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BOUAZZA ZOUBIDA

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Intitulée

Une contribution à l'étude de quelques classes de problèmes aux limites pour des équations différentielles fractionnaires d'ordre variable

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Devant le jury composé de :

Président : Mr. BENCHOHRA Mouffak, Prof, Université de SIDI BEL ABBES.

Examinateurs : Mme. LITIMEIN Sara, Prof, Université de SIDI BEL ABBES.
Mr. MAAZOUZ Kadda, MCA, Université de Tiaret.
Mr. MOKHTARI Mokhtar, MCA, Université de Tiaret.
Directeur de thèse : Mr. SOUID Mohammed Said, Prof, Université de Tiaret.
Co-Directeur de thèse : Mr. LAZREG Jamal Eddine, Prof, Université de SIDI BEL ABBES.

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Dédicace

Je dédie cette thèse :
A mes très chers parents qui ont toujours été là pour moi, et qui m'ont encouragé et soutenu pendant mes études.
A la mémoire de ma chère grand-mère.
A mon mari, et mes enfants Moataz, Mohamed ismail et Soulef.
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Publications

1. **Z. Bouazza**, S. Etemad, M. S. Souid, S. Rezapour, F. Matinez and M. K. A. Kaabar, A Study on the Solutions of a Multiterm fractional boundary value problem of Variable Order, *Journal of Function Spaces*, **2021**(2021), 1-9.

2. **Z. Bouazza**, M. S. Souid and Hatira Günerhan, Multiterm Boundary Value Problem of Caputo Fractional Differential Equations of Variable Order, *Advances in Difference Equations*, **2021**(2021), 1-17.

3. S. Rezapour, **Z. Bouazza**, M. S. Souid, S. Etemad and M. K. A. Kaabar, Darbo Fixed Point Criterion on Solutions of a Hadamard Nonlinear Variable Order Problem and Ulam-Hyers-Rassias Stability, *Journal of Function Spaces*, **2022**(2022).

4. S. Rezapour, M. S. Souid, S. Etemad, **Z. Bouazza**, S. K. Ntouyas, S. Asawasamrit and J. Tariboon, Mawhin Continuation Technique for a Nonlinear BVP of Variable Order at Resonance via Piece-wise Constant Functions, *Fractal and Fractional*, **5**(2021), 216-230.

5. S. Rezapour, M. S. Souid, **Z. Bouazza**, A. Hussain, S. Etemad, On the Fractional Variable Order Thermostat Model : Existence Theory on Cones via Piece-Wise Constant Functions, *Journal of Function Spaces*, **2022**(2022).

6. **Z. Bouazza**, M. S. Souid, V. Rakocevic, On Ulam-Hyers-Rassias stability of the boundary value problem of Hadamard fractional differential equations of variable order, *Afrika Matematika*, **33.1**(2022), 1-17.

7. S. Hristova, **Z. Bouazza**, M. S. Souid, Nonlinear implicit differential equations of fractional order at resonance, *AIP Conference Proceedings*, Vol. **2459**, No. **1**, (2022).

Abstract

In this thesis, we investigate the existence, uniqueness and stability of solutions for some classes of nonlinear boundary value problem involving the Riemann-Liouville, Caputo and Hadamard fractional differential equations of variable order.
All results in this study are established by means of fixed point theorems, Mawhin's continuation theorem, technique of measure of noncompactness and with the help of piece-wise constant function, we convert the Riemann-Liouville, Caputo and Hadamard fractional variable order to an equivalent standard Riemann-Liouville, Caputo and Hadamard of the fractional constant order. Further, we examine the stability of the obtained solutions in the sense of Ulam-Hyers-Rassias and in the sense of Ulam-Hyers.

Keywords : Fractional differential equations of variable order, Boundary value problem, Piecewise constant functions, Fixed point theorem, Green's function, Kuratowski measure of noncompactness, Stability, Resonance, Mawhin's continuation theorem.

AMS (MOS) Subject Classifications : 26A33, 34A08, 34A37, 34A60.

Résumé

Dans cette thèse, nous étudions l'existence, l'unicité et la stabilité de solutions de certaines classes de problèmes aux limites non linéaires associés à des équations différentielles fractionnaires (Riemann-Liouville, Caputo et Hadamard) d'ordre variable.
Tous les résultats de cette étude sont basés sur les théorèmes de points fixes, du théorème de continuation de Mawhin, la technique des mesures de non-compactité et à l'aide de fonction constante par morceaux, nous convertissons l'ordre des variables fractionnaires de Riemann-Liouville, Caputo et Hadamard en un standard équivalent Riemann-Liouville, Caputo et Hadamard de l'ordre des constantes fractionnaires. De plus, nous examinons la stabilité des solutions obtenues au sens de Ulam-Hyers-Rassias et au sens de Ulam-Hyers.

Mots clés : Equations différentielles fractionnaires d'ordre variable, Problème de valeur aux limites, Fonctions constantes par morceaux, Théorème du point fixe, Fonction de Green, Mesure de non-compactité de Kuratowski, Stabilité, Résonance, Théorème de continuation de Mawhin.

Classifcations (AMS) : 26A33, 34A08, 34A37, 34A60.

الملخص:

في هذه الرسالة ، قمنا بدراسة وجود وحدانية واستقرار الحلول لبعض فئات مسائل القيم الحدودية التي تتضمن معادلات تفاضلية غير الخطية ذات الترتيب الكسري المتغير لكل من ريمان-ليوفيل وكابوتو و هادامارد.

تم إنشاء جميع النتائج في هذه الدراسة عن طريق نظريات النقاط الثابتة، نظرية استمرار ماوين ، تقنية قياس عدم الانضغاط و بمساعدة الدالة الثابتة بالجزء، نقوم بتحويل ترتيب المتغير الجزئي ريمان-ليوفيل وكابوتو وهادامار إلى ما يعادله. معيار ريمان-ليوفيل وكابوتو و هادامار للترتيب الثابت الكسري. علاوة على ذلك، نقوم بفحص استقرار الحلول التي تم الحصول عليها بإستعمال مقياس إلام- هارس- راسياس و إلام- هارس .

الكلمات المفتاحية: المعادلات التفاضلية الكسرية ذات الترتيب المتغير، مشكلة القيمة الحدودية، الدوال الثابتة بالجزء، نظرية النقطة الصامدة، دالة غرين، مقياس عدم التراص، الاستقرار، الرنين، نظرية استمرار ماوين.

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INTRODUCTION

The main idea of fractional calculus is to constitute the natural numbers in the order of derivation operators with rational ones. Although this idea is preliminary and simple, it involves remarkable effects and outcomes which describe some physical, dynamics, modeling, control theory, bioengineering, and biomedical applications phenomena.

The subject of fractional calculus has gained considerable popularity and importance due to its frequent appearance in different research areas and engineering, such as physics, chemistry, control of dynamical systems etc.

The operators of variable order, which fall into a more complex operator category, are the derivatives and integrals whose order is the function of certain variables. The variable order fractional derivative is an extension of constant order fractional derivative. In recent years, the operator and differential equations of variable order have been applied in engineering more and more frequently, for the examples and details, see [1, 3, 18, 33, 35, 36, 43, 52, 54, 55, 56, 57, 58, 60, 61, 64, 66].

Recently, the Hadamard-type operators originally introduced in [25] and later generalized to variable fractional order have been investigated in [6, 7].

In the last years, many people paid attention to the existence and uniqueness of solutions to boundary value problems for fractional differential equations. Although the existing literature on solutions of boundary value problems of fractional order (constant order) is quite wide, on the contrary, few papers deal with the existence of solutions to boundary value problems of variable order, see, e.g., [1, 56, 57, 58, 64].

In general, it is usually difficult to solve boundary value problems of fractional boundary variable order and obtain their analytical solution. Therefore, some methods are introduced for the approximation of solutions to different fractional boundary value problem of variable order. In relation to the study of the existence theory to fractional boundary value problem of variable order, we point out some of them. In [65], Zhang studied solutions of a two-point boundary value problem of fractional variable order involving singular fractional differential equations. After some years, Zhang and Hu [67] established the existence results for approximate solutions of variable order fractional initial value problems on the half line.

While several research studies have been performed on investigating the existence solutions of the fractional constant-order problems, the existence solutions of the variable order problems are rarely discussed in literature ; we refer to [58, 65, 67, 68, 69].

In 2021, Bouazza et al. [12] considered a multiterm fractional boundary value problem of variable order and derived their results by terms of fixed point methods, Hristova et al. [27] turned to investigation of the Hadamard fractional boundary value problem of variable order by means of Kuratowski MNC method. For more details on other instances, refer to [53, 57] and the references therein.

The stability theory of functional equations has developed very rapidly during the past decades. In 1940, Ulam posed the problem of stability of functional equations at the University of Wisconsin, see [59]. A year later, Hayers [26] gave the first answer to the Ulam problem in the case of Banach spaces. Therefore, this type of stability came to be called the Ulam-Hyers stability. In 1978, Rassias [50] provided a generalization of the Ulam-Hyers stability. After that, the study of these two types of stabilities, the Ulam-Hyers stability and the Ulam-Hyers-Rassias stability has grown to be one of the most essential subjects in the field of mathematical analysis and especially the stability of differential equations, see e.g. [28, 30, 31, 44, 46, 51, 62].

The technique of measures of noncompactness which is often used in several branches of nonlinear analysis. Especially, that technique turns out to be a very useful tool in existence for several types of integral equations ; details are found in Akhmerov et al. [5], Alvarez [2], Banas et al. [8, 9, 15, 16, 17], Guo et al. [23].

In 1970, Gaines and Mawhin introduced the theory of the degree of coincidence in the analysis of functional and differential equations. Mawhin has made important contributions since then, and this theory is also known as Mawhin's theory of coincidence. Coincidence theory is considered to be the very powerful technique, especially with regard to questions about the existence of solutions in nonlinear differential equations. Furthermore, many researchers have used it to solve boundary value problems at resonance, see [37, 38, 39, 40, 41, 42, 48].

In the following we give an outline of our thesis organization, consisting of **5 chapters**.

The **first chapter** gives some notations, definitions, lemmas, fixed point theorems and coincidence degree theory which are used throughout this thesis.

In Chapter 2, we study the existence of solutions to the proposed multiterm boundary value problem (BVP) for the nonlinear fractional differential equation of variable order in the format

$$\begin{cases} D_{0^+}^{u(t)}x(t) + f(t, x(t), I_{0^+}^{u(t)}x(t)) = 0, \ \ t \in J := [0, T], \\ x(0) = 0, \ \ x(T) = 0, \end{cases} \tag{1}$$

where $0 < T < +\infty$, $1 < u(t) \leq 2$, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $D_{0^+}^{u(t)}$, $I_{0^+}^{u(t)}$ are the Riemann-Liouville fractional derivative and Riemann-Liouville fractional integral of variable-order $u(t)$.

In Chapter 3, we deal with the boundary value problem

$$\begin{cases} {}^cD_{0^+}^{u(t)}x(t) + f(t, x(t), I_{0^+}^{u(t)}x(t)) = 0, \ \ t \in J := [0, T], \\ x(0) = 0, \ \ x(T) = 0, \end{cases} \tag{2}$$

where $1 < u(t) \leq 2$, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and ${}^cD_{0^+}^{u(t)}$, $I_{0^+}^{u(t)}$ are the Caputo fractional derivative and integral Riemann-Liouville of variable-order $u(t)$. Further, we study the stability of the obtained solution in the sense of Ulam-Hyers.

In Chapter 4, we investigate the existence of solutions for the nonlinear Hadamard fractional boundary value problem of variable order as follows :

$$\begin{cases} {}^HD_{1^+}^{u(t)}x(t) + f(t, x(t)) = 0, \ \ t \in J := [1, T], \\ x(1) = x(T) = 0, \end{cases} \tag{3}$$

where $1 < T < +\infty$, $1 < u(t) \leq 2$, $f : J \times X \to X$ is a continuous function (X is a Banach space) and ${}^HD_{1^+}^{u(t)}$ specifies the Hadamard derivative of variable order $u(t)$. Further, we study the stability of the obtained solution in the sense of Ulam-Hyers-Rassias.

In Chapter 5, we shall investigate a nonlinear boundary value problem of variable order which takes a structure as follows

$$\begin{cases} {}^cD_{a^+}^{u(t)}y(t) = f(t, y(t)), \ t \in J, \\ \\ y(a) = y(T), \end{cases} \tag{4}$$

where $J = [a, T]$, $0 \leq a < T < \infty$, $u(t) : J \to (0, 1]$ is the variable order of the fractional derivatives, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function and ${}^cD_{a^+}^{u(t)}$ is the Caputo fractional derivative of variable-order $u(t)$.

Chapitre 1

Preliminaries

We introduce in this chapter notations, definitions, fixed point theorems and coincidence degree theory which are used throughout this thesis.

1.1 Notations and definitions

Let $(X; \|.\|)$ be a Banach space. We denote by $C(J,X)$ the space of X-valued continuous functions on J with the usual supremum norm

$$\|y\| = sup\{\|y(t)\| : t \in J\},$$

for any $y \in C(J,X)$.

and for each $i \in \{1,2,...,n\}$, the symbol $E_i = C(J_i,X)$, indicated the Banach space of continuous functions $y : J_i \to X$ equipped with the norm

$$\|y\|_{E_i} = \sup_{t \in J_i} \|y(t)\|.$$

A measurable function $y : J \to X$ is Bochner integrable if and only if $\|y\|$ is Lebesgue integrable.
Let $L^1(J,X)$ denote the Banach space of measurable functions $y : J \to X$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^T \|y(t)\|dt.$$

For properties of the Bochner integrable functions, see [63].

1.2 Fractional calculus.

1.2.1 Fractional calculus of constant-order

Definition 1.1 *([34, 49]). The left Riemann-Liouville fractional integral of the function $h \in L^1([a,b],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by*

$$I_a^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)ds,$$

where $\Gamma(.)$ is the gamma function.

Definition 1.2 *([34, 49]). The left Riemann-Liouville fractional derivative of order $\alpha > 0$ of function $h \in L^1([a,b],\mathbb{R}_+)$, is given by*

$$(D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{d}{dt}\Big)^n \int_a^t (t-s)^{n-\alpha-1} h(s)ds,$$

here $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α. If $\alpha \in (0,1]$, then

$$(D_{a+}^\alpha h)(t) = \frac{d}{dt} I_{a+}^{1-\alpha} h(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{ds} \int_a^t (t-s)^{-\alpha} h(s)ds.$$

The following properties are some of the main ones of the fractional derivatives and integrals.

Lemma 1.1 *([34]). Let $\alpha > 0$, $a \geq 0$, $h \in L^1(a,b)$, $D_{a^+}^\alpha h \in L^1(a,b)$. Then, the differential equation*

$$D_{a^+}^\alpha h = 0$$

has unique solution

$$h(t) = \omega_1 (t-a)^{\alpha-1} + \omega_2 (t-a)^{\alpha-2} + ... + \omega_\ell (t-a)^{\alpha-\ell} + ... + \omega_n (t-a)^{\alpha-n},$$

where $n = [\alpha]+1$, $\omega_\ell \in \mathbb{R}$, $\ell = 1,2,...,n$.

Lemma 1.2 *([34]). Let $\alpha > 0$, $a \geq 0$, $h \in L^1(a,b)$, $D_{a^+}^\alpha h \in L^1(a,b)$. Then,*

$$I_{a^+}^\alpha D_{a^+}^\alpha h(t) = h(t) + \omega_1 (t-a)^{\alpha-1} + \omega_2 (t-a)^{\alpha-2} + ... + \omega_\ell (t-a)^{\alpha-\ell} + ... + \omega_n (t-a)^{\alpha-n},$$

where $n = [\alpha]+1$, $\omega_\ell \in \mathbb{R}$, $\ell = 1,2,...,n$.

Lemma 1.3 *([34]). Let $\alpha > 0$, $a \geq 0$, $h \in L^1(a,b)$, $D_{a^+}^\alpha h \in L^1(a,b)$. Then,*

$$D_{a^+}^\alpha I_{a^+}^\alpha h(t) = h(t).$$

Lemma 1.4 *([34]). Let $\alpha,\ \beta > 0$, $a \geq 0$, $h \in L^1(a,b)$. Then,*

$$I^{\alpha}_{a^+} I^{\beta}_{a^+} h(t) = I^{\beta}_{a^+} I^{\alpha}_{a^+} h(t) = I^{\alpha+\beta}_{a^+} h(t).$$

Definition 1.3 *([34, 49]). The left Caputo fractional derivative of order $\alpha > 0$ of function $h \in L^1([a,b],\mathbb{R}_+)$, is given by*

$${}^{c}D^{\alpha}_{a+} h(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds,$$

where $n = [\alpha] + 1$.

Lemma 1.5 *([34]). Let $\alpha > 0$, $a \geq 0$, $h \in L^1(a,b)$, ${}^{c}D^{\alpha}_{a^+} h \in L^1(a,b)$. Then, the differential equation*

$${}^{c}D^{\alpha}_{a^+} h = 0$$

has unique solution

$$h(t) = \omega_0 + \omega_1(t-a) + \omega_2(t-a)^2 + ... + \omega_\ell(t-a)^\ell + ... + \omega_{n-1}(t-a)^{n-1},$$

where $n = [\alpha] + 1$, $\omega_\ell \in \mathbb{R}$, $\ell = 0, 1, ..., n-1$.

Lemma 1.6 *([34]). Let $\alpha > 0$, $a \geq 0$, $h \in L^1(a,b)$, ${}^{c}D^{\alpha}_{a^+} h \in L^1(a,b)$. Then,*

$$I^{\alpha}_{a^+}\ {}^{c}D^{\alpha}_{a^+} h(t) = h(t) + \omega_0 + \omega_1(t-a) + \omega_2(t-a)^2 + ... + \omega_\ell(t-a)^\ell + ... + \omega_{n-1}(t-a)^{n-1},$$

where $n = [\alpha] + 1$, $\omega_\ell \in \mathbb{R}$, $\ell = 0, 1, ..., n-1$.

Lemma 1.7 *([34]). Let $\alpha > 0$, $a \geq 0$, $h \in L^1(a,b)$, ${}^{c}D^{\alpha}_{a^+} h \in L^1(a,b)$. Then,*

$${}^{c}D^{\alpha}_{a^+} I^{\alpha}_{a^+} h(t) = h(t).$$

Definition 1.4 *([34, 49]). The left Hadamard fractional integral of the function $h \in L^1([a,b],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by*

$${}^{H}I^{\alpha}_{a^+} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (\ln \frac{t}{s})^{\alpha-1} \frac{h(s)}{s} ds, \quad t > a.$$

Definition 1.5 *([34, 49]). The left Hadamard fractional derivative of order $\alpha > 0$ of function $h \in L^1([a,b],\mathbb{R}_+)$, is given by*

$${}^{H}D^{\alpha}_{a^+} h(t) = \frac{1}{\Gamma(n-\alpha)} (t\frac{d}{dt})^n \int_a^t (\ln \frac{t}{s})^{n-\alpha-1} \frac{h(s)}{s} ds, \quad t > a,$$

where $n = [\alpha] + 1$.

The following properties are some of the main ones of the fractional derivatives and integrals.

Lemma 1.8 *([34]). Assume that $a \geq 1$, $\alpha > 0$, $h \in L^1(a,b)$, ${}^H D^{\alpha}_{a^+} h \in L^1(a,b)$. Then, the homogeneous differential equation*

$$ {}^H D^{\alpha}_{a^+} h = 0 $$

has unique solution

$$ h(t) = \omega_1 (\ln \frac{t}{a})^{\alpha-1} + \omega_2 (\ln \frac{t}{a})^{\alpha-2} + ... + \omega_\ell (\ln \frac{t}{a})^{\alpha-\ell} + ... + \omega_n (\ln \frac{t}{a})^{\alpha-n}, $$

and

$$ {}^H I^{\alpha}_{a^+} ({}^H D^{\alpha}_{a^+}) h(t) = h(t) + \omega_1 (\ln \frac{t}{a})^{\alpha-1} + \omega_2 (\ln \frac{t}{a})^{\alpha-2} + ... + \omega_\ell (\ln \frac{t}{a})^{\alpha-\ell} + ... + \omega_n (\ln \frac{t}{a})^{\alpha-n}, $$

with $n = [\alpha] + 1$, $\omega_\ell \in \mathbb{R}$, $\ell = 1, 2, ..., n$.

Lemma 1.9 *([34]). Let $\alpha > 0$, $a \geq 1$, $h \in L^1(a,b)$, $D^{\alpha}_{a^+} h \in L^1(a,b)$. Then,*

$$ {}^H D^{\alpha}_{a^+} ({}^H I^{\alpha}_{a^+}) h(t) = h(t). $$

Lemma 1.10 *([34]). Let $\alpha,\ \beta > 0$. Then,*

$$ {}^H I^{\alpha}_{a^+} ({}^H I^{\beta}_{a^+}) h(t) = {}^H I^{\beta}_{a^+} ({}^H I^{\alpha}_{a^+}) h(t) = {}^H I^{\alpha+\beta}_{a^+} h(t). $$

1.2.2 Fractional calculus of variable-order

Definition 1.6 *([54, 61]). For $-\infty < a < b < +\infty$, we consider the mapping $u(t) : [a,b] \to (0, +\infty)$. Then, the left Riemann-Liouville fractional integral (RLInVo) of variable-order $u(t)$ for function $h(t)$ is expressed by*

$$ I^{u(t)}_{a^+} h(t) = \int_a^t \frac{(t-s)^{u(s)-1}}{\Gamma(u(s))} h(s) ds, \quad t > a, \tag{1.1} $$

where the gamma function is denoted by $\Gamma(.)$.

Definition 1.7 *([54, 61]). For $-\infty < a < b < +\infty$, we consider the mapping $u(t) : [a,b] \to (n-1, n)$, $n \in \mathbb{N}$. Then, the left Riemann-Liouville fractional derivative of variable-order $u(t)$ for function $h(t)$ is expressed by*

$$ D^{u(t)}_{a^+} h(t) = \Big(\frac{d}{dt}\Big)^n I^{n-u(t)}_{a^+} h(t) = \Big(\frac{d}{dt}\Big)^n \int_a^t \frac{(t-s)^{n-u(s)-1}}{\Gamma(n-u(s))} h(s) ds, \quad t > a. \tag{1.2} $$

Obviously, if the order $u(t)$ is a constant function u, then the Riemann-Liouville fractional derivative of variable-order (1.2) and Riemann-Liouville fractional integral of variable-order(1.1) are the usual Riemann-Liouville fractional derivative and Riemann-Liouville fractional integral, respectively ; see [54, 55, 34].

Remark ([69, 65]). Generally, for functions $u(t)$ and $v(t)$, the semigroup property does not hold, i.e.,

$$I_{a^+}^{u(t)} I_{a^+}^{v(t)} h(t) \neq I_{a^+}^{u(t)+v(t)} h(t).$$

Example : Let

$$u(t) = \begin{cases} 2, & t \in [0,1], \\ 1, & t \in]1,3], \end{cases} \qquad v(t) = \begin{cases} 1, & t \in [0,1], \\ 2, & t \in]1,3], \end{cases}$$

and $\quad h(t) = t, \quad t \in [0,3]$.

$$\begin{aligned} I_{0^+}^{u(t)} I_{0^+}^{v(t)} h(t) &= \int_0^1 \frac{(t-s)^{u(s)-1}}{\Gamma(u(s))} \int_0^s \frac{(s-\tau)^{v(\tau)-1}}{\Gamma(v(\tau))} h(\tau) d\tau ds \\ &+ \int_1^t \frac{(t-s)^{u(s)-1}}{\Gamma(u(s))} \int_0^s \frac{(s-\tau)^{v(\tau)-1}}{\Gamma(v(\tau))} h(\tau) d\tau ds, \\ &= \int_0^1 \frac{(t-s)^1}{\Gamma(2)} \int_0^s \frac{(s-\tau)^0}{\Gamma(1)} \tau d\tau ds \\ &+ \int_1^t \frac{(t-s)^0}{\Gamma(1)} [\int_0^1 \frac{(s-\tau)^0}{\Gamma(1)} \tau d\tau + \int_1^s \frac{(s-\tau)^1}{\Gamma(2)} \tau d\tau] ds, \\ &= \int_0^1 \frac{(t-s)s^2}{2\Gamma(2)} ds + \int_1^t \frac{s^3}{6} - \frac{s}{2} + \frac{5}{6} ds, \end{aligned}$$

$$I_{0^+}^{u(t)+v(t)} h(t) = \int_0^t \frac{(t-s)^{u(s)+v(s)-1}}{\Gamma(u(s)+v(s))} h(s) ds,$$

we see that

$$\begin{aligned} I_{0^+}^{u(t)} I_{a^+}^{v(t)} h(t)|_{t=2} &= \int_0^1 \frac{(2-s)s^2}{2\Gamma(2)} ds + \int_1^2 \frac{s^3}{6} - \frac{s}{2} + \frac{5}{6} ds, \\ &= \frac{5}{24} + \frac{17}{24} = \frac{22}{24}, \end{aligned}$$

$$I_{0^+}^{u(t)+v(t)} h(t)|_{t=2} = \int_0^1 \frac{(2-s)^{2+1-1}}{\Gamma(2+1)} s ds + \int_1^2 \frac{(2-s)^{1+2-1}}{\Gamma(1+2)} s ds = \frac{11}{24} + \frac{5}{24} = \frac{16}{24}.$$

Therefore, we obtain

$$I_{0^+}^{u(t)} I_{0^+}^{v(t)} h(t)|_{t=2} \neq I_{0^+}^{u(t)+v(t)} h(t)|_{t=2}.$$

Definition 1.8 *([54, 55, 61]). For $-\infty < a < b < +\infty$, we consider the mapping $v(t) : [a,b] \to (n-1,n)$. Then, the left Caputo fractional derivative of variable-order $v(t)$ for function $h(t)$ is*

$$^{c}D_{a^+}^{v(t)} h(t) = \int_a^t \frac{(t-s)^{n-v(t)-1}}{\Gamma(n-v(t))} h^{(n)}(s) ds, \quad t > a. \tag{1.3}$$

As anticipated, in case of $v(t)$ is constant, then Caputo fractional derivative of variable-order is coincide with the standard Caputo fractional derivative, see e.g. [34, 54, 55].

