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Application de la méthode variationnelle pour certaine classe d'équations et inclusions différentielles.

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ملخص

في هذه الأطروحة، ندرس وجود حل لفئات من المعادلات التفاضلية المندفعة والمعادلات التفاضلية للأنظمة الكسرية ذات الشروط الحدية. تستند نتائجنا إلى نظريات الطريقة المتغيرة ونظريات النقطة الثابتة في فضاءات باناخ المعممة، ونقوم بدراسة وجود وتعدد الحلول الإيجابية للمعادلات التفاضلية الكسرية في المجالات المحدودة.

Abstract

In this thesis, we study the existence of solution for a classes of impulsive differential equation and system of fractional differential equations with boundary conditions. Our results are based on variational methods theorems and fixed point theorems in generalized Banach spaces, and we study the existence and multiplicity of positive solutions for fractional differential equations on bounded domains.

Résumé

Dans cette thèse, nous étudions l'existence des solution pour une classe d'équations différentielles impulsive et système d'équations différentielles fractionnaires avec conditions aux limites. Nos résultats sont basés sur des théorèmes de la méthode variationnelle et des théorèmes de point fixe dans les espaces de Banach généralisés , et on étudie l'existence et la multiplicité des solutions positives pour les équations différentielles fractionnaires sur des domaines bornés

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Publications

- **A. Mahdjouba**, A. Dellal, J. J. Nieto, A. Ouahab, Existence results for systems of second impulsive differential equations via variational method, *International Journal of Difference Equations*, Vol. 2020 (2020), No 2, pp. 116.

- **A. Mahdjouba**, J. J. Nieto, A. Ouahab, System of fractional boundary value problem with p-Laplacian and advanced arguments, *Advances in Difference Equations* , Vol. 2021 (2021), No 352.

Contents

Introduction

The theory of impulsive differential equations goes back to 1960 in a paper of Milman and Myshkis [59, 60]. After a period of active research, mostly in Eastern Europe from 1960-1970, culminating with the monograph by Halanay and Wexler [38].

Several mathematical schools were created, continuing the scientific research on the fundamental and qualitative theory of impulsive differential equations and their applications in the early eighties and then see for example ([3, 8, 25, 63, 67]).
In recent years, a great deal of work has been done in the study of the existence of solutions for impulsive boundary-value problems, by which a number of chemotherapy, population dynamics, optimal control, ecology, industrial robotics and physics phenomena are described. Systems of ordinary impulsive boundary value problems have been studied by a number of authors (see [10, 13, 52, 92]) Various mathematical results (existence, structure of solutions set, asymptotic behavior,...) have been obtained so far (see [7, 8, 22, 23, 30]), and many authors have studied impulsive differential equations using a variety of methods , such as fixed point theory, topological degree theory (including continuation method and coincidence degree theory), comparison methods (including upper and lower solutions methods and monotone iterative method) (see [78, 80–82, 97, 106, 107]). Recently in [67, 93] the authors studied the existence and multiplicity of solutions of some class of second order impulsive problems by variational method have become a powerful tool.

The initiative idea of non-integer order derivatives is quite old and history of fractional calculus spans on three centuries. Since in the mid twentieth century, and latter decades the number of papers devoted to fractional calculus increased rapidly. One of the reasons for the significant interest in the field of fractional calculus is that verity of physical [42], chemical [71] and biological [54] phenomena can be described with fractional differential equations. The field of fractional calculus can be considered as new branch of applied mathematics. A fair amount of basic mathematical theory related to the study of fractional calculus attributed to Leibniz, Caputo, Liouville, Riemann, Euler and many others. However, in the past few decades more and more convincing applications in different fields of engineering and sciences have been found. It is notable that a larger part of research work is committed to the existence theory of fractional differential equations(FDEs)(see [66,99]). Recently, there have been some papers dealing with the existence and multiplicity of solution (or positive solution) of nonlinear initial fractional differential equation by the use of techniques of nonlinear analysis, (see [6, 17, 18, 47, 96]).

In this thesis, we study the existence of some classes of impulsive differential equation and system of fractional differential equations with boundary conditions. Our results are based on variational methods theorems and fixed point theorems in generalized Banach spaces. We have arranged this thesis as follows:

In **chapter 1**, we introduce notations, definitions, lemmas and critical point theorems and fixed point theorems which are used throughout this thesis.

In **chapter 2**, we present results on the existence results for systems of the second order impulsive differential equations via variational method

$$\begin{cases} -\ddot{u}+m^2u &= f(t,u,v), & t\neq t_k, k=1,\dots,p,\ t\in J,\\ -\ddot{v}+m^2v &= g(t,u,v), & t\neq t_k, k=1,\dots,p,\ t\in J,\\ \dot{u}(t_k^+)-\dot{u}(t_k^-) &= I_k(u(t_k)),\ k=1,\dots,p, & \\ \dot{v}(t_k^+)-\dot{v}(t_k^-) &= \overline{I}_k(u(t_k)),\ k=1,\dots,p, & \\ u(0)=u(b) &= v(0)=v(b)=0, & \end{cases} \tag{0.0.1}$$

where $J:=[0,b],\ m\neq 0, f,g: J\times\mathbb{R}^2\to\mathbb{R}$ are two functions, $I_k,\overline{I}_k\in C(\mathbb{R},\mathbb{R}),\ k\in\{1,2,\dots,n\}$ $\dot{u}(t_k^+)$ and $\dot{u}(t_k^+)$ denote the right and the left limits respectively of $\dot{u}$ at t_k for $0\leq k\leq p,\ 0=t_0<t_1,\dots,t_k<t_p<b,\ p\in\mathbb{N}$.

We shall provide sufficient conditions ensuring some existence and uniqueness results for system (0.0.1) via an application of the Nash-type equilibrium method in vector Banach spaces.

In **chapter 3**, we aim to study the following impulsive boundary value problem with a second-order p-Laplacian on $\sigma(T)$ periodic time scales $\mathbb{T}$:

$$\begin{cases} (\varphi_p(u^\Delta(t)))^\Delta = f(\sigma(t), u^\sigma(t)), & \Delta - \text{a.e. } t \in [0, \sigma(T)]^k_{\mathbb{T}}, t \neq t_j, \\ \varphi_p(u^\Delta(t_j^+)) - \varphi_p(u^\Delta(t_j^-)) = I_j(u(t_j)), & j = 1, \dots, n \\ u(0) - u(\sigma(T)) = u^\Delta(0) - u^\Delta(\sigma(T)) = 0, \end{cases} \tag{0.0.2}$$

where Δ is the derivative on the time scale $\mathbb{T}$, and σ is the forward jump operator, $0, T \in \mathbb{T}$ with $\sigma(t) \in \mathbb{T}^k$ and $f(t, x) : [0, \sigma(T)]^k_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is Lebesgue integrable in $\mathbb{T}$ for each $x \in \mathbb{R}$, and continuously Δ-differentiable with respect to x for any $\Delta - a.e. t \in [0, \sigma(T)]^k_{\mathbb{T}}$, $I_j(u(t_j)) \in C(\mathbb{R}, \mathbb{R})$, $\varphi_p(u^\Delta(t_j^+))$ and $\varphi_p(u^\Delta(t_j^-))$ denote the right and the left limits respectively of u^Δ at t_k for $0 \leq k \leq p,\ 0 = t_0 < t_1, \dots, t_k < t_p < T,\ p \in \mathbb{N}$.

The main tool employed here is the critical point theorem. three examples are also given to illustrate this work.

In **chapter 4**, we discuss the existence and multiplicity of positive solutions for system of fractional differential equations with boundary conditions and advanced arguments:

$$\begin{cases} (\varphi_p(D^\alpha_{0^+} u(t)))' + a_1(t) f(u(\theta_1(t)), v(\theta_2(t))) = 0, & 0 < t < 1, \\ (\varphi_{\tilde{p}}(D^\alpha_{0^+} v(t)))' + a_2(t) g(u(\theta_1(t)), v(\theta_2(t))) = 0, & 0 < t < 1, \\ D^\alpha_{0^+} u(0) = u(0) = u'(0) = 0, D^\beta_{0^+} u(1) = \gamma D^\beta_{0^+} u(\eta), \\ D^\alpha_{0^+} v(0) = v(0) = v'(0) = 0, \quad D^\beta_{0^+} v(1) = \gamma D^\beta_{0^+} v(\eta), \end{cases} \tag{0.0.3}$$

where $\eta \in (0, 1), \gamma \in (0, \frac{1}{\eta^{\alpha-\beta-1}}), D^\alpha_{0^+}, D^\beta_{0^+},$ are the standard Riemann-Liouville fractional derivatives with $\alpha \in (2, 3), \beta \in (1, 2)$ such that $\alpha \geq \beta + 1$, the p-Laplacian operator is defined as $\varphi_p(s) = |s|^{p-2}s,\ p > 1$, and the functions $f, g \in C(\mathbb{R}^2, \mathbb{R}), a_i \in L^1[0, 1], \theta_i \in C([0, 1], [0, 1])$ for $i = 1, 2$.

In the first section, the existence result proved via Leray-Schauder's fixed point theorem type in vector Banach space.
In the second section, our goal is to establish positive solutions for the system

(0.0.3).
In third section, we present the following general existence, multiplicity and localization result.

key words and phrases: Weak solutions, Sobolev spaces, critical point, impulses, variational methods, fixed point, energy functional, Nash-type equilibrium, fractional differential equations, p-Laplacian operator, cone, fixed point theorem, positive solutions, multiplicity of solutions.

Chapter 1

Preliminaries

In this chapter, we will introduce some notations and concept, essential to the development of other chapter. We will recall some critical point theorems and fixed point theorems for a better presentation of the demonstrations of the results of the our work .

1.1 Sobolev Spaces

In this section we discus Sobolev spaces. The following elements have been gathered from several analysis and some specialized books and the most is in the following bibliography [1, 14, 19].
Let $I = (a, b)$ be an open interval, possibly unbounded, and let $p \in \mathbb{R}$ with $1 \leq p < \infty$.

Definition 1.1.1. *We set*

$$L^p(I) = \{f \colon I \to \mathbb{R} : \ f \text{ is measurable and } |f|^p \in L^1(I)\},$$

equipped with the norm

$$\|f\|_{L^p} = \Big[\int_I |f(x)|^p d\mu\Big]^{1/p}.$$

Definition 1.1.2. *We set*

$$L^\infty(I) = \Big\{f \colon I \to \mathbb{R} : \ f \text{ is measurable and } \ \|f\|_{L^\infty} < +\infty\Big\},$$

where

$$\|f\|_{L^\infty} = \inf\{C, \ |f(x)| \leq C \quad a.e. \ on \ I\}.$$

Notation 1.1.1. *Let $1 \leq p \leq \infty$: we denote by p' the conjugate exponent,*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Theorem 1.1.1. *Assume that $f \in L^p$ and $g \in L^{p'}$ with $1 \leq p \leq \infty$. Then*

$$\int |fg| \leq \|f\|_{L^p} \|g\|_{L^{p'}}. \tag{1.1.1}$$

Theorem 1.1.2. *Let (f_n) be a sequence in L^p and let $f \in L^p$ be such that $\|f_n - f\|_p \to 0$. Then, there exist a subsequence (f_{nk}) and a function $h \in L^p$ such that:*

- $f_{nk}(x) \to f(x)$ *a.e. on I,*
- $|f_{nk}(x)| \leq h(x)\ \forall k,$ *a.e. on I.*

Definition 1.1.3. *A function $f \in L^1(I)$ is said to be weakly differentiable if there exists $g \in L^1(I)$ such that:*

$$\int_I \phi(s) g(s) ds = -\int_I f(s) \phi'(s) ds, \quad \forall \phi \in C_0^\infty(I),$$

where $C_0^\infty(I)$ is the space of smooth functions with compact support.

Remark 1.1.1. *There are functions which are weakly differentiable, but not differentiable in the classical sense.*

Example 1.1.1. *The absolute value function $u : \mathbb{R} \to \mathbb{R}_+,\ u(t) = |t|$, which is not differentiable at $t = 0$ has a weak derivative $v : \mathbb{R} \to \mathbb{R}$ known as the sign function, and given by*

$$v(t) = \begin{cases} 1 & if \quad t > 0; \\ 0 & if \quad t = 0; \\ -1 & if \quad t < 0. \end{cases}$$

Definition 1.1.4. *The Sobolev space $W^{1,p}(I)$ is defined by*

$$W^{1,p}(I) = \left\{ u \in L^p(I),\ \exists g \in L^p(I) \text{ such that } \int_I u\varphi' = -\int_I g\varphi \ \ \forall \varphi \in C_0^1(I) \right\}.$$

We set

$$H^1(I) = W^{1,2}(I).$$

Notation 1.1.2. *The space $W^{1,p}$ is equipped with the norm*

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \|u'\|_{L^p},$$

or sometimes, if $1 \leq p \leq \infty$, with the equivalent norm $(\|u\|_{L^p}+\|u'\|_{L^p})^{1/p}$. The space H^1 is equipped with the scalar product

$$(u,v)_{H^1} = (u,v)_{L^2} + (u',v')_{L^2} = \int_a^b (uv + u'v')$$

and with the associated norm

$$\|u\|_{H^1} = \left(\|u\|_{L^2}^2 + \|u'\|_{L^2}^2\right)^{1/2}. \tag{1.1.2}$$

Proposition 1.1.3. *The space $W^{1,p}$ is a Banach space for $1 \leq p \leq \infty$. It is reflexive and separable for $1 \leq p \leq \infty$. The space H^1 is a separable Hilbert space.*

Theorem 1.1.4. *There exists a constant C (depending only on $|I| \leq \infty$) such that*

$$\|u\|_{L^\infty(I)} \leq C\|u\|_{W^{1,p}(I)} \quad \forall u \in W^{1,p}(I), \quad \forall 1 \leq p \leq \infty. \tag{1.1.3}$$

In other words, $W^{1,p}(I) \subset L^\infty(I)$ with continuous injection for all $1 \leq p \leq \infty$.
Further, if I is bounded then

$$\textit{the injection } W^{1,p}(I) \subset C(\overline{I}) \textit{is compact for all } 1 < p \leq \infty, \tag{1.1.4}$$

$$\textit{the injection } W^{1,p}(I) \subset L^q(I) \textit{is compact for all } 1 \leq q < \infty. \tag{1.1.5}$$

Definition 1.1.5. *Given $1 \leq p \leq \infty$, denote by $W_0^{1,p}(I)$ the closure of $C_c^1(I)$ in $W^{1,p}(I)$. Set*

$$H_0^1(I) = W_0^{1,2}(I).$$

The space $W_0^{1,p}(I)$ is equipped with the norm of $W^{1,p}(I)$, and the space H_0^1 is equipped with the scalar product of H^1.
The space $W_0^{1,p}$ is a separable Banach space. Moreover, it is reflexive for $p > 1$. The space H_0^1 is a separable Hilbert space.

Remark 1.1.2. *When $I = \mathbb{R}$ we know that $C_c^1(\mathbb{R})$ is dense in $W^{1,p}(\mathbb{R})$ (by theorem 8.7 see [14]) and therefore $W_0^{1,p}(\mathbb{R}) = W^{1,p}(\mathbb{R})$.*

Theorem 1.1.5. *Let $u \in W^{1,p}(I)$. Then $u \in W_0^{1,p}(I)$ if and only if $u = 0$ on ∂I.*

Proposition 1.1.6. *Suppose I is a bounded interval. Then there exists a constant C (depending on $|I| < \infty$) such that*

$$\|u\|_{W^{1,p}(I)} \leq C\|u'\|_{L^p(I)} \quad \forall u \in W_0^{1,p}(I). \tag{1.1.6}$$

In other words, on $W_0^{1,p}$, the quantity $\|u'\|_{L^p(I)}$ is a norm equivalent to the $W^{1,p}$ norm.

The dual space of $W_0^{1,p}(I)$, $(1 \leq p < \infty)$ is denoted by $W^{-1,p'}(I)$ and the dual space of $H_0^1(I)$ is denoted by $H^{-1}(I)$.
We identify L^2 and its dual, but we do not identify H_0^1 and its dual.We have the inclusions

$$H_0^1 \subset L^2 \subset H^{-1},$$

where these injections are continuous and dense (i.e., they have dense ranges).
If I is a bounded interval we have

$$W_0^{1,p} \subset L^2 \subset W^{-1,p'} \text{ for all } \ 1 \leq p < \infty,$$

with continuous injections (and dense injections when $1 \leq p < \infty$).
If I is unbounded we have only

$$W_0^{1,p} \subset L^2 \subset W^{-1,p'} \text{ for all } \ 1 \leq p \leq 2,$$

with continuous injections,then we may write

$$C_0^\infty(I) \subset H_0^1(I) \subset H^1(I) \subset L^2(I) \subset H^{-1}(I).$$

We define the following isometry operator

$$L : H^{-1}(I) \to H_0^1(I), \quad h \mapsto Lh := u_h$$

where u_h is the unique element of $H_0^1(I)$ guaranteed by Riesz's representation theorem, satisfying the identity

$$\langle u_h, v\rangle_{H_0^1(I)} = \langle h, v\rangle, \quad v \in H_0^1(I). \tag{1.1.7}$$

Here, by $\langle h, v\rangle$ we mean the value at v of the functional h from (1.1.2), one has

$$\|u_h\|^2_{H_0^1(I)} = \langle u_h, u_h\rangle_{H_0^1(I)} = \langle h, u_h\rangle \leq \|h\|_{H^{-1}(I)}\|u_h\|_{H_0^1(I)}$$

whence

$$\|u_h\|_{H_0^1(I)} \leq \|h\|_{H^{-1}(I)}.$$

On the other hand,

$$\begin{aligned}\|h\|_{H^{-1}(I)} &= \sup_{v\neq 0}\frac{|\langle h, v\rangle|}{\|v\|_{H_0^1(I)}} = \sup_{v\neq 0}\frac{|\langle u_h, v\rangle_{H_0^1(I)}|}{\|v\|_{H_0^1(I)}} \\ &\leq \sup_{v\neq 0}\frac{\|u_h\|_{H_0^1(I)}\|v\|_{H_0^1(I)}}{\|v\|_{H_0^1(I)}}.\end{aligned}$$

These two inequalities show that L is an isometry between $H^{-1}(I)$ and $H_0^1(I)$.

Lemma 1.1.7. *[27] There exists $c > 0$ such that, if $u \in H^{1,p}_{per}(I, \mathbb{R})$, $\ 1 < p < \infty$, then*

$$\|u\|_\infty \leq c\|u\|_{H^{1,p}_{per}}.$$

Moreover, if $\int_0^b u(t)dt = 0$, then

$$\|u\|_\infty \leq c\|u'\|_{L^p},$$

where

$$H^{1,p}_{per}(I, \mathbb{R}) = \{u \in H^{1,p}(I, \mathbb{R}) : u(0) = u(b),\ u'(0) = u'(b)\}.$$

Lemma 1.1.8. *[27] If $u \in H^{1,p}_{per}(I, \mathbb{R})$ $(p \in (1, \infty))$ and $\int_0^b u(t)dt = 0$, then*

$$\|u\|_\infty \leq b^{\frac{1}{p'}}\|u'\|_{L^p},\ \text{with}\ \frac{1}{p} + \frac{1}{p'} = 1.$$

1.2 Generalized Banach space

In this part, we consider the notation and definition of generalized Banach space ,and we introduce definitions, lemmas and theorems concerning to matrice convergent (see [108, 109]).

1.2.1 Some definitions

Definition 1.2.1. *Let E be a vector space metric on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A map $\|\cdot\|\colon E \to \mathbb{R}^n_+$ is called an norm on E if it satisfies the following properties* :

- $\|x\| = 0$ *then* $x = (0, \cdots, 0)$,
- $\|\lambda x\| = |\lambda|\|x\|$ *for* $x \in E, \lambda \in \mathbb{K}$,
- $\|x + y\| \leq \|x\| + \|y\|$ *for every* $x, y \in E$.

Remark 1.2.1. *The pair $(E, \|\cdot\|)$ is called a generalized normed space. If the generalized metric generated by $\|\cdot\|$ (i.e $d(x;y) = \|x-y\|$) is complete then the space $(E, \|\cdot\|)$ is called a generalized Banach space, where*

$$\|x-y\| = \begin{pmatrix} \|x-y\|_1 \\ \vdots \\ \|x-y\|_n \end{pmatrix}.$$

Definition 1.2.2. *Let E be a nonempty set and let $\|\cdot\| : E \to \mathbb{R}^n_+$ be a norm on E. Then, the pair $(E, \|\cdot\|)$ is called a generalized normed space. If moreover, $(E, \|\cdot\|)$ has the property that any Cauchy sequence from X is convergent in norm, then we say that $(E, \|\cdot\|)$ is a generalized Banach space.*

Let $C(I, \mathbb{R}) \times C(I, \mathbb{R})$ be endowed with the vector norm $\|\cdot\|$ defined by $\|v\|_\infty = (\|u_1\|_\infty; \|u_2\|_\infty)$ for $v = (u_1, u_2)$. It is clear that $(C(I, \mathbb{R}) \times C(I, \mathbb{R}), \|\cdot\|_\infty)$ is a generalized Banach space.

Definition 1.2.3. *Let $(E, \|\cdot\|)$ be a generalized Banach space. A subset $A \subset E$ is called open if, for any $x \in A$, there exists $r := (r_1, \ldots, r_m) \in \mathbb{R}^m_+$, such that $B(x_0, x) \subset A$, where*

$$B(x_0, r) = \{x \in E : \|x - x_0\| < r\},$$

denote the the open ball centered in x_0 with radius r, and

$$\overline{B(x_0, r)} = \{x \in E : \|x - x_0\| \leq r\},$$

the closed ball centered at x_0 with radius r.

Definition 1.2.4. *[36] Let X be a real Banach space. A nonempty closed convex set $P \subset X$ is called cone if*

1) *If $x \in P, \lambda \geq 0$ then $\lambda x \in P$,*

2) *if $x \in P, -x \in P$ then $x = 0$.*

Definition 1.2.5. *Let $(E, \|\cdot\|)$ be a generalized Banach space a sequence (x_n) in E is called the Cauchy sequence, if for each $\epsilon > 0$ there exist $N \in \mathbb{N}$ such that for any $n, m \geq N : \|x_n - x_m\| < \epsilon$.*

Definition 1.2.6. *An generalized Banach space $(E, \|\cdot\|)$ is called complete if each Cauchy sequence in E converges to a limit in E.*

Definition 1.2.7. *Let $(E, \|\cdot\|)$ be a generalized Banach space, we say that a subset $F \subset E$ is a closed if $(x_n) \subset F$ and $x_n \to x$ imply $x \in F$.*

Let $I = [a, b]$ be an interval of $\mathbb{R}$. Let $(E, \|.\|)$ be a real Banach space.

Definition 1.2.8. *A map $f \colon I \times E \to E$ is said to be L^1- Carathéodory if:*

1. *$t \to f(., y)$ is measurable for all $y \in E$,*

2. *$y \to f(t, .)$ is continuous for almost each $t \in [a, b]$,*

3. *for each $r > 0$, there exists $h_r \in L^1([a, b]; \mathbb{R}_+)$ such that*

$$|f(t, y)| \leq h_r(t) \text{ for all } |y| \leq r \text{ for almost each } t \in [a, b].$$

Definition 1.2.9. *A subset A of a real Banach space E is called uniformly bounded in E if and only if:*

$$\exists M > 0, \ \forall f \in A, \quad \|f\|_\infty \leq M.$$

Definition 1.2.10. *Let f be a differentiable function defined from X in $\mathbb{R}$. A point $u \in X$ is said to be critical point f if and only if $Df(u) = 0$, where Df is the differentiable of f.*

Definition 1.2.11. *Let C be a convex subspace of real vector space V,and F a mapping of C into $\mathbb{R}$. F is said to be convex if, for every u and v in C, we have:*

$$F(\lambda u + (1 - \lambda)v) \leq \lambda F(u) + (1 - \lambda)F(v) \quad \forall \lambda \in [0, 1].$$

Definition 1.2.12. *Let real vector space V, we recall that a function $F : V \to \overline{\mathbb{R}}$ is said to be lower semi-continuous on V (l.s.c.), if it satisfies the two equivalent conditions:*

$$\begin{array}{ll} \forall a \in \mathbb{R} & \{u \in V | F(u) \leq a\} \text{ is closed,} \\ \forall \overline{u} \in V & \underline{\lim} F(u) \geq F(\overline{u}). \end{array}$$

Definition 1.2.13. *A functional $E : D \subset X \to \overline{\mathbb{R}}$ defined on an unbounded set D, is said to be coercive if $E(u) \to +\infty$ as $|u| \to \infty$.*

Definition 1.2.14. *A functional $E : D \subset X \to \overline{\mathbb{R}}$ defined on an unbounded set D, is said to be anti-coercive if $E(u) \to -\infty$ as $|u| \to \infty$.*

Definition 1.2.15. *Let E be a Banach space. A subset A of E is equicontinuous on I if*

$$\forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall x, y \in I; |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon,$$

for all $f \in A$.

Definition 1.2.16. *Let E be a Banach space and $\Omega \subset E$. The operator $T : \Omega \to E$ is called completely continuous if T is continuous and if for all bounded subset B of Ω, $T(B)$ is relatively compact on E.*

Lemma 1.2.1. *Let B be a subset of $E = C([a, b], \mathbb{R}^n)$, B is relatively compact if:*

1. *B is equicontinuous on $[a, b]$;*

2. *B is uniformly bounded in E.*

Definition 1.2.17. *Let F be a Banach space. The mapping $f \colon E \to F$ is called compact if :*

1. *f is continuous in E;*

2. *$f(E)$is relatively compact in F.*

1.2.2 Matrix convergent

In this section, we introduce definitions, lemmas and theorems concerning to matrice convergent.

Definition 1.2.18. *A square matrix M of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1.*

Lemma 1.2.2. *Let $M \in M_{n,n}(\mathbb{R}^+)$, and I denote the identity matrix in $M_{n,n}(\mathbb{R}^+)$. The following statements are equivalent:*

- *M is a matrix convergent to zero.*
- *The eigenvalues of M are in open disc, i.e., $|\mu| < 1$, for every $\mu \in \mathbb{C}$ with $det(M - \mu I) = 0$.*
- *$M^n \to 0$ as $n \to \infty$.*
- *The matrix $I - M$ is nonsingular and $(I-M)^{-1} = I + M + \ldots + M^n + \ldots$.*
- *The matrix $I - M$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.*

Some examples of matrices convergent to zero, $A \in M_{n,n}(\mathbb{R})$, which also satisfies the property $(I - A)^{-1}|I - A| \leq I$ are:

1. $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ where $a, b \in \mathbb{R}^+$ and $\max(a, b) < 1$,
2. $A = \begin{pmatrix} a & -c \\ 0 & b \end{pmatrix}$ where $a, b, c \in \mathbb{R}^+$ and $a + b < 1, c < 1$,
3. $A = \begin{pmatrix} a & -a \\ b & -b \end{pmatrix}$ where $a, b \in \mathbb{R}^+$ and $|a - b| < 1, a > 1, b > 0$.

Lemma 1.2.3. *If $A \in M_{n\times n}(\mathbb{R}_+)$ is a matrix with $\rho(A) < 1$, then $\rho(A + B) < 1$ for every matrix $B \in M_{n\times n}(\mathbb{R}_+)$ whose elements are small enough.*

1.2.3 Fixed point results

Fixed point theory plays a major role in many of our existence principles, therefore we shall state the fixed point theorems in generalized Banach spaces.

Theorem 1.2.4. *[31, 102] Let X be a generalized Banach space and let $N : X \to X$ be a completely continuous operator. Then, either*

(i) the equation $N(x) = x$ has a at least one solution, or

(ii) the set $\mathcal{M} = \{x \in X| \ \mu N(x) = x, \mu \in (0,1)\}$ is unbounded.

