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Intitulée

Stabilisation de quelques équations d'évolution par approche diffusive.

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Résumé

Dans cette thèse, nous étudions l'existence globale et le comportement asymptotique de solutions de léquation des ondes dégénérée avec un contrôle frontière de type fractionnaire ou dissipation frontière dynamique de type dérivé fractionnaire. Les outils utilisés sont méthode d'analyse spectrale, semigroupe, C_0 -semigroupe, le théorème de Borichev et Tomilov, théorème de Hille-Yosida et le théorème de Rouché.

Premièrement, nous nous intressons à létude de la stabilisation d'équation d'onde unidimensionnelle faiblement dégénérée $u_{tt} - (x^{\gamma}u_x)_x = 0$ with $x \in (0, 1)$ and $\gamma \in [0, 1)$, contrôlée par un feedback fractionnaire au bord agissant $x = 0$. Stabilisation forte, uniforme et non uniforme sont obtenus avec une estimation explicite de la décroissance de l'énergie dans des espaces appropriés. Les résultats sont obtenus à travers une estimation de la résolvante du générateur associé au semigroupe. On utilise une méthode spectrale, nous établissons la vitesse de dcroissance polynomial optimal de l'énergie du systme.

Ensuite, nous considérons une équation d'onde dégénérée avec une condition de contrôle frontière de type dérivé fractionnaire. Nous montrons que le problème n'est pas uniformément stable par une méthode spectrale et nous étudions la stabilité polynomiale l'aide de la théorie des opérateurs linéaires basée sur le semigroupe.

Enfin, nous nous intressons létude de lexistence globale des solutions déquations unidimensionnelles faiblement dégénérée $u_{tt} - (x^{\gamma}u_x)_x = 0$ avec $x \in (0, 1)$ et $\gamma \in [0, 1)$, avec une condition de contrle frontière dynamique de type dérivé fractionnaire.

Mots Clés:

Équation d'onde dégénérée, dissipation frontière dynamique de type dérivé fractionnaire, la vitesse de décroissance optimal, fonctions de Bessel, contrôle aux limites fractionnaires, stabilité polynomiale, , stabilité polynomiale, C_0 -semigroupe.

Abstract

In this, thesis we study the global existence and asymptotic behavior in time of solutions to degenerate wave equation with fractional boundary control or Dynamic boundary dissipation of fractional derivative type. However, using a spectrum method, semigroup, C_0-semigroup, Borichev and Tomilov , Hille-Yosida and Rouché's theorems.

First, we consider a degenerate wave equation with a boundary control condition of fractional derivative type. We show that the problem is not uniformly stale by a spectrum method and we study the polynomial stability using the semigroup theory of linear operators.

Next, we are concerned with the study of stabilization of one-dimensional weakly degenerate wave equation $u_{tt} - (x^\gamma u_x)_x = 0$ with $x \in (0, 1)$ and $\gamma \in [0, 1)$, controlled by a fractional boundary feedback acting at $x = 0$. Strong, uniform, and nonuniform stabilization are obtained with explicit decay estimates in appropriate spaces. The results are obtained through an estimate on the resolvent of the generator associated with the semigroup. However, using a spectral method, we establish the optimal polynomial decay rate of the energy of the system.

Finally, we are concerned with the study of global existence of solutions of one-dimensional weakly degenerate wave equation $u_{tt} - (x^\gamma u_x)_x = 0$ with $x \in (0, 1)$ and $\gamma \in [0, 1)$, with a dynamic boundary control condition of fractional derivative type.

Keywords:

Degenerate wave equation, Dynamic boundary dissipation of fractional derivative type, optimal decay rate, Bessel functions, fractional boundary control, Polynomial stability, polynomial stability, C_0-semigroup.

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Introduction

Control theory is the study of the process of controlling the behavior of an operator system to achieve a certain target. Its application ranges widely from earthquake engineering and seismology to fluid transfer, cooling water and noise reduction in cavities, vehicles, such as pipe systems. Acoustics, aeronautics, hydraulics, are also some of the diverse disciplines where control theory is applied.
Of the most important notions in modern systems and control theory we mention controllability, stabilizability and observability. Various types of those notions have been introduced for abstract systems defined on Banach or Hilbert spaces and the relations between them has been extensively explored by several authors.
The boundary feedback under the consideration in this thesis are of fractional type and are described by the fractional derivatives

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\, ds,\ \eta \geq 0.$$

The order of our derivatives is between 0 and 1. Very little is known in the literature. In addition to being nonlocal, fractional derivatives involve singular and non-integrable kernels ($t^\alpha, 0 < \alpha < 1$). This makes the problem more delicate. It has been shown (see [31]) that, as ∂_t, the fractional derivative ∂_t^α forces the system to become dissipative and the solution to approach the equilibrium state. Therefore, when applied on the boundary, we can consider them as controllers which help to reduce the vibrations.
In the recent years, fractional calculus has been applied successfully in various areas to modify many existing models of physical processes such as heat conduction, diffusion, viscoelasticity, wave propagation, electronics etc. Caputo and Mainardi [10] have established the relation between fractional derivative and theory of viscoelasticity. The generalization of the concept of derivative and integral to a non-integer order has been subjected to several approaches and some various alternative definition of fractional derivative appeared in [17, 19].
We study stability of the system using the semigroup theory of linear operators and a result obtained by Borichev and Tomilov.

This thesis is divided into 3 Chapter.

CHAPTER 1: PRELIMINARIES

In this Chapter, firstly, we present some well known results on Sobolev spaces and some basic definitions and theorems . Secondly, we recall some results on a C0-semigroup, including some theorems on strong, exponential and polynomial stability of a C0-semigroup. Next, we display a brief historical introduction to fractional derivatives and we define the fractional derivative operator and we present some physical interpretations . After that, we present the Bessel functions and their basic definitions . Finally, we present an appendix that contains almost all the secondary calculations used in this Thesis.

CHAPTER 2: DECAY ESTIMATES FOR A DEGENERATE WAVE EQUATION WITH A DYNAMIC FRACTIONAL FEEDBACK ACTING ON THE DEGENERATE BOUNDARY

In this Chapter, we are concerned with the system
(P_1)
$\{ u_{tt}(x,t) - (x^\gamma u_x(x,t))_x = 0$ in $(0,1)\times(0,+\infty), -mu_{tt}(0,t) + (x^\gamma u_x)(0,t) = \varrho\partial_t^{\alpha,\eta}u(0,t)$ in $(0,+\infty), u$

where $m > 0$, $\gamma \in [0,1)$ and $\varrho > 0$ .The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α, $(0 < \alpha \leq 1)$, with respect to the time variable.It is defined as follows

$$\partial_t^{\alpha,\eta}w(t) = \begin{cases} w_t & \text{for } \alpha = 1, \eta \geq 0 \\ \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^t (t-s)^{-\alpha}e^{-\eta(t-s)}\frac{dw}{ds}(s)\,ds, & \text{for } \alpha \neq 1, \eta \geq 0. \end{cases}$$

Where we discuss and establish the existence, the uniqueness of solution, and we prove lack of exponential stability by spectral analysis by using Bessel functions and we show an optimal decay rate.

CHAPTER 3: DECAY ESTIMATES FOR A DEGENERATE WAVE EQUATION WITH TWO BOUNDARY FRACTIONAL FEEDBACKS IN THE PRESENCE OF DIPLACEMENT:

In this chapter, we are concerned with the dynamic boundary stabilization of fractional type for degenerate wave equation of the form
(P_2)
$\{ u_{tt}(x,t) - (x^\gamma u_x(x,t))_x + \beta u = 0$ in $(0,1)\times(0,+\infty), (x^\gamma u_x)(0,t) = \varrho\partial_t^{\alpha,\eta}u(0,t)$ in $(0,+\infty), u_x(1,t) =$

where $\gamma \in [0,1), \varrho > 0, \tilde{\varrho} > 0$ and $\beta > 0$. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α, $(0 < \alpha \leq 1)$, with respect to the time variable.It is defined as follows

$$\partial_t^{\alpha,\eta}w(t) = \begin{cases} w_t & \text{for } \alpha = 1, \eta \geq 0 \\ \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^t (t-s)^{-\alpha}e^{-\eta(t-s)}\frac{dw}{ds}(s)\,ds, & \text{for } \alpha \neq 1, \eta \geq 0. \end{cases}$$

Where Strong, uniform, and nonuniform stabilization are obtained with explicit decay estimates in appropriate spaces. The results are obtained through an estimate on the resolvent of the generator associated with the semigroup.

Chapter 1

PRELIMINARIES

1.1 Sobolev spaces

We denote by Ω an open domain in $\mathbb{R}^n, n \geq 1$, with a smooth boundary $\Gamma = \partial\Omega$. In general, some regularity of Ω will be assumed. We will suppose that either

$$\Omega \text{ is Lipschitz,}$$

i.e., the boundary Γ is locally the graph of a Lipschitz function, or

$$\Omega \text{ is of class } \mathcal{C}^r, r \geq 1,$$

i.e., the boundary Γ is a manifold of dimension $n \geq 1$ of class $\mathcal{C}^r$. In both cases we assume that Ω is totally on one side of Γ. These definitions mean that locally the domain Ω is below the graph of some function ψ, the boundary Γ is represented by the graph of ψ and its regularity is determined by that of the function ψ. Moreover, it is necessary to note that a domain with a continuous boundary is never on both sides of its boundary at any point of this boundary and that a Lipschitz boundary has almost everywhere a unit normal vector ν.

We will also use the following multi-index notation for partial differential derivatives of a function:

$$\partial_i^k u = \frac{\partial^k u}{\partial x_i^k} \text{ for all } k \in \mathbb{N} \text{ and } i = 1, ..., n,$$
$$D^\alpha u = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_n^{\alpha_n} u = \frac{\partial^{\alpha_1 + \ldots + \alpha_n} u}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}},$$
$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n, |\alpha| = \alpha_1 + \ldots + \alpha_n.$$

We denote by $\mathcal{C}(D)$ (respectively $\mathcal{C}^k(D), k \in \mathbb{N}$ or $k = +\infty$) the space of real continuous functions on D (respectively the space of k times continuously differentiable functions on D), where D plays the role of Ω or its closure $\overline{\Omega}$. The space of real $\mathcal{C}^\infty$ functions on Ω with a compact support in Ω is denoted by $\mathcal{C}_0^\infty(\Omega)$ or $\mathcal{D}(\Omega)$ as in the distributions theory of Schwartz.The distributions space on Ω is denoted by $\mathcal{D}'(\Omega)$, i.e., the space of continuous linear form over $\mathcal{D}(\Omega)$.

For $1 \leq p \leq \infty$, we call $L^p(\Omega)$ the space of measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p dx \right)^{1/p} < +\infty \quad \text{for} \quad p < +\infty$$

$$\|f\|_{L^\infty(\Omega)} \quad = \quad \sup_{\Omega} |f(x)| < +\infty \qquad\qquad \text{for} \quad p = +\infty$$

The space $L^p(\Omega)$ equipped with the norm $f \longrightarrow \|f\|_{L^p}$ is a Banach space: it is reflexive and separable for $1 < p < \infty$ (its dual is $L^{\frac{p}{p-1}}(\Omega)$), separable but not reflexive for $p = 1$ (its dual is $L^\infty(\Omega)$), and not separable, not reflexive for $p = \infty$ (its dual contains strictly $L^1(\Omega)$). In particular the space $L^2(\Omega)$ is a Hilbert space equipped with the scalar product defined by

$$(f, g)_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx.$$

We denote by $L^p_{loc}(\Omega)$ the space of functions which are L^p on any bounded sub-domain of Ω.

Similar space can be defined on any open set other than Ω, in particular, on the cylinder set $\Omega \times]a, b[$ or on the set $\Gamma \times]a, b[$, where $a, b \in \mathbb{R}$ and $a < b$.

Let U be a Banach space, $1 < p < +\infty$ and $-\infty \leq a < b \leq +\infty$, then $L^p(a, b; U)$ is the space of L^p functions f from (a, b) into U which is a Banach space for the norm

$$\|f\|_{L^p(a,b;U)} = \left(\int_a^b \|f(x)\|_U^p \, dt \right)^{1/p} < +\infty \quad \text{for} \quad p < +\infty$$

and for the norm

$$\|f\|_{L^\infty(a,b;U)} = \sup_{t \in (a,b)} \|f(x)\|_U < +\infty \quad \text{for} \quad p = +\infty$$

Similarly, for a Banach space $U, k \in \mathbb{N}$ and $-\infty < a < b < +\infty$, we denote by $C([a, b]; U)$ (respectively $C^k([a, b]; U)$) the space of continuous functions (respectively the space of k times continuously differentiable functions) f from $[a, b]$ into U, which are Banach spaces, respectively, for the norms

$$\|f\|_{\mathcal{C}(a,b;U)} = \sup_{t \in (a,b)} \|f(x)\|_U, \quad \|f\|_{\mathcal{C}^k(a,b;U)} = \sum_{i=0}^{k} \left\| \frac{\partial^i f}{\partial t^i} \right\|_{\mathcal{C}(a,b;U)}$$

1.1.1 Definition of Sobolev Spaces

Now, we will introduce the Sobolev spaces: The Sobolev space $W^{k,p}(\Omega)$ is defined to be the subset of L^p such that function f and its weak derivatives up to some order k have a finite L^p norm, for given $p \geq 1$.

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega); D^\alpha f \in L^p(\Omega). \;\; \forall \alpha; |\alpha| \leq k\} \;\; ,$$

With this definition, the Sobolev spaces admit a natural norm,

$$f \longrightarrow \|f\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq m} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p} \;, \text{ for } p < +\infty$$

and

$$f \longrightarrow \|f\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha|\le m} \|D^\alpha f\|_{L^\infty(\Omega)} \ , \ \text{for } p = +\infty$$

Space $W^{k,p}(\Omega)$ equipped with the norm $\| \ . \ \|_{W^{k,p}}$ is a Banach space. Moreover is a reflexive space for $1 < p < \infty$ and a separable space for $1 \le p < \infty$. Sobolev spaces with $p = 2$ are especially important because of their connection with Fourier series and because they form a Hilbert space. A special notation has arisen to cover this case:

$$W^{k,2}(\Omega) = H^k(\Omega)$$

the H^k inner product is defined in terms of the L^2 inner product:

$$(f,g)_{H^k(\Omega)} = \sum_{|\alpha|\le k} (D^\alpha f, D^\alpha g)_{L^2(\Omega)} \ .$$

The space $H^m(\Omega)$ and $W^{k,p}(\Omega)$ contain $\mathcal{C}^\infty(\overline{\Omega})$ and $\mathcal{C}^m(\overline{\Omega})$. The closure of $\mathcal{D}(\Omega)$ for the $H^m(\Omega)$ norm (respectively $W^{m,p}(\Omega)$ norm) is denoted by $H_0^m(\Omega)$ (respectively $W_0^{k,p}(\Omega)$).

Now, we introduce a space of functions with values in a space X (a separable Hilbert space).

The space $L^2(a,b;X)$ is a Hilbert space for the inner product

$$(f,g)_{L^2(a,b;X)} = \int_a^b (f(t), g(t))_X \, dt$$

We note that $L^\infty(a,b;X) = (L^1(a,b;X))'$.

Now, we define the Sobolev spaces with values in a Hilbert space X

For $k \in \mathbb{N}$, $p \in [1,\infty]$, we set:

$$W^{k,p}(a,b;X) = \left\{ v \in L^p(a,b;X); \frac{\partial v}{\partial x_i} \in L^p(a,b;X). \ \ \forall i \le k \right\} \ ,$$

The Sobolev space $W^{k,p}(a,b;X)$ is a Banach space with the norm

$$\begin{aligned} \|f\|_{W^{k,p}(a,b;X)} &= \left(\sum_{i=0}^k \left\| \frac{\partial f}{\partial x_i} \right\|_{L^p(a,b;X)}^p \right)^{1/p} , \ \text{for } p < +\infty \\ \|f\|_{W^{k,\infty}(a,b;X)} &= \sum_{i=0}^k \left\| \frac{\partial v}{\partial x_i} \right\|_{L^\infty(a,b;X)} , \qquad \text{for } p = +\infty \end{aligned}$$

The spaces $W^{k,2}(a,b;X)$ form a Hilbert space and it is noted $H^k(0,T;X)$. The $H^k(0,T;X)$ inner product is defined by:

$$(u,v)_{H^k(a,b;X)} = \sum_{i=0}^k \int_a^b \left(\frac{\partial u}{\partial x^i}, \frac{\partial v}{\partial x^i} \right)_X dt \ .$$

Theorem 1.1.1 *Let $1 \leq p \leq n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$$

where p^* is given by $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}$ (where $p = n, p^* = \infty$). Moreover there exists a constant $C = C(p,n)$ such that

$$\|u\|_{L^{p^*}} \leq C\|\nabla u\|_{L^p(\mathbb{R}^n)} \forall u \in W^{1,p}(\mathbb{R}^n).$$

Corollary 1.1.1 *Let $1 \leq p < n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \quad \forall q \in [p, p^*]$$

with continuous imbedding.

For the case $p = n$, we have

$$W^{1,n}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \quad \forall q \in [n, +\infty[$$

Theorem 1.1.2 *Let $p > n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$$

with continuous imbedding.

Corollary 1.1.2 *Let Ω a bounded domain in $\mathbb{R}^n$ of C^1 class with $\Gamma = \partial\Omega$ and $1 \leq p \leq \infty$. We have*

$$\begin{array}{ll} if & 1 \leq p < \infty,\ then\ W^{1,p}(\Omega) \subset L^{p^*}(\Omega)\ where\ \dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}. \\ if & p = n,\ then\ W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [p, +\infty[. \\ if & p > n,\ then\ W^{1,p}(\Omega) \subset L^\infty(\Omega) \end{array}$$

with continuous imbedding.
Moreover, if $p > n$, we have: $\forall u \in W^{1,p}(\Omega)$,

$$|u(x) - u(y)| \leq C|x - y|^\alpha \|u\|_{W^{1,p}(\Omega)} \ a.e\ x, y \in \Omega$$

with $\alpha = 1 - \dfrac{n}{p} > 0$ and C is a constant which depend on p, n and Ω. In particular $W^{1,p}(\Omega) \subset C(\overline{\Omega})$.

Corollary 1.1.3 *Let Ω a bounded domain in $\mathbb{R}^n$ of C^1 class with $\Gamma = \partial\Omega$ and $1 \leq p \leq \infty$. We have*

$$\begin{array}{ll} if & p < n,\ then\ W^{1,p}(\Omega) \subset L^q(\Omega) \forall q \in [1, p^*[\ where\ \dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}. \\ if & p = n,\ then\ W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [p, +\infty[. \\ if & p > n,\ then\ W^{1,p}(\Omega) \subset C(\overline{\Omega}) \end{array}$$

with compact imbedding.

Remark 1.1.1 *We remark in particular that*

$$W^{1,p}(\Omega) \subset L^q(\Omega)$$

with compact imbedding for $1 \leq p \leq \infty$ *and for* $p \leq q < p^*$.

Corollary 1.1.4

$$\begin{array}{ll} if & \dfrac{1}{p} - \dfrac{m}{n} > 0, \ then\ W^{m,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)\ where\ \dfrac{1}{q} = \dfrac{1}{p} - \dfrac{m}{n}. \\ if & \dfrac{1}{p} - \dfrac{m}{n} = 0, \ then\ W^{m,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \forall q \in [p, +\infty[. \\ if & \dfrac{1}{p} - \dfrac{m}{n} < 0, \ then\ W^{m,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n) \end{array}$$

with continuous imbedding.

1.2 Weak convergence

Let $(E; \|.\|_E)$ a Banach space and E' its dual space, i.e., the Banach space of all continuous linear forms on E endowed with the norm $\|.\|'_E$ defined by

$$\|f\|_{E'} =: \sup_{x \neq 0} \frac{|\langle f, x\rangle|}{\|x\|}$$

; where $\langle f, x\rangle$; denotes the action of f on$x$, i.e.$\langle f, x\rangle := f(x)$. In the same way, we can define the dual space of E' that we denote by E''. (The Banach space E'' is also called the bi-dual space of E.) An element x of E can be seen as a continuous linear form on E' by setting $x(f) := \langle x, f\rangle$, which means that $E \subset E''$:

Definition 1.2.1 *The Banach space E is said to be reflexive if* $E = E''$.

Definition 1.2.2 *The Banach space E is said to be separable if there exists a countable subset D of E which is dense in E, i.e.* $\overline{D} = E$.

Theorem 1.2.1 *(Riesz). If* $(H; \langle ., .\rangle)$ *is a Hilbert space,* $\langle ., .\rangle$ *being a scalar product on H, then* $H' = H$ *in the following sense: to each* $f \in H'$ *there corresponds a unique* $x \in H$ *such that* $f = \langle x, .\rangle$ *and* $\|f\|'_H = \|x\|_H$

Remark : From this theorem we deduce that $H'' = H$. This means that a Hilbert space is reflexive.

Proposition 1.2.1 *If E is reflexive and if F is a closed vector subspace of E, then F is reflexive.*

Corollary 1.2.1 *The following two assertions are equivalent: (i) E is reflexive; (ii)* E' *is reflexive.*

1.2.1 Weak and strong convergence

Definition 1.2.3 *(Weak convergence in E). Let $x \in E$ and let $\{x_n\} \subset E$. We say that $\{x_n\}$ weakly converges to x in E, and we write $x_n \rightharpoonup x$ in E, if*

$$\langle f, x_n \rangle \to \langle f, x \rangle$$

for all $f \in E'$.

Definition 1.2.4 *(weak convergence in E'). Let $f \in E'$ and let $\{f_n\} \subset E'$. We say that $\{f_n\}$ weakly converges to f in E', and we write $f_n \rightharpoonup f$ in E', if*

$$\langle f_n, x \rangle \to \langle f, x \rangle$$

for all $x \in E''$.

Definition 1.2.5 *(strong convergence). Let $x \in E$(resp. $f \in E'$) and let $\{x_n\} \subset E$ (resp $\{f_n\} \subset E'$). We say that $\{x_n\}$ (resp. $\{f_n\}$) strongly converges to x (resp. f), and we write $x_n \to x$ in E (resp. $f_n \to f$ in E'), if*

$$\lim_n \|x_n - x\|_E = 0; (\textit{resp. } \lim_n \|f_n - f\|'_E = 0)$$

Proposition 1.2.2 *Let $x \in E$, let $\{x_n\} \subset E$, let $f \in E'$ and let $\{f_n\} \subset E'$.*

i. If $x_n \to x$ in E then $x_n \rightharpoonup x$ in E.

ii. If $x_n \rightharpoonup x$ in E then $\{x_n\}$ is bounded.

iii. If $x_n \rightharpoonup x$ in E then $\liminf\limits_{n\to\infty} \|x_n\|_E \geq \|x\|_E$

iv. If $f_n \to f$ in E' then $f_n \rightharpoonup f$ inE' (and so $f_n \overset{*}{\rightharpoonup} f$ in E').

v. If $f_n \rightharpoonup f$ in E' then $\{f_n\}$ is bounded.

vi. If $f_n \rightharpoonup f$ in E' then then $\liminf\limits_{n\to\infty} \|f_n\|'_E \geq \|f\|'_E$

Proposition 1.2.3 *(finite dimension). If $\dim E < \infty$ then strong, weak and weak star convergence are equivalent.*

1.2.2 Bounded and Unbounded linear operators

Let $(E, \|.\|_E)$ and $(F, \|.\|_F)$ be two Banach spaces over $\mathbb{C}$, and H will always denote a Hilbert space equipped with the scalar product $< .,. >_H$ and the corresponding norm $\|.\|_H$. A linear operator $T : E \longrightarrow F$ is a transformation which maps linearly E in F, that is

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v), \ \ \forall u, v \in E \text{ and } \alpha, \beta \in \mathbb{C}.$$

Definition 1.2.6 *A linear operator $T : E \longrightarrow F$ is said to be bounded if there exists $C \geq 0$ such that*

$$\|Tu\|_F \leq C\|u\|_E \ \ \forall u \in E.$$

The set of all bounded linear operators from E into F is denoted by $\mathcal{L}(E, F)$. Moreover, the set of all bounded linear operators from E into E is denoted by $\mathcal{L}(E)$.

Definition 1.2.7 *A bounded operator $T \in \mathcal{L}(E, F)$ is said to be compact if for each sequence $(x_n)_{n\in\mathbb{N}} \in E$ with $\|x_n\|_E = 1$ for each $n \in \mathbb{N}$, the sequence $(Tx_n)_{n\in\mathbb{N}}$ has a subsequence which converges in F.*
The set of all compact operators from E into F is denoted by $\mathcal{K}(E, F)$. For simplicity one writes $\mathcal{K}(E) = \mathcal{K}(E, F)$.

Definition 1.2.8 *Let $T \in \mathcal{L}(E, F)$ we define*
• *Range of T by*

$$\mathcal{R}(T) = \{Tu : u \in E\} \subset F.$$

• *Kernel of T by*

$$ker(T) = \{u \in E : Tu = 0\} \subset E.$$

Theorem 1.2.2 (Fredholm alternative)
If $T \in \mathcal{K}(E)$, then
• *$ker(I - T)$ is finite dimension, (I is the identity operator on E) .*
• *$\mathcal{R}(I - T)$ is closed.*
• *$ker(I - T) = 0 \Leftrightarrow \mathcal{R}(I - T) = E$.*

Definition 1.2.9 *An unbounded linear operator T from E into F is a pair $(T, D(T))$, consisting of a subspace $D(T) \subset E$ (called the domain of T) and a linear transformation.*

$$T : D(T) \subset E \mapsto F.$$

In the case when $E = F$ then we say $(T, D(T))$ is an unbounded linear operator on E. If $D(T) = E$ then $T \in \mathcal{L}(E, F)$.

Definition 1.2.10 *Let $T : D(T) \subset E \mapsto F$ be an unbounded linear operator.*
• *The range of T is defined by*

$$\mathcal{R}(T) = \{Tu : u \in D(T)\} \subset F.$$

• *The Kernel of T is defined by*

$$ker(T) = \{u \in D(T) : Tu = 0\} \subset E.$$

• *The graph of T is defined by*

$$G(T) = \{(u, Tu) : u \in D(T)\} \subset E \times F.$$

Definition 1.2.11 *A map T is said to be closed if $G(T)$ is closed in $E \times F$. The closedness of an unbounded linear operator T can be characterize as following if $u_n \in D(T)$ such that $u_n \to u$ in E and $Tu_n \to v$ in F, then $u \in D(T)$ and $Tu = v$.*

Definition 1.2.12 *Let $T : D(T) \subset E \mapsto F$ be a closed unbounded linear operator.*
• *The resolvent set of T is defined by*

$$\rho(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is bijective from } D(T) \text{ onto } F\}.$$

• *The resolvent of T is defined by*

$$\mathcal{R}(\lambda, T) = \{(\lambda I - T)^{-1} : \lambda \in \rho(T)\}$$

• *The spectrum set of T is the complement of the resolvent set in $\mathbb{C}$, denoted by*

$$\sigma(T) = \mathbb{C}/\rho(T)$$

Definition 1.2.13 *Let $T : D(T) \subset E \mapsto F$ be a closed unbounded linear operator. we can split the spectrum $\sigma(T)$ of T into three disjoint sets, given by*
• *The punctual spectrum of T is define by*

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : ker(\lambda I - T) \neq \{0\}\}$$

in this case λ is called an eigenvalue of T.
• *The continuous spectrum of T is define by*

$$\sigma_c(T) = \{\lambda \in \mathbb{C} \ : \ ker(\lambda I - T) = 0, \ \overline{\mathcal{R}(\lambda I - T)} = F \text{ and } (\lambda I - T)^{-1} \text{ is not bounded } \}.$$

• *The residual spectrum of T is define by*

$$\sigma_r(T) = \{\lambda \in \mathbb{C} \ : \ ker(\lambda I - T) = 0, \text{ and } \mathcal{R}(\lambda I - T) \text{ is not dense in } F\}$$

Definition 1.2.14 *Let $T : D(T) \subset E \longrightarrow F$ be a closed unbounded linear operator and let λ be an eigevalue of A. non-zero element $e \in E$ is called a generalized eigenvector of T associated with the eigenvalue value λ, if there exists $n \in \mathbb{N}^*$ such that*

$$(\lambda I - T)^n e = 0 \quad \text{and} \quad (\lambda I - T)^{n-1} e \neq 0.$$

if $n = 1$, then e is called an eigenvector.

Definition 1.2.15 *Let $T : D(T) \subset E \longrightarrow F$ be a closed unbounded linear operator. We say that T has compact resolvent, if there exist $\lambda_0 \in \rho(T)$ such that $(\lambda_0 I - T)^{-1}$ is compact.*

Theorem 1.2.3 *Let $(T, D(T))$ be a closed unbounded linear operator on H then the space $(D(T), \|.\|_{D(T)})$ where $\|u\|_{D(T)} = \|Tu\|_H + \|u\|_H \quad \forall u \in D(T)$ is Banach space .*

Theorem 1.2.4 *Let $(T, D(T))$ be a closed unbounded linear operator on H then, $\rho(T)$ is an open set of $\mathbb{C}$.*

1.3 Semigroups, Existence and uniqueness of solution

The vast majority of the evolution equations can be reduced to the form

$$\begin{cases} U_t(t) = AU(t), & t > 0, \\ U(0) = U_0, \end{cases} \tag{1.1}$$

where $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $S(t)$ over a Hilbert space H. Lets start by basic definitions and theorems.
Let $(X, \|.\|_X)$ be a Banach space, and H be a Hilbert space equipped with the inner product $< ., . >_H$ and the induced norm $\|.\|_H$.

Definition 1.3.1 *Let X be a Banach space and let $I : X \to X$ its identity operator.*

1. *A one parameter family $(S(t))_{t\geq 0}$, of bounded linear operators from X into X is a semigroup of bounded linear operators on X if*
   - **(i)** $S(0) = I$;
   - **(ii)** $S(t + s) = S(t)S(s)$ *for every* $s, t \geq 0$.
2. *A semigroup of bounded linear operators, $(S(t))_{t\geq 0}$, is uniformly continuous if*
$$\lim_{t\to 0} \|S(t) - I\| = 0.$$
3. *A semigroup $(S(t))_{t\geq 0}$ of bounded linear operators on X is a strongly continuous semigroup of bounded linear operators or a C_0-semigroup if*
$$\lim_{t\to 0} S(t)x = x$$
4. *The linear operator $\mathcal{A}$ defined by*
$$\mathcal{A}x = \lim_{t\to 0} \frac{S(t)x - x}{t}, \quad \forall x \in D(\mathcal{A})$$
where
$$D(\mathcal{A}) = \Big\{x \in X;\ \lim_{t\to 0} \frac{S(t)x - x}{t} \ \textit{exists}\Big\}$$
is the infinitesimal generator of the semigroup $(S(t))_{t\geq 0}$.

Some properties of semigroup and its generator operator $\mathcal{A}$ are given in the following theorems:

Theorem 1.3.1 (Pazy) *Let A be the infinitesimal generator of a C_0- semigroup of contractions $(S(t))_{t\geq 0}$. Then, the resolvent $(\lambda I - \mathcal{A})^{-1}$ of $\mathcal{A}$ contains the open right half-plane, i.e., $\rho(\mathcal{A}) \subset \{\lambda : \mathcal{R}(\lambda) > 0\}$ and for such λ we have*

$$\|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(H)} \leq \frac{1}{\mathcal{R}(\lambda)}.$$

Theorem 1.3.2 (Kato) *Let $\mathcal{A}$ be a closed operator in a Banach space X such that the resolvent $(I-\mathcal{A})^{-1}$ of $\mathcal{A}$ exists and is compact. Then the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ consists entirely of isolated eigenvalues with finite multiplicities.*

Theorem 1.3.3 (Pazy) *Let $(S(t))_{t\geq 0}$ be a C_0-semigroup on a Hilbert space H. Then there exist two constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|S(t)\|_{\mathcal{L}(H)} \leq Me^{\omega t}, \ \ \forall t \geq 0.$$

If $\omega = 0$, the semigroup $(S(t))_{t\geq 0}$ is called uniformly bounded and if moreover $M = 1$, then it is called a C_0-semigroup of contractions. For the existence of solution of problem (1.1), we typically use the following Lumer-Phillips and Hille-Yosida theorems :

Theorem 1.3.4 (Lumer-Phillips) *Let $\mathcal{A}$ be a linear operator with dense domain $D(A)$ in a Hilbert space H. If*

(i) *$\mathcal{A}$ is dissipative, i.e., $< \mathcal{R}(< \mathcal{A}x, x >_H) \leq 0, \ \ \forall x \in D(\mathcal{A})$*
and if

(ii) *there exists a $\lambda_0 > 0$ such that the range $\mathcal{R}(\lambda_0 I - \mathcal{A}) = H$,*
then $\mathcal{A}$ generates a C_0-semigroup of contractions on H.

