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THESE

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Intitulée

Théorèmes limites asymptotiques des estimations fonctionnelles, applications aux modèles linéaires

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Chapter 1

Introduction

The interpretation of a continuous time stochastic process as a random element in a function space has been proved to be useful in limit theory and in statistical inference for stochastic processes. Especially useful is the prediction of a continuous time random process,for knowing its values up to the present arises naturally in many applications.

1.1 Stochastics process:

1.1.1 Notations:

Definition 1.1.1 • *A real stochastic process $X_T = (X_t, t \in T)$ is a random variables family defined on the same space of probability $(\Omega, \mathcal{A}, \mathbb{P})$ with value in (E, B_E).*

•$(\Omega, \mathcal{A}, \mathbb{P})$ *is said a base space, where Ω is a no empty set, $\mathcal{A}$ is a $\sigma-$algeba of subsets of Ω and $\mathbb{P}$ is a probability measure on $\mathcal{A}$.*

•T *is a set of time.*

• *for all ω fixed on Ω, the application:*

$t \longrightarrow X_t(\omega)$ *is the realization of the process on the point ω.*

• *For $t \in T$, $\omega \longmapsto X_t(\omega)$ is a form of the process at the moment t.*

The process $(X_t, t \in T)$ can be consider like a random variable X_T with value in (E^T, B_{E^T}):

$$\begin{aligned} X_T : (\Omega, \mathcal{A}, \mathbb{P}) &\longrightarrow (E^T, B_{E^T}) \\ \omega \quad &\longmapsto (X_t(\omega))_{t \in T} \end{aligned}$$

where B_{E^T} is a Borel algebra of E^T.

Remark 1.1.1 • *If $E = \mathbb{R}$: X_t is a random variables.*
• *If $E = \mathbb{R}^p; p > 1$: X_t is a random vectors.*
• *If $E =$ space : X_t is a random elements.*

1.1.2 Distribution of the Process:

The distribution $L(X_T)$ of process $X_T = (X_t, t \in T)$ is a probability $\mathbb{P}_{X_T}$ on B_{E^T} defined by:

$$\mathbb{P}_{X_T}(S) = \mathbb{P}(X_T^{-1}(S)) \qquad \text{for } S \in B_{E^T}$$

1.1.3 Stationarity:

1) The process $(X_t, t \in \mathbb{Z})$ is strictaly stationary if, for all a finite part $\{t_1, ..., t_n\}$ of T and all $s > 0$: $L\left(X_{t_1+s}, ..., X_{t_n+s}\right) = L\left(X_{t_1}, ..., X_{t_n}\right)$.

2) A real process X_T where all the moments $\mathbb{E}(X_t^2)$ exists , is lowly stationary ,or stationary of second order on $\mathbb{Z}$ if it's covariance on $\mathbb{Z} \times \mathbb{Z}$ defined by:

$$C(s,t) = \mathbb{E}\left(X_s - \mathbb{E}(X_s)\right)\left(X_t - \mathbb{E}(X_t)\right)$$

depend only at the difference $s - t$ of its arguments

1.2 Ergodicity

The strong mixing property used by **Rosenblatt** have an increasing interest in inference statistical and limit theorems for a large class of process.

For vectorial or reals autoregressives process, Markov chains and lineary process, many results are known (strong mixing, absolute regularity). For example **K.C.Chanda** [19] and **C.S.Whithers** [64] witch has establish this properties; under some specifics assumptions; for the lineary process $Y_n = \sum_{j=0}^{+\infty} g_j e_{n-j}$ with $(g_j)_{j \in \mathbb{N}}$ a real sequence and $(e_n, n \in \mathbb{Z})$ a sequence of a reals random variables i.i.d. In an other hand, **V.V.Corodetskii** [34] has establish the same property; under an others assumptions for the same class of process. **D.Pham** and **T.Tram** [53] has obtained the same results for lineary process, with values on $\mathbb{R}^p$. **B.Atherya** and **G.Pantula** [4] has obtained a sufficient conditions of a low mixing property for a real autoregressive process of first order and has establish the strong mixing property for Markov chains with values on mesurable space. **A.Mokkadem** [48] has obtain a sufficient conditions for recurrence and geometric mixed of Markov chains with values on finite and separabale topological space with $\sigma-$finite and finite measure on any compact.

In the case of polynomials autoregressives process, he suppose that the dimension's space is finite. As applications; he obtain the geometric strong mixing for vectorials process ARMA.

In general, ergodic property and Harris recurrence are establish to deduce the mixing properties for the studie's process. **R.L.Tweedie** [63] has obtain a sufficients conditions for geometric ergodicity of irreductibl Markov chains. This points use the notion of small sets descript on [52], as well as , in many cases of finite dimension, as a compacts sets.

For the autoregressives pocess with values on Hilbert spaces and Banach spaces of infinite dimension; the mixe property known an interest on estimation problems and limit theorems [14], [49], [50]. D.Bosq (1995) obtain a result on strong mixing of a Gaussian, hilbertian autoregressives process of first order.

1.2.1 Mixing process

Let $(X_t, t \in \mathbb{Z})$ a strictly stationary process defined on $(\Omega, \mathcal{A}, \mathbb{P})$

Let be T the transformation "translating at left" defined on the set of an infinite sequences by:

$$T(., ., ., X_0, ., ., .) = (., ., ., X_1, ., ., .)$$

T^{-1} is the inverse transformation of T.

Definition 1.2.1 *We say that the sequence $(X_t, t \in \mathbb{Z})$ verify the mixing assumption in sens (1) if:*

$$\lim_{j \longrightarrow \infty} \mathbb{P}\left(B \cap T^{-j}(A)\right) = \mathbb{P}(B).\mathbb{P}(A) \tag{1.1}$$

For all evenements A et B.

Remark 1.2.1 *The mixing condition is a form of an asymptotic independence. In other way, we have the following assumption :*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{P}\left(B \cap T^{-j}(A)\right) = \mathbb{P}(B).\mathbb{P}(A) \tag{1.2}$$

is a sufficient and necessary condition for ergodicity of the process $(X_t, t \in \mathbb{Z})$.

We deduce that all mixing process in sens (1) is ergodic.

1.2.2 A strong mixing process

A property of mixing is a very important notion on statistics of process.

A strictly stationary process $X_n, n \in \mathcal{Z}$ is said $\alpha-$mixing (strongly mixing) (respectively $\phi-$mixing (weakly mixing) or $\beta-$mixing (absolutely regular)) if:

$$\alpha(m) = \sup_{A \in F_{-\infty}^{T}, B \in F_{T+h}^{+\infty}} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| \underset{h \longrightarrow +\infty}{\longrightarrow} 0$$

$$\phi(m) = \sup_{A \in F_{-\infty}^{T}, B \in F_{T+h}^{+\infty}} |\mathbb{P}(A|B) - \mathbb{P}(A)\mathbb{P}(B)| \underset{h \longrightarrow +\infty}{\longrightarrow} 0$$

$$\beta(m) = \sup_{A \in F_{T+h}^{+\infty}} \mathbf{E}|\mathbb{P}\left(A|F_{-\infty}^{T}\right) - \mathbb{P}(A)| \underset{h \longrightarrow +\infty}{\longrightarrow} 0$$

where F_h^n is the $\sigma-$ field generated by the random variables $(X_i, h \leq i \leq n)$. It is know that $\alpha(h) \leq \beta(h) \leq \phi(h)$.

These properties have an increasing interest in the **limit theorem** and in statistical inference for processes.

An assumptions of regularitie were establish by **Kolmogorov et Rosanov**, on spectral density of a guaussian stationary process to be strong mixing.

Theorem 1.2.1 *If the spectral density* $f(\lambda)$ *of a gaussian stationary process is continuous and positive strictly for* $-\pi \leq \lambda \leq +\pi$*; then; the process is strong mixing.*

Then; we can say that ; a gaussian stationary autoregressive process is strong mixing.

1.2.3 lineary process

If $(Z_t, t \in \mathbb{Z})$ is a sequence of an independent random variables , with the same law , with zero-mean and with variance σ^2, then the sequence $(Z_t, t \in \mathbb{Z})$ is mixe in sens (1). Then the lineary process:

$$X_t = \sum_{j=1}^{\infty} a_j Z_{t-j}, \quad \text{where} \quad \sum_{j=1}^{\infty} a_j^2 < \infty$$

is mixe in sens (1) and consequently ergodic .

An autoregressive having the Wold decomposition is consequently ergodic (ie: If all the roots of polynomial equation are with module strictly inferior to 1, then X_t is writing as follow:

$$X_t = \sum_{r=0}^{\infty} \delta_t \epsilon_{t-r}, \qquad t \in \mathbb{Z}$$

We call this decomposition a Wold representation of a reguliar process.

1.2.4 The mixe of a lineary process

Let a lineary process $Y_t = \sum_{k=0}^{\infty} g_k Z_{t-k}$,

where the sequence $(Z_j, j \in \mathbb{Z})$ is constituting of random variables and with density $f_j(x)$.

Gorodetski has done somme assumptions wich under its a lineary process is strong mixing. With the following notations:

$$S_i(\delta) = \sum_{j=i}^{\infty} |g_j|^{\delta}$$

$$\beta(k) = \sum_{i=k}^{\infty} (S_i(\delta))^{\frac{1}{1+\delta}}, \qquad \delta < 2$$

$$\beta(k) = \sum_{i=k}^{\infty} max \left\{ (S_i(\delta))^{1/1+\delta}, \sqrt{S_i(2)|LogS_i(2)|} \right\}, \qquad \delta \geq 2$$

$f_i(x)$ is the density of the random variable Z_i, we have the following theorem:

Theorem 1.2.2 *If*

(i) $\displaystyle\int_{-\infty}^{+\infty} |f_i(x) - f_i(x+\alpha)|\, dx \leq c_1|\alpha|.$

(ii) $\mathbb{E}(|Z_i^\delta|) \leq c_2 < \infty$ *for one* $\delta > 0$*;*
We suppose that $\mathbb{E}(Z_i) = 0$ *if* $\delta \geq 1$*,*
and $Var(Z_i) = 1$ *if* $\delta \geq 2$*.*

(iii) $g(z) = \displaystyle\sum_{k=0}^{\infty} g_k z^k \neq 0$ *for* $|z| < 1$*.*

(iv) $\beta(0) < \infty$

Then $(Y_t, t \in \mathbb{Z})$ *satisfy the property of strong mixing.*

Remark 1.2.2 *The strong mixing is little than the mixing in sens* (1).

The general framework presented in this thesis is to try to apply some limit theorems to a certain process models ; especially to the linear autorogressive process. We have divided the work into 4 parts. The second part is a reminder of autoregressive process of order d (AR(d)). The third part is devoted to the estimation of the parameters of this type of process.

In the fourth chapter we demonstrate a new application of the probability inequality for LNQD sequences and we obtain a result of this application by demonstrating the complete convergence for conditionally lineary negative quadrant dependent random variables sequence application to AR(1) model generated by LNQD errors , the body of which is constituted by published article titled **New exponential probability inequality and complete convergence for conditionally lineary negative quadrant dependent random** . It is well know that, the concept of complete convergence of a sequence of random variables was introduced by (**Hsu and Robins, 1947**) as follows. A sequence $(X_n, n \geq 1)$ of random variables converges completely to the constant C if

$$\sum_{n=1}^{\infty} \mathbb{P}\left(|X_n - C| > \epsilon\right) < \infty \text{ for all } \epsilon > 0$$

By Borel-Cantelli lemma, this implies $X_n \longrightarrow C$ almost surely (a.s), and the converse implication is true if $(X_n, n \geq 1)$ are independent. Complete convergence for the sequence of random variables plays a central role in the area of limit theorems in probability theory and mathematical statistics. Conditions of independence and identical distribution of random variables are basic in historic results due to Bernoulli, Borel and Kolmogorov. Since then, serious attempts have been made to relax these strong conditions.

The fifth chapter is devoted to the study of the complete convergence for weighted sums of WOD random variables with application to the statistics model, the body of which is constituted by the article in preparation titled **Probability inequalities and complete convergence for weighted sums of WOD random variables with application to first order autoregressive process model**.

And the sixth one, we will give; under suitable conditions;the almost complete convergence (a.co) rate of the M-estimator with regression function kernel weights when the observations are independent and identically distributed. For exampe, the concentration hypothesis (H_1) is less restrictive than the strict positivity of the explanatory variables density X which is usually assumed in most of the previous works in the finite-dimensional case see ([18]) and ([41]) . Moreover, it is checked for a great class of continuous time processes see ([11]) for a gaussian measure and ([45]) for a general gaussian process. Remarks that the functional character of our model is well exploited in this work. Indeed, hypothesis (H_2) is a regularity condition which characterizes the functional space. Finally, in this work, we consider a family of $\psi-$functions indexed by x, in order to cover most of the M-estimate classes see ([18]) for some examples of ψ_x. It is also worth noting that we keep the same conditions on the function ψ_x (assumption (H_3)) as were given by ([18]) in the multivariante case. Furthermore, the boundedness assumption on ψ is made only for the simplicity of the proof. It can be dropped while using truncation methods as to those used in ([41].)

Chapter 2

Generality on the autoregressive process

2.1 Generalities on the autoregressive process:

2.1.1 Definitions and generals results:

At the following we consider that $X_T = (X_t, t \in \mathbb{Z})$ is a real process with zero-mean.

Definition 2.1.1 *i) We call the process $(\varepsilon_t, t \in \mathbb{Z})$ a* **low white noise** *if:*

$$\mathbb{E}(\varepsilon_t) = 0 \quad \textit{and} \quad \mathbb{E}(\varepsilon_t \varepsilon_s) = \delta_{st}\sigma^2$$

where δ_{st} is the kronecher symbol and $\sigma^2 > 0$.

ii) The process $(\varepsilon_t, t \in \mathbb{Z})$ is said a **strong white noise** *if the random variables are zero-means, independents with the same distribution and with variance $\sigma^2 > 0$.*

Definition 2.1.2 *The process $(X_t, t \in \mathbb{Z})$ is an autoregressive of order k if it verify for $k > 0$:*

$$\begin{aligned} X_t &= a_1 X_{t-1} + a_2 X_{t-2} + ... + a_k X_{t-k} + \varepsilon_t \\ a_k &\neq 0 \end{aligned}$$

where $a_1, ..., a_k$ are reals numbers ,the random variables $(\varepsilon_t, t \in \mathbb{Z})$ constitute a low white noise such that:

$$\mathbb{E}(\varepsilon_t X_s) = 0 \qquad \textit{for} \qquad s < t \tag{2.1}$$

the condition (2.1) imply the uniquency of the decomposition in (2.1).

Definition 2.1.3 *Let consider the set of reals sequences and on this set we will define the delay operators B, and the advance operators F (we conserve the English symbols B for Backward and F for Farward).*

a) The delay operator B:

It is define by:

$$Bz_t = z_{t-1} \qquad \forall t$$

• *The linearity is evident:*

$$B(\alpha y_t + \beta z_t) = \alpha y_{t-1} + \beta z_{t-1} = \alpha B y_t + \beta B z_t$$

•*We put $B^0 z_t = 1.z_t$ (identity operator)*

•*The operator (αB) is define by :$(\alpha B)z_t = \alpha B z_t = \alpha z_{t-1}$.*

•*The operator B^n is defined by: $B(B^{n-1}z_t) = z_{t-n}$*

•*The sum of the operators is defined by:*

$(\alpha_1 B^{n_1} + ... + \alpha_p B^{n_p})z_t = (\alpha_1 B^{n_1} z_t + ... + \alpha_p B^{n_p} z_t) = \alpha_1 z_{t-n_1} + ... + \alpha_p z_{t-n_p}$

b) The advance operator F:

It is defined by :

$$Fz_t = z_{t+1} \qquad \forall t$$

Remark 2.1.1 *All definitions of B can be applied at F.*

Definition 2.1.4 *We call polynomial equation associated at AR(k), the equation:*

$$P(z) = z_k - \sum_{i=1}^{k} a_i B^i z_k = 0 \tag{2.2}$$

Theorem 2.1.1 *One condition is necessary and sufficient for existence of an autoregressive process stationary low verify (2.1):*

is that the roots of polynomial equation associated $P(z) = 0$ are with module strictly inferior to 1

Proof:

For $k = 2$:

$X_t = a_1 X_{t-1} + a_2 X_{t-2} + \varepsilon_t$

we put: $\begin{pmatrix} X_t \\ X_{t-1} \end{pmatrix} = \widetilde{X}_t; \begin{pmatrix} a_1 & a_2 \\ 1 & 0 \end{pmatrix} = A; \begin{pmatrix} \varepsilon_t \\ 0 \end{pmatrix} = \widetilde{\varepsilon}_t$

(2.1.1) $\iff \widetilde{X}_t = A\widetilde{X}_{t-1} + \widetilde{\varepsilon}_t; (\widetilde{X}_t, t \in \mathbb{R})$ is an AR(1) on $\mathbb{R}^2$. wich call **Markovian representation.**

Condition N:

Let be $(X_t)_{t\in\mathbb{Z}}$ a low stationary process verify 2.1.1 we have:

$\widetilde{X}_t\widetilde{X}_t' = \left(A\widetilde{X}_{t-1} + \widetilde{\varepsilon}_t\right)\left(A\widetilde{X}_{t-1} + \widetilde{\varepsilon}_t\right)'$

where $\widetilde{X}_t' =^t (\widetilde{X}_t)$

Let be: Γ the covariance matrix of $\widetilde{X}_t$ (ie: $\Gamma = \mathbb{E}\left(\widetilde{X}_t\widetilde{X}_t'\right)$)