Theorem 1.2.5. *[75] Let $(X, \|\cdot\|)$ be a normed space, $P_1, P_2 \subset X$ two cones; $P := P_1 \times P_2, r, R \in \mathbb{R}^2_+, P_{r,R} := \{u \in P_i : r_i \leq \|u_i\| \leq R_i\}$ with $0 < r < R$, and let $N : P_{r,R} \to P, N = (N_1, N_2)$ a compact map. Assume that for each $i \in \{1, 2\}$, one of the following conditions is satisfied in $P_{r,R}$:*

1. *$N_i(u_i) \not\prec u_i$ if $\|u_i\| = r_i$, and $N_i(u_i) \not\succ u_i$ if $\|u_i\| = R_i$,*
2. *$N_i(u_i) \not\succ u_i$ if $\|u_i\| = r_i$, and $N_i(u_i) \not\prec u_i$ if $\|u_i\| = R_i$.*

Then N has a fixed point u in P with $r_i \leq \|u_i\| \leq R_i$ for $i \in \{1, 2\}$, where $\preceq$, namely $u \preceq v$ if and only if $v - u \in P$. We shall say that $u \prec v$ if $v - u \in P \setminus \{0\}$.

Remark 1.2.2. *[75] In Theorem (1.2.5) four cases are possible for $u \in P_{r,R}$:*

$$\begin{array}{ll} (c_1): & N_1(u) \not\prec u_1 \text{ if } \|u_1\| = r_1, \text{ and } N_1(u) \not\succ u_1 \text{ if } \|u_1\| = R_1, \\ & N_2(u) \not\prec u_2 \text{ if } \|u_2\| = r_2, \text{ and } N_2(u) \not\succ u_2 \text{ if } \|u_2\| = R_2. \\ (c_2): & N_1(u) \not\prec u_1 \text{ if } \|u_1\| = r_1, \text{ and } N_1(u) \not\succ u_1 \text{ if } \|u_1\| = R_1, \\ & N_2(u) \not\succ u_2 \text{ if } \|u_2\| = r_2, \text{ and } N_2(u) \not\prec u_2 \text{ if } \|u_2\| = R_2. \\ (c_3): & N_1(u) \not\succ u_1 \text{ if } \|u_1\| = r_1, \text{ and } N_1(u) \not\prec u_1 \text{ if } \|u_1\| = R_1, \\ & N_2(u) \not\prec u_2 \text{ if } \|u_2\| = r_2, \text{ and } N_2(u) \not\succ u_2 \text{ if } \|u_2\| = R_2. \\ (c_4) & N_1(u) \not\succ u_1 \text{ if } \|u_1\| = r_1, \text{ and } N_1(u) \not\prec u_1 \text{ if } \|u_1\| = R_1, \\ & N_2(u) \not\succ u_2 \text{ if } \|u_2\| = r_2, \text{ and } N_2(u) \not\prec u_2 \text{ if } \|u_2\| = R_2. \end{array}$$

Theorem 1.2.6. *[76] Let $(X, \|\cdot\|)$ be a Banach space, $P_1, P_2 \subset X$ two cones and $P := P_1 \times P_2$ the corresponding cone of $X^2 = X \times X$, and let $\alpha_i, \beta_i > 0$ we denote:*

$$U_{\alpha_i} = \{u \in P_i : \|u\| < \alpha_i\}, \text{ and } V_{\beta_i} = \{u \in P_i : \|u\| < \beta_i\},$$

with $\alpha_i \neq \beta_i, r_i = \min\{\alpha_i, \beta_i\}$ and $R_i = \max\{\alpha_i, \beta_i\}$ for $i = 1, 2$. Assume that $N : \overline{W_1 \times W_2} \to P, N = (N_1, N_2)$, is a compact map (where $W_i =$

$U_{\alpha_i} \cup V_{\beta_i}$ for $i = 1, 2$) and there exist $h_i \in P_i \setminus \{0\}, i = 1, 2,$ such that for each $i \in \{1, 2\}$ the following condition is satisfied in $\overline{W_1 \times W_2}$:

$$\lambda u_i \neq N_i u \text{ for } \|u_i\| = \alpha_i \text{ and } \lambda \geq 1, \tag{1.2.1}$$

$$u_i \neq N_i u + \mu h_i \text{ for } \|u_i\| = \beta_i \text{ and } \mu \geq 0. \tag{1.2.2}$$

Then

1. *N has at least one fixed point $u = (u_1, u_2)$ in P such that $u_i \in U_{\alpha_i} \setminus \overline{V_{\beta_i}}$ for $i = 1, 2$ if $\alpha_i > \beta_i$ for $i = 1, 2$,*

2. *N has at least two fixed points located in $(U_{\alpha_1} \setminus \overline{V_{\beta_1}}) \times U_{\alpha_2}$ and $(U_{\alpha_1} \setminus \overline{V_{\beta_1}}) \times (V_{\beta_2} \setminus U_{\alpha_2})$ if $\beta_1 < \alpha_1$ and $\beta_2 > \alpha_2$,*

3. *N has at least two fixed points located in $U_{\alpha_1} \times (U_{\alpha_2} \setminus \overline{V_{\beta_2}})$ and $(V_{\beta_1} \setminus \overline{U_{\alpha_1}}) \times (U_{\alpha_2} \setminus \overline{V_{\beta_2}})$ if $\beta_1 > \alpha_1$ and $\beta_2 < \alpha_2$,*

4. *N has at least four (three non-trivial) fixed points in $U_{\alpha_1} \times U_{\alpha_2}, U_{\alpha_1} \times (V_{\beta_2} \setminus \overline{U_{\alpha_2}}), (V_{\beta_1} \setminus \overline{U_{\alpha_1}}) \times U_{\alpha_2},$ and $(V_{\beta_1} \setminus \overline{U_{\alpha_1}}) \times (V_{\beta_2} \setminus \overline{U_{\alpha_2}})$ if $\alpha_i < \beta_i$ for $i = 1, 2$.*

Remark 1.2.3. *[76] Our previous results can be easily generalized to systems of n operator equations.*

1.3 Elements of calculus on time scales

A time scale is an arbitrary nonempty closed subset of real numbers. The study of dynamic equations on time scales is a fairly new subject, and research in this area is rapidly growing. The theory of dynamic equations unifies the theories of differential equations and difference equations. We suppose that the reader is familiar with the basic concepts concerning the calculus on time scales for dynamic equations. Otherwise one can find in Bohner and Peterson books [11,12,48] most of the material needed to read this work. We start by giving some definitions necessary for our work. All the definitions, theorem, notations, and basic results that are used in this section can be found in [11, 12].

1.3.1 Description of time scales

Definition 1.3.1. *A time scale is an arbitrary nonempty closed subset of the set of real numbers* $\mathbb{R}$ *is denoted by* $\mathbb{T}$.

Example 1.3.1. *The reals* $\mathbb{R}$, *the integers* $\mathbb{Z}$, *the positive integers* $\mathbb{N}$, *and the nonnegative integers* $\mathbb{N}_0$ *are a time scales. The most common time scales are* $\mathbb{T} = \mathbb{R}$ *for continuous calculus,* $\mathbb{T} = \mathbb{Z}$ *for discrete calculus, and* $\mathbb{T} = q^{\mathbb{N}_0} = \{q^n : n \in \mathbb{N}_0\}$, *where* $q > 1$, *for quantum calculus.*

Example 1.3.2. *The rational numbers* $\mathbb{Q}$, *the irrational numbers* $\mathbb{R}\backslash\mathbb{Q}$, *the complex numbers* $\mathbb{C}$, *and the open interval* $(0, 1)$, *are not time scales.*

We Assume throughout that a time scale $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology.

Definition 1.3.2.

- *If* $\mathbb{T}$ *has a right-scattered minimum* m, *we define* $\mathbb{T}_k = \mathbb{T} - \{m\}$; *otherwise, we set* $\mathbb{T}_k = \mathbb{T}$.
- *If* $\mathbb{T}$ *has a left-scattered maximum* M, *we define* $\mathbb{T}^k = \mathbb{T} - \{M\}$; *otherwise, we set* $\mathbb{T}^k = \mathbb{T}$.

Definition 1.3.3. *The forward jump operator* $\sigma(t) : \mathbb{T} \to \mathbb{T}$ *and the backward jump operator* $\rho(t) : \mathbb{T} \to \mathbb{T}$ *are given by*

$$\sigma(t) = \inf_{s \in \mathbb{T}} \{s > t\},$$

$$\rho(t) = \sup_{s \in \mathbb{T}} \{s < t\}.$$

The graininess function $\mu(t) : \mathbb{T} \to [0, \infty)$ *is given by* $\mu(t) = \sigma(t) - t$. *A point* $t \in \mathbb{T}$ *is called right-dense if* $t < \sup \mathbb{T}$ *and* $\sigma(t) = t$, *right-scattered if* $\sigma(t) > t$, *left-dense if* $t > \inf \mathbb{T}$ *and* $\rho(t) = t$, *and left-scattered if* $\rho(t) < t$.

Here it is assumed that $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ contains the maximal element t) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ contains the minimal element t).
Points that are right-scattered and left-scattered at the same time are called isolated. Points that are right-dense and left-dense at the same time are called dense.

Example 1.3.3. *1. If $\mathbb{T} = \mathbb{R}$, $\sigma(t) = t$, $\rho(t) = t$ and $\mu(t) = 0$ for all $t \in \mathbb{T}$. Hence every point $t \in \mathbb{R}$ is dense.*

2. *If $\mathbb{T} = h\mathbb{Z}$ $(h \neq 0)$, $\sigma(t) = t+h, \rho(t) = t-h$ and $\mu(t) = h$ for all $t \in \mathbb{T}$. Hence if $h > 0$ every point $t \in \mathbb{Z}$ is isolated.*

3. *If $\mathbb{T} = q^{N_0} = \{q^n : n \in \mathbb{N}_0\}, (h \neq 0), \sigma(t) = qt, \rho(t) = \frac{1}{q}$ and $\mu(t) = q(t-1)$ for all $t \in \mathbb{T}$. Hence if $q > 1$ every point $t \in q^{N_0}$ is isolated.*

4. *If $\mathbb{T} = \{\frac{1}{n} : n \in \mathbb{N}\} \bigcup \{0\}, \sigma(t) = \frac{t}{1-t}, \rho(t) = \frac{t}{t+1}$ and $\mu(t) = \frac{t^2}{1-t}$ for all $t \in \mathbb{T} - \{1\}$.*

5. *If $\mathbb{T} = \mathbb{N}^2 = \{n^2 : n \in \mathbb{N}_0\}, \sigma(t) = (\sqrt{t}+1)^2, \rho(t) = (\sqrt{t}-1)^2$ and $\mu(t) = 2\sqrt{t}+1$ for all $t \in \mathbb{T}$.*

Definition 1.3.4. *If $f : \mathbb{T} \to \mathbb{R}$ we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by*

$$f^\sigma(t) = f(\sigma(t)) \quad \text{for all } t \in \mathbb{T},$$

i.e., $f^\sigma = f \circ \sigma$.

The notion of periodic time scales is introduced in Kaufmann and Raffoul [46].

Definition 1.3.5. *Suppose that T_1 is a positive number. If for any $t \in \mathbb{T}$, $t + T_1 \in \mathbb{T}$, holds then $\mathbb{T}$ is called a T_1 periodic time scale and T_1 is a period.*

Definition 1.3.6. *Suppose that $\mathbb{T}$ is a T_1 periodic time scale, and $u : \mathbb{T} \to \mathbb{R}$ is a function. If for any $t \in \mathbb{T}$, $t+T_1 \in \mathbb{T}$ holds and satisfies $u(t+T_1) = u(t)$, then u is called a periodic function and T_1 is a period of u. The smallest period of a periodic function is called the fundamental period of the function.*

Remark 1.3.1. *The definition of periodic functions implies that if u is a periodic function with the period T_1 then $u(t) = u(t + mT_1)$ holds for all t, $mT_1 + t \in \mathbb{T}$ and any integer m.*

1.3.2 Differentiation on time scales

The theory of dynamic equations at time scales was introduced in 1988 by Stefan Hilger in his doctoral dissertation where he defined the Δ-derivated as follows (see [41]).

Definition 1.3.7. *Let $g : \mathbb{T} \to \mathbb{R}$ is a function and $t \in \mathbb{T}^k$. Define $g^\Delta(t)$ to be the number (when it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e. $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that*

$$|(g^\sigma(t) - g(s)) - g^\Delta(t)(\sigma(t) - s)| \le \epsilon|\sigma(t) - s|, \text{ for all } s \in U.$$

We call $g^\Delta(t)$ the delta derivative or Hilger derivative of g at t.

The real number $g^\nabla(t)$ is called nabla derivative of g at the point $t \in \mathbb{T}_k$ if for any $\epsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$|g(\rho(t)) - g(s) - g^\nabla(t)(\rho(t) - s)| \le \epsilon|\rho(t) - s|,$$

for all $s \in U$.

Remark 1.3.2.

- *If $\mathbb{T} = \mathbb{R}$, we can easy show that $g^\Delta(t) = g'(t) = g^\nabla(t)$.*
- *If $\mathbb{T} = \mathbb{Z}$, then $g^\Delta(t) = g(t+1) - g(t)$ and $g^\nabla(t) = g(t) - g(t-1)$.*

Example 1.3.4.

1. *If $g : \mathbb{T} \to \mathbb{R}$ is defined by $g(t) = a$ for all $t \in \mathbb{T}$, where $a \in \mathbb{R}$ is constant, then $g^\Delta(t) = 0$. This is clear because for any $\epsilon > 0$,*

$$|(g^\sigma(t) - g(s)) - 0(\sigma(t) - s)| = |a - a| = 0 \le \epsilon|\sigma(t) - s|, \text{ for all } s \in U.$$

2. *If $g : \mathbb{T} \to \mathbb{R}$ is defined by $g(t) = t$ for all $t \in \mathbb{T}$, then $g^\Delta(t) = 1$. This is clear because for any $\epsilon > 0$,*

$$|(g^\sigma(t) - g(s)) - 1(\sigma(t) - s)| = 0 \le \epsilon|\sigma(t) - s|, \text{ for all } s \in U.$$

Theorem 1.3.1. *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we have the following:*

- *If f is differentiable at t, then f is continuous at t.*
- *If f is continuous at t and t is right-scattered, then f is differentiable at t with*
$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

- *If t is right-dense, then f is differentiable at t with*

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

- *If f is differentiable at t, then*

$$f(\sigma(t)) = \mu(t) f^{\Delta}(t) + f(t).$$

Example 1.3.5. *1. If $\mathbb{T} = \mathbb{R}$, then $f^{\Delta}(t) = f'(t)$ for all $t \in \mathbb{R}$.*

2. If $\mathbb{T} = h\mathbb{Z}$, then $f^{\Delta}(t) = \frac{f(t+h) - f(t)}{h} = \Delta f$ for all $t \in \mathbb{T}$.

3. If $\mathbb{T} = q^{N_0}$ and $f(t) = t^2$, then $f^{\Delta}(t) = qt + t$ for all $t \in \mathbb{T}$.

4. If $\mathbb{T} = q^{N_0}$ $(q > 1)$ and $f(t) = \log(t)$, then $f^{\Delta}(t) = \frac{\log(q)}{t(q-1)}$ for all $t \in \mathbb{T}$.

Theorem 1.3.2. *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}$. Then*

1. *The sum $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$(f + g)^{\Delta} = f^{\Delta}(t) + g^{\Delta}(t).$$

2. *For any constant a, $af : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$(af)^{\Delta}(t) = af^{\Delta}(t).$$

3. *The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$\begin{aligned} (fg)^{\Delta}(t) &= f^{\Delta}(t) g(\sigma(t)) + f(t) g^{\Delta}(t) \\ &= f^{\Delta}(t) g(t) + f(\sigma(t)) g^{\Delta}(t). \end{aligned}$$

4. *If $f(t) f(\sigma(t)) \neq 0$, then $\frac{1}{f}$ is differentiable at t with*

$$(\frac{1}{f})^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t) f(\sigma(t))}.$$

5. *If $g(t) g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is differentiable at t with*

$$(\frac{f}{g})^{\Delta}(t) = \frac{f^{\Delta}(t) g(t) - f(t) g^{\Delta}(t)}{g(t) g(\sigma(t))}.$$

Example 1.3.6. *1. Let $a \in \mathbb{R}$ and $m \in \mathbb{N}$, $f : \mathbb{T} \to \mathbb{R}$ defined by $f(t) = (t-a)^m$ we have*

$$f^{\Delta}(t) = \sum_{k=0}^{m-1} (\sigma(t) - a)^k (t-a)^{m-k-1}.$$

2. Let $a \in \mathbb{R}$ and $m \in \mathbb{N}, g : \mathbb{T} - \{a\} \to \mathbb{R}$ defined by $g(t) = \frac{1}{(t-a)^m}$ provided $(t-a)(\sigma(t)-a) \neq 0$ we have

$$gf^{\Delta}(t) = \sum_{k=0}^{m-1} \frac{1}{(\sigma(t) - a)^{m-k}(t-a)^{k+1}}.$$

Theorem 1.3.3. *Suppose $f : \mathbb{T} \to \mathbb{R}$ is differentiable at $t_0 \in \mathbb{T} \backslash \{\max \mathbb{T}\}$. If $f^{\Delta}(t_0) > 0$, then f is right-increasing. If $f^{\Delta}(t_0) < 0$, then f is right-decreasing.*

1.3.3 Integration on time scales

Definition 1.3.8. *A function $f : \mathbb{T} \to \mathbb{R}$ is called regulated provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.*

Definition 1.3.9. *A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at every right-dense point $t \in \mathbb{T}$ and its left-sided limits exist, and is finite at every left-dense point $t \in \mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by*

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by

$$C^1_{rd} = Cr^1_d(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R}).$$

Theorem 1.3.4. *Let f be regulated. Then there exists a function F which is pre-differentiable with region of differentiation D such that*

$$F^{\Delta}(t) = f(t) \quad \textit{holds for all } t \in D.$$

Definition 1.3.10. *Assume $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Any function F is called a pre-antiderivative of f. We define the indefinite integral of a regulated function f by*

$$\int f^{\Delta}(t)\Delta t = F(t) + c,$$

where c is an arbitrary constant and F is a pre-antiderivative of f. For all $a, b \in \mathbb{T}$, the Cauchy integral is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided

$$F^{\Delta}(t) = f(t) \text{ holds for all } t \in \mathbb{T}.$$

Lemma 1.3.5. *[16] A function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ is absolutely continuous on $[a, b]_{\mathbb{T}}$ if and only if f is Δ-differentiable almost everywhere on $[a, b)_{\mathbb{T}}$, $f^{\Delta} \in L^1_{\Delta}([a, b)_{\mathbb{T}}, \mathbb{R})$ and*

$$f(t) = f(a) + \int_{[a,b)_{\mathbb{T}}} f^{\Delta}(s)\Delta s, \quad \forall t \in [a, b]_{\mathbb{T}}.$$

Lemma 1.3.6. *[2] Let f and $g : [a, b]_{\mathbb{T}} \to \mathbb{R}$ are absolutely continuous functions on $[a, b]_{\mathbb{T}}$, then fg is absolutely continuous on $[a, b]_{\mathbb{T}}$ and we have*

$$\int_{[a,b)_{\mathbb{T}}} (f^{\Delta}g + f^{\sigma}g^{\Delta})(t)\Delta t = f(b)g(b) - f(a)g(a) = \int_{[a,b)_{\mathbb{T}}} (fg^{\Delta} + f^{\Delta}g^{\sigma})(t)\Delta t.$$

Theorem 1.3.7. *If $a, b, c \in \mathbb{T}, \alpha \in \mathbb{R}$, and $f, g \in C_{rd}$, then*

- $\displaystyle\int_a^b [f(r) + g(r)]\Delta r = \int_a^b f(r)\Delta r + \int_a^b g(r)\Delta r,$
- $\displaystyle\int_a^b [\alpha f(r)]\Delta r = \alpha \int_a^b f(r)\Delta r,$
- $\displaystyle\int_a^b f(r)\Delta r = -\int_b^a f(r)\Delta r,$
- $\displaystyle\int_a^b f(r)\Delta r = \int_a^c f(r)\Delta r + \int_c^b f(r)\Delta r,$

- $\displaystyle\int_a^b f(\sigma(r))g^{\Delta}(r)\Delta r = (fg)(b) - (fg)(a) + \int_a^b f^{\Delta}(r)g(r)\Delta r,$

- $\displaystyle\int_a^b f(r)g^{\Delta}(r)\Delta r = (fg)(b) - (fg)(a) + \int_a^b f^{\Delta}(r)g(\sigma(r))\Delta r,$

- $\displaystyle\int_a^a f(r)\Delta r = 0,$

- *if* $f(t) \geq 0$ *for all* $a \leq t < b$, *then* $\displaystyle\int_a^b f(r)\Delta r \geq 0,$

- *if* $f(t) \leq g(t)$ *on* $[a, b)$ *then,* $\displaystyle\left|\int_a^b f(r)\Delta r\right| \leq \int_a^b g(r)\Delta r.$

Definition 1.3.11. *Infinite integrals are defined as*

$$\int_a^{\infty} f(r)\Delta r = \lim_{t\to\infty} \int_a^t f(r)\Delta r.$$

Theorem 1.3.8. *If* $f : \mathbb{T} \to \mathbb{R}$ *is an arbitrary function and* $t \in \mathbb{T}$, *then*

$$\int_t^{\sigma(t)} f(r)\Delta r = \mu(t)f(t).$$

1.3.4 Sobolev's spaces on time scales

In this section, we define the Sobolev's spaces on time scales and study their important properties (see [2, 37]).

Definition 1.3.12. *Let* $E \subset \mathbb{T}$ *be a* Δ*-measurable set and let* $p \in \overline{\mathbb{R}} = [-\infty, +\infty]$ *be such that* $p \geq 1$ *and let* $f : E \to \overline{\mathbb{R}}$ *be a* Δ*-measurable function. Say that* f *belongs to* $L^p_{\Delta}(E, \mathbb{R})$ *provided that either* $\displaystyle\int_E |f(t)|^p \Delta t < +\infty$ *if* $p \in \mathbb{R}$, *or there exists a constant* $C \in \mathbb{R}$ *such that* $|f| \leq C$ Δ*-a.e. on* E *if* $p = +\infty$.

Lemma 1.3.9. *Let* $p \in \overline{\mathbb{R}}$ *be such that* $p \geq 1$. *Then the set* $L^p_{\Delta}([a, b)_{\mathbb{T}}, \mathbb{R})$ *is a Banach space together with the norm defined for* $f \in L^p_{\Delta}([a, b)_{\mathbb{T}}, \mathbb{R})$ *as*

$$\|f\|_{L^p_\Delta([a,b)_\mathbb{T},\mathbb{R})} = \begin{cases} \Big(\displaystyle\int_{[a,b)_\mathbb{T}} |f(t)|^p \Delta t\Big)^{\frac{1}{p}}, & p \in \mathbb{R}; \\ \inf\{C \in \mathbb{R} : |f| \leq C \ \Delta - a.e. \ on \ [a,b)_\mathbb{T}\}, & p = +\infty. \end{cases} \tag{1.3.1}$$

Moreover, $L^2_\Delta([a,b)_\mathbb{T},\mathbb{R})$ *is a Hilbert space together with the inner product given for every* $(f,g) \in L^p_\Delta([a,b)_\mathbb{T},\mathbb{R}) \times L^p_\Delta([a,b)_\mathbb{T},\mathbb{R}) by :$

$$< f, g >_{L^2_\Delta([a,b)_\mathbb{T},\mathbb{R})} = \int_{[a,b)_\mathbb{T}} f(t)g(t)\Delta t.$$

Proposition 1.3.10. *Suppose* $p \in \mathbb{R}$ *and* $p \geq 1$*. Let* $p' \in \overline{\mathbb{R}}$ *be such that* $\frac{1}{p} + \frac{1}{p'} = 1$*. Then , if* $f \in L^p_\Delta([a,b)_\mathbb{T},\mathbb{R})$ *and* $gf \in L^{p'}_\Delta([a,b)_\mathbb{T},\mathbb{R})$*, then* $f.g \in L^1_\Delta([a,b)_\mathbb{T},\mathbb{R})$ *and*

$$\|f.g\|_{L^1_\Delta([a,b)_\mathbb{T},\mathbb{R})} \leq \|f\|_{{}^p_\Delta([a,b)_\mathbb{T},\mathbb{R})}.\|g\|_{{}^{p'}_\Delta([a,b)_\mathbb{T},\mathbb{R})}.$$

For $p \in \mathbb{R}, p \geq 1$*, with the norm* $\|f\|_{L^p_\Delta([a,b)_\mathbb{T},\mathbb{R})} = \Big(\displaystyle\int_{[a,b)_\mathbb{T}} |f(t)|^p \Delta t\Big)^{\frac{1}{p}}$.

Definition 1.3.13. *Let* $p \in \mathbb{R}$ *be such that* $p > 1$. *and* $u : [0,\sigma(t)]_\mathbb{T} \to \mathbb{R}$*. We say that* $u \in W^{1,p}_{\Delta,T}([0,\sigma(t)]_\mathbb{T},\mathbb{R})$ *if and only if* $u \in L^p_\Delta([0,\sigma(t))_\mathbb{T},\mathbb{R})$ *and there exists* $g \in L^p_\Delta(([0,\sigma(T))_\mathbb{T},\mathbb{R})$ *such that*

$$\int_{[0,\sigma(T))_\mathbb{T}} u(s)\varphi(s)\Delta s = -\int_{[0,\sigma(T))_\mathbb{T}} g(s)\varphi^\sigma(s)\Delta s$$

for $\forall\varphi \in \mathcal{C}^1_{0,rd}([0,\sigma(t))^k_\mathbb{T},\mathbb{R})$*. where*

$\mathcal{C}^1_{0,rd}([0,\sigma(t)]^k_\mathbb{T},\mathbb{R}) = \{u : [0,\sigma(t)]_\mathbb{T} \to \mathbb{R} | u \in \mathcal{C}^1_{rd}([0,\sigma(t)]^k_\mathbb{T},\mathbb{R}), u(0) = u(\sigma(T)) = 0\}.$

$W^{1,p}_\Delta([0,\sigma(T))_\mathbb{T},\mathbb{R})$ is a Banach space which is equivalent to the following functional space

$$AC_\mathbb{T}([0,\sigma(T)]_\mathbb{T},\mathbb{R}) = \{u : [0,\sigma(T)]_\mathbb{T} \to \mathbb{R} | u \text{ is absolutely continuous and } u^\Delta \in L^p_\Delta(([0,\sigma(T))_\mathbb{T},\mathbb{R})\}.$$

Thus

$$W^{1,p}_\Delta([0,\sigma(T)]_\mathbb{T},\mathbb{R}) = AC_\mathbb{T}([0,\sigma(T)]_\mathbb{T},\mathbb{R}),$$

is a reflexive Banach with the norm

$$\|u\| = \|u\|_{W^{1,p}_{\Delta,T}} = \Big(\int_{[0,\sigma(T))_\mathbb{T}} |u|^p \Delta t + \int_{[0,\sigma(T))_\mathbb{T}} |u^\Delta|^p \Delta t\Big)^{\frac{1}{p}}, \quad u \in W^{1,p}_\Delta([0,\sigma(T)]_\mathbb{T},\mathbb{R}).$$

Proposition 1.3.11. *[2] Let $p \in \overline{\mathbb{R}}$ be such that $p \geq 1$, and J equipped with the Lebesgue $\Delta-$ measure. Then, the following statements are true:*

1. *If $p > 1$, then the immersion $W_{\Delta}^{1,p}(J) \hookrightarrow C(J)$ is compact.*
2. *If $p = 1$, then the immersion $W_{\Delta}^{1,p}(J) \hookrightarrow C(J)$ is compact if and only if every point of J is isolated.*

Corollary 1.3.12. *Let $p \in \overline{\mathbb{R}}$ be such that $p > 1$, and J equipped with the Lebesgue $\Delta-$ measure, let $\{x_m\}_{m\in\mathbb{N}} \subset W_{\Delta}^{1,p}(J)$, and let $x \in W_{\Delta}^{1,p}(J)$. If $\{x_m\}_{m\in\mathbb{N}}$ converges weakly in $W_{\Delta}^{1,p}(J)$ to x, then $\{x_m\}_{m\in\mathbb{N}}$ converges strongly in $C(J)$ to x.*

Lemma 1.3.13. *There exists a constant $c \geq 0$ such that, if $u \in W_{\Delta}^{1,p}([0,\sigma(T)]_{\mathbb{T}}, \mathbb{R})$ and $\int_{t\in[0,\sigma(T))_{\mathbb{T}}} u(t)\Delta t = 0$, then*

$$\|u\|_{\infty} \leq c\|u^{\Delta}\|_{L^p_{\Delta}}.$$

Proposition 1.3.14. *[89] Let $\Omega_{\mathbb{T}}$ be an open subset of $\mathbb{T}^n$ and $|\Omega_{\mathbb{T}}| < \infty$. Suppose that $1 \leq p, r < \infty$ and $f \in \mathcal{C}(\Omega_{\mathbb{T}} \times \mathbb{R}, \mathbb{R})$ satisfies*

$$|f(t,x)| \leq c(1 + |x|^{\frac{p}{r}}). \tag{1.3.2}$$

Then for any $x \in L^p_{\Delta}(\Omega_{\mathbb{T}}, \mathbb{R})$, $f(\cdot, x) \in L^r_{\Delta}(\Omega_{\mathbb{T}} \times \mathbb{R}, \mathbb{R})$ and the operator

$$\mathcal{N} : L^p_{\Delta}(\Omega_{\mathbb{T}}, \mathbb{R}) \to L^r_{\Delta}(\Omega_{\mathbb{T}} \times \mathbb{R}, \mathbb{R}) : x \mapsto f(t,x)$$

is continuous.

