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Publications

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Abstract

Fractional differential equations occur in a variety of areas of biological, physical and engineering applications. Such equations have received much attention in recent years. This thesis discusses the existence of solutions for fractional differential inclusions and random system of fractional differential equations with nonlocal fractional integral boundary conditions. Our results will be obtained by means of fixed points theorems.

Key words and phrases : Differential equations, Hadamard-Caputo fractional derivative, Hadamard fractional integral, fractional differential inclusions, existence, random fractional differential equation, fixed point, vector metric space.

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Contents

Introduction

Fractional calculus is an extension of the classical notions of primitive and derivation of non-zero integer order to any real order. Although fractional derivation has been defined by several approaches with the names of Grunwald-Letnikov, Riemann-Liouville, Caputo. This notion was introduced in the 17th century when Gottfried Leibniz defined the symbol of the derivation of positive integer order, Guillaume l'Hospital questioned him about the possibility of having an order derivative. This question has attracted the attention of mathematicians including Euler or Lagrange in the 18th century followed by Liouville in 1837, Riemann in 1847 as well as Grunwald in 1867 and Letnikov in 1868. For more historical details, one can consult [41, 49].

Recently, fractional differential equations have been shown to be very useful in the study of models of many phenomena in various fields of science and engineering, such as physics, chemistry, biology, signal and image processing, biophysics, blood flow phenomena, control theory, economics, aerodynamics and fitting of experimental data. For examples and recent development of the topic, see [52, 31] and references cited therein. However, the literature on Hadamard type fractional differential equations is not enriched yet. The fractional derivative due to Hadamard, introduced in 1892 [17], differs from the Riemann-Liouville and Caputo type fractional derivatives in the sense that the kernel of the integral in the definition of Hadamard derivative contains logarithmic function of arbitrary exponent. A detailed description of Hadamard fractional derivative and integral can be found in [52, 7, 8] and references cited therein.

The Langevin equation (first formulated by Langevin in 1908) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [12]. For some new developments on the fractional Langevin equation, see, for example, [32, 1].

Probabilistic functional analysis is an important mathematical area of research due to its applications to probabilistic models in applied problems. Random operator theory is needed for the study of various classes of random equations. Indeed, in many cases the mathematical models or equations used to describe phenomena in the biological, physical, engineering and systems sciences contain certain parameters or coefficients which have specific interpretations, but whose values are unknown. Therefore, it is more realistic to consider such equations as random operator equations. These equations are much more difficult to handle mathematically than deterministic equations. Important contributions to the study of the mathematical aspects of such random equations have been undertaken in [5, 58, 43] among others.

In the following we give an outline of our thesis organization.
The first chapter gives some notations, definitions, lemmas and fixed point theorems which are used throughout this thesis.

In chapter 2, we establish the existence of solutions for a problem of Langevin fractional differential inclusions with nonlocal fractional integral conditions via Caputo-Hadamard derivative. Both cases of convex and nonconvex valued right hand sides are considered. We consider a problem of fractional differential inclusions as follows:

$$\begin{cases} D^{\alpha}(D^{\beta}+\lambda)\,x\,(t) & \in & F(t,\,x(t)) \\ \sum\limits_{i=1}^{m}\theta_i I^{\mu_i}x(\eta_i) & = & \sum\limits_{j=1}^{n}\phi_j I^{\gamma_j}x(\xi_j) \\ \sum\limits_{k=1}^{p}\varepsilon_k I^{\varsigma_k}x(\psi_k) & = & \sum\limits_{l=1}^{q}\Xi_l I^{\tau_l}x(\varphi_l) \end{cases} \tag{1}$$

where D^ρ denotes the Hadamard Caputo-type fractional derivative of order $\rho \in \{\alpha, \beta\}$ with $0 < \alpha, \beta < 1, 1 < \alpha + \beta < 2, \lambda$ is a given constant, I^r is the Hadamard fractional integral of order $r > 0, r \in \{\mu_i,\, \gamma_j,\, \varsigma_k,\, \tau_l\}$ the constants $\eta_i,\, \xi_j,\, \psi_k,\, \varphi_l \in (1,\, e)$ and $\theta_i,\, \phi_j,\, \varepsilon_k,\, \Xi_l \in \mathbb{R}$, for all $i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,$
$k = 1, 2, \ldots, p,\ l = 1, 2, \ldots, q$ and $F : [1, e] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a given function.

In chapter 3, we give some variants of random Perov, Schauder, Kras noselskii and Leray-Schauder-type fixed point theorems in generalized Banach spaces. The results are used to prove the existence of solution for random differential equations with initial and boundary conditions.

In the last chapter, we consider the system of random fractional differential equations with boundary conditions in the following form:

$$\begin{cases} D^{\alpha}(D^{\beta} + \lambda_1)\, x\,(t,\omega) &= f(t,\, x(t,\omega),\, y(t,\omega), \omega) \\ D^{\gamma}(D^{\sigma} + \lambda_2)\, y\,(t,\omega) &= g(t,\, x(t,\omega),\, y(t,\omega), \omega) \\ \sum_{i=1}^{m} \theta_i I^{\mu_i} x(\eta_i, \omega) &= \sum_{j=1}^{n} \phi_j I^{\gamma_j} x(\xi_j, \omega) \\ \sum_{k=1}^{p} \varepsilon_k I^{\varsigma_k} x(\psi_k, \omega) &= \sum_{l=1}^{q} \nu_l I^{\tau_l} x(\varphi_l, \omega) \\ \sum_{i=1}^{m} \overline{\theta_i} I^{\overline{\mu_i}} y(\overline{\eta_i}, \omega) &= \sum_{j=1}^{n} \overline{\phi_j} I^{\overline{\gamma_j}} y(\overline{\xi_j}, \omega) \\ \sum_{k=1}^{p} \overline{\varepsilon_k} I^{\overline{\varsigma_k}} y(\overline{\psi_k}, \omega) &= \sum_{l=1}^{q} \overline{\nu_l} I^{\overline{\tau_l}} y(\overline{\varphi_l}, \omega) \end{cases} \tag{2}$$

where D^ρ denotes the Hadamard Caputo-type fractional derivative of order $\rho \in \{\alpha, \beta, \gamma, \sigma\}$ with $0 < \alpha, \beta, \gamma, \sigma < 1, 1 < \alpha + \beta < 2, 1 < \gamma + \sigma < 2,\ \lambda_1, \lambda_2$ are given constants, I^r is the Hadamard fractional integral of order $r > 0,\ r \in \{\mu_i,\, \gamma_j,\, \varsigma_k,\, \tau_l,\, \overline{\mu_i},\, \overline{\gamma_j},\, \overline{\varsigma_k},\, \overline{\tau_l}\}$ the constants $\eta_i,\, \xi_j,\, \psi_k,\, \varphi_l,\, \overline{\eta_i},\, \overline{\xi_j},\, \overline{\psi_k},\, \overline{\varphi_l} \in (1,\, e)$ and $\theta_i,\, \phi_j,\, \varepsilon_k,\, \nu_l,\, \overline{\theta_i},\, \overline{\phi_j},\, \overline{\varepsilon_k},\, \overline{\nu_l} \in \mathbb{R}$, for all $i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,\ k = 1, 2, \ldots, p,\ l = 1, 2, \ldots, q$ and $f, g :$ $[1, e] \times \mathbb{R}^m \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m$ are given fonctions. $(\Omega, \mathcal{A})$ is a measurable space.

Chapter 1

Preliminaries

In this chapter, we introduce notations, definitions and preliminary facts that will be used in the remainder of this survey paper.

1.1 Some Notations and Definitions

Let $J = [1, e]$. By $C(J, \mathbb{R})$, we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\}.$$

$L^1(J, \mathbb{R})$ denotes the Banach space of measurable functions $y : J \to \mathbb{R}$ that are Lebesgue integrable with the norm

$$\|y\|_{L^1} = \int_0^T |y(t)| dt.$$

$AC(J,\mathbb{R})$ denotes the space of functions $y : J \longrightarrow \mathbb{R}$ that are absolutely continuous and $AC^1(J,\mathbb{R})$ is the space of differentiable functions whose first derivative y' is absolutely continuous.

1.2 Multi-valued analysis

Let $(E, \|\cdot\|)$ be a Banach space. We define the following subsets of $\mathcal{P}(E)$:

$P_{cl}(E) = \{Y \in \mathcal{P}(E) : Y \text{ is closed}\}$,

$P_b(E) = \{Y \in \mathcal{P}(E) : Y \text{ is bounded}\}$,

$P_{cp}(E) = \{Y \in \mathcal{P}(E) : Y \text{ is compact}\}$

$P_{cv}(E) = \{Y \in \mathcal{P}(E) : Y \text{ is convex}\}$

$P_{cp,cv}(E) = P_{cp}(E) \cap P_{cv}(E)$.

Definition 1.2.1. *A multivalued map $G : E \to \mathcal{P}(E)$ is said to be convex (closed) valued if $G(x)$ is convex (closed) for all $x \in E$. A multivalued map G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in E for all $B \in P_b(E)$ (i.e. $\sup\limits_{x\in B}\{\sup\{|y| : y \in G(x)\}$ exists).*

Definition 1.2.2. *A multivalued map $G : E \to \mathcal{P}(E)$ is called upper semi-continuous (u.s.c.) on E if for each $x_0 \in E$, the set $G(x_0)$ is a nonempty closed subset of E, and for each open set N of E containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subset N$. G is said to be completely continuous if $G(B)$ is relatively compact for every $B \in P_b(\mathbb{R})$.*

Definition 1.2.3. *Let $G : X \to \mathcal{P}(E)$ be completely continuous with nonempty compact values. Then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \to x_*, y_n \to y_*, y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in E$ such that $x \in G(x)$.*

We denote by $FixG$ the fixed point set of the multivalued operator G.

Definition 1.2.4. *A multivalued map $G : J \to P_{cl}(E)$ is said to be measurable if for every $y \in E$, the function:*

$$t \to d(y, G(t)) = \inf\{\|y - z\| : z \in G(t)\}$$

is measurable.

Definition 1.2.5. *A multivalued map $F : J \times E \to \mathcal{P}(E)$ is said to be Carathéodory if:*

(1) $t \to F(t, u)$ is measurable for each $u \in E$

(2) $u \to F(t, u)$ is upper semicontinuous for almost all $t \in J$.

F is said to be L^1-Carathéodory if $(1), (2)$ and the following condition holds:

(3) For each $q > 0$, there exists $\varphi_q \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t, u)\|_{\mathcal{P}} = \sup\{\|v\| : v \in F(t, u)\} \leq \varphi_q \;\; ; for\ all\ \|u\| \leq q \quad and\ for\ a.e.\, t \in J.$$

For each $y \in C(J, \mathbb{R})$, define the set of selections of F by

$$S_{F,y} = \{v \in L^1(J) : v(t) \in F(t, y(t)) \text{ a.e. } t \in J\}.$$

Lemma 1.2.1.

Let E be a Banach space. Let $F : J \times E \longrightarrow P_{cp,c}(E)$ be an L^1-Carathéodory multivalued map with $S_{F,y} \neq 0$ and let Γ be a linear continuous mapping from $L^1(J, E)$ into $C(J, E)$, then the operator

$$\begin{array}{rcl} \Gamma \circ S_F : C(J, E) & \longrightarrow & P_{cp,c}(C(J, E)), \\ y & \longmapsto & (\Gamma \circ S_F)(y) := \Gamma(S_{F,y}) \end{array}$$

is a closed graph operator in $C(J,E)\times C(J,E)$.

Lemma 1.2.2. *[66] Let G be a completely continuous multivalued map with nonempty compact values, then G is u.s.c. if and only if G has a closed graph.*

Lemma 1.2.3. *(Nonlinear alternative for Kakutani maps).*

Let E *be a Banach space,* C *a closed convex subset of* E, U *an open subset of* C *and* $0\in U$. *Suppose that* $N:U\longrightarrow P_{cp,c}(C)$ *is an upper semicontinuous compact map. Then either*

(i) N *has a fixed point in* U, *or*

(ii) *there is a* $y\in\partial U$ *and* $\lambda\in(0,1)$ *with* $y\in\lambda N(y)$;

Let (E,d) be a metric space induced from the normed space $(\|\cdot\|)$. The function $H_d:\mathcal{P}(E)\times\mathcal{P}(E)\to\mathbb{R}_+\cup\{\infty\}$ given by:

$$H_d(A,B)=\max\{\sup_{a\in A}d(a,B),\sup_{b\in B}d(A,b)\}$$

is known as the Hausdorff-Pompeiu metric.

Definition 1.2.6.

A multivalued operator $N:E\longrightarrow P_{cl}(E)$ *is called*

(i) γ*-Lipschitz if and only if there exists* $\gamma>0$ *such that*

$$H_d(N(x),N(y))\leq\gamma d(x,y),\quad for\,each\,x,y\in E,$$

(ii) *a contraction if and only if it is* γ*-Lipschitz with* $\gamma<1$.

Lemma 1.2.4. *(Covitz and Nadler [10])*

Let (E,d) *be a complete metric space. If* $N:X\longrightarrow P_{cl}(E)$ *is a contraction, then* $FixN\neq\varnothing$.

For more details on multivalued maps see the books of Aubin and Cellina [3], Aubin and Frankowska [13] and Castaing and Valadier [14].

1.3 Fractional Calculus

The Gamma Function

One of the basic tools of fractional calculus is the Gamma function which naturally extends the factorial to positive real numbers (and even to complex numbers with positive real parts).

Definition 1.3.1.

Let $x \in \mathbb{R}^{+} - \{0\}$, the Gamma function is given by:

$$F(x) = \int_{0}^{+\infty} e^{-t} t^{x-1} dt,$$

(this integral is convergent for all r> 0).

Proposition 1.3.1.

For all $x > 0$, and for all $n \in \mathbb{N} - \{0\}$, we have:

1) $\Gamma(0_{+}) = +\infty,$

2) $\Gamma(x+1) = x\Gamma(x),$

3) $\Gamma(n) = (n-1)!,$

4) $\Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\Pi}}{2^{n}},$

5) $\Gamma(x) = \lim\limits_{n \longrightarrow +\infty} \frac{n! n^{x}}{x(x+1)\cdots(x+n)}.$

Special cases :

6) $\Gamma(1) = \Gamma(2) = \int_0^{+\infty} e^{-t} t^{1-1} dx = 1.$

7) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\Pi}.$

The Beta Function

Among the basic functions of fractional compute Beta function, this function plays an important role specially in some combination with Gamma function.

Definition 1.3.2. *The function* $(x, y) \longrightarrow \beta(x, y)$ *,* $Re(x) > 0$*,* $Re(y) > 0$*, defined by* $\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$ *is called the Beta function.*

There is relation between Gamma and Beta functions given in the relation :

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

It should also be mentioned that the Beta function is symmetric, i.e.,

$$\beta(x, y) = \beta(y, x).$$

The Mittag-Leffler Function

The exponential function e^z plays a very important role in the theory of integer-order differential equations. Its one-parameter generalization, the Mittag-Leffler function which is denoted by :

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

Definition 1.3.3. *A two-parameter function of the Mittag-Leffler type is defined by the series expansion :*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}.$$

It follows from the definition that

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z.$$

Theorem 1.3.1. ***(Cauchy formula for repeated integration.)***

Let f be some continuous function on the interval $[a,b]$. The n-th repeated integral of f based at a,

$$f^{(-n)}(t) = \int_a^t \int_a^{s_1} \cdots \int_a^{s_{n-1}} f(s_n)\, ds_n ds_{n-1} \cdots ds_2 ds_1,$$

is given by single integration

$$f^{(-n)}(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s)\, ds.$$

From this formula the definition of fractional integral is constructed, so we can take an integral of any real degree. Replacing $(n-1)!$ by $\Gamma(n)$ and the power in the integrand with some $\alpha \in \mathbb{R}$, we have Riemann-Liouville fractional integral.

Definition 1.3.4. *[22]*

The fractional (arbitrary) order integral of the function $h \in L^1[a,b]$ of order $\alpha \in \mathbb{R}$ is defined by

$$I_a^{(\alpha)} h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds,$$

where Γ *is the gamma function. When* $a = 0$ *we write* $I^{(\alpha)}h(t) = h(t) * \varphi_\alpha(t)$, *where* $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ *for* $t > 0$, *and* $\varphi_\alpha(t) = 0$ *for* $t \leq 0$, *and* $\varphi_\alpha \longrightarrow \delta(t)$ *as* $\alpha \longrightarrow 0$, *where* δ *is the delta function.*

Theorem 1.3.2.

For $h \in L^1[a, b]$, *the fractional Riemann-Liouville integral has the following property:*

$$I_a^{(\alpha)}\left[I_a^{(\beta)}h(x)\right] = I_a^{(\alpha+\beta)}h(x) \; four \, \alpha > 0, \, \beta > 0$$

Proof.

The proof follows directly from the definition

$$I_a^{(\alpha)}\left[I_a^{(\beta)}h(x)\right] = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x \frac{dt}{(s-t)^{\alpha-1}}\int_a^x \frac{h(u)}{(t-u)^{1-\beta}}du.$$

Or $h \in L^1[a, b]$, according to Fubini's theorem and by the change $t = u + s(x - u)$ we get

$$I_a^{(\alpha)}\left[I_a^{(\beta)}h(x)\right] = \frac{\mathcal{B}(\alpha, \beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x \frac{h(u)}{(t-u)^{1-\beta}}du = I_a^{(\alpha+\beta)}h(x)$$

where $\mathcal{B}(\alpha, \beta)$ denotes the Beta function, the proof is complete. □

Proposition 1.3.2.

- $I_a^0 h(t) = h(t)$
- *the integral operator* I_a^0 *is linear.*

Example:

Let $h(t) = (t-a)^m$ where $m > -1$

$$\begin{aligned} I_a^{(\alpha)} h(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)\, ds \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} (s-a)^m\, ds. \end{aligned}$$

Using a change of variable $s = a + (t-a)x$ we obtain,

$$\begin{aligned} I_a^{\alpha} h(t) &= \frac{(t-a)^{m+\alpha}}{\Gamma(\alpha)} \int_a^t (1-x)^{\alpha-1} x^m ds \\ &= \frac{(t-a)^{m+\alpha}}{\Gamma(\alpha)} \mathcal{B}(\alpha, m+1) \\ &= \frac{(t-a)^{m+\alpha}\, \Gamma(\alpha)\, \Gamma(m+1)}{\Gamma(\alpha)\, \Gamma(\alpha+m+1)}, \end{aligned}$$

then

$$I_a^{(\alpha)} h(t) = \frac{\Gamma(m+1)}{\Gamma(\alpha+m+1)} (t-a)^{m+\alpha}.$$

Definition 1.3.5. ***(Riemann-Liouville fractional derivative).***

Let $f \in L^1[a,b]$ be an integrable function on $[a,b]$, the fractional derivative in the Riemann-Liouville sens of the function of f order α is noted ${}^{RL}D_a^{\alpha} f$ and defined by:

$$\begin{aligned} {}^{RL}D_a^{\alpha} f(x) &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-t)^{n-\alpha-1} f(t)\, dt \\ &= \left(\frac{d}{dx}\right)^n \left(I^{n-\alpha} f(t)\right). \end{aligned}$$

with $n > \alpha$ a natural integer.

Notation: We denote the operator D^n, $n \in \mathbb{N}$, differentiation of the operator of integer order i. e:

$$D^n = \frac{d^n}{dt^n}.$$

Remark 1.3.1.

Fractional derivative of Riemann-Liouville is non-commutative i.e:

$$^{RL}D^{m RL}D^{\alpha} f(t) = ^{RL}D^{\alpha+m} f(t) \neq ^{RL}D^{\alpha RL}D^{m} f(t).$$

Theorem 1.3.3.

Let the functions f and g whose Riemann-Liouville fractional derivatives of order α exist. So for $\lambda, \mu \in \mathbb{R}$, $^{RL}D_a^{\alpha}(\lambda f + \mu g)$ exist and we have:

$$^{RL}D_a^{\alpha}(\lambda f + \mu g) = \lambda^{RL}D_a^{\alpha} f + \mu^{RL}D_a^{\alpha} g.$$

Proof.

Let $f, g \in L^1[a,b], \lambda \in \mathbb{R}$, we have:

$$^{RL}D_a^{\alpha} f(t) = D_a^n I^{n-\alpha} f(t)$$

$$^{RL}D_a^{\alpha}(\lambda f(t) + g(t)) = D_a^n I^{n-\alpha}[\lambda f(t) + g(t)]$$

$$= \lambda D_a^n I^{n-\alpha}[(f+g)(t)].$$

Since the n-th derivative and the integral are linear then.

$$^{RL}D_a^{\alpha}(\lambda f(t) + g(t)) = \lambda D_a^n I^{n-\alpha} f(t) + D_a^n I^{n-\alpha} g(t)$$

$$= \lambda^{RL}D_a^{\alpha} f(t) + {}^{RL}D_a^{\alpha} g(t).$$

The proof is complete. □

Example: The Riemann-Liouville fractional derivative of a power function is:

$$^{RL}D^{\alpha}t^{p}=\frac{\Gamma\left(p+1\right)}{\Gamma\left(p-\alpha+1\right)}t^{p-\alpha},\quad n-1<\alpha<n,\quad p>-1,p\in\mathbb{R}.$$

Proof.

$$^{RL}D^{\alpha}t^{p}=\frac{1}{\Gamma\left(n-\alpha\right)}\frac{d^{n}}{dt^{n}}\int_{0}^{t}\left(t-x\right)^{n-\alpha-1}x^{p}dx.$$

By changing the variable $x=\lambda t$, we will have:

$$\begin{aligned}
^{RL}D^{\alpha}t^{p}&=\frac{1}{\Gamma\left(n-\alpha\right)}\frac{d^{n}}{dt^{n}}\int_{0}^{t}\left(t\left(1-\lambda\right)\right)^{n-\alpha-1}\left(\lambda t\right)^{p}td\lambda\\
&=\frac{1}{\Gamma\left(n-\alpha\right)}\frac{d^{n}}{dt^{n}}t^{n-\alpha+p}\int_{0}^{t}\left(1-\lambda\right)^{n-\alpha-1}\lambda^{p}d\lambda\\
&=\frac{\Gamma\left(n-\alpha+p+1\right)\mathcal{B}\left(n-\alpha,p+1\right)}{\Gamma\left(n-\alpha\right)}t^{p-\alpha}\\
&=\frac{\Gamma\left(n-\alpha+p+1\right)\Gamma\left(n-\alpha\right)\Gamma\left(p+1\right)}{\Gamma\left(n-\alpha\right)\Gamma\left(p-\alpha+1\right)\Gamma\left(n-\alpha+1\right)}t^{p-\alpha}\\
&=\frac{\Gamma\left(p+1\right)}{\Gamma\left(p-\alpha+1\right)}t^{p-\alpha}.
\end{aligned}$$

The proof is complete. □

Definition 1.3.6. *[22]*

For a function h given on the interval $[a,b]$, the Caputo fractional-order derivative of h, is defined by

$$\left(^{c}D_{a^{+}}^{\alpha}h\right)\left(t\right)=\frac{1}{\Gamma\left(n-\alpha\right)}\int_{a}^{t}\left(t-s\right)^{n-\alpha-1}h^{(n)}\left(s\right)ds,$$

where $n = [\alpha] + 1$.

Lemma 1.3.1. *[69]*

Let $\alpha > 0$, *then the differential equation*

$$^{c}D^{\alpha}h\left(t\right) = 0,$$

has solutions $h\left(t\right) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}$, $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [\alpha] + 1$.

Lemma 1.3.2. *[69]*

Let $\alpha > 0$, *then*

$$I^{\alpha c}D^{\alpha}h\left(t\right) = h\left(t\right) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [\alpha] + 1$.

Lemma 1.3.3. ***(Linearity)***

Let $n - 1 < \alpha < n, n \in \mathbb{N}$, $\lambda \in \mathbb{C}$ *and let* f *and* g *such that* $^{c}D^{\alpha}f\left(t\right)$ *and* $^{c}D^{\alpha}g\left(t\right)$ *exists. Caputo fractional derivative is a linear operator i.e:*

$$^{c}D^{\alpha}\left(\lambda f\left(t\right) + g\left(t\right)\right) = \lambda ^{c}D^{\alpha}f\left(t\right) + ^{c}D^{\alpha}g\left(t\right), \quad t > 0.$$

Proof.

