*N° d'ordre :*


REPUBLIQUE ALGERIENNE DEMOCRATIQUE & POPULAIRE

MINISTERE DE L'ENSEIGNEMENT SUPERIEUR & DE LA RECHERCHE SCIENTIFIQUE

UNIVERSITE DJILLALI LIABES
FACULTE DES SCIENCES EXACTES
SIDI BEL ABBES


# THÈSE DE DOCTORAT EN SCIENCES

*Présentée par*


***Benia Kheireddine***


*Spécialité : Mathématiques*
*Option : Analyse fonctionnelle*

*Intitulée*

## *Contribution to the study of measure of non-compactness and fractional differential equations and inclusions.*

*Soutenue le 17 Mai 2022*

*Devant le jury composé de :*

| | | |
|---|---|---|
| ***Président :*** | *Lazreg jamal Eddine* | *Pr. Univ. Djillali Liabes SBA* |
| ***Examinateurs :*** | *Ouahab Abdelghani* | *Pr. Univ. Djillali Liabes SBA* |
| | *Ghomari Kaoutar* | *Pr. Ecole Nat. Poly. d'Oran* |
| | *Larabi Abderahmanne* | *MCA. Univ. Ibn Khaldoun Tiaret* |
| ***Directeur de thèse :*** | *Hedia Benaouda* | *Pr. Univ. Ibn Khaldoun Tiaret* |
| ***Co-Directeur de thèse :*** | *Miloudi Mostefa* | *Pr. Univ. Djillali Liabes SBA* |

*Année universitaire : 2021/2022*

# Remerciements

La réalisation de cette Thèse a été possible grâce aux efforts de plusieurs Professeurs à qui je voudrais témoigner toute ma gratitude.

En premier temps je remercie, mon directeur de thèse Monsieur Hedia Benaouda, professeur à l'université de Tiaret, pour sa patience, sa disponibilité et surtout ses judicieux conseils, qui ont contribué à alimenter ma réflexion. Travailler avec lui fut pour moi un grand honneur. Cette occasion m'a permis de profiter de son esprit vif et de sa grande culture scientifique.

Je remercie également mon co-directeur Monsieur Miloudi Mostéfa, Professeur à l'université de Sidi Bel Abbès, Djillali Liabes, pour la qualité de son encadrement et pour sa patience tout au long de cette thèse, ses conseils pertinents m'ont permis de finaliser ce travail.

Je remercie aussi Monsieur Lazreg Jamal Eddine, Professeur à l'université de Sidi Bel Abbès, Djillali Liabes, qui m'a fait l'honneur de présider le jury pour la soutenance de cette thèse.

Je remercie Madame Ghomari Kaoutar, Professeur à l'Ecole Nat. Poly. d'Oran , d'avoir accepté d'examiner ma thèse et de faire partie du jury.

Je remercie Monsieur Ouahab Abdelghani, Professeur à l'université de Sidi Bel Abbès, d'avoir accepté d'examiner ma thèse et de faire partie du jury.

Je remercie Monsieur Larabi Abderrahmane, Professeur à l'université de Tiaret, d'avoir accepté d'examiner ma thèse et de faire partie du jury.

Je tiens à témoigner toute ma reconnaissance à Monsieur Beddani, pour son aide dans la réalisation de cette thèse.

A Mon Cher Père, pour le soutien constant et ses encouragements.

À tous ces intervenants, je présente mes remerciements, mon respect et ma gratitude.

# Contents

## Publications

1. K. Benia, E. Ait Dads, M. Beddani and B. Hedia, Nonlocal Fractional Differential Equation On The Half Line in Banach Space, *J. ATNAA.* **6** (1) 2022, 118-134.
2. K. Benia, M. Beddanin M.Feckan and B. Hedia, Existence result for a problem involving $\psi$-Riemann-Liouville fractional derivative on unbbounded domain, *J. Differ. Equ. Appl.* (to appear).
3. K. Benia and B. Hedia, Hybrid Implicit Fractional Integro-differential equations with Hadamard integral boundary conditions, *J. Dynamics of Continuous, Discrete and Impulsive Systems.* **28** (3) 2021, 207-222.

# Abstract

This thesis deals with the existence of solution sets and its topological structure for three different problems, a fractional differential equation with $\psi$-Riemann-Liouville fractional derivative on $(0;\infty)$ in a special Banach space, non-local fractional differential equations on the half-line in a Banach space and a hybrid fractional integro-differential equations with Hadamard integral boundary conditions. Our approach in the three cases is based on a fixed point theorem combined with measure of non-compactness. An example is given to show the applicability.

**Key words and phrases:** Fractional differential equations, Riemann-Liouville fractional derivative, $\psi$-Riemann-Liouville fractional derivative, Hadamard fractional derivative, Hybrid differential equation, boundary domain, unbounded domain, Nonlocal boundary value problem, fixed points theorems,Meir-Keeler condensing operators, Dhage fixed point theorem, Banach algebra, special Banach space, measure of non-compactness.

# Résumé

Cette thèse traite l'existence de l'ensemble des solutions et sa structure topologique, de trois différents problèmes, une équation différentielle fractionnaire avec $\psi$-dérivée fractionnaire de Riemann-Liouville sur $(0;\infty)$ dans un espace de Banach spécial, une équation différentielle fractionnaire non locale sur la demi droite dans un espace de Banach et une équation intégro-différentielle fractionnaire hybride avec des conditions aux limites intégrales de type Hadamard. Notre approche dans les trois cas est basée sur les théorèmes du point fixe combiné à une mesure de non-compacité. Un exemple est donné pour chaque cas afin de vérifier l'applicabilité de nos résultats.
**Mots clés:** Equations différentielles fractionnaires, dérivée fractionnaire de Riemann-Liouville, $\psi$-dérivée fractionnaire de Riemann-Liouville, dérivée fractionnaire de Hadamardn, équation différentielle hybride, domaine borné, domaine non borné, problème borné de valeur non-local, théorèmes de points fixes, opérateur condensé de Meir-Keeler, théorèmes de points fixes de Dhage, espace de Banach, algèbre de Banach, espace de Banach spécial, mesure de non-compacité.

# ملخص

تتناول هذه الأطروحة إيجاد مجموعة الحلول و بنيتها الطوبولوجية لثلاثة مسائل مختلفة، معادلة تفاضلية كسرية مع مشتق كسري $\psi$ ريمان ليوفيل على $(0;\infty)$ في فضاء بناخي مميز، معادلة تفاضلية كسرية غير محلية على $(0;\infty)$ في فضاء بناخي و معادلة تفاضلية تكاملية كسرية هجينة مع شروط حدودية من نوع تكامل هدمار. النهج المعتمد في الحلات الثلاث هو نظرية النقطة الصامدة مع القياس الامتراص. في كل حالة يتم اعطاء مثال يوضح قابلية التطبيق.

**الكلمات و العبارات المفتاحية** : المعادلات التفاضلية الكسرية، المشتقة الكسرية لريمان ليوفيل، المشتق الكسري $\psi$ ريمان ليوفيل، المشتقة الكسرية لهدمار، المعادلات التفاضلية الهجينة، المجال المحدود، المجال الغير محدود، القيم الغير محلية، نظريات النقطة الثابتة، نظرية النقطة الثابتة لداهدج، عامل التكثيف مايلر كيلر، المقاييس الامتراصة.

# Introduction

Fractional calculus can be seen as a generalization of the ordinary differentiation and integration to arbitrary non integer order, and has been recognized as one of the most powerful tools to describe long memory processes in the last decades. The concept of fractional calculus initially start from communications between Marquis de L'Hôpital (1661-1704) to Gottfried Wilhelm Leibniz (1646-1716), where Marquis de L'Hôpital ask Gottfried Wilhelm Leibniz the meaning of $\frac{d^n y}{dx^n}$ for the derivative of order $n = \frac{1}{2}$ (What if $n = \frac{1}{2}$ ?). Subsequent mention of fractional derivatives was made, in some context or the other, by (for example) Euler in 1730 , Lagrange in 1772 , Laplace in 1812 , Lacroix in 1819, Fourier in 1822, Liouville in 1832, Riemann in 1847, Greer in 1859, Holmgren in 1865, Grünwald in 1867, Letnikov in 1868, Sonin in 1869, Laurent in 1884, Nekrassov in 1888, Krug in 1890, and Weyl in 1917. For a long time, the theory of fractional Calculus developed only as a pure theoretical field of mathematics. However, in the last decades, it was found that fractional derivatives and integrals provide, in some situations. Indeed, fractional order models can be found to be more adequate than integer order models in some real world problems as fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes [1]. Applications of fractional Calculus include modeling viscoelastic and viscoplastic materials [49], chemical processes [66], and a wide range of engineering problems. The mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electro dynamics of complex medium, polymer rheology, etc..., involves derivatives of fractional order.

As a consequence, the subject of fractional differential equations is gaining more importance and attention. There has been significant development in ordinary and partial differential equations involving both Riemann-Liouville and Caputo fractional derivatives, see [5, 8]and [7], [9]. Another kind of fractional derivative appears side by side to Riemann Liouville and Caputo derivatives in the literature is Hadamard fractional derivative introduced in 1892 [44], which differs from the preceding ones in the sense that the kernel of the integral (in the definition of Hadamard derivative) contains logarithmic function of arbitrary exponent. Details and properties of Hadamard fractional derivative and integral can be found in [27, 28, 29, 48, 50, 61].

Fixed point theory is one of the most effective and fruitful tool used in nonlinear analysis to solve functional integral equations. It's concerned with the conditions for the existence of one or more fixed points of a mapping from a topological space into itself. Brouwer established a fixed point result what has become the well-known Brouwer's fixed point theorem for finite dimensional spaces. While in 1922, Banach introduced his celebrated Banach contraction principle for complete metric spaces which guarantee the existence and uniqueness of fixed point. Afterwards, in 1930 , Schauder extended the Brouwer's fixed point theorem to infinite dimensional spaces using the condition of compactness. There are many developments in fixed point theory in various directions,, Kuratowski in 1930, opened up a new direction of research with the introduction of the concept of a measure of noncompactness, which gives the degree of noncompactness for bounded sets. The measure of noncompactness can also be used in the study of single-valued and multivalued mappings, especially in metric and topological fixed point theory. The measure of noncompactness combining with some algebraic arguments is beneficial for studying mathematical formulations, especially solving the existence of solutions of some nonlinear problems under certain situations.

Quadratic perturbations of nonlinear differential equations have attracted much attention. We call such differential equations hybrid differential equations. There have been many works on the theory of hybrid differential equations, we refer the readers to the articles [13, 26, 60, 65]. Integro-differential and integrals equations of fractional order have also proved to be valuable tools in the modelling of many phenomena in various fields of science and engineering, the kind of this area of research have been studied by many authors [41, 42, 63].

Non-local conditions were initiated by Byszewaski [31] where he proved the existence and uniqueness of mild and classical solutions of non-local Cauchy problems. As remarked by Byszewaski [30], the non-local conditions can be more useful than the standard condition to describe some physical phenomena.
Very recently, many research papers have appeared concerning the fractional differential equations in Banach spaces, some of them investigated the existence results of solutions on finite intervals and unbounded domain by classical tools from functional analysis and measure of non compactness see, for example the following references: [10, 18, 22, 45, 53, 59].

We have organized this thesis as follows :

**Chapter 1.** Groups together the notations, the definitions of the concepts used throughout this manuscript. We introduce some important notions for fractional calculus, non-compactness measure and fixed point theory.

**Chapter 2.** we show the existence solution of the following boundary value problem :

$$\begin{cases} {}^{RL}\mathcal{D}_{0^+}^{\alpha,\psi} y(t) = f(t, y(t)), \quad t \in (0, +\infty), \\ {}^{RL}\mathfrak{J}_{0^+}^{2-\alpha,\psi} y(0^+) = a, \\ {}^{RL}\mathcal{D}_{0^+}^{\alpha-1,\psi} y(\infty) = b, \end{cases}$$

where ${}^{RL}\mathcal{D}^{\alpha,\psi}$ denote the left-sided $\psi$-Riemann-Liouville fractional derivative with
$1 < \alpha < 2$.The operator $\mathfrak{J}_{0^+}^{(2-\alpha),\psi}$ denotes the left-sided $\psi$- Riemann-Liouville fractional integral, $E$ is a Banach space with the norm $\|.\|$, $a, b \in E$, $f : (0, \infty) \times E \to E$. We prove the existence of solutions, by applying the fixed point theorem combined with the technique of measure of non-compactness. Finally an example.

**Chapter 3.** In Chapter three we deal with the existence of solution sets and its topological structure for fractional differential equations on unbounded domain with the non-local conditions for the problem :

$$\begin{cases} {}^{\mathcal{RL}}\mathcal{D}^{\alpha}_{0^+}y(t) = f(t, y(t)),\ t \in J = (0, +\infty), \\ \mathcal{I}^{2-\alpha}_{0^+}y(0^+) = \sum_{i=1}^{m} \lambda_i y(\tau_i), \\ {}^{\mathcal{RL}}\mathcal{D}^{\alpha-1}_{0^+}y(\infty) = y_\infty, \end{cases}$$

where ${}^{\mathcal{RL}}\mathcal{D}^{\alpha}_{0^+}$ denotes the Riemann-Liouville fractional derivative of order $\alpha$, $1 < \alpha \leq 2$. The operator $\mathcal{I}^{2-\alpha}_{0^+}$ denotes the Riemann-Liouville fractional integral, the state $y(\cdot)$ takes values in a Banach space $E$, $f : (0, \infty) \times E \to E$, $\tau_i$, $i = 1, 2, \dots, m$ are pre-fixed points satisfying $0 < \tau_1 \leq \cdots \leq \tau_m$, $\lambda_i \in \mathbb{R}^*_+$ and

$$\Gamma(\alpha - 1) \neq \sum_{i=1}^{m} \lambda_i \tau_i^{\alpha-2},\ \text{where } \Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt.$$

We prove the existence of solutions, by using the fixed point theorem for Meir-Keeler condensing operators via measure of non-compactness. We also present an illustrative example.

**Chapter 4.** We discuss in this Chapter the following hybrid implicit integro-differential boundary fractional value problem with Hadamard integral boundary conditions,

$$\begin{cases} {}_{H}D^{\alpha}\Big(\dfrac{x(t)}{f(t, x(t), G(t, x(t))}\Big) = g\Big(t, x(t), {}_{H}D^{\alpha}\Big(\dfrac{x(t)}{f(t, x(t), G(t, x(t)))}\Big)\Big), \\ \qquad\qquad\qquad\qquad t \in J = [1, T],\ 0 < \alpha \leq 1, \\ a_H J^{1-\alpha} x(t)|_{t=1} + b_H J^{1-\alpha} x(t)|_{t=T} = c, \end{cases}$$

where ${}_{H}D^{\alpha}$ is the Hadamard fractional derivative of order $\alpha$, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \setminus \{0\})$, $g : J \times \mathbb{R} \times R \to \mathbb{R}$ are continuous functions, $G$ is a nonlinear integral operator defined by

$$G(t, x(t)) = \int_0^t \psi(t, \tau) h(\tau, x(\tau)) d\tau,$$

$\psi$ and $h$ are functions that will be specified later, ${}_{H}J^{\alpha}$ is the Hadamard fractional integral of order $\alpha$, $a, b$ and $c$ are real numbers such that $a + b \neq 0$. In this Chapter we present the existence result for the problem, using Dhage fixed point theorem and an illustrative example.

# Chapter 1

# Preliminaries

This chapter contains used notations, definitions of functional spaces, fractional integrals and fractional derivatives. We also give the definitions and proprieties of the measure of non-compactness and fixed points theorems which are used throughout this thesis.

## 1.1 Spaces of integrable, absolutely continuous, and continuous functions

In this section we present different used functional spaces.

Let $J = [a, b]$, $0 < a < b$ be a finite or infinite interval of the real positive axes $[0, \infty)$ and $E$ be a real Banach space with the norm $\| \cdot \|$. We denote by $L^1(J, \mathbb{R})$ the space of Lebesgue integrable functions $y$ on $J$ with the norm

$$\|y\|_{L^1} = \int_a^b |y(t)|dt,$$

A measurable function $y : J \to E$ is Bochner integrable if and only if $\|y\|$ is Lebesgue integrable.

Let $L^1(J; E)$ be the space of $E$-valued Bochner integrable functions on $J$ with the norm

$$\|y\|_{L^1} = \int_a^b \|y(t)\|dt,$$

Let now $J = [a, b]$, $0 \leq a < b < \infty$ be a finite interval and $\mathcal{AC}(J, E)$ the space of $E$-valued functions which are absolutely continuous in $J$. For $n \in \mathbb{N}^*$, we denote by $\mathcal{AC}^n(J, E)$ the Banach space of functions from the interval $J$ into $E$ which is defined as:

$$\mathcal{AC}^n(J, E) = \{y : J \to E : y \in C^n(J, E) \text{ with } D^n y \in \mathcal{AC}(J, E)\}.$$

Let $J = [a, b]$, $0 < a < b$ be a finite or infinite interval of the real positive axes $[0, \infty)$ and $n \in \mathbb{N}^*$. We denote by $C^n(J, E)$ a space of $E$-valued functions $y$ which are $n$ times continuously differentiable on $J$ with the norm

$$\|y\|_{C^n} = \sum_{k=0}^{n} \|y^{(k)}\|_\infty = \sum_{k=0}^{n} \sup_{x \in J} \|y^{(k)}(x)\|.$$

In particular, for $n = 0$, $C^0(J, E) = C(J, E)$ is the space of continuous $E$-valued functions $y$ on $J$ with the norm

$$\|y\|_\infty = \sup_{x \in J} \|y(x)\|.$$

When $J = [a, b]$, $0 < a < b$ be a finite or infinite interval of the real positive axes and $\alpha > 0$ we introduce the weighted space $C_{\alpha,\log}(J)$ of real valued functions $y$, given on $(a, b]$ and such that $[\log(x/a)]^\alpha y(x) \in C(J)$ and

$$\|y\| = \left\| \left( \ln \frac{x}{a} \right)^\alpha y(x) \right\|_\infty$$

We consider the following Banach space

$$C_\alpha([0,\infty), E) = \{y \in C((0,\infty), E) : \lim_{t \to 0^+} t^{2-\alpha} y(t) \text{ and } \lim_{t \to \infty} \frac{t^{2-\alpha} y(t)}{1 + t^\alpha} \text{ exist and are finite}\}.$$

A norm in this space is given by

$$\|y\|_\alpha = \sup_{t \in J} \frac{t^{2-\alpha}\|y(t)\|}{1 + t^\alpha}.$$

For $y \in C_\alpha([0,\infty), E)$, we define $y_\alpha$ by

$$y_\alpha(t) = \begin{cases} \dfrac{t^{2-\alpha} y(t)}{1 + t^\alpha}, & t \in (0,\infty), \\ \lim\limits_{t \to 0} \dfrac{t^{2-\alpha} y(t)}{1 + t^\alpha}, & t = 0. \end{cases}.$$

It is clear that $y_\alpha \in C([0,\infty), E)$.

**Definition 1.1.** *Let $p, q > 0$, then the Beta function $\beta(p, q)$ is defined as*

$$\beta(p, q) = \int_0^1 x^{p-1}(1 - x)^{q-1} dx.$$

**Remark 1.2.** *For $p, q > 0$, the following identity holds,*

$$\beta(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p + q)},$$

*where $\Gamma$ is Gamma function defined by*

$$\Gamma(q) = \int_0^\infty e^{-s} s^{q-1} ds.$$

## 1.2 Fractional integrals and fractional derivatives

This section contains the definitions and some properties of fractional integrals and fractional derivatives of used types.

We begin with some definitions and lemmas of the theory of fractional calculus.
Let $J = [a, b]$, $a, b > 0$ and $(E, \|.\|)$ be a real Banach space.