1.4 Theorems of variational method

In this section, we state some theorems which are used to prove our main results.

Theorem 1.4.1. *[74] Let $(X_i, |\cdot|_i)$, $i = 1, 2$ be Hilbert spaces identified to their duals and let $X = X_1 \times X_2$. In addition, assume that $E_1(\cdot, v)$ and $E_2(u, \cdot)$ are bounded from below for every $u \in X_1, v \in X_2$, and the following boundedness condition holds: there are $R, a > 0$ such that*

$$\text{either } E_1(u,v) \geq \inf_{X_1} E_1(\cdot, v) + a \quad \text{for } |u|_1 \geq R \text{ and all } v \in X_2 \tag{1.4.1}$$

$$\text{or } E_2(u,v) \geq \inf_{X_2} E_2(u,\cdot) + a \quad \text{for } |v|_2 \geq R \text{ and all } u \in X_1. \tag{1.4.2}$$

Then the unique fixed point (u^*, v^*) *of* (N_1, N_2) *is a Nash-type equilibrium of the pair of functionals* (E_1, E_2)*, i.e.,*

$$\begin{aligned} E_1(u^*, v^*) &= \inf_{X_1} E_1(\cdot, v^*) \\ E_2(u^*, v^*) &= \inf_{X_2} E_2(u^*, \cdot). \end{aligned}$$

Definition 1.4.1. *Let* E *be a real Banach space,* $\varphi \in \mathcal{C}^1(E, \mathbb{R})$*, for any sequence* $\{u_n\} \subset E$ *such that* $\varphi(u_n)$ *is bounded and*

$$\varphi'(u_n) \to 0 \;\; as \;\; n \to \infty,$$

then, $\{u_n\}$ *is called a Palais Smale sequence (PS sequence, for short). If a PS sequence of* φ *possesses a convergent subsequence, we say that* φ *satisfies the PS condition.*

Lemma 1.4.2. *[79] Let* $E = V \oplus X$*, where* X *is a Banach space and* $V \neq 0$ *is finite dimensional vectorial space. Suppose that* $\varphi \in \mathcal{C}^1(E, \mathbb{R})$ *satisfies the PS condition, and*

1) *there are a constant* α *and a bounded neighborhood* D *of* 0 *in* V *such that* $\varphi|_{\partial D} \leq \alpha$,

2) *there is a constant* $\beta > \alpha$ *such that* $\varphi|_X \geq \beta$.

Then, φ *possesses a critical value* $c^* \geq \beta$*. Moreover,* c^* *can be characterized as*

$$c^* = \inf_{h \in \Gamma} \max_{u \in \bar{D}} \varphi(h(u)),$$

where $\Gamma = \{h \in \mathcal{C}(\bar{D}, E) | h = id \text{ on } \partial D\}$.

Lemma 1.4.3. *[56] If* φ *is weakly lower semi-continuous on a reflexive Banach space* X^* *and has a bounded minimizing sequence, then* φ *has a minimum on* X^*.

Lemma 1.4.4. *[15] Let* E *be a Banach space such that*

$$E = X_1 \oplus X_2,$$

where X_1, X_2 are two subspaces of E and $\dim X_2 = m < \infty$. Let φ be a $\mathcal{C}^1$ functional on E which satisfies the PS condition. Assume that for some $r > 0$ we have

$$\varphi(u) \geq 0 \; for \; u \in X_1 \; and \; \|u\| \leq r,$$

and

$$\varphi(u) \leq 0 \; for \; u \in X_2 \; and \; \|u\| \leq r.$$

Assume that φ is bounded below and $\inf_E \varphi < 0$. Then, φ has at least two nonzero critical points.

1.5 Basic definitions on fractional calculus

In this chapter, we present some definitions that have an important role in the theory of fractional calculus see [21, 58, 112].

1.5.1 Riemann-Liouville Integrals

In this section we give the definitions of Riemann-Liouville fractional integrals.

Definition 1.5.1. *The function $\Gamma : (0, \infty) \to \mathbb{R}$, defined by*

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt,$$

is called Eulers Gamma function (or Eulers integral of the second kind).

Remark 1.5.1. *For $n \in \mathbb{N}$, we have $(n-1)! = \Gamma(n)$.*

Definition 1.5.2. *Let $\alpha \in \mathbb{R}_+$. The operator I_a^α, defined on $L_1[a, b]$ by*

$$I_a^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt$$

for $a \leq x \leq b$, is called the Riemann-Liouville fractional integral operator of order α.
For $\alpha = 0$, we set $I_a^0 := I$, the identity operator.

The definition for $\alpha = 0$ is quite convenient for future manipulations. It is evident that the Riemann-Liouville fractional integral coincides with the classical definition of I_a^α in the case $\alpha \in \mathbb{N}$, except for the fact that we have extended the domain from Riemann integrable functions to Lebesgue integrable functions (which will not lead to any problems in our development). Moreover, in the case $\alpha \geq 1$ it is obvious that the integral $I_a^\alpha f(x)$ exists for every $x \in [a, b]$ because the integrand is the product of an integrable function f and the continuous function $(x - .)^{\alpha-1}$. In the case $0 < \alpha < 1$ though, the situation is less clear at first sight. However, the following result asserts that this definition is justified.

Theorem 1.5.1. *Let $f \in L^1[a, b]$ and $\alpha > 0$. Then, the integral $I_a^\alpha f(x)$ exists for almost every $x \in [a, b]$. Moreover, the function $I_a^\alpha f$ itself is also an element of $L^1[a, b]$.*

One important property of integer-order integral operators is preserved by our generalization:

Theorem 1.5.2. *Let $\beta, \alpha \geq 0$ and $\phi \in L^1[a, b]$. Then,*

$$I_a^\beta I_a^\alpha \phi = I_a^{\beta+\alpha} \phi,$$

holds almost everywhere on $[a, b]$. If additionally $\phi \in C[a, b]$ or $\beta + \alpha \geq 1$, then the identity holds everywhere on $[a, b]$.

Corollary 1.5.3. *Under the assumptions of Theorem 1.5.2,*

$$I_a^\beta I_a^\alpha \phi = I_a^\alpha I_a^\beta \phi.$$

Example 1.5.1. *Let $f(x) = (x - a)^\beta$ for some $\beta > -1$ and $\alpha > 0$. Then,*

$$\begin{aligned} I_a^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (t - a)^\beta (x - t)^{\alpha-1} dt, \\ &= \frac{1}{\Gamma(\alpha)} (x - a)^{\alpha+\beta} \int_0^1 s^\beta (1 - s)^{\alpha-1} ds, \\ I_a^\alpha f(x) &= \frac{\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 1)} (x - a)^{\alpha+\beta}. \end{aligned}$$

Theorem 1.5.4. *Let $\alpha > 0$. Assume that $(f_k)_{k=1}^\infty$ is a uniformly convergent sequence of continuous functions on $[a, b]$. Then we may interchange the fractional integral operator and the limit process, i.e.*

$$(I_a^\alpha \lim_{k\to\infty} f_k)(x) = (\lim_{k\to\infty} I_a^\alpha f_k)(x).$$

In particular, the sequence of functions $(I_a^\alpha f_k)_{k=1}^\infty$ *is uniformly convergent.*

Theorem 1.5.5. *Let* $1 \leq p < \infty$ *and let* $(\beta_k)_{k=1}^\infty$ *be a convergent sequence of nonnegative numbers with limit* β*. Then, for every* $f \in L^p[a,b]$*,*

$$\lim_{k\to\infty} I_a^{\beta_k} f = I_a^\beta f,$$

where the convergence is in the sense of the $L^p[a,b]$ *norm.*

1.5.2 Riemann-Liouville Derivatives

Having established these fundamental properties of Riemann-Liouville integral operators, we now come to the corresponding differential operators.

Definition 1.5.3. *Let* $\alpha \in \mathbb{R}_+$ *and* $\beta = [\alpha] + 1$*. The operator* D_a^α*, defined by*

$$D_a^\alpha f(x) = D^\beta I_a^{\beta-\alpha} f(x)$$

is called the Riemann-Liouville fractional differential operator of order α*.*

For $\alpha = 0$, we set $D_a^0 := I$, the identity operator.

Theorem 1.5.6. *Assume that* $\alpha_1, \alpha_2 \geq 0$*. Moreover let* $\phi \in L^1[a,b]$ *and* $f = I_a^{\alpha_1+\alpha_2}\phi$*. Then,*

$$D_a^{\alpha_1} D_a^{\alpha_2} f = D_a^{\alpha_1+\alpha_2} f.$$

Note that in order to apply this identity we do not need to know the function ϕ *explicitly, it is sufficient to know that such a function exists.*

Example 1.5.2. *Let* $f(x) = x^{-1/2}$*, and* $\alpha_1 = \alpha_2 = 1/2$*.*

$$D_0^{\alpha_1} f(x) = D_0^{\alpha_2} f(x) = 0,$$

and hence also

$$D_0^{\alpha_1} D_0^{\alpha_2} f(x) = 0,$$

but

$$D_0^{\alpha_1+\alpha_2} f(x) = D_0^1 f(x) = -(2x^{3/2})^{-1}.$$

Example 1.5.3. *Let* $f(x) = x^{1/2}, \alpha_1 = 1/2$ *and* $\alpha_2 = 3/2$.

$$D_0^{\alpha_1} f(x) = \sqrt{\pi}/2 \text{ and } D_0^{\alpha_2} f(x) = 0,$$

This implies

$$D_0^{\alpha_1} D_0^{\alpha_2} f(x) = 0,$$

but

$$D_0^{\alpha_2} D_0^{\alpha_1} f(x) = -x^{-3/2}/4 = D_0^2 f(x) = D_0^{\alpha_1+\alpha_2} f(x).$$

In other words, the first of these two examples shows that it is possible to have

$$D_a^{\alpha_1} D_a^{\alpha_2} f = D_a^{\alpha_2} D_a^{\alpha_1} f \neq D_a^{\alpha_1+\alpha_2} f,$$

whereas the second one exemplifies the case where

$$D_a^{\alpha_1} D_a^{\alpha_2} f \neq D_a^{\alpha_2} D_a^{\alpha_1} f = D_a^{\alpha_1+\alpha_2} f,$$

holds.

Theorem 1.5.7. *Let* $\alpha \geq 0$. *Then, for every* $f \in L^1[a,b]$,

$$D_a^{\alpha} I_a^{\alpha} f = f, \text{ almost everywhere.}$$

Theorem 1.5.8. *Let* $\alpha > 0$. *Assume that* $(f_k)_{k=1}^{\infty}$ *is a uniformly convergent sequence of continuous functions on* $[a,b]$, *and that* $D_a^{\alpha} f_k$ *exists for every* k. *Moreover assume that* $(D_a^{\alpha} f_k)_{k=1}^{\infty}$ *converges uniformly on* $[a+\epsilon, b]$ *for every* $\epsilon > 0$. *Then, for every* $x \in (a,b]$, *we have*

$$(\lim_{k\to\infty} D_a^{\alpha} f_k)(x) = (D_a^{\alpha} \lim_{k\to\infty} f_k)(x).$$

Theorem 1.5.9. *Let* f_1 *and* f_2 *be two functions defined on* $[a,b]$ *such that* $D_a^{\alpha} f_1$ *and* $D_a^{\alpha} f_2$ *exist almost everywhere. Moreover, let* $c_1, c_2 \in \mathbb{R}$. *Then,*$D_a^{\alpha}(c_1 f_1 + c_2 f_2)$ *exists almost everywhere, and*

$$D_a^{\alpha}(c_1 f_1 + c_2 f_2) = c_1 D_a^{\alpha} f_1 + c_2 D_a^{\alpha} f_2.$$

Lemma 1.5.10. *Let* $\alpha > 0, \alpha \notin \mathbb{N}$, *and* $\beta = \lceil \alpha \rceil$. *Assume that* $f \in C^{\beta}[a,b]$ *and* $x \in [a,b]$. *Then,*

$$D_a^{\alpha} f(x) = \frac{1}{\Gamma(-\alpha)} \int_a^x (x-t)^{-\alpha-1} f(t)dt.$$

Definition 1.5.4. *For a function $h \in AC^n(J)$, the Riemann-Liouville fractional order derivative of order $\alpha > 0$ of h, is defined by*

$$D_{0^+}^{\alpha} h(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{h(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the real number α.

Remark 1.5.2. *[6]*

1. *If $\lambda > -1$*

$$D_{0^+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha},$$

and

$$D_{0^+}^{\alpha} t^{\alpha-m} = 0, \quad m = 1, 2, \ldots, n, \text{ where } n = [\alpha] + 1.$$

2. *$D_{0^+}^{\alpha} I_{0^+}^{\alpha} u(t) = u(t)$ for all $u \in C(0,1) \cap L^1(0,1)$.*

3. *If $u \in L^1(0,1), \alpha > \beta > 0$, then*

$$D_{0^+}^{\beta} I_{0^+}^{\alpha} u(t) = I_{0^+}^{\alpha-\beta} u(t).$$

Lemma 1.5.11. *[6] If we assume that $u \in C(0,1) \cap L^1(0,1)$, then the fractional differential equation*

$$D_{0^+}^{\alpha} u(t) = 0, \quad \alpha > 0,$$

has $u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \ldots + C_n t^{\alpha-n}, C_i \in \mathbb{R},\ i = 1, 2, \ldots, n$, as unique solution, where $n = [\alpha] + 1$.

Lemma 1.5.12. *[6] Suppose that $u \in C(0,1) \cap L^1(0,1)$, such that $D_{0^+}^{\alpha} u \in C(0,1) \cap L^1(0,1)$. Then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \ldots + C_n t^{\alpha-n},$$

for some $C_i \in \mathbb{R},\ i = 1, 2, \ldots, n$, where $n = [\alpha] + 1$.

Lemma 1.5.13. *[18] If $x, y \geq 0, \gamma > 0$, then*

$$(x+y)^{\gamma} \leq \max\{2^{\gamma-1}, 1\}(x^{\gamma} + y^{\gamma}).$$

Lemma 1.5.14. *[18] Let $c > 0, \gamma > 0$. For any $x, y \in [0, c]$, we have that*

1. *If $\gamma > 1$, then $|x^{\gamma} - y^{\gamma}| \leq \gamma c^{\gamma-1} |x - y|$,*

2. *if $0 < \gamma \leq 1$, then $|x^{\gamma} - y^{\gamma}| \leq |x - y|^{\gamma}$.*

Chapter 2

Systems of second impulsive differential equations

In this chapter we study the existence results for systems of second impulsive differential equations via variational method. More precisely we consider:

$$\left\{\begin{array}{lcl} -\ddot{u}+m^2u & = & f(t,u,v),\ \ t\neq t_k, k=1,\dots,p,\ \ t\in J,\\ -\ddot{v}+m^2v & = & g(t,u,v),\ \ t\neq t_k, k=1,\dots,p,\ \ t\in J,\\ \dot{u}(t_k^+)-\dot{u}(t_k^-) & = & I_k(u(t_k)),\ \ k=1,\dots,p,\\ \dot{v}(t_k^+)-\dot{v}(t_k^-) & = & \overline{I}_k(u(t_k)),\ \ k=1,\dots,p,\\ u(0)=u(b) & = & v(0)=v(b)=0, \end{array}\right. \tag{2.0.1}$$

where $J:=[0,b],\ m\neq 0,\ f,g:J\times\mathbb{R}^2\to\mathbb{R}$ are two functions, $I_k,\overline{I}_k\in C(\mathbb{R},\mathbb{R}),\ k\in\{1,2,\dots,n\}$, $\dot{u}(t_k^+)$ and $\dot{u}(t_k^+)$ denote the right and the left limits respectively of $\dot{u}$ at t_k for $0\le k\le p,\ 0=t_0<t_1\dots,t_k<t_p<b,\ p\in\mathbb{N}$.

We shall provide sufficient conditions ensuring some existence and uniqueness results for system (2.0.1) via an application of the Nash-type equilibrium method in vector Banach spaces.
Let $(X_i,|\cdot|_i),\ i=1,2$ be Hilbert spaces identified to their duals and let $X=X_1\times X_2$. Consider the system

$$\left\{\begin{array}{lcl} u & = & N_1(u,v)\\ v & = & N_2(u,v) \end{array}\right. \tag{2.0.2}$$

where $(u,v)\in X$. Assume that each equation of the system has a variational form, i.e., that there exist continuous functionals $E_1,E_2:X\longrightarrow\mathbb{R}$ such that $E_1(\cdot, v)$ is Fréchet differentiable for every $v \in X_2$, $E_2(u, \cdot)$ is Fréchet differentiable for every $u \in X_1$, and

$$\begin{array}{rcl} E_{11}(u,v) & = & u - N_1(u,v), \\ E_{22}(u,v) & = & v - N_2(u,v). \end{array} \tag{2.0.3}$$

Here, by $E_{11}(u, v), E_{22}(u, v)$ we mean the Fréchet derivative of $E_1(\cdot, v)$ and $E_2(u, \cdot)$, respectively.

Define Sobolev space

$$H_0^1(J) = \{u \in L^2(J) : \dot{u} \in L^2(J), u(0) = u(b) = 0\},$$

We endow $H_0^1(J)$ with the scalar product

$$\langle u, v\rangle_{H_0^1(J)} = \int_0^b (\dot{u}\dot{v} + m^2 uv)dt = m^2\langle u, v\rangle_{L^2(J)} + \langle \dot{u}, \dot{v}\rangle_{L^2(J)}$$

and the corresponding equivalent norm

$$\|u\|_{H_0^1(J)} = (m^2\|u\|^2_{L^2(J)} + \|\dot{u}\|^2_{L^2(J)})^{1/2}. \tag{2.0.4}$$

The advantage of using the norm (2.0.4) on $H_0^1(J)$ is that a Poincaré-type inequality holds in connection to the embedding $H_0^1(J) \subset L^2(J)$, namely, the obvious relation

$$\|u\|_{L^2(J)} \leq \frac{1}{m}\|u\|_{H_0^1(J)}, \qquad u \in H_0^1(J) \tag{2.0.5}$$

A similar Poincaré inequality holds for the inclusion $L^2(J) \subset H^{-1}(J)$. Indeed, if $h \in L^2(J)$, then using (2.0.5) and the above isometry, we obtain

$$\begin{array}{rclcl} \|h\|^2_{H^{-1}(J)} & = & \|u_h\|^2_{H_0^1(J)} = \langle h, u_h\rangle & = & \langle h, u_h\rangle_{L^2(J)} \\ & \leq & \|h\|_{L^2(J)}\|u_h\|_{L^2(J)} & \leq & \frac{1}{m}\|u_h\|_{H_0^1(J)}\|h\|_{L^2(J)} \\ & = & \frac{1}{m}\|h\|_{H^{-1}(J)}\|h\|_{L^2(J)} & & \end{array}$$

which gives

$$\|h\|_{H^{-1}(J)} \leq \frac{1}{m}\|h\|_{L^2(J)}. \tag{2.0.6}$$

2.1 Variational formulation of the problem

Lemma 2.1.1. *The function* $E = (E_1, E_2) : H_0^1(J) \times H_0^1(J) \longrightarrow \mathbb{R}$ *defined by*

$$E_1(u,v) = \int_0^b \left[\frac{1}{2}(\dot{u}^2 + m^2u^2) - F(t,u,v)\right] dt + \sum_{k=1}^{p} \int_0^{u(t_k)} I_k(s)ds,$$

$$E_2(u,v) = \int_0^b \left[\frac{1}{2}(\dot{v}^2 + m^2v^2) - G(t,u,v)\right] dt + \sum_{k=1}^{p} \int_0^{v(t_k)} \overline{I}_k(s)ds,$$

where

$$F(t,u,v) = \int_0^u f(t,s,v)ds,$$

and

$$G(t,u,v) = \int_0^v g(t,u,s)ds,$$

are the functionals energy of the system (2.0.1).

Proof. Let $w \in C_0^\infty(J)$, then
For $t \in [0, t_1]$

$$\int_0^{t_1} -\ddot{u}wdt + \int_0^{t_1} m^2uwdt = \int_0^{t_1} wf(t,u,v)dt.$$

By integration of above equation, we get

$$(-\dot{u}w)(t_1) + \int_0^{t_1} \dot{u}\dot{w}dt + \int_0^{t_1} m^2uwdt = \int_0^{t_1} wf(t,u,v)dt.$$

For $t \in (t_1, t_2]$, we have

$$\int_{t_1}^{t_2} -\ddot{u}wdt + \int_{t_1}^{t_2} m^2uwdt = \int_{t_1}^{t_2} wf(t,u,v)dt.$$

By integration of least equation, and using the jump definition of $u'(t_1^+)$, we obtain

$$(-\dot{u}w)(t_2^-) + (wu')(t_1^+) + \int_{t_1}^{t_2} \dot{u}\dot{w}dt + \int_{t_1}^{t_2} m^2uwdt = \int_{t_1}^{t_2} wf(t,u,v)dt.$$

For $t \in (t_p, b)$: We continue the same calculus and we find

$$\int_{t_p}^{b} -\ddot{u}wdt + \int_{t_p}^{b} m^2uwdt = \int_{t_p}^{b} wf(t,u,v)dt.$$

Then

$$\begin{aligned}
(-\dot{u}w)(b) + (\dot{u}w)(t_p^+) + \int_{t_p}^{b} \dot{u}\dot{w}dt + \int_{t_p}^{b} m^2uwdt &= \int_{t_p}^{b} wf(t,u,v)dt \\
(\dot{u}w)(t_p^-) + (wI_p)(t_p) + \int_{t_p}^{b} \dot{u}\dot{w}dt + \int_{t_p}^{b} m^2uwdt &= \int_{t_p}^{b} wf(t,u,v)dt.
\end{aligned}$$

Observe that

$$\dot{u}(t_p)w(t_p^-) = \sum_{k=1}^{p} w(t_k)I_k(u(t_k), v(t_k)) + \int_0^{t_p} \dot{u}\dot{w}dt + \int_0^{t_p} m^2uwdt - \int_0^{t_p} wf(t,u,v)dt.$$

Consequently,

$$\sum_{k=1}^{p} w(t_k)I_k(u(t_k))) + \int_0^{b} \dot{u}\dot{w}dt + \int_0^{b} m^2uwdt = \int_0^{b} wf(t,u,v)dt.$$

For $w = u$, we obtain

$$\sum_{k=1}^{p} u(t_k)I_k(u(t_k)) + \int_0^{b} \dot{u}^2dt + \int_0^{b} m^2u^2dt = \int_0^{b} uf(t,u,v)dt.$$

Finally we define the energy functional $E = (E_1, E_2)$ by:

$$E_1(u,v) = \int_0^b \left[\frac{1}{2}(\dot{u}^2 + m^2u^2) - F(t,u,v)\right] dt + \sum_{k=1}^{p} \int_0^{u(t_k)} I_k(s)ds,$$

$$E_2(u,v) = \int_0^b \left[\frac{1}{2}(\dot{v}^2 + m^2v^2) - G(t,u,v)\right] dt + \sum_{k=1}^{p} \int_0^{v(t_k)} \overline{I}_k(s)ds.$$

where

$$F(t,u,v) = \int_0^u f(t,s,v)ds, \qquad G(t,u,v) = \int_0^v g(t,u,s)ds.$$

$\square$

Now, we define what we mean by a solution of problem (2.0.1).

Definition 2.1.1. *A pair of function $(u,v) \in H_0^1(J) \times H_0^1(J)$ is said to be a weak solution of problem (2.0.1) if*

$$\sum_{k=1}^{p} w(t_k) I_k(u(t_k)) + \int_0^b \dot{u}\dot{w}dt + \int_0^b m^2 uwdt = \int_0^b wf(t,u,v)dt,$$

and

$$\sum_{k=1}^{p} w(t_k) I_k(u(t_k)) + \int_0^b \dot{u}^2 dt + \int_0^b m^2 u^2 dt = \int_0^b wf(t,u,v)dt,$$

for every $w \in H_0^1(J)$.

2.2 Existence result

We assume that the following conditions are satisfied:

(H_1) $f, g : J \times \mathbb{R}^2 \to \mathbb{R}$ are Carathéodory functions.

(H_2) $f(\cdot,0,0), g(\cdot,0,0) \in L^2(J)$ and there exist $m_{ij} \in \mathbb{R}_+$ $(i,j = 1,2)$ such that

$$\begin{array}{rcl} |f(t,u,v) - f(t,\overline{u},\overline{v})| & \leq & m_{11}|u-\overline{u}| + m_{12}|v-\overline{v}| \\ |g(t,u,v) - g(t,\overline{u},\overline{v})| & \leq & m_{21}|u-\overline{u}| + m_{22}|v-\overline{v}| \end{array}$$

for all $u, \overline{u}, v, \overline{v} \in \mathbb{R}$, a.e. $t \in J$.

(H_3) There exist $\overline{m}_{ij} \in \mathbb{R}_+(i,j = 1,2)$ such that

$$|I_k(x) - I_k(y)| \leq \overline{m}_{k_{11}}|x-y|, \quad |\overline{I}_k(x) - \overline{I}_k(y)| \leq \overline{m}_{k_{22}}|x-y|, \text{ for all } x,y \in \mathbb{R}.$$

Lemma 2.2.1. *Assume that the conditions (H_1) and (H_2) hold. Then*

$$\begin{array}{rcl} -\int_0^b |F(s,u,v)|ds & \geq & -\frac{m_{11}}{2}\|u\|^2_{L^2(J)} - m_{12}\|v\|_{L^2(J)}\|u\|_{L^2(J)} \\ & & -\|f(t,0,0)\|_{L^2(J)}\|u\|_{L^2(J)}. \end{array} \tag{2.2.1}$$

Proof. Indeed, from (H_2) we have

$$|f(t,u,v)| \leq m_{11}|u| + m_{12}|v| + |f(t,0,0)|, \tag{2.2.2}$$

whence, since $f(\cdot,0,0) \in L^2(J)$, $f(\cdot,u(\cdot),v(\cdot)) \in L^2(J)$ whenever $(u,v) \in L^2(\mathbb{R}_+) \times L^2(\mathbb{R}_+)$. Also (2.2.2) gives

$$|F(t,u,v)| \leq \frac{m_{11}}{2}|u|^2 + m_{12}|v||u| + |f(t,0,0)||u|.$$

Then

$$\begin{aligned} -\int_0^b |F(s,u,v)|ds &\geq -\int_0^b \left[\frac{m_{11}}{2}|u(s)|^2 + m_{12}|v(s)||u(s)| + |f(s,0,0)||u(s)|\right] ds \\ &\geq -\frac{m_{11}}{2}\|u\|^2_{L^2(J)} - m_{12}\|v\|_{L^2(J)}\|u\|_{L^2(J)} \\ &\quad -\|f(\cdot,0,0)\|_{L^2(J)}\|u\|_{L^2(J)}. \end{aligned}$$

□

We assume that the spectral radius of matrix

$$M = \frac{1}{m^2}\begin{bmatrix} m_{11} + \sum_{k=1}^{p} m^3\overline{m}_{k_{11}}\sqrt{bt_k} & m_{12} \\ m_{21} & m_{22} + \sum_{k=1}^{p} m^3\overline{m}_{k_{22}}\sqrt{bt_k} \end{bmatrix} \tag{2.2.3}$$

is strictly less than one.