We have

$$^{c}D^{\alpha}f\left(t\right) = I^{n-\alpha}D^{n}f\left(t\right)$$

$$^{c}D^{\alpha}\left(\lambda f\left(t\right) + g\left(t\right)\right) = I^{n-\alpha}D^{n}\left[\lambda f\left(t\right) + g\left(t\right)\right]$$

$$= \lambda I^{n-\alpha}D^{n}\left[\left(f + g\right)\left(t\right)\right].$$

The n-th derivative and the integral are linear

$$^{c}D^{\alpha}\left(\lambda f\left(t\right)+g\left(t\right)\right)=\lambda I^{n-\alpha}D^{n}f\left(t\right)+I^{n-\alpha}D^{n}g\left(t\right)$$

$$=\lambda{}^{c}D^{\alpha}f\left(t\right)+{}^{c}D^{\alpha}g\left(t\right).$$

The proof is complete. □

Lemma 1.3.4. ***(Non-commutativity)***

We suppose that $n-1<\alpha<n, m,n\in\mathbb{N},\alpha\in\mathbb{R}$ *and let the function* f *such that* ${}^{c}D^{\alpha}f\left(t\right)$ *exists, then:*

$$^{c}D^{\alpha}\ {}^{c}D^{m}f\left(t\right)={}^{c}D^{\alpha+m}f\left(t\right)\neq{}^{c}D^{m}\ {}^{c}D^{\alpha}f\left(t\right). \tag{1.1}$$

Corollary 1.3.1.

Suppose that $n-1<\alpha<n,\beta=\alpha-\left(n-1\right),\left(0<\beta<1\right),n\in\mathbb{N},\alpha,\beta\in\mathbb{R}$ *and let the function* f *be such that* ${}^{c}D^{\alpha}f\left(t\right)$ *exists, then:*

$$^{c}D^{\alpha}f\left(t\right)={}^{c}D^{\beta}D^{n-1}f\left(t\right).$$

Proof.

We replace β by α and $n-1$ by m in (1.1), then:

$$^{c}D^{\beta}D^{n-1}f\left(t\right)={}^{c}D^{\beta+n-1}f\left(t\right)={}^{c}D^{\alpha-(n-1)+n-1}f\left(t\right)={}^{c}D^{\alpha}f\left(t\right).$$

The proof is complete. □

Definition 1.3.7. *[22]*

The Hadamard fractional integral of order α for a function y; $t \in [1, +\infty)$ is defined as

$$I^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(log\frac{t}{s}\right)^{\alpha-1} \frac{y(s)}{s} ds, \quad \alpha > 0,$$

provided the integral exists.

Example 1.3.1. *Let $q > 0$, then*

$$I^{q} \log t = \frac{1}{\Gamma(q+2)} (\log t)^{1+q}; \quad t \in [1, +\infty).$$

Definition 1.3.8. *[22].The Hadamard derivative of fractional order α for a function y: $[1, \infty) \longrightarrow \mathbb{R}$ is defined as*

$${}^{H}D^{\alpha}y(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^{n} \int_1^t \left(log\frac{t}{s}\right)^{n-\alpha-1} \frac{y(s)}{s} ds, \quad n-1 < \alpha < n, \quad n = [\alpha]+1$$

where $[\alpha]$ denotes the integer part of the real number α and $log(.) = log_e(.)$.

Definition 1.3.9. *[21]. **(The Caputo-Hadamard fractional derivative).***

For at least n-times differentiable function y: $[1, \infty) \longrightarrow \mathbb{R}$, the Caputo-type Hadamard derivative of fractional order α is defined as

$$D^{\alpha}y(t) = \frac{1}{\Gamma(n-\alpha)} \int_1^t \left(log\frac{t}{s}\right)^{n-\alpha-1} \delta^{n}y(s)\frac{s}{ds}, \quad n-1 < \alpha < n, \quad n = [\alpha]+1$$

where $\delta = t\frac{d}{dt}$, $[\alpha]$ denotes the integer part of the real number α and $log(.) = log_e(.)$.

Lemma 1.3.5. *[21]*

Let $u \in AC_{\delta}^{n}[a,b]$ or $C_{\delta}^{n}[a,b]$ and $\alpha \in \mathbb{C}$, where

$X_{\delta}^{n}[a,b] = \{F : [a,b] \to \mathbb{C} : \delta^{(n-1)} F(t) \in X[a,b]\}$. *Then, we have*

$$I^{\alpha}(D^{\alpha})\,u(t) = u(t) - \sum_{k=0}^{n-1} c_k (log\, t)^k ,$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n-1, (n = [\alpha] + 1)$.

Lemma 1.3.6.

Suppose $\alpha > 0$, $a(t)$ *and* $u(t)$ *are nonnegative functions and locally integrable on* $1 \leq t < T$ *(some* $T \leq +\infty$ *) and* $F(t)$ *is a nonnegative, nondecreasing, continuous function defined on* $1 \leq t < T$, $F(t) \leq M$ *(constant). If the following inequality*

$$u(t) \leq a(t) + F(t) \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} u(s) \frac{ds}{s}, \quad 1 \leq t < T$$

holds, then

$$u(t) \leq a(t) + \int_1^t \left[\sum_{n=1}^{\infty} \frac{(F(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} (\ln \frac{t}{s})^{n\alpha-1} a(s) \right] \frac{ds}{s}, \quad 1 \leq t < T.$$

Chapter 2

Langevin fractional differential inclusions with nonlocal fractional integral conditions

2.1 Introduction

We consider a problem of fractional differential inclusions as follows:

$$\begin{cases} D^{\alpha}(D^{\beta}+\lambda)\,x\,(t) & \in & F(t,\, x(t)) \\ \sum\limits_{i=1}^{m}\theta_i I^{\mu_i}x(\eta_i) & = & \sum\limits_{j=1}^{n}\phi_j I^{\gamma_j}x(\xi_j) \\ \sum\limits_{k=1}^{p}\varepsilon_k I^{\varsigma_k}x(\psi_k) & = & \sum\limits_{l=1}^{q}\Xi_l I^{\tau_l}x(\varphi_l) \end{cases} \tag{2.1}$$

where D^{ρ} denotes the Hadamard Caputo-type fractional derivative of order ρ, $\rho\in\{\alpha,\beta\}$ with $0<\alpha,\beta<1, 1<\alpha+\beta<2, \lambda$ is a given constant, I^{r} is the Hadamard fractional integral of order $r>0, r\in\{\mu_i,\, \gamma_j,\, \varsigma_k,\, \tau_l\}$ the constants $\eta_i,\, \xi_j,\, \psi_k,\, \varphi_l\in(1,\, e)$ and $\theta_i,\, \phi_j,\, \varepsilon_k,\, \Xi_l\in\mathbb{R}$, for all $i\,=1,2,\ldots,m,\ j\,=1,2,\ldots,n,$

$k = 1, 2, \ldots, p$, $l = 1, 2, \ldots, q$ and $F : [1, e] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$; where F is a multifonction.

2.2 Main results

Definition 2.2.1. *A function $x \in AC_\delta^2(J)$ is said to be a solution of 2.1 if there exist a function $v \in L^1(J, \mathbb{R})$ such that $v(t) \in F(t, x(t))$ a.e $t \in J$ and satisfies the equation $D^\alpha(D^\beta + \lambda)\, x(t) = v(t)$, for each, $t \in J$, and conditions $\sum_{i=1}^{m} \theta_i I^{\mu_i} x(\eta_i) = \sum_{j=1}^{n} \phi_j I^{\gamma_j} x(\xi_j)$, and $\sum_{k=1}^{p} \varepsilon_k I^{\varsigma_k} x(\psi_k) = \sum_{l=1}^{q} \Xi_l I^{\tau_l} x(\varphi_l)$ are satisfied.*

Consider the constants

$$\begin{aligned}
\Omega_1 &= \sum_{i=1}^{m} \theta_i \frac{(\log \eta_i)^{\mu_i}}{\Gamma(\mu_i + 1)} - \sum_{j=1}^{n} \phi_j \frac{(\log \xi_j)^{\gamma_j}}{\Gamma(\gamma_j + 1)}, \\
\Omega_2 &= \sum_{i=1}^{m} \theta_i \frac{(\log \eta_i)^{\beta+\mu_i}}{\Gamma(\beta + \mu_i + 1)} - \sum_{j=1}^{n} \phi_j \frac{(\log \xi_j)^{\beta+\gamma_j}}{\Gamma(\beta + \gamma_j + 1)}, \\
\Omega_3 &= \sum_{k=1}^{p} \varepsilon_k \frac{(\log \psi_k)^{\varsigma_k}}{\Gamma(\varsigma_k + 1)} - \sum_{l=1}^{q} \nu_l \frac{(\log \varphi_l)^{\tau_l}}{\Gamma(\tau_l + 1)}, \\
\Omega_4 &= \sum_{k=1}^{p} \varepsilon_k \frac{(\log \psi_k)^{\beta+\varsigma_k}}{\Gamma(\beta + \varsigma_k + 1)} - \sum_{l=1}^{q} \nu_l \frac{(\log \varphi_l)^{\beta+\tau_l}}{\Gamma(\beta + \tau_l + 1)} \qquad (2.2)
\end{aligned}$$

and

$$\Omega = \Omega_1\Omega_4 - \Omega_2\Omega_3. \qquad (2.3)$$

Lemma 2.2.1. *[63] Let $\Omega \neq 0$, $0 < \alpha, \beta \leq 1, 1 < \alpha + \beta \leq 2$, λ is a given constant, $\mu_i, \gamma_j, \varsigma_k, \tau_l > 0$, constants $\eta_i, \xi_j, \psi_k, \varphi_l \in (1, e)$ and $\theta_i, \phi_j, \varepsilon_k, \nu_l \in \mathbb{R}$, for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, p$, $l = 1, 2, \ldots, q$.*

Then the problem

$$\begin{cases} D^{\alpha}(D^{\beta}+\lambda)\,x\,(t) & = & h(t) \\ \sum\limits_{i=1}^{m}\theta_i I^{\mu_i}x(\eta_i) & = & \sum\limits_{j=1}^{n}\phi_j I^{\gamma_j}x(\xi_j) \\ \sum\limits_{k=1}^{p}\varepsilon_k I^{\varsigma_k}x(\psi_k) & = & \sum\limits_{l=1}^{q}\nu_l I^{\tau_l}x(\varphi_l) \end{cases} \tag{2.4}$$

has a unique solution given by

$$\begin{aligned} x\,(t) &= \frac{1}{\Omega}\Bigg[\left(\Omega_4-\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_3\right)\left(\sum_{j=1}^{n}\phi_j\left[I^{\alpha+\beta+\gamma_j}h\,(\xi_j)\right)-\lambda I^{\beta+\gamma_j}x\,(\xi_j)\right] \\ &- \sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}h(\eta_i)-\lambda I^{\beta+\mu_i}x\,(\eta_i)\right]\Bigg) \\ &+ \left(\frac{(\log\,t)^{\beta}}{\Gamma(\beta+1)}\Omega_1-\Omega_2\right)\left(\sum_{l=1}^{q}\upsilon_l\left[I^{\alpha+\beta+\tau_i}h\,(\varphi_l)-\lambda I^{\beta+\tau_i}x\,(\varphi_l)\right]\right. \\ &- \left.\sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\varsigma_k}h(\psi_k)-\lambda I^{\beta+\varsigma_k}x(\psi_k)\right]\right)\Bigg] \\ &+ I^{\alpha+\beta}h(t)-\lambda I^{\beta}x(t). \end{aligned} \tag{2.5}$$

Proof.

Using definition (2.2.1) can be expressed as an equivalent integral equation:

$$\left(D^{\beta}+\lambda\right)x\,(t)=I^{\alpha}h\,(t)+c_0 \tag{2.6}$$

It follows that

$$x\,(t)=I^{\alpha+\beta}h\,(t)-\lambda I^{\beta}x(t)+c_0\frac{(\log t)^{\beta}}{\Gamma\,(\beta+1)}+c_1, \tag{2.7}$$

for some $c_0, c_1 \in \mathbb{R}$.

Taking the Hadamard fractional integral of order $x > 0$ for (2.7), we have

$$I^{\kappa}x\left(t\right) = I^{\alpha+\beta+\kappa}h\left(t\right) - \lambda I^{\beta+\kappa}x(t) + c_0\frac{\left(\log t\right)^{\beta+\kappa}}{\Gamma\left(\beta+\kappa+1\right)} + c_1\frac{\left(\log t\right)^{\kappa}}{\Gamma\left(\kappa+1\right)}. \tag{2.8}$$

Substituting $\kappa = \mu_i, \gamma_j, \sigma_k, \tau_l$ and putting $t = \eta_i, \omega_j, \psi_k, \varphi_l$ in (2.8), respectively, and using conditions of the problem (2.1), we get the system of linear equations:

$$\begin{aligned}\Omega_1 c_1 + \Omega_2 c_0 =& \sum_{j=1}^{n}\left[I^{\alpha+\beta+\gamma_j}h\left(\omega_j\right) - \lambda I^{\beta+\gamma_j}x(\omega_j)\right]\\ &- \sum_{i=1}^{m}\left[I^{\alpha+\beta+\mu_j}h\left(\eta_j\right) - \lambda I^{\beta+\mu_j}x(\eta_j)\right],\end{aligned}$$

$$\begin{aligned}\Omega_3 c_1 + \Omega_4 c_0 =& \sum_{l=1}^{p}\upsilon_l\left[I^{\alpha+\beta+\tau_l}h\left(\varphi_l\right) - \lambda I^{\beta+\tau_l}x(\varphi_l)\right]\\ &- \sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\sigma_k}h\left(\psi_k\right) - \lambda I^{\beta+\sigma_k}x(\psi_k)\right].\end{aligned}$$

Solving the system of linear equations for constants c_0, c_1we have

$$\begin{aligned}c_0 = \frac{1}{\Omega}\Bigg[&\Omega_1\Bigg(\sum_{l=1}^{p}\upsilon_l\left[I^{\alpha+\beta+\tau_l}h\left(\varphi_l\right) - \lambda I^{\beta+\tau_l}x(\varphi_l)\right]\\ &- \sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\sigma_k}h\left(\psi_k\right) - \lambda I^{\beta+\sigma_k}x(\psi_k)\right]\Bigg)\\ &- \Omega_3\Bigg(\sum_{j=1}^{n}\phi_j\left[I^{\alpha+\beta+\gamma_j}h\left(\omega_j\right) - \lambda I^{\beta+\gamma_j}x(\omega_j)\right]\\ &- \sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}h\left(\eta_i\right) - \lambda I^{\beta+i}x(\eta_i)\right]\Bigg)\Bigg],\end{aligned}$$

$$c_1 = \frac{1}{\Omega}\Bigg[\Omega_4\Bigg(\sum_{j=1}^{n} \phi_j \left[I^{\alpha+\beta+\gamma_j} h\left(\omega_j\right) - \lambda I^{\beta+\gamma_j} x(\omega_j)\right]$$

$$- \sum_{i=1}^{m} \theta_i \left[I^{\alpha+\beta+\mu_i} h\left(\eta_i\right) - \lambda I^{\beta+i} x(\eta_i)\right]\Bigg)\Bigg]$$

$$- \Omega_2\Bigg(\sum_{l=1}^{p} \upsilon_l \left[I^{\alpha+\beta+\tau_l} h\left(\varphi_l\right) - \lambda I^{\beta+\tau_l} x(\varphi_l)\right]$$

$$- \sum_{k=1}^{p} \varepsilon_k \left[I^{\alpha+\beta+\sigma_k} h\left(\psi_k\right) - \lambda I^{\beta+\sigma_k} x(\psi_k)\right]\Bigg)\Bigg].$$

Substituting constants c_0 and c_1 into (2.7), we obtain (2.1) as required, the proof is complete.

□

Let us set the constant

$$\begin{aligned} \Lambda(u) &= \frac{1}{|\Omega|}\Bigg[\Bigg(|\Omega_4| + \frac{|\Omega_3|}{\Gamma(\beta+1)}\Bigg)\Bigg(\sum_{j=1}^{n}|\phi_j|\frac{(log\,\xi_j)^{u+\beta+\gamma_j}}{\Gamma(u+\beta+\gamma_j+1)} \\ &+ \sum_{i=1}^{m}|\theta_i|\frac{(log\,\eta_i)^{u+\beta+\mu_i}}{\Gamma(u+\beta+\mu_i+1)}\Bigg) \\ &+ \Bigg(\frac{|\Omega_1|}{\Gamma(\beta+1)} + |\Omega_2|\Bigg)\Bigg(\sum_{l=1}^{q}|\upsilon_l|\frac{(log\,\varphi_l)^{u+\beta+\tau_l}}{\Gamma(u+\beta+\tau_l+1)} \\ &+ \sum_{k=1}^{p}|\varepsilon_k|\frac{(log\,\psi_k)^{u+\beta+\varsigma_k}}{\Gamma(u+\beta+\varsigma_k+1)}\Bigg)\Bigg] + \frac{1}{\Gamma(u+\beta+1)}. \end{aligned}$$

The following hypotheses will be used in the sequel:

$(S1)$ The multifonction $F : J \times \mathbb{R} \longrightarrow P_{cp,c}(\mathbb{R})$ is Carathéodory;

$(S2)$ There exist a function $\gamma \in C(J, \mathbb{R}_+)$ and a continuous non-decreasing function $\Psi : [0, \infty) \longrightarrow (0, \infty)$ such that:

$$\|F(t, y)\| \leq \gamma(t)\,\Psi(|y|) \text{ for all } t \in J \text{and} \;\; y \in \mathbb{R}.$$

Set

$$\gamma^* = \sup_{t\in I} \gamma(t).$$

$(S3)$ There exists a constant $M > 0$ such that

$$\frac{M}{\gamma^* \Psi(M)\Lambda(\alpha) + |\lambda| M \Lambda(0)} > 1.$$

Theorem 2.2.1. *Assume that $(S1)$-$(S3)$ hold. Then the problem 4.1 has at least one solution.*

Proof. Let the operator $N : C([1,e],\mathbb{R}) \longrightarrow P(C([1,e],\mathbb{R}))$ defined by

$$N(x) = \Bigg\{ h \in C([1,e],\mathbb{R}) : h(t) = \frac{1}{\Omega}\Bigg[\left(\Omega_4 - \frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_3\right)$$

$$\Bigg(\sum_{j=1}^{n} \phi_j \left[I^{\alpha+\beta+\gamma_j} v(\xi_j) - \lambda I^{\beta+\gamma_j} x(\xi_j)\right]$$

$$- \sum_{i=1}^{m} \theta_i \left[I^{\alpha+\beta+\mu_i} v(\eta_i) - \lambda I^{\beta+\mu_i} x(\eta_i)\right]\Bigg)$$

$$+ \left(\frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_1 - \Omega_2\right)\Bigg(\sum_{l=1}^{q} \Xi_l \left[I^{\alpha+\beta+\tau_l} v(\varphi_l) - \lambda I^{\beta+\tau_l} x(\varphi_l)\right]$$

$$- \sum_{k=1}^{p} \varepsilon_k \left[I^{\alpha+\beta+\varsigma_k} v(\psi_k) - \lambda I^{\beta+\varsigma_k} x(\psi_k)\right]\Bigg)\Bigg] + I^{\alpha+\beta} v(t) - \lambda I^\beta x(t),\ v \in S_{F,x} \Bigg\}$$

We shall show that N satisfies the assumptions of nonlinear alternative of Leray-Schauder type.

Claim 1: $N(x)$ is convex for each $x \in C([1,e],\mathbb{R})$. Indeed, if h_1, h_2 belong to $N(x)$, then there exist $v_1, v_2 \in S_{F,x}$ such that for each $t \in [1,e]$, we have

$$
\begin{aligned}
h_i(t) &= \tfrac{1}{\Omega}\Bigg[\left(\Omega_4 - \tfrac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_3\right)\left(\sum_{j=1}^{n}\phi_j\left[I^{\alpha+\beta+\gamma_j}v_i(\xi_j) - \lambda I^{\beta+\gamma_j}x(\xi_j)\right]\right.\\
&- \left.\sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}v_i(\eta_i) - \lambda I^{\beta+\mu_i}x(\eta_i)\right]\right)\\
&+ \left(\tfrac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_1 - \Omega_2\right)\left(\sum_{l=1}^{q}\Xi_l\left[I^{\alpha+\beta+\tau_l}v_i(\varphi_l) - \lambda I^{\beta+\tau_l}x(\varphi_l)\right]\right.\\
&- \left.\sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\varsigma_k}v_i(\psi_k) - \lambda I^{\beta+\varsigma_k}x(\psi_k)\right]\right)\Bigg] + I^{\alpha+\beta}v_i(t) - \lambda I^{\beta}x(t),\ i=1,2.
\end{aligned}
$$

Let $0 \le d \le 1$. Then, for each $t \in [1,e]$, we have

$$
\begin{aligned}
(dh_1 + (1-d)h_2)(t) = \frac{1}{\Omega}\Bigg[&\left(\Omega_4 - \frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_3\right)\\
&\left(\sum_{j=1}^{n}\phi_j\left[I^{\alpha+\beta+\gamma_j}(dv_1(\xi_j) + (1-d)v_2(\xi_j)) - \lambda I^{\beta+\gamma_j}x(\xi_j)\right]\right.\\
&-\left.\sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}(dv_1(\eta_i) + (1-d)v_2(\eta_i)) - \lambda I^{\beta+\mu_i}x(\eta_i)\right]\right)\\
&+\left(\frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_1 - \Omega_2\right)\\
&\left(\sum_{l=1}^{q}\Xi_l\left[I^{\alpha+\beta+\tau_l}(dv_1(\varphi_l) + (1-d)v_2(\varphi_l)) - \lambda I^{\beta+\tau_l}x(\varphi_l)\right]\right.\\
&-\left.\sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\varsigma_k}(dv_1(\psi_k) + (1-d)v_2(\psi_k)) - \lambda I^{\beta+\varsigma_k}x(\psi_k)\right]\right)\Bigg]\\
&+ I^{\alpha+\beta}(dv_1(t) + (1-d)v_2(t)) - \lambda I^{\beta}x(t).
\end{aligned}
$$

Since $S_{F,x}$ is convex (because F has convex values), then $dh_1 + (1-d)h_2 \in N(x)$.