D the covariance matrix of $\widetilde{\varepsilon}_t$ $\left(ie : D = \mathbb{E}\left(\widetilde{\varepsilon}_t\widetilde{\varepsilon}_t'\right)\right)$.

$$\begin{aligned}
\mathbb{E}\left(\widetilde{X}_t\widetilde{X}_t'\right) &= \mathbb{E}\left[\left(A\widetilde{X}_{t-1} + \widetilde{\varepsilon}_t\right)\left(\widetilde{X}_{t-1}'A' + \widetilde{\varepsilon}_t'\right)\right] \\
&= \mathbb{E}\left[\left(A\widetilde{X}_{t-1}\widetilde{X}_{t-1}'A'\right) + \left(A\widetilde{X}_{t-1}\widetilde{\varepsilon}_t'\right) + \left(\widetilde{\varepsilon}_t\widetilde{X}_{t-1}'A'\right) + \left(\widetilde{\varepsilon}_t\widetilde{\varepsilon}_t'\right)\right] \\
&= \mathbb{E}\left(A\widetilde{X}_{t-1}\widetilde{X}_{t-1}'A'\right) + \mathbb{E}\left(A\widetilde{X}_{t-1}\widetilde{\varepsilon}_t'\right) + \mathbb{E}\left(A'\widetilde{\varepsilon}_t\widetilde{X}_{t-1}'\right) + \mathbb{E}\left(\widetilde{\varepsilon}_t\widetilde{\varepsilon}_t'\right) \quad . \\
&= A\mathbb{E}\left(\widetilde{X}_{t-1}\widetilde{X}_{t-1}'\right)A' + A\mathbb{E}\left(\widetilde{X}_{t-1}\widetilde{\varepsilon}_t'\right) + A'\mathbb{E}\left(\widetilde{\varepsilon}_t\widetilde{X}_{t-1}'\right) + \mathbb{E}\left(\widetilde{\varepsilon}_t\widetilde{\varepsilon}_t'\right) \\
&= A\Gamma A' + D
\end{aligned}$$

then:

$$\Gamma = A\Gamma A' + D \tag{2.3}$$

Now we search the roots of $P(z)$:

$\mid A - \lambda\mathcal{I} \mid = \begin{vmatrix} a_1 - \lambda & a_2 \\ 1 & -\lambda \end{vmatrix} = -\lambda(a_1 - \lambda) - a_2 = \lambda^2 - a_1\lambda - a_2 = P(\lambda)$

The roots of $P(z)$ are exactaly the eigenvalues of A

Let v an eigen vector of A associated at the eigenvalue $\lambda : Av = \lambda v$

$$
\begin{aligned}
vDv' &= v(\Gamma - A\Gamma A')v' \\
&= v\Gamma v' - vA\Gamma A'v' \\
&= v\Gamma v' - vA\Gamma(vA)' \\
&= v\Gamma v' - \lambda v\Gamma(\lambda v)' \\
&= v\Gamma v' - \lambda v\Gamma \lambda v' \\
&= v\Gamma v' - \lambda^2 v\Gamma v' \\
&= (1-\lambda^2)(v\Gamma v')
\end{aligned} .
$$

Then $vDv' = (1-\lambda^2)(v\Gamma v')$. But the matrix D and Γ are defined positives in case no degenerate, where:

$$1 - \lambda^2 > 0 \Longrightarrow \lambda^2 < 1 \Longrightarrow |\lambda| < 1$$

Then the roots of $P(z)$ verify $|z| < 1$.

Condition S:

The roots of $P(z)$ verify $|z| < 1$.

Let $\widetilde{X}_t = A\widetilde{X}_{t-1} + \widetilde{\epsilon}_t$

where $\widetilde{X}_t = (X_t, X_{t-1})'$, $\quad \widetilde{\epsilon}_t = (\epsilon_t, 0)'$

$$
\begin{aligned}
\widetilde{X}_t &= A\widetilde{X}_{t-1} + \widetilde{\epsilon}_t \\
&= A\left(A\widetilde{X}_{t-2} + \widetilde{\epsilon}_{t-1}\right) + \widetilde{\epsilon}_t \\
&= A^2\widetilde{X}_{t-2} + A\widetilde{\epsilon}_{t-1} + \widetilde{\epsilon}_t \\
&= A^2\left(A\widetilde{X}_{t-3} + \widetilde{\epsilon}_{t-2} + A\widetilde{\epsilon}_{t-1}\right) + \widetilde{\epsilon}_t \\
&= A^3\widetilde{X}_{t-3} + A^2\widetilde{X}_{t-2}A\widetilde{\epsilon}_{t-1} + \widetilde{\epsilon}_t \\
&= \ldots\ldots \\
&= \ldots\ldots \\
&= A^s\widetilde{X}_{t-s} + \sum_{i=0}^{s-1} A^i\widetilde{\epsilon}_{t-i}
\end{aligned}
$$

Then; $\widetilde{X}_t - \sum_{i=0}^{s-1} A^i \widetilde{\epsilon}_{t-i} = A^s \widetilde{X}_{t-s}$

$$\begin{aligned}\left(\widetilde{X}_t - \sum_{i=0}^{s-1} A^i \widetilde{\epsilon}_{t-i}\right)\left(\widetilde{X}_t - \sum_{i=0}^{s-1} A^i \widetilde{\epsilon}_{t-i}\right)^{'} &= \left(A^s \widetilde{X}_t\right)\left(A^s \widetilde{X}_{t-s}\right)^{'} \\ &= A^s \widetilde{X}_{t-s} \widetilde{X}^{'}_{t-s} A^{s^{'}}\end{aligned}$$

wich implies that:

$$\mathbb{E}\left(\widetilde{X}_t - \sum_{i=0}^{s-1} A^i \widetilde{\epsilon}_{t-i}\right)\left(\widetilde{X}_t - \sum_{i=0}^{s-1} A^i \widetilde{\epsilon}_{t-i}\right)^{'} = A^s \mathbb{E}\left(\widetilde{X}_{t-s} \widetilde{X}^{'}_{t-s}\right) A^{s^{'}} = A^s \Gamma A^{s^{'}}$$

The elements of Γ are finites because of $\mathbb{E}(X_t^2) < \infty$.

And we have the following result of linear algebra :

If

$$|\lambda| < 1 \Longrightarrow A^s \underset{s \longrightarrow \infty}{\longrightarrow} 0$$

where λ is the eigenvalue of A.

Then we obtain:

$$\mathbb{E}\left(\widetilde{X}_t - \sum_{i=0}^{s-1} A^i \widetilde{\epsilon}_{t-i}\right)\left(\widetilde{X}_t - \sum_{i=0}^{s-1} A^i \widetilde{\epsilon}_{t-i}\right)^{'} \underset{s \longrightarrow \infty}{\longrightarrow} 0.$$

And we have:

$$\widetilde{X}_t = \sum_{i=0}^{s-1} A^i \widetilde{\epsilon}_{t-i} \qquad \text{in quadratic mean.}$$

□

2.1.2 Banach space valued autoregressive process of first order.

Let $(X_n, n \in \mathbb{Z})$ be a strictly stationnary autoregressive process of order one with valued on banach space $\mathbb{B}$ defined by:

$$X_n = \rho X_{n-1} + \varepsilon_n \tag{2.4}$$

where ρ is a linear borned operator defined on $\mathbb{B}$.

The process $(X_n, n \in \mathbb{Z})$ is an homogenuous Markov chain with transition probability given by:

$$\forall A \in \mathbb{B}, \quad \mathbb{P}(x, A) = \mathbb{P}\left(X_1 \in A / X_0 = x\right) = \mathbb{P}\left(\varepsilon_1 + \rho x \in A\right)$$

We have the following definitions:

(1) if we design μ the invariante measure of the process (X_n), a set $A \in \mathbb{B}$ with measure $\mu(A) > 0$ is a small set , if forall $C \in \mathbb{B}$ with measure $\mu(C) > 0$, they exist an integar i_0 wich is $\inf_{x \in A} \sum_{i=1}^{i_0} \mathbb{P}^i\left(x, C\right) > 0$.

(2) A Markov process with transition $\mathbb{P}$ is geometricly ergodic if they exist $\xi, \quad 0 \leq \xi \leq 1$, such that $\forall x \in \mathbb{B}, \xi^{-n} ||\mathbb{P}^n\left(x, .\right) - \mu\left(.\right)|| \underset{n \longrightarrow \infty}{\longrightarrow} 0$ where $||.||$ is the invariante norm.

We denote by λ is a $\sigma-$ finite measure on $(\mathbb{B}, \mathcal{B})$

We impose the following assumptions:

(H_1) : $||\rho|| < 1$.

(H_2) : the probability law $\mathbb{P}_{\varepsilon_1}$ is absolutely continuous with respect to a $\sigma-$finite measure λ on $B, \mathbb{B}$ with density f and the set of positivity of f defined by $E = \{f > 0\}$ is an open set such that the origin $0 \in E$.

(H_3) : the measure λ is such that for all $A \in \mathbb{B}$ with $\lambda(A) > 0$ they exists an open subset $U \in A$ satisfying $\lambda(U) > 0$.

•(H_1) is sufficient condition for the strict stationarity of $(X_n; n \in \mathbb{Z})$.

•(H_2) is a technical assumption and permits to the measure $\mathbb{P}_{\varepsilon_1}$, to charge the open sets of the origin 0. It is satisfied if we take $\mathbb{P}_{\epsilon_1} = \mathbb{P}_W$ the Wiener measure on $\mathbb{B} = C_{[0,1]}$ the space of all continuous functions on $[0, 1]$ and the reference measure λ is translate of $\mathbb{P}_W$ by an element of the reproducing space of the covariance function of the Wiener process.

•(H_3) is an assumption of the nonatomicity of the measure λ. The Gaussian measures on B satisfy this condition as well as some Radon's measures. This is true in the finite dimensional case.

We will produce the technic of **Mokkadem** [48]. Let be consider the following representation of the process $(X_n) : X_n = \phi\left(X_{n-1}, \varepsilon_n\right); n \in \mathbb{Z}$ Where the transformation ϕ is defined by $\phi : \mathbb{B} \times \mathbb{B} \longrightarrow \mathbb{B}$ and $\phi(x, y) = \rho x + y$.

Remark 2.1.2 *On the space of the trajectory* $\{\omega \in \Omega/\varepsilon_1(\omega) = 0\}$*, one property is important it's such that : If* $\phi(t,0) = t$*, then* $t = 0$*.*

we put $\phi^1(x, e_1) = \phi(x, e_1)$*, then for* $j > 1$: $\phi^j : \mathbb{B} \times \mathbb{B}^j \longrightarrow \mathbb{B}$ *such that:*

$$\phi^j(x, e_1, e_2, ..., e_j) = \phi\left(\phi^{j-1}(x, e_1, e_2, ..., e_{j-1}), e_j\right)$$
$$= \rho^j x + \rho^{j-1} e_1 + \rho^{j-2} e_2 + ... + e_j$$

Under (H_1)*, we have* $X_n = \sum_{i=0}^{+\infty} \rho^i \varepsilon_{n-1}$ *p.s and on* $\mathcal{L}^2_{\mathbb{B}}$ *and with the precedent notation* $X_{n+1} = \phi^{j+1}(X_{n-j}, \varepsilon_{n-j+1}, \varepsilon_{n-j+2}, ..., \varepsilon_{n+1})$

We put $D_j = \phi^j(0, E^j)$ *where* $E^j = E \times E \times ..... \times E$($j$ *times*) *and a set* E *is define in* (H_2)

We have the following result:

Theorem 2.1.2 *Under the assumptions* $(H_1), (H_2)$ *and* (H_3) *the process* $(X_n, n \in \mathbb{Z})$ *defined by (2.4) is absolutely regular geometrically.*

We need the following lemmas to proof this theorem.

Lemma 2.1.1 *Under* $(H_1), (H_2)$*, the sequence* $(D_j)_{j \in \mathbb{N}^*}$ *is an increasent sequence of an open's sets and we have:*

$$\forall n \in \mathbb{Z}, \qquad X_n \in \overline{\underset{j \in \mathbb{N}^*}{\cup} D_{j \in \mathbb{N}^*}} \qquad a.s$$

Lemma 2.1.2 *Let* $j \in \mathbb{N}^*$ *and* $y_0 \in D_j$*. Under* $(H_1), (H_2)$*, they exist* M_0 *neighborhood of* 0 *on* $\mathbb{B}$ *such that for all* $t \in M_0$*, there is an open neighborhood of* y_0 *in* $\phi^j(t, E^j)$ *.*

Lemma 2.1.3 *Under* (H_2) *and for all* $j \in \mathbb{N}^*$*, they exist* y_0 *on* D_j *such that forall an open neighborhood* V *of* y_0 *on* D_j *we have* $\lambda(V) > 0$*.*

Lemma 2.1.4 *Under* $(H_1), (H_2)$ *and* (H_3) *we have:*

$$\forall n \in \mathbb{Z}, \quad \mathbb{P}\left(X_n \in \underset{j \in \mathbb{N}^*}{\cup} D_j\right) > 0$$

Lemma 2.1.5 *Under* $(H_1), (H_2), (H_3)$ *we have:*

$$\forall n \in \mathbb{Z}, \qquad \mathbb{P}\left(X_n \in \underset{j \in \mathbb{N}^*}{\cup} D_j\right) = 1$$

Lemma 2.1.6 *Under* $(H_1), (H_2), (H_3)$ *and for all a borelian set* A *on* $\mathbb{B}$ *in* $\underset{j\in\mathbb{N}^*}{\cup} D_j$ *such that* $\lambda(A) > 0$*, if* K *is a compact set on* $\mathbb{B}$ *in* $\underset{j\in\mathbb{N}^*}{\cup} D_j$ *, then they exist* $(j, l) \in \mathbb{N}^{*2}$ *such that for all* $r \geq l$*:*

$$\forall z \in K, \qquad \mathbb{P}^{j+r}(z, A) > 0$$

Lemma 2.1.7 *Under* $(H_1), (H_2), (H_3)$ *the chain* $(X_n, n \in \mathbb{Z})$ *defined by (2.4) is* $\lambda-$*irredictibl and aperiodic*

Lemma 2.1.8 *Under* $(H_1), (H_2), (H_3)$ *the chain* $(X_n, n \in \mathbb{Z})$ *defined by (2.4) is Harris reccurent.*

Lemma 2.1.9 *Under* $(H_1), (H_2), (H_3)$ *the chain* $(X_n, n \in \mathbb{Z})$ *defined by (2.4) is geometricly ergodic*

Now we can use the results of **E.Nummelin. P. Tuominen** [52], and **Y.A. Davydov** [27] to say that the process (X_n) defined by (2.4) is geometricly absolutly regular. □

2.1.3 Banach space valued autoregressive process of superior order

To study the mixing property of the autoregressive process of a superior order defined by (2.1) we use the following Markovienne representation on the product space $\mathbb{B}^p, p \in \mathbb{N}^*$:

$$Y_n = AY_{n-1} + \widetilde{A}\varepsilon_n \tag{2.5}$$

Where $Y_n = (X_n, ..., X_{n-p+1})', \widetilde{A} = (I, 0, ..., 0)'$ and A is the matrix operator from $\mathbb{B}^p$ to it's self defined by:

$$\begin{pmatrix} \rho_1 & \rho_2 & . & . & . & \rho_{p-1} & \rho_p \\ I & 0 & . & . & . & 0 & 0 \\ 0 & I & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & . & . & . & 0 & I & 0 \end{pmatrix}$$

I is the identity operator of $\mathbb{B}$

Now we we have to put the following assymtion:

$(H_1)'$: $\exists j_0 \geq 1$ such that $||A^{j_0}|| < 1$.

Remark 2.1.3 *In general the norm of A is higher than 1, but we can found a degree of A with norm inferior than 1* **([14], ch.3.2 or [49], ch.9)**.

We conserve the assumptions $(H_2), (H_3)$. With a similar study of ARB(1), we can deduce that (Y_n) defined by (2.5), is geometricly, absolutly regular. For what; (X_n) defined by (2.1) is geometricy absolutly regular.

Theorem 2.1.3 *Under $(H_1)', (H_2), (H_3)$. The process (X_n) defined by (2.4) is geometricly absolutly regular.*

2.2 Autoregressive process in Hilbert space

Let H be a separable Hilbert space with norm $||.||$ and scalar product $< .,. >$ and Borel $\sigma-$algebra $\mathcal{B}_H$.

Definition 2.2.1 *A sequence $\varepsilon = (\varepsilon_n, n \in \mathbb{Z})$ of $H-$ random variables is said to be an $H-$ white noise (WN) if*

1) $0 < \mathbb{E}||\varepsilon_n||^2 = \sigma^2 < \infty, \quad \mathbb{E}(\varepsilon_n) = 0$

2) *ε_n is orthogonal to ε_m; $n, m \in \mathbb{Z}; n \neq m$;*

$$\mathbb{E}\left(< \varepsilon_n, x >< \varepsilon_m, y >\right) \quad x, y \in H$$

ε is said to be an $H-$ strong white noise (SWN) if it satisfies $1)$ and

$2')$ *(ε_n) is a sequence of i.i.d. $H-$random variables*

An SWN is a WN and the converse fails.

Example 2.2.1 *Let $H = L^2\left([0,1], \mathcal{B}_{[0,1]}, \lambda\right)$, where λ is the Lebesgue measure, consider a measurable bilateral Wiener process W, and put*

$$\varepsilon_n(t) = W_{n+1} - W_n, \quad 0 \leq t \leq 1, \quad n \in \mathbb{Z}$$

(ε_n) defines a sequence of $H-$random variables. Since increments of W are independent stationary, ε is a strong white noise.