Lemma 2.2.2. *The energy E of the problem has a Fréchet derivative.*

Proof. Direct computation shows that the derivative of E at any u, after the direction $w \in H_0^1(\mathbb{R})$, is given by

$$\begin{aligned} (E_1'(u,v),w) &= \lim_{\lambda\to 0}(E_1(u+\lambda w,v) - E_1(u,v))\lambda^{-1} \\ (E_1'(u,v),w) &= \lim_{\lambda\to 0}\Big[\int_0^b \left[\frac{1}{2}((\dot{u}+\lambda\dot{w})^2 + m^2(u+\lambda w)^2) - F(t,u+\lambda w,v)\right]dt \\ &\quad + \sum_{k=1}^{p}\int_0^{u(t_k)+\lambda w(t_k)} I_k(s)ds - \int_0^b \left[\frac{1}{2}(\dot{u}^2 + m^2u^2) - F(t,u,v)\right]dt \\ &\quad - \sum_{k=1}^{p}\int_0^{u(t_k)} I_k(s)ds\Big]\lambda^{-1} \\ &= \int_0^b [\dot{u}\dot{w} + m^2u^2w^2 - f(t,u,v)w]dt + \sum_{k=1}^{p} w(t_k)I_k(u(t_k)) \\ &= \langle u,w\rangle_{H_0^1(J)} - \langle f(.,u,v),w\rangle_{L^2(J)} + \sum_{k=1}^{p} w(t_k)I_k(u(t_k)). \end{aligned}$$

Hence, the Fréchet derivative of E_1 at any $u \in H_0^1(J)$ is given by

$$\begin{aligned} E_{11}(u,v) &= u - Lf(\cdot,u,v) + \sum_{k=1}^{p} I_k(u(t_k)) \\ &= u - N_1(u,v) \end{aligned}$$

and

$$\begin{aligned} E_{22}(u,v) &= v - Lg(\cdot,u,v) + \sum_{k=1}^{p} \overline{I}_k(v(t_k)) \\ &= v - N_2(u,v) \end{aligned}$$

where $N_1, N_2 : H_0^1(J) \times H_0^1(J) \to H_0^1(J)$ are defined by

$$N_1(u,v) = Lf(\cdot,u,v) - \sum_{k=1}^{p} I_k(u(t_k)),$$

and

$$N_2(u,v) = Lg(\cdot,u,v) - \sum_{k=1}^{p} \overline{I}_k(v(t_k)).$$

$\square$

This shows that the weak solutions of (2.0.1) are the critical points of the functional E.

Theorem 2.2.3. *Assume that the conditions* (H_1), (H_2) *and* (H_3) *hold. In addition, assume that there exist two functions* $\widetilde{g}, \widetilde{g}_1 : J \times \mathbb{R} \to \mathbb{R}$ *such that* $\widetilde{g}(t,\cdot)$, $\widetilde{g}_1(t,\cdot)$ *are coercive and satisfied*

$$\widetilde{g}_1(t,y) \le G(t,x,y) \le \widetilde{g}(t,y), \text{ for all } x,y \in \mathbb{R}, a.e.\ t \in J, \tag{2.2.4}$$

and

$$I_k(x) \ge 0, \quad \overline{I}_k(y) \ge 0, \text{ for all } x,y \in \mathbb{R}. \tag{2.2.5}$$

Then the system (2.0.1) has a unique solution $(u^*, v^*) \in H_0^1(J) \times H_0^1(J)$ *which is a Nash-type equilibrium of the pair of functionals* (E_1, E_2) *associated to the system, i.e.,*

$$\begin{aligned} E_1(u^*,v^*) &= \inf_{H_0^1(J)} E_1(.,v^*) \\ E_2(u^*,v^*) &= \inf_{H_0^1(J)} E_2(u^*,\cdot). \end{aligned}$$

Proof. We shall apply Theorem 1.4.1. First using the Lipschitz conditions (H_2) we can obtain that $E_1(\cdot, v), E_2(u, .)$ are bounded for each $u, v \in H_0^1(J)$

$$\begin{aligned} E_1(u,v) &= \int_0^b \left[\frac{1}{2}(\dot{u}^2 + m^2u^2) - F(t,u,v)\right] dt + \sum_{k=1}^{p} \int_0^{u(t_k)} I_k(s)ds \\ &= \frac{1}{2}\|u\|^2_{H_0^1(J)} - \int_0^b F(s,u,v)ds + \sum_{k=1}^{p} \int_0^{u(t_k)} I_k(s)ds. \end{aligned}$$

Using the inequality (2.2.1) and (2.2.5), we obtain

$$\begin{aligned} E_1(u,v) \geq\ & \frac{1}{2}\|u\|^2_{H_0^1(J)} - \frac{m_{11}}{2}\|u\|^2_{L^2(J)} - m_{12}\|v\|_{L^2(J)}\|u\|_{L^2(J)} \\ & -\|f(\cdot,0,0)\|_{L^2(J)}\|u\|_{L^2(J)}. \end{aligned}$$

By Poincaré inequality, we get

$$\begin{aligned} E_1(u,v) &\geq \frac{1}{2}\|u\|^2_{H_0^1(J)} - \frac{m_{11}}{2m^2}\|u\|^2_{H_0^1(J)} - \frac{m_{12}}{m}\|v\|_{L^2(J)}\|u\|_{H_0^1(J)} \\ &\quad -\frac{1}{m}\|f(\cdot,0,0)\|_{L^2(J)}\|u\|_{H_0^1(J)} \\ &\geq \frac{1}{2}\left(1 - \frac{m_{11}}{2m^2}\right)\|u\|^2_{H_0^1(J)} - \frac{m_{12}}{m}\|v\|_{L^2(J)}\|u\|_{H_0^1(J)} \\ &\quad -\frac{1}{m}\|f(\cdot,0,0)\|_{L^2(J)}\|u\|_{H_0^1(J)}. \end{aligned}$$

Similarly, we can get

$$\begin{aligned} E_2(u,v) \geq\ & \frac{1}{2}\left(1 - \frac{m_{22}}{2m^2}\right)\|v\|^2_{H_0^1(J)} - \frac{m_{12}}{m}\|u\|_{L^2(J)}\|v\|_{H_0^1(J)} \\ & -\frac{1}{m}\|g(\cdot,0,0)\|_{L^2(J)}\|v\|_{H_0^1(J)}. \end{aligned}$$

Consequently, the functionals $E_1(\cdot, v)$ and $E_2(u, \cdot)$ are bounded from below for each $u, v \in H_0^1(J)$. In addition, we use the inequality from (2.2.4) to obtain

$$\begin{aligned} E_2(u,v) &= \frac{1}{2}\|v\|^2_{H_0^1(J)} - \int_0^b G(t,u(t),v(t))dt + \sum_{k=1}^{p} \int_0^{v(t_k)} \overline{I}_k(s)ds \\ &\geq \frac{1}{2}\|v\|^2_{H_0^1(J)} - \int_0^b \widetilde{g}(t,v(t))dt \end{aligned}$$

we use the inequality from (2.2.5) to obtain

$$E_2(u,v) \;\; \geq \;\; \phi(v), \quad \text{for all, } \; v \in H_0^1(J) \tag{2.2.6}$$

where

$$\phi(v) = \frac{1}{2}\|v\|^2_{H_0^1(J)} - \int_0^b \widetilde{g}(t, v(t))dt.$$

Since g is coercive function, ϕ is bounded from below and thus $E_2(u, \cdot)$ is bounded from below uniformly with respect to u. Next,

$$\begin{aligned} E_2(u,v) &= \frac{1}{2}\|v\|^2_{H_0^1(J)} - \int_0^b G(t,u(t),v(t))dt + \sum_{k=1}^{p}\int_0^{v(t_k)} \overline{I}_k(s)ds \\ &\leq \frac{1}{2}\|v\|^2_{H_0^1(J)} - \int_0^b \widetilde{g}_1(t,v(t))dt + \sum_{k=1}^{p}\int_0^{v(t_k)} \overline{I}_k(s)ds. \end{aligned}$$

Then

$$E_2(u,v) \leq \phi_1(v), \;\; \text{for all } v \in H_0^1(J), \tag{2.2.7}$$

where

$$\phi_1(v) = \frac{1}{2}\|v\|^2_{H_0^1(J)} - \int_0^b \widetilde{g}_1(t,v(t))dt + \sum_{k=1}^{p}\int_0^{v(t_k)} \overline{I}_k(s)ds.$$

From (2.2.6) and (2.2.7)

$$\phi(v) \leq E_2(u,v) \leq \phi_1(v), \;\; \text{for all } u, v \in H_0^1(J). \tag{2.2.8}$$

Since ϕ is coercive, for each $\lambda > 0$, there is R_λ such that

$$\phi(v) \geq \lambda \text{ for } \|v\|_{H_0^1(J)} \geq R_\lambda. \tag{2.2.9}$$

Let $a > 0$ and $\lambda = \inf_{v\in H_0^1(J)} \phi_1(v) + a$ for $\|v\|_{H_0^1(J)} \geq R_\lambda$ and any $u \in H_0^1(J)$ we have

$$E_2(u,v) \geq \phi(v) \geq \inf_{v\in H_0^1(J)} \phi_1(v) + a. \tag{2.2.10}$$

From the first inequality of (2.2.8), we have

$$\inf_{v\in H_0^1(J)} E_2(u,v) + a \leq \inf_{v\in H_0^1(J)} \phi_1(v) + a = \lambda. \tag{2.2.11}$$

But (2.2.9) and (2.2.11), imply that

$$E_2(u,v) \geq \inf_{v\in H_0^1(J)} E_2(u,v) + a \quad \|v\|_{H_0^1(J)} \geq R_\lambda \ \forall u \in H_0^1(J).$$

This shows that E_2 satisfies the condition (1.4.2). Finally we prove that $N = (N_1, N_2)$ is a Perov contraction. Indeed, for any $u, v, \overline{u}, \overline{v} \in H_0^1(J)$, using the fact that L is an isometry between $H^{-1}(J)$ and $H_0^1(J)$, the relations (2.0.5), (2.0.6) and the Lipschitz condition (H_3), we obtain

$$\begin{aligned}
\|N_1(u,v) - N_1(\overline{u},\overline{v})\|_{H_0^1(J)} &\leq \|Lf(\cdot,u,v) - Lf(\cdot,\overline{u},\overline{v})\|_{H_0^1(J)} \\
&\quad + \sum_{k=1}^{p} \|I_k(u(t_k)) - I_k(\overline{u}(t_k))\|_{H_0^1(J)} \\
&\leq \|f(\cdot,u,v) - f(\cdot,\overline{u},\overline{v})\|_{H^{-1}(J)} \\
&\quad + \sum_{k=1}^{p} \|I_k(u(t_k)) - I_k(\overline{u}(t_k))\|_{H_0^1(J)} \\
&\leq \frac{1}{m}\|f(\cdot,u,v) - f(\cdot,\overline{u},\overline{v})\|_{L^2(J)} \\
&\quad + \sum_{k=1}^{p} \|I_k(u(t_k)) - I_k(\overline{u}(t_k))\|_{H_0^1(J)}.
\end{aligned}$$

For each $k \in \{1, \ldots, p\}$ we have

$$\begin{aligned}
\|I_k(u(t_k)) - I_k(\overline{u}(t_k))\|_{H_0^1(J)} &\leq m\left(\int_0^b |I_k(u(t_k)) - I_k(\overline{u}(t_k))|^2 dt\right)^{\frac{1}{2}} \\
&= m\sqrt{b}|I_k(u(t_k)) - I_k(\overline{u}(t_k))| \\
&\leq m\overline{m}_{k_{11}}\sqrt{b}|u(t_k) - \overline{u}(t_k)| \\
&\leq m\overline{m}_{k_{11}}\sqrt{b}\int_0^{t_k} |u'(t) - \overline{u}'(t)|dt \\
&\leq m\overline{m}_{k_{11}}\sqrt{bt_k}\left(\int_0^{t_k} |u'(t) - \overline{u}'(t)|^2 dt\right)^{\frac{1}{2}}.
\end{aligned}$$

Hence

$$\|I_k(u(t_k)) - I_k(\overline{u}(t_k))\|_{H_0^1(J)} \leq m\overline{m}_{k_{11}}\sqrt{bt_k}\|u - \overline{u}\|_{H_0^1(J)} \quad k = 1, \ldots, p.$$

Then

$$\begin{aligned}\|N_1(u,v)-N_1(\overline{u},\overline{v})\|_{H_0^1(J)} &\leq \frac{m_{11}}{m}\|u-\overline{u}\|_{L^2(J)}+\frac{m_{12}}{m}\|v-\overline{v}\|_{L^2(J)}\\ &+\sum_{k=1}^{p} m\overline{m}_{k_{11}}\sqrt{bt_k}\|u-\overline{u}\|_{H_0^1(J)}.\end{aligned}$$

Therefore

$$\begin{aligned}\|N_1(u,v)-N_1(\overline{u},\overline{v})\|_{H_0^1(J)} &\leq \frac{1}{m^2}\left(m_{11}+\sum_{k=0}^{p} m^3\overline{m}_{k_{11}}\sqrt{bt_k}\right)\|u-\overline{u}\|_{H_0^1(J)}\\ &+\frac{m_{12}}{m^2}\|v-\overline{v}\|_{H_0^1(J)}.\end{aligned}$$

Similar for N_2 we have

$$\begin{aligned}\|N_2(u,v)-N_2(\overline{u},\overline{v})\|_{H_0^1(J)} &\leq \frac{1}{m^2}\left(m_{22}+\sum_{k=0}^{p} m^3\overline{m}_{k_{22}}\sqrt{bt_k}\right)\|v-\overline{v}\|_{H_0^1(J)}\\ &+\frac{m_{21}}{m^2}\|u-\overline{u}\|_{H_0^1(J)}.\end{aligned}$$

Hence, N is a Perov contraction with the Lipschitz matrix M given by (2.2.3). Therefore, Theorem 1.4.1 can be applied. □

Remark 2.2.1. *Notice that the theory for systems of two equations can easily be extended to the general case of $n-$dimensional systems.*

2.3 Example

We conclude this chapter with an illustrative example.

Example 2.3.1. *Consider the following system*

$$\left\{\begin{array}{lcll} -\ddot{u}+m^2u &=& f(t,u,v), & t\in[0,b]\\ -\ddot{v}+m^2v &=& g(t,u,v), & t\in[0,b]\\ \dot{u}(t_1^+)-\dot{u}(t_1^-) &=& \frac{1}{a_1}|u(t_1)|, & a_1>0, t_1\neq 0, \quad t_1\in(0,b)\\ \dot{v}(t_1^+)-\dot{v}(t_1^-) &=& \frac{1}{a_2}|v(t_1)|, & a_2>0\\ u(0)=u(b) &=& v(0)=v(b)=0, & \end{array}\right. \tag{2.3.1}$$

where $m \neq 0$; $\alpha_i, \beta_i \in C([0,b], \mathbb{R}_+)$, $\sigma_i \in L^2([0,b], \mathbb{R}_+)(i = 1,2)$ *and*

$$f(t,u,v) = \alpha_1(t)\cos u(t) + \beta_1(t)\sin u(t)\sin v(t) + \sigma_1(t),$$

$$g(t,u,v) = \alpha_2(t)\sin v(t) + \beta_2(t)\cos u(t)\sin v(t) + \sigma_2(t).$$

In this case,

$$F(t,x,y) = \alpha_1(t)\sin x + \beta_1(t)(1-\cos x)\sin y + \sigma_1(t)x,$$

$$G(t,x,y) = \alpha_2(t)(1-\cos y) + \beta_2(1-cosy)cosx + \sigma_2(t)y.$$

If the spectral radius of the matrix

$$M = \frac{1}{m^2}\begin{bmatrix} \|\alpha_1\|_\infty + \|\beta_1\|_\infty + \frac{m^3\sqrt{bt_1}}{a_1} & \|\beta_1\|_\infty \\ \|\beta_2\|_\infty & \|\alpha_2\|_\infty + \|\beta_2\|_\infty + \frac{m^3\sqrt{bt_1}}{a_2} \end{bmatrix}$$

is less than one, where then the system (2.3.2) has a unique solution, which is a Nash-type equilibrium of the corresponding pair of energy functionals. In particular, the result holds for the following system on $[0,1]$

$$\begin{cases} -\ddot{u} + u &= \frac{2}{15}\cos u(t) + \frac{1}{5}\sin u(t)\sin v(t) + \sigma_1(t) \\ -\ddot{v} + v &= \frac{1}{6}\sin v(t) + \frac{1}{6}\cos u(t)\sin v(t) + \sigma_2(t) \\ \dot{u}(t_1^+) - \dot{u}(t_1^-) &= |u(t_1)|, \quad t_1 = \frac{1}{9} \\ \dot{v}(t_1^+) - \dot{v}(t_1^-) &= |v(t_1)|, \quad t_1 = \frac{1}{9} \\ u(0) = u(1) &= v(0) = v(1) = 0, \end{cases} \tag{2.3.2}$$

where $\sigma_i \in L^2([0,1], \mathbb{R}_+)(i = 1,2)$.*In this case, the matrix* M *is*

$$M = \begin{bmatrix} \frac{2}{3} & \frac{1}{5} \\ \frac{1}{6} & \frac{2}{3} \end{bmatrix} \tag{2.3.3}$$

and one can easily see that its spectral radius is less than one.

Chapter 3

Impulsive p-Laplacian boundary value problem

In this chapter, we aim to study the following boundary impulsive value problem with a second-order p-Laplacian on $\sigma(T)$ periodic time scales $\mathbb{T}$:

$$\begin{cases} (\varphi_p(u^{\Delta}(t)))^{\Delta} & = & f(\sigma(t), u^{\sigma}(t)),\ \Delta-\text{a.e. } t \in [0, \sigma(T)]_{\mathbb{T}}^{k}, t \neq t_j, \\ \varphi_p(u^{\Delta}(t_j^{+})) - \varphi_p(u^{\Delta}(t_j^{-})) & = & I_j(u(t_j)),\ j = 1, \ldots, n \\ u(0) = u(\sigma(T)) = u^{\Delta}(\sigma(T)) & = & u^{\Delta}(0) = 0, \end{cases} \tag{3.0.1}$$

where $\varphi_p(u^{\Delta}(t)) = |u^{\Delta}(t)|^{p-2}u^{\Delta}(t)$, and Δ is the derivative on the time scale $\mathbb{T}$, and σ is the forward jump operator, $0, T \in \mathbb{T}$ with $\sigma(t) \in \mathbb{T}^k$ and $f(t, x) : [0, \sigma(T)]_{\mathbb{T}}^{k} \times \mathbb{R} \to \mathbb{R}$ is Lebesgue integrable in $\mathbb{T}$ for each $x \in \mathbb{R}$, and continuously Δ-differentiable with respect to x for any $\Delta - a.e.t \in [0, \sigma(T)]_{\mathbb{T}}^{k}$, $I_j(u(t_j)) \in C(\mathbb{R}, \mathbb{R})$, $\varphi_p(u^{\Delta}(t_j^{+}))$ and $\varphi_p(u^{\Delta}(t_j^{-}))$ denote the right and the left limits respectively of u^{Δ} at t_k for $0 \leq k \leq p,\ 0 = t_0 < t_1, \ldots, t_k < t_p < T,\ p \in \mathbb{N}$.
It is notable that in our study the function f is not required to be periodic in t.
Throughout this Chapter, we make the following assumption:

(H_0) There exist two functions $a \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$ and $b \in L^1([0, \sigma(t)]_{\mathbb{T}}, \mathbb{R}^+)$ such that

$$\left| \int_0^x f(t, s) ds \right| \leq a(|x|)b(t), \quad |f(t, x)| \leq a(|x|)b(t)$$

for all $x \in \mathbb{R}$ and $\Delta - a.e.t \in [0, \sigma(T)]_{\mathbb{T}}$.

In the cases where $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$ the problem (3.0.1) take the following forms:

$$(\varphi_p(u'(t)))' = f(t, u(t)), \quad t \in [0, T], t \neq t_j, \tag{3.0.2}$$

and

$$\Delta(\varphi_p(\Delta u(k))) = f(k+1, u(k+1)), \quad k \in \{0, 1, 2, \ldots, N\}, \tag{3.0.3}$$

where $N \in \mathbb{Z}^+$ and Δ denotes the difference operator defined by

$$\Delta u(k) = u(k+1) - u(k), \quad k \in \{1, 2, \ldots, N\}.$$

Also in this work, we consider the following impulsive problem with the mixed derivatives on time scales of the form:

$$\left\{ \begin{array}{lcl} (\varphi_p(u^{\Delta}(t)))^{\nabla} & = & f(t, u(t)),\ \Delta - \text{a.e. } t \in [0, T]_{\mathbb{T}}, t \neq t_j, \\ \varphi_p(u^{\Delta}(t_j{}^{+})) - \varphi_p(u^{\Delta}(t_j{}^{-})) & = & I_j(u(t_j)),\ j = 1, \ldots, n, \\ u(0) = u(\sigma(T)) = u^{\Delta}(\sigma(T)) & = & u^{\Delta}(0) = 0, \end{array} \right. \tag{3.0.4}$$

3.1 Variational formulation of the problem

In this section, we defined the functionals energy of the problem (3.0.1).
We consider the following Banach spaces

$$\widetilde{W}^{1,p}_{\Delta,T}([0, \sigma(T)]_{\mathbb{T}}, \mathbb{R}) = \{u \in W^{1,p}_{\Delta,T}([0, \sigma(T))_{\mathbb{T}}, \mathbb{R}) : \bar{u} = 0\},$$

and

$$W^{1,p}_{\Delta,T}([0, \sigma(T)]_{\mathbb{T}}, \mathbb{R}) = \widetilde{W}^{1,p}_{\Delta,T}([0, \sigma(T)]_{\mathbb{T}}, \mathbb{R}) \oplus \mathbb{R},$$

with

$$\tilde{u} = u - \bar{u},$$

where

$$\bar{u} = \frac{1}{\sigma(T)} \int_{[0,\sigma(T))_{\mathbb{T}}} u(t) \Delta t.$$

Proposition 3.1.1. *[79] Assume that f satisfies the condition (H_0), the sequence*

$$\{u_n\} \subset W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})$$

satisfies $\varphi\prime(u_n) \to 0$ and $\{u_n\}$ is bounded in $W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})$. Then,$\{u_n\}$ has a convergent subsequence in $W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})$

Lemma 3.1.2. *The functionals energy of the problem (3.0.1) is the function $\psi : W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R}) \to \mathbb{R}$ defined by:*

$$\psi(u) = \frac{1}{p}\int_0^{\sigma(T)} |u^{\Delta}(s)|^p \Delta t + \int_0^{\sigma(T)} F(\sigma(t), u^{\sigma}(t))\Delta t + \sum_{k=1}^{n}\int_0^{u(t_k)} I_k(t)\Delta t \tag{3.1.1}$$

where

$$F(t,\xi) = \int_0^{\xi} f(t,s)ds. \tag{3.1.2}$$

Proof. Let $v \in W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})$, then
For $t \in [0,t_1]$

$$\int_0^{t_1} v(s)(\varphi_p(u^{\Delta}(s)))^{\Delta}\Delta s = \int_0^{t_1} f(\sigma(s), u^{\sigma}(s))v(s)\Delta s.$$

We use the integration by part, we get

$$v(t_1)\varphi_p(u^{\Delta}(t_1)) - v(0)\varphi_p(u^{\Delta}(0)) - \int_0^{t_1} v^{\Delta}(s)\varphi_p(u^{\Delta}(s))\Delta s = \int_0^{t_1} f(\sigma(s), u^{\sigma}(s))v(s)\Delta s.$$

This implies that

$$v(t_1)\varphi_p(u^{\Delta}(t_1)) = v(0)\varphi_p(u^{\Delta}(0)) + \int_0^{t_1} v^{\Delta}(s)\varphi_p(u^{\Delta}(s))\Delta s + \int_0^{t_1} f(\sigma(s), u^{\sigma}(s))v(s)\Delta s.$$

For $t \in (t_1,t_2]$, we have

$$\int_{t_1}^{t_2} v(s)(\varphi_p(u^{\Delta}(s)))^{\Delta}\Delta s = \int_{t_1}^{t_2} f(\sigma(s), u^{\sigma}(s))v(s)\Delta s.$$

By integration of least equation, and using the jump condition of $u^{\Delta}(t_1^+)$, we obtain

$$v(t_2)\varphi_p(u^\Delta(t_2)) = v(t_1)I_1(u(t_1)) + v(0)\varphi_p(u^\Delta(0)) + \int_0^{t_2} v^\Delta(s)\varphi_p(u^\Delta(s))\Delta s$$
$$+ \int_0^{t_2} f(\sigma(s), u^\sigma(s))v(s)\Delta s.$$

For $t \in (t_n, \sigma(T)]$: We continue the same calculus and we find

$$\int_{t_n}^{\sigma(T)} v(s)(\varphi_p(u^\Delta(s)))^\Delta \Delta s = \int_{t_n}^{\sigma(T)} f(\sigma(s), u^\sigma(s))v(s)\Delta s.$$

By integration by part,we get

$$\begin{aligned}
\int_{t_n}^{\sigma(T)} f(\sigma(s), u^\sigma(s))v(s)\Delta s &= v(\sigma(T))\varphi_p(u^\Delta(\sigma(T)) - v(t_n)\varphi_p(u^\Delta(t_n)\\
&- \int_{t_n}^{\sigma(T)} v^\Delta(s)\varphi_p(u^\Delta(s))\Delta s\\
&= v(\sigma(T))\varphi_p(u^\Delta(\sigma(T)) - v(t_n)(\varphi_p(t_n^-) + v(t_n)(I_n(u(t_n)))\\
&- \int_{t_n}^{\sigma(T)} v^\Delta(s)\varphi_p(u^\Delta(s))\Delta s.
\end{aligned}$$

Observe that

$$v(t_n)\varphi_p(t_n^-) = \sum_{0<t_k<t_n} v(t_k)I_k(u(t_k)) + \int_0^{t_n} v^\Delta(s)\varphi_p(u^\Delta(s))\Delta s - \int_0^{t_n} f(\sigma(s), u^\sigma(s))v(s)\Delta s.$$

Consequently,

$$v(\sigma(T))\varphi_p(u^\Delta(\sigma(T)) - v(t_1)I_1(u(t_1)) - v(t_2)I_2(u(t_2)) - \ldots - v(t_n)I_n(u(t_n))$$
$$-\int_0^{t_n} v^\Delta(s)\varphi_p(u^\Delta(s))\Delta s - \int_0^{t_n} f(\sigma(s), u^\sigma(s))v(s)\Delta s - \int_{t_n}^{\sigma(T)} v^\Delta(s)\varphi_p(u^\Delta(s))\Delta s$$
$$= \int_{t_n}^{\sigma(T)} f(\sigma(s), u^\sigma(s))v(s)\Delta s.$$

Then

$$\int_0^{\sigma(T)} |u^\Delta(s)|^{p-2}u^\Delta(s)v^\Delta(s)\Delta s + \int_0^{\sigma(T)} f(\sigma(s), u^\sigma(s))v(s)\Delta s + \sum_{k=1}^{n} v(t_k)I_k(u(t_k)) = 0,$$

for $v = u$

$$\int_0^{\sigma(T)} |u^{\Delta}(s)|^p \Delta s + \int_0^{\sigma(T)} f(\sigma(s), u^{\sigma}(s))u(s)\Delta s + \sum_{k=1}^{n} u(t_k)I_k(u(t_k)) = 0.$$

Then

$$\int_0^{\sigma(T)} |u^{\Delta}(s)|^p \Delta t + \int_0^{\sigma(T)} F(\sigma(t), u^{\sigma}(t))\Delta t + \sum_{k=1}^{n} u(t_k)I_k(u(t_k)) = 0.$$

Finally we define the energy functional ψ by:

$$\psi(u) = \frac{1}{p}\int_0^{\sigma(T)} |u^{\Delta}(s)|^p \Delta t + \int_0^{\sigma(T)} F(\sigma(t), u^{\sigma}(t))\Delta t + \sum_{k=1}^{n} u(t_k)I_k(u(t_k))$$

i.e

$$\psi(u) = \frac{1}{p}\int_0^{\sigma(T)} |u^{\Delta}(s)|^p \Delta t + \int_0^{\sigma(T)} F(\sigma(t), u^{\sigma}(t))\Delta t + \sum_{k=1}^{n} \int_0^{u(t_k)} I_k(t)\Delta t$$

where

$$F(t, \xi) = \int_0^{\xi} f(t, s)ds.$$

$\square$

Now, we define what we mean by a solution of problem (3.0.1).