Claim 2: N maps bounded sets into bounded sets in $C([1,e],\mathbb{R})$. Indeed, it is enough to show that for any $q > 0$, there exists a positive constant l such that for each $x \in B_q =$

$\left\{x \in C([1,e],\mathbb{R}) : \left\|x\right\|_{\infty} \leq q\right\}$, we have $\left\|N(x)\right\|_{\infty} \leq l$. Then for each $h \in N(x)$, there exists $v \in S_{F,x}$ such that

$$\begin{aligned} h(t) &= \frac{1}{\Omega}\Bigg[\left(\Omega_4 - \frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_3\right)\left(\sum_{j=1}^{n}\phi_j\left[I^{\alpha+\beta+\gamma_j}v(\xi_j) - \lambda I^{\beta+\gamma_j}x(\xi_j)\right]\right. \\ &- \left.\sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}v(\eta_i) - \lambda I^{\beta+\mu_i}x(\eta_i)\right]\right) \\ &+ \left(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_1 - \Omega_2\right)\left(\sum_{l=1}^{q}\Xi_l\left[I^{\alpha+\beta+\tau_l}v(\varphi_l) - \lambda I^{\beta+\tau_l}x(\varphi_l)\right]\right. \\ &- \left.\sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\varsigma_k}v(\psi_k) - \lambda I^{\beta+\varsigma_k}x(\psi_k)\right]\right)\Bigg] + I^{\alpha+\beta}v(t) - \lambda I^{\beta}x(t). \end{aligned}$$

By $(S2)$ we have for each $t \in [1,e]$

$$\begin{aligned} |h(t)| &\leq \frac{1}{|\Omega|}\Bigg[\left(|\Omega_4| + \frac{(\log t)^{\beta}}{\Gamma(\beta+1)}|\Omega_3|\right)\left(\sum_{j=1}^{n}|\phi_j|\left[I^{\alpha+\beta+\gamma_j}|v(\xi_j)| + |\lambda| I^{\beta+\gamma_j}|x(\xi_j)|\right]\right. \\ &+ \left.\sum_{i=1}^{m}|\theta_i|\left[I^{\alpha+\beta+\mu_i}|v(\eta_i)| + |\lambda| I^{\beta+\mu_i}|x(\eta_i)|\right]\right) \\ &+ \left(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}|\Omega_1| + |\Omega_2|\right)\left(\sum_{l=1}^{q}|\Xi_l|\left[I^{\alpha+\beta+\tau_l}|v(\varphi_l)| + |\lambda| I^{\beta+\tau_l}|x(\varphi_l)|\right]\right. \\ &+ \left.\sum_{k=1}^{p}|\varepsilon_k|\left[I^{\alpha+\beta+\varsigma_k}|v(\psi_k)| + |\lambda| I^{\beta+\varsigma_k}|x(\psi_k)|\right]\right)\Bigg] + I^{\alpha+\beta}|v(t)| + |\lambda| I^{\beta}|x(t)|, \end{aligned}$$

then

$$\begin{aligned} |h(t)| &\leq \frac{1}{|\Omega|}\Bigg[\left(|\Omega_4| + \frac{(\log t)^{\beta}}{\Gamma(\beta+1)}|\Omega_3|\right)\left(\sum_{j=1}^{n}|\phi_j|\left[I^{\alpha+\beta+\gamma_j}\gamma(t)\Psi(|x(\xi_j)|) + |\lambda| I^{\beta+\gamma_j}|x(\xi_j)|\right]\right. \\ &+ \left.\sum_{i=1}^{m}|\theta_i|\left[I^{\alpha+\beta+\mu_i}\gamma(t)\Psi(|x(\eta_i)|) + |\lambda| I^{\beta+\mu_i}|x(\eta_i)|\right]\right)\Bigg] \\ &+ \left(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}|\Omega_1| + |\Omega_2|\right)\left(\sum_{l=1}^{q}|\Xi_l|\left[I^{\alpha+\beta+\tau_l}\gamma(t)\Psi(|x(\varphi_l)|) + |\lambda| I^{\beta+\tau_l}|x(\varphi_l)|\right]\right. \\ &+ \left.\sum_{k=1}^{p}|\varepsilon_k|\left[I^{\alpha+\beta+\varsigma_k}\gamma(t)\Psi(|x(\psi_k)|) + |\lambda| I^{\beta+\varsigma_k}|x(\psi_k)|\right]\right)\Bigg] \\ &+ I^{\alpha+\beta}\gamma(t)\Psi(|x(t)|) + |\lambda| I^{\beta}|x(t)| \\ &\leq \gamma^{*}\Lambda(\alpha)\Psi(\|x\|) + |\lambda|\,\|x\|\Lambda(0) \leq \gamma^{*}\Lambda(\alpha)\Psi(q) + |\lambda| q\Lambda(0) := l. \end{aligned}$$

Claim 3: N maps bounded sets into equicontinuons sets of $C([1,e],\mathbb{R})$ Let $t_1, t_2 \in$

$[1,e]$, $t_1 < t_2$, B_q a bounded set of $C([1,e],\mathbb{R})$ as in Claim 2 and let $x \in B_q$ and $h \in N(x)$. Then

$$\begin{aligned}
|h(t_2) - h(t_1)| &\leq \frac{1}{|\Omega|}\Bigg[\left(\frac{(\log t_2)^{\beta} - (\log t_1)^{\beta}}{\Gamma(\beta+1)} \mid \Omega_3 \mid\right)\left(\sum_{j=1}^{n} \mid \phi_j \mid \left[I^{\alpha+\beta+\gamma_j}\gamma(t)\Psi(|x(\xi_j)|)\right.\right. \\
&+ \left.\left.\mid \lambda \mid I^{\beta+\gamma_j}|x(\xi_j)|\right] + \sum_{i=1}^{m} \mid \theta_i \mid \left[I^{\alpha+\beta+\mu_i}\gamma(t)\Psi(|x(\eta_i)|) + \mid \lambda \mid I^{\beta+\mu_i}|x(\eta_i)|\right]\right) \\
&+ \left(\frac{(\log t_2)^{\beta} - (\log t_1)^{\beta}}{\Gamma(\beta+1)} \mid \Omega_1 \mid\right) \\
&\left(\sum_{l=1}^{q} \mid \Xi_l \mid \left[I^{\alpha+\beta+\tau_l}\gamma(t)\Psi(|x(\varphi_l)|) + \mid \lambda \mid I^{\beta+\tau_l}|x(\varphi_l)|\right]\right. \\
&+ \left.\sum_{k=1}^{p} \mid \varepsilon_k \mid \left[I^{\alpha+\beta+\varsigma_k}\gamma(t)\Psi(|x(\psi_k)|) + \mid \lambda \mid I^{\beta+\varsigma_k}|x(\psi_k)|\right]\right)\Bigg] \\
&+ \frac{\gamma(t)\Psi(\|x\|)}{\Gamma(\alpha+\beta+1)}\left|(\log t_2)^{\alpha+\beta} - (\log t_1)^{\alpha+\beta}\right| \\
&+ \frac{\mid \lambda \mid \|x\|}{\Gamma(\beta+1)}\left|(\log t_2)^{\beta} - (\log t_1)^{\beta} + 2\left(\log\frac{t_2}{t_1}\right)^{\beta}\right| \\
&\leq \frac{1}{|\Omega|}\Bigg[\left(\frac{(\log t_2)^{\beta} - (\log t_1)^{\beta}}{\Gamma(\beta+1)} \mid \Omega_3 \mid\right) \\
&\left(\sum_{j=1}^{n} \mid \phi_j \mid \left[\frac{\gamma^*\Psi(q)(\log \xi_j)^{\alpha+\beta+\gamma_j}}{\Gamma(\alpha+\beta+\gamma_j+1)} + \frac{\mid \lambda \mid q(\log \xi_j)^{\beta+\gamma_j}}{\Gamma(\beta+\gamma_j+1)}\right]\right. \\
&+ \left.\sum_{i=1}^{m} \mid \theta_i \mid \left[\frac{\gamma^*\Psi(q)(\log \eta_i)^{\alpha+\beta+\mu_i}}{\Gamma(\alpha+\beta+\mu_i+1)} + \frac{\mid \lambda \mid q(\log \eta_i)^{\beta+\mu_i}}{\Gamma(\beta+\mu_i+1)}\right]\right) \\
&+ \left(\tfrac{(\log t_2)^{\beta} - (\log t_1)^{\beta}}{\Gamma(\beta+1)} \mid \Omega_1 \mid\right) \\
&\left(\sum_{l=1}^{q} \mid \Xi_l \mid \left[\frac{\gamma^*\Psi(q)(\log \varphi_l)^{\alpha+\beta+\tau_l}}{\Gamma(\alpha+\beta+\tau_l+1)} + \frac{\mid \lambda \mid q(\log \varphi_l)^{\beta+\tau_l}}{\Gamma(\beta+\tau_l+1)}\right]\right. \\
&+ \left.\sum_{k=1}^{p} \mid \varepsilon_k \mid \left[\frac{\gamma^*\Psi(q)(\log \psi_k)^{\alpha+\beta+\varsigma_k}}{\Gamma(\alpha+\beta+\varsigma_k+1)} + \frac{\mid \lambda \mid q(\log \psi_k)^{\beta+\varsigma_k}}{\Gamma(\beta+\varsigma_k+1)}\right]\right)\Bigg] \\
&+ \frac{\gamma^*\Psi(q)}{\Gamma(\alpha+\beta+1)}\left|(\log t_2)^{\alpha+\beta} - (\log t_1)^{\alpha+\beta}\right| \\
&+ \frac{\mid \lambda \mid q}{\Gamma(\beta+1)}\left|(\log t_2)^{\beta} - (\log t_1)^{\beta} + 2\left(\log\frac{t_2}{t_1}\right)^{\beta}\right|.
\end{aligned}$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Claims 1 to 3 together with the Arzela-Ascoli theorem, we can conclude that operator $N : C([1,e], \mathbb{R}) \longrightarrow C([1,e], \mathbb{R})$ is completely continuous.

Claim 4: N has a closed graph Let $x_n \longrightarrow x_*$, $h_n \in N(x_n)$ and $h_n \longrightarrow h_*$. We need to show that $h_* \in N(x_*)$. $h_n \in N(x_n)$ means that there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [1,e]$

$$h_n(t) = \frac{1}{\Omega}\Bigg[\left(\Omega_4 - \frac{(log\,t)^{\beta}}{\Gamma(\beta+1)}\Omega_3\right)\left(\sum_{j=1}^{n}\phi_j\left[I^{\alpha+\beta+\gamma_j}v_n(\xi_j) - \lambda I^{\beta+\gamma_j}x_n(\xi_j)\right]\right.$$
$$\left. - \sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}v_n(\eta_i) - \lambda I^{\beta+\mu_i}x_n(\eta_i)\right]\right)$$
$$+ \left(\frac{(log\,t)^{\beta}}{\Gamma(\beta+1)}\Omega_1 - \Omega_2\right)\left(\sum_{l=1}^{q}\Xi_l\left[I^{\alpha+\beta+\tau_l}v_n(\varphi_l) - \lambda I^{\beta+\tau_l}x_n(\varphi_l)\right]\right.$$
$$\left.\left. - \sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\varsigma_k}v_n(\psi_k) - \lambda I^{\beta+\varsigma_k}x_n(\psi_k)\right]\right)\right] + I^{\alpha+\beta}v_n(t) - \lambda I^{\beta}x_n(t).$$

We must show that there exists $v_* \in S_{F,x_*}$ such that, for each $t \in [1,e]$

$$h_*(t) = \frac{1}{\Omega}\Bigg[\left(\Omega_4 - \frac{(log\,t)^{\beta}}{\Gamma(\beta+1)}\Omega_3\right)\left(\sum_{j=1}^{n}\phi_j\left[I^{\alpha+\beta+\gamma_j}v_*(\xi_j) - \lambda I^{\beta+\gamma_j}x_*(\xi_j)\right]\right.$$
$$\left. - \sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}v_*(\eta_i) - \lambda I^{\beta+\mu_i}x_*(\eta_i)\right]\right)$$
$$+ \left(\frac{(log\,t)^{\beta}}{\Gamma(\beta+1)}\Omega_1 - \Omega_2\right)\left(\sum_{l=1}^{q}\Xi_l\left[I^{\alpha+\beta+\tau_l}v_*(\varphi_l) - \lambda I^{\beta+\tau_l}x_*(\varphi_l)\right]\right.$$
$$\left.\left. - \sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\varsigma_k}v_*(\psi_k) - \lambda I^{\beta+\varsigma_k}x_*(\psi_k)\right]\right)\right] + I^{\alpha+\beta}v_*(t) - \lambda I^{\beta}x_*(t).$$

Consider the continuous linear operator

$$\Theta : L^1[1,e] \longrightarrow C([1,e],\mathbb{R})$$

defined by

$$\begin{aligned} v \longmapsto (\Theta v)(t) = \frac{1}{\Omega}\Bigg[& \left(\Omega_4 - \frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_3\right)\left(\sum_{j=1}^{n}\phi_j\left[I^{\alpha+\beta+\gamma_j}v(\xi_j) - \lambda I^{\beta+\gamma_j}x(\xi_j)\right]\right. \\ & \left. - \sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}v(\eta_i) - \lambda I^{\beta+\mu_i}x(\eta_i)\right]\right) \\ & + \left(\frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_1 - \Omega_2\right)\left(\sum_{l=1}^{q}\Xi_l\left[I^{\alpha+\beta+\tau_l}v(\varphi_l) - \lambda I^{\beta+\tau_l}x(\varphi_l)\right]\right. \\ & \left. - \sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\varsigma_k}v(\psi_k) - \lambda I^{\beta+\varsigma_k}x(\psi_k)\right]\right)\Bigg] + I^{\alpha+\beta}v(t) - \lambda I^\beta x(t). \end{aligned}$$

Clearly, $\|h_n - h_*\|_\infty \longrightarrow 0$ as $n \longrightarrow \infty$. From Lemma 1.2.3, it follows that $\Theta \circ S_F$ is a closed graph operator. Moreover, we have $h_n(t) \in \Theta(S_{F,x_n})$ Since $x_n \longrightarrow x_*$, it follows from Lemma 1.2.3 that

$$\begin{aligned} h_*(t) = \frac{1}{\Omega}\Bigg[& \left(\Omega_4 - \frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_3\right)\left(\sum_{j=1}^{n}\phi_j\left[I^{\alpha+\beta+\gamma_j}v_*(\xi_j) - \lambda I^{\beta+\gamma_j}x(\xi_j)\right]\right. \\ & \left. - \sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}v_*(\eta_i) - \lambda I^{\beta+\mu_i}x(\eta_i)\right]\right) \\ & + \left(\frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_1 - \Omega_2\right)\left(\sum_{l=1}^{q}\Xi_l\left[I^{\alpha+\beta+\tau_l}v_*(\varphi_l) - \lambda I^{\beta+\tau_l}x(\varphi_l)\right]\right. \\ & \left. - \sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\varsigma_k}v_*(\psi_k) - \lambda I^{\beta+\varsigma_k}x(\psi_k)\right]\right)\Bigg] + I^{\alpha+\beta}v_*(t) - \lambda I^\beta x(t). \end{aligned}$$

Claim 5: Let $x \in C([1,e],\mathbb{R})$ be such that $x \in \lambda N(x)$ for some $\lambda \in (0,1)$. Then, there exists $v \in L^1([1,e],\mathbb{R})$ with $v \in S_{F,x}$ such that, for each $t \in J$,

$$\begin{aligned} x(t) &= \tfrac{1}{\Omega}\Bigg[\left(\Omega_4 - \tfrac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_3\right)\left(\sum_{j=1}^{n}\phi_j\left[I^{\alpha+\beta+\gamma_j}v(\xi_j) - \lambda I^{\beta+\gamma_j}x(\xi_j)\right]\right.\\ &- \left.\sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}v(\eta_i) - \lambda I^{\beta+\mu_i}x(\eta_i)\right]\right)\\ &+ \left(\tfrac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_1 - \Omega_2\right)\left(\sum_{l=1}^{q}\Xi_l\left[I^{\alpha+\beta+\tau_l}v(\varphi_l) - \lambda I^{\beta+\tau_l}x(\varphi_l)\right]\right.\\ &- \left.\sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\varsigma_k}v(\psi_k) - \lambda I^{\beta+\varsigma_k}x(\psi_k)\right]\right)\Bigg] + I^{\alpha+\beta}v(t) - \lambda I^\beta x(t). \end{aligned}$$

Then

$$\begin{aligned} |x(t)| &\le \tfrac{1}{|\Omega|}\Bigg[\left(|\Omega_4| + \tfrac{(\log t)^\beta}{\Gamma(\beta+1)}|\Omega_3|\right)\left(\sum_{j=1}^{n}|\phi_j|\left[I^{\alpha+\beta+\gamma_j}|v(\xi_j)| + |\lambda| I^{\beta+\gamma_j}|x(\xi_j)|\right]\right.\\ &+ \left.\sum_{i=1}^{m}|\theta_i|\left[I^{\alpha+\beta+\mu_i}|v(\eta_i)| + |\lambda| I^{\beta+\mu_i}|x(\eta_i)|\right]\right)\\ &+ \left(\tfrac{(\log t)^\beta}{\Gamma(\beta+1)}|\Omega_1| + |\Omega_2|\right)\left(\sum_{l=1}^{q}|\Xi_l|\left[I^{\alpha+\beta+\tau_l}|v(\varphi_l)| + |\lambda| I^{\beta+\tau_l}|x(\varphi_l)|\right]\right.\\ &+ \left.\sum_{k=1}^{p}|\varepsilon_k|\left[I^{\alpha+\beta+\varsigma_k}|v(\psi_k)| + |\lambda| I^{\beta+\varsigma_k}|x(\psi_k)|\right]\right)\Bigg] + I^{\alpha+\beta}|v(t)| + |\lambda| I^\beta|x(t)|. \end{aligned}$$

From $(S2)$, we get

$$\begin{aligned} |x(t)| &\le \tfrac{1}{|\Omega|}\Bigg[\left(|\Omega_4| + \tfrac{(\log t)^\beta}{\Gamma(\beta+1)}|\Omega_3|\right)\left(\sum_{j=1}^{n}|\phi_j|\left[I^{\alpha+\beta+\gamma_j}\gamma(t)\Psi(|x(\xi_j)|) + |\lambda| I^{\beta+\gamma_j}|x(\xi_j)|\right]\right.\\ &+ \left.\sum_{i=1}^{m}|\theta_i|\left[I^{\alpha+\beta+\mu_i}\gamma(t)\Psi(|x(\eta_i)|) + |\lambda| I^{\beta+\mu_i}|x(\eta_i)|\right]\right)\\ &+ \left(\tfrac{(\log t)^\beta}{\Gamma(\beta+1)}|\Omega_1| + |\Omega_2|\right)\left(\sum_{l=1}^{q}|\Xi_l|\left[I^{\alpha+\beta+\tau_l}\gamma(t)\Psi(|x(\varphi_l)|) + |\lambda| I^{\beta+\tau_l}|x(\varphi_l)|\right]\right.\\ &+ \left.\sum_{k=1}^{p}|\varepsilon_k|\left[I^{\alpha+\beta+\varsigma_k}\gamma(t)\Psi(|x(\psi_k)|) + |\lambda| I^{\beta+\varsigma_k}|x(\psi_k)|\right]\right)\Bigg]\\ &+ I^{\alpha+\beta}\gamma(t)\Psi(|x(t)|) + |\lambda| I^\beta|x(t)| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\mid\Omega\mid}\Bigg[\left(\left(\mid\Omega_4\mid+\frac{1}{\Gamma(\beta+1)}\mid\Omega_3\mid\right)\left(\sum_{j=1}^{n}\mid\phi_j\mid\left[I^{\alpha+\beta+\gamma_j}\gamma^*\Psi\left(\|x\|\right)\mid\lambda\mid I^{\beta+\gamma_j}\|x\|\right]\right.\right.\\
&+\left.\left.\sum_{i=1}^{m}\mid\theta_i\mid\left[I^{\alpha+\beta+\mu_i}\gamma^*\Psi\left(\|x\|\right)+\mid\lambda\mid I^{\beta+\mu_i}\|x\|\right]\right)\right.\\
&+\left(\tfrac{1}{\Gamma(\beta+1)}\mid\Omega_1\mid+\mid\Omega_2\mid\right)\left(\sum_{l=1}^{q}\mid\Xi_l\mid\left[I^{\alpha+\beta+\tau_l}\gamma^*\Psi\left(\|x\right)\Big\|\right)+\mid\lambda\mid I^{\beta+\tau_l}\|x\|\right]\\
&+\left.\sum_{k=1}^{p}\mid\varepsilon_k\mid\left[I^{\alpha+\beta+\varsigma_k}\gamma^*\Psi\left(\|x\|\right)+\mid\lambda\mid I^{\beta+\varsigma_k}\|x\|\right]\right)\Bigg]\\
&+I^{\alpha+\beta}\gamma^*\Psi\left(\|x\|\right)+\mid\lambda\mid I^{\beta}\|x\|.
\end{aligned}$$

Then

$$|x(t)| \leq \gamma^*\Psi(\|x\|_\infty)\Lambda(\alpha)+|\lambda|\Lambda(0)\|x\|_\infty.$$

Thus

$$\frac{\|x\|_\infty}{\gamma^*\Psi(\|x\|_\infty)\Lambda(\alpha)+|\lambda|\Lambda(0)\|x\|_\infty}\leq 1.$$

By $(S3)$, it follow that $\|x\|_\infty \neq M$. Set

$$U=\left\{x\in C\left([1,e],\mathbb{R}\right):\|x\|_\infty<K+1\right\}.$$

From the choice of U, there is no $x\in\partial U$ such that $u\in\lambda N(u)$ for some $\lambda\in(0,1)$. As a consequence of the nonlinear alternative of leray-Schauder type, we deduce that N has a fixed point u in U which is a solution to the problem (2.1).

□

We present now a result for the problem (4.1) with a nonconvex valued right hand side.

Our consideration are based on the fixed point result in Lemma 1.2.4. So, let us introduce the following hypotheses

$(S4)$ $F : J \times \mathbb{R} \longrightarrow \mathbb{P}_{cp}(\mathbb{R})$ has the property that, $F(.,u) : J \longrightarrow P_{cp}(\mathbb{R})$ is measurable for each $u \in C(J,\mathbb{R})$.

$(S5)$ There exists $L \in C(J,\mathbb{R})$ such that

$$H_d(F(t,u) - F(t,\overline{u})) \leq L(t)|u - \overline{u}| \quad for\, every\, u,\, \overline{u} \in \mathbb{R}.$$

Theorem 2.2.2. *Assume that* $(S4)$ *and* $(S5)$ *are hold.*
If

$$L^{*}\Lambda(\alpha) + |\lambda|\Lambda(0) < 1, \tag{2.9}$$

then the problem (2.1) has at least one solution on J.

Proof. We shall show that N satisfies the assumptions of Lemma 1.2.4. The proof will be given in two steps.

Step 1: $N(x) \in P_{cl}(C(J,\mathbb{R}))$ for each $x \in C(J,\mathbb{R})$.

Indeed, let $(h_n)_{n\geq 0} \subset N(x)$ be such that $h_n \longrightarrow \widetilde{h}$ in $C(J,\mathbb{R})$. Then $\overline{x}$ in $C(J,\mathbb{R})$ and there exists $v_n \in S_{F,x}$ such that for each $t \in J$,

$$\begin{aligned} h_n(t) = \frac{1}{\Omega}\Bigg[&\left(\Omega_4 - \frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_3\right)\left(\sum_{j=1}^{n}\phi_j\left[I^{\alpha+\beta+\gamma_j}v_n(\xi_j) - \lambda I^{\beta+\gamma_j}x(\xi_j)\right]\right.\\ &\left.-\sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}v_n(\eta_i) - \lambda I^{\beta+\mu_i}x(\eta_i)\right]\right)\\ &+\left(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_1 - \Omega_2\right)\left(\sum_{l=1}^{q}\Xi_l\left[I^{\alpha+\beta+\tau_l}v_n(\varphi_l) - \lambda I^{\beta+\tau_l}x(\varphi_l)\right]\right.\\ &\left.-\sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\varsigma_k}v_n(\psi_k) - \lambda I^{\beta+\varsigma_k}x(\psi_k)\right]\right)\Bigg] + I^{\alpha+\beta}v_n(t) - \lambda I^{\beta}x(t). \end{aligned}$$

Using the fact that F has compact values and from $(S2)$ we may pass to a subsequence to see that $v_n \longrightarrow v$ in $L^1(J)$. Then for each $t \in J$

$$h_n(t) \longrightarrow \widetilde{h}(t) = \frac{1}{\Omega}\Bigg[\left(\Omega_4 - \frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_3\right)\left(\sum_{j=1}^{n}\phi_j\left[I^{\alpha+\beta+\gamma_j}v(\xi_j) - \lambda I^{\beta+\gamma_j}x(\xi_j)\right]\right.$$
$$\left. - \sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}v(\eta_i) - \lambda I^{\beta+\mu_i}x(\eta_i)\right]\right)$$
$$+\left(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_1 - \Omega_2\right)\left(\sum_{l=1}^{q}\Xi_l\left[I^{\alpha+\beta+\tau_l}v(\varphi_l) - \lambda I^{\beta+\tau_l}x(\varphi_l)\right]\right.$$
$$\left. - \sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\varsigma_k}v(\psi_k) - \lambda I^{\beta+\varsigma_k}x(\psi_k)\right]\right)\Bigg] + I^{\alpha+\beta}v(t) - \lambda I^{\beta}x(t).$$

So, $\widetilde{h} \in N(x)$.

