N° d'ordre :

REPUBLIQUE ALGERIENNE DEMOCRATIQUE & POPULAIRE

MINISTERE DE L'ENSEIGNEMENT SUPERIEUR & DE LA RECHERCHE SCIENTIFIQUE

UNIVERSITE DJILLALI LIABES
FACULTE DES SCIENCES EXACTES
SIDI BEL ABBES

THESE DE DOCTORAT EN SCIENCES

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Spécialité : Mathématiques
Option : Analyse fonctionnelle et numérique

Intitulée

Application de la méthode variationnelle pour un système d'équations et inclusions différentielles avec impulsions

Soutenue le : 25/05/2022
Devant le jury composé de :

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Année universitaire : 2021/2022

Acknowledgements

First and foremost, I would like to express my sincere gratitude to my advisor Prof. Abdelghani OUAHAB for proposing this subject and accepting to supervise my thesis. I am thankful for his patience, valuable guidance and for responding to my questions and queries so promptly.

Beside my advisor, I wish to thank the professor Kaïs AMMARI from Faculty of Sciences of Monastir-Tunisia who did me the honor of hosting at the laboratory of "Analysis and Control of PDEs" during my internship. I thank him very much for the guidance, support and time given to me.

I sincerely thank the chairman of the jury committee, Prof. Mouffak BENCHOHRA, and the jury members, Prof. Toufik MOUSSAOUI, Prof. Said ABBAS, Prof. Jamal Eddine LAZEG and Prof. Mohammed Said SOUID for their attention to this thesis and for their participation in the jury.

My sincere thanks go to the Prof. Toufik MOUSSAOUI from ENS-Kouba-Algiers for his support and guidance. God bless him always.

Finally, I must express my very profound gratitude to all of my family and friends for providing me with unfailing support and continuous encouragement throughout my years of study. Thank you.

Abstract

In this work we study the existence of weak solutions for a class of nonlinear differential equations with periodic boundary conditions and impulses. The approach is based on variational methods and critical point theory.

In the first chapter we recall some basic tools of elementary functional analysis and some general results on critical point theory.

The second chapter is devoted to the question of existence of the solutions to a class of nonlinear differential equations with instantaneous impulses by means of variational methods.

In the third chapter we consider a class of nonlinear differential equations with non-instantaneous impulses and obtain the existence of solutions.

For the last chapter we generalize the model studied in the foregoing chapter.

Keywords: Nonlinear differential equation with impulses. Instantaneous impulses. Non-instantaneous impulses. Variational method. Weak solution. Critical point.

ملخص

تناولنا في هذه الأطروحة دراسة وجود الحلول الضعيفة لثلاثة مسائل مكونة من معادلات تفاضلية عادية غير خطية من الرتبة الثانية، مرفقة بنبضات لحظية أو غير لحظية، معتمدين على طريقة المقاربة التغايرية ونظرية النقطة الحرجة.

جزئنا العمل إلى أربعة فصول: في الفصل الأول قدمنا بعض المفاهيم والأدوات الأساسية والضرورية لدراستنا، إهتممنا في الفصل الثاني بدراسة نموذج لمعادلات تفاضلية غير خطية مرفقة بنبضات لحظية، أما في الفصل الثالث تطرقنا لدراسة نموذج مكون من معادلات تفاضلية غير خطية مرفقة بنبضات غير لحظية، وأخيرا في الفصل الرابع قمنا بتعميم للنموذج المدروس في الفصل السابق.

كلمات مفتاحية: المعادلة التفاضلية العادية غير الخطية المرفقة بنبضات. نبضات لحظية. نبضات غير لحظية. المقاربة التغايرية. الحل الضعيف. النقطة الحرجة.

Résumé

L'objectif de ce travail est l'étude de l'existence de solutions faibles pour une certaine classe d'équations différentielles non linéaires avec impulsions. L'approche utilisée est basé sur la méthode variationnelle et la théorie du point critique classique.

Dans le premier chapitre nous présentons des outils de base nécessaires à l'étude des trois principales parties qui constituent cette thèse.

Le deuxième chapitre s'attache à l'étude de l'existence de solutions faibles pour un système d'équations différentielles non linéaires avec impulsions de type instantanées.

On s'intéresse dans le troisième chapitre à l'existence de solutions faibles mais avec impulsions de type non-instantanées.

Le dernier chapitre est consacré à une généralisation pour le dernier modèle.

Mots clés: Équation différentielle non linéaire avec impulsions. Impulsions de type instantanées. Impulsions de type non-instantanées. Méthode variationnelle. Solution faible. Point critique.

List of publications

1. R. Nesraoui, A. Dellal, J. J. Nieto and A. Ouahab, Variational approach to non-instantaneous impulsive system of differential equations, *Nonlinear Stud.*, **28**, (2021), 563-573.

2. R. Nesraoui and A. Ouahab, Variational approach to instantaneous impulsive differential system, Preprint, 2021.

3. R. Nesraoui and A. Ouahab, Variational approach to non-instantaneous impulsive differential system, Submitted, 2021.

Contents

Notations

In what follows, we use the following notations

t_i	Impulse points, such that $0 = t_0 < t_1 < t_2 < \ldots < t_m < t_{m+1} = T$.
$(t_i, s_i]$	Impulse intervals, such that $0 = s_0 < t_1 < s_1 < t_2 < s_2 < \ldots < t_m < s_m < t_{m+1} = T$.
I_i, J_i	Impulse functions defined on $\mathbb{R}$.
α_i, β_i	Impulse functions defined on $\mathbb{R}$. But in this work we took them constants.
$u'(t_i^+)$	The right derivative of u at t_i, i.e., $u'(t_i^+) = \lim_{t \to t_i^+} u'(t)$.
$u'(t_i^-)$	The left derivative of u at t_i, i.e., $u'(t_i^-) = \lim_{t \to t_i^-} u'(t)$.

$\Delta u'(t_i) := u'(t_i^+) - u'(t_i^-)$.

f_u, f_v	The derivatives of $f(t, u, v)$ at u and v respectively.
$D_u f_i, D_v f_i$	The derivatives of $f_i(t, u, v)$ at u and v respectively.
$X^\star$	Dual space.
$X^{\star\star}$	Bidual space.
$\langle \cdot, \cdot \rangle_{X^\star, X}$	Scalar product in the duality $X^\star, X$.
$\sigma(X, X^\star)$	Weak topology on X.
$\rightharpoonup$	Weak convergence.
$\to$	Strong convergence.
$\liminf$	The limit inferior.
J	Canonical injection from X into $X^{\star\star}$.
dim	Dimension of a vector space.
$\overline{A}$	Closure of the set A.

$\lvert A\rvert$	Measure of the set A.
B_X	Closed unit ball in X, i.e. $B_X = \Big\{x \in X;\ \|x\|_X \leq 1\Big\}$.
$\Omega \subset \mathbb{R}^n$	Open set in $\mathbb{R}^n$.
I	Open interval in $\mathbb{R}$.
∂I	Boundary of I.

$L^p(\Omega) = \Big\{u : \Omega \to \mathbb{R},\ u \text{ is measurable and } \displaystyle\int_\Omega |u(x)|^p dx < \infty\Big\},\ 1 \leq p < \infty.$

p'	Conjugate exponent of p, i.e., $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.
a.e.	Almost everywhere.

$L^\infty(\Omega) = \Big\{u : \Omega \to \mathbb{R},\ u \text{ is measurable and } \ |u(x)| \leq C \text{ a.e. in } \Omega \text{ for some canstant } C\Big\}.$

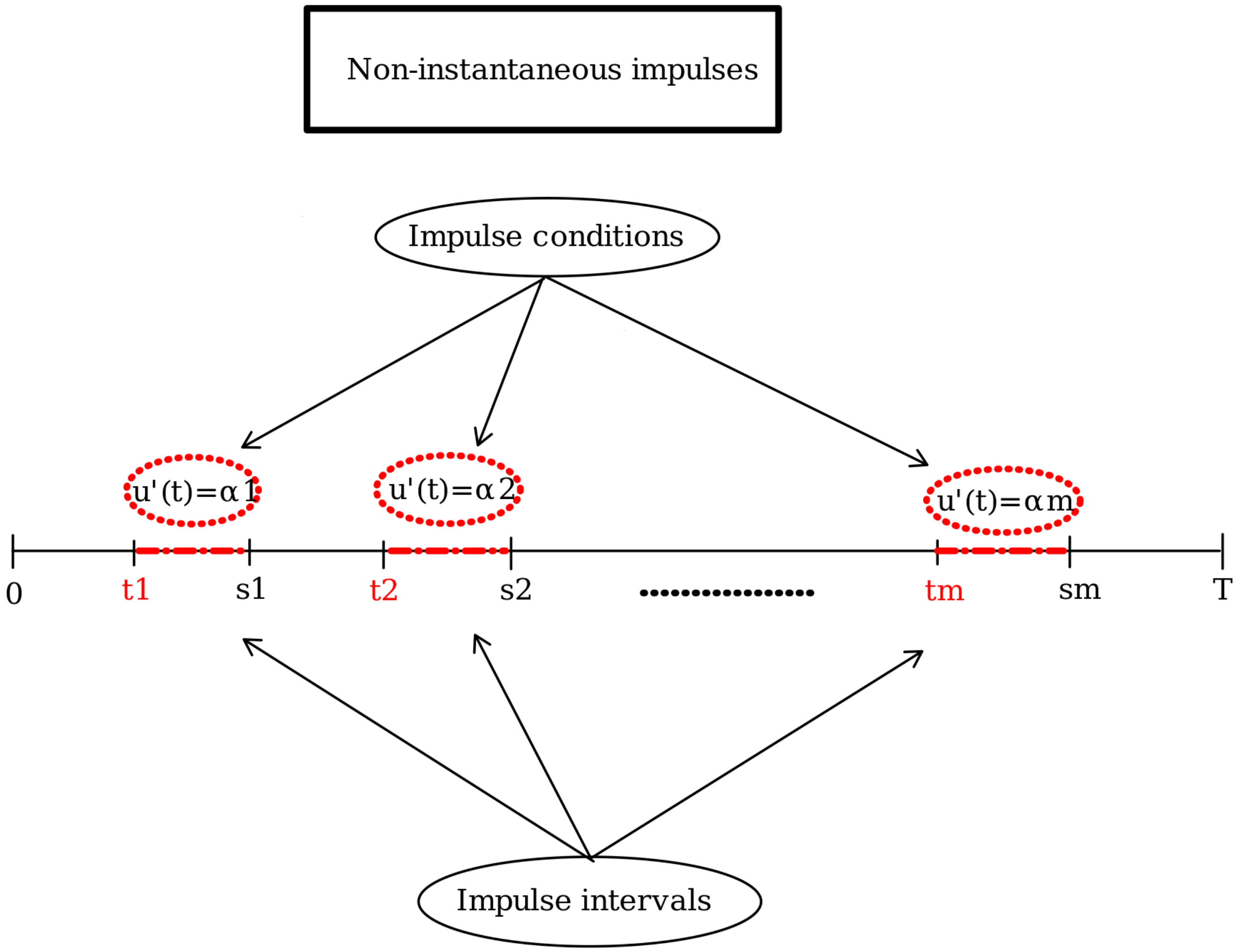
inf	Infimum.
$ess\inf$	Essential infimum.
$supp\ f$	Support of the function f, i.e., $supp\ f = \overline{\big\{x \in I,\ f(x) \neq 0\big\}}$.
$\mathcal{C}(\overline{I})$	Space of continuous functions on $\overline{I}$.
$\mathcal{C}_c^\infty(I)$	Space of infinity times differentiable functions with compact support in I.
max	Maximum.
min	Minimum.
$D^\alpha u$	Successive derivatives of the function u, i.e. $D^\alpha u = u^{(\alpha)}$.
$W^{m,p}, W_0^{m,p}, H^m, H_0^m, W_T^{1,p}$	Sobolev spaces.
$\mathcal{L}(X,Y)$	Space of linear bounded operators from X into Y.
$o(h)$	Landau notation, a little quantity such that its norm when it divided by the norm of h will tend to zero.
$DF(a)$ or $F'(a)$	Differential of F at the point a.
DF	Differential of F.
$\mathcal{C}^1(U)$	Class of functions differentiable on U with the differential must be continuous on U.
$\nabla F(a)$	Gradient of F at the point a.
l.s.c.	Lower semi-continuous.
w.l.s.c.	Weakly lower semi-continuous.

General Introduction

The aim of this thesis is the study of some impulsive problems with regard to the existence of weak solutions, where our results are based on the variational methods and the classical critical point theory.

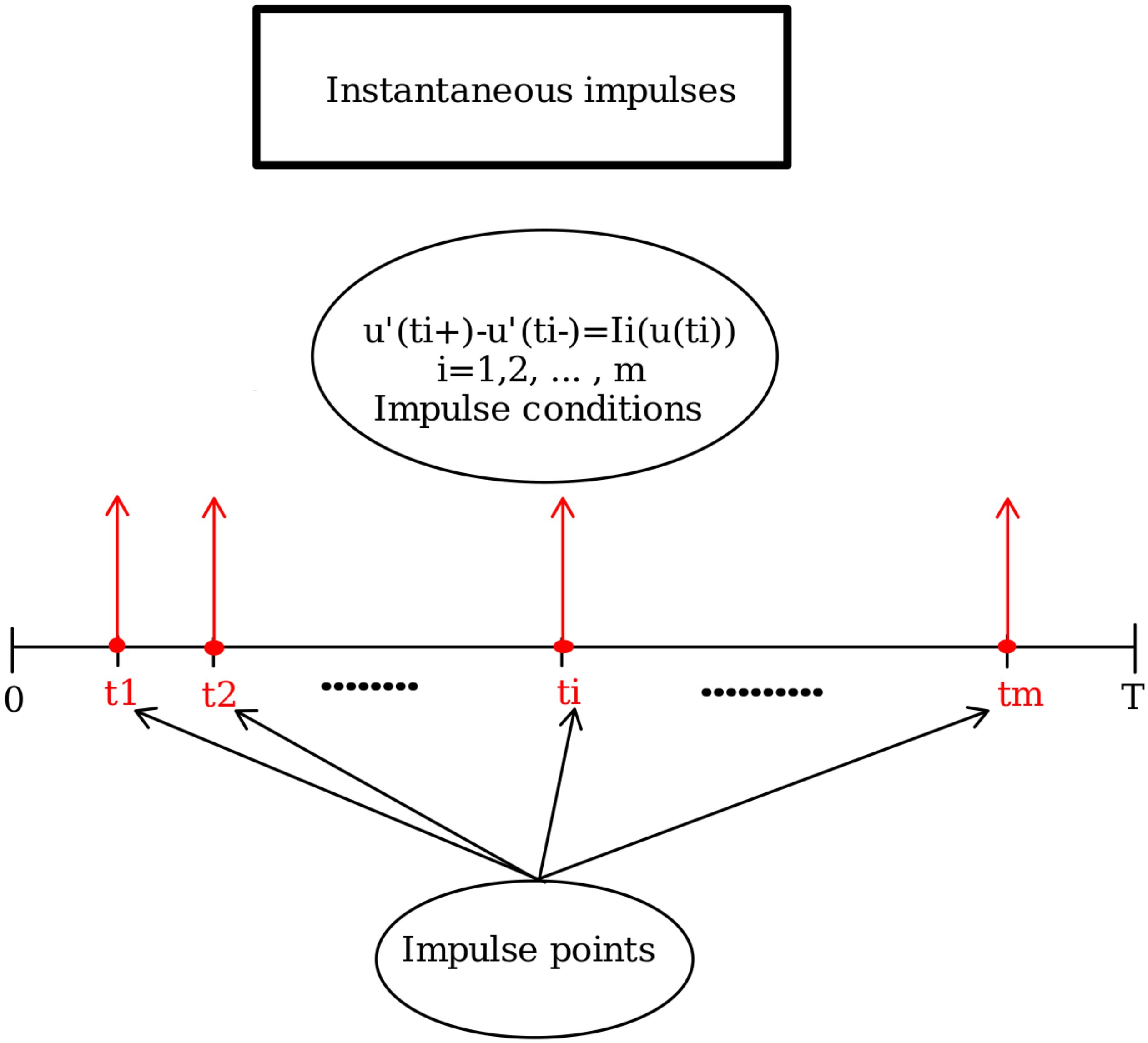
The theory of impulsive differential equations describes processes which experience a sudden change of their state at certain moments. Processes with such a character arise naturally and often, especially in phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. A comprehensive introduction to the basic theory is well developed in the monographs see for example the books [6, 10, 13, 15, 21] and in the references therein.

If a sudden change in the behavior of the phenomenon is happening and take some time before returning to the initial situation, in this case we speak of non-instantaneous impulses, as the figure below illustrates.



In the above figure the impulses start abruptly at points $t_k, k = 1, 2, \cdots, m$, and keep the derivative constant on a finite time interval $(t_k, s_k], k = 1, 2, \cdots, m$.

Now if the change of the state happens quickly and again quickly nothing happens, here we are talking about instantaneous impulses, as the figure below shows.



The impulses in the above figure start abruptly at points $t_k, k = 1, 2, \cdots, m$.

Recently Hernández and O'Regan [14] initially the theory of non-instantaneous impulsive differential equations. For example, impulses start abruptly at the instant t_k and their action continue on a finite time interval $(t_k, s_k]$. This type of problem motivates to study certain dynamical changes of evolution processes in pharmacotheraphy [20, 12, 24]. The existence of solutions of non-instantaneous impulsive problem has been studied via some approaches, such as fixed point theory and theory of C_0- semigroup, see, for example, [11, 9, 20]. Important contributions to the study of the mathematical aspects of such equations have been undertaken in [2].

Many problems can be understood and solved by minimization of a functional, usually related to the energy, in an appropriate space of functions. Recently, variational methods have been widely used to study impulsive problems. This method was initiated by Tian and Ge [23] and Nieto and O'Regan [19]. For some recent works see, for example, [1, 3, 4, 5, 7, 16, 22, 25, 26] and the references therein.