**Definition 1.3.** *[49] The Riemann-Liouville fractional integral of order $\alpha > 0$ of the function $h$ is defined almost everywhere in $[a, b]$ by*

$$I_{a^+}^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s) ds,$$

*where $\Gamma$ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = [h * \varphi_\alpha](t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$ and $\varphi_\alpha(t) = 0$ for $t \leq 0$. The equality holds everywhere if $h \in C([a, b], E)$.*

**Definition 1.4.** *[49] Let $\alpha > 0$ and $n$ be the smallest integer greater than or equal to $\alpha$ and $h : [a, b] \to E$ be a function such that $I^{n-\alpha} h \in \mathcal{AC}^n([a, b], E)$. Then, the Riemann-Liouville fractional derivative of order $\alpha$ of the function $h$ is defined almost every where in $[a, b]$ by*

$$^{RL}D_{a^+}^{\alpha} h(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt} \left( \int_a^t (t-s)^{n-\alpha-1} h(s) ds \right).$$

**Lemma 1.5.** *[64] Let $\alpha > 0$ and $h \in C(J, E) \cap L^1(J, E)$. Then the differential equation*

$$^{RL}D_{a^+}^{\alpha} h(t) = 0,$$

*has solutions $h(t) = c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \ldots + c_n(t-a)^{\alpha-n}$, for some $c_i \in \mathbb{R}$, $i = 1 \ldots n$, where $n = [\alpha] + 1$.*

**Lemma 1.6.** *[64] Let $\alpha > 0$. Suppose that $h \in C(J, E) \cap L^1(J, E)$ with a fractional derivative of order $\alpha$ belonging to $C(J, E) \cap L^1(J, E)$. Then*

$$I^{\alpha RL} D_{a^+}^{\alpha} h(t) = h(t) + c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \ldots + c_n(t-a)^{\alpha-n},$$

*for some $c_i \in \mathbb{R}$, $i = 1, \ldots, n$, where $n = [\alpha] + 1$.*

**Remark 1.7.** *[49] For $\alpha > 0$, $k > -1$, we have*

$$I_{0+}^{\alpha} t^k = \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)} t^{\alpha+k} \text{ and } {}^{RL}D_{0+}^{\alpha} t^k = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} t^{k-\alpha}, \ t > 0,$$

*giving in particular ${}^{RL}D_{0+}^{\alpha} t^{\alpha-m} = 0$, $m = 1, \ldots, n$, where $n$ is the smallest integer greater than or equal to $\alpha$.*

**Remark 1.8.** *If $h$ is suitabe function (see for instance [49, 54] ), we have*

$$^{RL}D_{0+}^{\alpha} I_{0+}^{\alpha} h(t) = h(t), \ \alpha > 0 \text{ and } {}^{RL}D_{0+}^{\alpha} I_{0+}^{k} h(t) = I_{0+}^{k-\alpha} h(t), \ k > \alpha > 0, \ t > 0.$$

**Definition 1.9.** *[49, 62] Let $h$ be an integrable function defined on $(a, b]$. Then,*

*(i) the $\psi$-Riemann-Liouville fractional integral of order $\alpha > 0$ of the function $h$ is defined by*

$$\mathfrak{I}_{a^+}^{\alpha,\psi} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) \psi(t, s)_{\alpha-1} h(s) ds,$$

*(ii) the $\psi$-Riemann- Liouville fractional derivative of order $\alpha > 0$ of the function $h$ is defined by*

$$^{RL}\mathcal{D}_{a^+}^{\alpha,\psi}h(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n\left(\int_a^t \psi'(s)\psi(t,s)_{n-\alpha-1}h(s)ds\right),$$

*where $\Gamma$ is the gamma function and $\psi_\eta(t,s) = (\psi(t)-\psi(s))^\eta$, for all $\eta > -1$and $t \geq s \geq 0$.*

**Lemma 1.10.** *[49, 62] Let $\alpha, \beta \in \mathbb{R}_+^*$. We have then*

1. $I_{0^+}^{\alpha,\psi}\psi_{\beta-1}(t,0) = \dfrac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\psi_{\alpha+\beta-1}(t,0)$.
2. *If $1 < \alpha < 2$, we have*
   *($i_1$)* $^{RL}\mathcal{D}_{a^+}^{\alpha-1,\psi}\psi_{\alpha-1}(t,a) = \Gamma(\alpha)$ *and* $^{RL}\mathcal{D}_{a^+}^{\alpha-1,\psi}\psi_{\alpha-2}(t,a) = 0$,
   *($i_2$)* $^{RL}\mathcal{D}_{a^+}^{\alpha,\psi}\psi_{\alpha-1}(t,a) = ^{RL}\mathcal{D}_{a^+}^{\alpha,\psi}\psi_{\alpha-2}(t,a) = 0$.

**Definition 1.11.** *[49] Let $\alpha > 0$ and $n$ be the smallest integer greater than or equal to $\alpha$. Let $h \in L^1(J,\mathbb{R})$, the Hadamard fractional integral of order $\alpha > 0$ is defined almost everywhere by*

$$_HI^\alpha h(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds,$$

*where $\ln(\cdot) = \ln_e(\cdot)$.*

**Definition 1.12.** *[49] Let $\alpha > 0$ and $n$ be the smallest integer greater than or equal to $\alpha$. Let $h : J \to R$ be a function such that $_HI^{n-\alpha}h \in AC_\delta^n(J;R)$. Then Hadamard derivative of fractional order $\alpha$ is defined by*

$$\begin{aligned} _HD^\alpha h(t) &= \delta^{(n)}(I^{n-\alpha}h(t))(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n\int_1^t\left(\ln\frac{t}{s}\right)^{n-\alpha-1}\frac{h(s)}{s}ds. \end{aligned}$$

**Lemma 1.13.** *[49] If $Re(\alpha), Re(\beta) > 0$, and $0 < a < b < \infty$, then*

1) $(\ _HI_{a+}^\alpha(\ln\frac{\tau}{a})^{\beta-1})(t) = \dfrac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\ln\frac{t}{a})^{\beta+\alpha-1}$

2) $(\ _HD_{a+}^\alpha(\ln\frac{\tau}{a})^{\beta-1})(t) = \dfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\ln\frac{t}{a})^{\beta-\alpha-1}$

3) $(\ _HI_{b^-}^\alpha(\ln\frac{b}{\tau})^{\beta-1})(t) = \dfrac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\ln\frac{b}{t})^{\beta+\alpha-1}$

4) $(\ _HD_{b^-}^\alpha(\ln\frac{b}{\tau})^{\beta-1})(t) = \dfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\ln\frac{b}{t})^{\beta-\alpha-1}$.

**Lemma 1.14.** *[49]Let $\alpha > 0$ and $x \in C[1, +\infty) \cap L^1[1, +\infty)$. Then the solution of Hadamard fractional differential equation ${}_H D^\alpha x(t) = 0$ is given by*

$$x(t) = \sum_{i=1}^{n} c_i \ (\ln t)^{\alpha - i}$$

*and the following formula holds :*

$${}_H I_H^\alpha D^\alpha x(t) = x(t) + \sum_{i=1}^{n} c_i (\ln t)^{\alpha - i},$$

*where $c_i \in \mathbb{R},\ i = 1, \dots, n$ are arbitrary constants and $n - 1 < \alpha \leq n$.*

**Lemma 1.15.** *[49] Let $\alpha > \beta > 0$*

- *If $0 < a < b < \infty$ and $1 \leq p \leq \infty$, then for $h \in L^p(a, b)$*

$${}_H I_{a^+}^\alpha \ {}_H I_{a^+}^\beta h = \ {}_H I_{a^+}^{\alpha+\beta} h$$

$${}_H D_{a^+}^\beta \ {}_H I_{a^+}^\alpha h = \ {}_H I_{a^+}^{\alpha-\beta} h$$

- *In Particular if $\beta = m \in \mathbb{N}$, then*

$${}_H D_{a^+}^m \ {}_H I_{a^+}^\alpha h = \ {}_H I_{a^+}^{\alpha-m} h$$

## 1.3 Measure of non-compactness and condensing operators

We start this section some definitions and properties of measure of non-compactness. For more details, we refer the reader to Deimling [36] and Kamenskii [47]. We introduce an important measure of non-compactness called the Kuratowski measure of non-compactness.

**Definition 1.16.** *[19, 43] Let $D$ a bounded subset of $E$. The Kuratowski measure of non-compactness $\vartheta$ of the subset $D$ is defined as :*

$$\vartheta(D) = \inf\{e > 0 : \ D = \bigcup_{i=1}^{n} D_i \text{ for some } D_i \text{ with } diam(D_i) \leq e \text{ for } 1 \leq i \leq n < \infty\}.$$

*Here, $diam(D)$ denotes the diameter of a set $D \subset E$, that is,*

$$diam(D) := \sup\{d(x, y) | x, y \in D\}.$$

Another important measure of noncompactness is the so-called Hausdorff (or ball) measure of noncompactness defined as follow

**Definition 1.17.** *[19, 43] Let $D$ a bounded subset of $E$. The Hausdorff measure of non-compactness $\chi$ of the subset $D$ is defined as :*

$$\chi(D) = \inf\{\varepsilon : D \text{ has a finite } \varepsilon\text{-net in } E\}$$

**Remark 1.18.** *[19] The measures $\chi$ and $\vartheta$ are equivalent, that is, for any bounded subset $D$ of $E$, the following estimate holds,*

$$\chi(D) \leq \vartheta(D) \leq 2\chi(D)$$

Let us now recall the definition of the measure of non-compactness in the sense of Kuratowski and its properties. For all $G \subseteq E$, we denote by $S_b(G)$ the set of all bounded subsets of $G$.

**Lemma 1.19.** *[19, 43] Let $A, B \in S_b(E)$. The following properties hold:*
($\mathbf{i}_1$) *$v(A) = 0$ if and only if $A$ is relatively compact,*
($\mathbf{i}_2$) *$v(A) = v(\overline{A})$, where $\overline{A}$ denotes the closure of $A$,*
($\mathbf{i}_3$) *$v(A + B) \leq v(A) + v(B)$,*
($\mathbf{i}_4$) *$A \subset B$ implies $v(A) \leq v(B)$,*
($\mathbf{i}_5$) *$v(a.A) = \|a\|.v(A)$ for all $a \in E$,*
($\mathbf{i}_6$) *$v(\{a\} \cup A) = v(A)$ for all $a \in E$,*
($\mathbf{i}_7$) *$v(A) = v(Conv(A))$, where $Conv(A)$ is the smallest convex that contains $A$.*

**Lemma 1.20.** *[38] Let $D \in S_b(E)$ and $\varepsilon > 0$. Then, there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subset D$, such that*

$$\vartheta(D) \leq 2\vartheta(\{x_n, n \in \mathbb{N}\}) + \varepsilon.$$

**Lemma 1.21.** *[43] If $D$ is an equicontinuous and bounded subset of $\mathcal{C}(J, E)$, then $\vartheta(D(.)) \in \mathcal{C}(J, \mathbb{R}^+)$*

$$\vartheta_{\mathcal{C}}(D) = \max_{r\in[a,b]} \vartheta(D(r)), \ \vartheta\left(\left\{ \int_a^b w(r)dr : w \in D\right\}\right) \leq \int_a^b \vartheta(D(r))dr,$$

*where $D(r) = \{w(r) : w \in D\}$ and $\vartheta_{\mathcal{C}}$ is the measure of non-compactness on the space $\mathcal{C}([a, b], E)$.*

**Definition 1.22.** *[19] Let $E_1$ and $E_2$ be two Banach spaces and $\mu_1$ and $\mu_2$ be arbitrary measures of noncompactness on $E_1$ and $E_2$, respectively. An operator $F$ from $E_1$ to $E_2$ is called a $(\mu_1, \mu_2)$-condensing operator if it is continuous and for every bounded noncompact set $D \subset E_1$, the following inequality holds,*

$$\mu_2(F(D)) < \mu_1(D).$$

Whenever $E_1 = E_2$ and $\mu_1 = \mu_2$ we shall simply say, $F$ is an $\mu$-condensing operator.

Meir-Keeler has been introduced since 1969 the notion of Meir-Keeler contraction mapping in a metric space.

**Definition 1.23.** *[11] Let $\mu$ be an arbitrary measure of non-compactness on $E$ and $G$ be a nonempty subset of $E$. Let $F$ be an operator from $G$ to $G$. $F$ is said Meir-Keeler condensing operator if*

$$\forall\varepsilon > 0, \ \exists k(\varepsilon) > 0, \forall D \in S_b(G) : \varepsilon \leq \mu(D) < \varepsilon + k \Longrightarrow \kappa(FD) < \varepsilon.$$

## 1.4 Some fixed point theorems

Most recently in 2015, the authors [11] introduced the following fixed point theorem.

**Theorem 1.24.** *[11] Let $\mu$ be an arbitrary measure of non-compactness on $E$ and $D$ a closed, bounded and convex subset of $E$. Let $F$ be an operator from $D$ to $D$, assume that $F$ is a Meir-Keeler condensing operator and continuous, then the set $\{w \in D : F(w) = w\}$ is nonempty and compact.*

The following result was improved by Dhage.

**Theorem 1.25.** *[39] Let $D$ be a non-empty, closed convex and bounded subset of the Banach algebra $E$ let $F : E \to E$ and $H : D \to E$ be two operators such that*

*(A1)* *$F$ is Lipschitzian with a Lipschitz constant $K$.*

*(A2)* *$H$ is completely continuous.*

*(A3)* *$x = FxHy \Rightarrow x \in D$ for all $y \in D$.*

*(A4)* *$MK \leq 1$ where $M = \sup_{x \in D} |H(x)|$. Then the operator equation $x = FxHx$ has at least one solution.*

# Chapter 2

# Existence result for a problem involving $\psi$-Riemann-Liouville fractional derivative on unbounded domain

## 2.1 Introduction

This chapter study the existence of solutions on unbounded domain of the following problem :

$$\begin{cases} {}^{RL}\mathcal{D}_{0^+}^{\alpha,\psi} y(t) = f(t, y(t)), \quad t \in (0, +\infty), & (2.1) \\ {}^{RL}\mathfrak{I}_{0^+}^{2-\alpha,\psi} y(0^+) = a, & (2.2) \\ {}^{RL}\mathcal{D}_{0^+}^{\alpha-1,\psi} y(\infty) = b, & (2.3) \end{cases}$$

where ${}^{RL}\mathcal{D}^{\alpha,\psi}$ denote the left-sided $\psi$-Riemann-Liouville fractional derivative with $1 < \alpha < 2$. The operator $\mathfrak{I}_{0^+}^{(2-\alpha),\psi}$ denotes the left-sided $\psi$- Riemann-Liouville fractional integral, $E$ is a Banach space with the norme $\|.\|$, $a, b \in E$, $f : (0, \infty) \times E \to E$ a function satisfying some specified conditions and $\psi \in \mathcal{C}^1([0, \infty), \mathbb{R}^+)$ satisfied $\psi'(t) > 0$, for all $t \in [0, \infty)$.

## 2.2 Background and basic results

Let $I \subset (0, \infty)$ be a compact interval and denote by $\mathcal{C}(I, E)$ the Banach space of continuous functions $y : I \to E$ with the usual norm

$$\|y\|_\infty = \sup\{\|y(t)\|, \ t \in I\}.$$

$L^1(J, E)$ the space of $E$ valued Bochner integrable functions on $J$ with the norm

$$\|f\|_{L^1} = \int_0^{+\infty} \|f(t)\| dt.$$

For all $\eta > -1$ and $s, t \in [0, \infty)$ with $t \geq s$, we pose $\psi_\eta(t,s) = (\psi(t) - \psi(s))^\eta$. We consider the following Banach space

$$\mathcal{C}_{\alpha,\psi}([0,\infty), E) = \{y \in \mathcal{C}((0,\infty), E) : \lim_{t\to 0} \psi_{2-\alpha}(t,0)y(t) \text{ and } \lim_{t\to\infty} \frac{\psi_{2-\alpha}(t,0)y(t)}{1+\psi_\alpha(t,0)} \text{ exists and finite}\},$$

equipped with the norm

$$\|y\|_\alpha^\psi = \sup\left\{\frac{\psi_{2-\alpha}(t,0)\|y(t)\|}{1+\psi_\alpha(t,0)},\ t \in (0,\infty)\right\}.$$

**Definition 2.1.** *A function $y \in \mathcal{C}_{\alpha,\psi}([0,+\infty))$ is said to be solution of the problem (2.1)-(2.3) if $y$ satisfies the equation ${}^{RL}\mathcal{D}^\alpha_{0^+}y(t) = f(t, y(t))$ and the conditions (2.2)-(2.3).*

Before we give our result we introduce the following four hypotheses which are needed :

**($\mathbf{H_1}$)** $f : (0,\infty) \times E \to E$ is a continuous function and for all $x, y$ and $(0,T] \subset (0,\infty)$:

$$\|f(t,x) - f(t,y)\| \leq A\psi_{2-\alpha}(t,0)\|x-y\|, \text{ for all } t \in (0,T],$$

where $A \in \mathbb{R}^+$.

**($\mathbf{H_2}$)** There exists functions $a, b \in \mathcal{C}([0,\infty), \mathbb{R}^+)$ such that

$$\|f(t,u)\| \leq a(t) + \psi_{2-\alpha}(t,0)b(t)\|u\| \text{ for all } t \in (0,\infty) \text{ and } u \in E,$$

with

$$\int_0^\infty \psi'(s)[1+\psi_\alpha(s,0)]b(s)dt < \Gamma(\alpha)$$

and

$$\int_0^\infty \psi'(s)a(s)dt < \infty.$$

**($\mathbf{H_3}$)** There exists a function $\ell \in \mathcal{C}([0,\infty), \mathbb{R}^+)$ such that for each nonempty, bounded set $\Omega \subset C_{\alpha,\psi}((0,\infty), E)$

$$\vartheta(f(t,\Omega(t))) \leq \ell(t)\psi_{2-\alpha}(t,0)\vartheta(\Omega(t)), \quad \text{for all } t \in (0,\infty)$$

with,

$$\int_0^\infty \psi'(s)(1+\psi_\alpha(s,0))\ell(s)ds \leq \frac{\Gamma(\alpha)}{2}.$$

**($\mathbf{H_4}$)** There exists $R > 0$ such that

$$R > \frac{\|b\| + (\alpha-1)\|a\| + \int_0^\infty \psi'(s)a(s)ds}{\Gamma(\alpha) - \int_0^\infty \psi'(s)(1+\psi_\alpha(s,0))b(s)ds}.$$

Let
$$B = \{y \in \mathcal{C}_{\alpha,\psi}([0,\infty), E) : \|y\|_{\psi}^{\alpha} \leq R\},$$
such that $R$ is a strictly positive real.

**Remark 2.2.** *There exists a positive real number $M$ such that*
$$\int_0^{\infty} \psi'(s)\|f(s, y(s))\|ds \leq M, \text{ for any } y \in B.$$

**Lemma 2.3.** *Any solution $y \in B$ of the following integral equation*
$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\alpha)}\left[b - \int_0^{\infty} \psi'(s)f(s, y(s)ds\psi_{\alpha-1}(t,0)\right] + \frac{a\psi_{\alpha-2}(t,0)}{\Gamma(\alpha-1)} \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^{t} \psi'(s)\psi_{\alpha-1}(t,s)f(s,y(s)ds \end{aligned} \tag{2.4}$$
*is a solution of the problem (2.1)-(2.3).*

*Proof.* Let $y \in B$ be a solution of (2.4). Applying $\mathfrak{I}_{0^+}^{2-\alpha,\psi}$ to both sides of (2.4) and utilizing Lemma 1.10, we get
$$\mathfrak{I}_{0^+}^{2-\alpha,\psi}y(t) = \frac{1}{\Gamma(\alpha)}[b - \int_0^{\infty} \psi'(t)f(t, y(t)dt]\psi_1(t,0) + a + \mathfrak{I}_{0^+}^{2,\psi}f(t, y(t)).$$
By taking $t$ tends to 0, we get
$$\mathfrak{I}_{0^+}^{1-\alpha,\psi}y(0) = a.$$
By applying ${}^{RL}\mathcal{D}_{0^+}^{\alpha-1,\psi}$ to both sides of (2.4) and using Lemma 1.10, we have
$${}^{RL}\mathcal{D}_{0^+}^{\alpha-1,\psi}y(t) = b - \int_0^{\infty} \psi'(t)f(t, y(t)dt + I_{0^+}^{1,\psi}f(t, y(t)).$$
As $t \longrightarrow \infty$, we get
$${}^{RL}\mathcal{D}_{0^+}^{\alpha-1,\psi}y(\infty) = b.$$
Next, by applying ${}^{RL}\mathcal{D}_{0^+}^{\alpha,\psi}$ to both sides of (2.4) and by using Lemma 1.10, we obtain
$${}^{RL}\mathcal{D}_{0^+}^{\alpha,\psi}y(t) = f(t, y(t)).$$
The results are proved completely. □

## 2.3 Existence result

The theorem below is the main result.