Step 2: There exists $\gamma < 1$ such that $H_d(N(x), N(\overline{x})) \leq \gamma \|x - \overline{x}\|_{\infty}$ for each $x, \overline{x} \in C(J, \mathbb{R})$. Let $x, \overline{x} \in C(J, \mathbb{R})$ and $h_1 \in N(x)$. Then there exists $v_1 \in F(t, x(t))$ such that for each $t \in J$,

$$h_1(t) = \frac{1}{\Omega}\Bigg[\left(\Omega_4 - \frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_3\right)\left(\sum_{j=1}^{n}\phi_j\left[I^{\alpha+\beta+\gamma_j}v_1(\xi_j) - \lambda I^{\beta+\gamma_j}x(\xi_j)\right]\right.$$
$$\left. - \sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}v_1(\eta_i) - \lambda I^{\beta+\mu_i}x(\eta_i)\right]\right)$$
$$+\left(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_1 - \Omega_2\right)\left(\sum_{l=1}^{q}\Xi_l\left[I^{\alpha+\beta+\tau_l}v_1(\varphi_l) - \lambda I^{\beta+\tau_l}x(\varphi_l)\right]\right.$$
$$\left. - \sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\varsigma_k}v_1(\psi_k) - \lambda I^{\beta+\varsigma_k}x(\psi_k)\right]\right)\Bigg] + I^{\alpha+\beta}v_1(t) - \lambda I^{\beta}x(t).$$

From (S_5) it follows that

$$H_d\left(F\left(t,x\left(t\right)\right),F\left(t,\overline{x}\left(t\right)\right)\right)\leq L\left(t\right)\left|x\left(t\right)-\overline{x}\left(t\right)\right|.$$

Hence, there exists $\omega\in F\left(t,\overline{x}\left(t\right)\right)$ such that

$$\left|v_1\left(t\right)-\omega\right|\leq L\left(t\right)\left|x\left(t\right)-\overline{x}\left(t\right)\right|,t\in J.$$

Consider $U:J\longrightarrow\mathcal{P}\left(\mathbb{R}\right)$ given by

$$U\left(t\right)=\left\{\omega\in\mathbb{R}:\left|v_1\left(t\right)-\omega\right|\leq L\left(t\right)\left|x\left(t\right)-\overline{x}\left(t\right)\right|\right\}.$$

Since the multivalued operator $V\left(t\right)=U\left(t\right)\bigcap F\left(t,\overline{x}\left(t\right)\right)$ is measurable, there exists a function v_2 which is a measurable selection for V, so $v_2\in F\left(t,\overline{x}\left(t\right)\right)$ and for each $t\in J$,

$$\left|v_1\left(t\right)-v_2\left(t\right)\right|\leq L\left(t\right)\left|x\left(t\right)-\overline{x}\left(t\right)\right|,t\in J.$$

Let us define for each $t\in J$,

$$\begin{aligned}h_2\left(t\right)=\frac{1}{\Omega}\Bigg[&\left(\Omega_4-\frac{\left(\log t\right)^{\beta}}{\Gamma\left(\beta+1\right)}\Omega_3\right)\left(\sum_{j=1}^{n}\phi_j\left[I^{\alpha+\beta+\gamma_j}v_2\left(\xi_j\right)-\lambda I^{\beta+\gamma_j}\overline{x}\left(\xi_j\right)\right]\right.\\&\left.-\sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}v_2\left(\eta_i\right)-\lambda I^{\beta+\mu_i}\overline{x}\left(\eta_i\right)\right]\right)\\&+\left(\frac{\left(\log t\right)^{\beta}}{\Gamma\left(\beta+1\right)}\Omega_1-\Omega_2\right)\left(\sum_{l=1}^{q}\Xi_l\left[I^{\alpha+\beta+\tau_l}v_2\left(\varphi_l\right)-\lambda I^{\beta+\tau_l}\overline{x}\left(\varphi_l\right)\right]\right.\\&\left.-\sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\varsigma_k}v_2\left(\psi_k\right)-\lambda I^{\beta+\varsigma_k}\overline{x}\left(\psi_k\right)\right]\right)\Bigg]+I^{\alpha+\beta}v_2\left(t\right)-\lambda I^{\beta}\overline{x}\left(t\right).\end{aligned}$$

Then for each $t \in J$

$$
\begin{aligned}
\left\| h_1(t) - h_2(t) \right\| \leq \frac{1}{|\Omega|} \Bigg[& \left(|\Omega_4| + \frac{(log\, t)^{\beta}}{\Gamma(\beta+1)} |\Omega_3| \right) \\
& \left(\sum_{j=1}^{n} |\phi_j| \left[I^{\alpha+\beta+\gamma_j} \left\| (v_1 - v_2)(\xi_j) \right\| + \lambda I^{\beta+\gamma_j} \left\| (x - \overline{x})(\xi_j) \right\| \right] \right. \\
& \left. + \sum_{i=1}^{m} |\theta_i| \left[I^{\alpha+\beta+\mu_i} \left\| (v_1 - v_2)(\eta_i) \right\| + \lambda I^{\beta+\mu_i} \left\| (x - \overline{x})(\eta_i) \right\| \right] \right) \\
& + \left(\frac{(log\, t)^{\beta}}{\Gamma(\beta+1)} |\Omega_1| + |\Omega_2| \right) \\
& \left(\sum_{l=1}^{q} |\Xi_l| \left[I^{\alpha+\beta+\tau_l} \left\| (v_1 - v_2)(\varphi_l) \right\| + \lambda I^{\beta+\tau_l} \left\| (x - \overline{x})(\varphi_l) \right\| \right] \right. \\
& \left. + \sum_{k=1}^{p} |\varepsilon_k| \left[I^{\alpha+\beta+\varsigma_k} \left\| (v_1 - v_2)(\psi_k) \right\| + \lambda I^{\beta+\varsigma_k} \left\| (x - \overline{x})(\psi_k) \right\| \right] \right) \Bigg] \\
& + I^{\alpha+\beta} \left\| (v_1 - v_2)(t) \right\| + \lambda I^{\beta} \left\| (x - \overline{x})(t) \right\| \\
\leq & \frac{1}{|\Omega|} \Bigg[\left(|\Omega_4| + \frac{(log\, t)^{\beta}}{\Gamma(\beta+1)} |\Omega_3| \right) \\
& \left(\sum_{j=1}^{n} |\phi_j| \left[I^{\alpha+\beta+\gamma_j} \gamma^* \left\| (x - \overline{x}) \right\|_\infty + |\lambda| I^{\beta+\gamma_j} \left\| (x - \overline{x}) \right\|_\infty \right] \right. \\
& \left. + \sum_{i=1}^{m} |\theta_i| \left[I^{\alpha+\beta+\mu_i} \gamma^* \left\| (x - \overline{x}) \right\|_\infty + |\lambda| I^{\beta+\mu_i} \left\| (x - \overline{x}) \right\|_\infty \right] \right) \\
& + \left(\frac{(log\, t)^{\beta}}{\Gamma(\beta+1)} |\Omega_1| + |\Omega_2| \right) \\
& \left(\sum_{l=1}^{q} |\Xi_l| \left[I^{\alpha+\beta+\tau_l} \gamma^* \left\| (x - \overline{x}) \right\|_\infty + |\lambda| I^{\beta+\tau_l} \left\| (x - \overline{x}) \right\|_\infty \right] \right. \\
& \left. + \sum_{k=1}^{p} |\varepsilon_k| \left[I^{\alpha+\beta+\varsigma_k} \gamma^* \left\| (x - \overline{x}) \right\|_\infty + |\lambda| I^{\beta+\varsigma_k} \left\| (x - \overline{x}) \right\|_\infty \right] \right) \Bigg] \\
& + I^{\alpha+\beta} \gamma^* \left\| (x - \overline{x}) \right\|_\infty + |\lambda| I^{\beta} \left\| (x - \overline{x}) \right\|_\infty \\
\leq & \gamma^* \Lambda(\alpha) \left\| (x - \overline{x}) \right\|_\infty + |\lambda| \Lambda(0) \left\| (x - \overline{x}) \right\|_\infty \\
\leq & \left(L^* \Lambda(\alpha) + |\lambda| \Lambda(0) \right) \left\| (x - \overline{x}) \right\|_\infty
\end{aligned}
$$

So N is a contraction and thus, by Lemma 1.2.4, N has a fixed point x which is solution to the problem (2.1). □

2.3 An Example

Consider the following problem of Caputo-Hadamard fractional differential inclusion with nonlocal fractional integral conditions:

$$\begin{cases} D^{\frac{2}{3}}(D^{\frac{2}{5}}+\lambda)\,x\,(t) \in F(t,\,x(t)) \\ \frac{2}{5}I^{\frac{2}{3}}y(\frac{e+2}{4}) + \frac{1}{2}I^{\frac{1}{2}}y(\frac{e+3}{4}) + \frac{4}{5}I^{\frac{3}{2}}y(\frac{e+4}{4}) = 4I^{\frac{2}{3}}y(\frac{3e}{4}) \\ \frac{5}{7}I^{\frac{1}{2}}y(\frac{3e}{5}) = \frac{1}{5}I^{\frac{3}{4}}y(\frac{4e}{5}) \end{cases} \tag{2.10}$$

Here $\alpha = \frac{2}{3}, \beta = \frac{2}{5}, \lambda = \frac{1}{6\Lambda(0)}, \Lambda(\frac{2}{3}) \approx 6.598, \Lambda(0) \approx 5,945.$
Set

$$F(t, x(t)) = \left\{ v \in \mathbb{R} : 0 \le v \le e^{-4t}(\|x\| + 1) \right\}.$$

For each $t \in [1, e]$ and $u \in \mathbb{R}$

$$\|F(t, u)\| \le e^{-4t}(|u| + 1).$$

By putting $\gamma(t) = e^{-4t}$ and $\Psi(u) = |u| + 1$, we can show that

$$\frac{M}{\gamma^*\Psi(M)\Lambda(\alpha) + |\lambda| M\Lambda(0)} > 1,$$

which implies that $M > 2,68584$. Hence, by theorem 2.2.1, the problem (2.10) has at least one solution on $[1, e]$.

Chapter 3

Generalized metric space and random variables

3.1 Introduction

Random differential equations and random integral equations have been studied sys tematically by Ladde and Lakshmikantham [25] and Bharucha-Reid [5], respectively. They are good models in various branches of science and engineering since random factors and uncertainties have been taken into consideration. Hence, the study of the fractional differential equations with random parameters seem to be a natural one. We refer the reader to the monographs [53, 60], and the references therein. Very recently fractional differential equations with random parameters have been studied by Lupulescu et al [30] and Lupulescu and Ntouyas [29].

3.2 Generalized metric space

In this section we recall from the literature some notations, definitions, and auxiliary results which will be used throughout this chapter.

Let $x, y \in \mathbb{R}^m$ with $x = (x_1, x_2, \ldots, x_m),\ y = (y_1, y_2, \ldots, y_m)$. By $x \leq y$ we mean $x_i \leq y_i,\ i = 1, \ldots, m$. Also $|x| = (|x_1|, |x_2|, \ldots, |x_m|)$, $\max(x, y) = (\max(x_1, y_1), \ldots, \max(x_m, y_m))$, and $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x_i \in \mathbb{R}_+,\ i = 1, \ldots, m\}$. If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c,\ i = 1, \ldots, m$.

Definition 3.2.1.

Let X be a nonempty set. By a vector-valued metric on X we mean a map $d : X \times X \to \mathbb{R}^m$ with the following properties:

(i) $d(x, y) \geq 0$ *for all* $x, y \in X$, *and if* $d(x, y) = 0$, *then* $x = y$;

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in X$.

We call the pair (X, d) a generalized metric space with $d(x, y) := \begin{pmatrix} d_1(x, y) \\ d_2(x, y) \\ . \\ . \\ . \\ d_m(x, y) \end{pmatrix}$.

Notice that d is a generalized metric space on X if and only if $d_i,\ i = 1, \ldots, m$, are metrics on X.

For $r = (r_1, \ldots, r_2) \in \mathbb{R}^n_+$, we will denote by

$$B(x_0, r) = \{x \in X : d(x_0, x) < r\} = \{x \in X : d_i(x_0, x) < r_i, i = 1, \ldots, n\}$$

the open ball centered in x_0 with radius r and

$$\overline{B(x_0,r)} = \{x \in X : d(x_0,x) < r\} = \{x \in X : d_i(x_0,x) < r_i, i = 1,\dots,n\}$$

the closed ball centered in x_0 with radius r. We mention that for generalized metric space, the notions of open subset, closed set, convergence, Cauchy sequence and completeness are similar to those in usual metric spaces.

Definition 3.2.2. *[67]*

A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc, i.e. $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$, where I denotes the unit matrix of $M_{m\times m}(\mathbb{R})$.

Example 3.2.1.

The matrix $A \in M_{2\times 2}(\mathbb{R})$ defined by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

converges to zero in the following cases:

(1) $b = c = 0,\ a, d > 0$ *and* $\max\{a, d\} < 1$.

(2) $c = 0,\ a, d > 0,\ a + d < 1$ *and* $-1 < b < 0$.

(3) $a + b = c + d = 0,\ a > 1,\ c > 0$ *and* $|a - c| < 1$.

Theorem 3.2.1. *([67],* ***p.12, p.88****)*

Let $M \in M_{n\times n}(\mathbb{R}_+)$ The following assertions are equivalent:

(i) M is convergent towards zero;

(ii) $M^k \longrightarrow 0 \, as \, k \longrightarrow \infty$;

(iii) The matrix $(I - M)$ is nonsingular and

$$(I - M)^{-1} = I + M + M^2 + \ldots + M^k + \ldots,$$

(iv) The matrix $(I - M)$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.

Definition 3.2.3.

Let (X, d) be a generalized metric space. An operator $N : X \longrightarrow X$ is said to be contractive if there exists a convergent to zero matrix M such that

$$d\left(N\left(x\right), N\left(y\right)\right) \leq M d\left(x, y\right) \; for \, all \, x, \, y \, \in X.$$

Theorem 3.2.2. *[46]*

Let (X, d) be a complete generalized metric space and $N : X \longrightarrow X$ a contractive operator with Lipschitz matrix M. Then N has a unique fixed point x_* and for each $x_0 \in X$ we have

$$d\left(N^k\left(x_0\right), x_*\right) \leq M^k \left(I - M\right)^{-1} d\left(x_0, N\left(x_0\right)\right) \; for \, all \, k \in \mathbb{N}.$$

3.3 Random operators

Let $\left(\widetilde{\Omega}, \mathcal{F}\right)$ be a measurable space; that is, a set $\widetilde{\Omega}$ with a σ-algebra of subsets of $\widetilde{\Omega}$. A probability measure $\mathbb{P}$ is a measure with $\mathbb{P}\left(\varOmega\right) = 1$. Then $(\varOmega, \mathcal{F}, \mathbb{P})$ is called a probability space. In the following, assume that $\left(\widetilde{\Omega}, \mathcal{F}, \mathbb{P}\right)$ is a complete probability space. Let X be a metric space, $B(X)$ will be the σ-algebra of all Borel subsets of X. A measurable function $x : \widetilde{\Omega} \to X$ is called a random element in X.

Let X, Y are two locally compact, metric spaces and $f : \widetilde{\Omega} \times X \to Y$. By $C(X,Y)$ we denote the space of continuous functions from X into Y endowed with the compact-open topology.

Definition 3.3.1.

A random operator $T : \widetilde{\Omega} \times X \to X$ *is said to be continuous at* $x_0 \in X$ *if* $\underset{n\to\infty}{lim} \|x_n - x_0\| = 0$ *implies* $\underset{n\to\infty}{lim} \|T(\omega, x_n) - T(\omega, x_0)\| = 0.$

Lemma 3.3.1. *[42]*

f *is a Carathéodory function if and only if* $\omega \to r(\omega)(.) = f(\omega, .)$ *is a measurable function from* $\Omega \to C(X,Y)$.

Proposition 3.3.1.

If $f : [1,e] \times \Omega \to \mathbb{R}^m$ *is a Carathéodory function, then the function* $(t,\omega) \longmapsto I^{\alpha} f(t,\omega)$ *is also a Carathéodory function.*

Proof.

Clear that $I^{\alpha} : C([1,e], \mathbb{R}^m) \to C([1,e], \mathbb{R}^m)$ is a continuous operator, let $L : \Omega \to C([1,e], \mathbb{R}^m)$ defined by $L(\omega)(.) = f(., \omega)$. Then $L(.)$ is measurable. Then the operator $\omega \to (I^{\alpha} \circ L)(\omega)(.)$ is measurable. Since the continuous function $t \to I^{\alpha} f(t,\omega)$. Hence $(t,\omega) \to I^{\alpha} f(t,\omega)$ is a Carathéodory function, the proof is complete. □

Theorem 3.3.1. *[16, 47, 54]*

Let $(\Omega, \mathcal{F}, \mu)$ *be a probability space,* X *be a real separable generalized Banach space and* $F : \Omega \times X \to X$ *be a continuous random operator, and let* $M(\omega) \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ *be a random variable matrix such that for every* $\omega \in \Omega$, *the matrix* $M(\omega)$ *converges to* 0 *and*

$$d(F(\omega, x_1), F(\omega, x_2)) \leq M(\omega) d(x_1, x_2); \ for\ each\ x_1, x_2 \in X\ and\ w \in \Omega.$$

Then there exists a random variable $x : \Omega \to X$ which is the unique random fixed point of F.

Theorem 3.3.2.

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, X be a real separable generalized Banach space and $T : \Omega \times X \to X$ be a continuous random operator, and let $M(\omega) \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ be a nonnegative real matrix random variable such that $\rho\left(M\left(\omega\right)\right) < 1\, a.s.$ and

$$\|T(\omega, x_1) - T(\omega, x_2)\| \leq M(\omega)d(x_1, x_2); \; for \; each \; x_1, x_2 \in X \; and \; \omega \in \Omega.$$

Then there exists a random variable $x : \Omega \to X$ which is the unique random fixed point of T.

Theorem 3.3.3. *[15, 54]*

Let X be a real separable generalized Banach space and $T : \widetilde{\Omega} \times X \to X$ be a completely continuous random operator. Then, either of the following holds

(i) The random equation $T\left(\omega, x\right) = x$ has a random solution, i.e., there is a measurable function $x : \widetilde{\Omega} \to X$ such that $T\left(\omega, x\left(\omega\right)\right) = x\left(\omega\right)$ for all $\omega \in \widetilde{\Omega}$,

(ii) The set $M = \left\{x : \widetilde{\Omega} \to X \; is\, measurable\, \lambda(\omega)T(\omega, x) = x\right\}$ is unbounded for some measurable $\lambda : \widetilde{\Omega} \to X$ with $0 < \lambda\left(\omega\right) < 1$ on $\widetilde{\Omega}$.

Chapter 4

System of boundary random fractional differential equations

[1]

4.1 Introduction

Random differential equations, as natural extensions of deterministic ones, arise in many applications and have been investigated by many mathematicians. We refer the reader to the monographs ([5, 25, 65]). The initial value problems for fractional differential with random parameters have been studied by Lupulescu and Ntouyas [29]. The basic tool in the study of the problems for random fractional differential equations is to treat it as a fractional differential equation in some appropriate Banach space.

In 2008, Precup [48] proved the role of matrix convergence in the study of semilinear operator systems. Recently, many authors studied the existence of solutions for systems of differential equations and fractional differential equations and inclusions by using vector version fixed point theorems; see [6, 37, 35, 36, 50] and in the references therein.

[1]Zakaria Malki, Farida Berhoun and Abdelghani Ouahab, System of boundary random fractional differential equations via Hadamard derivative, *Paedagog. Crac. Stud. Math. J* **20** (2021), 17-41

Motived by such works, we consider the system of random fractional differential equations with nonlocal boundary conditions in the following form:

$$\begin{array}{rcl} D^{\alpha}(D^{\beta}+\lambda_1)\,x\,(t,\omega) & = & f(t,\,x(t,\omega),\,y(t,\omega),\omega) \\ D^{\gamma}(D^{\sigma}+\lambda_2)\,y\,(t,\omega) & = & g(t,\,x(t,\omega),\,y(t,\omega),\omega) \\ \sum\limits_{i=1}^{m}\theta_i I^{\mu_i}x(\eta_i,\omega) & = & \sum\limits_{j=1}^{n}\phi_j I^{\gamma_j}x(\xi_j,\omega) \\ \sum\limits_{k=1}^{p}\varepsilon_k I^{\varsigma_k}x(\psi_k,\omega) & = & \sum\limits_{l=1}^{q}\nu_l I^{\tau_l}x(\varphi_l,\omega) \\ \sum\limits_{i=1}^{m}\overline{\theta_i} I^{\overline{\mu_i}}y(\overline{\eta_i},\omega) & = & \sum\limits_{j=1}^{n}\overline{\phi_j} I^{\overline{\gamma_j}}y(\overline{\xi_j},\omega) \\ \sum\limits_{k=1}^{p}\overline{\varepsilon_k} I^{\overline{\varsigma_k}}y(\overline{\psi_k},\omega) & = & \sum\limits_{l=1}^{q}\overline{\nu_l} I^{\overline{\tau_l}}y(\overline{\varphi_l},\omega) \end{array} \tag{4.1}$$

where D^{ρ} denotes the Hadamard Caputo-type fractional derivative of order $\rho\in\{\alpha,\beta,\gamma,\sigma\}$ with $0<\alpha,\beta,\gamma,\sigma<1, 1<\alpha+\beta<2, 1<\gamma+\sigma<2$, λ_1,λ_2 are given constants, I^r is the Hadamard fractional integral of order $r>0$, $r\in\{\mu_i,\,\gamma_j,\,\varsigma_k,\,\tau_l,\,\overline{\mu_i},\,\overline{\gamma_j},\,\overline{\varsigma_k},\,\overline{\tau_l}\}$ the constants $\eta_i,\,\xi_j,\,\psi_k,\,\varphi_l,\,\overline{\eta_i},\,\overline{\xi_j},\,\overline{\psi_k},\,\overline{\varphi_l}\in(1,\,e)$; $\theta_i,\,\phi_j,\,\varepsilon_k,\,\nu_l,\,\overline{\theta_i},\,\overline{\phi_j},\,\overline{\varepsilon_k},\,\overline{\nu_l}\in\mathbb{R}$, for all $i=1,2,\dots,m$, $j=1,2,\dots,n$, $k=1,2,\dots,p$, $l=1,2,\dots,q$; and $f,\,g:[1,e]\times\mathbb{R}^m\times\mathbb{R}^m\times\Omega\to\mathbb{R}^m$ are given function, $(\Omega,\mathcal{A})$ is a measurable space.