Definition 2.2.2 • *An $H-$valued second order process $X = (X_n, n \in \mathbb{Z})$ is a Markov process in the wide sence if*

$$\pi^{\mathcal{G}^k_{n-1}}(X_n) = \pi^{\mathcal{G}^1_{n-1}}(X_n), \quad n \in \mathbb{Z}, k \geq 2$$

where $\pi^{\mathcal{G}^k_{n-1}}(k \geq 1)$ is the orthogonal projector over the hermetically closed sub space of $L^2_H(\Omega, \mathcal{A}, \mathbb{P})$ generated by $X_{n-1}, ..., X_{n-k}$

• *An $H-$valued process $X = (X_n, n \in \mathbb{Z})$ is a Markov process in the strict sense if*

$$\mathbb{P}^{\mathcal{A}^k_{n-1}}(X_n \in A) = \mathcal{A}^{\nVdash}_{\ltimes-\nVdash}(X_n \in A), A \in \mathcal{B}_H, n \in \mathbb{Z}, k \geq 2$$

where $\mathbb{P}^{\mathcal{A}^k_{n-1}}, (k \geq 1)$ denotes conditional probability with respect to the $\sigma-$algebra $\mathcal{A}^k_{n-1} = \sigma(X_{n-1}, ..., X_{n-k})$

Definition 2.2.3 *A sequence $X_n = (X_n, n \in \mathbb{Z})$ of $H-$random variables is called an autoregressive hilbertian process of order 1 ($ARH(1)$) associated with (μ, ε, ρ) if it is statinary and such that*

$$X_n - \mu = \rho(X_{n-1} - \mu) + \varepsilon_n, \qquad n \in \mathbb{Z} \tag{2.6}$$

where $\mu \in H, \rho$ is a bounded linear operator and $\varepsilon = (\varepsilon_n, n \in \mathbb{Z})$ is an $H-$white noise.

Existence of such a process is ensured by the following conditions
(c_0) There exists an integer $j_0 \geq 1$ such that $||\rho^{j_0}||_{\mathcal{L}} < 1$.
and
(c_1) There exist $a > 0$ and $0 < b < 1$ such that $||\rho^j||_{\mathcal{L}} \leq ab^j, j \geq 0$.

Lemma 2.2.1 *(c_0) and (c_1) are equivalent*

Proof:

It is obvious that (c_1) yields (c_0).

Let us show that (c_0) implies (c_1)

Clearly it is suffices to prove (c_1) for $j > j_0$ and $0 < ||\rho^{j_0}||_{\mathcal{L}} < 1$. For such a j we may write the result of its euclidian division by j_0 under the form

$$j = j_0 q + r \tag{2.7}$$

where $q \geq 1$ and $0 \leq r < j_0$

Now the properties of $||.||_{\mathcal{L}}$ entail

$$||\rho^j||_{\mathcal{L}} \leq ||\rho^{j_0}||_{\mathcal{L}}^q ||\rho^r||_{\mathcal{L}}$$

and since $q = \frac{j}{j_0} - 1$ and $0 < ||\rho^{j_0}||_{\mathcal{L}} < 1$ it follows that

$$||\rho^j||_{\mathcal{L}} \leq ab^j, \quad j > j_0$$

where $a = ||\rho^{j_0}||_{\mathcal{L}}^{-1} \max\limits_{0 \leq r \leq j_0} ||\rho^r||_{\mathcal{L}}$ and $b = ||\rho^{j_0}||_{\mathcal{L}}^{\frac{1}{j_0}} < 1$ □

Theorem 2.2.1 *If (c_0) holds, then (2.6) has a unique stationary solution given by*

$$X_n = \mu + \sum_{j=0}^{\infty} \rho^j(\varepsilon_{n-j}), \quad n \in \mathbb{Z} \tag{2.8}$$

where the series converges in $L_H^2(\Omega, \mathcal{A}, \mathbb{P})$ and with probability 1.

Proof:
We may and do assume that $\mu = 0$. Now orthogonality of the ε_n's entails

$$\delta_m^{m'} = || \sum_{j=m}^{m'} \rho^j\left(\varepsilon_{n-j}\right) ||_{\mathcal{L}_H^2(\mathbb{P})}^2 = \sum_{j=m}^{m'} ||\rho^j\left(\varepsilon_{n-j}\right)||_{\mathcal{L}_H^2(\mathbb{P})}^2$$

$1 \leq m \leq m'$. On the other hand

$$||\rho^j\left(\varepsilon_{n-j}\right)||_{\mathcal{L}_H^2(\mathbb{P})} = \mathbb{E} < \rho^j\left(\varepsilon_{n-j}\right), \rho^j\left(\varepsilon_{n-j}\right) >$$
$$\leq \sigma^2 ||\rho^j||_{\mathcal{L}}^2$$

hence the lemma as above yields

$$\delta_m^{m'} \leq \sigma^2 \sum_{j=m}^{m'} ||\rho^j||_{\mathcal{L}}^2 \underset{m,m' \longrightarrow \infty}{\longrightarrow} 0$$

Thus from the Cauchy criterion it follows that the series in (2.8) converge in $L_H^2(\mathbb{P})$.

In fact, since $\mathbb{E}\left(\sum_{j=0}^{\infty} ||\rho^j|| ||\varepsilon_{n-j}||\right)^2 < \infty$, it follows that

$\sum_{j=0}^{\infty} ||\rho^j|| ||\varepsilon_{n-j}|| < \infty a.s.$ and the series also converge almost surely.

Let us now consider the stationary process

$$Y_n = \sum_{j=0}^{\infty} \rho^j \left(\varepsilon_{n-j}\right), \quad n \in \mathbb{Z}$$

by using boundedness of ρ we see that

$$Y_n - \rho(Y_{n-1}) = \sum_{j=0}^{\infty} \rho^j \left(\varepsilon_{n-j}\right) - \sum_{j=0}^{\infty} \rho^{j+1} \left(\varepsilon_{n-1-j}\right)$$

$$= \varepsilon_n, \quad n \in \mathbb{Z}$$

which means that (Y_n) is a solution of equation (2.6)

Convesely, let (X_n) be a stationary solution of (2.6). A straightforward induction gives

$$X_n = \sum_{j=0}^{k} \rho^j \left(\varepsilon_{n-j}\right) + \rho^{k+1} \left(X_{n-k-1}\right), \quad k \geq 1 \tag{2.9}$$

Therefore

$$\mathbb{E}||X_n - \sum_{j=0}^{k} \rho^j \left(\varepsilon_{n-j}\right)^2 || \leq ||\rho^{k+1}||_{\mathcal{L}}^2 \mathbb{E}||X_{n-k-1}||^2$$

By stationary, $\mathbb{E}||X_{n-k-1}||^2$ remains constant and the previously lemma yields $||\rho^{k+1}||_{\mathcal{L}}^2 \underset{k \longrightarrow \infty}{\longrightarrow} 0 \quad a.s.$ Consequently

$$X_n = \sum_{j=0}^{\infty} \rho^j \left(\varepsilon_{n-j}\right), \quad n \in \mathbb{Z}$$

This poves uniqueness. □

Example 2.2.2 *Consider the Hilbert space $H = L^2\left([0,1], \mathcal{B}_{[0,1]}, \lambda\right)$ and $\rho = l_k$, a Kernel operator associated with with a Kernel K such that*

$$\int_{[0,1]^2} K^2 (s,t) \, dsdt < 1$$

Take a white noise $(\varepsilon_n)^{\phi}$ *given by*

$\varepsilon_n^{\phi}(t) = \int_{n}^{n+1} \phi(n+t-s)\,dW(s), \quad 0 \le t \le 1, \quad n \in \mathbb{Z}$ *where* $\phi \in H$ *and* $\int_0^1 \phi^2(u)du > 0$. *Conditions in the last theorem are then satisfied and one obtains the ARH(1) process*

$$X_n = \sum_{j=0}^{\infty} l_K^j \left(\varepsilon_{n-j}^{(\phi)} \right), \quad n \in \mathbb{Z}.$$

In order to state a corollary concerning uniqueness of (μ, ε, ρ), let us recall that the support S_Z of distribution of a random variable Z is defined by

$$S_Z = \{x : x \in H, \mathbb{P}(||Z - x|| < \alpha) > 0 \quad \forall \alpha > 0\}$$

Corollary 2.2.1 *If* X *is an ARH(1) associated with* (μ, ε, ρ) *and* (c_0) *holds, then* (μ, ε) *is unique, and* ρ *is unique over*

$$S = \overline{sp} \bigcup_{n \in \mathbb{Z}} \left(S_{X_n - \mu} \cup S_{\varepsilon_n} \right)$$

Proof:

Uniqueness of (μ, ε) is obvious since $\mathbb{E}(X_n) = \mu$ and ε is the innovation of $(X_n - \mu)$

Now if $\rho_1 \in \mathcal{L}$ satisfies (c_0) and

$$X_n = \mu + \sum_{j=0}^{\infty} \rho_1^j (\varepsilon_{n-j}), \quad n \in \mathbb{Z} \tag{2.10}$$

Then (2.6) implies

$$\rho(X_{n-1} - \mu) = \rho_1(X_{n-1} - \mu), \quad (a.s), \quad n \in \mathbb{Z}$$

Which in turn implies $\rho_1 = \rho$ over $S_{X_{n-1} - \mu}, \quad \forall n$

On the other hand, (2.8) and (2.10) entail

$$(\rho - \rho_1)(\varepsilon_{n-1}) = \sum_{j=0}^{\infty} \left(\rho_1^j - \rho^j \right) (\varepsilon_{n-j})$$

Then, from

$$(\rho - \rho_1)(\varepsilon_{n-1}) \sum_{j=0}^{\infty} \left(\rho_1^j - \rho^j\right)(\varepsilon_{n-j})$$

it follows that

$$(\rho - \rho_1)(\varepsilon_{n-1}) = 0, \quad n \in \mathbb{Z}$$

This implies equality of ρ and ρ_1 over $S_{\varepsilon_{n-1}}, \forall n$

Finally, by linearity and continuity of ρ and ρ_1, one obtains uniqueness of ρ over S. □

2.3 Autoregressive Hilbrtian processes of order p

The Markovian character of the ARH(1) model induces some limits to its eff-ciency for applications to statistics in continuous time. In this paragraph we introduce the more flexible autoregressive model of order p.

Definition 2.3.1 *Let H be a separable Hilbert space.*
A sequence $X = (X_n, n \in \mathbb{Z})$ of $H-$random variables is said to be an autoregressive hilbertian process of order p (ARH(p)) associated with $(\mu, \varepsilon, \rho_1, ..., \rho_p)$ if it is stationary and such that

$$X_n - \mu = \rho_1 (X_{n-1} - \mu) + ... + \rho_p (X_{n-p} - \mu) + \epsilon_n, \quad n \in \mathbb{Z} \tag{2.11}$$

where $(\varepsilon_n, n \in \mathbb{Z})$ is an $H-$white noise, $\mu \in H$, and $\rho_1, ..., \rho_p \in \mathcal{L}$, with $\rho_p \neq 0$

2.3.1 Markovian representation of an ARH(p)

Let H^p be the cartesian product of p copies of H.

H^p is a separable Hilbert space if it is equipped with the scalar product

$$< (x_1, ..., x_p), (y_1, ..., y_p) >_p = \sum_{j=1}^{p} < x_j, y_j > \tag{2.12}$$

with $x_j, y_j \in H, j = 1, ..., p$

Then we denote by:

- $||.||_p$ the norm in H^p.
- $\mathcal{L}_p$ the space of bounded linear operators over H^p.

- S_p the space of Hilbert-schmidt operators over H^p. and the corresponding norms and scalar products by $||.||_{S_p}, \quad <.,.>_{S_p}$.
- $Y = (Y_n, n \in \mathbb{Z})$, where $Y_n = (X_n, ..., X_{n-p+1}), n \in \mathbb{Z}$;

$\mu^{'} = (\mu, ..., \mu) \in H^p$; $\varepsilon^{'} = \left(\varepsilon^{'}_n, n \in \mathbb{Z}\right)$ with $\varepsilon^{'}_n = (\varepsilon_n, 0, ..., 0)$ and consider the operator on H^p defined as

$$\rho^{'} = \begin{pmatrix} \rho_1 & \rho_2 & . & . & . & \rho_{p-1} & \rho_p \\ I & 0 & . & . & . & 0 & 0 \\ 0 & I & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & . & . & . & 0 & I & 0 \end{pmatrix}$$

where I is the identity operator.

We have the following lemma

Lemma 2.3.1 *If X is an ARH(p) associated with $(\mu, \varepsilon, \rho_1, ..., \rho_p)$, then Y is an $ARH^p(1)$ associated with $\left(\mu^{'}, \varepsilon^{'}, \rho^{'}\right)$*

The existence and uniqueness of X are gives by the following theorem.

Theorem 2.3.1 *Under the assumption*

$$(c^{'}_0) \qquad ||\rho^{'j_0}||_{\mathcal{L}_p} < 1 \text{ for some } j_0 \geq 1$$

*equation (**??**) has a unique stationary solution given by*

$$X_n = \mu + \sum_{j=0}^{\infty} \left(\pi \rho^{'j}\right) \left(\varepsilon^{'}_{n-j}\right), \quad n \in \mathbb{Z} \tag{2.13}$$

where π is the natural projector of H^p on H defined by

$$\pi (x_1, ..., x_p) = x_1, \quad (x_1, ..., x_p) \in H^p$$

and the series converges in $L^2_H(\Omega, \mathcal{L}, \mathbb{P})$, with probability 1

Now, we will introduce a condition that is directly associated with the operators $\rho_1, ..., \rho_p$

$$Q(z) = z^p I - z^{p-1}\rho_1 - ...z\rho_{p-1} - \rho_p, \quad z \in \mathbb{C}$$

For every z, $Q(z)$ is a bounded linear operator over the complex extension $H^{'}$ of H.

Then we have the following theorem

Theorem 2.3.2 *Suppose that the following condition holds:*

$$Q(z)\, notinversible \implies |z| < 1 \tag{2.14}$$

Than $(c_0^{'})$ *holds. THerefore (**??**) has a unique stationary solution given by (2.13)*

Proof:

Let us consider the operators on H^p defined as

$$N(z) = \begin{pmatrix} I & zI & z^2I & . & . & . & z^{p-1}I \\ 0 & I & zI & . & . & . & z^{p-2}I \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & . & . & . & . & I & zI \\ 0 & . & . & . & . & 0 & I \end{pmatrix}$$

and

$$M(z) = \begin{pmatrix} 0 & -I & 0 & . & . & . & 0 \\ 0 & 0 & -I & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & . & . & . & . & 0 & -I \\ Q_0(z) & Q_1(z) & . & . & . & . & Q_{p-1}z \end{pmatrix}$$

where $Q_0(z) = I$ and $Q_j(z) = zQ_{j-1}(z) - \rho_j, \quad j = 1, ..., p$

It's easy to see that:

$$M(z)\left(zI_p - \rho^{'}\right)N(z) = \begin{pmatrix} I & 0 & . & . & . & 0 & 0 \\ 0 & I & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & I & 0 \\ 0 & . & . & . & . & 0 & Q(z) \end{pmatrix} \tag{2.15}$$

where I_p is the identity of H^p.

Now, due to their special form, $N(z)$ and $M(z)$ are inversible for all z.

Then from (2.15) it follows that

$E = \{z, zI - \rho^{'}$ is not inversible $\} \subset \{z, Q(z)$ is not inversible $\}$

and using (2.14) we get

$$E \subset \{z, |z| < 1\} \tag{2.16}$$

E is the so-called spectrum of $\rho^{'}$ over $H^{'}$. It is a closed set and

$$r_{\rho^{'}} = \sup_{z \in E} |z| = \lim_{j \longrightarrow \infty} ||\rho^{'j}||_{\mathcal{L}_p}^{1/j} \tag{2.17}$$

Then, from (2.16) and (2.17) we deduce that there exist an integer $j_1, \alpha \in]0, 1[$ and $k > 0$ such that

$$||\rho^{'j}||_{\mathcal{L}_p} \leq k\alpha^j, \quad j \geq j_1$$

Thus $(c_0^{'})$ holds and the proof is complete. □

Note that it is possible to show that $r_{\rho^{'}} \leq ||\rho^{'}||_{\mathcal{L}_p}$, but $r_{\rho^{'}}$ does not entails $||\rho^{'}||_{\mathcal{L}_p} < 1$. On the other hand, if H is finite dimensional, (2.14) is equivalent to " determinant of $Q(z) = 0 \Longrightarrow |z| < 1$."

Corollary 2.3.1 *If* $\sum_{j=1}^{p} ||\rho_j||_{\mathcal{L}} < 1$, *then (2.14) holds*

and

$$Q(z) = 0 \Longrightarrow |z| < 1$$

Example 2.3.1 *Take* $H = \left(L^2[0,1], \mathcal{B}_{[0,1]}, \lambda\right)$ *and* ε *such that*

$$\varepsilon_n^{\phi}(t) = \int_{n}^{n+1} \phi\,(n + t - s)\, dW(s), \quad 0 \leq t \leq 1, \quad n \in \mathbb{Z}$$

where W *is a Wiener process and* $\phi \in H$ *and* $\int_0^1 \phi^2(u)du > 0$ *and let* $\rho_j = l_k; \quad j = 1, ..., p$ *be kernel operators associated with kernels* $K_1, ..., K_p$ *such that*

$$\sum_{j=1}^{p} \left(\int_{[0,1]^2} K_j^2(s,t)dsdt \right)^{1/2} < 1$$

*Then from the corollary as above we show that (**??**) has a unique stationary solution* (X_n) *which satisfies*

$$X_n(t) = \int_0^1 \left(\sum_{j=1}^{p} K_j(s,t) X_{n-j}(s) \right) ds + \int_n^{n+1} \phi(n + t - s)dW(s)$$

$0 \leq t \leq 1, \quad n \in \mathbb{Z}$

We finally indicate a result concerning projection of an ARH(p) process.