Our purpose in this work is to show that the existence of solutions of the impulsive differential equations considered is a problem equivalent to minimize some energy functional. Also, the critical points of the functional are indeed solutions of the impulsive differential equations problem. The goal of this work is to solve some class of boundary value problems for impulsive differential equations by using critical point theory. The variational structure of general non-instantaneous impulsive problem has been study in first time by Bai and Nieto [4].

Our work is made up of four chapters, and is organized as follows

In the first one we presented some well-known basic tools of elementary functional analysis and some general results on critical point theory which were useful for the following.

The second chapter is devoted to the question of existence of the solutions to a class of nonlinear differential equations with instantaneous impulses by means of variational methods.

$$\begin{cases} -u''(t) = f_u(t,u,v), \ t \in (0,T) \setminus \{t_1,...,t_m\}, \\ -v''(t) = f_v(t,u,v), \ t \in (0,T) \setminus \{t_1,...,t_m\}, \\ u(0) = u(T) = v(0) = v(T) = 0, \\ \Delta u'(t_k) = u'(t_k^+) - u'(t_k^-) = I_k\big(u(t_k)\big), \ k = 1,2,...,m, \\ \Delta v'(t_k) = v'(t_k^+) - v'(t_k^-) = J_k\big(v(t_k)\big), \ k = 1,2,...,m, \end{cases}$$

where the impulses start abruptly at points t_k.

In the third chapter we considered a class of nonlinear differential equations with non-instantaneous impulses and obtained the existence of solutions.

$$\begin{cases} -u''(t) &= D_u f_i\Big(t, u(t)-u(t_{i+1}), v(t)-v(t_{i+1})\Big),\ t\in(s_i,t_{i+1}],\ i=0,1,\ldots,m,\\ -v''(t) &= D_v f_i\Big(t, u(t)-u(t_{i+1}), v(t)-v(t_{i+1})\Big),\ t\in(s_i,t_{i+1}],\ i=0,1,\ldots,m,\\ u'(t) &= \alpha_i,\ t\in(t_i,s_i],\ i=1,2,\ldots,m,\\ v'(t) &= \beta_i,\ t\in(t_i,s_i],\ i=1,2,\ldots,m,\\ u'(s_i^+) &= u'(s_i^-),\ i=1,2,\ldots,m,\\ v'(s_i^+) &= v'(s_i^-),\ i=1,2,\ldots,m,\\ u'(0^+) &= \alpha_0,\quad v'(0^+)=\beta_0,\\ u(0) &= u(T)=v(0)=v(T)=0. \end{cases}$$

Here the impulses start abruptly at points t_i and keep the derivative constant on a finite time interval $]t_i, s_i]$.

For the last chapter we generalized the model studied in the third chapter, by adding the terms $\eta_i(t)\Big(u(t)-u(t_{i+1})\Big)$ and $\theta_i(t)\Big(v(t)-v(t_{i+1})\Big)$ to the principal equations.

$$\begin{cases} -u''(t)+\eta_i(t)\Big(u(t)-u(t_{i+1})\Big) &= D_u f_i\Big(t, u(t)-u(t_{i+1}), v(t)-v(t_{i+1})\Big),\\ & \quad t\in(s_i,t_{i+1}],\ i=0,1,\ldots,m,\\ -v''(t)+\theta_i(t)\Big(v(t)-v(t_{i+1})\Big) &= D_v f_i\Big(t, u(t)-u(t_{i+1}), v(t)-v(t_{i+1})\Big),\\ & \quad t\in(s_i,t_{i+1}],\ i=0,1,\ldots,m,\\ u'(t) &= \alpha_i,\ t\in(t_i,s_i],\ i=1,2,\ldots,m,\\ v'(t) &= \beta_i,\ t\in(t_i,s_i],\ i=1,2,\ldots,m,\\ u'(s_i^+) &= u'(s_i^-),\ i=1,2,\ldots,m,\\ v'(s_i^+) &= v'(s_i^-),\ i=1,2,\ldots,m,\\ u'(0^+) &= \alpha_0,\quad v'(0^+)=\beta_0,\\ u(0) &= u(T)=v(0)=v(T)=0. \end{cases}$$

Chapter 1

Some basic tools

In this section we introduce some notations and definitions which are used throughout this work.

1.1 Some functional spaces

1.1.1 Weak topology $\sigma(X, X^\star)$

Let X be a Banach space and $X^\star$ be the dual space with norm

$$\|f\|_{X^\star} = \sup_{\substack{x \in X \\ \|x\|_X \leq 1}} \big|\langle f, x\rangle_{X^\star, X}\big|.$$

Let $f \in X^\star$, we denote by $\varphi_f : X \longrightarrow \mathbb{R}$ the linear functional $\varphi_f(x) = \langle f, x\rangle_{X^\star, X}$. As f runs through $X^\star$ we obtain a collection $(\varphi_f)_{f\in X^\star}$ of maps from X into $\mathbb{R}$. We now ignore the usual topology on X (associated to $\|\cdot\|_X$) and define a new topology on the set X as follows:

Definition 1.1. *(Weak topology) The weak topology on X is the coarsest (or weakest) topology on X, denoted by $\sigma(X, X^\star)$, that makes all the maps $(\varphi_f)_{f\in X^\star}$ continuous.*

Remark 1.1. *The open sets of the weak topology $\sigma(X, X^\star)$ are obtained by considering first $\cap_{finite}$ of sets of the form $\varphi_f^{-1}(\omega)$, ω is an open set in $\mathbb{R}$, and then $\cup_{arbitrary}$.*

We have in the following some properties of the weak topology.

Proposition 1.1. *The weak topology $\sigma(X, X^\star)$ is Hausdorff (i.e. given $x_1, x_2 \in X$ with $x_1 \neq x_2$ there are two open sets O_1 and O_2 for the weak topology $\sigma(X, X^\star)$ such that $x_1 \in O_1, x_2 \in O_2$, and $O_1 \cap O_2 = \emptyset$).*

Notation. If a sequence (x_n) in X converges to x in the weak topology $\sigma(X, X^\star)$ we shall write

$$x_n \rightharpoonup x.$$

To avoid any confusion we shall sometimes say, "$x_n \rightharpoonup x$ weakly in $\sigma(X, X^\star)$". In order to be totally clear we shall sometimes emphasize strong convergence by saying, "$x_n \longrightarrow x$ strongly", meaning that $\|x_n - x\|_X \overset{n\to\infty}{\longrightarrow} 0$.

Proposition 1.2. *Let (x_n) be a sequence in X. Then*

1. $\Big(x_n \rightharpoonup x$ *weakly in* $\sigma(X, X^\star)\Big) \Longleftrightarrow \Big(\langle f, x_n\rangle_{X^\star,X} \longrightarrow \langle f, x\rangle_{X^\star,X},\ \forall f \in X^\star\Big)$.
2. $\Big(x_n \longrightarrow x$ *strongly*$\Big) \Longrightarrow \Big(x_n \rightharpoonup x$ *weakly in* $\sigma(X, X^\star)\Big)$.
3. $\Big(x_n \rightharpoonup x$ *weakly in* $\sigma(X, X^\star)\Big) \Longrightarrow \Big(\big(\|x_n\|_X\big)$ *is bounded and* $\|x\|_X \leq \liminf \|x_n\|_X\Big)$.
4. $\Big(x_n \rightharpoonup x$ *weakly in* $\sigma(X, X^\star)$ *and* $f_n \longrightarrow f$ *strongly in* $X^\star$ $\big[$*i.e.* $\|f_n - f\|_{X^\star} \overset{n\to\infty}{\longrightarrow} 0\big]\Big) \Longrightarrow \Big(\langle f_n, x_n\rangle_{X^\star,X} \longrightarrow \langle f, x\rangle_{X^\star,X}\Big)$.

Remark 1.2. *In a Hilbert space H with the scalar product $\langle\cdot,\cdot\rangle_H$, we have through Riesz-Fréchet representation theorem:*

$$x_n \rightharpoonup x \Longleftrightarrow \lim_{n\to+\infty} \langle y, x_n\rangle_H = \langle y, x\rangle_H,\ \forall y \in H.$$

Proposition 1.3. *When X is finite-dimensional, the weak topology $\sigma(X, X^\star)$ and the usual topology are the same. In particular, a sequence (x_n) converges weakly if and only if it converges strongly.*

Remark 1.3. *Open (resp. closed) sets in the weak topology $\sigma(X, X^\star)$ are always open (resp. closed) in the strong topology. In any infinite-dimensional space the weak topology is strictly coarser than the strong topology, i.e., there exist open (resp. closed) sets in the strong topology that are not open (resp. closed) in the weak topology. Here are two examples:*

1. *The unit sphere* $S = \Big\{x \in X;\ \|x\|_X = 1\Big\}$*, with* X *infinite-dimensional, is never closed in the weak topology* $\sigma(X, X^\star)$*. More precisely, we have*

$$\overline{S}^{\sigma(X,X^\star)} = B_X,$$

where $\overline{S}^{\sigma(X,X^\star)}$ *denotes the closure of* S *in the topology* $\sigma(X, X^\star)$ *and* B_X *denotes the closed unit ball in* X*,*

$$B_X = \Big\{x \in X;\ \|x\|_X \leq 1\Big\}.$$

2. *The unit ball* $U = \Big\{x \in X;\ \|x\|_X < 1\Big\}$*, with* X *infinite-dimensional, is never open in the weak topology* $\sigma(X, X^\star)$*.*

Every weakly closed set is strongly closed and the converse is false in infinite-dimensional spaces. However, it is very useful to know that for convex sets, weakly closed = strongly closed.

Theorem 1.1. *Let* C *be a convex subset of a Banach space* X*. Then* C *is closed in the weak topology* $\sigma(X, X^\star)$ *if and only if it is closed in the strong topology.*

Corollary 1.1. *(Mazur) Assume* (x_n) *converges weakly to* x*. Then there exists a sequence* (y_n) *made up of convex combinations of the* x_n*'s:*

$$y_n = \sum_{k=1}^{k=n} \alpha_{n_k} x_k,\ \sum_{k=1}^{k=n} \alpha_{n_k} = 1,\ \ \alpha_{n_k} \geq 0\ (n \in \mathbb{N}^*),$$

such that (y_n) *converges strongly to* x*.*

1.1.2 Reflexive spaces

Let X be a Banach space and $X^\star$ be the dual space with norm

$$\|f\|_{X^\star} = \sup_{\substack{x \in X \\ \|x\|_X \leq 1}} \big|\langle f, x\rangle_{X^\star, X}\big|.$$

The bidual $X^{\star\star}$ is the dual of $X^\star$ with norm

$$\|\xi\|_{X^{\star\star}} = \sup_{\substack{f \in X^\star \\ \|f\|_{X^\star} \leq 1}} \big|\langle \xi, f\rangle_{X^{\star\star}, X^\star}\big|.$$

There is a canonical injection from X into $X^{\star\star}$ defined as follows

$$\begin{array}{rcccl} J: & X & \longrightarrow & X^{\star\star} & \\ & x & \longmapsto & J_x: & X^{\star} \longrightarrow \mathbb{R} \\ & & & & f \longmapsto \langle J_x, f\rangle_{X^{\star\star},X^{\star}} = \langle f, x\rangle_{X^{\star},X}. \end{array}$$

It is clear that J is linear and that J is an isometry, that is, $\|J_x\|_{X^{\star\star}} = \|x\|_X$. It may happen that J is not surjective from X onto $X^{\star\star}$. However, it is convenient to identify X with a subspace of $X^{\star\star}$ using J. If J turns out to be surjective then one says that X is reflexive, and $X^{\star\star}$ is identified with X.

Definition 1.2. *(Reflexive Space) The space X is said to be reflexive if the canonical injection J from X into $X^{\star\star}$ is surjective, i.e., $J(X) = X^{\star\star}$.*

When X is reflexive, $X^{\star\star}$ is usually identified with X.

Remark 1.4. *Many important spaces in analysis are reflexive. Clearly, finite-dimensional spaces are reflexive (since $\dim X = \dim X^{\star} = \dim X^{\star\star}$). L^p spaces are reflexive for $1 < p < \infty$. Hilbert spaces are reflexive. However, equally important spaces in analysis are not reflexive, for example L^1 and L^∞.*

The next result describes the basic properties of reflexive spaces.

Theorem 1.2. *The following statements are equivalent:*

1. *X is reflexive.*
2. *The closed unit ball in X*
$$B_X = \Big\{x \in X;\ \|x\|_X \leq 1\Big\},$$
is compact in the weak topology $\sigma(X, X^{\star})$.
3. *For every bounded sequence (x_n) in X, there exists a subsequence (x_{n_k}) that converges in the weak topology $\sigma(X, X^{\star})$.*

In order to clarify the connection among the above equivalent, it is maybe useful to recall the following facts:

- If X is a metric space, then

$$X \text{ is compact} \Longleftrightarrow \text{every sequence in } X \text{ admits a convergent subsequence.}$$

- There exist compact topological spaces X and some sequences in X without any convergent subsequence.
- If X is a topological space with the property that every sequence admits a convergent subsequence, then X need not be compact.

Here are some further properties of reflexive spaces.

Proposition 1.4. *Assume that X is a reflexive Banach space and let $A \subset X$ be a closed vector subspace of X. Then A is reflexive.*

Corollary 1.2. *A Banach space X is reflexive if and only if its dual space $X^\star$ is reflexive.*

1.1.3 L^p Spaces

Let $\Omega \subset \mathbb{R}^n$ with $n \in \mathbb{N}^*$.

Definition 1.3. *Let $p \in \mathbb{R}$ with $1 \leq p < \infty$, we set*

$$L^p(\Omega) = \Big\{u : \Omega \to \mathbb{R},\ u \textit{ is measurable and } \int_\Omega |u(x)|^p dx < \infty\Big\},$$

with norm

$$\|u\|_{L^p(\Omega)} = \left[\int_\Omega |u(x)|^p dx\right]^{\frac{1}{p}}.$$

Definition 1.4. *We set*

$$L^\infty(\Omega) = \Big\{u : \Omega \to \mathbb{R},\ u \textit{ is measurable and } |u(x)| \leq C \textit{ a.e. in } \Omega \textit{ for some canstant } C\Big\},$$

with the norm

$$\|u\|_{L^\infty(\Omega)} = \inf\Big\{C,\ |u(x)| \leq C \textit{ a.e. on } \Omega\Big\}.$$

We have the following properties:

1. L^p is a Banach space for any p, $1 \leq p \leq \infty$.

2. The dual of L^p is $L^{p'}$, for any p, $1 < p < \infty$, where p' is the conjugate exponent of p, i.e. $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. The dual of L^1 is L^∞. The dual of L^∞ is strictly bigger than L^1.

3. Hölder's inequality:
   Assume that $u \in L^p$ and $v \in L^{p'}$ with $1 \leq p, p' \leq \infty$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. Then $uv \in L^1$ and
   $$\int_\Omega \big|u(x)v(x)\big|dx \leq \|u\|_{L^p}\|v\|_{L^{p'}}.$$

4. L^p is reflexive for any p, $1 < p < \infty$.

5. L^1 and L^∞ are not reflexive spaces.

6. L^2 equipped with the scalar product
   $$(u, v) = \int_\Omega u(x)v(x)dx,$$
   is the unique Hilbert space among all L^p spaces.

7. If $u \in L^\infty$ then we have
   $$|u(x)| \leq \|u\|_{L^\infty} \text{ a.e. on } \Omega.$$

1.1.4 Sobolev spaces

Let $I = (a, b)$ be an open interval, possibly unbounded, and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$.

1.1.4.1 The space $W^{1,p}(I)$

Definition 1.5. *The Sobolev space $W^{1,p}(I)$ is defined to be*
$$W^{1,p}(I) = \Big\{u \in L^p(I) \ / \ u' \in L^p(I)\Big\}.$$
We set $H^1(I) = W^{1,2}(I)$.

It is essential to notice here that the derivative taken above the function u is in the sense of distributions, i.e.

$$u' \in L^p(I) \Longleftrightarrow \exists v \in L^p(I) : \int_I u\varphi' \ dx = -\int_I v\varphi \ dx, \ \forall \varphi \in \mathcal{C}_c^\infty(I),$$

where $\mathcal{C}_c^\infty(I)$ is the space of infinity times differentiable functions with compact support in I,

$$supp \ \varphi = \overline{\{x \in I, \ \varphi(x) \neq 0\}}.$$

The space $W^{1,p}(I)$ is equipped with the norm

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \|u'\|_{L^p},$$

or with the equivalent norm

$$\|u\|_{W^{1,p}} = \Big(\|u\|_{L^p}^p + \|u'\|_{L^p}^p\Big)^{\frac{1}{p}}, \ 1 < p < \infty.$$

The space $H^1(I)$ is equipped with the scalar product

$$(u,v)_{H^1} = (u,v)_{L^2} + (u',v')_{L^2} = \int_I (uv + u'v') \ dx,$$

and with the associated norm

$$\|u\|_{H^1} = \Big(\|u\|_{L^2}^2 + \|u'\|_{L^2}^2\Big)^{\frac{1}{2}}.$$

Here some properties of Sobolev space $W^{1,p}(I)$:

1. The space $W^{1,p}$ is a Banach space for $1 \leq p \leq \infty$.

2. $W^{1,p}$ is reflexive for $1 < p < \infty$.

3. H^1 is the unique Hilbert space among all $W^{1,p}$ spaces.

4. For any $u \in W^{1,p}(I)$ with $1 \leq p \leq \infty$, and I bounded or unbounded, then there exists a function $\widetilde{u} \in \mathcal{C}(\overline{I})$ $\Big(\mathcal{C}(\overline{I})$: space of all continuous functions on $\overline{I}$ with norm $\|v\|_\infty = \max_{t\in\overline{I}} |v(t)|\Big)$, such that

$$u = \widetilde{u} \ \text{ a.e. on } I,$$

and

$$\widetilde{u}(x) - \widetilde{u}(y) = \int_y^x u'(t)dt, \ \forall x, y \in \overline{I}.$$

We note that if one function u belongs to $W^{1,p}$ then all functions v such that $v = u$ a.e. on I also belong to $W^{1,p}$. Every function $u \in W^{1,p}$ admits one (and only one) continuous representative on $\overline{I}$, i.e. there exists a continuous function on $\overline{I}$ that belongs to the equivalence class of u ($v \sim u$ if $v = u$ a.e.). When it is useful (for example, in order to give a meaning to $u(x)$ for every $x \in \overline{I}$) we replace u by its continuous representative.