4.2 Main results

Consider the constants

$$\begin{aligned} \Omega_1 &= \sum_{i=1}^{m}\theta_i\frac{(\log\,\eta_i)^{\mu_i}}{\Gamma(\mu_i+1)}-\sum_{j=1}^{n}\phi_j\frac{(\log\,\xi_j)^{\gamma_j}}{\Gamma(\gamma_j+1)}, \\ \Omega_2 &= \sum_{i=1}^{m}\theta_i\frac{(\log\,\eta_i)^{\beta+\mu_i}}{\Gamma(\beta+\mu_i+1)}-\sum_{j=1}^{n}\phi_j\frac{(\log\,\xi_j)^{\beta+\gamma_j}}{\Gamma(\beta+\gamma_j+1)}, \\ \Omega_3 &= \sum_{k=1}^{p}\varepsilon_k\frac{(\log\,\psi_k)^{\varsigma_k}}{\Gamma(\varsigma_k+1)}-\sum_{l=1}^{q}\nu_l\frac{(\log\,\varphi_l)^{\tau_l}}{\Gamma(\tau_l+1)}, \end{aligned}$$

$$\Omega_4 = \sum_{k=1}^{p} \varepsilon_k \frac{(\log \psi_k)^{\beta+\varsigma_k}}{\Gamma(\beta+\varsigma_k+1)} - \sum_{l=1}^{q} \nu_l \frac{(\log \varphi_l)^{\beta+\tau_l}}{\Gamma(\beta+\tau_l+1)} \tag{4.2}$$

and

$$\Omega = \Omega_1\Omega_4 - \Omega_2\Omega_3. \tag{4.3}$$

Similarly, we set

$$\overline{\Omega_1} = \sum_{i=1}^{m} \overline{\theta_i} \frac{(\log \overline{\eta_i})^{\overline{\mu_i}}}{\Gamma(\overline{\mu_i}+1)} - \sum_{j=1}^{n} \overline{\phi_j} \frac{\left(\log \overline{\xi_j}\right)^{\overline{\gamma_j}}}{\Gamma(\overline{\gamma_j}+1)},$$

$$\overline{\Omega_2} = \sum_{i=1}^{m} \overline{\theta_i} \frac{(\log \overline{\eta_i})^{\sigma+\overline{\mu_i}}}{\Gamma(\sigma+\overline{\mu_i}+1)} - \sum_{j=1}^{n} \overline{\phi_j} \frac{\left(\log \overline{\xi_j}\right)^{\sigma+\overline{\gamma_j}}}{\Gamma(\sigma+\overline{\gamma_j}+1)},$$

$$\overline{\Omega_3} = \sum_{k=1}^{p} \overline{\varepsilon_k} \frac{\left(\log \overline{\psi_k}\right)^{\overline{\varsigma_k}}}{\Gamma(\overline{\varsigma_k}+1)} - \sum_{l=1}^{q} \overline{\nu_l} \frac{(\log \overline{\varphi_l})^{\overline{\tau_l}}}{\Gamma(\overline{\tau_l}+1)},$$

$$\overline{\Omega_4} = \sum_{k=1}^{p} \overline{\varepsilon_k} \frac{\left(\log \overline{\psi_k}\right)^{\sigma+\overline{\varsigma_k}}}{\Gamma(\sigma+\overline{\varsigma_k}+1)} - \sum_{l=1}^{q} \overline{\nu_l} \frac{(\log \overline{\varphi_l})^{\sigma+\overline{\tau_l}}}{\Gamma(\sigma+\overline{\tau_l}+1)} \tag{4.4}$$

and

$$\overline{\Omega} = \overline{\Omega_1\Omega_4} - \overline{\Omega_2\Omega_3}. \tag{4.5}$$

Lemma 4.2.1. *[63] Let $\widetilde{\Omega} \neq 0$, $0 < \alpha, \beta \leq 1, 1 < \alpha + \beta \leq 2$, λ_1 is a given constant, $\mu_i, \gamma_j, \varsigma_k, \tau_l > 0$, the constants $\eta_i, \xi_j, \psi_k, \varphi_l \in (1, e)$ and $\theta_i, \phi_j, \varepsilon_k, \nu_l \in \mathbb{R}$, for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, p$, $l = 1, 2, \ldots, q$. Then the problem*

$$\begin{cases} D^{\alpha}(D^{\beta}+\lambda_1)\, x(t,\omega) = f(t,\, x(t,\omega),\, y(t,\omega), \omega) \\ \sum\limits_{i=1}^{m} \theta_i I^{\mu_i} x(\eta_i, \omega) = \sum\limits_{j=1}^{n} \phi_j I^{\gamma_j} x(\xi_j, \omega) \\ \sum\limits_{k=1}^{p} \varepsilon_k I^{\varsigma_k} x(\psi_k, \omega) = \sum\limits_{l=1}^{q} \nu_l I^{\tau_l} x(\varphi_l, \omega) \end{cases} \tag{4.6}$$

is equivalent to the problem

$$
\begin{aligned}
x(t) &= \tfrac{1}{\Omega}\Bigg[\Bigg(\Omega_4-\tfrac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_3\Bigg)\Bigg(\sum_{j=1}^{n}\phi_j\left[I^{\alpha+\beta+\gamma_j}f\left(\xi_j,\,x\left(\xi_j,\omega\right),\,y\left(\xi_j,\omega\right),\,\omega\right)-\lambda_1 I^{\beta+\gamma_j}x\left(\xi_j,\omega\right)\right]\\
&- \sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}f\left(\eta_i,\,x\left(\eta_i,\omega\right),\,y\left(\eta_i,\omega\right),\,\omega\right)-\lambda_1 I^{\beta+\mu_i}x\left(\eta_i,\omega\right)\right]\Bigg)\\
&+ \left(\tfrac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_1-\Omega_2\right)\left(\sum_{l=1}^{q}\upsilon_l\left[I^{\alpha+\beta+\tau_l}f\left(\varphi_l,\,x\left(\varphi_l,\omega\right),\,y\left(\varphi_l,\omega\right),\,\omega\right)-\lambda_1 I^{\beta+\tau_l}x\left(\varphi_l,\omega\right)\right]\right.\\
&- \left.\sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\varsigma_k}f\left(\psi_k,\,x\left(\psi_k,\omega\right),\,y\left(\psi_k,\omega\right),\,\omega\right)-\lambda_1 I^{\beta+\varsigma_k}x\left(\psi_k,\omega\right)\right]\right)\Bigg]\\
&+ I^{\alpha+\beta}f(t,\,x(t,\omega),\,y(t,\omega),\,\omega)-\lambda_1 I^{\beta}x(t,\omega).
\end{aligned}
\tag{4.7}
$$

Similarly

Let $\overline{\Omega}\neq 0, 0<\gamma,\sigma\leq 1, 1<\gamma+\sigma\leq 2, \lambda_2$ *is a given constant,* $\overline{\mu_i}, \overline{\gamma_j}, \overline{\varsigma_k}, \overline{\tau_l}>0$*, the constants* $\overline{\eta_i}, \overline{\xi_j}, \overline{\psi_k}, \overline{\varphi_l}\in(1,\,e)$ *and* $\overline{\theta_i}, \overline{\phi_j}, \overline{\varepsilon_k}, \overline{\nu_l}\in\mathbb{R}$*, for* $i=1,2,\ldots,m,\ j=1,2,\ldots,n,\ k=1,2,\ldots,p,\ l=1,2,\ldots,q$*. Then the problem*

$$
\begin{cases}
D^{\gamma}(D^{\sigma}+\lambda_2)\,y\,(t,\omega) &= g(t,\,x(t,\omega),\,y(t,\omega),\omega)\\
\sum\limits_{i=1}^{m}\overline{\theta_i}I^{\overline{\mu_i}}y(\overline{\eta_i},\omega) &= \sum\limits_{j=1}^{n}\overline{\phi_j}I^{\overline{\gamma_j}}y(\overline{\xi_j},\omega)\\
\sum\limits_{k=1}^{p}\overline{\varepsilon_k}I^{\overline{\varsigma_k}}y(\overline{\psi_k},\omega) &= \sum\limits_{l=1}^{q}\overline{\nu_l}I^{\overline{\tau_l}}y(\overline{\varphi_l},\omega)
\end{cases}
\tag{4.8}
$$

is equivalent to the problem

$$
\begin{aligned}
y(t) &= \tfrac{1}{\overline{\Omega}}\Bigg[\Bigg(\overline{\Omega_4}-\tfrac{(\log t)^{\sigma}}{\Gamma(\sigma+1)}\overline{\Omega_3}\Bigg)\Bigg(\sum_{j=1}^{n}\overline{\phi_j}\left[I^{\gamma+\sigma+\overline{\gamma_j}}g\left(\overline{\xi_j},\,x\left(\overline{\xi_j},\omega\right),\,y\left(\overline{\xi_j},\omega\right),\,\omega\right)-\lambda_2 I^{\sigma+\overline{\gamma_j}}y\left(\overline{\xi_j},\omega\right)\right]\\
&- \sum_{i=1}^{m}\overline{\theta_i}\left[I^{\gamma+\sigma+\overline{\mu_i}}g\left(\overline{\eta_i},\,x\left(\overline{\eta_i},\omega\right),\,y\left(\overline{\eta_i},\omega\right),\,\omega\right)-\lambda_2 I^{\sigma+\overline{\mu_i}}y\left(\overline{\eta_i},\omega\right)\right]\Bigg)\\
&+ \left(\tfrac{(\log t)^{\sigma}}{\Gamma(\sigma+1)}\overline{\Omega_1}-\overline{\Omega_2}\right)\left(\sum_{l=1}^{q}\overline{\upsilon_l}\left[I^{\gamma+\sigma+\overline{\tau_l}}g\left(\overline{\varphi_l},\,x\left(\overline{\varphi_l},\omega\right),\,y\left(\overline{\varphi_l},\omega\right),\,\omega\right)-\lambda_2 I^{\sigma+\overline{\tau_l}}y\left(\overline{\varphi_l},\omega\right)\right]\right.\\
&- \left.\sum_{k=1}^{p}\overline{\varepsilon_k}\left[I^{\gamma+\sigma+\overline{\varsigma_k}}g\left(\overline{\psi_k},\,x\left(\overline{\psi_k},\omega\right),\,y\left(\overline{\psi_k},\omega\right),\,\omega\right)-\lambda_2 I^{\sigma+\overline{\varsigma_k}}y\left(\overline{\psi_k},\omega\right)\right]\right)\Bigg]\\
&+ I^{\gamma+\sigma}g\left(t,\,x\left(t,\omega\right),\,y\left(t,\omega\right),\,\omega\right)-\lambda_2 I^{\sigma}y\left(t,\omega\right).
\end{aligned}
\tag{4.9}
$$

Let us set the constants

$$\begin{aligned}\Lambda_1(u) &= \frac{1}{\mid\Omega\mid}\Bigg[\left(\mid\Omega_4\mid+\frac{\mid\Omega_3\mid}{\Gamma(\beta+1)}\right)\left(\sum_{j=1}^{n}\mid\phi_j\mid\frac{(\log\xi_j)^{u+\beta+\gamma_j}}{\Gamma(u+\beta+\gamma_j+1)}\right.\\&+\left.\sum_{i=1}^{m}\mid\theta_i\mid\frac{(\log\eta_i)^{u+\beta+\mu_i}}{\Gamma(u+\beta+\mu_i+1)}\right)\\&+\left(\frac{\mid\Omega_1\mid}{\Gamma(\beta+1)}+\mid\Omega_2\mid\right)\left(\sum_{l=1}^{q}\mid\upsilon_l\mid\frac{(\log\varphi_l)^{u+\beta+\tau_l}}{\Gamma(u+\beta+\tau_l+1)}\right.\\&+\left.\sum_{k=1}^{p}\mid\varepsilon_k\mid\frac{(\log\psi_k)^{u+\beta+\varsigma_k}}{\Gamma(u+\beta+\varsigma_k+1)}\right)\Bigg]+\frac{1}{\Gamma(u+\beta+1)}\end{aligned}$$

and

$$\begin{aligned}\Lambda_2(u) &= \frac{1}{\mid\overline{\Omega}\mid}\Bigg[\left(\mid\overline{\Omega_4}\mid+\frac{\mid\overline{\Omega_3}\mid}{\Gamma(\sigma+1)}\right)\left(\sum_{j=1}^{n}\mid\overline{\phi_j}\mid\frac{\left(\log\overline{\xi_j}\right)^{u+\sigma+\overline{\gamma_j}}}{\Gamma(u+\sigma+\overline{\gamma_j}+1)}\right.\\&+\left.\sum_{i=1}^{m}\mid\overline{\theta_i}\mid\frac{(\log\overline{\eta_i})^{u+\sigma+\overline{\mu_i}}}{\Gamma(u+\sigma+\overline{\mu_i}+1)}\right)\\&+\left(\frac{\mid\overline{\Omega_1}\mid}{\Gamma(\sigma+1)}+\mid\overline{\Omega_2}\mid\right)\left(\sum_{l=1}^{q}\mid\overline{\upsilon_l}\mid\frac{(\log\overline{\varphi_l})^{u+\sigma+\overline{\tau_l}}}{\Gamma(u+\sigma+\overline{\tau_l}+1)}\right.\\&+\left.\sum_{k=1}^{p}\mid\overline{\varepsilon_k}\mid\frac{\left(\log\overline{\psi_k}\right)^{u+\sigma+\overline{\varsigma_k}}}{\Gamma(u+\sigma+\overline{\varsigma_k}+1)}\right)\Bigg]+\frac{1}{\Gamma(u+\sigma+1)}.\end{aligned}$$

Our main first result is the existence and uniqueness of random solution of the problem (4.1)

Theorem 4.2.1. *Let $f, g : [1, e] \times \mathbb{R}^m \times \mathbb{R}^m \times \widetilde{\Omega} \to \mathbb{R}^m$ are two Carathéodory functions. Assume that the following condition*

(H) *There exists random variables $p_1, p_2, p_3, p_4 : \widetilde{\Omega} \to \mathbb{R}_+$ such that*

$$\|f(t,x,y,\omega)-f(t,\tilde{x},\tilde{y},\omega)\| \leq p_1(\omega)\|x-\tilde{x}\|+p_2(\omega)\|y-\tilde{y}\|,\ \forall x,y,\tilde{x},\tilde{y}\in\mathbb{R}^m$$

and

$$\|g(t,x,y,\omega)-g(t,\tilde{x},\tilde{y},\omega)\| \le p_3(\omega)\|x-\tilde{x}\|+p_4(\omega)\|y-\tilde{y}\|,\ \forall x,y,\tilde{x},\tilde{y}\in\mathbb{R}^m$$

holds.
If for every $\omega \in \widetilde{\Omega}$, $\widetilde{M}(\omega)$ *converge to* 0, *where*

$$\widetilde{M}(\omega) = \begin{pmatrix} \Lambda_1(\alpha)p_1(\omega)+|\lambda_1|\Lambda_1(0) & \Lambda_1(\alpha)p_2(\omega) \\ \Lambda_2(\gamma)p_3(\omega) & \Lambda_2(\gamma)p_4(\omega)+|\lambda_2|\Lambda_2(0) \end{pmatrix},$$

then problem (4.1) has unique random solution.

Proof. Consider the operator $N : C([1,e],\mathbb{R}^m)\times C([1,e],\mathbb{R}^m)\times\Omega \to C([1,e],\mathbb{R}^m)\times C([1,e],\mathbb{R}^m)$,

$$(x(.,\omega),y(.,\omega),\omega)\mapsto (N_1(t,x(t,\omega),y(t,\omega),\omega),N_2(t,x(t,\omega),y(t,\omega),\omega))$$

where

$$\begin{aligned} N_1(x(t,\omega),y(t,\omega),\omega) &= \frac{1}{\Omega}\Bigg[\left(\Omega_4-\frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_3\right) \\ &\quad \Bigg(\sum_{j=1}^{n}\phi_j\left[I^{\alpha+\beta+\gamma_j}f(\xi_j,x(\xi_j,\omega),y(\xi_j,\omega),\omega)-\lambda_1 I^{\beta+\gamma_j}x(\xi_j,\omega)\right] \\ &\quad - \sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}f(\eta_i,\,x(\eta_i,\omega),\,y(\eta_i,\omega),\,\omega)-\lambda_1 I^{\beta+\mu_i}x(\eta_i,\omega)\right]\Bigg) \\ &\quad + \left(\frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_1-\Omega_2\right) \\ &\quad \Bigg(\sum_{l=1}^{q}\upsilon_l\left[I^{\alpha+\beta+\tau_l}f(\varphi_l,x(\varphi_l,\omega),y(\varphi_l,\omega),\omega)-\lambda_1 I^{\beta+\tau_l}x(\varphi_l,\omega)\right] \\ &\quad - \sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\varsigma_k}f(\psi_k,\,x(\psi_k,\omega),\,y(\psi_k,\omega),\,\omega)-\lambda_1 I^{\beta+\varsigma_k}x(\psi_k,\omega)\right]\Bigg)\Bigg] \\ &\quad + I^{\alpha+\beta}f(t,\,x(t,\omega),\,y(t,\omega),\,\omega)-\lambda_1 I^\beta x(t,\omega) \end{aligned}$$

and

$$\begin{aligned}
N_2\left(x(t,\omega),y(t,\omega),\omega\right) &= \tfrac{1}{\Omega}\Bigg[\left(\overline{\Omega_4}-\tfrac{(\log t)^{\sigma}}{\Gamma(\sigma+1)}\overline{\Omega_3}\right)\\
&\quad\left(\sum_{j=1}^{n}\overline{\phi_j}\left[I^{\gamma+\sigma+\overline{\gamma_j}}g\left(\overline{\xi_j},\,x\left(\overline{\xi_j},\omega\right),\,y\left(\overline{\xi_j},\omega\right),\,\omega\right)-\lambda_2 I^{\sigma+\overline{\gamma_j}}y\left(\overline{\xi_j},\omega\right)\right]\right.\\
&-\left.\sum_{i=1}^{m}\overline{\theta_i}\left[I^{\gamma+\sigma+\overline{\mu_i}}g\left(\overline{\eta_i},\,x\left(\overline{\eta_i},\omega\right),\,y\left(\overline{\eta_i},\omega\right),\,\omega\right)-\lambda_2 I^{\sigma+\overline{\mu_i}}y\left(\overline{\eta_i},\omega\right)\right]\right)\\
&+\left(\tfrac{(\log t)^{\sigma}}{\Gamma(\sigma+1)}\overline{\Omega_1}-\overline{\Omega_2}\right)\\
&\quad\left(\sum_{l=1}^{q}\overline{\upsilon_l}\left[I^{\gamma+\sigma+\overline{\tau_l}}g\left(\overline{\varphi_l},\,x\left(\overline{\varphi_l},\omega\right),\,y\left(\overline{\varphi_l},\omega\right),\,\omega\right)-\lambda_2 I^{\sigma+\overline{\tau_l}}y\left(\overline{\varphi_l},\omega\right)\right]\right)\\
&-\left.\sum_{k=1}^{p}\overline{\varepsilon_k}\left[I^{\gamma+\sigma+\overline{\varsigma_k}}g\left(\overline{\psi_k},\,x\left(\overline{\psi_k},\omega\right),\,y\left(\overline{\psi_k},\omega\right),\,\omega\right)-\lambda_2 I^{\sigma+\overline{\varsigma_k}}y\left(\overline{\psi_k},\omega\right)\right]\right)\Bigg]\\
&+ I^{\gamma+\sigma}g(t,\,x(t,\omega),\,y(t,\omega),\,\omega)-\lambda_2 I^{\sigma}y(t,\omega).
\end{aligned}$$

First we show that N is a random operator on $C\left([1,e],\mathbb{R}^m\right)\times C\left([1,e],\mathbb{R}^m\right)$. Since f and g are Carathéodory functions, then $\omega\to f(t,x,y,\omega)$ and $\omega\to g(t,x,y,\omega)$ are measurable maps in view of proposition 3.3.1 we concluded that, the maps
$\omega\to N_1(x(t,\omega),y(t,\omega),\omega),\omega\to N_2(x(t,\omega),y(t,\omega),\omega)$ are measurable. As a result, N is a random operator on $C\left([1,e],\mathbb{R}^m\right)\times C([1,e],\mathbb{R}^m)\times\Omega$ into $C([1,e],\mathbb{R}^m)\times C([1,e],\mathbb{R}^m)$. We show that N satisfies all the conditions of theorem 3.3.1 on $C([1,e],\mathbb{R}^m)\times C([1,e],\mathbb{R}^m)$. Let $(x,y),(\tilde{x},\tilde{y})\in C([1,e],\mathbb{R}^m)\times C([1,e],\mathbb{R}^m)$, then

$$\begin{aligned}
&\left\|N_1(x(t,\omega),y(t,\omega),\omega)-N_1(\tilde{x}(t,\omega),\tilde{y}(t,\omega),\omega)\right\|\\
=&\left\|\tfrac{1}{\Omega}\left[\left(\Omega_4-\tfrac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_3\right)\left(\sum_{j=1}^{n}\phi_j\Big[I^{\alpha+\beta+\gamma_j}f\left(\xi_j,\,x\left(\xi_j,\omega\right),\,y\left(\xi_j,\omega\right)\right)\right.\right.\right.\\
-&\ I^{\alpha+\beta+\gamma_j}f\left(\xi_j,\,\tilde{x}\left(\xi_j,\omega\right),\,\tilde{y}\left(\xi_j,\omega\right)\right)-\lambda_1\left[I^{\beta+\gamma_j}x\left(\xi_j,\omega\right)-I^{\beta+\gamma_j}\tilde{x}\left(\xi_j,\omega\right)\right]\Big]\\
-&\sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}f\left(\eta_i,\,x\left(\eta_i,\omega\right),\,y\left(\eta_i,\omega\right)\right)-I^{\alpha+\beta+\mu_i}f\left(\eta_i,\,\tilde{x}\left(\eta_i,\omega\right),\,\tilde{y}\left(\eta_i,\omega\right)\right)\right]\\
-&\left.\lambda_1\left[I^{\beta+\mu_i}x\left(\eta_i,\omega\right)-I^{\beta+\mu_i}\tilde{x}\left(\eta_i,\omega\right)\right]\Big]\right)\\
+&\left(\tfrac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_1-\Omega_2\right)\left(\sum_{l=1}^{q}\upsilon_l\Big[I^{\alpha+\beta+\tau_l}f\left(\varphi_l,\,x\left(\varphi_l,\omega\right),\,y\left(\varphi_l,\omega\right)\right)\right.\\
-&\ I^{\alpha+\beta+\tau_l}f\left(\varphi_l,\,\tilde{x}\left(\varphi_l,\omega\right),\,\tilde{y}\left(\varphi_l,\omega\right)\right)-\lambda_1\left[I^{\beta+\tau_l}x\left(\varphi_l,\omega\right)-I^{\beta+\tau_l}\tilde{x}\left(\varphi_l,\omega\right)\right]\Big]\\
-&\sum_{k=1}^{p}\varepsilon_k\Big[I^{\alpha+\beta+\varsigma_k}f\left(\psi_k,\,x\left(\psi_k,\omega\right),\,y\left(\psi_k,\omega\right)\right)-I^{\alpha+\beta+\varsigma_k}f\left(\psi_k,\,\tilde{x}\left(\psi_k,\omega\right),\,\tilde{y}\left(\psi_k,\omega\right)\right)
\end{aligned}$$

$$
\begin{aligned}
- & \lambda_1 \left[I^{\beta+\varsigma_k} x(\psi_k,\omega) - I^{\beta+\varsigma_k} \tilde{x}(\psi_k,\omega) \right] \Big) \Big] \\
+ & I^{\alpha+\beta} f(t,\, x(t,\omega)\,,\, y(t,\omega)) - I^{\alpha+\beta} f(t,\, \tilde{x}(t,\omega)\,,\, \tilde{y}(t,\omega)) - \lambda_1 \left[I^{\beta} x(t,\omega) - I^{\beta} \tilde{x}(t,\omega) \right] \Bigg\| \\
\leq & \frac{1}{\Omega} \Bigg[\left| \Omega_4 - \frac{(\log\, t)^{\beta}}{\Gamma(\beta+1)} \Omega_3 \right| \Bigg(\sum_{j=1}^{n} \sum \phi_j \Bigg[I^{\alpha+\beta+\gamma_j} \Big\| f(\xi_j,\, x(\xi_j,\omega)\,,\, y(\xi_j,\omega)) \\
- & f(\xi_j,\, \tilde{x}(\xi_j,\omega)\,,\, \tilde{y}(\xi_j,\omega)) \Big\| + \lambda_1 I^{\beta+\gamma_j} \Big\| x(\xi_j,\omega) - \tilde{x}(\xi_j,\omega) \Big\| \Bigg] \\
+ & \sum_{i=1}^{m} \theta_i \Bigg[I^{\alpha+\beta+\mu_i} \Big\| f(\eta_i,\, x(\eta_i,\omega)\,,\, y(\eta_i,\omega)) \\
- & f(\eta_i,\, \tilde{x}(\eta_i,\omega)\,,\, \tilde{y}(\eta_i,\omega)) \Big\| + \lambda_1 I^{\beta+\mu_i} \Big\| x(\eta_i,\omega) - \tilde{x}(\eta_i,\omega) \Big\| \Bigg] \Bigg) \\
+ & \left| \frac{(\log\, t)^{\beta}}{\Gamma(\beta+1)} \Omega_1 - \Omega_2 \right| \Bigg(\sum_{l=1}^{q} \upsilon_l \Bigg[I^{\alpha+\beta+\tau_l} \Big\| f(\varphi_l,\, x(\varphi_l,\omega)\,,\, y(\varphi_l,\omega)) \\
- & f(\varphi_l,\, \tilde{x}(\varphi_l,\omega)\,,\, \tilde{y}(\varphi_l,\omega)) \Big\| + \lambda_1 I^{\beta+\tau_l} \Big\| x(\varphi_l,\omega) - \tilde{x}(\varphi_l,\omega) \Big\| \Bigg] \\
+ & \sum_{k=1}^{p} \varepsilon_k \Bigg[I^{\alpha+\beta+\varsigma_k} \Big\| f(\psi_k,\, x(\psi_k,\omega)\,,\, y(\psi_k,\omega)) - f(\psi_k,\, \tilde{x}(\psi_k,\omega)\,,\, \tilde{y}(\psi_k,\omega)) \Big\| \Bigg] \\
+ & \lambda_1 I^{\beta+\varsigma_k} \Big\| x(\psi_k,\omega) - \tilde{x}(\psi_k,\omega) \Big\| \Bigg] \Bigg) \Bigg] \\
+ & I^{\alpha+\beta} \Big\| f(t,\, x(t,\omega)\,,\, y(t,\omega)) - f(t,\, \tilde{x}(t,\omega)\,,\, \tilde{y}(t,\omega)) \Big\| + \lambda_1 I^{\beta} \Big\| x(t,\omega) - \tilde{x}(t,\omega) \Big\| \\
\leq & p_1(\omega) \Bigg\{ \frac{1}{|\Omega|} \Bigg[\left(|\Omega_4| + \frac{|\Omega_3|}{\Gamma(\beta+1)} \right) \Bigg(\sum_{j=1}^{n} |\phi_j| \frac{(\log\, \xi_j)^{\alpha+\beta+\gamma_j}}{\Gamma(\alpha+\beta+\gamma_j+1)} \\
+ & \sum_{i=1}^{m} |\theta_i| \frac{(\log\, \eta)^{\alpha+\beta+\mu_i}}{\Gamma(\alpha+\beta+\mu_i+1)} \Bigg) \\
+ & \left(\frac{(\log\, t)^{\beta}}{\Gamma(\beta+1)} |\Omega_1| + |\Omega_2| \right) \Bigg(\sum_{l=1}^{q} |\upsilon_l| \frac{(\log\, \varphi_l)^{\alpha+\beta+\tau_l}}{\Gamma(\alpha+\beta+\tau_l+1)} \\
+ & \sum_{k=1}^{p} |\varepsilon_k| \frac{(\log\, \psi_k)^{\alpha+\beta+\varsigma_k}}{\Gamma(\alpha+\beta+\varsigma_k+1)} \Bigg] + \frac{1}{\Gamma(\alpha+\beta+1)} \Bigg\} \Big\| x(.,\omega) - \tilde{x}(.,\omega) \Big\| \\
+ & |\lambda_1| \Bigg\{ \frac{1}{|\Omega|} \Bigg[\left(|\Omega_4| + \frac{|\Omega_3|}{\Gamma(\beta+1)} \right) \Bigg(\sum_{j=1}^{n} |\phi_j| \frac{(\log\, \xi_j)^{\beta+\gamma_j}}{\Gamma(\beta+\gamma_j+1)} \\
+ & \sum_{i=1}^{m} |\theta_i| \frac{(\log\, \eta)^{\beta+\mu_i}}{\Gamma(\beta+\mu_i+1)} \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\left|\Omega_1\right|+\left|\Omega_2\right|\right)\left(\sum_{l=1}^{q}\left|\upsilon_l\right| \frac{(\log \varphi_l)^{\beta+\tau_l}}{\Gamma(\beta+\tau_l+1)}\right.\\
&+ \left.\left.\sum_{k=1}^{p}\left|\varepsilon_k\right| \frac{(\log \psi_k)^{\beta+\varsigma_k}}{\Gamma(\beta+\varsigma_k+1)}\right]+\frac{1}{\Gamma(\beta+1)}\right\}\left\|x(\cdot,\omega)-\tilde{x}(\cdot,\omega)\right\|\\
&+ p_2(\omega)\left\{\frac{1}{|\Omega|}\left[\left(\left|\Omega_4\right|+\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\left|\Omega_3\right|\right)\left(\sum_{j=1}^{n}\left|\phi_j\right| \frac{(\log \xi_j)^{\alpha+\beta+\gamma_j}}{\Gamma(\alpha+\beta+\gamma_j+1)}\right.\right.\right.\\
&+ \left.\sum_{i=1}^{m}\left|\theta_i\right| \frac{(\log \eta)^{\alpha+\beta+\mu_i}}{\Gamma(\alpha+\beta+\mu_i+1)}\right)\\
&+ \left(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\left|\Omega_1\right|+\left|\Omega_2\right|\right)\left(\sum_{l=1}^{q}\left|\upsilon_l\right| \frac{(\log \varphi_l)^{\alpha+\beta+\tau_l}}{\Gamma(\alpha+\beta+\tau_l+1)}\right.\\
&+ \left.\left.\left.\sum_{k=1}^{p}\left|\varepsilon_k\right| \frac{(\log \psi_k)^{\alpha+\beta+\varsigma_k}}{\Gamma(\alpha+\beta+\varsigma_k+1)}\right]+\frac{1}{\Gamma(\alpha+\beta+1)}\right\}\left\|y(\cdot,\omega)-\tilde{y}(\cdot,\omega)\right\|\\
&+ \left|\lambda_1\right|\left\{\frac{1}{|\Omega|}\left[\left(\left|\Omega_4\right|+\frac{\left|\Omega_3\right|}{\Gamma(\beta+1)}\right)\left(\sum_{j=1}^{n}\left|\phi_j\right| \frac{(\log \xi_j)^{\beta+\gamma_j}}{\Gamma(\beta+\gamma_j+1)}\right.\right.\right.\\
&+ \left.\sum_{i=1}^{m}\left|\theta_i\right| \frac{(\log \eta)^{\beta+\mu_i}}{\Gamma(\beta+\mu_i+1)}\right)\\
&+ \left(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\left|\Omega_1\right|+\left|\Omega_2\right|\right)\left(\sum_{l=1}^{q}\left|\upsilon_l\right| \frac{(\log \varphi_l)^{\beta+\tau_l}}{\Gamma(\beta+\tau_l+1)}\right.\\
&+ \left.\left.\left.\sum_{k=1}^{p}\left|\varepsilon_k\right| \frac{(\log \psi_k)^{\beta+\varsigma_k}}{\Gamma(\beta+\varsigma_k+1)}\right]+\frac{1}{\Gamma(\beta+1)}\right\}\left\|y(\cdot,\omega)-\tilde{y}(\cdot,\omega)\right\|\\
&\le \left(p_1(\omega)\Lambda_1(\alpha)+|\lambda_1|\Lambda_1(0)\right)\|x(\cdot,\omega)-\tilde{x}(\cdot,\omega)\|_{\infty}+p_2(\omega)\Lambda(\alpha)\|y(\cdot,\omega)-\tilde{y}(\cdot,\omega)\|_{\infty},
\end{aligned}
$$

