mathbb{R}} H^2(u) du \right) \\
&= \mathcal{O}\left(h_K^{b_1} + h_H^{b_2}\right) + h_H f(\theta, y, x) \int_{\mathbb{R}} H^2(u) du. \qquad (5.3)
\end{aligned}$$

As for J_2,

$J_2^{'} = \mathbb{E}\left(H_1(y)| < \theta, X_1 >\right) = \int_{\mathbb{R}} H\left(\frac{y-v}{h_H}\right) f(\theta, y, X_1) dv.$

Moreover, by changing variables, we have :

$$J_2^{'} = h_H \int_{\mathbb{R}} H(u)(f(\theta, y - uh_H, X_1 - f(\theta, y, x)) du + h_H f(\theta, y, x) \int_{\mathbb{R}} H(u) du,$$

the last equality is due to the fact that H is a probability density, thus, we have :

$$J_2^{'} = \mathcal{O}\left(h_k^{b_1} + h_H^{b_2}\right) + h_H f(\theta, y, x).$$

Finally, we get $J_2 \underset{n \to \infty}{\longrightarrow} 0$.

Concerning J_{2n}, by (H1)-(H3), we obtain that $J_{2n} \underset{n \to \infty}{\longrightarrow} 0$.

Meanwhile, by (H1)-(H3), it follows that

$$\frac{\phi_{\theta,x}(h_K)\mathbb{E}K_1^2(\theta, x)}{\mathbb{E}^2 K_1(\theta, x)} \underset{n\to\infty}{\longrightarrow} \frac{\alpha_2(\theta, x)}{(\alpha_1(\theta, x))^2}.$$

By combining equations (5.2) and (5.3), it yields

$$V_n(\theta, t, x) \longrightarrow \frac{\alpha_2(\theta, x) f(\theta, y, x)}{(\alpha_1(\theta, x))^2} \int_{\mathbb{R}} H^2(u) du.$$

□

Lemma 5.3.2. *If the assumptions (H1)-(H3)-(H4) and (H6) are satisfied, then we have as $n \longrightarrow \infty$,*

$$\sqrt{nh_H \phi_{\theta,x}(h_K)} B_n(\theta, y, x) \longrightarrow 0, \quad \textit{in probability.}$$

Proof. We have

$$\sqrt{nh_H\phi_{\theta,x}(h_K)}B_n(\theta,y,x) = \frac{\sqrt{nh_H\phi_{\theta,x}(h_K)}}{\widehat{f}_D(\theta,x)}\Big\{\mathbb{E}\widehat{f}_N(\theta,y,x) - \alpha_1(\theta,x)f(\theta,y,x) + f(\theta,y,x)(\alpha_1(\theta,x) - \mathbb{E}\widehat{f}_D(\theta,x))\Big\}.$$

Firstly, observed that the results below

$$\frac{1}{\phi_{\theta,x}(h_K)}\mathbb{E}\left[K^l\left(\frac{< x - X_i, \theta >}{h_K}\right)\right] \longrightarrow \alpha_l(\theta,x), \quad \text{as} \quad n\to\infty, \quad \text{for} \quad l = 1,2, \tag{5.4}$$

$$\mathbb{E}\left[\widehat{f}_D(\theta,x)\right] \longrightarrow \alpha_1(\theta,x) \ \text{ and } \mathbb{E}\left[\widehat{f}_N(\theta,y,x)\right] \longrightarrow \alpha_1(\theta,x)f(\theta,y,x), \quad \text{as} \quad n\to\infty, \tag{5.5}$$

can be proved in the same way as in Ezzahrioui and Ould-Saïd (2008) [11] corresponding to their lemmas Lemma 1 and Lemma 3, and then their proofs are omitted.

Secondly, on the one hand, making use of (5.4) and (5.5), we have

$$\Big\{\mathbb{E}\widehat{f}_N(\theta,y,x) - \alpha_1(\theta,x)f(\theta,y,x) + f(\theta,y,x)(\alpha_1(\theta,x) - \mathbb{E}\widehat{f}_D(\theta,x))\Big\} \underset{n\to\infty}{\longrightarrow} 0.$$

On other hand,

$$\frac{\sqrt{nh_H\phi_{\theta,x}(h_K)}}{\widehat{f}_D(\theta,x)} = \frac{\sqrt{nh_H\phi_{\theta,x}(h_K)}\widehat{f}(\theta,y,x)}{\widehat{f}_D(\theta,x)\widehat{f}(\theta,y,x)} = \frac{\sqrt{nh_H\phi_{\theta,x}(h_K)}\widehat{f}(\theta,y,x)}{\widehat{f}_N(\theta,y,x)}.$$

Since $K(\cdot)$ and $H(\cdot)$ are continuous with support on $[0,1]$, then by (H3) and (H4) $\exists m = \inf_{[0,1]} K(t)H(t)$, if follows that

$$\widehat{f}_N(\theta,y,x) \geq \frac{m}{h_H\phi_{\theta,x}(h_K)}, \quad \text{which gives} \quad \frac{\sqrt{nh_H\phi_{\theta,x}(h_K)}}{\widehat{f}_N(\theta,y,x)} \leq \frac{\sqrt{nh_H^3\phi_{\theta,x}^3(h_K)}}{m}.$$

Finally, using (H6)-(iii), the proof of Lemma 5.3.2 is completed.

□

5.4 Application: The conditional mode in functional single-index model

The main objective of this section is to establish the asymptotic normality of the kernel estimator of the conditional mode $M_\theta(x)$ defined as

$$\widehat{M}_\theta(x) = \arg\sup_{y\in\mathcal{S}_\mathbb{R}} \widehat{f}_\theta^x(y). \tag{5.6}$$

In order to present the estimation of the conditional mode in the functional single-index model, we introduce the following additional smoothness condition.

(U1) $f(\theta, \cdot, x)$ is twice continuously differentiable around the point $M_\theta(x)$ with $f^{(1)}(\theta, M_\theta(x), x) = 0$ and $f^{(2)}(\theta, \cdot, x)$ is uniformly continuous on $S_{\mathbb{R}}$ such that $f^{(2)}(\theta, M_\theta(x), x) \neq 0$, where $f^{(j)}(\theta, \cdot, x)$ $(j = 1, 2)$ is the jth order derivative of the conditional density $f(\theta, y, x)$.

(U2) $\forall \varepsilon > 0$, $\exists \eta > 0$, $\forall y \in S_{\mathbb{R}}$

$$|M_\theta(x) - y| \geq \varepsilon \Rightarrow |f(\theta, M_\theta(x), x) - f(\theta, y, x)| \geq \eta.$$

(U3) The conditional density function $f(\theta, y, x)$ satisfies: $\exists \beta_0 > 0,\ \forall (y_1, y_2) \in S_{\mathbb{R}} \times S_{\mathbb{R}}$,

$$|f^{(j)}(\theta, y_1, x) - f^{(j)}(\theta, y_2, x)| \leq C\big(|y_1 - y_2|^{\beta_0}\big),\ \forall j = 1, 2.$$

(U4) $H^{'}$and $H^{''}$ are bounded respectively with

$$\int \left(H^{'}(t)\right)^2 dt < \infty\,, \int |t|^{\beta_0} H(t) dt < \infty.$$

Theorem 5.4.1. *Suppose that hypotheses (H1)-(H6) and (U1)-(U4) are satisfied. If*

$$n h_H^3 \phi_{\theta,x}(h_K) \underset{n \to \infty}{\longrightarrow} 0, \tag{5.7}$$

we have as $n \to \infty$

$$\sqrt{\frac{n h_H^3 \phi_{\theta,x}(h_K)}{\nu^2(\theta, M_\theta(x), x)}} (\widehat{M}_\theta(x) - M_\theta(x)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1), \tag{5.8}$$

where

$$\nu^2(\theta, M_\theta(x), x) = \frac{\alpha_2(\theta, x) f(\theta, M_\theta(x), x)}{\left(\alpha_1(\theta, x) f^{(2)}(\theta, M_\theta(x), x)\right)^2} \int (H^{'}(t))^2 dt.$$

Proof. Writing the first order Taylor expansion for $f^{(1)}(\theta, y, x)$ at point $M_\theta(x)$ leads to the existence of some $M_\theta^*(x)$ between $\widehat{M}_\theta(x)$ and $M_\theta(x)$, and by the fact that $f^{(1)}(\theta, M_\theta(x), x) = 0$ (condition (U1)), we obtain

$$\sqrt{n h_H^3 \phi_{\theta,x}(h_K)} (\widehat{M}_\theta(x) - M_\theta(x)) = \frac{-\sqrt{n h_H^3 \phi_{\theta,x}(h_K)} \widehat{f}^{(1)}(\theta, M_\theta(x), x)}{\widehat{f}^{(2)}(\theta, M_\theta^*(x), x)}.$$

In order to prove (5.8), we only need to show that

$$-\sqrt{n h_H^3 \phi_{\theta,x}(h_K)} \widehat{f}^{(1)}(\theta, M_\theta(x), x) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \nu_0^2(\theta, M_\theta(x), x)), \tag{5.9}$$

and

$$\widehat{f}^{(2)}(\theta, M_\theta^*(x), x) \longrightarrow f^{(2)}(\theta, M_\theta(x), x) \neq 0, \text{ in probability,} \tag{5.10}$$

where

$$\nu_0^2(\theta, M_\theta(x), x) = \frac{\alpha_2(\theta, x) f(\theta, M_\theta(x), x)}{\Big(\alpha_1(\theta, x)\Big)^2} \int (H^{'}(t))^2 dt.$$

In fact, because the continuity of the function $f(\theta, y, x)$ and by (U2) and the definitions of $\widehat{M_\theta}(x)$ and $M_\theta(x)$, we have, for all $\varepsilon > 0$, $\exists \eta(\varepsilon) > 0$ such that:

$$\begin{aligned} \mathbb{P}\left(|\widehat{M_\theta}(x) - M_\theta(x)| \geq \varepsilon\right) \;&\leq\; \mathbb{P}\left(|f(\theta, M_\theta(x), x) - \widehat{f}(\theta, M_\theta(x), x)| \geq \frac{\eta(\varepsilon)}{2}\right) \\ &+\; \mathbb{P}\left(|\widehat{f}(\theta, \widehat{M_\theta}(x), x) - f(\theta, \widehat{M_\theta}(x), x)| \geq \frac{\eta(\varepsilon)}{2}\right). \end{aligned} \tag{5.11}$$

Thus, similar to Ferraty and Vieu (2006) [15], by (H1)-(H4) and (H6)-(i), we have $\widehat{f}(\theta, y, x) \longrightarrow f(\theta, y, x)$ in probability, which implies that $\widehat{M_\theta}(x) \longrightarrow M_\theta(x)$ in probability by (5.11). Similarly, the methodology can be also applied to obtain $\widehat{f}^{(2)}(\theta, y, x) \longrightarrow f^{(2)}(\theta, y, x)$ in probability as $n \to \infty$ by (H1), (H4), (H6), (U3) and (U4). Therefore, (5.10) is valid by the fact that $f^{(2)}(\theta, y, x)$ is uniformly continuous with respect to y on $\mathcal{S}_{\mathbb{R}}$. Next, we prove (5.9). In fact, since

$$\begin{aligned} \widehat{f}^{(1)}(\theta, M_\theta(x), x) \;&=\; \frac{1}{\widehat{f}_D(\theta, x)}\left(\widehat{f}_N^{(1)}(\theta, M_\theta(x), x) - \mathbb{E}\widehat{f}_N^{(1)}(\theta, M_\theta(x), x)\right) \\ &- \frac{1}{\widehat{f}_D(\theta, x)}\left(f^{(1)}(\theta, M_\theta(x), x) - \mathbb{E}\widehat{f}_N^{(1)}(\theta, M_\theta(x), x)\right). \end{aligned} \tag{5.12}$$

By (U1), (U3)-(U4) and (5.12), similar to the proof of Lemmas, Lemma 5.3.1 and Lemma 5.3.2 respectively, (5.9) follows directly. Then, the proof of Theorem 5.4.1 is completed. □

5.4.1 Application and Confidence bands

The asymptotic variances $\sigma^2(\theta, y, x)$ and $\nu^2(\theta, M_\theta(x), x)$ in Theorem 5.3.1 and Theorem 5.4.1 depend on some unknown quantities including $\alpha_1(\theta, x)$, $\alpha_2(\theta, x)$, $M_\theta(x)$ and $f(\theta, \cdot, x)$. So, $M_\theta(x)$ and $f(\theta, \cdot, x)$ should be replaced by their estimators $\widehat{M_\theta}(x)$ and $\widehat{f}(\theta, \cdot, x)$ previously given in (5.6) and (5.1), respectively.

By the assumptions (H1)-(H4), $\alpha_l(\theta, x)$ for $l = 1, 2$, can be estimated by $\widehat{\alpha}_l(\theta, x)$, which is defined as :

$$\widehat{\alpha}_l(\theta, x) = \frac{1}{n\widehat{\phi}_{\theta,x}(h_K)} \sum_{i=1}^{n} K_i^l(\theta, x), \text{ for } l = 1, 2, \qquad \text{where } \widehat{\phi}_{\theta,x}(h_K) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{|<x-X_i,\theta>|<h_K\}},$$

with $\mathbf{1}_{\{.\}}$ being the indicator function.

By applying the kernel estimator of $f(\theta, y, x)$ given above, the quantity $\sigma^2(\theta, y, x)$ can be estimated finally by:

$$\widehat{\sigma}^2(\theta, y, x) = \frac{\widehat{\alpha}_2(\theta, x)\widehat{f}(\theta, y, x)}{\widehat{\alpha}_1^2(\theta, x)} \int H^2(t)dt.$$

Hence, we can derive the following corollary:

Corollary 5.4.1. *Under the assumptions of Theorem 5.3.1, we have as $n \to \infty$*

$$\sqrt{\frac{nh_H\widehat{\phi}_{\theta,x}(h_K)}{\widehat{\sigma}^2(\theta, y, x)}}\left(\widehat{f}(\theta, y, x) - f(\theta, y, x)\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1).$$

Proof. Observe that

$$\begin{aligned}
\Sigma &= \frac{\widehat{\alpha}_1(\theta, x)}{\sqrt{\widehat{\alpha}_2(\theta, x)}}\sqrt{\frac{nh_H\widehat{\phi}_{\theta,x}(h_K)}{\widehat{f}(\theta, y, x)}}\left(\widehat{f}(\theta, y, x) - f(\theta, y, x)\right) \\
&= \frac{\widehat{\alpha}_1(\theta, x)\sqrt{\alpha_2(\theta, x)}}{\alpha_1(\theta, x)\sqrt{\widehat{\alpha}_2(\theta, x)}}\sqrt{\frac{nh_H\widehat{\phi}_{\theta,x}(h_K)f(\theta, y, x)}{\widehat{f}(\theta, y, x)nh_H\phi_{\theta,x}(h_K)}} \\
&\quad \times \frac{\alpha_1(\theta, x)}{\sqrt{\alpha_2(\theta, x)}}\sqrt{\frac{nh_H\phi_{\theta,x}(h_K)}{f(\theta, y, x)}}\left(\widehat{f}(\theta, y, x) - f(\theta, y, x)\right).
\end{aligned}$$

Via Theorem 5.3.1, we have

$$\frac{\alpha_1(\theta, x)}{\sqrt{\alpha_2(\theta, x)}}\sqrt{\frac{nh_H\phi_{\theta,x}(h_K)}{f(\theta, y, x)}}\left(\widehat{f}(\theta, y, x) - f(\theta, y, x)\right) \longrightarrow \mathcal{N}(0,\, 1).$$

Next, by Laib and Louani (2010) [22], we can prove that

$$\widehat{\alpha}_1(\theta, x) \xrightarrow{\mathbb{P}} \alpha_1(\theta, x),\;\; \widehat{\alpha}_2(\theta, x) \xrightarrow{\mathbb{P}} \alpha_2(\theta, x), \text{ and } \frac{\widehat{\phi}_{\theta,x}(h_K)}{\phi_{\theta,x}(h_K)} \xrightarrow{\mathbb{P}} 1, \quad \text{as} \quad n \to \infty. \tag{5.13}$$

Therefore, we obtain

$$\frac{\widehat{\alpha}_1(\theta, x)\sqrt{\alpha_2(\theta, x)}}{\alpha_1(\theta, x)\sqrt{\widehat{\alpha}_2(\theta, x)}}\sqrt{\frac{nh_H\widehat{\phi}_{\theta,x}(h_K)f(\theta, y, x)}{\widehat{f}(\theta, y, x)nh_H\phi_{\theta,x}(h_K)}} \longrightarrow 1, \text{ as } n \to \infty.$$

This yields the proof of Corollary 5.4.1. □

Now, in order to show the asymptotic $(1 - \xi)$ confidence interval of $M_\theta(x)$, we need to consider the estimator of $\nu^2(\theta, M_\theta(x), x)$ as follows :

$$\widehat{\nu}^2(\theta, \widehat{M}_\theta(x), x) = \frac{\widehat{\alpha}_2(\theta, x)\widehat{f}(\theta, \widehat{M}_\theta(x), x)}{\left(\widehat{\alpha}_1(\theta, x)\widehat{f}^{(2)}(\theta, \widehat{M}_\theta(x), x)\right)^2} \int (H^{'}(t))^2 dt.$$

Thus, the following corollary is obtained.