Theorem 2.3.3 *Let (X_n) be an ARH(p) zero-mean process associated with $(\rho_1, ..., \rho_p; \varepsilon)$. Suppose that there exist $v \in H$ and $\alpha_1, ..., \alpha_p \in \mathbb{R}, (\alpha_p \neq 0)$ such that*

$$\rho_j^*(v) = \alpha_j v, \quad j = 1, ..., p.$$

and

$$\mathbb{E}\left(< \varepsilon_0, v >^2\right) > 0.$$

Then $(< X_n, v >, n \in \mathbb{Z})$ is an AR(p) process that satisfies

$$< X_n, v >= \sum_{j=1}^{p} \alpha_j < X_{n-j}, v > + < \varepsilon_n, v >, \quad n \in \mathbb{Z} \tag{2.18}$$

Chapter 3

Estimation of an autoregressive parameters

3.1 The autocovariance and autocorrelation functions

Let $(X_t, t \in \mathbb{Z})$ be a real stationary process of second order (not degenerate).

Definition 3.1.1 *The autocovariace function of the process* $(X_t, t \in \mathbb{Z})$ *is defined by:*

$$R(h) = cov(X_t, X_{t+h}) \quad \textit{where} \quad h \in \mathbb{Z}.$$

The autocorrelation function is defined by:

$$\rho(h) = \frac{R(h)}{R(0)}, \qquad h \in \mathbb{Z}$$

.

Propriete 3.1.1 *i)* $\bullet R(0) = \sigma^2, (ie : R(0) = cov(X_t, X_t))$.

$\bullet R(h) < R(0)$.

$\bullet R(h) = R(-h), (ie : R(-h) = cov(X_t, X_{t-h}) = cov(X_t, X_{t+h}) = R(h))$.

The function $R(h)$ *is positive*

(ie: $\forall t_1, ..., t_n \in \mathbb{Z}$ *, and* $\forall a_1, ..., a_n$: $\sum_{r=1}^{n} \sum_{s=1}^{n} a_r a_s R(t_r - t_s) \geq 0$*)*

ii) $\bullet\rho(0) = 1, \left(ie : \rho(0) = \frac{R(0)}{R(0)} = 1\right).$

$\bullet|\rho(h)| < 1, \left(ie : |\rho(h)| = \frac{|R(h)|}{|R(0)|} < 1 \quad \textit{car } R(h) < R(0)\right).$

$\bullet\rho(-h) = \rho(h), \left(ie : \rho(-h) = \frac{R(-h)}{R(0)} = \frac{R(h)}{R(0)} = \rho(h)\right).$

Theorem 3.1.1 *Autocovariance of a process AR(k) verify the equations:*

$$\begin{aligned} \sum_{i=1}^{k} a_i R(h-i) &= R(h), \qquad\qquad k = 1, 2, \\ \sum_{i=1}^{k} a_i R(i) + \sigma_\epsilon^2 &= R(0) \end{aligned}$$

At the same way, we establish for the autocorelation:

$$\rho(h) - \sum_{i=1}^{k} a_i \rho(h-i) = 0 \Longleftrightarrow \rho(h) = \sum_{i=1}^{k} a_i \rho(h-i) \tag{3.1}$$

$$\Longleftrightarrow \begin{pmatrix} \rho(1) \\ \rho(2) \\ .. \\ .. \\ \rho(k) \end{pmatrix} = \begin{pmatrix} 1 & \rho(1) & \rho(2) & . & . & \rho(k-1) \\ \rho(1) & 1 & \rho(1) & . & . & \rho(k-2) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \rho(k-1) & . & . & . & \rho(1) & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ a_k \end{pmatrix} (*)$$

Remark 3.1.1 *From the system $(*)$ we can obtain the a_i function of $\rho(1), \ldots, \rho(k)$ (the matrix is defined positive)*

3.2 Function of partial autocorelation

Definition 3.2.1 *$(X_t, t \in \mathbb{Z})$ is a stationary process of second order , we call the function of partial autocorelation the function:*

$$r(h) = \frac{cov(X_t - X_t^*, X_{t-h} - X_{t-h}^*)}{Var(X_t - X_t^*)}, \qquad h > 1$$

where X_t^*(resp X_{t-h}^*) is the regression affine of X_t(resp X_{t-h}) on $X_{t-1}, ..., X_{t-h+1}$.

Remark 3.2.1 *$r(h)$ can be seen like the coefficient of correlation of X_t, X_{t-h} when we have supprime the influence of $X_{t-1}, ..., X_{t-h+1}$ on X_t and X_{t-h}.*

The sequence of partial autocorrelation of a process AR(k) has an importante propriete :

Propriete 3.2.1 *If $(X_t, t \in \mathbb{Z})$ is an AR(k), then:*

$$r(k) = a_k \quad \textit{and} \quad r(p) = 0 \quad \textit{for} \quad p > k.$$

where a_k is the last coefficient of autoregressive AR(k).

Remark 3.2.2 *1) From the system (5), we can see that $r(k)($ or $a_k)$ is function of $\rho(1), ..., \rho(k)$, and we can proof that $\rho(k)$, is function of $r(1), ..., r(k)$.*

From this we can show that knowing $\rho(k)$ is equivalent at knowing $r(k)$.

2) This propriete of the sequence $(r(h), h \geq 2)$ will be used to identify an observed autoregressive .

3.3 Built the sequence of the partial autocorrelation

Soit $(X_t, t \in \mathbb{Z})$ a real process zero-mean stationary of second order. We suppose that:

$R(0) = \sigma^2 = 1$ and the sequence $(\rho(h), h \geq 1)$ is defined positive.

$\left(ie : \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_i \delta_j \left(\rho(t_i - t_j)\right) > 0 \right)$.

For all reals not all equal to zero, and $t_1, ..., t_n \in \mathbb{Z}$ and with $\rho(h) = \mathbb{E}\left(X_t X_{t+h}\right)$ for all t and h in $\mathbb{Z}$. In this case we have: $\rho(h) = R(h)$.

If we project X_t on the vectorial space constituting by $\{X_{t-1}, ..., X_{t-l}, l \geq 1\}$ we have: $X_t = \sum_{i=1}^{l} a_i(l) X_{t-j} + \varepsilon_t$

where the random variable ε_t are zero-mean and orthogonals at $X_{t-1}, ..., X_{t-l}$

Then; the sequence of partial autocorrelation $(r(h), h \geq 1)$ is determines by the rezolution of the following sequence of matrix equations:

$$R_l a^{(l)} = \rho^{(l)} \quad \text{for} \quad l \geq 1 \tag{3.2}$$

where:

The vector: $\left(a^{(l)}\right)' = \left(a_1^{(l)}, a_2^{(l)}, ..., a_k^{(l)}\right)$
The matrix: $R_l = (\rho(|i-j|))\, , i, j = 1, ..., l$
The vector: $\rho^{(l)} = \left(\rho^{(1)}, ..., \rho^{(l)}\right)$
And the sequence $(r(h), h \geq 1)$ is defined by:

$$r(l) = a_l(l) \quad \text{for} \quad l \geq 1 \tag{3.3}$$

The resolution of the last equation is taking as following:

$$\begin{array}{rrcl} (D.1) & r(1) & = & a_1(1) = \rho(1) \\ (D.2) & \sigma^2(1) & = & 1 - r^2(1) \\ (D.3) & \rho(l+1) & = & a_{l+1}(l+1) = \dfrac{\rho(l+1) - \sum_{j=1}^{l} a_j(l)\rho(l+1-j)}{\sigma^2(l)} \\ & \text{où } : \sigma^2(l) & = & 1 - \sum_{j=1}^{l} a_j(l)\rho(j) \\ (D.4) & a_j(l+1) & = & a_j(l) - r(l+1)a_{l+1-j}(l) \qquad (j = 1, 2, ..., l) \\ (D.5) & \sigma^2(l+1) & = & \sigma^2(l)(1 - r^2(l+1)) \end{array} \qquad (D)$$

With the conditions: $\sigma^2(l) \neq 0$ and $|r^2(l) < 1|$

3.4 Partial autocorrelation of an autoregressive process

Let $(X_t, t \in \mathbb{Z})$ a gaussian process stationary, zero-mean, having for function of covariance, and partial correlation respectively, $R(h), r(h)$

Theorem 3.4.1 *The covariance function $R(h)$ of process $(X_t, t \in \mathbb{Z})$ is defined positive (ie:$\exists c_1, ..., c_n$ such that : $\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j R(i-j) > 0$ with the minimum $c_i \neq 0$) If and only if:*

$$|r(h)| < 1 \qquad \textit{for all} \qquad h > 1$$

Theorem 3.4.2 *If $(X_t, t \in \mathbb{Z})$ is an autoregressive process, then its covariance function is defined positive.*

From the two last theorems , we deduce the following corollary:

Corollary 3.4.1 *The sequence of partial autocorrelation $(r(h), h \geq 1)$ of an autoregressive process is such that:*

$$|r(h)| < 1 \quad \textit{for all} \quad h \geq 1$$

From the following theorem, we can gate informations about $\sigma^2(l)$ which was introduce in Durbin rezolution .

Theorem 3.4.3 *If $(X_t, t \in \mathbb{Z})$ is an autoregressive process with an associated white noise $(\varepsilon_t)_{t\in\mathbb{Z}}$, then $\sigma^2(l)$ is such that:*

$$\sigma^2(l) \geq \sigma_\varepsilon^2 \quad \textit{for} \quad l \geq 1$$

$$\sigma^2(l) \longrightarrow \sigma_\varepsilon^2 \quad \textit{when} \quad l \longrightarrow \infty$$

And exactaly if $(X_t, t \in \mathbb{Z})$ is an autoregressive process of order k_0 we obtain:

$$\sigma^2(l) = \sigma_\varepsilon^2 \quad \textit{for} \quad l \geq k_0$$

where $\sigma_\varepsilon^2 = Var(\varepsilon_t) > 0$

3.5 Estimation of autocovariance operators for ARH(1)

3.6 Construct of estimators

Let be $(X_t, t \in \mathbb{Z})$ a gaussian, stationary and zero-mean autoregressive process of order k wich verify:

$$X_t = a_1 X_{t-1} + a_2 X_{t-2} + ... + a_k X_{t-k} + \varepsilon_t$$

where the parameters of estimate are:$a_1, ..., a_k$ and σ^2. with a knowing order k.

The $(\varepsilon_t, t \in \mathbb{Z})$ constitute a sequence of random variable wich are independents and with the same law $\mathcal{N}(0, \sigma_\varepsilon^2)$.

Remark 3.6.1 *The frequentaly used estimators are the estimators of moindres carrés and the maximum likelihood.*

The first estimators are obtains by regression of X_t on $X_{t-1}, ..., X_{t-k}$ for $t = 1, ..., N$. wich recall to minimise the quantity:

$$Q(a_1, ..., a_k) = \sum_{t=1}^{N}(X_t - a_1X_{t-1} - ... - a_kX_{t-k})^2$$

For the second estimators , we obtain the maximum likelihood estmators of the logarithm under normality assumption of (ε_t)

$$L(a_1, ..., a_k) = (-N-k)Log(\sigma_\varepsilon\sqrt{2\pi}) - \frac{1}{2\sigma_\varepsilon^2}\sum_{t=1}^{N}(X_t - a_1X_{t-1} - ... - a_kX_{t-k})^2$$

Then we have: to maximise $L(a_1, ..., a_k)$ we must minimise its second term wich is $Q(a_1, ..., a_k)$

And the two estimators are similars and its done by the equations of **Yule-Walker**:

$$\begin{pmatrix} \widehat{R}(1,1) & . & . & . & \widehat{R}(1,k) \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \widehat{R}(k,1) & . & . & . & \widehat{R}(k,k) \end{pmatrix} \begin{pmatrix} \widehat{a_1}(k) \\ . \\ . \\ . \\ \widehat{a_k}(k) \end{pmatrix} = \begin{pmatrix} \widehat{R}(0,1) \\ . \\ . \\ . \\ \widehat{R}(0,k) \end{pmatrix}$$

where: $\widehat{R}(i,j) = \frac{1}{N}\sum_{t=1}^{N} X_{t-i}X_{t-j}$ with $i, j = 0, 1, ..., k$

The $\widehat{a_1}(k), ..., \widehat{a_k}(k)$ are the estimators of $a_1, ..., a_k$.

The approched maximum likelihood estimator of σ_ε^2 is:

$$\widehat{\sigma_\varepsilon^2}(k) = \frac{1}{N}\sum_{t=1}^{N}(X_t - \widehat{a_1}(k)X_{t-1} - ... - \widehat{a_k}(k)X_{t-k})^2$$

.

3.6.1 Convergence and limit law of estimators

(a) Probability Convergence:

Theorem 3.6.1 *Under assumption* (ϵ_t) *are independents, zero-mean, with a same law and such that:*

$\mathbb{E}(\epsilon_t^4) < \infty$, *then the random vector:*

$(\sqrt{N}(\widehat{a}_1(k) - a_1), ..., \sqrt{N}(\widehat{a}_k(k) - a_k)) \rightarrow_L \mathcal{N}(0, \sigma_\epsilon^2 R_k^{-1})$

where R_k *is a covariance matrix of* X_t*.*

Its evident that from the last theorem we can obtain:

$\widehat{a}_i(k) \longrightarrow a_i$ in probability for $i = 1, 2, ..., k$

We can see that from the following theorem wich is establish by **Anderson** we obtain the same result but under low assumptions on (ϵ_t).

Theorem 3.6.2 *If the* ϵ_t *are independents with* $\mathbb{E}(\epsilon_t) = 0$ *and* $\mathbb{E}(\epsilon_t^2) = \sigma_\epsilon^2 > 0$*.*

If the ϵ_t *are with the same law, verify:*

$\mathbb{E}(|\epsilon_t|^{2+\epsilon}) < m$ *with* $t = 1, 2, ..., \epsilon > 0$ *and* $m > 0$*, then we have:*

$$\widehat{a}_i(k) \rightarrow_P a_i \qquad \text{for } i = 1, ..., k$$
$$\widehat{\sigma_\epsilon^2}(k) \rightarrow_P \sigma_\epsilon^2 \qquad \text{when } N \longrightarrow \infty$$

Remark 3.6.2 *The first theorem is the principal result of* **Mann et Wald** *wich hade establish the first results on the convergence and the limits law of estimators(des moindres carrés) .*

This results are used principly for built tests and confiance intervals for the parameters to estimate in case of a big size.

(b)convergence presque sure:

Hannan et Rissanen have establish the convergence presque sure of maximum likelihood estimators. Under some regularity assumption on the space of parameter and on the function of likelihood , we have:

$\widehat{a}_i(k) \longrightarrow a_i \quad p.s$ for $i = 1, ..., k$ $\widehat{\sigma}_\epsilon^2(k) \longrightarrow \sigma_\epsilon^2 \quad p.s$ when $N \longrightarrow \infty$.

3.7 Estimate of autocorrelation and partial autocorrelation

3.7.1 The empirical autocorrelation

A natural estimator of autocorrelation $\rho(h)$ is:

$$\widehat{\rho}(h) = \frac{\sum_{t=0}^{N-k}(X_t - \overline{X}_N)(X_{t+h} - \overline{X}_N)}{\sum_{t=0}^{N}(X_t - \overline{X_N})^2} \qquad h \geq 0$$

with $\overline{X}_N = \frac{1}{N}\sum_{t=1}^{N} X_t$.

Theorem 3.7.1 *If $(X_t, t \in \mathbb{Z})$ has the representation:*

$$X_t = \sum_{j=-\infty}^{+\infty} \psi_j \epsilon_{t-j}$$

where $\sum_j |\psi_j| < \infty$ and the random variables $(\epsilon_t)_{t\in\mathbb{Z}}$ are zero-mean, independents, with variance σ^2 and admette a moments of order 6 (with $\mathbb{E}(\epsilon_t^6) = \tau\sigma^6$)

Under this assumptions, for h and q fixed, we have:

(1)

$$\begin{aligned} Ncov(\widehat{\rho}(h), \widehat{\rho}(q)) &\longrightarrow \sum_{j=-\infty}^{+\infty}[\rho(j)\rho(j-h+q) + \rho(j+q)\rho(j-h) \\ &- 2\rho(q)\rho(j)\rho(j-h) - 2\rho(h)\rho(j-q)\rho(j) + 2\rho(h)\rho(q)\rho^2(j)] \\ &= \phi_{h,q}. \end{aligned}$$

(2) $\left(\sqrt{N}(\widehat{\rho}(1) - \rho(1)), ..., \sqrt{N}(\widehat{\rho}(k) - \rho(k))\right) \rightarrow_L \mathcal{N}(0, \sum), \forall k$ *fixed and where the covariance matrix* $\sum$ *has the indix term* (h, q) *the quantity* $\phi_{h,q}$

Corollary 3.7.1 *Under the same assumptions of the last theorem, we have:*

For $h \geq q > 0$:

$$\mathbb{E}(\widehat{\rho}(h)) = -\frac{N-h}{N(N-1)} + 0(N^{-2})$$

3.7.2 Empirical partial autocorrelation

To estimate the sequence of partial autocorrelation $(r(l), l \geq 2)$ of the autoregressive of order k, we must use that $r(l)$ is a function of $\rho(l)$.