**Theorem 2.4.** *Suppose that conditions* $(\mathbf{H_1}) - (\mathbf{H_4})$ *are valid. Then the problem* $(2.1) - (2.3)$ *has at least one solution.*

Consider the operator $N : \mathcal{C}_{\alpha,\psi}([0,\infty),E) \to \mathcal{C}_{\alpha,\psi}([0,\infty),E)$ defined by

$$Ny(t) = \frac{1}{\Gamma(\alpha)}\left[b - \int_0^\infty \psi'(t)f(t,y(t)dt\right]\psi_{\alpha-1}(t,0) + \frac{a\psi_{\alpha-2}(t,0)}{\Gamma(\alpha-1)} + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)\psi_{\alpha-1}(t,s)f(s,y(s)ds.$$

*Proof.* From the definition of the operator $N$ and Lemma 2.3, we see that the fixed points of $N$ are solutions of problem (2.1) – (2.3).
For this reason, it suffices to verify the conditions of Theorem 1.24, it is done in four steps.
**Step 1:** $N$ is bounded on $B$.
Let $y \in \mathcal{C}_{\alpha,\psi}([0,\infty),E)$, from $(\mathbf{H_2})$ it is easy to deduce that $Ny \in \mathcal{C}_{\alpha,\psi}([0,\infty),E)$. Using $(\mathbf{H_2})$, for all $y \in B$ and $t \in (0,\infty)$ we get

$$\frac{\psi_{2-\alpha}(t,0)\|N(y)(t)\|}{1+\psi_\alpha(t,0)} \leq \frac{\|b\|+M}{\Gamma(\alpha)} + \frac{\|a\|}{\Gamma(\alpha-1)} + \frac{1}{\Gamma(\alpha)}\int_0^\infty \psi'(s)\|f(s,y(s))\|ds \leq \frac{\|b\|+2M+(\alpha-1)\|a\|}{\Gamma(\alpha)}.$$

Hence, $NB$ is bounded.
**Step 2:** $N$ is continuous.
We rewrite $N$ as follows

$$Ny(t) = \frac{b\psi_{\alpha-1}(t,0)}{\Gamma(\alpha)} + \frac{a\psi_{\alpha-2}(t,0)}{\Gamma(\alpha-1)} - \frac{\psi_{\alpha-1}(t,0)}{\Gamma(\alpha)}\int_t^\infty \psi'(s)f(s,y(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)[\psi_{\alpha-1}(t,s)-\psi_{\alpha-1}(t,0)]f(s,y(s)ds.$$

Let $\{y_n\}_{n\in\mathbb{N}}$ converges to $y$ in $\mathcal{C}_{\alpha,\psi}([0,\infty),E)$ and $\varepsilon > 0$, by noticing that the functions $y_n, n \in \mathbb{N}$ and $y$ are bounded, it implies that there exists $M > 0$ such that

$$\|y_n\|_\alpha^\psi \leq M, n \in \mathbb{N}$$

and

$$\|y\|_\alpha^\psi \leq M.$$

Hypothese $(\mathbf{H_2})$ assume that there exists $L > 0$, such that

$$\int_L^\infty \psi'(s)a(t)dt < \frac{\Gamma(\alpha)}{6}\varepsilon$$

and

$$\int_L^\infty \psi'(s)(1+\psi_\alpha(t,0))b(t)dt < \frac{\Gamma(\alpha)}{6}\varepsilon,$$

and from $(\mathbf{H_1})$ there exists $m \in \mathbb{N}$ such that, for all $n \geq m$ and $t \in (0,L]$, we have

$$\|f(t,y_n(t)) - f(t,y(t))\| < \frac{\Gamma(\alpha)}{3\psi_1(L,0)}\varepsilon. \tag{2.5}$$

Then for all $t \in (0, \infty)$ and $n > m$, we have

$$\begin{aligned}\frac{\psi_{2-\alpha}(t,0)}{1+\psi_\alpha(t,0)}\|N(y_n)(t)-N(y)(t)\| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)\|f(s,y_n(s))-f(s,y(s))\|ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_t^\infty \psi'(s)\|f(s,y_n(s))-f(s,y(s))\|ds \\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^L \psi'(s)\|f(s,y_n(s))-f(s,y(s))\|ds \\ &+ \frac{2M}{\Gamma(\alpha)}\int_L^\infty \psi'(s)[1+\psi_\alpha(s,0)]b(s)ds \\ &+ \frac{2}{\Gamma(\alpha)}\int_L^\infty \psi'(s)a(s)ds \leq \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.\end{aligned}$$

So,

$$\|Ny_n - Ny\|_\alpha^\psi \to 0 \text{ as } n \to \infty.$$

**Step 3:** $NB$ is equicontinuous on any compact $[c,d]$ of $(0,\infty)$.
Let $y \in B$ and $t_1,\ t_2 \in [c,d]$, where $t_2 > t_1$. Then

$$\begin{aligned}&\left\|\frac{\psi_{2-\alpha}(t_2,0)N(y)(t_2)}{1+\psi_\alpha(t_2,0)} - \frac{\psi_{2-\alpha}(t_1,0)N(y)(t_1)}{1+\psi_\alpha(t_1,0)}\right\| \\ &\leq \frac{\|b\|+M}{\Gamma(\alpha)}\left|\frac{\psi_1(t_2,0)}{1+\psi_\alpha(t_2,0)} - \frac{\psi_1(t_1,0)}{1+\psi_\alpha(t_1,0)}\right| \\ &+ \frac{\|a\|}{\Gamma(\alpha)}\left|\frac{1}{1+\psi_\alpha(t_2,0)} - \frac{1}{1+\psi_\alpha(t_1,0)}\right| \\ &+ \frac{1}{\Gamma(\alpha)}\left\|\int_0^{t_2}\psi'(s)\psi_{\alpha-1}(t_2,s)f(s,y(s))ds \right. \\ &\left. - \int_0^{t_1}\psi'(s)\psi_{\alpha-1}(t_1,s)f(s,y(s))ds\right\| \\ &\leq \frac{\|b\|+M}{\Gamma(\alpha)}\left|\frac{\psi_1(t_2,0)}{1+\psi_\alpha(t_2,0)} - \frac{\psi_1(t_1,0)}{1+\psi_\alpha(t_1,0)}\right| \\ &+ \frac{\|a\|}{\Gamma(\alpha)}\left|\frac{1}{1+\psi_\alpha(t_2,0)} - \frac{1}{1+\psi_\alpha(t_1,0)}\right| \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\psi'(s)[\psi_{\alpha-1}(t_2,s)-\psi_{\alpha-1}(t_1,s)] \\ &\|f(s,y(s))\|ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\psi'(s)\psi_{\alpha-1}(t_2,s)\|f(s,y(s))\|ds\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\|b\|+M}{\Gamma(\alpha)}\left|\frac{\psi_1(t_2,0)}{1+\psi_\alpha(t_2,0)}-\frac{\psi_1(t_1,0)}{1+\psi_\alpha(t_1,0)}\right| \\
&+\frac{\|a\|}{\Gamma(\alpha)}\left|\frac{1}{1+\psi_\alpha(t_2,0)}-\frac{1}{1+\psi_\alpha(t_1,0)}\right| \\
&+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\psi'(s)[\psi_{\alpha-1}(t_2,s)-\psi_{\alpha-1}(t_1,s)]a(s)ds \\
&+\frac{R}{\Gamma(\alpha)}\int_0^{t_1}\psi'(s)[\psi_{\alpha-1}(t_2,s)-\psi_{\alpha-1}(t_1,s)] \\
&(1+\psi_\alpha(s,0))b(s)ds \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\psi'(s)\psi_{\alpha-1}(t_2,s)a(s)ds \\
&+\frac{R}{\Gamma(\alpha)}\int_{t_1}^{t_2}\psi'(s)\psi_{\alpha-1}(t_2,s)(1+\psi_\alpha(s,0))b(s)ds \\
&\leq \frac{\|b\|+M}{\Gamma(\alpha)}\left|\frac{\psi_1(t_2,0)}{1+\psi_\alpha(t_2,0)}-\frac{\psi_1(t_1,0)}{1+\psi_\alpha(t_1,0)}\right| \\
&+\frac{\|a\|}{\Gamma(\alpha)}\left|\frac{1}{1+\psi_\alpha(t_2,0)}-\frac{1}{1+\psi_\alpha(t_1,0)}\right| \\
&+\frac{a^*+b^*R}{\Gamma(\alpha)}\left(\int_0^{t_1}\psi'(s)[\psi_{\alpha-1}(t_2,s)-\psi_{\alpha-1}(t_1,s)]ds\right) \\
&+\frac{a^*+b^*R}{\Gamma(\alpha)}\int_{t_1}^{t_2}\psi'(s)\psi_{\alpha-1}(t_2,s)ds \\
&+\frac{2b^*R}{\Gamma(\alpha)}\left(\int_0^{t_2}\psi'(s)\psi_{\alpha-1}(t_2,s)\psi_\alpha(s,0)ds\right. \\
&\left.-\int_0^{t_1}\psi'(s)\psi_{\alpha-1}(t_1,s)\psi_\alpha(s,0)ds\right) \\
&\leq \frac{\|b\|+M}{\Gamma(\alpha)}\left|\frac{\psi_1(t_2,0)}{1+\psi_\alpha(t_2,0)}-\frac{\psi_1(t_1,0)}{1+\psi_\alpha(t_1,0)}\right| \\
&+\frac{\|a\|}{\Gamma(\alpha)}\left|\frac{1}{1+\psi_\alpha(t_2,0)}-\frac{1}{1+\psi_\alpha(t_1,0)}\right| \\
&+\frac{a^*+b^*R}{\Gamma(1+\alpha)}\left(\psi_\alpha(t_2,0)-\psi_\alpha(t_1,0)-\psi_\alpha(t_2,t_1)\right) \\
&+\frac{a^*+b^*R}{\Gamma(1+\alpha)}\psi_\alpha(t_2,t_1)+\frac{2b^*R\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\psi_{2\alpha}(t_2,t_1),
\end{aligned}
$$

where $a^*=\max\limits_{t\in[c,d]}a(t)$ and $b^*=\max\limits_{t\in[c,d]}b(t)$. As $t_2\to t_1$ the right-hand side of the above inequality tends to zero.

Then $NB$ is equicontinuous on any compact $[c,d]$ of $(0,\infty)$.

**Step 4:** We verify that $N$ satisfies the assumptions of theorem 1.24.

First, we now show that $N$ is defined from $B$ to $B$, Indeed, for any $y\in B$, by above conditions

$(\mathbf{H_2}), (\mathbf{H_4})$ and by according to a little calculation, we have

$$\begin{aligned}
\|\frac{\psi_{2-\alpha}(t,0)N(y)(t)}{1+\psi_\alpha(t,0)}\| &\leq \frac{\|b\|}{\Gamma(\alpha)} + \frac{\|a\|}{\Gamma(\alpha-1)} + \frac{1}{\Gamma(\alpha)}\int_0^\infty \psi'(s)\|f(t,y(t))\|dt \\
&\leq \frac{1}{\Gamma(\alpha)}\left(\|b\| + (\alpha-1)\|a\| + \int_0^\infty \psi'(s)a(s)ds\right. \\
&\left. + R\int_0^\infty \psi'(s)(1+\psi_\alpha(s,0))b(s)ds\right) \\
&< R.
\end{aligned}$$

We put $D = \overline{conv}(NB)$, it is clear that $D$ is a closed, bounded and convex subset of $B$. Knowing that $ND \subset NB \subset D$, then $N$ remains defined from $D$ to $D$. We denote by $\vartheta_{(\alpha,\psi)}$ the Kuratowski measure of non-compactness on $C_{\alpha,\psi}([0,\infty), E)$, we will show the following equality

$$\vartheta_{(\alpha,\psi)}(NV) = \sup\left\{\vartheta\left(\frac{\psi_{2-\alpha}(t,0)NV(t)}{1+\psi_\alpha(t,0)}\right),\ t\in(0,\infty)\right\},\ \text{for all } V\subset D. \tag{2.6}$$

Let us first show that for all $\varepsilon > 0$, there is a real number $T_\infty > 0$ such that, for any $t_1, t_2 \geq T_\infty$ and $y \in V$, we have

$$\|\frac{\psi_{2-\alpha}(t_2,0)Ny(t_2)}{1+\psi_\alpha(t_2,0)} - \frac{\psi_{2-\alpha}(t_1,0)Ny(t_1)}{1+\psi_\alpha(t_1,0)}\| < \varepsilon. \tag{2.7}$$

We have

$$\begin{aligned}
&\|\frac{\psi_{2-\alpha}(t_2,0)N(y)(t_2)}{1+\psi_\alpha(t_2,0)} - \frac{\psi_{2-\alpha}(t_1,0)N(y)(t_1)}{1+\psi_\alpha(t_1,0)}\| \\
&\leq \frac{\|b\|+M}{\Gamma(\alpha)}\left|\frac{\psi_1(t_2,0)}{1+\psi_\alpha(t_2,0)} - \frac{\psi_1(t_1,0)}{1+\psi_\alpha(t_1,0)}\right| \\
&+ \frac{\|a\|}{\Gamma(\alpha)}\left|\frac{1}{1+\psi_\alpha(t_2,0)} - \frac{1}{1+\psi_\alpha(t_1,0)}\right| \\
&+ \frac{1}{\Gamma(\alpha)}\left\|\frac{\psi_1(t_2,0)}{1+\psi_\alpha(t_2,0)} - \frac{\psi_1(t_1,0)}{1+\psi_\alpha(t_1,0)}\right\|\int_0^\infty \psi'(s)\left\|f(s,y(s))\right\|ds.
\end{aligned}$$

We distinguish two cases. If $\lim \psi_1(t,0)) = \infty$, we obtain

$$\lim_{t\to\infty}\frac{\psi_1(t,0))}{1+\psi_\alpha(t,0)} = 0,$$

and

$$\lim_{t\to\infty}\frac{1}{1+\psi_\alpha(t,0)} = 0,$$

then, this shows that

$$\|\frac{\psi_{2-\alpha}(t_2,0)Ny(t_2)}{1+\psi_\alpha(t_2,0)} - \frac{\psi_{2-\alpha}(t_1,0)Ny(t_1)}{1+\psi_\alpha(t_1,0)}\| \to 0 \text{ as } t_1,t_2\to\infty. \tag{2.8}$$

If $\lim \psi_1(t,0)) = l < \infty$, by noticing the inequality

$$\left\|\frac{\psi_1(t_2,0)}{1+\psi_\alpha(t_2,0)} - \frac{\psi_1(t_1,0)}{1+\psi_\alpha(t_1,0)}\right\| \leq \left\|\frac{\psi_1(t_2,0)}{1+\psi_\alpha(t_2,0)} - \frac{l}{1+l^\alpha}\right\| + \left\|\frac{\psi_1(t_1,0)}{1+\psi_\alpha(t_1,0)} - \frac{l}{1+l^\alpha}\right\|,$$

we easily obtain the estimate (2.8). In the same way, we verify that for all $\varepsilon > 0$, there is a real number $0 < T_0 << T_\infty$ such that, for any $t_1, t_2 \leq T_0$ and $y \in V$, we have

$$\left\|\frac{\psi_{2-\alpha}(t_2,0)Ny(t_2)}{1+\psi_\alpha(t_2,0)} - \frac{\psi_{2-\alpha}(t_1,0)Ny(t_1)}{1+\psi_\alpha(t_1,0)}\right\| < \varepsilon. \tag{2.9}$$

We come back to show equality (2.6), we show first

$$\vartheta_{(\alpha,\psi)}(NV) \leq \sup_{(0,\infty)} \vartheta\left(\frac{\psi_{2-\alpha}(t,0)NV(t)}{1+\psi_\alpha(t,0)}\right).$$

Let $NV|_K$ the restriction of $NV$ on the interval $K = [T_0, T_\infty]$ and let $\varepsilon$ be a strictly positive real number, by utilizing Lemma 1.21 and the third step, we get

$$\vartheta_{(\alpha,\psi)}(NV|_K) = \sup_K \vartheta\left(\frac{\psi_{2-\alpha}(t,0)NV(t)}{1+\psi_\alpha(t,0)}\right) \leq \sup_{(0,\infty)} \vartheta\left(\frac{\psi_{2-\alpha}(t,0)NV(t)}{1+\psi_\alpha(t,0)}\right),$$

this implies that there exists a finite partition $NV_i$ of $NV$ so that

$$NV = \cup_i NV_i$$

and

$$diam(NV_i|_K) < \sup_{(0,\infty)} \vartheta\left(\frac{\psi_{2-\alpha}(t,0)NV(t)}{1+\psi_\alpha(t,0)}\right) + \varepsilon, \quad i = 0, 1, \cdots, k. \tag{2.10}$$

Consequently, using inequalities (2.7) and (2.10), we get, for all $Ny_1, Ny_2$ of $NV_i$ and $t \geq T_\infty$

$$\begin{aligned}
&\|\frac{\psi_{2-\alpha}(t,0)Ny_2(t)}{1+\psi_\alpha(t,0)} - \frac{\psi_{2-\alpha}(t,0)Ny_1(t))}{1+\psi_\alpha(t,0)}\| \\
&\leq \|\frac{\psi_{2-\alpha}(t,0)Ny_2(t)}{1+\psi_\alpha(t,0)} - \frac{\psi_{2-\alpha}(T_\infty,0)Ny_2(T_\infty)}{1+\psi_\alpha(T_\infty,0)}\| \\
&+ \|\frac{\psi_{2-\alpha}(T_\infty,0)Ny_2(T_\infty)}{1+\psi_\alpha(T_\infty,0)} - \frac{\psi_{2-\alpha}(T_\infty,0)Ny_1(T_\infty)}{1+\psi_\alpha(T_\infty,0)}\| \\
&+ \|\frac{\psi_{2-\alpha}(t,0)Ny_1(t)}{1+\psi_\alpha(t,0)} - \frac{\psi_{2-\alpha}(T_\infty,0)Ny_2(T_\infty)}{1+\psi_\alpha(T_\infty,0)}\| \\
&< 3\varepsilon + \sup_{(0,\infty)} \vartheta\left(\frac{\psi_{2-\alpha}(t,0)NV(t)}{1+\psi_\alpha(t,0)}\right).
\end{aligned}$$

So,

$$\|\frac{\psi_{2-\alpha}(t,0)Ny_2(t)}{1+\psi_\alpha(t,0)} - \frac{\psi_{2-\alpha}(t,0)Ny_1(t))}{1+\psi_\alpha(t,0)}\| \leq 3\varepsilon + \sup_{(0,\infty)} \vartheta\left(\frac{\psi_{2-\alpha}(t,0)NV(t)}{1+\psi_\alpha(t,0)}\right). \tag{2.11}$$

By the same procedure and using inequalities (2.9) and (2.10), we easily show that the inequality (2.11) is also true for all $Ny_1, Ny_2$ of $NV_i$ and $t \leq T_0$. Then, from (2.10) and (2.11), we obtain

$$diam(NV_i) < \sup_{(0,\infty)} \vartheta\left(\frac{\psi_{2-\alpha}(t,0)NV(t)}{1+\psi_\alpha(t,0)}\right) + 3\varepsilon, \quad i = 0, 1, \cdots, k.$$

Thus,

$$\vartheta_{(\alpha,\psi)}(NV) < \sup_{(0,\infty)} \vartheta\left(\frac{\psi_{2-\alpha}(t,0)NV(t)}{1+\psi_\alpha(t,0)}\right) + 3\varepsilon.$$

Since $\varepsilon$ is arbitrary, this leads us to the result.
Conversely. we show that $\sup_{(0,\infty)} \vartheta\left(\frac{\psi_{2-\alpha}(t,0)NV(t)}{1+\psi_\alpha(t,0)}\right) \leq \vartheta_{(\alpha,\psi)}(NV)$.
According to the definition of Kuratowski MNC, we have, for all $\varepsilon > 0$ we can find a finite partition $NV = \cup_i NV_i$ such that

$$diam(NV_i) < \vartheta_{(\alpha,\psi)}(NV) + \varepsilon,$$

then for all $y_1, y_2 \in V$ and $t \in (0,\infty)$, we obtain

$$\|\frac{\psi_{2-\alpha}(t,0)Ny_2(t)}{1+\psi_\alpha(t,0)} - \frac{\psi_{2-\alpha}(t,0)Ny_1(t)}{1+\psi_\alpha(t,0)}\| \leq \|Ny_2 - Ny_1\|_\alpha^\psi < \vartheta_{(\alpha,\psi)}(NV) + \varepsilon.$$

According to $NV(t) = \cup_i NV_i(t)$, we get

$$\vartheta\left(\frac{\psi_{2-\alpha}(t,0)NV(t)}{1+\psi_\alpha(t,0)}\right) < \vartheta_{(\alpha,\psi)}(NV) + \varepsilon,$$

since $\varepsilon$ is arbitrary, we then have

$$\vartheta\left(\frac{\psi_{2-\alpha}(t,0)NV(t)}{1+\psi_\alpha(t,0)}\right) \leq \vartheta_{(\alpha,\psi)}(NV).$$

So,

$$\sup_{(0,\infty)} \vartheta\left(\frac{\psi_{2-\alpha}(t,0)NV(t)}{1+\psi_\alpha(t,0)}\right) \leq \vartheta_{(\alpha,\psi)}(NV).$$

That's all he would like to show.
Next, it remains to prove that $N$ is a Meir-Keeler condensing operator via the measure of non-compactness $\vartheta_{(\alpha,\psi)}$, this is equivalent to demonstrating the following implication

$$\forall \epsilon > 0, \exists \varrho(\epsilon) : \epsilon \leq \vartheta_{(\alpha,\psi)}(V) < \epsilon + \varrho \Longrightarrow \vartheta_{(\alpha,\psi)}(NV) < \epsilon, \text{ for any } V \subset D. \tag{2.12}$$

Let $\epsilon$ be a strictly positive real, $V \subset D$ and $t \in (0,\infty)$, for all $\iota, \kappa \in \mathbb{R}_+^*$ verifying $0 < \iota \leq t \leq \kappa$, we define the auxiliary operator $N_{\iota,\kappa}$ by

$$N_{\iota,\kappa}y(t) = \frac{b\psi_{\alpha-1}(t,0)}{\Gamma(\alpha)} + \frac{a\psi_{\alpha-2}(t,0)}{\Gamma(\alpha-1)} - \frac{\psi_{\alpha-1}(t,0)}{\Gamma(\alpha)}\int_t^\kappa \psi'(s)f(s,y(s)ds + \frac{1}{\Gamma(\alpha)}\int_\iota^t \psi'(s)[\psi_{\alpha-1}(t,s) - \psi_{\alpha-1}(t,0)]f(s,y(s)ds.$$

Using the properties of $\vartheta$, we get

$$\vartheta\left(\frac{\psi_{2-\alpha}(t,0)N_{\iota,\kappa}V(t)}{1+\psi_\alpha(t,0)}\right) \to \left(\frac{\psi_{2-\alpha}(t,0)NV(t)}{1+\psi_\alpha(t,0)}\right) \text{ as } \iota \to 0 \text{ and } \kappa \to \infty. \tag{2.13}$$