In order to simplify the notation we also write u for its continuous representative. We finally point out that the property "u has a continuous representative" is not the same as "u is continuous a.e.".

5. Sobolev inequality (Sobolev embedding):

   (a) There exists a constant C (depending only on $|I| \leq \infty$) such that

   $$\|u\|_{W^{\infty}(I)} \leq C\|u\|_{W^{1,p}(I)}, \ \forall u \in W^{1,p}(I), \ \forall \ 1 \leq p \leq \infty.$$

   In other words, $W^{1,p}(I) \subset W^{\infty}(I)$ with continuous injection for all $1 \leq p \leq \infty$.

   (b) If I is bounded then

   - The injection $W^{1,p}(I) \subset \mathcal{C}(\overline{I})$ is compact for all $1 < p \leq \infty$.
   - The injection $W^{1,1}(I) \subset L^{q}(I)$ is compact for all $1 \leq q < \infty$.

Remark 1.5. *The injection $W^{1,1}(I) \subset \mathcal{C}(\overline{I})$ is continuous but it is never compact, even if I is a bounded interval.*

Remark 1.6. *(Continuous embedding / Compact embedding) If X and Y be two normed vector spaces, with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively, such that $X \subseteq Y$.*

► *We say that X is continuously embedded in Y if the identity function*

$$\begin{aligned} i : X &\longrightarrow Y, \\ x &\longmapsto x, \end{aligned}$$

is continuous, i.e. if there exists a constant $C \geq 0$ such that

$$\|x\|_Y \leq C\|x\|_X, \ \forall x \in X.$$

► *We say that X is compactly embedded in Y if*

- *X is continuously embedded in Y.*
- *The identity function i of X into Y is a compact operator, i.e. any bounded subset in X is relatively compact subset in Y (or in other words for any bounded sequence (x_n) in X, there exists a subsequence (x_{n_k}) that converges in Y.).*

6. Suppose that I is an unbounded interval and $u \in W^{1,p}(I)$ with $1 \leq p < \infty$. Then

$$\lim_{\substack{x \in I \\ |x| \to \infty}} u(x) = 0.$$

1.1.4.2 The space $W_0^{1,p}(I)$

Let $1 \leq p < \infty$.

Definition 1.6. *We denote to the closure of $\mathcal{C}_c^\infty(I)$ in $W^{1,p}(I)$ by $W_0^{1,p}(I)$, i.e.*

$$\overline{\mathcal{C}_c^\infty(I)}^{W^{1,p}(I)} = W_0^{1,p}(I) \subseteq W^{1,p}(I).$$

We set $H_0^1(I) = W_0^{1,2}(I)$.

The space $W_0^{1,p}(I)$ is equipped with the norm of $W^{1,p}(I)$, and the space $H_0^1(I)$ is equipped with the scalar product of $H^1(I)$.

Here some properties of the space $W_0^{1,p}(I)$:

1. $W_0^{1,p}$ is a Banach space for $1 \leq p < \infty$, and it is reflexive if $1 < p < \infty$.

2. H_0^1 is a Hilbert space.

3. If $I = \mathbb{R}$, we have $\overline{\mathcal{C}_c^\infty(\mathbb{R})}^{W^{1,p}(\mathbb{R})} = W^{1,p}(\mathbb{R})$, and therefore $W_0^{1,p}(\mathbb{R}) = W^{1,p}(I)$.

4. The following property is a basic characterization of functions in $W_0^{1,p}(I)$:
   If $u \in W^{1,p}(I)$, then we have

   $$u \in W_0^{1,p}(I) \Longleftrightarrow u = 0 \text{ on } \partial I,$$

   where ∂I denotes the boundary of I.

5. Poincaré's inequality:

   Suppose I is a bounded interval. Then there exists a constant C (depending on $|I| < \infty$) such that

$$\|u\|_{L^p(I)} \leq C\|u'\|_{L^p(I)}, \ \forall u \in W_0^{1,p}(I).$$

   As a consequence of Poincaré's inequality, the quantity $\|u'\|_{L^p(I)}$ is a norm equivalent to the $W^{1,p}(I)$ norm.

6. If I is bounded, the expression $(u', v')_{L^2(I)} = \displaystyle\int_I u'v' \, dx$, defines a scalar product on $H_0^1(I)$, and the associated norm, i.e., $\|u'\|_{L_2(I)}$ is equivalent to the $H^1(I)$ norm.

1.1.4.3 The space $W^{m,p}(I)$

Let $m \geq 0$ an integer.

Definition 1.7. *The Sobolev space $W^{m,p}(I)$ is defined by*

$$W^{m,p}(I) = \Big\{ u \in L^p(I) \ / \ D^\alpha u \in L^p(I), \ \forall \alpha \in \{0, 1, 2, \cdots, m\} \Big\},$$

where $D^\alpha u, \ \alpha \in \{0, 1, 2, \cdots, m\}$ denote the successive derivatives of u, i.e.

$$D^0 u = u, \ D^1 u = u', \ D^2 u = u'', \cdots, \ D^m u = u^{(m)}.$$

We set $H^m(I) = W^{m,2}(I)$.

We have for any $\alpha \in \{0, 1, 2, \cdots, m\}$:

$$D^\alpha u \in L^p(I) \Longleftrightarrow \exists v_\alpha \in L^p(I) : \int_I u D^\alpha \varphi \, dx = (-1)^\alpha \int_I v_\alpha \varphi \, dx, \ \forall \varphi \in \mathcal{C}_c^\infty(I).$$

The space $W^{m,p}(I)$ is equipped with the norm

$$\|u\|_{W^{m,p}} = \sum_{\alpha=0}^{m} \|D^\alpha u\|_{L^p},$$

or sometimes with the equivalent norm

$$\|u\|_{W^{m,p}} = \left(\sum_{\alpha=0}^{m} \|D^\alpha u\|_{L^p}^p \right)^{\frac{1}{p}}, \ 1 < p < \infty.$$

The space $H^m(I)$ is equipped with the scalar product

$$(u,v)_{H^m} = \sum_{\alpha=0}^{m} (D^\alpha u, D^\alpha v)_{L^2} = \sum_{\alpha=0}^{m} \int_I D^\alpha u D^\alpha v \ dx,$$

and with the associated norm

$$\|u\|_{H^1} = \left(\sum_{\alpha=0}^{m} \|D^\alpha u\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

It is easily show

$$\cdots \subset W^{3,p} \subset W^{2,p} \subset W^{1,p} \subset W^{0,p} = L^p.$$

We can extend to the space $W^{m,p}$ $(m \geq 2)$, all the properties shown for $W^{1,p}$. For example, if I is bounded, $W^{m,p}(I)$ is continuously embedded in $\mathcal{C}^{m-1}(\overline{I})$ for $1 \leq p \leq \infty$, and it is compactly embedded if $1 < p \leq \infty$.

1.1.4.4 The space $W_0^{m,p}(I)$

Let $1 \leq p < \infty$, and let $m \geq 2$ an integer.

Definition 1.8. *$W_0^{m,p}(I)$ is defined as the closure of $\mathcal{C}_c^\infty(I)$ in $W^{m,p}(I)$, i.e.*

$$\overline{\mathcal{C}_c^\infty(I)}^{W^{m,p}(I)} = W_0^{m,p}(I) \subseteq W^{m,p}(I).$$

We set $H_0^m(I) = W_0^{m,2}(I)$.

Remark 1.7.

1. *We can also define $W_0^{m,p}(I)$ as follow*

$$W_0^{m,p}(I) = \left\{ u \in W^{m,p}(I) \ / \ D^\alpha u = 0 \ on \ \partial I, \ \forall \alpha \in \{0,1,\cdots,m-1\} \right\}.$$

2. *It is necessary to note the distinction between:*

$$W_0^{2,p}(I) = \left\{ u \in W^{2,p}(I) \ / \ u = Du = 0 \ on \ \partial I \right\},$$

and

$$W^{2,p}(I) \cap W_0^{1,p}(I) = \left\{ u \in W^{2,p}(I) \ / \ u = 0 \ on \ \partial I \right\}.$$

We finish this section by adding the following result

Proposition 1.5. *(Proposition 1.2. of [17]) Let $1 < p < \infty$, and let $T > 0$. If the sequence (x_n) converges weakly to x in $W_T^{1,p}(0,T)$, then (x_n) converges uniformly to x on $[0,T]$.*
Here the Sobolev space $W_T^{1,p}(0,T)$ is the space of functions $u \in W^{1,p}(0,T)$ with $u(0) = u(T)$, i.e.

$$W_T^{1,p}(0,T) = \left\{ u \in W^{1,p}(0,T) \;/\; u(0) = u(T) \right\}.$$

1.2 Some classical definitions and results on critical point theory

Let X and Y two normed spaces.

1.2.1 Differentiable maps

We begin by defining two kinds of differentiability, the notion of Fréchet derivative and then recall the definition of Gateaux derivative.

Definition 1.9. *(Bounded operator) A linear operator $F : X \longrightarrow Y$ is called bounded (or continuous) if there is a constant $C \geq 0$ such that*

$$\|F(x)\|_Y \leq C\|x\|_X, \ \forall x \in X.$$

We denote by $\mathcal{L}(X,Y)$ to the space of all linear bounded operators from X into Y.

The norm of a bounded operator is defined by

$$\|F\|_{\mathcal{L}(X,Y)} = \sup_{\substack{x\in X \\ x\neq 0}} \frac{\|F(x)\|_Y}{\|x\|_X} = \sup_{\substack{x\in X \\ \|x\|_X\leq 1 \\ x\neq 0}} \|F(x)\|_Y = \sup_{\substack{x\in X \\ \|x\|_X=1}} \|F(x)\|_Y.$$

Definition 1.10. *(Fréchet derivative) Let U a nonempty open subset in X.*
An operator $F : U \subset X \longrightarrow Y$ is called Fréchet differentiable at a point $a \in U$, if there exists a function $G_a \in \mathcal{L}(X,Y)$ such that

$$F(a+h) - F(a) = G_a(h) + o(h), \ h \in X,$$

where the remainder $o(h)$ (Little o -Landau notation-) is in Y and satisfies

$$\lim_{\|h\|_X \to 0} \frac{\|o(h)\|_Y}{\|h\|_X} = 0.$$

It is essential to notice if $a \in U$, and since U is open, then there exists $r > 0$ such that the ball $B(a, r) = \Big\{x \in X \ / \ \|x - a\|_X < r\Big\} \subset U$. Hence we can choose $h \in X$ with $\|h\|_X < r$ to ensure $a + h \in U$, therefore $F(a + h)$ will be well defined.

Usually the operator G_a will be denoted by

$$DF(a),\ dF(a),\ d_aF \text{ or } F'(a),$$

and it is called the Fréchet derivative (differential) of F at the point $a \in U$.

The image of $h \in X$ under $DF(a)$ $\Big(DF(a)(h) \in Y\Big)$ is called the Fréchet differential of F at the point a taken at h.

We say F is Fréchet differentiable on U if F is Fréchet differentiable at each point of U. In this case, the mapping $a \longmapsto DF(a)$ is called Fréchet derivative (differential) of F on U, and it is denoted by DF, hence

$$\begin{aligned} DF : U &\longrightarrow \mathcal{L}(X, Y) \\ a &\longmapsto DF(a). \end{aligned}$$

F is said to be of class $\mathcal{C}^1$ on U if the differential DF is continuous on U, and we write $F \in \mathcal{C}^1(U, F) = \mathcal{C}^1(U)$.

Definition 1.11. *(Gateaux derivative) Let U a nonempty open subset in X.*
An operator $F : U \subset X \longrightarrow Y$ is called Gateaux differentiable at a point $a \in U$, if there exists $G_a \in \mathcal{L}(X, Y)$ such that

$$\lim_{t \to 0} \frac{F(a + th) - F(a)}{t} = G_a(h),\ \forall h \in X,$$

where the limit is taken for real t and convergence in the norm of Y is meant.

As in the notion of Fréchet derivative G_a will be denoted by

$$DF(a),\ dF(a),\ d_aF \text{ or } F'(a),$$

and is called the Gateaux derivative of F at the point $a \in U$.

The image of $h \in X$ under $DF(a)$ $\Big(DF(a)(h) \in Y\Big)$ is called the Gateaux differential of F at the point a in the direction h.

We say F is Gateaux differentiable on U if F is Gateaux differentiable at each point of U. In this case, the mapping $a \longmapsto DF(a)$ is called Gateaux derivative of F on U, and it is denoted by DF, hence $DF : U \longrightarrow \mathcal{L}(X, Y)$.

Remark 1.8.

1. *It follows immediately from the definitions, if F is Fréchet differentiable at $a \in U$, then F is Gateaux differentiable at a, and the Fréchet and Gateaux derivatives of F at a are the same. The converse is not always true, but we have the following*
   If the Gateaux derivative DF exists in some neighborhood $U(a) \subset U$ of the point $a \in U$, and is continuous at a, then the Fréchet derivative $DF(a)$ exists and the Fréchet and Gateaux derivatives of F at a are the same. In other words, a continuous Gateaux derivative is a Fréchet derivative.

2. *If F is Fréchet differentiable at $a \in U$, then F is continuous at a.*

1.2.1.1 Some basic examples

In the following some basic examples which are used in the next chapters (regarding the construction of the energy functional).

Example 1.1. *Let H be a Hilbert space with the scalar product $(\cdot, \cdot)_H$. Consider the functional F defined as following*

$$\begin{aligned} F : H &\longrightarrow \mathbb{R} \\ u &\longmapsto F(u) = \frac{1}{2}\|u\|_H^2 = \frac{1}{2}(u, u)_H. \end{aligned}$$

Since

$$F(u + \varphi) - F(u) = \frac{1}{2}\|u + \varphi\|_H^2 - \frac{1}{2}\|u\|_H^2 = (u, \varphi)_H + \frac{1}{2}\|\varphi\|_H^2,$$

so F is Fréchet differentiable at any point $u \in H$, and we have

$$\begin{aligned} DF(u) : H &\longrightarrow \mathbb{R} \\ \varphi &\longmapsto DF(u)(\varphi) = (u, \varphi)_H, \end{aligned}$$

and $o(\varphi) = \dfrac{1}{2}\|\varphi\|_H^2$.

Example 1.2. *Let $F:\mathbb{R}^n \longrightarrow \mathbb{R}$ be a real function and $n \geq 1$.*
If F is differentiable (in the sense of Fréchet) at the point $a \in \mathbb{R}^n$, then all the partial derivatives of F at the point a are exist, and we have

$$\begin{aligned} DF(a):\mathbb{R}^n &\longrightarrow \mathbb{R} \\ h &\longmapsto DF(a)(h) = \langle \nabla F(a), h\rangle = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(a)h_i, \end{aligned}$$

where $\nabla F(a)$ is the gradient of F at the point a, and $\langle \cdot,\cdot\rangle$ denotes the canonical scalar product in $\mathbb{R}^n$.

The converse is not always true, but we have the following
If the partial derivatives of F at the point a are exist and continuous, then F is differentiable at the point a.

Example 1.3. *If $F:\mathbb{R} \longrightarrow \mathbb{R}$ be a derivable function. So for any $x \in \mathbb{R}$ we have $DF(x)(h) = F'(x)h,\ h \in \mathbb{R}$.*

As a special case if $f:\mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function, and F defined as follows:

$$\begin{aligned} F:\mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto F(x) = \int_0^x f(s)ds, \end{aligned}$$

then F is derivable, and we have $DF(x)(h) = f(x)h,\ h \in \mathbb{R}$.

1.2.2 Extreme points

Let F be a real functional defined on a nonempty open subset $U \subseteq X$ $\big(F:U \subseteq X \longrightarrow \mathbb{R}\big)$.

Definition 1.12. *(Extremum point) A point $x_0 \in U$ is called an extremum of F if there exists an open neighborhood $U(x_0) \subseteq X$ of x_0 such that*

$$F(x) \leq F(x_0),\ \text{for every } x \in U(x_0) \cap U,\ \text{i.e., } F \text{ is maximal at } x_0,$$

or

$$F(x) \geq F(x_0),\ \text{for every } x \in U(x_0) \cap U,\ \text{i.e., } F \text{ is minimal at } x_0.$$

Definition 1.13. *(Critical point -Stationary point-) If F is differentiable at $x_0 \in U$, then x_0 is called a critical point (or stationary point) of F if*

$$DF(x_0) = 0.$$

In the following result a necessary condition for x_0 to be an extremum.

Theorem 1.3. *Suppose F is differentiable at $x_0 \in U$.*
If x_0 is an extremum of F, then $DF(x_0) = 0$ $\big($i.e., x_0 is a critical point of $F\big)$. In other words a necessary condition for x_0 to be an extremum is that it is critical.

Remark 1.9.