Estimate $\rho(l)$ by $\widehat{\rho}(l)$, and using the built of sequence $(r(l), l \geq 2)$ seen in section I-5, we obtain an estimator $\widehat{r}(l)$ of $r(l)$ from the equation as bellow:

$$\widehat{r}(l) = \widehat{a}_l(l) \qquad \text{for } l \geq 1 \tag{3.4}$$

where the estimator $\widehat{a}_l(l)$ is obtained from the system (7) using $\widehat{\rho}(i), i = 1, ...$

From the Durbin rezolution we can obtain the sequence $\widehat{r}(l), l \geq 1$:

$$\widehat{r}(1) = \widehat{a}_1(1) = \widehat{\rho}(1)$$

$$\widehat{r}(l+1) = \frac{\widehat{\rho}(l+1) - \sum_{j=1}^{l} \widehat{a}_j(l)\widehat{\rho}(l+1-j)}{\widehat{\sigma}^2(l)}$$

where:

$$\widehat{\sigma}^2(l) = 1 - \sum_{j=1}^{l} \widehat{a}_j(l)\widehat{\rho}(l)$$

and we have also:

$$\widehat{\sigma}^2(1) = 1 - \widehat{r}^2(1)$$

$$\widehat{\sigma}^2(l+1) = \widehat{\sigma}^2(l)(1 - \widehat{r}^2(l+1))$$

in each time that we have: $\widehat{\sigma}^2(l) \neq 0$ and $|\widehat{r}^2(l)| < 1$.

Remark 3.7.1 *If the process $(X_t, t \in \mathbb{Z})$ is an autoregressive of order k, we have:*

$$\widehat{r}(k) = \widehat{a}_k(k) = \widehat{a}_k$$

where $\widehat{a}_k$ is an estimator of the last coefficient a_k of the autoregressive and $\widehat{\sigma}^2_\epsilon(k) = \widehat{\sigma}^2_\epsilon$ where $\widehat{\sigma}^2_\epsilon = 1 - \sum_{j=1}^{k} \widehat{a}_j(l)\widehat{\rho}(j)$ is the estimator of σ^2_ϵ defined by the equation of **Yule-Walker** *putting $\sigma^2_x = 1$.*

Chapter 4

New exponential probability inequality and complete convergence for conditionally linearly negative quadrant dependent random variables sequence, application to AR(1) model generated by $\mathcal{F}$-LNQD errors

Abstract. The exponential probability inequalities have been important tools in probability and statistics. In this paper, We prove a new tail probability inequality for the distributions of sums of conditionally linearly negative quadrant dependent ($\mathcal{F}$-LNQD, in short) random variables, and obtain a result dealing with conditionally complete convergence of first-order autoregressive processes with identically distributed ($\mathcal{F}$-LNQD) innovations.

4.1 Introduction

The exponential inequality plays an important role in various proofs of limit theorems. In particular, it provides a measure of the complete convergence for partial sums.

Firstly, we will recall the definitions of conditionally negative quadrant dependent, conditionally negatively associated, and conditionally linearly negative quadrant dependent sequence.
Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and all random variables in this paper are defined on it unless otherwise mentioned. Let $\mathcal{F}$ be a sub-algebra of $\mathcal{A}$, two random variables ζ_1 and ζ_2 are said to be conditionally negative quadrant dependent given $\mathcal{F}$($\mathcal{F}$-NQD, in short) if, for all $\epsilon_1, \epsilon_2 \in \mathbb{R}$

$$\mathbb{P}^{\mathcal{F}}(\zeta_1 \leq \epsilon_1, \zeta_2 \leq \epsilon_2) \leq \mathbb{P}^{\mathcal{F}}(\zeta_1 \leq \epsilon_1)\mathbb{P}^{\mathcal{F}}(\zeta_2 \leq \epsilon_2). \tag{4.1}$$

One of the many possible multivariate generalizations of conditionally negative quadrant dependence is conditionally negatively association introduced by Yuan et al.[68] .
A finite collection of random variables $\zeta_1, \zeta_2, ..., \zeta_n$ is said to be conditionally negatively associated ($\mathcal{F}$-NA, in short) if for every pair of disjoint subsets A, B of $\{1, 2, ..., n\}$

$$Cov^{\mathcal{F}}(f(\zeta_i : i \in A), g(\zeta_j : j \in B)) \leq 0,$$

whenever f and g are coordinatewise nondecreasing such that this covariance exists. An infinite sequence $\{\zeta_n, n \geq 1\}$ is $\mathcal{F}$-NA if every finite subcollection is $\mathcal{F}$-NA.
We now propose another multivariate generalization of conditionally negative quadrant dependence called conditionally linearly negative quadrant dependence, which is weaker than $\mathcal{F}$-NA property.

Definition 4.1.1 *A finite sequence of random variables $\{\zeta_n, n \geq 1\}$ is said to be conditionally linearly negative quadrant dependent given ($\mathcal{F}$-LNQD, in short) if for any disjoint subsets $A, B \subset \mathcal{Z}$ and positive $r_j^{'}s$,*

$$\sum_{k\in A} r_k \zeta_k \textit{ and } \sum_{j \in B} r_j \zeta_j \textit{ are } \mathcal{F} - NQD.$$

As mentioned earlier, it can be shown that the concepts of linearly negative quadrant dependent and conditional linearly negative quadrant dependent are not equivalent. See, for example, Yuan and Xie [69], where various of counterexamples are given.

A concrete example where conditional limit theorems are useful is the study of statistical inference for non-ergodic models as discussed in Bassawa and Prakasa Rao [5] and Basawa and Scott [6]. For instance, if one wants to estimate the mean off-spring for a Galton-Watson branching process, the asymptotic properties of the maximum likelihood estimator depend on the set of non-extinction.

As it was pointed out earlier, the conditional LNQD property does not imply the LNQD property and the opposite implication is also not true. Hence one does have to derive limit theorems under conditioning if there is a need for such results even through the results and proofs of such results may be analogous to those under the non-conditioning setup. This one of the reasons for developing results for sequences of $\mathcal{F}$-LNQD random variables in this chapter.

As mentioned earlier, large numbers of results for LNQD random variables have been achieved. However, nothing is variable for conditional LNQD random variables. Yuan and Wu [71] extended many results from negative association to asymptotically negative association, Yuan and Yang [72] extended many results from association to conditional association, Yuan et al [68] extended many results from negative association to conditional negative association, and these motivate our original interest in conditional LNQD.

On the other hand, the concept of complet convergence of a sequence of random variables was introduced by [37]. Note that complete convergence implies almost sure convergence in view of the Borel-Cantelli lemma. Now we extend this concept a conditionally converge completely given $\mathcal{F}$ to a constant a if $\sum_{i=1}^{\infty} \mathrm{P}(|X_i - a| > \varepsilon/\mathcal{F}) < \infty$ for every $\varepsilon > 0$, and we whrite $X_n \to a$ conditionally completely given $\mathcal{F}$.

The main purpose of this chapter is to establish a new probability inequality and conditional complete convergence for the $\mathcal{F} - LNQD$

random variables and to extend and improve the results of Wang et al [65].

Throughout the paper, let $S_n = \sum_{i=1}^{n} X_i$ for a sequence $\{X_n, n \geq 1\}$ of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{F}$ is a sub-σ-algebra of $\mathcal{A}$, $\{X_n, n \geq 1\}$ will be called $\mathcal{F}$-centered if $\mathbb{E}^{\mathcal{F}} X_n = 0$

for every $n \geq 1$. Denote $B_n = \sum_{i=1}^{n} \mathbb{E}^{\mathcal{F}} |X_i|^2$ for each $1 \leq i \leq n$.

4.2 Some lemmas

Lemma 4.2.1 *[69] Let random variables X and Y be $\mathcal{F}$-NQD. Then*
(i) $\mathbb{E}^{\mathcal{F}}(XY) \leq \mathbb{E}^{\mathcal{F}}(X)\mathbb{E}^{\mathcal{F}}(Y)$;
(ii) $\mathbb{P}^{\mathcal{F}}(X > x, Y > y) \leq \mathbb{P}^{\mathcal{F}}(X > x)\mathbb{P}^{\mathcal{F}}(Y > y)$;
(iii) *If f and g are both nondecreasing (or both nonincreasing) functions, then $f(X)$ and $g(Y)$ are $\mathcal{F}$-NQD.*

Corollary 4.2.1 *Let $\{X_n, n \geq 1\}$ be a sequence of $\mathcal{F}$-LNQD random variables and $t > 0$, then for each $n \geq 1$,*

$$\mathbb{E}^{\mathcal{F}}\left[\sum_{i=1}^{n} \exp(tX_i)\right] \leq \prod_{i=1}^{n} \mathbb{E}^{\mathcal{F}}(\exp(tX_i)) \tag{4.2}$$

Proof. For $t > 0$, it is easy to see that tX_i and $t\sum_{j=i+1}^{n} X_j$ are $\mathcal{F}$-NQD by the definition of $\mathcal{F}$-LNQD, which implies that $\exp(tX_i)$ and $\exp(t\sum_{j=i+1}^{n} X_j)$ are also $\mathcal{F}$-NQD for $i = 1, 2, ..., n-1$ by Lemma 4.2.1(iii). It follows from Lemma 4.2.1(i) and induction that

$$\begin{aligned}
\mathbb{E}^{\mathcal{F}}\left[\sum_{i=1}^{n} \exp(tX_i)\right] &= \mathbb{E}^{\mathcal{F}}\left[\exp(tX_1)\exp\lambda(\sum_{i=2}^{n} tX_i\rho)\right] \\
&\leq \mathbb{E}^{\mathcal{F}}[\exp(tX_1)]\,\mathbb{E}^{\mathcal{F}}\left[\exp\lambda(\sum_{i=2}^{n} tX_i\rho)\right] \\
&= \mathbb{E}^{\mathcal{F}}[\exp(tX_1)]\,\mathbb{E}^{\mathcal{F}}\left[\exp(tX_2)\exp\lambda(\sum_{i=3}^{n} tX_i\rho)\right] \\
&\leq \mathbb{E}^{\mathcal{F}}[\exp(tX_1)]\,\mathbb{E}^{\mathcal{F}}[\exp(tX_2)]\,\mathbb{E}^{\mathcal{F}}\left[\exp\lambda(\sum_{i=3}^{n} tX_i\rho)\right] \\
&\leq \prod_{i=1}^{n} \mathbb{E}^{\mathcal{F}}(\exp(tX_i)).
\end{aligned}$$

This completes the proof of the lemma.

Lemma 4.2.2 *[20] For any $x \in \mathbb{R}$, we have*

$$\exp(x) \le 1 + x + \frac{|x|}{2}\ln(1+|x|)\exp(2|x|).$$

Lemma 4.2.3 *Let $\{X_n, n \ge 1\}$ be a sequence of $\mathcal{F}$-LNQD random variables with $\mathbb{E}^{\mathcal{F}}(X_n) = 0$ for each $n \ge 1$. If there exists a sequence of positive numbers $\{c_n, n \ge 1\}$ such that $|X_i| \le c_i$ for each $i \ge 1$, then for any $t > 0$,*

$$\mathbb{E}^{\mathcal{F}} \exp\left\{ t\sum_{i=1}^{n} X_i \right\} \le \exp\left\{ \frac{t^2}{2}\sum_{i=1}^{n} e^{2tc_i}\mathbb{E}^{\mathcal{F}}|X_i|^2 \right\}. \tag{4.3}$$

Proof. By lemma 4.2.2, for all $x \in \mathbb{R}$, $\exp(x) \le 1 + x + \frac{|x|}{2}\ln(1+|x|)\exp(2|x|)$. Thus, by $\mathbb{E}^{\mathcal{F}}(X_i) = 0$ and $|X_i| \le c_i$ for each $i \ge 1$, we have

$$\begin{aligned}
\mathbb{E}^{\mathcal{F}} \exp(tX_i) &\le \mathbb{E}^{\mathcal{F}}\left\{1 + tX_i + \frac{t}{2}|X_i|\ln(1+|tX_i|)\exp(2|tX_i|)\right\} \\
&= 1 + t\mathbb{E}^{\mathcal{F}}(X_i) + \frac{t}{2}\mathbb{E}^{\mathcal{F}}\left\{|X_i|\ln(1+|tX_i|)\exp(2|tX_i|)\right\} \\
&= 1 + \frac{t}{2}\mathbb{E}^{\mathcal{F}}\left\{|X_i|\ln(1+|tX_i|)\exp(2|tX_i|)\right\} \\
&\le 1 + \frac{t}{2}\mathbb{E}^{\mathcal{F}}\left\{|X_i|\ln(1+|tX_i|)\exp(2tc_i)\right\} \\
&= 1 + \frac{t}{2}\exp(2tc_i)\mathbb{E}^{\mathcal{F}}\left\{t|X_i|^2\right\} \\
&= 1 + \frac{t^2}{2}\exp(2tc_i)\mathbb{E}^{\mathcal{F}}\left\{t|X_i|^2\right\} \\
&\le \exp\left\{\frac{t^2}{2}\exp(2tc_i)\mathbb{E}^{\mathcal{F}}\left\{|X_i|^2\right\}\right\} (\text{ using } 1 + y \le \exp(y) \text{ for all } y \in \mathbb{R})
\end{aligned} \tag{4.4}$$

for any $t > 0$. By Lemma 4.2.1 and (4.4), we have can see that

$$\mathbb{E}^{\mathcal{F}} \exp\left\{ t\sum_{i=1}^{n} X_i \right\} \le \prod_{i=1}^{n} \mathbb{E}^{\mathcal{F}} \exp\{tX_i\} \tag{4.5}$$

$$\le \exp\left\{ \frac{t^2}{2}\sum_{i=1}^{n} e^{2tc_i}\mathbb{E}^{\mathcal{F}}|X_i|^2 \right\}. \tag{4.6}$$

The lemma is thus proved.

Lemma 4.2.4 *Let $\{X_n, n \geq 1\}$ be a sequence of $\mathcal{F}$-LNQD random variables with $\mathbb{E}^{\mathcal{F}}(X_n) = 0$ for each $n \geq 1$. If there exists a sequence of positive numbers $\{c_n, n \geq 1\}$ such that $|X_i| \leq c_i$ for each $i \geq 1$, then for any $t > 0$ and $\varepsilon > 0$*

$$\mathbb{P}^{\mathcal{F}}(|\sum_{i=1}^{n} X_i| \geq \varepsilon) \leq \exp\left\{-t\varepsilon + \frac{t^2}{2}\sum_{i=1}^{n} e^{2tc_i}\mathbb{E}^{\mathcal{F}}|X_i|^2\right\}. \tag{4.7}$$

Proof. By Markov's inequality and lemma 4.2.3, we can see that

$$\begin{aligned}
\mathbb{P}^{\mathcal{F}}(\sum_{i=1}^{n} X_i \geq \varepsilon) &\leq \exp(-t\varepsilon)\mathbb{E}^{\mathcal{F}} \exp\left\{t\sum_{i=1}^{n} X_i\right\} \\
&\leq \exp(t\varepsilon)\prod_{i=1}^{n} \mathbb{E}^{\mathcal{F}} \exp\{tX_i\} \\
&\leq \exp\left\{-t\varepsilon + \frac{t^2}{2}\sum_{i=1}^{n} e^{2tc_i}\mathbb{E}^{\mathcal{F}}|X_i|^2\right\}. \qquad (4.8)
\end{aligned}$$

The desired result follows by remplacing X_i by $-X_i$ in (4.8). This completes the proof of the lemma.

4.3 Main Results and Proofs

Theorem 4.3.1 *Let $\{X_n, n \geq 1\}$ be a sequence of $\mathcal{F}$-LNQD random variables with $\mathbb{E}^{\mathcal{F}}(X_i) = 0$. If there exists a positive numbers c such that*

$|X_i| \leq c_i, i \geq 1$, where $B_n = \sum_{i=1}^{n} \mathbb{E}^{\mathcal{F}}|X_i|^2$, then for any $\varepsilon > 0$ and $n \geq 1$, then

$$\mathbb{P}^{\mathcal{F}}(S_n/B_n \geq \varepsilon) \leq \exp\left\{\frac{1}{q}b^{q/p}e\right\}\exp\left\{-\left(\frac{\varepsilon 2^{p-1}bp}{B_n^{p-1}}\right)^{\frac{1}{2p-1}}\varepsilon B_n\left(1-\frac{1}{p-1}\right)\right\} \tag{4.9}$$

Proof.By Markov's inequality, we have that for any $t > 0$,

$$\begin{aligned}
\mathbb{P}^{\mathcal{F}}(S_n/B_n \geq \varepsilon) &= \mathbb{P}^{\mathcal{F}}(e^{tS_n} \geq e^{t\varepsilon B_n}), \\
&\leq e^{-t\varepsilon B_n}\mathbb{E}^{\mathcal{F}}\left(\prod_{i=1}^{n} e^{tX_i}\right), \\
&\leq \exp\left\{-t\varepsilon B_n + \frac{t^2}{2}e^{2t\max_{1\leq i\leq n}c_i}B_n\right\}.
\end{aligned} \tag{4.10}$$

Let $p > 1$. It is well known that

$$uv = \inf_{b>0}\left\{\frac{1}{pb}u^p + \frac{1}{q}b^{q/p}v^q\right\} \text{ for } u > 0, v > 0 \text{ and } \frac{1}{p} + \frac{1}{q} = 1.$$

This yields the inequality

$$\frac{t^2}{2}e^{2t\max_{1\leq i\leq n}c_i}B_n \leq \frac{1}{pb}\frac{t^{2p}}{2^p}B_n^p + \frac{1}{q}b^{q/p}e^{2tq\max_{1\leq i\leq n}c_i}. \tag{4.11}$$

We can thus conclude that for every $p > 1$, there for all $t > 0$, such that

$$\begin{aligned}
\mathbb{P}^{\mathcal{F}}(S_n/B_n \geq \varepsilon) &\leq \exp\left\{-t\varepsilon B_n + \frac{1}{pb}\frac{t^{2p}}{2^p}B_n^p\right\} \\
&\times \exp\left\{\frac{1}{q}b^{q/p}e^{2tq\max_{1\leq i\leq n}c_i}\right\} \\
&= \exp\left\{\frac{1}{q}b^{q/p}e^{2tq\max_{1\leq i\leq n}c_i}\right\}\exp(\Phi(t,n)).
\end{aligned} \tag{4.12}$$

The equation $\frac{\partial\Phi(t,n)}{\partial t} = 0$ has the unique solution

$$t = \left(\frac{\varepsilon 2^{p-1}bp}{B_n^{p-1}}\right)^{\frac{1}{2p-1}} \tag{4.13}$$

which minimizes $\Phi(t,n)$. Then from 4.12,4.13 and taking $2tq\max_{1\leq i\leq n}c_i \leq 1$ we obtain 4.9.