Theorem 1.3.5 (Hille-Yosida) *Let $\mathcal{A}$ be a linear operator on a Banach space X and let $\omega \in \mathbb{R}$, $M \geq 1$ be two constants. Then the following properties are equivalent*

(i) *$\mathcal{A}$ generates a C_0-semigroup $(S(t))_{t\geq 0}$, satisfying*

$$\|S(t)\|_{\mathcal{L}(H)} \leq Me^{\omega t}, \ \ \forall t \geq 0.$$

(ii) *$\mathcal{A}$ is closed, densely defined, and for every $\lambda > \omega$ one has $\lambda \in \rho(\mathcal{A})$ and*

$$\|(\lambda - \omega)^n (\lambda - \mathcal{A})^{-n}\| \leq M, \ \ \forall n \in \mathbb{N}.$$

(iii) *$\mathcal{A}$ is closed, densely defined, and for every $\lambda \in \mathbb{C}$ with $\mathcal{R} > \omega$, one has $\lambda \in \rho(\mathcal{A})$ and*

$$\|(\lambda - \mathcal{A})^{-n}\| \leq \frac{M}{(\mathcal{R}(\lambda) - \omega)^n}, \ \ \forall n \in \mathbb{N}.$$

Consequently, $\mathcal{A}$ is maximal dissipative operator on a Hilbert space H if and only if it generates a C_0-semigroup of contractions $(S(t))_{t\geq 0}$ on H. Thus, the existence of solution is justified by the following corollary which follows from Lumer-Phillips theorem.

Corollary 1.3.1 *Let H be a Hilbert space and let $\mathcal{A}$ be a linear operator defined from $D(\mathcal{A}) \subset H$ into H. If $\mathcal{A}$ is maximal dissipative operator then the initial value problem (1.1) has a unique solution $U(t) = S_A(t)U0$ such that $U \in C([0, +1), H)$, for each initial datum $U_0 \in H$. Moreover, if $U_0 \in D(\mathcal{A})$, then*

$$U \in C([0, +1), D(\mathcal{A})) \cap C^1([0, +1), H).$$

Finally, we also recall the following theorem concerning a perturbations by a bounded linear operators

Theorem 1.3.6 *Let X be a Banach space and let $\mathcal{A}$ be the infinitesimal generator of a C_0-semigroup $(S(t))_{t\geq 0}$ on X, satisfying $\|S_{\mathcal{A}}(t)\|_{\mathcal{L}(H)} \leq Me^{\omega t}$ for all $t \geq 0$. If $\mathcal{B}$ is a bounded linear operator on X , then the operator $\mathcal{A}+\mathcal{B}$ becomes the infinitesimal generator of a C_0-semigroup $(S_{\mathcal{A}+\mathcal{B}}(t))_{t\geq 0}$ on X, satisfying $\|S_{\mathcal{A}+\mathcal{B}}(t)\|_{\mathcal{L}(H)} \leq Me^{(\omega+M\|\mathcal{B}\|)t}$ for all $t \geq 0$.*

1.4 Stability of semigroup

In this section we start by introducing some definition about strong, exponential and polynomial stability of a C_0-semigroup. Then we collect some results about the stability of C_0-semigroup. Let $(X, \|.\|_X)$ be a Banach space, and H be a Hilbert space equipped with the inner product $< .,. >_H$ and the induced norm $\|.\|_H$.

Definition 1.4.1 *Assume that $\mathcal{A}$ is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on X. We say that the C_0-semigroup $(S(t))_{t\geq 0}$ is*

1. *Strongly stable if*
$$\lim_{t\to+\infty} \|S(t)u\|_X = 0, \quad \forall u \in X.$$
2. *Uniformly stable if*
$$\lim_{t\to+\infty} \|S(t)\|_{\mathcal{L}(X)} = 0.$$
3. *Exponentially stable if there exist two positive constants M and ϵ such that*
$$\|S(t)u\|_X \leq Me^{-\epsilon t}\|u\|_X, \quad \forall t > 0,\ \forall u \in X.$$
4. *Polynomially stable if there exist two positive constants C and α such that*
$$\|S(t)u\|_X \leq Ct^{-\alpha}\|u\|_X, \quad \forall t > 0,\ \forall u \in X.$$

Proposition 1.4.1 *Assume that $\mathcal{A}$ is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on X. The following statements are equivalent*

- *$(S(t))_{t\geq 0}$ is uniformly stable.*
- *$(S(t))_{t\geq 0}$ is exponentially stable.*

First, we look for the necessary conditions of strong stability of a C_0-semigroup. The result was obtained by Arendt and Batty.

Theorem 1.4.1 *(Arendt and Batty) Assume that $\mathcal{A}$ is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on a reflexive Banach space X. If*

(i) *$\mathcal{A}$ has no pure imaginary eigenvalues.*

(ii) *$\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable.*

Then $S(t)$ is strongly stable.

Remark 1.4.1 *If the resolvent $(I-T)^{-1}$ of T is compact, then $\sigma(T)=\sigma_p(T)$. Thus, the state of Theorem (...) lessens to $\sigma_p(\mathcal{A})\cap i\mathbb{R}=\emptyset$. Next, when the C_0-semigroup is strongly stable, we look for the necessary and sufficient conditions of exponential stability of a C_0-semigroup. In fact, exponential stability results are obtained using different methods like: multipliers method, frequency domain approach, Riesz basis approach, Fourier analysis or a combination of them .*

Theorem 1.4.2 *(Huang-Pruss)Assume that $\mathcal{A}$ is the generator of a strongly continuous semi-group of contractions $(S(t))_{t\geq 0}$ on H. $S(t)$ is uniformly stable if and only if*

1. $i\mathbb{R}\subset\rho(\mathcal{A})$.
2. $\sup_{\beta\in\mathbb{R}}\|(i\beta I-\mathcal{A})^{-1}\|_{\mathcal{L}(H)}<+\infty$.

The second one, is a classical method based on the spectrum analysis of the operator $\mathcal{A}$

In the case when the C_0-semigroup is not exponentially stable we look for a polynomial one. In general, polynomial stability results also are obtained using different methods like : multipliers method, frequency domain approach, Riesz basis approach, Fourier analysis or a combination of them .

Theorem 1.4.3 *(Batty , A.Borichev and Y.Tomilov, Z. Liu and B. Rao.)Assume that $\mathcal{A}$ is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on H. If $i\mathbb{R}\subset\rho(\mathcal{A})$, then for a fixed $l>0$ the following conditions are equivalent*

1. $\lim_{|\lambda|\to+\infty}\sup\frac{1}{\lambda^l}\|(\lambda I-\mathcal{A})^{-1}\|_{\mathcal{L}(H)}<+\infty$.
2. $\|S(t)U_0\|_H\leq\frac{C}{t^{l-1}}\|U_0\|_{D(\mathcal{A})}\ \forall t>0,\ U_0\in D(\mathcal{A}),$ *for some* $C>0$.

1.5 Lax-Milgrame Theorem:

Let H be a Hilbert space equipped with the inner product $(.,.)_H$ and the induced norm $\|.\|_H$.

Definition 1.5.1 *A bilinear form*

$$a:H\times H\to\mathbb{R}$$

is said to be

- *(i) continuous if there is a constant C such that*

$$|a(u,v)|\leq C\|u\|\|v\|,\ \ \forall u,v\in H$$

- *(ii) coercive if there is a constant $\alpha>0$ such that*

$$|a(u,u)|\geq\alpha\|u\|^2,\ \ \forall u\in H$$

Theorem 1.5.1 (Lax-Milgrame Theorem) *Assume that $a(.,.)$ is a continuous coercive bilinear form on H. Then, given any $L\in\mathcal{L}(\mathcal{H},\mathbb{C})$, there exists a unique element $u\in H$ such that*

$$a(u,v)=L(v),\ \ \forall v\in H$$

1.6 Fractional Derivative Control

In this part, we introduce the necessary elements for the good understanding of this manuscript. It includes a brief reminder of the basic elements of the theory of fractional computation. The concept of fractional computation is a generalization of ordinary derivation and integration to an arbitrary order. Derivatives of non-integer order are now widely applied in many domains, for example in economics, electronics, mechanics, biology, probability and viscoelasticity. A particular interest for fractional derivation is related to the mechanical modeling of gums and rubbers. In short, all kinds of materials that preserve the memory of previous deformations in particular viscoelastic. Indeed, the fractional derivation is introduced naturally.
The fractional calculus is an important developing field in both pure and applied mathematics. Many real world problems have been investigated within the fractional derivatives, particularly Caputo fractional derivative is extensively and successfully used in many branches of sciences and engineering.

1.6.1 Some history of fractional calculus:

In a letter dated **September 30th, 1695** LHospital wrote to Leibniz asking him about the meaning of $d^n y/dx^n$ if $n = 1/2$, that is "what if n is fractional?". Leibnizs response: An apparent paradox, from which one day useful consequences will be drawn.
In **1819** S. F. Lacroix [100] was the first to mention in some two pages a derivative of arbitrary order.Thus for $y = x^a$, $a \in \mathbb{R}_+$, he showed that

$$\frac{d^{1/2}y}{dx^{1/2}} = \frac{\Gamma(a+1)}{\Gamma(1+1/2)} x^{a-1/2}.$$

In particular he had $(d/dx)^{1/2} x = 2\sqrt{x/\pi}$.
In **1822** J. B. J. Fourier derived an integral representation for f(x),

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} f(\alpha) d\alpha \int_{\mathbb{R}} \cos p(x-\alpha) dp,$$

obtained (formally) the derivative version

$$\frac{d^\nu}{dx^\nu} f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} f(\alpha) d\alpha \int_{\mathbb{R}} p^\nu \cos[p(x-\alpha) + \frac{\nu\pi}{2}] dp$$

where "the number v will be regarded as any quantity whatever, positive ornegative".
In **1823** Abel resolved the integral equation arising from the brachistochrone problem, namely

$$\frac{1}{\Gamma(\alpha)} \int_0^x \frac{g(u)}{(x-u)^{1-\alpha}} du = f(x), \qquad 0 < \alpha < 1$$

with the solution

$$g(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{f(u)}{(x-u)^\alpha} du$$

Abel never solved the problem by fractional calculus but, in **1832** Liouville [103], did solve this integral equation.
Perhaps the first serious attempt to give a logical definition of a fractional derivative is due to Liouville; he published nine papers on the subject between **1832** and **1837**, the last in the field in **1855**. They grew out of Liouville's early work on electromagnetism. There is further work of George Peacock (**1833**), D. F. Gregory (**1841**), Augustus de Morgan (**1842**), P. Kelland (**1846**), William Center (**1848**). Especially basic is Riemann's student paper of **1847** [139].
After the participation of Riemann and the work of Cayley in **1880** see [127], among the mathematicians spearheading research in the broad area of fractional calculus until **1941** were S.F. Lacroix, J.B.J. Fourier, N.H. Abel, J. Liouville, A. De Morgan, B. Riemann, Hj. Holmgren, K. Griinwald, A.V. Letnikov, N.Ya. Sonine, J. Hadamard, G.H. Hardy, H. Weyl, M. Riesz, H.T. Davis, A. Marchaud, J.E. Littlewood, E.L. Post, E.R. Love, B.Sz.-Nagy, A. Erdelyi and H. Kober.
Fractional calculus has developed especially intensively since **1974** when the first international conference in the field took place.It was organized by Bertram Ross [144].
Samko et al in their encyclopedic volume [153, p. xxxvi] state and we cite: "We pay tribute to investigators of recent decades by citing the names of mathematicians who have made a valuable scientific contribution to fractional calculus development from **1941** until the present [**1990**]. These are M.A. Al- Bassam, L.S. Bosanquet, P.L. Butzer, M.M. Dzherbashyan, A. Erdelyi, T.M. Flett, Ch. Fox, S.G. Gindikin, S.L. Kalla, LA. Kipriyanov, H. Kober, P.I. Lizorkin, E.R. Love, A.C. McBride, M. Mikolas, S.M. Nikol'skii, K. Nishimoto, LI. Ogievetskii, R.O. O'Neil, T.J. Osier, S. Owa, B. Ross, M. Saigo, I.N. Sneddon, H.M. Srivastava, A.F. Timan, U. Westphal, A. Zygmund and others". To this list must of course be added the names of the authors of Samko et al [153] and many other mathematicians, particularly those of the younger generation. Books especially devoted to fractional calculus include K.B. Oldham and J. Spanier [133], S.G. Samko, A.A. Kilbas and O.I. Marichev [153], V.S. Kiryakova [91], K.S. Miller and B. Ross [121], B. Rubin [147]. Books containing a chapter or sections dealing with certain aspects of fractional calculus include H.T. Davis [37], A. Zygmund [181], M.M.Dzherbashyan [45], I.N. Sneddon [159], P.L. Butzer and R.J. Nessel [25], P.L. Butzer and W. Trebels [28], G.O. Okikiolu [132], S. Fenyo and H.W. Stolle [55], H.M. Srivastava and H.L. Manocha [162], R. Gorenfio and S. Vessella [65].

1.6.2 Various approaches of fractional derivatives

There exists a many mathematical definitions of fractional order integration and derivation. These definitions do not always lead to identical results but are equivalent for a wide large of functions. We introduce the fractional integration operator as well as the two most definitions of fractional derivatives, used, namely that Riemann-Liouville, Caputo and Hadamard.

From the classical fractional calculus, we recall

Definition 1.6.1 *The left Riemann-Liouville fractional integral of order $\alpha > 0$ starting from a has the following form*

$$({}_aI^\alpha f)(n) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt.$$

Definition 1.6.2 *The right Riemann-Liouville fractional integral of order $\alpha > 0$ ending at $b > a$ is defined by*

$$(I_b^\alpha f)(n) = \frac{1}{\Gamma(\alpha)} \int_x^b (x-t)^{\alpha-1} f(t)dt.$$

Definition 1.6.3 *The left Riemann-Liouville fractional derivative of order $\alpha > 0$ starting at a is given below*

$$({}_aD^\alpha f)(x) = (\frac{d}{dx})^n (aI^{n-\alpha} f)(x), \;\; n = [\alpha] + 1.$$

Definition 1.6.4 *The right Riemann-Liouville fractional derivative of order $\alpha > 0$ ending at b becomes*

$$(D_b^\alpha f)(x) = (-\frac{d}{dx})^n (I_b^{n-\alpha} f)(x).$$

Definition 1.6.5 *The left Caputo fractional of order $\alpha > 0$ starting from a has the following form*

$$({}_aD^\alpha f)(x) = (aI^{n-\alpha} f^{(n)})(x), \;\; n = [\alpha] + 1.$$

Definition 1.6.6 *The right Caputo fractional derivative of order $\alpha > 0$ ending at b becomes*

$$(D_b^\alpha f)(x) = (I_b^{n-\alpha} (-1)^n f^{(n)})(x).$$

The Hadamard type fractional integrals and derivatives were introduced in [15] as:

Definition 1.6.7 *The left Hadamard fractional integral of order $\alpha > 0$ starting from a has the following form*

$$({}_aI^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (\ln x - \ln t)^{\alpha-1} f(t)dt$$

Definition 1.6.8 *The right Hadamard fractional integral of order $\alpha > 0$ ending at $b > a$ is defined by*

$$(I_b^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\ln t - \ln x)^{\alpha-1} f(t)dt$$

Definition 1.6.9 *The left Hadamard fractional derivative of order $\alpha > 0$ starting at a is given below*

$$({}_aD^\alpha f)(x) = (x\frac{d}{dx})^n (aI^{n-\alpha} f)(x), \;\; n = [\alpha] + 1.$$

Definition 1.6.10 *The right Hadamard fractional derivative of order $\alpha > 0$ ending at b becomes*

$$(D_b^\alpha f)(x) = (-x\frac{d}{dx})^n (I_b^{n-\alpha} f)(x).$$

Definition 1.6.11 *The fractional derivative of order α, $0 < \alpha < 1$, in sense of Caputo, is defined by*

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{df}{ds}(s)ds.$$

Definition 1.6.12 *The fractional integral of order α, $0 < \alpha < 1$, in sense Riemann-Liouville, is defined by*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds.$$

Remark 1.6.1 *From the above definitions, clearly*

$$D^\alpha f = I^{\alpha-1} Df, \;\; 0 < \alpha < 1.$$

Lemma 1.6.1

$$I^\alpha D^\alpha f(t) = f(t) - f(0), \;\; 0 < \alpha < 1.$$

Lemma 1.6.2 *If*

$$D^\beta f(0) = 0.$$

then

$$D^\alpha D^\beta f = D^{\alpha+\beta} f, \;\; 0 < \alpha < 1, \;\; 0 < \beta < 1.$$

Now, we give the definitions of the generalized Caputo's fractional derivative and the generalized fractional integral. These exponentially modified fractional integro-differential operators were first proposed in [].

Definition 1.6.13 *The generalized Caputo's fractional derivative is given by*

$$D^{\alpha,\eta} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{df}{ds}(s)\, ds, \;\; 0 < \alpha < 1, \eta \geq 0.$$

Remark 1.6.2 *The operators D^α and $D^{\alpha,\eta}$ differ just by their kernels.*

Definition 1.6.14 *The generalized fractional integral is given by*

$$I^{\alpha,\eta} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\eta(t-s)} f(s)\, ds, \;\; 0 < \alpha < 1, \eta \geq 0.$$

Remark 1.6.3 *We have*

$$D^{\alpha,\eta} f = I^{1-\alpha,\eta} Df, \;\; 0 < \alpha < 1, \eta \geq 0.$$

1.7 Bessel functions

We will discuss a class of functions known as Bessel functions. These are named after the German mathematician and astronomer Friedrich Bessel. Bessel functions occur in many other physical problems, usually in a cylindrical geometry.

Definition 1.7.1 *Bessel's equation can be written in the form*

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0, \tag{1.2}$$

with ν real and positive. note that (1.2) has a regular singular point at $x = 0$.

1.7.1 The Gamma Function and Pockhammer Symbol:

Definition 1.7.2 *The gamma function is defined by*

$$\Gamma(x) = \int_0^\infty e^{-q} q^{x-1} dq, \quad for\ x > 0. \tag{1.3}$$

Note that the integration is over the dummy variable q and x is treated as constant during the integration

Definition 1.7.3 *The pockhammer symbol is a simple way of writing down long products. It is defined as*

$$(\alpha)_r = \alpha(\alpha+1)(\alpha+2)...(\alpha+r-1)$$

So that, for example, $(\alpha)_1 = \alpha$ and $(\alpha)_2 = \alpha(\alpha+1)$.
Note that $(1)_n = n!$

The relationship between the gamma function and the pockhammer symbol is

$$\Gamma(x)(x)_n = \Gamma(x+n)$$

1.7.2 Series solutions of Bessel's Equation:

b.Fundamental solutions of Bessel's equation when $\nu \notin \mathbb{N}$:
We can now proceed to consider a Frobenius solution,

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+c}$$

Where we have used the Pockhammer symbol to simplify the expression. So we have

$$y(x) = a_0 x^{\pm\nu} \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{2^{2m}(1\pm\nu)_m m!}$$

With a suitable choice of a_0 we can write this as

$$y(x) = A\frac{x^{\pm\nu}}{2^{\pm\nu}\Gamma(1\pm\nu)} \sum_{m=0}^{\infty} (-1)^m \frac{(\frac{x^2}{4})^m}{(1\pm\nu)_m m!} = AJ_{\pm\nu}(x).$$

These are the Bessel functions of order $\pm\nu$. The general solution of Bessel's equation (1.2), is therefore

$$y(x) = AJ_{+\nu}(x) + BJ_{-\nu}(x),$$

for arbitrary constants A and B, with the first of the two series converges for all values of x and defines the so-called Bessel function of order ν and of the first kind which is denoted by J_ν

$$J_\nu(x) = \frac{x^\nu}{2^\nu\Gamma(1+\nu)}\sum_{m=0}^{\infty}(-1)^m\frac{(\frac{x^2}{4})^m}{(1+\nu)_m m!} = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+\nu+1)}(\frac{x}{2})^{2m+\nu} = \sum_{m=0}^{\infty}c^+_{\nu,m}x^{2m+\nu}\,, x \geq 0. \tag{1.4}$$

The second series converges for all positive values of x and is evidently $J_{-\nu}$

$$J_{-\nu}(x) = \frac{x^{-\nu}}{2^{-\nu}\Gamma(1-\nu)}\sum_{m=0}^{\infty}(-1)^m\frac{(\frac{x^2}{4})^m}{(1-\nu)_m m!} = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m-\nu+1)}(\frac{x}{2})^{2m-\nu} = \sum_{m=0}^{\infty}c^-_{\nu,m}x^{2m-\nu}\,, x > 0. \tag{1.5}$$

c.Fundamental solutions of Bessel's equation when $\nu = n \in \mathbb{N}$:
Assume that $\nu = n \in \mathbb{N}$. When looking for solutions of (1.2)of the form of series of ascending powers of x, one sees that J_n and J_{-n} are still solutions of (1.2), where J_n is still by (1.5) and J_{-n} is given by (1.5); when $\nu = n \in \mathbb{N}$, J_{-n} can be written .

$$J_n(x) = \sum_{m\geq n}\frac{(-1)^m}{m!\Gamma(m-n+1)}(\frac{x}{2})^{-n+2m}. \tag{1.6}$$

However now $J_{-n}(x) = (-1)^n J_n(y)$, hence J_n and J_{-n} are linearly dependent. The determination of a fundamental system of solutions in this case requires further investigation. In this purpose, one introduces the Bessel's functions of order ν and of the second kind: among the several definitions of Bessel's functions of second order, we recall here the definition by Weber. The Bessel's functions of order ν and of second kind are denoted by Y_ν and defined by

$$\begin{cases} \forall\nu \notin \mathbb{N}, & Y_\nu(y) := \frac{J_\nu(x)cos(\nu\pi) - J_{-\nu}(x)}{sin(\nu\pi)}, \\ \forall n \in \mathbb{N}, & Y_n(y) := \lim_{\nu\to n} Y_\nu(x), \end{cases}$$

For any $\nu \in \mathbb{R}_+$, the two functions J_ν and Y_ν always are linearly independent. In particular, in the case $\nu = n \in \mathbb{N}$, the pair (J_n, Y_n) forms a fundamental system of solutions of the Bessels equation for functions of order n.

1.7.3 Differential and Recurrence Relations Between Bessel functions:

It is often useful to find relationships between Bessel functions with different indices. We will derive two such relationships. We start with (1.5),we multiply by x^ν and differentiate to obtain

$$\frac{d}{dx}\{x^\nu J_\nu(x)\} = x^\nu J_{\nu-1}(x) \tag{1.7}$$

and consequently

$$\frac{d}{dx}\{x^{-\nu} J_\nu(x)\} = -x^{-\nu} J_{\nu+1}(x) \tag{1.8}$$

We can use these relationships to derive recurrence relations between the Bessel functions. We expand the differentials in each expression to give the equations

$$J'_\nu(x) + \frac{\nu}{x} J_\nu(x) = J_{\nu-1}(x), \tag{1.9}$$

where we have divided through by x^ν, and

$$J'_\nu(x) - \frac{\nu}{x} J_\nu(x) = -J_{\nu+1}(x), \tag{1.10}$$

where this time we have multiplied by x^ν. By adding these expressions we find that

$$J'_\nu(x) = \frac{1}{2}\{J_{\nu-1}(x) - J_{\nu+1}(x)\}, \tag{1.11}$$

and by subtracting then

$$\frac{2\nu}{x} J_\nu(x) = J_{\nu-1}(x) + J_{\nu+1}(x), \tag{1.12}$$

which is a pure recurrence relationship. These results can also be used when integrating Bessel functions.

1.7.4 Inhomogeneous Terms in Bessel's Equation:

The Inhomogeneous version of Bessel's equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = f(x), \tag{1.13}$$

and the solution can be written as

$$y(x) = AJ_\nu(x) + BJ_{-\nu}(x) + \frac{2\nu}{sin\nu\pi} \int_0^x \frac{f(s)}{s}\Big(J_\nu(s)J_{-\nu}(x) - J_\nu(x)Y_{-\nu}(s)\Big)ds. \tag{1.14}$$

1.8 Appendix

Theorem 1.8.1 *Let μ be the function:*

$$\mu(\xi) = |\xi|^{\frac{2\alpha-d}{2}}, \quad \xi \in \mathbb{R}^d \quad and \quad 0 < \alpha < 1. \tag{1.15}$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t \omega(\xi, t) + (|\xi|^2 + \eta)\omega(\xi, t) - U(t)\mu(\xi) = 0, \quad \xi \in \mathbb{R}^d, t \in \mathbb{R}^+ \quad and \quad \eta \geq 0 \tag{1.16}$$

$$\omega(\xi, 0) = 0, \tag{1.17}$$

$$O(t) = \frac{2\sin(\alpha\pi)\Gamma(\frac{d}{2}+1)}{d\pi^{\frac{d}{2}+1}} \int_{\mathbb{R}^d} \mu(\xi)\omega(\xi, t)d\xi, \tag{1.18}$$

is given by

$$O = I^{1-\alpha,\eta}U = D^{\alpha,\eta}U. \tag{1.19}$$

Proof Step 1. Take $\eta = 0$, the from equation (1.16) and (1.17),we have

$$\omega(\xi,t) = \int_0^t \mu(\xi)e^{-|\xi|^2(t-\tau)}U(\tau)d\tau \tag{1.20}$$

Then from equations (1.18) and (1.20), we get

$$O(t) = \delta \int_{\mathbb{R}^d} |\xi|^{2\alpha-d}[\int_0^t \mu(\xi)e^{-|\xi|^2(t-\tau)}U(\tau)d\tau]d\xi \tag{1.21}$$

where $\delta = \frac{2\sin(\alpha\pi)\Gamma(\frac{d}{2}+1)}{d\pi^{\frac{d}{2}+1}}$. Next, using the spherical coordinates defined by,

$$\begin{cases} \xi_1 = \rho sin(\phi_1)sin(\phi_2)...sin(\phi_{d-3})sin(\phi_{d-2})sin(\phi_{d-1}), \\ \xi_2 = \rho sin(\phi_1)sin(\phi_2)...sin(\phi_{d-3})sin(\phi_{d-2})cos(\phi_{d-1}), \\ \xi_3 = \rho sin(\phi_1)sin(\phi_2)...sin(\phi_{d-3})cos(\phi_{d-2}), \\ \xi_4 = \rho sin(\phi_1)sin(\phi_2)...cos(\phi_{d-3}), \\ . \\ . \\ . \\ \xi_{d-1} = \rho sin(\phi_1)cos(\phi_2), \\ \xi_d = \rho cos(\phi_1). \end{cases} \tag{1.22}$$

where,$\rho = |\xi| = \sqrt{\sum_{i=1}^d |\xi_i|^2}$, $\phi_j \in [0,\pi]$ if $1 \leq j \leq d-2$ and $\phi_{d-2} \in [0, 2\pi]$. The jacobian J is defined by

$$J = \rho^{d-1}\prod_{j=1}^{d-2} sin^{d-1-j}(\phi_j) \tag{1.23}$$

Since the integrating is a function which depends only on $|\xi| = \rho$, thus we can integrate on all the angles and the calculation reduces that of a simple integral on the positive real axis. Then, from equations (1.21)-(1.23) we get

$$O(t) = \delta \int_0^{+\infty} \rho^{2\alpha-1}\prod_{j=1}^{d-2}(\int_0^{\pi} \ sin^{d-1-j}(\phi_j)d\phi_j) \int_0^{2\pi} d\phi_{d-1}[\int_0^t e^{-\rho^2(t-\tau)}U(\tau)d\tau]d\rho \tag{1.24}$$

By induction, it easy to see that

$$\prod_{j=1}^{d-2}(\int_0^{\pi} \ sin^{d-1-j}(\phi_j)d\phi_j) \int_0^{2\pi} d\phi_{d-1} = \frac{d\Pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)} \tag{1.25}$$

Inserting equation (1.25) in equation (1.24), we get

$$O(t) = \frac{sin(\alpha\pi)}{\pi}\int_0^t 2[\int_0^{+\infty} \rho^{2\alpha-1}e^{-\rho^2(t-\tau)}d\rho]U(\tau)d\tau. \tag{1.26}$$

Thus,

$$O(t) = \frac{sin(\alpha\pi)}{\pi}\int_0^t [(t-\tau)^{-\alpha}\Gamma(\alpha)]U(\tau)d\tau. \tag{1.27}$$

Using the fact that $\frac{sin(\alpha\pi)}{\pi} = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}$ in equation, we obtain

$$O(t) = \int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} U(\tau) d\tau. \tag{1.28}$$

It follows that, from equation (1.28) we have

$$O = I^{1-\alpha} U \tag{1.29}$$

Step 1. By simply effecting the following change of function

$$\omega(\xi, t) := e^{-\eta t} \varphi(\xi, t)$$

in equations (1.16) and (1.18), we directly obtain

$$\partial_t \omega(\xi, t) + (|\xi|^2 + \eta)\omega(\xi, t) - U(t)\mu(\xi) = 0, \quad \xi \in \mathbb{R}^N, t \in \mathbb{R}^+ \quad and \quad \eta \geq 0, \tag{1.30}$$

$$\omega(\xi, 0) = 0, \tag{1.31}$$

$$O(t) = \delta e^{-\eta t} \int_{\mathbb{R}^d} \mu(\xi)\omega(\xi, t) d\xi \tag{1.32}$$

Hence, from **Step 1**, (1.30)-(1.32) yield the desired result

$$O(t) = e^{-\eta t} \int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} e^{\eta\tau} U(\tau) d\tau$$

The proof has been completed.

Lemma 1.8.1 *If* $\lambda \in D = \{\lambda \in \mathbb{C} : Re\lambda + \eta > 0\} \cup \{\lambda \in \mathbb{C} : Im\lambda \neq 0\}$ *then*

$$F_1(\lambda) = \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2} d\xi = \frac{\pi}{\sin \alpha\pi} (\lambda + \eta)^{\alpha-1}$$

and

$$F_2(\lambda) = \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{(\lambda + \eta + \xi^2)^2} d\xi = (1-\alpha) \frac{\pi}{\sin \alpha\pi} (\lambda + \eta)^{\alpha-2}.$$

Proof Let us set

$$f_\lambda(\xi) = \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2}.$$

We have

$$\left| \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2} \right| \leq \frac{\mu^2(\xi)}{Re\lambda + \eta + \xi^2}.$$

Then the function f_λ is integrable. Moreover

$$\left| \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2} \right| \leq \begin{cases} \dfrac{\mu^2(\xi)}{\eta_0 + \eta + \xi^2} \text{ for all } Re\lambda \geq \eta_0 > -\eta \\ \dfrac{\mu^2(\xi)}{\tilde{\eta}_0 + \xi^2} \text{for all } |Im\lambda| \geq \tilde{\eta}_0 > 0 \end{cases}$$

From theorem 1.16.1 in [?], the function

$$f_\lambda : D \to \mathbb{C} \text{ is holomorphe.}$$

For a real number $\lambda > -\eta$, we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\, d\xi = \int_{-\infty}^{+\infty} \frac{|\xi|^{2\alpha-1}}{\lambda+\eta+\xi^2}\, d\xi = \int_0^{+\infty} \frac{x^{\alpha-1}}{\lambda+\eta+x}\, dx \ (\text{ with } \xi^2 = x)\\
&= (\lambda+\eta)^{\alpha-1} \int_1^{+\infty} y^{-1}(y-1)^{\alpha-1}\, dy \ (\text{ with } y = x/(\lambda+\eta)+1)\\
&= (\lambda+\eta)^{\alpha-1} \int_0^1 z^{-\alpha}(1-z)^{\alpha-1}\, dz \ (\text{ with } z = 1/y)\\
&= (\lambda+\eta)^{\alpha-1} B(1-\alpha,\alpha) = (\lambda+\eta)^{\alpha-1}\Gamma(1-\alpha)\Gamma(\alpha) = (\lambda+\eta)^{\alpha-1}\frac{\pi}{\sin \pi\alpha}.
\end{aligned}$$

Both holomorphic functions f_λ and $\lambda \mapsto (\lambda+\eta)^{\alpha-1}\frac{\pi}{\sin \pi\alpha}$ coincide on the half line $]-\infty, -\eta[$, hence on D following the principle of isolated zeroes.