An argument similar to that of third step, we show that the $N_{\iota,\kappa}V$ is equicontinuous and bounded on $[\iota,\kappa]$. From Lemmas 1.19, 1.20,1.21, ($\mathbf{H_3}$) and the previous steps, we have, there exists a sequence $\{\mu_n\}_{n=0}^\infty \subset V$ such that

$$\begin{aligned}\vartheta\left(\frac{\psi_{2-\alpha}(t,0)N_{\iota,\kappa}V(t)}{1+\psi_\alpha(t,0)}\right) &\le \frac{\epsilon}{2} + \frac{1}{\Gamma(\alpha)}\vartheta\left\{\int_t^\kappa \psi'(s)f(s,\mu_n(s))ds, n\in\mathbb{N}\right\}\\ &+ \frac{1}{\Gamma(\alpha)}\vartheta\left\{\int_\iota^t \psi'(s)f(s,\mu_n(s))ds, n\in\mathbb{N}\right\}\\ &\le \frac{\epsilon}{2} + \frac{1}{\Gamma(\alpha)}\int_\iota^\kappa \psi'(s)\vartheta\left\{f(s,\mu_n(s)), n\in\mathbb{N}\right\}ds\\ &\le \frac{\epsilon}{2} + \frac{\vartheta_{(\alpha,\psi)}(V)}{\Gamma(\alpha)}\int_0^\infty \psi'(s)[1+\psi_\alpha(s,0)]\ell(s)ds.\end{aligned}$$

From (2.13), we know that

$$\vartheta_{(\alpha,\psi)}(NV) \le \frac{\epsilon}{2} + \frac{\vartheta_{(\alpha,\psi)}(V)}{\Gamma(\alpha)}\int_0^\infty \psi'(s)[1+\psi_\alpha(s,0)]\ell(s)ds.$$

If

$$\vartheta_{(\alpha,\psi)}(NV) \le \frac{\epsilon}{2} + \frac{\vartheta_{(\alpha,\psi)}(V)}{\Gamma(\alpha)}\int_0^\infty \psi'(s)[1+\psi_\alpha(s,0)]\ell(s)ds < \epsilon,$$

this implies that

$$\vartheta_{(\alpha,\psi)}(V) < \frac{\Gamma(\alpha)}{2\int_0^\infty \psi'(s)[1+\psi_\alpha(s,0)]\ell(s)ds}\epsilon,$$

so that implication (2.12) is fulfilled, we take

$$\varrho = \frac{\Gamma(\alpha) - 2\int_0^\infty \psi'(s)[1+\psi_\alpha(s,0)]\ell(s)ds}{2\int_0^\infty \psi'(s)[1+\psi_\alpha(s,0)]\ell(s)ds}\epsilon.$$

So, $N$ is a Meir-Keeler condensing operator via $\vartheta_{(\alpha,\psi)}$, finally all the hypotheses of the theorem 1.24 are fulfilled, which ensures us that the solution sets of problem (2.1) – (2.3 is nonempty and compact. □

## 2.4 Example

As an application of our results we consider the following fractional differential equation.

$$\begin{cases} {}^{RL}\mathcal{D}^{\frac{3}{2},\psi}y(t) = \left(\dfrac{\sqrt{\psi_{0.5}(t,0)}y_n(t)}{1+\psi_{1.5}(t,0)} + \dfrac{\sin(t)}{1+e^{2t}}\right)_{n=1}^{\infty}, \quad t \in (0,+\infty), & (2.14) \\ {}^{RL}\mathfrak{I}_{0^+}^{\frac{1}{2},\psi}y(0) = (1,0,\ldots,0,\ldots), & (2.15) \\ {}^{RL}\mathcal{D}_{0^+}^{\frac{1}{2},\psi}y(\infty) = (1,0,\ldots,0,\ldots). & (2.16) \end{cases}$$

where $\psi(t) = -\arctan(\frac{1}{1+t})$, this implies that

$$\psi'(t) = \frac{1}{1+(1+t)^2}$$

and

$$\psi_\eta(t,0) = [\psi(t) + \frac{\pi}{4}]^\eta.$$

Let

$$E = \{(y_1, y_2, \ldots, y_n, \ldots) : \sup_n |y_n| < \infty\},$$

with the norm $\|y\| = \sup_n |y_n|$, then $(E, \|.\|)$ consists a Banach space, by comparing with the $(2.1)-(2.3)$, we notice that

$$\alpha = 1.5$$

and

$$f(t, y(t)) = (f(t, y_1(t)), \ldots, f(t, y_n(t)), \ldots),$$

where

$$f(t, y_n(t)) = \frac{\sqrt{\psi_{0.5}(t,0)}y_n(t)}{1+\psi_{1.5}(t,0)} + \frac{\sin(t)}{1+e^{2t}}, \quad n \in \mathbb{N}^*.$$

We shall verify the conditions $(\mathbf{H_1})$ and $(\mathbf{H_2})$. Evidently, $f$ is continuous function in $(0,\infty)\times E$ and

$$\|f(t, y(t))\| \leq \frac{\sqrt{\psi_{0.5}(t,0)}}{1+\psi_{1.5}(t,0)}\|y(t)\| + \frac{1}{1+e^{2t}}.$$

With the aid of simple computation we find that

$$\int_0^\infty \psi'(t)b(t)[1+\psi_{1.5}(t,0)]dt = \int_0^\infty \frac{dt}{1+(1+t)^2} = \frac{\pi}{4} < \Gamma(1.5) \text{ and}$$

$$\int_0^\infty \psi'(t)a(t)dt = \int_0^\infty \frac{dt}{(1+e^{2t})(1+(1+t)^2)} \leq \frac{\pi}{2} < \infty.$$

Finally, we verify condition $(H_3)$. For any bounded set $\Omega \subset \mathcal{C}_{\alpha,\psi}((0,\infty), E)$, we have

$$f(t, \Omega(t)) = \frac{\sqrt{\psi_{0.5}(t,0)}}{1+\psi_{1.5}(t,0)}\Omega(t) + \left\{\frac{\sin(t)}{1+e^{2t}}\right\}.$$

Then

$$\vartheta(f(t,\Omega(t)) \leq \frac{\sqrt{\psi_{0.5}(t,0)}}{1+\psi_{1.5}(t,0)}\vartheta(\Omega(t)).$$

Since $\int_0^\infty \psi'(t)\ell(t)[1+\psi_{1.5}(t,0)]dt \leq \frac{\Gamma(1.5)}{2}$, we conclude that condition $(\mathbf{H_3})$ is satisfied. Therefore, Theorem 2.4 ensures that the solution sets of problem (2.14)-(2.16) is nonempty and compact.

# Chapter 3

# Non-local Fractional Differential Equation On The Half Line in Banach Space

## 3.1 Introduction

In this Chapter we deal with the existence of solution sets and its topological structure for fractional differential equations on unbounded domain with the non-local conditions. We consider the following non-local boundary-value problem :

$$\begin{cases} {}^{\mathcal{RL}}\mathcal{D}^{\alpha}_{0^+}y(t) = f(t, y(t)),\ t \in J = (0, +\infty), & (3.1)\\ \mathcal{I}^{2-\alpha}_{0^+}y(0^+) = \sum_{i=1}^{m} \lambda_i y(\tau_i), & (3.2)\\ {}^{\mathcal{RL}}\mathcal{D}^{\alpha-1}_{0^+}y(\infty) = y_\infty, & (3.3) \end{cases}$$

where ${}^{\mathcal{RL}}\mathcal{D}^{\alpha}_{0^+}$ denotes the Riemann-Liouville fractional derivative of order $\alpha$, $1 < \alpha \leq 2$. The operator $\mathcal{I}^{2-\alpha}_{0^+}$ denotes the Riemann-Liouville fractional integral, the state $y(\cdot)$ takes values in a Banach space $E$, $f : (0, \infty) \times E \to E$ will be specified. $\tau_i$, $i = 1, 2, \ldots, m$ are pre-fixed points satisfying $0 < \tau_1 \leq \cdots \leq \tau_m$, $\lambda_i \in \mathbb{R}^*_+$ and

$$\Gamma(\alpha - 1) \neq \sum_{i=1}^{m} \lambda_i \tau_i^{\alpha-2},\ \text{where}\ \Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1}e^{-t}dt. \tag{3.4}$$

## 3.2 Background and basic results

Let $I \subset J = (0, \infty)$ be a compact interval and denote by $C(I, E)$ the Banach space of continuous functions $y : I \to E$ with the usual norm

$$\|y\|_\infty = \sup\{\|y(t)\|,\ t \in I\}.$$

$L^1(J,E)$ the space of $E$ valued Bochner integrable functions on $J$ with the norm

$$\|f\|_{L^1} = \int_0^{+\infty} \|f(t)\|dt.$$

We consider the following Banach space

$$C_\alpha([0,\infty),E) = \{y \in C((0,\infty),E) : \lim_{t\to 0^+} t^{2-\alpha}y(t) \text{ and } \lim_{t\to\infty} \frac{t^{2-\alpha}y(t)}{1+t^\alpha} \text{ exist and are finite}\}.$$

A norm in this space is given by

$$\|y\|_\alpha = \sup_{t\in J} \frac{t^{2-\alpha}\|y(t)\|}{1+t^\alpha}.$$

For $y \in C_\alpha([0,\infty),E)$, we define $y_\alpha$ by

$$y_\alpha(t) = \begin{cases} \dfrac{t^{2-\alpha}y(t)}{1+t^\alpha}, & t \in (0,\infty), \\ \lim\limits_{t\to 0} \dfrac{t^{2-\alpha}y(t)}{1+t^\alpha}, & t = 0. \end{cases}$$

$y_\alpha \in C([0,\infty),E)$.

In the sequel we denote

$$T = \frac{1}{\Gamma(\alpha-1) - \sum\limits_{i=1}^{m} \lambda_i \tau_i^{\alpha-2}}.$$

**Definition 3.1.** *A function $y \in \mathcal{C}_\alpha([0,+\infty))$ is said to be a solution of the problem (3.1)-(3.3) if $y$ satisfies the equation ${}^{\mathcal{RL}}\mathcal{D}^\alpha_{0^+}y(t) = f(t,y(t))$ and the conditions* $(3.2)-(3.3)$.

**Lemma 3.2.** *Let $1<\alpha<2$. A function $y$ is a solution of the fractional integral equation*

$$\begin{aligned} y(t) \;&= \frac{y_\infty - \int_0^\infty f(s,y(s))ds}{\Gamma(\alpha)} \left[ t^{\alpha-1} + T\left(\sum_{i=1}^{m} \lambda_i \tau_i^{\alpha-1}\right) t^{\alpha-2}\right] \\ &+ \frac{Tt^{\alpha-2}}{\Gamma(\alpha)} \sum_{i=1}^{m} \lambda_i \int_0^{\tau_i} (\tau_i - s) f(s,y(s))ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,y(s))ds \end{aligned} \tag{3.5}$$

*if and only if $y$ is a solution of the problem :*

$$\begin{cases} {}^{\mathcal{RL}}\mathcal{D}^\alpha_{0^+}y(t) = f(t,y(t)), \quad t \in J = (0,+\infty), & (3.6) \\ \mathcal{I}^{2-\alpha}_{0^+}y(0^+) = \sum\limits_{i=1}^{m} \lambda_i y(\tau_i), & (3.7) \\ {}^{\mathcal{RL}}\mathcal{D}^{\alpha-1}_{0^+}y(\infty) = y_\infty. & (3.8) \end{cases}$$

*Proof.* Assume that $y$ satisfies the problem (3.6)-(3.8). We may apply Lemma 1.20 to reduce equation (3.6) to an equivalent integral equation

$$y(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \mathcal{I}^{\alpha}_{0^+} f(t, y(t)), \tag{3.9}$$

for some $c_1, c_2 \in \mathbb{R}$. Applying $\mathcal{I}^{2-\alpha}_{0^+}$ to both sides of (3.9), we have

$$\mathcal{I}^{2-\alpha}_{0^+} y(t) = c_1 \mathcal{I}^{2-\alpha}_{0^+} t^{\alpha-1} + c_2 \mathcal{I}^{2-\alpha}_{0^+} t^{\alpha-2} + \mathcal{I}^{2-\alpha}_{0^+} \mathcal{I}^{\alpha}_{0^+} f(t, y(t)).$$

From Remark 1.7, we then get

$$\mathcal{I}^{2-\alpha}_{0^+} y(t) = \frac{c_1 \Gamma(\alpha)}{\Gamma(2)} t + c_2 \Gamma(\alpha-1) + \frac{1}{\Gamma(2)} \int_0^t (t-s) f(s, y(s)) ds.$$

Taking $t \longrightarrow 0$, we obtain

$$c_2 = \frac{\mathcal{I}^{2-\alpha}_{0^+} y(0^+)}{\Gamma(\alpha-1)}.$$

Applying ${}^{\mathcal{RL}}\mathcal{D}^{\alpha-1}_{0^+}$ to both sides of (3.9), we obtain

$${}^{\mathcal{RL}}\mathcal{D}^{\alpha-1}_{0^+} y(t) = c_1 {}^{\mathcal{RL}}\mathcal{D}^{\alpha-1}_{0^+} t^{\alpha-1} + c_2 {}^{\mathcal{RL}}\mathcal{D}^{\alpha-1}_{0^+} t^{\alpha-2} + {}^{\mathcal{RL}}\mathcal{D}^{\alpha-1}_{0^+} \mathcal{I}^{\alpha}_{0^+} f(t, y(t)).$$

From Remark 1.7 and Remark 1.8, we get

$${}^{\mathcal{RL}}\mathcal{D}^{\alpha-1}_{0^+} y(t) = c_1 \Gamma(\alpha) + \frac{1}{\Gamma(1)} \int_0^t f(s, y(s)) ds.$$

Hence

$$c_1 = \frac{1}{\Gamma(\alpha)} \left[ y_\infty - \int_0^\infty f(s, y(s))) ds \right].$$

Thus, we have

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\alpha)} \left[ y_\infty - \int_0^\infty f(s, y(s))) ds \right] t^{\alpha-1} + \frac{\mathcal{I}^{2-\alpha}_{0^+} y(0^+)}{\Gamma(\alpha-1)} t^{\alpha-2} \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) ds. \end{aligned} \tag{3.10}$$

Next, we substitute $t$ by $\tau_i$ into the above equation,

$$\begin{aligned} y(\tau_i) &= \tfrac{1}{\Gamma(\alpha)} \left[ y_\infty - \int_0^\infty f(s, y(s))) ds \right] \tau_i^{\alpha-1} + \frac{\mathcal{I}^{2-\alpha}_{0^+} y(0^+)}{\Gamma(\alpha-1)} \tau_i^{\alpha-2} \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} f(s, y(s)) ds, \end{aligned}$$

by multiplying both sides of the equality by $\lambda_i$, we obtain

$$\begin{aligned} \lambda_i y(\tau_i) &= \frac{1}{\Gamma(\alpha)} \left[ y_\infty - \int_0^\infty f(s, y(s))) ds \right] \lambda_i \tau_i^{\alpha-1} + \frac{\mathcal{I}^{2-\alpha}_{0^+} y(0^+)}{\Gamma(\alpha-1)} \lambda_i \tau_i^{\alpha-2} \\ &+ \frac{\lambda_i}{\Gamma(\alpha)} \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} f(s, y(s)) ds. \end{aligned}$$

From (3.2), we have

$$\mathcal{I}_{0^+}^{2-\alpha}y(0^+) = \frac{1}{\Gamma(\alpha)}\left[y_\infty - \int_0^\infty f(s,y(s)))ds\right]\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-1} + \frac{\mathcal{I}_{0^+}^{2-\alpha}y(0^+)}{\Gamma(\alpha-1)}\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-2}$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}f(s,y(s))ds,$$

which implies

$$\begin{aligned}\mathcal{I}_{0^+}^{2-\alpha}y(0^+) &= \frac{T}{\alpha-1}\Bigg[(y_\infty - \int_0^\infty f(s,y(s)))ds)\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-1} \\ &+\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}f(s,y(s))ds.\Bigg]\end{aligned} \tag{3.11}$$

Substituting (3.11) into (3.10), we derive that (3.5).
Conversely, assume that $y$ satisfies the integral equation (3.5). Applying $\mathcal{I}_{0^+}^{2-\alpha}$ to both sides of (3.5) and using Remark 1.7, we have

$$\mathcal{I}_{0^+}^{2-\alpha}y(t) = \left(y_\infty - \int_0^\infty f(s,y(s)))ds\right)\left(t + \frac{T}{\alpha-1}\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-1}\right)$$
$$+\frac{T}{\alpha-1}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}f(s,y(s))ds + \mathcal{I}_{0^+}^{2}f(t,y(t))).$$

As $t \longrightarrow 0$, we get

$$\mathcal{I}_{0^+}^{2-\alpha}y(0^+) = \frac{T}{\alpha-1}(\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})\left(y_\infty - \int_0^\infty f(s,y(s)))ds\right)$$
$$+\frac{T}{\alpha-1}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}f(s,y(s))ds,$$

$t$ by $\tau_i$ into (3.5), we have

$$y(\tau_i) = \frac{y_\infty - \int_0^\infty f(s,y(s)))ds}{\Gamma(\alpha)}\left(\tau_i^{\alpha-1} + T(\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})\tau_i^{\alpha-2}\right)$$
$$+\frac{T\tau_i^{\alpha-2}}{\Gamma(\alpha)}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}f(s,y(s))ds + \frac{1}{\Gamma(\alpha)}\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}f(s,y(s))ds.$$

So, we derive

$$\sum_{i=1}^{m}\lambda_i y(\tau_i) = \frac{y_\infty - \int_0^\infty f(s,y(s)))ds}{\Gamma(\alpha)}\left(\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-1} + T(\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-2}\right)$$

$$
\begin{aligned}
&+\frac{T\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-2}}{\Gamma(\alpha)}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}f(s,y(s))ds\\
&+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}f(s,y(s))ds\\
&=\frac{(y_\infty-\int_0^\infty f(s,y(s)))ds)\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1}}{\Gamma(\alpha)}\left(1+T\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-2}\right)\\
&+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}f(s,y(s))ds\left(1+T\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-2}\right).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\sum_{i=1}^{m}\lambda_i y(\tau_i)=&\frac{T}{\alpha-1}(\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})\left(y_\infty-\int_0^\infty f(s,y(s)))ds\right)\\
&+\frac{T}{\alpha-1}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}f(s,y(s))ds=\mathcal{I}_{0^+}^{2-\alpha}y(0^+)
\end{aligned}
$$

Now by applying ${}^{\mathcal{RL}}\mathcal{D}_{0^+}^{\alpha-1}$ to both sides of (3.5) and using Remark 1.7, Remark 1.8, we have

$$
{}^{\mathcal{RL}}\mathcal{D}_{0^+}^{\alpha-1}y(t)=y_\infty-\int_0^\infty f(s,y(s)))ds+\mathcal{I}_{0^+}^{1}f(t,y(t)).
$$

Let $t\longrightarrow\infty$, then we get

$$
{}^{\mathcal{RL}}\mathcal{D}_{0^+}^{\alpha-1}y(\infty)=y_\infty.
$$

Next, by applying ${}^{\mathcal{RL}}\mathcal{D}_{0^+}^{\alpha}$ to both sides of (3.5) and using Remark 1.7, Remark 1.8, we obtain

$$
{}^{\mathcal{RL}}\mathcal{D}_{0^+}^{\alpha}y(t)=f(t,y(t)).
$$

Which ends the proof. □

Now, we are in a position to give the main result of this work. Let

$$
B=\{y\in\mathcal{C}_\alpha([0,\infty),E):\|y\|_\alpha\le R\}.
$$

**Remark 3.3.** *We can write Equation (3.5) in the following form,*

$$
\begin{aligned}
y(t)=&\frac{y_\infty}{\Gamma(\alpha)}\left[t^{\alpha-1}+T(\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})t^{\alpha-2}\right]+\frac{Tt^{\alpha-2}}{\Gamma(\alpha)}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}f(s,y(s))ds\\
&-\frac{1}{\Gamma(\alpha)}\int_0^t[t^{\alpha-1}+Tt^{\alpha-2}\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-1}-(t-s)^{\alpha-1}]f(s,y(s))ds\\
&-\frac{1}{\Gamma(\alpha)}\int_t^\infty(t^{\alpha-1}+Tt^{\alpha-2}\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})f(s,y(s))ds.
\end{aligned}
$$

## 3.3 Existence result

Before we give our main result we need the following assumptions.

($\mathbf{H_1}$) There exist nonnegative continuous functions $a, b$ such that

$$\begin{cases} \|f(t,u)\| \leq a(t) + t^{2-\alpha} b(t)\|u\| \,, & \text{for all } t \in J \text{ and } \; u \in E, \\ \int_0^\infty (1+t^\alpha) b(t)dt \leq \dfrac{\Gamma(\alpha)}{3(1+|T|\sum\limits_{i=1}^m \lambda_i \tau_i^{\alpha-1})}, & \\ \int_0^\infty a(t)dt < \infty. & \end{cases}$$

($\mathbf{H_2}$) $f : (0,\infty) \times E \to E$ is a continuous function and for all $x, y$ and $(0,b] \subset (0,\infty)$:

$$\|f(t,x) - f(t,y)\| \leq \alpha t^{2-\alpha}\|x-y\|, \text{ for all } t \in (0,b],$$

with $\alpha \in \mathbb{R}^+$.