Definition 3.1.1. *A critical points $u \in W^{1,p}_{\Delta,T}([0, \sigma(T)]_{\mathbb{T}}, \mathbb{R})$ of the functional ψ defined by (3.1.1) is said to be a weak solution of Problem (3.0.1) if*

$$-\int_0^{\sigma(T)} |u^{\Delta}(t)|^{p-2}u^{\Delta}(t)w^{\Delta}\Delta t - \int_0^{\sigma(T)} f(\sigma(t), u^{\sigma}(t))w^{\sigma}\Delta t - \sum_{k=1}^{N} w_k I_k(t_k) = 0,$$

for every $w \in W^{1,p}_{\Delta,T}([0, \sigma(T)]_{\mathbb{T}}, \mathbb{R})$.

Proposition 3.1.3. *The functional $\psi \in \mathcal{C}^1(W^{1,p}_{\Delta,T}([0, \sigma(T))_{\mathbb{T}}, \mathbb{R}), \mathbb{R})$.*

Proof. **Firstly**, we show the existence of the Gâteaux derivative.
Direct computation shows that the derivative of ψ at any u, after the direction $v \in W^{1,p}_{\Delta,T}([0,\sigma(T))_{\mathbb{T}}, \mathbb{R})$ and $\epsilon \in \mathbb{R}$ $(0 < |\epsilon| < 1)$, is given by

$$\begin{aligned}\frac{\psi(u+\epsilon v)-\psi(u)}{\epsilon} &= \int_{[0,\sigma(T))_{\mathbb{T}}} \frac{1}{\epsilon p}[|u^\Delta + \epsilon v^\Delta|^p - |u^\Delta|^p]\Delta t \\ &+ \int_{[0,\sigma(T))_{\mathbb{T}}} \frac{F(\sigma(t), u^\sigma + \epsilon v^\sigma) - F(\sigma(t), u^\sigma)}{\epsilon}\Delta t \\ &+ \frac{1}{\epsilon}\sum_{k=1}^{n}\int_0^{u(t_k)+\epsilon v(t_k)} I_k(t)\Delta t - \sum_{k=1}^{n}\int_0^{u(t_k)} I_k(t)\Delta t.\end{aligned}$$

where F is defined as formula (3.1.2).
For $u \in \mathbb{R}$, it follows from the mean value theorem that there exists $\rho_1 \in (0,1)$ such that

$$\begin{aligned}\frac{||u^\Delta+\epsilon v^\Delta|^p - |u^\Delta|^p|}{|\epsilon|} &= p[|u^\Delta + \epsilon\rho_1 v^\Delta|^{p-2}|u^\Delta + \epsilon\rho_1 v^\Delta|]|v^\Delta| \\ &\le Cp[|u^\Delta| + |v^\Delta|]^{p-1}|v^\Delta|,\end{aligned} \tag{3.1.3}$$

where C is a positive constant,$|u^\Delta|$ and $|v^\Delta| \in L^p_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}}$, we use Hölder inequality on time scales we find

$$\int_{[0,\sigma(T)]_{\mathbb{T}}} [|u^\Delta|+|v^\Delta|]^{p-1}|v^\Delta|\Delta t \le \Big[\int_{[0,\sigma(T)]_{\mathbb{T}}} [|u^\Delta|+|v^\Delta|]^p \Delta t\Big]^{\frac{p-1}{p}} \Big[\int_{[0,\sigma(T)]_{\mathbb{T}}} [|v^\Delta|]^p\Delta t\Big]^{\frac{1}{p}}$$

which implies

$$[|u^\Delta| + |v^\Delta|]^{p-1}|v^\Delta| \in L^1_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}}, \mathbb{R}).$$

Given $u \in \mathbb{R}$, by the mean value theorem, there exists $\rho_2 \in (0,1)$ such that

$$\begin{aligned}\tfrac{1}{|\epsilon|}|F(\sigma(t), u^\sigma + \epsilon v^\sigma) - F(\sigma(t), u^\sigma)| &= \tfrac{1}{|\epsilon|}|\tfrac{\partial F}{\partial \xi}|_{(\sigma(t), u^\sigma + \rho_2\epsilon v^\sigma)} |\epsilon v^\sigma| \\ &= |f(\sigma(t), u^\sigma + \rho_2\epsilon v^\sigma)||v^\sigma|.\end{aligned} \tag{3.1.4}$$

Note that

$$|f(\sigma(t), u^\sigma + \rho_2\epsilon v^\sigma)||v^\sigma| \in L^1_{\Delta,T}([0,\sigma(T))_{\mathbb{T}}, \mathbb{R}).$$

Therefore, for $u \in \mathbb{R}$, the mean value theorem indicates that there exists $\rho_3 \in (0,1)$ such that

$$\begin{aligned}
\frac{\sum_{k=1}^{n}\int_0^{u_k+\epsilon v_k} I_k(t)\Delta t - \sum_{k=1}^{n}\int_0^{u_k} I_k(t)\Delta t}{\epsilon} &= \tfrac{1}{\epsilon}\Big(\sum_{k=1}^{n}\frac{\partial \int_0^x I_k(s)ds}{\partial x}\Big|_{u_k+\epsilon\rho_3 v_k}\Big)|\epsilon v_k| \\
&= \sum_{k=1}^{n} I_k(u_k+\epsilon\rho_3 v_k)|v_k| \\
&= \sum_{k=1}^{n} v_k I_k(u_k) \in C(\mathbb{R},\mathbb{R}) \quad \text{for } \epsilon \to 0,
\end{aligned} \tag{3.1.5}$$

$$\langle \psi\prime(u), v\rangle = \lim_{\epsilon\to 0}\frac{1}{\epsilon}[\psi(u+\epsilon v)-\psi(u)]$$

By formulas (3.1.3),(3.1.4),(3.1.5) and the Lebesgues dominated convergence theorem on time scales, we derive that

$$\langle \psi\prime(u), v\rangle = \int_0^{\sigma(T)}\Big(p|u^{\Delta}|^{p-2}u^{\Delta}v^{\Delta} + f(\sigma(t),u^{\sigma})v^{\sigma}\Big)\Delta t + \sum_{k=1}^{n} v_k I_k(u_k) \tag{3.1.6}$$

Secondly, we consider the continuity of the Gâteaux derivative.
Assume that the sequence $\{u_n\} \subset W^{1,p}_{\Delta,T}([0,\sigma(T))_{\mathbb{T}},\mathbb{R})$ satisfies $u_n \to u$ as $n \to \infty$ in $W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})$. By proposition (1.3.11) tells us that

$$W^{1,p}_{\Delta,T}([0,\sigma(T))_{\mathbb{T}}),\mathbb{R} \hookrightarrow C([0,\sigma(T))_{\mathbb{T}},\mathbb{R}) \text{ is compact.}$$

Then, $u_n \to u$ as $n \to \infty$ in $L^{q_1}_{\Delta,T}([0,\sigma(T))_{\mathbb{T}},\mathbb{R})$.
Let $f(x) = p|x^{\Delta}|^{p-2}x^{\Delta}$. From proposition (1.3.14) we get

$$\||u_n^{\Delta}|^{p-2}u_n^{\Delta} - |u^{\Delta}|^{p-2}u^{\Delta}\|_{L^{q_1}_{\Delta}} \to 0 \ \ as \ \ n \to \infty$$

where $q_1 = \frac{p}{p-1}$. Using (H_0) and Lebesgue dominated convergence theorem on time scales yields

$$\int_{[0,\sigma(T))_{\mathbb{T}}} |f(\sigma(t),u_n^{\sigma}) - f(\sigma(t),u^{\sigma})|\Delta t \to 0 \ \ as \ \ n \to \infty.$$

By continuity of I

$$\sum_{k=1}^{N} |I_{kn}(u(t_{kn})) - I_k(u(t_k))| \to 0 \ \ as \ \ n \to \infty.$$

For an arbitrary $v \in W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}}, \mathbb{R})$, there holds

$$\begin{aligned}
\langle \psi'(u_n) - \psi'(u), v\rangle &= \int_{[0,\sigma(T))_{\mathbb{T}}} \Big((|u_n^\Delta|^{p-2}u_n^\Delta - |u^\Delta|^{p-2}u^\Delta)v^\Delta\Big)\Delta t \\
&+ \int_{[0,\sigma(T))_{\mathbb{T}}} \Big(f(\sigma(t), u_n^\sigma) - f(\sigma(t), u^\sigma)v^\sigma\Big)\Delta t \\
&+ \sum_{k=1}^{n} \Big(I_{kn}(u(t_{kn})) - I_k(u(t_k))\Big)v_k.
\end{aligned}$$

Using the Hölder inequality on time scales again gives:

$$\begin{aligned}
|\langle \psi'(u_n) - \psi(u), v\rangle| &\leq \int_{[0,\sigma(T))_{\mathbb{T}}} \Big||u_n^\Delta|^{p-2}u_n^\Delta - |u^\Delta|^{p-2}u^\Delta)\Big|v^\Delta \Delta t \\
&+ \int_{[0,\sigma(T))_{\mathbb{T}}} \Big|(f(\sigma(t), u_n^\sigma) - f(\sigma(t), u^\sigma), v^\sigma)\Big|\Delta t \\
&+ \sum_{k=1}^{n} \Big|I_{kn}(u(t_{kn})) - I_k(u(t_k))\Big||v_k| \\
&\leq \Big(\int_{[0,\sigma(T))_{\mathbb{T}}} \Big||u_n^\Delta|^{p-2}u_n^\Delta - |u^\Delta|^{p-2}u^\Delta\Big|^{\frac{p}{p-1}}\Delta t\Big)^{\frac{p-1}{p}} \\
&\times \Big(\int_{[0,\sigma(T))_{\mathbb{T}}} |v^\Delta|^p \Delta t\Big)^{\frac{1}{p}} + \int_{[0,\sigma(T))_{\mathbb{T}}} |(f(\sigma(t), u_n^\sigma) \\
&- f(\sigma(t), u^\sigma), v^\sigma)|\Delta t + \sum_{k=1}^{n} \Big|I_{kn}(u(t_{kn})) - I_k(u(t_k))\Big||v_k| \\
&\leq \Big\||u_n^\Delta|^{p-2}u_n^\Delta - |u^\Delta|^{p-2}u^\Delta\Big\|_{L^{q_1}_\Delta}\|v\| \\
&+ \|v^\sigma\| \int_{[0,\sigma(T))_{\mathbb{T}}} |f(\sigma(t), u_n^\sigma) - f(\sigma(t), u^\sigma)|\Delta t \\
&+ \sum_{k=1}^{n} \Big|I_{kn}(u(t_{kn})) - I_k(u(t_k))\Big||v|.
\end{aligned}$$

Hence, we have,

$$\begin{aligned}\|\psi'(u_n)-\psi(u)\| &\leq \left\| |u_n^\Delta|^{p-2}u_n^\Delta - |u^\Delta|^{p-2}u^\Delta)\right\|_{L_\Delta^{q_1}} \\ &+ \frac{\|v^\sigma\|}{\|v\|}\int_{[0,\sigma(T))_{\mathbb{T}}} |f(\sigma(t),u_n^\sigma)-f(\sigma(t),u^\sigma)|\Delta t \\ &+ \frac{|v|}{\|v\|}\sum_{k=1}^{n}\left|I_{kn}(u(t_{kn}))-I_k(u(t_k))\right| \to 0 \ as \ n\to\infty.\end{aligned}$$

which implies $\psi'(u_n)\to\psi(u) as \ n\to\infty$

□

3.2 Main results

In this section, we state and prove our results on the existence of periodic solution of equation (3.0.1) on time scales.

Theorem 3.2.1. *Suppose that the condition* (H_0) *and that the following condition holds:*

(H_1) *let* $\{u_n\}\subset W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})$ *such that* $\|u_n\|\to\infty$ *and* $\frac{|\overline{u_n}|\sigma(T)^{\frac{1}{p}}}{\|u_n\|}\to 1$, *and there holds*

$$\lim_{n\to\infty}\inf\int_{[0,\sigma(T))_{\mathbb{T}}} f(\sigma(t),u_n^\sigma)\frac{\overline{u_n^\sigma}}{|\overline{u_n^\sigma}|}\Delta t+\sum_{k=1}^{n}I_k(u_{nk})\frac{\overline{u_{nk}}}{|\overline{u_{nk}}|}<0,$$

(H_2) *the inequalities* $\liminf_{|x|\to\infty}\frac{\displaystyle\int_0^x f(t,s)ds}{|x|^{p-1}} > -\frac{1}{pR_p}$ *holds uniformly for* $\Delta - a.e.\ t\in[0,\sigma(T))_{\mathbb{T}}$, *where*

$$R_p = sup\{\|u\|^p_{L^p_\Delta}\Big|\|u^\Delta\|^p_{L^p_\Delta}=1\},$$

(H_3) *the inequalities* $\liminf_{x\to\infty}\frac{\displaystyle\int_0^x I_k(t)\Delta t}{|x^\Delta|^{p-1}} > -\frac{1}{k_p}$ *where*

$$k_p = sup\{\|u\|^p_\infty\Big|\|u^\Delta\|^p_{L^p_\Delta}\}.$$

Then, the boundary value problem (3.0.1) has at least one periodic solution in $W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})$.

Proof. We shall apply Lemma (1.4.2). The proof will be given in several steps.

Step 1 To show that ψ satisfies the PS condition.

Assume that there exist a sequence $\{u_n\}\subset W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})$ and a constant c such that

$$\psi\prime(u_n)\to 0,\quad \psi(u_n)\le c,\qquad n=1,2,\dots \tag{3.2.1}$$

We show that $\{u_n\}$ is bounded in $W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})$.

Suppose to the contrary there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) which is unbounded. That is,

$$\|u_n\|\to\infty \;\; as \;\; n\to\infty. \tag{3.2.2}$$

Let $v_n=\frac{u_n}{\|u_n\|}$, then $\{v_n\}$ is bounded in $W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})$. It follows from Proposition (1.3.11) that $W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})\hookrightarrow C([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})$ is compact. Hence, there exist a point $v_0\in W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})$ and a subsequence of $\{v_n\}$ such that

$$v_n\rightharpoonup v_0 \;in\; W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R}),$$

and

$$v_n\to v_0 \;\; strongly\; in\; L^q_{\Delta}([0,\sigma(T))_{\mathbb{T}},\mathbb{R}). \tag{3.2.3}$$

In addition, there exists a function $\omega\in L^q_{\Delta}([0,\sigma(T))_{\mathbb{T}},\mathbb{R})$ such that $|v_n|\le\omega$ for $\Delta-a.e\;\; t\in[0,\sigma(T)]_{\mathbb{T}}$.

By (H_2), there exist constants $0<\epsilon_0<\frac{1}{pR_p}$ and $M>0$ such that

$$\begin{aligned}\int_0^{u^\sigma(t)} f(\sigma(t),s)ds &> (-\frac{1}{pR_p}+\epsilon_0)\min_{t\in[0,\sigma(T))_{\mathbb{T}}}|u^\sigma(t)|^{p-1}\\ &\ge -\frac{1}{pR_p}|u(t)|^{p-1}+\epsilon_0|u(t)|^{p-1},\end{aligned}$$

for all $u^\sigma\in\mathbb{R}$ with $|u^\sigma|>M$ and $\Delta-a.e\;\; t\in[0,\sigma(T)]_{\mathbb{T}}$.

Let $a_M=\max_{|u|\le M}a(|u|)$. It follows from (H_0) that

$$\int_0^{u^\sigma(t)} f(\sigma(t),s)ds\ge -a_M b(\sigma(t)) \tag{3.2.4}$$

holds for all $u^\sigma \in \mathbb{R}$ with $|u^\sigma| > M$ and $\Delta - a.e\ \ t \in [0, \sigma(T)]_{\mathbb{T}}$. Hence, we deduce that

$$\int_0^{u^\sigma(t)} f(\sigma(t), s)ds > -\frac{1}{pR_p}|u(t)|^{p-1} + \epsilon_0|u(t)|^{p-1} - a_M b(\sigma(t)) \quad (3.2.5)$$

for all $u^\sigma \in \mathbb{R}$ and $\Delta - a.e\ \ t \in [0, \sigma(T))_{\mathbb{T}}$.
By (H_3), there exist constants $0 < \epsilon_1 < \frac{1}{k_p}$ such that

$$\int_0^x I_k(t)\Delta t > (-\frac{1}{k_p} + \epsilon_1)\min_t |x(t)|^{p-1},$$

$$\int_0^x I_k(t)\Delta t > -\frac{|x|^{p-1}}{k_p} + \epsilon_1|x|^{p-1}, \quad (3.2.6)$$

It follows from(3.2.1),(3.2.5),and (3.2.6) that

$$\begin{aligned}
\frac{c}{\|u_n\|^p} &\geq \frac{\psi(u_n)}{\|u_n\|^p} \\
&= \frac{1}{p}\int_{[0,\sigma(T))_{\mathbb{T}}} \frac{|(u_n)^\Delta|^p}{\|u_n\|^p}\Delta t + \frac{1}{\|u_n\|^p}\int_0^{\sigma(T)}\int_0^{u^\sigma(t)} f(\sigma(s), u^\sigma(s))u(s)\Delta s \\
&+ \frac{1}{\|u_n\|^p}\sum_{k=1}^{n}\int_0^{u_{nk}} I_{kn}(t)\Delta t \\
&> \frac{1}{p}\int_0^{\sigma(T)} |(v_n)^\Delta|^p\Delta t + \frac{1}{\|u_n\|^p}\int_0^{\sigma T}\Big[-\frac{1}{pR_p}|u_n|^{p-1} + \epsilon_0|u_n|^{p-1} \\
&- a_M b(\sigma(t))\Big]\Delta t + \frac{1}{\|u_n\|^p}\sum_{k=1}^{n}\Big[-\frac{|u_{nk}|^{p-1}}{k_p} + \epsilon_1|u_{nk}|^{p-1}\Big] \\
&> \frac{1}{p}\int_0^{\sigma(T)} |(v_n)^\Delta|^p\Delta t + \frac{1}{\|u_n\|^p}\int_0^{\sigma T}\Big[-\frac{1}{pR_p}|u_n|^{p-1} - a_M b(\sigma(t))\Big]\Delta t \\
&- \frac{1}{\|u_n\|^p}\sum_{k=1}^{n}\frac{|u_{nk}|^{p-1}}{k_p} \\
&> \frac{1}{p}\int_0^{\sigma(T)} |(v_n)^\Delta|^p\Delta t - \frac{1}{pR_p\|u_n\|}\int_0^{\sigma T}|v_n|^{p-1} - \frac{1}{\|u_n\|^p}\int_0^{\sigma(T)} a_M b(\sigma(t))\Delta t \\
&- \sum_{k=1}^{n}\frac{|v_{nk}|^{p-1}}{k_p\|u_n\|}
\end{aligned}$$

$$\begin{aligned}\frac{c}{\|u_n\|^p} &> \frac{1}{p}[1-\int_{[0,\sigma(T))_{\mathbb{T}}}|v_n|^p\Delta t] - \frac{1}{pR_p\|u_n\|}\int_{[0,\sigma(T))_{\mathbb{T}}}|v_n|^{p-1} \\ &- \frac{1}{\|u_n\|^p}\int_{[0,\sigma(T))_{\mathbb{T}}} a_M b(\sigma(t))\Delta t - \sum_{k=1}^{n}\frac{|v_{nk}|^{p-1}}{k_p\|u_n\|}.\end{aligned}$$

Using this inequality together with (3.2.2) and (3.2.3) gives

$$0 \geq \frac{1}{p}[1-\int_{[0,\sigma(T))_{\mathbb{T}}}|v_0|^p\Delta t],$$

then

$$\int_{[0,\sigma(T))_{\mathbb{T}}}|v_0|^p\Delta t \geq 1. \tag{3.2.7}$$

The weakly lower semi-continuity of the norm means that

$$\Big[\int_{[0,\sigma(T))_{\mathbb{T}}}|v_0|^p\Delta t + \int_{[0,\sigma(T))_{\mathbb{T}}}|(v_0)^{\Delta}|^p\Delta t\Big]^{\frac{1}{p}} = \|v_0\| \leq \liminf \|v_n\| = 1,$$

that is

$$\|v_0\| \leq 1. \tag{3.2.8}$$

From (3.2.7) and (3.2.8), we deduce that

$$\int_{[0,\sigma(T))_{\mathbb{T}}}|(v_0)^{\Delta}|^p\Delta t = 0,$$

that is

$$|v_0| \equiv \text{ constant}.$$

Then, we obtain

$$\int_{[0,\sigma(T))_{\mathbb{T}}}|v_0|^p\Delta t = 1,$$

which gives

$$|v_0|^p = \frac{1}{\sigma(T)}.$$

Thus, we have (by definition of $\overline{u}$)

$$\frac{|\overline{u_n}|^p}{\|u_n\|^p} = \Big|\frac{1}{\sigma(T)}\int_{[0,\sigma(T))_{\mathbb{T}}}\frac{u_n}{\|u_n\|}\Delta t\Big|^p$$

$$\begin{aligned}\frac{|\overline{u_n}|^p}{\|u_n\|^p} &= \left|\frac{1}{\sigma(T)}\int_{[0,\sigma(T))_{\mathbb{T}}}|v_n|\Delta t\right|^p \longrightarrow \left|\frac{1}{\sigma(T)}\int_{[0,\sigma(T))_{\mathbb{T}}}|v_0|\Delta t\right|^p \\ &= \frac{1}{\sigma(T)},\end{aligned}$$

which implies that

$$\frac{|\overline{u_n}|\sigma(T)^{\frac{1}{p}}}{\|u_n\|} \to 1.$$

It follows from (H_1) that

$$\liminf_{n\to\infty}\int_{[0,\sigma(T))_{\mathbb{T}}} f(\sigma(t),u_n^\sigma)\frac{|\overline{u_n}|}{|\overline{u_n}|}\Delta t+\sum_{k=1}^{n} I_k(u_{nk})\frac{\overline{u_{nk}}}{|\overline{u_{nk}}|}<0.$$

However, by (3.1.6) and (3.2.1), we obtain

$$\int_{[0,\sigma(T))_{\mathbb{T}}} f(\sigma(t),u_n^\sigma)\frac{\overline{u_n^\sigma}}{|\overline{u_n^\sigma}|}\Delta t+\sum_{k=1}^{n} I_k(u_{nk})\frac{\overline{u_{nk}}}{|\overline{u_{nk}}|}=\langle\psi'(u_n),\frac{\overline{u_n}}{|\overline{u_n}|}\rangle\to 0 \;\; as \;\; n\to\infty,$$

which yields a contradiction.
Consequently, $\{u_n\}$ is bounded in $W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})$. It follows from Proposition (3.1.1) that $\{u_n\}$ has a convergent subsequence in $W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})$. Namely, ψ satisfies the PS condition.
Since

$$W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})=\mathbb{R}\oplus\widetilde{W}^{1,p}_{\Delta,T}([0,\sigma(T))_{\mathbb{T}},\mathbb{R}).$$

Step 2 We prove that ψ is anti-coercive on $\mathbb{R}$. That is,

$$\psi(x)\to-\infty \;\; as \;\; |x|\to\infty \;\; for \;\; x\in\mathbb{R}, \tag{3.2.9}$$

which implies that the condition (I_1) of Lemma (1.4.2) is fulfilled.
In order to prove (3.2.9), we need to show that there exist $\delta_1>0$ and $\rho_1>0$ such that

$$\int_{[0,\sigma(T))_{\mathbb{T}}} f(\sigma(t),x)x\Delta t\le-\delta_1|x| \;\; for\ all\ x\in\mathbb{R}\ with\ |x|\ge\rho_1. \tag{3.2.10}$$

Otherwise, there is a sequence $\{x_n\}\subset\mathbb{R}$ with $|x_n|\to\infty$ such that

$$\int_{[0,\sigma(T))_{\mathbb{T}}} f(\sigma(t),x_n)\frac{\overline{x_n}}{|\overline{x_n}|}\Delta t>-\frac{1}{n} \;\; for\ n\ge1.$$

This contradicts with (H_1) because of the fact $\frac{|\overline{x_n}|\sigma(T)^{\frac{1}{p}}}{|x_n|} \to 1$. So the inequality (3.2.10) is true.From (H_0), we have

$$\begin{aligned} |\int_{[0,\sigma(T))_{\mathbb{T}}}[\int_0^{\frac{\rho_1}{|x|}} f(\sigma(t),sx)xds]\Delta t| &\leq \int_{[0,\sigma(T))_{\mathbb{T}}}[\int_0^{\frac{\rho_1}{|x|}} |f(\sigma(t),sx)||x|ds]\Delta t \\ &\leq \int_{[0,\sigma(T))_{\mathbb{T}}}[\int_0^{\frac{\rho_1}{|x|}} a(|sx|)b(\sigma(t))|x|ds]\Delta t \\ &\leq \int_{[0,\sigma(T))_{\mathbb{T}}}[\int_0^{\frac{\rho_1}{|x|}} a_{\rho_1}b(\sigma(t))|x|ds]\Delta t \\ &= \int_{[0,\sigma(T))_{\mathbb{T}}} \rho_1 a_{\rho_1} b(\sigma(t))\Delta t, \end{aligned}$$

where $a_{\rho_1} = \max_{|x|\leq\rho_1} a(|x|)$.

According to (3.2.10), we obtain

$$\begin{aligned} \int_{[0,\sigma(T))_{\mathbb{T}}}[\int_{\frac{\rho_1}{|x|}}^1 f(\sigma(t),sx)xds]\Delta t &= \int_{\frac{\rho_1}{|x|}}^1 \frac{1}{s}[\int_{[0,\sigma(T))_{\mathbb{T}}} f(\sigma(t),sx)sx\Delta t]ds| \\ &\leq -\delta_1|x|(1-\frac{\rho_1}{|x|}) \\ &= -\delta_1|x| + \delta_1\rho_1. \end{aligned}$$

We need to show that there exist $\delta_2 > 0$ and $\rho_2 > 0$ such that

$$\sum_{k=1}^{n} x_k I_k(x_k) \leq -\delta_2|x| \text{ for all } x \in \mathbb{R} \text{ with } |x| \geq \rho_2 \text{ and for } k = 1, \cdots, n. \tag{3.2.11}$$

For an arbitrary $x \in \mathbb{R}$ with $|x| > \min\{\rho_1, \rho_2\}$ there holds

$$\begin{aligned} \psi(x) &= \frac{1}{p}\int_{[0,\sigma(T))_{\mathbb{T}}} |x^{\Delta}|^p\Delta t + \int_{[0,\sigma(T))_{\mathbb{T}}} F(\sigma(t),x)\Delta t + \sum_{k=1}^{n} x_k I_k(x_k) \\ &= \int_{[0,\sigma(T))_{\mathbb{T}}}[\int_0^1 f(\sigma(t),sx)xds]\Delta t + \sum_{k=1}^{n} x_k I_k(x_k) \end{aligned}$$

$$\begin{aligned}\psi(x) &= \int_{[0,\sigma(T))_{\mathbb{T}}} \Big(\int_0^{\frac{\rho_1}{|x|}} f(\sigma(t), sx)xds + \int_{\frac{\rho_1}{|x|}}^1 f(\sigma(t), sx)xds\Big)\Delta t \\ &+ \sum_{k=1}^n x_k I_k(x_k)\end{aligned}$$

$$\begin{aligned}\psi(x) &\le -\delta_1|x| + \delta_1\rho_1 + \int_{[0,\sigma(T))_{\mathbb{T}}} \rho_1 a_{\rho_1} b(\sigma(t))\Delta t - \delta_2|x| \longrightarrow -\infty \\ &\quad \text{as } |x| \to \infty,\end{aligned}$$

which implies that ψ is anti-coercive on $\mathbb{R}$.