1. *The foregoing theorem shows us if we need to find the extremum points of a function F we have to look for them among the critical points.*

2. *Not always a critical point is an extremum point. For example the function F defined on $\mathbb{R}$ by $F(x) = x^3$ has a critical point at 0, but 0 not an extremum point.*

3. *In the case $X = \mathbb{R}^n$, we have the following equivalents*

$$\begin{aligned} DF(x) = 0 \iff & \nabla F(x) = 0, \\ \iff & \frac{\partial F}{\partial x_i}(x) = 0, \ \forall i = 1, 2, \cdots, n, \end{aligned}$$

so the problem of finding the critical points returns to solve a system of n algebraic equations

$$\begin{cases} \dfrac{\partial F}{\partial x_i}(x_1, x_2, \cdots, x_n) = 0, \\ i = 1, 2, \cdots, n. \end{cases}$$

1.2.3 Minimizing sequence / Weakly lower semi-continuous functions / Coercive functions

Definition 1.14. *(Minimizing sequence) A minimizing sequence of a functional $F : X \longrightarrow \mathbb{R}$, is a sequence $(x_j) \subset X$, such that*

$$\lim_{j \longrightarrow \infty} F(x_j) = \inf_{x \in X} F(x).$$

Definition 1.15. *(Lower semi-continuous / Weakly lower semi-continuous) A functional $F : X \longrightarrow \mathbb{R}$, is lower semi-continuous (resp. weakly lower semi-continuous), if*

$$\forall (x_j) \subset X : x_j \longrightarrow x \Longrightarrow \liminf_{j \longrightarrow \infty} F(x_j) \geq F(x),$$

$$\left(\textit{resp.} \ \ \forall (x_j) \subset X : x_j \rightharpoonup x \Longrightarrow \liminf_{j \longrightarrow \infty} F(x_j) \geq F(x) \right).$$

The following properties are easy consequences of the definition:

1. The sum of two l.s.c. (resp. w.l.s.c.) functions is l.s.c. (resp. w.l.s.c.).
2. The product of a l.s.c. (resp. w.l.s.c.) function by a positive constant is l.s.c. (resp. w.l.s.c.).
3. If $(\varphi_\lambda)_{\lambda \in \Lambda}$ is a family of l.s.c. (resp. w.l.s.c.) functions, the function $\sup_{\lambda \in \Lambda} \varphi_\lambda$ defined by

$$\left(\sup_{\lambda \in \Lambda} \varphi_\lambda \right)(u) = \sup_{\lambda \in \Lambda} \varphi_\lambda(u),$$

is l.s.c. (resp. w.l.s.c.).

Remark 1.10.

1. *If the functional F is w.l.s.c. then F is l.s.c.*
2. *If the functional F is continuous then F is l.s.c.*

Definition 1.16. *(Coercive) A functional $F : X \longrightarrow \mathbb{R}$, is called coercive if, for every $x \in X$,*

$$F(x) \to +\infty \quad \textit{if} \quad \|x\|_X \to +\infty.$$

Now we are interested to find conditions that ensure that a functional, defined on all of a Banach space, achieves its extremum. So we have the following result

Theorem 1.4. *(Th. 1.1 of [17]) Let F be a functional defined on a reflexive Banach space X. If F satisfies*

1. *F is w.l.s.c.*
2. *F has a bounded minimizing sequence.*

Then, F has a minimum on X. In other words F is bounded from below on X and achieves its infimum at some point $x_0 \in X$. If moreover F is differentiable at x_0, then $DF(x_0) = 0$.

Proof. Let (x_j) be a bounded minimizing sequence. Going if necessary to a subsequence, we can assume, by the reflexivity of X, that (x_j) converges weakly to some $x \in X$. Thus,

$$F(x) \leq \liminf_{j \longrightarrow \infty} F(x_j) = \lim_{j \longrightarrow \infty} F(x_j) = \inf_{y \in X} F(y),$$

so that $\inf_{y \in X} F(y) = F(x)$. □

Remark 1.11. *The existence of a bounded minimizing sequence will be in particular insured when F is coercive.*

Corollary 1.3. *If F satisfies*

1. *F is w.l.s.c.*
2. *F is coercive.*

Then, F has a minimum on X.

1.2.4 Convex functions

Definition 1.17 (Convex function)**.** *A functional $F : X \longrightarrow \mathbb{R}$, is convex if*

$$F\big((1-\lambda)x + \lambda y\big) \leq (1-\lambda)F(x) + \lambda F(y),$$

for all $\lambda \in (0,1)$, $x, y \in X$.

The following properties are easy consequences of the definition:

1. The sum of two convex functions is a convex function.
2. The product of a convex function by a positive constant is a convex function.
3. If $(\varphi_\lambda)_{\lambda \in \Lambda}$ is a family of convex functions then $\sup_{\lambda \in \Lambda} \varphi_\lambda$ is a convex function.

In view of theorem 1.4, it is important to obtain sufficient conditions for weak lower semi-continuity.

Theorem 1.5. *(Th. 1.2. of [17]) If X is a normed space and $F : X \longrightarrow \mathbb{R}$, is l.s.c. and convex, then F is w.l.s.c.*

Proof. Assume that $x_j \rightharpoonup x$ and let $c > \liminf_{j\to+\infty} F(x_j)$. Going if necessary to a subsequence, we can assume that $c > F(x_j)$ for all $j \in \mathbb{N}^*$. By Mazur's theorem (corollary 1.1), there exists a sequence (y_j) with

$$y_j = \sum_{k=1}^{j} \alpha_{j_k} x_k,\ \sum_{k=1}^{j} \alpha_{j_k} = 1,\ \alpha_{j_k} \geq 0,$$

such that $y_j \to x$. Since F is l.s.c. and convex, we obtain

$$\begin{aligned}
F(x) &\leq \liminf_{j\to+\infty} F(y_j), \\
&= \liminf_{j\to+\infty} F\left(\sum_{k=1}^{j} \alpha_{j_k} x_k\right), \\
&\leq \liminf_{j\to+\infty} \sum_{k=1}^{j} \alpha_{j_k} F(x_k), \\
&\leq \liminf_{j\to+\infty} \left(c \sum_{k=1}^{j} \alpha_{j_k}\right), \\
&= \liminf_{j\to+\infty} \big(c\big), \\
&= c.
\end{aligned}$$

Since $c > \liminf_{j\to+\infty} F(x_j)$ is arbitrary, we have $F(x) \leq \liminf_{j\to+\infty} F(x_j)$, so that F is w.l.s.c. $\square$

Corollary 1.4. *If the functional F is continuous and convex on a normed space X, then F is w.l.s.c. In particular, for every sequence $(x_j) \subset X$ converging weakly to x, we have*

$$\liminf_{j\longrightarrow\infty} \|x_j\|_X \geq \|x\|_X.$$

Chapter 2

Variational approach to instantaneous impulsive differential system

In this chapter we consider a nonlinear Dirichlet problem with instantaneous impulses and obtain the existence of solutions by means of variational methods.

$$\begin{cases} -u''(t) = f_u(t,u,v),\ t \in (0,T) \setminus \{t_1,...,t_m\}, \\ -v''(t) = f_v(t,u,v),\ t \in (0,T) \setminus \{t_1,...,t_m\}, \\ u(0) = u(T) = v(0) = v(T) = 0, \\ \Delta u'(t_k) = u'(t_k^+) - u'(t_k^-) = I_k\big(u(t_k)\big),\ k = 1,2,...,m, \\ \Delta v'(t_k) = v'(t_k^+) - v'(t_k^-) = J_k\big(v(t_k)\big),\ k = 1,2,...,m, \end{cases} \tag{2.1}$$

where $0 = t_0 < t_1 < t_2 < ... < t_m < t_{m+1} = T$, the impulses start abruptly at points t_k, here $u'(t_k^{\pm}) = \lim_{t \to t_k^{\pm}} u'(t)$. The nonlinear functions f_u, f_v (the derivatives of $f(t,u,v)$ with respect to u and v respectively) are Carathéodory functions on $(0,T) \times \mathbb{R}^2$, and $I_k, J_k,\ (k = 1,...,m)$, are continuous functions on $\mathbb{R}$.

2.1 Functional space framework

We define the following functional spaces:

$\mathcal{C}[0,T]$ be the space of all continuous functions on $[0,T]$ with the norm

$$\|u\|_\infty = \max_{t\in[0,T]} |u(t)|.$$

$H_0^1(0,T)$ is the Sobolev space with the inner products

$$(u,v)_1 = \int_0^T u'(t)v'(t)\ dt,$$

and

$$(u,v)_2 = \int_0^T u(t)v(t)\ dt + \int_0^T u'(t)v'(t)\ dt,$$

which induce the corresponding norms

$$\|u\|_1 = \left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{1}{2}},$$

and

$$\|u\|_2 = \left(\int_0^T \left|u(t)\right|^2 dt + \int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{1}{2}}.$$

By Poincare's inequality,

$$\left(\int_0^T u^2(t)\ dt\right)^{\frac{1}{2}} \le \frac{1}{\sqrt{\lambda_1}}\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{1}{2}},\ \text{for any } u\in H_0^1(0,T),$$

we easily obtain that the norms $\|.\|_1$ and $\|.\|_2$ are equivalent. Here, $\lambda_1 = \dfrac{\pi^2}{T^2}$ is the first eigenvalue of the Dirichlet problem

$$-u''(t) = \lambda u(t),\ t\in(0,T);\ u(0)=u(T)=0.$$

Set $H = H_0^1(0,T)\times H_0^1(0,T)$, in the Hilbert space H, for any $(u,v)\in H$, we set the norm

$$\|(u,v)\|_H = \left(\|u\|_1^2 + \|v\|_1^2\right)^{\frac{1}{2}}.$$

Lemma 2.1. *There exists $\gamma>0$ such that, if $(u,v)\in H$, then*

$$\|u\|_\infty,\ \|v\|_\infty \le \gamma\big\|(u,v)\big\|_H.$$

Proof. For any $(u, v) \in H$, it follows from the mean value theorem that

$$u(\tau) = \frac{1}{T}\int_0^T u(s)\ ds,$$

for some $\tau \in (0, T)$. Hence, for $t \in [0, T]$, using Hölder's inequality and Poincare's inequality

$$\begin{aligned}
|u(t)| &= \left|u(\tau) + \int_\tau^t u'(s)\ ds\right|, \\
&\leq |u(\tau)| + \int_0^T |u'(s)|\ ds, \\
&\leq \frac{1}{T}\int_0^T |u(s)|\ ds + \int_0^T |u'(s)|\ ds, \\
&\leq \frac{1}{\sqrt{T}}\|u\|_{L^2} + \sqrt{T}\|u'\|_{L^2}, \\
&\leq \frac{1}{\sqrt{\lambda_1 T}}\|u'\|_{L^2} + \sqrt{T}\|u'\|_{L^2}, \\
&= \left(\frac{1}{\sqrt{\lambda_1 T}} + \sqrt{T}\right)\|u'\|_{L^2}, \\
&\leq \left(\frac{1}{\sqrt{\lambda_1 T}} + \sqrt{T}\right)\Big\|(u, v)\Big\|_H.
\end{aligned}$$

Then, there exists $\gamma = \dfrac{1}{\sqrt{\lambda_1 T}} + \sqrt{T} > 0$, such that

$$\|u\|_\infty \leq \gamma\Big\|(u, v)\Big\|_H.$$

Similarly, we can get

$$\|v\|_\infty \leq \gamma\Big\|(u, v)\Big\|_H.$$

□

2.2 Variational approach

In the following, we are concerned with problem (2.1) subject to impulses in the derivative at the prescribed instantsts $t_k, k = 1, 2, ..., m$. We are interested in the solution (u, v) of problem (2.1) satisfying the impulse conditions,

$$\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-) = I_k(u(t_k)), \tag{2.2}$$

and

$$\Delta v'(t_k) = v'(t_k^+) - v'(t_k^-) = J_k(v(t_k)),\ \ k = 1, 2, ..., m. \tag{2.3}$$

For $u, v \in H^2(0, T)$, we have that u, v, u' and v' are both absolutely continuous. Meanwhile, $u'', v'' \in L^2(0, T)$. Hence, $\Delta u'(t) = u'(t^+) - u'(t^-) = 0$ and $\Delta v'(t) = v'(t^+) - v'(t^-) = 0$ for any $t \in [0, T]$.

If $u, v \in H_0^1(0, T)$, then u, v are absolutely continuous and $u', v' \in L^2(0, T)$. In this case, the one-sided derivatives $u'(t^+), u'(t^-), v'(t^+)$ and $v'(t^-)$ may not exist.

Thus, we need to introduce a concept of solution which is different from a classical solution. We say that (u, v) is a classical solution of problem (2.1) if it satisfies the corresponding equations a.e. on $[0, T]$, the limits $u'(t_k^+), u'(t_k^-), v'(t_k^+)$ and $v'(t_k^-), k = 1, 2, ..., m$, exist and (2.2), (2.3) hold.

Taking $(\varphi, \psi) \in H$ and multiplying the two sides of the equalities

$$-u''(t) = f_u(t, u, v),$$

and

$$-v''(t) = f_v(t, u, v),$$

by φ and ψ respectively, then integrating from 0 to T, we have

$$-\int_0^T u''(t)\varphi(t)\ dt = \int_0^T f_u(t, u, v)\varphi(t)\ dt, \tag{2.4}$$

and

$$-\int_0^T v''(t)\psi(t)\ dt = \int_0^T f_v(t, u, v)\psi(t)\ dt. \tag{2.5}$$

The first terms of (2.4) and (2.5) are now

$$\begin{aligned} -\int_0^T u''(t)\varphi(t)\ dt &= -\sum_{k=0}^{m}\int_{t_k}^{t_{k+1}} u''(t)\varphi(t)\ dt, \\ &= \sum_{k=1}^{m} I_k\big(u(t_k)\big)\varphi(t_k) + \int_0^T u'(t)\varphi'(t)\ dt, \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} -\int_0^T v''(t)\psi(t)\ dt &= -\sum_{k=0}^{m}\int_{t_k}^{t_{k+1}} v''(t)\psi(t)\ dt, \\ &= \sum_{k=1}^{m} J_k\big(v(t_k)\big)\psi(t_k) + \int_0^T v'(t)\psi'(t)\ dt. \end{aligned} \tag{2.7}$$

In connection with (2.4), (2.5), (2.6) and (2.7), we have

$$\int_0^T u'(t)\varphi'(t)\ dt + \int_0^T v'(t)\psi'(t)\ dt + \sum_{k=1}^{m} I_k\big(u(t_k)\big)\varphi(t_k) + \sum_{k=1}^{m} J_k\big(v(t_k)\big)\psi(t_k)$$
$$= \int_0^T f_u(t,u,v)\varphi(t)\ dt + \int_0^T f_v(t,u,v)\psi(t)\ dt. \quad (2.8)$$

Based on equality (2.8), we introduce the concept of weak solution for problem (2.1).

Definition 2.1. *We say that a pair of functions $(u, v) \in H$ is a weak solution for problem (2.1) if identity (2.8) holds for any $(\varphi, \psi) \in H$.*

We consider the energy functional corresponding to problem (2.1)

$$\Phi : H \longrightarrow \mathbb{R},$$

defined by

$$\Phi(u,v) = \frac{1}{2}\int_0^T \Big(u'(t)\Big)^2\ dt + \frac{1}{2}\int_0^T \Big(v'(t)\Big)^2 dt + \sum_{k=1}^{m}\int_0^{u(t_k)} I_k(t)dt + \sum_{k=1}^{m}\int_0^{v(t_k)} J_k(t)dt$$
$$- \int_0^T f(t,u,v)\ dt,$$

for more details about the construction of Φ, see the subsection 1.2.1.1.

Therefore

$$\Phi(u,v) \quad = \quad \frac{1}{2}\Big\|(u,v)\Big\|_H^2 \ + \ \sum_{k=1}^{m}\int_0^{u(t_k)} I_k(t)\ \ dt \ + \ \sum_{k=1}^{m}\int_0^{v(t_k)} J_k(t)\ \ dt \ - \ \int_0^T f(t,u,v)\ \ dt. \quad (2.9)$$

Proposition 2.1. *The functional $\Phi : H \longrightarrow \mathbb{R}$, defined by (2.9) is continuously Fréchet-differentiable and weakly lower semi-continuous. Moreover, the critical points of Φ are weak solutions of (2.1).*

Proof. Using the continuity of f_u, f_v, I_k and $J_k,\ k = 1, 2, ..., m$, we easily obtain that the functional $\Phi \in \mathcal{C}^1(H, \mathbb{R})$. Furthermore, we have the differential of Φ at $(u, v) \in H$

$$\Phi'(u,v) : H \longrightarrow \mathbb{R},$$

is defined by

$$\Phi'(u,v)(\varphi,\psi) = \int_0^T u'(t)\varphi'(t)\ dt + \int_0^T v'(t)\psi'(t)\ dt$$
$$+ \sum_{k=1}^{m} I_k\big(u(t_k)\big)\varphi(t_k) + \sum_{k=1}^{m} J_k\big(v(t_k)\big)\psi(t_k)$$
$$- \int_0^T f_u(t,u,v)\varphi(t)\ dt - \int_0^T f_v(t,u,v)\psi(t)\ dt.$$

This shows that the critical points of Φ give us the weak solutions of (2.1).

To show that Φ is weakly lower semi-continuous, let $\{(u_j, v_j)\} \subset H$, with $(u_j, v_j) \rightharpoonup (u, v)$, then we have that $\{u_j\}$ and $\{v_j\}$ converge uniformly to u and v on $[0, T]$ respectively (Proposition 1.5). In connection with the fact that $\liminf_{j\longrightarrow\infty} \|(u_j, v_j)\|_H \geq \|(u, v)\|_H$ (Corollary 1.4), one has

$$\begin{aligned}
\liminf_{j\longrightarrow\infty} \Phi(u_j, v_j) &= \liminf_{j\longrightarrow\infty} \left\{ \frac{1}{2}\left\|(u_j, v_j)\right\|_H^2 + \sum_{k=1}^{m} \int_0^{u_j(t_k)} I_k(t)\ dt \right. \\
&\quad \left. + \sum_{k=1}^{m} \int_0^{v_j(t_k)} J_k(t)\ dt - \int_0^T f(t, u_j, v_j)\ dt \right\}, \\
&\geq \frac{1}{2}\left\|(u, v)\right\|_H^2 + \sum_{k=1}^{m} \int_0^{u(t_k)} I_k(t)\ dt \\
&\quad + \sum_{k=1}^{m} \int_0^{v(t_k)} J_k(t)\ dt - \int_0^T f(t, u, v)\ dt, \\
&= \Phi(u, v).
\end{aligned}$$

This implies that the functional Φ is weakly lower semi-continuous. □

2.3 Main results

Theorem 2.1. *Suppose that f_u, f_v verify the following condition:*

(H_1) *There exist $M > 0$, such that*

$$\begin{cases} |f_u(t, u, v)| \leq M, \ \text{for every } (t, u, v) \in (0, T) \times \mathbb{R}^2, \\ |f_v(t, u, v)| \leq M, \ \text{for every } (t, u, v) \in (0, T) \times \mathbb{R}^2, \end{cases}$$

and the impulsive functions $I_k, J_k, k = 1, 2, ..., m$, verify

(H_2) *There exist $M_k > 0, k = 1, 2, ..., m$, such that*

$$\begin{cases} |I_k(u)| \leq M_k, \ \text{for every } u \in \mathbb{R}, \\ |J_k(v)| \leq M_k, \ \text{for every } v \in \mathbb{R}. \end{cases}$$

Then there is a critical point of Φ, and (2.1) has at least one solution.