$\square$

Chapter 2

DECAY ESTIMATES FOR A DEGENERATE WAVE EQUATION WITH A DYNAMIC FRACTIONAL FEEDBACK ACTING ON THE DEGENERATE BOUNDARY

abstract: We consider a one-dimensional weakly degenerate wave equation with a dynamic nonlocal boundary feedback of fractional type acting at a degenerate point. First We show well-posedness by using the semigroup theory. Next, we show that our system is not uniformly stable by spectral analysis. Hence, we look for a polynomial decay rate for a smooth initial data by using a result due Borichev and Tomilov which reduces the problem of estimating the rate of energy decay to finding a growth bound for the resolvent of the generator associated with the semigroup. This analysis proves that the degeneracy affect the energy decay rates.

2.1 Introduction

We are concerned with the dynamic boundary stabilization of fractional type for degenerate wave equation of the form

$$(P) \qquad \begin{cases} u_{tt}(x,t) - (x^\gamma u_x(x,t))_x = 0 & \text{in } (0,1)\times(0,+\infty), \\ -mu_{tt}(0,t) + (x^\gamma u_x)(0,t) = \varrho \partial_t^{\alpha,\eta} u(0,t) & \text{in } (0,+\infty), \\ u(1,t) = 0 & \text{in } (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } (0,1), \end{cases}$$

where $(x,t) \in (0,1)\times(0,+\infty), \gamma \in [0,1), m > 0$ and $\varrho > 0$. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α, $(0 < \alpha \leq 1)$, with respect to the time variable (see **[19]**). It is defined as follows

$$\partial_t^{\alpha,\eta} w(t) = \begin{cases} w_t(t) & \text{for } \alpha = 1, \eta \geq 0, \\ \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\, ds, & \text{for } 0 < \alpha < 1, \eta \geq 0. \end{cases}$$

The problem (P) describes the motion of a pinched vibration cable with tip mass $m > 0$ (see **[39]** and **[28]**). The situations where the coefficients are variables arise in engineering problems that generally use non-homogeneous materials such as smart materials.

The bibliography of works concerning the stabilization of nondegenerate non-homogeneous wave equation with different types of dampings is truly long (see e.g. **[17]**, **[20]**, **[16]** and the references therein). D'Andrea-Novel and al. in **[20]** studied the wave equation with one feedback depending only on the boundary velocities and the boundary displacement i.e, they considered the following problem

$$\begin{cases} u_{tt}(x,t) - (a(x)u_x)_x = 0, & 0 < x < 1,\ t > 0, \\ (au_x)(0,t) = 0, & t > 0 \\ (au_x)(1,t) = -ku(1,t) - u_t(1,t), & t > 0\ k > 0, \end{cases}$$

where $a(x) = a_1 x + a_0$. They have established aymptotics stabilization. Chentouf and al. in **[16]** considered the following modelization of a flexible torque arm controlled by two feedbacks depending only on the boundary velocities:

$$\begin{cases} u_{tt}(x,t) - (a(x)u_x)_x + \alpha u_t(x,t) + \beta y(x,t) = 0,\ 0 < x < 1,\ t > 0, \\ (a(x)u_x)(0) = k_1 u_t(0,t), t > 0, \\ (a(x)u_x)(1) = -k_2 u_t(1,t), t > 0, \end{cases}$$

where

$$\begin{cases} \alpha \geq 0, \beta > 0, k_1, k_2 \geq 0, k_1 + k_2 \neq 0, \\ a \in W^{1,\infty}(0,1), a(x) \geq a_0 \text{ for all } x \in [0,1]. \end{cases}$$

They establish the exponential decay of the solutions.

Let us mention here that the case $\gamma = 0$ and $\alpha = 1$ in (P) corresponds to a classical boundary damping and it has been extensively studied by many authors (see, for instance, **[34]**, **[26]**, and references therein). It has been proved, in particular that solutions exist globally with an optimal decay rate that is $E(t) \sim c/t$ by using Riesz basis property of the generalized eigenvector of the system.

Recently in **[9]**, Benaissa and Benkhedda considered the stabilization for the following wave equation with dynamic boundary feedback of fractional derivative type (CF):

$$(PF) \qquad \begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0 & \text{in } (0,1) \times (0,+\infty) \\ u(0,t) = 0 & \text{in } (0,+\infty) \\ mu_{tt}(1,t) + u_x(1,t) = -\varrho \partial_t^{\alpha,\eta} u(1,t) & \text{in } (0,+\infty) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } (0,1) \end{cases}$$

They proved that the decay of the energy is not exponential but polynomial that is $E(t) \leq C1/t^{(2-\alpha)}$.

Very recently in **[18]**, Cheheb and al. considered the stabilization for the following wave equation with a general dynamic boundary feedback of diffusive type (CF):

$$(P)\qquad \begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0 & \text{in } (0,1)\times(0,+\infty),\\ u(0,t) = 0 & \text{in } (0,+\infty),\\ mu_{tt}(1,t) + u_x(1,t) = -\zeta \displaystyle\int_{-\infty}^{+\infty} \nu(\xi)\varphi(\xi,t)\,d\xi & \text{in } (0,+\infty),\\ \partial_t\varphi(\xi,t) + (\xi^2+\eta)\varphi(\xi,t) - u_t(1,t)\nu(\xi) = 0 & \text{in } (-\infty,\infty)\times(0,+\infty),\\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } (0,1),\\ \varphi(\xi,0) = \varphi_0 & \text{in } (-\infty,\infty). \end{cases}$$

They proved that the decay of the energy is not exponential. Moreover, they obtained a precise and optimal energy decay estimate for a general feedback of diffusive type, from which the usual feedback of fractional derivative type is a special case.

Very recently in **[8]**, Benaissa and Aichi studied the degenerate wave equation of the type

$$u_{tt}(x,t) - (a(x)u_x(x,t))_x = 0 \text{ in } (0,1)\times(0,+\infty), \tag{2.1}$$

where the coefficient a is a positive function on $]0,1]$ but vanishes at zero. The degeneracy of (2.1) at $x=0$ is measured by the parameter μ_a defined by

$$\mu_a = \sup_{0<x\le 1} \frac{x|a'(x)|}{a(x)} \tag{2.2}$$

and the initial conditions are

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \tag{2.3}$$

followed by the boundary conditions

$$(P1)\qquad \begin{cases} \begin{cases} u(0,t) = 0 & \text{if } 0\le\mu_a<1\\ (au_x)(0,t) = 0 & \text{if } 1\le\mu_a<2 \end{cases} & \text{in } (0,+\infty),\\ u_x(1,t) + \varrho\partial_t^{\alpha,\eta}u(1,t) + \beta u(1,t) = 0 & \text{in } (0,+\infty). \end{cases}$$

They proved an optimal polynomial decay rate. It is proved that the presence of feedback of fractional time derivative type and located at a nondegenerate point $x=1$ has no effect on the stabilisation results in **[8]**.

Here we want to focus on the following remarks:

- The method based on the theory of Riesz basis property of the generalized eigenvector of the system does not seem to be work in the presence of a fractional feedback.

- The frequency method based on multiplier techniques used in **[8]** and the enegy method based on multiplier techniques used in **[31]** do not seem to be work in the case of a feedback located at a degenerate point $x=0$.

In this chapter, we explain the influence of the relation between the tip mass term, the degenerate coefficient and the fractional feedback on decay estimates. We prove a sharp polynomial decay rate depending on parameters γ, α. To the best of our knowledge, there is no result concerning the stabilization of a degenerate wave equation with the presence of a dynamic fractional feedback acting on the degenerate boundary.

This chapter is organized as follows. In section 2, we give preliminaries results and we reformulate the system (P) into an augmented system by coupling the degenerate wave equation with a suitable diffusion equation and we show the well-posedness of our problem by semigroup theory. In section 3, we prove lack of exponential stability by spectral analysis and by using Bessel functions. In the last section, we prove an optimal decay rate. Our approach is based on a result due to Borichev and Tomilov, which reduces the problem of estimating the rate of energy decay to finding a growth bound for the resolvent of the semigroup generator using an explicit representation of the resolvent by the help of Bessel functions.

2.2 Preliminary results

Now, we introduce the following weighted Sobolev spaces:

$$H^1_{0,\gamma}(0,1) = \Big\{ u \text{ is locally absolutely continuous in } (0,1] : x^{\gamma/2}u_x \in L^2(0,1)/\ u(1) = 0 \Big\},$$

$$H^1_{\gamma}(0,1) = \Big\{ u \text{ is locally absolutely continuous in } (0,1] : x^{\gamma/2}u_x \in L^2(0,1) \Big\}.$$

We remark that $H^1_\gamma(0,1)$ is a Hilbert space with the scalar product

$$(u,v)_{H^1_\gamma(0,1)} = \int_0^1 (u\overline{v} + x^\gamma u'(x)\overline{v'(x)})\,dx, \quad \forall u,v \in H^1_\gamma(0,1).$$

Let us also set

$$|u|_{H^1_{0,\gamma}(0,1)} = \left(\int_0^1 x^\gamma |u'(x)|^2\,dx\right)^{1/2} \quad \forall u \in H^1_\gamma(0,1).$$

Actually, $|\cdot|_{H^1_{0,\gamma}(0,1)}$ is an equivalent norm on the closed subspace $H^1_{0,\gamma}(0,1)$ to the norm of $H^1_\gamma(0,1)$. This fact is a simple consequence of the following version of Poincaré's inequality.

Proposition 2.2.1 *There is a positive constant $C_* = C(\gamma)$ such that*

$$\|u\|_{L^2(\Omega)} \leq C_* |u|_{H^1_{0,\gamma}(0,1)} \quad \forall u \in H^1_{0,\gamma}(0,1). \tag{2.4}$$

Proof. Let $u \in H^1_{0,\gamma}(0,1)$. For any $x \in]0,1]$ we have that

$$|u(x)| = \left|\int_x^1 u'(s)\,ds\right| \leq |u|_{H^1_{0,\gamma}(0,1)} \left\{\int_0^1 \frac{1}{x^\gamma}\,dx\right\}^{1/2}.$$

Therefore

$$\int_0^1 |u(x)|^2\,dx \leq \frac{1}{1-\gamma}|u|^2_{H^1_{0,\gamma}(0,1)}.$$

Next, we define

$$H^2_\gamma(0,1) = \{u \in H^1_\gamma(0,1) : \ x^\gamma u' \in H^1(0,1)\},$$

where $H^1(0,1)$ denotes the classical Sobolev space. In this section we reformulate (P) into an augmented system. For that, we need the following proposition.

Remark 2.2.1 *Notice that if $u \in H^2_\gamma(0,1), \gamma \in [1,2)$, we have $(x^\gamma u_x)(0) \equiv 0$. Indeed, if $x^\gamma u_x(x) \to L$ when $x \to 0$, then $x^\gamma |u_x(x)|^2 \sim L/x^\gamma$ and therefore $L = 0$ otherwise $u \notin H^1_\gamma(0,1)$.*

Proposition 2.2.2 (see [37]) *Let ν be the function:*

$$\nu(\xi) = |\xi|^{(2\alpha-1)/2}, \quad -\infty < \xi < +\infty, \ 0 < \alpha < 1. \tag{2.5}$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t \varphi(\xi,t) + (\xi^2 + \eta)\varphi(\xi,t) - U(t)\nu(\xi) = 0, \quad -\infty < \xi < +\infty, \eta \geq 0, t > 0, \tag{2.6}$$

$$\varphi(\xi, 0) = 0, \tag{2.7}$$

$$O(t) = (\pi)^{-1} \sin(\alpha\pi) \int_{-\infty}^{+\infty} \nu(\xi)\varphi(\xi,t)\, d\xi \tag{2.8}$$

is given by

$$O = I^{1-\alpha,\eta} U. \tag{2.9}$$

where

$$[I^{\alpha,\eta} f](t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{-\eta(t-\tau)} f(\tau)\, d\tau.$$

Lemma 2.2.1 (see [1]) *If $\lambda \in D_\eta = \mathbb{C} \backslash]-\infty, -\eta]$ then*

$$F(\lambda) = \int_{-\infty}^{+\infty} \frac{\nu^2(\xi)}{\lambda + \eta + \xi^2}\, d\xi = \frac{\pi}{\sin \alpha\pi} (\lambda + \eta)^{\alpha - 1}.$$

Using now Proposition 2.2.2 and relation (2.9), system (P) may be recast into the following augmented system

$$(P') \qquad \begin{cases} u_{tt}(x,t) - (x^\gamma u_x(x,t))_x = 0, \\ \varphi_t(\xi,t) + (\xi^2 + \eta)\varphi(\xi,t) - u_t(0,t)\nu(\xi) = 0, & -\infty < \xi < +\infty, \quad t > 0, \\ -m u_{tt}(0,t) + (x^\gamma u_x)(0,t) = \zeta \displaystyle\int_{-\infty}^{+\infty} \nu(\xi)\varphi(\xi,t)\, d\xi, \\ u(1,t) = 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \end{cases}$$

where $\zeta = \varrho(\pi)^{-1} \sin(\alpha\pi)$. Thus, we shall consider problem (P') instead of (P).

2.3 Well-posedness

In this section, we will use the semigroup approach to study the well-posedness of system (P'). To define the semigroup associated with (P'), we consider the right-end boundary condition

$$u_t(0,t) = \theta(t), t > 0,$$

where θ solve the equation

$$-m\theta_t(t) + (x^\gamma u_x)(0,t) - \zeta \int_{-\infty}^{+\infty} \nu(\xi)\varphi(\xi,t)\, d\xi = 0. \tag{2.10}$$

with the initial condition

$$\theta(0) = u_1(0) = \theta_0. \tag{2.11}$$

Considering $U := (u, u_t, \varphi, \theta)^T$ and $U_0 = (u_0, u_1, 0, \theta_0)^T$, system (P') can be written in the following abstract framework

$$\partial_t U = \mathcal{P}U, \qquad U(0) = U_0, \tag{2.12}$$

where the operator $\mathcal{P}$ is given by

$$\mathcal{P}\begin{pmatrix} u \\ v \\ \varphi \\ \theta \end{pmatrix} = \begin{pmatrix} v \\ (x^\gamma u_x)_x \\ -(\xi^2+\eta)\varphi + v(0)\nu(\xi) \\ \dfrac{1}{m}(x^\gamma u_x)(0) - \dfrac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty} \nu(\xi)\varphi(\xi)\, d\xi \end{pmatrix}. \tag{2.13}$$

This operator will be defined in an appropriate subspace of the Hilbert space

$$\mathcal{H} = H^1_{0,\gamma}(0,1) \times L^2(0,1) \times L^2(-\infty,+\infty) \times \mathbb{C},$$

endowed with the inner product

$$\left\langle \begin{pmatrix} u \\ v \\ \varphi \\ \theta \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\varphi} \\ \tilde{\theta} \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_0^1 x^\gamma u_x \overline{\tilde{u}}_x dx + \int_0^1 v\overline{\tilde{v}} dx + \zeta \int_{-\infty}^{+\infty} \varphi\overline{\tilde{\varphi}}\, d\xi + m\theta\overline{\tilde{\theta}}.$$

We choose the domain for the operator $\mathcal{P}$ as

$$D(\mathcal{P}) = \left\{ \begin{array}{l} (u,v,\varphi,\theta) \text{ in } \mathcal{H} : u \in H^2_\gamma(0,1) \cap H^1_{0,\gamma}(0,1), v \in H^1_{0,\gamma}(0,1), \theta \in \mathbb{C}, \\ -(\xi^2+\eta)\varphi + v(0)\nu(\xi) \in L^2(-\infty,+\infty), v(0) = \theta, \\ |\xi|\varphi \in L^2(-\infty,+\infty) \end{array} \right\}. \tag{2.14}$$

Our main result is giving by the following theorem.

Theorem 2.3.1 *The operator $\mathcal{P}$ defined by (2.13) and (2.14), generates a C_0-semigroup of contractions $e^{t\mathcal{P}}$ in the Hilbert space $\mathcal{H}$.*

Proof. To prove this result we shall use the Lumer-Phillips theorem. Since for every $U = (u, v, \varphi, \theta) \in D(\mathcal{P})$ we have

$$\Re\langle \mathcal{P}U, U\rangle_{\mathcal{H}} = -\zeta \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\varphi(\xi)|^2 \, d\xi. \tag{2.15}$$

then the operator $\mathcal{P}$ is dissipative.

Let $\lambda > 0$. we prove that the operator $(\lambda I - \mathcal{P})$ is a surjection. Let $F = (f_1, f_2, f_3, f_4) \in \mathcal{H}$, the vector $U = (u, v, \varphi, \theta) \in D(\mathcal{P})$ is a solution of the system $\lambda I - \mathcal{P}U = F$ if its components satisfy

$$\begin{cases} \lambda u - v = f_1, \\ \lambda v - (x^\gamma u_x)_x = f_2, \\ \lambda\varphi + (\xi^2 + \eta)\varphi - v(0)\nu(\xi) = f_3. \\ \lambda\theta - \frac{1}{m}(x^\gamma u_x)(0) + \frac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty} \nu(\xi)\varphi(\xi)\, d\xi = f_4. \end{cases} \tag{2.16}$$

Suppose u is found with the appropriate regularity. Then, $(2.16)_1$ and $(2.16)_3$ yield

$$v = \lambda u - f_1 \in H^1_{0,\gamma}(0,1), \tag{2.17}$$

$$\varphi(\xi) = \frac{f_3(\xi)}{\xi^2 + \eta + \lambda} + \frac{\lambda u(0)\nu(\xi)}{\xi^2 + \eta + \lambda} - \frac{f_1(0)\nu(\xi)}{\xi^2 + \eta + \lambda}. \tag{2.18}$$

By using (2.16) and (2.17) it can easily be shown that u satisfies

$$\lambda^2 u - (x^\gamma u_x)_x = f_2 + \lambda f_1. \tag{2.19}$$

Solving equation (2.19) is equivalent to finding $u \in H^2_\gamma(0,1) \cap H^1_{0,\gamma}(0,1)$ such that

$$\int_0^1 (\lambda^2 u\overline{w} - (x^\gamma u_x)_x \overline{w})\, dx = \int_0^1 (f_2 + \lambda f_1)\overline{w}\, dx, \tag{2.20}$$

for all $w \in H^1_{0,\gamma}(0,1)$. By using (2.20), the boundary condition $(2.16)_4$, the fact that $\theta = v(0)$ and (2.18), the function u satisfying the following equation

$$\begin{cases} \displaystyle\int_0^1 (\lambda^2 u\overline{w} + x^\gamma u_x \overline{w}_x)\, dx + \lambda(m\lambda + \tilde{\zeta})u(0)\overline{w}(0) \\ = \displaystyle\int_0^1 (f_2 + \lambda f_1)\overline{w}\, dx - \zeta \int_{-\infty}^{+\infty} \frac{\nu(\xi)}{\xi^2 + \eta + \lambda} f_3(\xi)\, d\xi \overline{w}(0) + (m\lambda + \tilde{\zeta})f_1(0)\overline{w}(0) - mf_4\overline{w}(0). \end{cases} \tag{2.21}$$

where $\tilde{\zeta} = \zeta \displaystyle\int_{-\infty}^{+\infty} \frac{\nu^2(\xi)}{\xi^2 + \eta + \lambda}\, d\xi$. Problem (2.21) is of the form

$$\mathcal{B}(u, w) = \mathcal{L}(w), \qquad \forall w \in H^1_{0,\gamma}(0,1), \tag{2.22}$$

where $\mathcal{B} : [H^1_{0,\gamma}(0,1) \times H^1_{0,\gamma}(0,1)] \to \mathbb{C}$ is the the sesquilinear form defined by

$$\mathcal{B}(u, w) = \int_0^1 (\lambda^2 u\overline{w} + x^\gamma u_x \overline{w}_x)\, dx + \lambda(m\lambda + \tilde{\zeta})u(0)\, \overline{w}(0)$$

and $\mathcal{L} : H^1_{0,\gamma}(0,1) \to \mathbb{C}$ is the antilinear form given by

$$\mathcal{L}(w) = \int_0^1 (f_2 + \lambda f_1)\overline{w}\, dx - \zeta \int_{-\infty}^{+\infty} \frac{\nu(\xi)}{\xi^2+\eta+\lambda} f_3(\xi)\, d\xi \overline{w}(0) + (m\lambda + \tilde{\zeta}) f_1(0)\overline{w}(0) - m f_4 \overline{w}(0).$$

It is easy to verify that $\mathcal{B}$ is continuous and coercive, and $\mathcal{L}$ is continuous. Therefore, using the Lax-Milgram Theorem, we conclude that (2.22) has a unique solution $u \in H^1_{0,\gamma}(0,1)$. By classical regularity arguments, we conclude that the solution u of (2.22) belongs into $H^2_\gamma(0,1)$. Therefore, the operator $\lambda I - \mathcal{P}$ is surjective for any $\lambda > 0$.

□

As a consequence of Theorem 2.3.1, the system (P') is well-posed in the energy space $\mathcal{H}$ and we have the following proposition.

Proposition 2.3.1 *For $(u_0, u_1, 0, \theta_0) \in \mathcal{H}$, the problem (P') admits a unique weak solution*

$$(u, u_t, \varphi, \theta) \in C^0(\mathbb{R}_+, \mathcal{H}).$$

and for $(u_0, u_1, 0, \theta_0) \in D(\mathcal{P})$, the problem (P') admits a unique strong solution

$$(u, u_t, \varphi, \theta) \in C^0(\mathbb{R}_+, D(\mathcal{P})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

Moreover, from the density $D(\mathcal{P})$ in $\mathcal{H}$ the energy of $(u(t), \varphi(t))$ at time $t \geq 0$ by

$$E(t) = \frac{1}{2}\int_0^1 (|u_t|^2 + x^\gamma |u_x|^2) dx + \frac{m}{2}|u_t(0,t)|^2 + \frac{\zeta}{2}\int_{-\infty}^{+\infty} |\varphi(\xi,t)|^2\, d\xi \tag{2.23}$$

decays as follow

$$E'(t) = -\zeta \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\varphi(\xi,t)|^2\, d\xi \leq 0. \tag{2.24}$$

Proof of Proposition 2.3.1. Noting that the regularity of the solution of the problem (P') is consequence of the semigroup properties. We have just to prove (2.24).

Multiplying the first equation in (P') by $\overline{u}_t$, integrating over $(0,1)$ and using integration by parts, we get

$$\int_0^1 u_{tt}(x,t)\overline{u}_t dx - \int_0^1 (x^\gamma u_x(x,t))_x \overline{u}_t dx = 0.$$

Then

$$\frac{d}{dt}\left(\frac{1}{2}\int_0^1 |u_t(x,t)|^2 dx\right) + \frac{1}{2}\frac{d}{dt}\int_0^1 x^\gamma |u_x(x,t)|^2\, dx - \Re\Big[(x^\gamma u_x)(x,t)\overline{u}_t\Big]_0^1 = 0.$$

Then

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left(|u_t(x,t)|^2 + x^\gamma |u_x(x,t)|^2\right) dx + \frac{m}{2}|u_t(0,t)|^2 + \zeta \Re \overline{u}_t(0,t) \int_{-\infty}^{+\infty} \nu(\xi)\varphi(\xi,t)\, d\xi = 0. \tag{2.25}$$

Multiplying the second equation in (P') by $\zeta\overline{\varphi}$ and integrating over $(-\infty, +\infty)$, to obtain:

$$\zeta \int_{-\infty}^{+\infty} \varphi_t(\xi,t)\overline{\varphi}(\xi,t) d\xi + \zeta \int_{-\infty}^{+\infty} (\xi^2+\eta)|\varphi(\xi,t)|^2 d\xi - \zeta u_t(0,t)\int_{-\infty}^{+\infty} \nu(\xi)\overline{\varphi}(\xi,t) d\xi = 0.$$

Hence

$$\frac{\zeta}{2}\frac{d}{dt}\int_{-\infty}^{+\infty}|\varphi(\xi,t)|^2d\xi+\zeta\int_{-\infty}^{+\infty}(\xi^2+\eta)|\varphi(\xi,t)|^2d\xi-\zeta\Re u_t(0,t)\int_{-\infty}^{+\infty}\nu(\xi)\overline{\varphi}(\xi,t)d\xi=0. \tag{2.26}$$

From (2.23), (2.25) and (2.26) we get

$$E'(t)=-\zeta\int_{-\infty}^{+\infty}(\xi^2+\eta)|\varphi(\xi,t)|^2\,d\xi\leq 0.$$

This completes the proof of the lemma.

Remark 2.3.1 • *We can easily extend the global existence result for a general function $a(x)$ instead of x^γ with $0<\mu_a<1$ (see (2.2)).*
• *In the case $\alpha=1$, we take $\varrho u_t(0,t)$ instead of $\varrho\partial_t^{\alpha,\eta}u(0,t)$. We do not need to introduce a diffusive representation technique to bring the problem back into the semigroup theory. Indeed the operator $\mathcal{P}$ takes the form*

$$\tilde{\mathcal{P}}\begin{pmatrix}u\\ v\\ \theta\end{pmatrix}=\begin{pmatrix}v\\ (x^\gamma u_x)_x\\ \frac{1}{m}(x^\gamma u_x)(0)-\frac{\varrho}{m}\theta\end{pmatrix}. \tag{2.27}$$

with domain

$$D(\tilde{\mathcal{P}})=\left\{\begin{array}{l}(u,v,\theta)\ \text{in}\ \mathcal{H}: u\in H^2_\gamma(0,1)\cap H^1_{0,\gamma}(0,1), v\in H^1_{0,\gamma}(0,1),\theta\in\mathbb{C},\\ v(0)=\theta,\end{array}\right\}. \tag{2.28}$$

where

$$\mathcal{H}=H^1_{0,\gamma}(0,1)\times L^2(0,1)\times\mathbb{C},$$

with the inner product

$$\left\langle\begin{pmatrix}u\\ v\\ \theta\end{pmatrix},\begin{pmatrix}\tilde{u}\\ \tilde{v}\\ \tilde{\theta}\end{pmatrix}\right\rangle_{\mathcal{H}}=\int_0^1x^\gamma u_x\overline{\tilde{u}}_xdx+\int_0^1v\overline{\tilde{v}}dx+m\theta\overline{\tilde{\theta}}.$$

The well-posedness result follows exactly as in the case $0<\alpha<1$. Moreover, the energy function is defined as

$$\tilde{E}(t)=\frac{1}{2}\int_0^1(|u_t|^2+x^\gamma|u_x|^2)dx+\frac{m}{2}|u(0,t)|^2 \tag{2.29}$$

and decays as follows

$$\tilde{E}'(t)=-\varrho|u_t(0,t)|^2\leq 0.$$

□

2.4 Strong stability and lack of exponential stability

2.4.1 Strong Stability

We need the following Theorem to prove strong stability of solutions.

Theorem 2.4.1 ([35]) *Let $\mathcal{P}$ be the generator of a uniformly bounded C_0-semigroup $\{S(t)\}_{t\geq 0}$ on a Hilbert space $\mathcal{X}$. If:*

(i) $\mathcal{P}$ does not have eigenvalues on $i\mathbb{R}$.

(ii) The intersection of the spectrum $\sigma(\mathcal{P})$ with $i\mathbb{R}$ is at most a countable set,

then the semigroup $\{S(t)\}_{t\geq 0}$ is asymptotically stable, i.e, $\|S(t)z\|_{\mathcal{X}} \to 0$ as $t \to \infty$ for any $z \in \mathcal{X}$.

Our main result is the following theorem:

Theorem 2.4.2 *The C_0-semigroup $e^{t\mathcal{P}}$ is strongly stable in $\mathcal{H}$; i.e., for all $U_0 \in \mathcal{H}$, the solution of (2.12) satisfies*

$$\lim_{t\to\infty} \|e^{t\mathcal{P}}U_0\|_{\mathcal{H}} = 0.$$

For the proof of Theorem 2.4.2, we need the following two lemmas.

Lemma 2.4.1 *$\mathcal{P}$ does not have eigenvalues on $i\mathbb{R}$.*

Proof.
We make a distinction between $i\lambda = 0$ and $i\lambda \neq 0$.
Step 1. Solving for $\mathcal{P}U = 0$ leads to the system

$$(2.30)\qquad \begin{cases} v = 0, \\ (x^\gamma u_x)_x = 0, \\ (\xi^2+\eta)\varphi - v(0)\nu(\xi) = 0. \\ -\frac{1}{m}(x^\gamma u_x)(0) + \frac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty} \nu(\xi)\varphi(\xi)\, d\xi = 0. \end{cases}$$

Then $v = 0, \varphi = 0, (x^\gamma u_x)(0) = 0$ and

$$(x^\gamma u_x)(x) = c.$$

As $(x^\gamma u_x)(0) = 0$, we have $(x^\gamma u_x)(x) = 0$. Hence

$$u_x(x) = 0 \text{ for } x \in (0,1).$$

As $u(1) = 0$, then $u = 0$. we have $U = 0$. Hence, $i\lambda = 0$ is not an eigenvalue of $\mathcal{P}$.

Step 2. Let $\lambda \in \mathbb{R} - \{0\}$. We prove that $i\lambda$ is not an eigenvalue of $\mathcal{P}$ by proving that the unique solution $U \in D(\mathcal{P})$ of the equation

$$\mathcal{P}U = i\lambda U \tag{2.31}$$

is $U = 0$. Let $U = (u, v, \varphi, \theta)^T$. The equation (2.31) means that

$$\begin{cases} i\lambda u - v = 0, \\ i\lambda v - (x^\gamma u_x)_x = 0, \\ i\lambda\varphi + (\xi^2 + \eta)\varphi - v(0)\nu(\xi) = 0. \\ i\lambda\theta - \frac{1}{m}(x^\gamma u_x)(0) + \frac{\zeta}{m}\int_{-\infty}^{+\infty} \nu(\xi)\varphi(\xi)\, d\xi = 0. \end{cases} \tag{2.32}$$

Using (2.15) and (2.31), we find

$$\varphi \equiv 0, \tag{2.33}$$

then, using the third equation in (2.32), we deduce that

$$v(0) = 0. \tag{2.34}$$

Therefore, from the first and last equation in (2.32), we find

$$u(0) = 0 \quad \text{and} \quad (x^\gamma u_x)(0) = 0. \tag{2.35}$$

Thus, by eliminating v, the system (2.32) implies that

$$\begin{cases} \lambda^2 u + (x^\gamma u_x)_x = 0 \text{ on } (0,1), \\ u(0) = u(1) = 0, \\ (x^\gamma u_x)(0) = 0. \end{cases} \tag{2.36}$$

The solution of the equation (2.36) is given by

$$u(x) = C_1\Phi_+(x) + C_2\Phi_-(x),$$

where Φ_+ and Φ_- are defined by

$$\Phi_+(x) = x^{\frac{1-\gamma}{2}} J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right), \quad \Phi_-(x) = x^{\frac{1-\gamma}{2}} J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right). \tag{2.37}$$

From boundary conditions $(2.36)_2$ and $(2.36)_3$, we deduce that

$$u \equiv 0.$$

Therefore $U = 0$. Consequently, $\mathcal{P}$ does not have purely imaginary eigenvalues.

Lemma 2.4.2
If $\lambda \neq 0$, the operator $i\lambda I - \mathcal{P}$ is surjective.
If $\lambda = 0$ and $\eta \neq 0$, the operator $i\lambda I - \mathcal{P}$ is surjective.