Definition 1.9 *([6, 7]). For $1 \leq a < b < +\infty$, we consider the mapping $u(t) : [a, b] \to (0, +\infty)$. Then, the left Hadamard fractional integral of variable order $u(t)$ for function $h(t)$ is*

$$ {}^{H}I_{a^+}^{u(t)}h(t) = \frac{1}{\Gamma(u(t))}\int_a^t (\ln\frac{t}{s})^{u(t)-1}\frac{h(s)}{s}ds, \quad t > a. \tag{1.4}$$

Definition 1.10 *([6, 7]). For $1 \leq a < b < +\infty$, we consider the mapping $v(t) : [a, b] \to (n-1, n)$. Then, the left Hadamard fractional derivative of variable order $v(t)$ for function $h(t)$ is*

$$ {}^{H}D_{a^+}^{v(t)}h(t) = \frac{1}{\Gamma(n - v(t))}(t\frac{d}{dt})^n\int_a^t (\ln\frac{t}{s})^{n-v(t)-1}\frac{h(s)}{s}ds, \quad t > a. \tag{1.5}$$

Obviously, in case of $u(t)$ and $v(t)$ are constant, then both above Hadamard variable order operators are in coincidence with the usual Hadamard constant order operators (see [34, 55, 54]).

Remark 1.1 . *The semigroup property is not fulfilled for the functions $u(t)$ and $v(t)$, i.e.,*

$$ {}^{H}I_{a^+}^{u(t)}({}^{H}I_{a^+}^{v(t)})h(t) \neq {}^{H}I_{a^+}^{u(t)+v(t)}h(t).$$

Example : Let

$$u(t) = \begin{cases} 1, & t \in [1, 2], \\ 2, & t \in]2, 4], \end{cases} \qquad v(t) = \begin{cases} 3, & t \in [1, 2], \\ 4, & t \in]2, 4], \end{cases} \qquad h(t) = 2t^2, \quad t \in [1, 4].$$

We obtain

$$\begin{aligned}
{}^{H}I_{1^+}^{u(t)}({}^{H}I_{1^+}^{v(t)})h(t) &= \frac{1}{\Gamma(u(t))}\int_1^t \frac{1}{s}(\ln\frac{t}{s})^{u(t)-1}\Big[\frac{1}{\Gamma(v(s))}\int_1^s (\ln\frac{s}{\tau})^{v(s)-1}\frac{h(\tau)}{\tau}d\tau\Big]ds \\
&= \frac{1}{\Gamma(u(t))}\int_1^2 \frac{1}{s}(\ln\frac{t}{s})^{u(t)-1}\Big[\frac{1}{\Gamma(v(s))}\int_1^s (\ln\frac{s}{\tau})^{v(s)-1}\frac{h(\tau)}{\tau}d\tau\Big]ds \\
&+ \frac{1}{\Gamma(u(t))}\int_2^t \frac{1}{s}(\ln\frac{t}{s})^{u(t)-1}\Big[\frac{1}{\Gamma(v(s))}\int_1^s (\ln\frac{s}{\tau})^{v(s)-1}\frac{h(\tau)}{\tau}d\tau\Big]ds \\
&= \frac{1}{\Gamma(1)}\int_1^2 \frac{1}{s}(\ln\frac{t}{s})^0\int_1^s \frac{1}{\Gamma(3)}(\ln\frac{s}{\tau})^2 2\tau d\tau ds \\
&+ \frac{1}{\Gamma(2)}\int_2^t \frac{1}{s}(\ln\frac{t}{s})\Big[\frac{1}{\Gamma(3)}\int_1^2 (\ln\frac{s}{\tau})^2 2\tau d\tau + \frac{1}{\Gamma(4)}\int_2^s (\ln\frac{s}{\tau})^3 2\tau d\tau\Big]ds
\end{aligned}$$

$$
\begin{aligned}
&= \int_1^2 \Big(\frac{s}{4} - \frac{1}{2s}(\ln s)^2 - \frac{1}{2s}(\ln s) - \frac{1}{4s}\Big)ds \\
&+ \int_2^t \frac{1}{s}(\ln\frac{t}{s})\Big[-\frac{2}{3}(\ln\frac{s}{2})^3 + (\ln\frac{s}{2})^2 \\
&+ (\ln\frac{s}{2}) - \frac{1}{2}(\ln s)^2 - \frac{1}{2}(\ln s) + \frac{1}{8}s^2 + \frac{1}{4}]ds,
\end{aligned}
$$

and

$$
{}^H I_{1^+}^{u(t)+v(t)} h(t) = \frac{1}{\Gamma(u(t)+v(t))}\int_1^t (\ln\frac{t}{s})^{u(t)+v(t)-1}\frac{h(s)}{s}ds.
$$

So,

$$
\begin{aligned}
{}^H I_{1^+}^{u(t)}({}^H I_{1^+}^{v(t)})h(t)|_{t=3} &= -\frac{1}{30}(\ln\frac{3}{2})^5 + \frac{1}{24}(\ln\frac{3}{2})^4 + \frac{1}{12}(\ln\frac{3}{2})^3 + \frac{1}{8}(\ln\frac{3}{2})^2 \\
&- \frac{1}{4}(\ln\frac{3}{2}) - \frac{1}{6}(\ln 2)^2(\ln\frac{3}{2})^2 - \frac{1}{6}(\ln 2)(\ln\frac{3}{2})^3 - \frac{(\ln 2)^3}{6} \\
&- \frac{(\ln 2)^2}{4} - \frac{\ln 2}{4} - \frac{1}{4}(\ln 2)(\ln\frac{3}{2})^2 + \frac{17}{32} \simeq 0.0522.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
{}^H I_{1^+}^{u(t)+v(t)} h(t)|_{t=3} &= \int_1^2 \frac{1}{\Gamma(4)}(\ln\frac{3}{s})^3 2sds + \int_2^3 \frac{1}{\Gamma(6)}(\ln\frac{3}{s})^5 2sds \\
&= -\frac{1}{30}(\ln\frac{3}{2})^5 - \frac{1}{12}(\ln\frac{3}{2})^4 + \frac{1}{3}(\ln\frac{3}{2})^3 + \frac{3}{4}(\ln\frac{3}{2})^2 \\
&+ \frac{3}{4}(\ln\frac{3}{2}) - \frac{1}{6}(\ln 3)^3 - \frac{1}{4}(\ln 3)^2 - \frac{1}{4}(\ln 3) + \frac{17}{32} \simeq 0.1809.
\end{aligned}
$$

Therefore, we obtain

$$
{}^H I_{1^+}^{u(t)}({}^H I_{1^+}^{v(t)})h(t)|_{t=3} \neq^H I_{1^+}^{u(t)+v(t)} h(t)|_{t=3}.
$$

Lemma 1.11 *([70]). Let* $u : J := [0,T] \to (1,2]$ *be a continuous function, then for* $h \in C_\delta(J,X) = \{h(t) \in C(J,X),\ t^\delta h(t) \in C(J,X)\},\ (0 \le \delta \le \min_{t\in J}|u(t)|)$, *the variable order fractional integral* $I_{0^+}^{u(t)}h(t)$ *exists for any points on* J.

Lemma 1.12 *([70]). Let* $u : J := [0,T] \to (1,2]$ *be a continuous function, then* $I_{0^+}^{u(t)}h(t) \in C(J,X)$ *for* $h \in C(J,X)$.

Lemma 1.13 *([54]). If* $u : J := [1,T] \to (1,2]$ *be a continuous function, then for* $h \in C_\delta(J,\mathbb{R}) = \{h(t) \in C(J,\mathbb{R}),\ (lnt)^\delta h(t) \in C(J,\mathbb{R})\}$, $0 \le \delta \le 1$, *the variable order fractional integral* ${}^H I_{1^+}^{u(t)}h(t)$ *exists for any points on* J.

Proof. Taking the continuity of $\Gamma(u(t))$ into account, we shall claim that $M_u = \max_{t\in J} |\frac{1}{\Gamma(u(t))}|$ exists. We let $u^* = \max_{t\in J} |(u(t))|$. Thus, for $1 \le s \le t \le T$, we have

$$(\ln\frac{t}{s})^{u(t)-1} \le 1, \qquad if \quad 1 \le \frac{t}{s} \le e$$

$$(\ln\frac{t}{s})^{u(t)-1} \le (\ln\frac{t}{s})^{u^*-1}, \ if \quad \frac{t}{s} > e$$

Then, for $1 \le \frac{t}{s} < +\infty$, we know

$$(\ln\frac{t}{s})^{u(t)-1} \le \max\{1, (\ln\frac{t}{s})^{u^*-1}\} = M^*.$$

For $h \in C_\delta(J,\mathbb{R})$, by the definition of (1.4), we deduce that

$$\begin{aligned}
|{}^H I_{1^+}^{u(t)} h(t)| &= \frac{1}{\Gamma(u(t))}\int_1^t (\ln\frac{t}{s})^{u(t)-1}\frac{|h(s)|}{s}ds \\
&\le M_u \int_1^t (\ln\frac{t}{s})^{u(t)-1}(\ln s)^{-\delta}(\ln s)^{\delta}\frac{|h(s)|}{s}ds \\
&\le M_u M^* \int_1^t \frac{1}{s}(\ln s)^{-\delta}\max_{s\in J}(\ln s)^{\delta}|h(s)|ds \\
&\le M_u M^* \max_{s\in J}(\ln s)^{\delta} h^\star \int_1^t \frac{1}{s}(\ln s)^{-\delta}ds \\
&\le M_u M^* \max_{s\in J}(\ln s)^{\delta} h^\star \frac{(\ln T)^{1-\delta}}{1-\delta} < \infty,
\end{aligned}$$

where $h^\star = \max_{t\in J}|h(t)|$. It yields that the variable order fractional integral ${}^H I_{1^+}^{u(t)} h(t)$ exists for any points on J.

Lemma 1.14 *([54]). Let $u : J := [1,T] \to (1,2]$ be a continuous function, then*

$${}^H I_{1^+}^{u(t)} h(t) \in C(J,\mathbb{R}), \ \text{for } h \in C(J,\mathbb{R}).$$

Proof. For $t, t_0 \in J$, $t_0 \leq t$ and $h \in C(J,\mathbb{R})$, we obtain

$$\begin{aligned}
&\Big|{}^H I_{1^+}^{u(t)} h(t) - {}^H I_{1^+}^{u(t_0)} h(t_0)\ \Big| = \Big|\int_1^t \frac{1}{\Gamma(u(t))}(\ln\frac{t}{s})^{u(t)-1}\frac{h(s)}{s}ds \\
&-\int_1^{t_0}\frac{1}{\Gamma(u(t_0))}(\ln\frac{t_0}{s})^{u(t_0)-1}\frac{h(s)}{s}ds\Big| \\
=&\ \Big|\int_0^1 \frac{1}{\Gamma(u(t))}\frac{(t-1)}{r(t-1)+1}(\ln\frac{t}{r(t-1)+1})^{u(t)-1}h(r(t-1)+1)dr \\
&-\int_0^1\frac{1}{\Gamma(u(t_0))}\frac{(t_0-1)}{r(t_0-1)+1}(\ln\frac{t_0}{r(t_0-1)+1})^{u(t_0)-1}h(r(t_0-1)+1)dr\Big| \\
=&\ \Big|\int_0^1\Big[\frac{1}{\Gamma(u(t))}\frac{(t-1)}{r(t-1)+1}(\ln\frac{t}{r(t-1)+1})^{u(t)-1}h(r(t-1)+1) \\
&-\frac{1}{\Gamma(u(t))}\frac{(t_0-1)}{r(t_0-1)+1}(\ln\frac{t}{r(t-1)+1})^{u(t)-1}h(r(t-1)+1)\Big]dr \\
&+\int_0^1\Big[\frac{1}{\Gamma(u(t))}\frac{(t_0-1)}{r(t_0-1)+1}(\ln\frac{t}{r(t-1)+1})^{u(t)-1}h(r(t-1)+1) \\
&-\frac{1}{\Gamma(u(t))}\frac{(t_0-1)}{r(t_0-1)+1}(\ln\frac{t_0}{r(t_0-1)+1})^{u(t_0)-1}h(r(t-1)+1)\Big]dr \\
&+\int_0^1\Big[\frac{1}{\Gamma(u(t))}\frac{(t_0-1)}{r(t_0-1)+1}(\ln\frac{t_0}{r(t_0-1)+1})^{u(t_0)-1}h(r(t-1)+1) \\
&-\frac{1}{\Gamma(u(t_0))}\frac{(t_0-1)}{r(t_0-1)+1}(\ln\frac{t_0}{r(t_0-1)+1})^{u(t_0)-1}h(r(t-1)+1)\Big]dr \\
&+\int_0^1\Big[\frac{1}{\Gamma(u(t_0))}\frac{(t_0-1)}{r(t_0-1)+1}(\ln\frac{t_0}{r(t_0-1)+1})^{u(t_0)-1}h(r(t-1)+1) \\
-&\ \frac{1}{\Gamma(u(t_0))}\frac{(t_0-1)}{r(t_0-1)+1}(\ln\frac{t_0}{r(t_0-1)+1})^{u(t_0)-1}h(r(t_0-1)+1)\Big]dr\Big| \\
\leq&\ h^\star\int_0^1\frac{1}{\Gamma(u(t))}(\ln\frac{t}{r(t-1)+1})^{u(t)-1}\Big|\frac{(t-1)}{r(t-1)+1}-\frac{(t_0-1)}{r(t_0-1)+1}\Big|dr \\
&+h^\star\int_0^1\frac{1}{\Gamma(u(t))}\frac{(t_0-1)}{r(t_0-1)+1}\Big|(\ln\frac{t}{r(t-1)+1})^{u(t)-1}-(\ln\frac{t_0}{r(t_0-1)+1})^{u(t)-1}\Big|dr
\end{aligned}$$

$$\begin{aligned}
&+h^\star\int_0^1\frac{(t_0-1)}{r(t_0-1)+1}(\ln\frac{t_0}{r(t_0-1)+1})^{u(t_0)-1}\Big|\frac{1}{\Gamma(u(t))}-\frac{1}{\Gamma(u(t_0))}\Big|dr \\
&+\int_0^1\frac{1}{\Gamma(u(t_0))}\frac{(t_0-1)}{r(t_0-1)+1}(\ln\frac{t_0}{r(t_0-1)+1})^{u(t_0)-1}\Big|h(r(t-1)+1)-h(r(t_0-1)+1)\Big|dr,
\end{aligned}$$

where $h^\star = \max_{t\in J}|h(t)|$. On account of the continuity of functions ln, h, we get that the integral ${}^H I_{1^+}^{u(t)}h(t)$ is continuous at the point t_0, then ${}^H I_{1^+}^{u(t)}h(t) \in C(J,\mathbb{R})$ for $h(t) \in C(J,\mathbb{R})$.

Definition 1.11 *([32, 66]). A generalized interval is a subset I of $\mathbb{R}$ which is either an interval (i.e. a set of the form $[a,b],(a,b),[a,b)$ or $(a,b]$), a point $\{a\}$, or the empty set $\emptyset$.*

Definition 1.12 *([32, 66]). If I is a generalized interval. A partition of I is a finite set $\mathcal{P}$ of generalized intervals contained in I, such that every x in I lies in exactly one of the generalized intervals J in $\mathcal{P}$.*

Example : The set $\mathcal{P} = \{\{1\},(1,6),[6,7),\{7\},(7,8]\}$ of generalized intervals is a partition of $[1,8]$.

Definition 1.13 *([32, 66]). Let I be a generalized interval, let $f : I \to \mathbb{R}$ be a function, and let $\mathcal{P}$ a partition of I. f is said to be piecewise constant with respect to $\mathcal{P}$ if for every $J \in \mathcal{P}$, f is constant on J.*

Example : The function $f : [1,6] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 3, & 1 \leq x < 3 \\ 4, & x = 3 \\ 5, & 3 < x < 6 \\ 2, & x = 6 \end{cases}$$

is piecewise constant with respect to the partition $\Big\{[1,3),\{3\},(3,6),\{6\}\Big\}$ of $[1,6]$.

Definition 1.14 *([32, 66]). Let I be a generalized interval. The function $f : I \to \mathbb{R}$ is called piecewise constant on I, if there exists a partition $\mathcal{P}$ of I such that f is piecewise constant with respect to $\mathcal{P}$.*

1.3 Measure of noncompactness

We define in this section the Kuratowski measure of noncompactness and give their basic properties in Banach space.

Definition 1.15 *([9]). Let X be a Banach space and Ω_X is a bounded set in X. The **KMNC** is the function $\mu : \Omega_X \to [0,\infty]$ which is constructed as follows :*

$$\mu(D) = \inf\{\epsilon > 0 : \exists (D_i)_{i=1,2,\ldots,n} \subset X,\ D \subseteq \cup_{i=1}^{n} D_i,\ diam(D_i) \leq \epsilon\},$$

where

$$diam(D_i) = sup\{||x-y|| :\ x,\ y \in D_i\}.$$

Proposition 1.1 *([9, 15]). Let X be a Banach space, $D,\ D_1,\ D_2$ are bounded subsets of X, then*

(a) D *is relatively compact* $\Longleftrightarrow \mu(D) = 0$.

(b) $\mu(\emptyset) = 0$.

(c) $\mu(D) = \mu(\overline{D})$.

(d) $D_1 \subset D_2 \Longrightarrow \mu(D_1) \leq \mu(D_2)$.

(e) $\mu(D_1 + D_2) \leq \mu(D_1) + \mu(D_2)$.

(f) $\mu(\lambda D) = |\lambda|\mu(D), \lambda \in \mathbb{R}$.

(g) $\mu(D_1 \cup D_2) = max\{\mu(D_1), \mu(D_2)\}$.

(h) $\mu(D_1 \cap D_2) = min\{\mu(D_1), \mu(D_2)\}$.

(i) $\mu(D + x_0) = \mu(D)$ *for any* $x_0 \in X$.

Lemma 1.15 *([23]). If the bounded set* $U \subset C(J, X)$ *is equicontinuous, then*

(i) *the function* $\mu(U(t))$ *is continuous for* $t \in J$*, and*

$$\widehat{\mu}(U) = \sup_{t \in J} \mu(U(t)).$$

(ii) $\mu\left(\int_0^T x(\theta)d\theta : x \in U\right) \leq \int_0^T \mu(U(\theta))d\theta$,

where

$$U(s) = \{x(s) : \ x \in U\}, \ s \in J.$$

1.4 Coincidence degree theory

Definition 1.16 *([24, 39]). Let* X *and* Y *be normed spaces. A linear operator* $L : domL \subset X \to Y$ *is said to be a Fredholm operator of index zero provided that*

1. $imgL$ *is a closed subset of* Y *;*
2. $\dim \ker L = codim\ imgL < +\infty$.

It follows from Definition 1.16 that there exist continuous projections $P : X \to X$ and $Q : Y \to Y$ such that

$$imgP = \ker L, \quad \ker Q = imgL, \quad X = \ker L \oplus \ker P, \quad Y = imgL \oplus imgQ.$$

This implies that the restriction of L to $domL \cap \ker P$, which we will denote by L_P, is an isomorphism onto its image.

Definition 1.17 *([24, 39]). Let* L *be a Fredholm operator of index zero and let* $\Omega \subseteq X$ *be a bounded set with* $domL \cap \Omega \neq \emptyset$*. The operator* $N : \overline{\Omega} \to Y$ *is L-compact in* $\overline{\Omega}$ *if*

1. *the mapping* $QN : \overline{\Omega} \to Y$ *is continuous and* $QN(\overline{\Omega}) \subseteq Y$ *is bounded,*
2. *the mapping* $(L_P)^{-1}(I - Q)N : \overline{\Omega} \to X$ *is completely continuous.*

Theorem 1.1 ***(Mawhin's continuation Theorem)****[47]. Let X and Y be Banach spaces and let $\Omega \subset X$ be a bounded open symmetric set with $0 \in \Omega$.
Let $L : domL \subset X \to Y$ be a Fredholm operator of index zero with $domL \cap \overline{\Omega} \neq \emptyset$ and $N : X \to Y$ be an L-compact operator on $\overline{\Omega}$.
Assume that*

$$Lx - Nx \neq -\lambda(Lx + N(-x))$$

for all $x \in domL \cap \partial\Omega$ and all $\lambda \in (0, 1]$, where $\partial\Omega$ is the boundary of Ω with respect to X. Then the equation $Lx = Nx$ has at least one solution on $domL \cap \overline{\Omega}$.

1.5 Some fixed point theorems

Definition 1.18 *. Let $T : M \subset X \longrightarrow X$ be a bounded operator from a Banach space X into itself. The operator T is called a k-set contraction if there is a number k $(0 \leq k < 1)$ such that*

$$\mu(T(A)) \leq k\mu(A)$$

for all bounded sets A in M. The bounded operator T is called condensing if $\mu(T(A)) < \mu(A)$ for all bounded sets A in M with $\mu(M) > 0$.

Obviously, every k-set contraction for $0 \leq k < 1$ is condensing. Every compact map T is a k-set contraction with $k = 0$.

Theorem 1.2 *(Banach's fixed point theorem [21]). Let C be a non-empty closed subset of a Banach space X, then any contraction mapping T of C into itself has a unique fixed point.*

Theorem 1.3 *(Schauder fixed point theorem [19]). Let X a Banach space and Q be a convex subset of X and $T : Q \longrightarrow Q$ is compact, and continuous map. Then T has at least one fixed point in Q.*

Theorem 1.4 *(Darbo's fixed point theorem [9]). Let M be nonempty, bounded, convex and closed subset of a Banach space X and $T : M \longrightarrow M$ is a continuous operator satisfying $\mu(TA) \leq k\mu(A)$ for any nonempty subset A of M and for some constant $k \in [0, 1)$. Then T has at least one fixed point in M.*

1.6 Types of stability

Theorem 1.5 *([14, 51]). The boundary value problem is Ulam-Hyers stable if there exists $c_f > 0$, such that for any $\epsilon > 0$ and for every solution $z \in C(J, \mathbb{R})$ of the following inequality*

$$|^cD_{0^+}^{u(t)}z(t) + f(t, z(t), I_{0^+}^{u(t)}z(t))| \leq \epsilon, \ t \in J, \tag{1.6}$$

there exists a solution $x \in C(J, \mathbb{R})$ of boundary value problem with

$$|z(t) - x(t)| \leq c_f\epsilon, \ t \in J.$$

Theorem 1.6 *([14, 51]). The Hadamard fractional boundary value problem of variable order is Ulam-Hyers-Rassias stable with respect to $\vartheta \in C(J,X)$ if there exists a real number $c_f > 0$, such that for each $\epsilon > 0$ and for each solution $z \in C(J,X)$ of the inequality*

$$\|{}^H D_{1^+}^{u(t)} z(t) + f(t, z(t))\| \leq \epsilon \vartheta(t), \ t \in J, \tag{1.7}$$

there exists a solution $x \in C(J,X)$ of Hadamard fractional boundary value problem of variable order with

$$\|z(t) - x(t)\| \leq c_f \epsilon \vartheta(t), \ t \in J.$$

Chapitre 2

A Study on the Solutions of a Multiterm Fractional Boundary Value Problem of Variable Order

2.1 Introduction and motivations

In [13], Bai *et al.* studied the existence of solutions for the following nonlinear fractional differential equations of constant order

$$\begin{cases} {}^cD^{\alpha}_{a^+}x(t) = f(t, x(t), I^{\alpha}_{a^+}x(t)), \;\; t \in [a, b], \;\; \alpha \in]0, 1], \\ x(a) = x_a, \end{cases}$$

where ${}^cD^{\alpha}_{a^+}$ and $I^{\alpha}_{a^+}$ stand for the Caputo-Hadamard derivative and Hadamard integral operators, respectively, $f : [a, b] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, $x_a \in \mathbb{R}$, and $0 < a < b < \infty$.