Corollary 5.4.2. *Under conditions of Theorem 5.4.1, as $n \to \infty$ we have*

$$\sqrt{\frac{nh_H^3 \widehat{\phi}_{\theta,x}(h_K)}{\widehat{\nu}^2(\theta, \widehat{M}_\theta(x), x)}}(\widehat{M}_\theta(x) - M_\theta(x)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1).$$

Proof. Observe that

$$\begin{aligned}
\Sigma' &= \frac{\widehat{\alpha}_1(\theta, x)\widehat{f}^{(2)}(\theta, \widehat{M}_\theta(x), x)}{\sqrt{\widehat{\alpha}_2(\theta, x)}} \sqrt{\frac{nh_H^3 \widehat{\phi}_{\theta,x}(h_K)}{\widehat{f}(\theta, \widehat{M}_\theta(x), x)}} \left(\widehat{M}_\theta(x) - M_\theta(x)\right) \\
&= \frac{\widehat{\alpha}_1(\theta, x)\sqrt{\alpha_2(\theta, x)}}{\alpha_1(\theta, x)\sqrt{\widehat{\alpha}_2(\theta, x)}} \sqrt{\frac{nh_H^3 \widehat{\phi}_{\theta,x}(h_K) f(\theta, M_\theta(x), x)}{\widehat{f}(\theta, \widehat{M}_\theta(x), x) nh_H^3 \phi_{\theta,x}(h_K)}} \frac{\widehat{f}^{(2)}(\theta, \widehat{M}_\theta(x), x)}{f^{(2)}(\theta, M_\theta(x), x)} \\
&\quad \times \frac{\alpha_1(\theta, x)}{\sqrt{\alpha_2(\theta, x)}} \sqrt{\frac{nh_H^3 \phi_{\theta,x}(h_K)}{f(\theta, M_\theta(x), x)}} f^{(2)}(\theta, M_\theta(x), x) \left(\widehat{M}_\theta(x) - M_\theta(x)\right).
\end{aligned}$$

Making use of Theorem 5.4.1, we obtain

$$\frac{\alpha_1(\theta, x)}{\sqrt{\alpha_2(\theta, x)}} \sqrt{\frac{nh_H^3 \phi_{\theta,x}(h_K)}{f(\theta, M_\theta(x), x)}} f^{(2)}(\theta, M_\theta(x), x) \left(\widehat{M}_\theta(x) - M_\theta(x)\right) \longrightarrow \mathcal{N}(0,\, 1).$$

Further, by considering (5.10) and (5.13), we obtain

$$\frac{\widehat{\alpha}_1(\theta, x)\sqrt{\alpha_2(\theta, x)}}{\alpha_1(\theta, x)\sqrt{\widehat{\alpha}_2(\theta, x)}} \sqrt{\frac{nh_H^3 \widehat{\phi}_{\theta,x}(h_K) f(\theta, M_\theta(x), x)}{\widehat{f}(\theta, \widehat{M}_\theta(x), x) nh_H^3 \phi_{\theta,x}(h_K)}} \frac{\widehat{f}^{(2)}(\theta, \widehat{M}_\theta(x), x)}{f^{(2)}(\theta, M_\theta(x), x)} \xrightarrow{\mathbb{P}} 1, \quad \text{as } n \to \infty.$$

Hence, the proof is completed. □

Remark 5.4.1. *Thus, following the corollaries, Corollary 5.4.1 and Corollary 5.4.2, the asymptotic $(1 - \xi)$ confidence interval of $f(\theta, y, x)$ and $M_\theta(x)$ are given by*

$$\widehat{f}(\theta, y, x) \pm \tau_{\xi/2} \times \frac{\widehat{\sigma}(\theta, x)}{\sqrt{nh_H \widehat{\phi}_{\theta,x}(h_K)}} \quad \text{and } \widehat{M}_\theta(x) \pm \tau_{\xi/2} \times \frac{\widehat{\nu}(\theta, \widehat{M}_\theta(x), x)}{\sqrt{nh_H^3 \widehat{\phi}_{\theta,x}(h_K)}},$$

where $\tau_{\xi/2}$ is the upper $\xi/2$ quantile of standard Normal $\mathcal{N}(0, 1)$.

5.5 Simulation study

To study the behavior of our conditional mode estimator, we consider in this part two examples of simulation, where we compare our model FSIM (fonctional single index model) with that of NPFDA (non-parametric functional data analysis).
The best way to know the behavior of the estimator of conditional density is to compute its mean square error. So, in this part of paper we compare between the conditional density estimation in the FSIM which is our model and the conditional density estimation in the NPFDA defined in (5.14).

$$\widehat{f}_n(y|x) = \frac{h_H^{-1} \sum_{i=1}^{n} K\left(h_K^{-1} d(x, X_i)\right) H\left(h_H^{-1}(y - Y_i)\right)}{\sum_{i=1}^{n} K\left(h_K^{-1} d(x, X_i)\right)}, \tag{5.14}$$

where we estimate the conditional mode function $\widehat{M}(x)$ such that

$$M(x) = \arg \sup_{y \in \mathcal{S}_{\mathbb{R}}} f(y|x) \ \text{ and } \ \widehat{M}(x) = \arg \sup_{y \in \mathcal{S}_{\mathbb{R}}} \widehat{f}_n(y|x).$$

In the following, our purpose consists in assessing the performance in terms of prediction of $\widehat{M}_\theta(x)$ and $\widehat{M}(x)$. For each given predictor $(X_j)_{j \in \mathcal{J}}$ in the testing subsample, we are interested in the prediction of the response variable $(Y_j)_{j \in \mathcal{J}}$ via the single functional index conditional mode $\widehat{M}_\theta(x)$ and the fully nonparametric conditional mode $\widehat{M}(x)$ so as to compare the finite-sample behavior of the estimator. As assessment tool, we consider the mean square error (MSE) defined as follows:

$$MSE = \frac{1}{|\mathcal{J}|} \sum_{j \in \mathcal{J}} \left(Y_j - \widehat{Y}_j\right)^2, \tag{5.15}$$

where $\widehat{Y}_j$ is a predictor of Y_j obtained either semi-parametrically by $\widehat{M}_\theta(x)$ or nonparametrically via $\widehat{M}(x)$.

Furthermore, some tuning parameters have to be specified. The kernel $K(\cdot)$ is chosen to be the quadratic function defined as $K(u) = \frac{3}{2}(1 - u^2)\,\mathbf{1}_{[0,1]}$, and the cumulative df $H(u) = \int_{-\infty}^{u} \frac{3}{4}(1 - z^2)\,\mathbf{1}_{[-1,1]}(z)\,dz$.

The semi-metric $d(\cdot, \cdot)$ will be specified according to the choice of the functional space $\mathcal{H}$ discussed in the scenarios below. It is well-known that one of the crucial parameters in semi-parametric models is the smoothing parameters which are involved in defining the shape of the link function between the response and the covariate.

Using the result given in Theorem 5.4.1, the variance of our estimator is obtained as

$$CV = \frac{\alpha_2(\theta,x)f(\theta,M_\theta(x),x)}{nh_H^3\phi_{\theta,x}(h_K)\Big(\alpha_1(\theta,x)f^{(2)}(\theta,M_\theta(x),x)\Big)^2}.$$

The idea is to choose the parameters h_K and h_H so that the variance is minimal. Since the variance (CV) depends on several unknown parameters that must be estimated, the calculus becomes tedious. Thus, by replacing the unknown parameters by their respective estimators $\widehat{\alpha}_1(\theta,x)$, $\widehat{\alpha}_2(\theta,x)$, $\widehat{M_\theta}(x)$, $\widehat{f}(\theta,\cdot,x)$, and $\widehat{\phi}_{\theta,x}(h_K)$, we obtain

$$(h_K,h_H) = \arg\min_{h_K,h_H} CV\,(h_K,h_H) = \arg\min_{h_K,h_H} \frac{\widehat{\alpha}_2(\theta,x)\widehat{f}(\theta,\widehat{M_\theta}(x),x)}{nh_H^3\widehat{\phi}_{\theta,x}(h_K)\Big(\widehat{\alpha}_1(\theta,x)\widehat{f}^{(2)}(\theta,\widehat{M_\theta}(x),x)\Big)^2}.$$

Now, for simplifying the implementation of our methodology, we take the bandwidths $h_H \sim h_K = h$, where h will be chosen by the cross-validation method on the k-nearest neighbors (see Ferraty and Vieu (2006) [15], p. 102).

Example 5.5.1. ***Case of smooth curves:***

Let us consider the following regression model, where the covariate is a curve and the response is a scalar:

$$T_i = R\,(X_i) + \epsilon_i,\ i = 1,\ldots,n,$$

where ϵ_i is a sequence of i.i.d. random variables normally distributed with a variance equal to 0.1.

The functional covariate X is assumed to be a diffusion process defined on $[0,\ 1]$ *and generated by the following equation:*

$$X_i(t) = a_i\cos(b_i + \pi W_i t) + c_i\sin(d_i + \pi W_i t) + (1-a_i)\sin(\pi t W_i),\ t\in[0,\ 1]\,,$$

where W_i, b_i and d_i are independent of normal distributions $\rightsquigarrow \mathcal{N}(0,\ 1)$, $\rightsquigarrow \mathcal{N}(0,\ 0.03)$ *and* $\rightsquigarrow \mathcal{N}(0,\ 0.05)$, *respectively. The variables a_i and c_i are Bernoulli's laws* $\rightsquigarrow$ *Bernoulli*(0.5). *Figure 5.1 depicts a sample of* 200 *curves representing a realization of the functional random variable X.*

Take into account of the smoothness of the curves $X_i(t)$ (see Figure 5.1), we choose the distance $deriv_1$ (the semi-metric based on the first derivatives of the curves) in $\mathcal{H}$ as:

$$d\,(\chi_1,\chi_2) = \left(\int_0^1 \left(\chi_1'\,(t) - \chi_2'\,(t)\right)^2 dt\right)^{1/2}.$$

as semi-metric.

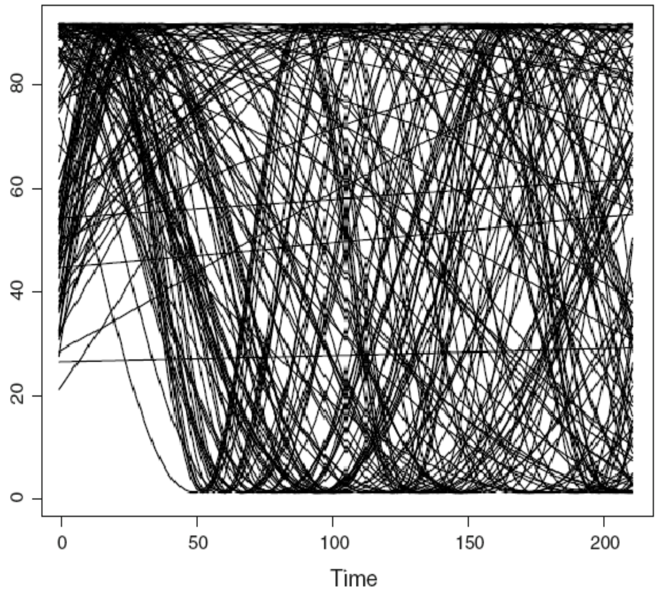

Figure 5.1: A sample of 200 curves $X_{i=1,\ldots,200}(t)$, $t \in [0, 1]$

Then, we consider a nonlinear regression function defined as

$$R(X) = 4\log\left\{1/\left(\int_0^1 (X'(t))^2\, dt + \left[\int_0^1 X'(t)dt\right]^2\right)\right\}.$$

Given $X = x$, we have $T \rightsquigarrow \mathcal{N}(R(x), 0.2)$.
The computation of our estimator is based on the observed data $(X_i, Y_i)_{i=1,\ldots,n}$, and the single index θ which is unknown and has to be estimated.

In practice, this parameter can be selected by cross-validation approach (see Aït-Saïdi et al. *(2008) [2]). In this passage, it may be that one can select the real-valued function $\theta(t)$ among the eigenfunctions of the covariance operator $\mathbb{E}[(X' - \mathbb{E}X') < X', \cdot >_{\mathcal{H}}]$, where $X(t)$ is a diffusion processes defined on a real interval $[a, b]$ and $X'(t)$ its first derivative (see Attaoui and Ling (2016) [6]). Hence, for the chosen training sample $\mathcal{L}$, by applying the principal component analysis (PCA) method, the computation of the eigenvectors of the covariance operator estimated by its empirical covariance operator: $\frac{1}{\mathcal{L}}\sum_{i\in\mathcal{L}}(X_i' - \mathbb{E}X')^t(X_i' - \mathbb{E}X')$ will be the one best approximation of our functional parameter θ. Now, let us denote $\theta^\star$ the first eigenfunction corresponding to the first higher eigenvalue of the empirical covariance operator, which will replace θ during the simulation step.*

In the following graphs, the covariance operator for $\mathcal{L} = \{1, \ldots, 125\}$ gives the discretization of the first eigenfunction θ (presented by a continuous curve), twenty, and all the eigenfunctions $\theta_i(t)$ (Figure 5.2, Figure 5.3 and Figure 5.4).

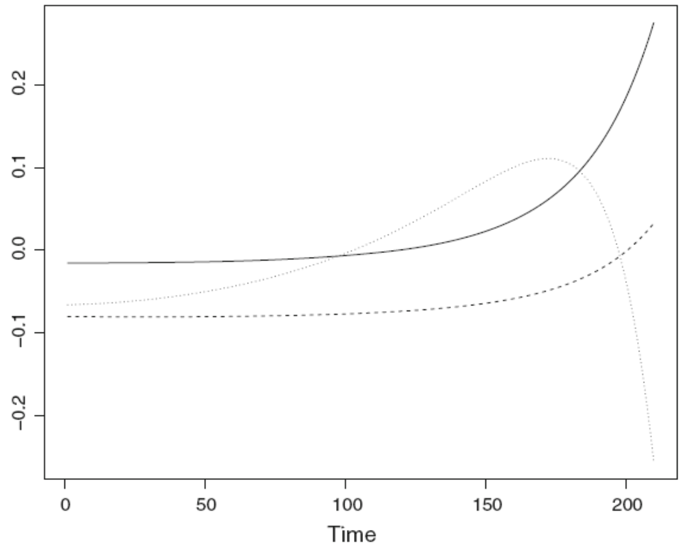

Figure 5.2: The curves $\theta_{i=1,2,3}(t)$, $t \in [0,\ 1]$

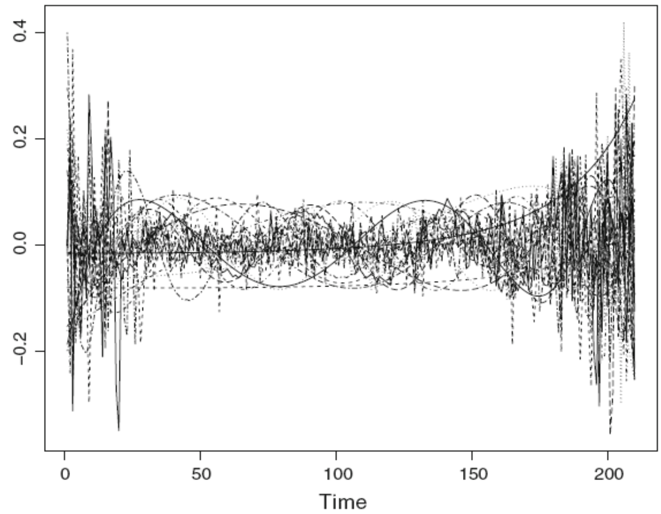

Figure 5.3: The curves $\theta_{i=1,\ldots,20}(t)$, $t \in [0,\ 1]$

In this simulation part, we divide our sample of size 200 *into two parts. The first one from* 1 *to* 125 *(learning sample) and the second from* 126 *to* 200 *(test sample).*

We follow the following steps:

Step 1. *Simulate the responses variables* Y_i*.*

Step 2. *For each* j *in the test sample* $\mathcal{J} = \{126, \ldots, 200\}$*, we compute:* $\widehat{Y}_j = \widehat{M}_{\theta^\star}(X_j)$ *and* $\widehat{Y}_j = \widehat{M}(X_j)$*.*

Finally, we present the results by plotting the predicted values versus the true values and compute the mean square error (MSE) defined by (5.15).