Proof.
Case 1: $\lambda \neq 0$**.** Let $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$ be given, and let $U = (u, v, \varphi, \theta)^T \in D(\mathcal{P})$ be such that

$$(i\lambda I - \mathcal{P})U = F. \tag{2.38}$$

Equivalently, we have

$$\begin{cases} i\lambda u - v = f_1, \\ i\lambda v - (x^\gamma u_x)_x = f_2, \\ i\lambda\varphi + (\xi^2 + \eta)\varphi - v(0)\nu(\xi) = f_3. \\ i\lambda\theta - \frac{1}{m}(x^\gamma u_x)(0) + \frac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty} \nu(\xi)\varphi(\xi)\, d\xi = f_4 \end{cases} \tag{2.39}$$

with boundary conditions. Then we deduce from these equations a weak formulation (similar computation as in Theorem 2.3.1):

$$\mathcal{B}(u, w) = l(w), \quad \forall w \in H^1_{0,\gamma}(0, 1), \tag{2.40}$$

where

$$\mathcal{B}(u, w) = \mathcal{B}_1(u, w) + \mathcal{B}_2(u, w)$$

with

$$\begin{cases} \mathcal{B}_1(u, w) = \displaystyle\int_0^1 x^\gamma u_x \overline{w}_x\, dx + i\lambda\varrho(i\lambda + \eta)^{\alpha-1} u(0)\overline{w}(0), \\ \mathcal{B}_2(u, w) = -\displaystyle\int_0^1 \lambda^2 u\overline{w}\, dx - m\lambda^2 u(0)\overline{w}(0), \end{cases} \tag{$*$}$$

and

$$l(w) = \int_0^1 (f_2 + i\lambda f_1)\overline{w}\, dx - \zeta \int_{-\infty}^{+\infty} \frac{\nu(\xi)}{\xi^2 + \eta + i\lambda} f_3(\xi)\, d\xi \overline{w}(0) + (mi\lambda + \varrho(i\lambda + \eta)^{\alpha-1}) f_1(0)\overline{w}(0) - m f_4 \overline{w}(0).$$

Let $(H^1_{0,\gamma}(0, 1))'$ be the dual space of $H^1_{0,\gamma}(0, 1)$. Let us define the following operators

$$\begin{array}{ll} B : H^1_{0,\gamma}(0, 1) \to (H^1_{0,\gamma}(0, 1))' & B_i : H^1_{0,\gamma}(0, 1) \to (H^1_{0,\gamma}(0, 1))' \quad i \in \{1, 2\} \\ \qquad u \mapsto Bu & \qquad u \mapsto B_i u \end{array} \tag{$**$}$$

such that

$$\begin{array}{l} (Bu)w = \mathcal{B}(u, w),\ \forall w \in H^1_{0,\gamma}(0, 1), \\ (B_i u)w = \mathcal{B}_i(u, w),\ \forall w \in H^1_{0,\gamma}(0, 1), i \in \{1, 2\}. \end{array} \tag{$***$}$$

We need to prove that the operator B is an isomorphism. For this aim, we divide the proof into three steps:
Step 1. In this step, we want to prove that the operator B_1 is an isomorphism. For this aim, it is easy to see that $\mathcal{B}_1$ is sesquilinear, continuous form on $H^1_{0,\gamma}(0, 1)$. Furthermore

$$\begin{array}{rcl} \Re\mathcal{B}_1(u, u) & = & \|x^{\gamma/2} u_x\|_2^2 + \varrho\lambda\Re\left(i(i\lambda + \eta)^{\alpha-1}\right)|u(0)|^2 \\ & \geq & \|x^{\gamma/2} u_x\|_2^2, \end{array}$$

where we have used the fact that

$$\varrho\lambda\Re\left(i(i\lambda+\eta)^{\alpha-1}\right)=\zeta\lambda^2\int_{-\infty}^{+\infty}\frac{\mu(\xi)^2}{\lambda^2+(\eta+\xi^2)^2}\,d\xi>0.$$

Thus $\mathcal{B}_1$ is coercive. Then, from $(**)$ and Lax-Milgram theorem, the operator B_1 is an isomorphism.

Step 2. In this step, we want to prove that the operator B_2 is compact. For this aim, from $(*)$ and $(***)$, we have

$$|\mathcal{B}_2(u,w)|\leq c\|u\|_{L^2(0,1)}\|w\|_{L^2(0,1)}+c'|u|_{H^1_{0,\gamma}(0,1)}|w|_{H^1_{0,\gamma}(0,1)},$$

and consequently, using the compact embedding from $H^1_{0,\gamma}(0,1)$ to $L^2(0,1)$ we deduce that B_2 is a compact operator. Therefore, from the above steps, we obtain that the operator $B=B_1+B_2$ is a Fredholm operator of index zero. Now, following Fredholm alternative, we still need to prove that the operator B is injective to obtain that the operator B is an isomorphism.

Step 3. Let $u\in ker(B)$, then

$$\mathcal{B}(u,w)=0\quad\forall w\in H^1_{0,\gamma}(0,1).\tag{2.41}$$

In particular for $w=u$, it follows that

$$\lambda^2\|u\|^2_{L^2(0,1)}+m\lambda^2|u(0)|^2-i\varrho\lambda(i\lambda+\eta)^{\alpha-1}|u(0)|^2=\|x^{\gamma/2}u_x\|^2_{L^2(0,1)}.$$

Hence, we have

$$u(0)=0.\tag{2.42}$$

From (2.41), we obtain

$$(x^{\gamma/2}u_x)(0)=0\tag{2.43}$$

and then

$$\begin{cases}-\lambda^2u-(x^\gamma u_x)_x=0,\\ u(0)=(x^{\gamma/2}u_x)(0)=0,\\ u(1)=0.\end{cases}\tag{2.44}$$

Then, according to Lemma 2.4.1, we deduce that $u=0$ and consequently $Ker(B)=\{0\}$. Finally, from Step 3 and Fredholm alternative, we deduce that the operator B is isomorphism. It is easy to see that the operator l is a antilinear and continuous form on $H^1_{0,\gamma}(0,1)$. Consequently, (2.40) admits a unique solution $u\in H^1_{0,\gamma}(0,1)$. By using the classical elliptic regularity, we deduce that $U\in D(\mathcal{P})$ is a unique solution of (2.38). Hence $i\lambda-\mathcal{P}$ is surjective for all $\lambda\in\mathbb{R}^*$.

Case 2: $\lambda=0$ **and** $\eta\neq 0$. Using Lax-Milgram Lemma, we obtain the result.

Taking account of Lemmas 2.4.1, 2.4.2 and from Theorem 2.4.1 The C_0-semigroup $e^{t\mathcal{P}}$ is strongly stable in $\mathcal{H}$.

$\square$

2.4.2 Lack of exponential stability

This section will be devoted to the study of the lack of exponential decay of solutions associated with the system (2.12). In order to state and prove our stability results, we need some lemmas.

Theorem 2.4.3 ([42]) *Let $S(t)$ be a C_0-semigroup of contractions on Hilbert space $\mathcal{X}$ with generator $\mathcal{P}$. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{P}) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R}$$

and

$$\overline{\lim_{|\beta|\to\infty}} \|(i\beta I - \mathcal{P})^{-1}\|_{\mathcal{L}(\mathcal{X})} < \infty.$$

Our main result is the following.

Theorem 2.4.4 *The semigroup generated by the operator $\mathcal{P}$ is not exponentially stable.*

Proof. We will examine two cases.
•**Case 1** $\eta = 0$: We shall show that $i\lambda = 0$ is not in the resolvent set of the operator $\mathcal{P}$. Indeed, noting that $F = (\sin(x-1), 0, 0, 0)^T \in \mathcal{H}$, and suppose that there exists $U = (u, v, \varphi, \theta)^T \in D(\mathcal{P})$ such that $-\mathcal{P}U = F$. We get $\varphi(\xi) = |\xi|^{\frac{2\alpha-5}{2}} \sin 1$. But, then $\varphi \notin L^2(-\infty, +\infty)$, since $\alpha \in]0, 1[$. So $(u, v, \varphi, \theta)^T \notin D(\mathcal{P})$ and the operator $\mathcal{P}$ is not invertible.
• **Case 2** $\eta \neq 0$:

We aim to show that an infinite number of eigenvalues of $\mathcal{P}$ approach the imaginary axis which prevents the system (P) from being exponentially stable. Indeed we first compute the characteristic equation that gives the eigenvalues of $\mathcal{P}$. Let λ be an eigenvalue of $\mathcal{P}$ with associated eigenvector $U = (u, v, \varphi, \theta)^T$. Then $\mathcal{P}U = \lambda U$ is equivalent to

$$\begin{cases} \lambda u - v = 0, \\ \lambda v - (x^\gamma u_x)_x = 0, \\ \lambda\varphi + (\xi^2 + \eta)\varphi - v(0)\nu(\xi) = 0, \\ \lambda\theta - \frac{1}{m}(x^\gamma u_x)(0) + \frac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty} \nu(\xi)\varphi(\xi)\, d\xi = 0. \end{cases} \tag{2.45}$$

It is well-known that Bessel functions play an important role in this type of problem. From $(2.45)_1 - (2.45)_2$ for such λ, we find

$$\lambda^2 u - (x^\gamma u_x)_x = 0. \tag{2.46}$$

Using the boundary conditions and $(2.45)_3$, we deduce that

$$\begin{cases} \lambda^2 u - (x^\gamma u_x)_x = 0, \\ (x^\gamma u_x)(0) - (m\lambda^2 + \varrho\lambda(\lambda + \eta)^{\alpha-1})u(0) = 0, \\ u(1) = 0. \end{cases} \tag{2.47}$$

Assume that u is a solution of (2.47) associated to eigenvalue $-\lambda^2$, then one easily checks that the function

$$u(x) = x^{\frac{1-\gamma}{2}} \Psi\left(\frac{2}{2-\gamma} i\lambda x^{\frac{2-\gamma}{2}}\right)$$

is a solution of the following problem:

$$y^2\Psi''(y) + y\Psi'(y) + (y^2 - (\frac{\gamma-1}{2-\gamma})^2)\Psi(y) = 0. \tag{2.48}$$

We have

$$u(x) = c_+\Phi_+ + c_-\Phi_-, \tag{2.49}$$

where Φ_+ and Φ_- are defined by

$$\Phi_+(x) := x^{\frac{1-\gamma}{2}} J_{\nu_\gamma}\left(\frac{2}{2-\gamma} i\lambda x^{\frac{2-\gamma}{2}}\right)$$

and

$$\Phi_-(x) := x^{\frac{1-\gamma}{2}} J_{-\nu_\gamma}\left(\frac{2}{2-\gamma} i\lambda x^{\frac{2-\gamma}{2}}\right),$$

where

$$J_\nu(y) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\nu+1)} \left(\frac{y}{2}\right)^{2m+\nu} = \sum_{m=0}^{\infty} c^+_{\nu,m} y^{2m+\nu}, \tag{2.50}$$

$$J_{-\nu}(y) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m-\nu+1)} \left(\frac{y}{2}\right)^{2m-\nu} = \sum_{m=0}^{\infty} c^-_{\nu,m} y^{2m-\nu} \tag{2.51}$$

$$\nu_\gamma = \frac{1-\gamma}{2-\gamma}$$

and J_{ν_γ} and $J_{-\nu_\gamma}$ are Bessel functions of the first kind of order ν_γ and $-\nu_\gamma$. As $\nu_\gamma \notin \mathbb{N}$, so J_{ν_γ} and $J_{-\nu_\gamma}$ are linearly independent and therefore the pair $(J_{\nu_\gamma}, J_{-\nu_\gamma})$ (classical result) forms a fundamental system of solutions (2.48).

Then, using the series expansion of J_{ν_α} and $J_{-\nu_\alpha}$, one obtains

$$\Phi_+(x) = \sum_{m=0}^{\infty} \tilde{c}^+_{\nu_\gamma,m} x^{1-\gamma+(2-\gamma)m}, \quad \Phi_-(x) = \sum_{m=0}^{\infty} \tilde{c}^-_{\nu_\gamma,m} x^{(2-\gamma)m},$$

with

$$\tilde{c}^+_{\nu_\gamma,m} = c^+_{\nu_\gamma,m} \left(\frac{2}{2-\gamma} i\lambda\right)^{2m+\nu_\gamma}, \quad \tilde{c}^-_{\nu_\gamma,m} = c^-_{\nu_\gamma,m} \left(\frac{2}{2-\gamma} i\lambda\right)^{2m-\nu_\gamma}.$$

Next one easily verifies that $\Phi_+, \Phi_- \in H^1_\gamma(0,1)$: indeed,

$$\Phi_+(x) \sim_0 \tilde{c}^+_{\nu_\gamma,0} x^{1-\gamma}, \quad x^{\gamma/2}\Phi'_+(x) \sim_0 (1-\gamma)\tilde{c}^+_{\nu_\gamma,0} x^{-\gamma/2},$$
$$\Phi_-(x) \sim_0 \tilde{c}^-_{\nu_\gamma,0}, \quad x^{\gamma/2}\Phi'_-(x) \sim_0 (2-\gamma)\tilde{c}^-_{\nu_\gamma,0} x^{1-\gamma/2},$$

where we have used the following relation

$$xJ'_\mu(x) = \mu J_\mu(x) - xJ_{\mu+1}(x). \tag{2.52}$$

Hence, given c_+ and c_-, $u(x) = c_+\Phi_+(x) + c_-\Phi_-(x) \in H^1_\gamma(0,1)$ with the following boundary conditions

$$\begin{cases} (x^\gamma u_x)(0) - (m\lambda^2 + \varrho\lambda(\lambda+\eta)^{\alpha-1})u(0) = 0, \\ u(1) = 0. \end{cases}$$

Then

$$M(\lambda)C(\lambda) = \begin{pmatrix} (1-\gamma)\tilde{c}^+_{\nu_\gamma,0} & -(m\lambda^2 + \varrho\lambda(\lambda+\eta)^{\alpha-1})\tilde{c}^-_{\nu_\gamma,0} \\ J_{\nu_\gamma}\left(\frac{2}{2-\gamma}i\lambda\right) & J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}i\lambda\right) \end{pmatrix} \begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.53}$$

Hence, a non-trivial solution u exists if and only if the determinant of $M(\lambda)$ vanishes. Set $f(\lambda) = \det M(\lambda)$. Thus the characteristic equation is $f(\lambda) = 0$.

Our purpose in the sequel is to prove, thanks to Rouché's Theorem, that there is a subsequence of eigenvalues for which their real part tends to 0.

Since $\mathcal{P}$ is dissipative, we study the asymptotic behavior of the large eigenvalues λ of $\mathcal{P}$ in the strip $-\alpha_0 \le \Re(\lambda) \le 0$, for some $\alpha_0 > 0$ large enough and for such λ, we remark that Φ_+, Φ_- remains bounded.

Lemma 2.4.3 *There exists $N \in \mathbb{N}$ sufficiently large and a sequence $(\lambda_k)_{k\in\mathbf{Z}^*,|k|\ge N}$ of simple roots of* $\det M$ *(that are also simple eigenvalues of $\mathcal{P}$) and satisfying the following asymptotic behavior:*

$$\begin{aligned} \lambda_k \quad = \quad & -\frac{2-\gamma}{2}i\left(k + \frac{\nu_\gamma}{2} + \frac{3}{4}\right)\pi - i\frac{1-\gamma}{m}\left(\frac{2}{2-\gamma}\right)\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\frac{\sin\nu_\gamma\pi}{(k\pi)^{2-2\nu_\gamma}} \\ & + i\frac{1-\gamma}{m}\left(\frac{2}{2-\gamma}\right)\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\frac{(2-2\nu_\gamma)(\frac{\nu_\gamma}{2}+\frac{3}{4})}{\pi^{2-2\nu_\gamma}k^{3-2\nu_\gamma}}\sin\nu_\gamma\pi \\ & - \left(\frac{1-\gamma}{m}\right)^2\left(\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\right)^2\frac{8}{(2-\gamma)^3}\frac{\sin\nu_\gamma\cos\nu_\gamma}{(\pi k)^{4-4\nu_\gamma}}i \\ & - i\left(\frac{2}{2-\gamma}\right)^{3-\alpha}\frac{\varrho(1-\gamma)}{m^2}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\frac{\sin\nu_\gamma\pi\sin(1-\alpha)\frac{\pi}{2}}{\pi^{4-\alpha-2\nu_\gamma}}\frac{1}{k^{4-\alpha-2\nu_\gamma}} \\ & - \left(\frac{2}{2-\gamma}\right)^{3-\alpha}\frac{\varrho(1-\gamma)}{m^2}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\frac{\sin\nu_\gamma\pi\cos(1-\alpha)\frac{\pi}{2}}{\pi^{4-\alpha-2\nu_\gamma}}\frac{1}{k^{4-\alpha-2\nu_\gamma}} + o\left(\frac{1}{k^\omega}\right) \end{aligned} \tag{2.54}$$

$$\lambda_k = \overline{\lambda_{-k}} \text{ if } k \le -N,$$

where $\omega = \max\{4-\alpha-2\nu_\gamma, 4-4\nu_\gamma\}$. Moreover for all $|k| \ge N$, the eigenvalues λ_k are simple.

Proof. We look at the roots of $f(\lambda)$. From (2.53), we have

$$f(\lambda) = (1-\gamma)\tilde{c}^+_{\nu_\gamma,0}J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}i\lambda\right) + (m\lambda^2 + \varrho\lambda(\lambda+\eta)^{\alpha-1}))\tilde{c}^-_{\nu_\gamma,0}J_{\nu_\gamma}\left(\frac{2}{2-\gamma}i\lambda\right) = 0.$$

We will use the following classical asymptotic development (see **[32]** p. 122, (5.11.6)): for all $\delta > 0$, the following development holds when $|argz| \leq \pi - \delta$:

$$J_\mu(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos\left(z - \mu\frac{\pi}{2} - \frac{\pi}{4}\right)\left(1 + O(\frac{1}{|z|^2})\right) - \left(\frac{2}{\pi z}\right)^{1/2} \sin\left(z - \mu\frac{\pi}{2} - \frac{\pi}{4}\right) O\left(\frac{1}{|z|^2}\right). \tag{2.55}$$

We divide the proof into five steps:
Step 1. First, using the asymptotic expansion, we get

$$\frac{1}{(\lambda+\eta)^{1-\alpha}} = \frac{1}{\lambda^{1-\alpha}}(1 + O(\lambda^{-1})) \tag{2.56}$$

Next, using (2.55) and (2.56), we get

$$f(\lambda) = m\left(\frac{2}{\pi\tilde{z}}\right)^{1/2} \lambda^{2-\nu_\gamma} c^-_{\nu_\gamma,0}\left(\frac{2}{2-\gamma}i\right)^{-\nu_\gamma} \frac{e^{-i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}}{2}\tilde{f}(\lambda), \tag{2.57}$$

where

$$\tilde{z} = \frac{2}{2-\gamma}i\lambda$$

and

$$\begin{aligned}\tilde{f}(\lambda) &= (e^{2i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})} + 1) + \frac{1-\gamma}{m}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\left(\frac{2}{2-\gamma}i\right)^{2\nu_\gamma}\frac{e^{2i(\tilde{z}-\frac{\pi}{4})} + e^{-i\nu_\gamma\pi}}{\lambda^{2-2\nu_\gamma}} \\ &\qquad + \frac{\varrho}{m}\frac{e^{2i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}+1}{\lambda^{2-\alpha}} + O\left(\frac{1}{\lambda^2}\right) \\ &= f_0(\lambda) + \frac{f_1(\lambda)}{\lambda^{2-2\nu_\gamma}} + \frac{f_2(\lambda)}{\lambda^{2-\alpha}} + O\left(\frac{1}{\lambda^2}\right),\end{aligned} \tag{2.58}$$

where

$$f_0(\lambda) = e^{2i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})} + 1, \tag{2.59}$$

$$f_1(\lambda) = \frac{1-\gamma}{m}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\left(\frac{2}{2-\gamma}i\right)^{2\nu_\gamma}(e^{2i(\tilde{z}-\frac{\pi}{4})} + e^{-i\nu_\gamma\pi}), \tag{2.60}$$

$$f_2(\lambda) = \frac{\varrho}{m}(e^{2i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})} + 1). \tag{2.61}$$

Note that f_0, f_1 and f_2 remain bounded in the strip $-\alpha_0 \leq \Re(\lambda) \leq 0$.
Step 2. We look at the roots of f_0. From (2.59), f_0 has one family of roots that we denote λ^0_k.

$$f_0(\lambda) = 0 \Leftrightarrow e^{2i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})} + 1 = 0$$

Hence

$$2i\left(\frac{2}{2-\gamma}i\lambda - \nu_\gamma\frac{\pi}{2} - \frac{\pi}{4}\right) = i(2k+1)\pi, \quad k \in \mathbf{Z},$$

i.e.,

$$\lambda^0_k = -\frac{2-\gamma}{2}i\left(k + \frac{\nu_\gamma}{2} + \frac{3}{4}\right)\pi, \quad k \in \mathbf{Z}.$$

Now with the help of Rouché's Theorem and the asymptotic Equation (2.58), we will show that the roots of $\tilde{f}$ are close to those of f_0. Let us start with the first family. Changing in (2.58) the unknown λ by $u = 2iz$ then (2.58) becomes

$$\tilde{f}(u) = (e^u + 1) + O\left(\frac{1}{u^{\varpi}}\right) = f_0(u) + O\left(\frac{1}{u^{\varpi}}\right),$$

where $\varpi = \max\{2 - 2\nu_\gamma, 2 - \alpha\}$. The roots of f_0 are $u_k = -\frac{2-\gamma}{2} i \left(k + \frac{\nu_\gamma}{2} + \frac{3}{4}\right)\pi, k \in \mathbf{Z}$, and setting $u = u_k + re^{it}, t \in [0, 2\pi]$, we can easily check that there exists a constant $C > 0$ independent of k such that $|e^u + 1| \geq Cr$ for r small enough. This allows to apply Rouché's Theorem. Consequently, there exists a subsequence of roots of $\tilde{f}$ which tends to the roots u_k of f_0. Equivalently, it means that there exists $N \in \mathbb{N}$ and a subsequence $\{\lambda_k\}_{|k|\geq N}$ of roots of $f(\lambda)$, such that $\lambda_k = \lambda_k^0 + o(1)$ which tends to the roots $-\frac{2-\gamma}{2} i \left(k + \frac{\nu_\gamma}{2} + \frac{3}{4}\right)\pi$ of f_0. Finally for $|k| \geq N, \lambda_k$ is simple since λ_k^0 is.

Step 3. From Step 2, we can write

$$\lambda_k = -\frac{2-\gamma}{2} i \left(k + \frac{\nu_\gamma}{2} + \frac{3}{4}\right)\pi + \varepsilon_k. \tag{2.62}$$

Using (2.62), we get

$$\begin{aligned} e^{2i((\frac{2}{2-\gamma} i\lambda_k) - \nu_\gamma \frac{\pi}{2} - \frac{\pi}{4})} &= -e^{-\frac{4}{2-\gamma}\varepsilon_k} \\ &= -1 + \tfrac{4}{2-\gamma}\varepsilon_k + O(\varepsilon_k^2). \end{aligned} \tag{2.63}$$

Substituting (2.63) into (2.58), using that $\tilde{f}(\lambda_k) = 0$, we get:

$$\tilde{f}(\lambda_k) = \frac{4}{2-\gamma}\varepsilon_k + \frac{1-\gamma}{m} \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}} \left(\frac{2}{2-\gamma}\right)^2 \frac{2i \sin \nu_\gamma \pi}{(k\pi)^{2-2\nu_\gamma}} + o(\varepsilon_k) + o\left(\frac{1}{k^{2-2\nu_\gamma}}\right) = 0 \tag{2.64}$$

and hence

$$\varepsilon_k = -\frac{1-\gamma}{m} \left(\frac{2}{2-\gamma}\right) \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}} i \frac{\sin \nu_\gamma \pi}{(k\pi)^{2-2\nu_\gamma}}.$$

Step 4. From Step 3, we can write

$$\lambda_k = -\frac{2-\gamma}{2} i \left(k + \frac{\nu_\gamma}{2} + \frac{3}{4}\right)\pi - i\frac{1-\gamma}{m} \left(\frac{2}{2-\gamma}\right) \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}} \frac{\sin \nu_\gamma \pi}{(k\pi)^{2-2\nu_\gamma}} + \varepsilon_k. \tag{2.65}$$

Using (2.62), we get

$$\begin{aligned} e^{2i((\frac{2}{2-\gamma} i\lambda_k) - \nu_\gamma \frac{\pi}{2} - \frac{\pi}{4})} &= -e^{-\frac{4}{2-\gamma}\varepsilon_k + \frac{4c}{2-\gamma}\frac{1}{k^{2-2\nu_\gamma}}} \\ &= -1 + \tfrac{4}{2-\gamma}\varepsilon_k - \tfrac{4c}{2-\gamma}\tfrac{1}{k^{2-2\nu_\gamma}} + O(\varepsilon_k^2), \end{aligned} \tag{2.66}$$

where

$$c = \frac{1-\gamma}{m} \left(\frac{2}{2-\gamma}\right) \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}} \frac{\sin \nu_\gamma \pi}{\pi^{2-2\nu_\gamma}} i.$$

Substituting (2.66) into (2.58), using that $\tilde{f}(\lambda_k)=0$, we get:

$$
\begin{aligned}
\tilde{f}(\lambda_k) = \tfrac{4}{2-\gamma}\varepsilon_k - i\frac{1-\gamma}{m}\left(\frac{8}{(2-\gamma)^2}\right)\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\frac{(2-2\nu_\gamma)(\frac{\nu_\gamma}{2}+\frac{3}{4})}{\pi^{2-2\nu_\gamma}k^{3-2\nu_\gamma}}\sin\nu_\gamma\pi \\
+o(\varepsilon_k)+o\left(\tfrac{1}{k^{3-2\nu_\gamma}}\right)=0
\end{aligned}
\tag{2.67}
$$

and hence

$$
\varepsilon_k = i\frac{1-\gamma}{m}\left(\frac{2}{2-\gamma}\right)\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\frac{(2-2\nu_\gamma)(\frac{\nu_\gamma}{2}+\frac{3}{4})}{\pi^{2-2\nu_\gamma}k^{3-2\nu_\gamma}}\sin\nu_\gamma\pi.
$$

Step 5. From Step 4, we can write

$$
\begin{aligned}
\lambda_k &= -\frac{2-\gamma}{2}i\left(k+\frac{\nu_\gamma}{2}+\frac{3}{4}\right)\pi - i\frac{1-\gamma}{m}\left(\frac{2}{2-\gamma}\right)\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\frac{\sin\nu_\gamma\pi}{(k\pi)^{2-2\nu_\gamma}} \\
&\quad +i\frac{1-\gamma}{m}\left(\frac{2}{2-\gamma}\right)\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\frac{(2-2\nu_\gamma)(\frac{\nu_\gamma}{2}+\frac{3}{4})}{\pi^{2-2\nu_\gamma}k^{3-2\nu_\gamma}}\sin\nu_\gamma\pi+\varepsilon_k.
\end{aligned}
\tag{2.68}
$$

Using (2.62), we get

$$
\begin{aligned}
e^{2i((\frac{2}{2-\gamma}i\lambda_k)-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})} &= -e^{-\frac{4}{2-\gamma}\varepsilon_k+\frac{4c}{2-\gamma}\frac{1}{k^{2-2\nu_\gamma}}-\frac{4\tilde{c}}{2-\gamma}\frac{1}{k^{3-2\nu_\gamma}}} \\
&= -1+\tfrac{4}{2-\gamma}\varepsilon_k-\tfrac{4c}{2-\gamma}\tfrac{1}{k^{2-2\nu_\gamma}}+\tfrac{4\tilde{c}}{2-\gamma}\tfrac{1}{k^{3-2\nu_\gamma}}-\tfrac{1}{2}(\tfrac{4c}{2-\gamma})^2\tfrac{1}{k^{4-4\nu_\gamma}}+O(\varepsilon_k^2),
\end{aligned}
\tag{2.69}
$$

where

$$
\begin{aligned}
c &= \frac{1-\gamma}{m}\left(\frac{2}{2-\gamma}\right)\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\frac{\sin\nu_\gamma\pi}{\pi^{2-2\nu_\gamma}}i, \\
\tilde{c} &= i\frac{1-\gamma}{m}\left(\frac{2}{2-\gamma}\right)\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\frac{(2-2\nu_\gamma)(\frac{\nu_\gamma}{2}+\frac{3}{4})}{\pi^{2-2\nu_\gamma}k^{3-2\nu_\gamma}}\sin\nu_\gamma\pi.
\end{aligned}
$$

Substituting (2.69) into (2.58), using that $\tilde{f}(\lambda_k)=0$, we get:

$$
\begin{aligned}
\tilde{f}(\lambda_k) &= \tfrac{4}{2-\gamma}\varepsilon_k-\tfrac{4c}{2-\gamma}\tfrac{1}{k^{2-2\nu_\gamma}}+\tfrac{4\tilde{c}}{2-\gamma}\tfrac{1}{k^{3-2\nu_\gamma}} \\
&\quad -\tfrac{1}{2}(\tfrac{4c}{2-\gamma})^2\tfrac{1}{k^{4-4\nu_\gamma}}-2i\frac{\tilde{\tilde{c}}}{\delta^{2-2\nu_\gamma}}\frac{\sin\nu_\gamma\pi}{k^{2-2\nu_\gamma}} \\
&\quad +2i(2-2\nu_\gamma)(\tfrac{\nu_\gamma}{2}+\tfrac{3}{4})\frac{\tilde{\tilde{c}}}{\delta^{2-2\nu_\gamma}}\frac{\sin\nu_\gamma\pi}{k^{3-2\nu_\gamma}}-\frac{\tilde{\tilde{c}}c}{\delta^{2-2\nu_\gamma}}\frac{4}{2-\gamma}\frac{e^{i\nu_\gamma\pi}}{k^{4-4\nu_\gamma}}-\frac{\varrho}{m}\frac{4}{2-\gamma}\frac{c}{\delta^{2-\alpha}}\frac{1}{k^{4-\alpha-2\nu_\gamma}} \\
&\quad +o(\varepsilon_k)+o\left(\tfrac{1}{k^{\omega}}\right) \\
&= \tfrac{4}{2-\gamma}\varepsilon_k+\left(\frac{1-\gamma}{m}\right)^2\left(\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\right)^2\frac{32}{(2-\gamma)^4}\frac{\sin\nu_\gamma\cos\nu_\gamma}{(\pi k)^{4-4\nu_\gamma}}i-\frac{\varrho}{m}\frac{4}{2-\gamma}\frac{c}{\delta^{2-\alpha}}\frac{1}{k^{4-\alpha-2\nu_\gamma}} \\
&\quad +o(\varepsilon_k)+o\left(\tfrac{1}{k^{\omega}}\right)=0,
\end{aligned}
\tag{2.70}
$$

where $\omega=\max\{4-\alpha-2\nu_\gamma,4-4\nu_\gamma\}$ and

$$
\delta=-\frac{2-\gamma}{2}i\pi,\quad \tilde{\tilde{c}}=\frac{1-\gamma}{m}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\left(\frac{2}{2-\gamma}i\right)^{2\nu_\gamma}
$$

and hence

$$\begin{aligned}\varepsilon_k &= -\left(\frac{1-\gamma}{m}\right)^2\left(\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\right)^2 \frac{8}{(2-\gamma)^3}\frac{\sin\nu_\gamma\cos\nu_\gamma}{(\pi k)^{4-4\nu_\gamma}}i\\ &\quad -i\left(\frac{2}{2-\gamma}\right)^{3-\alpha}\frac{\varrho(1-\gamma)}{m^2}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\frac{\sin\nu_\gamma\pi\sin(1-\alpha)\frac{\pi}{2}}{\pi^{4-\alpha-2\nu_\gamma}}\frac{1}{k^{4-\alpha-2\nu_\gamma}}\\ &\quad -\left(\frac{2}{2-\gamma}\right)^{3-\alpha}\frac{\varrho(1-\gamma)}{m^2}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\frac{\sin\nu_\gamma\pi\cos(1-\alpha)\frac{\pi}{2}}{\pi^{4-\alpha-2\nu_\gamma}}\frac{1}{k^{4-\alpha-2\nu_\gamma}}+o\left(\frac{1}{k^\omega}\right).\end{aligned}$$

As (2.54) shows that the eigenvalues λ_k of $\mathcal{P}$ approach the imaginary axis as k goes to infinity, clearly system (2.12) is not uniformly stable. From (2.54), we have

$$|k|^{4-\alpha-2\nu_\gamma}\Re\lambda_k \approx -\left(\frac{2}{2-\gamma}\right)^{3-\alpha}\frac{\varrho(1-\gamma)}{m^2}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\frac{\sin\nu_\gamma\pi\cos(1-\alpha)\frac{\pi}{2}}{\pi^{4-\alpha-2\nu_\gamma}}.$$

The operator $\mathcal{P}$ has a non exponential decaying branche of eigenvalues. Thus the proof is complete.