In this chapter we deal with the existence of solutions for multiterm boundary value problem for the nonlinear fractional differential equation of variable order in the format

$$\begin{cases} D^{u(t)}_{0^+}x(t) + f(t, x(t), I^{u(t)}_{0^+}x(t)) = 0, \;\; t \in J := [0, T], \\ x(0) = 0, \;\; x(T) = 0, \end{cases} \tag{2.1}$$

where $0 < T < +\infty$, $1 < u(t) \leq 2$, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $D^{u(t)}_{0^+}$, $I^{u(t)}_{0^+}$ are the Riemann-Liouville fractional derivative and Riemann-Liouville fractional integral of variable-order $u(t)$.

This chapter is divided into the following sections : two important results are as follows : one is relied on Schauder fixed-point theorem, and the other one is relied on the Banach contraction principle, which are provided in Section 2.2. In Section 2.3, a numerical example is provided to validate and apply our theoretical results.

Z. Bouazza, S. Etemad, M. S. Souid, S. Rezapour, F. Matinez and M. K. A. Kaabar, A Study on the Solutions of a Multiterm fractional boundary value problem of Variable Order, *Journal of Function Spaces*, **2021**(2021), 1-9.

2.2 Existence of solutions

All our original main results in this chapter are discussed in this section. Some assumptions are presented as follows :

(H1) Let $n \in \mathbb{N}$ be an integer and the finite sequence of points $\{T_k\}_{k=0}^{n}$ be given such that $0 = T_0 < T_k < T_n = T$, $k = 1, ..., n-1$.
Denote $J_k := (T_{k-1}, T_k]$, $k = 1, 2, ..., n$. Then $\mathcal{P} = \bigcup_{k=1}^{n} J_k$ is a partition of the interval J.
Let $u(t) : J \to (1, 2]$ be a piecewise constant function with respect to $\mathcal{P}$ as follows :

$$u(t) = \sum_{i=1}^{n} u_i I_i(t) = \begin{cases} u_1, & if\ t \in J_1, \\ u_2, & if\ t \in J_2, \\ . & \\ . & \\ . & \\ u_n, & if\ t \in J_n, \end{cases}$$

where $1 < u_i \leq 2$ are constants and I_i is an indicator of the interval $J_i, i = 1, 2, ..., n$:

$$I_i(t) = \begin{cases} 1, & for\ t \in J_i, \\ 0, & for\ elsewhere. \end{cases}$$

(H2) Let $t^{\delta} f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function $(0 < \delta < 1)$. There exist constants $c_j > 0, j = 1, 2, 3$ and $0 < \gamma < 1,\ 0 < \eta < 1$, such that

$$t^{\delta} |f(t, y, z)| \leq c_1 + c_2 |y|^{\gamma} + c_3 |z|^{\eta},$$

for any $y, z \in \mathbb{R}$ and $t \in J$.

(H3) There exist constants $K,\ L > 0,\ 0 < \delta < 1$, such that

$$t^{\delta} |f(t, y_1, z_1) - f(t, y_2, z_2)| \leq K |y_1 - y_2| + L |z_1 - z_2|,$$

for any $y_1,\ y_2,\ z_1,\ z_2 \in \mathbb{R}$ and $t \in J$.

To get our original results, let us first perform an essential analysis to our proposed BVP(2.1).
By (1.2), the equation of the BVP(2.1) can be written as

$$\frac{d^2}{dt^2} \int_0^t \frac{(t-s)^{1-u(s)}}{\Gamma(2-u(s))} x(s) ds + f(t, x(t), I_{0^+}^{u(t)} x(t)) = 0, \quad t \in J. \tag{2.2}$$

According to $(H1)$, equation (2.2) on the interval $J_i, i = 1, 2, ..., n$ can be written as

$$\frac{d^2}{dt^2} \Big(\int_0^{T_1} \frac{(t-s)^{1-u_1}}{\Gamma(2-u_1)} x(s) ds + ... + \int_{T_{i-1}}^{t} \frac{(t-s)^{1-u_i}}{\Gamma(2-u_i)} x(s) ds \Big) + f(t, x(t), I_{0^+}^{u_i} x(t)) = 0, \tag{2.3}$$

for $t \in J_i$. Let us now define the solution to the BVP (2.1), which is essential in this chapter.

Definition 2.1 . *BVP (2.1) has a solution, if there are functions $x_i, i = 1, 2, ..., n$, so that $x_i \in C([0, T_i], \mathbb{R})$ fulfilling equation (2.3) and $x_i(0) = 0 = x_i(T_i)$.*

From our previous analysis above, BVP (2.1) can be expressed as equation (2.2), which can be written on the interval $J_i, i \in \{1, 2, ..., n\}$ as (2.3).
For $0 \leq t \leq T_{i-1}$, by taking $x(t) \equiv 0$, then (2.3) is written as follows :

$$D^{u_i}_{T^+_{i-1}} x(t) + f(t, x(t), I^{u_i}_{T^+_{i-1}} x(t)) = 0, \quad t \in J_i.$$

Let us consider the following boundary value problem :

$$\begin{cases} D^{u_i}_{T^+_{i-1}} x(t) + f(t, x(t), I^{u_i}_{T^+_{i-1}} x(t)) = 0, \quad t \in J_i, \\ x(T_{i-1}) = 0, \quad\quad x(T_i) = 0. \end{cases} \tag{2.4}$$

For the existence of solutions for the problem (2.4), an auxiliary lemma is needed as follows :

Lemma 2.1 *The function $x \in E_i$ is a solution of problem (2.4) if and only if x satisfies the integral equation as follows :*

$$x(t) = \int_{T_{i-1}}^{T_i} G_i(t, s) f(s, x(s), I^{u_i}_{T^+_{i-1}} x(s)) ds, \tag{2.5}$$

where $G_i(t, s)$ is Green's function defined by :

$$G_i(t, s) = \begin{cases} \frac{1}{\Gamma(u_i)} \Big[(T_i - T_{i-1})^{1-u_i} (t - T_{i-1})^{u_i - 1} (T_i - s)^{u_i - 1} - (t - s)^{u_i - 1} \Big], \\ \qquad\qquad\qquad\qquad T_{i-1} \leq s \leq t \leq T_i, \\ \frac{1}{\Gamma(u_i)} (T_i - T_{i-1})^{1-u_i} (t - T_{i-1})^{u_i - 1} (T_i - s)^{u_i - 1}, \\ \qquad\qquad\qquad\qquad T_{i-1} \leq t \leq s \leq T_i, \end{cases}$$

where $i = 1, 2, ..., n$.

Proof. Let $x \in E_i$ be a solution of the BVP (2.4). Now, let us apply the operator $I^{u_i}_{T^+_{i-1}}$ to both sides of the equation of the supposed BVP (2.4).
By Lemma 1.2, we obtain

$$x(t) = w_1 (t - T_{i-1})^{u_i - 1} + w_2 (t - T_{i-1})^{u_i - 2} - \frac{1}{\Gamma(u_i)} \int_{T_{i-1}}^{t} (t - s)^{u_i - 1} f(s, x(s), I^{u_i}_{T^+_{i-1}} x(s)) ds, \quad t \in J_i.$$

By $x(T_{i-1}) = 0$, we obtain $w_2 = 0$.
Let $x(t)$ satisfies $x(T_i) = 0$. Thus, we get $w_1 = (T_i - T_{i-1})^{1-u_i} I^{u_i}_{T^+_{i-1}} f(T_i, x(T_i), I^{u_i}_{T^+_{i-1}} x(T_i))$.
Then, we have

$$x(t) = (T_i - T_{i-1})^{1-u_i} (t - T_{i-1})^{u_i - 1} I^{u_i}_{T^+_{i-1}} f(T_i, x(T_i), I^{u_i}_{T^+_{i-1}} x(T_i)) - I^{u_i}_{T^+_{i-1}} f(t, x(t), I^{u_i}_{T^+_{i-1}} x(t)), \quad t \in J_i,$$

by the continuity of Green's function which implies that

$$x(t) = \int_{T_{i-1}}^{T_i} G_i(t,s) f(s, x(s), I^{u_i}_{T^+_{i-1}} x(s)) ds.$$

Conversely, let $x \in E_i$ be a solution of integral equation (2.5) ; then, by the continuity of function $t^{\delta} f$ and Lemma 1.3, we can easily get that x is the solution of BVP (2.4). The following Proposition will be needed :

Proposition 2.1 *([68]). Let $0 < \delta < 1$ and assume that $t^{\delta} f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, $u(t) : J \to (1,2]$ satisfies $(H1)$. Then, Green's function of BVP (2.4) satisfies the following properties :*

(1) $G_i(t,s) \geq 0$ *for all* $T_{i-1} \leq t, s \leq T_i$,

(2) $\max\limits_{t \in J_i} G_i(t,s) = G_i(s,s), s \in J_i$,

(3) $G_i(s,s)$ *has unique maximum given by*

$$\max_{s \in J_i} G_i(s,s) = \frac{1}{\Gamma(u_i)} \Big(\frac{T_i - T_{i-1}}{4} \Big)^{u_i - 1},$$

where $i = 1, 2, ..., n$.

The first existence result is relied on Theorem 1.3.

Theorem 2.1 *Suppose that (H1)-(H3) hold ; then, the BVP(2.1) possesses at least one solution in E_i.*

Proof. Problem (2.4) can be transformed into a fixed point problem. Let us construct the following operator

$$W : E_i \to E_i,$$

formulated by :

$$Wx(t) = \int_{T_{i-1}}^{T_i} G_i(t,s) f(s, x(s), I^{u_i}_{T^+_{i-1}} x(s)) ds, \quad t \in J_i. \tag{2.6}$$

It follows from the properties of fractional integrals and from the continuity of function $t^{\delta} f$ that the operator $W : E_i \to E_i$ defined in (2.6) is well-defined.
We consider the set

$$B_{R_i} = \{x \in E_i, \|x\|_{E_i} \leq R_i\},$$

where

$$R_i = \max \Big\{ \frac{3c_1}{\Gamma(u_i)} \Big(\frac{T_i - T_{i-1}}{4} \Big)^{u_i - 1} \Big(\frac{T_i^{1-\delta} - T_{i-1}^{1-\delta}}{1-\delta} \Big), \Big(\frac{3c_2}{\Gamma(u_i)} \Big(\frac{T_i - T_{i-1}}{4} \Big)^{u_i - 1} \Big(\frac{T_i^{1-\delta} - T_{i-1}^{1-\delta}}{1-\delta} \Big) \Big)^{\frac{1}{1-\gamma}},$$

$$\Big(\frac{3c_3}{\Gamma(u_i)} \Big(\frac{T_i - T_{i-1}}{4} \Big)^{u_i - 1} \Big(\frac{T_i^{1-\delta} - T_{i-1}^{1-\delta}}{1-\delta} \Big) \Big(\frac{(T_i - T_{i-1})^{u_i}}{\Gamma(u_i + 1)} \Big)^{\eta} \Big)^{\frac{1}{1-\eta}} \Big\}.$$

Clearly, B_{R_i} is nonempty, convex, bounded, and closed.
Now, we prove in the following three steps that W satisfies the hypotheses of Theorem 1.3.

Step 1 : $W(B_{R_i}) \subseteq (B_{R_i})$.
For $x \in B_{R_i}$, by Proposition 2.1 and (H2), we get

$$\begin{aligned}
|Wx(t)| &= \Big|\int_{T_{i-1}}^{T_i} G_i(t,s) f(s,x(s), I^{u_i}_{T^+_{i-1}} x(s))ds\Big| \\
&\leq \int_{T_{i-1}}^{T_i} G_i(t,s)\Big|f(s,x(s), I^{u_i}_{T^+_{i-1}} x(s))ds\Big| \\
&\leq \int_{T_{i-1}}^{T_i} G_i(t,s)s^{-\delta}(c_1 + c_2|x(s)|^\gamma + c_3|I^{u_i}_{T^+_{i-1}} x(s)|^\eta)ds \\
&\leq \frac{1}{\Gamma(u_i)}\Big(\frac{T_i - T_{i-1}}{4}\Big)^{u_i-1}\int_{T_{i-1}}^{T_i} s^{-\delta}\Big(c_1 + c_2|x(s)|^\gamma + c_3\Big(\frac{(T_i - T_{i-1})^{u_i}}{\Gamma(u_i+1)}\Big)^\eta |x(s)|^\eta\Big)ds \\
&\leq \frac{1}{\Gamma(u_i)}\Big(\frac{T_i - T_{i-1}}{4}\Big)^{u_i-1}\Big(\frac{T_i^{1-\delta} - T_{i-1}^{1-\delta}}{1-\delta}\Big)\Big(c_1 + c_2R_i^\gamma + c_3\Big(\frac{(T_i - T_{i-1})^{u_i}}{\Gamma(u_i+1)}\Big)^\eta R_i^\eta\Big) \\
&\leq \frac{R_i}{3} + \frac{R_i}{3} + \frac{R_i}{3} = R_i.
\end{aligned}$$

Which means that $W(B_{R_i}) \subseteq B_{R_i}$.
Step 2 : W is continuous.
We presume that the sequence (x_n) converges to x in E_i. We vertify that

$$\|(Wx_n) - (Wx)\|_{E_i} \to 0, \quad n \to \infty.$$

Indeed, for $t \in J_i$, by Proposition 2.1 and (H3), we obtain
$|(Wx_n)(t) - (Wx)(t)|$

$$\begin{aligned}
&\leq \int_{T_{i-1}}^{T_i} G_i(t,s)\Big|f(s,x_n(s), I^{u_i}_{T^+_{i-1}} x_n(s)) - f(s,x(s), I^{u_i}_{T^+_{i-1}} x(s))\Big|ds \\
&\leq \frac{1}{\Gamma(u_i)}\Big(\frac{T_i - T_{i-1}}{4}\Big)^{u_i-1}\int_{T_{i-1}}^{T_i} \Big|f(s,x_n(s), I^{u_i}_{T^+_{i-1}} x_n(s)) - f(s,x(s), I^{u_i}_{T^+_{i-1}} x(s))\Big|ds \\
&\leq \frac{1}{\Gamma(u_i)}\Big(\frac{T_i - T_{i-1}}{4}\Big)^{u_i-1}\int_{T_{i-1}}^{T_i} s^{-\delta}\Big(K|x_n(s) - x(s)| + LI^{u_i}_{T^+_{i-1}}|x_n(s) - x(s)|\Big)ds
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(u_i)}\Big(\frac{T_i - T_{i-1}}{4}\Big)^{u_i-1}\Big[K\|x_n - x\|_{E_i}\int_{T_{i-1}}^{T_i} s^{-\delta}ds + L\|I^{u_i}_{T^+_{i-1}}(x_n - x)\|_{E_i}\int_{T_{i-1}}^{T_i} s^{-\delta}ds\Big] \\
&\leq \frac{({T_i}^{1-\delta} - {T_{i-1}}^{1-\delta})(T_i - T_{i-1})^{u_{i-1}}}{4^{u_i-1}(1-\delta)\Gamma(u_i)}\Big(K + \frac{L(T_i - T_{i-1})^{u_i}}{\Gamma(u_i+1)}\Big)\|x_n - x\|_{E_i},
\end{aligned}$$

so

$$\|(Wx_n) - (Wx)\|_{E_i} \to 0 \ \ as \ \ n \to \infty.$$

Consequently, W is a continuous operator on E_i.

Step 3 : W is compact.
Now, we will prove that $W(B_{R_i})$ is relatively compact, meaning that W is compact. Clearly, $W(B_{R_i})$ is uniformly bounded because by Step 1, we have

$$W(B_{R_i}) = \{W(x) : x \in B_{R_i}\} \subset B_{R_i}.$$

Thus, for each $x \in B_{R_i}$, we have $\|W(x)\|_{E_i} \leq R_i$ which means that $W(B_{R_i})$ is uniformly bounded. It remains to prove that $W(B_{R_i})$ is equicontinuous.
For $t_1, t_2 \in J_i,\ t_1 < t_2$ and $x \in B_{R_i}$ and $f^\star = \sup\limits_{s \in J_i} f(s, 0, 0)$, we have :

$$\begin{aligned}
&|(Wx)(t_2) - (Wx)(t_1)| \\
&\quad = \Big| \int_{T_{i-1}}^{T_i} G_i(t_2, s) f(s, x(s), I^{u_i}_{T^+_{i-1}} x(s)) ds - \int_{T_{i-1}}^{T_i} G_i(t_1, s) f(s, x(s), I^{u_i}_{T^+_{i-1}} x(s)) ds \Big| \\
&\quad \leq \int_{T_{i-1}}^{T_i} |(G_i(t_2, s) - G_i(t_1, s)) f(s, x(s), I^{u_i}_{T^+_{i-1}} x(s)) ds| \\
&\quad \leq \int_{T_{i-1}}^{T_i} |(G_i(t_2, s) - G_i(t_1, s))| |f(s, x(s), I^{u_i}_{T^+_{i-1}} x(s)) - f(s, 0, 0) + f(s, 0, 0)| ds \\
&\quad \leq \int_{T_{i-1}}^{T_i} |(G_i(t_2, s) - G_i(t_1, s))| |f(s, x(s), I^{u_i}_{T^+_{i-1}} x(s)) - f(s, 0, 0)| ds \\
&\quad + \int_{T_{i-1}}^{T_i} |(G_i(t_2, s) - G_i(t_1, s))| |f(s, 0, 0)| ds \\
&\quad \leq \int_{T_{i-1}}^{T_i} |(G_i(t_2, s) - G_i(t_1, s))| \Big[s^{-\delta} \Big(K|x(s)| + L|I^{u_i}_{T^+_{i-1}}(x(s))| \Big) \Big] ds \\
&\quad + f^\star \int_{T_{i-1}}^{T_i} |(G_i(t_2, s) - G_i(t_1, s))| ds,
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{T_{i-1}}^{T_i} |(G_i(t_2, s) - G_i(t_1, s))| \Big[s^{-\delta} \Big(K|x(s)| + \frac{L(T_i - T_{i-1})^{u_i}}{\Gamma(u_i + 1)} |x(s)| \Big) \Big] ds \\
&+ f^\star \int_{T_{i-1}}^{T_i} |(G_i(t_2, s) - G_i(t_1, s))| ds \\
&\leq T_{i-1}^{-\delta} \Big(K + \frac{L(T_i - T_{i-1})^{u_i}}{\Gamma(u_i + 1)} \Big) R_i \int_{T_{i-1}}^{T_i} |(G_i(t_2, s) - G_i(t_1, s))| ds \\
&+ f^\star \int_{T_{i-1}}^{T_i} |(G_i(t_2, s) - G_i(t_1, s))| ds,
\end{aligned}$$

by the continuity of Green's function G_i. Hence $|(Wx)(t_2)-(Wx)(t_1)| \to 0$ as $|t_2-t_1| \to 0$. It implies that $W(B_{R_i})$ is equicontinuous.
From Steps 1 to 3 and the Arzela-Ascoli theorem, it can be concluded that W is completely continuous.
Now, from Theorem 1.3, problem (2.4) possesses at least a solution $\widetilde{x}_i$ in B_{R_i}.
We let

$$x_i = \begin{cases} 0, & t \in [0, T_{i-1}], \\ \widetilde{x}_i, & t \in J_i, \end{cases} \tag{2.7}$$

we know that $x_i \in C([0,T_i],\mathbb{R})$ defined by (2.7) satisfies the following equation :

$$\frac{d^2}{dt^2}\Big(\int_0^{T_1} \frac{(t-s)^{1-u_1}}{\Gamma(2-u_1)} x_i(s)ds + ... + \int_{T_{i-1}}^{t} \frac{(t-s)^{1-u_i}}{\Gamma(2-u_i)} x_i(s)ds\Big) + f(s, \widetilde{x}_i(s), I_{0^+}^{u_i}\widetilde{x}_i(s)) = 0,$$

for $t \in J_i$, which means that x_i is a solution of (2.3) with $x_i(0) = 0,\ x_i(T_i) = \widetilde{x}_i(T_i) = 0$.
In consequence, we figure out that the BVP (2.1) admits at least a solution defined by :

$$x(t) = \begin{cases} x_1(t), & t \in J_1, \\ x_2(t) = \begin{cases} 0, & t \in J_1, \\ \widetilde{x}_2, & t \in J_2, \end{cases} & . \\ . \\ . \\ . \\ x_n(t) = \begin{cases} 0, & t \in [0, T_{n-1}], \\ \widetilde{x}_n, & t \in J_n, \end{cases} \end{cases} \tag{2.8}$$

and the argument is ended.
Let us discuss our second result which is relied on the Banach contraction principle.

Theorem 2.2 *Suppose that* $(H1)$ *and* $(H3)$ *hold and if*

$$\frac{({T_i}^{1-\delta} - {T_{i-1}}^{1-\delta})(T_i - T_{i-1})^{u_i-1}}{4^{u_i-1}(1-\delta)\Gamma(u_i)}\Big(K + L\frac{(T_i - T_{i-1})^{u_i}}{\Gamma(u_i+1)}\Big) < 1, \tag{2.9}$$

then the BVP (2.1) has a unique solution in E_i.

Proof. Let us use the Banach contraction principle to show that W defined in (2.6) has a unique fixed point.

By Proposition 2.1 and $(H3)$, and for $x(t), y(t) \in E_i$:

$$\begin{aligned}
& |(Wx)(t) - (Wy)(t)| \\
= \ & \Big| \int_{T_{i-1}}^{T_i} G_i(t,s) f(s, x(s), I^{u_i}_{T^+_{i-1}} x(s)) ds - \int_{T_{i-1}}^{T_i} G_i(t,s) f(s, y(s), I^{u_i}_{T^+_{i-1}} y(s)) ds \Big| \\
\leq \ & \int_{T_{i-1}}^{T_i} G_i(t,s) \Big| f(s, x(s), I^{u_i}_{T^+_{i-1}} x(s)) - f(s, y(s), I^{u_i}_{T^+_{i-1}} y(s)) \Big| \\
\leq \ & \frac{1}{\Gamma(u_i)} \Big(\frac{T_i - T_{i-1}}{4} \Big)^{u_i - 1} \int_{T_{i-1}}^{T_i} \Big| f(s, x(s), I^{u_i}_{T^+_{i-1}} x(s)) - f(s, y(s), I^{u_i}_{T^+_{i-1}} y(s)) \Big| \\
\leq \ & \frac{1}{\Gamma(u_i)} \Big(\frac{T_i - T_{i-1}}{4} \Big)^{u_i - 1} \int_{T_{i-1}}^{T_i} s^{-\delta} \Big(K|x(s) - y(s)| + L I^{u_i}_{T^+_{i-1}} |x(s) - y(s)| \Big) ds \\
\leq \ & \frac{1}{\Gamma(u_i)} \Big(\frac{T_i - T_{i-1}}{4} \Big)^{u_i - 1} \Big[K \|x - y\|_{E_i} \int_{T_{i-1}}^{T_i} s^{-\delta} ds + L \frac{(T_i - T_{i-1})^{u_i}}{\Gamma(u_i + 1)} \|x - y\|_{E_i} \int_{T_{i-1}}^{T_i} s^{-\delta} ds \Big] \\
\leq \ & \frac{({T_i}^{1-\delta} - {T_{i-1}}^{1-\delta})(T_i - T_{i-1})^{u_i - 1}}{4^{u_i - 1}(1 - \delta)\Gamma(u_i)} \Big(K + L \frac{(T_i - T_{i-1})^{u_i}}{\Gamma(u_i + 1)} \Big) \|x - y\|_{E_i}.
\end{aligned}$$

Consequently by (2.9), the operator W is a contraction.
Thus, by Banach contraction principle, W has a unique fixed point $\widetilde{x}_i \in E_i$, which is a unique solution of the BVP (2.4). We let

$$x_i = \begin{cases} 0, & t \in [0, T_{i-1}], \\ \widetilde{x}_i, & t \in J_i. \end{cases} \tag{2.10}$$

By assuming $C([0, T_i], \mathbb{R})$ as the set of all continuous functions from $[0, T_i]$ into $\mathbb{R}$, we know that $x_i \in C([0, T_i], \mathbb{R})$ defined by (2.10) satisfies the following equation :

$$\frac{d^2}{dt^2} \Big(\int_0^{T_1} \frac{(t-s)^{1-u_1}}{\Gamma(2-u_1)} x_i(s) ds + ... + \int_{T_{i-1}}^{t} \frac{(t-s)^{1-u_i}}{\Gamma(2-u_i)} x_i(s) ds \Big) + f(s, \widetilde{x}_i(s), I^{u_i}_{0^+} \widetilde{x}_i(s)) = 0,$$

for $t \in J_i$, which means that x_i is a unique solution of (2.3) with $x_i(0) = 0$ and $x_i(T_i) = \widetilde{x}_i(T_i) = 0$. Then,

$$x(t) = \begin{cases} x_1(t), \quad t \in J_1, \\ x_2(t) = \begin{cases} 0, & t \in J_1, \\ \widetilde{x}_2, & t \in J_2, \end{cases} \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) = \begin{cases} 0, & t \in [0, T_{n-1}], \\ \widetilde{x}_n, & t \in J_n, \end{cases} \end{cases} . \tag{2.11}$$

is a unique solution of the BVP (2.1).

2.3 Example

An illustrative numerical example is given in this section to apply and validate all our theoretical results.