We see that the mean square error (MSE) of our method Functional-Single-Index-Model (FSIM) is less than the one of the Non-Parametric-Functional-Data-Analysis (NPFDA).

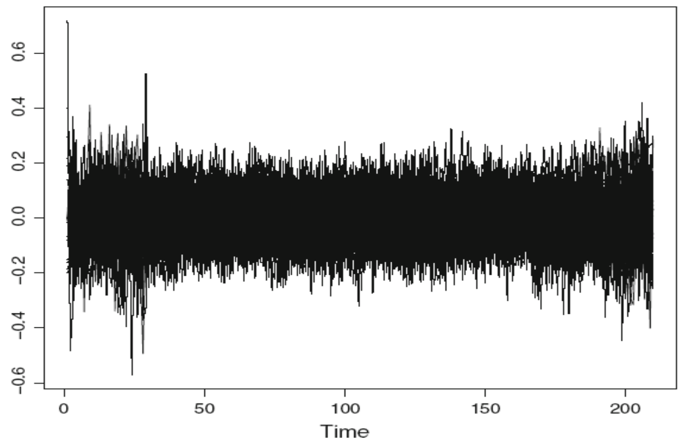

Figure 5.4: The curves $\theta_{i=1,\ldots,125}(t),\ t \in [0,\ 1]$

This is confirmed by the following graphs, when we compare the conditional mode by (FSIM) against the conditional mode by (NPFDA) (Figure 5.5). Our estimator is so acceptable. As intuitively expected, it is well observed that the mean square errors of our estimator are smaller than that of NPFDA. Thus, again, the FSIM model produces much more accurate estimation accuracies than NPFDA model in all criteria.

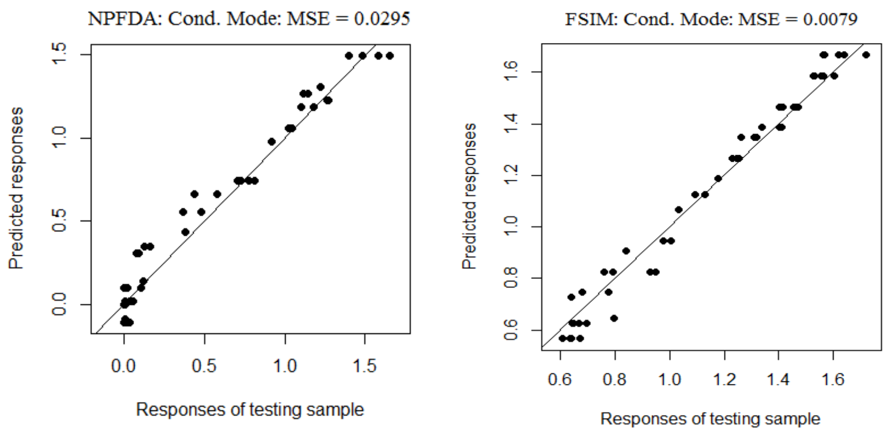

Figure 5.5: Comparison between the NPFDA and the FSIM via the conditional mode

In order to construct conditional confidence bands we proceed by the following algorithm:

Step 1. *We compute the inner product:* $< \theta^\star, X_1 >, \cdots, < \theta^\star, X_{200} >$.

Step 2. *For each* i *in the training sample, we calculate the estimator:* $\widehat{Y}_i = \widehat{M}_{\theta^\star}(X_i)$.

Step 3. *For each* X_j *in the test sample* $\mathcal{J} = 126, \ldots, 200$, *we set:* $i^\star := \arg\min_{i \in \mathcal{L}} d_\theta(X_i, X_j)$.

Step 4. *For each* j *in the test sample* $\mathcal{J} = 126, \ldots, 200$, *we define the confidence bands by*

$$\left[\widehat{M}_{\theta^\star}(X_{i^\star}) - \tau_{0.975} \times \left(\frac{\widehat{\nu}(\theta^\star, X_{i^\star})}{\sqrt{\mathcal{L} h_H^3 \widehat{\phi}_{\theta^\star,x}(h_K)}}\right),\ \widehat{M}_{\theta^\star}(X_{i^\star}) + \tau_{0.975} \times \left(\frac{\widehat{\nu}(\theta^\star, X_{i^\star})}{\sqrt{\mathcal{L} h_H^3 \widehat{\phi}_{\theta^\star,x}(h_K)}}\right)\right].$$

We obtain the following figure (Figure 5.6) which gathers asymptotic confidence bands study.

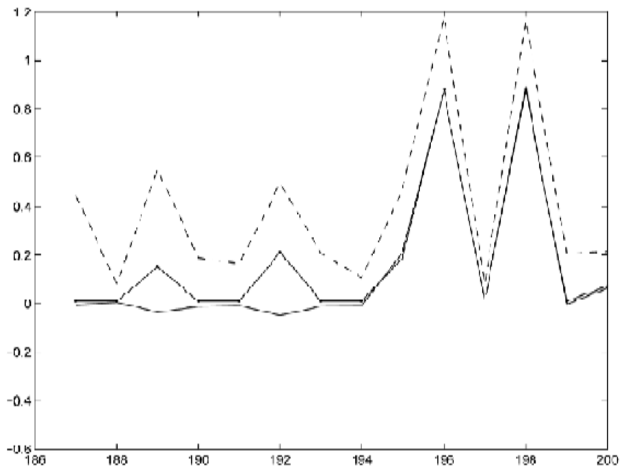

Figure 5.6: The 95% conditional predictive bands. The solid curve connects the true values. The crossed curve joins the predicted values. The dashed curves connects the lower and upper predicted values

For making a decision, we choose an other Example 5.5.2 in which the distribution of the model is known and usual.

Example 5.5.2. *Let $X_1, \dots, X_n$ be a standard Brownian movements in $[0,1]$, with $n =$ 100. Our study focuses on the linear model with defined functional index by:*

$$Y_i = \frac{|<\theta,\, X_i>|}{150} + 0.5\,\epsilon_i,$$

where $(\epsilon_i)_i$ are i.i.d and $\epsilon_i \rightsquigarrow \mathcal{N}\,(0,\,1)$. We keep the values of $\theta^\star$ and $(X_i)_{i=1,\dots,100}$ of the precedent example (θ is replaced by $\theta^\star$).

According to this model, it is clear that, when $X = x$, the variable $Y \rightsquigarrow \mathcal{N}\left(\frac{|<\theta,\, x>|}{150}, 4\right)$. In this study, as the curves are rough (see Figure 5.7), we use the semi-metric pca.

Table 5.1: Estimation accuracy of the conditional mode function between the functional single index model and the nonparametric functional model for different values of ξ

Error	Model	Semi-metric	$n = 100$		
			$\xi = 0.05$	$\xi = 0.50$	$\xi = 0.95$
MSE	FSIM	pca	0.0116	0.0112	0.0127
	NPFDA	pca	0.0634	0.0621	0.0641

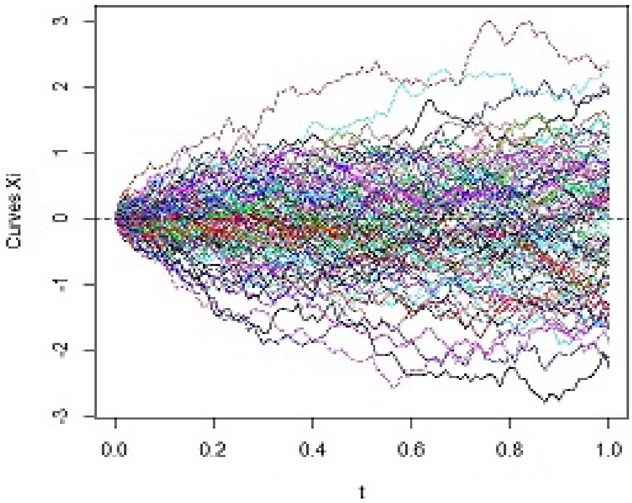

Figure 5.7: Standard Brownian motions

Table 5.1 presents the MSE of FSIM and NPFDA models by considering different values of ξ*, with* 100 *replications. From the obtained results presented in Table 5.1, we can confirm that our FSIM estimator of conditional mode is better than that of NPFDA. It gives a smaller mean square error. So it allows for a more accurate estimation.*

After the calculation of the errors, we find for our method an error $MSE = 0.0225$*. The NPFDA method gives an error* $MSE = 0.0763$*, while the real error (knowing that* $Y \rightsquigarrow \mathcal{N}\left(\frac{|<\theta, x>|}{150}, 4\right)$*) is equal to* $MSE = 1.938\,10^{-31}$ *(see Figure 5.8). This confirms once again that our estimator is much better than that of NPFDA case. So, in the context of i.i.d data, our estimator is much preferable.*

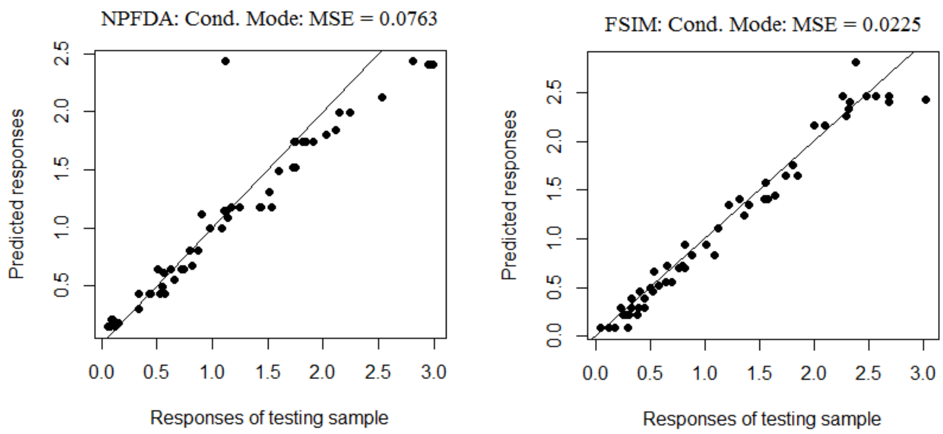

Figure 5.8: Comparison between the NPFDA and the FSIM via the conditional mode

5.6 Conclusion

This paper focused on the nonparametric estimation of the conditional mode in the single functional index model for independent data. Both the asymptotic normality as well as a confidence interval of the resulted estimator were derived. The proofs are based on a combination of existing techniques. The study's prime aim was to improve the performance of the single-index model for the conditional mode with a scalar response variable conditioned by a functional Hilbertian regressor under the independent property. Through a series of simulations, this model out performs the nonparametric functional estimator. The contribution in this study is focused on the estimation of the conditional density function for complete data in a functional framework. This approach is used for the estimation of the conditional mode. Then on parametric aspect is properly exploited in the first two sections by the given hypotheses. The proposed estimators are consistent an asymptotically distributed under appropriate conditions. Note that this approach is more significant in the presence of a simple single functional index. Then, the estimation and forecast accuracies between the FSIM and NPFDA models were evaluated and compared, and via empirical analysis, it was shown that the considered estimator has good finite sample behavior for the prediction, and provides improved estimation and prediction accuracy compared to the NPFDA estimator.

In addition, in order to explore the effectiveness of our method in real situations, we can apply our approach to data constituting hourly electricity demand as well as spectrometric data. Another real example is forecasting the daily peak in electricity demand, accurate prediction of daily peak load demand is very important for decision in the energy sector. In fact, short-term load forecasts enable effective load shifting between transmission substations, scheduling of startup times of peak stations, load flow analysis and power system security studies. Other real data application (Maximum Ozone Concentration, Peak electricity demand) can be highlighted several attractive features of in functional prediction context, with unknown scale parameter estimator.

Research in the nonparametric field remains an open question which will be the subject of several future studies in order to improve and highlight the results obtained in this work. Extend our study of estimation of the conditionals mode to the estimation of the conditional models of a MAR (missing at random) response to the independent case and the dependent case. Another type of dependency could be considered such as the quasi-associated case. Develop the asymptotic properties of a kernel estimator of the k-nearest neighbors. Generalize the obtained results by using other families of semi-metrics in order to improve the prediction performance of our estimators so the choice of the smoothing window is important.

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Chapter 6

CLT for single functional index quantile regression under dependence structure

This chapter is the topic of an international publication in
Acta Universitatis Sapientiae, Mathematica.

CLT for single functional index quantile regression under dependence structure

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Abstract: In this paper, we investigate the asymptotic properties of a nonparametric conditional quantile estimation in the single functional index model for dependent functional data and censored at random responses are observed. First of all, we establish asymptotic properties for a conditional distribution estimator from which we derive an central limit theorem (CLT) of the conditional quantile estimator. Simulation study is also presented to illustrate the validity and finite sample performance of the considered estimator. Finally, the estimation of the functional index via the pseudo-maximum likelihood method is discussed, but not tackled.

Keywords: Conditional quantile, Censored data, Functional random variable, Kernel estimator, Nonparametric estimation, Probabilities of small balls, Strong mixing processes, Single index model.

2010 Mathematics Subject Classification: Primary 62G05, Secondary 62G99, Thirdly 62M10.

6.1 Introduction

Multivariate regression analysis is a powerful statistical tool in biomedical research and many fields of life (Muharisa *et al.* (2018) [27]) with numerous applications. While linear regression can be used to model the expected value (ie, mean) of a continuous outcome given the covariates in the model, quantile regression can be used to compare the entire distribution of a continuous response or a specific quantile of the response between groups.

Despite the regression function is of interest, other statistics such as quantile and mode regression might be important from a theoretical and a practical point of view. Quantile regression is a common way to describe the dependence structure between a response variable T and some covariate X. Unlike the regression function that relies only on the central tendency of the data, the conditional quantile function allows the analyst to estimate the functional dependence between variables for all portions of the conditional distribution of the response variable. Moreover, it is well known that conditional quantiles can give a good description of the data (see Chaudhuri *et al.* (1997) [9]), such as robustness to heavy-tailed error distributions and outliers to ordinary mean-based regression. As a particular case, note that the conditional median is useful for asymmetric distributions.

Quantile regression(QR) is one of the major statistical tools and is gradually developing into a comprehensive strategy for completing the regression prediction. It is emerging as a popular statistical approach, which complements the estimation of conditional mean models. While the latter only focuses on one aspect of the conditional distribution of the dependent variable, the mean, quantile regression provides more detailed insights by modeling conditional quantiles. Her can therefore detect whether the partial effect of a regressor on the conditional quantiles is the same for all quantiles or differs across quantiles, and can provide evidence for a statistical relationship between two variables even if the mean regression model does not. In many fields of applications like quantitative finance, econometrics, marketing and also in medical and biological sciences, QR is a fundamental element for data analysis, modeling and inference. An application in finance is the analysis of conditional Value-at-Risk, moreover, her is the development of statistical tools used to explain the relationship between response and predictor variables (see Yanuar *et al.* (2019) [37]). The quantile method is a technique of dividing a group of data into several parts after the data is sorted from the smallest to the largest Yanuar *et al.* (2017) [36]. QR enjoys some very appealing features. Apart from enabling some very exible patterns of partial effects, quantile regressions are also interesting because they satisfy some equivariance and robustness principles.