2.5 Polynomial Stability (for $\eta \neq 0$)

To state and prove our stability results, we need some results from semigroup theory.

Theorem 2.5.1 ([10]) *Let $S(t)$ be a bounded C_0-semigroup on a Hilbert space $\mathcal{X}$ with generator $\mathcal{P}$. If*

$$i\mathbb{R} \subset \rho(\mathcal{P}) \text{ and } \overline{\lim_{|\beta|\to\infty}}\frac{1}{\beta^l}\|(i\beta I-\mathcal{P})^{-1}\|_{\mathcal{L}(\mathcal{X})} < \infty$$

for some l, then there exist c such that

$$\|e^{\mathcal{P}t}U_0\|^2 \leq \frac{c}{t^{\frac{2}{l}}}\|U_0\|^2_{D(\mathcal{A})}.$$

Our main result is the following.

Theorem 2.5.2 *The semigroup $S_{\mathcal{P}}(t)_{t\geq 0}$ is polynomially stable and*

$$E(t) = \|S_{\mathcal{P}}(t)U_0\|^2_{\mathcal{H}} \leq \frac{1}{t^{\frac{2}{(4-\alpha-2\nu_\gamma)}}}\|U_0\|^2_{D(\mathcal{P})}.$$

Proof. We will need to study the resolvent equation $(i\lambda-\mathcal{P})U = F$, for $\lambda \in \mathbb{R}$, namely

$$\begin{cases} i\lambda u - v = f_1, \\ i\lambda v - (x^\gamma u_x)_x = f_2, \\ i\lambda\varphi + (\xi^2+\eta)\varphi - v(0)\nu(\xi) = f_3. \\ i\lambda\theta - \frac{1}{m}(x^\gamma u_x)(0) + \frac{\zeta}{m}\int_{-\infty}^{+\infty}\nu(\xi)\varphi(\xi)\,d\xi = f_4. \end{cases} \tag{2.71}$$

where $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$. From $(2.71)_1$ and $(2.71)_2$, we have

$$\lambda^2 u + (x^\gamma u_x)_x = -(f_2 + i\lambda f_1) \tag{2.72}$$

with

$$\begin{cases} \lambda^2 u - (x^\gamma u_x)_x = 0, \\ -(x^\gamma u_x)(0) + (-m\lambda^2 + i\varrho\lambda(i\lambda+\eta)^{\alpha-1})u(0) \\ \qquad\qquad = mf_4 - \zeta \displaystyle\int_{-\infty}^{+\infty} \frac{\nu(\xi) f_3(\xi)}{i\lambda + \eta + \xi^2}\, d\xi + (mi\lambda + \varrho(i\lambda+\eta)^{\alpha-1}) f_1(0), \\ u(1) = 0. \end{cases} \tag{2.73}$$

Assume that Φ is a solution of (2.72), then one easily checks that the function Ψ defined by

$$\Phi(x) = x^{\frac{1-\gamma}{2}} \Psi\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right) \tag{2.74}$$

is solution of the following inhomogeneous Bessel equation:

$$\begin{aligned} &y^2\Psi''(y) + y\Psi'(y) + \left(y^2 - \left(\frac{\gamma-1}{2-\gamma}\right)^2\right)\Psi(y) = \\ &-(\tfrac{2}{2-\gamma})^2(\tfrac{2-\gamma}{2}\tfrac{1}{\lambda}y)^{\frac{3-\gamma}{2-\gamma}}\left(f_2\left((\tfrac{2-\gamma}{2}\tfrac{1}{\lambda}y)^{\frac{2}{2-\gamma}}\right) + i\lambda f_1\left((\tfrac{2-\gamma}{2}\tfrac{1}{\lambda}y)^{\frac{2}{2-\gamma}}\right)\right). \end{aligned} \tag{2.75}$$

The general solution of (2.75) is easily seen to be

$$\Psi(y) = AJ_{\nu_\gamma}(y) + BJ_{-\nu_\gamma}(y) - \frac{\pi}{2\sin\nu_\gamma\pi}\int_0^y \frac{f(s)}{s}\left(J_{\nu_\gamma}(s)J_{-\nu_\gamma}(y) - J_{\nu_\gamma}(y)J_{-\nu_\gamma}(s)\right) ds,$$

where A and B are constants free to be determined later and

$$f(s) = -(\frac{2}{2-\gamma})^2(\frac{2-\gamma}{2}\frac{1}{\lambda}s)^{\frac{3-\gamma}{2-\gamma}}\left(f_2\left((\frac{2-\gamma}{2}\frac{1}{\lambda}s)^{\frac{2}{2-\gamma}}\right) + i\lambda f_1\left((\frac{2-\gamma}{2}\frac{1}{\lambda}s)^{\frac{2}{2-\gamma}}\right)\right).$$

Thus,

$$\begin{aligned} &u(x) = Ax^{\frac{1-\gamma}{2}}J_{\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right) + Bx^{\frac{1-\gamma}{2}}J_{-\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right) \\ &+\frac{\pi}{2\sin\nu_\gamma\pi}\left(\frac{2}{2-\gamma}\right)x^{\frac{1-\gamma}{2}}\int_0^x s^{\frac{1-\gamma}{2}}(f_2(s) + i\lambda f_1(s))\left(J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda s^{\frac{2-\gamma}{2}}\right)J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right)\right. \\ &\left.-J_{\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right)J_{-\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda s^{\frac{2-\gamma}{2}}\right)\right) ds. \end{aligned}$$

Therefore,

$$\begin{aligned} u(x) &= A\Phi_+(x) + B\Phi_-(x) \\ &\quad+\frac{\pi}{2\sin\nu_\gamma\pi}\left(\frac{2}{2-\gamma}\right)\int_0^x (f_2(s) + i\lambda f_1(s))(\Phi_+(s)\Phi_-(x) - \Phi_+(x)\Phi_-(s))\, ds, \end{aligned} \tag{2.76}$$

where Φ_+ and Φ_- are defined by

$$\Phi_+(x) = x^{\frac{1-\gamma}{2}}J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right), \quad \Phi_-(x) = x^{\frac{1-\gamma}{2}}J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right). \tag{2.77}$$

We thus have

$$
\begin{aligned}
u_x(x) &= A\Phi'_+(x) + B\Phi'_-(x) \\
&\quad + \frac{\pi}{2\sin\nu_\gamma\pi}\left(\frac{2}{2-\gamma}\right)\int_0^x (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi'_-(x) - \Phi'_+(x)\Phi_-(s))\,ds.
\end{aligned} \tag{2.78}
$$

It remains to determine the constants A and B. Using $(2.73)_2$, (2.78) and (2.76), we conclude that

$$
\begin{aligned}
&(1-\gamma)\tilde{c}^+_{\nu_\gamma,0}A - (-m\lambda^2 + \varrho i\lambda(i\lambda+\eta)^{\alpha-1})\tilde{c}^-_{\nu_\gamma,0}B \\
&\quad = -mf_4 + \zeta\int_{-\infty}^{+\infty}\frac{\nu(\xi)f_3(\xi)}{i\lambda+\eta+\xi^2}\,d\xi - (mi\lambda + \varrho(i\lambda+\eta)^{\alpha-1})f_1(0),
\end{aligned} \tag{2.79}
$$

$$
A\Phi_+(1) + B\Phi_-(1) = -\frac{\pi}{2\sin\nu_\gamma\pi}\left(\frac{2}{2-\gamma}\right)\int_0^1 (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi_-(1) - \Phi_+(1)\Phi_-(s))\,ds, \tag{2.80}
$$

where

$$
\tilde{c}^+_{\nu_\gamma,m} = c^+_{\nu_\gamma,m}\left(\frac{2}{2-\gamma}\lambda\right)^{2m+\nu_\gamma}, \quad \tilde{c}^-_{\nu_\gamma,m} = c^-_{\nu_\gamma,m}\left(\frac{2}{2-\gamma}\lambda\right)^{2m-\nu_\gamma}
$$

and

$$
\Phi_+(1) = J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda\right), \qquad \Phi_-(1) = J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda\right).
$$

We write equations (2.79) and (2.80) in matrix form as

$$
\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} C \\ \tilde{C} \end{pmatrix}, \tag{2.81}
$$

where

$$
\begin{aligned}
r_{11} &= (1-\gamma)\tilde{c}^+_{\nu_\gamma,0}, \\
r_{12} &= (m\lambda^2 - \varrho i\lambda(i\lambda+\eta)^{\alpha-1})\tilde{c}^-_{\nu_\gamma,0}, \\
r_{21} &= J_{\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda\right), \\
r_{22} &= J_{-\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda\right),
\end{aligned}
$$

$$
\begin{aligned}
C &= -mf_4 + \zeta\int_{-\infty}^{+\infty}\frac{\nu(\xi)f_3(\xi)}{i\lambda+\eta+\xi^2}\,d\xi - (mi\lambda + \varrho(i\lambda+\eta)^{\alpha-1})f_1(0), \\
\tilde{C} &= -\frac{\pi}{2\sin\nu_\gamma\pi}\left(\frac{2}{2-\gamma}\right)\int_0^1 (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi_-(1) - \Phi_+(1)\Phi_-(s))\,ds.
\end{aligned}
$$

Let the determinant of the linear system given in (2.81) be denoted by D. Then Note that

$$
\begin{aligned}
D &= (1-\gamma)\tilde{c}^+_{\nu_\gamma,0}J_{-\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda\right) - (m\lambda^2 - \varrho i\lambda(i\lambda+\eta)^{\alpha-1})\tilde{c}^-_{\nu_\gamma,0}J_{\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda\right) \\
&= (1-\gamma)c^+_{\nu_\gamma,0}\left(\tfrac{2}{2-\gamma}\right)^{\nu_\gamma}\lambda^{\nu_\gamma}\left[\left(\frac{2-\gamma}{\pi\lambda}\right)^{1/2}\cos\left(\tfrac{2}{2-\gamma}\lambda + \nu_\gamma\tfrac{\pi}{2} - \tfrac{\pi}{4}\right) + O\left(\frac{1}{\lambda^{5/2}}\right)\right] \\
&\quad -(m\lambda^2 - \varrho i\lambda(i\lambda+\eta)^{\alpha-1})c^-_{\nu_\gamma,0}\left(\tfrac{2}{2-\gamma}\right)^{-\nu_\gamma}\lambda^{-\nu_\gamma}\left[\left(\frac{2-\gamma}{\pi\lambda}\right)^{1/2}\cos\left(\tfrac{2}{2-\gamma}\lambda - \nu_\gamma\tfrac{\pi}{2} - \tfrac{\pi}{4}\right) + O\left(\frac{1}{\lambda^{5/2}}\right)\right] \\
&= -mc^-_{\nu_\gamma,0}\left(\tfrac{2}{2-\gamma}\right)^{-\nu_\gamma}\left(\frac{2-\gamma}{\pi}\right)^{1/2}\lambda^{2-\nu_\gamma-\frac{1}{2}}\cos\left(\tfrac{2}{2-\gamma}\lambda - \nu_\gamma\tfrac{\pi}{2} - \tfrac{\pi}{4}\right) \\
&\quad +(1-\gamma)c^+_{\nu_\gamma,0}\left(\tfrac{2}{2-\gamma}\right)^{\nu_\gamma}\left(\frac{2-\gamma}{\pi}\right)^{1/2}\lambda^{\nu_\gamma-\frac{1}{2}}\cos\left(\tfrac{2}{2-\gamma}\lambda + \nu_\gamma\tfrac{\pi}{2} - \tfrac{\pi}{4}\right) \\
&\quad +\varrho i^\alpha c^-_{\nu_\gamma,0}\left(\tfrac{2}{2-\gamma}\right)^{-\nu_\gamma}\left(\frac{2-\gamma}{\pi}\right)^{1/2}\lambda^{\alpha-\nu_\gamma-\frac{1}{2}}\cos\left(\tfrac{2}{2-\gamma}\lambda - \nu_\gamma\tfrac{\pi}{2} - \tfrac{\pi}{4}\right) + O\left(\frac{1}{\lambda^{3/2+\nu_\gamma-\alpha}}\right).
\end{aligned}
$$

As $D \neq 0$ for all $\lambda \neq 0$, then A and B are uniquely determined by (2.81).

Now, it is easy to prove that

$$|D| \geq c|\lambda|^{-5/2+\nu_\gamma+\alpha} \text{ for large } \lambda. \tag{2.82}$$

In the following lemma we will prove some technical inequalities which will be useful for showing the optimal polynomial decay of the solution.

Lemma 2.5.1
(I) *for all* $\lambda \in \mathbb{R} - \{0\}$ *large, we have*

$$\|\Phi_+\|_{L^2(0,1)}, \|\Phi_-\|_{L^2(0,1)} \leq \frac{c}{\sqrt{|\lambda|}}. \tag{2.83}$$

(II)

$$\left\| x^{-\frac{1}{2}} J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right)\right\|_{L^2(0,1)}, \left\| x^{-\frac{1}{2}} J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right)\right\|_{L^2(0,1)} \leq c\sqrt{|\lambda|}. \tag{2.84}$$

(III) *There exists a constant* $C > 0$ *such that, for all* $f_1 \in H^1_{0,\gamma}(0,1), f_2 \in L^2(0,1)$ *and* $\lambda \in \mathbb{R} - \{0\}$,

$$\left|\int_0^1 (f_2(s) + i\lambda f_1(s))(\Phi_+(s)\Phi_-(1) - \Phi_+(1)\Phi_-(s))\, ds\right| \leq \frac{1}{|\lambda|}\left(\|f_1\|_{H^1_{0,\gamma}(0,1)} + \|f_2\|_{L^2(0,1)}\right). \tag{2.85}$$

Proof. Suppose that $\lambda \neq 0$. It is enough to consider $\lambda > 0$. We will use the following results (see [**23**]).

Lemma 2.5.2 *If* $a \neq b$ *are complex numbers and* $\Re\vartheta > -1$, *we have*

$$\begin{aligned} (a^2 - b^2)\int_0^x tJ_\vartheta(at)J_\vartheta(bt)\, dt &= x\left(J_\vartheta(ax)\frac{d}{dx}(J_\vartheta(bx)) - J_\vartheta(bx)\frac{d}{dx}(J_\vartheta(ax))\right), \\ 2a^2\int_0^x t(J_\vartheta(at))^2\, dt &= (a^2x^2 - \vartheta^2)(J_\vartheta(ax))^2 + \left(x\frac{d}{dx}(J_\vartheta(ax))\right)^2, \end{aligned} \tag{2.86}$$

$$\begin{aligned} \frac{d}{dx}(x^\vartheta J_\vartheta(x)) &= x^\vartheta J_{\vartheta-1}(x), \\ \frac{d}{dx}(x^{-\vartheta} J_\vartheta(x)) &= -x^{-\vartheta} J_{\vartheta+1}(x). \end{aligned} \tag{2.87}$$

(I)

$$\|\Phi_+\|^2_{L^2(0,1)} = \int_0^1 x^{1-\gamma}\left(J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right)\right)^2 dx. \tag{2.88}$$

Let $z = \frac{2}{2-\gamma}\lambda s^{\frac{2-\gamma}{2}}$ in equation (2.88), we get

$$\begin{aligned} \|\Phi_+\|^2_{L^2(0,1)} &= \frac{2-\gamma}{2\lambda^2}\int_0^{\frac{2}{2-\gamma}\lambda} z\left(J_{\nu_\gamma}(z)\right)^2 dz \\ &= \frac{2-\gamma}{4\lambda^2}\left[\left(\left(\frac{2\lambda}{2-\gamma}\right)^2 - \nu_\gamma^2\right)\left(J_{\nu_\gamma}\left(\frac{2\lambda}{2-\gamma}\right)\right)^2 + \left(\frac{2\lambda}{2-\gamma}J'_{\nu_\gamma}\left(\frac{2\lambda}{2-\gamma}\right)\right)^2\right]. \end{aligned}$$

Using (2.86), (2.52) and (2.55), we deduce

$$\|\Phi_+\|_{L^2(0,1)} \leq \frac{c}{\sqrt{\lambda}}.$$

Similarly, we prove that

$$\|\Phi_-\|_{L^2(0,1)} \leq \frac{c}{\sqrt{\lambda}}.$$

(II)

$$\begin{aligned}\left\| x^{-\frac{1}{2}} J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right)\right\|^2_{L^2(0,1)} &= \int_0^1 x^{-1}\left(J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right)\right)^2 dx \\ &= \frac{2}{2-\gamma}\int_0^{\frac{2}{2-\gamma}\lambda} z^{-1}\left(J_{\nu_\gamma}(z)\right)^2 dz\end{aligned}$$

Now, using (2.52), we have

$$\begin{aligned} I &= \int_0^{\frac{2}{2-\gamma}\lambda} z^{-1}\left(J_{\nu_\gamma}(z)\right)^2 dz = \frac{1}{\nu_\gamma}\int_0^{\frac{2}{2-\gamma}\lambda} J_{\nu_\gamma}(z)(J'_{\nu_\gamma}(z) - J_{\nu_\gamma+1}(z))\, dz \\ &= \frac{1}{2\nu_\gamma}\left(J_{\nu_\gamma}\left(\frac{2\lambda}{2-\gamma}\right)\right)^2 - \frac{1}{\nu_\gamma}\int_0^{\frac{2}{2-\gamma}\lambda} J_{\nu_\gamma}(z)J_{\nu_\gamma+1}(z)\, dz \\ &\leq \frac{1}{2\nu_\gamma}\left(J_{\nu_\gamma}\left(\frac{2\lambda}{2-\gamma}\right)\right)^2 + \frac{\varepsilon}{2}I + \frac{1}{2\varepsilon\nu_\gamma^2}\int_0^{\frac{2}{2-\gamma}\lambda} z(J_{\nu_\gamma+1}(z))^2\, dz.\end{aligned}$$

for every $\varepsilon > 0$. Choosing ε small enough and using (2.86) and (2.55), we obtain

$$\begin{aligned} I &\leq c(J_{\nu_\gamma}(\tfrac{2\lambda}{2-\gamma}))^2 + c'\int_0^{\frac{2}{2-\gamma}\lambda} z(J_{\nu_\gamma+1}(z))^2\, dz. \\ &\leq c\lambda.\end{aligned}$$

Hence

$$\left\| x^{-\frac{1}{2}} J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right)\right\|_{L^2(0,1)} \leq c\sqrt{\lambda}.$$

Similarly, we prove that

$$\left\| x^{-\frac{1}{2}} J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right)\right\|_{L^2(0,1)} \leq c\sqrt{\lambda}.$$

(III) Let $f_1 \in H^1_{0,\gamma}(0,1)$ and $f_2 \in L^2(0,1)$. First we estimate $I = \displaystyle\int_0^1 f_1(s)\Phi_+(s)\,ds$. We have

$$
\begin{aligned}
I &= \frac{1}{\lambda}\left(\frac{2-\gamma}{2\lambda}\right)^{\frac{1}{2-\gamma}} \int_0^{\frac{2}{2-\gamma}\lambda} f_1\left(\left(\frac{2-\gamma}{2\lambda}z\right)^{\frac{2}{2-\gamma}}\right) z^{\frac{1}{2-\gamma}} J_{\nu_\gamma}(z)\,dz \\
&= -\frac{1}{\lambda}\left(\frac{2-\gamma}{2\lambda}\right)^{\frac{1}{2-\gamma}} \int_0^{\frac{2}{2-\gamma}\lambda} f_1\left(\left(\frac{2-\gamma}{2\lambda}z\right)^{\frac{2}{2-\gamma}}\right) \frac{d}{dz}\left(z^{\frac{1}{2-\gamma}} J_{-\frac{1}{2-\gamma}}(z)\right)\,dz \\
&= \frac{1}{\lambda}\left(\frac{2-\gamma}{2\lambda}\right)^{\frac{1}{2-\gamma}} \left[c^-_{\frac{1}{2-\gamma},0} f_1(0) + \int_0^{\frac{2}{2-\gamma}\lambda} \frac{d}{dz}\left(f_1\left(\left(\frac{2-\gamma}{2\lambda}z\right)^{\frac{2}{2-\gamma}}\right)\right) z^{\frac{1}{2-\gamma}} J_{-\frac{1}{2-\gamma}}(z)\,dz\right] \\
&= \frac{1}{\lambda}\left(\frac{2-\gamma}{2\lambda}\right)^{\frac{1}{2-\gamma}} c^-_{\frac{1}{2-\gamma},0} f_1(0) + \frac{1}{\lambda}\frac{2}{2-\lambda}\left(\frac{2-\gamma}{2\lambda}\right)^{\frac{3}{2-\gamma}} \int_0^{\frac{2}{2-\gamma}\lambda} f_1'\left(\left(\frac{2-\gamma}{2\lambda}z\right)^{\frac{2}{2-\gamma}}\right) z^{\frac{\gamma+1}{2-\gamma}} J_{-\frac{1}{2-\gamma}}(z)\,dz \\
&= \frac{1}{\lambda}\left(\frac{2-\gamma}{2\lambda}\right)^{\frac{1}{2-\gamma}} c^-_{\frac{1}{2-\gamma},0} f_1(0) + \frac{1}{\lambda}\int_0^1 f_1'(s) s^{\frac{1}{2}} J_{-\frac{1}{2-\gamma}}\left(\frac{2}{2-\gamma}\lambda s^{\frac{2-\gamma}{2}}\right)\,ds.
\end{aligned}
$$

Using (2.86) and the fact that $|f_1(0)| \leq \|f_1\|_{L^\infty(0,1)} \leq \frac{1}{\sqrt{1-\gamma}}\|f_1\|_{H^1_{0,\gamma}(0,1)}$, we deduce that

$$
\begin{aligned}
|I| &\leq \frac{1}{\lambda}\left(\frac{2-\gamma}{2\lambda}\right)^{\frac{1}{2-\gamma}} c^-_{\frac{1}{2-\gamma},0}|f_1(0)| + \frac{1}{\lambda}\left(\frac{2-\gamma}{2\lambda^2}\right)^{1/2} \|f_1\|_{H^1_{0,\gamma}(0,1)} \left(\int_0^{\frac{2}{2-\gamma}\lambda} z(J_{-\frac{1}{2-\gamma}}(z))^2\,dz\right)^{1/2} \\
&\leq c\frac{1}{|\lambda|^{\frac{3-\gamma}{2-\gamma}}}\|f_1\|_{H^1_{0,\gamma}(0,1)} + c'\frac{1}{|\lambda|^{\frac{3}{2}}}\|f_1\|_{H^1_{0,\gamma}(0,1)} \leq c\frac{1}{|\lambda|^{\frac{3}{2}}}\|f_1\|_{H^1_{0,\gamma}(0,1)}.
\end{aligned}
$$

Then

$$\left|i\lambda\int_0^1 f_1(s)\Phi_+(s)\Phi_-(1)\,ds\right| \leq \frac{1}{|\lambda|}\|f_1\|_{H^1_{0,\gamma}(0,1)}.$$

Also, we have

$$\left|\int_0^1 f_2(s)\Phi_+(s)\Phi_-(1)\,ds\right| \leq |\Phi_-(1)|\|f_2\|_{L^2(0,1)}\|\Phi_+(s)\|_{L^2(0,1)} \leq c\frac{1}{|\lambda|}\|f_2\|_{L^2(0,1)}.$$

In the same way, we can check that

$$\left|\int_0^1 f_1(s)\Phi_-(s)\,ds\right| \leq c\frac{1}{|\lambda|^{\frac{3}{2}}}\|f_1\|_{H^1_{0,\gamma}(0,1)}$$

and

$$\left|\int_0^1 f_2(s)\Phi_-(s)\Phi_+(1)\,ds\right| \leq c\frac{1}{|\lambda|}\|f_2\|_{L^2(0,1)}.$$

Consequently, we get (2.85). Thus, the proof of the Lemma (2.5.1) is complete.

□

Now, inverting the matrix in (2.81) we obtain

$$
\begin{cases}
A = \dfrac{1}{D}(Cr_{22} - \tilde{C}r_{12}) \\
B = \dfrac{1}{D}(-Cr_{21} + \tilde{C}r_{11})
\end{cases}
$$

Considering only the dominant terms of λ, the following is obtained:

$$\begin{array}{rcl} |D||A| & \leq & c_1|\lambda|^{\frac{1}{2}} + c_2|\lambda|^{1-\nu_\gamma} \leq c_3|\lambda|^{1-\nu_\gamma}, \\ |D||B| & \leq & c_1|\lambda|^{\frac{1}{2}} + c_2|\lambda|^{\nu_\gamma - 1} \leq c|\lambda|^{\frac{1}{2}}. \end{array}$$

Hence, using (2.82), we deduce that

$$|A| \leq c|\lambda|^{\frac{7}{2}-\alpha-2\nu_\gamma} \tag{2.89}$$

$$|B| \leq c|\lambda|^{3-\alpha-\nu_\gamma}. \tag{2.90}$$

Also, we have

$$\begin{aligned} &\left\| \int_0^x f_2(s)\Phi_\pm(x)\Phi_\mp(s)\, ds \right\|_{L^2(0,1)} \leq \|f_2\|_{L^2(0,1)}\|\Phi_\pm\|_{L^2(0,1)}\|\Phi_\mp\|_{L^2(0,1)} \leq \frac{c}{|\lambda|}, \\ &\left\| i\lambda \int_0^x f_1(s)\Phi_\pm(x)\Phi_\mp(s)\, ds \right\|_{L^2(0,1)} \leq \|f_1\|_{L^2(0,1)}\|\Phi_\pm\|_{L^2(0,1)}\|\Phi_\mp\|_{L^2(0,1)} \leq c. \end{aligned} \tag{2.91}$$

Then, from (2.76), (2.89), (2.90) and (2.91), we get

$$\|u\|_{L^2(0,1)} \leq c|\lambda|^{3-\alpha-2\nu_\gamma} \left(\|f_1\|_{H^1_{0,\gamma}(0,1)} + \|f_2\|_{L^2(0,1)} + \|f_3\|_{L^2(-\infty,+\infty)} \right),$$

consequently, from $(2.71)_2$ and (2.76), we get

$$\|v\|_{L^2(0,1)} \leq c|\lambda|^{4-\alpha-2\nu_\gamma} \left(\|f_1\|_{H^1_{0,\gamma}(0,1)} + \|f_2\|_{L^2(0,1)} + \|f_3\|_{L^2(-\infty,+\infty)} \right).$$

Using (2.77) and (2.52), we obtain

$$\left\{ \begin{array}{rcl} x^{\gamma/2}\Phi_+'(x) & = & (\frac{1-\gamma}{2} + \frac{2-\gamma}{2}\nu_\gamma)x^{-1/2}J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right) - \lambda x^{\frac{1-\gamma}{2}} J_{1+\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right), \\ x^{\gamma/2}\Phi_-'(x) & = & (\frac{1-\gamma}{2} - \frac{2-\gamma}{2}\nu_\gamma)x^{-1/2}J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right) - \lambda x^{\frac{1-\gamma}{2}} J_{1-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right). \end{array} \right.$$

Then from (2.78), (2.83) and (2.84), we can get

$$\|x^{\gamma/2}u_x\|_{L^2(0,1)} \leq c|\lambda|^{4-\alpha-2\nu_\gamma} \left(\|f_1\|_{H^1_{0,\gamma}(0,1)} + \|f_2\|_{L^2(0,1)} + \|f_3\|_{L^2(-\infty,+\infty)} \right).$$

Moreover from (2.15), we have

$$\|\varphi\|^2_{L^2(-\infty,\infty)} \leq \frac{1}{\eta} \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\varphi(\xi)|^2\, d\xi \leq c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

Thus, we conclude that

$$\|(i\lambda I - \mathcal{P})^{-1}\|_{\mathcal{H}} \leq c|\lambda|^{4-\alpha-2\nu_\gamma} \text{ as } |\lambda| \to \infty.$$

The conclusion then follows by applying Theorem 2.5.1.

Besides, we prove that the decay rate is optimal. Indeed, the decay rate is consistent with the asymptotic expansion of eigenvalues.

□

Remark 2.5.1 *We can extend the results of this chapter to more general measure density (see* **[18]***) instead of (2.5). Indeed, let us suppose that ν is an even nonnegative measurable function such that*

$$\int_{-\infty}^{\infty} \frac{\nu(\xi)^2}{1+\xi^2}\, d\xi < \infty. \tag{2.92}$$

We easily obtain the following Theorem.

Theorem 2.5.3 *Let*

$$\Lambda(\lambda) = \frac{|\lambda|^{3-2\nu_\gamma}}{(\Re S(i\lambda))},$$

where $S(i\lambda) = \int_{-\infty}^{+\infty} \frac{\nu(\xi)^2}{i\lambda+\eta+\xi^2}\, d\xi$. Then the semigroup $S_{\mathcal{P}}(t)_{t\geq 0}$ associated to (P') satisfies the following decay estimate

$$\|e^{\mathcal{P}t}U_0\| \leq C\frac{1}{\Lambda^{-1}(t)}\|U_0\|_{D(\mathcal{P})}, \quad t \to \infty,$$

where Λ^{-1} is any asymptotic inverse of Λ.

Open problem

It seems to be interesting to study a qualitatve propreties of (P) with $a(x)$ instead of x^γ (see (2.2)) with $0 < \mu_a < 1$.

Chapter 3

DECAY ESTIMATES FOR A DEGENERATE WAVE EQUATION WITH TWO BOUNDARY FRACTIONAL FEEDBACKS IN THE PRESENCE OF DIPLACEMENT

3.1 Introduction

In this chapter, we are concerned with the boundary stabilization of fractional type for degenerate wave equation of the form

$$(P) \qquad \begin{cases} u_{tt}(x,t) - (x^\gamma u_x(x,t))_x + \beta u = 0 & \text{in } (0,1)\times(0,+\infty), \\ (x^\gamma u_x)(0,t) = \varrho \partial_t^{\alpha,\eta} u(0,t) & \text{in } (0,+\infty), \\ u_x(1,t) = -\tilde{\varrho}\partial_t^{\tilde{\alpha},\eta} u(1,t) & \text{in } (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } (0,1), \end{cases}$$

where $\gamma \in [0,1)$, $\varrho > 0$, $\tilde{\varrho} > 0$ and $\beta > 0$. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α, $(0 < \alpha \leq 1)$, with respect to the time variable (see **[19]**). It is defined as follows

$$\partial_t^{\alpha,\eta} w(t) = \begin{cases} w_t(t) & \text{for } \alpha = 1, \eta \geq 0, \\ \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\, ds, & \text{for } 0 < \alpha < 1, \eta \geq 0. \end{cases}$$

The degenerate wave equation (P) (i.e $\gamma \neq 0$) can describe the vibration problem of an elastic string. In a neighborhood of an endpoint $x = 0$ of this string, the elastic is sufficiently small or the linear density is large enough.

The bibliography of works concerning the stabilization of nondegenerate wave equation with different types of dampings is truly long (see e.g. **[17]**, **[20]**, **[16]** and the references therein).

In **[20]**, for $a(x) = a_1x + a_0$: the authors have established aymptotics stabilization with the following boundary damping

$$\begin{cases} (au_x)(0,t) = 0, \\ (au_x)(1,t) = -ku(1,t) - u_t(1,t), \ k > 0. \end{cases}$$

In **[16]**, the authors considered the following modelization of a flexible torque arm controlled by two feedbacks depending only on the boundary velocities:

$$\begin{cases} u_{tt}(x,t) - (a(x)u_x)_x + \alpha u_t(x,t) + \beta y(x,t) = 0, \ 0 < x < 1, \ t > 0, \\ (a(x)u_x)(0) = k_1u_t(0,t), t > 0, \\ (a(x)u_x)(1) = -k_2u_t(1,t), t > 0, \end{cases}$$

where

$$\begin{cases} \alpha \geq 0, \beta > 0, k_1, k_2 \geq 0, k_1 + k_2 \neq 0, \\ a \in W^{1,\infty}(0,1), a(x) \geq a_0 \text{ for all } x \in [0,1]. \end{cases}$$

They proved the exponential decay of the solutions.