Proof. From the theorem 1.4, the remark 1.11 and the proposition 2.1, to get the result, we just show that Φ is coercive.

For any $(u,v)\in H$, we have

$$\begin{aligned}\Phi(u,v) &= \frac{1}{2}\Big\|(u,v)\Big\|_H^2+\sum_{k=1}^m\int_0^{u(t_k)} I_k(t)\ dt+\sum_{k=1}^m\int_0^{v(t_k)} J_k(t)\ dt-\int_0^T f(t,u,v)\ dt,\\ &\geq \frac{1}{2}\Big\|(u,v)\Big\|_H^2-\sum_{k=1}^m\int_0^{u(t_k)} M_k\ dt-\sum_{k=1}^m\int_0^{v(t_k)} M_k\ dt-\int_0^T\Big(M|u|+M|v|\Big)\ dt,\\ &\geq \frac{1}{2}\Big\|(u,v)\Big\|_H^2-m\max_k\{M_k\}\|u\|_\infty-m\max_k\{M_k\}\|v\|_\infty-MT\Big(\|u\|_\infty+\|v\|_\infty\Big),\end{aligned}$$

from the lemma 2.1, we have

$$\Phi(u,v)\ \geq\ \frac{1}{2}\Big\|(u,v)\Big\|_H^2-2m\gamma\max_k\{M_k\}\Big\|(u,v)\Big\|_H-2MT\gamma\Big\|(u,v)\Big\|_H.$$

This implies that $\Phi(u,v)\to\infty$ if $\|(u,v)\|_H\to\infty$, then Φ is coercive on H. □

Remark 2.1. *We can relax the conditions (H_1) and (H_2) by the condition (H_3) and (H_4) to obtain the generalized result.*

(H_3) *There exist $a,b>0$, and $\alpha_1,\alpha_2\in[0,1)$, such that*

$$\begin{cases}|f_u(t,u,v)|\leq a+b|u|^{\alpha_1},\ \text{for every } (t,u,v)\in(0,T)\times\mathbb{R}^2,\\ |f_v(t,u,v)|\leq a+b|v|^{\alpha_2},\ \text{for every } (t,u,v)\in(0,T)\times\mathbb{R}^2.\end{cases}$$

(H_4) *There exist $a_k,b_k>0$, and $\beta_k\in[0,1), k=1,2,...,m$, such that*

$$\begin{cases}|I_k(u)|\leq a_k+b_k|u|^{\beta_k},\ \text{for every } u\in\mathbb{R},\\ |J_k(v)|\leq a_k+b_k|v|^{\beta_k},\ \text{for every } v\in\mathbb{R}.\end{cases}$$

Theorem 2.2. *Assume that (H_3) and (H_4) are hold, then the problem (2.1) has at least one solution.*

Proof. For any $(u,v)\in H$, we have

$$\begin{aligned}\Phi(u,v) &= \frac{1}{2}\Big\|(u,v)\Big\|_H^2+\sum_{k=1}^m\int_0^{u(t_k)} I_k(t)\ dt+\sum_{k=1}^m\int_0^{v(t_k)} J_k(t)\ dt-\int_0^T f(t,u,v)\ dt,\\ &\geq \frac{1}{2}\Big\|(u,v)\Big\|_H^2-\sum_{k=1}^m\int_0^{u(t_k)}\Big(a_k+b_k|t|^{\beta_k}\Big)\ dt-\sum_{k=1}^m\int_0^{v(t_k)}\Big(a_k+b_k|t|^{\beta_k}\Big)\ dt\\ &\quad-\int_0^T\Big(a|u|+a|v|+b|u|^{\alpha_1+1}+b|v|^{\alpha_2+1}\Big)\ dt,\\ &\geq \frac{1}{2}\Big\|(u,v)\Big\|_H^2-m\max_k\{a_k\}\|u\|_\infty-\max_k\{b_k\}\sum_{k=1}^m\|u\|_\infty^{\beta_k+1}\\ &\quad-m\max_k\{a_k\}\|v\|_\infty-\max_k\{b_k\}\sum_{k=1}^m\|v\|_\infty^{\beta_k+1}\\ &\quad-aT\Big(\|u\|_\infty+\|v\|_\infty\Big)-bT\Big(\|u\|_\infty^{\alpha_1+1}+\|v\|_\infty^{\alpha_2+1}\Big),\end{aligned}$$

now using the lemma 2.1, we get

$$\begin{aligned}\Phi(u,v) \;\geq\; & \frac{1}{2}\Big\|(u,v)\Big\|_H^2 - 2m\gamma\max_k\{a_k\}\Big\|(u,v)\Big\|_H - 2\max_k\{b_k\}\sum_{k=1}^{m}\gamma^{\beta_k+1}\Big\|(u,v)\Big\|_H^{\beta_k+1}\\ & -2aT\gamma\Big\|(u,v)\Big\|_H - bT\gamma^{\alpha_1+1}\Big\|(u,v)\Big\|_H^{\alpha_1+1} - bT\gamma^{\alpha_2+1}\Big\|(u,v)\Big\|_H^{\alpha_2+1}.\end{aligned}$$

Because $\alpha_1+1, \alpha_2+1, \beta_k+1 < 2,\ k = 1, 2, ..., m$, we have that

$$\lim_{\|(u,v)\|_H\to\infty}\Phi(u,v) = \infty,$$

it follows that the functional Φ is coercive on H. □

Example 2.1. *Let $T = \pi, t_1 = 1$. We consider the following problem with impulses*

$$\begin{cases} -u''(t) = t^2 + \sqrt[5]{u(t)},\ t \in (0,\pi)\setminus\{t_1\},\\ -v''(t) = t + \sqrt[3]{v(t)},\ t \in (0,\pi)\setminus\{t_1\},\\ u(0) = u(\pi) = v(0) = v(\pi) = 0,\\ \Delta u'(t_1) = u'(t_1^+) - u'(t_1^-) = 2 + \sqrt[3]{u(t_1)},\\ \Delta v'(t_1) = v'(t_1^+) - v'(t_1^-) = t_1 + \sqrt[3]{v(t_1)}, \end{cases} \tag{2.10}$$

where the functions $f : (0,\pi)\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ and $I_1, J_1 : \mathbb{R} \longrightarrow \mathbb{R}$ are defined by

$$f(t,u,v) = t^2u + tv + \frac{5}{6}u^{\frac{6}{5}} + \frac{3}{4}v^{\frac{4}{3}},$$

$$I_1(u) = 2 + \sqrt[3]{u},$$

$$J_1(v) = 1 + \sqrt[3]{v}.$$

We can see that

$$|f_u(t,u,v)| = |t^2 + \sqrt[5]{u}| \leq \pi^2 + |u|^{\frac{1}{5}},$$

$$|f_v(t,u,v)| = |t + \sqrt[3]{v}| \leq \pi + |v|^{\frac{1}{3}},$$

and

$$|I_1(u)| = |2 + \sqrt[3]{u}| \leq 2 + |u|^{\frac{1}{3}},$$

$$|J_1(v)| = |1 + \sqrt[3]{v}| \leq 1 + |v|^{\frac{1}{3}}.$$

Taking $a = \pi^2, b = 1, \alpha_1 = \frac{1}{5}$ and $\alpha_2 = \frac{1}{3}$ so (H_3) holds, for $a_1 = 2, b_1 = 1, \beta_1 = \frac{1}{3}$ then (H_4) holds. By Theorem 2.2, the instantaneous impulsive problem (2.10) has at least one solution.

Chapter 3

Variational approach to non-instantaneous impulsive differential system

In this chapter we present the variational structure associated to the following nonlinear problem with no instantaneous impulses

$$\begin{cases} -u''(t) &=& D_u f_i\Big(t, u(t)-u(t_{i+1}), v(t)-v(t_{i+1})\Big),\ t\in(s_i,t_{i+1}],\ i=0,1,\ldots,m,\\ -v''(t) &=& D_v f_i\Big(t, u(t)-u(t_{i+1}), v(t)-v(t_{i+1})\Big),\ t\in(s_i,t_{i+1}],\ i=0,1,\ldots,m,\\ u'(t) &=& \alpha_i,\ t\in(t_i,s_i],\ i=1,2,\ldots,m,\\ v'(t) &=& \beta_i,\ t\in(t_i,s_i],\ i=1,2,\ldots,m,\\ u'(s_i^+) &=& u'(s_i^-),\ i=1,2,\ldots,m,\\ v'(s_i^+) &=& v'(s_i^-),\ i=1,2,\ldots,m,\\ u'(0^+) &=& \alpha_0, \quad v'(0^+)=\beta_0,\\ u(0) &=& u(T)=v(0)=v(T)=0, \end{cases} \tag{3.1}$$

where $0=s_0<t_1<s_1<t_2<s_2<\ldots<t_m<s_m<t_{m+1}=T$, the impulses start abruptly at points $t_i, i=0,1,2,\ldots,m$, and keep the derivative constant on a finite time interval $(t_i,s_i]$. Here $u'(s_i^{\pm})=\lim_{s\to s_i^{\pm}}u'(s)$, and $\alpha_i,\beta_i, i=0,1,2,\ldots,m$, are given constants. For each $i=0,1,2,\ldots,m$, the nonlinear functions D_uf_i, D_vf_i $\big($the derivatives of $f_i(t,u,v)$ with respect to u and v respectively$\big)$ are Carathéodory functions on $(s_i,t_{i+1}]\times\mathbb{R}^2$.

3.1 Functional space framework

We need to define as like as the previous chapter the following functional spaces:
$\mathcal{C}[0,T]$ be the space of all continuous functions on $[0,T]$ with the norm

$$\|u\|_\infty = \max_{t\in[0,T]} |u(t)|.$$

$H_0^1(0,T)$ is the Sobolev space with the inner product

$$(u,v)_1 = \int_0^T u'(t)v'(t)\ dt,$$

and the corresponding norm

$$\|u\|_1 = \left(\int_0^T \left|u'(t)\right|^2\ dt\right)^{\frac{1}{2}}.$$

Set $H = H_0^1(0,T) \times H_0^1(0,T)$, in the Hilbert space H, for any $(u,v) \in H$, we set the norm

$$\|(u,v)\|_H = \left(\|u\|_1^2 + \|v\|_1^2\right)^{\frac{1}{2}}.$$

By Hölder's inequality and Poincare's inequality we have the following lemma

Lemma 3.1. *There exists $\gamma > 0$ such that, if $(u,v) \in H$, then*

$$\|u\|_\infty,\ \|v\|_\infty \leq \gamma \big\|(u,v)\big\|_H.$$

For the proof we can see the previous chapter.

3.2 Variational formula

We use the ideas of the variational approach of the problem (3.1), for each $(\varphi, \psi) \in H$, we have

$$\begin{aligned}
-\int_0^T u''(t)\varphi(t)\ dt &= -\int_0^{t_1} u''(t)\varphi(t)\ dt - \sum_{i=1}^{m}\int_{t_i}^{s_i} u''(t)\varphi(t)\ dt \\
&\quad -\sum_{i=1}^{m-1}\int_{s_i}^{t_{i+1}} u''(t)\varphi(t)\ dt - \int_{s_m}^{T} u''(t)\varphi(t)\ dt \\
&= \int_0^T u'(t)\varphi'(t)\ dt - \sum_{i=1}^{m}\Big(u'(t_i^-) - u'(t_i^+)\Big)\varphi(t_i) \\
&\quad -\sum_{i=1}^{m}\Big(u'(s_i^-) - u'(s_i^+)\Big)\varphi(s_i) \\
&= \int_0^T u'(t)\varphi'(t)\ dt - \sum_{i=1}^{m}\Big(u'(t_i^-) - \alpha_i\Big)\varphi(t_i).
\end{aligned}$$

To determine $u'(t_i^-), i = 1, 2, \ldots, m$, on $(s_{i-1}, t_i]$ we have

$$-\int_{s_{i-1}}^{t_i} u''(t)dt = \int_{s_{i-1}}^{t_i} D_u f_{i-1}\Big(t, u(t) - u(t_i), v(t) - v(t_i)\Big)dt$$

then

$$-u'(t_i^-) + u'(s_{i-1}^+) = \int_{s_{i-1}}^{t_i} D_u f_{i-1}\Big(t, u(t) - u(t_i), v(t) - v(t_i)\Big)dt,$$

and as $u'(s_{i-1}^+) = u'(s_{i-1}^-) = \alpha_{i-1}, i = 2, \ldots, m,$ $\Big($for $i = 1$, we have $u'(s_0^+) = u'(0^+) = \alpha_0\Big)$, we obtain

$$u'(t_i^-) = \alpha_{i-1} - \int_{s_{i-1}}^{t_i} D_u f_{i-1}\Big(t, u(t) - u(t_i), v(t) - v(t_i)\Big)dt, i = 1, 2, \ldots, m.$$

Therefore

$$\begin{aligned}
-\int_0^T u''(t)\varphi(t)\ dt &= \int_0^T u'(t)\varphi'(t)\ dt - \sum_{i=1}^{m}\Big(\alpha_{i-1} - \alpha_i\Big)\varphi(t_i) \\
&\quad + \sum_{i=0}^{m-1}\left(\int_{s_i}^{t_{i+1}} D_u f_i\Big(t, u(t) - u(t_{i+1}), v(t) - v(t_{i+1})\Big)dt\right)\varphi(t_{i+1}).
\end{aligned} \tag{3.2}$$

On the other hand,

$$\begin{aligned}
-\int_0^T u''(t)\varphi(t)\ dt &= -\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} u''(t)\varphi(t)\ dt - \sum_{i=1}^{m}\int_{t_i}^{s_i} u''(t)\varphi(t)\ dt \\
&= \sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} D_u f_i\Big(t, u(t) - u(t_{i+1}), v(t) - v(t_{i+1})\Big)\varphi(t)\ dt \\
&\quad -\sum_{i=1}^{m}\int_{t_i}^{s_i} \frac{d}{dt}(\alpha_i)\varphi(t)\ dt.
\end{aligned}$$

Hence

$$-\int_0^T u''(t)\varphi(t)\,dt = \sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} D_u f_i\Big(t,u(t)-u(t_{i+1}),v(t)-v(t_{i+1})\Big)\varphi(t)\,dt. \tag{3.3}$$

Thus, in view of $\varphi(t_{m+1})=\varphi(T)=0$, (3.2), and (3.3), we find that

$$\begin{aligned}&\int_0^T u'(t)\varphi'(t)\,dt-\sum_{i=1}^{m}\Big(\alpha_{i-1}-\alpha_i\Big)\varphi(t_i)\\ &=\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} D_u f_i\Big(t,u(t)-u(t_{i+1}),v(t)-v(t_{i+1})\Big)\Big(\varphi(t)-\varphi(t_{i+1})\Big)dt.\end{aligned} \tag{3.4}$$

Similarly

$$\begin{aligned}&\int_0^T v'(t)\psi'(t)\,dt-\sum_{i=1}^{m}\Big(\beta_{i-1}-\beta_i\Big)\psi(t_i)\\ &=\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} D_v f_i\Big(t,u(t)-u(t_{i+1}),v(t)-v(t_{i+1})\Big)\Big(\psi(t)-\psi(t_{i+1})\Big)dt.\end{aligned} \tag{3.5}$$

We combined (3.4) and (3.5), we obtain

$$\begin{aligned}&\int_0^T u'(t)\varphi'(t)\,dt+\int_0^T v'(t)\psi'(t)\,dt-\sum_{i=1}^{m}\Big(\alpha_{i-1}-\alpha_i\Big)\varphi(t_i)-\sum_{i=1}^{m}\Big(\beta_{i-1}-\beta_i\Big)\psi(t_i)\\ &=\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} D_u f_i\Big(t,u(t)-u(t_{i+1}),v(t)-v(t_{i+1})\Big)\Big(\varphi(t)-\varphi(t_{i+1})\Big)dt\\ &+\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} D_v f_i\Big(t,u(t)-u(t_{i+1}),v(t)-v(t_{i+1})\Big)\Big(\psi(t)-\psi(t_{i+1})\Big)dt.\end{aligned} \tag{3.6}$$

Now, we introduce the concept of weak solution for problem (3.1).

Definition 3.1. *We say that a pair of functions $(u,v)\in H$ is a weak solution for problem (3.1) if identity (3.6) holds for any $(\varphi,\psi)\in H$.*

We consider the energy functional corresponding to problem (3.1)

$$\Phi : H \longrightarrow \mathbb{R},$$

is defined by

$$
\begin{aligned}
\Phi(u,v) &= \frac{1}{2}\int_0^T \Big(u'(t)\Big)^2 dt + \frac{1}{2}\int_0^T \Big(v'(t)\Big)^2 dt - \sum_{i=1}^{m}\Big(\alpha_{i-1}-\alpha_i\Big)u(t_i) - \sum_{i=1}^{m}\Big(\beta_{i-1}-\beta_i\Big)v(t_i) \\
&\quad -\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} f_i\Big(t, u(t)-u(t_{i+1}), v(t)-v(t_{i+1})\Big)dt, \\
&= \frac{1}{2}\Big\|(u,v)\Big\|_H^2 - \sum_{i=1}^{m}\Big(\alpha_{i-1}-\alpha_i\Big)u(t_i) - \sum_{i=1}^{m}\Big(\beta_{i-1}-\beta_i\Big)v(t_i) \\
&\quad -\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} f_i\Big(t, u(t)-u(t_{i+1}), v(t)-v(t_{i+1})\Big)dt, \qquad (3.7)
\end{aligned}
$$

for more details about the construction of Φ, see the subsection 1.2.1.1.