The advantage of the QR methodology is that it allows for understanding relationships between variables outside of the conditional mean of the response; it is useful for understanding an outcome at its various quantiles and comparing groups or levels of an exposure on those quantiles. QR is a common way to describe the dependence structure between a response variable T and some covariate X. Unlike the regression function (which is defined as the conditional mean) that relies only on the central tendency of the data, the conditional quantile function allows the analysts to estimate the functional dependence between variables for all portions of the conditional distribution of the response variable. Moreover, quantiles are well known for their robustness to heavy-tailed error distributions and outliers which allow to consider them as a useful alternative to the regression function Chaouch and Khardani (2015) [8].

Moreover, it is a statistical technique intended to estimate, and conduct inference about, conditional quantile functions. Just as classical linear regression methods based on minimizing sums of squared residuals enable one to estimate models for conditional mean functions, quantile regression methods offer a mechanism for estimating models for the conditional median function, and the full range of other conditional quantile functions. By supplementing the estimation of conditional mean functions with techniques for estimating an entire family of conditional quantile functions, quantile regression is capable of providing a more complete statistical analysis of the stochastic relationships among random variables.

For example, QR has been used in a broad range of application settings. Reference growth curves for children's height and weight have a long history in pediatric medicine; quantile regression methods may be used to estimate upper and lower quantile reference curves as a function of age, sex, and other covariates without imposing stringent parametric assumptions on the relationships among these curves. In ecology, theory often suggests how observable covariates affect limiting sustainable population sizes, and quantile regression has been used to directly estimate models for upper quantiles of the conditional distribution rather than inferring such relationships from models based on conditional central tendency. In survival analysis, and event history analysis more generally, there is often also a desire to focus attention on particular segments of the conditional distribution, for example survival prospects of the oldest-old, without the imposition of global distributional assumptions.

In recent years, estimating conditional quantiles has received increasing interest in the literature, for both independent and dependent data; Samanta (1989) [31] established a nonparametric estimation of conditional quantiles, Wang and Zhao (1999) [35] presented a kernel estimator for conditional t-quantiles for mixing samples and established its strong uniform convergence. Ferraty *et al.* (2005) [15] studied the estimation of a conditional quantiles for functional dependent data with application to the climatic El Ninö phenomenon. Ezzahrioui & Elias Ould-Saïd (2008) [14] considered the estimation of the conditional quantile function when the covariates take values in some abstract function space, the almost complete convergence and the asymptotic normality of the kernel estimator of the conditional quantile under the α-mixing assumption were established.

In life time data analysis, nonparametrically estimated conditional survival curves (such as the conditional Kaplan-Meier estimate) are useful for assessing the influence of risk factors, predicting survival probabilities, and checking goodness-of-fit of various survival regression models. It is well known that in medical studies the observation on the survival time of a patient is often incomplete due to right censoring. Classical examples of the causes of this type of censoring are that the patient was alive at the termination of the study, that the patient withdrew alive during the study, or that the patient died from other causes than those under study. The censored quantile regression model is derived from the censored model. This method is used to overcome problems in modeling censored data as well as to overcome the assumptions of linear models that are not met, in this linear models Sarmada and Yanuar (2020) [32] have compared the results of the analysis of the quantile regression method with the censored quantile regression method for censored data. In the context of censored data, Gannoun *et al.* (2003)

[17] introduced a local linear (LL) estimator of the quantile regression and established its almost sure consistency (without rate) as well as its asymptotic normality in the independent and identically distributed (i.i.d.) case. El Ghouch and Van Keilegom (2009) [13] considered the LL estimation of the quantile regression and its first derivative under an α-mixing assumption and studied their asymptotic properties. Ould-Saïd (2006) [28] constructed a kernel estimator of the conditional quantile under an i.i.d. censorship model and established its strong uniform convergence rate. Under an α-mixing assumption, Liang and Alvarez (2011) [21] established the strong uniform convergence (with rate) of the conditional quantile function as well as its asymptotic distribution.

The single index model is a natural extension of the linear regression model for applications in which linearity does not hold. This last approach is widely applied in econometrics as a reasonable compromise between nonparametric and parametric models. In the past few recent years, the single functional index models have received much attention, and it has been studied extensively in both statistical and econometric literatures. Interesting to this methods, many authors worked on this sort of problems, see for instance Ait-Saïdi *et al.* ((2005) [1], (2008) [2]). Attaoui *et al.* (2011) [4] investigated the kernel estimator of the conditional density of a scalar response variable T, given a Hilbertian random variable X when the observations are from a single functional index model. Ling *et al.* (2014) [23] reconsidered the kernel estimator of the conditional density when the scalar response variable T and the Hilbertian random variable X also come from the single functional index model. The asymptotic results such as pointwise almost complete consistency and the uniform almost complete convergence of the kernel estimation with rates in the setting of the α mixing functional data are also obtained, which extend the i.i.d. case in Attaoui *et al.* (2011) [4] to the dependence setting. Ling & Xu (2012) [24] investigated the estimation of conditional density function based on the single-index model for functional time series data. Under α-mixing condition, the asymptotic normality of the conditional density estimator and the conditional mode estimator where obtained. Attaoui (2014) [3] studied a nonparametric estimation of the conditional density of a scalar response variable given a random variable taking values in separable Hilbert space when the variables satisfy the strong mixing dependency, based on the single-index structure.

Inspired by all the papers above, our work in this paper aims to contribute to the research on functional nonparametric regression model, by giving an alternative estimation of QR estimation in the single functional index model with randomly right-censored data under α-mixing conditions whose definition is given below.

Recall that a process $(X_i, T_i)_{i\geq 1}$ is called α-mixing or strongly mixing (see Lin and Lu (1996) [22] for more details and examples), if

$$\sup_k \sup_{A\in\mathcal{F}_1^k} \sup_{B\in\mathcal{F}_{n+k}^\infty} |\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)| = \alpha(n) \to 0 \text{ as } n\to\infty,$$

where $\mathcal{F}_j^k$ denotes the σ-field generated by the random variables $\{(X_i,\ T_i),\ j\leq i\leq k\}$. The process $\{(X_i,\ T_i),\ i\geq 1\}$ is said to be arithmetically α mixing with order $a>0$, if

$\exists C > 0,\ \alpha(n) \leq Cn^{-a}$.

The strong-mixing condition is reasonably weak and has many practical applications (see, e.g., Cai (2011) [6], Doukhan (1994) [12], Dedecker *et al.* (2007) [10] Ch. 1, for more details). In particular, Masry and Tøjstheim (1995) [26] proved that, both ARCH processes and nonlinear additive autoregressive models with exogenous variables, which are particularly popular in finance and econometrics, are stationary and α-mixing.

This article is organized as follows: In Section 6.2, we describe our model and construct precisely the QR estimator based on the functional stationary data under censorship model. In Section 6.3, we build up asymptotic theorems for our model. Section 6.4 illustrates those asymptotic properties through some simulated data. Finally, the proofs of the main results are postponed to Section 6.5.

6.2 Notations and estimators of the semi-parametric framework

6.2.1 The model

Let (X, T) be a pair of random variables where T is a real-valued random variable and X takes its values in a separable Hilbert space $\mathcal{H}$ with the norm $\|\cdot\|$ generated by an inner product $< \cdot, \cdot >$. Let C be a censoring variable with common continuous distribution function G. The continuity of G allows to use the convergence results for the Kaplan and Meier estimator of G. ((1958)[19]).

From now on, we suppose that (X, T) and C are independent. It is plausible whenever the censoring is independent of the characteristics of the patients under study. In the right censorship model, the pair (T, C) is not directly observed and the corresponding available information is given by $Y = \min(T, C)$ and $\delta = \mathbf{1}_{\{T \leq C\}}$, where $\mathbf{1}_A$ is the indicator function of the set A.

Such censorship models have been amply studied in the literature for real or multi-dimensional random variables, and in nonparametric frameworks the kernel techniques are particularly used (see Tanner and Wong (1983) [33], Padgett (1988) [29], Lecoutre and Ould-Saïd (1995) [20] and Van Keilegom and Veraverbeke (2001) [34], for a necessarily non-exhaustive sample of literature in this area).

Furthermore, let $(X_i, T_i)_{1 \leq i \leq n}$ be the statistical sample of pairs which are identically distributed like (X, T), but not necessarily independent, $(C_i)_{1 \leq i \leq n}$ is a sequence of i.i.d. random variables which is independent of $(X_i, T_i)_{1 \leq i \leq n}$. Therefore, we assume that the sample $\{(X_i, \delta_i, Y_i), i = 1, \ldots, n\}$ is at our disposal. Moreover, we consider $d_\theta(\cdot, \cdot)$ a semi-metric associated with the single index $\theta \in \mathcal{H}$ defined by $d_\theta(x_1, x_2) := | < x_1 - x_2, \theta > |$, for x_1 and x_2 in $\mathcal{H}$.

For a fixed x in $\mathcal{H}$, the conditional cumulative distribution function (*cond-cdf*) of T given $< \theta, X > = < \theta, x >$, is defined as follows:

$$\forall t \in \mathbb{R},\ F(\theta, t, x) := \mathbb{P}(T \leq t | < X, \theta > = < x, \theta >).$$

Saying that, we are implicitly assuming the existence of a regular version for the conditional distribution of T given $< \theta, X >$. Now, let $\zeta_\theta(\gamma, x)$ be the γth-conditional quantile of the distribution of Y given $< \theta, X >=< \theta, x >$. Formally, $\zeta_\theta(\gamma, x)$ is defined as:

$$\zeta_\theta(\gamma, x) := \inf\{t \in \mathbb{R} : \; F(\theta, t, x) \geq \gamma\}, \qquad \forall \gamma \in (0, 1).$$

In order to simplify our framework and to focus on the main interest of our paper (the functional feature of $< \theta, X >$), we assume that $F(\theta, \cdot, x)$ is strictly increasing and continuous in a neighborhood of $\zeta_\theta(\gamma, x)$. This is insuring that the conditional quantile $\zeta_\theta(\gamma, x)$ is uniquely defined by:

$$\zeta_\theta(\gamma, x) \;=\; F^{-1}(\theta, \gamma, x) \;\text{ equivalently } F(\theta, \zeta_\theta(\gamma, x)\;, x) = \gamma. \tag{6.1}$$

Next, in all what follows, we assume only smoothness restrictions for the *cond-cdf* $F(\theta, \cdot, x)$ through nonparametric modeling. Assume also that $(X_i, T_i)_{i\in\mathbb{N}}$ is an α-mixing sequence, which is one among the most general mixing structures.

6.2.2 The estimators

The kernel estimator $F_n(\theta, \cdot, x)$ of $F(\theta, \cdot, x)$ is presented as follows:

$$F_n(\theta, t, x) = \frac{\sum_{i=1}^{n} K\left(h_K^{-1}(< x - X_i, \theta >)\right) H\left(h_H^{-1}(t - T_i)\right)}{\sum_{i=1}^{n} K\left(h_K^{-1}(< x - X_i, \theta >)\right)}, \tag{6.2}$$

where K is a kernel function, H a cumulative distribution function and $h_K = h_{K,n}$ (resp. $h_H = h_{H,n}$) a sequence of positive real numbers. Note that using similar ideas, Roussas (1969) [30] introduced some related estimates but in the special case when X is real, while Samanta [31] (1989) produced previous asymptotic study.

As a by-product of (6.1) and (6.2), it is easy to derive an estimator $\zeta_{\theta,n}(\gamma, x)$ of $\zeta_\theta(\gamma, x)$:

$$\zeta_{\theta,n}(\gamma, x) \;=\; F_n^{-1}(\theta, \gamma, x). \tag{6.3}$$

Such an estimator is unique as soon as H is an increasing continuous function. Such an approach has been largely used in the case where the variable X is of finite dimension (see *e.g* Whang and Zhao (1999) [35], Cai (2002) [7], Zhou and Liang (2003) [38] or Gannoun *et al.* (2003) [17]).

The objective of this section is to adapt these ideas under functional random variable X, and build a kernel type estimator of the conditional distribution $F(\theta, \cdot, X)$ adapted for censored samples. In the censoring case, based on the observed sample $(X_i, \delta_i, Y_i)_{i=1,\dots,n}$ we define the following "pseudo-estimator" of $F(\theta, \cdot, X)$ which is used as an intermediate estimator. Thus, we have

$$\widetilde{F}(\theta,t,x)=\frac{\displaystyle\sum_{i=1}^{n}\frac{\delta_i}{\bar{G}(Y_i)}K\left(h_K^{-1}(<x-X_i,\theta>)\right)H\left(h_H^{-1}(t-Y_i)\right)}{\displaystyle\sum_{i=1}^{n}K\left(h_K^{-1}(<x-X_i,\theta>)\right)}. \tag{6.4}$$

In practice $\bar{G}(\cdot)=1-G(\cdot)$ is unknown, hence it is impossible to use the estimator (6.4). Then, we replace $\bar{G}(\cdot)$ by its Kaplan and Meier (1958) [19] estimate $\bar{G}_n(\cdot)$ given by

$$\bar{G}_n(t)=1-G_n(t)=\begin{cases}\displaystyle\prod_{i=1}^{n}\left(1-\frac{1-\delta_{(i)}}{n-i+1}\right)^{\mathbf{1}_{\{Y_{(i)}\le t\}}}, & \text{if } t<Y_{(n)};\\ 0, & \text{if } t\ge Y_{(n)},\end{cases} \tag{6.5}$$

where $Y_{(1)}<Y_{(2)}<\ldots<Y_{(n)}$ are the order statistics of Y_i and $\delta_{(i)}$ is the concomitant of $Y_{(i)}$. Therefore, a full estimator of the conditional distribution function $F(\theta,\cdot,x)$ is defined as:

$$\widehat{F}(\theta,t,x)=\frac{\displaystyle\sum_{i=1}^{n}\frac{\delta_i}{\bar{G}_n(Y_i)}K\left(h_K^{-1}(<x-X_i,\theta>)\right)H\left(h_H^{-1}(t-Y_i)\right)}{\displaystyle\sum_{i=1}^{n}K\left(h_K^{-1}(<x-X_i,\theta>)\right)}. \tag{6.6}$$

Consequently, a natural estimator of $\zeta_\theta(\gamma,x)$ is given by

$$\begin{aligned}\widehat{\zeta}_\theta(\gamma,x) &= \widehat{F}^{-1}(\theta,\gamma,x)\\ &= \inf\{t\in\mathbb{R}:\ \widehat{F}(\theta,t,x)\ge\gamma\},\end{aligned} \tag{6.7}$$

which satisfies

$$\widehat{F}(\theta,\widehat{\zeta}_\theta(\gamma,x)\ ,x)=\gamma. \tag{6.8}$$

6.3 Assumptions and results

6.3.1 Assumptions on the functional variable

Let N_x be a fixed neighborhood of x and let $B_\theta(x,h_K)$ be the ball of center x and radius h, namely $B_\theta(x,h_K)=\{f\in\mathcal{H}/0<|<x-f,\theta>|<h_K\}$. Assume that, $(C_i)_{i\ge1}$ and $(T_i)_{i\ge1}$ are independent and we assume that $\tau_G:=\sup\{t:\ G(t)<1\}$ and let τ be a positive real number such that $\tau<\tau_G$.