($\mathbf{H_3}$) There exists nonnegative function $\ell \in L^1(J, \mathbb{R}^+)$ such that for each non empty, bounded set $\Omega \subset C_\alpha(J,E)$

$$\begin{cases} \gamma(f(t,\Omega(t))) \leq t^{2-\alpha}\ell(t)\gamma(\Omega(t)), & \text{for all } t \in J, \\ \int_0^\infty (1+t^\alpha)\ell(t)dt \leq \dfrac{\Gamma(\alpha)}{4(1+|T|\sum\limits_{i=1}^m \lambda_i \tau_i^{\alpha-1})}. & \end{cases}$$

($\mathbf{H_4}$) There exists strictly positive real number $R$ such that

$$R > \frac{\|y_\infty\| + 3\int_0^\infty a(t)dt}{\dfrac{\Gamma(\alpha)}{(1+|T|(\sum\limits_{i=1}^m \lambda_i \tau_i^{\alpha-1}))} - 3\int_0^\infty (1+t^\alpha)b(t)dt}.$$

**Theorem 3.4.** *Assume that conditions* ($\mathbf{H_1}$), ($\mathbf{H_2}$), ($\mathbf{H_3}$) *and* ($\mathbf{H_4}$) *are satisfied. Then, the problem (3.1)-(3.3) has at least one solution.*

*Proof.* Let the operator $N : \mathcal{C}_\alpha([0,\infty), E) \to \mathcal{C}_\alpha([0,\infty), E)$ be defined as

$$\begin{aligned} N(y)(t) = {} & \frac{y_\infty}{\Gamma(\alpha)}[t^{\alpha-1} + T(\sum_{i=1}^m \lambda_i \tau_i^{\alpha-1})t^{\alpha-2}] + \frac{Tt^{\alpha-2}}{\Gamma(\alpha)}\sum_{i=1}^m \lambda_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} f(s,y(s))ds \\ & - \frac{1}{\Gamma(\alpha)}\int_0^t [t^{\alpha-1} + T(\sum_{i=1}^m \lambda_i \tau_i^{\alpha-1})t^{\alpha-2} - (t-s)^{\alpha-1}] f(s,y(s))ds \\ & - \frac{1}{\Gamma(\alpha)}\int_t^\infty (t^{\alpha-1} + T(\sum_{i=1}^m \lambda_i \tau_i^{\alpha-1})t^{\alpha-2}) f(s,y(s))ds. \end{aligned}$$

From the definition of the operator $N$ and Lemma 3.2, we see that the fixed points of $N$ are solutions of problem (3.1)-(3.3). For this reason, it suffices to verify the axioms of Theorem 1.24, which is done in four steps.

**Step1:** We start to prove that $N$ is bounded.

Let $y \in C_\alpha([0,\infty),E)$, from $(\mathbf{H_1})$ it is easy to deduce that $Ny \in C_\alpha(J,E)$. Using $(\mathbf{H_1})$, for all $y \in B$ and $t \in (0,\infty)$, we get

$$
\begin{aligned}
\frac{t^{2-\alpha}\|N(y)(t)\|}{1+t^\alpha} &\leq \frac{\|y_\infty\|\left(1+|T|\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1}\right)}{\Gamma(\alpha)} \\
&+ \frac{|T|}{\Gamma(\alpha)}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}\|f(s,y(s))\|ds \\
&+ \frac{2+|T|\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1}}{\Gamma(\alpha)}\int_0^t \|f(s,y(s))\|ds \\
&+ \frac{1+|T|\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1}}{\Gamma(\alpha)}\int_t^\infty \|f(s,y(s))\|ds \\
&\leq \frac{\|y_\infty\|(1+|T|\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})}{\Gamma(\alpha)} + \frac{3|T|\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1}+3}{\Gamma(\alpha)}\int_0^\infty a(t)dt \\
&+ \frac{3|T|\|y\|_\alpha\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1}+3}{\Gamma(\alpha)}\int_0^\infty (1+t^\alpha)b(t)dt.
\end{aligned}
$$

Hence, $N : C_\alpha(J,E) \to C_\alpha(J,E)$ is bounded.

**Step2:** We will show that $N$ is continuous.

Let $\{y_n\}_{n=1}^\infty \subset C_\alpha(J,E)$ and $y \in C_\alpha(J,E)$ such that $y_n \to y$ as $n \to \infty$. Then, $\{y_n\}_{n=1}^\infty$ is a bounded set of $C_\alpha(J,E)$, i.e. there exists $M > 0$ such that $\|y_n\|_\alpha \leq M$, for $n > 1$. We also have by taking the limit that $\|y\|_\alpha \leq M$. In view of condition $(\mathbf{H_1})$, for all $\varepsilon > 0$, there exists $L > \tau_m$ such that

$$
\int_L^\infty a(t)dt < \frac{\Gamma(\alpha)\varepsilon}{3\left[\ |T|\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1}+4\right]},
$$

and

$$
\int_L^\infty (1+t^\alpha)b(t)dt < \frac{\Gamma(\alpha)\varepsilon}{3\left[4|T|\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1}+4\right]M}, \tag{3.12}
$$

and from $(\mathbf{H_2})$, there exists $\widetilde{N} \in \mathbb{N}$ such that, for all $n \geq \widetilde{N}$ and $t \in (0,L]$, we have

$$\|f(t, y_n(t)) - (t, y(t))\| < \frac{\Gamma(\alpha)}{3\left[2|T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1}) + 3\right]L}\varepsilon. \tag{3.13}$$

Therefore, for all $t \in J$ and $n > \widetilde{N}$, we have

$$\begin{aligned}\frac{t^{2-\alpha}}{1+t^\alpha}\|N(y_n)(t) - N(y)(t)\| \leq & \frac{|T|}{\Gamma(\alpha)}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i - s)^{\alpha-1}\|f(s, y_n(s)) - (s, y(s))\|ds \\ & + \frac{2+|T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})}{\Gamma(\alpha)}\int_0^t \|f(s, y_n(s)) - f(s, y(s))\|ds \\ & + \frac{1+|T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})}{\Gamma(\alpha)}\int_t^\infty \|f(s, y_n(s)) - f(s, y(s))\|ds.\end{aligned}$$

If $t \leq L$ and $n > \widetilde{N}$, we have

$$\begin{aligned}\frac{t^{2-\alpha}}{1+t^\alpha}\|N(y_n)(t) - N(y)(t)\| \leq & \frac{|T|}{\Gamma(\alpha)}\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-1}\int_0^L \|f(s, y_n(s)) - f(s, y(s))\|ds \\ & + \frac{3+2|T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})}{\Gamma(\alpha)}\int_0^L \|f(s, y_n(s)) - f(s, y(s))\|ds \\ & + \frac{2+2|T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})}{\Gamma(\alpha)}\int_L^\infty a(s)ds \\ & + \frac{(2+2|T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1}))M}{\Gamma(\alpha)}\int_L^\infty (1+s^\alpha)b(s)ds.\end{aligned}$$

From (3.3) and (3.13), we obtain,

$$\frac{t^{2-\alpha}}{1+t^\alpha}\|N(y_n)(t) - N(y)(t)\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

The case when $t > L$ and $n > \widetilde{N}$ is treated similarly. Thus we conclude that,

$$\|y_n - y\|_\alpha \to 0 \text{ as } n \to \infty.$$

So, $N$ is continuous.

**Step3:** We prove the following results :
**(i)** $NB_\alpha = \{(Ny)_\alpha : y \in B_\alpha\}$ is equicontinuous on any compact $[0, d]$ of $[0, \infty)$.

**(ii)** For given $\varepsilon > 0$, there exists a constant $n_1 > 0$ such that

$$\|\frac{N(y)_\alpha(t_1)}{1+t_1^\alpha} - \frac{N(y)_\alpha(t_2)}{1+t_2^\alpha}\| < \varepsilon,$$

for any $t_1, t_2 \geq n_1$ and $y(.) \in B_\alpha$.
We have, from $(\mathbf{H_1})$ and the boundedness of $B$, there exists $M > 0$ such that

$$\int_0^\infty \|f(t, y(t))\|dt \leq M \text{ for any } y \in B. \tag{3.14}$$

Let us show the equicontinuity of $NB_\alpha$ on any compact $[0, d]$. Indeed, let $y \in B$ and $t_1,\ t_2 \in [0, d]$, where $t_2 > t_1$.Then

$$\begin{aligned}
\|\frac{t_1^{2-\alpha}N(y)(t_1)}{1+t_1^\alpha} - \frac{t_2^{2-\alpha}N(y)(t_2)}{1+t_2^\alpha}\| \leq\ & \frac{\|y_\infty\| + M}{\Gamma(\alpha)}\left(|\frac{t_1}{1+t_1^\alpha} - \frac{t_2}{1+t_2^\alpha}| + |\frac{1}{1+t_1^\alpha} - \frac{1}{1+t_2^\alpha}|\right) \\
& + \left[\frac{|T|}{\Gamma(\alpha)}\sum_{i=1}^m \lambda_i \int_0^{\tau_i}(\tau_i - s)^{\alpha-1}\|f(s, y(s))\|ds\right]\left|\frac{1}{1+t_1^\alpha} - \frac{1}{1+t_2^\alpha}\right| \\
& + \frac{1}{\Gamma(\alpha)}\left\|\int_0^{t_1}(t_1 - s)^{\alpha-1}f(s, y(s))ds \right. \\
& \left. - \int_0^{t_2}(t_2 - s)^{\alpha-1}f(s, y(s))ds\right\| \\
\leq\ & \frac{\|y_\infty\| + M}{\Gamma(\alpha)}\left|\frac{t_1}{1+t_1^\alpha} - \frac{t_2}{1+t_2^\alpha}\right| \\
& + \frac{\|y_\infty\| + |T|(\sum\limits_{i=1}^m \lambda_i \tau_i^{\alpha-1})M + M}{\Gamma(\alpha)}\left|\frac{1}{1+t_1^\alpha} - \frac{1}{1+t_2^\alpha}\right| \\
& + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}|(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}|\|f(s, y(s))\|ds \\
& + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2 - s)^{\alpha-1}\|f(s, y(s))\|ds \\
\leq\ & \frac{\|y_\infty\| + M}{\Gamma(\alpha)}\left|\frac{t_1}{1+t_1^\alpha} - \frac{t_2}{1+t_2^\alpha}\right| \\
& + \frac{\|y_\infty\| + |T|(\sum\limits_{i=1}^m \lambda_i \tau_i^{\alpha-1})M + M}{\Gamma(\alpha)}\left|\frac{1}{1+t_1^\alpha} - \frac{1}{1+t_2^\alpha}\right| \\
& + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}|(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}|a(s)ds \\
& + \frac{R}{\Gamma(\alpha)}\int_0^{t_1}|(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}|(1 + s^\alpha)b(s)ds
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} a(s) ds \\
&+ \frac{R}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} (1 + s^\alpha) b(s) ds \\
&\leq \frac{\|y_\infty\| + M}{\Gamma(\alpha)} \left| \frac{t_1}{1 + t_1^\alpha} - \frac{t_2}{1 + t_2^\alpha} \right| \\
&+ \frac{\|y_\infty\| + |T| (\sum_{i=1}^{m} \lambda_i \tau_i^{\alpha-1}) M + M}{\Gamma(\alpha)} \left| \frac{1}{1 + t_1^\alpha} - \frac{1}{1 + t_2^\alpha} \right| \\
&+ \frac{a^* + b^* R}{\Gamma(\alpha)} \left( \int_0^{t_1} (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} ds \right) \\
&+ \frac{a^* + b^* R}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds \\
&+ \frac{2b^* R}{\Gamma(\alpha)} \left( \int_0^{t_2} (t_2 - s)^{\alpha-1} s^\alpha ds - \int_0^{t_1} (t_1 - s)^{\alpha-1} s^\alpha ds \right) \\
&\leq \frac{\|y_\infty\| + M}{\Gamma(\alpha)} \left| \frac{t_1}{1 + t_1^\alpha} - \frac{t_2}{1 + t_2^\alpha} \right| \\
&+ \frac{\|y_\infty\| + |T| (\sum_{i=1}^{m} \lambda_i \tau_i^{\alpha-1}) M + M}{\Gamma(\alpha)} \left| \frac{1}{1 + t_1^\alpha} - \frac{1}{1 + t_2^\alpha} \right| \\
&+ \frac{a^* + b^* R}{\Gamma(1 + \alpha)} \left( t_2^\alpha - t_1^\alpha - (t_2 - t_1)^\alpha \right) \\
&+ \frac{a^* + b^* R}{\Gamma(1 + \alpha)} (t_2 - t_1)^\alpha + \frac{2b^* R \mathcal{B}(\alpha, \alpha + 1)}{\Gamma(\alpha)} (t_2^{2\alpha} - t_1^{2\alpha}),
\end{aligned}
$$

where $a^* = \max\limits_{t \in [a,b]} a(t)$ and $b^* = \max\limits_{t \in [a,b]} b(t)$. As $t_2 \to t_1$ the right-hand side of the above inequality tends to zero. Then $\left\{ t \longrightarrow \dfrac{t^{2-\alpha} N(y)(t)}{1 + t^\alpha},\ y \in B \right\}$ is equicontinuous on $[0, d]$.

Next, let us show the equiconvergence of $NB_\alpha$. In fact, let $\varepsilon > 0$, we have

$$
\begin{aligned}
\left\| \frac{t_1^{2-\alpha} N(y)(t_1)}{1 + t_1^\alpha} - \frac{t_2^{2-\alpha} N(y)(t_2)}{1 + t_2^\alpha} \right\| &\leq \frac{\|y_\infty\| + M}{\Gamma(\alpha)} \left| \frac{t_1}{1 + t_1^\alpha} - \frac{t_2}{1 + t_2^\alpha} \right| \\
&+ \frac{\|y_\infty\| + |T| (\sum_{i=1}^{m} \lambda_i \tau_i^{\alpha-1}) M + M}{\Gamma(\alpha)} \left| \frac{1}{1 + t_1^\alpha} - \frac{1}{1 + t_2^\alpha} \right| \\
&+ \frac{1}{\Gamma(\alpha)} \left\| \int_0^{t_1} \frac{t_1^{2-\alpha} (t_1 - s)^{\alpha-1}}{1 + t_1^\alpha} f(s, y(s)) ds \right. \\
&\left. - \int_0^{t_2} \frac{t_2^{2-\alpha} (t_2 - s)^{\alpha-1}}{1 + t_2^\alpha} f(s, y(s)) ds \right\|.
\end{aligned}
$$

It suffices to show that

$$\left\|\int_0^{t_1}\frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^\alpha}f(s,y(s))ds-\int_0^{t_2}\frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^\alpha}f(s,y(s))ds\right\|\leq\varepsilon.$$

Relation (3.14) yields that there exits $N_0>0$ such that

$$\int_{N_0}^{\infty}\|f(t,y(t))\|dt\leq\frac{\varepsilon}{3}\text{ for any }y\in B. \tag{3.15}$$

On the other hand, since $\lim_{t\to\infty}\dfrac{t^{2-\alpha}(t-N_0)^{\alpha-1}}{1+t^\alpha}=0$, there exists $N_1>N_0$ such that, for any $t_1,t_2\geq N_1$ and $s\in[0,N_0]$, we have

$$\left|\frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^\alpha}-\frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^\alpha}\right|<\frac{\varepsilon}{3M}. \tag{3.16}$$

Now taking $t_1,t_2\geq N_1$, from (3.15), (3.16), we can arrive at

$$\begin{aligned}&\left\|\int_0^{t_1}\frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^\alpha}f(s,y(s))ds-\int_0^{t_2}\frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^\alpha}f(s,y(s))ds\right\|\\
&\leq\int_0^{N_1}\left|\frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^\alpha}-\frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^\alpha}\right|\|f(s,y(s))\|ds\\
&+\int_{N_1}^{t_1}\frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^\alpha}\|f(s,y(s))\|ds+\int_{N_1}^{t_2}\frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^\alpha}\|f(s,y(s))\|ds\\
&<\frac{\varepsilon}{3M}\int_0^{\infty}\|f(s,y(s))\|ds+2\int_{N_1}^{\infty}\|f(s,y(s))\|ds<\varepsilon.\end{aligned}$$

Thus, $NB_\alpha$ is equiconvergent.

**Step 4:** Now, let us show that $N$ satisfies the assumptions of Theorem. 1.24

First, we now show that $N$ is defined from $B$ to $B$, Indeed, for any $y\in B$, by above conditions $(\mathbf{H_1}),(\mathbf{H_4})$ and according to a little calculation, we have

$$\begin{aligned}\|\frac{t^{2-\alpha}N(y)(t)}{1+t^\alpha}\|\leq&\frac{\|y_\infty\|}{\Gamma(\alpha)}(1+|T|(\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-1}))\\
&+\frac{|T|}{\Gamma(\alpha)}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}\|f(s,y(s))\|ds\\
&+\frac{2+|T|(\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})}{\Gamma(\alpha)}\int_0^{t}\|f(t,y(t))\|dt\\
&+\frac{1+|T|(\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})}{\Gamma(\alpha)}\int_0^{\infty}\|f(t,y(t))\|dt\end{aligned}$$

$$\leq \frac{(1+|T|(\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-1}))}{\Gamma(\alpha)}\Bigg(\|y_\infty\|+3\int_0^\infty a(t)dt$$
$$+3R\int_0^\infty(1+t^\alpha))b(t)dt\Bigg)$$
$$< R.$$

Hence, $\|Ny\|_\alpha \leq R$, we conclude that $N : B \to B$.
We put $D = \overline{conv}(NB)$, it is clear that $D$ is a closed, bounded and convex subset of $B$. As we know that $ND \subset NB \subset D$, then $N$ remains defined from $D$ to $D$.
We denote by $\gamma_\alpha$ the Kuratowski measure of non-compactness on $C_\alpha([0,\infty),E)$. Let us first show that $\gamma_\alpha$ satisfies the following equality

$$\gamma_\alpha(NV) = \sup\left\{\gamma\left(\frac{t^{2-\alpha}NV(t)}{1+t^\alpha}\right),\ t\in(0,\infty)\right\},\ \text{for all } V\subset D. \tag{3.17}$$

**Remark 3.5.** *From the definitions of $C_\alpha([0,\infty),E)$, we see that*

$$\gamma_\alpha(\Omega) = \gamma_\varphi(\Omega_\alpha),\quad \textit{for all bounded subset } \Omega \textit{ of } C_\alpha([0,\infty),E).$$

We show first

$$\gamma_\alpha(NV) \leq \sup_{(0,\infty)}\gamma\left(\frac{t^{2-\alpha}NV(t)}{1+t^\alpha}\right).$$

Let $\varepsilon$ be a strictly positive real number. From the equiconvergence of $NV_\alpha$, there exists $A > 0$ such that

$$\|\frac{t_2^{2-\alpha}Ny(t_2)}{1+t_2^\alpha} - \frac{t_1^{2-\alpha}Ny(t_1)}{1+t_1^\alpha}\| < \varepsilon,\quad t_1,t_2 > A. \tag{3.18}$$

Let $NV_\alpha|_K$ be the restriction of $NV_\alpha$ on the interval $K = [0,A]$, by using Lemma 1.21 and the third step, we get

$$\gamma_\alpha(NV_\alpha|_K) = \sup_K\gamma\left(\frac{t^{2-\alpha}NV(t)}{1+t^\alpha}\right) \leq \sup_{t\in(0,\infty)}\gamma\left(\frac{t^{2-\alpha}NV(t)}{1+t^\alpha}\right),$$

this implies that there exists a finite partition $NV_\alpha^i$ of $NV_\alpha$ so that

$$NV_\alpha = \cup_i NV_\alpha^i$$

and

$$diam(NV_\alpha^i|_K) < \sup_{t\in(0,\infty)}\gamma\left(\frac{t^{2-\alpha}NV(t)}{1+t^\alpha}\right)+\varepsilon,\quad i = 0,1,\cdots,k. \tag{3.19}$$

Consequently, using inequalities (3.18) and (3.19), for all $Ny_1, Ny_2$ of $NV_i$ and $t \geq A$, we have

$$\begin{aligned}\|\frac{t^{2-\alpha}Ny_2(t)}{1+t^\alpha}-\frac{t^{2-\alpha}Ny_1(t))}{1+t^\alpha}\| &\leq \|\frac{t^{2-\alpha}Ny_2(t)}{1+\psi_\alpha(t,0)}-\frac{A^{2-\alpha}Ny_2(A)}{1+A^\alpha}\| \\ &+\|\frac{A^{2-\alpha}Ny_2(A)}{1+A^\alpha}-\frac{A^{2-\alpha}Ny_1(A)}{1+A^\alpha}\| \\ &+\|\frac{t^{2-\alpha}Ny_1(t)}{1+\psi_\alpha(t,0)}-\frac{A^{2-\alpha}Ny_2(A)}{1+A^\alpha}\| \\ &< 3\varepsilon+\sup_{t\in(0,\infty)}\gamma\left(\frac{t^{2-\alpha}NV(t)}{1+t^\alpha}\right).\end{aligned}$$

So,

$$\|\frac{t^{2-\alpha}Ny_2(t)}{1+t^\alpha}-\frac{t^{2-\alpha}Ny_1(t))}{1+t^\alpha}\| \leq 3\varepsilon+\sup_{t\in(0,\infty)}\gamma\left(\frac{t^{2-\alpha}NV(t)}{1+t^\alpha}\right). \tag{3.20}$$