Theorem 4.3.2 *Let $\{X_n, n \geq 1\}$ be a sequence of $\mathcal{F}$-LNQD random variables with $\mathbb{E}^{\mathcal{F}}(X_i) = 0$. If there exists a positive numbers c such that $|X_i| \leq c_i, i \geq 1$, then for any $\varepsilon > 0$ and $n \geq 1$,*

$$\mathbb{P}^{\mathcal{F}}(|S_n| \geq \varepsilon) \leq 2\exp\left\{\frac{1}{q}b^{q/p}e\right\}\exp\left\{-\left(\frac{\varepsilon 2^{p-1}bp}{B_n^{p-1}}\right)^{\frac{1}{2p-1}}\varepsilon\left(1-\frac{1}{p-1}\right)\right\} \tag{4.14}$$

Proof.From conditions $\mathbb{E}^{\mathcal{F}}(X_i) = 0$ and $|X_i| < c_i$ for each $i \geq 1$. By Markov's inequality and Lemma 4.2.3, Corollary 4.2.1 with the fact that $1 + x \leq e^x$, then

$$\begin{aligned}
\mathbb{P}^{\mathcal{F}}(S_n \geq \varepsilon) &= e^{-t\varepsilon}\mathbb{E}^{\mathcal{F}}(e^{tS_n}), \\
&\leq e^{-t\varepsilon}\prod_{i=1}^{n}\exp\left(\frac{t^2}{2}e^{2tc_i}\mathbb{E}^{\mathcal{F}}|X_i|^2\right), \\
&\leq \exp\left\{-t\varepsilon + \frac{t^2}{2}e^{2t\max_{1\leq i\leq n}c_i}B_n\right\}.
\end{aligned} \tag{4.15}$$

Let $p > 1$. It is well known that

$$uv = \inf_{b>0}\left\{\frac{1}{pb}u^p + \frac{1}{q}b^{q/p}v^q\right\} \text{ for } u > 0, v > 0 \text{ and } \frac{1}{p} + \frac{1}{q} = 1.$$

This yields the inequality

$$\frac{t^2}{2}e^{2t\max_{1\leq i\leq n}c_i}B_n \leq \frac{1}{pb}\frac{t^{2p}}{2^p}B_n^p + \frac{1}{q}b^{q/p}e^{2tq\max_{1\leq i\leq n}c_i}. \tag{4.16}$$

We can thus conclude that for every $p > 1$, there for all $t > 0$, such that

$$\begin{aligned}
\mathbb{P}^{\mathcal{F}}(|S_n| \geq \varepsilon) &\leq 2\exp\left\{-t\varepsilon + \frac{1}{pb}\frac{t^{2p}}{2^p}B_n^p\right\} \\
&\times \exp\left\{\frac{1}{q}b^{q/p}e^{2tq\max_{1\leq i\leq n}c_i}\right\} \\
&= 2\exp\left\{\frac{1}{q}b^{q/p}e^{2tq\max_{1\leq i\leq n}c_i}\right\}\exp(\Phi(t,n)).
\end{aligned} \tag{4.17}$$

The equation $\frac{\partial \Phi(t,n)}{\partial t} = 0$ has the unique solution

$$t = \left(\frac{\varepsilon 2^{p-1} bp}{B_n^p} \right)^{\frac{1}{2p-1}} \tag{4.18}$$

which minimizes $\Phi(t,n)$. Then from 4.17,4.18 and taking $2tq \max_{1 \le i \le n} c_i \le 1$ we obtain upper bound for the tail probability as

$$\mathbb{P}^{\mathcal{F}}(|S_n| \ge \varepsilon) \le 2\exp\{\frac{1}{q} b^{q/p} e\} \exp\left\{ - \left(\frac{\varepsilon 2^{p-1} bp}{B_n^p} \right)^{\frac{1}{2p-1}} \varepsilon \left(1 - \frac{1}{p-1} \right) \right\} \tag{4.19}$$

Since $\{-X_n, n \ge 1\}$ is also a sequense of $\mathcal{F}$-LNQD random variables it follows from 4.19 that

$$\begin{aligned} \mathbb{P}^{\mathcal{F}}(S_n \le -\varepsilon) = \mathbb{P}^{\mathcal{F}}(-S_n \ge \varepsilon) \;\; & \le \;\; \exp\left\{ \frac{1}{q} b^{q/p} e \right\} \\ & \times \;\; \exp\left\{ - \left(\frac{\varepsilon 2^{p-1} bp}{B_n^p} \right)^{\frac{1}{2p-1}} \varepsilon \left(1 - \frac{1}{p-1} \right) \right\} \end{aligned} \tag{4.20}$$

From 4.19 and 4.20 we obtain

$$\begin{aligned} \mathbb{P}^{\mathcal{F}}(|S_n| \ge \varepsilon) = \mathbb{P}^{\mathcal{F}}(S_n \ge -\varepsilon) + \mathbb{P}^{\mathcal{F}}(S_n \le \varepsilon) \;\; & \le \;\; 2\exp\left\{ \frac{1}{q} b^{q/p} e \right\} \\ & \times \;\; \exp\left\{ - \left(\frac{\varepsilon 2^{p-1} bp}{B_n^p} \right)^{\frac{1}{2p-1}} \varepsilon \left(1 - \frac{1}{p-1} \right) \right\} \end{aligned} \tag{4.21}$$

Theorem 4.3.3 *Let $\{X_n, n \ge 1\}$ be a sequence of $\mathcal{F}$-LNQD random variables with mean zero and finite variances. If there exists a positive numbers c such that $|X_i| \le c_i, i \ge 1,$ where $c_n, n \ge 1$ is a sequence of positive numbers. Then for any $\varepsilon > 0$ and $n \ge 1$,*

$$\mathbb{P}^{\mathcal{F}}(|S_n - \mathbb{E}^{\mathcal{F}} S_n| \ge \varepsilon) \le 2\exp\left\{ \frac{1}{q} b^{q/p} e \right\} \exp\left\{ - \left(\frac{\varepsilon 2^{p-1} bp}{B_n^{p-1}} \right)^{\frac{1}{2p-1}} \varepsilon B_n \left(1 - \frac{1}{p-1} \right) \right\} \tag{4.22}$$

Proof.By Markov's inequality and Lemma 4.2.2, we have that for any $t > 0$,

$$\begin{aligned} \mathbb{P}^{\mathcal{F}}(S_n - \mathbb{E}^{\mathcal{F}}S_n \geq \varepsilon) &\leq e^{-t\varepsilon}\mathbb{E}^{\mathcal{F}}[\exp(t\sum_{i=1}^{n}(X_i - \mathbb{E}^{\mathcal{F}}X_i))], \\ &\leq e^{-t\varepsilon}\mathbb{E}^{\mathcal{F}}\prod_{i=1}^{n}\left[e^{t(X_i - \mathbb{E}^{\mathcal{F}}X_i)}\right], \\ &\leq \exp\left\{-t\varepsilon + \frac{t^2}{2}e^{2t\max_{1\leq i\leq n} c_i}B_n\right\}. \end{aligned} \tag{4.23}$$

Let $p > 1$. It is well known that

$$uv = \inf_{b>0}\left\{\frac{1}{pb}u^p + \frac{1}{q}b^{q/p}v^q\right\} \text{ for } u > 0, v > 0 \text{ and } \frac{1}{p} + \frac{1}{q} = 1.$$

This yields the inequality

$$\frac{t^2}{2}e^{2t\max_{1\leq i\leq n} c_i}B_n \leq \frac{1}{pb}\frac{t^{2p}}{2^p}B_n^p + \frac{1}{q}b^{q/p}e^{2tq\max_{1\leq i\leq n} c_i}. \tag{4.24}$$

We can thus conclude that for every $p > 1$, there for all $t > 0$, such that

$$\begin{aligned} \mathbb{P}^{\mathcal{F}}(|S_n - \mathbb{E}^{\mathcal{F}}S_n| \geq \varepsilon) &\leq 2\exp\left\{-t\varepsilon + \frac{1}{pb}\frac{t^{2p}}{2^p}B_n^p\right\} \\ &\times \exp\left\{\frac{1}{q}b^{q/p}e^{2tq\max_{1\leq i\leq n} c_i}\right\} \\ &= 2\exp\left\{\frac{1}{q}b^{q/p}e^{2tq\max_{1\leq i\leq n} c_i}\right\}\exp(\Phi(t,n)). \end{aligned} \tag{4.25}$$

The equation $\frac{\partial\Phi(t,n)}{\partial t} = 0$ has the unique solutio

Taking $t = \left(\frac{\varepsilon 2^{p-1}bp}{B_n^p}\right)^{\frac{1}{2p-1}}$. Hence it follows from 4.23 that

$$\mathbb{P}^{\mathcal{F}}(S_n - \mathbb{E}^{\mathcal{F}}S_n \geq \varepsilon) \leq \exp\left\{\frac{1}{q}b^{q/p}e\right\}\exp\left\{-\left(\frac{\epsilon 2^{p-1}bp}{B_n^p}\right)^{\frac{1}{2p-1}}\varepsilon\left(1 - \frac{1}{p-1}\right)\right\} \tag{4.26}$$

Let $-S_n = T_n = \sum_{i=1}^{n}(-X_n)$. Since $\{-X_n, n \geq 1\}$ is also a sequence of $\mathcal{F}$-LNQD random variables we also have

$$\mathbb{P}^{\mathcal{F}}(S_n - \mathbb{E}^{\mathcal{F}} S_n \leq -\varepsilon) = \mathbb{P}^{\mathcal{F}}(T_n - \mathbb{E}^{\mathcal{F}} T_n \geq \varepsilon) \leq \exp\left\{\frac{1}{q} b^{q/p} e\right\} \times \exp\left\{-\left(\frac{\varepsilon 2^{p-1} bp}{B_n^p}\right)^{\frac{1}{2p-1}} \varepsilon\left(1 - \frac{1}{p-1}\right)\right\} \tag{4.27}$$

by Combining (4.26) and(4.27) we get (4.22)

Corollary 4.3.1 *Let $\{X_n, n \geq 1\}$ be a sequence of $\mathcal{F}$-LNQD random variables. Assume that there exists a positive integer n_0 such that $|X_i| \leq c_n$, for each $1 \leq i \leq n, n \geq n_0$, where $\{c_n, n \geq 1\}$ is a sequence of positive numbers. Then for any $\varepsilon > 0$*

$$\mathbb{P}^{\mathcal{F}}(|S_n - \mathbb{E}^{\mathcal{F}} S_n| \geq n\varepsilon) \leq 2\exp\left\{\frac{1}{q} b^{q/p} e\right\} \exp\left\{-\left(\frac{n\varepsilon 2^{p-1} bp}{B_n^p}\right)^{\frac{1}{2p-1}} n\varepsilon\left(1 - \frac{1}{p-1}\right)\right\} \tag{4.28}$$

Theorem 4.3.4 *Let $\{X_n, n \geq 1\}$ be a sequence of $\mathcal{F}$-LNQD random variables with $\mathbb{E}^{\mathcal{F}}(X_i) = 0$. If there exists a positive numbers c such that $|X_i| \leq c_i, i \geq 1$. Then for any $r > 0$*

$$n^{-r} S_n \to 0 \quad completely, \quad n \to \infty. \tag{4.29}$$

Proof.Let $B = \sum_{n=1}^{\infty} \mathbb{E}^{\mathcal{F}}(X_n)^2 \leq \infty$. For any $\varepsilon > 0$, it follows from Theoreme 4.3.2 we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathbb{P}^{\mathcal{F}}(|S_n| \geq n^r\varepsilon) &\leq 2\sum_{n=1}^{\infty} \exp\left\{\frac{1}{q} b^{q/p} e\right\} \exp\left\{-\left(\frac{n^r\varepsilon 2^{p-1} bp}{B_n^p}\right)^{\frac{1}{2p-1}} \varepsilon n^r\left(1 - \frac{1}{p-1}\right)\right\} \\
&\leq 2\sum_{n=1}^{\infty} \exp\left\{\frac{1}{q} b^{q/p} e\right\} \exp\left\{-\left(\frac{\varepsilon 2^{p-1} bp}{B_n^p}\right)^{\frac{1}{2p-1}} \varepsilon\left(1 - \frac{1}{p-1}\right)\right\}^{n^{\frac{2rp}{2p-1}}} \\
&\leq 2\exp\left\{\frac{1}{q} b^{q/p} e\right\} \sum_{n=1}^{\infty} [\exp(-c)]^{n^{\frac{2rp}{2p-1}}}.
\end{aligned} \tag{4.30}$$

where C is positive number not depending on n. (by the inequality $e^{-y} \leq (\frac{a}{ey})^a$), choosing $a = \frac{2p-1}{rp}$, since $a > 0, y > 0$. Then the right-hand side of 4.30 become

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathbb{P}^{\mathcal{F}}(|S_n| \geq n^r \varepsilon) &\leq 2\exp\{\frac{1}{q} b^{q/p} e\} \sum_{n=1}^{\infty} \left(\frac{a}{ec}\right)^a \left(\frac{1}{n}\right)^{\left(\frac{2rp}{2p-1}\right)^a} \\
&\leq 2\exp\left\{\frac{1}{q} b^{q/p} e\right\} \frac{a^a}{(ec)^a} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{2rpa}{2p-1}}} \\
&\leq 2\exp\left\{\frac{1}{q} b^{q/p} e\right\} \frac{a^a}{(ec)^a} \sum_{n=1}^{\infty} \frac{1}{n^2}, \\
&= 2\exp\left\{\frac{1}{q} b^{q/p} e\right\} \frac{a^a}{(ec)^a} \frac{\pi^2}{6}, \\
&< \infty
\end{aligned} \tag{4.31}$$

Theorem 4.3.5 *Let $\{X_n, n \geq 1\}$ be a sequence of $\mathcal{F}$-LNQD random variables. Assume that there exists a positive integer n_0 such that $|X_i| \leq c_n$, for each $1 \leq i \leq n, n \geq n_0$, where $\{c_n, n \geq 1\}$ is a sequence of positive numbers.*

$$\sum_{n=1}^{\infty} \mathbb{P}^{\mathcal{F}} \left(\frac{1}{n} |S_n - \mathbb{E}^{\mathcal{F}} S_n| \geq \varepsilon_n \right) < \infty. \tag{4.32}$$

Theorem 4.3.6 *Let $\{X_n, n \geq 1\}$ be a sequence of $\mathcal{F}$-LNQD random variables with $\mathbb{E}^{\mathcal{F}}(X_i) = 0$. If there exists a positive numbers c such that $|X_i| \leq c_i, i \geq 1$. Then for any $r > 0$*

$$n^{-r}(S_n - \mathbb{E}^{\mathcal{F}} S_n) \to 0 \quad completely, n \to \infty. \tag{4.33}$$

Proof. For any $\varepsilon > 0$, it follows from Corollary 4.3.1 that

$$
\begin{aligned}
\sum_{n=1}^{\infty} \mathbb{P}^{\mathcal{F}}(|S_n - \mathbb{E}^{\mathcal{F}} S_n| \geq n^r \varepsilon) &\leq 2 \sum_{n=1}^{\infty} \exp\left\{\frac{1}{q} b^{q/p} e\right\} \exp\left\{-\left(\frac{n^r \varepsilon 2^{p-1} bp}{B_n^p}\right)^{\frac{1}{2p-1}} \varepsilon n^r \left(1 - \frac{1}{p-1}\right)\right\} \\
&\leq 2 \sum_{n=1}^{\infty} \left[\exp\left\{\frac{1}{q} b^{q/p} e\right\}\right] \\
&\times \left[\exp\left\{-\left(\frac{\varepsilon 2^{p-1} bp}{B_n^p}\right)^{\frac{1}{2p-1}} \varepsilon \left(1 - \frac{1}{p-1}\right)\right\}\right]^{n^{\frac{2rp}{2p-1}}}
\end{aligned}
\tag{4.34}
$$

After this result we get 4.33.

4.4 Applications to the results to AR(1) model

The basic object of this section is applying the results to first-order autoregressive processes(AR(1)).