Now, let's consider the following hypotheses:

(H1) $\forall h_K>0,\ \mathbb{P}\left(X\in B_\theta(x,h_K)\right)=\phi_{\theta,x}(h_K)>0.$

(H2) $(X_i, Y_i)_{i\in\mathbb{N}}$ is an α-mixing sequence whose the coefficients of mixture verify:

$$\exists a > 0,\ \exists c > 0 :\ \forall n \in \mathbb{N},\ \alpha(n) \le cn^{-a}.$$

(H3) $0 < \sup_{i\neq j} \mathbb{P}\left((X_i, X_j) \in B_\theta(x, h_K) \times B_\theta(x, h_K)\right) \ = \ \mathcal{O}\left(\frac{(\phi_{\theta,x}(h_K))^{(a+1)/a}}{n^{1/a}}\right).$

6.3.2 The nonparametric model

As usually in nonparametric estimation, we suppose that the cond-cdf $F(\theta, \cdot, x)$ verifies some smoothness constraints. Let β_1 and β_2 be two positive numbers; such that:

(H4) $\forall (x_1, x_2) \in N_x \times N_x$, $\forall (t_1, t_2) \in \mathcal{S}_{\mathbb{R}}^2$,

(i) $|F(\theta, t_1, x_1) - F(\theta, t_2, x_2)| \le C_{\theta,x}\left(\|x_1 - x_2\|^{\beta_1} + |t_1 - t_2|^{\beta_2}\right)$,

(ii) $\int_{\mathbb{R}} tf(\theta, t, x)dt < \infty$ for all $\theta, x \in \mathcal{H}$.

(H5) $\forall (t_1, t_2) \in \mathbb{R}^2,\ |H(t_1) - H(t_2)| \le C|t_1 - t_2|$ with $\int H^{(1)}(t)dt = 1$,

$\int H^2(t)dt < \infty$ and $\int |t|^{\beta_2} H^{(1)}(t)dt < \infty$.

(H6) K is a positive bounded function with support $[0, 1]$.

(H7) The df of the censored random variable G has bounded first derivative G'.

(H8) For all $u \in [0, 1]$, $\lim_{h\to 0} \frac{\phi_{\theta,x}(uh)}{\phi_{\theta,x}(h)} = \lim_{h\to 0} \xi_h^{\theta,x}(u) = \xi_0^{\theta,x}(u)$.

(H9) The bandwidth h_H satisfies,

(i) $nh_H^2\phi_{\theta,x}^2(h_K) \longrightarrow \infty$, and $\frac{nh_H^3\phi_{\theta,x}(h_K)}{\log^2 n} \longrightarrow \infty$ as $n \to \infty$,

(ii) $nh_H^2\phi_{\theta,x}^3(h_K) \longrightarrow 0$, as $n \to \infty$.

(H10) There exist sequences of integers (u_n) and (v_n) increasing to infinity such that $(u_n + v_n) \le n$, satisfying

(i) $v_n = o((n\phi_{\theta,x}(h_K))^{1/2})$ and $\left(\frac{n}{\phi_{\theta,x}(h_K)}\right)^{1/2} \alpha(v_n) \to 0$ as $n \to \infty$,

(ii) $q_n v_n = o((n\phi_{\theta,x}(h_K))^{1/2})$ and $q_n \left(\frac{n}{\phi_{\theta,x}(h_K)}\right)^{1/2} \alpha(v_n) \to 0$ as $n \to \infty$,

where q_n is the largest integer such that $q_n(u_n + v_n) \le n$.

6.3.3 Comments of the assumptions

(H1) can be interpreted as a concentration hypothesis acting on the distribution of the *f.r.v.* X, while (H3) concerns the behavior of the joint distribution of the pairs (X_i, X_j). Indeed, this hypothesis is equivalent to assume that, for n large enough

$$\sup_{i\neq j} \frac{\mathbb{P}\left((X_i, X_j) \in B_\theta(x, h_K) \times B_\theta(x, h_K)\right)}{\mathbb{P}\left(X \in B_\theta(x, h_K)\right)} \leq C \left(\frac{\phi_{\theta,x}(h_K)}{n}\right)^{1/a}.$$

This is one way to control the local asymptotic ratio between the joint distribution and its margin. Remark that the upper bound increases with a. In other words, more the dependence is strong, (H3) is more restrictive. The hypothesis (H2) specifies the asymptotic behavior of the α-mixing coefficients. Assumptions (H5), (H6) and (H7) are classical in nonparametric estimation. To establish the asymptotic normality dealing with strong mixing random variables (under (H2)), we use the well-known sectioning device introduced by Doob (1953) [11] in (H10).

This part of paper is devoted to the main result, the asymptotic normality of $\widehat{F}(\theta, t, x)$ and $\widehat{\zeta}_\theta(\gamma, x)$.

Theorem 6.3.1. *Under assumptions (H1)-(H10), we have*

$$\left(\frac{n\phi_{\theta,x}(h_K)}{\sigma^2(\theta, t, x)}\right)^{1/2} \left(\widehat{F}(\theta, t, x) - F(\theta, t, x)\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1), \tag{6.9}$$

where $\sigma^2(\theta, t, x) = \dfrac{a_2(\theta, x)}{(a_1(\theta, x))^2} F(\theta, t, x) \left(\dfrac{1}{\overline{G}(t)} - F(\theta, t, x)\right)$,

with $a_l(\theta, x) = K^l(1) - \int_0^1 (K^l)'(u)\xi_h^{\theta,x}(u)\, du$, *for* $l = 1, 2$,

and " $\xrightarrow{\mathcal{D}}$ " *means the convergence in distribution.*

Theorem 6.3.2. *If the assumptions (H1)-(H10) are satisfied, and γ is the unique order of the quantile such that $\gamma = F(\theta, \zeta_\theta(\gamma, x), x) = \widehat{F}(\theta, \widehat{\zeta}_\theta(\gamma, x), x)$,*

$$\left(\frac{n\phi_{\theta,x}(h_K)}{\Sigma^2(\theta, \zeta_\theta(\gamma, x), x)}\right)^{1/2} \left(\widehat{\zeta}_\theta(\gamma, x) - \zeta_\theta(\gamma, x)\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1), \tag{6.10}$$

where $\Sigma(\theta, \zeta_\theta(\gamma, x), x) = \dfrac{\sigma(\theta, \zeta_\theta(\gamma, x), x)}{f(\theta, \zeta_\theta(\gamma, x), x)}$.

As one can see, the asymptotic variance $\Sigma(\theta, \zeta_\theta(\gamma, x), x)$ depends on some unknown functions $f(\theta, \zeta_\theta(\gamma, x), x)$ and $\phi_{\theta,x}(h_K)$ and other theoretical quantities $F(\theta, t, x)$, $\overline{G}(\cdot)$, $a_l(\theta, x)$ for $l = 1, 2$, and $\zeta_\theta(\gamma, x)$ that have to be estimated in practice. Therefore, $\overline{G}(\cdot)$, $F(\theta, t, x)$, $f(\theta, \zeta_\theta(\gamma, x), x)$ and $\zeta_\theta(\gamma, x)$ should be replaced, respectively, by the Kaplan-Meier's estimator $\overline{G}_n(\cdot)$, the kernel-type estimators of the joint distribution $\widehat{F}(\theta, t, x)$ and of the joint density $\widehat{f}(\theta, \zeta_\theta(\gamma, x), x)$, $\widehat{\zeta}_\theta(\gamma, x)$ the conditional quantile estimator given by equation (6.7). Moreover, by assumption (H1), one can estimate $\phi_{\theta,x}(h_K)$ by $F_{x,n}(h_K) = 1/n \sum_{i=1}^{n} \mathbf{1}_{\{X_i \in B_\theta(x,h_K)\}}$. The quantity $a_l(\theta, x)$ for $l = 1, 2$ must be estimated by $\widehat{a}_l(\theta, x)$.

The corollary below allows one to obtain a confidence interval in practice since all quantities are known.

6.3.4 Confidence intervals

Corollary 6.3.1. *Using the same hypotheses of Theorem 6.3.2, one gets*

$$\left(\frac{nF_{x,n}(h_K)}{\widehat{\Sigma}^2(\theta, \widehat{\zeta}_\theta(\gamma, x), x)} \right)^{1/2} \left(\widehat{\zeta}_\theta(\gamma, x) - \zeta_\theta(\gamma, x) \right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1),$$

where $\widehat{\Sigma}(\theta, \widehat{\zeta}_\theta(\gamma, x), x) = \dfrac{\widehat{\sigma}(\theta, \widehat{\zeta}_\theta(\gamma, x), x)}{\widehat{f}(\theta, \widehat{\zeta}_\theta(\gamma, x), x)}$.

Now, based on the quantities estimation, we easily get a plug-in estimator $\widehat{\Sigma}(\theta, \widehat{\zeta}_\theta(\gamma, x), x)$ of $\Sigma(\theta, \zeta_\theta(\gamma, x), x)$. The Corollary 6.3.1 can be used to provide the $100(1-\gamma)\%$ confidence bands for $\zeta_\theta(\gamma, x)$ which is given, for $x \in \mathcal{H}$, by

$$\left[\widehat{\zeta}_\theta(\gamma, x) - c_{\gamma/2} \frac{\widehat{\Sigma}(\theta, \widehat{\zeta}_\theta(\gamma, x), x)}{\sqrt{nF_{x,n}(h_K)}}, \widehat{\zeta}_\theta(\gamma, x) + c_{\gamma/2} \frac{\widehat{\Sigma}(\theta, \widehat{\zeta}_\theta(\gamma, x), x)}{\sqrt{nF_{x,n}(h_K)}} \right],$$

where $c_{\gamma/2}$ is the upper $\gamma/2$ quantile of the distribution of $\mathcal{N}(0, 1)$.

6.4 Finite sample performance

This section considers simulated data study to assess the finite-sample performance of the proposed estimator and compare it to its competitor. More precisely, we are interested in comparing the conditional quantile estimator based on single functional index model (SFIM) to the kernel-type conditional quantile estimator (NP) introduced in Chaouch and Khardani (2015) [8], when the data is dependent and the response variable is subject to a random right-censorship phenomena. Throughout the simulation part, the n i.i.d. random variables $(C_i)_i$ (censured variables) are simulated through the exponential distribution $\mathcal{E}(1.5)$. Similarly, in the real data applications, the censored variables are simulated according to the aforementioned exponential law.

The single functional index $\theta \in \mathcal{H}$ is usually unknown and has to be estimated in practice. This topic was discussed in single functional regression model literature and an estimation approaches based on cross-validation or maximum-likelihood methods were discussed, for instance, in Ait Saïdi *et al.* (2008) [2] and the references therein. Another alternative which will be adopted in this section consists in selecting $\theta(t)$ among the eigenfunctions of the covariance operator $\mathbb{E}\left[(X' - \mathbb{E}(X')) < X', . >_{\mathcal{H}}\right]$, where $X(t)$ is, for instance, a diffusion-type process defined on a real interval $[a,\, b]$ and $X'(t)$ its first derivative (see, for instance, Attaoui and Ling (2016) [5]). Given a training sample $\mathcal{L}$, the covariance operator can be estimated by its empirical version $\frac{1}{|\mathcal{L}|}\sum_{i\in\mathcal{L}}(X_i' - \mathbb{E}X')^{t}(X_i' - \mathbb{E}X')$. Consequently, one can obtain a discretized version of the eigenfunctions $\theta_i(t)$ by applying the principle component analysis method. Let $\theta^{\star}$ be the first eigenfunction corresponding to the highest eigenvalue of the empirical covariance operator, which will replace θ in the simulation steps to calculate the estimator of the conditional distribution as well as the conditional quantiles.

6.4.1 Simulation study

We generate n copies, say $(X_i, \delta_i, Y_i)_{i=1,\ldots,n}$, of (X, δ, Y), where X and Y are simulated according to the following functional regression model.

$$T_i = R\,(X_i) + \epsilon_i,\ i = 1, \ldots, n,$$

where ϵ_i is the error assumed to be generated according to an autoregressive model defined as:

$$\epsilon_i = 1/\sqrt{2}\epsilon_{i-1} + \eta_i,\ i = 1, \ldots, n,$$

where $(\eta_i)_i$ is a sequence of i.i.d. random variables normally distributed with a variance equal to 0.1. The functional covariate X is assumed to be a diffusion process defined on $[0,\, 1]$ and generated by the following equation:

$$X_i\,(t) = A_i\,(2 - \cos\,(\pi t W_i)) + (1 - A_i)\cos(\pi t W_i),\ t \in [0,\, 1]\,,$$

where $W_i \rightsquigarrow \mathcal{N}\,(0,\, 1)$ and $A_i \rightsquigarrow \text{Bernoulli}(1/2)$.

Figure 6.1 depicts a sample of 100 realizations of the functional random variable X sampled in 100 equidistant points over the interval $[0, 1]$.

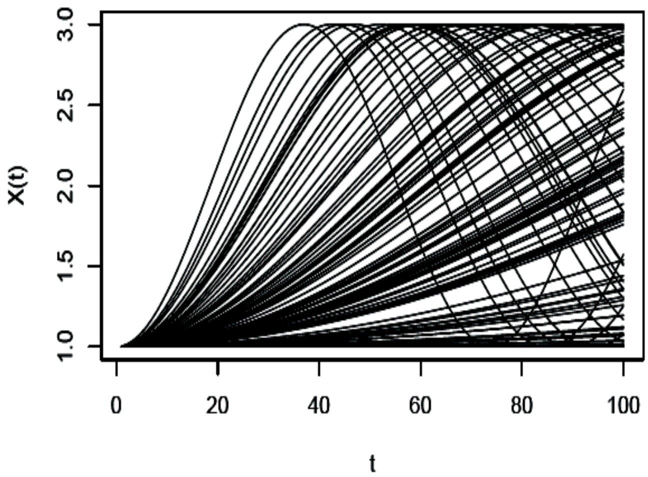

Figure 6.1: A sample of 100 curves $\{X_i\,(t)\,,\, t \in [0, 1]\}_{i=1,\ldots,100}$

On the other side, a nonlinear functional regression defined as follows is considered

$$R(X) = \frac{1}{4}\int_0^1 \left(X'(t)\right)^2 dt.$$

The computation of our estimator is based on the observed data $(X_i,\ \delta_i, Y_i)_{i=1,\cdots,n}$, where $Y_i = \min(T_i,\ C_i)$, $\delta_i = \mathbf{1}_{\{T_i \le C_i\}}$.

To assess the accuracy of the proposed estimator, we split the generated data into a training ($\mathcal{L}$) and a testing ($\mathcal{J}$) subsamples. The training subsample is used to estimate the single functional index and to select the smoothing parameters h_k and h_H. Whereas the testing subsample is used to assess and compare the single functional index model based on the estimator of the conditional quantile, namely $\widehat{\zeta}_\theta(\gamma,\cdot)$, to the kernel-type conditional quantile estimator, say $\widehat{\zeta}(\gamma,\cdot)$, which is introduced in Chaouch and Khardani (2015) [8] as follows:

$$\widehat{\zeta}(\gamma, x) = \inf\left\{t \in \mathbb{R},\ \widehat{F}^x(t) \ge \gamma\right\},$$

where

$$\widehat{F}^x(t) = \frac{\sum_{i=1}^n \frac{\delta_i}{\bar{G}_n(Y_i)} K\left(h_K^{-1} d(x,\ X_i)\right) H\left(h_H^{-1}(t - Y_i)\right)}{\sum_{i=1}^n K\left(h^{-1} d(x,\ X_i)\right)}, \quad \forall t \in \mathbb{R}.$$

Figure 6.2 displays the first two eigenfunctions calculated from the estimated covariance operator using the data in the training subsample.