Consider the fractional boundary value problem :

$$\begin{cases} D_{0^+}^{u(t)}x(t) + \frac{t^{-\frac{1}{5}}}{1+|x(t)|+|I_{0^+}^{u(t)}x(t)|} = 0, \quad t \in J := [0,2], \\ x(0) = 0, \qquad x(2) = 0. \end{cases} \tag{2.12}$$

Let

$$f(t,y,z) = \frac{t^{-\frac{1}{5}}}{1+|y|+|z|}, \; (t,y,z) \in [0,2] \times [0,+\infty) \times [0,+\infty);$$

$$u(t) = \begin{cases} 1.7, & t \in J_1 := [0,1], \\ 1.8, & t \in J_2 :=]1,2]. \end{cases} \tag{2.13}$$

We see that $u(t)$ satisfies condition (H1). We have :

$$\begin{aligned} t^{\frac{1}{5}}|f(t,y_1,z_1) - f(t,y_2,z_2)| &= \left| t^{\frac{1}{5}} \left(\frac{t^{-\frac{1}{5}}}{1+|y_1|+|z_1|} - \frac{t^{-\frac{1}{5}}}{1+|y_2|+|z_2|} \right) \right|; \\ &= \left| \frac{|y_2|+|z_2|-|y_1|-|z_1|}{(1+|y_1|+|z_1|)(1+|y_2|+|z_2|)} \right|; \end{aligned}$$

$$\le |y_1 - y_2| + |z_1 - z_2|.$$

Thus, (H3) holds with $\delta = \frac{1}{5}$ and $K = L = 1$.
By (2.13), the equation of problem (2.12) is divided into two expressions as follows

$$\begin{cases} D_{0^+}^{1.7}x(t) + \frac{t^{-\frac{1}{5}}}{1+|x(t)|+|I_{0^+}^{1.7}x(t)|} = 0, \quad t \in J_1, \\ D_{1^+}^{1.8}x(t) + \frac{t^{-\frac{1}{5}}}{1+|x(t)|+|I_{1^+}^{1.8}x(t)|} = 0, \quad t \in J_2. \end{cases}$$

For $t \in J_1$, the BVP (2.12) is corresponding to the following boundary value problem :

$$\begin{cases} D_{0^+}^{1.7}x(t) + \frac{t^{-\frac{1}{5}}}{1+|x(t)|+|I_{0^+}^{1.7}x(t)|} = 0, \quad t \in J_1, \\ x(0) = 0, \qquad x(1) = 0. \end{cases} \tag{2.14}$$

We shall check that condition (2.9) is satisfied as follows :

$$\frac{({T_1}^{1-\delta} - {T_0}^{1-\delta})(T_1 - T_0)^{u_1 - 1}}{4^{u_1-1}(1-\delta)\Gamma(u_1)} \Big(K + L \frac{(T_1 - T_0)^{u_1}}{\Gamma(u_1+1)} \Big) = \frac{5}{4^{1.7}\Gamma(1.7)} \Big(1 + \frac{1}{\Gamma(2.7)} \Big) \simeq 0.8587 < 1.$$

By Theorem 2.2, the BVP(2.14) has a unique solution $\widetilde{x}_1 \in E_1$.
For $t \in J_2$, problem (2.12) can been written as follows :

$$\begin{cases} D_{1^+}^{1.8}x(t) + \frac{t^{-\frac{1}{5}}}{1+|x(t)|+|I_{1^+}^{1.8}x(t)|} = 0, \quad t \in J_2, \\ x(1) = 0, \qquad x(2) = 0. \end{cases} \tag{2.15}$$

We see that

$$\frac{({T_2}^{1-\delta} - {T_1}^{1-\delta})(T_2 - T_1)^{u_2-1}}{4^{u_2-1}(1-\delta)\Gamma(u_2)}\Big(K+L\frac{(T_2-T_1)^{u_2}}{\Gamma(u_2+1)}\Big) = \frac{5(2^{\frac{4}{5}}-1)}{4^{1.8}\Gamma(1.8)}\Big(1+\frac{1}{\Gamma(2.8)}\Big) \simeq 0.5237 < 1.$$

Thus, condition (2.9) is satisfied. Therefore, by Theorem 2.2, the BVP (2.15) has a unique solution $\widetilde{x}_2 \in E_2$.
Then, by Theorem 2.2, the BVP (2.12) has a unique solution defined by :

$$x(t) = \begin{cases} \widetilde{x}_1(t), \quad t \in J_1, \\ x_2(t), \quad t \in J_2, \end{cases}$$

where

$$x_2(t) = \begin{cases} 0, \quad t \in J_1, \\ \widetilde{x}_2(t), \quad t \in J_2. \end{cases}$$

Chapitre 3

Multiterm Boundary Value Problem of Caputo Fractional Differential Equations of Variable Order

3.1 Introduction

In this chapter we deal with the existence of solutions and the stability of the obtained solution in the sense of Ulam-Hyers for the boundary value problem (BVP)

$$\begin{cases} {}^{c}D_{0^+}^{u(t)}x(t) + f(t, x(t), I_{0^+}^{u(t)}x(t)) = 0, \ \ t \in J := [0, T], \\ x(0) = 0, \ \ x(T) = 0, \end{cases} \tag{3.1}$$

where $1 < u(t) \leq 2$, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and ${}^{c}D_{0^+}^{u(t)}$, $I_{0^+}^{u(t)}$ are the Caputo fractional derivative and integral Riemann-Liouville of variable-order $u(t)$.

In this chapter, we shall look for a solution of (3.1). Further, we study the stability of the obtained solution of (3.1) in the sense of Ulam-Hyers .

3.2 Existence of solutions

Let us introduce the following assumption :

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(H1) Let $n \in \mathbb{N}$ be an integer and the finite sequence of points $\{T_k\}_{k=0}^{n}$ be given such that $0 = T_0 < T_k < T_n = T$, $k = 1, ..., n-1$.
Denote $J_k := (T_{k-1}, T_k]$, $k = 1, 2, ..., n$. Then $\mathcal{P} = \bigcup_{k=1}^{n} J_k$ is a partition of the interval J.
Let $u(t) : J \to (1, 2]$ be a piecewise constant function with respect to $\mathcal{P}$ as follows :

$$u(t) = \sum_{i=1}^{n} u_i I_i(t) = \begin{cases} u_1, & if\ t \in J_1, \\ u_2, & if\ t \in J_2, \\ . \\ . \\ . \\ u_n, & if\ t \in J_n, \end{cases}$$

where $1 < u_i \leq 2$ are constants and I_i is an indicator of the interval $J_i, i = 1, 2, ..., n$:

$$I_i(t) = \begin{cases} 1, & for\ t \in J_i, \\ 0, & for\ elsewhere. \end{cases}$$

Then, for any $t \in J_i,\ i = 1, 2, ..., n$, the left Caputo fractional derivative of variable order $u(t)$ for function $x(t) \in C(J, \mathbb{R})$, defined by (1.3), could be presented as a sum of left caputo fractional derivatives of constant-orders $u_i,\ i = 1, 2, ..., n$.

$${}^{c}D_{0^+}^{u(t)} x(t) = \int_0^{T_1} \frac{(t-s)^{1-u_1}}{\Gamma(2-u_1)} x^{(2)}(s) ds + ... + \int_{T_{i-1}}^{t} \frac{(t-s)^{1-u_i}}{\Gamma(2-u_i)} x^{(2)}(s) ds. \tag{3.2}$$

Thus, according to (3.2), the equation of the BVP (3.1) can be written for any $t \in J_i,\ i = 1, 2, ..., n$ in the form

$$\int_0^{T_1} \frac{(t-s)^{1-u_1}}{\Gamma(2-u_1)} x^{(2)}(s) ds + ... + \int_{T_{i-1}}^{t} \frac{(t-s)^{1-u_i}}{\Gamma(2-u_i)} x^{(2)}(s) ds + f(t, x(t), I_{0^+}^{u_i} x(t)) = 0, \quad t \in J_i. \tag{3.3}$$

In what follows we shall introduce the solution to the BVP (3.1).

Definition 3.1 . *BVP (3.1) has a solution, if there are functions* $x_i, i = 1, 2, ..., n$, *so that* $x_i \in C([0, T_i], \mathbb{R})$ *fulfilling equation (3.3) and* $x_i(0) = 0 = x_i(T_i)$.

Let the function $x \in C(J, \mathbb{R})$ be such that $x(t) \equiv 0$ on $t \in [0, T_{i-1}]$ and it solves integral equation (3.3). Then (3.3) is reduced to

$${}^{c}D_{T_{i-1}^+}^{u_i} x(t) + f(t, x(t), I_{T_{i-1}^+}^{u_i} x(t)) = 0, \quad t \in J_i.$$

We shall deal with following BVP

$$\begin{cases} {}^{c}D_{T_{i-1}^+}^{u_i} x(t) + f(t, x(t), I_{T_{i-1}^+}^{u_i} x(t)) = 0, \quad t \in J_i, \\ x(T_{i-1}) = 0, \ \ x(T_i) = 0. \end{cases} \tag{3.4}$$

For the existence of solutions for the BVP(3.4), an auxiliary lemma is needed as follows :

Lemma 3.1 *Let $i \in \{1, 2, ..., n\}$ be a natural number, $f \in C(J_i \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and there exists a number $\delta \in (0, 1)$ such that $t^\delta f \in C(J_i \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.*
Then, the function $x \in E_i$ is a solution of BVP (3.4) if and only if x solves the integral equation

$$x(t) = \int_{T_{i-1}}^{T_i} G_i(t, s) f(s, x(s), I_{T_{i-1}^+}^{u_i} x(s)) ds, \tag{3.5}$$

where $G_i(t, s)$ is the Green's function defined by :

$$G_i(t,s) = \begin{cases} \frac{1}{\Gamma(u_i)} \Big[(T_i - T_{i-1})^{-1} (t - T_{i-1})(T_i - s)^{u_i - 1} - (t - s)^{u_i - 1} \Big], \\ \qquad\qquad T_{i-1} \le s \le t \le T_i, \\ \frac{1}{\Gamma(u_i)} (T_i - T_{i-1})^{-1} (t - T_{i-1})(T_i - s)^{u_i - 1}, \\ \qquad\qquad T_{i-1} \le t \le s \le T_i, \end{cases}$$

where $i = 1, 2, ..., n$.

Proof. We presume that $x \in E_i$ is solution of BVP (3.4). Employing the operator $I_{T_{i-1}^+}^{u_i}$ to both sides of (3.4) and regarding Lemma 1.6, we find

$$x(t) = \omega_1 + \omega_2 (t - T_{i-1}) - I_{T_{i-1}^+}^{u_i} f(t, x(t), I_{T_{i-1}^+}^{u_i} x(t)), \quad t \in J_i.$$

By $x(T_{i-1}) = 0$, we get $\omega_1 = 0$.

Let $x(t)$ satisfy $x(T_i) = 0$. So, we observe that

$$\omega_2 = (T_i - T_{i-1})^{-1} I_{T_{i-1}^+}^{u_i} f(T_i, x(T_i), I_{T_{i-1}^+}^{u_i} x(T_i)).$$

Then, we find

$$x(t) = (T_i - T_{i-1})^{-1} (t - T_{i-1}) I_{T_{i-1}^+}^{u_i} f(T_i, x(T_i), I_{T_{i-1}^+}^{u_i} x(T_i)) - I_{T_{i-1}^+}^{u_i} f(t, x(t), I_{T_{i-1}^+}^{u_i} x(t)), t \in J_i,$$

by the continuity of the Green's function which implies that

$$x(t) = \int_{T_{i-1}}^{T_i} G_i(t, s) f(s, x(s), I_{T_{i-1}^+}^{u_i} x(s)) ds.$$

Conversely, let $x \in E_i$ be solution of integral equation (3.5).
Regarding the continuity of function $t^\delta f$ and Lemma 1.7, we deduce that x is the solution of BVP (3.4).
The following Proposition will be needed.

Proposition 3.1 *Let $0 < \delta < 1$ and assume that $t^{\delta} f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous function, $u(t) : J \to (1, 2]$ satisfies $(H1)$, then the Green functions of BVP (3.4) satisfy the following properties :*

(1) $G_i(t, s) \geq 0$ *for all* $T_{i-1} \leq t, s \leq T_i$,

(2) $\max_{t \in J_i} G_i(t, s) = G_i(s, s), \ s \in J_i$,

(3) $G_i(s, s)$ *has one unique maximum given by :*

$$\max_{s \in J_i} G_i(s, s) = \frac{1}{\Gamma(u_i + 1)} \Big[(T_i - T_{i-1})(1 - \frac{1}{u_i})\Big]^{u_i - 1},$$

where $i = 1, 2, ..., n$.

Proof. Let $\varphi(t, s) = (T_i - T_{i-1})^{-1}(t - T_{i-1})(T_i - s)^{u_i - 1} - (t - s)^{u_i - 1}$.
We see that

$$\begin{aligned}
\varphi_t(t, s) &= (T_i - T_{i-1})^{-1}(T_i - s)^{u_i - 1} - (u_i - 1)(t - s)^{u_i - 2}, \\
&\leq (T_i - T_{i-1})^{-1}(T_i - T_{i-1})^{u_i - 1} - (T_i - T_{i-1})^{u_i - 2}, \\
&= 0,
\end{aligned}$$

which means that $\varphi(t, s)$ is nonincreasing with respect to t, so $\varphi(t, s) \geq \varphi(T_i, s) = 0$, for $T_{i-1} \leq s \leq t \leq T_i$.
Thus, together this with the expression of $G_i(t, s)$, we have $G_i(t, s) \geq 0$, for any $T_{i-1} \leq t, \ s \leq T_i, \ i = 1, ..., n$.

Since $\varphi(t, s)$ is nonincreasing with respect to t, then $\varphi(t, s) \leq \varphi(s, s)$ for $T_{i-1} \leq s \leq t \leq T_i$.
On the other hand, for $T_{i-1} \leq t \leq s \leq T_i$, we get

$$(T_i - T_{i-1})^{-1}(t - T_{i-1})(T_i - s)^{u_i - 1} \leq (T_i - T_{i-1})^{-1}(s - T_{i-1})(T_i - s)^{u_i - 1}.$$

These assure that $\max_{t \in [T_{i-1}, T_i]} G_i(t, s) = G_i(s, s), \ s \in [T_{i-1}, T_i], \ i = 1, ..., n$.

Further, we verify (3) of Proposition 3.1. Clearly, the maximum points of $G_i(s, s)$ are not T_{i-1} and T_i, $i = 1, ..., n$.

For $s \in [T_{i-1}, T_i], \ i = 1, ..., n$, we have

$$\begin{aligned}
\frac{dG_i(s, s)}{ds} &= \frac{1}{\Gamma(u_i)}(T_i - T_{i-1})^{-1}\Big[(T_i - s)^{u_i - 1} - (s - T_{i-1})(u_i - 1)(T_i - s)^{u_i - 2}\Big], \\
&= \frac{1}{\Gamma(u_i)}(T_i - T_{i-1})^{-1}(T_i - s)^{u_i - 2}\Big[(T_i - s) - (s - T_{i-1})(u_i - 1)\Big], \\
&= \frac{1}{\Gamma(u_i)}(T_i - T_{i-1})^{-1}(T_i - s)^{u_i - 2}\Big[T_i + (u_i - 1)T_{i-1} - u_i s\Big],
\end{aligned}$$

which indicates that the maximum points of $G_i(s,s)$ is $s=\frac{T_i+(u_i-1)T_{i-1}}{u_i},\ i=1,...,n.$
Hence, for $i=1,...,n$,

$$\max_{s\in[T_{i-1},T_i]} G_i(s,s)=G_i\Big(\frac{T_i+(u_i-1)T_{i-1}}{u_i},\frac{T_i+(u_i-1)T_{i-1}}{u_i}\Big)=\frac{1}{\Gamma(u_i+1)}\Big[(T_i-T_{i-1})(1-\frac{1}{u_i})\Big]^{u_i-1}.$$

We will prove the existence results for the BVP(3.4). First result is based on Theorem 1.3.

Theorem 3.1 *Let the conditions of Lemma 3.1 be satisfied and there exist constants $K,\ L>0$, such that,*
$t^\delta|f(t,y_1,z_1)-f(t,y_2,z_2)|\le K|y_1-y_2|+L|z_1-z_2|$, *for any* $y_i,\ z_i\in\mathbb{R}$, $i=1,2$, $t\in J_i$. *and the inequality*

$$\frac{(T_i^{1-\delta}-T_{i-1}^{1-\delta})\Big((T_i-T_{i-1})(1-\frac{1}{u_i})\Big)^{u_i-1}}{(1-\delta)\Gamma(u_i+1)}(K+L\frac{(T_i-T_{i-1})^{u_i}}{\Gamma(u_i+1)})<1, \tag{3.6}$$

holds.

Then, BVP (3.4) possesses at least one solution in E_i.

Proof. We construct the operator

$$W:E_i\to E_i$$

as follow :

$$Wx(t)=\int_{T_{i-1}}^{T_i}G_i(t,s)f(s,x(s),I_{T_{i-1}^+}^{u_i}x(s))ds,\quad t\in J_i. \tag{3.7}$$

It follows from the properties of fractional integrals and from the continuity of function $t^\delta f$ that the operator $W:E_i\to E_i$ defined in (3.7) is well defined.
Let

$$R_i \ \ge\ \frac{\frac{f^\star}{\Gamma(u_i+1)}(T_i-T_{i-1})^{u_i}(1-\frac{1}{u_i})^{u_i-1}}{1-\frac{(T_i^{1-\delta}-T_{i-1}^{1-\delta})\Big((T_i-T_{i-1})(1-\frac{1}{u_i})\Big)^{u_i-1}}{(1-\delta)\Gamma(u_i+1)}(K+L\frac{(T_i-T_{i-1})^{u_i}}{\Gamma(u_i+1)})},$$

with

$$f^\star=\sup_{t\in J_i}|f(t,0,0)|.$$

We consider the set

$$B_{R_i}=\{x\in E_i,\ \|x\|_{E_i}\le R_i\}.$$

Clearly B_{R_i} is nonempty, closed, convex and bounded.

Now, we demonstrate that W satisfies the assumption of the Theorem 1.3. We shall prove it in three steps.

Step 1 : $W(B_{R_i}) \subseteq (B_{R_i})$.

For $x \in B_{R_i}$, by Proposition 3.1, we have

$$\begin{aligned}
|Wx(t)| &= \Big| \int_{T_{i-1}}^{T_i} G_i(t,s) f\Big(s, x(s), I^{u_i}_{T^+_{i-1}} x(s)\Big) ds \Big| \\
&\leq \int_{T_{i-1}}^{T_i} G_i(t,s) \Big| f\Big(s, x(s), I^{u_i}_{T^+_{i-1}} x(s)\Big)\Big| ds \\
&\leq \tfrac{1}{\Gamma(u_i+1)} \Big((T_i - T_{i-1})(1-\tfrac{1}{u_i})\Big)^{u_i-1} \int_{T_{i-1}}^{T_i} \Big| f\Big(s, x(s), I^{u_i}_{T^+_{i-1}} x(s)\Big) - f(s,0,0)\Big| ds \\
&+ \tfrac{1}{\Gamma(u_i+1)} \Big((T_i - T_{i-1})(1-\tfrac{1}{u_i})\Big)^{u_i-1} \int_{T_{i-1}}^{T_i} |f(s,0,0)| ds \\
&\leq \tfrac{1}{\Gamma(u_i+1)} \Big((T_i - T_{i-1})(1-\tfrac{1}{u_i})\Big)^{u_i-1} \int_{T_{i-1}}^{T_i} s^{-\delta}(K|x(s)| + L|I^{u_i}_{T^+_{i-1}} x(s)|) ds \\
&+ \frac{f^\star}{\Gamma(u_i+1)} (T_i - T_{i-1})^{u_i} (1-\frac{1}{u_i})^{u_i-1} \\
&\leq \frac{(T_i^{1-\delta} - T_{i-1}^{1-\delta})\Big((T_i - T_{i-1})(1-\tfrac{1}{u_i})\Big)^{u_i-1}}{(1-\delta)\Gamma(u_i+1)} (K + L\frac{(T_i - T_{i-1})^{u_i}}{\Gamma(u_i+1)}) R_i \\
&+ \frac{f^\star}{\Gamma(u_i+1)} (T_i - T_{i-1})^{u_i} (1-\frac{1}{u_i})^{u_i-1} \\
&\leq R_i,
\end{aligned}$$

which means that $W(B_{R_i}) \subseteq B_{R_i}$.
Step 2 : W is continuous.
We presume that the sequence (x_n) converges to x in E_i and $t \in J_i$. Then,

$$\begin{aligned}
& |(Wx_n)(t) - (Wx)(t)| \\
\leq\ & \int_{T_{i-1}}^{T_i} G_i(t,s) |f(s, x_n(s), I^{u_i}_{T^+_{i-1}} x_n(s)) - f(s, x(s), I^{u_i}_{T^+_{i-1}} x(s))| ds \\
\leq\ & \frac{1}{\Gamma(u_i+1)} \Big((T_i - T_{i-1})(1-\frac{1}{u_i})\Big)^{u_i-1} \int_{T_{i-1}}^{T_i} s^{-\delta}(K|x_n(s) - x(s)| + LI^{u_i}_{T^+_{i-1}} |x_n(s) - x(s))|) ds \\
\leq\ & \tfrac{K}{\Gamma(u_i+1)} \Big((T_i - T_{i-1})(1-\tfrac{1}{u_i})\Big)^{u_i-1} \|x_n - x\|_{E_i} \int_{T_{i-1}}^{T_i} s^{-\delta} ds \\
+\ & \tfrac{L}{\Gamma(u_i+1)} \Big((T_i - T_{i-1})(1-\tfrac{1}{u_i})\Big)^{u_i-1} \|I^{u_i}_{T^+_{i-1}} (x_n - x)\|_{E_i} \int_{T_{i-1}}^{T_i} s^{-\delta} ds \\
\leq\ & \tfrac{K({T_i}^{1-\delta} - {T_{i-1}}^{1-\delta})\Big((T_i - T_{i-1})(1-\frac{1}{u_i})\Big)^{u_i-1}}{(1-\delta)\Gamma(u_i+1)} \|x_n - x\|_{E_i} + \tfrac{L(T_i - T_{i-1})^{2u_i-1}(1-\frac{1}{u_i})^{u_i-1}({T_i}^{1-\delta} - {T_{i-1}}^{1-\delta})}{(1-\delta)\Big(\Gamma(u_{i+1})\Big)^2} \|x_n - x\|_{E_i} \\
\leq\ & \tfrac{({T_i}^{1-\delta} - {T_{i-1}}^{1-\delta})\Big((T_i - T_{i-1})(1-\frac{1}{u_i})\Big)^{u_i-1}}{(1-\delta)\Gamma(u_i+1)} \Big(K + L\tfrac{(T_i - T_{i-1})^{u_i}}{\Gamma(u_i+1)}\Big) \|x_n - x\|_{E_i}.
\end{aligned}$$

i.e., we obtain

$$\|(Wx_n) - (Wx)\|_{E_i} \to 0 \quad as \quad n \to \infty.$$

Then, the operator W is a continuous on E_i.

Step 3 : W is compact.
Now, we will show that $W(B_{R_i})$ is relatively compact, meaning that W is compact. Clearly $W(B_{R_i})$ is uniformly bounded because by Step 1, we have $W(B_{R_i}) = \{W(x) : x \in B_{R_i}\} \subset W(B_{R_i})$ thus for each $x \in B_{R_i}$ we have $\|W(x)\|_{E_i} \leq R_i$ which means that $W(B_{R_i})$ is bounded. It remains to indicate that $W(B_{R_i})$ is equicontinuous.
For $t_1, t_2 \in J_i,\ t_1 < t_2$ and $x \in B_{R_i}$, we have

$$
\begin{aligned}
&|(Wx)(t_2) - (Wx)(t_1)| \\
&= \Big| \int_{T_{i-1}}^{T_i} G_i(t_2,s) f(s, x(s), I^{u_i}_{T^+_{i-1}} x(s)) ds - \int_{T_{i-1}}^{T_i} G_i(t_1,s) f(s, x(s), I^{u_i}_{T^+_{i-1}} x(s)) ds \Big| \\
\leq &\int_{T_{i-1}}^{T_i} |G_i(t_2,s) - G_i(t_1,s)| |f(s, x(s), I^{u_i}_{T^+_{i-1}} x(s))| ds \\
\leq &\int_{T_{i-1}}^{T_i} |G_i(t_2,s) - G_i(t_1,s)| |f(s, x(s), I^{u_i}_{T^+_{i-1}} x(s)) - f(s,0,0)| ds \\
+ &\int_{T_{i-1}}^{T_i} |G_i(t_2,s) - G_i(t_1,s)| |f(s,0,0)| ds \\
\leq &\int_{T_{i-1}}^{T_i} |G_i(t_2,s) - G_i(t_1,s)| \Big[s^{-\delta} \Big(K|x(s)| + L|I^{u_i}_{T^+_{i-1}} x(s)| \Big) \Big] ds \\
+ &\ f^\star \int_{T_{i-1}}^{T_i} |G_i(t_2,s) - G_i(t_1,s)| ds \\
\leq &\Big(K\|x\|_{E_i} + L\|I^{u_i}_{T^+_{i-1}} x\|_{E_i} \Big) \int_{T_{i-1}}^{T_i} s^{-\delta} |G_i(t_2,s) - G_i(t_1,s)| ds \\
+ &\ f^\star \int_{T_{i-1}}^{T_i} |G_i(t_2,s) - G_i(t_1,s)| ds \\
\leq &\ T_{i-1}^{-\delta} \Big(K + L \frac{(T_i - T_{i-1})^{u_i}}{\Gamma(u_i+1)} \Big) \|x\|_{E_i} \int_{T_{i-1}}^{T_i} |G_i(t_2,s) - G_i(t_1,s)| ds \\
+ &\ f^\star \int_{T_{i-1}}^{T_i} |G_i(t_2,s) - G_i(t_1,s)| ds,
\end{aligned}
$$

by the continuity of the Green's function G_i. Hence $\|(Wx)(t_2) - (Wx)(t_1)\|_{E_i} \to 0$ as $|t_2 - t_1| \to 0$. It implies that $W(B_{R_i})$ is equicontinuous.
Therefore, all conditions of Theorem 1.3 are fulfilled and thus, there exists $\widetilde{x}_i \in B_{R_i}$, such that $W\widetilde{x}_i = \widetilde{x}_i$, which is a solution of the BVP (3.4). Since $B_{R_i} \subset E_i$, the step of Theorem 3.1 is proved.