We consider an autoregressive time series of first order AR(1) defined by

$$X_{n+1} = \theta X_n + \zeta_{n+1},\ n = 1, 2, ..., \tag{4.35}$$

where $\{\zeta_n, n \geq 0\}$ is a sequence of identically distributed $\mathcal{F}$-LNQD random variables with $\zeta_0 = X_0 = 0$, $0 < \mathbb{E}^{\mathcal{F}} \zeta_k^4 < \infty$, $k = 1, 2, ...$ and where θ is a parameter with $|\theta| < 1$. Here, we can rewrite X_{n+1} in 4.35 as follows:

$$X_{n+1} = \theta^{n+1} X_0 + \theta^n \zeta_1 + \theta^{n-1} \zeta_2 + ... + \zeta_{n+1}. \tag{4.36}$$

The coefficient θ is fitted least squares, giving the estimator

$$\widehat{\theta}_n = \frac{\sum_{j=1}^{n} X_j X_{j-1}}{\sum_{j=1}^{n} X_{j-1}^2} \tag{4.37}$$

It immediately follows from (4.35) and (4.37) that

$$\widehat{\theta}_n - \theta = \frac{\sum_{j=1}^{n} \zeta_j X_{j-1}}{\sum_{j=1}^{n} X_{j-1}^2} \tag{4.38}$$

Theorem 4.4.1 *Let the conditions of Theorem 4.3.3 be satisfied then for any* $\frac{(\mathbb{E}^{\mathcal{F}}\zeta_1^2)^{1/2}}{\rho^2} < \xi$ *positive, we have*

$$\begin{aligned} \mathbb{P}^{\mathcal{F}}(\sqrt{n}|\widehat{\theta}_n - \theta| > \rho) \quad &\leq \quad 2\exp\left\{-\left(\frac{(\rho^2\xi^2 - \mathbb{E}^{\mathcal{F}}\zeta_1^2)n2^{p-1}bp}{B_n^p}\right)^{\frac{1}{2p-1}} (\rho^2\xi^2 - \mathbb{E}^{\mathcal{F}}\zeta_1^2)n\left(1 - \frac{1}{p-1}\right)\right\} \\ &\times \quad \exp\left\{\frac{1}{q}b^{q/p}e\right\} + \exp\left\{-\frac{1}{2}n\frac{(K_1 - n\xi^2)^2}{K_2}\right\} \end{aligned} \tag{4.39}$$

where $K_1 = \mathbb{E}^{\mathcal{F}}(X_i^2) < \infty, K_2 = \mathbb{E}^{\mathcal{F}}(X_i^4) < \infty.$

Proof. Firstly, we notice that :

$$\widehat{\theta}_n - \theta = \frac{\sum_{j=1}^{n} \zeta_j X_{j-1}}{\sum_{j=1}^{n} X_{j-1}^2}.$$

It follows that

$$\mathbb{P}^{\mathcal{F}}(\sqrt{n}|\widehat{\theta}_n - \theta| > \rho) = \mathbb{P}^{\mathcal{F}}\left(|\frac{\frac{1}{\sqrt{n}}\sum_{j=1}^{n} \zeta_j X_{j-1}}{\frac{1}{n}\sum_{j=1}^{n} X_{j-1}^2}| > \rho\right)$$

By virtue of the probability properties and Hölder's inequality, we have for any ξ positive

$$\begin{aligned}
\mathbb{P}^{\mathcal{F}}(\sqrt{n}|\widehat{\theta}_n-\theta|>\rho) &\leq \mathbb{P}^{\mathcal{F}}\left(\frac{1}{n}\sum_{j=1}^{n}\zeta_j^2 \geq \rho^2\xi^2\right)+\mathbb{P}^{\mathcal{F}}\left(\frac{1}{n^2}\sum_{j=1}^{n}X_{j-1}^2 \leq \xi^2\right)\\
&= \mathbb{P}^{\mathcal{F}}\left(\sum_{j=1}^{n}\zeta_j^2 \geq (\rho^2\xi^2)n\right)+\mathbb{P}^{\mathcal{F}}\left(\sum_{j=1}^{n}X_{j-1}^2 \leq n^2\xi^2\right)\\
&= I_{1n}+I_{2n}.
\end{aligned}$$

Next we estimate I_{1n} and I_{2n}.

$$\begin{aligned}
I_{1n} &= \mathbb{P}^{\mathcal{F}}\left(\sum_{j=1}^{n}\zeta_j^2 \geq (\rho^2\xi^2)n\right)\\
&= \mathbb{P}^{\mathcal{F}}\left(\sum_{j=1}^{n}(\zeta_j^2-\mathbb{E}^{\mathcal{F}}\zeta_j^2+\mathbb{E}^{\mathcal{F}}\zeta_j^2) \geq (\rho^2\xi^2)n\right)\\
&= \mathbb{P}^{\mathcal{F}}\left(\sum_{j=1}^{n}(\zeta_j^2-\mathbb{E}^{\mathcal{F}}\zeta_j^2) \geq (\rho^2\xi^2-\mathbb{E}^{\mathcal{F}}\zeta_1^2)n\right)\\
&\leq \mathbb{P}^{\mathcal{F}}\left(\left|\sum_{j=1}^{n}(\zeta_j^2-\mathbb{E}^{\mathcal{F}}\zeta_j^2)\right| \geq (\rho^2\xi^2-\mathbb{E}^{\mathcal{F}}\zeta_1^2)n\right)
\end{aligned} \tag{4.40}$$

By using the Theorem 4.3.3 the right hand side of 4.40 become

$$\begin{aligned}
I_{1n} &= \mathbb{P}^{\mathcal{F}}\left(\sum_{j=1}^{n}\zeta_j^2 \geq (\rho^2\xi^2)n\right)\\
&\leq 2\exp\left\{-\left(\frac{(\rho^2\xi^2-\mathbb{E}^{\mathcal{F}}\zeta_1^2)n2^{p-1}bp}{B_n^p}\right)^{\frac{1}{2p-1}}(\rho^2\xi^2-\mathbb{E}^{\mathcal{F}}\zeta_1^2)n\left(1-\frac{1}{p-1}\right)\right\}\\
&\times \exp\left\{\frac{1}{q}b^{q/p}e\right\}
\end{aligned} \tag{4.41}$$

We will bound now, the second probability of the right-hand side of the expression I_{2n}.According to the Markov's inequality, it follows for any t positive

$$
\begin{aligned}
I_{2n} &= \mathbb{P}^{\mathcal{F}}\left(\frac{1}{n^2}\sum_{i=1}^{n} X_{i-1}^2 \leq \xi^2\right) \\
&= \mathbb{P}^{\mathcal{F}}\left(n^2\xi^2 - \sum_{i=1}^{n} X_{i-1}^2 \geq 0\right) \\
&= \mathbb{E}^{\mathcal{F}}\left(\mathbb{I}_{\{n\epsilon^2 - \sum_{i=1}^{n} X_{i-1}^2 \geq 0\}}\right) \\
&\leq \mathbb{E}^{\mathcal{F}}\left(\exp t\left(n^2\xi^2 - \sum_{i=1}^{n} X_{i-1}^2\right)\right) \qquad (t > 0) \\
&\leq e^{tn^2\xi^2}\mathbb{E}^{\mathcal{F}}\left(\exp -t\sum_{i=1}^{n} X_{i-1}^2\right) \\
&\leq e^{tn^2\xi^2}\prod_{i=1}^{n}\mathbb{E}^{\mathcal{F}}\left(\exp -tX_{i-1}^2\right).
\end{aligned}
$$

Since

$$
I_{2n} \leq e^{tn^2\xi^2}\prod_{i=1}^{n}\mathbb{E}^{\mathcal{F}}\left(\exp -tX_{i-1}^2\right).
$$

we first claim that for $x \geq 0$

$$
e^{-x} \leq 1 - x + \frac{1}{2}x^2. \tag{4.42}
$$

To see this let $\psi(x) = e^{-x}$ and $\phi(x) = 1 - x + \frac{1}{2}x^2$, ($\psi'(x) = -e^{-x}$) and recall that for every x

$$
e^{x} \geq 1 + x \quad \forall x, \tag{4.43}
$$

so that $\psi'(x) = -e^{-x} \leq -1 + x = \phi'(x)$. Since $\psi(0) = 1 = \phi(0)$ this implies $\psi(x) \leq \phi(x)$ for all $x \geq 0$ and 4.42 is claimed.
From 4.42 and 4.43 it follows that for $t > 0$

$$
\begin{aligned}
e^{tn\epsilon^2}\prod_{i=1}^{n}\mathbb{E}^{\mathcal{F}}\left(\exp(-tX_{i-1}^2)\right) &\le e^{tn^2\xi^2}\left(1-tK_1+\frac{t^2}{2}K_2\right)^n\\
&\le e^{tn^2\xi^2}\left(\exp\left(-tK_1+\frac{t^2}{2}K_2\right)\right)^n\\
&\le e^{tn^2\xi^2}\exp\left(-ntK_1+\frac{t^2}{2}nK_2\right)
\end{aligned}
$$

where $K_1=\mathbb{E}^{\mathcal{F}}(X_i^2)<\infty, K_2=\mathbb{E}^{\mathcal{F}}(X_i^4)<\infty$.
Hence

$$
I_{2n}=\mathbb{P}^{\mathcal{F}}\left(\sum_{i=1}^{n}X_{i-1}^2\le n^2\xi^2\right)\le\exp\left[t\left(n^2\xi^2-nK_1\right)+\frac{nt^2K_2}{2}\right]. \qquad (4.44)
$$

With $h(t)=n^2\xi^2-nK_1+\frac{nt^2K_2}{2}$ and $t>0$, the equation $h^{'}(t)=0$ has the unique solution $t=\frac{K_1-n\xi^2}{K_2}$ which minimize $h(t)$. Hence

$$
\mathbb{P}^{\mathcal{F}}\left(\sum_{i=1}^{n}X_{i-1}^2\le n^2\xi^2\right)\le\exp\left\{-\frac{1}{2}n\frac{(K_1-n\xi^2)^2}{K_2}\right\} \qquad (4.45)
$$

Then for every $\rho>0$, $K_1<\infty$, $K_2<\infty$, and by the assumption

$$
\begin{aligned}
\mathbb{P}^{\mathcal{F}}(\sqrt{n}|\widehat{\theta}_n-\theta|>\rho) \;\le\; & 2\exp\left\{-\left(\frac{(\rho^2\xi^2-\mathbb{E}^{\mathcal{F}}\zeta_1^2)n2^{p-1}bp}{B_n^p}\right)^{\frac{1}{2p-1}}(\rho^2\xi-\mathbb{E}^{\mathcal{F}}\zeta_1^2)n\left(1-\frac{1}{p-1}\right)\right\}\\
\times\; & \exp\left\{\frac{1}{q}b^{q/p}e\right\}+\exp\left\{-\frac{1}{2}n\frac{(K_1-n\xi^2)^2}{K_2}\right\}. \qquad (4.46)
\end{aligned}
$$

These complete the proof.

Corollary 4.4.1 *The sequence $(\widehat{\theta}_n)_{n\in\mathbb{N}}$ is completely converges to the parameter θ of autoregressive process AR(1) model. Then we have*

$$
\sum_{n=1}^{+\infty}\mathbb{P}^{\mathcal{F}}(\sqrt{n}|\widehat{\theta}_n-\theta|>\rho)<+\infty. \qquad (4.47)
$$

Proof. By using Theorem 4.3.6 and $\mathbb{E}^{\mathcal{F}}(X_i^2)<\infty,\mathbb{E}^{\mathcal{F}}(X_i^4)<\infty$ we get the result of 4.47 immediately.

Chapter 5

Probability type inequalities and complete convergence for weighted partial sums of WOD random variables

5.1 Introduction

The laws of large numbers for weighted sums of random variables has been studied in the last decades by many authors such as [21], [23], [59], [60], [**?**] and [62]. These autors established the almost sure convergence of

$$\sum_{k=1}^{n} a_{n,k} X_k \tag{5.1}$$

under the traditional assumption of independence and identical distribution (i.i.d) of the sequence of random variables $\{X_n, n \geq 1\}$ and imposing an asymptotic condition on the triangular array o real numbers $\{a_{n,k}, 1 \leq k \leq n, n \geq 1\}$. Among them [22] stands out where the rate of convergence obtained for $a_{n,k}$ are the same order off magnitude as the sums of i.i.d. random variables in the classical Kolmogorv strong law of large numbers,

$$\max_{1\leq k\leq n} |a_{n,k}| = O(\frac{1}{n}), \ \ n \longrightarrow \infty$$

we also highlight the [59] where interesting rate of convergence for $a_{n,k}$ is considered admitting finite pth absolute moment for i.i.d. random variables $(1 < p < 2)$

wich, as a matter of fact, improved previous results of [61]. In the inal eighties [3] established te almost sure convergence of

$$\sum_{k=1}^{n} a_{n,k} \left(X_k - \mathbb{E}X_k\right) \tag{5.2}$$

For a special tpe of weights, that is when $a_{n,k} = \frac{\alpha_k}{\beta_n}$. The study of the almost sure convergence of 5.1 for this sort of weights continues until today under weaker assumptions of the random variables [26], [73]

The importance of the limiting behavior of 5.1 is well illustrated in many statistical problems such as least-squares estimators, nonparametric regression function estimators or jackknife estimators among others, which emoldens us to study this challenging topic.The main purpose of this chapter is to obtain the complete convergence of the weighted sum 5.2 under weak assumptions on the sequence of random variables $(X_n, n \geq 1)$, on the other hand, keeping alive the best results known to the rate of convergence of 5.1 in the i.i.d. scenario, on the other. More precisely, we relax the assumption of identical distribution to stochastic doinance which states that a random sequence $(X_n, n \geq 1)$ is stocchastically dominated by a random variable X if there exists a constant $C > 0$ such that

$$\sup_{n \geq 1} \mathbb{P}\{|X_n| > t\} \leq C\mathbb{P}\{|X| > t\}$$

for each $t > 0$ (any identical distributed random sequence $(X_n, n \geq 1)$ is of course; stochastically dominated by X_1). In particular, this is the only assumption on the random sequence that we need to obtain the almost complete convergence of 5.1 when $0 < p < 1$. For $1 \leq p \leq 2$ some additional condition on the random sequence shall be required; indeed supposing that $(X_n, n \geq 1)$ is **widely dependent random variables**

Definition 5.1.1 *It is well know that various dependent random variables (r.v.s) have been put forward successively. Based on the notion of negatively orthant dependence structure of r.v. s, Wang et al [66] introduced the notion of widl orthant dependence structure of r.v.s. By definition* $(X_i, i \geq 1)$ *are said to be widely upper orthant dependent (WUOD) if for each* $n \geq 1$*, there exists a positive number* $g_u(n)$ *such that, for all* $x_i \in \mathbb{R}, i = 1, ..., n$

$$\mathbb{P}\left(\cap_{i=1}^{n}\{X_i > x_i\}\right) \leq g_u(n) \prod_{i=1}^{n} \mathbb{P}(X_i > x_i) \tag{5.3}$$

they are said to be widely lower orthant dependent (WLOD) if for each $n \geq 1$, there exists a positive number $g_L(n)$ such that, for all $x_i \in \mathbb{R}, i = 1, ..., n$

$$\mathbb{P}\left(\cap_{i=1}^{n}\{X_i \leq x_i\}\right) \leq g_L(n) \prod_{i=1}^{n} \mathbb{P}(X_i \leq x_i) \tag{5.4}$$

and they are said to be widely orthant dependent (WOD) if they are both WUOD and WLOD.
WUOD, WLOD and WOD r.v.s are called by joint name widely dependent (WD) r.v.s, and $g_U(n), g_L(n), n \geq 1$, are called dominating coefficients. Clearly, we have $g_U(n) \geq 1, g_L(n) \geq 1, n \geq 2$, and $g_U(1) = g_L(1) = 1$

Further, Wang et al [66] provided some examples of WD r.v.s , which showed that the WD structure may contain common negatively dependent r.v.s, some positively dependent r.v.s and some others.For example, when $g_U(n), g_L(n) = M$ for all $n \geq 1$ and some positive constant M, inequalities 5.3 and 5.4 describe extended negativel upper and lower orthant dependent (ENUOD and ENLOD) r.v.s, respectively. Random variables $(X_i, i \geq 1)$, are said to be extended negatively orthant dependent (ENOD), if they are both ENLOD and ENUOD. ENUOD, ENLOD and ENOD r.v.s are called collectively END r.v.s (see [43]). More specially, if $M = 1$, then we have correspondingly the notions of NUOD, NLOD, NOD and ND r.v.s (see [29]). hold for each $n \geq 1$ and $x_1, x_, ..., x_n$ [43], it will be established that the rate of convergence considered by [59] for weighted sums of i.i.d. random variables having finite pth absolute moment $(1 \leq p \leq 2)$, it is also sufficient to ensure the almost complete convergence of 5.2.

Associated to a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we shall consider the space $\mathcal{L}_p(p > 0)$ of all measurable functions X for which $\mathbb{E}|X|^p < \infty$. Moreover, the function $x \longrightarrow max\{1, logx\}$ will be denoted by $Logx$.