Step 3 We show that ψ is coercive in $\widetilde{W}^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}}, \mathbb{R})$. That is,

$$\psi(u) \to +\infty \; as \; \|u\| \to \infty \; in \; \widetilde{W}^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}}, \mathbb{R}).$$

which implies that the condition (I_2) of Lemma (1.4.2) is satisfied.
By the definition of R_p, we have

$$\|u\|^p_{L^p_\Delta} \le R_p \|u^\Delta\|^p_{L^p_\Delta} \quad for \; all \; u \subset \widetilde{W}^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}}, \mathbb{R}). \tag{3.2.12}$$

It follows from the Hölder inequality, inequality (3.2.5), (3.2.6) and Lemma (1.3.13) that

$$\begin{aligned}\psi(u) &= \frac{1}{p}\int_{[0,\sigma(T))_{\mathbb{T}}} |u^\Delta|^p \Delta t + \int_0^{\sigma(T)} \int_0^{u^\sigma(t)} f(\sigma, s) ds\Delta t + \sum_{k=1}^n u_k I_k(u_k) \\ &> \frac{1}{p}\int_0^{\sigma T} |u^\Delta|^p \Delta t + \int_0^{\sigma T} \Big(- \frac{1}{pR_p}|u|^{p-1} + \epsilon_0 |u|^{p-1} - a_M b(\sigma(t))\Big)\Delta t \\ &+ \sum_{k=1}^n \Big[- \frac{|u_k|^{p-1}}{k_p} + \epsilon_1 |u_k|^{p-1}\Big] \\ &> \frac{1}{p}\|u^\Delta\|^p_{L^p_\Delta} - \frac{1}{pR_p}(\sigma(T))^{\frac{1}{p}}\Big(\int_0^{\sigma(T)} |u|^p \Delta t\Big)^{\frac{p-1}{p}} - \int_0^{\sigma(T)} a_M b(\sigma(t)))\Delta t \\ &- \sum_{k=1}^n \frac{|u_k|^{p-1}}{k_p} \\ &> \frac{1}{p}\|u^\Delta\|^p_{L^p_\Delta} - \frac{1}{pR_p}(\sigma(T))^{\frac{1}{p}}\|u\|^{p-1}_{L^p_\Delta} - \int_0^{\sigma(T)} a_M b(\sigma(t)))\Delta t - \sum_{k=1}^n \frac{\|u\|^{p-1}_\infty}{k_p}\end{aligned}$$

$$\begin{aligned}
&> \frac{1}{p}\|u^{\Delta}\|^{p}_{L^{p}_{\Delta}} - \frac{1}{pR_p}(\sigma(T))^{\frac{1}{p}} R_p \|u^{\Delta}\|^{p-1}_{L^{p}_{\Delta}} - \int_0^{\sigma(T)} a_M b(\sigma(t)))\Delta t \\
&- \sum_{k=1}^{n} \frac{K^{p-1}\|u^{\Delta}\|^{p-1}_{L^{p}_{\Delta}}}{k_p},
\end{aligned}$$

for $u \in \widetilde{W}^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})$. Hence, ψ is coercive on $\widetilde{W}^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})$. Up to here, we see that all conditions of Lemma (1.4.2) are fulfilled. Consequently, the boundary value problem (3.0.1) has at least one periodic solution in $W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})$.

□

Theorem 3.2.2. *Suppose that the condition* (H_0) *and the following conditions are satisfied:*

(H_4) *the inequalities* $\liminf_{|x|\to\infty} \frac{\int_0^x f(t,s)ds}{|x|^p} > 0$ *holds uniformly for* $\Delta -$ *a.e.* $t \in [0,\sigma(T))_{\mathbb{T}}$,

(H_5) *the inequalities* $\liminf_{|x|\to\infty} \frac{\int_0^x I(s)ds}{|x|} > 0$ *for all* $x \in \mathbb{R}$,

(H_6) *let* $\{u_n\} \subset W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})$ *such that* $\|u_n\| \to \infty$ *and* $\frac{|\overline{u_n}|\sigma(T)^{\frac{1}{p}}}{\|u_n\|} \to 1$, *and*

$$\lim_{n\to\infty} \int_{[0,\sigma(T))_{\mathbb{T}}} \int_0^{u_n(t)} f(t,s)\Delta s \Delta t + \sum_{k=1}^{N} \int_0^{u_{nk}} I_{kn}(s)ds = +\infty.$$

Then, the boundary value problem (3.0.1) has at least one periodic solution in $W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})$.

Proof. Firstly, we prove that ψ is coercive in $W^{1,p}_{\Delta,T}([0,\sigma(T)])_{\mathbb{T}},\mathbb{R})$ namely,

$$\psi(u) \to +\infty \;\; as \;\; \|u\| \to \infty \;\; for \;\; u \in W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R}).$$

If not, there exist a sequence $\{u_n\} \subset W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})$ and a constant c such that

$$\|u_n\| \to \infty, \quad \psi(u_n) \le c, \qquad n = 1, 2, \ldots \tag{3.2.13}$$

Let $v_n = \frac{u_n}{\|u_n\|}$, then $\{v_n\}$ is bounded in $W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}}, \mathbb{R})$. Since

$$W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}}, \mathbb{R}) \hookrightarrow L^p_{\Delta}([0,\sigma(T))_{\mathbb{T}}, \mathbb{R}) \text{ is compact,}$$

there exist a point $v_0 \in W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}}, \mathbb{R})$ and a subsequence of $\{v_n\}$ such that

$$v_n \rightharpoonup v_0 \;\; in \;\; W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}}, \mathbb{R}),$$

and

$$v_n \to v_0 \;\; strongly \;\; in \;\; L^p_{\Delta}([0,\sigma(T))_{\mathbb{T}}, \mathbb{R}). \tag{3.2.14}$$

From (H_4), for any $\epsilon_0 > 0$ there exists a constant $M > 0$ such that

$$\int_0^{u^\sigma(t)} f(\sigma(t), s)ds > -\frac{\epsilon_0}{2} \min_{t\in[0,\sigma(T)]_{\mathbb{T}}} |u^\sigma(t)|^p \geq -\frac{\epsilon_0}{2}|u(t)|^p, \tag{3.2.15}$$

for all $u^\sigma \in \mathbb{R}$ with $|u^\sigma| > M$ and $\Delta-$ a.e $t \in [0,\sigma(T)]_{\mathbb{T}}$. From (3.2.4) and (3.2.15), it gives

$$\int_0^{u^\sigma(t)} f(\sigma(t), s)ds > -\frac{\epsilon_0}{2}|u(t)|^p - a_M b(\sigma(t)). \tag{3.2.16}$$

From (H_5), for any $\epsilon_1 > 0$ such that

$$\int_0^{x} I(s)ds > -\epsilon_1 x, \tag{3.2.17}$$

for all $u^\sigma \in \mathbb{R}$ and $\Delta-$ a.e $t \in [0,\sigma(T)]_{\mathbb{T}}$. In view of (3.2.13), (3.2.16) and (3.2.17) we have

$$\begin{aligned}
\frac{c}{\|u_n\|^p} &\geq \frac{\psi(u_n)}{\|u_n\|^p} \\
&= \frac{1}{2}\int_{[0,\sigma(T))_{\mathbb{T}}} \frac{|u_n^\Delta|^p}{\|u_n\|^p}\Delta t + \frac{1}{\|u_n\|^p}\int_{[0,\sigma(T))_{\mathbb{T}}}\int_0^{u_n^\sigma(t)} f(\sigma(t), s)ds\Delta t \\
&+ \frac{1}{\|u_n\|^p}\sum_{k=1}^{n}\int_0^{u_{nk}} I_k(s)ds \\
&> \frac{1}{2}\int_{[0,\sigma(T))_{\mathbb{T}}} |(v_n)^\Delta|^p\Delta t - \frac{1}{\|u_n\|^p}\int_{[0,\sigma(T))_{\mathbb{T}}}\Big[\frac{\epsilon_0}{2}|u_n|^p + a_M b(\sigma(t))\Big]\Delta t \\
&- \sum_{k=1}^{n}\frac{\epsilon_1|u_{nk}|}{\|u_n\|^p}
\end{aligned}$$

$$\begin{aligned}
\frac{c}{\|u_n\|^p} &\geq \frac{1}{2}\int_{[0,\sigma(T))_{\mathbb{T}}} |(v_n)^{\Delta}|^p \Delta t - \frac{\epsilon_0}{2}\int_{[0,\sigma(T))_{\mathbb{T}}} |v_n|^p \Delta t - \frac{a_M}{\|u_n\|^p}\int_{[0,\sigma(T))_{\mathbb{T}}} b(\sigma(t))\Delta t \\
&- \sum_{k=1}^{n} \frac{\epsilon_1 |v_{nk}|}{\|u_n\|^{p-1}} \\
&= \frac{1}{2} - \frac{1}{2}(1+\epsilon_0)\int_{[0,\sigma(T))_{\mathbb{T}}} |v_n|^p \Delta t - \frac{a_M}{\|u_n\|^p}\int_{[0,\sigma(T))_{\mathbb{T}}} b(\sigma(t))\Delta t - \sum_{k=1}^{n} \frac{\epsilon_1 v_{nk}}{\|u_n\|^{p-1}}.
\end{aligned}$$

According to (3.2.13) and (3.2.14), there is

$$\frac{1}{2}(1+\epsilon_0)\int_{[0,\sigma(T))_{\mathbb{T}}} |v_0|^p \Delta t \geq \frac{1}{2}.$$

Letting $\epsilon_0 \to 0$, we obtain

$$\int_{[0,\sigma(T))_{\mathbb{T}}} |v_0|^p \Delta t \geq 1.$$

By the weakly lower semi-continuity of the norm, we get

$$\|v_0\| \leq \liminf_{n\to\infty} \|v_n\| = 1.$$

That is

$$\|v_0\|^p = \int_{[0,\sigma(T))_{\mathbb{T}}} \left(|v_0|^p + |v_0^{\Delta}|^p\right)\Delta t \leq 1.$$

Using a similar argument to the proof of Theorem (3.2.1), we have $\frac{|\overline{u_n}|\sigma(T)^{\frac{1}{p}}}{\|u_n\|} \to 1$. Then, it follows from (H_6) that

$$\lim_{n\to\infty} \int_{[0,\sigma(T))_{\mathbb{T}}} \int_0^{u_n^{\sigma}(t)} f(\sigma(t),s)ds\Delta t + \sum_{k=1}^{N}\int_0^{u_{nk}} I_{kn}(s)ds = +\infty. \quad (3.2.18)$$

However, combining (3.2.13) with (3.2.18) gives

$$c \geq \lim_{n\to\infty} \psi(u_n) \geq \lim_{n\to\infty} \int_{[0,\sigma(T))_{\mathbb{T}}} \int_0^{u_n^{\sigma}(t)} f(\sigma(t),s)\Delta s\Delta t + \sum_{k=1}^{N}\int_0^{u_{nk}} I_k(s)ds = +\infty.$$

This yields a contradiction.
Since ψ is lower semi-continuous and coercive, we see that ψ is bounded

below and has a bounded minimizing sequence. From Lemma (1.4.3), we conclude that ψ has at least a single critical point in $W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})$. Consequently, the boundary value problem (3.0.1) has at least one single solution in $W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})$. $\square$

Theorem 3.2.3. *Suppose that the condition* $(H_0),(H_4),(H_5)$ *and* (H_6) *hold, and the following two condition are satisfied:*

(H_7) *there exists a constant* $\delta_1>0$ *such that*

$$-\frac{1}{2pR_p}|x|^p \le \int_0^x f(t,s)ds \le 0 \ \text{ for all } |x|\le\delta_1 \text{ and } \Delta\text{−a.e. } t\in[0,\sigma(T)]_{\mathbb{T}},$$

(H_8) *there exists a constant* $\delta_2>0$ *such that*

$$-\frac{1}{2pk_p}|x|^p \le \sum_{k=1}^{N}\int_0^x I(s)ds \le 0 \quad \text{for all } \ |x|\le\delta_2.$$

Then, the boundary value problem (3.0.1) has at least one periodic solution in $W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})$.

Proof. It follows from (H_7) and (H_8) that $u\equiv 0$ is a solution of the boundary value problem (3.0.1). In the following, we show that there exist at least two nontrivial distinct solutions of the boundary value problem (3.0.1) in $W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})$ according to Lemma (1.4.4). From the proof of Theorem (3.2.2), we know that ψ is coercive. Assume that

$$u_n \subset W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R}),$$

such that $\psi'(u_n)\to 0$ as $n\to\infty$, and $\psi(u_n)$ is bounded. From the proof of Theorem (3.2.1), we know that $\{u_n\}$ is bounded. By Proposition (3.1.1), we obtain that $\{u_n\}$ has a convergent subsequence in $W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})$. This implies that ψ satisfies the PS condition.

Since

$$W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R}) = \mathbb{R}\oplus\widetilde{W}^{1,p}_{\Delta,T}([0,\sigma(T))_{\mathbb{T}},\mathbb{R}),$$

according to (H_7) and (H_8) one has

$$\psi(u) = \int_{[0,\sigma(T))_{\mathbb{T}}}\int_0^{u^\sigma(t)} f(\sigma(t),s)ds\Delta t + \sum_{k=1}^{N}\int_0^{u_k} I_k(s)ds \le 0 \qquad (3.2.19)$$

for $u \in \mathbb{R}$ with $\|u\| \leq \min\{\delta_1, \delta_2\}$. In view of $(H_7), (H_8)$ inequality (3.2.12), and Lemma (1.3.13), we have

$$\begin{aligned}
\psi(u) &= \frac{1}{p}\int_{[0,\sigma(T))_{\mathbb{T}}} |u^{\Delta}|^p \Delta t + \int_{[0,\sigma(T))_{\mathbb{T}}} \int_0^{u^{\sigma}(t)} f(\sigma, s) ds \Delta t + \sum_{k=1}^{N} \int_0^{u_k} I_k(s) ds \\
&> \frac{1}{p}\|u^{\Delta}\|^p_{L^p_{\Delta}} - \frac{1}{2pR_p}\int_{[0,\sigma(T))_{\mathbb{T}}} |u^{\sigma}|^p \Delta t - \frac{1}{2pk_p}|u|^p \\
&= \frac{1}{p}\|u^{\Delta}\|^p_{L^p_{\Delta}} - \frac{1}{2p}\|u^{\Delta}\|^p_{L^p_{\Delta}} - \frac{1}{2pk_p}\sup |u|^p \\
&= \frac{1}{p}\|u^{\Delta}\|^p_{L^p_{\Delta}} - \frac{1}{2p}\|u^{\Delta}\|^p_{L^p_{\Delta}} - \frac{1}{2pk_p}\|u\|^p_{\infty} \\
&= \frac{1}{p}\|u^{\Delta}\|^p_{L^p_{\Delta}} - \frac{1}{2p}\|u^{\Delta}\|^p_{L^p_{\Delta}} - \frac{1}{2p}\|u^{\Delta}\|^p_{L^p_{\Delta}} \\
&= 0
\end{aligned}$$

for $u \in \widetilde{W}^{1,p}_{\Delta,T}([0,\sigma(T))_{\mathbb{T}}, \mathbb{R})$ with $\|u\| \leq \frac{\delta}{R_p}$.
Let $r = \min\{\delta, \frac{\delta}{R_p}\}$. There holds

$$\psi(u) \leq 0, \quad u \in \mathbb{R} \ \text{ with } \ \|u\| \leq r,$$

and

$$\psi(u) \geq 0, \quad u \in \widetilde{W}^{1,p}_{\Delta,T}([0,\sigma(T))_{\mathbb{T}}, \mathbb{R}) \text{ with } \ \|u\| \leq r.$$

It follows from the proof of Theorem (3.2.3) that ψ is bounded below.
If $\inf_{W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})} \psi < 0$, we observe that all conditions of lemma (1.4.4) are fulfilled.
Hence, it follows from lemma (1.4.4) that the boundary value problem (3.0.1) has at least three distinct periodic solutions in $W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}}, \mathbb{R})$.
If $\inf_{W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})} \psi \geq 0$, by (3.2.19), we have

$$\psi(x) = \inf_{W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})} \psi = 0 \ \ for\ all\ x \in \mathbb{R} \ \ with\ \|x\| \leq r,$$

which implies that all $x \in \mathbb{R}$ with $\|x\| \leq r$ are minima of ψ. That is, the boundary value problem (3.0.1) has infinitely many solutions in $W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}}, \mathbb{R})$.
Consequently, the boundary value problem (3.0.1) has at least three distinct solutions in $W^{1,p}_{\Delta,T}([0,\sigma(T)]_{\mathbb{T}}, \mathbb{R}^+)$.

$\square$

3.3 Examples

Example 3.3.1. *Let $T = 200, m \in \mathbb{Z}$, and*

$$\mathbb{T} = \{0, 5 + 200m, 120 + 200m, 190 + 200m\} \cup [190.5 + 200m, 200 + 200m].$$

Consider the following second-order p-Laplacian boundary value impulsive problem on time scales $\mathbb{T}$ of the form

$$\begin{cases} \left(|u^{\Delta}(t)|^3 |u^{\Delta}(t)|\right)^{\Delta} &= -3\sigma(t)(u^{\sigma}(t))^2, \Delta - a.e.t \in [0,200]_{\mathbb{T}}, \\ |u^{\Delta}(t_k^+)|^3 |u^{\Delta}(t_k^+) - |u^{\Delta}(t_k^-)|^3 |u^{\Delta}(t_k^-) &= \frac{1}{200} u^{\sigma}(200), \ k = 1, \dots, p, \\ u(0) - u(200) = u^{\Delta}(0) - u^{\Delta}(200) &= 0. \end{cases} \tag{3.3.1}$$

Since $\displaystyle\int_0^x f(t,s)ds = -tx^3$, one can check that the condition (H_0) and all conditions of Theorem (3.2.1) are fulfilled. It follows from Theorem (3.2.1) that the problem (3.3.1) has at least one periodic solution.

Example 3.3.2. *Let $m \in \mathbb{Z}, n \in \mathbb{N}, T = 1$, and*

$$\mathbb{T} = \{0\} \cup \left\{\frac{1}{3^n} + m\right\} \cup [0.4 + m, 1 + m].$$

Consider the second-order boundary value problem

$$\begin{cases} (|u^{\Delta}(t)|^3 |u^{\Delta(t)}|)^{\Delta} = -6(\sigma(T) + \sigma(t))(sgn(u^{\sigma}))(u^{\sigma(t)})^5, & \Delta - a.e.t \in [0,1]_{\mathbb{T}} \\ |u^{\Delta}(t_k^+)|^3 |u^{\Delta(t_k^+)}| - |u^{\Delta}(t_k^-)|^3 |u^{\Delta(t_k^-)}| = \frac{1}{a_1} u^{\sigma}(T), & k = 1, \dots, p, \quad a_1 > 0 \\ u(0) - u(1) = 0, u^{\Delta}(0) - u^{\Delta}(1) = 0. & \end{cases} \tag{3.3.2}$$

Since $\displaystyle\int_0^x f(t,s)ds = (\sigma(t) + t)|x|^6$, one can see that the condition (H_0) and all conditions of Theorem (3.2.2) are fulfilled. It follows from Theorem (3.2.2) that the problem (3.3.2) has at least one periodic solution.

Example 3.3.3. *Let $\mathbb{T} = \{0, 1, 2, 3\}$ and $T = 1$. Consider the boundary value problem*

$$\begin{cases} u^{\Delta\Delta}(t) - c &= 0, \quad \Delta - a.e. \ t \in [0,1]_{\mathbb{T}}^k, \quad t \neq t_k, k = 1, \dots, p, \ t \in J, \\ u^{\Delta}(t_k^+) - u^{\Delta}(t_k^-) &= -\frac{1}{a_2}, \quad k = 1, \dots, p, a_2 > 0 \\ u(0) - u(2) &= 0, u^{\Delta}(0) - u^{\Delta}(2) = 0, \end{cases} \tag{3.3.3}$$

where $f(t,s) = c \neq 0$ does not satisfy hypotheses $(H_1) - (H_5)$. Here, one can see that the boundary value problem does not have one periodic solution.

Chapter 4

$p-$ Laplacian fractional boundary value problem

In this chapter, we discuses the existence and multiplicity of positive solutions for system of fractional differential equations with boundary condition and advanced arguments. The existence result proved via Leary-Schauder's fixed point theorem type in vector Banach space. Further, by using new fixed point theorem order Banach space, we study the multiplicity of positive. Finally, some example are given to illustrate the result.

4.1 Existence result

This section, is concerned the existence of solutions for the system of fractional boundary value problem with $p-$laplacian conditions:

$$\begin{cases} (\varphi_p(D^{\alpha}_{0^+}u(t)))' + a_1(t)f(u(\theta_1(t)), v(\theta_2(t))) = 0, & 0 < t < 1, \\ (\varphi_{\tilde{p}}(D^{\alpha}_{0^+}v(t)))' + a_2(t)g(u(\theta_1(t)), v(\theta_2(t))) = 0, & 0 < t < 1, \\ D^{\alpha}_{0^+}u(0) = u(0) = u'(0) = 0, \\ D^{\alpha}_{0^+}v(0) = v(0) = v'(0) = 0, \\ D^{\beta}_{0^+}u(1) = \gamma D^{\beta}_{0^+}u(\eta), \\ D^{\beta}_{0^+}v(1) = \gamma D^{\beta}_{0^+}v(\eta), \end{cases} \tag{4.1.1}$$

where $\eta \in (0,1), \gamma \in (0, \frac{1}{\eta^{\alpha-\beta-1}}), D^{\alpha}_{0^+}, D^{\beta}_{0^+}$, are the standard Riemann-Liouville fractional derivatives with $\alpha \in (2,3), \beta \in (1,2)$ such that $\alpha \geq \beta + 1$, the p-Laplacian operator is defined as $\varphi_p(s) = |s|^{p-2}s,\ p > 1$, and the functions

$f, g \in C(\mathbb{R}^2, \mathbb{R})$.

In recent years, many authors studied the existence of solutions for systems of difference and differential equations with and without fractional derivative by using the vector version of the fixed point theorem (see [13, 33, 34, 40, 45, 65, 68, 70], the monograph of Graef *et al* [31], and the references therein).

For establish the existence and multiple positive solutions of problem (4.1.1), let us list the following assumptions:

(H_1) $a_i \in L^1[0,1]$ is nonnegative and $a_i(t) \not\equiv 0$ on any subinterval of $[0,1]$, for $i = 1, 2$.

(H_2) The advanced argument $\theta \in C((0,1),(0,1])$ and $0 \leq \theta(t) \leq 1, \forall t \in (0,1)$.

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}^+$ be the set of nonnegative real numbers. Denote by $C([0,1])$ the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm

$$\|u\| = \max\{|u(t)| : t \in [0,1]\}$$

Define the cone P in $C([0,1])$ as $P = \{u \in C([0,1]) : u(t) \geq 0, t \in [0,1]\}$.
Let $q > 1$ and $\tilde{q} > 1$ satisfy the relation $\frac{1}{p} + \frac{1}{q} = 1, \frac{1}{\tilde{p}} + \frac{1}{\tilde{q}} = 1$, where $p, \tilde{p}$ are given by (4.1.1).
To prove the existence of solutions to (4.1.1), we need the following auxiliary Lemma.

Lemma 4.1.1. *Given* $h_1, h_2 \in C[0,1], \eta \in (0,1), \gamma \in (0, \frac{1}{\eta^{\alpha-\beta-1}})$ *and* $\alpha \geq \beta + 1$, *the unique solution of the boundary value problem for a coupled system*

$$(\varphi_p(D^{\alpha}_{0^+}u(t)))' + h_1(t) = 0, \quad 0 < t < 1, \tag{4.1.2}$$

$$(\varphi_{\tilde{p}}(D^{\alpha}_{0^+}v(t)))' + h_2(t) = 0, \quad 0 < t < 1, \tag{4.1.3}$$

$$D^{\alpha}_{0^+}u(0) = u(0) = u'(0) = 0, \quad D^{\beta}_{0^+}u(1) = \gamma D^{\beta}_{0^+}u(\eta), \tag{4.1.4}$$

$$D^{\alpha}_{0^+}v(0) = v(0) = v'(0) = 0, \quad D^{\beta}_{0^+}v(1) = \gamma D^{\beta}_{0^+}v(\eta), \tag{4.1.5}$$

is $(u,v) \in C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$ *where*

$$\begin{aligned} u(t) &= \int_0^1 G_1(t,s)\varphi_q\left(\int_0^s h_1(\tau)d\tau\right)ds \\ &+\tfrac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1 G_2(\eta,s)\varphi_q\left(\int_0^s h_1(\tau)d\tau\right)ds, \end{aligned} \tag{4.1.6}$$

and

$$\begin{aligned} v(t) &= \int_0^1 G_1(t,s)\varphi_{\tilde{q}}\left(\int_0^s h_2(\tau)d\tau\right)ds \\ &+\tfrac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1 G_2(\eta,s)\varphi_{\tilde{q}}\left(\int_0^s h_2(\tau)d\tau\right)ds, \end{aligned} \tag{4.1.7}$$

where

$$G_1(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)} & 0 \le s \le t \le 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} & 0 \le t \le s \le 1, \end{cases}$$

$$G_2(\eta,s) = \begin{cases} \frac{[\eta(1-s)]^{\alpha-\beta-1}-(\eta-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} & 0 \le s \le \eta \le 1, \\ \frac{[\eta(1-s)]^{\alpha-\beta-1}}{\Gamma(\alpha)} & 0 \le \eta \le s \le 1. \end{cases}$$

Proof. Integrating the equation (4.1.2) from 0 to t, we have

$$\varphi_p\left(D_{0^+}^{\alpha}u(t)\right) - \varphi_p\left(D_{0^+}^{\alpha}u(0)\right) = \int_0^t h_1(s)ds$$

and so,

$$D_{0^+}^{\alpha}u(t) = -\varphi_q\left(\int_0^t h_1(s)ds\right).$$

From Lemma 1.5.12

$$\begin{aligned} u(t) &= -I_{0^+}^{\alpha}\varphi_q\left(\int_0^t h_1(s)ds\right) + At^{\alpha-1} + Bt^{\alpha-2} + Ct^{\alpha-3} \\ &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_q\left(\int_0^s h_1(\tau)d\tau\right)ds + At^{\alpha-1} + Bt^{\alpha-2} + Ct^{\alpha-3}. \end{aligned}$$

From (4.2.2), $B = C = 0$, and so

$$u(t) = -I_{0^+}^{\alpha}\varphi_q\left(\int_0^t h_1(s)ds\right) + At^{\alpha-1}. \tag{4.1.8}$$

Now, from Remark 1.5.2

$$
\begin{aligned}
D_{0^+}^{\beta}u(t) &= -I_{0^+}^{\alpha-\beta}\varphi_q\left(\int_0^t h_1(s)ds\right) + A\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}t^{\alpha-\beta-1}\\
&= -\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\varphi_q(\int_0^s h_1(\tau)d\tau)ds + A\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}t^{\alpha-\beta-1}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
D_{0^+}^{\beta}u(1) &= -\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-\beta-1}\varphi_q(\int_0^s h_1(\tau)d\tau)ds + A\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)},\\
\gamma D_{0^+}^{\beta}u(\eta) &= -\frac{\gamma}{\Gamma(\alpha-\beta)}\int_0^\eta (\eta-s)^{\alpha-\beta-1}\varphi_q(\int_0^s h_1(\tau)d\tau)ds + A\frac{\Gamma(\alpha)\gamma}{\Gamma(\alpha-\beta)}\\
&\quad \times t^{\alpha-\beta-1}\eta^{\alpha-\beta-1},
\end{aligned}
$$

by boundary condition (4.2.2), we have

$$
\begin{aligned}
A &= \frac{1}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\varphi_q\left(\int_0^s h_1(\tau)d\tau\right)ds\\
&- \frac{\gamma}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^\eta \frac{(\eta-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\varphi_q(\int_0^s h_1(\tau)d\tau)ds,
\end{aligned}
$$

and replacing in (4.1.8), we obtain

$$
\begin{aligned}
u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_q(\int_0^s h_1(\tau)d\tau)ds\\
&\quad +\frac{t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\varphi_q(\int_0^s h_1(\tau)d\tau)ds\\
&\quad +\frac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1 \frac{(\eta-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\varphi_q(\int_0^s h_1(\tau)d\tau)ds.
\end{aligned}
$$

Splitting the second integral in two parts of the form

$$
t^{\alpha-1} + \frac{k}{1-\gamma\eta^{\alpha-\beta-1}} = \frac{t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}
$$

we have $k = \gamma\eta^{\alpha-\beta-1}t^{\alpha-1}$, and thus,

$$\begin{aligned}
u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_q(\int_0^s h_1(\tau)d\tau)ds \\
&+ t^{\alpha-1}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\varphi_q(\int_0^s h_1(\tau)d\tau)ds \\
&+ \frac{\gamma\eta^{\alpha-\beta-1}t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\varphi_q(\int_0^s h_1(\tau)d\tau)ds \\
&= \int_0^t \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q(\int_0^s h_1(\tau)d\tau)ds \\
&\quad +\int_t^1 \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\varphi_q(\int_0^s h_1(\tau)d\tau)ds \\
&\quad +\frac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^\eta \frac{[\eta(1-s)]^{\alpha-\beta-1}-(\eta-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\times\varphi_q(\int_0^s h_1(\tau)d\tau)ds \\
&= \int_0^1 G_1(t,s)\varphi_q(\int_0^s h_1(\tau)d\tau)ds \\
&+ \frac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\times\int_0^1 G_2(\eta,s)\varphi_q(\int_0^s h_1(\tau)d\tau)ds.
\end{aligned}$$

This completes the proof. □

Lemma 4.1.2. *[101] Let $\rho \in (0,1)$ be fixed. The kernel $G_1(t,s)$ satisfies the following properties:*

1. $G_1(t,s) \in C([0,1]\times[0,1])$ *and* $G_1(t,s) > 0$ *for all* $s,t \in (0,1)$,
2. $G_1(t,s) \le G_1(1,s)$ *for all* $s \in (0,1)$,
3. $\min_{\rho\le t\le 1} G_1(t,s) \ge \rho^{\alpha-1}G_1(1,s)$ *for all* $s \in [0,1]$.