The second result is based on the Banach contraction principle.

Theorem 3.2 *Let the conditions of Theorem 3.1 be satisfied. Then, BVP (3.4) has a unique solution in* E_i.

Proof. We shall use the Banach contraction principle to prove that W be defined in (3.7) has a unique fixed point.
For $x(t), y(t) \in E_i$, by Proposition 3.1, we obtain that

$$\begin{aligned}
&|(Wx)(t)-(Wy)(t)| \\
=& \Big|\int_{T_{i-1}}^{T_i} G_i(t,s)f(s,x(s),I^{u_i}_{T^+_{i-1}}x(s)) - \int_{T_{i-1}}^{T_i} G_i(t,s)f(s,y(s),I^{u_i}_{T^+_{i-1}}y(s))ds\Big| \\
\le& \int_{T_{i-1}}^{T_i} G_i(t,s)\Big|f(s,x(s),I^{u_i}_{T^+_{i-1}}x(s)) - f(s,y(s),I^{u_i}_{T^+_{i-1}}y(s))\Big|ds \\
\le& \frac{1}{\Gamma(u_i+1)}\Big((T_i-T_{i-1})(1-\frac{1}{u_i})\Big)^{u_i-1}\int_{T_{i-1}}^{T_i} s^{-\delta}\Big(K|x(s)-y(s)|+LI^{u_i}_{T^+_{i-1}}|x(s)-y(s))|\Big)ds \\
\le& \frac{K}{\Gamma(u_i+1)}\Big((T_i-T_{i-1})(1-\frac{1}{u_i})\Big)^{u_i-1}\|x-y\|_{E_i}\int_{T_{i-1}}^{T_i} s^{-\delta}ds \\
+& \frac{L(T_i-T_{i-1})^{2u_i-1}(1-\frac{1}{u_i})^{u_i-1}}{\Big(\Gamma(u_i+1)\Big)^2}\|x-y\|_{E_i}\int_{T_{i-1}}^{T_i} s^{-\delta}ds \\
\le& \frac{1}{\Gamma(u_i+1)}\Big((T_i-T_{i-1})(1-\frac{1}{u_i})\Big)^{u_i-1}\Big(K+\frac{L(T_i-T_{i-1})^{u_i}}{\Gamma(u_i+1)}\Big)\|x-y\|_{E_i}\int_{T_{i-1}}^{T_i} s^{-\delta}ds \\
\le& \frac{(T_i^{1-\delta}-T_{i-1}^{1-\delta})\Big((T_i-T_{i-1})(1-\frac{1}{u_i})\Big)^{u_i-1}}{(1-\delta)\Gamma(u_i+1)}\Big(K+\frac{L(T_i-T_{i-1})^{u_i}}{\Gamma(u_i+1)}\Big)\|x-y\|_{E_i}.
\end{aligned}$$

Consequently by (3.6), the operator W is a contraction.
Hence, by Banach's contraction principal, W has a unique fixed point $\widetilde{x_i} \in E_i$, which is a unique solution of the problem (3.4).

Now, we will prove the existence result for BVP (3.1).
Introduce the following assumption :

(H2) Let $f \in C(J\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and there exists a number $\delta \in (0,1)$ and constants $K,\ L>0$, such that,

$$t^{\delta}|f(t,y_1,z_1)-f(t,y_2,z_2)| \le K|y_1-y_2|+L|z_1-z_2|, \text{for any } y_1,\ y_2,\ z_1,\ z_2 \in \mathbb{R} \text{ and } t \in J.$$

Theorem 3.3 *Let the conditions (H1), (H2) and inequality (3.6) be satisfied for all* $i \in \{1,2,...,n\}$. *Then, the problem (3.1) possesses at least one solution in* $C(J,\mathbb{R})$.

Proof. For any $i \in \{1,2,...,n\}$ according to Theorem 3.1 the BVP(3.4) possesses at least one solution $\widetilde{x_i} \in E_i$.

For any $i \in \{1, 2, ..., n\}$ we define the function

$$x_i = \begin{cases} 0, & t \in [0, T_{i-1}], \\ \widetilde{x}_i, & t \in J_i. \end{cases}$$

Thus, the function $x_i \in C([0, T_i], \mathbb{R})$ solves the integral equation (3.3) for $t \in J_i$ with $x_i(0) = 0,\ x_i(T_i) = \widetilde{x}_i(T_i) = 0$.

Then, the function

$$x(t) = \begin{cases} x_1(t), & t \in J_1, \\ x_2(t) = \begin{cases} 0, & t \in J_1, \\ \widetilde{x}_2, & t \in J_2, \end{cases} & . \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) = \begin{cases} 0, & t \in [0, T_{n-1}], \\ \widetilde{x}_n, & t \in J_n, \end{cases} \end{cases} \tag{3.8}$$

is a solution of the BVP (3.1) in $C(J, \mathbb{R})$.

3.3 Ulam-Hyers stability

Theorem 3.4 *Let the conditions (H1), (H2) and the inequality (3.6) be satisfied. Then, BVP* (3.1) *is Ulam-Hyers stable.*

Proof. Let $\epsilon > 0$ an arbitrary number and the function $z(t)$ from $C(J, \mathbb{R})$ satisfy the following inequality

$$|{}^cD_{0^+}^{u(t)} z(t) + f(t, z(t), I_{0^+}^{u(t)} z(t))| \leq \epsilon,\ t \in J. \tag{3.9}$$

For any $i \in \{1, 2, ..., n\}$ we define the functions $z_1(t) \equiv z(t), t \in [0, T_1]$ and for $i = 2, 3, ..., n$:

$$z_i(t) = \begin{cases} 0, & t \in [0, T_{i-1}], \\ z(t), & t \in J_i. \end{cases}$$

For any $i \in \{1, 2, ..., n\}$ according to equality (3.2), for $t \in J_i$ we get

$${}^cD_{0^+}^{u(t)} z_i(t) = \int_{T_{i-1}}^{t} \frac{(t-s)^{1-u_i}}{\Gamma(2-u_i)} z^{(2)}(s) ds.$$

Taking $I^{u_i}_{T^+_{i-1}}$ of both sides of the inequality (3.9), we obtain

$$\begin{aligned}\Big|z_i(t) &+ \int_{T_{i-1}}^{T_i} G_i(t,s)f(s,z_i(s),I^{u_i}_{T^+_{i-1}}z_i(s))ds\Big| \\ &\leq \epsilon\int_{T_{i-1}}^{t}\frac{(t-s)^{u_i-1}}{\Gamma(u_i)}ds \\ &\leq \epsilon\frac{(T_i-T_{i-1})^{u_i}}{\Gamma(u_i+1)}.\end{aligned}$$

According to Theorem 3.3, BVP(3.1) has a solution $x \in C(J,\mathbb{R})$ defined by $x(t) = x_i(t)$ for $t \in J_i,\ i = 1,2,...,n$, where

$$x_i = \begin{cases} 0, & t \in [0,T_{i-1}], \\ \widetilde{x}_i, & t \in J_i, \end{cases} \tag{3.10}$$

and $\widetilde{x}_i \in E_i$ is a solution of (3.4).
According to Lemma 3.1 the integral equation

$$\widetilde{x}_i(t) = \int_{T_{i-1}}^{T_i} G_i(t,s)f(s,\widetilde{x}_i(s),I^{u_i}_{T^+_{i-1}}\widetilde{x}_i(s))ds, \tag{3.11}$$

holds.

Let $t \in J_i,\ i = 1,2,...,n$. Then by equation (3.10) and (3.11) we get

$$\begin{aligned} & |z(t)-x(t)| = |z(t)-x_i(t)| = |z_i(t)-\widetilde{x}_i(t)| \\ = & \left|z_i(t) - \int_{T_{i-1}}^{T_i} G_i(t,s)f(s,\widetilde{x}_i(s),I^{u_i}_{T^+_{i-1}}\widetilde{x}_i(s))ds\right| \\ \leq & \left|z_i(t) - \int_{T_{i-1}}^{T_i} G_i(t,s)f(s,z_i(s),I^{u_i}_{T^+_{i-1}}z_i(s))ds\right| \\ + & \left|\int_{T_{i-1}}^{T_i} G_i(t,s)f(s,z_i(s),I^{u_i}_{T^+_{i-1}}z_i(s))ds - \int_{T_{i-1}}^{t} G_i(t,s)f(s,\widetilde{x}_i(s),I^{u_i}_{T^+_{i-1}}\widetilde{x}_i)ds\right| \\ \leq & \left|z_i(t) + \int_{T_{i-1}}^{T_i} G_i(t,s)f(s,z_i(s),I^{u_i}_{T^+_{i-1}}z_i(s))ds\right| \\ + & \left|\int_{T_{i-1}}^{T_i} G_i(t,s)f(s,z_i(s),I^{u_i}_{T^+_{i-1}}z_i(s))ds - \int_{T_{i-1}}^{t} G_i(t,s)f(s,\widetilde{x}_i(s),I^{u_i}_{T^+_{i-1}}\widetilde{x}_i)ds\right|ds \\ \leq & \epsilon\tfrac{(T_i-T_{i-1})^{u_i}}{\Gamma(u_i+1)} \\ + & \tfrac{1}{\Gamma(u_i+1)}\Big((T_i-T_{i-1})(1-\tfrac{1}{u_i})\Big)^{u_i-1}\int_{T_{i-1}}^{T_i}|f(s,z_i(s),I^{u_i}_{T^+_{i-1}}z_i(s))ds - f(s,\widetilde{x}_i(s),I^{u_i}_{T^+_{i-1}}\widetilde{x}_i)|ds \\ \leq & \epsilon\tfrac{(T_i-T_{i-1})^{u_i}}{\Gamma(u_i+1)} + \tfrac{1}{\Gamma(u_i+1)}\Big((T_i-T_{i-1})(1-\tfrac{1}{u_i})\Big)^{u_i-1}\int_{T_{i-1}}^{T_i} s^{-\delta}(K|z_i(s)-\widetilde{x}_i(s)| \\ + & LI^{u_i}_{T^+_{i-1}}|z_i(s)-\widetilde{x}_i(s)|)ds \end{aligned}$$

$$\begin{aligned}
&\leq \epsilon\tfrac{(T_i-T_{i-1})^{u_i}}{\Gamma(u_i+1)}+\tfrac{1}{\Gamma(u_i+1)}\Big((T_i-T_{i-1})(1-\tfrac{1}{u_i})\Big)^{u_i-1}(K\|z_i-\widetilde{x}_i\|_{E_i}+L\|I^{u_i}_{T^+_{i-1}}(z_i-\widetilde{x}_i)\|_{E_i})\int_{T_{i-1}}^{T_i}s^{-\delta}ds\\
&\leq \epsilon\tfrac{(T_i-T_{i-1})^{u_i}}{\Gamma(u_i+1)}+\tfrac{({T_i}^{1-\delta}-{T_{i-1}}^{1-\delta})\Big((T_i-T_{i-1})(1-\frac{1}{u_i})\Big)^{u_i-1}}{(1-\delta)\Gamma(u_i+1)}(K\|z_i-\widetilde{x}_i\|_{E_i}+L\tfrac{(T_i-T_{i-1})^{u_i}}{\Gamma(u_i+1)}\|z_i-\widetilde{x}_i\|_{E_i})\\
&\leq \epsilon\tfrac{(T_i-T_{i-1})^{u_i}}{\Gamma(u_i+1)}+\tfrac{({T_i}^{1-\delta}-{T_{i-1}}^{1-\delta})\Big((T_i-T_{i-1})(1-\frac{1}{u_i})\Big)^{u_i-1}}{(1-\delta)\Gamma(u_i+1)}(K+L\tfrac{(T_i-T_{i-1})^{u_i}}{\Gamma(u_i+1)})\|z_i-\widetilde{x}_i\|_{E_i}\\
&\leq \epsilon\tfrac{(T_i-T_{i-1})^{u_i}}{\Gamma(u_i+1)}+\mu\|z-x\|,
\end{aligned}$$

where

$$\mu=\max_{i=1,2,\ldots,n}\frac{({T_i}^{1-\delta}-{T_{i-1}}^{1-\delta})\Big((T_i-T_{i-1})(1-\frac{1}{u_i})\Big)^{u_i-1}}{(1-\delta)\Gamma(u_i+1)}(K+L\frac{(T_i-T_{i-1})^{u_i}}{\Gamma(u_i+1)}).$$

Then,

$$\|z-x\|(1-\mu)\leq\frac{(T_i-T_{i-1})^{u_i}}{\Gamma(u_i+1)}\epsilon.$$

We obtain, for each $t\in J_i$

$$|z(t)-x(t)|\leq\|z-x\|\leq\frac{(T_i-T_{i-1})^{u_i}}{(1-\mu)\Gamma(u_i+1)}\epsilon:=c_f\epsilon.$$

Therefore, by Theorem 1.5, the BVP(3.1) is Ulam-Hyers stable.

3.4 Example

Let us consider the following fractional boundary value problem,

$$\begin{cases} {}^cD^{u(t)}_{0^+}x(t)+\frac{t^{-\frac{1}{2}}}{4e^t(1+|x(t)|+|I^{u(t)}_{0^+}x(t)|)}=0, & t\in J:=[0,2],\\ x(0)=0, \ \ x(2)=0.\end{cases} \tag{3.12}$$

Let

$$f(t,y,z)=\frac{t^{-\frac{1}{2}}}{4e^t(1+y+z)},\ (t,y,z)\in[0,2]\times[0,+\infty)\times[0,+\infty).$$

$$u(t)=\begin{cases}\frac{7}{5}, & t\in J_1:=[0,1],\\ \frac{3}{2}, & t\in J_2:=]1,2].\end{cases} \tag{3.13}$$

Then, we have

$$\begin{aligned} t^{\frac{1}{2}}|f(t,y_1,z_1)-f(t,y_2,z_2)| &= \left|\frac{1}{4e^t}\left(\frac{1}{1+y_1+z_1}-\frac{1}{1+y_2+z_2}\right)\right| \\ &\leq \frac{(|y_1-y_2|+|z_1-z_2|)}{4e^t(1+y_1+z_1)(1+y_2+z_2)} \\ &\leq \frac{1}{4e^t}(|y_1-y_2|+|z_1-z_2|) \\ &\leq \frac{1}{4}|y_1-y_2|+\frac{1}{4}|z_1-z_2|. \end{aligned}$$

Hence the condition (H2) holds with $\delta=\frac{1}{2}$ and $K=L=\frac{1}{4}$.
By (3.13), according to (3.4) we consider two auxiliary boundary value problem for Caputo fractional differential equations of constant order

$$\begin{cases} {}^cD_{0^+}^{\frac{7}{5}}x(t)+\frac{t^{-\frac{1}{2}}}{4e^t(1+|x(t)|+|I_{0^+}^{\frac{7}{5}}x(t)|)}=0, \quad t\in J_1, \\ x(0)=0, \;\; x(1)=0. \end{cases} \tag{3.14}$$

and

$$\begin{cases} {}^cD_{1^+}^{\frac{3}{2}}x(t)+\frac{t^{-\frac{1}{2}}}{4e^t(1+|x(t)|+|I_{1^+}^{\frac{3}{2}}x(t)|)}=0, \quad t\in J_2, \\ x(1)=0, \;\; x(2)=0. \end{cases} \tag{3.15}$$

Next, we prove that the condition (3.6) is fulfilled for $i=1$. Indeed,

$$\frac{(T_1^{1-\delta}-T_0^{1-\delta})\Big((T_1-T_0)(1-\frac{1}{u_1})\Big)^{u_1-1}}{(1-\delta)\Gamma(u_1+1)}(K+L\frac{(T_1-T_0)^{u_1}}{\Gamma(u_1+1)})=\frac{(1-\frac{5}{7})^{\frac{2}{5}}}{\frac{1}{2}\Gamma(\frac{12}{5})}(\frac{1}{4}+\frac{1}{4\Gamma(\frac{12}{5})})\simeq 0.4402<1.$$

Accordingly the condition (3.6) is achieved. By Theorem 3.1, the problem (3.14) has a solution $\widetilde{x}_1\in E_1$.
We prove that the condition (3.6) is fulfilled for $i=2$. Indeed,

$$\frac{(T_2^{1-\delta}-T_1^{1-\delta})\Big((T_2-T_1)(1-\frac{1}{u_2})\Big)^{u_2-1}}{(1-\delta)\Gamma(u_2+1)}(K+L\frac{(T_2-T_1)^{u_2}}{\Gamma(u_2+1)})=\frac{(2^{\frac{1}{2}}-1)(1-\frac{2}{3})^{\frac{1}{2}}}{\frac{1}{2}\Gamma(\frac{5}{2})}(\frac{1}{4}+\frac{1}{4\Gamma(\frac{5}{2})})$$
$$\simeq 0.1576<1.$$

Thus, the condition (3.6) is satisfied.
According to Theorem 3.1, the BVP (3.15) possesses a solution $\widetilde{x}_2\in E_2$.
Then, by Theorem 3.3, the BVP (3.12) has a solution

$$x(t)=\begin{cases} \widetilde{x}_1(t), & t\in J_1, \\ x_2(t), & t\in J_2, \end{cases}$$

where

$$x_2(t)=\begin{cases} 0, & t\in J_1, \\ \widetilde{x}_2(t), & t\in J_2. \end{cases}$$

According to Theorem 3.4, BVP(3.12) is Ulam-Hyers stable.

Chapitre 4

Darbo Fixed Point Criterion on Solutions of a Hadamard Nonlinear Variable Order Problem and Ulam-Hyers-Rassias Stability

4.1 Introduction

In this chapter, we investigate the existence of solutions for the nonlinear **Hadamard** fractional boundary value problem of variable order as follows :

$$\begin{cases} {}^H D_{1^+}^{u(t)} x(t) + f(t, x(t)) = 0, & t \in J := [1, T], \\ x(1) = x(T) = 0, \end{cases} \tag{4.1}$$

where $1 < T < +\infty$, $1 < u(t) \leq 2$, $f : J \times X \to X$ is a continuous function (X is a Banach space) and ${}^H D_{1^+}^{u(t)}$ specifies the Hadamard derivative of variable order $u(t)$.
For the first time, as the novelty of this chapter, we here consider a fractional boundary value problem in the variable order Hadamard settings and establish the existence specifications of solutions to mentioned system on the generalized subintervals by combining the existing notions in relation to the Kuratowski measure of noncompactness in the context of Darbo fixed point criterion. The piece-wise constant functions will play a vital role in our study for converting the Hadamard fractional boundary value problem of variable order (4.1) to the standard Hadamard fractional boundary value problem. Lastly, another criterion of the behavior of solutions like the Ulam-Hyers-Rassias sta-

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bility is analyzed and a numerical illustrative example will complete the consistency of our findings.

4.2 Existence criterion of solutions

Let us introduce the following assumptions :

(H1) Let $n \in \mathbb{N}$ be an integer and the finite sequence of points $\{T_k\}_{k=0}^{n}$ be given such that $1 = T_0 < T_k < T_n = T$, $k = 1, ..., n-1$.
Denote $J_k := (T_{k-1}, T_k]$, $k = 1, 2, ..., n$. Then $\mathcal{P} = \bigcup_{k=1}^{n} J_k$ is a partition of the interval J.
Let $u(t) : J \to (1, 2]$ be a piecewise constant function with respect to $\mathcal{P}$ as follows :

$$u(t) = \sum_{i=1}^{n} u_i I_i(t) = \begin{cases} u_1, & if\ t \in J_1, \\ u_2, & if\ t \in J_2, \\ . & \\ . & \\ . & \\ u_n, & if\ t \in J_n, \end{cases}$$

where $1 < u_i \leq 2$ are constants and I_i is an indicator of the interval $J_i, i = 1, 2, ..., n$:

$$I_i(t) = \begin{cases} 1, & for\ t \in J_i, \\ 0, & for\ elsewhere. \end{cases}$$

(H2) Let $(\ln t)^{\delta} f : J \times X \to X$ be continuous, $(0 < \delta < 1)$, and there exists a $K > 0$, such that $(\ln t)^{\delta} \|f(t, y_1) - f(t, y_2)\| \leq K \|y_1 - y_2\|$, for any $y_1,\ y_2 \in X$ and $t \in J$.

Further, for a supposed set U of all mappings $u : J \to X$, define

$$U(t) = \{u(t), u \in U\},\ t \in J,$$

and

$$U(J) = \{U(t) : u \in U,\ t \in J\}.$$

Let us now establish the existence of solution for the Hadamard fractional boundary value problem of variable order (4.1) via KMNC and Darbo's criterion (Theorem 1.4).

Then, for any $t \in J_i,\ i = 1, 2, ..., n$, the left Hadamard fractional derivative of variable order $u(t)$ for function $x(t) \in C(J, X)$, could be presented as a sum of left Hadamard fractional derivatives of constant-orders $u_i,\ i = 1, 2, ..., n$.

$$\begin{aligned} {}^{H}D_{1^+}^{u(t)} x(t) = (t\frac{d}{dt})^2 \Big(\frac{1}{\Gamma(2-u_1)} \int_1^{T_1} (\ln \frac{t}{s})^{1-u_1} \frac{x(s)}{s} ds + \dots \\ + \frac{1}{\Gamma(2-u_i)} \int_{T_{i-1}}^{t} (\ln \frac{t}{s})^{1-u_i} \frac{x(s)}{s} ds \Big) ds. \end{aligned} \tag{4.2}$$

Thus, according to (4.2), the equation of the Hadamard fractional boundary value problem of variable order (4.1) can be written for any $t \in J_i,\ i = 1, 2, ..., n$ in the form

$$(t\frac{d}{dt})^2\Big(\frac{1}{\Gamma(2-u_1)}\int_1^{T_1}(\ln\frac{t}{s})^{1-u_1}\frac{x(s)}{s}ds+\dots$$
$$+\frac{1}{\Gamma(2-u_i)}\int_{T_{i-1}}^{t}(\ln\frac{t}{s})^{1-u_i}\frac{x(s)}{s}ds\Big)+f(t,x(t))=0. \tag{4.3}$$

Definition 4.1 . *The Hadamard fractional boundary value problem of variable order* (4.1) *has a solution, if there are functions $x_i, i = 1, 2, ..., n$, so that $x_i \in C([1, T_i], X)$ fulfilling equation (4.3) and $x_i(1) = 0 = x_i(T_i)$.*

Let the function $x \in C(J, X)$ be such that $x(t) \equiv 0$ on $t \in [1, T_{i-1}]$ and it solves integral equation (4.3). Then (4.3) is reduced to

$${}^HD^{u_i}_{T^+_{i-1}}x(t)+f(t,x(t))=0, \quad t\in J_i.$$

In this case, we follow our study by considering the standard Hadamard constant-order fractional boundary value problem as follows :

$$\begin{cases} {}^HD^{u_i}_{T^+_{i-1}}x(t)+f(t,x(t))=0, \quad t\in J_i,\\ x(T_{i-1})=0, \ \ x(T_i)=0.\end{cases} \tag{4.4}$$

The fundamental part of our analysis regarding solutions of the Hadamard constant-order fractional boundary value problem (4.4) is discussed below.

Lemma 4.1 *A function $x \in E_i$ is a solution of the Hadamard constant-order fractional boundary value problem* (4.4) *if and only if x fulfills the integral equation*

$$x(t)=\int_{T_{i-1}}^{T_i}\frac{1}{s}G_i(t,s)f(s,x(s))ds, t\in J_i, \tag{4.5}$$

where $G_i(t, s)$ stands for the Green function formulated by :

$$G_i(t,s)=\frac{1}{\Gamma(u_i)}\begin{cases}\Big(\ln\frac{T_i}{T_{i-1}}\Big)^{1-u_i}\Big[\Big(\ln\frac{t}{T_{i-1}}\Big)\Big(\ln\frac{T_i}{s}\Big)\Big]^{u_i-1}-\Big(\ln\frac{t}{s}\Big)^{u_i-1}, \\ \qquad\qquad T_{i-1}\le s\le t\le T_i,\\ \Big(\ln\frac{T_i}{T_{i-1}}\Big)^{1-u_i}\Big[\Big(\ln\frac{t}{T_{i-1}}\Big)\Big(\ln\frac{T_i}{s}\Big)\Big]^{u_i-1}, \\ \qquad\qquad T_{i-1}\le t\le s\le T_i,\end{cases} \tag{4.6}$$

where $i \in \{1, 2, ..., n\}$.