then

$$
\begin{aligned}
\left\|N_1(t,x,y,\omega)-N_1(t,\tilde{x},\tilde{y},\omega)\right\| \le {} & \left(p_1(\omega)\Lambda_1(\alpha)+|\lambda_1|\Lambda_1(0)\right)|x-\tilde{x}|\\
& + p_2(\omega)\Lambda_1(\alpha)|y-\tilde{y}|.
\end{aligned}
$$

Similarly, we obtain

$$\|N_2(t,x,y,\omega) - N_2(t,\tilde{x},\tilde{y},\omega)\| \leq p_3(\omega)\,\Lambda_2(\gamma)\,|x-\tilde{x}| + \Bigg(p_4(\omega)\,\Lambda_2(\gamma) + |\lambda_2|\Lambda_2(0)\Bigg)\,|y-\tilde{y}|\,.$$

Hence

$$d\left(N\left(x\left(\cdot,\omega\right),y\left(\cdot,\omega\right),\omega\right),N\left(\tilde{x}\left(\cdot,\omega\right),\tilde{y}\left(\cdot,\omega\right),\omega\right)\right) \leq \tilde{M}\left(\omega\right)d\left(\left(x\left(\cdot,\omega\right),y\left(\cdot,\omega\right)\right),\left(\tilde{x}\left(\cdot,\omega\right),\tilde{y}\left(\cdot,\omega\right)\right)\right),$$

where

$$d(x,y) = \begin{pmatrix} \|x(\cdot,\omega) & - & y(\cdot,\omega)\|_\infty \\ \|x(\cdot,\omega) & - & y(\cdot,\omega)\|_\infty \end{pmatrix},$$

and

$$\widetilde{M}(\omega) = \begin{pmatrix} \Lambda_1(\alpha)\,p_1(\omega) + |\lambda_1|\Lambda_1(0) & \Lambda_1(\alpha)\,p_2(\omega) \\ \Lambda_2(\gamma)\,p_3(\omega) & \Lambda_2(\gamma)\,p_4(\omega) + |\lambda_2|\Lambda_2(0) \end{pmatrix}.$$

Since for every $\omega \in \Omega$, $\widetilde{M}(\omega) \in M_{n\times n}(\mathbb{R}_+)$ converge to zero, then from theorem 3.3.1 there exists unique random solution of problem (4.1). This completes the proof. □

Now, we present an existence result without Lipschitz conditions. We consider the following hypotheses:

(H_1) For every $\omega \in \Omega$, the functions $f(\cdot,\cdot,\cdot,\omega)$ and $g(\cdot,\cdot,\cdot,\omega)$ are continuous and $\omega \to f(\cdot,\cdot,\cdot,\omega)$, $\omega \to g(\cdot,\cdot,\cdot,\omega)$ are measurable.

(H_2) There exists measurable and bounded functions γ_1, $\gamma_2 : \Omega \to \mathbb{R}_+$ such that

$$\|f(t,x,y,\omega)\| \leq \gamma_1(\omega)(\|x\| + \|y\| + 1),$$

$$\|g(t,x,y,\omega)\| \leq \gamma_2(\omega)(\|x\| + \|y\| + 1),$$

for all $t \in [1, e]$, $\omega \in \Omega$ and $x, y \in \mathbb{R}^m$.

We shall rely on Leray-Schauder random fixed point theorem type in generalized Banach space to prove our existence result.

Theorem 4.2.2.

Assume that the hypotheses (H_1), (H_2) *and the condition*

$$\Lambda_1(\alpha)\gamma_1(\omega) + \Lambda_2(\gamma)\gamma_2(\omega) + |\lambda_1|\Lambda_1(0) + |\lambda_2|\Lambda_2(0) < 1, \tag{4.10}$$

hold. Then the problem (4.1) has a random solution defined on $[1, e]$*. Moreover, the solution set*

$$S = \{(x, y) : \Omega \to C([1, e], \mathbb{R}^m) \times C([1, e], \mathbb{R}^m) : (x(\cdot, \omega), y(\cdot, \omega)), \omega \in \Omega \, is \, solution \, of \ (4.1)\}$$

is compact.

Proof.

Let $N : C([1, e], \mathbb{R}^m) \times C([1, e], \mathbb{R}^m) \times \Omega \to C([1, e], \mathbb{R}^m) \times C([1, e], \mathbb{R}^m)$ be a random operator defined in Theorem 4.2.1. In order to apply theorem 3.3.3, we first show that N is completely continuous. The proof will be given in several steps.

Step 1.

$N(\cdot, \cdot, \omega) = (N_1(\cdot, \cdot, \omega), N_2(\cdot, \cdot, \omega))$ is continuous.

Let (x_n, y_n) be a sequence such that $(x_n, y_n) \to (x, y) \in C([1, e], \mathbb{R}^m) \times C([1, e], \mathbb{R}^m)$ as $n \to \infty$. Since f is a continuous function, then

$$\sum_{j=1}^{n} |\phi_j| \| I^{\alpha+\beta+\gamma_j} f(\xi_j, \, x_n(\xi_j, \omega), \, y_n(\xi_j, \omega), \omega) - I^{\alpha+\beta+\gamma_j} f(\xi_j, \, x(\xi_j, \omega), \, y(\xi_j, \omega), \omega) \|_\infty \to 0 \quad \text{as } n \to \infty,$$
$$\sum_{i=1}^{m} |\theta_i| \| I^{\alpha+\beta+\mu_i} f\left(\eta_i, \, x_n(\eta_i, \omega), \, y_n(\eta_i, \omega)\right) - I^{\alpha+\beta+\mu_i} f\left(\eta_i, \, x(\eta_i, \omega), \, y(\eta_i, \omega)\right) \|_\infty \to 0 \quad \text{as } n \to \infty,$$
$$\sum_{l=1}^{q} |v_l| \| I^{\alpha+\beta+\tau_l} f\left(\varphi_l, \, x_n(\varphi_l, \omega), \, y_n(\varphi_l, \omega)\right) - I^{\alpha+\beta+\gamma_j} f\left(\varphi_l, \, x(\varphi_l, \omega), \, y(\varphi_l, \omega)\right) \|_\infty \to 0 \quad \text{as } n \to \infty.$$

and

$$\|I^{\alpha+\beta}(f(t,\, x_n(t,\omega),\, y_n(t,\omega),\omega) - f(t,\, x(t,\omega),\, y(t,\omega),\, \omega)\|_\infty$$
$$+|\lambda_1|\|I^{\beta}(x_n(t,\omega) - x(t,\omega)\|_\infty \to 0 \quad \text{as } n \to \infty$$

Thus

$$\left\|N_1\left(x_n\left(\cdot,\omega\right),y_n\left(\cdot,\omega\right),\omega\right) - N_1\left(x\left(\cdot,\omega\right),y\left(\cdot,\omega\right),\omega\right)\right\|_\infty \to 0 \;as\; n \to \infty.$$

Similarly

$$\left\|N_2\left(x_n\left(\cdot,\omega\right),y_n\left(\cdot,\omega\right),\omega\right) - N_2\left(x\left(\cdot,\omega\right),y\left(.,\omega\right),\omega\right)\right\|_\infty \to 0 \;as\; n \to \infty.$$

Thus N is continuous.

Step 2.

N maps bounded sets into bounded sets in $C([1,e],\mathbb{R}^m) \times C([1,e],\mathbb{R}^m)$. Indeed, it is enough to show that for any $q > 0$ there exists a positive constant l such that for each $(x,y) \in B_q = \{(x,y) \in C([1,e],\mathbb{R}) \times C([1,e],\mathbb{R}) : (\|x\|_\infty, \|y\|_\infty) \leq (q,q)\}$, we have $\left\|N\left(x,y,\omega\right)\right\|_\infty \leq l = (l_1, l_2)$.

Then for each $t \in [1,e]$; we get

$$\begin{aligned}
& \|N_1(x\left(t\right),y\left(t\right),\omega)\| \\
= & \Bigg\| \frac{1}{\Omega}\Bigg[\left(\Omega_4 - \frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_3\right)\Bigg(\sum_{j=1}^{n}\phi_j\Big[I^{\alpha+\beta+\gamma_j}f\left(\xi_j,\, x\left(\xi_j,\omega\right),\, y\left(\xi_j,\omega\right)\right) \\
- & \lambda_1 I^{\beta+\gamma_j}x\left(\xi_j,\omega\right)\Big] \\
- & \sum_{i=1}^{m}\theta_i\Big[I^{\alpha+\beta+\mu_i}f\left(\eta_i,\, x\left(\eta_i,\omega\right),\, y\left(\eta_i,\omega\right)\right) - \lambda_1 I^{\beta+\mu_i}x\left(\eta_i,\omega\right)\Big]\Bigg) \\
+ & \left(\frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_1 - \Omega_2\right)\Bigg(\sum_{l=1}^{q}\upsilon_l\Big[I^{\alpha+\beta+\tau_l}f\left(\varphi_l,\, x\left(\varphi_l,\omega\right),\, y\left(\varphi_l,\omega\right)\right) - \lambda_1 I^{\beta+\tau_l}x\left(\varphi_l,\omega\right)\Big] \\
- & \sum_{k=1}^{p}\varepsilon_k\Big[I^{\alpha+\beta+\varsigma_k}f\left(\psi_k,\, x\left(\psi_k,\omega\right),\, y\left(\psi_k,\omega\right)\right) - \lambda_1 I^{\beta+\varsigma_k}x\left(\psi_k,\omega\right)\Big]\Bigg)\Bigg] \\
+ & I^{\alpha+\beta}f\left(t,\, x\left(t,\omega\right),\, y\left(t,\omega\right)\right) - \lambda_1 I^{\beta}x\left(t,\omega\right)\Bigg\|
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \frac{1}{|\Omega|}\left[\left(\left(\mid \Omega_4 \mid + \frac{(\log t)^{\beta}}{\Gamma(\beta+1)} \mid \Omega_3 \mid\right)\left(\sum_{j=1}^{n} \mid \phi_j \mid \left[I^{\alpha+\beta+\gamma_j}\left\| f\left(\xi_j,\ x\left(\xi_j,\omega\right),\ y\left(\xi_j,\omega\right)\right)\right\|\right.\right.\right.\right.\\
+\ & \left.\mid \lambda_1 \mid I^{\beta+\gamma_j}\|x\left(\xi_j,\omega\right)\|\right]\\
+\ & \left.\sum_{i=1}^{m} \mid \theta_i \mid \left[I^{\alpha+\beta+\mu_i}\left\| f\left(\eta_i,\ x\left(\eta_i,\omega\right),\ y\left(\eta_i,\omega\right)\right)\right\| + \mid \lambda_1 \mid I^{\beta+\mu_i}\|x\left(\eta_i,\omega\right)\|\right]\right)\\
+\ & \left(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)} \mid \Omega_1 \mid + \mid \Omega_2 \mid\right)\left(\sum_{l=1}^{q} \mid \upsilon_l \mid \left[I^{\alpha+\beta+\tau_l}\left\| f\left(\varphi_l,\ x\left(\varphi_l,\omega\right),\ y\left(\varphi_l,\omega\right)\right)\right\|\right.\right.\\
+\ & \left.\mid \lambda_1 \mid I^{\beta+\tau_l}\|x\left(\varphi_l,\omega\right)\|\right]\\
+\ & \left.\left.\sum_{k=1}^{p} \mid \varepsilon_k \mid \left[I^{\alpha+\beta+\varsigma_k}\left\| f\left(\psi_k,\ x\left(\psi_k,\omega\right),\ y\left(\psi_k,\omega\right)\right)\right\| + \mid \lambda_1 \mid I^{\beta+\varsigma_k}\|x\left(\psi_k,\omega\right)\|\right]\right)\right]\\
& + I^{\alpha+\beta}\left\| f\left(t,\ x\left(t,\omega\right),\ y\left(t,\omega\right)\right)\right\| + \mid \lambda_1 \mid I^{\beta}\|x\left(t,\omega\right)\|\\
& \leq \frac{1}{|\Omega|}\left[\left(\left(\mid \Omega_4 \mid + \frac{(\log t)^{\beta}}{\Gamma(\beta+1)} \mid \Omega_3 \mid\right)\left(\sum_{j=1}^{n} \mid \phi_j \mid \left[I^{\alpha+\beta+\gamma_j}\gamma_1\left(\omega\right)\left(\|x(\xi_j,\omega)\| + \|y(\xi_j,\omega)\|\right)\right.\right.\right.\right.\\
& + \left.\mid \lambda_1 \mid I^{\beta+\gamma_j}\|x(\xi_j,\omega)\|\right]\\
& + \left.\left.\sum_{i=1}^{m} \mid \theta_i \mid \left[I^{\alpha+\beta+\mu_i}\gamma_1\left(\omega\right)\left(\|x(\eta_i,\omega)\| + \|y(\eta_i,\omega)\|\right) + \mid \lambda_1 \mid I^{\beta+\mu_i}\|x(\eta_i,\omega)\|\right]\right)\right]\\
& + \left(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)} \mid \Omega_1 \mid + \mid \Omega_2 \mid\right)\left(\sum_{l=1}^{q} \mid \upsilon_l \mid \left[I^{\alpha+\beta+\tau_l}\gamma_1\left(\omega\right)\left(\|x(\varphi_l,\omega)\| + \|y(\varphi_l,\omega)\|\right)\right.\right.\\
& + \left.\mid \lambda_1 \mid I^{\beta+\tau_l}\|x\left(\varphi_l,\omega\right)\|\right]\\
& + \left.\left.\sum_{k=1}^{p} \mid \varepsilon_k \mid \left[I^{\alpha+\beta+\varsigma_k}\gamma_1\left(\omega\right)\left(\|x\left(\psi_k,\omega\right)\| + \|y\left(\psi_k,\omega\right)\|\right) + \mid \lambda_1 \mid I^{\beta+\varsigma_k}\|x\left(\psi_k,\omega\right)\|\right]\right)\right]\\
& + I^{\alpha+\beta}\gamma_1\left(\omega\right)\left(\|x\left(t,\omega\right)\| + \|y\left(t,\omega\right)\|\right) + \mid \lambda_1 \mid I^{\beta}\|x\left(t,\omega\right)\|\\
& \leq \gamma_1\left(\omega\right)\Lambda_1\left(\alpha\right)\left(\|x\| + \|y\|\right) + \mid \lambda_1 \mid q\Lambda_1\left(0\right)\\
& \leq \gamma_1\left(\omega\right)\Lambda_1\left(\alpha\right)\left(2q\right) + \mid \lambda_1 \mid q\Lambda_1\left(0\right).
\end{aligned}
$$

Then

$$\|N_1\left(x\left(\cdot,\omega\right), y\left(\cdot,\omega\right), \omega\right)\|_{\infty} \leq \gamma_1\left(\omega\right)\Lambda_1\left(\alpha\right)\left(2q\right) + \mid \lambda \mid q\Lambda_1\left(0\right) \leq l_1\left(\omega\right).$$

Similarly, we have

$$\|N_2\left(x\left(\cdot,\omega\right), y\left(\cdot,\omega\right), \omega\right)\|_{\infty} \leq \gamma_2\left(\omega\right)\Lambda_2\left(\gamma\right)\left(2q\right) + \mid \lambda \mid q\Lambda_2\left(0\right) \leq l_2\left(\omega\right).$$

Step 3.

Next, we will show that N maps bounded sets into equicontinuous sets of $C([1,e],\mathbb{R}^m)\times C([1,e],\mathbb{R}^m)$. Let $B_r=\{(x,y)\in C([1,e],\mathbb{R}^m)\times C([1,e],\mathbb{R}^m):\|x\|\leq r,\ \|y\|\leq r\}$ be a bounded set in $C([1,e],\mathbb{R}^m)\times C([1,e],\mathbb{R}^m)$ as in Step 2.
Let $t_1,t_2\in[1,e]$ with $t_1<t_2$ and $(x,y)\in B_r$.
Then we have

$$
\begin{aligned}
& \left\| N_1(x(t_2,\omega),y(t_2,\omega),\omega)-N_1(x(t_1,\omega),y(t_1,\omega),\omega)\right\| \\
= & \left\| \frac{1}{\Omega}\left[\left(\frac{(\log t_2)^\beta-(\log t_1)^\beta}{\Gamma(\beta+1)}\Omega_3\right)\right.\right. \\
& \left(\sum_{j=1}^{n}\phi_j\left[I^{\alpha+\beta+\gamma_j}f(\xi_j,x(\xi_j,\omega),y(\xi_j,\omega),\omega)-\lambda_1 I^{\beta+\gamma_j}x(\xi_j,\omega)\right]\right. \\
- & \left.\sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}f(\eta_i,\ x(\eta_i,\omega),\ y(\eta_i,\omega),\ \omega)-\lambda_1 I^{\beta+\mu_i}x(\eta_i,\omega)\right]\right) \\
+ & \left(\frac{(\log t_2)^\beta-(\log t_1)^\beta}{\Gamma(\beta+1)}\Omega_1\right) \\
& \left(\sum_{l=1}^{q}\upsilon_l\left[I^{\alpha+\beta+\tau_l}f(\varphi_l,x(\varphi_l,\omega),y(\varphi_l,\omega),\omega)-\lambda_1 I^{\beta+\tau_l}x(\varphi_l,\omega)\right]\right. \\
- & \left.\left.\sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\varsigma_k}f(\psi_k,\ x(\psi_k,\omega),\ y(\psi_k,\omega),\ \omega)-\lambda_1 I^{\beta+\varsigma_k}x(\psi_k,\omega)\right]\right)\right] \\
+ & I^{\alpha+\beta}f(t,\ x(t_2,\omega),\ y(t_2,\omega),\ \omega)-\lambda_1 I^\beta x(t_2,\omega) \\
- & \left. I^{\alpha+\beta}f(t,\ x(t_1,\omega),\ y(t_1,\omega),\ \omega)+\lambda_1 I^\beta x(t_1,\omega)\right\| \\
\leq & \frac{1}{|\Omega|}\left[\left(\frac{(\log t_2)^\beta-(\log t_1)^\beta}{\Gamma(\beta+1)}\mid\Omega_3\mid\right)\left(\sum_{j=1}^{n}\mid\phi_j\mid\left[I^{\alpha+\beta+\gamma_j}\gamma_1(\omega)\left(\|x(\xi_j,\omega)\|+\|y(\xi_j,\omega)\|+1\right)\right.\right.\right. \\
+ & \left.\mid\lambda_1\mid I^{\beta+\gamma_j}\|x(\xi_j,\omega)\|\right] \\
+ & \left.\sum_{i=1}^{m}\mid\theta_i\mid\left[I^{\alpha+\beta+\mu_i}\gamma_1(\omega)(\|x(\eta_i,\omega)\|+\|y(\eta_i,\omega)\|+1)+\mid\lambda_1\mid I^{\beta+\mu_i}\|x(\eta_i,\omega)\|\right]\right) \\
+ & \left(\frac{(\log t_2)^\beta-(\log t_1)^\beta}{\Gamma(\beta+1)}\mid\Omega_1\mid\right)\left(\sum_{l=1}^{q}\mid\upsilon_l\mid\left[I^{\alpha+\beta+\tau_l}\gamma_1(\omega)\left(\|x(\varphi_l,\omega)\|+\|y(\varphi_l,\omega)\|+1\right)\right.\right. \\
+ & \left.\mid\lambda_1\mid I^{\beta+\tau_l}\|x(\varphi_l,\omega)\|\right] \\
+ & \left.\left.\sum_{k=1}^{p}\mid\varepsilon_k\mid\left[I^{\alpha+\beta+\varsigma_k}\gamma_1(\omega)\left(\|x(\psi_k,\omega)\|+\|y(\psi_k,\omega)\|\right)+\mid\lambda_1\mid I^{\beta+\varsigma_k}\|x(\psi_k,\omega)\|\right]\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{|\lambda_1| \|x\|}{\Gamma(\beta+1)} \left| (\log\, t_2)^{\beta} - (\log\, t_1)^{\beta} + 2\left(\log\, \frac{t_2}{t_1}\right)^{\beta} \right| \\
&\leq \quad \frac{1}{|\Omega|} \left[\left(\left(\frac{(\log\, t_2)^{\beta} - (\log\, t_1)^{\beta}}{\Gamma(\beta+1)} \mid \Omega_3 \mid \right) \left(\sum_{j=1}^{n} \mid \phi_j \mid \left[\frac{(2r+1)(\log\, \xi_j)^{\alpha+\beta+\gamma_j} \gamma_1(\omega)}{\Gamma(\alpha+\beta+\gamma_j+1)} \right.\right.\right.\right. \\
&+ \quad \left. \frac{|\lambda_1| r (\log\, \xi_j)^{\beta+\gamma_j}}{\Gamma(\beta+\gamma_j+1)} \right] \\
&+ \quad \left. \sum_{i=1}^{m} \mid \theta_i \mid \left[\frac{(2r+1)(\log\, \eta_i)^{\alpha+\beta+\mu_i} \gamma_1(\omega)}{\Gamma(\alpha+\beta+\mu_i+1)} + \frac{|\lambda_1| r (\log\, \eta_i)^{\beta+\mu_i}}{\Gamma(\beta+\mu_i+1)} \right] \right) \\
&+ \quad \left(\frac{(\log\, t_2)^{\beta} - (\log\, t_1)^{\beta}}{\Gamma(\beta+1)} \mid \Omega_1 \mid \right) \left(\sum_{l=1}^{q} \mid \upsilon_l \mid \left[\frac{(2r+1)(\log\, \varphi_l)^{\alpha+\beta+\tau_l} \gamma_1(\omega)}{\Gamma(\alpha+\beta+\tau_l+1)} \right.\right. \\
&+ \quad \left. \frac{|\lambda_1| r (\log\, \varphi_l)^{\beta+\tau_l}}{\Gamma(\beta+\tau_l+1)} \right] \\
&+ \quad \left.\left. \sum_{k=1}^{p} \mid \varepsilon_k \mid \left[\frac{(2r+1)(\log\, \psi_k)^{\alpha+\beta+\varsigma_k} \gamma_1(\omega)}{\Gamma(\alpha+\beta+\varsigma_k+1)} + \frac{|\lambda| r (\log\, \psi_k)^{\beta+\varsigma_k}}{\Gamma(\beta+\varsigma_k+1)} \right] \right) \right] \qquad . \\
&+ \quad \frac{\gamma_1(\omega)(2r+1)}{\Gamma(\alpha+\beta+1)} \left| (\log\, t_2)^{\alpha+\beta} - (\log\, t_1)^{\alpha+\beta} \right| \\
&+ \quad \frac{|\lambda_1| r}{\Gamma(\beta+1)} \left| (\log\, t_2)^{\beta} - (\log\, t_1)^{\beta} + 2(\log\, \frac{t_2}{t_1})^{\beta} \right|.
\end{aligned}
$$