Proposition 3.1. *The functional $\Phi : H \longrightarrow \mathbb{R}$, defined by (3.7) is continuously Fréchet-differentiable and weakly lower semi-continuous. Moreover, the critical points of Φ are weak solutions of (3.1).*

Proof. Using the continuity of $D_u f_i, D_v f_i, i = 0, 1, \dots, m$, we easily obtain that the functional $\Phi \in \mathcal{C}^1(H, \mathbb{R})$. Furthermore, we have the differential of Φ at $(u,v) \in H$

$$\Phi'(u,v) : H \longrightarrow \mathbb{R},$$

defined by

$$
\begin{aligned}
\Phi'(u,v)(\varphi,\psi) = \int_0^T u'(t)\varphi'(t)\, dt + \int_0^T v'(t)\psi'(t)\, dt - \sum_{i=1}^{m}\Big(\alpha_{i-1}-\alpha_i\Big)\varphi(t_i) - \sum_{i=1}^{m}\Big(\beta_{i-1}-\beta_i\Big)\psi(t_i) \\
-\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} D_u f_i\Big(t, u(t)-u(t_{i+1}), v(t)-v(t_{i+1})\Big)\Big(\varphi(t)-\varphi(t_{i+1})\Big)dt \\
-\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} D_v f_i\Big(t, u(t)-u(t_{i+1}), v(t)-v(t_{i+1})\Big)\Big(\psi(t)-\psi(t_{i+1})\Big)dt.
\end{aligned}
$$

This shows that the critical points of Φ give us the weak solutions of (3.1).

To show that Φ is weakly lower semi-continuous, let $\{(u_j, v_j)\} \subset H$, with $(u_j, v_j) \rightharpoonup (u,v)$, then we have that $\{u_j\}$ and $\{v_j\}$ converge uniformly to u and v on $[0,T]$ respectively (Proposition 1.5).

In connection with the fact that $\liminf_{j\longrightarrow\infty}\|(u_j,v_j)\|_H \geq \|(u,v)\|_H$ (Corollary 1.4), one has

$$\begin{aligned}
\liminf_{j\longrightarrow\infty}\Phi(u_j,v_j) &= \liminf_{j\longrightarrow\infty}\Bigg\{\frac{1}{2}\Big\|(u_j,v_j)\Big\|_H^2 - \sum_{i=1}^{m}\Big(\alpha_{i-1}-\alpha_i\Big)u_j(t_i) - \sum_{i=1}^{m}\Big(\beta_{i-1}-\beta_i\Big)v_j(t_i) \\
&\quad - \sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} f_i\Big(t,u_j(t)-u_j(t_{i+1}),v_j(t)-v_j(t_{i+1})\Big)dt\Bigg\} \\
&\geq \frac{1}{2}\Big\|(u,v)\Big\|_H^2 - \sum_{i=1}^{m}\Big(\alpha_{i-1}-\alpha_i\Big)u(t_i) - \sum_{i=1}^{m}\Big(\beta_{i-1}-\beta_i\Big)v(t_i) \\
&\quad - \sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} f_i\Big(t,u(t)-u(t_{i+1}),v(t)-v(t_{i+1})\Big)dt \\
&= \Phi(u,v).
\end{aligned}$$

This implies that the functional Φ is weakly lower semi-continuous. □

3.3 Main results

In this section we give the proofs of our main results in this chapter.

Theorem 3.1. *Suppose that $D_u f_i, D_v f_i$, verify the following condition:*

(H_1) *There exist $M_i > 0, i = 0,1,\ldots,m$, such that*

$$\begin{cases} |D_u f_i(t,u,v)| \leq M_i, \text{ for every } (t,u,v) \in (s_i,t_{i+1}] \times \mathbb{R}^2, \\ |D_v f_i(t,u,v)| \leq M_i, \text{ for every } (t,u,v) \in (s_i,t_{i+1}] \times \mathbb{R}^2. \end{cases}$$

Then there is a critical point of Φ, and (3.1) has at least one solution.

Proof. From the theorem 1.4, the remark 1.11 and the proposition 3.1, to get the result, we just show that Φ is coercive.

For any $(u,v)\in H$, we have

$$\begin{aligned}
\Phi(u,v) &= \frac{1}{2}\Big\|(u,v)\Big\|_H^2-\sum_{i=1}^{m}\Big(\alpha_{i-1}-\alpha_i\Big)u(t_i)-\sum_{i=1}^{m}\Big(\beta_{i-1}-\beta_i\Big)v(t_i)\\
&\quad-\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} f_i\Big(t,u(t)-u(t_{i+1}),v(t)-v(t_{i+1})\Big)dt\\
&\geq \frac{1}{2}\Big\|(u,v)\Big\|_H^2-\sum_{i=1}^{m}\Big(\alpha_{i-1}-\alpha_i\Big)u(t_i)-\sum_{i=1}^{m}\Big(\beta_{i-1}-\beta_i\Big)v(t_i)\\
&\quad-\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}\Big(M_i\big|u(t)-u(t_{i+1})\big|+M_i\big|v(t)-v(t_{i+1})\big|\Big)dt\\
&\geq \frac{1}{2}\Big\|(u,v)\Big\|_H^2-m\max_{i=1,\dots,m}\Big\{|\alpha_{i-1}-\alpha_i|\Big\}\|u\|_\infty-m\max_{i=1,\dots,m}\Big\{|\beta_{i-1}-\beta_i|\Big\}\|v\|_\infty\\
&\quad-2(m+1)T\max_{i=0,\dots,m}\Big\{M_i\Big\}\Big(\|u\|_\infty+\|v\|_\infty\Big),
\end{aligned}$$

from the lemma 3.1, we have

$$\begin{aligned}
\Phi(u,v) &\geq \frac{1}{2}\Big\|(u,v)\Big\|_H^2-m\gamma\max_{i=1,\dots,m}\Big\{|\alpha_{i-1}-\alpha_i|\Big\}\Big\|(u,v)\Big\|_H-m\gamma\max_{i=1,\dots,m}\Big\{|\beta_{i-1}-\beta_i|\Big\}\Big\|(u,v)\Big\|_H\\
&\quad-4(m+1)\gamma T\max_{i=0,\dots,m}\Big\{M_i\Big\}\Big\|(u,v)\Big\|_H.
\end{aligned}$$

This implies that $\Phi(u,v)\to\infty$ if $\|(u,v)\|_H\to\infty$, then Φ is coercive on H. □

Remark 3.1. *We can relax the condition (H_1) by the following condition*

(H_2) *There exist $a_i,b_i>0$, and $\gamma_1,\gamma_2\in[0,1), i=0,1,\dots,m$, such that*

$$\begin{cases}
|D_uf_i(t,u,v)|\leq a_i+b_i|u|^{\gamma_1}, \text{ for every } (t,u,v)\in(s_i,t_{i+1}]\times\mathbb{R}^2,\\
|D_vf_i(t,u,v)|\leq a_i+b_i|v|^{\gamma_2}, \text{ for every } (t,u,v)\in(s_i,t_{i+1}]\times\mathbb{R}^2.
\end{cases}$$

Theorem 3.2. *Assume that (H_2) holds, then the problem (3.1) has at least one solution.*

Proof. By the same argument of the above theorem, we show that the functional ϕ is coercive.

Let $(u,v) \in H$, then

$$\begin{aligned}
\Phi(u,v) &= \frac{1}{2}\Big\|(u,v)\Big\|_H^2 - \sum_{i=1}^{m}\Big(\alpha_{i-1}-\alpha_i\Big)u(t_i) - \sum_{i=1}^{m}\Big(\beta_{i-1}-\beta_i\Big)v(t_i) \\
&\quad - \sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} f_i\Big(t, u(t)-u(t_{i+1}), v(t)-v(t_{i+1})\Big)dt \\
&\geq \frac{1}{2}\Big\|(u,v)\Big\|_H^2 - \sum_{i=1}^{m}\Big(\alpha_{i-1}-\alpha_i\Big)u(t_i) - \sum_{i=1}^{m}\Big(\beta_{i-1}-\beta_i\Big)v(t_i) \\
&\quad - \sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \Bigg(a_i\Big|u(t)-u(t_{i+1})\Big| + a_i\Big|v(t)-v(t_{i+1})\Big| \\
&\quad + b_i\Big|u(t)-u(t_{i+1})\Big|^{\gamma_1+1} + b_i\Big|v(t)-v(t_{i+1})\Big|^{\gamma_2+1}\Bigg)dt.
\end{aligned}$$

Then

$$\begin{aligned}
\Phi(u,v) &\geq \frac{1}{2}\Big\|(u,v)\Big\|_H^2 - m\max_{i=1,\ldots,m}\Big\{|\alpha_{i-1}-\alpha_i|\Big\}\|u\|_\infty - m\max_{i=1,\ldots,m}\Big\{|\beta_{i-1}-\beta_i|\Big\}\|v\|_\infty \\
&\quad -2(m+1)T\max_{i=0,\ldots,m}\Big\{a_i\Big\}\Big(\|u\|_\infty + \|v\|_\infty\Big) \\
&\quad -2^{\gamma_1+1}(m+1)T\max_{i=0,\ldots,m}\Big\{b_i\Big\}\|u\|_\infty^{\gamma_1+1} - 2^{\gamma_2+1}(m+1)T\max_{i=0,\ldots,m}\Big\{b_i\Big\}\|v\|_\infty^{\gamma_2+1},
\end{aligned}$$

now using the lemma 3.1, we get

$$\begin{aligned}
\Phi(u,v) &\geq \frac{1}{2}\Big\|(u,v)\Big\|_H^2 - m\gamma\max_{i=1,\ldots,m}\Big\{|\alpha_{i-1}-\alpha_i|\Big\}\Big\|(u,v)\Big\|_H - m\gamma\max_{i=1,\ldots,m}\Big\{|\beta_{i-1}-\beta_i|\Big\}\Big\|(u,v)\Big\|_H \\
&\quad -4(m+1)\gamma T\max_{i=0,\ldots,m}\Big\{a_i\Big\}\Big\|(u,v)\Big\|_H \\
&\quad -(2\gamma)^{\gamma_1+1}(m+1)T\max_{i=0,\ldots,m}\Big\{b_i\Big\}\Big\|(u,v)\Big\|_H^{\gamma_1+1} \\
&\quad -(2\gamma)^{\gamma_2+1}(m+1)T\max_{i=0,\ldots,m}\Big\{b_i\Big\}\Big\|(u,v)\Big\|_H^{\gamma_2+1}.
\end{aligned}$$

Because $\gamma_1+1, \gamma_2+1 < 2$, we have that

$$\lim_{\|(u,v)\|_H \to \infty} \Phi(u,v) = \infty,$$

it follows that the functional Φ is coercive on H. $\square$

Example 3.1. *Let $T = 1$, we consider the following problem with non-instantaneous impulses*

$$\begin{cases} -u''(t) &= \dfrac{1-\big[u(t)-u(t_{i+1})\big]^2}{\Big(1+\big[u(t)-u(t_{i+1})\big]^2\Big)^2}, \qquad t\in(s_i,t_{i+1}],\ i=0,1,\ldots,m,\\ -v''(t) &= \dfrac{1-\big[v(t)-v(t_{i+1})\big]^2}{\Big(1+\big[v(t)-v(t_{i+1}\big]^2\Big)^2}, \qquad t\in(s_i,t_{i+1}],\ i=0,1,\ldots,m,\\ u'(t) &= \alpha_i, \qquad t\in(t_i,s_i],\ i=1,2,\ldots,m,\\ v'(t) &= \beta_i, \qquad t\in(t_i,s_i],\ i=1,2,\ldots,m,\\ u'(s_i^+) &= u'(s_i^-), \qquad i=1,2,\ldots,m,\\ v'(s_i^+) &= v'(s_i^-), \qquad i=1,2,\ldots,m,\\ u'(0^+) &= \alpha_0, \qquad v'(0^+)=\beta_0,\\ u(0) &= u(1)=v(0)=v(1)=0, \end{cases} \tag{3.8}$$

taking $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by

$$f(x,y)=\frac{x}{1+x^2}+\frac{y}{1+y^2}, \quad \forall x,y\in\mathbb{R}.$$

It's clear that

$$|D_xf(x,y)|=\left|\frac{1-x^2}{(1+x^2)^2}\right|\le 1, \quad \forall x,y\in\mathbb{R},$$

and

$$|D_yf(x,y)|=\left|\frac{1-y^2}{(1+y^2)^2}\right|\le 1, \quad \forall x,y\in\mathbb{R}.$$

Then all the conditions of Theorem 3.1, and thus Problem (3.8) has at least one solution.

Chapter 4

Variational approach to non-instantaneous impulsive differential generalized system

In this chapter we deal with the following not instantaneous impulsive differential system of the form

$$\begin{cases} -u''(t) + \eta_i(t)\Big(u(t) - u(t_{i+1})\Big) &= D_u f_i\Big(t, u(t) - u(t_{i+1}), v(t) - v(t_{i+1})\Big), \\ & \quad t \in (s_i, t_{i+1}],\ i = 0, 1, \ldots, m, \\ -v''(t) + \theta_i(t)\Big(v(t) - v(t_{i+1})\Big) &= D_v f_i\Big(t, u(t) - u(t_{i+1}), v(t) - v(t_{i+1})\Big), \\ & \quad t \in (s_i, t_{i+1}],\ i = 0, 1, \ldots, m, \\ u'(t) &= \alpha_i,\ t \in (t_i, s_i],\ i = 1, 2, \ldots, m, \\ v'(t) &= \beta_i,\ t \in (t_i, s_i],\ i = 1, 2, \ldots, m, \\ u'(s_i^+) &= u'(s_i^-),\ i = 1, 2, \ldots, m, \\ v'(s_i^+) &= v'(s_i^-),\ i = 1, 2, \ldots, m, \\ u'(0^+) &= \alpha_0, \quad v'(0^+) = \beta_0, \\ u(0) &= u(T) = v(0) = v(T) = 0, \end{cases} \tag{4.1}$$

where $0 = s_0 < t_1 < s_1 < t_2 < s_2 < \ldots < t_m < s_m < t_{m+1} = T$. For each $i = 0, 1, 2, \ldots, m$, $\eta_i, \theta_i \in L^\infty(s_i, t_{i+1}]$, the nonlinear functions $D_u f_i, D_v f_i$ (the derivatives of $f_i(t, u, v)$ with respect to u and v respectively) are Carathéodory functions on $(s_i, t_{i+1}] \times \mathbb{R}^2$. And for $i = 1, 2 \ldots, m$, $u'(s_i^\pm) =$

$\lim_{s\to s_i^{\pm}} u'(s)$, α_i, β_i are given constants where the impulses start abruptly at points t_i keep the derivative constant on a finite time interval $(t_i, s_i]$.

Throughout this chapter we need the following assumptions

(A_1) Assume that

$$\nu_i > -\lambda_i, \ \forall i = 0, 1, \ldots, m,$$

where $\nu_i = \min\left\{ess\inf_{t\in(s_i,t_{i+1}]} \eta_i(t), ess\inf_{t\in(s_i,t_{i+1}]} \theta_i(t)\right\}$ and $\lambda_i = \dfrac{2}{(t_{i+1}-s_i)^2}$ (see lemma 4.1).

(A_2) Suppose that $D_u f_i, D_v f_i$ verify the following condition:

There exist $a_i, b_i > 0$, and $\gamma_1, \gamma_2 \in [0, 1), i = 0, 1, \ldots, m$, such that

$$\begin{cases} |D_u f_i(t, u, v)| \leq a_i + b_i|u|^{\gamma_1}, \text{ for every } (t, u, v) \in (s_i, t_{i+1}] \times \mathbb{R}^2, \\ |D_v f_i(t, u, v)| \leq a_i + b_i|v|^{\gamma_2}, \text{ for every } (t, u, v) \in (s_i, t_{i+1}] \times \mathbb{R}^2. \end{cases}$$

4.1 Functional space framework

We define the following functional spaces:

$\mathcal{C}[0, T]$ be the space of all continuous functions on $[0, T]$ with the norm

$$\|u\|_\infty = \max_{t\in[0,T]} |u(t)|.$$

$H_0^1(0, T)$ is the Sobolev space with the inner product

$$(u, v)_1 = \int_0^T u'(t)v'(t)\ dt,$$

and the corresponding norm

$$\|u\|_1 = \left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{1}{2}}.$$

Set $H = H_0^1(0, T) \times H_0^1(0, T)$, in the Hilbert space H, for any $(u, v) \in H$, we set the norm

$$\|(u, v)\|_H = \left(\|u\|_1^2 + \|v\|_1^2\right)^{\frac{1}{2}}.$$

We need the following results

Lemma 4.1. *We have for each* $w \in H_0^1(0,T)$

$$\lambda_i \int_{s_i}^{t_{i+1}} \Big(w(t) - w(t_{i+1})\Big)^2 dt \le \int_{s_i}^{t_{i+1}} \big|w'(t)\big|^2 dt,\ \forall i = 0, 1, \ldots, m,$$

where $\lambda_i = \dfrac{2}{(t_{i+1} - s_i)^2}$.