Proof. Suppose that $x \in E_i$ satisfies the Hadamard constant-order fractional boundary value problem (4.4). Let us employ the operator ${}^H I^{u_i}_{T^+_{i-1}}$ on both sides (4.4) and using Lemma 1.8, we get :

$$x(t) = \omega_1(\ln \frac{t}{T_{i-1}})^{u_i-1} + \omega_2(\ln \frac{t}{T_{i-1}})^{u_i-2} -^H I^{u_i}_{T^+_{i-1}} f(t, x(t)), \quad t \in J_i, \ i \in \{1, 2, ..., n\}.$$

From definition of f along with $x(T_{i-1}) = 0$, we get $\omega_2 = 0$.
Suppose that x satisfies $x(T_i) = 0$. Hence,

$$\omega_1 = (\ln \frac{T_i}{T_{i-1}})^{1-u_i} \ {}^H I^{u_i}_{T^+_{i-1}} f(T_i, x(T_i)).$$

Thus,

$$x(t) = (\ln \frac{T_i}{T_{i-1}})^{1-u_i}(\ln \frac{t}{T_{i-1}})^{u_i-1} \ {}^H I^{u_i}_{T^+_{i-1}} f(T_i, x(T_i)) -^H I^{u_i}_{T^+_{i-1}} f(t, x(t)), \quad t \in J_i.$$

Then, the solution of the Hadamard constant-order fractional boundary value problem (4.4) is given by

$$\begin{aligned} x(t) &= (\ln \frac{T_i}{T_{i-1}})^{1-u_i}(\ln \frac{t}{T_{i-1}})^{u_i-1} \frac{1}{\Gamma(u_i)} \int_{T_{i-1}}^{T_i} (\ln \frac{T_i}{s})^{u_i-1} \frac{f(s, x(s))}{s} ds \\ &- \frac{1}{\Gamma(u_i)} \int_{T_{i-1}}^{t} (\ln \frac{t}{s})^{u_i-1} \frac{f(s, x(s))}{s} ds \\ &= \frac{1}{\Gamma(u_i)} \Big[\int_{T_{i-1}}^{t} \Big[(\ln \frac{T_i}{T_{i-1}})^{1-u_i}(\ln \frac{t}{T_{i-1}})^{u_i-1}(\ln \frac{T_i}{s})^{u_i-1} - (\ln \frac{t}{s})^{u_i-1} \Big] \frac{f(s, x(s))}{s} ds \\ &+ \int_{t}^{T_i} (\ln \frac{T_i}{T_{i-1}})^{1-u_i}(\ln \frac{t}{T_{i-1}})^{u_i-1}(\ln \frac{T_i}{s})^{u_i-1} \frac{f(s, x(s))}{s} ds \Big], \end{aligned}$$

and the continuity property of the Green function gives

$$x(t) = \int_{T_{i-1}}^{T_i} \frac{1}{s} G_i(t, s) f(s, x(s)) ds, \ t \in J_i.$$

Conversely, let $x \in E_i$ be a solution of integral equation (4.5). Because of the continuity of $(\ln\ t)^\delta f$ and by Lemma 1.9, it is simply verified that x is the solution of Hadamard constant-order fractional boundary value problem (4.4).

Proposition 4.1 *Let* $0 < \delta < 1$ *and assume that* $(\ln\ t)^\delta f : J \times X \to X$ *is continuous,* $u(t) : J \to (1, 2]$ *satisfies* $(H1)$*. Then, Green's function given by* (4.6) *satisfies the following properties :*

(1) $G_i(t, s) \geq 0$ *for all* $T_{i-1} \leq t, \ s \leq T_i$*,*

(2) $\max\limits_{t\in J_i} G_i(t,s) = G_i(s,s),\ s \in J_i,$

(3) $G_i(s,s)$ *has one unique maximum given by*

$$\max_{s\in J_i} G_i(s,s) = \frac{1}{\Gamma(u_i)}\Big(\frac{\ln T_i - \ln T_{i-1}}{4}\Big)^{u_i-1},$$

where $i = 1, 2, ..., n.$

Proof. Let $\varphi(t,s) = \Big(\ln\frac{T_i}{T_{i-1}}\Big)^{1-u_i}\Big[\Big(\ln\frac{t}{T_{i-1}}\Big)\Big(\ln\frac{T_i}{s}\Big)\Big]^{u_i-1} - \Big(\ln\frac{t}{s}\Big)^{u_i-1}.$
We see that

$$\begin{aligned}\varphi_t(t,s) &= \Big(\frac{u_i-1}{t}\Big)\Big(\ln\frac{T_i}{T_{i-1}}\Big)^{1-u_i}\Big(\ln\frac{T_i}{s}\Big)^{u_i-1}\Big(\ln\frac{t}{T_{i-1}}\Big)^{u_i-2} - \Big(\frac{u_i-1}{t}\Big)\Big(\ln\frac{t}{s}\Big)^{u_i-2}\\ &\le \Big(\frac{u_i-1}{t}\Big)\Big(\ln\frac{T_i}{s}\Big)^{1-u_i}\Big(\ln\frac{T_i}{s}\Big)^{u_i-1}\Big(\ln\frac{t}{s}\Big)^{u_i-2} - \Big(\frac{u_i-1}{t}\Big)\Big(\ln\frac{t}{s}\Big)^{u_i-2} = 0,\end{aligned}$$

which means that $\varphi(t,s)$ is nonincreasing with respect to t, so $\varphi(t,s) \ge \varphi(T_i,s) = 0$, for $T_{i-1} \le s \le t \le T_i$.
Thus, $G_i(t,s) \ge 0$ for any $T_{i-1} \le t,\ s \le T_i,\ i = 1, ..., n.$
Since $\varphi(t,s)$ is nonincreasing with respect to t, then $\varphi(t,s) \le \varphi(s,s)$ for $T_{i-1} \le s \le t \le T_i$.
On the other hand, for $T_{i-1} \le t \le s \le T_i$, we get

$$\Big(\ln\Big(\frac{T_i}{T_{i-1}}\Big)\Big)^{1-u_i}\Big(\ln\Big(\frac{t}{T_{i-1}}\Big)\ln\Big(\frac{T_i}{s}\Big)\Big)^{u_i-1} \le \Big(\ln\Big(\frac{T_i}{T_{i-1}}\Big)\Big)^{1-u_i}\Big(\ln\Big(\frac{s}{T_{i-1}}\Big)\ln\Big(\frac{T_i}{s}\Big)\Big)^{u_i-1}.$$

These confirm that $\max\limits_{t\in[T_{i-1},T_i]} G_i(t,s) = G_i(s,s),\ s \in [T_{i-1},T_i],\ i = 1, ..., n.$
Further, we verify (3) of Proposition 4.1. Clearly, the maximum points of $G_i(s,s)$ are not T_{i-1} and T_i, $i = 1, ..., n.$
For $s \in [T_{i-1},T_i],\ i = 1, ..., n$, we have

$$\begin{aligned}\frac{dG_i(s,s)}{ds} &= \Big(\frac{u_i-1}{s}\Big)\Big(\ln\frac{T_i}{T_{i-1}}\Big)^{1-u_i}\Big(\ln\frac{s}{T_{i-1}}\Big)^{u_i-2}\Big(\ln\frac{T_i}{s}\Big)^{u_i-2}\Big[\Big(\ln\frac{T_i}{s}\Big) - \Big(\ln\frac{s}{T_{i-1}}\Big)\Big],\\ &= \Big(\frac{u_i-1}{s}\Big)\Big(\ln\frac{T_i}{T_{i-1}}\Big)^{1-u_i}\Big(\ln\frac{s}{T_{i-1}}\Big)^{u_i-2}\Big(\ln\frac{T_i}{s}\Big)^{u_i-2}\Big[\ln(T_iT_{i-1}) - \ln(s^2)\Big],\end{aligned}$$

which indicates that the maximum points of $G_i(s,s)$ is $s = \sqrt{T_iT_{i-1}},\ i = 1, ..., n.$
Hence, for $i = 1, ..., n$,

$$\begin{aligned}\max_{s\in[T_{i-1},T_i]} G_i(s,s) &= G_i\Big(\sqrt{T_iT_{i-1}}, \sqrt{T_iT_{i-1}}\Big)\\ &= \frac{1}{\Gamma(u_i)}\Big(\frac{1}{4}\ln\frac{T_i}{T_{i-1}}\Big)^{u_i-1}\\ &= \frac{1}{\Gamma(u_i)}\Big(\frac{\ln T_i - \ln T_{i-1}}{4}\Big)^{u_i-1}.\end{aligned}$$

The existence of solutions for the Hadamard constant-order fractional boundary value problem (4.4) in this chapter depends on the hypotheses of Theorem 1.4 which we investigate them in this position.

Theorem 4.1 *Suppose that both (H1) and (H2) hold, and*

$$\frac{K\Big((\ln T_i)^{1-\delta}-(\ln T_{i-1})^{1-\delta}\Big)\Big(\ln T_i-\ln T_{i-1}\Big)^{u_i-1}}{4^{u_i-1}(1-\delta)\Gamma(u_i)}<1. \tag{4.7}$$

Then, the Hadamard constant-order fractional boundary value problem (4.4) *possesses at least one solution on* E_i.

Proof. We construct the operator

$$W:E_i\to E_i$$

as follow

$$Wx(t)=\int_{T_{i-1}}^{T_i}\frac{1}{s}G_i(t,s)f(s,x(s))ds,\quad t\in J_i. \tag{4.8}$$

It follows from the properties of fractional integrals and from the continuity of function $(\ln t)^\delta f$ that the operator $W:E_i\to E_i$ defined in (4.8) is well defined.

Let

$$R_i\geq\frac{\dfrac{f^\star(\ln T_i-\ln T_{i-1})^{u_i}}{4^{u_i-1}\Gamma(u_i)}}{1-\dfrac{K\Big((\ln T_i)^{1-\delta}-(\ln T_{i-1})^{1-\delta}\Big)\Big(\ln T_i-\ln T_{i-1}\Big)^{u_i-1}}{4^{u_i-1}(1-\delta)\Gamma(u_i)}},$$

with

$$f^\star=\sup_{t\in J_i}\|f(t,0)\|.$$

Let us consider the following set :

$$B_{R_i}=\{x\in E_i,\ \|x\|_{E_i}\leq R_i\}.$$

Clearly, B_{R_i} is nonempty, convex, bounded, and closed.
We shall show that W satisfies Theorem 1.4 in four steps.
Step 1 : $W(B_{R_i})\subseteq(B_{R_i})$.

For $x \in B_{R_i}$, by Proposition 4.1 and (H2), we get

$$\begin{aligned}
\|Wx(t)\| &= \Big\| \int_{T_{i-1}}^{T_i} \frac{1}{s} G_i(t,s) f(s,x(s)) ds \Big\| \\
&\leq \int_{T_{i-1}}^{T_i} \frac{1}{s} G_i(t,s) \|f(s,x(s))\| ds \\
&\leq \frac{1}{\Gamma(u_i)} \Big(\frac{\ln T_i - \ln T_{i-1}}{4} \Big)^{u_i - 1} \int_{T_{i-1}}^{T_i} \frac{1}{s} \|f(s,x(s))\| ds \\
&\leq \frac{1}{\Gamma(u_i)} \Big(\frac{\ln T_i - \ln T_{i-1}}{4} \Big)^{u_i - 1} \int_{T_{i-1}}^{T_i} \frac{1}{s} \Big\| f(s,x(s)) - f(s,0) \Big\| ds \\
&+ \frac{1}{\Gamma(u_i)} \Big(\frac{\ln T_i - \ln T_{i-1}}{4} \Big)^{u_i - 1} \int_{T_{i-1}}^{T_i} \frac{1}{s} \|f(s,0)\| ds \\
&\leq \frac{1}{\Gamma(u_i)} \Big(\frac{\ln T_i - \ln T_{i-1}}{4} \Big)^{u_i - 1} \int_{T_{i-1}}^{T_i} \frac{1}{s} (\ln s)^{-\delta} (K \|x(s)\|) ds \\
&+ \frac{f^\star \Big(\ln T_i - \ln T_{i-1} \Big)^{u_i}}{4^{u_i - 1} \Gamma(u_i)} \\
&\leq \frac{K}{\Gamma(u_i)} \Big(\frac{\ln T_i - \ln T_{i-1}}{4} \Big)^{u_i - 1} \|x\|_{E_i} \int_{T_{i-1}}^{T_i} \frac{1}{s} (\ln s)^{-\delta} ds \\
&+ \frac{f^\star \Big(\ln T_i - \ln T_{i-1} \Big)^{u_i}}{4^{u_i - 1} \Gamma(u_i)} \\
&\leq \frac{K}{\Gamma(u_i)} \Big(\frac{\ln T_i - \ln T_{i-1}}{4} \Big)^{u_i - 1} R_i \Big(\frac{(\ln T_i)^{1-\delta} - (\ln T_{i-1})^{1-\delta}}{1-\delta} \Big) \\
&+ \frac{f^\star \Big(\ln T_i - \ln T_{i-1} \Big)^{u_i}}{4^{u_i - 1} \Gamma(u_i)} \\
&\leq \frac{K \Big((\ln T_i)^{1-\delta} - (\ln T_{i-1})^{1-\delta} \Big) \Big(\ln T_i - \ln T_{i-1} \Big)^{u_i - 1}}{4^{u_i - 1} (1-\delta) \Gamma(u_i)} R_i
\end{aligned}$$

$$\begin{aligned}
&+ \frac{f^\star \Big(\ln T_i - \ln T_{i-1} \Big)^{u_i}}{4^{u_i - 1} \Gamma(u_i)} \\
&\leq R_i,
\end{aligned}$$

which means that $W(B_{R_i}) \subseteq B_{R_i}$.
Step 2 : W is continuous.
We presume that the sequence (x_n) converges to x in E_i and $t \in J_i$. Then,

$$\begin{aligned}
\|(Wx_n)(t)-(Wx)(t)\| &\leq \int_{T_{i-1}}^{T_i}\frac{1}{s}G_i(t,s)\Big\|f(s,x_n(s))-f(s,x(s))\Big\|ds\\
&\leq \frac{1}{\Gamma(u_i)}\Big(\frac{\ln T_i-\ln T_{i-1}}{4}\Big)^{u_i-1}\int_{T_{i-1}}^{T_i}\frac{1}{s}\Big\|f(s,x_n(s))-f(s,x(s))\Big\|ds\\
&\leq \frac{1}{\Gamma(u_i)}\Big(\frac{\ln T_i-\ln T_{i-1}}{4}\Big)^{u_i-1}\int_{T_{i-1}}^{T_i}\frac{1}{s}(\ln s)^{-\delta}(K\|x_n(s)-x(s)\|)ds\\
&\leq \frac{1}{\Gamma(u_i)}\Big(\frac{\ln T_i-\ln T_{i-1}}{4}\Big)^{u_i-1}(K\|x_n-x\|_{E_i})\int_{T_{i-1}}^{T_i}\frac{1}{s}(\ln s)^{-\delta}ds\\
&\leq \frac{K\Big((\ln T_i)^{1-\delta}-(\ln T_{i-1})^{1-\delta}\Big)\Big(\ln T_i-\ln T_{i-1}\Big)^{u_i-1}}{4^{u_i-1}(1-\delta)\Gamma(u_i)}\|x_n-x\|_{E_i},
\end{aligned}$$

i.e., we get

$$\|(Wx_n)-(Wx)\|_{E_i}\to 0 \text{ as } n\to\infty.$$

Then, the operator W is a continuous on E_i.
Step 3 : $W(B_{R_i})$ is bounded and equicontinous.
From Step 1, $W(B_{R_i}) = \{W(x) : x \in B_{R_i}\} \subset B_{R_i}$, thus for each $x \in B_{R_i}$, we get $\|W(x)\|_{E_i} \leq R_i$, in other ways, it means that $W(B_{R_i})$ is bounded. It remains to check the equicontinuity of $W(B_{R_i})$.
Now, $\forall\, t_1 < t_2 \in J_i,\ t_1 < t_2$ and $x \in B_{R_i}$, we write

$$\begin{aligned}
\|(Wx)(t_2) - (Wx)(t_1)\| &= \Big\|\int_{T_{i-1}}^{T_i}\frac{1}{s}G_i(t_2,s)f(s,x(s))ds-\int_{T_{i-1}}^{T_i}\frac{1}{s}G_i(t_1,s)f(s,x(s))ds\Big\|\\
&\leq \int_{T_{i-1}}^{T_i}\frac{1}{s}\Big\|\Big(G_i(t_2,s)-G_i(t_1,s)\Big)f(s,x(s))\Big\|ds\\
&\leq \int_{T_{i-1}}^{T_i}\frac{1}{s}\Big\|G_i(t_2,s)-G_i(t_1,s)\Big\|\|f(s,x(s))\|ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{T_{i-1}}^{T_i}\frac{1}{s}\Big\|G_i(t_2,s)-G_i(t_1,s)\Big\|\Big(\Big\|f(s,x(s))-f(s,0)\Big\|+\|f(s,0)\|\Big)ds\\
&\leq \int_{T_{i-1}}^{T_i}\frac{1}{s}\Big\|G_i(t_2,s)-G_i(t_1,s)\Big\|\Big[(\ln s)^{-\delta}(K\|x(s)\|)+f^\star\Big]ds\\
&\leq \int_{T_{i-1}}^{T_i}\Big\|G_i(t_2,s)-G_i(t_1,s)\Big\|\Big[\frac{1}{s}(\ln s)^{-\delta}\Big(K\|x\|_{E_i}\Big)+\frac{1}{s}f^\star\Big]ds\\
&\leq \frac{K(\ln T_{i-1})^{-\delta}}{T_{i-1}}\|x\|_{E_i}R_i\int_{T_{i-1}}^{T_i}\Big\|G_i(t_2,s)-G_i(t_1,s)\Big\|ds\\
&+ \frac{f^\star}{T_{i-1}}\int_{T_{i-1}}^{T_i}\Big\|G_i(t_2,s)-G_i(t_1,s)\Big\|ds,
\end{aligned}
$$

by the continuity of Green's function G_i. Hence, $\|(Wx)(t_2)-(Wx)(t_1)\|_{E_i}\to 0$ as $|t_2-t_1|\to 0$. It implies that $W(B_{R_i})$ is equicontinuous.

Remark 4.1 *([11]). Note that the inequality*

$$\mu\Big((\ln t)^\delta\|f(t,B_1)\|\Big)\leq K\mu(B_1),$$

is equivalent to (H2) for each $B_1\subset X$ and $t\in J$, where B_1 is bounded.

Step 4 : W is k-set contraction.
For $U\in B_{R_i}$, $t\in J_i$, we get,

$$
\begin{aligned}
\mu(W(U)(t)) &= \mu((Wx)(t), x\in U)\\
&\leq \Big\{\int_{T_{i-1}}^{T_i}\frac{1}{s}G_i(t,s)\mu f(s,x(s))ds, x\in U\Big\}.
\end{aligned}
$$

Remark 4.1 indicates that

$$\mu(W(U)(t)) \leq \Big\{\int_{T_{i-1}}^{T_i}\frac{1}{s}G_i(t,s)[K\mu(\{x(s),x\in U\})]\Big\}$$

$$
\begin{aligned}
&\leq \Big\{\frac{1}{\Gamma(u_i)}\Big(\frac{\ln T_i-\ln T_{i-1}}{4}\Big)^{u_i-1}\Big[K\widehat{\mu}(U)\int_{\mathcal{T}_{i-1}}^{T_i}\frac{1}{s}(\ln s)^{-\delta}ds\Big], x\in U\Big\}\\
&\leq \frac{K\Big((\ln T_i^{1-\delta})-(\ln T_{i-1}{}^{1-\delta})\Big)(\ln T_i-\ln T_{i-1})^{u_i-1}}{4^{u_i-1}(1-\delta)\Gamma(u_i)}\widehat{\mu}(U).
\end{aligned}
$$

Therefore,

$$\widehat{\mu}(WU) \leq \frac{K\Big((\ln T_i^{1-\delta}) - (\ln T_{i-1}^{1-\delta})\Big)(\ln T_i - \ln T_{i-1})^{u_i-1}}{4^{u_i-1}(1-\delta)\Gamma(u_i)}\widehat{\mu}(U).$$

Consequently by (4.7), we deduce that W is a k-set contraction.
Therefore, all conditions of Theorem 1.4 are fulfilled and thus, the Hadamard constant-order fractional boundary value problem (4.4) has at least solution $\widetilde{x}_i \in B_{R_i}$. Since $B_{R_i} \subset E_i$, the step of Theorem 4.1 is proved.
Now, we will prove the existence result for the Hadamard fractional boundary value problem of variable order (4.1).

Theorem 4.2 *Let the conditions (H1), (H2) and inequality (4.7) be satisfied for all $i \in \{1, 2, ..., n\}$. Then, the Hadamard fractional boundary value problem of variable order* (4.1) *possesses at least one solution in $C(J, X)$.*

Proof. According to Theorem 4.1, the Hadamard constant-order fractional boundary value problem (4.4) possesses at least one solution $\widetilde{x}_i \in E_i$, $i \in \{1, 2, ..., n\}$.
For any $i \in \{1, 2, ..., n\}$, we define the function

$$x_i = \begin{cases} 0, & t \in [1, T_{i-1}], \\ \widetilde{x}_i, & t \in J_i. \end{cases}$$

Thus, the function $x_i \in C([1, T_i], X)$ solves the integral equation (4.3) for $t \in J_i$ with $x_i(1) = 0,\ x_i(T_i) = \widetilde{x}_i(T_i) = 0$.

Then, the function

$$x(t) = \begin{cases} x_1(t), & t \in J_1, \\ x_2(t) = \begin{cases} 0, & t \in J_1, \\ \widetilde{x}_2, & t \in J_2, \end{cases} \\ . \\ . \\ . \\ x_n(t) = \begin{cases} 0, & t \in [1, T_{n-1}], \\ \widetilde{x}_n, & t \in J_n, \end{cases} \end{cases}.$$

gives the solution for the Hadamard fractional boundary value problem of variable order (4.1).

4.3 Ulam-Hyers-Rassias stability

Theorem 4.3 *Assume (H1), (H2), (4.7), and*

(H3) *The function $\vartheta \in C(J, X)$ is increasing and there exists $\lambda_\vartheta > 0$ such that, for each $t \in J_i$, we have*

$$^{H}I^{u_i}_{T_{i-1}^+}\vartheta(t) \leq \lambda_{\vartheta(t)}\vartheta(t),$$

then, the Hadamard fractional boundary value problem of variable order (4.1) is Ulam-Hyers-Rassias stable with respect to ϑ.