On the contrary, when the coefficient $a(x)$ is degenerate very little is known in the literature, even though many problems that are relevant for applications are described by hyperbolic equations degenerating at the boundary of the space domain (see **[27]**, [**?**] and [**?**]). In **[27]**, for any $0 < \gamma < 1$, the null controllability of the following degenerate wave equation was considered:

$$(PC) \qquad \begin{cases} u_{tt}(x,t) - (x^\gamma u_x(x,t))_x = 0 & \text{on } (0,1) \times (0,T), \\ u(0,t) = \theta(t), u(1,t) = 0 & \text{on } (0,T), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } (0,1), \end{cases}$$

where $\theta(t)$ is the control variable and it acts on the degenerate boundary. Recently, in [**?**] (see also [**?**]), the authors studied the null controllability problems of one-dimensional degenerate wave equations as in **[27]** but the control acts on the nondegenerate boundary. They proved that any initial value in state space is controllable. Also, an explicit expression for the controllability time is given.

Very recently, Alabau et al. [**?**] studied the degenerate wave equation of the type

$$u_{tt}(x,t) - (a(x)u_x(x,t))_x = 0 \text{ in } (0,1) \times (0,+\infty), \tag{3.1}$$

where the coefficient a is a positive function on $]0,1]$ but vanishes at zero. The degeneracy of (3.1) at $x = 0$ is measured by the parameter μ_a defined by

$$\mu_a = \sup_{0<x\leq 1} \frac{x|a'(x)|}{a(x)} \tag{3.2}$$

and the initial conditions are

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \tag{3.3}$$

followed by the boundary conditions

$$(P1) \qquad \begin{cases} \begin{cases} u(0,t) = 0 & \text{if } 0 \leq \mu_a < 1 \\ (au_x)(0,t) = 0 & \text{if } 1 \leq \mu_a < 2 \end{cases} & \text{in } (0,+\infty), \\ u_x(1,t) + u_t(1,t) + \beta u(1,t) = 0 & \text{in } (0,+\infty), \end{cases}$$

they obtained exponential stability of the solutions.

Here we want to focus on the following ramarks:

- System (3.1), (3.3) and $(P1)$ under study is different from one studied on [?]. Indeed, the control is located at $x = 0$.
- The fractional velocity feedbacks considered here provide a weaker damping than the velocity feedbacks (see [**37**]).
- The explicit representation of the resolvant gives us a sharp polynomial decay rate, however in [?], stabilization is done under the classical energy method based on multiplier techniques (see [**31**]). Unfortunately, this method does not seem to be applicable in the case of damping acting at $x = 0$.

In this chapter, we explain the influence of the relation between the degenerate coefficient and the fractional feedback on decay estimates.

This chapter is organized as follows. In section 2, we give preliminaries results and we reformulate the system (P) into an augmented system by coupling the degenerate wave equation with a suitable diffusion equation and we show the well-posedness of our problem by semigroup theory. In section 3, we prove lack of exponential stability by spectral analysis by using Bessel functions. In the last section, we prove an optimal decay rate. The proof heavily relies on Bessel equations and Borichev-Tomilov Theorem.

3.2 Preliminaries results

Now, we introduce, as in [**12**] or [?], the following weighted Sobolev spaces:

$$H^1_{0,\gamma}(0,1) = \Big\{ u \text{ is locally absolutely continuous in } (0,1] : x^{\gamma/2} u_x \in L^2(0,1)/\ u(1) = 0 \Big\}$$

$$H^1_{\gamma}(0,1) = \Big\{ u \text{ is locally absolutely continuous in } (0,1] : x^{\gamma/2} u_x \in L^2(0,1) \Big\}.$$

We remark that $H^1_\gamma(0,1)$ is a Hilbert space with the scalar product

$$(u,v)_{H^1_\gamma(0,1)} = \int_0^1 (u\overline{v} + x^\gamma u'(x)\overline{v'(x)})\, dx, \quad \forall u, v \in H^1_\gamma(0,1).$$

Let us also set

$$|u|_{H^1_{0,\gamma}(0,1)} = \left(\int_0^1 x^\gamma |u'(x)|^2\, dx \right)^{1/2} \qquad \forall u \in H^1_\gamma(0,1).$$

Actually, $|\cdot|_{H^1_{0,\gamma}(0,1)}$ is an equivalent norm on the closed subspace $H^1_{0,\gamma}(0,1)$ to the norm of $H^1_\gamma(0,1)$. This fact is a simple consequence of the following version of Poincaré's inequality.

Proposition 3.2.1 *There is a positive constant $C_* = C(\gamma)$ such that*

$$\|u\|^2_{L^2(\Omega)} \leq C_* |u|^2_{1,\gamma} \quad \forall u \in H^1_{0,\gamma}(0,1). \tag{3.4}$$

Proof. Let $u \in H^1_{0,\gamma}(0,1)$. For any $x \in]0,1]$ we have that

$$|u(x)| = \left| \int_x^1 u'(s)\, ds \right| \leq |u|_{1,\gamma} \left\{ \int_0^1 \frac{1}{x^\gamma}\, ds \right\}^{1/2}.$$

Therefore

$$\int_0^1 |u(x)|^2\, dx \leq \frac{1}{1-\gamma} |u|^2_{1,\gamma}.$$

Next, we define

$$H^2_\gamma(0,1) = \{u \in H^1_\gamma(0,1) :\ x^\gamma u' \in H^1(0,1)\},$$

where $H^1(0,1)$ denotes the classical Sobolev space.

Remark 3.2.1 *Notice that if $u \in H^2_\gamma(0,1), \gamma \in [1,2)$, we have $(x^\gamma u_x)_{x=0} \equiv 0$ since $1/x^\gamma$ is not integrable over neighbourhoods of 0. Hence the problem is not well-posed in terms of the semigroups in the Hilbert space.*

3.2.1 Augmented model

In this section we reformulate (P) into an augmented system. For that, we need the following proposition.

Proposition 3.2.2 (see [37]) *Let μ be the function:*

$$\mu(\xi) = |\xi|^{(2\alpha-1)/2}, \quad -\infty < \xi < +\infty,\ 0 < \alpha < 1. \tag{3.5}$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t \phi(\xi,t) + (\xi^2 + \eta)\phi(\xi,t) - U(t)\mu(\xi) = 0, \quad -\infty < \xi < +\infty, \eta \geq 0, t > 0, \tag{3.6}$$

$$\phi(\xi,0) = 0, \tag{3.7}$$

$$O(t) = (\pi)^{-1} \sin(\alpha\pi) \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,t)\, d\xi \tag{3.8}$$

is given by

$$O = I^{1-\alpha,\eta} U. \tag{3.9}$$

where

$$[I^{\alpha,\eta} f](t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{-\eta(t-\tau)} f(\tau)\, d\tau$$

Lemma 3.2.1 (see [1]) *If $\lambda \in D_\eta = \mathbb{C} \backslash]-\infty, -\eta]$ then*

$$F(\lambda) = \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2}\, d\xi = \frac{\pi}{\sin \alpha\pi} (\lambda + \eta)^{\alpha - 1}.$$

Using now Proposition 3.2.2 and relation (3.9), system (P) may be recast into the following augmented system

$$(P_2')\qquad \begin{cases} u_{tt}(x,t) - (x^\gamma u_x(x,t))_x + \beta u = 0, \\ \phi_t(\xi,t) + (\xi^2+\eta)\phi(\xi,t) - u_t(0,t)\mu(\xi) = 0, \quad -\infty < \xi < +\infty, \quad t>0, \\ \tilde{\phi}_t(\xi,t) + (\xi^2+\eta)\tilde{\phi}(\xi,t) - u_t(1,t)\tilde{\mu}(\xi) = 0, \quad -\infty < \xi < +\infty, \quad t>0, \\ (x^\gamma u_x)(0,t) = \zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,t)\,d\xi, \\ u_x(1,t) = -\tilde{\zeta}\int_{-\infty}^{+\infty} \tilde{\mu}(\xi)\tilde{\phi}(\xi,t)\,d\xi, \\ \phi(\xi,0) = \tilde{\phi}(\xi,0) = 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \end{cases}$$

where $\zeta = \varrho(\pi)^{-1}\sin(\alpha\pi)$ and $\tilde{\zeta} = \tilde{\varrho}(\pi)^{-1}\sin(\tilde{\alpha}\pi)$. For a solution $(u,\phi,\tilde{\phi})$ of (P'), we define the energy

$$(3.10)\; E(t) = \frac{1}{2}\int_0^1 (|u_t|^2 + x^\gamma |u_x|^2)dx + \beta|u|^2 + \frac{\zeta}{2}\int_{-\infty}^{+\infty} |\phi(\xi,t)|^2\,d\xi + \frac{\tilde{\zeta}}{2}\int_{-\infty}^{+\infty} |\tilde{\phi}(\xi,t)|^2\,d\xi$$

Lemma 3.2.2 *Let $(u,\phi,\tilde{\phi})$ be a regular solution of the problem (P'). Then, the energy functional defined by (3.10) satisfies*

$$(3.11)\qquad E'(t) = -\zeta\int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi,t)|^2\,d\xi - \tilde{\zeta}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\tilde{\phi}(\xi,t)|^2\,d\xi \le 0.$$

3.3 Well-posedness

In this section, we are interested in showing that system (P') is well posed in the sens of semigroups.

We introduce the Hilbert space $\mathcal{H} = H^1_{0,\gamma}(0,1)\times L^2(0,1)\times L^2(-\infty,+\infty)$ with inner product

$$\left\langle \begin{pmatrix} u \\ v \\ \phi_1 \\ \phi_2 \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_0^1 x^\gamma u_x \overline{\tilde{u}}_x dx + \int_0^1 v\overline{\tilde{v}}dx + \zeta\int_{-\infty}^{+\infty} \phi_1\overline{\tilde{\phi}_1} + \tilde{\zeta}\int_{-\infty}^{+\infty}\phi_2\overline{\tilde{\phi}_2}\,d\xi.$$

If we put $U = (u,u_t,\phi,\tilde{\phi})^T$ it is clear that (P') can be written as

$$(3.12)\qquad U' = \mathcal{A}U, \quad U(0) = U_0,$$

where $U_0 = (u_0,u_1,0,0)^T$ and $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is defined by

$$(3.13)\qquad \mathcal{A}\begin{pmatrix} u \\ v \\ \phi \\ \tilde{\phi} \end{pmatrix} = \begin{pmatrix} v \\ (x^\gamma u_x)_x - \beta u \\ -(\xi^2+\eta)\phi + v(0)\mu(\xi) \\ -(\xi^2+\eta)\tilde{\phi} + v(1)\tilde{\mu}(\xi) \end{pmatrix},$$

with domain

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (u, v, \phi, \tilde{\phi}) \text{ in } \mathcal{H} : u \in H^2_\gamma(0,1) \cap H^1_{0,\gamma}(0,1), v \in H^1_{0,\gamma}(0,1), \\ -(\xi^2+\eta)\phi + v(0)\mu(\xi) \in L^2(-\infty,+\infty), \\ -(\xi^2+\eta)\tilde{\phi} + v(1)\tilde{\mu}(\xi) \in L^2(-\infty,+\infty), \\ (x^\gamma u_x)(0) - \zeta \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi = 0, \\ u_x(1) + \tilde{\zeta} \displaystyle\int_{-\infty}^{+\infty} \tilde{\mu}(\xi)\tilde{\phi}(\xi)\, d\xi = 0, \\ |\xi|\phi \in L^2(-\infty,+\infty) \\ , |\xi|\tilde{\phi} \in L^2(-\infty,+\infty) \end{array} \right\}. \tag{3.14}$$

The well-posedness of problem (P') is ensured by the following theorem.

Theorem 3.3.1 (Existence and uniqueness)

(1) If $U_0 \in D(\mathcal{A})$, then system (3.12) has a unique strong solution

$$U \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

(2) If $U_0 \in \mathcal{H}$, then system (3.12) has a unique weak solution

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

Proof of Theorem 3.3.1. We show that $\mathcal{A}$ is monotone maximal. First, it is easy to see that we have

$$\Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\zeta \int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi)|^2\, d\xi - \tilde{\zeta} \int_{-\infty}^{+\infty} (\xi^2+\eta)|\tilde{\phi}(\xi)|^2\, d\xi. \tag{3.15}$$

For the maximality, let $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$ and look for $U = (u, v, \phi, \tilde{\phi})^T \in D(\mathcal{A})$ satisfying $\lambda U - \mathcal{A}U = F$ for $\lambda > 0$, that is,

$$\left\{ \begin{array}{l} \lambda u - v = f_1, \\ \lambda v - (x^\gamma u_x)_x + \beta u = f_2, \\ \lambda \phi + (\xi^2+\eta)\phi - v(0)\mu(\xi) = f_3, \\ \lambda \tilde{\phi} + (\xi^2+\eta)\tilde{\phi} - v(1)\mu(\xi) = f_4. \end{array} \right. \tag{3.16}$$

Suppose u is found with the appropriate regularity. Then, $(3.16)_1, (3.16)_3$ and $(3.16)_4$ yield

$$v = \lambda u - f_1 \in H^1_{0,\gamma}(0,1), \tag{3.17}$$

$$\phi = \frac{f_3(\xi) + \mu(\xi)v(0)}{\xi^2+\eta+\lambda}. \tag{3.18}$$

$$\tilde{\phi} = \frac{f_4(\xi) + \mu(\xi)v(1)}{\xi^2+\eta+\lambda}. \tag{3.19}$$

By using (3.16) and (3.17) it can easily be shown that u satisfies

$$\lambda^2 u - (x^\gamma u_x)_x + \beta u = f_2 + \lambda f_1. \tag{3.20}$$

Solving equation (3.20) is equivalent to finding $u \in H^2_\gamma(0,1) \cap H^1_{0,\gamma}(0,1)$ such that

$$\int_0^1 (\lambda^2 u\overline{w} - (x^\gamma u_x)_x \overline{w} + \beta u w)\, dx = \int_0^1 (f_2 + \lambda f_1)\overline{w}\, dx, \tag{3.21}$$

for all $w \in H^1_{0,\gamma}(0,1)$. By using (3.21), the boundary condition $(3.14)_3$ and (3.18) the function u satisfying the following equation

$$\begin{aligned} &\int_0^1 (\lambda^2 u\overline{w} + (x^\gamma u_x)\overline{w}_x + \beta u w)\, dx + \zeta_2 v(0)\overline{w}(0) + \zeta_1 v(1)\overline{w}(1) \\ &\quad = \int_0^1 (f_2 + \lambda f_1)\overline{w}\, dx - \tilde{\zeta} \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_3(\xi)\, d\xi \overline{w}(0) - \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_4(\xi)\, d\xi \overline{w}(1) \end{aligned} \tag{3.22}$$

where $\zeta_1 = \zeta \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2+\eta+\lambda}\, d\xi$, $\zeta_2 = \tilde{\zeta} \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2+\eta+\lambda}\, d\xi$. Using again (3.17), we deduce that

$$v(0) = \lambda u(0) - f_1(0), \tag{3.23}$$

$$v(1) = \lambda u(1) - f_1(1). \tag{3.24}$$

Inserting (3.23) into (3.22), we get

$$\begin{aligned} &\int_0^1 (\lambda^2 u\overline{w} + x^\gamma u_x\overline{w}_x + \beta u w)\, dx + \lambda\zeta_2 u(0)\overline{w}(0) + \lambda\zeta_1 u(1)\overline{w}(1) \\ &= \int_0^1 (f_2 + \lambda f_1)\overline{w}\, dx - \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_4(\xi)\, d\xi \overline{w}(1) - \tilde{\zeta} \int_{-\infty}^{+\infty} \\ &\frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_3(\xi)\, d\xi \overline{w}(0) + \zeta_2 f_1(0)\overline{w}(0) + \zeta_1 f_1(1)\overline{w}(1). \end{aligned} \tag{3.25}$$

Problem (3.43) is of the form

$$\mathcal{B}(u,w) = \mathcal{L}(w), \tag{3.26}$$

where $\mathcal{B} : [H^1_{0,\gamma}(0,1) \times H^1_{0,\gamma}(0,1)] \to \mathbb{C}$ is the bilinear form defined by

$$\mathcal{B}(u,w) = \int_0^1 (\lambda^2 u\overline{w} + x^\gamma u_x\overline{w}_x + \beta u w)\, dx + \lambda\zeta_2 u(0)\overline{w}(0) + \lambda\zeta_1 u(1)\overline{w}(1)$$

and $\mathcal{L} : H^1_{0,\gamma}(0,1) \to \mathbb{C}$ is the linear functional given by

$$\begin{aligned} \mathcal{L}(w) = &\int_0^1 (f_2 + \lambda f_1)\overline{w}\, dx - \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_4(\xi)\, d\xi \overline{w}(1) - \tilde{\zeta} \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_3(\xi)\, d\xi \overline{w}(0) \\ &+ \zeta_2 f_1(0)\overline{w}(0) + \zeta_1 f_1(1)\overline{w}(1). \end{aligned}$$

It is easy to verify that $\mathcal{B}$ is continuous and coercive, and $\mathcal{L}$ is continuous. Consequently, by the Lax-Milgram Lemma, system (3.26) has a unique solution $u \in H^1_{0,\gamma}(0,1)$. By the regularity theory for the linear elliptic equations, it follows that $u \in H^2_\gamma(0,1)$. Therefore, the operator $\lambda I - \mathcal{A}$ is surjective for any $\lambda > 0$.

$\square$

3.4 Strong stability of the system

We use a general criteria of Arendt-Batty and Lyubich-Vu (see [**?**] or [**35**]), following which a C_0-semigroup of contractions $e^{t\mathcal{A}}$ in a Banach space is strongly stable, if $\mathcal{A}$ has no pure imaginary eigenvalues and $\sigma(\mathcal{A}) \cap i\mathbb{R}$ contains only a countable number of elements. Our main result is the following theorem.

Theorem 3.4.1 *The C_0-semigroup $e^{t\mathcal{A}}$ is strongly stable in $\mathcal{H}$; i.e, for all $U_0 \in \mathcal{H}$, the solution of (3.12) satisfies*

$$\lim_{t\to\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0.$$

For the proof of Theorem 3.4.1, we need the following two lemmas.

Lemma 3.4.1 *$\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.*

Proof
e make a distinction between $i\lambda = 0$ and $i\lambda \neq 0$.
Step 1. Solving for $\mathcal{A}U = 0$ leads to the system

$$\begin{cases} v = 0, \\ (x^\gamma u_x)_x - \beta u = 0, \\ -(\xi^2+\eta)\phi + v(0)\mu(\xi) = 0. \\ -(\xi^2+\eta)\tilde{\phi} + v(1)\mu(\xi) = 0. \end{cases} \tag{3.27}$$

Then $v = 0, \phi = 0, \tilde{\phi} = 0, (x^\gamma u_x)(0) = (x^\gamma u_x)(1) = 0$. Multiplying the second equation in (25) by $\overline{u}$, using Green formula,and the boundary conditions, we get

$$\int_0^1 (x^\gamma |u_x|^2)dx + \int_0^1 (\beta|u|^2)dx = 0 \tag{3.28}$$

then $u = 0$. We have $U = 0$. Hence, $i\lambda = 0$ is not an eigenvalue of $\mathcal{A}$.

Step 2. Let $\lambda \in \mathbb{R} - \{0\}$. We prove that $i\lambda$ is not an eigenvalue of $\mathcal{A}$ by proving that the unique solution $U \in D(\mathcal{A})$ of the equation

$$\mathcal{A}U = i\lambda U \tag{3.29}$$

is $U = 0$. Let $U = (u, v, \phi, \tilde{\phi})^T$. The equation (27) means that

$$\begin{cases} i\lambda u - v = 0, \\ i\lambda v - (x^\gamma u_x)_x + \beta u = 0, \\ i\lambda\phi + (\xi^2+\eta)\phi - v(0)\mu(\xi) = 0, \\ i\lambda\tilde{\phi} + (\xi^2+\eta)\tilde{\phi} - v(1)\mu(\xi) = 0 \end{cases} \tag{3.30}$$

Using (3.15) and (3.29), we find

$$\phi \equiv 0, \tilde{\phi} \equiv 0 \tag{3.31}$$

then, using the third and four equations in (3.30), we deduce that

$$v(0) = 0, \; v(1) = 0. \tag{3.32}$$

Therefore, from the first and last equation in (3.30), we find

$$\begin{array}{lll} u(0) = 0 & \text{and} & (x^\gamma u_x)(0) = \beta u(0) = 0, \\ u(1) = 0 & \text{and} & (x^\gamma u_x)(1) = \beta u(1) = 0 \end{array} \tag{3.33}$$

Thus, by eliminating v, the system (3.30) implies that

$$\begin{cases} \lambda^2 u + (x^\gamma u_x)_x - \beta u = 0 \text{ on } (0,1), \\ u(0) = u(1) = 0, \\ (x^\gamma u_x)(0) = (x^\gamma u_x)(1) = 0. \end{cases} \tag{3.34}$$

The solution of the equation (3.34) is given by

$$u(x) = C_1 \Phi_+(x) + C_2 \Phi_-(x).$$

where Φ_+ and Φ_- are defined by

$$\Phi_+(x) = x^{\frac{1-\gamma}{2}} J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\sqrt{\lambda^2 - \beta}\, x^{\frac{2-\gamma}{2}}\right), \quad \Phi_-(x) = x^{\frac{1-\gamma}{2}} J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\sqrt{\lambda^2 - \beta}\, x^{\frac{2-\gamma}{2}}\right). \tag{3.35}$$

From boundary conditions $(3.34)_2$ and $(3.34)_3$, we deduce that

$$u \equiv 0.$$

Therefore $U = 0$. Consequently, $\mathcal{P}$ does not have purely imaginary eigenvalues.

□

Lemma 3.4.2
If $\lambda \neq 0$, the operator $i\lambda I - \mathcal{A}$ is surjective.
If $\lambda = 0$ and $\eta \neq 0$, the operator $i\lambda I - \mathcal{A}$ is surjective.

Proof.
Case 1: $\lambda \neq 0$. Let $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$ be given, and let $U = (u, v, \phi, \tilde{\phi})^T \in D(\mathcal{A})$ be such that

$$(i\lambda I - \mathcal{A})U = F. \tag{3.36}$$

Equivalently, we have

$$\begin{cases} i\lambda u - v = f_1, \\ i\lambda v - (x^\gamma u_x)_x = f_2, \\ i\lambda \phi + (\xi^2 + \eta)\phi - \mu(\xi) v(0) = f_3 \\ i\lambda \tilde{\phi} + (\xi^2 + \eta)\tilde{\phi} - \tilde{\mu}(\xi) v(1) = f_4 \end{cases} \tag{3.37}$$

together with the conditions (3.52).

Inserting $(3.37)_1$ into $(3.37)_2$, we get

$$-\lambda^2 u - (x^\gamma u_x)_x + \beta u = f_2 + \lambda f_1. \tag{3.38}$$

Solving system (3.38) is equivalent to finding $u \in H^2_\gamma \cap H^1_{0,\gamma}(0,1)$ such that

$$\int_0^1 (-\lambda^2 u\overline{w} - (x^\gamma u_x)_x \overline{w} + \beta u\overline{w})\, dx = \int_0^1 (f_2 + i\lambda f_1)\overline{w}\, dx, \tag{3.39}$$

for all $w \in H^1_{0,\gamma}(0,1)$. By using $(3.37)_3$ and $(3.37)_1$ the function u satisfies the following system

$$\begin{aligned}&\int_0^1 (-\lambda^2 u\overline{w} + (x^\gamma u_x)\overline{w}_x + \beta u\overline{w})\, dx + \zeta_2 v(0)\overline{w}(0) + \zeta_1 v(1)\overline{w}(1)\\ &\quad = \int_0^1 (f_2 + i\lambda f_1)\overline{w}\, dx - \tilde{\zeta}\int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+i\lambda} f_3(\xi)\, d\xi \overline{w}(0) - \zeta \int_{-\infty}^{+\infty} \frac{\tilde{\mu}(\xi)}{\xi^2+\eta+i\lambda} f_4(\xi)\, d\xi \overline{w}(1)\end{aligned} \tag{3.40}$$

Using again (3.17), we deduce that

$$v(0) = i\lambda u(0) - f_1(0), \tag{3.41}$$

$$v(1) = i\lambda u(1) - f_1(1). \tag{3.42}$$

Inserting (3.41) into (3.22), we get

$$\begin{aligned}&\int_0^1 (-\lambda^2 u\overline{w} + x^\gamma u_x\overline{w}_x + \beta u\overline{w})\, dx + i\lambda\zeta_2 u(0)\overline{w}(0) + i\lambda\zeta_1 u(1)\overline{w}(1)\\ &= \int_0^1 (f_2 + i\lambda f_1)\overline{w}\, dx - \zeta\int_{-\infty}^{+\infty} \frac{\tilde{\mu}(\xi)}{\xi^2+\eta+i\lambda} f_4(\xi)\, d\xi \overline{w}(1)\\ &-\tilde{\zeta}\int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+i\lambda} f_3(\xi)\, d\xi \overline{w}(0) + \zeta_2 f_1(0)\overline{w}(0) + \zeta_1 f_1(1)\overline{w}(1).\end{aligned} \tag{3.43}$$

We can rewrite (3.40) as

$$\mathcal{B}(u,w) = l(w), \quad \forall w \in H^1_{0,\gamma}(0,1), \tag{3.44}$$

where

$$\mathcal{B}(u,w) = \mathcal{B}_1(u,w) + \mathcal{B}_2(u,w)$$

with

$$\begin{cases} \mathcal{B}_1(u,w) = \displaystyle\int_0^1 (x^\gamma u_x\overline{w}_x + \beta u\overline{w})\, dx\, dx + i\varrho\lambda(i\lambda+\eta)^{\alpha-1}u(0)\overline{w}(0) + i\tilde{\varrho}\lambda(i\lambda+\eta)^{\tilde{\alpha}-1}u(1)\overline{w}(1),\\ \mathcal{B}_2(u,w) = -\displaystyle\int_0^1 \lambda^2 u\overline{w}\, dx, \end{cases} \tag{$*$}$$

and

$$\begin{aligned}l(w) = \int_0^1 (f_2 + i\lambda f_1)\overline{w}\, dx - \zeta\int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+i\lambda} f_3(\xi)\, d\xi\ \overline{w}(0) - \tilde{\zeta}\int_{-\infty}^{+\infty} \frac{\tilde{\mu}(\xi)}{\xi^2+\eta+i\lambda} f_4(\xi)\, d\xi\ \overline{w}(1)\\ +\varrho(i\lambda+\eta)^{\alpha-1}f_1(0)\overline{w}(0) + \tilde{\varrho}(i\lambda+\eta)^{\tilde{\alpha}-1}f_1(1)\overline{w}(1).\end{aligned}$$

Let $(H^1_{0,\gamma}(0,1))'$ be the dual space of $H^1_{0,\gamma}(0,1)$. Let us define the following operators

$$(**)\qquad \begin{array}{ll} B : H^1_{0,\gamma}(0,1) \to (H^1_{0,\gamma}(0,1))' & B_i : H^1_{0,\gamma}(0,1) \to (H^1_{0,\gamma}(0,1))' \quad i \in \{1,2\} \\ \qquad u \mapsto Bu & \qquad u \mapsto B_i u \end{array}$$

such that

$$(***)\qquad \begin{array}{l} (Bu)w = \mathcal{B}(u,w),\ \forall w \in H^1_{0,\gamma}(0,1), \\ (B_i u)w = \mathcal{B}_i(u,w),\ \forall w \in H^1_{0,\gamma}(0,1), i \in \{1,2\}. \end{array}$$

We need to prove that the operator B is an isomorphism. For this aim, we divide the proof into three steps:

Step 1. In this step, we want to prove that the operator B_1 is an isomorphism. For this aim, it is easy to see that $\mathcal{B}_1$ is sesquilinear, continuous form on $H^1_{0,\gamma}(0,1)$. Furthermore

$$\begin{array}{rcl} \Re\mathcal{B}_1(u,u) & = & \|x^{\gamma/2}u_x\|_2^2 + \beta\|u\|_2^2 + \varrho\lambda\Re\left(i(i\lambda+\eta)^{\alpha-1}\right)|u(0)|^2 + \tilde{\varrho}\lambda\Re\left(i(i\lambda+\eta)^{\tilde{\alpha}-1}\right)|u(1)|^2 \\ & \geq & \|x^{\gamma/2}u_x\|_2^2 + \beta\|u\|_2^2, \end{array}$$

where we have used the fact that

$$\varrho\lambda\Re\left(i(i\lambda+\eta)^{\alpha-1}\right) = \zeta\lambda^2\int_{-\infty}^{+\infty}\frac{\mu(\xi)^2}{\lambda^2+(\eta+\xi^2)^2}\,d\xi > 0.$$

Thus $\mathcal{B}_1$ is coercive. Then, from $(**)$ and Lax-Milgram theorem, the operator B_1 is an isomorphism.

Step 2. In this step, we want to prove that the operator B_2 is compact. For this aim, from $(*)$ and $(***)$, we have

$$|\mathcal{B}_2(u,w)| \leq c\|u\|_{L^2(0,1)}\|w\|_{L^2(0,1)},$$

and consequently, using the compact embedding from $H^1_{0,\gamma}(0,1)$ to $L^2(0,1)$ (see **[2]**) we deduce that B_2 is a compact operator. Therefore, from the above steps, we obtain that the operator $B = B_1 + B_2$ is a Fredholm operator of index zero. Now, following Fredholm alternative, we still need to prove that the operator B is injective to obtain that the operator B is an isomorphism.

Step 3. Let $u \in ker(B)$, then

$$\mathcal{B}(u,w) = 0 \quad \forall w \in H^1_{0,\gamma}(0,1). \tag{3.45}$$

In particular for $w = u$, it follows that

$$\lambda^2\|u\|^2_{L^2(0,1)} - i\varrho\lambda(i\lambda+\eta)^{\alpha-1}|u(0)|^2 - i\tilde{\varrho}\lambda(i\lambda+\eta)^{\tilde{\alpha}-1}|u(1)|^2 = \|x^{\gamma/2}u_x\|^2_{L^2(0,1)} + \beta\|u\|_2^2.$$

Hence, we have

$$u(0) = u(1) = 0. \tag{3.46}$$

From (3.45), we obtain

$$(x^{\gamma/2}u_x)(0) = 0, u_x(1) = 0 \tag{3.47}$$

and then

$$\begin{cases} -\lambda^2 u - (x^\gamma u_x)_x + \beta u = 0, \\ u(0) = (x^{\gamma/2} u_x)(0) = 0, \\ u(1) = 0. \end{cases} \tag{3.48}$$

Then, according to Lemma 3.4.1, we deduce that $u = 0$ and consequently $Ker(B) = \{0\}$. Finally, from Step 3 and Fredholm alternative, we deduce that the operator B is isomorphism. It is easy to see that the operator l is a antilinear and continuous form on $H^1_{0,\gamma}(0,1)$. Consequently, (3.44) admits a unique solution $u \in H^1_{0,\gamma}(0,1)$. By using the classical elliptic regularity, we deduce that $U \in D(\mathcal{A})$ is a unique solution of (3.36). Hence $i\lambda - \mathcal{A}$ is surjective for all $\lambda \in \mathbb{R}^*$.

Case 2: $\lambda = 0$ and $\eta \neq 0$. Using Lax-Milgram Lemma, we obtain the result.

Taking account of Lemmas 3.4.1, 3.4.2 and from Theorem **??** The C_0-semigroup $e^{t\mathcal{A}}$ is strongly stable in $\mathcal{H}$.

□

3.5 Spectral analysis and lack of uniform stability

This section will be devoted to the study of the lack of exponential decay of solutions associated with the system (3.12). To do this, we shall use the following well-known result from semigroup theory.

Theorem 3.5.1 ([42]-[30]) *Let $S(t)$ be a C_0-semigroup of contractions on Hilbert space $\mathcal{X}$ with generator $\mathcal{A}$. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R} \tag{3.49}$$

and

$$\overline{\lim_{|\beta|\to\infty}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{X})} < \infty. \tag{3.50}$$

Our main result is the following.

Theorem 3.5.2 *The semigroup generated by the operator $\mathcal{A}$ is not exponentially stable if $\eta = 0$ or $\alpha \neq 2\nu_\gamma$.*

Proof. We will examine two cases.
•**Case 1** $\eta = 0$ and $\alpha \neq 1$: We shall show that $i\lambda = 0$ is not in the resolvent set of the operator $\mathcal{A}$. Indeed, noting that $F = (\sin x, 0, 0, 0)^T \in \mathcal{H}$, and assume that there exists $U = (u, v, \phi)^T \in D(\mathcal{A})$ such that $-\mathcal{A}U = F$. It follows

$$\begin{cases} -v = \sin x, \\ -(x^\gamma u_x)_x + \beta u = 0, \\ \xi^2 \phi - v(0)\mu(\xi) = 0. \\ \xi^2 \tilde{\phi} - v(1)\tilde{\mu}(\xi) = 0. \end{cases}$$

We see that $\tilde{\phi}(\xi) = -|\xi|^{\frac{2\tilde{\alpha}-5}{2}} \sin 1$. But, then $\tilde{\phi} \notin L^2(-\infty, +\infty)$, since $\tilde{\alpha} \in]0,1[$. So $(u, v, \phi, \tilde{\phi})^T \notin D(\mathcal{A})$. Then the operator $-\mathcal{A}$ is not invertible.