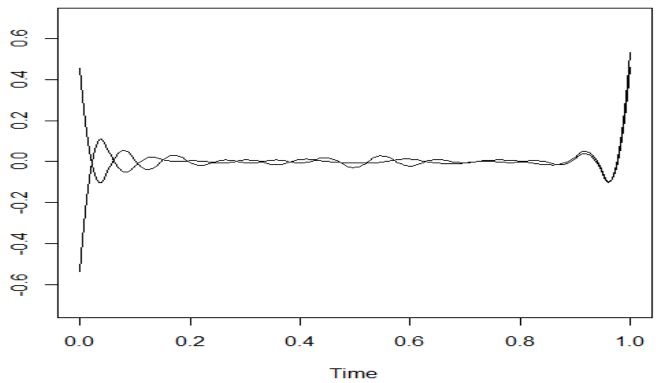

Figure 6.2: The first two eigenfunctions $\theta_i(t), i = 1, 2$

Given a fixed curve $X = x$, we can observe that the random variable T has a normal distribution with mean equal to $R(x)$ and standard deviation equal to 0.2. Therefore, the conditional median is equal to $R(x)$. A 500 Monte-Carlo simulations are performed in order to assess the estimation accuracy of $R(x)$ using the conditional median estimation by the single functional index approach and by the nonparametric approach. The simulations were performed for two sample sizes $n = 100, 500$, and for two Censorship Rates $CR = 60\%, 30\%$. Furthermore, some tuning parameters have to be specified. The kernel $K(\cdot)$ is chosen to be the quadratic function defined as $K(u) = \frac{3}{2}(1 - u^2)\mathbf{1}_{[0,1]}$, and the cumulative

Table 6.1: First, second and third quartile of the Absolute errors ($AE_{k,\theta}$ and AE_k, $k = 1, \ldots, 500$) obtained for CR=60% and CR=30%(between parentheses).

	n=100		n=500	
	NP	SFIM	NP	SFIM
1st quartile of AE	0.709 (0.29)	0.69 (0.212)	0.62 (0.136)	0.53 (0.097)
Median of AE	0.955 (0.557)	0.93 (0.573)	0.95 (0.584)	0.75 (0.346)
3rd quartile of AE	1.085 (0.73)	1.08 (0.76)	1.07 (0.718)	0.92 (0.624)

distribution function $H(u) = \int_{-\infty}^{u} \frac{3}{4}\left(1 - z^2\right) \mathbf{1}_{[-1,1]}(z)\, dz$. As shown in Figure 6.1, the covariate is a smooth process and the regression function $R(\cdot)$ is defined as the integral of the derivative of the functional random variable X. Consequently, according to Ferraty and Vieu (2006) [16], the appropriate choice of the semi-metric is the L_2 distance between the first derivatives of the curves. In this section, we assume that $h := h_K = h_H$, is selected using a cross-validation method based on the k-nearest neighbors as described in Ferraty and Vieu (2006) [16], p. 102.

We consider the absolute error (AE) as a measure of accuracy of the estimators:

$$AE_{k,\theta} = |\widehat{\zeta}_\theta(0.5, x) - R(x)| \quad \text{and} \quad AE_k = |\widehat{\zeta}(0.5, x) - R(x)|, \quad k = 1, \ldots, 500,$$

where $\widehat{\zeta}_\theta(0.5, x)$ and $\widehat{\zeta}(0.5, x)$ are, respectively, the estimators of the conditional median using the single functional index model and the nonparametric approach. Table 6.1 shows that the SFIM estimator performs better that the NP one in estimating $R(x)$. Higher is the sample size and lower is the censorship rate better will be the accuracy of the SFIM compared to the NP one. Moreover, even when CR=60% and $n = 100$, the SFIM estimator is still performing better than the NP one.

The next phase of this simulation study consists in comparing the accuracy of the SFIM and the NP approaches in terms of prediction. For this purpose, a sample of 550 observations was simulated according to the previous functional regression model defined above. A subsample of size 500 is considered for training and the remaining 50 observations are used for prediction assessment. The purpose consists in predicting the response variable Y_i in the test sample using the conditional median which is estimated either by SFIM or NP approach. An overall assessment of the predictions is performed using the median square error, where the square error (SE) is defined as follows: $SE_{j,\theta} := (Y_j - \widehat{\zeta}_\theta(0.5, x))$ and $SE_j := (Y_j - \widehat{\zeta}(0.5, x)), j = 1, \ldots, 50$. Two censorship rates are considered here: $CR = 45\%$ and $CR = 2\%$.

Figures 6.3 and 6.4 show that the SFIM estimator performs better than the NP estimator in predicting the response variable in the testing subsample. The accuracy increases when the censorship rate decreases. Indeed when $CR = 45\%$, the median square error is equal to 0.011 using the SFIM approach and 0.055 for the NP one. whereas, when

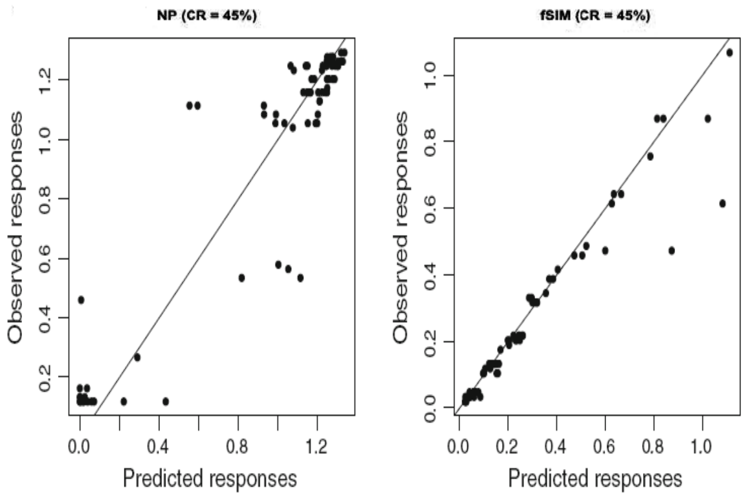

Figure 6.3: Prediction of $(Y_j)_{j=1,\ldots,50}$ in the test subsample when $CR = 45\%$.

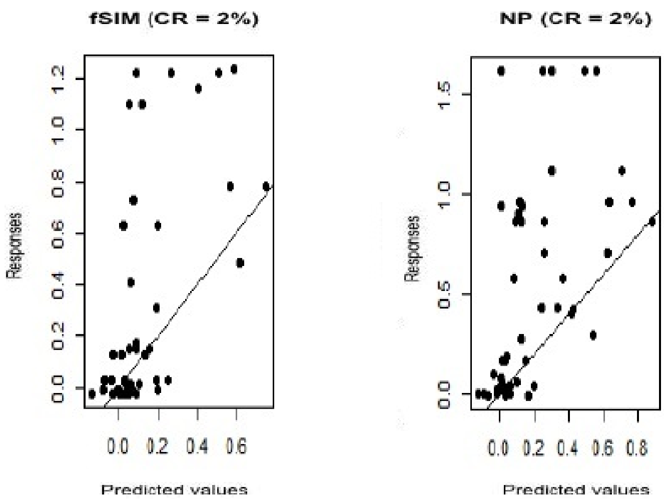

Figure 6.4: Prediction of $(Y_j)_{j=1,\ldots,50}$ in the test subsample when $CR = 2\%$.

$CR = 2\%$, the median square error is equal to 0.008 for the SFIM and 0.012 for the NP approach.

6.5 Proofs

In order to prove our results, let's first introduce some further notations.
Observe that (6.6) can be rewritten as:

$$\widehat{F}(\theta, t, x) = \frac{\widehat{F}_N(\theta, t, x)}{\widehat{F}_D(\theta, x)}, \tag{6.11}$$

with

$$\widehat{F}_N(\theta, t, x) = \frac{1}{n\mathbb{E}(K_1(\theta, x))} \sum_{i=1}^{n} \frac{\delta_i}{\bar{G}_n(Y_i)} K_i(\theta, x) H_i(t);$$

$$\widetilde{F}_N(\theta,t,x) = \frac{1}{n\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^{n}\frac{\delta_i}{\bar{G}(Y_i)}K_i(\theta,x)H_i(t);$$

$$\widehat{F}_D(\theta,x) = \frac{1}{n\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^{n}K_i(\theta,x),$$

where,

$$K_i(\theta,x) = K(h_K^{-1}(< x - X_i, \theta >)) \ , \ H_i(t) = H(h_h^{-1}(t - Y_i)).$$

Now, we consider the following decomposition

$$\begin{aligned}
\widehat{F}(\theta,t,x) - F(\theta,t,x) &= \frac{\widehat{F}_N(\theta,t,x)}{\widehat{F}_D(\theta,x)} - \frac{a_1(\theta,x)F(\theta,t,x)}{a_1(\theta,x)} \\
&= \frac{1}{\widehat{F}_D(\theta,x)}\left(\widehat{F}_N(\theta,t,x) - \mathbb{E}\widehat{F}_N(\theta,t,x)\right) \\
&\quad - \frac{1}{\widehat{F}_D(\theta,x)}\left(a_1(\theta,x)F(\theta,t,x) - \mathbb{E}\widehat{F}_N(\theta,t,x)\right) \\
&\quad + \frac{F(\theta,t,x)}{\widehat{F}_D(\theta,x)}\left(a_1(\theta,x) - \mathbb{E}\left[\widehat{F}_D(\theta,x)\right]\right) \\
&\quad - \frac{F(\theta,t,x)}{\widehat{F}_D(\theta,x)}\left(\widehat{F}_D(\theta,x) - \mathbb{E}\widehat{F}_D(\theta,x)\right) \\
&= \frac{1}{\widehat{F}_D(\theta,x)}A_n(\theta,t,x) + B_n(\theta,t,x), \qquad (6.12)
\end{aligned}$$

where

$$\begin{aligned}
A_n(\theta,t,x) &= \frac{1}{n\mathbb{E}K_1(x,\theta)}\sum_{i=1}^{n}\Big\{\left(\frac{\delta_i}{\bar{G}_n}H_i(t) - F(\theta,t,x)\right)K_i(\theta,x) \\
&\qquad - \mathbb{E}\left[\left(\frac{\delta_i}{\bar{G}_n}H_i(t) - F(\theta,t,x)\right)K_i(\theta,x)\right]\Big\} \\
&= \frac{1}{n\mathbb{E}K_1(x,\theta)}\sum_{i=1}^{n}N_i(\theta,t,x).
\end{aligned}$$

It follows that,

$$\begin{aligned} n\phi_{\theta,x}(h_K)Var\left(A_n(\theta,t,x)\right) &= \frac{\phi_{\theta,x}(h_K)}{\mathbb{E}^2K_1(x,\theta)}Var(N_1) \\ &\quad + \frac{\phi_{\theta,x}(h_K)}{n\mathbb{E}^2K_1(x,\theta)}\sum_{|i-j|>0}^{n}\sum Cov(N_i,N_j) \\ &= V_n(\theta,t,x) \\ &\quad + \frac{\phi_{\theta,x}(h_K)}{n\mathbb{E}^2K_1(x,\theta)}\sum_{|i-j|>0}^{n}\sum Cov(N_i,N_j), \qquad (6.13) \end{aligned}$$

where $N_i = N_i(\theta,t,x)$ and $N_j = N_j(\theta,t,x)$.

Lemma 6.5.1. *Under hypotheses (H1)-(H4) and (H5)-(H6), as $n \to \infty$ we have*

$$n\phi_{\theta,x}(h_K)Var\left(A_n(\theta,t,x)\right) \longrightarrow V(\theta,t,x),$$

where $V(\theta,t,x) = \dfrac{a_2(\theta,x)}{(a_1(\theta,x))^2}F(\theta,t,x)\left(\dfrac{1}{\bar{G}(t)} - F(\theta,t,x)\right)$.

Lemma 6.5.2. *Under hypotheses (H1)-(H3), (H6) and (H8)-(H10), as $n \to \infty$ we have*

$$\left(\frac{n\phi_{\theta,x}(h_K)}{V(\theta,t,x)}\right)^{1/2} A_n(\theta,t,x) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1),$$

where $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution.

Lemma 6.5.3. *Under assumptions (H1)-(H3) and (H6)-(H9), as $n \to \infty$ we have*

$$\sqrt{n\phi_{\theta,x}(h_K)}B_n(\theta,t,x) \longrightarrow 0 \quad \textit{in Probabilty.}$$

Next, making use of Proposition 3.2 for $l = 1$ and Theorem 3.1 in *Kadiri et al.* (2018) [18], we get the following corollary.

Corollary 6.5.1. *Under hypotheses of Lemma 6.5.3, as $n \to \infty$ we have*

$$\frac{\left(n\phi_{\theta,x}(h_K)\right)^{1/2} B_n(\theta,t,x)}{\widehat{f}\left(\theta,\zeta^*_{\theta,n}(\gamma,x),x\right)} \longrightarrow 0 \quad \textit{in Probabilty.}$$

Proof of Theorem 6.3.1. To prove Theorem 6.3.1, it suffices to use (6.12). Applying Lemmas Lemma 6.5.1 and Lemma 6.5.3, we get the result. □

Proof of Theorem 6.3.2. For Theorem 6.3.2, making use of (6.12), we have

$$\begin{aligned}\sqrt{n\phi_{\theta,x}(h_K)}\left(\zeta_\theta(\gamma,x)-\zeta_{\theta,n}(\gamma,x)\right) &= \sqrt{n\phi_{\theta,x}(h_K)}\frac{F_n(\theta,\zeta_\theta(\gamma,x),x)}{F_n'(\theta,\zeta^*_{\theta,n}(\gamma,x),x)}\\ &\quad -\sqrt{n\phi_{\theta,x}(h_K)}\frac{F(\theta,\zeta_\theta(\gamma,x),x)}{F_n'(\theta,\zeta^*_{\theta,n}(\gamma,x),x)}\\ &= \frac{\sqrt{n\phi_{\theta,x}(h_K)}A_n(\theta,t,x)}{F_n'(\theta,\zeta^*_{\theta,n}(\gamma,x),x)}\\ &\quad -\frac{\sqrt{n\phi_{\theta,x}(h_K)}B_n(\theta,t,x)}{F_n'(\theta,\zeta^*_{\theta,n}(\gamma,x),x)}.\end{aligned}$$

Then, using Theorem 6.3.1, Corollary 6.5.1 and Lemma 6.5.3, we obtain the result. □

Proof of Lemma 6.5.1.

$$V_n(\theta,t,x)=\frac{\phi_{\theta,x}(h_K)}{\mathbb{E}^2K_1(\theta,x)}\mathbb{E}\left[K_1^2(\theta,x)\left(\frac{\delta_1}{\bar G(Y_1)}H_1(t)-F(\theta,t,x)\right)^2\right]$$

$$=\frac{\phi_{\theta,x}(h_K)}{\mathbb{E}^2K_1(\theta,x)}\mathbb{E}\left[K_1^2(\theta,x)\mathbb{E}\left(\left(\frac{\delta_1H_1(t)}{\bar G(Y_1)}-F(\theta,t,x)\right)^2|<\theta,X_1>\right)\right]. \tag{6.14}$$

Using the definition of conditional variance, we have

$$\mathbb{E}\left[\left(\frac{\delta_1}{\bar G(Y_1)}H(h_H^{-1}(t-Y_1))-F(\theta,t,x)\right)^2|<\theta,X_1>\right]=J_{1n}+J_{2n},$$

where $J_{1n}=Var\left(\frac{\delta_1}{\bar G(Y_1)}H(h_H^{-1}(t-Y_1))|<\theta,X_1>\right)$,

$J_{2n}=\left[\mathbb{E}\left(\frac{\delta_1}{\bar G(Y_1)}H(h_H^{-1}(t-Y_1))|<\theta,X_1>\right)-F(\theta,t,x)\right]^2.$

Concerning J_{1n},

$$\begin{aligned}J_{1n} &= \mathbb{E}\left[\frac{\delta_1}{\bar G^2(Y_1)}H^2\left(\frac{t-Y_1}{h_H}\right)|<\theta,x>\right]\\ &\quad -\left(\mathbb{E}\left[\frac{\delta_1}{\bar G(Y_1)}H\left(\frac{t-Y_1}{h_H}\right)|<\theta,X_1>\right]\right)^2\\ &= \mathcal{J}_1+\mathcal{J}_2.\end{aligned}$$

As for $\mathcal{J}_1$, by the property of double conditional expectation, we get

$$\begin{aligned}
\mathcal{J}_1 &= \mathbb{E}\left\{\mathbb{E}\left[\frac{\delta_1}{\bar{G}^2(Y_1)}H^2\left(\frac{t-Y_1}{h_H}\right) \mid <\theta, X_1>, T_1\right]\right\} \\
&= \mathbb{E}\left\{\frac{\delta_1}{\bar{G}^2(T_1)}H^2\left(\frac{t-T_1}{h_H}\right)\mathbb{E}\left[\mathbf{1}_{T_1\leq C_1}|T_1\right] \mid <\theta, X_1>\right\} \\
&= \mathbb{E}\left(\frac{1}{\bar{G}(T_1)}H^2\left(\frac{t-T_1}{h_H}\right) \mid <\theta, X_1>\right) \\
&= \int_{\mathbb{R}}\frac{1}{\bar{G}(v)}H^2\left(\frac{t-v}{h_H}\right)dF(\theta, v, X_1) \\
&= \int_{\mathbb{R}}\frac{1}{\bar{G}(t-uh_H)}H^2(u)dF(\theta, t-uh_H, X_1). \qquad (6.15)
\end{aligned}$$

By the first order Taylor's expansion of the function $\bar{G}^{-1}(\cdot)$ around zero, one gets

$$\begin{aligned}
\mathcal{J}_1 &= \int_{\mathbb{R}}\frac{1}{\bar{G}(t)}H^2(u)dF(\theta, t-uh_H, X_1) \\
&\quad +\frac{h_H^2}{\bar{G}^2(t)}\int_{\mathbb{R}}uH(u)\bar{G}^{(1)}(t^*)f(\theta, t-uh_H, X_1)du + o(1),
\end{aligned}$$

where t^* is between t and $t-uh_H$.