From (3.18) and (3.19), we obtain

$$diam(NV_i)<\sup_{t\in(0,\infty)}\gamma\left(\frac{\psi_t^{2-\alpha}NV(t)}{1+t^\alpha}\right)+3\varepsilon, \quad i=0,1,\cdots,k.$$

Thus,

$$\gamma_\alpha(NV)<\sup_{t\in(0,\infty)}\gamma\left(\frac{t^{2-\alpha}NV(t)}{1+t^\alpha}\right)+3\varepsilon.$$

Since $\varepsilon$ is arbitrary, this leads us to the desired result.
Conversely, we show that

$$\sup_{t\in(0,\infty)}\gamma\left(\frac{t^{2-\alpha}NV(t)}{1+t^\alpha}\right)\leq\gamma_\alpha(NV).$$

According to the definition of Kuratowski MNC, we have, for all $\varepsilon>0$, we can find a finite partition $NV_\alpha=\cup_i NV_\alpha^i$ such that

$$diam(NV_\alpha^i)<\gamma_\alpha(NV)+\varepsilon,$$

then for all $y_1,y_2\in V$ and $t\in(0,\infty)$, we obtain,

$$\|\frac{t^{2-\alpha}Ny_2(t)}{1+t^\alpha}-\frac{t^{2-\alpha}Ny_1(t)}{1+t^\alpha}\|\leq\|Ny_2-Ny_1\|_\alpha<\gamma_\alpha(NV)+\varepsilon.$$

According to $NV_\alpha(t)=\cup_i NV_\alpha^i(t)$, we get

$$\gamma\left(\frac{t^{2-\alpha}NV(t)}{1+t^\alpha}\right)<\gamma_\alpha(NV)+\varepsilon,$$

since $\varepsilon$ is arbitrary, we then have

$$\gamma\left(\frac{t^{2-\alpha}NV(t)}{1+t^\alpha}\right)\leq\gamma_\alpha(NV).$$

So,

$$\sup_{t\in(0,\infty)} \gamma\left(\frac{t^{2-\alpha}NV(t)}{1+t^{\alpha}}\right) \leq \gamma_{\alpha}(NV).$$

Finally we need to prove the following implication

$$\forall\varepsilon > 0,\ \exists\varrho(\varepsilon) : \varepsilon \leq \gamma(V) < \varepsilon + \varrho \Longrightarrow \gamma_{(\alpha,\psi)}(NV) < \varepsilon,\ \text{for any } V \subset D. \tag{3.21}$$

Let $\varepsilon$ be a strictly positive real number, $V \subset D$ and $t \in (0,\infty)$, for all $\kappa \in \mathbb{R}_{+}^{*}$ satisfying $t \leq \kappa$, we define the auxiliary operator $N_\kappa$ by

$$\begin{aligned}
N_\kappa(y)(t) &= \frac{y_\infty}{\Gamma(\alpha)}[t^{\alpha-1} + T(\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})t^{\alpha-2}] + \frac{Tt^{\alpha-2}}{\Gamma(\alpha)}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i - s)^{\alpha-1}f(s,y(s))ds\\
&+ \frac{1}{\Gamma(\alpha)}\int_0^t [t^{\alpha-1} + T(\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})t^{\alpha-2} - (t-s)^{\alpha-1}]f(s,y(s))ds\\
&+ \frac{1}{\Gamma(\alpha)}\int_t^{\kappa}(t-s)^{\alpha-1}f(s,y(s))ds.
\end{aligned}$$

Then from($\mathbf{H_1}$), we obtain

$$\begin{aligned}
\frac{t^{2-\alpha}}{1+t^{\alpha}}\|N_\kappa(y)(t) - N(y)(t)\| &\leq \frac{1+|T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})}{\Gamma(\alpha)}\int_\kappa^\infty \|f(t,y(t))\|dt\\
&\leq \frac{1+|T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})}{\Gamma(\alpha)}\left(\int_\kappa^\infty a(t)dt + R\int_k^\infty(1+t^{\alpha}))b(t)dt\right),
\end{aligned}$$

this shows that

$$H_d\left(\frac{t^{2-\alpha}N_k(V)(t)}{1+t^{\alpha}}, \frac{t^{2-\alpha}N(V)(t)}{1+t^{\alpha}}\right) \to 0,$$

as $k \to \infty,\ t \in J$. Where $H_d$ denotes the Hausdorff metric in space $E$. By the property of non-compactness measure, we get

$$\lim_{\kappa\to\infty}\gamma\left(\frac{t^{2-\alpha}N_\kappa(V)(t)}{1+t^{\alpha}}\right) = \gamma\left(\frac{t^{2-\alpha}N(V)(t)}{1+t^{\alpha}}\right). \tag{3.22}$$

By a similar argument as the one of third step, we show that the $N_\kappa V_\alpha$ is equicontinuous and bounded on $[0,\kappa]$. From the Lemmas 1.19, 1.20,1.21, ($\mathbf{H_3}$) and the previous steps, it follows, that there exists a sequence $\{u_n\}_{n=0}^{\infty} \subset V$ such that

$$\begin{aligned}
\gamma\left(\frac{t^{2-\alpha}N_\kappa V(t)}{1+t^{\alpha}}\right) &\leq \frac{\varepsilon}{2} + \frac{2+2|T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})}{\Gamma(\alpha)}\int_0^\kappa \gamma\left\{f(s,u_n(s)),\ n\in\mathbb{N}\right\}ds\\
&\leq \frac{\varepsilon}{2} + \frac{2+2|T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})}{\Gamma(\alpha)}\int_0^\kappa (1+s^{\alpha})\ell(s)\gamma_\alpha(N(V))ds.
\end{aligned}$$

From (3.22), we know that

$$\gamma\left(\frac{t^{2-\alpha}N(V)(t)}{1+t^{\alpha}}\right) \leq \frac{\varepsilon}{2} + \frac{2[1+|T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})]\gamma_\alpha(N(V))}{\Gamma(\alpha)}\int_0^\infty (1+s^\alpha)\ell(s)ds.$$

Thus,

$$\gamma_\alpha(N(V)) \leq \frac{\varepsilon}{2} + \frac{2[1+|T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})]\vartheta_\alpha(N(V))}{\Gamma(\alpha)}\int_0^\infty (1+s^\alpha)\ell(s)ds.$$

If

$$\gamma_\alpha(N(V)) \leq \frac{\varepsilon}{2} + \frac{2[1+|T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})]\gamma_\alpha(N(V))}{\Gamma(\alpha)}\int_0^\infty (1+s^\alpha)\ell(s)ds < \varepsilon,$$

this implies that

$$\gamma_\alpha(N(V)) < \frac{\Gamma(\alpha)}{4[1+|T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})]\int_0^\infty (1+s^\alpha)\ell(s)ds}\varepsilon,$$

so that implication (3.21) is fulfilled, we take

$$\varrho = \frac{\Gamma(\alpha) - 4[1+|T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})]\int_0^\infty (1+s^\alpha)\ell(s)ds}{4[1+|T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})]\int_0^\infty (1+s^\alpha)\ell(s)ds}\varepsilon.$$

So, $N$ is a Meir-Keeler condensing operator via $\gamma_{(\alpha,\psi)}$, thus all the hypotheses of the Theorem 1.24 are fulfilled. Then, the problem $(3.1)-(3.3)$ is non-empty and compact. □

## 3.4 Example

As an application of our results, we consider the following fractional differential equation.

$$\begin{cases} {}^{\mathcal{RL}}\mathcal{D}^{\frac{3}{2}}y(t) = \left(\dfrac{\sqrt{t}y_n(t)}{(1+t^{\frac{3}{2}})e^{10t}} + \dfrac{\sin(t)}{1+t^2}\right)_{n=1}^{\infty}, \quad t \in J = (0,+\infty), & (3.23)\\ \mathcal{I}_{0^+}^{\frac{1}{2}}y(t) = \dfrac{1}{2}y(1) + y(4), & (3.24)\\ {}^{\mathcal{RL}}\mathcal{D}_{0^+}^{\frac{1}{2}}y(\infty) = y_\infty. & (3.25) \end{cases}$$

Let

$$E = \{(y_1, y_2, \dots, y_n, \dots) : \sup|y_n| < \infty\},$$

with the norm $\|y\| = \sup_n |y_n|$, then $E$ is a Banach space and problem (3.23)-(3.25) can be regaded as an abstract problem (3.1)-(3.3), with

$$\alpha = \frac{3}{2},\ T \simeq 0.5642 \text{ and } f(t, y(t)) = (f(t, y_1(t)), \ldots, f(t, y_n(t)), \ldots),$$

where

$$f(t, y_n(t)) = \frac{\sqrt{t} y_n(t)}{(1 + t^{\frac{3}{2}})e^{10t}} + \frac{\sin(t)}{1+t^2},\ n \in \mathbb{N}^*.$$

We shall verify the conditions $(\mathbf{H_1}) - (\mathbf{H_3})$. Evidently, $f$ is continuous in $J \times E$ and

$$\|f(t, y(t))\| \leq \frac{\sqrt{t}}{(1 + t^{\frac{3}{2}})e^{10t}} \|y(t)\| + \frac{1}{1+t^2}.$$

With the help of simple computation, we find that

$$\int_0^{\infty} e^{-10t} dt = \frac{1}{10} < \frac{\Gamma(\alpha)}{3(1 + |T| \sum_{i=1}^{m} \lambda_i \tau_i^{\alpha-1})} \simeq 0.2451 \text{ and } \int_0^{\infty} \frac{1}{1+t^2} dt = \frac{\pi}{2} < \infty.$$

Finally, we verify condition $(\mathbf{H_3})$. For any bounded set $B \subset E$, we have

$$f(t, B(t)) = \frac{\sqrt{t}}{(1 + t^{\frac{3}{2}})e^{10t}} B(t) + \frac{\sin(t)}{1+t^2}.$$

Then

$$\gamma(f(t, B(t)) \leq \frac{\sqrt{t}}{(1 + t^{\frac{3}{2}})e^{10t}} \gamma(B(t)).$$

Since

$$\int_0^{\infty} e^{-10t} dt = \frac{1}{10} < \frac{\Gamma(\alpha)}{4(1 + |T| \sum_{i=1}^{m} \lambda_i \tau_i^{\alpha-1})} \simeq 0.3676,$$

we conclude that condition $(\mathbf{H_3})$ is satisfied. Therefore, Theorem 3.4 ensures that problem (3.23)-(3.25) is non-empty and compact.

# Chapter 4

# Hybrid Implicit Fractional Integro-differential equations with Hadamard integral boundary conditions

## 4.1 Introduction

We deal in this chapter with the existence of solutions for the problem :

$$\begin{cases} {}_HD^{\alpha}\Big(\dfrac{x(t)}{f(t,x(t),G(t,x(t))}\Big) = g\Big(t, x(t), {}_HD^{\alpha}\Big(\dfrac{x(t)}{f(t,x(t),G(t,x(t)))}\Big)\Big),\ t \in J = (1,T], & (4.1)\\ a_H J^{1-\alpha}x(t)|_{t=1} + b_H J^{1-\alpha}x(t)|_{t=T} = c, & (4.2)\end{cases}$$

where ${}_HD^{\alpha}$ is the Hadamard fractional derivative of order $0 < \alpha \leq 1$, $f : J\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}\setminus\{0\})$, $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $G$ is a nonlinear integral operator defined by

$$G(t, x(t)) = \int_0^t \psi(t,\tau)h(\tau, x(\tau))d\tau,$$

$\psi$ and $h$ are functions that will be specified, ${}_HJ^{\alpha}$ is the Hadamard fractional integral of order $\alpha$, $a, b$ and $c$ are real numbers such that $a + b \neq 0$.

## 4.2 Background and basic results

For $0 < \alpha \leq 1$ let $C_\alpha(J,\mathbb{R})$ denote the weighted space of continuous functions defined by

$$C_\alpha(J,\mathbb{R}) = \{x : J \to \mathbb{R} : (\ln(.))^{1-\alpha}x(.) \in C(J,\mathbb{R})\},$$

with the norm

$$\|x\|_\alpha = \sup_{t\in J}(\ln t)^{1-\alpha}|x(t)|.$$

The$(C_\alpha(J,\mathbb{R}), \|.\|_\alpha)$ is a Banach space.

We introduce the following assumptions :

(1) $h : J \times \mathbb{R} \to \mathbb{R}$ is continuous on J and there exists a positive bounded function $L_h$ with bound $\|L_h\|$ such that For all $x_1,\ x_2 \in C(J, \mathbb{R})$ one has,

$$|h(t, x_1(t)) - h(t, x_2(t))| \leq L_h(t)|x_1(t) - x_2(t)|,$$

and

(2) $\psi \in C(J^2, \mathbb{R})$.

Finally properties of integral operator $G$ are derived,

**Lemma 4.1.** *The integral operator $G$ has the following properties*

*(1) $G : J \times \mathbb{R} \to \mathbb{R}$ is continuous,*
*and*

*(2) For all $x_1,\ x_2 \in C(J, \mathbb{R})$ we have,*

$$\|G(., x_1(.)) - G(., x_2(.)\|_\infty \leq T\ \|L_h\|\|\psi\|\ \|x_1 - x_2\|_\infty.$$

*Proof.* We shall prove that $G$ is continuous.
Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $C(J, \mathbb{R})$. For each $t \in J$, we have

$$\begin{aligned} |G(t, x_1(t)) - G(t, x_2(t)| &= |\int_0^t \psi(t, s)\, [h(s, x_n(s)) - h(s, x(s))]\, ds| \\ &\leq \|L_h\|\|\psi\| \int_0^t |h(s, x_n(s)) - h(s, x(s))| ds \\ &\leq T\|L_h\|\|\psi\|\|x_1 - x_2\|_\infty. \end{aligned}$$

taking the supremum over the both side one has

$$\|G(., x_1(.)) - G(., x_2(.)\|_\infty \leq T\|L_h\|\|\psi\|\|x_1 - x_2\|_\infty,$$

Since $x_n \to x$ in $C(J, \mathbb{R})$ one deduce,

$$\|G(., x_1(.)) - G(., x_2(.)\|_\infty \to 0 \quad \text{as} \quad n \to +\infty.$$

Thus $G$ is continuous.

From the above one has,

$$\|G(., x_1(.)) - G(., x_2(.)\|_\infty \leq T\|L_h\|\|\psi\|\|x_1 - x_2\|_\infty.$$

□

**Definition 4.2.** *A function $x \in C_\alpha(J,\mathbb{R})$ whose $\alpha$-derivative exists on $J$ is said to be a solution of (4.1)-(4.2) if $x$ satisfies the equation*

$$_HD^\alpha\Big(\frac{x(t)}{f(t,x(t),G(t,x(t))}\Big) = g\Big(t, x(t), _H D^\alpha(\frac{x(t)}{f(t,x(t),G(t,x(t)))})\Big),$$

*and condition (4.2).*

As a consequence of Lemma 1.6 and Lemma 1.14 we have the following result which is useful in what follows.

**Lemma 4.3.** *Given $\rho \in C(J,\mathbb{R})$, $x$ is a solution of the the integral equation*

$$x(t) = \frac{c(\ln t)^{\alpha-1}}{\Gamma(\alpha)(a+b)} - \frac{b(\ln t)^{\alpha-1}}{\Gamma(\alpha)(a+b)}\int_1^T \frac{\rho(s)}{s}ds + \frac{1}{\Gamma(\alpha)}\int_1^t \Big(\ln\frac{t}{s}\Big)^{\alpha-1}\frac{\rho(s)}{s}ds \tag{4.3}$$

*if and only if $x$ is a solution of the boundary value problem*

$$_HD^\alpha x(t) = \rho(t), \quad t \in J \tag{4.4}$$

$$_HJ^{1-\alpha}x(t)|_{t=1} + b_H J^{1-\alpha}x(t)|_{t=T} = c. \tag{4.5}$$

*Proof.* Apply $_HJ^\alpha$ to both side of (4.4) we obtain

$$_HJ^\alpha_H D^\alpha x(t) =_H J^\alpha \rho(t),$$

by lemma 1.14 we have,

$$x(t) + d\Big(\ln t\Big)^{\alpha-1} =_H J^\alpha\rho)(t).$$

Hence,

$$x(t) =_H J^\alpha \rho(t) - d\Big(\ln t\Big)^{\alpha-1}. \tag{4.6}$$

Now making use properties 1.13, 1.15 and apply $_HJ^{1-\alpha}$ to both side of (4.6) one obtains,

$$_HJ^{1-\alpha}x(t) =_H J^1\rho(t) - d\ \Gamma(\alpha),$$

let us tend $t$ to 1 one obtains

$$_HJ^{1-\alpha}x(1) = -d\ \Gamma(\alpha),$$

let us again tend $t$ to $T$ one has

$$_HJ^{1-\alpha}x(T) = -d\ \Gamma(\alpha) + \int_1^T \frac{\rho(s)}{s}ds,$$

then

$$d = -\frac{1}{\Gamma(\alpha)(a+b)}\Big(c - b\int_1^T \frac{\rho(s)}{s}ds\Big).$$

Finally

$$x(t) = \frac{c(\ln t)^{\alpha-1}}{\Gamma(\alpha)(a+b)} - \frac{b(\ln t)^{\alpha-1}}{\Gamma(\alpha)(a+b)} \int_1^T \frac{\rho(s)}{s} ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_1^t \Big(\ln \frac{t}{s}\Big)^{\alpha-1} \frac{\rho(s)}{s} ds.$$

Conversely if we apply ${}_H D^{\alpha}$ to both sides of (4.3) it follows,

$${}_H D^{\alpha} x(t) = (\frac{c_H D^{\alpha}(\ln t)^{\alpha-1}}{\Gamma(\alpha)(a+b)} + \frac{b_H D^{\alpha}(\ln t)^{\alpha-1}}{\Gamma(\alpha)(a+b)} \int_1^T \frac{\rho(s)}{s} ds)$$
$$+_H D^{\alpha}(\frac{1}{\Gamma(\alpha)} \int_1^t \Big(\ln \frac{t}{s}\Big)^{\alpha-1} \frac{\rho(s)}{s} ds),$$

one has by using property 1.15

$${}_H D^{\alpha}(\frac{1}{\Gamma(\alpha)} \int_1^t \Big(\ln \frac{t}{s}\Big)^{\alpha-1} \frac{\rho(s)}{s} ds) = \rho(t),$$

in another leg by taking account property 1.13 and bearing in the mind that $\frac{1}{\Gamma(0)} = 0$, we deduce easily ${}_H D^{\alpha}(\log t)^{\alpha-1} = 0$, which yields

$$(\frac{c_H D^{\alpha}(\ln t)^{\alpha-1}}{\Gamma(\alpha)(a+b)} + \frac{b_H D^{\alpha}(\ln t)^{\alpha-1}}{\Gamma(\alpha)(a+b)} \int_1^T \frac{\rho(s)}{s} ds) = 0.$$

Hence

$${}_H D^{\alpha} x(t) = \rho(t).$$

For the second condition we have

$$x(t) = \frac{c(\ln t)^{\alpha-1}}{\Gamma(\alpha)(a+b)} - \frac{(b\ln t)^{\alpha-1}}{\Gamma(\alpha)(a+b)} \int_1^T \frac{\rho(s)}{s} ds + \frac{1}{\Gamma(\alpha)} \int_1^t \Big(\ln \frac{t}{s}\Big)^{\alpha-1} \frac{\rho(s)}{s} ds$$
$$= \frac{(\ln t)^{\alpha-1}}{(a+b)\Gamma(\alpha)}[c - b\int_T^1 \frac{\rho(s)}{s} ds] +_H J^{\alpha}(t)$$

$$J^{1-\alpha} x(t)|_t = \frac{1}{(a+b)}[c - b\textstyle\int_1^T \frac{\rho(s)}{s} ds] +_H J^1(t)$$

$$aJ^{1-\alpha}x(t)|_t + bJ^{1-\alpha}x(t)|_t = \frac{a}{(a+b)}[c - b\int_1^T \frac{\rho(s)}{s} ds]$$
$$+ \frac{b}{(a+b)}[c - b\int_1^T \frac{\rho(s)}{s} ds] + ({}_H J^1 \rho)(T).$$
$$= c$$

Consequently (4.5) yields, proof is complete. □

**Lemma 4.4.** *[49] Given $x \in C(J, \mathbb{R})$, the integral solution of boundary value problem (4.1)-(4.2) is given by*

$$x(t) = f(t, x(t), G(t, x(t)))(\frac{c(\ln t)^{\alpha-1}}{\Gamma(\alpha)(a+b)} - \frac{b(\ln t)^{\alpha-1}}{\Gamma(\alpha)(a+b)} \int_1^T \frac{\varphi(s)}{s} ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{\varphi(s)}{s} ds),$$

*where $\varphi$ is the solution of the implicit equation $\varphi(s) = g(s, x(s), \varphi(s))$.*

## 4.3 Existence result

We shall use Theorem 1.25 to prove our main result.