We are now ready to present our main result in this section we give an existence result based on the non linear alternative of Leray-Schauder type.

Theorem 4.1.3. *Assume (H_1)-(H_2), and that the following condition holds:*

(H_3) There exist functions $p, q, h, \breve{p}, \breve{q}$, and $\bar{h} \in L^1([0,1],\mathbb{R}^+)$ and constants $\alpha_1, \alpha_2, \alpha_3$, and $\alpha_4 \in [0,1)$ such that

$$|f(u,v)| \le p(t)|u|^{\alpha_1} + q(t)|v|^{\alpha_2} + h(t) \quad \textit{for each } t \in [0,1] \textit{ and } u,v \in \mathbb{R}$$

and

$$|g(u,v)| \leq \breve{p}(t)|u|^{\alpha_3} + \breve{q}(t)|v|^{\alpha_4} + \breve{h}(t) \quad \textit{for each } t \in [0,1] \textit{ and } u,v \in \mathbb{R}.$$

If $\alpha_1 p, \alpha_2 p, \alpha_3 q,$ and $\alpha_4 q \in [0,1)$. Then the system (4.1.1) has at least one solution.

Proof. Let N be the operator defined

$$N : C(0,1) \times C(0,1) \to C(0,1) \times C(0,1)$$

defined by

$$N(u,v) = (N_1(u,v), N_2(u,v)),$$

where

$$\begin{aligned} N_1(u,v)(t) = &\int_0^1 G_1(t,s)\varphi_q(\int_0^s a_1(\tau)f(u(\theta_1(\tau)), v(\theta_2(\tau)))d\tau)ds \\ +\tfrac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}} &\int_0^1 G_2(\eta,s)\varphi_q(\int_0^s a_1(\tau)f(u(\theta_1(\tau)), v(\theta_2(\tau)))d\tau)ds, \end{aligned} \tag{4.1.9}$$

and

$$\begin{aligned} N_2(u,v)(t) = &\int_0^1 G_1(t,s)\varphi_{\tilde{q}}(\int_0^s a_2(\tau)g(u(\theta_1(\tau)), v(\theta_2(\tau))))d\tau)ds \\ +\tfrac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}} &\int_0^1 G_2(\eta,s)\varphi_{\tilde{q}}(\int_0^s a_2(\tau)g(u(\theta_1(\tau)), v(\theta_2(\tau)))d\tau)ds. \end{aligned} \tag{4.1.10}$$

We shall use Leray-Schauder fixed point theorem to prove that N has a fixed point. The proof will be given in several steps.

Step 1 To show that N is continuous let (u_n, v_n) be a sequence such that $(u_n, v_n) \to (u,v) \in C[0,1] \times C[0,1]$ as $n \to \infty$. Then we have

$$\begin{aligned} |N_1(u_n,v_n)(t) - N_1(u,v)(t)| = \Bigg| &\int_0^1 G_1(t,s)\varphi_q(\int_0^s a_1(\tau)f(u_n(\theta_1(\tau)), v_n(\theta_2(\tau)))d\tau)ds \\ &+\tfrac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}} \int_0^1 G_2(\eta,s)\varphi_q(\int_0^s a_1(\tau)f(u_n(\theta_1(\tau)), v_n(\theta_2(\tau)))d\tau)ds \\ &-\Bigg[\int_0^1 G_1(t,s)\varphi_q\left(\int_0^s a_1(\tau)f(u(\theta_1(\tau)), v(\theta_2(\tau)))d\tau\right)ds \\ &+\int_0^1 \frac{\gamma t^{\alpha-1}G_2(\eta,s)}{1-\gamma\eta^{\alpha-\beta-1}}\ \varphi_q\left(\int_0^s a_1(\tau)f(u(\theta_1(\tau)), v(\theta_2(\tau)))d\tau\right)ds\Bigg]\Bigg| \end{aligned}$$

by lemma 4.1.2 and $t \in [0,1]$

$$\begin{aligned}|N_1(u_n,v_n)(t)-N_1(u,v)(t)| \leq & \int_0^1 G_1(1,s)\left(\int_0^s |a_1(\tau)f(u_n(\theta_1(\tau)),v_n(\theta_2(\tau)))|^{q-1}\right.\\ & -|a_1(\tau)f(u(\theta_1(\tau)),v(\theta_2(\tau)))|^{q-1}d\tau)\,ds\\ & +\tfrac{\gamma}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1 G_2(\eta,s)\left(\int_0^s |a_1(\tau)f(u_n(\theta_1(\tau)),v_n(\theta_2(\tau)))d\tau|^{q-1}\right.\\ & -|a_1(\tau)f(u(\theta_1(\tau)),v(\theta_2(\tau)))|^{q-1}d\tau)\,ds.\end{aligned}$$

On the other hand, Since f is continue function combined with the fact that

$$\|u_n-u\| \to 0, \text{ as } n\to\infty,$$

then there exists $N \geq 1$ such that for all $\tau \in [0,1]$ the following estimate

$$|f(u_n(\theta_1(\tau)),v_n(\theta_2(\tau)))d\tau)-f(u(\theta_1(\tau)),v(\theta_2(\tau)))| < \epsilon,$$

holds for $n \geq N$. By the Lebesgue dominated convergence theorem,then

$$\|N_1(u_n,v_n)-N_1(u,v)\| \to 0, \text{ as } n\to\infty.$$

Similarly,

$$\|N_2(u_n,v_n)-N_2(u,v)\| \to 0, \text{ as } n\to\infty.$$

Consequently, N is continuous.

Step 2 N maps bounded sets into bounded sets in $C[0,1]\times C[0,1]$, it suffices to show that for any $r>0$ there exists a positive constant vector $l=(l_1;l_2)$ such that for each
$(u,v)\in B_r=\{(u,v)\in C[0,1]\times C[0,1]: \|u\|\leq r, \|v\|\leq r\}$, we have

$$\|N(u,v)\| \leq l.$$

For each $t\in[0,1]$, we have

$$\begin{aligned}|N_1(u,v)(t)| &\leq \int_0^1 |G_1(t,s)|\varphi_q\left(\int_0^s |a_1(\tau)f(u(\theta_1(\tau)),v(\theta_2(\tau)))d\tau|\right)ds\\ &+ \frac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1 |G_2(\eta,s)|\varphi_q\left(\int_0^s |a_1(\tau)f(u(\theta_1(\tau)),v(\theta_2(\tau)))d\tau|\right)ds\\ &\leq \max\{2^{q-1},1\}\int_0^1 G_1(1,s)\int_0^s |a_1(\tau)|^{q-1}|p(\tau)|^{q-1}|u(\theta_1(\tau))|^{\alpha_1(q-1)}\\ &+ |a_1(\tau)|^{q-1}|q(\tau)|^{q-1}|v(\theta_2(\tau))|^{\alpha_2(q-1)}+|a_1(\tau)|^{q-1}|h(\tau)|^{q-1}d\tau ds\\ &+ \frac{\gamma\max\{2^{q-1},1\}t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1 |G_2(\eta,s)|\int_0^s |a_1(\tau)|^{q-1}|p(\tau)|^{q-1}|u(\theta_1(\tau))|^{\alpha_1(q-1)}\\ &+ |a_1(\tau)|^{q-1}|q(\tau)|^{q-1}|v(\theta_2(\tau))|^{\alpha_2(q-1)}+|a_1(\tau)|^{q-1}|h(\tau)|^{q-1}d\tau ds.\end{aligned}$$

Thus

$$\begin{aligned}
\|N_1(u,v)\| &\leq \max\{2^{q-1},1\}(\|u\|^{\alpha_1(q-1)}\|a_1\|_{L_1}^{q-1}\|p\|_{L_1}^{q-1} + \|v\|^{\alpha_2(q-1)}\|a_1\|_{L_1}^{q-1}\|q\|_{L_1}^{q-1} \\
&\quad +\|a_1\|_{L_1}^{q-1}\|h\|_{L_1}^{q-1})\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} + \frac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}(\|u\|^{\alpha_1(q-1)} \\
&\quad \|a_1\|_{L_1}^{q-1}\|p\|_{L_1}^{q-1} + \|v\|^{\alpha_2(q-1)}\|a_1\|_{L_1}^{q-1}\|q\|_{L_1}^{q-1} + \|a_1\|_{L_1}^{q-1}\|h\|_{L_1}^{q-1}) \\
&\quad \times \int_0^1 \frac{\eta^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} \\
&\leq \frac{\max\{2^{q-1},1\}}{(\alpha-\beta)\Gamma(\alpha)}(r^{\alpha_1(q-1)}\|a_1\|_{L_1}^{q-1}\|p\|_{L_1}^{q-1} + r^{\alpha_2(q-1)}\|a_1\|_{L_1}^{q-1}\|q\|_{L_1}^{q-1} \\
&\quad +\|a_1\|_{L_1}^{q-1}\|h\|_{L_1}^{q-1}) + \frac{\max\{2^{q-1},1\}\eta^{\alpha-\beta-1}\gamma}{(1-\gamma\eta^{\alpha-\beta-1})(\alpha-\beta)\Gamma(\alpha)}(r^{\alpha_1(q-1)}\|a_1\|_{L_1}^{q-1}\|p\|_{L_1}^{q-1} \\
&\quad +r^{\alpha_2(q-1)}\|a_1\|_{L_1}^{q-1}\|q\|_{L_1}^{q-1} + \|a_1\|_{L_1}^{q-1}\|h\|_{L_1}^{q-1}).
\end{aligned}$$

Hence

$$\begin{aligned}
\|N_1(u,v)\| &\leq \frac{\max\{2^{q-1},1\}}{(\alpha-\beta)\Gamma(\alpha)}\big(r^{\alpha_1(q-1)}\|a_1\|_{L_1}^{q-1}\|p\|_{L_1}^{q-1} + r^{\alpha_2(q-1)}\|a_1\|_{L_1}^{q-1}\|q\|_{L_1}^{q-1} \\
&\quad +\|a_1\|_{L_1}^{q-1}\|h\|_{L_1}^{q-1}\big) + \frac{\max\{2^{q-1},1\}\eta^{\alpha-\beta-1}\gamma}{(1-\gamma\eta^{\alpha-\beta-1})(\alpha-\beta)\Gamma(\alpha)}(r^{\alpha_1(q-1)}\|a_1\|_{L_1}^{q-1}\|p\|_{L_1}^{q-1} \\
&\quad +r^{\alpha_2(q-1)}\|a_1\|_{L_1}^{q-1}\|q\|_{L_1}^{q-1} + \|a_1\|_{L_1}^{q-1}\|h\|_{L_1}^{q-1}) := l_1.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\|N_2(u,v)\| &\leq \frac{\max\{2^{\tilde{q}-1},1\}}{(\alpha-\beta)\Gamma(\alpha)}\Big(r^{\alpha_3(\tilde{q}-1)}\|a_2\|_{L_1}^{\tilde{q}-1}\|\breve{p}\|_{L_1}^{\tilde{q}-1} + r^{\alpha_4(\tilde{q}-1)}\|a_2\|_{L_1}^{\tilde{q}-1}\|\breve{q}\|_{L_1}^{\tilde{q}-1} \\
&\quad +\|a_2\|_{L_1}^{\tilde{q}-1}\|\breve{h}\|_{L_1}^{\tilde{q}-1}\Big) + \frac{\max\{2^{\tilde{q}-1},1\}\eta^{\alpha-\beta-1}\gamma}{(1-\gamma\eta^{\alpha-\beta-1})(\alpha-\beta)\Gamma(\alpha)}\Big(r^{\alpha_3(\tilde{q}-1)}\|a_2\|_{L_1}^{\tilde{q}-1}\|\breve{p}\|_{L_1}^{\tilde{q}-1} \\
&\quad +r^{\alpha_4(\tilde{q}-1)}\|a_2\|_{L_1}^{\tilde{q}-1}\|\breve{q}\|_{L_1}^{\tilde{q}-1} + \|a_2\|_{L_1}^{\tilde{q}-1}\|\breve{h}\|_{L_1}^{\tilde{q}-1}\Big) := l_2.
\end{aligned}$$

Step 3 N maps bounded sets into equicontinuous. Let $u \in B_r$, be a bounded set as in Step 2, $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, from (4.1.6) and lemma 1.5.13 we have

$$
\begin{aligned}
|N_1(u,v)(t_2)-N_1(u,v)(t_1)| &\le \int_0^1 |G_1(t_2,s)-G_1(t_1,s)|\varphi_q(\int_0^s |a_1(\tau)\\
&\times f(u(\theta_1(\tau)),v(\theta_2(\tau)))d\tau)|ds) + \frac{\gamma|t_2-t_1|^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1 |G_2(\eta,s)|\\
&\times \varphi_q(\int_0^s |a_1(\tau)f(u(\theta_1(\tau)),v(\theta_2(\tau)))d\tau)|ds\\
&\le \int_0^1 |G_1(t_2,s)-G_1(t_1,s)|\int_0^s |a_1(\tau)[p(\tau)|u(\theta_1|^{\alpha_1}\\
&+ q(\tau)|v(\theta_2(\tau))|^{\alpha_2}+h(\tau)d\tau ds]|^{q-1} + \frac{\gamma|t_2-t_1|^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1 |G_2(\eta,s)|\\
&\times \int_0^s |a_1(\tau)[p(\tau)|u(\theta_1|^{\alpha_1}+q(\tau)|v(\theta_2(\tau))|^{\alpha_2}+h(\tau)d\tau ds]|^{q-1}\\
&\le \max\{2^{q-1},1\}\int_0^1 |G_1(t_2,s)-G_1(t_1,s)|\int_0^s |a_1(\tau)|^{q-1}|p(\tau)|^{q-1}\\
&\quad |u(\theta_1(\tau))|^{\alpha_1(q-1)}+|a_1(\tau)|^{q-1}|q(\tau)|^{q-1}|v(\theta_2(\tau))|^{\alpha_2(q-1)}+|a_1(\tau)|^{q-1}\\
&\quad |h(\tau)|^{q-1}d\tau ds+\max\{2^{q-1},1\}\frac{\gamma|t_2-t_1|^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1 |G_2(\eta,s)|\\
&\quad \int_0^s |a_1(\tau)|^{q-1}|p(\tau)|^{q-1}u(\theta_1(\tau))|^{\alpha_1(q-1)}\\
&\quad +|a_1(\tau)|^{q-1}|q(\tau)|^{q-1}|v(\theta_2(\tau))|^{\alpha_2(q-1)}+|a_1(\tau)|^{q-1}|h(\tau)|^{q-1}d\tau ds.
\end{aligned}
$$

By lemma 1.5.14 we obtain

$$
\begin{aligned}
|N_1(u,v)(t_2)-N_1(u,v)(t_1)| &\le \max\{2^{q-1},1\}(r^{\alpha_1(q-1)}\|a_1\|_{L_1}^{q-1}\|p\|_{L_1}^{q-1}+r^{\alpha_2(q-1)}\|a_1\|_{L_1}^{q-1}\|q\|_{L_1}^{q-1}\\
&\quad +\|a_1\|_{L_1}^{q-1}\|h\|_{L_1}^{q-1})\int_0^1 |G_1(t_1,s)-G_1(t_2,s)|ds\\
&\quad +\frac{\max\{2^{q-1},1\}\gamma(\alpha-1)|t_2-t_1|}{(1-\gamma\eta^{\alpha-\beta-1})}(r^{\alpha_1(q-1)}\|a_1\|_{L_1}^{q-1}\|p\|_{L_1}^{q-1}\\
&\quad +r^{\alpha_2(q-1)}\|a_1\|_{L_1}^{q-1}\|q\|_{L_1}^{q-1}+\|a_1\|_{L_1}^{q-1}\|h\|_{L_1}^{q-1})\int_0^1 |G_2(\eta,s)|ds.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
|N_2(u,v)(t_1)-N_2(u,v)(t_2)| \leq\ & \max\{2^{\tilde{q}-1},1\}(r^{\alpha_3(q-1)}\|a_2\|_{L_1}^{\tilde{q}-1}\|\breve{p}\|_{L_1}^{\tilde{q}-1})\int_0^1 |G_1(t_1,s)\\
& -G_1(t_2,s)|ds+\max\{2^{\tilde{q}-1},1\}(r^{\alpha_4(\tilde{q}-1)}\|a_2\|_{L_1}^{\tilde{q}-1}\|\breve{q}\|_{L_1}^{\tilde{q}-1}\\
& +\|a_2\|_{L_1}^{\tilde{q}-1}\|\breve{h}\|_{L_1}^{\tilde{q}-1})\int_0^1 |G_1(t_1,s)-G_1(t_2,s)|ds\\
& +\frac{\max\{2^{\tilde{q}-1},1\}\gamma(\alpha-1)|t_2-t_1|}{(1-\gamma\eta^{\alpha-\beta-1})}(r^{\alpha_3(\tilde{q}-1)}\|a_2\|_{L_1}^{\tilde{q}-1}\|\breve{p}\|_{L_1}^{\tilde{q}-1}\\
& +r^{\alpha_4(\tilde{q}-1)}\|a_2\|_{L_1}^{\tilde{q}-1}\|\breve{q}\|_{L_1}^{\tilde{q}-1}+\|a_2\|_{L_1}^{\tilde{q}-1}\|\breve{h}\|_{L_1}^{\tilde{q}-1})\\
& \times\int_0^1 |G_2(\eta,s)|ds
\end{aligned}
$$

The continuity of G_1 implies that the right-side of the above inequality tends to zero if $t_2 \to t_1$. Therefore, by Arzela-Ascoli N is completely continuous.

Step 4 A priori bounds. Now it remains to show that the set

$$\mathcal{M}=\{(u,v)\in C([0,1],\mathbb{R})\times C([0,1],\mathbb{R}):(u,v)=\lambda N(u,v)\ \ 0<\lambda<1\},$$

is bounded. Let $(u,v)\in\mathcal{M}$, then there exists $0<\lambda<1$ such that $u=\lambda N_1(u,v)$ and $v=\lambda N_2(u,v)$. Thus, for $t\in[0,1]$, we have

$$
\begin{aligned}
|u(t)| \leq\ & \int_0^1 |G_1(t,s)|\varphi_q(\int_0^s |a_1(\tau)f(u(\theta_1(\tau)),v(\theta_2(\tau)))d\tau)|ds\\
& +\frac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1 |G_2(\eta,s)|\varphi_q(\int_0^s |a_1(\tau)f(u(\theta_1(\tau)),v(\theta_2(\tau)))d\tau)|ds\\
\leq\ & \max\{2^{q-1},1\}[\|p\|_{L^1}^{q-1}\|a_1\|_{L^1}^{q-1}\|u\|^{\alpha_1(q-1)}+\|q\|_{L^1}^{q-1}\|a_1\|_{L^1}^{q-1}\|v\|^{\alpha_2(q-1)}\\
& +\|a_1\|_{L^1}^{q-1}\|h\|_{L^1}^{q-1}]\int_0^1 G_1(1,s)ds+\frac{\gamma}{1-\gamma\eta^{\alpha-\beta-1}}[\|p\|_{L^1}^{q-1}\|a_1\|_{L^1}^{q-1}\|u\|^{\alpha_1(q-1)}\\
& +\|q\|_{L^1}^{q-1}\|a_1\|_{L^1}^{q-1}\|v\|^{\alpha_2(q-1)}+\|a_1\|_{L^1}^{q-1}\|h\|_{L^1}^{q-1}]\int_0^1 |G_2(\eta,s)|ds\\
\|u\| \leq\ & \max\{2^{q-1},1\}[\|p\|_{L^1}^{q-1}\|a_1\|_{L^1}^{q-1}\|u\|^{\alpha_1(q-1)}+\|q\|_{L^1}^{q-1}\|a_1\|_{L^1}^{q-1}\|v\|^{\alpha_2(q-1)}\\
& +\|a_1\|_{L^1}^{q-1}\|h\|_{L^1}^{q-1}]\left[\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}+\frac{\gamma}{1-\gamma\eta^{\alpha-\beta-1}}\frac{\eta^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}ds\right]\\
\|u\| \leq\ & \max\{2^{q-1},1\}[\|p\|_{L^1}^{q-1}\|a_1\|_{L^1}^{q-1}\|u\|^{\alpha_1(q-1)}+\|q\|_{L^1}^{q-1}\|a_1\|_{L^1}^{q-1}\|v\|^{\alpha_2(q-1)}\\
& +\|a_1\|_{L^1}^{q-1}\|h\|_{L^1}^{q-1}]\frac{1}{(1-\gamma\eta^{\alpha-\beta-1})\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\beta-1}ds.
\end{aligned}
$$

Hence,

$$\begin{aligned}\|u\| &\leq \frac{\max\{2^{q-1},1\}}{(\alpha-\beta)(1-\gamma\eta^{\alpha-\beta-1})\Gamma(\alpha)}[\|p\|_{L^1}^{q-1}\|a_1\|_{L^1}^{q-1}\|u\|^{\alpha_1(q-1)}+\|q\|_{L^1}^{q-1}\|a_1\|_{L^1}^{q-1}\|v\|^{\alpha_2(q-1)}\\ &+\|a_1\|_{L^1}^{q-1}\|h\|_{L^1}^{q-1}].\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\|v\| &\leq \frac{\max\{2^{q-1},1\}}{(\alpha-\beta)(1-\gamma\eta^{\alpha-\beta-1})\Gamma(\alpha)}[\|\breve{p}\|_{L^1}^{\tilde{q}-1}\|a_2\|_{L^1}^{\tilde{q}-1}\|u\|^{\alpha_3(\tilde{q}-1)}+\|\breve{q}\|_{L^1}^{\tilde{q}-1}\|a_2\|_{L^1}^{\tilde{q}-1}\|v\|^{\alpha_4(\tilde{q}-1)}\\ &+\|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{h}\|_{L^1}^{\tilde{q}-1}]\end{aligned}$$

Notice that if $\epsilon \leq \delta$ and $\|u\| > 1$, then $\|u\|^{\epsilon} \leq \|u\|^{\delta}$ Thus,$\|u\|^{\epsilon} \leq 1+\|u\|^{\delta}$ for all u. We then have

$$\begin{aligned}\|u\|+\|v\| &\leq \frac{\max\{2^{q-1},1\}}{(\alpha-\beta)(1-\gamma\eta^{\alpha-\beta-1})\Gamma(\alpha)}[\|p\|_{L^1}^{q-1}\|a_1\|_{L^1}^{q-1}\|u\|^{\alpha_1(q-1)}\\ &+\|q\|_{L^1}^{q-1}\|a_1\|_{L^1}^{q-1}\|v\|^{\alpha_2(q-1)}+\|a_1\|_{L^1}^{q-1}\|h\|_{L^1}^{q-1}]\\ &+\frac{\max\{2^{q-1},1\}\varphi_q(\int_0^1 a_1(\tau)d\tau)}{(\alpha-\beta)(1-\gamma\eta^{\alpha-\beta-1})\Gamma(\alpha)}[\|\breve{p}\|_{L^1}^{\tilde{q}-1}\|a_2\|_{L^1}^{\tilde{q}-1}\|u\|^{\alpha_3(\tilde{q}-1)}\\ &+\|\breve{q}\|_{L^1}^{\tilde{q}-1}\|a_2\|_{L^1}^{\tilde{q}-1}\|v\|^{\alpha_4(\tilde{q}-1)}+\|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{h}\|_{L^1}^{\tilde{q}-1}]\\ &\leq \frac{\max\{2^{q-1},1\}}{(\alpha-\beta)(1-\gamma\eta^{\alpha-\beta-1})\Gamma(\alpha)}(\|p\|_{L^1}^{q-1}\|a_1\|_{L^1}^{q-1}+\|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{q}\|_{L^1}^{\tilde{q}-1})\times\\ &(\|u\|^{\alpha_1(q-1)}+\|v\|^{\alpha_4(\tilde{q}-1)})\\ &+(\|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{p}\|_{L^1}^{\tilde{q}-1}+\|a_1\|_{L^1}^{q-1}\|q\|_{L^1}^{q-1})(\|u\|^{\alpha_3(\tilde{q}-1)}+\|v\|^{\alpha_2(q-1)})\\ &+(\|a_1\|_{L^1}^{q-1}\|h\|_{L^1}^{q-1}+\|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{h}\|_{L^1}^{\tilde{q}-1})\\ &\leq \frac{\max\{2^{q-1},1\}}{(\alpha-\beta)(1-\gamma\eta^{\alpha-\beta-1})\Gamma(\alpha)}\Big(\|a_1\|_{L^1}^{q-1}\|p\|_{L^1}^{q-1}+\|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{q}\|_{L^1}^{\tilde{q}-1}\\ &+\|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{p}\|_{L^1}^{\tilde{q}-1}+\|a_1\|_{L^1}^{q-1}\|q\|_{L^1}^{q-1}\Big)(\|u\|^{\alpha_\star}+\|v\|^{\alpha_\star})\\ &+(\|a_1\|_{L^1}^{q-1}\|h\|_{L^1}^{q-1}+\|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{h}\|_{L^1}^{\tilde{q}-1})\\ &\leq \frac{\max\{2^{q-1},1\}}{(\alpha-\beta)(1-\gamma\eta^{\alpha-\beta-1})\Gamma(\alpha)}(\|p\|_{L^1}^{q-1}\|a_1\|_{L^1}^{q-1}+\|\breve{q}\|_{L^1}^{\tilde{q}-1}\|a_2\|_{L^1}^{\tilde{q}-1}+\|\breve{p}\|_{L^1}^{\tilde{q}-1}\\ &+\|a_1\|_{L^1}^{q-1}\|q\|_{L^1}^{q-1})(\|u\|+\|v\|)^{\alpha_\star}+(\|a_1\|_{L^1}^{q-1}\|h\|_{L^1}^{q-1}+\|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{h}\|_{L^1}^{\tilde{q}-1})\end{aligned}$$

where

$$\alpha_\star = \max\{\alpha_1(q-1),\alpha_2(q-1),\alpha_3(\tilde{q}-1),\alpha_4(\tilde{q}-1)\}$$