• **Case 2** $\eta \neq 0$ and $\alpha \neq 2\nu_\gamma$:

We aim to show that an infinite number of eigenvalues of $\mathcal{A}$ approach the imaginary axis which prevents the system (P) from being exponentially stable. Indeed we first compute the characteristic equation that gives the eigenvalues of $\mathcal{A}$. Let λ be an eigenvalue of $\mathcal{A}$ with associated eigenvector $U = (u, v, \phi)^T$. Then $\mathcal{A}U = \lambda U$ is equivalent to

$$\left\{\begin{array}{l} \lambda u - v = 0, \\ \lambda v - (x^\gamma u_x)_x + \beta u = 0, \\ \lambda \phi + (\xi^2 + \eta)\phi - v(0)\mu(\xi) = 0 \\ \tilde{\phi} + (\xi^2 + \eta)\tilde{\phi} - v(1)\tilde{\mu}(\xi) = 0 \end{array}\right. \tag{3.51}$$

with boundary conditions

$$\left\{\begin{array}{l} (x^\gamma u_x)(0) - \zeta \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi = 0. \\ u_x(1) + \tilde{\zeta} \displaystyle\int_{-\infty}^{+\infty} \tilde{\mu}(\xi)\tilde{\phi}(\xi)\, d\xi = 0. \end{array}\right. \tag{3.52}$$

Inserting $(3.51)_1$ into $(3.51)_2$ and $(3.51)_3$, we get

$$\left\{\begin{array}{l} \lambda^2 u - (x^\gamma u_x)_x + \beta u = 0, \\ (\lambda + \xi^2 + \eta)\phi - \lambda u(0)\mu(\xi) = 0. \\ (\lambda + \xi^2 + \eta)\tilde{\phi} - \lambda u(1)\tilde{\mu}(\xi) = 0. \end{array}\right. \tag{3.53}$$

From the condition $(3.52)_2$, $(3.53)_2$ and Lemma 3.2.1, we obtain that

$$\left\{\begin{array}{l} (x^\gamma u_x)(0) - \varrho\lambda(\lambda + \eta)^{\alpha-1} u(0) = 0, \\ u_x(1) + \tilde{\varrho}\lambda(\lambda + \eta)^{\tilde{\alpha}-1} u(1) = 0, \end{array}\right. \tag{3.54}$$

Finally, we get the following problem

$$\left\{\begin{array}{l} \lambda^2 u - (x^\gamma u_x)_x + \beta u = 0, \\ (x^\gamma u_x)(0) - \varrho\lambda(\lambda + \eta)^{\alpha-1} u(0) = 0. \\ u_x(1) + \tilde{\varrho}\lambda(\lambda + \eta)^{\tilde{\alpha}-1} u(1) = 0. \end{array}\right. \tag{3.55}$$

It is well-known that Bessel functions play an important role in this type of problem. Assume that u is a solution of $(3.55)_1$ associated to eigenvalue $-(\lambda^2 + \beta)$, then one easily checks that the function

$$u(x) = x^{\frac{1-\gamma}{2}} \Psi\left(\frac{2}{2-\gamma} i\sqrt{\lambda^2 + \beta} x^{\frac{2-\gamma}{2}}\right)$$

is a solution of the following problem:

$$y^2\Psi''(y) + y\Psi'(y) + (y^2 - (\frac{\gamma - 1}{2-\gamma})^2)\Psi(y) = 0. \tag{3.56}$$

We have

$$u(x) = c_+\tilde{\Phi}_+ + c_-\tilde{\Phi}_-, \tag{3.57}$$

where $\tilde{\Phi}_+$ and $\tilde{\Phi}_-$ are defined by

$$\tilde{\Phi}_+(x) := x^{\frac{1-\gamma}{2}} J_{\nu_\gamma}\left(\frac{2}{2-\gamma} i\sqrt{\lambda^2+\beta}\, x^{\frac{2-\gamma}{2}}\right)$$

and

$$\tilde{\Phi}_-(x) := x^{\frac{1-\gamma}{2}} J_{-\nu_\gamma}\left(\frac{2}{2-\gamma} i\sqrt{\lambda^2+\beta}\, x^{\frac{2-\gamma}{2}}\right),$$

where

$$J_\nu(y) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\nu+1)} \left(\frac{y}{2}\right)^{2m+\nu} = \sum_{m=0}^{\infty} c^+_{\nu,m} y^{2m+\nu}, \tag{3.58}$$

$$J_{-\nu}(y) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m-\nu+1)} \left(\frac{y}{2}\right)^{2m-\nu} = \sum_{m=0}^{\infty} c^-_{\nu,m} y^{2m-\nu}, \tag{3.59}$$

$$\nu_\gamma = \frac{1-\gamma}{2-\gamma}.$$

J_{ν_γ} and $J_{-\nu_\gamma}$ are Bessel functions of the first kind of order ν_γ and $-\nu_\gamma$. As $\nu_\gamma \notin \mathbb{N}$, so J_{ν_γ} and $J_{-\nu_\gamma}$ are linearly independent and therefore the pair $(J_{\nu_\gamma}, J_{-\nu_\gamma})$ (classical result) forms a fundamental system of solutions (3.56).

Then, using the series expansion of J_{ν_α} and $J_{-\nu_\alpha}$, one obtains

$$\tilde{\Phi}_+(x) = \sum_{m=0}^{\infty} \tilde{c}^+_{\nu_\gamma,m} x^{1-\gamma+(2-\gamma)m}, \quad \tilde{\Phi}_-(x) = \sum_{m=0}^{\infty} \tilde{c}^-_{\nu_\gamma,m} x^{(2-\gamma)m}$$

with

$$\tilde{c}^+_{\nu_\gamma,m} = c^+_{\nu_\gamma,m} \left(\frac{2}{2-\gamma} i\sqrt{\lambda^2+\beta}\right)^{2m+\nu_\gamma}, \quad \tilde{c}^-_{\nu_\gamma,m} = c^-_{\nu_\gamma,m} \left(\frac{2}{2-\gamma} i\sqrt{\lambda^2+\beta}\right)^{2m-\nu_\gamma}.$$

Next one easily verifies that $\Phi_+, \Phi_- \in H^1_{0,\gamma}(0,1)$: indeed,

$$\begin{aligned} &\tilde{\Phi}_+(x) \sim_0 \tilde{c}^+_{\nu_\gamma,0} x^{1-\gamma}, \quad x^{\gamma/2}\tilde{\Phi}'_+(x) \sim_0 (1-\gamma)\tilde{c}^+_{\nu_\gamma,0} x^{-\gamma/2}, \\ &\tilde{\Phi}_-(x) \sim_0 \tilde{c}^-_{\nu_\gamma,0}, \quad x^{\gamma/2}\tilde{\Phi}'_-(x) \sim_0 (2-\gamma)\tilde{c}^-_{\nu_\gamma,0} x^{1-\gamma/2}, \end{aligned}$$

where we have used the following relation

$$xJ'_\nu(x) = \nu J_\nu(x) - xJ_{\nu+1}(x). \tag{3.60}$$

Hence, given c_+ and c_-, $u(x) = c_+\tilde{\Phi}_+(x) + c_-\tilde{\Phi}_-(x) \in H^1_\gamma(0,1)$ with the following boundary condition

$$\begin{cases} (x^\gamma u_x)(0) - \varrho\lambda(\lambda+\eta)^{\alpha-1}u(0) = 0, \\ u_x(1) + \tilde{\varrho}\lambda(\lambda+\eta)^{\tilde{\alpha}-1}u(1) = 0. \end{cases}$$

Then

$$M(\lambda)C(\lambda) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.61}$$

where

$$M(\lambda) = \begin{pmatrix} (1-\gamma)\tilde{c}^{+}_{\nu_\gamma,0} & -\varrho\lambda(\lambda+\eta)^{\alpha-1}\tilde{c}^{-}_{\nu_\gamma,0} \\ (1-\gamma+\tilde{\varrho}\lambda(\lambda+\eta)^{\tilde{\alpha}-1})J_{\nu_\gamma}(\tilde{\lambda}) - i\sqrt{\lambda^2+\beta}J_{1+\nu_\gamma}(\tilde{\lambda}) & \tilde{\varrho}\lambda(\lambda+\eta)^{\tilde{\alpha}-1}J_{-\nu_\gamma}(\tilde{\lambda}) - i\sqrt{\lambda^2+\beta}J_{1-\nu_\gamma}(\tilde{\lambda
\end{pmatrix}$$

$$C(\lambda) = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}.$$

Hence, a non-trivial solution u exists if and only if the determinant of $M(\lambda)$ vanishes. Set $f(\lambda) = detM(\lambda)$ thus the characteristic equation is $f(\lambda) = 0$.

Our purpose is to prove, thanks to Rouché's Theorem, that there is a subsequence of eigenvalues for which their real part tends to 0.

In the sequel, since $\mathcal{A}$ is dissipative, we study the asymptotic behavior of the large eigenvalues λ of $\mathcal{A}$ in the strip $-\alpha_0 \leq \Re(\lambda) \leq 0$, for some $\alpha_0 > 0$ large enough and for such λ, we remark that Φ_+, Φ_- remain bounded.

Lemma 3.5.1 *There exists $N \in \mathbb{N}$ such that*

$$\{\lambda_k\}_{k\in\mathbf{Z}^*,|k|\geq N} \subset \sigma(\mathcal{A}), \tag{3.62}$$

where

- *If $\gamma = 0$ and $\alpha = 1$, then*

$$\lambda_k = \left\{ \begin{array}{ll} \ln\sqrt{\dfrac{\varrho-1}{\varrho+1}} + ik\pi & \text{if } \rho > 1 \\ \ln\sqrt{\dfrac{1-\varrho}{\varrho+1}} + i\left(k+\dfrac{1}{2}\right)\pi & \text{if } \rho < 1 \end{array} \right\}, \quad k \in \mathbf{Z}.$$

- *If $\alpha = 2\nu_\gamma$, then*

$$\lambda_k = -i\frac{2-\gamma}{4}\left(2k\pi + \theta + \frac{\pi}{2}\right) - \frac{2-\gamma}{4}\ln\frac{1+\tilde{A}}{\sqrt{1+\tilde{A}^2+2\tilde{A}\cos 2\nu_\gamma\pi}} + O\left(\frac{1}{k^{1-\tilde{\alpha}}}\right), \quad k \in \mathbf{Z},$$

$$\lambda_k = \overline{\lambda_{-k}} \text{ if } k \leq -N,$$

where

$$\tilde{A} = \frac{1}{1-\gamma}\left(\frac{2}{2-\gamma}\right)^{-2\nu_\gamma}\frac{c^{-}_{\nu_\gamma,0}}{c^{+}_{\nu_\gamma,0}}$$

and θ is such that

$$\left\{ \begin{array}{rcl} \cos\theta & = & \dfrac{(1+\tilde{A})\cos\nu_\gamma\pi}{\sqrt{1+\tilde{A}^2+2\tilde{A}\cos 2\nu_\gamma\pi}}, \\ \sin\theta & = & \dfrac{(1-\tilde{A})\sin\nu_\gamma\pi}{\sqrt{1+\tilde{A}^2+2\tilde{A}\cos 2\nu_\gamma\pi}}. \end{array} \right.$$

- *If $\alpha > 2\nu_\gamma$ and $\alpha + \tilde{\alpha} > 1 + 2\nu_\gamma$, then*

$$\lambda_k = -\frac{2-\gamma}{2}i\left(k + \frac{\nu_\gamma}{2} + \frac{1}{4}\right)\pi + \frac{\tilde{\alpha}}{k^{1-\tilde{\alpha}}} + \frac{\beta}{k^{1-\tilde{\alpha}}} + o\left(\frac{1}{k^{1-\tilde{\alpha}}}\right), k \geq N, \tilde{\alpha} \in i\mathbb{R}, \beta \in \mathbb{R}, \beta < 0,$$

$$\lambda_k = \overline{\lambda_{-k}} \textit{ if } k \leq -N,$$

where

$$\beta = -\left(\frac{2-\gamma}{2}\right)^{\tilde{\alpha}} \frac{\rho\tilde{\rho}}{\pi^{1-\tilde{\alpha}}} \cos(1-\tilde{\alpha})\frac{\pi}{2}.$$

• *If* $\alpha > 2\nu_\gamma$ *and* $\alpha + \tilde{\alpha} < 1 + 2\nu_\gamma$, *then*

$$\lambda_k = -\frac{2-\gamma}{2} i\left(k - \frac{\nu_\gamma}{2} + \frac{3}{4}\right)\pi + \frac{\tilde{\alpha}}{k^{2\nu_\gamma - \alpha}} + \frac{\beta}{k^{2\nu_\gamma-\alpha}} + o\left(\frac{1}{k^{2\nu_\gamma-\alpha}}\right), k \geq N, \tilde{\alpha} \in i\mathbb{R}, \beta \in \mathbb{R}, \beta < 0,$$

$$\lambda_k = \overline{\lambda_{-k}} \textit{ if } k \leq -N,$$

where

$$\beta = -\frac{\varrho}{1-\gamma} \frac{c^-_{\nu_\gamma,0}}{c^+_{\nu_\gamma,0}} \left(\frac{2-\gamma}{2}\right)^{1+\alpha} \frac{\cos(1-\alpha)\frac{\pi}{2}\sin\nu_\gamma\pi}{\pi^{\alpha-2\nu_\gamma}}.$$

Moreover for all $|k| \geq N$, *the eigenvalues* λ_k *are simple.*

• *If* $\alpha < 2\nu_\gamma$, *then*

$$\lambda_k = -\frac{2-\gamma}{2} i\left(k - \frac{\nu_\gamma}{2} + \frac{3}{4}\right)\pi + \frac{\tilde{\alpha}}{k^{2\nu_\gamma - \alpha}} + \frac{\beta}{k^{2\nu_\gamma-\alpha}} + o\left(\frac{1}{k^{2\nu_\gamma-\alpha}}\right), k \geq N, \tilde{\alpha} \in i\mathbb{R}, \beta \in \mathbb{R}, \beta < 0,$$

$$\lambda_k = \overline{\lambda_{-k}} \textit{ if } k \leq -N,$$

where

$$\beta = -\frac{\varrho}{1-\gamma} \frac{c^-_{\nu_\gamma,0}}{c^+_{\nu_\gamma,0}} \left(\frac{2-\gamma}{2}\right)^{1+\alpha} \frac{\cos(1-\alpha)\frac{\pi}{2}\sin\nu_\gamma\pi}{\pi^{\alpha-2\nu_\gamma}}.$$

Moreover for all $|k| \geq N$, *the eigenvalues* λ_k *are simple.*

The proof of Lemma 3.5.1 will be given in Appendix A.

Now, setting $\tilde{U}_k = (\lambda_k^0 - \mathcal{A})U_k$, where U_k is a normalized eigenfunction associated to λ_k. We then have

$$\begin{aligned}
\|(\lambda_k^0 - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} = \sup_{U \in \mathcal{H}, U \neq 0} \frac{\|(\lambda_k^0 - \mathcal{A})^{-1}U\|_{\mathcal{H}}}{\|U\|_{\mathcal{H}}} &\geq \frac{\|(\lambda_k^0 - \mathcal{A})^{-1}\tilde{U}_k\|_{\mathcal{H}}}{\|\tilde{U}_k\|_{\mathcal{H}}} \\
&\geq \frac{\|U_k\|_{\mathcal{H}}}{\|(\lambda_k^0 - \mathcal{A})U_k\|_{\mathcal{H}}}.
\end{aligned}$$

Hence, by Lemma 3.5.1, we deduce that

$$\|(\lambda_k^0 - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \geq c \begin{cases} |k|^{\alpha - 2\nu_\gamma} & \text{if } \alpha > 2\nu_\gamma, \\ |k|^{2\nu_\gamma - \alpha} & \text{if } \alpha < 2\nu_\gamma. \end{cases}$$

Thus, (3.50) is not satisfied for $\alpha \neq 2\nu_\gamma$. So that, the semigroup $e^{t\mathcal{A}}$ is not exponentially stable. Thus the proof is complete.

□

Proof.
• $\gamma = 0$ and $\alpha = 1$.
System (3.55) becomes
$$\begin{cases} \lambda^2 u - u_{xx} = 0, \\ u_x(0) = \varrho\lambda u(0), \\ u(1) = 0. \end{cases}$$
The solution u is given by
$$u = c_1 e^{\lambda x} + c_2 e^{-\lambda x}.$$
Thus, the boundary conditions give
$$e^{2\lambda} = \frac{\varrho - 1}{\varrho + 1}.$$
If $\varrho > 1$ and if we set $\lambda = x + iy$, then
$$e^{2x} = \frac{\varrho - 1}{\varrho + 1} \text{ and } e^{2iy} = 1.$$
Hence
$$x = \frac{1}{2}\ln\frac{\varrho - 1}{\varrho + 1} \text{ and } y = k\pi, \quad k \in \mathbf{Z}.$$
Then
$$\lambda = \frac{1}{2}\ln\frac{\varrho - 1}{\varrho + 1} + ik\pi, \quad k \in \mathbf{Z}.$$
Now if $\varrho < 1$, we have
$$e^{2x} = \frac{1 - \varrho}{\varrho + 1} \text{ and } e^{2iy} = -1.$$
Hence
$$x = \frac{1}{2}\ln\frac{1 - \varrho}{\varrho + 1} \text{ and } y = (k + \frac{1}{2})\pi, \quad k \in \mathbf{Z}.$$
Then
$$\lambda = \frac{1}{2}\ln\frac{1 - \varrho}{\varrho + 1} + i(k + \frac{1}{2})\pi, \quad k \in \mathbf{Z}.$$
• $\alpha > 2\nu_\gamma$ and $\alpha + \tilde{\alpha} > 1 + 2\nu_\gamma$.
Step 1. From (3.61), our aim is to solve the equation
$$f(\lambda) = (1-\gamma)\tilde{\rho}\tilde{c}^+_{\nu_\gamma,0}\lambda^{\tilde{\alpha}} J_{-\nu_\gamma}\left(\tfrac{2}{2-\gamma} i\sqrt{\lambda^2+\beta}\right) - (1-\gamma) i\tilde{c}^+_{\nu_\gamma,0}\lambda J_{1-\nu_\gamma}\left(\tfrac{2}{2-\gamma} i\sqrt{\lambda^2+\beta}\right)$$
$$+\rho\tilde{\rho}\tilde{c}^-_{\nu_\gamma,0}\lambda^{\alpha+\tilde{\alpha}} J_{\nu_\gamma}\left(\tfrac{2}{2-\gamma} i\sqrt{\lambda^2+\beta}\right) - i\tilde{c}^-_{\nu_\gamma,0}\lambda^{1+\alpha} J_{1+\nu_\gamma}\left(\tfrac{2}{2-\gamma} i\sqrt{\lambda^2+\beta}\right)$$
We will use the following classical development (see **[32]** p. 122, (5.11.6)): for all $\delta > 0$, the following development holds when $|\arg z| < \pi - \delta$:
$$(3.63)\; J_\nu(z) = \left(\frac{2}{\pi z}\right)^{1/2}\left[\cos\left(z - \nu\frac{\pi}{2} - \frac{\pi}{4}\right) - \frac{(\nu - \frac{1}{2})(\nu + \frac{1}{2})}{2}\frac{\sin\left(z - \nu\frac{\pi}{2} - \frac{\pi}{4}\right)}{z} + O\left(\frac{1}{|z|^2}\right)\right].$$

Then

$$f(\lambda) = -\left(\frac{2}{\pi\tilde{z}}\right)^{1/2} \lambda^{1+\alpha-\nu_\gamma} c^-_{\nu_\gamma,0} \left(\frac{2}{2-\gamma} i\right)^{-\nu_\gamma} \frac{e^{-i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}}{2} \tilde{f}(\lambda), \tag{3.64}$$

where

$$\tilde{z} = \frac{2}{2-\gamma} i\sqrt{\lambda^2+\beta}$$

and

$$\begin{aligned} \tilde{f}(\lambda) &= (e^{2i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}-1) - \rho\tilde{\rho}\frac{e^{2i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}+1}{\lambda^{1-\tilde{\alpha}}} + o\left(\frac{1}{\lambda^{1-\tilde{\alpha}}}\right) \\ &= f_0(\lambda) + \frac{f_1(\lambda)}{\lambda^{1-\tilde{\alpha}}} + o\left(\frac{1}{\lambda^{1-\tilde{\alpha}}}\right), \end{aligned} \tag{3.65}$$

where

$$f_0(\lambda) = e^{2i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})} - 1. \tag{3.66}$$

$$f_1(\lambda) = -\rho\tilde{\rho}(e^{2i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}+1). \tag{3.67}$$

Note that f_0 and f_1 remain bounded in the strip $-\alpha_0 \leq \Re(\lambda) \leq 0$.
Step 2. We look at the roots of f_0. From (3.66), f_0 has one family of roots that we denote λ_k^0.

$$f_0(\lambda) = 0 \Leftrightarrow e^{2i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})} - 1 = 0$$

Hence

$$2i\left(\frac{2}{2-\gamma} i\lambda - \nu_\gamma\frac{\pi}{2} - \frac{\pi}{4}\right) = 2ik\pi, \quad k \in \mathbf{Z},$$

i.e.,

$$\lambda_k^0 = -\frac{2-\gamma}{2} i\left(k + \frac{\nu_\gamma}{2} + \frac{1}{4}\right)\pi, \quad k \in \mathbf{Z}.$$

We will now use Rouché's Theorem. Let $B_k(\lambda_k^0, r_k)$ be the ball of centrum λ_k^0 and radius $r_k = \dfrac{1}{k^{(1-\tilde{\alpha})/2}}$ and $\lambda \in \partial B_k$ (i.e $\lambda = \lambda_k^0 + r_k e^{i\theta}, \theta \in [0, 2\pi]$). Then we successively have:

$$f_0(\lambda) = \frac{4}{2-\lambda} r_k e^{i\theta} + O(r_k^2).$$

It follows that there exists a positive constant c such that

$$\forall \lambda \in \partial B_k, |f_0(\lambda)| \geq c r_k = \frac{c}{k^{(1-\tilde{\alpha})/2}}.$$

Then we deduce from (3.67) that $|\tilde{f}(\lambda) - f_0(\lambda)| = O\left(\frac{1}{\lambda^{(1-\tilde{\alpha})}}\right) = O\left(\frac{1}{k^{(1-\tilde{\alpha})}}\right)$. It follows that, for k large enough

$$\forall \lambda \in \partial B_k, |\tilde{f}(\lambda) - f_0(\lambda)| < |f_0(\lambda)|,$$

Then $\tilde{f}$ and f_0 have the same number of zeros in B_k. Consequently, there exists a subsequence of roots of $\tilde{f}$ which tends to the roots λ_k^0 of f_0. Equivalently, it means that there exists $N \in \mathbb{N}$ and a subsequence $\{\lambda_k\}_{|k|\geq N}$ of roots of $f(\lambda)$, such that $\lambda_k = \lambda_k^0 + o(1)$ which tends to the roots $-\dfrac{2-\gamma}{2} i\left(k + \dfrac{\nu_\gamma}{2} + \dfrac{1}{4}\right)\pi$ of f_0. Finally for $|k| \geq N, \lambda_k$ is simple since λ_k^0 is.

Step 3. From Step 2, we can write

$$\lambda_k = -\frac{2-\gamma}{2} i \left(k + \frac{\nu_\gamma}{2} + \frac{1}{4}\right)\pi + \varepsilon_k. \tag{3.68}$$

Using (3.68), we get

$$\begin{aligned} e^{2i\left(\frac{2}{2-\gamma} i\lambda_k - \nu_\gamma \frac{\pi}{2} - \frac{\pi}{4}\right)} &= e^{-\frac{4}{2-\gamma}\varepsilon_k} \\ &= 1 - \tfrac{4}{2-\gamma}\varepsilon_k + O(\varepsilon_k^2). \end{aligned} \tag{3.69}$$

Substituting (3.69) into (3.66), using that $\tilde{f}(\lambda_k) = 0$, we get:

$$\tilde{f}(\lambda_k) = -\frac{4}{2-\gamma}\varepsilon_k - \rho\tilde{\rho}\frac{2}{(-\frac{2-\gamma}{2} ik\pi)^{1-\tilde{\alpha}}} + o(\varepsilon_k) + o\left(\frac{1}{k^{1-\tilde{\alpha}}}\right) = 0 \tag{3.70}$$

and hence

$$\varepsilon_k = -\left(\frac{2-\gamma}{2}\right)^{\tilde{\alpha}} \frac{\rho\tilde{\rho}}{(k\pi)^{1-\tilde{\alpha}}}(\cos(1-\tilde{\alpha})\frac{\pi}{2} + i\sin(1-\tilde{\alpha})\frac{\pi}{2}) + o\left(\frac{1}{k^{1-\tilde{\alpha}}}\right). \tag{3.71}$$

From (3.71) we have in that case $|k|^{1-\tilde{\alpha}}\Re\lambda_k \sim \beta$ with

$$\beta = -\left(\frac{2-\gamma}{2}\right)^{\tilde{\alpha}} \frac{\rho\tilde{\rho}}{\pi^{1-\tilde{\alpha}}} \cos(1-\tilde{\alpha})\frac{\pi}{2}.$$

- $\alpha > 2\nu_\gamma$ and $\alpha + \tilde{\alpha} < 1 + 2\nu_\gamma$.
  From (3.61), our aim is to solve the equation

$$f(\lambda) = (1-\gamma)\tilde{c}^+_{\nu_\gamma,0} J_{-\nu_\gamma}\left(\frac{2}{2-\gamma} i\lambda\right) + \varrho\lambda(\lambda+\eta)^{\alpha-1}\tilde{c}^-_{\nu_\gamma,0} J_{\nu_\gamma}\left(\frac{2}{2-\gamma} i\lambda\right) = 0$$

Then

$$f(\lambda) = -\left(\frac{2}{\pi\tilde{z}}\right)^{1/2} \lambda^{1+\alpha-\nu_\gamma} c^-_{\nu_\gamma,0}\left(\frac{2}{2-\gamma} i\right)^{-\nu_\gamma} \frac{e^{-i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}}{2}\tilde{f}(\lambda), \tag{3.72}$$

where

$$\begin{aligned} \tilde{f}(\lambda) &= (e^{2i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})} - 1) + (1-\gamma)\left(\frac{2}{2-\gamma} i\right)^{2\nu_\gamma} \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}} \frac{e^{2i(\tilde{z}-\frac{\pi}{4})} - e^{-i\nu_\gamma\pi}}{\lambda^{\alpha-2\nu_\gamma}} + o\left(\frac{1}{\lambda^{\alpha-2\nu_\gamma}}\right) \\ &= f_0(\lambda) + \tfrac{f_1(\lambda)}{\lambda^{\alpha-2\nu_\gamma}} + o\left(\tfrac{1}{\lambda^{\alpha-2\nu_\gamma}}\right). \end{aligned} \tag{3.73}$$

where

$$f_0(\lambda) = e^{2i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})} - 1. \tag{3.74}$$

$$f_1(\lambda) = (1-\gamma)\left(\frac{2}{2-\gamma} i\right)^{2\nu_\gamma} \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}(e^{2i(\tilde{z}-\frac{\pi}{4})} - e^{-i\nu_\gamma\pi}). \tag{3.75}$$

we can write

$$\lambda_k = -\frac{2-\gamma}{2} i \left(k + \frac{\nu_\gamma}{2} + \frac{1}{4}\right)\pi + \varepsilon_k. \tag{3.76}$$

and substituting into (3.66) and using that $\tilde{f}(\lambda_k)=0$, we get:

$$(3.77)\quad \tilde{f}(\lambda_k) = -\frac{4}{2-\gamma}\varepsilon_k + (1-\gamma)\left(\frac{2}{2-\gamma}i\right)^{2\nu_\gamma}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\frac{2i\sin\nu_\gamma\pi}{(-\frac{2-\gamma}{2}ik\pi)^{\alpha-2\nu_\gamma}} + o(\varepsilon_k) + o\left(\frac{1}{k^{\alpha-2\nu_\gamma}}\right) = 0$$

and hence

$$(3.78)\qquad \varepsilon_k = -(1-\gamma)\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\left(\frac{2-\gamma}{2}\right)^{1-\alpha}\frac{\sin\nu_\gamma\pi}{(k\pi)^{\alpha-2\nu_\gamma}}(\sin\alpha\frac{\pi}{2} - i\cos\alpha\frac{\pi}{2}) + o\left(\frac{1}{k^{\alpha-2\nu_\gamma}}\right).$$

From (3.71) we have in that case $|k|^{\alpha-2\nu_\gamma}\Re\lambda_k \sim \beta$ with

$$\beta = -(1-\gamma)\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\left(\frac{2-\gamma}{2}\right)^{1-\alpha}\frac{\sin\nu_\gamma\pi\sin\alpha\frac{\pi}{2}}{\pi^{\alpha-2\nu_\gamma}}.$$

• $\alpha < 2\nu_\gamma$ and $\tilde{\alpha} + 2\nu_\gamma > 1+\alpha$.

step 1.

$$(3.79)\qquad f(\lambda) = -\left(\frac{2}{\pi\tilde{z}}\right)^{1/2}(1-\gamma)\lambda^{1+\nu_\gamma}c^+_{\nu_\gamma,0}\left(\frac{2}{2-\gamma}i\right)^{\nu_\gamma}\frac{e^{-i(\tilde{z}+\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}}{2}\tilde{f}(\lambda),$$

$$(3.80)\qquad \begin{aligned}\tilde{f}(\lambda) &= (e^{2i(\tilde{z}+\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}-1) - \tilde{\varrho}\frac{e^{2i(\tilde{z}+\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}+1}{\lambda^{1-\tilde{\alpha}}} + o\left(\frac{1}{\lambda^{1-\tilde{\alpha}}}\right)\\ &= f_0(\lambda) + \frac{f_1(\lambda)}{\lambda^{1-\tilde{\alpha}}} + o\left(\frac{1}{\lambda^{1-\tilde{\alpha}}}\right),\end{aligned}$$

where

$$(3.81)\qquad f_0(\lambda) = e^{2i(\tilde{z}+\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})} - 1.$$

$$(3.82)\qquad f_1(\lambda) = -\tilde{\varrho}(e^{2i(\tilde{z}+\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}+1).$$

We look at the roots of f_0. From (3.89), f_0 has one family of roots that we denote λ_k^0.

$$2i\left(\frac{2}{2-\gamma}i\lambda + \nu_\gamma\frac{\pi}{2} - \frac{\pi}{4}\right) = 2k\pi i, \quad k\in\mathbf{Z},$$

i.e.,

$$\lambda_k^0 = -\frac{2-\gamma}{2}i\left(k - \frac{\nu_\gamma}{2} + \frac{1}{4}\right)\pi, \quad k\in\mathbf{Z}.$$

Step 2. From Step 1, we can write

$$(3.83)\qquad \lambda_k = -\frac{2-\gamma}{2}i\left(k - \frac{\nu_\gamma}{2} + \frac{1}{4}\right)\pi + \varepsilon_k.$$

Using (3.91), we get

$$(3.84)\qquad \begin{aligned}e^{2i\left(\frac{2}{2-\gamma}i\lambda_k + \nu_\gamma\frac{\pi}{2} - \frac{\pi}{4}\right)} &= e^{-\frac{4}{2-\gamma}\varepsilon_k}\\ &= 1 - \tfrac{4}{2-\gamma}\varepsilon_k + O(\varepsilon_k^2).\end{aligned}$$