**Theorem 4.5.** *Assume that the following assumptions hold.*

*(H1) The function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is bounded continuous and there exists a positive bounded functions $\phi_1, \phi_2$ with bound $\|\phi_1\|,\ |\phi_2\|$ such that*

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq \phi_1(t)|u_1 - v_1| + \phi_2(t)|u_2 - v_2|,$$

*for each $t \in J$ and for all $u_1, v_1, u_2, v_2 \in \mathbb{R}$.*

*$(H2)$ There exist a positive constant $k$ and a constant $0 < l < 1$ such that*

$$|g(t, u_2, v_2) - g(t, u_1, v_1)| \leq k\ (\ln t)^{1-\alpha}|u_1 - v_1| + l|u_2 - v_2|,$$

*for each $t \in R,\ u_1,\ v_1,\ u_2,\ v_2 \in \mathbb{R}$.*

*$(H3)$ There exists a constant $r > 0$ such that*

$$\frac{L\ |c|}{\Gamma(\alpha)|a+b|} + \frac{bL \ln T(k\ r + g^*)}{1-l}\Big[\frac{|b|}{\Gamma(\alpha)|a+b|} + \frac{1}{\Gamma(\alpha+1)}\Big] \leq r, \tag{4.7}$$

*and*

$$\frac{|c|}{\Gamma(\alpha)|a+b|}(\|\phi_1\|_\infty + T\|\phi_2\|L_h\|\psi\|) + \frac{\ln T(k\ r + g^*)}{1-l}\Big[\frac{|b|}{\Gamma(\alpha)|a+b|} + \frac{1}{\Gamma(\alpha+1)}\Big] \leq 1, \tag{4.8}$$

*where $|f(t, u, v)| \leq L,\quad \forall (t, u, v) \in J \times \mathbb{R} \times \mathbb{R},\ g^* = \sup_{s \in J} g(s, 0, 0)$.*

*Then the boundary-value problem (4.1)-(4.2) has at least one solution on $J$.*

*Proof.* Notice that the space $C_\alpha(J, \mathbb{R})$ has also structure of Banach algebra, where the multiplication is defined as the usual product of real functions. Set $X = C_\alpha(J, \mathbb{R})$ and define a subset $S$ of $X$ as

$$S = \{x \in X : \|x\|_\alpha \leq r\},$$

where $r$ satisfies inequality (4.7). Clearly $S$ is closed, convex and bounded subset of the Banach space $X$. By Lemma 1.6, the boundary-value problem (4.1)-(4.2) is equivalent to the integral equation

$$\begin{aligned} x(t) &= (f(t,x(t)),G(t,x(t)))\left(\frac{c(\ln t)^{\alpha-1}}{\Gamma(\alpha)(a+b)} - \frac{b(\ln t)^{\alpha-1}}{\Gamma(\alpha)(a+b)}\int_1^T \frac{\varphi(s)}{s}ds \right.\\ &\left. + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{\varphi(s)}{s}ds\right). \end{aligned}$$

For $t \in J$, define two operators $\mathcal{A} : X \to X$ by

$$\mathcal{A}x(t) = f(t,x(t),G(t,x(t))), \quad t \in J,$$

and $\mathcal{B} : S \to X$ by

$$\begin{aligned} \mathcal{B}x(t) = &\frac{c(\ln t)^{\alpha-1}}{\Gamma(\alpha)(a+b)} - \frac{b(\ln t)^{\alpha-1}}{\Gamma(\alpha)(a+b)}\int_1^T \frac{\varphi(s)}{s}ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{\varphi(s))}{s}ds. \end{aligned}$$

Clearly, from Lemma 4.3, fixed points of $\mathcal{AB}$ are solutions to $(4.1)-(4.2)$. We shall show that $\mathcal{A}$, $\mathcal{B}$ satisfies the assumptions of Dhage fixed point theorem 1.25 [39]. The proof will be given in several steps.

For the sake of clarity, we split the proof into a sequence of steps.

**Step 1.**

We first prove that $\mathcal{A}$ is a Lipschitz operator on $X$. Let $x, y \in X$. Then by (H1) and lemma 4.1 we have

$$\begin{aligned} |(\ln t)^{1-\alpha}\mathcal{A}x(t) - (\ln t)^{1-\alpha}\mathcal{A}y(t)| &= (\ln t)^{1-\alpha}|f(t,x(t),G(t,x(t)) - f(t,y(t),G(t,y(t)))| \\ &\leq (\ln t)^{1-\alpha}\phi_1(t)|x(t)-y(t)| \\ &+ (\ln t)^{1-\alpha}\phi_2(t)|G(t,x(t)) - G(t,y(t))| \\ &\leq \|\phi_1\|_\infty\|x-y\|_\alpha + \|\phi_2\|L_h\|\psi\|T\|x-y\|_\alpha \\ &\leq (\|\phi_1\|_\infty + T\|\phi_2\|L_h\|\psi\|)|x-y\|_\alpha. \end{aligned}$$

Finally for all $x, y \in X$.

$$\|\mathcal{A}x - \mathcal{A}y\|_\alpha \leq (\|\phi_1\|_\infty + T\|\phi_2\|L_h\|\psi\|)\|x-y\|_\alpha,$$

which means that $\mathcal{A}$ is a Lipschitz operator on $X$ with Lipschitz constant

$$K = \|\phi_1\|_\infty + T\|\phi_2\|L_h\|\psi\|.$$

and then $(A1)$ of theorem 1.25 holds.

**Step 2.**

The operator $\mathcal{B}$ is completely continuous on $X$.

**Claim1.** $\mathcal{B}$ is continuous

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $X$. Then for $t \in J$ we have

$$\begin{aligned}\mathcal{B}u_n(t) - \mathcal{B}u(t) =& \frac{-b(\ln t)^{\alpha-1}}{\Gamma(\alpha)(a+b)} \int_1^T \frac{\varphi_n(s) - \varphi(s)}{s} ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_1^t \left( \ln \frac{t}{s} \right)^{\alpha-1} \frac{\varphi_n(s) - \varphi(s)}{s} ds,\end{aligned}$$

where

$$\varphi_n(s) = g(s, u_n(s), \varphi_n(s)),$$
$$\varphi(s) = g(s, u(s), \varphi(s)),$$

and

$$|\varphi_n(s) - \varphi(s)| \leq k \ (\ln s)^{1-\alpha} |u_n(s) - u(s)| + l |\varphi_n(s) - \varphi(s)|.$$

Thus

$$\|\varphi_n - \varphi\|_\infty \leq \frac{k}{1-l} \|u_n - u\|_\alpha.$$

On the other hand,

$$\begin{aligned}|(\ln t)^{1-\alpha}\mathcal{B}x_n(t) - (\ln t)^{1-\alpha}\mathcal{B}x(t)| \leq& \frac{|b|}{\Gamma(\alpha)|a+b|} \int_1^T \frac{|\varphi_n(s) - \varphi(s)|}{s} ds \\ &+ \frac{(\ln t)^{1-\alpha}}{\Gamma(\alpha)} \int_1^t \left( \ln \frac{t}{s} \right)^{\alpha-1} \frac{|\varphi_n(s) - \varphi(s)|}{s} ds \\ \leq& \frac{k|b| \ln T}{(1-l)\Gamma(\alpha)|a+b|} \|u_n - u\|_\alpha \\ &+ \frac{k(\ln t)^{1-\alpha}}{(1-l)\Gamma(\alpha)} \|u_n - u\|_\alpha \Big( \int_1^t \left( \ln \frac{t}{s} \right)^{\alpha-1} \frac{1}{s} ds \Big) \\ \leq& \frac{k \ln T}{(1-l)\Gamma(\alpha)} (\frac{|b|}{|a+b|} + 1) \|u_n - u\|_\alpha,\end{aligned}$$

which implies

$$\|\mathcal{B}x_n(t) - \mathcal{B}x(t)\|_\alpha \leq \frac{k \ln T}{(1-l)\Gamma(\alpha)} (\frac{|b|}{|a+b|} + 1) \|u_n - u\|_\alpha.$$

From the fact that $u_n \to u$ one deduce the continuity of $\mathcal{B}$.

**Claim2.**

$\mathcal{B}$ maps bounded sets in S into bounded sets in $X$. Indeed, it is enough to show that for any $B \subset S$, there exists a positive constant $\ell$ such that for each $x \in B \subset S$, we have $\|\mathcal{B}(x)\|_\alpha \leq \ell$. We have for each $t \in J$,

$$\begin{aligned}|(\ln t)^{1-\alpha}\mathcal{B}(x)(t)| \leq& \frac{|c|}{\Gamma(\alpha)|a+b|} + \frac{|b|}{\Gamma(\alpha)|a+b|} \int_1^T \frac{|\varphi(s)|}{s} ds \\ &+ \frac{(\ln t)^{1-\alpha}}{\Gamma(\alpha)} \int_1^t \left( \ln \frac{t}{s} \right)^{\alpha-1} \frac{|\varphi(s)|}{s} ds.\end{aligned}$$

From condition $(H2)$ yields

$$\begin{aligned}|\varphi(s)| &= |g(s,x(s),\varphi(s))|\\ &\leq |g(s,x(s),\varphi(s))-g(s,0,0)|+|g(s,0,0)|\\ &\leq k(\ln t)^{1-\alpha}|x(s)|+l|\varphi(s)|+|g(s,0,0)|\\ &\leq \frac{k(\ln t)^{1-\alpha}|x(s)|+|g(s,0,0)|}{1-l}\\ &\leq \frac{k\|x\|_{\alpha}+g^*}{1-l}\\ &\leq \frac{kr+g^*}{1-l}.\end{aligned}$$

Then

$$\begin{aligned}|(\ln t)^{1-\alpha}\mathcal{B}(x)(t)| &\leq \frac{|c|}{\Gamma(\alpha)|a+b|}+\frac{|b|\ln T}{\Gamma(\alpha)|a+b|}\frac{kr+g^*}{1-l}\\ &+\frac{k\ r+g^*}{1-l}\Big(\frac{(\ln t)^{1-\alpha}}{\Gamma(\alpha)}\int_1^t\Big(\ln\frac{t}{s}\Big)^{\alpha-1}\frac{ds}{s}\Big)\\ &\leq \frac{|c|}{\Gamma(\alpha)|a+b|}+\frac{|b|\ln T}{\Gamma(\alpha)|a+b|}\frac{kr+g^*}{1-l}+\frac{\ln T(k\ r+g^*)}{(1-l)\Gamma(\alpha+1)}.\end{aligned}$$

Thus

$$\|\mathcal{B}(x)\|_{\alpha}\leq \frac{|c|}{\Gamma(\alpha)|a+b|}+\frac{|b|\ln T}{\Gamma(\alpha)|a+b|}\frac{kr+g^*}{1-l}+\frac{\ln T(k\ r+g^*)}{(1-l)\Gamma(\alpha+1)}:=\ell.$$

**Claim3.**

$F$ maps bounded sets into equi-continuous sets of $X$. Let $\tau_1,\tau_2\in J$, $\tau_1<\tau_2$, $B$ be a set such that $B\subset S$ as in Claim 2, and let $x\in B$. Then

$$\begin{aligned}\Big|(\ln t)^{1-\alpha}\mathcal{B}(x)(\tau_2)-(\ln t)^{1-\alpha}\mathcal{B}(x)(\tau_1)\Big| &= \Big|\frac{(\ln\tau_2)^{1-\alpha}}{\Gamma(\alpha)}\int_1^{\tau_2}\Big(\ln\frac{\tau_2}{s}\Big)^{\alpha-1}\frac{\varphi(s)}{s}ds\\ &-\frac{(\ln\tau_2)^{1-\alpha}}{\Gamma(\alpha)}\int_1^{\tau_1}\Big(\ln\frac{\tau_2}{s}\Big)^{\alpha-1}\varphi(s)sds\Big|\\ &+\Big|\frac{(\ln\tau_2)^{1-\alpha}}{\Gamma(\alpha)}\int_1^{\tau_1}\Big(\ln\frac{\tau_2}{s}\Big)^{\alpha-1}\frac{\varphi(s)}{s}ds\\ &-\frac{(\ln\tau_2)^{1-\alpha}}{\Gamma(\alpha)}\int_1^{\tau_1}\Big(\ln\frac{\tau_1}{s}\Big)^{\alpha-1}\frac{\varphi(s)}{s}ds\Big|\\ &+\Big|\frac{(\ln\tau_2)^{1-\alpha}}{\Gamma(\alpha)}\int_1^{\tau_1}\Big(\ln\frac{\tau_1}{s}\Big)^{\alpha-1}\frac{\varphi(s)}{s}ds\Big)\\ &-\frac{(\ln\tau_1)^{1-\alpha}}{\Gamma(\alpha)}\int_1^{\tau_1}\Big(\ln\frac{\tau_1}{s}\Big)^{\alpha-1}\frac{\varphi(s)}{s}ds\Big|\\ &\leq \frac{(\ln\tau_2)^{1-\alpha}}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\Big(\ln\frac{\tau_2}{s}\Big)^{\alpha-1}\frac{|\varphi(s)|}{s}ds\end{aligned}$$

$$
\begin{aligned}
&+\frac{(\ln\tau_2)^{1-\alpha}}{\Gamma(\alpha)}\int_1^{\tau_1}\Big(\ln\frac{\tau_2}{s}\Big)^{\alpha-1}-\Big(\ln\frac{\tau_1}{s}\Big)^{\alpha-1}\frac{|\varphi(s)|}{s}ds\\
&+\frac{(\ln\tau_2)^{1-\alpha}-(\ln\tau_1)^{1-\alpha}}{\Gamma(\alpha)}\int_1^{\tau_1}\Big(\ln\frac{\tau_1}{s}\Big)^{\alpha-1}\frac{|\varphi(s)|}{s}ds\\
&\leq\frac{(\ln\tau_2)^{1-\alpha}(k\ r+g^*)}{(1-l)\Gamma(\alpha+1)}(\ln\frac{\tau_2}{\tau_1})^{\alpha}\\
&+\frac{(\ln\tau_2)^{1-\alpha}(k\ r+g^*)}{(1-l)\Gamma(\alpha+1)}(\ln\tau_2^{\alpha}-\ln\tau_1^{\alpha}+(\ln\frac{\tau_2}{\tau_1})^{\alpha})\\
&+\frac{((\ln\tau_2)^{1-\alpha}-(\ln\tau_1)^{1-\alpha})(k\ r+g^*)}{(1-l)\Gamma(\alpha+1)}(\ln\tau_1)^{\alpha}.
\end{aligned}
$$

As $\tau_1\to\tau_2$, the right-hand side of the above inequality tends to zero. As a consequence of Claims 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $\mathcal{B}:S\to X$ is completely continuous. Hence, condition $(A2)$ holds.

**Step 3.**

Next we shall prove that assumption $(A3)$ of theorem 1.25 is satisfied. Let $x\in X$ and $y\in S$ be arbitrary elements such that $x=\mathcal{A}x\mathcal{B}y$. Then we have

$$
\begin{aligned}
(\ln t)^{1-\alpha}|x(t)| &= (\ln t)^{1-\alpha}|\mathcal{A}x(t)||\mathcal{B}y(t)|\\
&=\Big|f(t,x(t),G(t,x(t)))\Big|\Big|\frac{c}{\Gamma(\alpha)|a+b|}-\frac{|b|}{\Gamma(\alpha)|a+b|}\int_1^T\frac{\varphi(s)}{s}ds\\
&+\frac{(\ln t)^{1-\alpha}}{\Gamma(\alpha)}\int_1^t\Big(\ln\frac{t}{s}\Big)^{\alpha-1}\frac{\varphi(s)}{s}ds\Big|\\
&\leq\frac{L\ |c|}{\Gamma(\alpha)|a+b|}+\frac{L|b|\ln T}{\Gamma(\alpha)|a+b|}\frac{k\ r+g^*}{1-l}+\frac{L\ln T(k\ r+g^*)}{(1-l)\Gamma(\alpha+1)},
\end{aligned}
$$

where $\varphi$ is solution of the implicit equation $\varphi(s)=g(s,y(s),\varphi(s))$ Taking into account condition (H3) inequality (4.7) one obtains,

$$
\begin{aligned}
\|x\|_\alpha &\leq\frac{L\ |c|}{\Gamma(\alpha)|a+b|}+\frac{L\ln T(k\ r+g^*)}{1-l}\Big[\frac{|b|}{\Gamma(\alpha)|a+b|}+\frac{1}{\Gamma(\alpha+1)}\Big]\\
&\leq r.
\end{aligned}
$$

which yields $x\in S$.

**Step 4.**

Now we shall show that $MK\leq 1$, where $M=\|B(S)\|=\sup_{x\in S}|B(x)|$ and $K$ the constant Lipschitz $K=\|\phi_1\|_\infty+T\|\phi_2\|L_h\|\psi\|$, from condition (H2) one obtain,

$$
\begin{aligned}
M=\|B(S)\| &=\sup_{x\in S}\mathcal{B}(x)\\
&\leq\frac{|c|}{\Gamma(\alpha)|a+b|}+\frac{b\ln T}{\Gamma(\alpha)|a+b|}\frac{k\ r+g^*}{1-l}+\frac{\ln T(k\ r+g^*)}{(1-l)\Gamma(\alpha+1)},
\end{aligned}
$$

it follows then from condition $(H_3)$ inequality (4.8)

$$\begin{aligned} M\ K = K\|B(S)\| &= \sup_{x\in S}\|\mathcal{B}x\| \\ &\leq K\frac{|c|}{\Gamma(\alpha)|a+b|} + \frac{b\ln T}{\Gamma(\alpha)|a+b|}\frac{k\ r+g^*}{1-l} + \frac{\ln T(k\ r+g^*)}{(1-l)\Gamma(\alpha+1)} \\ &\leq (\|\phi_1\|_\infty + T\|\phi_2\|) \\ &\left(\frac{|c|}{\Gamma(\alpha)|a+b|} + \frac{b\ln T}{\Gamma(\alpha)|a+b|}\frac{k\ r+g^*}{1-l} + \frac{\ln T(k\ r+g^*)}{(1-l)\Gamma(\alpha+1)}\right) \\ &\leq 1. \end{aligned}$$

Therefore $(A4)$ of theorem 1.25 holds.

Consequently, the operators $\mathcal{A}$ is Lipschitzian $\mathcal{B} : S \to X$ is completely continuous and satisfied condition (A4) of theorem 1.25. As a consequence of Dhage fixed point theorem 1.25 [39], we deduce that $\mathcal{AB}$ has a fixed point $x$ in $S$ which is a solution of problem (4.1)-(4.2), proof is complete. □

**Remark 4.6.** *It is easy for the reader to verify the main result by using a classical fixed point theorem such as Shauder's one. If we check to give existence and uniqueness of the solution for (4.1)-(4.2), Banach fixed point theorem can not be used in the space $C(J,R)$ on account of the unboundedness of the term $\frac{c(\ln t)^{\alpha-1}}{\Gamma(\alpha)(a+b)}$, however we must considered a weight space $C_\alpha(J,R)$ equipped with the norm $\|.\|_\alpha$which is indeed a Banach space.*

**Theorem 4.7.** *Assume that the condition $(H2)$ hold and $f$ is bounded function with $L$ its bound,*

$$\frac{k\ L\ln T}{(1-l)\Gamma(\alpha)}\left(\frac{|b|}{|a+b|}+1\right) < 1, \tag{4.9}$$

*then the problem (4.1)-(4.2) has a unique solution on J.*

*Proof.* We considered the operator $N$ defined by

$$\begin{aligned} Nx(t) &= (f(t,x(t)), G(t,x(t)))\times \\ &\Big(\frac{c(\ln t)^{\alpha-1}}{\Gamma(\alpha)(a+b)} - \frac{b(\ln t)^{\alpha-1}}{\Gamma(\alpha)(a+b)}\int_1^T \frac{g(s,x(s),\varphi(s))}{s}ds \\ &+ .\frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{g(s,x(s),\varphi(s))}{s}ds\Big). \\ &\left|(\ln t)^{1-\alpha}Nx(t) - (\ln t)^{1-\alpha}Ny(t)\right| \\ &= |(f(t,x(t)), G(t,x(t)))\times \\ &\Big(\frac{c}{\Gamma(\alpha)(a+b)} - \frac{b}{\Gamma(\alpha)(a+b)}\int_1^T \frac{g(s,x(s),\varphi(s))}{s}ds \\ &+ .\frac{(\ln t)^{1-\alpha}}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{g(s,x(s),\varphi(s))}{s}ds\Big) \end{aligned}$$

$$
\begin{aligned}
&-(f(t,y(t)),G(t,y(t)))\times\\
&(\frac{c}{\Gamma(\alpha)(a+b)}-\frac{b}{\Gamma(\alpha)(a+b)}\int_1^T\frac{g(s,x(s),\varphi(s))}{s}ds\\
&-.\frac{(\ln t)^{1-\alpha}}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{g(s,y(s),\varphi(s))}{s}ds)\Big|\\
&\leq L(\frac{|b|}{\Gamma(\alpha)|a+b|}\int_1^T\frac{|g(s,x(s),\varphi(s))-g(s,y(s),\varphi(s))}{s}ds\\
&+\frac{(\ln t)^{1-\alpha}}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{|g(s,x(s),\varphi(s))-g(s,y(s),\varphi(s))|}{s}ds)\\
&\leq\frac{k\ L}{(1-l)\Gamma(\alpha)}(\frac{|b|\ln T}{|a+b|}+\ln\ T)\|x-y\|_\alpha.
\end{aligned}
$$

Thus

$$\|Nx-Ny\|_\alpha\leq\frac{k\ L\ln T}{(1-l)\Gamma(\alpha)}(\frac{|b|}{|a+b|}+1)\|x-y\|_\alpha.$$

Consequently by (4.9), N is a contraction. As a consequence of Banach fixed point theorem, we deduce that N has a fixed point which is a solution of the problem (4.1)-(4.2). □

## 4.4 Example

In this section we give an example to illustrate the usefulness of our main result. Let us consider the implicit hybrid fractional boundary value -value problem,

$$
\begin{aligned}
&{}_HD^{\frac{1}{2}}(\frac{x(t)}{f(t,x(t),G(t,x(t)))})=\\
&\frac{e^{-t}\sqrt{\ln T}(|x(t)|+\left|{}_HD^\alpha(\frac{x(t)}{f(t,x(t),G(t,x(t)))})\right|}{(9+e^t)(1+|x(t)|+\left|{}_HD^\alpha(\frac{x(t)}{f(t,x(t),G(t,x(t)))})\right|}\\
&t\in J:=[1,4],\ \alpha=\frac{1}{2},
\end{aligned}
\tag{4.10}
$$

$$ {}_HJ^{1-\alpha}x(t)|_{t=1}+{}_HJ^{1-\alpha}x(t)|_{t=4}=1, \tag{4.11}$$

where,

$$f(t,x(t),G(t,x(t)))=e^{-2\ t}x(t)+\frac{1}{16}\int_0^t(t-\tau)^2e^{-(4\ t+|x(\tau)|)}d\tau$$

$$G(t,x(t))=\frac{1}{16}\int_0^t(t-\tau)^2e^{-(2\ t+|x(\tau)|)}d\tau,$$

$$\psi(t,\tau)=\frac{1}{16}(t-\tau)^2,\ h(t,x(\tau))=e^{-(2\ t+|x(\tau)|)}.$$

It obvious to see that $f$ is a continuous function, let us now verify that $f$ is bounded.

$$\begin{aligned}|f(t,x(t),G(t,x(t)))| =&|e^{-2\ t}x(t)| + \frac{1}{16}\int_0^t (t-\tau)^2|e^{-(4\ t+|x(\tau)|)}|d\tau \\ &\leq \frac{1}{e^2}(\sup_{x\in J}|x(t)| + \frac{\sup_{x\in J} e^{|x(\tau)|)}|}{16}\int_0^t (t-\tau)^2 d\tau)\end{aligned}$$

$$\leq \frac{1}{e^2}(\sup_{x\in J}|x(t)| + \frac{4\sup_{x\in J} e^{|x(\tau)|)}|}{3e^2} := \ell^*.$$

Thus

$$\|f(.,x(.),G(.,x(.)))\|_\infty \leq \ell^*.$$

Set

$$f(t,x,y) = e^{-2\ t}(x+y),$$

$$\Big|f(t,x,y) - f(t,\overline{x},\overline{y})\Big| \leq e^{-2t}(\Big|x-\overline{x}\Big| + \Big|y-\overline{y}\Big|).$$

Thus

$$\Big|f(t,x,y) - f(t,\overline{x},\overline{y})\Big| \leq \phi_1(t)\Big|x-\overline{x}\Big| + \phi_2(t)\Big|y-\overline{y}\Big|,$$

where

$$\phi_1(t) = e^{-2\ t},\ \phi_2(t) = e^{-2\ t}.$$

It easily to verify the properties concerning the nonlinear integral operator $G$.