Under hypothesis (H7) and using hypothesis (H4), we get

$$\mathcal{J}_1' = \frac{h_H^2}{\bar{G}^2(t)}\int_{\mathbb{R}}uH^2(t)\bar{G}^{(1)}(t^*)f(\theta, t-uh_H, X_1)du = o(h_H^2).$$

Indeed,

$$\mathcal{J}_1' \leq h_H^2\left(\sup_{u\in\mathbb{R}}|G'(u)|/\bar{G}^2(t)\right)\int_{\mathbb{R}}uf(\theta, t-uh_H, x)du.$$

On the other hand, by integrating by part and under assumption (H5), we have

$$\begin{aligned}
\int_{\mathbb{R}}\frac{H^2(u)}{\bar{G}(t)}dF(\theta, t-uh_H, X_1) &= \frac{1}{\bar{G}(t)}\int_{\mathbb{R}}2H(u)H'(u)F(\theta, t-uh_H, X_1)du \\
&\quad -\frac{1}{\bar{G}(t)}\int_{\mathbb{R}}2H(u)H'(u)F(\theta, t, x)du \\
&\quad +\frac{1}{\bar{G}(t)}\int_{\mathbb{R}}2H(u)H'(u)F(\theta, t, x)du.
\end{aligned}$$

Clearly, we have

$$\int_{\mathbb{R}}2H(u)H'(u)F(\theta, t, x)du = \Big[H^2(u)F(\theta, t, x)\Big]_{-\infty}^{+\infty} = F(\theta, t, x), \qquad (6.16)$$

thus,

$$\int_{\mathbb{R}} \frac{1}{\bar{G}(t)} H^2(u) dF(\theta, t-uh_H, X_1) = \frac{F(\theta,t,x)}{\bar{G}(t)} + \mathcal{O}(h_K^{\beta_1} + h_H^{\beta_2}). \tag{6.17}$$

- Concerning $\mathcal{J}_2$

$$\begin{aligned}
\mathcal{J}_2' &= \mathbb{E}\left[\frac{\delta_1}{\bar{G}(Y_1)} H_1(t) | < \theta, X_1 >\right] \\
&= \mathbb{E}\left(\mathbb{E}\left[\frac{\delta_1}{\bar{G}(Y_1)} H_1(t) | < \theta, X_1 >, T_1\right]\right) \\
&= \mathbb{E}\left(\frac{1}{\bar{G}(T_1)} H\left(\frac{t-T_1}{h_H}\right) \mathbb{E}\left[\mathbf{1}_{T_1 \le C_1} | T_1\right] | < \theta, X_1 >\right) \\
&= \mathbb{E}\left(H\left(\frac{t-T_1}{h_H}\right) | < \theta, X_1 >\right) \\
&= \int H\left(\frac{t-v}{h_H}\right) f(\theta, t, X_1) dv.
\end{aligned}$$

Moreover, we have by integration by parts and changing variables

$$\mathcal{J}_2' = F(\theta,t,x) \int H'(u) du + \int H'(u) \left(F(\theta, t-uh_H, x) - F(\theta,t,x)\right) du,$$

the last equality is due to the fact that H' is a probability density.

Thus we have:

$$\mathcal{J}_2' = F(\theta,t,x) + \mathcal{O}\left(h_K^{\beta_1} + h_H^{\beta_2}\right). \tag{6.18}$$

Finally, by hypothesis (H5), we get $\mathcal{J}_2 \underset{n\to\infty}{\longrightarrow} F^2(\theta,t,x)$

As for J_{2n}, by (H2), (H4) and (H5), and using Lemma 3.2 in Kadiri *et al.* (2018) [18], we obtain that

$$J_{2n} \longrightarrow 0, \quad \text{as } n \to \infty.$$

Meanwhile, by (H1), (H4), (H6) and (H8), it follows that:

$$\frac{\phi_{\theta,x}(h_K)\mathbb{E}K_1^2(\theta,x)}{\mathbb{E}^2 K_1(\theta,x)} \underset{n\to\infty}{\longrightarrow} \frac{a_2(\theta,x)}{(a_1(\theta,x))^2}.$$

Thus, by combining equations (6.14)-(6.18), it yields

$$V_n(\theta,t,x) \underset{n\to\infty}{\longrightarrow} \frac{a_2(\theta,x)}{(a_1(\theta,x))^2} F(\theta,t,x)\left(\frac{1}{\bar{G}(t)} - F(\theta,t,x)\right). \tag{6.19}$$

Secondly, by the boundness of H and conditioning on $(<\theta, X_i>, <\theta, X_j>)$, we have

$$\begin{aligned}
\mathbb{E}\left(|N_i N_j|\right) &= \mathbb{E}\left[\left(\Omega_i\right)\left(\Omega_j\right) K_i(\theta,x) K_j(\theta,x)\right] \\
&= \mathbb{E}\Big(\mathbb{E}\Big[\left(\Omega_i\right)\left(\Omega_j\right) \mid <\theta, X_i>, <\theta, X_j> \Big] K_i(\theta,x) K_j(\theta,x)\Big) \\
&\leq \left(1+\frac{1}{\bar{G}(\tau_F)}\right)^2 \mathbb{E}(K_i(\theta,x) K_j(\theta,x)) \\
&\leq C\mathbb{P}\left((X_i, X_j) \in B_\theta(x,h) \times B_\theta(x,h)\right) \\
&\leq C\left(\left(\frac{\phi_{\theta,x}(h_K)}{n}\right)^{1/a} \phi_{\theta,x}(h_K)\right),
\end{aligned}$$

where $\Omega_i = \dfrac{\delta_i}{\bar{G}_i} H_i(t) - F(\theta, t, x)$.

Then, taking

$$\begin{aligned}
\frac{\phi_{\theta,x}(h_K)}{n\mathbb{E}^2 K_1(x,\theta)} \sum\sum_{|i-j|>0}^{n} Cov(N_i, N_j) &= \frac{\phi_{\theta,x}(h_K)}{n\mathbb{E}^2 K_1(x,\theta)} \sum_{0<|i-j|\leq m_n}^{n} Cov(N_i, N_j) \\
&\quad + \frac{\phi_{\theta,x}(h_K)}{n\mathbb{E}^2 K_1(x,\theta)} \sum_{|i-j|>m_n}^{n} Cov(N_i, N_j) \\
&= K_{1n} + K_{2n}.
\end{aligned}$$

Therefore,

$$K_{1n} \leq C\, m_n \left\{\left(\frac{\phi_{\theta,x}(h_K)}{n}\right)^{1/a}\right\}, \quad \forall i \neq j.$$

Now, choose $m_n = \left(\frac{\phi_{\theta,x}(h_K)}{n}\right)^{-1/a}$, we get $K_{1n} = o(1)$.

For K_{2n} : since the variable $(\Delta_i)_{1\leq i\leq n}$ is bounded (i.e, $\|\Delta_i\|_\infty < \infty$), we can use the Davydov-Rio's inequality. So, we have for all $i \neq j$,

$$|Cov(\Delta_i, \Delta_j)| \leq C\alpha(|i-j|).$$

By the fact that $\displaystyle\sum_{k\geq m_n+1} k^{-a} \leq \int_{m_n}^{\infty} v^{-a} dv = \frac{m_n^{-a+1}}{a-1}$, we get by applying (H1)

$$K_{2n} \leq \sum_{|i-j|\geq m_n+1} |i-j|^{-a} \leq \frac{n m_n^{-a+1}}{a-1}.$$

With the same choice of m_n, we get $K_{2n} = o(1)$.

Finally, by

$$\frac{\phi_{\theta,x}(h_K)}{n\mathbb{E}^2K_1(x,\theta)}\sum\sum_{|i-j|>0}^{n} Cov(N_i,N_j)=o(1), \tag{6.20}$$

we complete the proof of the lemma. □

Proof of Lemma 6.5.2. We will establish the asymptotic normality of $A_n(\theta,t,x)$ suitably normalized. We have

$$\begin{aligned}\sqrt{n\phi_{\theta,x}(h_K)}A_n(\theta,t,x) &= \frac{\sqrt{n\phi_{\theta,x}(h_K)}}{n\mathbb{E}K_1(\theta,x)}\sum_{i=1}^{n}N_i(\theta,t,x)\\ &= \frac{\sqrt{\phi_{\theta,x}(h_K)}}{\sqrt{n}\mathbb{E}K_1(\theta,x)}\sum_{i=1}^{n}N_i(\theta,t,x)\\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Xi_i(\theta,t,x)=\frac{1}{\sqrt{n}}S_n.\end{aligned}$$

Now, we can write $\Xi_i=\dfrac{\sqrt{\phi_{\theta,x}(h_K)}}{\mathbb{E}K_1(\theta,x)}N_i$, we have

$$Var(\Xi_i)=\frac{\phi_{\theta,x}(h_K)}{\mathbb{E}^2K_1(\theta,x)}Var(N_i)=V_n(\theta,t,x).$$

Note that by (6.19), we have $Var(\Xi_i)\longrightarrow V(\theta,t,x)$ as n goes to infinity and by (6.20), we get

$$\sum_{|i-j|>0}|Cov(\Xi_i,\Xi_j)|=\frac{\phi_{\theta,x}(h_K)}{\mathbb{E}^2K_1(x,\theta)}\sum_{|i-j|>0}^{n}|Cov(N_i,N_j)|=o(n). \tag{6.21}$$

Obviously, we have

$$\sqrt{\frac{n\phi_{\theta,x}(h_K)}{V(\theta,t,x)}}\left(A_n(\theta,t,x)\right)=\left(nV(\theta,t,x)\right)^{-1/2}S_n.$$

Thus, the asymptotic normality of $(nV(\theta,t,x))^{-1/2}S_n$, is sufficient to show the proof of this Lemma. This last is shown by the blocking method, where the random variables Ξ_i are grouped into blocks of different sizes defined.

We consider the classical big- and small-block decomposition. We split the set $\{1,2,\ldots,n\}$ into $2k_n+1$ subsets, with large blocks of size u_n and small blocks of size v_n and put

$$k_n := \left[\frac{n}{u_n+v_n}\right].$$

Now, assumption (H10)-(ii) allows us to define the large block size by

$$u_n =: \left[\left(\frac{n\phi_{\theta,x}(h_K)}{q_n} \right)^{1/2} \right].$$

Using assumption (H10) and by simple algebra, it yields

$$\frac{v_n}{u_n} \to 0, \quad \frac{u_n}{n} \to 0, \quad \frac{u_n}{\sqrt{n\phi_{\theta,x}(h_K)}} \to 0, \quad \text{and} \quad \frac{n}{u_n}\alpha(v_n) \to 0. \tag{6.22}$$

Let Υ_j, Υ'_j and Υ''_j be defined as follows:

$$\Upsilon_j(\theta,t,x) = \Upsilon_j = \sum_{i=j(u+v)+1}^{j(u+v)+u} \Xi_i(\theta,t,x), \quad 0 \le j \le k-1,$$

$$\Upsilon'_j(\theta,t,x) = \Upsilon'_j = \sum_{i=j(u+v)+u+1}^{(j+1)(u+v)} \Xi_i(\theta,t,x), \quad 0 \le j \le k-1,$$

$$\Upsilon''_j(\theta,t,x) = \Upsilon''_j = \sum_{i=k(u+v)+1}^{n} \Xi_i(\theta,t,x), \quad 0 \le j \le k-1.$$

Clearly, we can write

$$\begin{aligned} S_n(\theta,t,x) = S_n &= \sum_{j=1}^{k-1} \Upsilon_j + \sum_{j=1}^{k-1} \Upsilon'_j + \Upsilon''_k \\ &= \Psi_n(\theta,t,x) + \Psi'_n(\theta,t,x) + \Psi''_n(\theta,t,x) \\ &= \Psi_n + \Psi'_n + \Psi''_n. \end{aligned}$$

We prove that

$$(i)\ \frac{1}{n}\mathbb{E}(\Psi'_n)^2 \longrightarrow 0, \quad (ii)\ \frac{1}{n}\mathbb{E}(\Psi''_n)^2 \longrightarrow 0, \tag{6.23}$$

$$\left| \mathbb{E}\left\{ \exp\left(izn^{-1/2}\Psi_n \right) \right\} - \prod_{j=0}^{k-1} \mathbb{E}\left\{ \exp\left(izn^{-1/2}\Upsilon_j \right) \right\} \right| \longrightarrow 0, \tag{6.24}$$

$$\frac{1}{n}\sum_{j=0}^{k-1} \mathbb{E}\left(\Upsilon_j^2 \right) \longrightarrow V(\theta,t,x), \tag{6.25}$$

$$\frac{1}{n}\sum_{j=0}^{k-1} \mathbb{E}\left(\Upsilon_j^2 \mathbf{1}_{\{|\Upsilon_j| > \varepsilon\sqrt{nV(\theta,t,x)}\}} \right) \longrightarrow 0 \text{ for every } \varepsilon > 0. \tag{6.26}$$

Expression (6.23) shows that the terms Ψ'_n and Ψ''_n are asymptotically negligible, while equations (6.24) and (6.25) show that the Υ_j are asymptotically independent, verifying that the sum of their variances tends to $V(\theta, t, x)$. Expression (6.26) is the Lindeberg-Feller's condition for a sum of independent terms. The asymptotic normality of S_n is a consequence of equations (6.23)-(6.26).