Substituting (3.92) into (3.88), using that $\tilde{f}(\lambda_k)=0$, we get:

$$(3.85)\qquad \tilde{f}(\lambda_k) = -\frac{4}{2-\gamma}\varepsilon_k - \frac{2\tilde{\varrho}}{(-\frac{2-\gamma}{2}ik\pi)^{1-\tilde{\alpha}}} + o(\varepsilon_k) + o\left(\frac{1}{k^{1-\tilde{\alpha}}}\right) = 0$$

and hence

$$
\begin{aligned}
\varepsilon_k &= -\left(\tfrac{2-\gamma}{2}\right)^{\tilde\alpha}\frac{\tilde\varrho}{(k\pi)^{1-\tilde\alpha}}(-i)^{\tilde\alpha-1}+o\left(\frac{1}{k^{1-\tilde\alpha}}\right)\\
&= -\left(\tfrac{2-\gamma}{2}\right)^{\tilde\alpha}\frac{\tilde\varrho}{(k\pi)^{1-\tilde\alpha}}(\cos(1-\tilde\alpha)\frac{\pi}{2}+i\sin(1-\tilde\alpha)\frac{\pi}{2})+o\left(\frac{1}{k^{1-\tilde\alpha}}\right)
\end{aligned}
\tag{3.86}
$$

From (3.94) we have in that case $|k|^{1-\tilde\alpha}\Re\lambda_k\sim\beta$ with

$$\beta=-\left(\frac{2-\gamma}{2}\right)^{\tilde\alpha}\frac{\tilde\varrho}{\pi^{1-\tilde\alpha}}\cos(1-\tilde\alpha)\frac{\pi}{2}.$$

• $\alpha<2\nu_\gamma$ and $\tilde\alpha+2\nu_\gamma<1+\alpha$.

step 1.

$$f(\lambda)=-\left(\frac{2}{\pi\tilde z}\right)^{1/2}(1-\gamma)\lambda^{1+\nu_\gamma}c^+_{\nu_\gamma,0}\left(\frac{2}{2-\gamma}i\right)^{\nu_\gamma}\frac{e^{-i(\tilde z+\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}}{2}\tilde f(\lambda), \tag{3.87}$$

$$
\begin{aligned}
\tilde f(\lambda) &= (e^{2i(\tilde z+\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}-1)+\left(\frac{2}{2-\gamma}i\right)^{-2\nu_\gamma}\frac{1}{1-\gamma}\frac{c^-_{\nu_\gamma,0}}{c^+_{\nu_\gamma,0}}\frac{e^{2i(\tilde z-\frac{\pi}{4})}-e^{i\nu_\gamma\pi}}{\lambda^{2\nu_\gamma-\alpha}}+o\left(\frac{1}{\lambda^{1-\tilde\alpha}}\right)\\
&= f_0(\lambda)+\tfrac{f_1(\lambda)}{\lambda^{1-\tilde\alpha}}+o\left(\tfrac{1}{\lambda^{1-\tilde\alpha}}\right),
\end{aligned}
\tag{3.88}
$$

where

$$f_0(\lambda)=e^{2i(\tilde z+\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}-1. \tag{3.89}$$

$$f_1(\lambda)=-\tilde\varrho(e^{2i(\tilde z+\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}+1). \tag{3.90}$$

We look at the roots of f_0. From (3.89), f_0 has one family of roots that we denote λ_k^0.

$$2i\left(\frac{2}{2-\gamma}i\lambda+\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4}\right)=2k\pi i,\quad k\in\mathbf{Z},$$

i.e.,

$$\lambda_k^0=-\frac{2-\gamma}{2}i\left(k-\frac{\nu_\gamma}{2}+\frac{1}{4}\right)\pi,\quad k\in\mathbf{Z}.$$

Step 2. From Step 1, we can write

$$\lambda_k=-\frac{2-\gamma}{2}i\left(k-\frac{\nu_\gamma}{2}+\frac{1}{4}\right)\pi+\varepsilon_k. \tag{3.91}$$

Using (3.91), we get

$$
\begin{aligned}
e^{2i\left(\frac{2}{2-\gamma}i\lambda_k+\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4}\right)} &= e^{-\frac{4}{2-\gamma}\varepsilon_k}\\
&= 1-\tfrac{4}{2-\gamma}\varepsilon_k+O(\varepsilon_k^2).
\end{aligned}
\tag{3.92}
$$

Substituting (3.92) into (3.88), using that $\tilde f(\lambda_k)=0$, we get:

$$\tilde f(\lambda_k)=-\frac{4}{2-\gamma}\varepsilon_k-\frac{2\tilde\varrho}{(-\frac{2-\gamma}{2}ik\pi)^{1-\tilde\alpha}}+o(\varepsilon_k)+o\left(\frac{1}{k^{1-\tilde\alpha}}\right)=0 \tag{3.93}$$

and hence

$$\begin{aligned} \varepsilon_k &= -\left(\tfrac{2-\gamma}{2}\right)^{\tilde{\alpha}} \frac{\tilde{\varrho}}{(k\pi)^{1-\tilde{\alpha}}}(-i)^{\tilde{\alpha}-1} + o\left(\frac{1}{k^{1-\tilde{\alpha}}}\right) \\ &= -\left(\tfrac{2-\gamma}{2}\right)^{\tilde{\alpha}} \frac{\tilde{\varrho}}{(k\pi)^{1-\tilde{\alpha}}}(\cos(1-\tilde{\alpha})\frac{\pi}{2} + i\sin(1-\tilde{\alpha})\frac{\pi}{2}) + o\left(\frac{1}{k^{1-\tilde{\alpha}}}\right) \end{aligned} \tag{3.94}$$

From (3.94) we have in that case $|k|^{1-\tilde{\alpha}}\Re\lambda_k \sim \beta$ with

$$\beta = -\left(\frac{2-\gamma}{2}\right)^{\tilde{\alpha}} \frac{\tilde{\varrho}}{\pi^{1-\tilde{\alpha}}}\cos(1-\tilde{\alpha})\frac{\pi}{2}.$$

• $\alpha = 2\nu_\gamma$.

step 1.

$$f(\lambda) = -\left(\frac{2}{\pi\tilde{z}}\right)^{1/2}(1-\gamma)\lambda^{1+\nu_\gamma}c^+_{\nu_\gamma,0}\left(\frac{2}{2-\gamma}i\right)^{\nu_\gamma}\frac{e^{-i(\tilde{z}+\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}}{2}\tilde{f}(\lambda), \tag{3.95}$$

$$\begin{aligned} \tilde{f}(\lambda) &= (e^{2i(\tilde{z}+\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}-1) + \left(\frac{2}{2-\gamma}i\right)^{-2\nu_\gamma}\frac{1}{1-\gamma}\frac{c^-_{\nu_\gamma,0}}{c^+_{\nu_\gamma,0}}(e^{2i(\tilde{z}-\frac{\pi}{4})}-e^{i\nu_\gamma\pi}) + o\left(\frac{1}{\lambda^{1-\tilde{\alpha}}}\right) \\ &= f_0(\lambda) + O\left(\tfrac{1}{\lambda^{1-\tilde{\alpha}}}\right), \end{aligned} \tag{3.96}$$

We look at the roots of f_0. From (3.66), f_0 has one family of roots that we denote λ_k^0.

$$f_0(\lambda) = 0 \Leftrightarrow e^{2i\tilde{z}} = i\frac{1+\tilde{A}}{e^{i\nu_\gamma\pi}+\tilde{A}e^{-i\nu_\gamma\pi}},$$

where

$$\tilde{A} = \frac{1}{1-\gamma}\left(\frac{2}{2-\gamma}\right)^{-2\nu_\gamma}\frac{c^-_{\nu_\gamma,0}}{c^+_{\nu_\gamma,0}}.$$

Let us set $\lambda = x+iy$. Then, we have

$$\begin{cases} e^{-\frac{4}{2-\gamma}x} = \dfrac{1+\tilde{A}}{\sqrt{1+\tilde{A}^2+2\tilde{A}\cos 2\nu_\gamma\pi}}, \\ -\frac{4}{2-\gamma}y = 2k\pi + \dfrac{\pi}{2} + \theta, \quad k \in \mathbf{Z}, \end{cases}$$

where θ is such that

$$\begin{cases} \cos\theta = \dfrac{(1+\tilde{A})\cos\nu_\gamma\pi}{\sqrt{1+\tilde{A}^2+2\tilde{A}\cos 2\nu_\gamma\pi}}, \\ \sin\theta = \dfrac{(1-\tilde{A})\sin\nu_\gamma\pi}{\sqrt{1+\tilde{A}^2+2\tilde{A}\cos 2\nu_\gamma\pi}}. \end{cases}$$

Hence

$$\begin{cases} x = -\frac{2-\gamma}{4}\ln\dfrac{1+\tilde{A}}{\sqrt{1+\tilde{A}^2+2\tilde{A}\cos 2\nu_\gamma\pi}}, \\ y = -\frac{2-\gamma}{4}(2k\pi + \dfrac{\pi}{2} + \theta), \quad k \in \mathbf{Z}. \end{cases}$$

Now with the help of Rouché's Theorem, we conclude.

Next, by an explicit representation of the resolvent of the generator on the imaginary axis and the use of Theorem **??**, we prove an optimal decay rate. Our main result is the following.

Theorem 3.5.3 *If $\eta \neq 0$, then the global solution of the problem (P) has the following energy decay property*

$$E(t) = \|S_{\mathcal{A}}(t)U_0\|^2_{\mathcal{H}} \leq \begin{cases} \dfrac{c}{t^{\frac{2}{1-\tilde{\alpha}}}}\|U_0\|^2_{D(\mathcal{A})} & \text{if } \alpha > 2\nu_\gamma \text{ and } \alpha+\tilde{\alpha} > 1+2\nu_\gamma, \\ \dfrac{c}{t^{\frac{2}{1-\alpha-2\nu_\gamma}}}\|U_0\|^2_{D(\mathcal{A})} & \text{if } \alpha > 2\nu_\gamma \text{ and } \alpha+\tilde{\alpha} < 1+2\nu_\gamma, \\ \dfrac{c}{t^{\frac{2}{1-\tilde{\alpha}}}}\|U_0\|^2_{D(\mathcal{A})} & \text{if } \alpha < 2\nu_\gamma \text{ and } \tilde{\alpha}+2\nu_\gamma > 1+\alpha, \\ \dfrac{c}{t^{\frac{2}{2\nu_\gamma-\alpha}}}\|U_0\|^2_{D(\mathcal{A})} & \text{if } \alpha < 2\nu_\gamma \text{ and } \tilde{\alpha}+2\nu_\gamma < 1+\alpha, \\ ce^{-\omega t}\|U_0\|^2_{\mathcal{H}} & \text{if } \alpha = 2\nu_\gamma. \end{cases}$$

Moreover, the rate of energy decay is optimal for general initial data in $D(\mathcal{A})$.

Proof.
Let us consider the resolvent equation

$$\begin{cases} i\lambda u - v = f_1, \\ i\lambda v - (x^\gamma u_x)_x + \beta u = f_2, \\ i\lambda\phi + (\xi^2+\eta)\phi - v(0)\mu(\xi) = f_3, \\ i\lambda\tilde{\phi} + (\xi^2+\eta)\tilde{\phi} - v(1)\tilde{\mu}(\xi) = f_4. \end{cases} \tag{3.97}$$

where $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$. From $(3.97)_1$ and $(3.97)_2$, we have

$$\lambda^2 u + (x^\gamma u_x)_x - \beta u = -(f_2 + i\lambda f_1) \tag{3.98}$$

with

$$\begin{cases} (x^\gamma u_x)(0) = \zeta \displaystyle\int_{-\infty}^{\infty} \mu(\xi)\phi(\xi)\, d\xi, \\ u_x(1) = -\tilde{\zeta} \displaystyle\int_{-\infty}^{\infty} \tilde{\mu}(\xi)\tilde{\phi}(\xi)\, d\xi, \\ . \end{cases} \tag{3.99}$$

The substitution of ϕ and $\tilde{\phi}$ given by $(3.97)_3$ and $(3.97)_4$ into $(3.99)_1$ and $(3.99)_2$ gives us

$$\begin{cases} (x^\gamma u_x)(0) = \varrho(i\lambda+\eta)^{\alpha-1}v(0) + \zeta \displaystyle\int_{-\infty}^{+\infty} \frac{\mu(\xi)f_3(\xi)}{i\lambda+\xi^2+\eta}\, d\xi, \\ u_x(1) = -\tilde{\varrho}(i\lambda+\eta)^{\tilde{\alpha}-1}v(1) - \tilde{\zeta} \displaystyle\int_{-\infty}^{+\infty} \frac{\tilde{\mu}(\xi)f_4(\xi)}{i\lambda+\xi^2+\eta}\, d\xi, \end{cases} \tag{3.100}$$

Moreover, from $(3.97)_1$, we have

$$\begin{cases} v(0) = i\lambda u(0) - f_1(0), \\ v(1) = i\lambda u(1) - f_1(1), \end{cases}$$

Then, the condition (3.100) become

$$\begin{cases} (x^\gamma u_x)(0) - \varrho i\lambda(i\lambda+\eta)^{\alpha-1}u(0) = -\varrho(i\lambda+\eta)^{\alpha-1}f_1(0) + \zeta \displaystyle\int_{-\infty}^{+\infty} \frac{\mu(\xi)f_3(\xi)}{i\lambda+\xi^2+\eta}\, d\xi, \\ u_x(1) + \tilde{\varrho} i\lambda(i\lambda+\eta)^{\tilde{\alpha}-1}u(1) = \tilde{\varrho}(i\lambda+\eta)^{\tilde{\alpha}-1}f_1(1) - \tilde{\zeta} \displaystyle\int_{-\infty}^{+\infty} \frac{\tilde{\mu}(\xi)f_4(\xi)}{i\lambda+\xi^2+\eta}\, d\xi, \end{cases} \tag{3.101}$$

Assume that Φ is a solution of (3.98), then one easily checks that the function Ψ defined by

$$\Phi(x) = x^{\frac{1-\gamma}{2}}\Psi\left(\frac{2}{2-\gamma}\sqrt{\lambda^2-\beta}x^{\frac{2-\gamma}{2}}\right) \tag{3.102}$$

is solution of the following inhomogeneous Bessel equation:

$$\begin{aligned} &y^2\Psi''(y) + y\Psi'(y) + \left(y^2 - \left(\frac{\gamma-1}{2-\gamma}\right)^2\right)\Psi(y) = \\ &-(\tfrac{2}{2-\gamma})^2(\tfrac{2-\gamma}{2}\tfrac{1}{\tilde\lambda}y)^{\frac{3-\gamma}{2-\gamma}}\left(f_2\left((\tfrac{2-\gamma}{2}\tfrac{1}{\tilde\lambda}y)^{\frac{2}{2-\gamma}}\right) + i\lambda f_1\left((\tfrac{2-\gamma}{2}\tfrac{1}{\tilde\lambda}y)^{\frac{2}{2-\gamma}}\right)\right). \end{aligned} \tag{3.103}$$

where $\tilde\lambda = \sqrt{\lambda^2-\beta}$. The solution can be written as

$$\Psi(y) = AJ_{\nu_\gamma}(y) + BJ_{-\nu_\gamma}(y) - \frac{\pi}{2\sin\nu_\gamma\pi}\int_0^y \frac{f(s)}{s}\left(J_{\nu_\gamma}(s)J_{-\nu_\gamma}(y) - J_{\nu_\gamma}(y)J_{-\nu_\gamma}(s)\right)ds,$$

where

$$f(s) = -(\frac{2}{2-\gamma})^2(\frac{2-\gamma}{2}\frac{1}{\tilde\lambda}s)^{\frac{3-\gamma}{2-\gamma}}\left(f_2\left((\frac{2-\gamma}{2}\frac{1}{\tilde\lambda}s)^{\frac{2}{2-\gamma}}\right) + i\lambda f_1\left((\frac{2-\gamma}{2}\frac{1}{\tilde\lambda}s)^{\frac{2}{2-\gamma}}\right)\right)$$

Thus,

$$\begin{aligned} &u(x) = Ax^{\frac{1-\gamma}{2}}J_{\nu_\gamma}\left(\tfrac{2}{2-\gamma}\tilde\lambda x^{\frac{2-\gamma}{2}}\right) + Bx^{\frac{1-\gamma}{2}}J_{-\nu_\gamma}\left(\tfrac{2}{2-\gamma}\tilde\lambda x^{\frac{2-\gamma}{2}}\right) \\ &+\frac{\pi}{2\sin\nu_\gamma\pi}\left(\frac{2}{2-\gamma}\right)x^{\frac{1-\gamma}{2}}\int_0^x s^{\frac{1-\gamma}{2}}(f_2(s)+i\lambda f_1(s))\left(J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\tilde\lambda s^{\frac{2-\gamma}{2}}\right)J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\tilde\lambda x^{\frac{2-\gamma}{2}}\right)\right. \\ &\left.-J_{\nu_\gamma}\left(\tfrac{2}{2-\gamma}\tilde\lambda x^{\frac{2-\gamma}{2}}\right)J_{-\nu_\gamma}\left(\tfrac{2}{2-\gamma}\tilde\lambda s^{\frac{2-\gamma}{2}}\right)\right)ds. \end{aligned}$$

Therefore,

$$\begin{aligned} u(x) &= A\Phi_+(x) + B\Phi_-(x) \\ &+\frac{\pi}{2\sin\nu_\gamma\pi}\left(\frac{2}{2-\gamma}\right)\int_0^x (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi_-(x) - \Phi_+(x)\Phi_-(s))\,ds, \end{aligned} \tag{3.104}$$

where Φ_+ and Φ_- are defined by

$$\Phi_+(x) = x^{\frac{1-\gamma}{2}}J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\tilde\lambda x^{\frac{2-\gamma}{2}}\right), \quad \Phi_-(x) = x^{\frac{1-\gamma}{2}}J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\tilde\lambda x^{\frac{2-\gamma}{2}}\right). \tag{3.105}$$

Then

$$\begin{aligned} u_x(x) &= A\Phi'_+(x) + B\Phi'_-(x) \\ &+\frac{\pi}{2\sin\nu_\gamma\pi}\left(\frac{2}{2-\gamma}\right)\int_0^x (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi'_-(x) - \Phi'_+(x)\Phi_-(s))\,ds. \end{aligned} \tag{3.106}$$

From (3.101), (3.106) and (3.104), we conclude that

$$(1-\gamma)\tilde c^+_{\nu_\gamma,0}A - \varrho i\lambda(i\lambda+\eta)^{\alpha-1}\tilde c^-_{\nu_\gamma,0}B = -\varrho(i\lambda+\eta)^{\alpha-1}f_1(0) + \zeta\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_3(\xi)}{i\lambda+\xi^2+\eta}\,d\xi \tag{3.107}$$

$$
\begin{aligned}
&A(\Phi_+'(1) + \tilde{\varrho} i\lambda(i\lambda+\eta)^{\tilde{\alpha}-1}\Phi_+(1)) + B(\Phi_-'(1) + \tilde{\varrho} i\lambda(i\lambda+\eta)^{\tilde{\alpha}-1}\Phi_-(1)) = \\
&\qquad \tilde{\varrho}(i\lambda+\eta)^{\tilde{\alpha}-1}f_1(0) - \tilde{\zeta}\int_{-\infty}^{+\infty}\frac{\tilde{\mu}(\xi)f_4(\xi)}{i\lambda+\xi^2+\eta}\,d\xi \\
&-\tilde{\varrho} i\lambda(i\lambda+\eta)^{\tilde{\alpha}-1}\tfrac{\pi}{(2-\gamma)\sin\nu_\gamma\pi}\int_0^1 (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi_-(1)-\Phi_+(1)\Phi_-(s))\,ds, \\
&-\tfrac{\pi}{(2-\gamma)\sin\nu_\gamma\pi}\int_0^1 (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi_-'(x)-\Phi_+'(x)\Phi_-(s))\,ds
\end{aligned}
\tag{3.108}
$$

where

$$
\tilde{c}^+_{\nu_\gamma,m} = c^+_{\nu_\gamma,m}\left(\frac{2}{2-\gamma}\tilde{\lambda}\right)^{2m+\nu_\gamma}, \quad \tilde{c}^+_{\nu_\gamma,m} = c^-_{\nu_\gamma,m}\left(\frac{2}{2-\gamma}\tilde{\lambda}\right)^{2m-\nu_\gamma}
$$

and

$$
\Phi_+(1) = J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\tilde{\lambda}\right), \qquad \Phi_-(1) = J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\tilde{\lambda}\right).
$$

Using (3.107) and (3.108), a linear system in A and B is obtained

$$
\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} C \\ \tilde{C} \end{pmatrix}, \tag{3.109}
$$

where

$$
\begin{aligned}
r_{11} &= (1-\gamma)\tilde{c}^+_{\nu_\gamma,0}, \\
r_{12} &= -\varrho i\lambda(i\lambda+\eta)^{\alpha-1}\tilde{c}^-_{\nu_\gamma,0}, \\
r_{21} &= \Phi_+'(1) + \tilde{\varrho} i\lambda(i\lambda+\eta)^{\tilde{\alpha}-1}\Phi_+(1), \\
r_{22} &= \Phi_-'(1) + \tilde{\varrho} i\lambda(i\lambda+\eta)^{\tilde{\alpha}-1}\Phi_-(1),
\end{aligned}
$$

$$
\begin{aligned}
C &= -\varrho(i\lambda+\eta)^{\alpha-1}f_1(0) + \zeta\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_3(\xi)}{i\lambda+\xi^2+\eta}\,d\xi, \\
\tilde{C} &= \tilde{\varrho}(i\lambda+\eta)^{\tilde{\alpha}-1}f_1(0) - \tilde{\zeta}\int_{-\infty}^{+\infty}\frac{\tilde{\mu}(\xi)f_4(\xi)}{i\lambda+\xi^2+\eta}\,d\xi \\
&\quad -\tilde{\varrho} i\lambda(i\lambda+\eta)^{\tilde{\alpha}-1}\tfrac{\pi}{(2-\gamma)\sin\nu_\gamma\pi}\int_0^1 (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi_-(1)-\Phi_+(1)\Phi_-(s))\,ds, \\
&\quad -\tfrac{\pi}{(2-\gamma)\sin\nu_\gamma\pi}\int_0^1 (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi_-'(x)-\Phi_+'(x)\Phi_-(s))\,ds
\end{aligned}
$$

Let the determinant of the linear system given in (3.109) be denoted by D. Then

$$
\begin{aligned}
D &= (1-\gamma)\tilde{c}^+_{\nu_\gamma,0}\left[-\tilde{\lambda}J_{1-\nu_\gamma}\left(\tfrac{2}{2-\gamma}\tilde{\lambda}\right) + \tilde{\varrho} i\lambda(i\lambda+\eta)^{\tilde{\alpha}-1}J_{-\nu_\gamma}\left(\tfrac{2}{2-\gamma}\tilde{\lambda}\right)\right] \\
&+\varrho i\lambda(i\lambda+\eta)^{\alpha-1}\tilde{c}^-_{\nu_\gamma,0}\left[-\tilde{\lambda}J_{1+\nu_\gamma}\left(\tfrac{2}{2-\gamma}\tilde{\lambda}\right) + ((1-\gamma)\tilde{\varrho} i\lambda(i\lambda+\eta)^{\tilde{\alpha}-1})J_{\nu_\gamma}\left(\tfrac{2}{2-\gamma}\tilde{\lambda}\right)\right]
\end{aligned}
$$

As $D \neq 0$ for all $\lambda \neq 0$, then A and B are uniquely determined by (3.109).

Now, we consider the case $\alpha = 2\nu_\gamma$, then

$$
|D| \geq c|\lambda|^{\nu_\gamma+1/2} \text{ for large } \lambda. \tag{3.110}
$$

Now

$$
A = \frac{1}{D}(Cr_{22} - \tilde{C}r_{12}),
$$

$$B = \frac{1}{D}(-Cr_{21} + \tilde{C}r_{11}).$$

Considering only the dominant terms of λ, the following is obtained:

$$\begin{array}{rcl} |D||A| & \leq & c_1|\lambda|^{\alpha-\frac{1}{2}} + c_2|\lambda|^{\alpha+\tilde{\alpha}-\nu_\gamma-1}, \\ |D||B| & \leq & c_1|\lambda|^{\alpha-\frac{1}{2}} + c_2|\lambda|^{\tilde{\alpha}+\nu_\gamma-1} \end{array}$$

Then, we conclude that

$$|A| \leq c|\lambda|^{\nu_\gamma-1} + c'|\lambda|^{\tilde{\alpha}-\frac{3}{2}},$$

$$|B| \leq c|\lambda|^{\nu_\gamma-1} + c'|\lambda|^{\tilde{\alpha}-\frac{3}{2}}.$$

Then

$$\|u\|_{L^2(0,1)} \leq (c|\lambda|^{\nu_\gamma-\frac{3}{2}} + c'|\lambda|^{\tilde{\alpha}-2}) \left(\|f_1\|_{H^1_{0,\gamma}(0,1)} + \|f_2\|_{L^2(0,1)}\right)$$

Using $(3.97)_1$ and (3.104), we get

$$\|v\|_{L^2(0,1)} \leq (c|\lambda|^{\nu_\gamma-\frac{1}{2}} + c'|\lambda|^{\tilde{\alpha}-1}) \left(\|f_1\|_{H^1_{0,\gamma}(0,1)} + \|f_2\|_{L^2(0,1)}\right).$$

From (3.105) and (3.60), we have

$$\begin{cases} x^{\gamma/2}\Phi'_+(x) & = & (\frac{1-\gamma}{2} + \frac{2\nu_\gamma}{2-\gamma})x^{-1/2}J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\tilde{\lambda}\right) - \tilde{\lambda}x^{\frac{1-\gamma}{2}}J_{1+\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda\right), \\ x^{\gamma/2}\Phi'_-(x) & = & (\frac{1-\gamma}{2} - \frac{2\nu_\gamma}{2-\gamma})x^{-1/2}J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\tilde{\lambda}\right) - \tilde{\lambda}x^{\frac{1-\gamma}{2}}J_{1-\nu_\gamma}\left(\frac{2}{2-\gamma}\tilde{\lambda}\right). \end{cases}$$

Then from (3.106), we can get

$$\|x^{\gamma/2}u_x\|_{L^2(0,1)} \leq\leq (c|\lambda|^{\nu_\gamma-\frac{1}{2}} + c'|\lambda|^{\tilde{\alpha}-1}) \left(\|f_1\|_{H^1_{0,\gamma}(0,1)} + \|f_2\|_{L^2(0,1)}\right)$$

Now, taking inner product of (3.97) with U in $\mathcal{H}$ and using (3.15) we get

$$|Re\langle \mathcal{A}U, U\rangle| \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

This implies that

$$\zeta \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi(\xi)|^2 \, d\xi \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{3.111}$$

Since $\eta > 0$, we have

$$\|\phi\|^2_{L^2(-\infty,\infty)} \leq \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi(\xi)|^2 \, d\xi \leq c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

Thus, we conclude that

$$\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{H}} \leq C.$$

The conclusion then follows by applying Theorem Theorem 3.5.1. By a similar way, we prove the other cases.

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الملخص

في هذه الأطروحة ، ندرس الوجود العالمي والسلوك المقارب لحلول معادلة الموجة المتدهورة باستخدام تحكم حدودي من النوع الكسري أو تبديد حد ديناميكي من النوع المشتق. الأدوات المستخدمة هي طريقة التحليل الطيفي ، والمجموعة النصفية ، ومجموعة C_0-semigroup ، ونظرية Borichev و Tomilov ، ونظرية Hille-Yosda ، ونظرية Rouché. أولاً ، نحن مهتمون بدراسة استقرار معادلة الموجة أحادية البعد المتدهورة بشكل ضعيف u_tt - 〚(x ^ γ u_x)〛 _ x = 0 مع x∈ (0،1) و γ∈ [0 ؛ 1) ، متحكم فيها من خلال رد فعل كسري عند الحد الذي يعمل عند x = 0. يتم الحصول على استقرار قوي وموحد وغير منتظم مع تقدير صريح لانحلال الطاقة في المساحات المناسبة. يتم الحصول على النتائج من خلال تقدير لمحلل المولد المرتبط بشبه المجموعة. باستخدام طريقة طيفية ، نحدد معدل الانحلال الأمثل لطاقة النظام. بعد ذلك ، نعتبر معادلة موجة متدهورة بشرط تحكم حدودي من نوع مشتق كسري. أظهرنا أن المشكلة ليست مستقرة بشكل موحد بالطريقة الطيفية ودرسنا الاستقرار متعدد الحدود باستخدام نظرية المشغلين الخطيين على أساس نصف المجموعة. أخيرًا ، نحن مهتمون بدراسة الوجود العالمي لحلول المعادلات أحادية البعد المتدهورة بشكل ضعيف u_tt - 〚(x ^ γ u_x)〛 _ x = 0 مع x∈ (0،1) و γ∈ [0 ؛ 1) ، مع شرط تحكم حد ديناميكي لنوع مشتق كسري.

الكلمات المفتاحية : معادلة الموجة المنحلة ، التبديد الديناميكي للحدود لنوع المشتق الكسري ، معدل الانحلال الأمثل ، دوال بيسيل ، فحص الحدود الجزئية ، الاستقرار متعدد الحدود ، 〚C〛 _0-semigroup

Résumé

Dans cette thèse ,nous étudions l'éxistence globale et le comportement asymptotique de solutions de l'équation des ondes dégénérée avec un contrôle frontière de type fractionnaire ou dissipation frontière dynamique de type dérivé fractionnaire . Les outils utilisées sont méthode d'analyse spectrale, semi groupe, $C_0 - semigroupe$, le théorème de Borichev et Tomilov, théorème de Hille-Yosda et le théorème de Rouché. Premièrement, nous nous intéressons à l'étude de la stabilisation d'équation d'onde unidimensionnelle faiblement dégénérée $u_{tt} - (x^\gamma u_x)_x = 0$ avec $x \in (0,1) et\ \gamma \in [0;1)$, controlée par un feedback fractionnaire au bord agissant à x=0 .

Stabilisation forte, uniforme et non uniforme sont obtenus avec une estimation explicite de la décroissance de l'énergie dans des espaces appropriés. Les résultats sont obtenus à travers

une estimation de la résolvante du générateur associé au semigroupe. On utilise une méthode spectrale, nous établissons la vitesse de décroissance polynomial optimal de l'énergie du système.

Ensuite, nous considérons une équation d'onde dégénérée avec une condition de contrôle frontière de type dérivé fractionnaire. Nous montrons que le problème n'est pas uniformément stable par une méthode spectrale et nous étudions la stabilité polynomiale à l'aide de la théorie des opérateurs linéaires basée sur le semigroupe. Enfin, nous nous intéressons à l'étude de l'existence globale des solution d'équations unidimensionnelles faiblement dégénérée

$u_{tt} - (x^\gamma u_x)_x = 0$ avec $x \in (0,1) et\ \gamma \in [0;1)$, avec une condition de contrôle frontière dynamique de type dérivé fractionnaire.

Les mots clés :

Equation d'onde dégénérée ,dissipation frontière dynamique de type dérivé fractionnaire , la vitesse de décroissance optimal , fonctions de Bessel , contrôle aux limites fractionnaires , stabilité polynomiale , $C_0 - semigroupe$.

Abstract :

In this thesis, we study the global existence and the asymptotic behavior of solutions of the degenerate wave equation with a fractional type boundary control or fractional derivative type dynamic boundary dissipation. The tools used are spectral analysis method, semigroup, C_0-semigroup, Borichev and Tomilov theorem, Hille-Yosda theorem and Rouché theorem. First, we are interested in the study of the stabilization of weakly degenerate one-dimensional wave equation u_tt- 〚(x^γ u_x)〛 _x=0 with x∈(0,1) and γ∈[0;1) , controlled by a fractional feedback at the boundary acting at x=0. Strong, uniform and non-uniform stabilization are obtained with an explicit estimate of the energy decay in appropriate spaces. The results are obtained through an estimate of the resolvent of the generator associated with the semigroup. Using a spectral method, we establish the optimal polynomial decay rate of the energy of the systemNext, we consider a degenerate wave equation with a boundary control condition of fractional derivative type. We show that the problem is not uniformly stable by a spectral method and we study the polynomial stability using the theory of linear operators based on the semigroup. Finally, we are interested in the study of the global existence of solutions of weakly degenerate one-dimensional equations u_tt- 〚(x^γ u_x)〛 _x=0 with x∈(0,1) and γ∈[0;1), with a .dynamic boundary control condition of fractional derivative type

Keywords : Degenerate wave equation , dynamic boundary dissipation of fractional derivative type , optimal decay rate , Bessel functions , fractional boundary check , polynomial stability , 〚C〛 _0-semigroup