(i) $G$ is a continuous function,

(ii) $\|G(.,x_1) - G(.,x_2)\|_\infty \leq T\ \|\psi\|\|L_h\|\|x_1-x_2\|_\infty$

Hence $(H1)$ is satisfied. In another hand, set

$$g(t,x,y) = \frac{e^{-t}\sqrt{\ln t}(x+y)}{(9+e^t)(1+x+y)}, \quad (t,x,y) \in J\times[0,\infty)\times[0,+\infty).$$

It clear that $g$ is continuous, let $x,y\in R$ and $t\in J$ then

$$\begin{aligned}|g(t,x,y) - g(t,\overline{x},\overline{y})| &= \frac{e^{-t}\sqrt{\ln t}}{(9+e^t)}\Big|\frac{x+y}{1+x+y} - \frac{\overline{x}+\overline{y}}{1+\overline{x}+\overline{y}}\Big| \\ &= \frac{e^{-t}\sqrt{\ln t}}{(9+e^t}\Big|\frac{1}{1+\overline{x}+\overline{y}} - \frac{1}{1+x+y}\Big| \\ &\leq \frac{e^{-t}\sqrt{\ln t}}{(9+e^t)}\Big|x+y-\overline{x}-\overline{y}\Big| \\ &\leq \frac{1}{10}\sqrt{\ln t}(\Big|x-\overline{x}\Big|) + \frac{\sqrt{2\ln 2}}{10}(\Big|y-\overline{y}\Big|).\end{aligned}$$

Hence the condition $(H2)$ holds with $k = \frac{1}{10},\ l = \frac{\sqrt{2\ln 2}}{10}$. We shall check that condition $(H3)$ is satisfied. Indeed, since

$$L = \ell^*,\ g^* = 0,\ k = l = \frac{1}{10},\ \Gamma(\frac{1}{2}) = \sqrt{\pi},\ \Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi},$$

(4.7) becomes

$$\begin{aligned} &\ell^*(\frac{1}{2\Gamma(\frac{1}{2})} + \frac{\frac{\ln 4}{10}\ r}{\frac{10-\sqrt{2\ln 2}}{10}}\Big[\frac{1}{2\Gamma(\frac{1}{2})} + \frac{1}{\Gamma(\frac{3}{2})}\Big]) \\ &= \ell^*(\frac{1}{2\Gamma(\frac{1}{2})} + \frac{5r\ln 2}{\sqrt{\pi}(10-\sqrt{\ln 2})}\ ) \\ &\leq r. \end{aligned} \tag{4.12}$$

Since

$$\|\phi_1\|_\infty = \|\phi_2\| = \frac{1}{e^2},\ \|L_h\| = \frac{1}{e},\ \|\psi\| = 1,$$

(4.8) becomes then,

$$\frac{4+e}{e^3}(\frac{1}{2\Gamma(\frac{1}{2})} + \frac{5r\ln 2}{\sqrt{\pi}(10-\sqrt{2\ln 2})}) \leq 1. \tag{4.13}$$

Then by Theorem (4.5) the problem (4.10)-(4.11) has at least one solution on $[1,4]$ for values of $r$ and $\ell^*$ satisfying (4.12), (4.13).

# Conclusion

In this thesis we deal with the study of some class of fractional differential equations using measure of non-compactness, it contains four chapters, in the first chapter we introduce notations, definitions, fixed point theorems and preliminary facts from analysis tools which are used throughout this thesis.
In the second one we are concerned with the study of existence of solution sets and its topological structure for some fractional differential equation with $\psi$ Riemman Liouville derivative on an unbounded domain, which implies a lack of compactness, we avoid this obstruction by using a special Banach space. We prove that this constructed space is in a natural way, in the sense that, one recover the characterization of the relatively compact subset in the space $C(J,E)$ when J is compact. Our main result is based on tools from classical functionnal analysis and Meir-Keeler condensing operators combined with measure of non-compactness.
In the third chapter we deal with the problem concerning existence of solution sets and its topological structure for non-local Riemman-Liouville fractional differential equation modeled by equation (3.1)-(3.3) on the half line with Riemann-Liouville fractional integral and derivative boundary conditions involving the discontinuity of the state $y$ at $0^+$. Our main result is to prove the existence of solution sets and its topological structure for the problem (3.1)-(3.3) on unbounded domain with the non-local conditions. To overcome the difficulty of the problem, we have to define a special weight space of continuous functions $C_\alpha(J,E)$. The constructed space is in a natural way, in the sense that this space is endowed with a Banach structure. As far as we know, in our opinion, this problem has not been studied in the literature.
The assumed hypotheses have as goals:
**i)** In this work we have assumed a more general growth condition $(H_1)$ unlike the affine condition.
**ii)** Hypothesis $(H_2)$ being supposed to overcome the equiconvergence at infinity.
**iii)** Conditions $(H_3)$ and $(H_4)$ ensure the veracity of the Meir-Keeler fixed point theorem for condensing operator.
These conditions are optimal in the sense that no condition implies the other. We make use in our approach the Meir-Keeler fixed point theorem combined with tools from classical functional analysis and measure of non-compactness. The chapter concludes with an example to illustrate the feasibility of our main result.
In the last chapter we deal with the existence of Hybrid integro-differential equation involving Hadamard fractional integral, our approach is based on analysis tools combined with Dhage fixed point theorem in Banach algebra.

# Bibliography


[1] S. Abbas, Y. Xia, Existence and attractivity of k-almost automorphic solutions of model of cellular neutral network with delay, *Acta. Math. Sci.*, **1** (2013), 290-302.

[2] S. Abbas, M. Benchohra, J. Graef and J. Henderson, *Implicit Fractional Differential and Integral Equations; Existence and Stability*, De Gruyter, Berlin, 2018.

[3] S. Abbas, M. Benchohra and G.M. N'Guérékata, *Topics in Fractional Differential Equations*, Springer, New York, 2012.

[4] S. Abbas, M. Benchohra and G.M. N'Guérékata, *Advanced Fractional Differential and Integral Equations*, Nova Science Publishers, New York, 2015.

[5] H. Afshari, Solution of fractional differential equations in quasi-b- metric and b-metric-like spaces, *Adv. Differ. Equ.* 2018, 285 (2018). https://doi.org/10.1186/s13662-019-2227-9.

[6] H. Afshari, M. Atapour, E. Karapinar, A discussion on a generalized Geraghty multi-valued mappings and applications, *Adv. Differ. Equ.* 2020, **356** (2020). https://doi.org/10.1186/s13662-020-02819-2.

[7] H. Afshari, H. Hosseinpour, H.R. Marasi, Application of some new contractions for existence and uniqueness of differential equations involving Caputo Fabrizio derivative, *Adv. Differ. Equ.* 2021, **321** (2021).

[8] H. Afshari, E. Karapinar, A discussion on the existence of positive solutions of the boundary value problems via $\psi$- Hilfer fractional derivative on b-metric spaces, *Adv. Differ. Equ.* 2020, **616** (2020). https://doi. org/10.1186/s13662-020-03076-z.

[9] H. Afshari, S. Kalantari, D. Baleanu, Solution of fractional differential equations via $\alpha - \phi-$ Geraphty type mappings, *Adv. Differ. Equ.* 2018, **347** (2018). https://doi.org/10.1186/s13662-018-1807-4.

[10] R. P. Agarwal, B. Hedia and M. Beddani, Structure of solutions sets for impulsive fractional differential equation, *J. Fractional Cal. Appl Vol.* **9**(1) Jan. (2018), pp. 15-34.

[11] A. Aghajani, M. Mursaleen and A. Shole Haghighi, Fixed point theorems for Meir-Keeler condensing operators via measure of non-compactness, *Acta Math. Sci. Ser.* **35** (2015), 552-556.

[12] B. Ahmad, Existence of solutions for irregular boundary value problems of nonlinear fractional differential equations, *Appl. Math. Lett.* **23** (2010), 390-394.

[13] B. Ahmad, S. K. Ntouyas; An existence theorem for fractional hybrid differential inclusions of Hadamard type with Dirichlet boundary conditions, em Abstr. Appl. Anal. 2014, Art. ID 705809, 7 pp.

[14] M. Ammi, E. El Kinani, D. Torres; Existence and uniqueness of solutions to functional integro-differential fractional equations, *Electron. J. Diff. Equ.*, Vol. 2012 (2012), No. 103, pp. 1-9.

[15] A. Arara, M. Benchohra, N. Hamidi, J.J. Nieto, Fractional order differential equations on an unbounded domain, *Nonlinear Anal.* TMA **72** (2010), 580-586.

[16] K. Balachandran, S. Kiruthika, J.J. Trujillo, Existence results for fractional impulsive integrodifferential equations in Banach spaces, *Commun. Nonlinear Sci. Numer. Simul.* **16** (2011),1970-1977.

[17] K. Balachandran, J.Y. Park, Nonlocal Cauchy problem for abstract fractional semilinear evolution equations, *Nonlinear Anal.* TMA **71** (2009), 4471-4475.

[18] K. Balachandran, J.J. Trujillo, The nonlocal Cauchy problem for nonlinear fractional integrodifferential equations in Banach spaces, *Nonlinear Anal.* TMA **72** (2010), 4587-4593.

[19] J. Banaŝ, K. Goebel, Measures of non-compactness in Banach spaces, Lecture Note in Pure App. Math, **60**, Dekker, New York, 1980.

[20] J. Banaŝ, M. Mursaleen, Sequence spaces and measures of noncompactness with applications to differential and integral equations, Springer, New Delhi, 2014.

[21] J. Banas, B. Rzepka; Nondecreasing solutions of a quadratic singular Volterra integral equation, *Math. Comput. Modelling* **49** (2009), 488-496.

[22] M. Beddani and B. Hedia, Solution sets for fractional differential inclusions, *J. Fractional Calc. Appl.* **10** (2) July 2019, 273-289.

[23] **K. Benia, E. Ait Dads, M. Beddani and B. Hedia, Nonlocal Fractional Differential Equation On The Half Line in Banach Space, *J. ATNAA.* 6 (1) 2022, 118-134.**

[24] **K. Benia, M. Beddanin M.Feckan and B. Hedia, Existence result for a problem involving $\psi$-RiemannLiouville fractional derivative on unbbounded domain, *J. Differ. Equ. Appl.* 2022.**

[25] **K. Benia and B. Hedia, Hybrid Implicit Fractional Integro-differential equations with Hadamard integral boundary conditions, *J. Dynamics of Continuous, Discrete and Impulsive Systems.* 28 (3) 2021, 207-222.**

[26] F.Z.Berrabah, B.Hedia and J. Henderson, A fully Hadamard and Erdelyi Kober type integral boundary value problem of a coupled system of implicit differential equations, *Turk J Math*, **43** (2019), 1308-1329.

[27] P. L. Butzer, A. A. Kilbas, J. J. Trujillo; Compositions of Hadamard-type fractional integration operators and the semigroup property, *J. Math. Anal. Appl.* **269** (2002), 387-400.

[28] P. L. Butzer, A. A. Kilbas, J. J. Trujillo; Fractional calculus in the Mellin setting and Hadamard-type fractional integrals, *J. Math. Anal. Appl.* **269** (2002), 1-27.

[29] P. L. Butzer, A. A. Kilbas, J. J. Trujillo; Mellin transform analysis and integration by parts for Hadamard-type fractional integrals, *J. Math. Anal. Appl.* **270** (2002), 1-15.

[30] L. Byszewski, Existence and uniqueness of mild and classical solutions of semilinear functional differential evolution non-local Cauchy problem, Selected problems of mathematics,*50th Anniv. Cracow Univ. Technol. Anniv.* Issue **6**, Cracow Univ. Technol. Krakow, (1995), 25-33.

[31] L. Byszewski, V. Lakshmikantham, Theorem about the existence and uniqueness of a solution of a non-local abstract Cauchy problem in a Banach space, *Appl. Anal.* **40** (1991), 11-19.

[32] W. Cheung, J. Ren, P. J. Y. Wong, D. Zhao, Multiple positive solutions for discrete nonlocal boundary value problems, *J. Math. Anal. Appl.* **330** (2007), 900-915.

[33] G. Christopher, Existence and uniqueness of solutions to a fractional difference equation with non-local conditions, *Comput. Math. with Appl.* **61** (2011), 191-202.

[34] M. A. Darwish; On monotonic solutions of an integral equation of Abel type, *Math. Bohem.*, **133** (2008), 407-420.

[35] M. A. Darwish, B. Rzepka; Asymptotically Stable Solutions of a Generalized Fractional Quadratic Functional-Integral Equation of Erdélyi-Kober Type, *J. Funct. Spaces*, Volume 2014 (2014), Article ID 192542, 9 pages.

[36] K. Deimling, *Multivalued Differential Equations*, De Gruyter, Berlin, New York, 1992.

[37] K. Deng, Exponential decay of solutions of semilinear parabolic equations with non-local initial conditions, *J. Math. Anal. Appl.* **179** (1993), 630-637.

[38] C. Derbazi, Z. Baitiche, M. Benchohra, Cauchy problem with $\psi$-Caputo fractional derivative in Banach spaces, *Adv. Theory Nonlinear Anal. Appl.* **4** (2020), 349-361.

[39] B. C. Dhage; On a fixed point theorem in Banach algebras with applications, *Appl. Math. Lett.* **18** (2005), 273-280.

[40] K. Diethelm; *The analysis of fractional differential equations. An Application-Oriented Exposition Using Differential Operators of Caputo Type*, Lecture Notes in Mathematics, 2004. Springer-Verlag, Berlin, 2010.

[41] M. El Borai, M. Abbas; On some integro-differential equations of fractional orders involving Carathéodory nonlinearities, *Int. J. Mod. Math.* **2** (2007), 41-52.

[42] A. M. A. El-Sayed, H. H. G. Hashem; Existence results for nonlinear quadratic functional integral equations of fractional orders, *Miskolc Math. Notes* **14** (2013), 79-88.

[43] D. J. Guo, V. Lakshmikantham, X. Liu, *Nonlinear Integral Equations in Abstract Spaces*, Kluwer Academic Publishers, Dordrecht, 1996.

[44] J. Hadamard; Essai sur l'etude des fonctions donnees par leur developpment de Taylor, *J. Mat. Pure Appl. Ser.* **8** (1892), 101-186.

[45] B. Hedia, Non-local Conditions for Semi-linear Fractional Differential Equations with Hilfer Derivative, *Springer proceeding in mathematics and statistics 303*, ICFDA 2018, Amman, Jordan, July 16-18, 69-83.

[46] A. G. Ibrahim, N. A. Al Sarori, Mild Solutions for Nonlocal Impulsive Fractional Semilinear Differential Inclusions with Delay in Banach Spaces, *Applied Mathematics***4** (2013), 40-56.

[47] M. Kamenskii, V. Obukhovskii, P. Zecca, Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces, *Nonlin. Anal. Appl.* **7**, Walter de Gruyter, Berlin, New York (2001).

[48] A. A. Kilbas; Hadamard-type fractional calculus, *J. Korean Math. Soc.* **38** (2001), 1191-1204.

[49] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier B. V. Amsterdam, 2006.

[50] A. A. Kilbas, J. J. Trujillo; Hadamard-type integrals as G-transforms, *Integral Transform. Spec. Funct.* **14** (2003), 413-427.

[51] K.D. Kucche, A. D. Mali, On the Nonlinear $\psi$-Hilfer Hybrid Fractional Differential Equations, *Mathematics*, (2020) arXiv:2008.06306.

[52] D. Luo, J. R. Wang, M. Feckan, Applying fractional calculus to analyze economic growth modelling, *J. Appl. Math. Stat. Inform.*, **14** (2018), 25-36.

[53] G. M. N'Guerekata, A Cauchy problem for some fractional abstract differential equation with non-local conditions, *Nonlinear Anal.* TMA **70** (2009), 1873-1876.

[54] I. Podlubny, *Fractional Differential Equations, in: Mathematics in Science and Engineering*, vol. **198**, Academic Press, New York, London, Toronto, 1999.

[55] H.A.H. Salem, On the fractional calculus in abstract spaces and their applications to the Dirichlet-type problem of fractional order, *Comput. Math. Appl.* **59** (2010), 1278-1293.

[56] H.A.H. Salem, Multi-term fractional differential equation in reflexive Banach space, *Math. Comput. Modelling.* **49** (2009), 829-834.

[57] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[58] A. R. Seadawy, D. Lu and M. M. A. Khater, New wave solutions for the fractional-order biological population model, time fractional burgers, Drinfeld-Sokolov-Wilson and system of shallow water wave equations and their applications, *European Journal of Computational Mechanics*, **26** (2017), 508-524.

[59] X. Su, Solutions to boundary value problem of fractional order on unbounded domains in a Banach space, *Nonlinear Analysis* **74** (2011), 2844-2852.

[60] S. Sun, Y. Zhao, Z. Han, Y. Li; The existence of solutions for boundary value problem of fractional hybrid differential equations, *Commun. Nonlinear Sci. Numer. Simul.* **17** (2012), 4961-4967.

[61] P. Thiramanus, S. K. Ntouyas, J. Tariboon; Existence and Uniqueness Results for Hadamard-Type Fractional Differential Equations with Nonlocal Fractional Integral Boundary Conditions, *Abstr. Appl. Anal.* 2014, Art. ID 902054, 9 pp.

[62] J. Vanterler da C. Sousa, E. Capelas de Oliveira, On the $\psi$-Hilfer fractional derivative, *Commun. Nonlinear Sci. Numer. Simul.*, **60** (2018), 72-91.

[63] X. Yu, C. Zhu, J. Wang; On a weakly Singular Quadratic Integral Equations of Volterra Type in Banach Algebras, *Adv. Difference Equ.* (2014).

[64] S. Zhang, Positive solutions for boundary-value problems of nonlinear fractional differential equations, *Electron. J. Differential Equations*, **36** (2006), 1-12.

[65] Y. Zhao, S. Sun, Z. Han, Q. Li; Theory of fractional hybrid differential equations, *Comput. Math. Appl.* **62** (2011), 1312-1324.

[66] Y. Zhou, F. Jiao, J. Pecaric, On the Cauchy problem for fractional functional differential equations in Banach spaces. Topol. Methods Nonlinear Anal. 42 (2013), 119-136.