Proof. For $i \in \{0, 1, \ldots, m\}$ and $t \in (s_i, t_{i+1}]$, we have

$$\begin{aligned}
\Big(w(t) - w(t_{i+1})\Big)^2 &= \left(\int_t^{t_{i+1}} w'(s) ds\right)^2, \\
&\le \left(\int_t^{t_{i+1}} 1^2 ds\right)\left(\int_t^{t_{i+1}} \big|w'(s)\big|^2 ds\right), \\
&\le \Big(t_{i+1} - t\Big)\left(\int_{s_i}^{t_{i+1}} \big|w'(s)\big|^2 ds\right),
\end{aligned}$$

so

$$\int_{s_i}^{t_{i+1}} \Big(w(t) - w(t_{i+1})\Big)^2 dt \ \le\ \Big[-\frac{\big(t_{i+1} - t\big)^2}{2}\Big]_{s_i}^{t_{i+1}} \left(\int_{s_i}^{t_{i+1}} \big|w'(t)\big|^2 dt\right),$$

therfore

$$\int_{s_i}^{t_{i+1}} \Big(w(t) - w(t_{i+1})\Big)^2 dt \ \le\ \frac{\big(t_{i+1} - s_i\big)^2}{2}\left(\int_{s_i}^{t_{i+1}} \big|w'(t)\big|^2 dt\right).$$

□

By (A_1), we also introduce the norm

$$\begin{aligned}
\|(u,v)\|_H^* = \Bigg[&\int_0^T \big|u'(t)\big|^2 dt + \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \eta_i(t)\Big(u(t) - u(t_{i+1})\Big)^2 dt \\
&+ \int_0^T \big|v'(t)\big|^2 dt + \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \theta_i(t)\Big(v(t) - v(t_{i+1})\Big)^2 dt\Bigg]^{\frac{1}{2}},
\end{aligned}$$

clear that $\|\cdot\|_H^*$ is well defined, since for each $w \in H_0^1(0,T)$

$$\begin{aligned}
&\int_0^T \big|w'(t)\big|^2 dt + \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \eta_i(t)\Big(w(t) - w(t_{i+1})\Big)^2 dt \\
&\qquad \ge \int_0^T \big|w'(t)\big|^2 dt + \sum_{i=0}^m \nu_i \int_{s_i}^{t_{i+1}} \Big(w(t) - w(t_{i+1})\Big)^2 dt,
\end{aligned}$$

using (A_1), we obtain

$$\int_0^T \left|w'(t)\right|^2 dt + \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \eta_i(t)\Big(w(t) - w(t_{i+1})\Big)^2 dt \geq \int_0^T \left|w'(t)\right|^2 dt - \sum_{i=0}^m \lambda_i \int_{s_i}^{t_{i+1}} \Big(w(t) - w(t_{i+1})\Big)^2 dt,$$

from the lemma 4.1, we have

$$\int_0^T \left|w'(t)\right|^2 dt + \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \eta_i(t)\Big(w(t) - w(t_{i+1})\Big)^2 dt \geq \int_0^T \left|w'(t)\right|^2 dt - \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \left|w'(t)\right|^2 dt,$$

so

$$\int_0^T \left|w'(t)\right|^2 dt + \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \eta_i(t)\Big(w(t) - w(t_{i+1})\Big)^2 dt \geq \int_0^T \left|w'(t)\right|^2 dt - \int_0^T \left|w'(t)\right|^2 dt,$$

as a result

$$\int_0^T \left|w'(t)\right|^2 dt + \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \eta_i(t)\Big(w(t) - w(t_{i+1})\Big)^2 dt \geq 0.$$

Lemma 4.2. *Assume that assumption (A_1) holds, then, for the Sobolev space H, the norm $\|\cdot\|_H$ and the norm $\|\cdot\|_H^*$ are equivalent.*

Proof. Since $\nu_i > -\lambda_i$, there exists $\zeta_i \in (0,1)$ such that $\nu_i \geq -\lambda_i(1-\zeta_i)$, which implies that $\nu_i \geq -\lambda_i(1-\zeta)$, for $i = 0, \ldots, m$, where $\zeta = \min\{\zeta_i,\ i = 0, \ldots, m\}$.

For any $(u, v) \in H$, we have

$$\begin{aligned}
\|(u,v)\|_H^{*2} &= \int_0^T \left|u'(t)\right|^2 dt + \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \eta_i(t)\Big(u(t) - u(t_{i+1})\Big)^2 dt \\
&\quad + \int_0^T \left|v'(t)\right|^2 dt + \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \theta_i(t)\Big(v(t) - v(t_{i+1})\Big)^2 dt, \\
&\geq \int_0^T \left|u'(t)\right|^2 dt + \sum_{i=0}^m \nu_i \int_{s_i}^{t_{i+1}} \Big(u(t) - u(t_{i+1})\Big)^2 dt \\
&\quad + \int_0^T \left|v'(t)\right|^2 dt + \sum_{i=0}^m \nu_i \int_{s_i}^{t_{i+1}} \Big(v(t) - v(t_{i+1})\Big)^2 dt, \\
&\geq \int_0^T \left|u'(t)\right|^2 dt - (1-\zeta)\sum_{i=0}^m \lambda_i \int_{s_i}^{t_{i+1}} \Big(u(t) - u(t_{i+1})\Big)^2 dt \\
&\quad + \int_0^T \left|v'(t)\right|^2 dt - (1-\zeta)\sum_{i=0}^m \lambda_i \int_{s_i}^{t_{i+1}} \Big(v(t) - v(t_{i+1})\Big)^2 dt,
\end{aligned}$$

by lemma 4.1, we get

$$
\begin{aligned}
\|(u,v)\|_H^{*^2} &\geq \int_0^T \left|u'(t)\right|^2 dt-(1-\zeta)\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}\left|u'(t)\right|^2 dt \\
&\quad+\int_0^T \left|v'(t)\right|^2 dt-(1-\zeta)\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}\left|v'(t)\right|^2 dt, \\
&\geq \int_0^T \left|u'(t)\right|^2 dt-(1-\zeta)\int_0^T\left|u'(t)\right|^2 dt \\
&\quad+\int_0^T \left|v'(t)\right|^2 dt-(1-\zeta)\int_0^T\left|v'(t)\right|^2 dt, \\
&= \zeta\int_0^T \left|u'(t)\right|^2 dt+\zeta\int_0^T\left|v'(t)\right|^2 dt,
\end{aligned}
$$

therefore

$$
\|(u,v)\|_H^{*^2} \geq \zeta\|(u,v)\|_H^2.
$$

Moreover, one has

$$
\begin{aligned}
\|(u,v)\|_H^{*^2} &= \int_0^T \left|u'(t)\right|^2 dt+\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}\eta_i(t)\Big(u(t)-u(t_{i+1})\Big)^2 dt \\
&\quad+\int_0^T \left|v'(t)\right|^2 dt+\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}\theta_i(t)\Big(v(t)-v(t_{i+1})\Big)^2 dt, \\
&\leq \int_0^T \left|u'(t)\right|^2 dt+\sum_{i=0}^{m}\|\eta_i\|_\infty\int_{s_i}^{t_{i+1}}\Big(u(t)-u(t_{i+1})\Big)^2 dt \\
&\quad+\int_0^T \left|v'(t)\right|^2 dt+\sum_{i=0}^{m}\|\theta_i\|_\infty\int_{s_i}^{t_{i+1}}\Big(v(t)-v(t_{i+1})\Big)^2 dt,
\end{aligned}
$$

using lemma 4.1, we obtain

$$
\begin{aligned}
\|(u,v)\|_H^{*^2} &\leq \int_0^T \left|u'(t)\right|^2 dt+\sum_{i=0}^{m}\frac{\|\eta_i\|_\infty}{\lambda_i}\int_{s_i}^{t_{i+1}}\left|u'(t)\right|^2 dt \\
&\quad+\int_0^T \left|v'(t)\right|^2 dt+\sum_{i=0}^{m}\frac{\|\theta_i\|_\infty}{\lambda_i}\int_{s_i}^{t_{i+1}}\left|v'(t)\right|^2 dt, \\
&\leq \int_0^T \left|u'(t)\right|^2 dt+|\eta|_\infty\int_0^T\left|u'(t)\right|^2 dt \\
&\quad+\int_0^T \left|v'(t)\right|^2 dt+|\theta|_\infty\int_0^T\left|v'(t)\right|^2 dt,
\end{aligned}
$$

where $|\eta|_\infty=\max\left\{\frac{\|\eta_i\|_\infty}{\lambda_i},\ i=0,\ldots,m\right\}$, and $|\theta|_\infty=\max\left\{\frac{\|\theta_i\|_\infty}{\lambda_i},\ i=0,\ldots,m\right\}$.

Hence

$$\|(u,v)\|_H^{*^2} \le \Big(1+|\eta|_\infty\Big)\int_0^T \big|u'(t)\big|^2dt + \Big(1+|\theta|_\infty\Big)\int_0^T \big|v'(t)\big|^2dt,$$

so

$$\|(u,v)\|_H^{*^2} \le \Big(1+\max\Big\{|\eta|_\infty, |\theta|_\infty\Big\}\Big)\|(u,v)\|_H^2.$$

Thereby, the norm $\|\cdot\|_H$ and the norm $\|\cdot\|_H^*$ are equivalent. □

Lemma 4.3. *There exists $\gamma > 0$ such that, if $(u,v) \in H$, then*

$$\|u\|_\infty,\ \|v\|_\infty \le \gamma\big\|(u,v)\big\|_H^*.$$

Proof. By Holder's inequality and Poincare's inequality, there exists a constant δ such that (For more details about the proof see the previous chapters)

$$\|u\|_\infty \le \delta\big\|(u,v)\big\|_H.$$

Using Lemma 4.2, there exists γ such that

$$\|u\|_\infty \le \gamma\big\|(u,v)\big\|_H^*.$$

Similarly, we can get

$$\|v\|_\infty \le \gamma\big\|(u,v)\big\|_H^*.$$

□

4.2 Variational formula

Following the ideas of the variational approach of the problem (4.1), for each $(\varphi,\psi) \in H$, we have

$$\begin{aligned}
-\int_0^T u''(t)\varphi(t)\ dt &= -\int_0^{t_1} u''(t)\varphi(t)\ dt - \sum_{i=1}^m \int_{t_i}^{s_i} u''(t)\varphi(t)\ dt \\
&\quad -\sum_{i=1}^{m-1}\int_{s_i}^{t_{i+1}} u''(t)\varphi(t)\ dt - \int_{s_m}^T u''(t)\varphi(t)\ dt, \\
&= \int_0^T u'(t)\varphi'(t)\ dt - \sum_{i=1}^m \Big(u'(t_i^-) - u'(t_i^+)\Big)\varphi(t_i) \\
&\quad -\sum_{i=1}^m \Big(u'(s_i^-) - u'(s_i^+)\Big)\varphi(s_i), \\
&= \int_0^T u'(t)\varphi'(t)\ dt - \sum_{i=1}^m \Big(u'(t_i^-) - \alpha_i\Big)\varphi(t_i).
\end{aligned}$$

To determine $u'(t_i^-)$, $i=1,2,\dots,m$, on $(s_{i-1},t_i]$ we have

$$-\int_{s_{i-1}}^{t_i} u''(t)dt + \int_{s_{i-1}}^{t_i} \eta_{i-1}(t)\Big(u(t)-u(t_i)\Big)dt = \int_{s_{i-1}}^{t_i} D_u f_{i-1}\Big(t, u(t)-u(t_i), v(t)-v(t_i)\Big)dt,$$

then

$$-u'(t_i^-) + u'(s_{i-1}^+) + \int_{s_{i-1}}^{t_i} \eta_{i-1}(t)\Big(u(t)-u(t_i)\Big)dt = \int_{s_{i-1}}^{t_i} D_u f_{i-1}\Big(t, u(t)-u(t_i), v(t)-v(t_i)\Big)dt.$$

And as $u'(s_{i-1}^+) = u'(s_{i-1}^-) = \alpha_{i-1}$, $i=2,3,\dots,m$, for $i=1$, we have $u'(s_0^+) = u'(0^+) = \alpha_0$, we obtain

$$u'(t_i^-) = \alpha_{i-1} + \int_{s_{i-1}}^{t_i} \eta_{i-1}(t)\Big(u(t)-u(t_i)\Big)dt - \int_{s_{i-1}}^{t_i} D_u f_{i-1}\Big(t, u(t)-u(t_i), v(t)-v(t_i)\Big)dt,$$
$$\text{for: } i=1,2,\dots,m.$$

Therefore

$$\begin{aligned} -\int_0^T u''(t)\varphi(t)dt = \int_0^T u'(t)\varphi'(t)dt - \sum_{i=1}^{m}\Big(\alpha_{i-1}-\alpha_i\Big)\varphi(t_i) \\ + \sum_{i=0}^{m-1}\left(\int_{s_i}^{t_{i+1}} D_u f_i\Big(t, u(t)-u(t_{i+1}), v(t)-v(t_{i+1})\Big)dt\right)\varphi(t_{i+1}) \\ - \sum_{i=0}^{m-1}\left(\int_{s_i}^{t_{i+1}} \eta_i(t)\Big(u(t)-u(t_{i+1})\Big)dt\right)\varphi(t_{i+1}). \end{aligned} \tag{4.2}$$

On the other hand,

$$\begin{aligned} -\int_0^T u''(t)\varphi(t)\, dt &= -\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} u''(t)\varphi(t)dt - \sum_{i=1}^{m}\int_{t_i}^{s_i} u''(t)\varphi(t)dt, \\ &= \sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} D_u f_i\Big(t, u(t)-u(t_{i+1}), v(t)-v(t_{i+1})\Big)\varphi(t)dt \\ &\quad -\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \eta_i(t)\Big(u(t)-u(t_{i+1})\Big)\varphi(t)dt - \sum_{i=1}^{m}\int_{t_i}^{s_i}\frac{d}{dt}\Big(\alpha_i\Big)\varphi(t)dt, \end{aligned}$$

hence

$$\begin{aligned} -\int_0^T u''(t)\varphi(t)\, dt = \sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} D_u f_i\Big(t, u(t)-u(t_{i+1}), v(t)-v(t_{i+1})\Big)\varphi(t)dt \\ -\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \eta_i(t)\Big(u(t)-u(t_{i+1})\Big)\varphi(t)dt. \end{aligned} \tag{4.3}$$

Thus, in view of $\varphi(t_{m+1}) = \varphi(T) = 0$, (4.2), and (4.3), we find that

$$\begin{aligned}\int_0^T u'(t)\varphi'(t)dt - \sum_{i=1}^{m}\Big(\alpha_{i-1}-\alpha_i\Big)\varphi(t_i) \\ = \sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} D_u f_i\Big(t, u(t)-u(t_{i+1}), v(t)-v(t_{i+1})\Big)\Big(\varphi(t)-\varphi(t_{i+1})\Big)dt \\ - \sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \eta_i(t)\Big(u(t)-u(t_{i+1})\Big)\Big(\varphi(t)-\varphi(t_{i+1})\Big)dt.\end{aligned} \tag{4.4}$$

Similarly

$$\begin{aligned}\int_0^T v'(t)\psi'(t)dt - \sum_{i=1}^{m}\Big(\beta_{i-1}-\beta_i\Big)\psi(t_i) \\ = \sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} D_v f_i\Big(t, u(t)-u(t_{i+1}), v(t)-v(t_{i+1})\Big)\Big(\psi(t)-\psi(t_{i+1})\Big)dt \\ - \sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \theta_i(t)\Big(v(t)-v(t_{i+1})\Big)\Big(\psi(t)-\psi(t_{i+1})\Big)dt.\end{aligned} \tag{4.5}$$

We combined (4.4) and (4.5), we obtain

$$\begin{aligned}\int_0^T u'(t)\varphi'(t)dt + \int_0^T v'(t)\psi'(t)dt - \sum_{i=1}^{m}\Big(\alpha_{i-1}-\alpha_i\Big)\varphi(t_i) - \sum_{i=1}^{m}\Big(\beta_{i-1}-\beta_i\Big)\psi(t_i) \\ = \sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} D_u f_i\Big(t, u(t)-u(t_{i+1}), v(t)-v(t_{i+1})\Big)\Big(\varphi(t)-\varphi(t_{i+1})\Big)dt \\ + \sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} D_v f_i\Big(t, u(t)-u(t_{i+1}), v(t)-v(t_{i+1})\Big)\Big(\psi(t)-\psi(t_{i+1})\Big)dt \\ - \sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \eta_i(t)\Big(u(t)-u(t_{i+1})\Big)\Big(\varphi(t)-\varphi(t_{i+1})\Big)dt \\ - \sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \theta_i(t)\Big(v(t)-v(t_{i+1})\Big)\Big(\psi(t)-\psi(t_{i+1})\Big)dt.\end{aligned} \tag{4.6}$$

Based on equality (4.6), we introduce the concept of weak solution for problem (4.1).

Definition 4.1. *We say that a pair of functions $(u, v) \in H$ is a weak solution for problem (4.1) if identity (4.6) holds for any $(\varphi, \psi) \in H$.*

We consider the energy functional corresponding to problem (4.1)

$$\Phi : H \longrightarrow \mathbb{R},$$

defined by

$$\begin{aligned}\Phi(u,v) &= \frac{1}{2}\int_0^T \Big(u'(t)\Big)^2 dt + \frac{1}{2}\int_0^T \Big(v'(t)\Big)^2 dt \\ &+\sum_{i=1}^{m}\Big(\alpha_i-\alpha_{i-1}\Big)u(t_i)+\sum_{i=1}^{m}\Big(\beta_i-\beta_{i-1}\Big)v(t_i) \\ &-\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} f_i\Big(t,u(t)-u(t_{i+1}),v(t)-v(t_{i+1})\Big)dt \\ &+\frac{1}{2}\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}\eta_i(t)\Big(u(t)-u(t_{i+1})\Big)^2dt+\frac{1}{2}\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}\theta_i(t)\Big(v(t)-v(t_{i+1})\Big)^2dt,\end{aligned}$$

so

$$\begin{aligned}\Phi(u,v) &= \frac{1}{2}\Big\|(u,v)\Big\|_H^{*^2}+\sum_{i=1}^{m}\Big(\alpha_i-\alpha_{i-1}\Big)u(t_i)+\sum_{i=1}^{m}\Big(\beta_i-\beta_{i-1}\Big)v(t_i) \\ &\quad-\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} f_i\Big(t,u(t)-u(t_{i+1}),v(t)-v(t_{i+1})\Big)dt. \qquad (4.7)\end{aligned}$$

For more details about the construction of Φ, see the subsection 1.2.1.1.