Hence

$$
\begin{aligned}
& \quad \left\| N_1\left(x\left(t_2,\omega\right), y\left(t_2,\omega\right), \omega\right) - N_1\left(x\left(t_1,\omega\right), y\left(t_1,\omega\right), \omega\right) \right\| \\
&\leq \quad \frac{1}{|\Omega|} \left[\left(\left(\frac{(\log\, t_2)^{\beta} - (\log\, t_1)^{\beta}}{\Gamma(\beta+1)} \mid \Omega_3 \mid \right) \left(\sum_{j=1}^{n} \mid \phi_j \mid \left[\frac{(2r+1)(\log\, \xi_j)^{\alpha+\beta+\gamma_j} \gamma_1(\omega)}{\Gamma(\alpha+\beta+\gamma_j+1)} \right.\right.\right.\right. \\
&+ \quad \left. \frac{|\lambda_1| r (\log\, \xi_j)^{\beta+\gamma_j}}{\Gamma(\beta+\gamma_j+1)} \right] \\
&+ \quad \left. \sum_{i=1}^{m} \mid \theta_i \mid \left[\frac{(2r+1)(\log\, \eta_i)^{\alpha+\beta+\mu_i} \gamma_1(\omega)}{\Gamma(\alpha+\beta+\mu_i+1)} + \frac{|\lambda_1| r (\log\, \eta_i)^{\beta+\mu_i}}{\Gamma(\beta+\mu_i+1)} \right] \right) \\
&+ \quad \left(\frac{(\log\, t_2)^{\beta} - (\log\, t_1)^{\beta}}{\Gamma(\beta+1)} \mid \Omega_1 \mid \right) \left(\sum_{l=1}^{q} \mid \upsilon_l \mid \left[\frac{(2r+1)(\log\, \varphi_l)^{\alpha+\beta+\tau_l} \gamma_1(\omega)}{\Gamma(\alpha+\beta+\tau_l+1)} \right.\right. \\
&+ \quad \left. \frac{|\lambda_1| r (\log\, \varphi_l)^{\beta+\tau_l}}{\Gamma(\beta+\tau_l+1)} \right] \\
&+ \quad \left.\left. \sum_{k=1}^{p} \mid \varepsilon_k \mid \left[\frac{(2r+1)(\log\, \psi_k)^{\alpha+\beta+\varsigma_k} \gamma_1(\omega)}{\Gamma(\alpha+\beta+\varsigma_k+1)} + \frac{|\lambda| r (\log\, \psi_k)^{\beta+\varsigma_k}}{\Gamma(\beta+\varsigma_k+1)} \right] \right) \right] \\
&+ \quad \frac{\gamma_1(\omega)(2r+1)}{\Gamma(\alpha+\beta+1)} \left| (\log\, t_2)^{\alpha+\beta} - (\log\, t_1)^{\alpha+\beta} \right| \\
&+ \quad \frac{|\lambda_1| r}{\Gamma(\beta+1)} \left| (\log\, t_2)^{\beta} - (\log\, t_1)^{\beta} + 2(\log\, \frac{t_2}{t_1})^{\beta} \right|
\end{aligned}
$$

and

$$
\begin{aligned}
&\|N_2(x(t_2,\omega),y(t_2,\omega),\omega)-N_2(x(t_1,\omega),y(t_1,\omega),\omega)\|\\
\leq\ & \tfrac{1}{|\overline{\Omega}|}\Bigg[\Bigg(\tfrac{(\log\,t_2)^\sigma-(\log\,t_1)^\sigma}{\Gamma(\sigma+1)}\mid\overline{\Omega_3}\mid\Bigg)\Bigg(\sum_{j=1}^{n}\mid\overline{\phi_j}\mid\Big[\tfrac{(2r+1)\left(\log\,\overline{\xi_j}\right)^{\gamma+\sigma+\overline{\gamma_j}}\overline{\gamma_1}(\omega)}{\Gamma(\gamma+\sigma+\overline{\gamma_j}+1)}\\
+\ & \tfrac{|\lambda_2|r\left(\log\,\overline{\xi_j}\right)^{\sigma+\overline{\gamma_j}}}{\Gamma(\sigma+\overline{\gamma_j}+1)}\Big]\\
+\ & \sum_{i=1}^{m}\mid\overline{\theta_i}\mid\Big[\tfrac{(2r+1)(\log\,\overline{\eta_i})^{\gamma+\sigma+\overline{\mu_i}}\overline{\gamma_1}(\omega)}{\Gamma(\gamma+\sigma+\overline{\mu_i}+1)}+\tfrac{|\lambda_2|r(\log\,\overline{\eta_i})^{\sigma+\overline{\mu_i}}}{\Gamma(\sigma+\overline{\mu_i}+1)}\Big]\Bigg)\\
&\\
+\ & \Bigg(\tfrac{(\log\,t_2)^\sigma-(\log\,t_1)^\sigma}{\Gamma(\sigma+1)}\mid\overline{\Omega_1}\mid\Bigg)\Bigg(\sum_{l=1}^{q}\mid\overline{\upsilon_l}\mid\Big[\tfrac{2q(\log\,\overline{\varphi_l})^{\gamma+\sigma+\overline{\tau_l}}\overline{\gamma_1}(\omega)}{\Gamma(\gamma+\sigma+\overline{\tau_l}+1)}\\
+\ & \tfrac{|\lambda_2|r(\log\,\overline{\varphi_l})^{\sigma+\overline{\tau_l}}}{\Gamma(\sigma+\overline{\tau_l}+1)}\Big]\\
+\ & \sum_{k=1}^{p}\mid\overline{\varepsilon_k}\mid\Big[\tfrac{(2r+1)\left(\log\,\overline{\psi_k}\right)^{\gamma+\sigma+\overline{\varsigma_k}}\overline{\gamma_1}(\omega)}{\Gamma(\gamma+\sigma+\overline{\varsigma_k}+1)}+\tfrac{|\lambda_2|r(\log\,\overline{\psi_k})^{\sigma+\overline{\varsigma_k}}}{\Gamma(\sigma+\overline{\varsigma_k}+1)}\Big]\Bigg)\Bigg]\\
+\ & \tfrac{\overline{\gamma}(\omega)(2r+1)}{\Gamma(\gamma+\sigma+1)}\Big|(\log\,t_2)^{\gamma+\sigma}-(\log\,t_1)^{\gamma+\sigma}\Big|\\
+\ & \tfrac{|\lambda_2|r}{\Gamma(\sigma+1)}\Big|(\log\,t_2)^\sigma-(\log\,t_1)^\sigma+2(\log\,\tfrac{t_2}{t_1})^\sigma\Big|.
\end{aligned}
$$

As $t_2-t_1\to 0$, the right-hand side of the above inequality tends to zero independently of $u\in B_q$. Therefore by the Arzela-Ascoli theorem the operator $N : C([1,e],\mathbb{R}^m)\times C([1,e],\mathbb{R}^m)$ is completely continuous.

Step 4. It remains to show that

$$
\begin{aligned}
\mathcal{A}(\omega)\ &=\ \Big\{(x(\cdot,\omega),y(\cdot,\omega))\in C([1,e],\mathbb{R}^m)\times C([1,e],\mathbb{R}^m):(x(\cdot,\omega),y(\cdot,\omega))\\
&=\ \lambda(\omega)\,N\left(x(\cdot,\omega),y(\cdot,\omega),\omega\right),\lambda(\omega)\in(0,1)\Big\}
\end{aligned}
$$

is bounded.

Let $(x,y)\in\mathcal{A}(\omega)$. Then $x=\lambda(\omega)\,N_1(x,y)$ and $y=\lambda(\omega)\,N_2(x,y)$ for some $0<\lambda<1$. Thus, for $t\in[1,e]$, we have

$$
\begin{aligned}
&\|x(t,\omega)\| \\
\leq\ & \frac{1}{|\Omega|}\Bigg[\Bigg(\left(|\ \Omega_4\ | + \frac{(\log\ t)^{\beta}}{\Gamma(\beta+1)}\ |\ \Omega_3\ |\right)\left(\sum_{j=1}^{n}|\ \phi_j\ |\ \left[I^{\alpha+\beta+\gamma_j}\left\|f\left(\xi_j,\ x\left(\xi_j,\omega\right),\ y\left(\xi_j,\omega\right)\right)\right\|\right.\right.\\
+\ & |\ \lambda_1\ |\ I^{\beta+\gamma_j}\|x\left(\xi_j,\omega\right)\|\Big]\\
+\ & \sum_{i=1}^{m}|\ \theta_i\ |\ \left[I^{\alpha+\beta+\mu_i}\left\|f\left(\eta_i,\ x\left(\eta_i,\omega\right),\ y\left(\eta_i,\omega\right)\right)\right\| + |\ \lambda_1\ |\ I^{\beta+\mu_i}\|x\left(\eta_i,\omega\right)\|\right]\Bigg)\\
+\ & \left(\frac{(\log\ t)^{\beta}}{\Gamma(\beta+1)}\ |\ \Omega_1\ | + |\ \Omega_2\ |\right)\left(\sum_{l=1}^{q}|\ \upsilon_l\ |\ \left[I^{\alpha+\beta+\tau_l}\left\|f\left(\varphi_l,\ x\left(\varphi_l,\omega\right),\ y\left(\varphi_l,\omega\right)\right)\right\|\right.\right.\\
+\ & |\ \lambda_1\ |\ I^{\beta+\tau_l}\|x\left(\varphi_l,\omega\right)\|\Big]\\
+\ & \sum_{k=1}^{p}|\ \varepsilon_k\ |\ \left[I^{\alpha+\beta+\varsigma_k}\left\|f\left(\psi_k,\ x\left(\psi_k,\omega\right),\ y\left(\psi_k,\omega\right)\right)\right\| + |\ \lambda_1\ |\ I^{\beta+\varsigma_k}\|x\left(\psi_k,\omega\right)\|\right]\Bigg)\Bigg]\\
+\ & I^{\alpha+\beta}\left\|f\left(t,\ x\left(t,\omega\right),\ y\left(t,\omega\right)\right)\right\| + |\ \lambda_1\ |\ I^{\beta}\|x\left(t,\omega\right)\|\\
\leq\ & \frac{1}{|\Omega|}\Bigg[\Bigg(\left(|\ \Omega_4\ | + \frac{(\log\ t)^{\beta}}{\Gamma(\beta+1)}\ |\ \Omega_3\ |\right)\left(\sum_{j=1}^{n}|\ \phi_j\ |\ \left[I^{\alpha+\beta+\gamma_j}\gamma_1\left(\omega\right)\left(\|x(\xi_j,\omega)\| + \|y(\xi_j,\omega)\|\right)\right.\right.\\
+\ & |\ \lambda_1\ |\ I^{\beta+\gamma_j}\|x(\xi_j,\omega)\|\Big]\\
+\ & \sum_{i=1}^{m}|\ \theta_i\ |\ \left[I^{\alpha+\beta+\mu_i}\gamma_1\left(\omega\right)\left(\|x(\eta_i,\omega)\| + \|y(\eta_i,\omega)\|\right) + |\ \lambda_1\ |\ I^{\beta+\mu_i}\|x(\eta_i,\omega)\|\right]\Bigg)\Bigg]\\
+\ & \left(\frac{(\log\ t)^{\beta}}{\Gamma(\beta+1)}\ |\ \Omega_1\ | + |\ \Omega_2\ |\right)\left(\sum_{l=1}^{q}|\ \upsilon_l\ |\ \left[I^{\alpha+\beta+\tau_l}\gamma_1\left(\omega\right)\left(\|x(\varphi_l,\omega)\| + \|y(\varphi_l,\omega)\|\right)\right.\right.\\
+\ & |\ \lambda_1\ |\ I^{\beta+\tau_l}\|x\left(\varphi_l,\omega\right)\|\Big]\\
+\ & \sum_{k=1}^{p}|\ \varepsilon_k\ |\ \left[I^{\alpha+\beta+\varsigma_k}\gamma_1\left(\omega\right)\left(\|x\left(\psi_k,\omega\right)\| + \|y\left(\psi_k,\omega\right)\|\right) + |\ \lambda_1\ |\ I^{\beta+\varsigma_k}\|x\left(\psi_k,\omega\right)\|\right]\Bigg)\Bigg]\\
+\ & I^{\alpha+\beta}\gamma_1\left(\omega\right)\left(\|x\left(t,\omega\right)\| + \|y\left(t,\omega\right)\|\right) + |\ \lambda_1\ |\ I^{\beta}\|x\left(t,\omega\right)\|.
\end{aligned}
$$

Then

$$
|x\left(t,\omega\right)| \ \leq\ \gamma_1(\omega)(\|x(\cdot,\omega)\|_{\infty} + \|y(\cdot,\omega)\|_{\infty} + 1)\Lambda_1(\alpha) + |\lambda_1|\Lambda_1(0)(\|x(\cdot,\omega)\|_{\infty} + \|y(\cdot,\omega)\|_{\infty})
$$

We have also

$$
|y\left(t,\omega\right)| \ \leq\ \gamma_2(\omega)(\|x(\cdot,\omega)\|_{\infty} + \|y(\cdot,\omega)\|_{\infty} + 1)\Lambda_2(\gamma) + |\lambda_2|\Lambda_2(0)(\|x(\cdot,\omega)\|_{\infty} + \|y(\cdot,\omega)\|_{\infty})
$$

Therefore

$$|x\,(t,\omega)| + |y\,(t,\omega)| \;\; \leq \;\; C + K(\|x(\cdot,\omega)\|_{\infty} + \|y(\cdot,\omega)\|_{\infty})$$

where

$$C = \gamma_1^*(\omega) + \gamma_2^*(\omega), \;\; K = \Lambda_1(\alpha)\gamma_1(\omega) + \Lambda_2(\gamma)\gamma_2(\omega) + |\lambda_1|\Lambda_1(0) + |\lambda_2|\Lambda_2(0).$$

Hence, from (4.10), we get

$$\|x(\cdot,\omega)\|_{\infty} + \|y(\cdot,\omega)\|_{\infty} \leq \frac{\gamma_1^*(\omega) + \gamma_2^*(\omega)}{1-K} := K_*.$$

Consequently $\|x\| \leq K_*$ and $\|y\| \leq K_*$

This shows that $\mathcal{A}(\omega)$ is bounded. As a consequence of Theorem 3.3.3 we deduce that N has a random fixed point $\omega \to (x(\cdot,\omega), y(\cdot,\omega))$ which is a solution to the problem (4.1).

Step 5.

Compactness of the solution set.

Let $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subset S$ be a sequence. For every $n \in \mathbb{N}$ and for fixe $\omega \in \Omega$, we get

$$\begin{aligned}
x_n(t,\omega) &= \tfrac{1}{\Omega}\Bigg[\left(\Omega_4 - \tfrac{(\log\, t)^{\beta}}{\Gamma(\beta+1)}\Omega_3\right) \\
&\quad \left(\sum_{j=1}^{n} \phi_j \left[I^{\alpha+\beta+\gamma_j} f\left(\xi_j, x_n\left(\xi_j,\omega\right), y_n\left(\xi_j,\omega\right),\omega\right) - \lambda_1 I^{\beta+\gamma_j} x_n\left(\xi_j,\omega\right)\right]\right. \\
&- \left.\sum_{i=1}^{m} \theta_i \left[I^{\alpha+\beta+\mu_i} f\left(\eta_i,\, x_n\left(\eta_i,\omega\right),\, y_n\left(\eta_i,\omega\right),\, \omega\right) - \lambda_1 I^{\beta+\mu_i} x_n\left(\eta_i,\omega\right)\right]\right) \\
&+ \left(\tfrac{(\log\, t)^{\beta}}{\Gamma(\beta+1)}\Omega_1 - \Omega_2\right) \\
&\quad \left(\sum_{l=1}^{q} \upsilon_l \left[I^{\alpha+\beta+\tau_l} f\left(\varphi_l, x_n\left(\varphi_l,\omega\right), y_n\left(\varphi_l,\omega\right),\omega\right) - \lambda_1 I^{\beta+\tau_l} x_n\left(\varphi_l,\omega\right)\right]\right. \\
&- \left.\sum_{k=1}^{p} \varepsilon_k \left[I^{\alpha+\beta+\varsigma_k} f\left(\psi_k,\, x_n\left(\psi_k,\omega\right),\, y_n\left(\psi_k,\omega\right),\, \omega\right) - \lambda_1 I^{\beta+\varsigma_k} x_n\left(\psi_k,\omega\right)\right]\right)\Bigg] \\
&+ I^{\alpha+\beta} f\left(t,\, x_n\left(t,\omega\right),\, y_n\left(t,\omega\right),\, \omega\right) - \lambda_1 I^{\beta} x_n\left(t,\omega\right)
\end{aligned}$$

and

$$\begin{array}{rcl} y_n(t,\omega) & = & \frac{1}{\overline{\Omega}}\Bigg[\left(\overline{\Omega_4}-\frac{(\log t)^\sigma}{\Gamma(\sigma+1)}\overline{\Omega_3}\right) \\ & & \left(\sum_{j=1}^{n}\overline{\phi_j}\left[I^{\gamma+\sigma+\overline{\gamma_j}}g\left(\overline{\xi_j},\, x_n\left(\overline{\xi_j},\omega\right),\, y_n\left(\overline{\xi_j},\omega\right),\,\omega\right)-\lambda_2 I^{\sigma+\overline{\gamma_j}}y_n\left(\overline{\xi_j},\omega\right)\right]\right. \\ & - & \left.\sum_{i=1}^{m}\overline{\theta_i}\left[I^{\gamma+\sigma+\overline{\mu_i}}g\left(\overline{\eta_i},\, x_n\left(\overline{\eta_i},\omega\right),\, y_n\left(\overline{\eta_i},\omega\right),\,\omega\right)-\lambda_2 I^{\sigma+\overline{\mu_i}}y_n\left(\overline{\eta_i},\omega\right)\right]\right) \\ & + & \left(\frac{(\log t)^\sigma}{\Gamma(\sigma+1)}\overline{\Omega_1}-\overline{\Omega_2}\right) \\ & & \left(\sum_{l=1}^{q}\overline{\upsilon_l}\left[I^{\gamma+\sigma+\overline{\tau_l}}g\left(\overline{\varphi_l},\, x_n\left(\overline{\varphi_l},\omega\right),\, y_n\left(\overline{\varphi_l},\omega\right),\,\omega\right)-\lambda_2 I^{\sigma+\overline{\tau_l}}y_n\left(\overline{\varphi_l},\omega\right)\right]\right) \\ & - & \left.\sum_{k=1}^{p}\overline{\varepsilon_k}\left[I^{\gamma+\sigma+\overline{\varsigma_k}}g\left(\overline{\psi_k},\, x_n\left(\overline{\psi_k},\omega\right),\, y_n\left(\overline{\psi_k},\omega\right),\,\omega\right)-\lambda_2 I^{\sigma+\overline{\varsigma_k}}y_n\left(\overline{\psi_k},\omega\right)\right]\right)\Bigg] \\ & + & I^{\gamma+\sigma}g\left(t,\, x_n(t,\omega),\, y_n(t,\omega),\,\omega\right)-\lambda_2 I^{\sigma}y_n(t,\omega) \end{array}.$$

As in Steps 3, 4, we can prove that subsequence $\{(x_{nk}, y_{nk})\}_{k\in\mathbb{N}}$ of $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ converge to some $(x(\cdot,\omega), y(\cdot,\omega)) \in C([1,e],\mathbb{R}^m)\times C([1,e],\mathbb{R}^m)$, such that

$$\omega \to x(t,\omega), \quad \omega \to y(\cdot,\omega)$$

are measurable functions. Since $f(\cdot,\cdot,\cdot,\omega)$ and $g(\cdot,\cdot,\cdot,\omega)$ are continuous functions, then

$$\begin{array}{rcl} x(t,\omega) & = & \frac{1}{\Omega}\Bigg[\left(\Omega_4-\frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_3\right) \\ & & \left(\sum_{j=1}^{n}\phi_j\left[I^{\alpha+\beta+\gamma_j}f\left(\xi_j, x\left(\xi_j,\omega\right), y\left(\xi_j,\omega\right),\omega\right)-\lambda_1 I^{\beta+\gamma_j}x\left(\xi_j,\omega\right)\right]\right. \\ & - & \left.\sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}f\left(\eta_i,\, x\left(\eta_i,\omega\right),\, y\left(\eta_i,\omega\right),\,\omega\right)-\lambda_1 I^{\beta+\mu_i}x\left(\eta_i,\omega\right)\right]\right) \\ & + & \left(\frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_1-\Omega_2\right) \\ & & \left(\sum_{l=1}^{q}\upsilon_l\left[I^{\alpha+\beta+\tau_l}f\left(\varphi_l, x\left(\varphi_l,\omega\right), y\left(\varphi_l,\omega\right),\omega\right)-\lambda_1 I^{\beta+\tau_l}x\left(\varphi_l,\omega\right)\right]\right. \\ & - & \left.\sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\varsigma_k}f\left(\psi_k,\, x\left(\psi_k,\omega\right),\, y\left(\psi_k,\omega\right),\,\omega\right)-\lambda_1 I^{\beta+\varsigma_k}x\left(\psi_k,\omega\right)\right]\right)\Bigg] \\ & + & I^{\alpha+\beta}f\left(t,\, x(t,\omega),\, y(t,\omega),\,\omega\right)-\lambda_1 I^{\beta}x(t,\omega) \end{array}$$

and

$$
\begin{aligned}
y(t,\omega) &= \tfrac{1}{\Omega}\Bigg[\left(\overline{\Omega_4}-\tfrac{(\log t)^\sigma}{\Gamma(\sigma+1)}\overline{\Omega_3}\right)\\
&\quad \left(\sum_{j=1}^{n}\overline{\phi_j}\left[I^{\gamma+\sigma+\overline{\gamma_j}}g\left(\overline{\xi_j},\,x\left(\overline{\xi_j},\omega\right),\,y\left(\overline{\xi_j},\omega\right),\,\omega\right)-\lambda_2 I^{\sigma+\overline{\gamma_j}}y\left(\overline{\xi_j},\omega\right)\right]\right.\\
&- \left.\sum_{i=1}^{m}\overline{\theta_i}\left[I^{\gamma+\sigma+\overline{\mu_i}}g\left(\overline{\eta_i},\,x\left(\overline{\eta_i},\omega\right),\,y\left(\overline{\eta_i},\omega\right),\,\omega\right)-\lambda_2 I^{\sigma+\overline{\mu_i}}y\left(\overline{\eta_i},\omega\right)\right]\right)\\
&+ \left(\tfrac{(\log t)^\sigma}{\Gamma(\sigma+1)}\overline{\Omega_1}-\overline{\Omega_2}\right)\\
&\quad \left(\sum_{l=1}^{q}\overline{\upsilon_l}\left[I^{\gamma+\sigma+\overline{\tau_l}}g\left(\overline{\varphi_l},\,x\left(\overline{\varphi_l},\omega\right),\,y\left(\overline{\varphi_l},\omega\right),\,\omega\right)-\lambda_2 I^{\sigma+\overline{\tau_l}}y\left(\overline{\varphi_l},\omega\right)\right]\right)\\
&- \sum_{k=1}^{p}\overline{\varepsilon_k}\left[I^{\gamma+\sigma+\overline{\varsigma_k}}g\left(\overline{\psi_k},\,x\left(\overline{\psi_k},\omega\right),\,y\left(\overline{\psi_k},\omega\right),\,\omega\right)-\lambda_2 I^{\sigma+\overline{\varsigma_k}}y\left(\overline{\psi_k},\omega\right)\right]\Bigg)\Bigg]\\
&+ I^{\gamma+\sigma}g\left(t,\,x(t,\omega),\,y(t,\omega),\,\omega\right)-\lambda_2 I^{\sigma}y(t,\omega).
\end{aligned}
$$

So S is compact. □

4.3 Examples

In this section we consider two examples for illustrate our main results.