If $\|u\| + \|v\| > 1$, then

$$\begin{aligned}\frac{\|u\|+\|v\|}{(\|u\|+\|v\|)^{\alpha_\star}} \quad \leq \quad & \frac{\max\{2^{q-1},1\}}{(\alpha-\beta)(1-\gamma\eta^{\alpha-\beta-1})\Gamma(\alpha)}(\|a_1\|_{L^1}^{q-1}\|p\|_{L^1}^{q-1} + \|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{q}\|_{L^1}^{\tilde{q}-1} \\ & +\|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{p}\|_{L^1}^{\tilde{q}-1} + \|a_1\|_{L^1}^{q-1}\|q\|_{L^1}^{q-1}) \\ & +\frac{(\|a_1\|_{L^1}^{q-1}\|h\|_{L^1}^{q-1} + \|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{h}\|_{L^1}^{\tilde{q}-1})}{(\|u\|+\|v\|)^{\alpha_\star}}\end{aligned}$$

or

$$\begin{aligned}(\|u\|+\|v\|)^{1-\alpha_\star} \quad \leq \quad & \frac{\max\{2^{q-1},1\}}{(\alpha-\beta)(1-\gamma\eta^{\alpha-\beta-1})\Gamma(\alpha)}(\|a_1\|_{L^1}^{q-1}\|p\|_{L^1}^{q-1} + \|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{q}\|_{L^1}^{\tilde{q}-1} \\ & +\|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{p}\|_{L^1}^{\tilde{q}-1} + \|a_1\|_{L^1}^{q-1}\|q\|_{L^1}^{q-1}) + (\|a_1\|_{L^1}^{q-1}\|h\|_{L^1}^{q-1} \\ & +\|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{h}\|_{L^1}^{\tilde{q}-1})\end{aligned}$$

This implies that

$$\begin{aligned}\|u\|+\|v\| \leq [A(\|a_1\|_{L^1}^{q-1}\|p\|_{L^1}^{q-1}+\|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{q}\|_{L^1}^{\tilde{q}-1}+\|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{p}\|_{L^1}^{\tilde{q}-1}+\|a_1\|_{L^1}^{q-1}\|q\|_{L^1}^{q-1}) \\ +(\|a_1\|_{L^1}^{q-1}\|h\|_{L^1}^{q-1} + \|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{h}\|_{L^1}^{\tilde{q}-1})]^{1-\alpha_\star}\end{aligned}$$

then

$$\|u\| + \|v\| \leq [AB + C]^{1-\alpha_\star} := M_2$$

where

$$A = \frac{\max\{2^{q-1},1\}}{(\alpha-\beta)(1-\gamma\eta^{\alpha-\beta-1})\Gamma(\alpha)},$$

$$B = \|a_1\|_{L^1}^{q-1}\|p\|_{L^1}^{q-1} + \|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{q}\|_{L^1}^{\tilde{q}-1} + \|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{p}\|_{L^1}^{\tilde{q}-1} + \|a_1\|_{L^1}^{q-1}\|q\|_{L^1}^{q-1}$$

and

$$C = \|a_1\|_{L^1}^{q-1}\|h\|_{L^1}^{q-1} + \|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{h}\|_{L^1}^{\tilde{q}-1}.$$

As a consequence of Theorem 1.2.4, the operator N has a fixed point that is a solution of system (4.1.1). This completes the proof of the theorem. □

Example 4.1.1. *Consider the fractional differential equation with advanced argument for p-Laplacian*

$$\begin{cases} \varphi_{3/2}(D_{0^+}^{5/2}u(t))' + \frac{t^{-1/2}}{4}\frac{t}{1+t}(|u(\theta(t))|^{\frac{1}{4}} + |v(\theta(t))|^{\frac{1}{5}}) = 0, & 0 < t < 1 \\ \varphi_{3/2}(D_{0^+}^{5/2}v(t))' + \frac{7t^{-1/2}}{2}\frac{t^2}{1+t^2}(13 + |v(\theta(t))|^{1/4} + |u(\theta(t))|^{\frac{1}{6}}) = 0, & 0 < t < 1 \\ D_{0^+}^{5/2}u(0) = u(0) = u'(0) = 0, \quad D_{0^+}^{7/6}u(1) = \frac{7}{10}D_{0^+}^{7/6}u(\frac{1}{2}), & \\ D_{0^+}^{5/2}v(0) = v(0) = v'(0) = 0, \quad D_{0^+}^{7/6}v(1) = \frac{7}{10}D_{0^+}^{7/6}v(\frac{1}{2}), & \end{cases} \tag{4.1.11}$$

where $\alpha = \frac{3}{2}, \beta = \frac{7}{6}, \eta = \frac{7}{10}, p = \tilde{p} = \frac{3}{2}, q = \tilde{q} = 3, a_1(t) = \frac{t^{-1/2}}{4}, a_2(t) = \frac{7t^{-1/2}}{2}, \varphi_3\left(\int_0^1 a_1(t)dt\right) = \frac{1}{4}, \varphi_3\left(\int_0^1 a_2(t)dt\right) = \frac{\sqrt{7}}{2},\ \alpha_1 p = \alpha_3\tilde{p} = \frac{3}{8} \in (0,1)$, $\alpha_2 p = \frac{3}{10} \in (0,1),\ \alpha_4\tilde{p} = \frac{3}{12} \in (0,1)$

$$f(u(\theta(t)), v(\theta(t))) = \frac{t}{1+t}(|u(\theta(t))|^{\frac{1}{4}} + |v(\theta(t))|^{\frac{1}{5}}), \quad \theta(t) = t^\gamma,\ \gamma \in (0,1)$$

and

$$g(u(\theta(t)), v(\theta(t))) = \frac{t^2}{1+t^2}(13+|v(\theta(t))|^{1/4}+|u(\theta(t))|^{\frac{1}{6}}),\ \theta(t) = t^\gamma,\ \gamma \in (0,1)$$

It is clear that, for all $(t,u,v) \in [0,1] \times \mathbb{R}^2$,

$$\begin{cases} |f(u,v)| & \leq & t(|u|^{\frac{1}{4}} + |v|^{\frac{1}{5}}) \\ |g(u,v)| & \leq & t^2(13 + |v|^{1/4} + |u|^{\frac{1}{6}}). \end{cases}$$

Hence all the conditions of Theorem 4.1.3, hold, this implies that the problem (4.1.11) has at leat one solution.

4.2 Positive solutions

In this section, our goal is to establish positive solutions for the problem to the system (4.1.1). To this end, we first in this section we assumed the functions $f, g \in C(\mathbb{R}^2, \mathbb{R}_+)$ and define the operator on P as $N : P \times P \to P \times P$ be the completely continuous map $N = (N_1, N_2)$ given in the proof of theorem 4.1.3. Then (4.1.6) and (4.1.7)are equivalent to the fixed point problem

$$u = N(u) \quad u \in P^2.$$

If $v \in P$ and

$$u_i(t) = \int_0^1 G_1(t,s)\varphi_q(\int_0^s a_1(\tau)v(\tau)d\tau)ds$$

$$+\frac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1 G_2(\eta,s)\varphi_q(\int_0^s a_1(\tau)v(\tau)d\tau)ds, \quad i = 1,2.$$

and $u_i(t_i) = \|u_i\|$, by lemma 4.1.2 imply that, for any $t \in [\rho, 1]$,

$$\begin{aligned}
u_i(t) &= \int_0^1 G_1(t,s)\varphi_q(\int_0^s a_1(\tau)v(\tau)d\tau)ds \\
&\quad + \frac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}} \int_0^1 G_2(\eta,s)\varphi_q(\int_0^s a_1(\tau)v(\tau)d\tau)ds \\
u_i(t) &\geq \int_0^1 \min G_1(t,s)\varphi_q(\int_0^s a_1(\tau)v(\tau)d\tau)ds \\
&\quad + \frac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}} \int_0^1 G_2(\eta,s)\varphi_q(\int_0^s a_1(\tau)v(\tau)d\tau)ds \\
&\geq \int_0^1 \rho^{\alpha-1} G_1(1,s)\varphi_q(\int_0^s a_1(\tau)v(\tau)d\tau)ds \\
&\quad + \frac{\gamma \rho^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}} \int_0^1 G_2(\eta,s)\varphi_q(\int_0^s a_1(\tau)v(\tau)d\tau)ds \\
&\geq \rho^{\alpha-1}\left[\int_0^1 G_1(1,s)\varphi_q(\int_0^s a_1(\tau)v(\tau)d\tau)ds\right. \\
&\quad \left. + \frac{\gamma}{1-\gamma\eta^{\alpha-\beta-1}} \int_0^1 G_2(\eta,s)\varphi_q(\int_0^s a_1(\tau)v(\tau)d\tau)ds\right] \\
&\geq \rho^{\alpha-1}\left[\int_0^1 G_1(t,s)\varphi_q(\int_0^s a_1(\tau)v(\tau)d\tau)ds\right. \\
&\quad \left. + \frac{\gamma}{1-\gamma\eta^{\alpha-\beta-1}} \int_0^1 G_2(\eta,s)\varphi_q(\int_0^s a_1(\tau)v(\tau)d\tau)ds\right] \\
u_i(t) &\geq \rho^{\alpha-1}\|u_i\|.
\end{aligned}$$

Define the cone P_i for $i = 1, 2$ in P by

$$P_i = \left\{u_i \in P : u_i(t) \geq \rho^{\alpha-1}\|u_i\|, \text{ for all } t \in [\rho, 1]\right\},$$

and the product cone $P = P_1 \times P_2$, then $N(P) \subset P$. Before we state our main result we introduce the following notations: $\alpha_i, \beta_i > 0$ with $\alpha_i \neq \beta_i$ we let $r_i = \min\{\alpha_i, \beta_i\}$, $R_i = \max\{\alpha_i, \beta_i\}$ $i = 1, 2$.

$\gamma_1 = \min\{f(u_1(\theta_1(t)), u_2(\theta_1(t))) : \rho \leq t \leq 1, \rho^{\alpha-1}\beta_1 \leq u_1 \leq \beta_1, \rho^{\alpha-1}r_2 \leq u_2 \leq R_2\}$,

$\gamma_2 = \min\{g(u_1(\theta_1(t)), u_2(\theta_1(t))) : \rho \leq t \leq 1, \rho^{\alpha-1}r_1 \leq u_1 \leq R_1, \rho^{\alpha-1}\beta_2 \leq u_2 \leq \beta_2\}$,

$\Gamma_1 = \max\{f(u_1(\theta_1(t)), u_2(\theta_1(t))) : \rho \leq t \leq 1, \rho^{\alpha-1}\alpha_1 \leq u_1 \leq \alpha_1, \rho^{\alpha-1}r_2 \leq u_2 \leq R_2\}$,

$\Gamma_2 = \max\{g(u_1(\theta_1(t)), u_2(\theta_1(t))) : \rho \le t \le 1, \rho^{\alpha-1}r_1 \le u_1 \le R_1, \rho^{\alpha-1}\alpha_2 \le u_2 \le \alpha_2\}.$

Also, let
$$A = \min\{G_1(t,s) : \rho \le t \le 1, 0 \le s \le 1\},$$
and
$$B = \max\{G_1(t,s) : \rho \le t \le 1, 0 \le s \le 1\}.$$

Theorem 4.2.1. *Assume that there exist $\alpha_i, \beta_i > 0$ with $\alpha_i \ne \beta_i, i = 1, 2$, such that*
$$\begin{array}{lcl} B\Gamma_1^{q-1} \le \alpha_1 & , & A\gamma_1^{q-1} \ge \beta_1, \\ B\Gamma_2^{q-1} \le \alpha_2 & , & A\gamma_2^{q-1} \ge \beta_2. \end{array} \tag{4.2.1}$$

Then (4.1.1) has a positive solution $u = (u_1, u_2)$ with $r_i \le \|u_i\| \le R_i, i = 1, 2$, where $r_i = \min\{\alpha_i, \beta_i\}$, $R_i = \max\{\alpha_i, \beta_i\}$. Moreover, corresponding orbit of u is included in the rectangle $[\rho r_1, R_1] \times [\rho r_2, R_2]$.

Proof. First note that if $u \in P_{r,R}$, then $r_1 \le \|u_1\| \le R_1$ and $r_2 \le \|u_2\| \le R_2$ and by the definition of P,
$$\{\rho^{\alpha-1}r_1 \le u_1(t) \le R_1 \text{ and } \rho^{\alpha-1}r_2 \le u_2(t) \le R_2\},$$
for all t, showing that the orbit of u for $t \in [\rho, 1]$ is included in the rectangle $[\rho r_1, R_1] \times [\rho r_2, R_2]$.
Also, if we know for example that $\|u_1\| = \alpha_1$, then
$$\rho^{\alpha-1}\alpha_1 \le u_1(t) \le \alpha_1.$$

We now prove that for every $u \in P_{r,R}$ and $i \in \{1, 2\}$, the following properties holds:
$$\begin{array}{lll} \|u_i\| = \alpha_i & \text{implies} & u_i \not< N_i(u), \\ \|u_i\| = \beta_i & \text{implies} & u_i \not> N_i(u), \end{array} \tag{4.2.2}$$

guaranteeing the applicability of Theorem 1.2.5. Indeed, if $\|u_1\| = \alpha_1$ and we would have $u_1 \prec N_1(u)$, then

$$\begin{aligned}
u_1(t) &< N_1(u)(t) \\
&\leq \int_0^1 \max G_1(t,s) \max |a_1(t) f(u(\theta_1(t)), v(\theta_2(t)))|^{q-1} dt \\
&\quad + \frac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}} \int_0^1 G_2(\eta,s) \max |a_1(t) f(u(\theta_1(t)), v(\theta_2(t)))|^{q-1} dt \\
&\leq B\Gamma_1^{q-1} + \Gamma_1^{q-1} \frac{\gamma}{1-\gamma\eta^{\alpha-\beta-1}} \int_0^1 G_2(\eta,s) \\
&\leq B\Gamma_1^{q-1} \\
&\leq \alpha_1
\end{aligned}$$

for all t. This yields the contradiction $\alpha_1 < \alpha_1$.
Now if $\|u_1\| = \beta_1$ and $u_1 \succ N_1(u)$, then for $t \in [\rho, 1]$, we obtain

$$\begin{aligned}
u_1(t) &> N_1(u)(t) \\
&\geq \int_0^1 \min G_1(t,s) \min |a_1(t) f(u(\theta_1(t)), v(\theta_2(t)))|^{q-1} dt \\
&\quad + \frac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}} \int_0^1 G_2(\eta,s) \min |a_1(t) f(u(\theta_1(t)), v(\theta_2(t)))|^{q-1} dt \\
&\geq A\gamma_1^{q-1} \\
&\geq \beta_1.
\end{aligned}$$

Then we deduce that $\beta_1 > \beta_1$, which is a contradiction. Hence (4.2.2) holds for $i = 1$. Similarly, (4.2.2) is true for $i = 2$. By Theorem 1.2.5, we see that N has at least one fixed point in P. Therefore, system (4.1.1) has at least one positive solution. □

Example 4.2.1. *Consider the fractional differential equation with advanced argument for p-Laplacian*

$$\begin{cases}
\varphi_{3/2}(D_{0^+}^{5/2} u(t))' + \frac{t^{-1/2}}{4} f(u(\theta(t)), v(\theta(t))) = 0, & 0 < t < 1 \\
\varphi_{3/2}(D_{0^+}^{5/2} v(t))' + \frac{7t^{-1/2}}{2} g(u(\theta(t)), v(\theta(t))) = 0, & 0 < t < 1 \\
D_{0^+}^{5/2} u(0) = u(0) = u'(0) = 0, \quad D_{0^+}^{7/6} u(1) = \frac{7}{10} D_{0^+}^{7/6} u(\frac{1}{2}), & \\
D_{0^+}^{5/2} v(0) = v(0) = v'(0) = 0, \quad D_{0^+}^{7/6} v(1) = \frac{7}{10} D_{0^+}^{7/6} v(\frac{1}{2}), &
\end{cases} \tag{4.2.3}$$

where $f, g \in C(\mathbb{R}^2, \mathbb{R}_+)$ *are nondecreasing in* u *and* v, $\theta(t) = t^\gamma,\ \gamma \in (0,1)$. *Assume that*

$$\lim_{z\to\infty} \frac{f(z,z)}{z} = \lim_{z\to\infty} \frac{g(z,z)}{z} = 0 \tag{4.2.4}$$

and

$$\lim_{z\to 0} \frac{f(z,z)}{z} = \lim_{z\to 0} \frac{g(z,z)}{z} = \infty. \tag{4.2.5}$$

From the conditions (4.2.4) *and* (4.2.5), *we can prove that there exist* $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$,$\alpha_1 < \beta_1, \alpha_2 = \beta_1$ *and* $\beta_2 = \alpha_1$ *such that*

$$\begin{aligned} \tfrac{f(\rho^{\alpha-1}\beta_1, \rho^{\alpha-1}\alpha_1)}{\rho^{\alpha-1}\beta_1} &\geq \tfrac{1}{\rho^{\alpha-1}A}, \\ \tfrac{g(\rho^{\alpha-1}\alpha_1, \rho^{\alpha-1}\alpha_1)}{\rho^{\alpha-1}\beta_2} &\geq \tfrac{1}{\rho^{\alpha-1}A} \end{aligned} \tag{4.2.6}$$

and

$$\frac{f(\alpha_1, \beta_1)}{\alpha_1} \leq \frac{1}{B}, \quad \frac{g(\beta_1, \beta_1)}{\alpha_2} \leq \frac{1}{B}. \tag{4.2.7}$$

Then, we sets

$$r_i = \alpha_1,\ R_i = \beta_1 \ \textit{for}\ i \in \{1,2\},$$

$$\Gamma_1 = f(\alpha_1,\ \beta_1),\ \Gamma_2 = g(\beta_1, \beta_1),$$

and

$$\gamma_1 = f(\rho^{\alpha-1}\beta_1, \rho^{\alpha-1}\alpha_1),\ \gamma_2 = g(\rho^{\alpha-1}\alpha_1, \rho^{\alpha-1}\alpha_1).$$

We concluded that, (4.2.6) and (4.2.7) guarantees (4.2.1). Hence by theorem 4.2.1, the problem (4.2.3) has a positive solution.

4.3 Multiple positive solutions

Now we study the existence of multiple positive solutions for the systems of fractional boundary value problem with p-laplacian boundary conditions

(H_4) f, g are positive and increasing, i.e.

$$0 \leq u \leq x, 0 \leq v \leq y \ \text{imply}\ 0 \leq f(u,v) \leq f(x,y), 0 \leq g(u,v) \leq g(x,y)$$

We present the following general existence, multiplicity and localization result.

Theorem 4.3.1. *Let the conditions* $(H_1)-(H_2)-(H_4)$ *hold and assume that the norm* $\|\cdot\|$ *is monotone with respect to each cone* $P_i(i=1,2)$. *Moreover, suppose that there exist* $\alpha_i, \beta_i > 0$, *with* $\alpha_i \neq \beta_i, i=1,2$, *such that*

$$\|N_1(\alpha_1\rho^{\alpha-1}, R_2\rho^{\alpha-1})\| < \alpha_1, \quad \|N_2(R_1\rho^{\alpha-1}, \alpha_2\rho^{\alpha-1})\| < \alpha_2, \tag{4.3.1}$$

$$\|N_1(\beta_1\rho^{\alpha-1}, 0)\| > \beta_1, \quad \|N_2(0, \beta_2\rho^{\alpha-1})\| < \beta_2, \tag{4.3.2}$$

where $R_i = \max\{\alpha_i, \beta_i\}(i=1,2)$.
Then problem (4.1.1) has at least

(1) *one solution* $u=(u_1,u_2)$ *such that* $\beta_i < \|u_i\| < \alpha_i$, *for* $i=1,2$, *if* $\alpha_i > \beta_i$ *for* $i=1,2$;

(2) *two solutions* (u_1,u_2) *and* (v_1,v_2) *such that* $\beta_1 < \|u_1\| < \alpha_1$, $\beta_2 < \|u_2\| < \alpha_2$, $\beta_1 < \|v_1\| < \alpha_1$ *and* $\|v_2\| < \alpha_2$ *if* $\alpha_1 > \beta_1$ *and* $\alpha_2 < \beta_2$;

(3) *two solutions* (u_1,u_2) *and* (v_1,v_2) *such that* $\alpha_1 < \|u_1\| < \beta_1$, $\alpha_2 < \|u_2\| < \beta_2$, $\|v_1\| < \alpha_1$ *and* $\beta_2 < \|v_2\| < \alpha_2$ *if* $\alpha_1 < \beta_1$ *and* $\alpha_2 > \beta_2$;

(4) *four solutions* $(u_1,u_2),(v_1,v_2),(w_1,w_2)$ *and* (z_1,z_2) *such that* $\beta_i < \|u_i\| < \alpha_i, \alpha_1 < \|v_1\| < \beta_1$, *and* $\|v_2\| < \alpha_2, \|w_1\| < \alpha_1, \alpha_2 < \|w_1\| < \beta_2$, *and* $\|z_i\| < \alpha_i$, *if* $\alpha_i < \beta_i$ *for* $i=1,2$.

Proof. We shall apply Theorem 1.2.6 to the operator $N=(N_1,N_2)$ defined as in (4.1.9) and (4.1.10). Let us see that it satisfies conditions (1.2.1)-(1.2.2). First we prove that

$$\lambda u_1 \neq N_1(u) \text{ for every } \ u \in k \text{ with } \|u_1\| = \alpha_1, \|u_2\| \leq R_2 \ \text{ and all } \lambda \geq 1 \tag{4.3.3}$$

Indeed, if not,

$$\lambda\|u_1\| = \lambda\alpha_1 = \|N_1(u)\|.$$

From $0 \leq u_1 \leq \alpha_1\rho^{\alpha-1}$ and $0 \leq u_2 \leq R_2\rho^{\alpha-1}$, by $(H_1),(H_4)$ it follows that

$$0 \leq f(u_1,u_2) \leq f(\alpha_1\rho^{\alpha-1}, R_2\rho^{\alpha-1}),$$

$$0 \leq \varphi_q\left(\int_0^s a_1(\tau)f(u_1,u_2)d\tau\right) \leq \varphi_q\left(\int_0^s a_1(\tau)f(\alpha_1\rho^{\alpha-1}, R_2\rho^{\alpha-1})d\tau ds\right),$$

by lemma (4.1.2) we obtain,

$$0 \leq N_1(u_1, u_2) \leq N_1(\alpha_1 \rho^{\alpha-1}, R_2 \rho^{\alpha-1})$$

and the norm of X being monotone,

$$\|N_1(u_1, u_2)\| \leq \|N_1(\alpha_1 \rho^{\alpha-1}, R_2 \rho^{\alpha-1})\|.$$

By assumption (4.1.9),

$$\|N_1(\alpha_1 \rho^{\alpha-1}, R_2 \rho^{\alpha-1})\| < \alpha_1,$$

so we obtain the contradiction

$$\lambda \alpha_1 < \alpha_1 \text{ for some } \lambda \geq 1.$$

Hence (4.3.3) holds.
Now, we prove that
$u_1 \neq N_1(u) + \mu \rho^{\alpha-1}$ for every $u \in P$ with $\|u_1\| = \beta_1, \|u_2\| \leq R_2$ and all $\mu \geq 0$.
Assume the contrary, i.e.,$u_1 = N_1(u) + \mu \rho^{\alpha-1}$for some $u \in P$ with $\|u_1\| = \beta_1, \|u_2\| \leq R_2$ and some $\mu \geq 0$. Then $u_1 - N_1(u) \in P_1$, so $0 \leq N_1(u) \leq u_1$, and the norm of X being monotone

$$\|N_1(u)\| \leq \|u_1\| = \beta_1 \tag{4.3.4}$$

Also, from the condition (H_4), $0 \leq \beta_1 \rho^{\alpha-1} \leq u_1$ and $0 \leq u_2$, so we obtain

$$0 \leq f_1(\beta_1 \rho^{\alpha-1}, 0) \leq f(u_1, u_2)$$

then by (H_1) we obtain

$$0 \leq \varphi_q \left(\int_0^s a_1(\tau) f_1(\beta_1 \rho^{\alpha-1}, 0) d\tau \right) \leq \varphi_q \left(\int_0^s a_1(\tau) f(u_1, u_2) d\tau \right)$$

and by lemma 4.1.2 we conclude $0 \leq N_1(\beta_1 \rho^{\alpha-1}, 0) \leq N_1(u_1, u_2)$. Hence by monotone of the norm

$$\|N_1(\beta_1 \rho^{\alpha-1}, 0)\| \leq \|N_1(u_1, u_2)\|.$$

Now, from (4.3.4) we have

$$\|N_1(\beta_1 \rho^{\alpha-1}, 0)\| \leq \beta_1$$

which contradicts the assumption (4.1.10). Therefore, conditions (1.2.1)-(3.0.4) hold for i = 1. Similarly, they can be verified for $i = 2$. $\square$

Example 4.3.1. *Consider the fractional differential equation with advanced argument for p-Laplacian*

$$\begin{cases} \varphi_{3/2}(D_{0^+}^{5/2}u(t))' + \frac{t^{-1/2}}{4}f(u(\theta(t)), v(\theta(t))) = 0, & 0 < t < 1 \\ \varphi_{3/2}(D_{0^+}^{5/2}v(t))' + \frac{7t^{-1/2}}{2}g(u(\theta(t)), v(\theta(t))) = 0, & 0 < t < 1 \\ D_{0^+}^{5/2}u(0) = u(0) = u'(0) = 0, \quad D_{0^+}^{7/6}u(1) = \frac{7}{10}D_{0^+}^{7/6}u(\frac{1}{2}), \\ D_{0^+}^{5/2}v(0) = v(0) = v'(0) = 0, \quad D_{0^+}^{7/6}v(1) = \frac{7}{10}D_{0^+}^{7/6}v(\frac{1}{2}), \end{cases} \tag{4.3.5}$$

where $f, g \in C(\mathbb{R}^2, \mathbb{R}_+)$ *are nondecreasing in* u *and* v, $\theta(t) = t^\gamma$, $\gamma \in (0, 1)$.

Since f, g *are positive and increasing, we can easily shows that*

$$\frac{f(\rho^{\alpha-1}\beta_1, 0)}{\beta_1} \geq \frac{1}{A}, \quad \frac{f(\rho^{\alpha-1}\alpha_1, \rho^{\alpha-1}R_2)}{\alpha_1} < \frac{1}{B}$$
$$\frac{g(, 0, \rho^{\alpha-1}\beta_2)}{\beta_2} \leq \frac{1}{A}, \quad \frac{g(\rho^{\alpha-1}R_1, \alpha_2\rho^{\alpha-1})}{\alpha_2} < \frac{1}{B}.$$

Thus conditions (4.1.9) and (4.1.10) hold. Then by Theorem 4.3.1 the problem (4.3.5) has multiplicity solutions.

Conclusion and Perspectives

In this thesis, we studied the existence of solutions for boundary value systems for impulsive differential equation, and boundary impulsive value problem with a second-order p-Laplacian on time scale by variational methods theorems .

Also in this work we discussed some existence and multiplicity of solution for system of fractional differential equations, under various assumptions on the right hand-side nonlinearity. The main assumptions on the nonlinearity are the continuity and some Nagumo-Bernstein type growth conditions. We have used fixed point theory in vector metric spaces with general properties from functional analysis. We hope this thesis can provide contributions to the questions of existence, positivity and multiplicity of solutions for fractional differential equations on bounded domains.

We plan to look for the differential inclusions by variational methods theorems.
We will study the question of stability of this class for problem of fractional differential equation by Lyapunov method.

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