Proof. Let $\epsilon > 0$ be an arbitrary number and the function $z(t)$ from $C(J, X)$ satisfy the following inequality

$$\|{}^{H}D^{u(t)}_{1^+}z(t) + f(t, z(t))\| \leq \epsilon\vartheta(t), t \in J. \tag{4.9}$$

For any $i \in \{1, 2, ..., n\}$ we define the functions $z_1(t) \equiv z(t), t \in [1, T_1]$ and for $i = 2, 3, ..., n$:

$$z_i(t) = \begin{cases} 0, & t \in [1, T_{i-1}], \\ z(t), & t \in J_i. \end{cases}$$

For any $i \in \{1, 2, ..., n\}$ according to equality (4.2) for $t \in J_i$, we obtain

$$^{H}D^{u(t)}_{1^+}z_i(t) = \frac{1}{\Gamma(2 - u(t))}(t\frac{d}{dt})^2 \int_{T_{i-1}}^{t} (\ln\frac{t}{s})^{1-u_i}\frac{z(s)}{s}ds.$$

Taking $^{H}I^{u_i}_{T_{i-1}^+}$ on both sides (4.9), we get

$$\begin{aligned}\left\|z(t) + \int_{T_{i-1}}^{T_i} \frac{1}{s}G_i(t, s)f(s, z(s))ds\right\| &\leq \frac{\epsilon}{\Gamma(u_i)}\int_{T_{i-1}}^{t} \frac{1}{s}(\ln\frac{t}{s})^{u_i-1}\vartheta(s)ds \\ &\leq \epsilon\lambda_{\vartheta(t)}\vartheta(t).\end{aligned}$$

According to Theorem 4.1, the Hadamard fractional boundary value problem of variable order (4.1) has a solution $x \in C(J, X)$ defined by $x(t) = x_i(t)$ for $t \in J_i,\ i = 1, 2, ..., n$ where

$$x_i = \begin{cases} 0, & t \in [1, T_{i-1}], \\ \widetilde{x}_i, & t \in J_i, \end{cases} \tag{4.10}$$

and $\widetilde{x}_i \in E_i$ is a solution of the Hadamard constant-order fractional boundary value problem (4.4). According to Lemma 4.1, the integral equation

$$\begin{aligned}\widetilde{x}_i(t) = &-\frac{(T_i - T_{i-1})^{-1}(t - T_{i-1})}{\Gamma(u_i)}\int_{T_{i-1}}^{T_i} (T_i - s)^{u_i-1}f(s, \widetilde{x}_i(s))ds \\ &+ \frac{1}{\Gamma(u_i)}\int_{T_{i-1}}^{t} (t - s)^{u_i-1}f(s, \widetilde{x}_i(s))ds, \quad t \in J_i.\end{aligned} \tag{4.11}$$

holds.
Let $t \in J_i$, where $i \in \{1, 2, ..., n\}$. Then, by Eq (4.10) and (4.11), we get

$$
\begin{aligned}
\|z(t) - x(t)\| &= \|z(t) - x_i(t)\| = \|z_i(t) - \widetilde{x}_i(t)\| \\
&= \|z_i(t) - \int_{T_{i-1}}^{T_i} \frac{1}{s} G_i(t,s) f(s, \widetilde{x}_i(s)) ds\| \\
&\leq \left\| z_i(t) - \int_{T_{i-1}}^{T_i} \frac{1}{s} G_i(t,s) f(s, z_i(s)) ds \right\| + \int_{T_{i-1}}^{T_i} \frac{1}{s} G_i(t,s) \left\| f(s, z_i(s)) - f(s, \widetilde{x}_i(s)) \right\| ds \\
&\leq \left\| z_i(t) + \int_{T_{i-1}}^{T_i} \frac{1}{s} G_i(t,s) f(s, z_i(s)) ds \right\| + \int_{T_{i-1}}^{T_i} \frac{1}{s} G_i(t,s) \left\| f(s, z_i(s)) - f(s, \widetilde{x}_i(s)) \right\| ds \\
&\leq \lambda_{\vartheta(t)} \epsilon \vartheta(t) + \frac{1}{\Gamma(u_i)} \Big(\frac{\ln T_i - \ln T_{i-1}}{4} \Big)^{u_i - 1} \int_{T_{i-1}}^{T_i} (\ln s)^{-\delta} \frac{K \|z_i(s) - \widetilde{x}_i(s)\|}{s} ds \\
&\leq \lambda_{\vartheta(t)} \epsilon \vartheta(t) + \frac{K}{\Gamma(u_i)} \Big(\frac{\ln T_i - \ln T_{i-1}}{4} \Big)^{u_i - 1} \|z_i - \widetilde{x}_i\|_{E_i} \int_{T_{i-1}}^{T_i} \frac{1}{s} (\ln s)^{-\delta} ds \\
&\leq \lambda_{\vartheta(t)} \epsilon \vartheta(t) + \frac{K \Big((\ln T_i)^{1-\delta} - (\ln T_{i-1})^{1-\delta} \Big) \Big(\ln T_i - \ln T_{i-1} \Big)^{u_i - 1}}{(1-\beta) 4^{u_i - 1} \Gamma(u_i)} \|z_i - \widetilde{x}_i\|_{E_i} \\
&\leq \lambda_{\vartheta(t)} \epsilon \vartheta(t) + \mu \|z - x\|,
\end{aligned}
$$

where

$$
\mu = \max_{i=1,2,\dots,n} \frac{K \Big((\ln T_i)^{1-\delta} - (\ln T_{i-1})^{1-\delta} \Big) \Big(\ln T_i - \ln T_{i-1} \Big)^{u_i - 1}}{(1-\delta) 4^{u_i - 1} \Gamma(u_i)}.
$$

Then

$$
\|z - x\|(1 - \mu) \leq \lambda_{\vartheta(t)} \epsilon \vartheta(t),
$$

and so by assuming $c_f := \frac{\lambda_{\vartheta(t)}}{(1-\mu)}$,

$$
\|z(t) - x(t)\| \leq \frac{\lambda_{\vartheta(t)} \vartheta(t)}{(1-\mu)} \epsilon := c_f \epsilon \vartheta(t).
$$

Then, by Theorem 1.6, the Hadamard fractional boundary value problem of variable order (4.1) is Ulam-Hyers-Rassias stable with respect to ϑ.

4.4 Example

Consider the Hadamard fractional boundary value problem of variable order :

$$\begin{cases} {}^H D_{1^+}^{u(t)} x(t) + \frac{1}{(1+\ln t)} \Big(1 + \frac{\|x_n(t)\|}{1+\|x_n(t)\|}\Big) = 0, \quad t \in J = [1, e]\,, \\ x(1) = 0, \ \ x(e) = 0. \end{cases} \tag{4.12}$$

Let

$$E = l^1 = \{x = (x_1, x_2, ..., x_n, ...), \sum_{n=1}^{\infty} |x_n| < \infty\},$$

E is a Banach space with the norm $\|x\| = \sum_{n=1}^{\infty} |x_n|$,

and

$$f(t,x) = \frac{1}{(1+\ln t)} \Big(1 + \frac{\|x\|}{1+\|x\|}\Big), \quad t \in [1, e], \ x \in E,$$

$$u(t) = \begin{cases} 1.2, & t \in J_1 := [1, 2], \\ 1.6, & t \in J_2 :=]2, e]. \end{cases} \tag{4.13}$$

Then, we get

$$\begin{aligned} (\ln t)^{\frac{1}{4}} |f(t,x) - f(t,y)| &= \frac{(\ln t)^{\frac{1}{4}}}{(1+\ln t)} \left| \frac{\|x\|}{1+\|x\|} - \frac{\|y\|}{1+\|y\|} \right| \\ &\leq \frac{1}{2} \|x - y\|. \end{aligned}$$

(H2) holds with $\delta = \frac{1}{4}$ and $K = \frac{1}{2}$.
From (4.13), the Hadamard fractional boundary value problem of variable order (4.12) is classified into the following :

$$\begin{cases} {}^H D_{1^+}^{1.2} x(t) + \frac{1}{(1+\ln t)} \Big(1 + \frac{\|x_n(t)\|}{1+\|x_n(t)\|}\Big) = 0, \quad t \in J_1, \\ {}^H D_{2^+}^{1.6} x(t) + \frac{1}{(1+\ln t)} \Big(1 + \frac{\|x_n(t)\|}{1+\|x_n(t)\|}\Big) = 0, \quad t \in J_2. \end{cases}$$

For $t \in J_1$, the Hadamard fractional boundary value problem of variable order (4.12) is equivalent to the Hadamard constant-order fractional boundary value problem

$$\begin{cases} {}^H D_{1^+}^{1.2} x(t) + \frac{1}{(1+\ln t)} \Big(1 + \frac{\|x_n(t)\|}{1+\|x_n(t)\|}\Big) = 0, \quad t \in J_1, \\ x(1) = 0, \ \ x(2) = 0. \end{cases} \tag{4.14}$$

Let us now show that the condition (4.7) is satisfied. Clearly, the following value is obtained

$$\frac{K\Big((\ln T_1)^{1-\delta} - (\ln T_0)^{1-\delta}\Big)\Big(\ln T_1 - \ln T_0\Big)^{u_1 - 1}}{4^{u_1-1}(1-\delta)\Gamma(u_1)} = \frac{\frac{1}{2}(\ln 2)^{\frac{3}{4}}(\ln 2)^{0.2}}{(4^{0.2})\frac{3}{4}\Gamma(1.2)} \simeq 0.1941 < 1.$$

On the other hand, let $\vartheta(t) = (\ln t)^{\frac{1}{2}}$. In this case,

$$\begin{aligned} {}^{H}I_{1^+}^{u_1}\vartheta(t) &= \frac{1}{\Gamma(1.2)}\int_1^2 (\ln \frac{t}{s})^{1.2-1}\frac{(\ln t)^{\frac{1}{2}}}{s}ds \\ &\leq \frac{(\ln t)^{\frac{1}{2}}}{\Gamma(1.2)}\int_1^2 (\ln \frac{2}{s})^{0.2}\frac{ds}{s} \\ &\leq \frac{(\ln 2)^{1.2}}{\Gamma(2.2)}(\ln t)^{\frac{1}{2}} := \lambda_{\vartheta(t)}\vartheta(t). \end{aligned}$$

As a result, (H3) is fulfilled with $\vartheta(t) = \sqrt{(\ln t)}$ and $\lambda_{\vartheta(t)} = \dfrac{(\ln 2)^{1.2}}{\Gamma(2.2)} \in \mathbb{R}$.
Theorem (4.1) guarantees the existence of a solution for the Hadamard constant-order fractional boundary value problem (4.14) $\widetilde{x}_1 \in E_1$.
For $t \in J_2$, the Hadamard fractional boundary value problem of variable order (4.12) can be written as the following constant-order fractional boundary value problem, i.e.,

$$\begin{cases} {}^{H}D_{2^+}^{1.6}x(t) + \frac{1}{(1+\ln t)}\Big(1 + \frac{\|x_n(t)\|}{1+\|x_n(t)\|}\Big) = 0, \quad t \in J_2, \\ x(2) = 0, \;\; x(e) = 0. \end{cases} \tag{4.15}$$

We see that

$$\frac{K\Big((\ln T_2)^{1-\delta} - (\ln T_1)^{1-\delta}\Big)\Big(\ln T_2 - \ln T_1\Big)^{u_2-1}}{4^{u_2-1}(1-\delta)\Gamma(u_2)} = \frac{\frac{1}{2}\Big(1-(\ln 2)^{\frac{3}{4}}\Big)(1-\ln 2)^{0.6}}{(4^{0.6})\frac{3}{4}\Gamma(1.6)} \simeq 0.0191 < 1.$$

Accordingly the condition (4.7) is achieved on the subinterval J_2. Further,

$$\begin{aligned} {}^{H}I_{2^+}^{u_1}\vartheta(t) &= \frac{1}{\Gamma(1.6)}\int_2^e (\ln \frac{t}{s})^{1.6-1}\frac{(\ln t)^{\frac{1}{2}}}{s}ds \\ &\leq \frac{(\ln t)^{\frac{1}{2}}}{\Gamma(1.6)}\int_2^e (\ln \frac{e}{s})^{0.6}\frac{ds}{s} \\ &\leq \frac{(\ln \frac{e}{2})^{1.6}}{\Gamma(2.6)}(\ln t)^{\frac{1}{2}} := \lambda_{\vartheta(t)}\vartheta(t). \end{aligned}$$

As a result, (H3) is also valid with $\vartheta(t) = \sqrt{(\ln t)}$ and $\lambda_{\vartheta(t)} = \dfrac{(\ln \frac{e}{2})^{1.6}}{\Gamma(2.6)} \in \mathbb{R}$.
On account of Theorem (4.1), the Hadamard constant-order fractional boundary value problem (4.15) possesses a solution $\widetilde{x}_2 \in E_2$.

Thus, by Theorem (4.2), the Hadamard fractional boundary value problem of variable order (4.12) has a solution

$$x(t) = \begin{cases} \widetilde{x}_1(t), & t \in J_1, \\ x_2(t), & t \in J_2, \end{cases}$$

where

$$x_2(t) = \begin{cases} 0, & t \in J_1, \\ \widetilde{x}_2(t), & t \in J_2. \end{cases}$$

From Theorem 4.3, the Hadamard fractional boundary value problem of variable order given by (4.12) is Ulam-Hyers-Rassias stable with respect to ϑ.

Chapitre 5

Mawhin Continuation Technique for a Nonlinear Boundary Value Problem of Variable Order at Resonance via Piece-wise Constant Functions

5.1 Introduction and motivations

The objective of this chapter is the study the existence of solutions to the boundary value problem of fractional differential equations involving Caputo derivative of variable order, by applying the degree of coincidence of Mawhin.

The Mawhin theory permits the use of an approach of topological degree type to problems which can be written as an abstract operator equation of the form $Lx = Nx$, where L is a linear noninvertible operator and N is a nonlinear operator acting on a given Banach space.

In 1972, Mawhin has developed a method to solve this equation in his famous paper. Topological degree and boundary value problems for nonlinear differential equations [40], he assumed that L is a Fredholm operator of index zero. Hence he has developed a new theory of topological degree known as the degree of coincidence for $(L,\ N)$, that is also known as Mawhin's coincidence degree theory in honor of him. A boundary value

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problem is said to be at resonance if the corresponding linear homogenous problem has nontrivial solution, otherwise it's said to be at resonance. Many authors studied ordinary boundary value problems at resonance using Mawhin coincidence degree theory, we can cite Feng and Webb[20], Guezane-Lakoud and Frioui [22], Mawhin and Ward [45], Infante [29], and the references therein.

In particular, Benchohra et al. [10] studied the following nonlinear implicit differential equation of fractional constant order α at resonance

$$\begin{cases} {}^cD^{\alpha}_{0^+}y(t) = f(t, y(t), {}^cD^{\alpha}_{0^+}y(t)), \ \ t \in [0, T], \ , T > 0, \ \ \alpha \in]0, 1], \\ y(0) = y(T), \end{cases}$$

where ${}^cD^{u}_{0^+}$ is the Caputo fractional derivative of constant order α, and f is a given continuous function.

In this chapter we will study the boundary value problem (BVP) for the Caputo fractional differential equation of variable order

$$\begin{cases} {}^cD^{u(t)}_{a^+}y(t) = f(t, y(t)), \ t \in J, \\ y(a) = y(T), \end{cases} \tag{5.1}$$

where $J = [a, T]$, $0 \leq a < T < \infty$, $u(t) : J \to (0, 1]$ is the variable order of the fractional derivatives, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function and ${}^cD^{u(t)}_{a^+}$ is the Caputo fractional derivative of variable-order $u(t)$.

In Section 5.2, based on coincidence degree theory, a partition of the given interval J is applied, and by defining the relevant piece-wise constant functions, the existence results are derived for an equivalent constant-order BVP at resonance and accordingly, for the given BVP of Caputo variable order (5.1). This proof is completed in some steps. In Section 5.3, we give an example to illustrate the theoretical existence theorems.

5.2 Existence of solutions

Let us introduce the following assumptions :

(H1) Let $n \in \mathbb{N}$ be an integer and the finite sequence of points $\{T_k\}_{k=0}^{n}$ be given such that $a = T_0 < T_k < T_n = T$, $k = 1, ..., n-1$.
Denote $J_k := (T_{k-1}, T_k]$, $k = 1, 2, ..., n$. Then $\mathcal{P} = \bigcup_{k=1}^{n} J_k$ is a partition of the interval J.
Let $u(t) : J \to (0, 1]$ be a piecewise constant function with respect to $\mathcal{P}$ as follows :

$$u(t) = \sum_{i=1}^{n} u_i I_i(t) = \begin{cases} u_1, & if \ t \in J_1, \\ u_2, & if \ t \in J_2, \\ \ \ . & \\ \ \ . & \\ \ \ . & \\ u_n, & if \ t \in J_n, \end{cases}$$

where $0 < u_i \leq 1$ are constants and I_i is an indicator of the interval $J_i, i = 1, 2, ..., n$:

$$I_i(t) = \begin{cases} 1, & for\ t \in J_i, \\ 0, & for\ elsewhere. \end{cases}$$

(H2) Let $f \in C(J_i \times \mathbb{R}, \mathbb{R})$ and there exists a number $\delta \in (0,1)$ such that $t^\delta f \in C(J_i \times \mathbb{R}, \mathbb{R})$ and there exists a constant K with $0 < K < \min\left\{1, \frac{T_{i-1}^{\delta}\Gamma(u_i+1)}{(T_i - T_{i-1})^{u_i}}\right\}$, such that
$t^\delta |f(t, y_1) - f(t, y_2)| \leq K|y_1 - y_2|$, for any $y_1,\ y_2 \in \mathbb{R}$ and $t \in J_i$.

The left Caputo fractional derivative of variable order $u(t)$ for $y(t) \in C(J, \mathbb{R})$, defined as (1.3), can be formulated as a sum of the left Caputo fractional derivative operators of constant orders $u_k \in \mathbb{R}$ which takes the form

$$ {}^{c}D_{a^+}^{u(t)} y(t) = \sum_{k=1}^{i-1} \int_{T_{k-1}}^{T_k} \frac{(t-s)^{-u_k}}{\Gamma(1-u_k)} y'(s)ds + \int_{T_{i-1}}^{t} \frac{(t-s)^{-u_i}}{\Gamma(1-u_i)} y'(s)ds. \tag{5.2}$$

Thus, the equation of the Caputo fractional differential equation of variable order (5.1) can be reformulated for each $t \in J_i,\ i = 1, 2, ..., n$ in the following structure

$$\sum_{k=1}^{i-1} \int_{T_{k-1}}^{T_k} \frac{(t-s)^{-u_k}}{\Gamma(1-u_k)} y'(s)ds + \int_{T_{i-1}}^{t} \frac{(t-s)^{-u_i}}{\Gamma(1-u_i)} y'(s)ds = f(t, y(t)). \tag{5.3}$$

Let the function $\tilde{y} \in E_i$ be so that $\tilde{y}(t) \equiv 0$ on $t \in [a, T_{i-1}]$ and it satisfied the above integral equation (5.3). In such a situation, (5.3) is converted to the standard constant order fractional differential equation as

$$ {}^{c}D_{T_{i-1}^+}^{u_i} \tilde{y}(t) = f(t, \tilde{y}(t)), \quad t \in J_i.$$

In accordance with above equation for each $i = 1, 2, ..., n$, we have the auxillary fractional boundary value problem (FBVP) equipped with Caputo constant order operator

$$\begin{cases} {}^{c}D_{T_{i-1}^+}^{u_i} y(t) = f(t, y(t)), \quad t \in J_i, \\ y(T_{i-1}) = y(T_i). \end{cases} \tag{5.4}$$

A resonance problem is a boundary problem in which the corresponding homogeneous BVP has a non–trivial solution. Hence, we consider the homogeneous version of the given equivalent constant order FBVP (5.4) by

$$\begin{cases} {}^{c}D_{T_{i-1}^+}^{u_i} y(t) = 0, \quad t \in J_i, \\ y(T_{i-1}) = y(T_i). \end{cases} \tag{5.5}$$

The homogeneous constant order FBVP (5.5) has nontrivial solution $y(t) = c$ which converts the equivalent constant order FBVP (5.4) to a resonance fractional boundary

value problem.
Let $X = \{y \in E_i : y(t) = I^{u_i}_{T^+_{i-1}} v(t) : v \in E_i, t \in J_i\}$ with the norm

$$\|y\|_X = \|y\|_{E_i}.$$

Let $L : Dom(L) \subseteq X \to E_i$ an $N : X \to E_i$ are defined as

$$L[y(t)] := {}^cD^{u_i}_{T^+_{i-1}} y(t), \tag{5.6}$$

and

$$N[y(t)] := f(t, y(t)), \quad t \in J_i, \tag{5.7}$$

where

$$Dom(L) = \{y \in X : {}^cD^{u_i}_{T^+_{i-1}} y \in E_i \quad \text{and} \quad y(T_{i-1}) = y(T_i)\}.$$

Then the equivalent constant order resonance FBVP (5.4) can be reformulated by the equation $Ly = Ny$.

Theorem 5.1 *If the condition* (H2) *holds, then the equivalent constant order resonance FBVP* (5.4) *has at least one solution.*

Proof. The proof will be followed in a sequence of steps.
Step 1 : We show that

$$ker(L) = \{c : c \in \mathbb{R}\},$$

and

$$img(L) = \Big\{y \in E_i : \int_{T_{i-1}}^{T_i} (T_i - s)^{u_i - 1} y(s) ds = 0\Big\}.$$

Let L defined by (5.6) be such that for $t \in J_i$ and by Lemma 1.5, the equation $L[y(t)] = {}^cD^{u_i}_{T^+_{i-1}} y(t) = 0$ has the solution $y(t) = c,\ c \in \mathbb{R}$. Then

$$ker(L) = \{y(t) = c : c \in \mathbb{R}\}.$$

On the other hand, for $v \in img(L)$, there exists $y \in Dom(L)$ such that $v = Ly \in E_i$. By Lemma 1.6, for any $t \in J_i$, we have

$$y(t) = y(T_{i-1}) + \frac{1}{\Gamma(u_i)} \int_{T_{i-1}}^{t} (t - s)^{u_i - 1} v(s) ds.$$

Since $y \in Dom(L)$, v satisfies

$$\frac{1}{\Gamma(u_i)} \int_{T_{i-1}}^{T_i} (T_i - s)^{u_i - 1} v(s) ds = 0.$$

Also, assume that $v \in E_i$ satisfies

$$\int_{T_{i-1}}^{T_i} (T_i - s)^{u_i - 1} v(s) ds = 0.$$

Let $y(t) = I_{T_{i-1}^+}^{u_i} v(t)$. Then $v(t) = {}^c D_{T_{i-1}^+}^{u_i} y(t)$ and so $y \in Dom(L)$. Hence, $v \in img(L)$, so

$$img(L) = \big\{ y \in E_i : \int_{T_{i-1}}^{T_i} (T_i - s)^{u_i - 1} y(s) ds = 0 \big\}.$$

Step 2 : L is a Fredholm operator of index zero.
The linear continuous projector operators $P : X \to X$ and $Q : E_i \to E_i$ can be considered by the following forms

$$Py = y(T_{i-1}), \quad Qv = \frac{u_i}{(T_i - T_{i-1})^{u_i}} \int_{T_{i-1}}^{T_i} (T_i - s)^{u_i - 1} v(s) ds.$$

Clearly, $img(P) = ker(L)$ and $P^2 = P$. It follows that for any $y \in X$,

$$y = (y - Py) + Py,$$

i.e., $X = ker(P) + ker(L)$. A simple computation shows that $ker(P) \cap ker(L) = 0$. Therefore, $X = ker(P) \oplus ker(L)$. A similar argument shows that for every $v \in E_i$, $Q^2 v = Qv$ and $v = (v - Q(v)) + Q(v)$, where $(v - Q(v)) \in ker(Q) = img(L)$.

From $img(L) = ker(Q)$ and $Q^2 = Q$, we have

$$img(L) \cap img(Q) = 0.$$

Then, $E_i = img(L) \oplus img(Q)$.
In this case,

$$\dim(ker(L)) = \dim img(Q) = \operatorname{codim}(img(L)).$$

The obtained result shows that L is a Fredholm operator of index zero.

Step 3 : $L_P^{-1} = (L|_{Dom(L) \cap ker(P)})^{-1}$ (the inverse of $L|_{Dom(L) \cap ker(P)}$).
Clearly, $L_P^{-1} : img(L) \to Dom(L) \cap ker(P)$ satisfies

$$L_P^{-1}(v)(t) = I_{T_{i-1}^+}^{u_i} v(t).$$

Let $v \in img(L)$. Then

$$L L_P^{-1}(v) = {}^c D_{T_{i-1}^+}^{u_i} (I_{T_{i-1}^+}^{u_i} v) = v. \tag{5.8}$$

Furthermore, for $y \in Dom(L) \cap ker(P)$, we get

$$L_P^{-1}(L(y(t))) = I_{T_{i-1}^+}^{u_i} ({}^c D_{T_{i-1}^+}^{u_i} y(t)) = y(t) - y(T_{i-1}).$$

Since $y \in Dom(L) \cap ker(P)$, we know that $y(T_{i-1}) = 0$. Thus

$$L_P^{-1}(L(y(t))) = y(t). \tag{5.9}$$

Combining (5.8) and (5.9) shows that $L_P^{-1} = (L|_{Dom(L)\cap ker(P)})^{-1}$.

Step 4 : On every bounded and open set $\Omega \subset X$, N is L-compact.
Define $\Omega = \{y \in X : \|y\|_X < M\}$ as a bounded and open set, where $M > 0$.
The proof of this step will be done in three claims.

Claim 1 : QN is continuous.
This property for QN is derived due to the imposed conditions on the nonlinear function f and the Lebesgue dominated convergence criterion, immediately.

Claim 2 : $QN(\overline{\Omega})$ is bounded.
Now, for each $y \in \overline{\Omega}$ and for all $t \in J_i$, we have

$$\begin{aligned}
|QN(y)(t)| &\le \frac{u_i}{(T_i - T_{i-1})^{u_i}} \int_{T_{i-1}}^{T_i} (T_i - s)^{u_i - 1} |f(s, y(s))| ds \\
&\le \frac{u_i}{(T_i - T_{i-1})^{u_i}} \int_{T_{i-1}}^{T_i} (T_i - s)^{u_i - 1} |f(s, y(s)) - f(s, 0)| ds \\
&\quad + \frac{u_i}{(T_i - T_{i-1})^{u_i}} \int_{T_{i-1}}^{T_i} (T_i - s)^{u_i - 1} |f(s, 0)| ds \\
&\le f^* + \frac{u_i}{(T_i - T_{i-1})^{u_i}} \int_{T_{i-1}}^{T_i} (T_i - s)^{u_i - 1} s^{-\delta} (K|y(s)|) ds \\
&\le f^* + MKT_{i-1}^{-\delta},
\end{aligned}$$

by assuming $f^* = \sup\limits_{t\in J_i} |f(t, 0)|$. Thus,

$$\|QN(y)\|_{E_i} \le f^* + MKT_{i-1}^{-\delta} := R > 0.$$

This shows that $QN(\overline{\Omega}) \subseteq E_i$ is bounded.