Example 4.3.1.

Consider the following system of fractional differential equation:

$$
\begin{cases}
D^{\frac{1}{2}}(D^{\frac{2}{3}}+\lambda_1)\,x(t,\omega)=f(t,\,x(t,\omega),\,y(t,\omega),\omega)\\
D^{\frac{2}{3}}(D^{\frac{2}{5}}+\lambda_2)\,y(t,\omega)=g(t,\,x(t,\omega),\,y(t,\omega),\omega)\\
4I^{\frac{2}{3}}x(\frac{2e}{3},\omega)+I^{\frac{3}{5}}x(\frac{e+1}{3},\omega)=\frac{2}{5}I^{\frac{2}{3}}x(\frac{e+2}{3},\omega)\\
\frac{2}{3}I^{\frac{1}{5}}x(\frac{e}{2},\omega)=I^{\frac{2}{5}}x(\frac{2e}{5},\omega)+3I^{\frac{1}{4}}x(\frac{4e}{5},\omega)\\
\frac{2}{5}I^{\frac{2}{3}}y(\frac{e+2}{4},\omega)+\frac{1}{2}I^{\frac{1}{2}}y(\frac{e+3}{4},\omega)+\frac{4}{5}I^{\frac{3}{2}}y(\frac{e+4}{4},\omega)=4I^{\frac{2}{3}}y(\frac{3e}{4},\omega)\\
\frac{5}{7}I^{\frac{1}{2}}y(\frac{3e}{5},\omega)=\frac{1}{5}I^{\frac{3}{4}}y(\frac{4e}{5},\omega)
\end{cases}
\tag{4.11}
$$

where $\alpha=\frac{1}{2},\beta=\frac{2}{3},\ \gamma=\frac{2}{3},\ \sigma=\frac{2}{5},\ \lambda_1=\frac{1}{6\Lambda_1(0)},\ \lambda_2=\frac{1}{6\Lambda_2(0)},\ \mathcal{B}(\mathbb{R})$ *denote the Borel*

σ-algebra, $f, g : [1, e] \times \mathbb{R} \times \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to \mathbb{R}$ defined by

$$f(t, x, y, \omega) = \frac{\cos(x + y)}{6\Lambda_1(\alpha)} + \omega t, \quad g(t, x, y, \omega) = \frac{|x + y|}{6\Lambda_2(\gamma)} + \frac{\log t}{t} + \omega^2,$$

where

$$\Lambda_1(\alpha) \approx 101,544, \quad \Lambda_1(0) \approx 175,398$$

and

$$\Lambda_2(\gamma) \approx 6,598, \quad \Lambda_2(0) \approx 15,945.$$

We can easily show that

$$|f(t, x, y, \omega) - f(t, x, y, \omega)| \leq \frac{1}{6\Lambda_1(\alpha)}(|x - \bar{x}| + |y - \bar{y}|), \quad \forall x, \bar{x}, y, \bar{y} \in \mathbb{R}, \ t \in [1, e]$$

and

$$|g(t, x, y, \omega) - g(t, x, y, \omega)| \leq \frac{1}{6\Lambda_2(\alpha)}(|x - \bar{x}| + |y - \bar{y}|), \quad \forall x, \bar{x}, y, \bar{y} \in \mathbb{R}, \ t \in [1, e].$$

Hence

$$\widetilde{M}(\omega) = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{pmatrix}, \quad det(M - \lambda I) = \left(\lambda - \frac{1}{2}\right)\left(\lambda - \frac{1}{6}\right).$$

We observe that

$$|\rho(M(\omega))| = \frac{1}{2} < 1,$$

then

$$M(\omega), \ \textit{converge to } 0.$$

Therefore, all the conditions of theorem 4.2.1 are satisfied. Hence the problem (4.11) has a unique random solution.

Example 4.3.2.

Consider the following system of fractional differential equation:

$$\begin{cases} D^{\frac{1}{2}}(D^{\frac{2}{3}} + \lambda_1)\, x\,(t,\omega) = f(t,\, x(t,\omega),\, y(t,\omega),\omega) \\ D^{\frac{2}{3}}(D^{\frac{2}{5}} + \lambda_2)\, y\,(t,\omega) = g(t,\, x(t,\omega),\, y(t,\omega),\omega) \\ 4I^{\frac{2}{3}}x(\frac{2e}{3},\omega) + I^{\frac{3}{5}}x(\frac{e+1}{3},\omega) = \frac{2}{5}I^{\frac{2}{3}}x(\frac{e+2}{3},\omega) \\ \frac{2}{3}I^{\frac{1}{5}}x(\frac{e}{2},\omega) = I^{\frac{2}{5}}x(\frac{2e}{5},\omega) + 3I^{\frac{1}{4}}x(\frac{4e}{5},\omega) \\ \frac{2}{5}I^{\frac{2}{3}}y(\frac{e+2}{4},\omega) + \frac{1}{2}I^{\frac{1}{2}}y(\frac{e+3}{4},\omega) + \frac{4}{5}I^{\frac{3}{2}}y(\frac{e+4}{4},\omega) = 4I^{\frac{2}{3}}y(\frac{3e}{4},\omega) \\ \frac{5}{7}I^{\frac{1}{2}}y(\frac{3e}{5},\omega) = \frac{1}{5}I^{\frac{3}{4}}y(\frac{4e}{5},\omega) \end{cases} \tag{4.12}$$

where $\alpha = \frac{1}{2}, \beta = \frac{2}{3},\ \gamma = \frac{2}{3},\ \sigma = \frac{2}{5},\ \lambda_1 = \frac{1}{6\Lambda_1(0)},\ \lambda_2 = \frac{1}{6\Lambda_2(0)},$

Here

$$f(t,x,y,\omega) = \frac{t\omega^2 x^2}{2(1+\omega^2)(1+x^2+y^2)} + 1$$

and

$$g(t,x,y,\omega) = \frac{t\omega^2 y^2}{2(1+\omega^2)(1+x^2+y^2)} + 1.$$

Clearly, the map $(t,\omega) \mapsto f(t,x,y,\omega)$ *is jointly continuous for all* $x, y \in \mathbb{R}$. *Thus the functions* f *and* g *are Carathéodory on* $[1,e]\times\mathbb{R}\times\mathbb{R}\times\mathcal{F}$. *Firstly, we show that* f *and* g *are Lipschitz functions. then*

$$|f(t,x,y,\omega) \le \frac{\omega^2}{6\Lambda_1(\alpha)(1+\omega^2)}, \quad \forall x, y \in \mathbb{R}$$

and

$$|g(t,x,y,\omega) \le \frac{\omega^2}{6\Lambda_2(\gamma)(1+\omega^2)}, \quad \forall x, y \in \mathbb{R}.$$

Therefore, all the conditions of theorem 4.2.2 hold. Then the problem (4.12) *has at least one random solution.*

Conclusion and Perspective

In this thesis, we have considered the following

(1) Langevin fractional differential inclusion

$$\begin{cases} D^{\alpha}(D^{\beta}+\lambda)\,x\,(t) & \in & F(t,\,x(t)) \\ \sum\limits_{i=1}^{m}\theta_i I^{\mu_i}x(\eta_i) & = & \sum\limits_{j=1}^{n}\phi_j I^{\gamma_j}x(\xi_j) \\ \sum\limits_{k=1}^{p}\varepsilon_k I^{\varsigma_k}x(\psi_k) & = & \sum\limits_{l=1}^{q}\Xi_l I^{\tau_l}x(\varphi_l) \end{cases}$$

where D^{ρ} denotes the Hadamard Caputo-type fractional derivative of order $\rho \in \{\alpha,\beta\}$ with $0<\alpha,\beta<1, 1<\alpha+\beta<2, \lambda$ is a given constant, I^{r} is the Hadamard fractional integral of order $r>0, r\in\{\mu_i,\,\gamma_j,\,\varsigma_k,\,\tau_l\}$ the constants $\eta_i,\,\xi_j,\,\psi_k,\,\varphi_l\in(1,\,e)$ and $\theta_i,\,\phi_j,\,\varepsilon_k,\,\Xi_l\in\mathbb{R}$, for all $i=1,2,\ldots,m,\ j=1,2,\ldots,n,$ $k=1,2,\ldots,p,\ l=1,2,\ldots,q$ and $F:\,[1,e]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$; where F is a multifonction.

(2) System of boundary random fractional differential equations

$$\begin{cases} D^{\alpha}(D^{\beta}+\lambda_1)\,x\,(t,\omega) & = & f(t,\,x(t,\omega),\,y(t,\omega),\omega) \\ D^{\gamma}(D^{\sigma}+\lambda_2)\,y\,(t,\omega) & = & g(t,\,x(t,\omega),\,y(t,\omega),\omega) \\ \sum\limits_{i=1}^{m}\theta_i I^{\mu_i}x(\eta_i,\omega) & = & \sum\limits_{j=1}^{n}\phi_j I^{\gamma_j}x(\xi_j,\omega) \\ \sum\limits_{k=1}^{p}\varepsilon_k I^{\varsigma_k}x(\psi_k,\omega) & = & \sum\limits_{l=1}^{q}\nu_l I^{\tau_l}x(\varphi_l,\omega) \\ \sum\limits_{i=1}^{m}\overline{\theta_i} I^{\overline{\mu_i}}y(\overline{\eta_i},\omega) & = & \sum\limits_{j=1}^{n}\overline{\phi_j} I^{\overline{\gamma_j}}y(\overline{\xi_j},\omega) \\ \sum\limits_{k=1}^{p}\overline{\varepsilon_k} I^{\overline{\varsigma_k}}y(\overline{\psi_k},\omega) & = & \sum\limits_{l=1}^{q}\overline{\nu_l} I^{\overline{\tau_l}}y(\overline{\varphi_l},\omega) \end{cases}$$

where D^{ρ} denotes the Hadamard Caputo-type fractional derivative of order $\rho \in$

$\{\alpha, \beta, \gamma, \sigma\}$ with $0 < \alpha, \beta, \gamma, \sigma < 1, 1 < \alpha + \beta < 2, 1 < \gamma + \sigma < 2$, λ_1, λ_2 are given constants, I^r is the Hadamard fractional integral of order $r > 0$, $r \in \{\mu_i, \gamma_j, \varsigma_k, \tau_l, \overline{\mu_i}, \overline{\gamma_j}, \overline{\varsigma_k}, \overline{\tau_l}\}$ the constants $\eta_i, \xi_j, \psi_k, \varphi_l, \overline{\eta_i}, \overline{\xi_j}, \overline{\psi_k}, \overline{\varphi_l} \in (1, e)$ and $\theta_i, \phi_j, \varepsilon_k, \nu_l, \overline{\theta_i}, \overline{\phi_j}, \overline{\varepsilon_k}, \overline{\nu_l} \in \mathbb{R}$, for all $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, p$, $l = 1, 2, \ldots, q$ and $f, g : [1, e] \times \mathbb{R}^m \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m$, $(\Omega, \mathcal{A})$ is a measurable space.

In the futur, we plan to study the qualitative aspect of the solutions for the above mentioned problems, in particular, we will look for the stability and controllability of the above cited problems.

Bibliography

[1] Ahmad, B., Nieto, J.J., Alsaedi, A., El-Shahed, M.: A study of nonlinear Langevin equation involving two fractional orders in different intervals. *Nonlinear Anal. Real World Appl.* **13** (2012), 599-606

[2] T. N. Anh, Some general random coincidence point theorems, New Zealand *J. Math.* **41** (2011), 17-24.

[3] J. P. Aubin and A. Cellina, *Differential Inclusions Springer-Verlag, Berlin-Heidelberg*, New York, 1984.

[4] A. T. Bharucha-Reid. *Fixed point theorems in probabilistic analysis, Bull. Amer. Math. Soc.* **82** (1976), 641-657.

[5] A. T. Bharucha-Reid, *Random Integral Equations*, New York, Academic Press, 1972.

[6] O. Bolojan-Nica, G. Infante and R. Precup, Existence results for systems with coupled nonlocal initial conditions, *Nonlinear Anal.* **4** (2014), 231–242.

[7] Butzer, P.L., Kilbas, A.A., Trujillo, J.J.: Compositions of Hadamard-type fractional integration oper ators and the semigroup property. J. *Math. Anal. Appl.* **269** (2002), 387-400

[8] Butzer, P.L., Kilbas, A.A., Trujillo, J.J.: Mellin transform analysis and integration by parts for Hadamard-type fractional integrals. J. *Math. Anal. Appl.* **270** (2002), 1-15

[9] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics, 580, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[10] H. Covitz, SB. Jr. Nadler. Multivalued contraction mappings in generalized metric spaces. *Isr. J. Math.* **8** (1970), 5-11.

[11] R. Cristescu, Order structures in normed vector spaces., (in Romanian) Editura Stiin Tifica si Enciclopedica, Bucuresti, 1983.

[12] Coffey, W.T., Kalmykov, YuP. Waldron, J.T : The Langevin Equation, 2nd edn. World Scientific, Singapore (2004)

[13] K. Diethelm and A. D. Freed. On the solution of nonlinear fractional order differ ential equations used in the modeling of viscoplasticity, in Scientifice Computing in Chemical Engineering II-Computational Fluid Dynamics, Reaction Engineering and Molecular Properties (F. Keil. W. Mackens. H. Voss. and J. Werther, Eds.) , Springer-Verlag, Heidelberg. (1999) 217-224.

[14] L. Gaul, P. Klein and S. Kempfle. Damping description involving fractional oper ators. Mech. Systems gnal rocessing 5 (1991). 81-88..

[15] J. R. Graef, J. Henderson, and A. Ouahab, *Topological Methods for Differential Equations and Inclusions.* Monographs and Research Notes in Mathematics Series Profile. Boca Raton, FL: CRC Press, (2019).

[16] J.R. Graef, J. Henderson and A. Ouahab, Some Krasnosel'skii type random fixed point theorems, *J. Nonlinear Funct. Anal.* **2017** (2017), 1-34, Article ID 46.

[17] Hadamard, J.: Essai sur l'etude des fonctions donnees par leur developpment de Taylor. *J. Math. Pure Appl. Ser.* **8** (1892), 101-186.

[18] O. Hans, *Random operator equations, Proc. 4th Berkeley Sympos.* Math. Statist. and Prob., Univ. California Press, Berkeley, Calif., (1961), II 185-202.

[19] O. Hans, *Random fixed point theorems. 1957 Transactions of the first Prague conference on information theory, statistical decision functions, random pro cesses held at Liblice near Prague from November 28 to 30,* (1956) pp. 105-125 Publishing House of the Czechoslovak Academy of Sciences, Prague.

[20] O. Hans and A. Spacek, *Random fixed point approximation by differentiable trajectories.* (1960) Trans. 2nd Prague Conf. Information Theory pp. 203-213 Publ. House Czechoslovak Acad. Sci., Prague, Academic Press, New York.

[21] F. Jarad, T. Abdeljawad and D. Baleanu, Caputo-type modification of the Hadamard fractional derivatives, *Adv. Differ. Equ.* (2012), 142.

[22] A.A. Kilbas, Hari M. Srivastava, and Juan J. Trujillo, *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies , 204. Elsevier Science B.V., Amsterdam (2006).

[23] Kisielewicz, M: *Differential Inclusions and Optimal Control.* Kluwer, Dordrecht (1991)

[24] M. A. Krasnosel'skii, *Some problems of nonlinear analysis*, Amer. Math. Soc. Transl. Ser. (2) 10 (1958), 345-409.

[25] G.S. Ladde and V. Lakshmikantham, *Random Differential Inequalities*, Academic Press, New York, (1980).

[26] V. Lakshmikantham, Theory of fractional functional differential equations, *Nonlinear Analysis: TMA*, **69** (2008) 10, 3337-3343.

[27] V. Lakshmikantham and A.S. Vatsala, Theory of fractional differential inequalities and applications, *Commun. Math. Anal.* **2** (2007), 395-402.

[28] V. Lakshmikantham and A.S. Vatsala, Basic theory of fractional differential equations, *Nonlinear Anal.* **69** (2008), 2677-2682.

[29] V. Lupulescu and S.K. Ntouyas, Random fractional differential equations, *Int. Electron. J. Pure Appl. Math.* **4** (2012) 119-136.

[30] V. Lupulescu D. O'Regan and Ghaus ur Rahman, Existence results for random fractional differential equations, *Opuscula Math.* **34**, (2014), 813-825.

[31] Liu, X., Jia, M., Ge, W.: Multiple solutions of a p-Laplacian model involving a fractional derivative. *Adv. Differ. Equ.* **126**, (2013).

[32] Lim, S.C., Li, M., Teo, L.P: Langevin equation with two fractional orders. *Phys. Lett.* A **372** (2008), 6309-6320

[33] A. Mukherjea, Transformations aleatoires separables. Theoreme du point fixe aleatoire, C. R. *Acad. Sei. Paris Ser.* A-B **263** (1966), 393-395.

[34] A. Mukherjea, *Random Transformations of Banach Spaces*; Ph. D. Dissertation, Wayne State Univ. Detriot, Michigan, (1968).

[35] O. Nica, Existence results for second order three point boundary value problems. *Differ. Equ. Appl.* **4** (2012), 547–570.

[36] O. Nica, Initial-value problems for first-order differential systems with general nonlocal conditions. *Electron. J. Differential Equations*, Vol.(2012), No. 74, pp. 1-15.

[37] O. Nica and R. Precup, On the nonlocal initial value problem for first order differential systems. *Stud. Univ. Babeş -Bolyai Math.* **56** (2011), No. 3, 125-137.

[38] Ntouyas, S. K. and Tsamatos, P. G., A fixed point theorem of Krasnoselskii-nonlinear alternative type with applications to functional integral equations, Differential Equations Dynam. Systems 7(2). (1999), 139-146.

[39] D. O'Regan, R. Precup; Fixed point theorems for set-valued maps and existence principles for integral inclusions, *J. Math. Anal. Appl.* **245** (2000), 594-612.

[40] A. Ouahab, Some Perov's and Krasnosel'skii type fixed point results and ap plication, Comm. *Appl. Nonlinear Anal.* **19** (2015), 623-642.

[41] K.B. Oldham and J. Spanier. The fractional calculus. Academic Press, New York, 1974. (page 17.)

[42] N.S. Papageorgiou, Random fixed point theorems for measurable multifunctions in Banach spaces, *Proc. Amer. Math. Soc.* **97** (1986), 507-514.

[43] W. Padgett and C. Tsokos, Random Integral Equations with Applications to Life Science and Engineering, Academic Press, New York, (1976).

[44] B. L. S. Prakasa Rao, Stochastic integral equations of mixed type II. *J. Math ematical and Physical Sci*, **7**, (1973), 245-260.

[45] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, (1999).

[46] A.I. Perov, *On the Cauchy problem for a system of ordinary differential equa tions,* Priblizhen. Met. Reshen. Differ. Uvavn.. 2. (1964), 115-134 (in Russian).

[47] I.R. Petre and A. Petrusel, Krasnoselskii's theorem in generalized Banach spaces and applications, Electron. *J. Qual. Theory Differ. Equ.* **85** (2012), 20 pp.

[48] R. Precup, The role of matrices that are convergent to zero in the study of semilinear operator systems. *Math. Comp. Modelling* **49** (2009), 703-708.

[49] B. Ross. *Fractional calculus and its applications.* Lecture Notes in Mathematics, 457, Springer-Verlag, New York, (1975), page 17.

[50] B. Roummani, J. Henderson and A. Ouahab, Existence and solution sets for systems of impulsive differential inclusions *Mem. Differ. Equ. Math. Phys.* **82** (2021), 1-37

[51] SAMKO, S. G., KILBAS, A. A., and MARICHEV, O. I., Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach Science Publishers, Switzerland, (1993).

[52] G. Samko, A.A. Kilbas and O.I. Marichev, Fractional Integrals and Derivatives. Theory and Applications, Gordon and Breach, Yverdon, (1993).

[53] T.T. Soong. Random Differential Equations in Science and Engineering, Academic Press, New York, (1973).

[54] M.L. Sinacer, J. J Nieto and A. Ouahab, Random fixed point theorems in generalized Banach spaces and applications, *Random Oper. Stoch. Equ.* **24** (2016), 93-112.

[55] N. Shahzad, Random fixed point theorems for various classes of 1-set-ontractive maps in Banach spaces, *J. Math. Anal. Appl.* **203** (1996), 712-718.

[56] N. Shahzad and S. Latif, Random fixed points for several classes of 1-ball contractive and 1-set-contractive random maps, *J. Math. Anal. Appl.* **237** (1999), 83-92.

[57] A. Spacek, Zulfallige Gleichungen, Czechoslovak,*Math. J.*, **5** (1995), 462-466.

[58] A. Skorohod, Random Linear Operators, Reidel, Boston, (1985).

[59] S. Stanek, Limit properties of positive solutions of fractional boundary value problems, *Appl. Math. Comput*, **219**, (2012) 2361-2370.

[60] J.L Strand, Random ordinary differential equations, Reidel, Boston, (1985).

[61] Subramanyam, P. V. and Sundarsanam, S. K., A note on functional integral equations, Differrential Equations Dynam. Systems 4 (1996), 473-478.

[62] V. E. Tarasov, *Fractional Dynamics Application of Fractional Calculus to Dynamics of Particles*, Fields and Media, Springer, Heidelberg; Higher Education Press, Beijing, (2010).

[63] J. Tariboon, S.K., Ntouyas and C. Thaiprayoon, Nonlinear Langevin equation of Hadamard-Caputo type fractional derivatives with nonlocal fractional integral conditions. *Adv. Math. Phys.* (2014). Art. ID372749, 15 pp.

[64] H. D. Thang and P. T. Anh, Random fixed points of completely random oper ators. *Random Oper. Stoch. Equ.* **21** (2013), no. 1, 1-20.

[65] C.P. Tsokos and W.J. Padgett, *Random Integral Equations with Applications to Life Sciences and Engineering*, Academic Press, New York, (1974).

[66] Sh. Hu, N. Papageorgiou, *Handbook of Multivalued Analysis, Volume I: Theory*, Kluwer, Dordrecht, Boston, London, (1997).

[67] R. S. Varga, *Matrix Iterative Analysis.* Second revised and expanded edition. Springer Series in Computational Mathematics, **27**. Springer-Verlag, Berlin, (2000).

[68] J. R. L. Webb and G. Infante, Positive solutions of nonlocal boundary value problems: a unified approach, *J. London Math. Soc*, **74** (2006), 673-693.

[69] Zhang, S: Positive solutions for boundary-value problems of nonlinear fractional differential equations. Electron. *J. Differ. Equ.* **36** (2006), 1-12