- ***Proof of (6.23).*** Because of $\mathbb{E}(\Xi_j) = 0, \ \forall j$, we have that

$$\mathbb{E}(\Psi'_n)^2 = Var\left(\sum_{j=1}^{k-1} \Upsilon'_j\right) = \sum_{j=1}^{k-1} Var\left(\Upsilon'_j\right) + \sum_{|i-j|>0}^{k-1} Cov\left(\Upsilon'_i, \Upsilon'_j\right) := \Pi_1 + \Pi_2.$$

By the second-order stationarity and (6.21), we get

$$\begin{aligned} Var\left(\Upsilon'_j\right) &= Var\left(\sum_{i=j(u_n+v_n)+u_n+1}^{(j+1)(u_n+v_n)} \Xi_i(\theta, t, x)\right) \\ &= v_n Var(\Xi_1(x)) + \sum_{|i-j|>0}^{v_n} Cov\left(\Xi_i(\theta, t, x), \Xi_j(\theta, t, x)\right) \\ &= v_n Var(\Xi_1(x)) + o(v_n). \end{aligned}$$

Then,

$$\begin{aligned} \frac{\Pi_1}{n} &= \frac{kv_n}{n} Var(\Xi_1(\theta, t, x)) + \frac{k}{n} o(v_n) \\ &\leq \frac{kv_n}{n}\left\{\frac{\phi_{\theta,x}(h_K)}{\mathbb{E}^2 K_1(x)} Var\left(\Xi_1(x)\right)\right\} + \frac{k}{n} o(v_n) \\ &\leq \frac{kv_n}{n}\left\{\frac{1}{\phi_{\theta,x}(h_K)} Var\left(\Xi_1(x)\right)\right\} + \frac{k}{n} o(v_n). \end{aligned}$$

Simple algebra gives us

$$\frac{kv_n}{n} \cong \left(\frac{n}{u_n + v_n}\right)\frac{v_n}{n} \cong \frac{v_n}{u_n + v_n} \cong \frac{v_n}{u_n} \longrightarrow 0 \quad \text{as } n \to \infty.$$

Using equation (6.20), we have

$$\lim_{n\to\infty} \frac{\Pi_1}{n} = 0. \tag{6.27}$$

Now, let us turn to Π_2/n. We have

$$\begin{aligned}\frac{\Pi_2}{n} &= \frac{1}{n}\sum_{|i-j|>0}^{k-1} Cov\left(\Upsilon_i(x), \Upsilon_j(x)\right) \\ &= \frac{1}{n}\sum_{|i-j|>0}^{k-1}\sum_{l_1=1}^{v_n}\sum_{l_2=1}^{v_n} Cov\left(\Xi_{m_j+l_1}, \Xi_{m_j+l_2}\right),\end{aligned}$$

with $m_i = i(u_n+v_n)+u_n+1$. As $i \neq j$, we have $|m_i - m_j + l_1 - l_2| \geq u_n$. It follows that

$$\frac{\Pi_2}{n} \leq \frac{1}{n}\sum_{|i-j|\geq u_n}^{n} Cov\left(\Xi_i(x), \Xi_j(x)\right) = o(1),$$

then,

$$\lim_{n\to\infty}\frac{\Pi_2}{n} = 0. \tag{6.28}$$

By equations (6.27) and (6.28), we get part(i) of the equation(6.23).

As for (ii), we have

$$\begin{aligned}\frac{1}{n}\mathbb{E}\left(\Psi_n''\right)^2 &= \frac{1}{n}Var\left(\Upsilon_k''\right) \\ &= \frac{\vartheta_n}{n}Var\left(\Xi_1(x)\right) + \frac{1}{n}\sum_{|i-j|>0}^{\vartheta_n} Cov\left(\Xi_i(x), \Xi_j(x)\right),\end{aligned}$$

where $\vartheta_n = n - k_n(u_n+v_n)$; by the definition of k_n, we have $\vartheta_n \leq u_n + v_n$.

Then

$$\frac{1}{n}\mathbb{E}\left(\Psi_n''\right)^2 \leq \frac{u_n+v_n}{n}Var\left(\Xi_1(x)\right) + \frac{1}{n}\sum_{|i-j|>0}^{\vartheta_n} Cov\left(\Xi_i(x), \Xi_j(x)\right)$$

By the definitions of u_n and v_n, we achieve the proof of (ii) of equation (6.23).

- ***Proof of (6.24)***. We make use of Volkonskii and Rozanov's lemma (see the appendix in Masry (2005) [25]), and the fact that the process (X_i, X_j) is strong mixing.

Note that Υ_a is $\mathcal{F}_{i_a}^{j_a}$-mesurable with $i_a = a(u_n+v_n)+1$ and $j_a = a(u_n+v_n)+u_n$; hence, with $V_j = \exp\left(izn^{-1/2}\Psi_n\right)$, we have

$$\begin{aligned}\left|\mathbb{E}\left\{V_j\right\} - \prod_{j=0}^{k-1}\mathbb{E}\left\{\exp\left(izn^{-1/2}\Upsilon_j\right)\right\}\right| &\leq 16k_n\alpha(v_n+1) \\ &\cong \frac{n}{v_n}\alpha(v_n+1),\end{aligned}$$

which goes to zero by the last part of equation (6.22). Now, we establish equation (6.25).

- ***Proof of (6.25)***. Note that $Var(\Psi_n) \longrightarrow V(\theta,t,x)$ by equation (6.23) (by the definition of the Ξ_i). Then because

$$\mathbb{E}\left(\Psi_n\right)^2 = Var\left(\Psi_n\right) = \sum_{j=0}^{k-1} Var\left(\Upsilon_j\right) + \sum_{i=0}^{k-1} \sum_{\substack{j=1 \\ i\neq j}}^{k-1} Cov\left(\Upsilon_i, \Upsilon_j\right),$$

all we have to prove is that the double sum of covariances in the last equation tends to zero. Using the same arguments as those previously used for Π_2 in the proof of first term of Equation (6.23), we obtain

$$\frac{1}{n}\sum_{j=1}^{k-1} \mathbb{E}\left(\Upsilon_j^2\right) = \frac{ku_n}{n} Var\left(\Xi_1\right) + o(1).$$

As $Var\left(\Xi_1\right) \longrightarrow V(\theta,t,x)$ and $\dfrac{ku_n}{n} \longrightarrow 1$, we get the result.

Finally, we prove equation (6.26).

- ***Proof of (6.26)***. Recall that

$$\Upsilon_j = \sum_{i=j(u_n+v_n)+1}^{j(u_n+v_n)+u_n} \Xi_i.$$

To establish (6.26), it suffices to show for n large enough that the set $\{|\Upsilon_j| > \varepsilon\sqrt{nV(\theta,t,x)}\}$ is empty .

Making use of assumptions (H3) and (H5), we have

$$\left|\Xi_i\right| \leq C\left(\phi_{\theta,x}(h_K)\right)^{-1/2},$$

therefore,

$$\left|\Upsilon_j\right| \leq Cu_n\left(\phi_{\theta,x}(h_K)\right)^{-1/2},$$

which goes to zero as n goes to infinity by equation (6.22).

Since $|H_i(t) - F(\theta,t,x)| \leq 1$, then

$$\begin{aligned} \left|\Upsilon_j\right| &\leq \frac{u_n N_j}{\sqrt{\phi_{\theta,x}(h_K)}} \\ &\leq \frac{Cu_n}{\sqrt{\phi_{\theta,x}(h_K)}}. \end{aligned}$$

Thus,

$$\frac{1}{\sqrt{n}}\left|\Upsilon_j\right| \leq \frac{Cu_n}{\sqrt{n\phi_{\theta,x}(h_K)}}.$$

Then, for n large enough, the set $\left\{|\Upsilon_j| > \varepsilon\left(nV(\theta,t,x)\right)^{-1/2}\right\}$ becomes empty, this completes the proof and therefore that of the asymptotic normality of $(nV(\theta,t,x))^{-1/2}\, S_n$ and the Lemma 6.5.2.

□

Proof of Lemma 6.5.3. We have

$$\sqrt{n\phi_{\theta,x}(h_K)}B_n(\theta,t,x) = \frac{\sqrt{n\phi_{\theta,x}(h_K)}}{\widehat{F}_D(\theta,x)}\Big\{\mathbb{E}\widehat{F}_N(\theta,t,x) - a_1(\theta,x)F(\theta,t,x) + F(\theta,t,x)\left(a_1(\theta,x) - \mathbb{E}\widehat{F}_D(\theta,x)\right)\Big\}.$$

Firstly, observed that the results below

$$\frac{1}{\phi_{\theta,x}(h_K)}\mathbb{E}\left[K^l\left(\frac{< x - X_i, \theta >}{h_K}\right)\right] \longrightarrow a_l(\theta,x), \quad \text{for } l = 1,2, \tag{6.29}$$

$$\mathbb{E}\left[\widehat{F}_D(\theta,x)\right] \longrightarrow a_1(\theta,x), \tag{6.30}$$

and

$$\mathbb{E}\left[\widehat{F}_N(\theta,t,x)\right] \longrightarrow a_1(\theta,x)F(\theta,t,x), \tag{6.31}$$

can be proved in the same way as in Ezzahrioui and Ould-Saïd (2008) [14] corresponding to their Lemmas 5.1 and 5.2, and then their proofs are omitted.

Secondly, on the one hand, making use of (6.29), (6.30) and (6.31), we have as $n \to \infty$

$$\left\{\mathbb{E}\widehat{F}_N(\theta,t,x) - a_1(\theta,x)F(\theta,t,x) + F(\theta,t,x)\left(a_1(\theta,x) - \mathbb{E}\widehat{F}_D(\theta,x)\right)\right\} \longrightarrow 0.$$

On the other hand,

$$\frac{\sqrt{n\phi_{\theta,x}(h_K)}}{\widehat{F}_D(\theta,x)} = \frac{\sqrt{n\phi_{\theta,x}(h_K)}\widetilde{F}'(\theta,t,x)}{\widehat{F}_D(\theta,x)\widetilde{F}'(\theta,t,x)} = \frac{\sqrt{n\phi_{\theta,x}(h_K)}\widetilde{F}'(\theta,t,x)}{\widetilde{F}'_N(\theta,t,x)}. \tag{6.32}$$

Then, using Proposition 3.2 in Kadiri *et al.* (2018) [18], it suffices to show that $\frac{\sqrt{n\phi_{\theta,x}(h_K)}}{\widetilde{F}'_N(\theta,t,x)}$ tends to zero as n goes to infinity.

Indeed,

$$\widetilde{F}'_N(\theta,t,x) = \frac{1}{nh_H \mathbb{E}K_1(\theta,x)} \sum_{i=1}^{n} \frac{\delta_i}{\bar{G}(Y_i)} K\left(\frac{< x - X_i, \theta >}{h_K}\right) H'\left(\frac{t - Y_i}{h_H}\right).$$

Since $K(\cdot)H'(\cdot)$ is continuous with support on $[0,1]$, then by (H5) and (H6) $\exists\ m = \inf_{[0,1]} K(t)H'(t)$, it follows that

$$\widetilde{F}'_N(\theta,t,x) \geq \frac{m}{h_H \phi_{\theta,x}(h_K)},$$

which gives

$$\frac{n\phi_{\theta,x}(h_K)}{\widetilde{F}'_N(\theta,t,x)} \leq \frac{\sqrt{nh_H^2 \phi_{\theta,x}(h_K)^3}}{m}.$$

Finally, using (H9), the proof of Lemma 6.5.3 is achieved.

□

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General Conclusion and Prospects

Conclusion

The problem addressed in this thesis is the nonparametric estimation of some functional parameters by using the kernel approach. Many models were studied in this work in diverse cases, all dealt with the functional explanatory random variables (valued in infinite dimensional space) by treating two issues : complete data and incomplete data.

The obtained asymptotic results covered several functions, where we established in Chapter 3 the consistency rate of the conditional density and the conditional mode function as well as the asymptotic normality of these kernel estimators. Under general ergodic condition, the regression function operator was examined in Chapter 4 whenever missing at random responses were considered. Further, we investigated in Chapters 5 and 6 the central limit theorems of the functional estimators of the conditional density and the conditional mode function, and of the randomly censored conditional distribution and conditional quantile estimation, for independent and dependent data cases, respectively. Note that the asymptotic normality gained considerable interest in the statistical literature. It is used for the construction of confidence intervals and to make statistical tests .

The fact of assuming that the treated data are always independent is not realistic, for this, we sought to mitigate this independence hypothesis by adopting the dependent case. Precisely, we dealt in Chapter 6 with the α-mixing sequence, which is reasonably weak among various dependence process and has many practical applications such as in time series prediction. We also focused on the ergodic property in the Chapters 3 and 4. Recall that the ergodicity condition is very general, it is less restrictive than the α-mixing assumption. This kind of dependence has not been investigated much before, it allows to avoid the widely used strong mixing condition and its variants to measure the dependency and the very involved probabilistic calculations that it implies.

The single index methodology was also used in the most of our works (Chapters 4, 5 and 6). This sort of modeling is very popular in the econometrics community, as it addresses two important concerns. The first is the reduction of dimension, since this methodology can effectively solve ”curse of dimensionality”. The second is related to the interpretability of the index (parameter) introduced in these models.

Finally, we emphhasize the effectiveness and the superiority of our models which are based on a combination of pivotal approaches. This is confirmed by the simulation studies that was often generated in the presented contributions.

Some Prospects

Research in the nonparametric field remains an open question which will be the subject of several future studies in order to improve and highlight the results obtained in this work.

- The extension of our studies of the censored data and the missing data to the truncated data case is a logical suite to follow.
- Another type of dependency could be considered such as the quasi-associated case.
- It is possible to elaborate the asymptotic properties of our estimators to other alternative estimation methods, such as, the k nearest neighbor method and the local linear approach.
- The expansion of the introduced contributions to the robust treatment is another future prospect.
- Our properties can be examined for the recursive kernel estimation.
- Generalize the obtained results by using other families of semi-metrics in order to improve the prediction performance of our estimators so the choice of the smoothing window is important.
- Other open questions could be addressed in the long term, such as the case where the two variables (the response variable and the explanatory variable) are functional.

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ملخص

هذه الأطروحة مكرسة للنمذجة اللامعلمية لمتغير الإستجابة الحقيقي المشروط بمتغير وظيفي (في فضاء ذو بعد غير منتهي). بتعبير أدق، ندرس الخصائص المقاربة لبعض الدوال الوظيفية للبيانات الكاملة و البيانات غير الكاملة في مواقف مختلفة. في الجزء الأول، نركز على التنبؤ بعملية أرجديك عبر تقدير النواة الوظيفية للبيانات غير الكاملة. يتم التعامل مع التنبؤ بالكثافة الخاضعة للرقابة العشوائية كدراسة أولية لدالة المنوال الشرطي. علاوة على ذلك، نقوم بتقييم توقعات دالة الانحدار في إطار مؤشر وظيفي بسيط من خلال تكييف المنهجية اللامعلمية في اتجاهين : خاصية أرجديك و بيانات عشوائية مفقودة. ثانيًا، نتعامل مع نظرية الحد المركزي باستخدام نهج مؤشر بسيط. من ناحية، نقترح دراسة على مقدر الكثافة الشرطية مع تطبيق على المنوال الشرطي في حالة البيانات المستقلة. من ناحية اخرى، نفحص حالة البيانات المرتبطة من خلال دالة التوزيع الشرطي و دالة الكمية الشرطية الخاضعين للرقابة العشوائية. لتوضيح فعالية نماذجنا، غالبًا ما نقدم دراسة محاكاة.

Résumé

Cette thèse est consacrée à la modélisation non paramétrique d'une variable de réponse réelle conditionnée par une covariable fonctionnelle (à valeurs dans un espace de dimension infini (espace semi-métrique/espace de Hilbert)). Plus précisément, nous étudions les propriétés asymptotiques de certains paramètres fonctionnels pour des données complètes et incomplètes dans des différentes situations. Dans la première partie, nous nous intéressons à la prévision du processus ergodique via l'estimation à noyau fonctionnel pour des données incomplètes. La prédiction de la densité censurée aléatoirement est traitée comme étude préliminaire de la fonction du mode conditionnel. De plus, nous évaluons l'estimation de la fonction de régression dans un cadre à indice fonctionnel simple en adaptant la méthodologie non paramétrique dans deux directions: propriété ergodique et données manquantes au hasard (MAR). Deuxièmement, nous traitons le théorème central limite en utilisant une approche à indice unique. D'une part, nous proposons une étude sur l'estimateur de la densité conditionnelle avec une application au mode conditionnel dans le cas des données indépendantes. D'autre part, nous examinons le cas des données dépendantes à travers la distribution conditionnelle et le quantile conditionnel censurés aléatoirement. Pour illustrer l'efficacité de nos modèles, nous introduisons souvent des études de simulation.

Summary

This thesis is devoted to the nonparametric modelization of a real response variable conditioned by a functional covariate (valued in infinite dimensional space (semi metric space/ Hilbert space)). More precisely, we study the asymptotic properties of some functional parameters for complete and incomplete data in different situations. In the first part, we focus on the ergodic process forecasting via a functional kernel estimation for incomplete data. The randomly censored density prediction is treated as a preliminary study of the conditional mode function. Moreover, we evaluate the regression function expectation in a functional single index framework by adapting the nonparametric methodology in two directions: ergodic property and missing at random data (MAR). Secondly, we deal with the central limit theorem by using single index approach. On the one hand, we propose a study on the conditional density estimator with an application to the conditional mode in the independent data case. On the other hand, we examine the dependent data case through the randomly censored conditional distribution and conditional quantile functions. To illustrate the effectiveness of our models, we often introduce simulation studies.