Claim 3 : $L_P^{-1}(I - Q)N : \overline{\Omega} \to X$ is completely continuous.
By considering the existing hypotheses in relation to Ascoli-Arzelà theorem, it is necessary that we prove two properties of the boundedness and equi-continuity for

$L_P^{-1}(I-Q)N(\overline{\Omega}) \subset X$. At first, for each $y \in \overline{\Omega}$ and for all $t \in J_i$, we have

$$\begin{aligned} L_P^{-1}(I-Q)Ny(t) &= L_P^{-1}(Ny(t) - QNy(t)) \\ &= I^{u_i}_{T_{i-1}^+}\Big[f(t,y(t)) - \frac{u_i}{(T_i - T_{i-1})_i^u}\int_{T_{i-1}}^{T_i}(T_i - s)^{u_i-1}f(s,y(s))\Big]ds \\ &= \frac{1}{\Gamma(u_i)}\int_{T_{i-1}}^{t}(t-s)^{u_i-1}f(s,y(s))ds \\ &\quad - \frac{t^{u_i}}{(T_i - T_{i-1})^{u_i}\Gamma(u_i)}\int_{T_{i-1}}^{T_i}(T_i - s)^{u_i-1}f(s,y(s))ds. \end{aligned}$$

Further, for each $y \in \overline{\Omega}$ and for all $t \in J_i$, we get

$$\begin{aligned} |L_P^{-1}(I-Q)Ny(t)| &\leq \frac{2}{\Gamma(u_i)}\int_{T_{i-1}}^{T_i}(T_i - s)^{u_i-1}|f(s,y(s)) - f(s,0)|ds \\ &\quad + \frac{2}{\Gamma(u_i)}\int_{T_{i-1}}^{T_i}(T_i - s)^{u_i-1}|f(s,0)|ds \\ &\leq [f^* + MKT_{i-1}^{-\delta}]\frac{2(T_i - T_{i-1})^{u_i}}{\Gamma(u_i+1)} := B. \end{aligned}$$

So

$$\|L_P^{-1}(I-Q)Ny\|_{E_i} \leq B,$$

which gives the uniform boundedness of $L_P^{-1}(I-Q)N(\overline{\Omega})$ in X.

To prove the equi-continuity of $L_P^{-1}(I-Q)N(\overline{\Omega})$, notice that for $T_{i-1} \leq t_1 \leq t_2 \leq T_i$ and $y \in \overline{\Omega}$, we get

$$\begin{aligned} |L_P^{-1}(I-Q)Ny(t_2) - L_P^{-1}(I-Q)Ny(t_1)| &\leq \frac{f^* + T_{i-1}^{-\delta}MK}{\Gamma(u_i)}\Big[\int_{t_1}^{t_2}(t_2 - s)^{u_i-1}ds \\ &\quad + \int_{T_{i-1}}^{t_1}|(t_2 - s)^{u_i-1} - (t_1 - s)^{u_i-1}|ds\Big] + \Big[\frac{T_{i-1}^{-\delta}MK + f^*}{\Gamma(u_i+1)}\Big](t_2^{u_i} - t_1^{u_i}). \end{aligned}$$

The right-hand side of above inequality tends to zero as $t_1 \to t_2$. Thus, $L_P^{-1}(I-Q)N(\overline{\Omega})$ is equicontinuous in X.
On the basis of the Ascoli-Arzelà theorem, $L_P^{-1}(I-Q)N(\overline{\Omega})$ is relatively compact.
In accordance with the steps 1 to 3, we can follow that N is L-compact in $\overline{\Omega}$.

Step 5 : There exists $A > 0$ (not depending on λ) so that if

$$L(y) - N(y) = -\lambda[L(y) + N(-y)], \quad \lambda \in (0,1], \tag{5.10}$$

then $\|y\|_X \leq A$.
By the condition (H2) and for each $y \in X$ satisfying (5.10), we get

$$L(y) - N(y) = -\lambda L(y) - \lambda N(-y).$$

So

$$Ł(y) = \frac{1}{1+\lambda} N(y) - \frac{\lambda}{1+\lambda} N(-y). \tag{5.11}$$

By (5.11), and for all $t \in J_i$, we get

$$y(t) = \frac{1}{1+\lambda} L_P^{-1} N y(t) - \frac{\lambda}{1+\lambda} L_P^{-1} N(-y(t)),$$

and so

$$\begin{aligned}
|y(t)| \leq & \frac{1}{(1+\lambda)\Gamma(u_i)} \int_{T_{i-1}}^{t} (t-s)^{u_i - 1} |f(s, y(s)) - f(s, 0)| \, ds \\
& + \frac{\lambda}{(1+\lambda)\Gamma(u_i)} \int_{T_{i-1}}^{t} (t-s)^{u_i - 1} |f(s, -y(s)) - f(s, 0)| \, ds \\
& + \frac{f^*(T_i - T_{i-1})^{u_i}}{(1+\lambda)\Gamma(u_i + 1)} + \frac{\lambda f^*(T_i - T_{i-1})^{u_i}}{(1+\lambda)\Gamma(u_i + 1)} \\
\leq & \left(\frac{1}{1+\lambda} + \frac{\lambda}{1+\lambda}\right) \frac{T_{i-1}^{-\delta}(T_i - T_{i-1})^{u_i}}{\Gamma(u_i + 1)} (K \|y\|_{E_i}) + \left(\frac{1}{1+\lambda} + \frac{\lambda}{1+\lambda}\right) \frac{f^*(T_i - T_{i-1})^{u_i}}{\Gamma(u_i + 1)} \\
= & \frac{K T_{i-1}^{-\delta}(T_i - T_{i-1})^{u_i}}{\Gamma(u_i + 1)} \|y\|_{E_i} + \frac{f^*(T_i - T_{i-1})^{u_i}}{\Gamma(u_i + 1)}.
\end{aligned}$$

Hence,

$$\|y\|_{E_i} \leq \left(f^* + K T_{i-1}^{-\delta} \|y\|_{E_i}\right) \frac{(T_i - T_{i-1})^{u_i}}{\Gamma(u_i + 1)}, \tag{5.12}$$

and so

$$\|y\|_X \leq \frac{f^*}{\frac{\Gamma(u_i+1)}{(T_i - T_{i-1})^{u_i}} - K T_{i-1}^{-\delta}} := A.$$

Step 6 : There exists a bounded and open set $\Omega \subset X$ such that

$$L(y) - N(y) \neq -\lambda [L(y) + N(-y)],$$

for all $y \in \partial\Omega$ and all $\lambda \in (0, 1]$.

By the condition (H2) and Step 5, there exists $A > 0$ (independent of λ) such that, if y satisfies

$$L(y) - N(y) = -\lambda [L(y) + N(-y)], \;\; \lambda \in (0, 1],$$

then $\|y\|_X \leq A$. Thus, if

$$\Omega = \{y \in X : \|y\|_X < B\}, \tag{5.13}$$

then from the condition (H2), it is immediately obtained that the set Ω introduced by (5.13), is symmetric, $0 \in \Omega$, and $X \cap \overline{\Omega} = \overline{\Omega} \neq \emptyset$.

Furthermore, it is obtained that

$$L(y) - N(y) \neq -\lambda[L(y) - N(-y)],$$

for all $y \in \partial\Omega = \{y \in X : \|y\|_X = B\}$ and for all $\lambda \in (0, 1]$, where $B > A$.
This together with Theorem 1.1 imply that the equivalent constant order resonance FBVP (5.4) has at least one solution, and this completes the proof.

Now, we complete our deduction on the existence property for solutions of the given Caputo FBVP of variable order (5.1).

Theorem 5.2 *Let the conditions* (H1), (H2) *be satisfied for all* $i \in \{1, 2, ..., n\}$. *Then, the Caputo FBVP of variable order* (5.1) *possesses at least one solution in* $C(J, \mathbb{R})$.

Proof. For any $i \in \{1, 2, ..., n\}$ according to Theorem (5.1), the equivalent constant order resonance FBVP (5.4) possesses at least one solution $\widetilde{y}_i \in E_i$.
For any $i \in \{1, 2, ..., n\}$ we define the function

$$y_i = \begin{cases} 0, & t \in [a, T_{i-1}], \\ \widetilde{y}_i, & t \in J_i. \end{cases}$$

Thus, the function $y_i \in C([a, T_i], \mathbb{R})$ solves the integral equation (5.3) for $t \in J_i$, which means that $y_i(a) = 0, y_i(T_i) = \widetilde{y}_i(T_i) = 0$ and solves (5.3) for $t \in J_i,\ i \in \{1, 2, ..., n\}$. Then the function,

$$y(t) = \begin{cases} y_1(t), & t \in J_1, \\ y_2(t) = \begin{cases} 0, & t \in J_1, \\ \widetilde{y}_2, & t \in J_2, \end{cases} \\ \cdot \\ \cdot \\ \cdot \\ y_n(t) = \begin{cases} 0, & t \in [a, T_{n-1}], \\ \widetilde{y}_n, & t \in J_n, \end{cases} \end{cases} .$$

is a solution of Caputo fractional differential equation of variable order(5.1) in $C(J, \mathbb{R})$.

5.3 Example

Let us consider the following fractional boundary value problem,

$$\begin{cases} {}^cD_{0.5^+}^{u(t)} y(t) = \dfrac{\sin y(t) - (y(t) + 2)\cos t}{5\sqrt{1 + t}}, & t \in J := [0.5, 2], \\ y(0.5) = y(2). \end{cases} \tag{5.14}$$

Let

$$f(t,y)=\frac{\sin y-(y+2)\cos t}{5\sqrt{1+t}},\ (t,y)\in[0.5,2]\times[0,+\infty),$$

and

$$u(t)=\begin{cases}\frac{5}{7}, & t\in J_1:=[0.5,1],\\ \frac{2}{3}, & t\in J_2:=]1,2].\end{cases}\tag{5.15}$$

Then, we have

$$\begin{aligned}t^{\frac{1}{2}}|f(t,y_1)-f(t,y_2)| &= \left|\frac{t^{\frac{1}{2}}(\sin y_1-(y_1+2)\cos t)}{5\sqrt{1+t}}-\frac{t^{\frac{1}{2}}(\sin y_2-(y_2+2)\cos t)}{5\sqrt{1+t}}\right|\\ &\le \frac{1}{5}\sqrt{\frac{t}{1+t}}\Big(|\sin y_1-\sin y_2|+|\cos t||y_1-y_2|\Big)\\ &\le \frac{2}{5}|y_1-y_2|.\end{aligned}$$

By (5.15), according to (5.4) we consider two auxiliary for constant order resonance fractional boundary value problem

$$\begin{cases}{}^cD_{0.5^+}^{\frac{5}{7}}y(t)=\dfrac{\sin y(t)-(y(t)+2)\cos t}{5\sqrt{1+t}}, & t\in J_1,\\ y(0.5)=y(1),\end{cases}\tag{5.16}$$

and

$$\begin{cases}{}^cD_{1^+}^{\frac{2}{3}}y(t)=\dfrac{\sin y(t)-(y(t)+2)\cos t}{5\sqrt{1+t}}, & t\in J_2,\\ y(1)=y(2).\end{cases}\tag{5.17}$$

Hence the condition (H2) holds for $i=1$ with $\delta=\frac{1}{2}$ and $K=\frac{2}{5}$, and

$$0<K=\frac{2}{5}<\min\left\{1,\frac{T_0^{\delta}\Gamma(u_1+1)}{(T_1-T_0)^{u_1}}\right\}=1.$$

According to Theorem (5.1), the constant order resonance FBVP (5.16) has a solution $\widetilde{y}_1\in E_1$.

Next, the condition (H2) holds for $i=2$ with $\delta=\frac{1}{2}$ and $K=\frac{2}{5}$, and

$$0<K=\frac{2}{5}<\min\left\{1,\frac{T_1^{\delta}\Gamma(u_2+1)}{(T_2-T_1)^{u_2}}\right\}=\Gamma(\frac{5}{3})\simeq 0.9027.$$

According to Theorem (5.1), the constant order resonance FBVP (5.17) has a solution $\widetilde{y}_2 \in E_2$.
Then, by Theorem 5.2, the Caputo FBVP of variable order (5.14) has a solution as

$$y(t) = \begin{cases} \widetilde{y}_1(t), & t \in J_1, \\ y_2(t), & t \in J_2, \end{cases}$$

where

$$y_2(t) = \begin{cases} 0, & t \in J_1, \\ \widetilde{y}_2(t), & t \in J_2. \end{cases}$$

Conclusion and Perspectives

In this work we presented results about the existence and uniqueness of solutions for some classes of nonlinear boundary value problem involving the Riemann-Liouville, Caputo and Hadamard fractional differential equations of variable order, which is a piecewise constant function based on the essential difference about the variable order. By using the standard fixed point theorems (Banach contraction principle, Schauder's fixed point, Darbo's fixed point theorem), Kuratowski's measure of noncompactness and Mawhin's continuation theorem we established the existence and uniqueness of solutions and, we study the stability in the sense of Ulam-Hyers-Rassias and in the sense of Ulam-Hyers to our problems. Therefore, all results in this work show a great potential to be applied in various applications of multidisciplinary sciences.
The variable order is important and interesting to all researchers. In other words, in the near future we want to study various classes of implicit nonlinear fractional differential equations in the variable order settings via singular and nonsingular operators, thermostat model,... involving integral conditions or integro-derivative conditions.

Bibliographie

[1] G. F. J. Aguilar, *Analytical and numerical solutions of a nonlinear alcoholism model via variable-order fractional differential equations*, Physica A. **494**(2018), 52-75.

[2] J. C. Alvarez, *Measure of noncompactness and fixed points of nonexpansive condensing mappings in locally convex spaces*, Rev. Real. Acad. Cienc. Exact. Fis. Natur. Madrid, **79**(1985), 53-66.

[3] A. Atangana, *On the stability and convergence of the time-fractional variable order telegraph equation*, J. Comput. Phys, **293**(2015), 104-114.

[4] J. An and P. Chen, *Uniqueness of solutions to initial value problem of fractional differential equations of variable-order*, Dyn. Sys. Appl. **28**(2019), 607-623.

[5] R. R. Akhmerov, M. I. Kamenskii, A. S. Patapov, A. E. Rodkina and B. N. Sadovskii, *Measures of noncompactness and condensing operators*. Birkhauser , Basel, 1992, 1-52.

[6] R. Almeida and D. F. M. Torres, *Computing Hadamard type operators of variable fractional order*, Applied Mathematics and Computation, **257**(2015), 74-88.

[7] R. Almeida, D. Tavares and D. F. M. Torres, *The variable-order fractional calculus of variations*, Springer International Publishing, 2019.

[8] J. Banas, *Applications of measures of weak noncompactness and some classes of operators in the theory of functional equations in the Lebesgue space*, Nonlinear Analysis. **30**(1997), 3283-3293.

[9] J. Banas, *On measures of noncompactness in Banach spaces*, Commentationes Mathematicae Universitatis Carolinae, **21**(1980), 131-143.

[10] M. Benchohra, S. Bouriah and J. R. Graef, *Nonlinear implicit differential equations of fractional order at resonance*, Electron. J. Differential Equations. **324**(2016), 1-10.

[11] M. Benchohra, S. Bouriah, J. E. Lazreg and J. J. Nieto, *Nonlinear implicit Hadamard's fractional differential equations with delay in Banach space*, Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, **55**(2016), 15-26.

[12] Z. Bouazza, S. Etemad, M. S. Souid, S. Rezapour, F. Martinez and M. K. A. Kaabar, *A study on the solutions of a multiterm FBVP of variable order*, Journal of Function Spaces, **2021**(2021), 1-9.

[13] Y. Bai and H. Kong, *Existence of solutions for nonlinear Caputo-Hadamard fractional differential equations via the method of upper and lower solutions*, J. Nonlinear Sci. Appl, **10**(2017), 5744-5752.

[14] M. Benchohra and J. E. Lazreg, *Existence and Ulam stability for nonlinear implicit fractional differential equations with Hadamard derivative*, Stud. Univ. Babes-Bolyai Math, **62**(2017), 27-38.

[15] J. Banas and L. Olszowy, *Measures of noncompactness related to monotonicity*, In Annales Societatis Mathematicae Polonae. Seria 1. Commentationes Mathematicae. No.**41**(2001), 13-23.

[16] J. Banas and B. Rzepka, *An application of a measure of noncompactness in the study of asymptotique stability*, Appl. Math. Lett. **16**(2003), 1-6.

[17] J. Banas and K. Sadarangani, *On some measures of noncompactness in the space of continuous functions*, Nonlinear Anal. **68**(2008), 377-383.

[18] C. M. Chen, F. Liu, V. Anh and L. Turner, *Numberical schemes whith high spatial accuracy for a variable-order anomalous subdiffusion equation*, SIAM J. Sci. Comput., **32**(2010), 1740-1760.

[19] K. Deimling, *Nonlinear functional analysis*, Courier Corporation, 2010.

[20] W. Feng and J. R. L. Webb, *Solvability of three point boundary value problems at resonance*, Nonlinear Analysis : Theory, Methods & Applications, **30**(1997), 3227-3238.

[21] A. Granas and J. Dugundji, *Fixed point theory*, Springer, New York, 2003.

[22] A. Guezane-Lakoud and A. Frioui, *Third order boundary value problem with integral condition at resonance*, Theory and Applications of Mathematics & Computer Science, **3**(2013), 56-64.

[23] D. Guo, V. Lakshmikantham and X. Liu, *Nonlinear integral equations in abstract spaces*, Springer Science & Business Media, **373**(2013).

[24] R. E. Gaines and J. L. Mawhin, *Coincidence degree and nonlinear differential equations*, Springer. **568**(2006).

[25] J. Hadamard, *Essai sur l'étude des fonctions, données par leur développement de Taylor*. Gauthier-Villars, 1892.

[26] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA, **27**(1941), 222.

[27] S. Hristova, A. Benkerrouche, M. S. Souid and A. Hakem, *Boundary value problems of Hadamard fractional differential equations of variable order*. Symmetry, **13**(2021), 896.

[28] D. H. Hyers, G. Isac and T. Rassias, *Stability of functional equations in several variables*, Springer Science & Business Media, **34**(2012).

[29] G. Infante, and M. Zima, *Positive solutions of multi-point boundary value problems at resonance*, Nonlinear Analysis : Theory, Methods & Applications, **69**(2008), 2458-2465.

[30] S. M. Jung, *Hyers-Ulam-stability of liner differential equations of first order*, Appl. Math. Lett, **17**(2004), 1135-1140.

[31] S. M. Jung, *Hyers-Ulam-Rassias stability of functional equations in nonlinear analysis*, Springer Science & Business Media, **48**(2011).

[32] A. Jiahui and C. Pengyu, *uniqueness of solutions to initial value problem of fractional differential equations of variable-order*, Dyn. Syst. Appl, **28**(2019), 607-623.

[33] Y. Jia, M. Xu and Y. Z. Lin, *A numberical solution for variable order fractional functional differential equations*, Appl. Math. Lett., **64**(2017), 125-130.

[34] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, **204**, Elsevier, Amsterdam, 2006.

[35] Y. Kian, E. Soccorsi and M. Yamamoto, *On time-fractional diffusion equations with space-dependent variable order*, Ann. Henri Poincaré, **19**(2018), 3855-3881.

[36] X. Li and B. Wu, *A numerical technique for variable fractional functional boundary value problems*, Appl. Math. Lett., **43**(2015), 108-113.

[37] R. Ma, *Multiplicity results for a third order value problem at resonance*, Nonlinear Anal. **32**(1998), 493-499.

[38] J. Mawhin, *Equivalence theorems for nonlinear operator equations and coincidence degree theory for some mappings in locally convex topological vector spaces*, J. Differential Equations, **12**(1972), 610-636.

[39] J. Mawhin, *Topological degree methods in nonlinear boundary value problems*, CBMS Regional Conference Series in Mathematics, vol. **40**, American Mathematical Society, 1979.

[40] J. Mawhin, *Topological degree and boundary value problems for nonlinear differential equations*. In Topological Methods for ordinary differential equations. Springer, Berlin, Heidelberg, 1993, 74-142.

[41] J. Mawhin, *Leray-Schauder continuation theorems in the absence of a priori bounds*, Topological Methods in Nonlinear Analysis, **9**(1997), 179-200.

[42] J. Mawhin, *Leray-Schauder degree : A half century of extensions and applications*, Topological Methods in Nonlinear Analysis, **14**(1999),195-228.

[43] W. Malesza, M. Macias and D. Sierocuik, *Analysitical solution of fractional variable order differential equations*, J. Comput. Appl. Math., **348**(2019), 214-236.

[44] T. Miura, S. Miyajima and S. E. Takahasi, *A characterization of Hyers-Ulam stability of first order linear differential operators*, J. Math. Anal. Appl, **286**(2003), 136-146.

[45] J. Mawhin and J. R. Ward, *Periodic solutions of some forced Lienard differential equations at resonance*, Archiv der Mathematik, **41**(1983), 337-351.

[46] M. Obloza, *Connections between Hyers and Lyapunov stability of the ordinary differential equations*, Rocznik Nauk-Dydakt. Prace Mat, **14**(1997), 141-146.

[47] D. O'Regan, Y. J. Cho, and Y. Q. Chen, *Topological degree theory and application*, vol. **10**, Chapman and Hall/CRC, 2006.

[48] D. O'Regan and M. Zima, *Leggett-Williams norm-type theorems for coincidences*, Arch. Math. **87**(2006), 233-244.

[49] I. Podlubny, *Fractional differential equations, Academic Press*, New York, USA, 1999.

[50] T. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc, **72**(1978), 297-300.

[51] I. A. Rus, *Ulam stabilities of ordinary differential equations in a Banach space*, Carpathian Journal of Mathematics, **26**(2010), 103-107.

[52] A. Razminia, A. F. Dizaji, and V. J. Majd, *Solution existence for non-autonomous variable-order fractional differential equations*, Mathematical and Computer Modelling, **55**(2012), 1106-1117.

[53] A. Refice, M. S. Souid and I. Stamova. *On the boundary value problems of Hadamard fractional differential equations of variable order via Kuratowski MNC technique. Mathematics*, **9**(2021), 1134-1150.

[54] S. G. Samko, *Fractional integration and differentiation of variable order*, Analysis Mathematica, **21**(1995), 213-236.

[55] S. G. Samko and B. Boss, *Integration and differentiation to a variable fractional order*, Integral Transforms and Special Functions, **1**(1993), 277-300.

[56] H. G. Sun, W. Chen, H. Wei and Y. Q. Chen, *A comparative study of constant-order and variable-order fractional models in characterizing memory property of systems*, Eur. Phys. J. Special Topics, **193**(2011), 185-192.

[57] J. V. D. C. Sousa and E. C. de Oliveira, *Two new fractional derivatives of variable order with non-singular kernel and fractional differential equation*, Computational and Applied Mathematics, **37**(2018), 5375-5394.

[58] D. Tavares, R. Almeida and D. F. M. Torres, *Caputo derivatives of fractional variable order numerical approximations*, Communications in Nonlinear Science and Numerical Simulation, **35**(2016), 69-87.

[59] S. M. Ulam, *A collection of mathematical problems*, Interscience Publishers. Inc, New york, **8**(1960).

[60] S. Umarov and S. Steinberg, *Variable order differential equations and diffusion processes with changing modes*, Available from : http ://arxiv.org/abs/0903.2524v1.

[61] D. Valério and J. S. Costa, *Variable-order fractional derivatives and their numerical approximations*, Signal Processing, **91**(2011), 470-483.

[62] J. Wang, M. Fec and Y. Zhou, *Ulams type stability of impulsive ordinary differential equations*, J. Math. Anal. Appl, **395**(2012), 258-264.

[63] K. Yosida, *Functional Analysis*,6th ed. Springer-Verlag, Berlin, 1980.

[64] J. Yang, H. Yao and B. Wu, *An efficient numberical method for variable order fractional functional differential equation*, Appl. Math. Lett., **76**(2018), 221-226.

[65] S. Zhang, *Existence of solutions for two-point boundary-value problems with singular differential equations of variable order*, Electronic Journal of Differential Equations, **2013**(2013), 1-16.

[66] S. Zhang, *The uniqueness result of solutions to initial value problems of differential equations of variable-order*, Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Math, **112**(2018), 407-423.

[67] S. Zhang, L. Hu, *Unique existence result of approximate solution to initial value problem for fractional differential equation of variable order involving the derivative arguments on the half-axis*, Mathematics , **7**(2019), 286.

[68] S. Zhang and L. Hu, *The existence of solutions and generalized Lyapunov-type inequalities to boundary value problems of differential equations of variable order*, AIMS Mathematics, **5**(2020), 2923-2943.

[69] H. Zhang, S. Li and L. Hu, *The existeness and uniqueness result of solutions to initial value problems of nonlinear diffusion equations involving with the conformable variable derivative*, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas, **113**(2019), 1601-1623.

[70] S. Zhang, S. Sun and L. Hu, *Approximate solutions to initial value problem for differential equation of variable order*, Journal of Fractional Calculus and Applications, **9**(2018), 93-112.