Proposition 4.1. *The functional $\Phi : H \longrightarrow \mathbb{R}$, defined by (4.7) is continuously Fréchet-differentiable and weakly lower semi-continuous. Moreover, the critical points of Φ are weak solutions of (4.1).*

Proof. Using the continuity of $D_u f_i, D_v f_i, i = 0, 1, ..., m$, we easily obtain that the functional $\Phi \in \mathcal{C}^1(H,\mathbb{R})$. Furthermore, we have the differential of Φ at $(u,v) \in H$

$$\Phi'(u,v) : H \longrightarrow \mathbb{R},$$

is defined by

$$\begin{aligned}
\Phi'(u,v)(\varphi,\psi) = \int_0^T u'(t)\varphi'(t)dt + \int_0^T v'(t)\psi'(t)dt \\
-\sum_{i=1}^{m}\Big(\alpha_{i-1}-\alpha_i\Big)\varphi(t_i) - \sum_{i=1}^{m}\Big(\beta_{i-1}-\beta_i\Big)\psi(t_i) \\
-\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} D_u f_i\Big(t, u(t)-u(t_{i+1}), v(t)-v(t_{i+1})\Big)\Big(\varphi(t)-\varphi(t_{i+1})\Big)dt \\
-\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} D_v f_i\Big(t, u(t)-u(t_{i+1}), v(t)-v(t_{i+1})\Big)\Big(\psi(t)-\psi(t_{i+1})\Big)dt \\
+\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \eta_i(t)\Big(u(t)-u(t_{i+1})\Big)\Big(\varphi(t)-\varphi(t_{i+1})\Big)dt \\
+\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \theta_i(t)\Big(v(t)-v(t_{i+1})\Big)\Big(\psi(t)-\psi(t_{i+1})\Big)dt,
\end{aligned}$$

this shows that the critical points of Φ give us the weak solutions of (4.1).

To show that Φ is weakly lower semi-continuous, let $\{(u_j,v_j)\}\subset H$, with $(u_j,v_j)\rightharpoonup(u,v)$, then we have that $\{u_j\}$ and $\{v_j\}$ converge uniformly to u and v on $[0,T]$ respectively (Proposition 1.5). In connection with the fact that $\liminf_{j\longrightarrow\infty}\|(u_j,v_j)\|_H^* \geq \|(u,v)\|_H^*$ (Corollary 1.4), one has

$$\begin{aligned}
\liminf_{j\longrightarrow\infty}\Phi(u_j,v_j) &= \liminf_{j\longrightarrow\infty}\Bigg\{\frac{1}{2}\Big\|(u_j,v_j)\Big\|_H^{*^2} - \sum_{i=1}^{m}\Big(\alpha_{i-1}-\alpha_i\Big)u_j(t_i) - \sum_{i=1}^{m}\Big(\beta_{i-1}-\beta_i\Big)v_j(t_i) \\
&\quad -\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} f_i\Big(t, u_j(t)-u_j(t_{i+1}), v_j(t)-v_j(t_{i+1})\Big)dt\Bigg\}, \\
&\geq \frac{1}{2}\Big\|(u,v)\Big\|_H^{*^2} - \sum_{i=1}^{m}\Big(\alpha_{i-1}-\alpha_i\Big)u(t_i) - \sum_{i=1}^{m}\Big(\beta_{i-1}-\beta_i\Big)v(t_i) \\
&\quad -\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} f_i\Big(t, u(t)-u(t_{i+1}), v(t)-v(t_{i+1})\Big)dt, \\
&= \Phi(u,v).
\end{aligned}$$

This implies that the functional Φ is weakly lower semicontinuous. □

4.3 Main result

In this section we give the proof of our main result in this chapter.

Theorem 4.1. *Suppose that assumptions (A_1) and (A_2) are satisfied. Then there is a critical point of Φ, and (4.1) has at least one solution.*

Proof. From the theorem 1.4, the remark 1.11 and the proposition 4.1, to get the result, we just show that Φ is coercive.

For any $(u,v)\in H$, we have

$$\begin{aligned}
\Phi(u,v) &= \frac{1}{2}\Big\|(u,v)\Big\|_H^{*2} - \sum_{i=1}^m \Big(\alpha_{i-1}-\alpha_i\Big)u(t_i) - \sum_{i=1}^m \Big(\beta_{i-1}-\beta_i\Big)v(t_i) \\
&\quad - \sum_{i=0}^m \int_{s_i}^{t_{i+1}} f_i\Big(t, u(t)-u(t_{i+1}), v(t)-v(t_{i+1})\Big)dt, \\
&\geq \frac{1}{2}\Big\|(u,v)\Big\|_H^{*2} - \sum_{i=1}^m \Big(\alpha_{i-1}-\alpha_i\Big)u(t_i) - \sum_{i=1}^m \Big(\beta_{i-1}-\beta_i\Big)v(t_i) \\
&\quad - \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \Bigg(a_i\Big|u(t)-u(t_{i+1})\Big| + a_i\Big|v(t)-v(t_{i+1})\Big| \\
&\quad + b_i\Big|u(t)-u(t_{i+1})\Big|^{\gamma_1+1} + b_i\Big|v(t)-v(t_{i+1})\Big|^{\gamma_2+1}\Bigg)dt, \\
&\geq \frac{1}{2}\Big\|(u,v)\Big\|_H^{*2} - m\max_{i=1,\ldots,m}\Big\{|\alpha_{i-1}-\alpha_i|\Big\}\|u\|_\infty - m\max_{i=1,\ldots,m}\Big\{|\beta_{i-1}-\beta_i|\Big\}\|v\|_\infty \\
&\quad -2(m+1)T\max_{i=0,\ldots,m}\Big\{a_i\Big\}\Big(\|u\|_\infty + \|v\|_\infty\Big) \\
&\quad -2^{\gamma_1+1}(m+1)T\max_{i=0,\ldots,m}\Big\{b_i\Big\}\|u\|_\infty^{\gamma_1+1} - 2^{\gamma_2+1}(m+1)T\max_{i=0,\ldots,m}\Big\{b_i\Big\}\|v\|_\infty^{\gamma_2+1},
\end{aligned}$$

now using the lemma 4.3, we get

$$\begin{aligned}
\Phi(u,v) &\geq \frac{1}{2}\Big\|(u,v)\Big\|_H^{*2} - m\gamma\max_{i=1,\ldots,m}\Big\{|\alpha_{i-1}-\alpha_i|\Big\}\Big\|(u,v)\Big\|_H^{*} - m\gamma\max_{i=1,\ldots,m}\Big\{|\beta_{i-1}-\beta_i|\Big\}\Big\|(u,v)\Big\|_H^{*} \\
&\quad -4(m+1)\gamma T\max_{i=0,\ldots,m}\Big\{a_i\Big\}\Big\|(u,v)\Big\|_H^{*} \\
&\quad -(2\gamma)^{\gamma_1+1}(m+1)T\max_{i=0,\ldots,m}\Big\{b_i\Big\}\Big\|(u,v)\Big\|_H^{*\gamma_1+1} \\
&\quad -(2\gamma)^{\gamma_2+1}(m+1)T\max_{i=0,\ldots,m}\Big\{b_i\Big\}\Big\|(u,v)\Big\|_H^{*\gamma_2+1}.
\end{aligned}$$

Because $\gamma_1+1, \gamma_2+1 < 2$, we have that

$$\lim_{\|(u,v)\|_H^* \to\infty} \Phi(u,v) = \infty,$$

it follows that the functional Φ is coercive on H. □

Corollary 4.1. *Suppose that assumption* (A_1) *is satisfied and* $D_u f_i, D_v f_i, i = 0, 1, \ldots, m$, *are bounded. Then there is a critical point of* Φ, *and (4.1) has at least one solution.*

Example 4.1. *Let* $T = 1$, *we consider the following problem with non-instantaneous impulses*

$$\begin{cases} -u''(t) + \left(\dfrac{1}{t_{i+1}-s_i}(t-s_i)+1\right)\left(u(t)-u(t_{i+1})\right) &= t^2 + \sqrt{\left|u(t)-u(t_{i+1})\right|}, \\ & \quad t \in (s_i, t_{i+1}],\ i = 0, 1, \ldots, m, \\ -v''(t) + \left((t-s_i)+(t-s_i)^2\right)\left(v(t)-v(t_{i+1})\right) &= t + \sqrt[3]{\left|v(t)-v(t_{i+1})\right|}, \\ & \quad t \in (s_i, t_{i+1}],\ i = 0, 1, \ldots, m, \\ u'(t) &= \alpha_i,\ t \in (t_i, s_i],\ i = 1, 2, \ldots, m, \\ v'(t) &= \beta_i,\ t \in (t_i, s_i],\ i = 1, 2, \ldots, m, \\ u'(s_i^+) &= u'(s_i^-),\ i = 1, 2, \ldots, m, \\ v'(s_i^+) &= v'(s_i^-),\ i = 1, 2, \ldots, m, \\ u'(0^+) &= \alpha_0, \quad v'(0^+) = \beta_0, \\ u(0) &= u(1) = v(0) = v(1) = 0. \end{cases} \tag{4.8}$$

First we can see that for $i = 0, 1, \ldots, m$, $\eta_i(t) = \frac{1}{t_{i+1}-s_i}(t-s_i)+1$, $\theta_i(t) = (t-s_i)+(t-s_i)^2$, *and* $\nu_i = 0 > -\frac{2}{(t_{i+1}-s_i)^2}$, *then* (A_1) *holds. Next, taking* $a_i = 1, b_i = 1, \gamma_1 = \dfrac{1}{2}$ *and* $\gamma_2 = \dfrac{1}{3}$, $i = 0, 1, ..., m$, (A_2) *holds. Then, by Theorem 4.1, the non-instantaneous impulsive problem (4.8) has at least one nontrivial solution.*

Conclusion and perspectives

We divided this thesis into three parts. In the first one, we focused our attention on a class of nonlinear differential equations with instantaneous impulses, of the form

$$\begin{cases} -u''(t) = f_u(t,u,v), \ t \in (0,T) \setminus \{t_1,...,t_m\}, \\ -v''(t) = f_v(t,u,v), \ t \in (0,T) \setminus \{t_1,...,t_m\}, \\ u(0) = u(T) = v(0) = v(T) = 0, \\ \Delta u'(t_k) = u'(t_k^+) - u'(t_k^-) = I_k\big(u(t_k)\big), \ k = 1,2,...,m, \\ \Delta v'(t_k) = v'(t_k^+) - v'(t_k^-) = J_k\big(v(t_k)\big), \ k = 1,2,...,m, \end{cases} \tag{1}$$

where the nonlinear functions f_u, f_v are Carathéodory on $(0,T) \times \mathbb{R}^2$, and I_k, J_k are continuous on $\mathbb{R}$.

By means of a variational method we have shown the existence of weak solutions under the following two conditions on the functions f_u, f_v, I_k and J_k

1. There exist $a, b > 0$, and $\alpha_1, \alpha_2 \in [0,1)$, such that

$$\begin{cases} |f_u(t,u,v)| \le a + b|u|^{\alpha_1}, \ \text{for every } (t,u,v) \in (0,T) \times \mathbb{R}^2, \\ |f_v(t,u,v)| \le a + b|v|^{\alpha_2}, \ \text{for every } (t,u,v) \in (0,T) \times \mathbb{R}^2. \end{cases}$$

2. There exist $a_k, b_k > 0$, and $\beta_k \in [0,1), k = 1,2,...,m$, such that

$$\begin{cases} |I_k(u)| \le a_k + b_k|u|^{\beta_k}, \ \text{for every } u \in \mathbb{R}, \\ |J_k(v)| \le a_k + b_k|v|^{\beta_k}, \ \text{for every } v \in \mathbb{R}. \end{cases}$$

In the second part, we were interested in studying of a problem with non-instantaneous impulses, of the type

$$\begin{cases} -u''(t) &=& D_u f_i\Big(t, u(t)-u(t_{i+1}), v(t)-v(t_{i+1})\Big),\ t \in (s_i, t_{i+1}],\ i=0,1,\dots,m, \\ -v''(t) &=& D_v f_i\Big(t, u(t)-u(t_{i+1}), v(t)-v(t_{i+1})\Big),\ t \in (s_i, t_{i+1}],\ i=0,1,\dots,m, \\ u'(t) &=& \alpha_i,\ t \in (t_i, s_i],\ i=1,2,\dots,m, \\ v'(t) &=& \beta_i,\ t \in (t_i, s_i],\ i=1,2,\dots,m, \\ u'(s_i^+) &=& u'(s_i^-),\ i=1,2,\dots,m, \\ v'(s_i^+) &=& v'(s_i^-),\ i=1,2,\dots,m, \\ u'(0^+) &=& \alpha_0, \quad v'(0^+) = \beta_0, \\ u(0) &=& u(T) = v(0) = v(T) = 0, \end{cases} \tag{2}$$

where α_i, β_i are given constants. The nonlinear functions $D_u f_i, D_v f_i$ are Carathéodory on $(s_i, t_{i+1}] \times \mathbb{R}^2$.

As in the foregoing model based on a variation method we obtained the existence of weak solutions when the following condition is satisfied

- There exist $a_i, b_i > 0$, and $\gamma_1, \gamma_2 \in [0,1), i = 0,1,\dots,m$, such that

$$\begin{cases} |D_u f_i(t,u,v)| \le a_i + b_i |u|^{\gamma_1},\ \text{for every } (t,u,v) \in (s_i, t_{i+1}] \times \mathbb{R}^2, \\ |D_v f_i(t,u,v)| \le a_i + b_i |v|^{\gamma_2},\ \text{for every } (t,u,v) \in (s_i, t_{i+1}] \times \mathbb{R}^2. \end{cases}$$

Then in the third part, we generalized the previous model as follows

$$\begin{cases} -u''(t) + \eta_i(t)\Big(u(t)-u(t_{i+1})\Big) &=& D_u f_i\Big(t, u(t)-u(t_{i+1}), v(t)-v(t_{i+1})\Big), \\ && t \in (s_i, t_{i+1}],\ i=0,1,\dots,m, \\ -v''(t) + \theta_i(t)\Big(v(t)-v(t_{i+1})\Big) &=& D_v f_i\Big(t, u(t)-u(t_{i+1}), v(t)-v(t_{i+1})\Big), \\ && t \in (s_i, t_{i+1}],\ i=0,1,\dots,m, \\ u'(t) &=& \alpha_i,\ t \in (t_i, s_i],\ i=1,2,\dots,m, \\ v'(t) &=& \beta_i,\ t \in (t_i, s_i],\ i=1,2,\dots,m, \\ u'(s_i^+) &=& u'(s_i^-),\ i=1,2,\dots,m, \\ v'(s_i^+) &=& v'(s_i^-),\ i=1,2,\dots,m, \\ u'(0^+) &=& \alpha_0, \quad v'(0^+) = \beta_0, \\ u(0) &=& u(T) = v(0) = v(T) = 0, \end{cases} \tag{3}$$

where $\eta_i, \theta_i \in L^\infty(s_i, t_{i+1}]$, and the nonlinear functions $D_u f_i, D_v f_i$ are Carathéodory functions on $(s_i, t_{i+1}] \times \mathbb{R}^2$, α_i, β_i are given constants.

Under the following assumptions

(A_1) Assume that

$$\nu_i > -\lambda_i,\ \forall i = 0, 1, \ldots, m,$$

where $\nu_i = \min\left\{ess\inf_{t\in(s_i,t_{i+1}]}\eta_i(t), ess\inf_{t\in(s_i,t_{i+1}]}\theta_i(t)\right\}$ and $\lambda_i = \dfrac{2}{(t_{i+1}-s_i)^2}$.

(A_2) Suppose that $D_u f_i, D_v f_i$ verify the following condition:

There exist $a_i, b_i > 0$, and $\gamma_1, \gamma_2 \in [0, 1), i = 0, 1, \ldots, m$, such that

$$\begin{cases} |D_u f_i(t,u,v)| \leq a_i + b_i|u|^{\gamma_1},\ \text{for every } (t,u,v) \in (s_i, t_{i+1}] \times \mathbb{R}^2, \\ |D_v f_i(t,u,v)| \leq a_i + b_i|v|^{\gamma_2},\ \text{for every } (t,u,v) \in (s_i, t_{i+1}] \times \mathbb{R}^2, \end{cases}$$

there is at least one solution of the generalized problem.

This work raises a number of questions for researchers to explore in further studies. Several generalizations are considered

1. We can consider a more general case by taking functions f and g, instead of f_u and f_v derivatives of $f(t,u,v)$ at u and v respectively.

2. For models (2) and (3), there are significant difficulties when considering non-constant impulses. For that, we can consider α_i and β_i depend on t or on $u(t)$ and $v(t)$.

3. What may happens when T tends to infinity? How about the global solutions? And then the stability.

4. It would also be interesting to give a multivalued version to the previous problems. More precisely, we can consider the following impulsive differential inclusions problem

$$\begin{cases} -u''(t) \in F(t,u,v),\ t \in (0,T) \setminus \{t_1, ..., t_m\}, \\ -v''(t) \in G(t,u,v),\ t \in (0,T) \setminus \{t_1, ..., t_m\}, \\ u(0) = u(T) = v(0) = v(T) = 0, \\ \Delta u'(t_k) = u'(t_k^+) - u'(t_k^-) = I_k\big(u(t_k)\big),\ k = 1, 2, ..., m, \\ \Delta v'(t_k) = v'(t_k^+) - v'(t_k^-) = J_k\big(v(t_k)\big),\ k = 1, 2, ..., m, \end{cases}$$

where $F, G : [0,T] \times \mathbb{R}^2 \longrightarrow 2^{\mathbb{R}}$.

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