5.2 Somme Lemmas

Lemma 5.2.1 *Let the sequence $(X_n, n \geq 1)$ of random variables be stochastically dominated by random variables X. Then, for any $p > 0, t > 0$*

$$\mathbb{E}|X_n|^p \mathbb{I}_{\{|X_n| \leq t\}} \leq C\mathbb{E}|X|^p \mathbb{I}_{\{|X| \leq t\}} + Ct^p \mathbb{P}\{|X| > t\}$$

and

$$\mathbb{E}|X_n|^p \mathbb{I}_{\{|X_n| > t\}} \leq C\mathbb{E}|X|^p \mathbb{I}_{\{|X| > t\}}$$

Lemma 5.2.2 *If* $(X_n, n \geq 1)$ *are non-negative random variables stochastically dominated by a non-negative random variables* X *such that* $\mathbb{E}(X) < \infty$ *then* $\sum_{n=1}^{\infty} \frac{X_n^p}{n^p} < \infty$ *a.s for any* $p > 1$

Lemma 5.2.3 *Let* $\{a_n\}$ *be a positive sequence of real numbers and* $s_n = \sum_{k=1}^{n} a_k \longrightarrow \infty$. *Then for any random variable* $X \geq 0$ *a.s.*

$$\sum_{n=1}^{\infty} a_n \mathbb{P}\{X \geq s_n\} \leq \mathbb{E}X \leq \sum_{n=0}^{\infty} a_{n+1} \mathbb{P}\{X > s_n\}$$

Lemma 5.2.4 *If* $(X_n, n \geq 1)$ *are random variables stochastically dominated by a random variable* X *such that* $\mathbb{E}|X|^p < \infty$ *for some* $0 < p < 2$ *then*

$$\sum_{n=1}^{\infty} \frac{1}{n^{2/p} Log^{1-2/p} n} \mathbb{E}[|X_n|^2 \mathbb{I}_{\{|X_n| \leq \frac{n^{1/p}}{Log^{1/p} n}\}} + \frac{n^{2/p}}{Log^{2/p} n} \mathbb{I}_{\{|X_n| > \frac{n^{1/p}}{Log^{1/p} n}\}}] < \infty$$

Furthermore, if $p > 1$ *then*

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/p} Log^{1-1/p} n} \mathbb{E}[|X_n| \mathbb{I}_{\{|X_n| \leq \frac{n^{1/p}}{Log^{1/p} n}\}} + \frac{n^{1/p}}{Log^{1/p} n} \mathbb{I}_{\{|X_n| > \frac{n^{1/p}}{Log^{1/p} n}\}}] < \infty$$

Remark 5.2.1 *for the proof of the last Lemmas , we can see [44]*

Lemma 5.2.5 *[20] for any* $x \in \mathbb{R}$, *and* $0 < \alpha \leq 1$, *we have:*

$$exp(x) \leq 1 + x + |x|^{1+\alpha} exp(2|x|)$$

5.3 Main results

Theorem 5.3.1 *If* $(X_n, n \geq 1)$ *is a sequence of widely dependent random variables stochastically dominated by a random variable* $X \in \mathcal{L}_p$ *for some* $1 < p < 2$, *and* $\{a_{n,k}, 1 \leq k \leq n, n \geq 1\}$ *is an array of constants satisfying*

$$\max_{1 \leq k \leq n} |a_{n,k}| = O\left(\frac{1}{n^{1/p} Log^{1-1/p} n}\right) \quad n \longrightarrow \infty \tag{5.5}$$

then

$$\sum_{k=1}^{n} a_{n,k} (X_k - \mathbb{E}X_k) \longrightarrow 0$$

Proof:
By
$$\sum_{k=1}^{n} a_{n,k}X_k = \sum_{k=1}^{n} a_{n,k}^{+}X_k - \sum_{k=1}^{n} a_{n,k}^{-}X_k$$
where $a_{n,k}^{+} = max\{a_{n,k}, 0\} \geq 0$ and $a_{n,k}^{-} = max\{-a_{n,k}, 0\} \geq 0$
Setting
$$\begin{aligned} X_n^{`} &= X_n \mathbb{I}_{\{|X_n| \leq \frac{n1/p}{Log^{1/p}n}\}} - \mathbb{E}X_n \mathbb{I}_{\{|X_n| \leq \frac{n^{1/p}}{Log^{1/p}n}\}} \\ &+ \frac{n^{1/p}}{Log^{1/p}n}\mathbb{I}_{\{X_n > \frac{n^{1/p}}{Log^{1/p}n}\}} - \frac{n^{1/p}}{Log^{1/p}n}\mathbb{I}_{\{X_n < -\frac{n^{1/p}}{Log^{1/p}n}\}} \end{aligned}$$
and
$$\begin{aligned} X_n^{``} &= X_n \mathbb{I}_{\{|X_n| > \frac{n1/p}{Log^{1/p}n}\}} - \mathbb{E}X_n \mathbb{I}_{\{|X_n| > \frac{n^{1/p}}{Log^{1/p}n}\}} \\ &+ \frac{n^{1/p}}{Log^{1/p}n}\mathbb{I}_{\{X_n < -\frac{n^{1/p}}{Log^{1/p}n}\}} - \frac{n^{1/p}}{Log^{1/p}n}\mathbb{I}_{\{X_n > \frac{n^{1/p}}{Log^{1/p}n}\}} \end{aligned}$$
we have $X_n^{`} + X_n^{``} = X_n + \mathbb{E}X_n$. Using the inequality
$$e^x \leq 1 + x + |x|^{1+\alpha}e^{2x} \text{ for all } x \in \mathbb{R}, \text{and } 0 \leq \alpha \leq 1$$
we get, for each $t > 0$
$$\begin{aligned} exp(ta_{n,k}X_k^{`}) &\leq 1 + ta_{n,k}X_k^{`} + |ta_{n,k}X_k^{`}|^{1+\alpha}exp(2ta_{n,k}X_k^{`}) \\ &\leq 1 + ta_{n,k}X_k^{`} + C\frac{t^{1+\alpha}X_k^{`\,1+\alpha}}{n^{(1+\alpha)/p}Log^{(1+\alpha)-((1+\alpha)/p)}n}exp(2\frac{Ct}{Logn}) \end{aligned}$$
The random variables
$$X_n \mathbb{I}_{\{|X_n| > \frac{n1/p}{Log^{1/p}n}\}} + \frac{n^{1/p}}{Log^{1/p}n}\mathbb{I}_{\{X_n > \frac{n^{1/p}}{Log^{1/p}n}\}} - \frac{n^{1/p}}{Log^{1/p}n}\mathbb{I}_{\{X_n < -\frac{n^{1/p}}{Log^{1/p}n}\}}$$
widely dependent random variables, hense the sequences $\{X_n^{`}\}$ and $\{a_{n,k}X_k^{`}, 1 \leq k \leq n\}$ for every $n \geq 1$, are also widely dependent random variables (WOD) and we obtain

$$\begin{aligned}
\mathbb{E}exp\left(t\sum_{k=1}^{n}a_{n,k}X_k^{‘}\right) &\leq g(n)\prod_{k=1}^{n}\mathbb{E}exp\left(ta_{n,k}X_k^{‘}\right)\\
&\leq g(n)\prod_{k=1}^{n}\mathbb{E}\left[1+ta_{n,k}X_k^{‘}+C\frac{t^{1+\alpha}X_k^{‘\,1+\alpha}}{n^{(1+\alpha)/p}Log^{(1+\alpha)-((1+\alpha)/p)}n}+exp(2\frac{Ct}{Logn})\right]\\
&\leq g(n)\prod_{k=1}^{n}[1+\frac{Ctk^{1/p}}{n^{1/p}Log^{1-1/p}nLog^{1/p}k}\mathbb{P}\{|X|>\frac{k^{1/p}}{Log^{1/p}k}\}\\
&+ C\frac{t^{1+\alpha}X_k^{‘\,1+\alpha}}{n^{(1+\alpha)/p}Log^{(1+\alpha)-((1+\alpha)/p)}n}+exp(2\frac{Ct}{Logn})]\\
&\leq g(n)exp[\frac{2Ct}{n^{1/p}Log^{1-1/p}n}\sum_{k=1}^{n}\frac{k^{1/p}}{Log^{1/p}k}\mathbb{P}\{|X|>\frac{k^{1/p}}{Log^{1/p}k}\}\\
&+ C\frac{t^{1+\alpha}}{n^{(1+\alpha)/p}Log^{(1+\alpha)-((1+\alpha)/p)}n}exp(2\frac{Ct}{Logn})\sum_{k=1}^{n}\mathbb{E}X_k^{‘\,1+\alpha}]
\end{aligned}$$

For some positive number $g(n)$.Given $\epsilon>0$ and putting $t=2Logn/\epsilon$ we get from Chebyshev inequality ([46], p159).

$$\begin{aligned}
\mathbb{P}\{\sum_{k=1}^{n}a_{n,k}X_k^{‘}>\epsilon\} &\leq exp(-\epsilon t)\mathbb{E}exp\left(t\sum_{k=1}^{n}a_{n,k}X_k^{‘}\right)\\
&\leq g(n)exp(-\epsilon t)exp[\frac{2Ct}{n^{1/p}Log^{1-1/p}n}\sum_{k=1}^{n}\frac{k^{1/p}}{Log^{1/p}k}\mathbb{P}\{|X|>\frac{k^{1/p}}{Log^{1/p}k}\}\\
&+ C\frac{t^{1+\alpha}}{n^{(1+\alpha)/p}Log^{(1+\alpha)-((1+\alpha)/p)}n}exp(2\frac{Ct}{Logn})\sum_{k=1}^{n}\mathbb{E}X_k^{‘\,1+\alpha}]\\
&\leq \frac{g(n)}{n^2}exp[Logn\frac{4C}{\epsilon n^{1/p}Log^{1-1/p}n}\sum_{k=1}^{n}\frac{k^{1/p}}{Log^{1/p}k}\mathbb{P}\{|X|>\frac{k^{1/p}}{Log^{1/p}k}\}\\
&+ C\frac{2^{1+\alpha}log^{1+\alpha}n}{\epsilon^{1+\alpha}n^{(1+\alpha)/p}Log^{(1+\alpha)-((1+\alpha)/p)}n}exp(2\frac{C}{\epsilon})\sum_{k=1}^{n}\mathbb{E}X_k^{‘\,1+\alpha}]
\end{aligned}$$

According to lemma 3 we have

$$\sum_{k=1}^{\infty}\frac{1}{Logk}\mathbb{P}\{|X|>\frac{k^{1/p}}{Log^{1/p}k}\}<\infty$$

So thhat Kronecker's lemma implies

$$\frac{1}{n^{1/p}Log^{1-1/p}n}\sum_{k=1}^{n}\frac{k^{1/p}}{Log^{1/p}k}\mathbb{P}\{|X|>\frac{k^{1/p}}{Log^{1/p}k}\}\longrightarrow 0$$

Again Kronecker's lemma and lemma 4 give

$$\frac{1}{n^{1+\alpha}Log^{(1+\alpha)-((1+\alpha)/p)}n}\sum_{k=1}^{n}\mathbb{E}X_k^{`2}\longrightarrow 0$$

So we have

$$Logn\frac{4C}{\epsilon n^{1/p}Log^{1-1/p}n}\sum_{k=1}^{n}\frac{k^{1/p}}{Log^{1/p}k}\mathbb{P}\{|X|>\frac{k^{1/p}}{Log^{1/p}k}\}$$
$$+C\frac{2^{1+\alpha}log^{1+\alpha}n}{n^{(1+\alpha)/p}Log^{(1+\alpha)-((1+\alpha)/p)}n}exp(2\frac{C}{\epsilon})\sum_{k=1}^{n}\mathbb{E}X_k^{`\,1-\alpha}Logn$$

is bounded by $\delta Logn,\ \delta>0$ for n large enough. Hense, choosing $0<\delta<1$ $\sum_{n=1}^{\infty}\mathbb{P}\{\sum_{k=1}^{n}a_{n,k}X_k^{`}>\epsilon\}\leq C\sum_{n=1}^{\infty}\frac{1}{n^{2-\delta}}<\infty$ and Borel Cantelli lemma([76], p,61) give us

$$\lim_{n\longrightarrow\infty} sup\sum_{k=1}^{n}a_{n,k}X_k^{'}\leq 0 \qquad a.s \tag{5.6}$$

On the other hand, we have

$$\begin{aligned}\max_{2^i\leq n\leq 2^{i+1}}|\sum_{k=1}^{n}a_{n,k}X_k^{``}| &\leq C\max_{2^i\leq n\leq 2^{i+1}}\frac{1}{n^{1/p}Log^{1-1/p}n}\sum_{k=1}^{n}|X_k^{``}|\\ &\leq C\frac{1}{(2^{i+1})^{1/p}(Log2^{i+1})^{1-1/p}}\sum_{i+1}^{2^{i+1}}|X_k^{``}|\end{aligned}$$

and for any $\epsilon > 0$ we obtain from lemma4

$$\begin{aligned}\sum_{i=1}^{\infty}\mathbb{P}\{\frac{1}{(2^i)^{1/p}(Log2^i)^{1-1/p}}\sum_{k=1}2^i|X_k^{``}| > \epsilon &\leq \frac{1}{\epsilon}\sum_{i=1}^{\infty}\frac{1}{(2^i)^{1/p}(Log2^i)^{1-1/p}}\sum_{k=1}2^i\mathbb{E}|X_k^{``}|\\ &= \frac{1}{\epsilon}\sum_{k=1}^{\infty}\mathbb{E}|X_k^{``}|\sum_{\{i,^i\geq k\}}\frac{1}{(2^i)^{1/p}(Log2^i)^{1-1/p}}\\ &\leq \frac{1}{\epsilon}\sum_{k=1}^{\infty}\mathbb{E}|X_k^{``}|\frac{1}{Log^{1-1/p}k}\sum_{\{i,2^i\geq k\}}\frac{1}{(2^i)^{1/p}}\\ &\leq C\sum_{k=1}^{\infty}\frac{\mathbb{E}|X_k^{``}|}{k^{1/p}Log^{1-1/p}k} < \infty\end{aligned}$$

And we conclude from Borel-Cantelli lemma that

$$\max_{2^{i-1}\leq n\leq 2^i}|\sum_{k=1}^{n}a_{n,k}X_k{``}| \longrightarrow 0$$

and

$$\sum_{k=1}^{n}a_{n,k}X_k^{``} \longrightarrow 0 \tag{5.7}$$

from (5.6) and (5.7) we have

$$\lim_{n\longrightarrow\infty} sup\sum_{k=1}^{n}a_{n,k}\left(X_k - \mathbb{E}X_k\right) \leq 0 \quad a.s$$

replacing X_k by $-X_k$ we obtain

$$\lim_{n\longrightarrow\infty} inf\sum_{k=1}^{n}a_{n,k}\left(X_k - \mathbb{E}X_k\right) \leq 0 \quad a.s$$

wich completing the proof □

Theorem 5.3.2 *Let $(X_n, n \geq 1)$ be a sequence of widely dependent and identically distributed $\mathcal{L}_1$ random variables.*
If $\{a_{n,k}, 1 \leq k \leq n, n \geq 1\}$ is an array of constants satisfying

$$\max_{1\leq k\leq n}|a_{n,k}| = O(\frac{1}{n}),\ n \longrightarrow \infty \tag{5.8}$$

then

$$\sum_{k=1}^{n} a_{n,k} \left(X_k - \mathbb{E}X_k \right) \longrightarrow 0$$

Proof:see [44]

Chapter 6

On robust non parametric regression estimation for functional regressor

6.1 Introduction

Regression function estimation is the most important tool for addressing nonparametric prediction problems. The goal of this chapter is to study this functional parameter when the expplanatory variable is a curve by using a robust approach. The robust method used in this work belongs to the class of M-estimates introduced by **Huber (1964)**. The literature on this estimation method is quite important when the data are real (see for instance **Robinson(1984)**). **Collomb** and **Hardle(1986)** and **Boente** and **Fraiman (1989,1990)** for previous results and **Laib** and **Ould-Said (2000)** and **Boente** and **Rodriguez (2006)** for recent advances and references.For the functional case, this literature is relatively limited; indeed, **Cadre (2001)** studied the estimation of the $\mathcal{L}_1$ median of a banach space-valued random variable. **Cardot et al. (2004)** used this robust approach to study the linear regression model on quantiles with explanatory variable taking values in a Hilbert space. They established the $\mathcal{L}_2-$ convergence rate. We refer the reader to **Ferraty** and **Vieu (2006)** for the prediction problem in functional nonparametric statistics via the regression function, the conditional mode and the conditional quantiles estimation by the kernel method. The assmptotic normality of these parameters has been obtained by **Masry (2005)** and **Ezzahrioui** and **Ould-Said (2008)a.b** respectively.

Our interest in this chapter is to generalize to infinite dimension, the robust nonparametric estimation of the regression function developed by **Collomb** and **Hardle (1986)** in the real case. We establish, under suitable conditions, the almost complete convergence rate of the M-estimator with the regression function kernel weights when the observations are independent and identicall distributed. This rate is closely related to the concentration property on small balls of the functional variables probability measure. Thus, by using recent results in the probability theory of small balls, we can clarify our results for some continuous-time stochastic processes.

6.2 The model

Let (X, Y) be a pair of random variables in $\mathcal{F} * \mathbb{R}$, where the space $\mathcal{F}$ is dotted with a semi-metric $d(.,.)$ (this covers the case of normed spaces of possibly infinite dimension.) In this work, X can be a functional random variable. For any $x \in \mathcal{F}$, let ψ_x be a real-valued Borel function satisfying some regularity conditions to be stated below. The nonparametric parameter studied in this work, denoted y θ_x, is implicitly defined as a zero with respect to (w.r.t) t of the equation

$$\psi(t,x) = \mathbb{E}\left(\psi(Y-t)/X=x\right) = 0 \tag{6.1}$$

We suppose that, for all $x \in \mathcal{F}$, θ_x exists and is unique (see, for instance, **Boente** and **Fraiman (1989)**). The model θ, called ψ_x-regression in **Laib** and **Ould-Said (2000)**, is a generalization of the classical regression function. Indeed, if $\psi_x(t) = t$ we get $\theta_x = \mathbb{E}(Y/X = x)$

Let $(X_1, Y_1), \ldots.., (X_n, Y_n)$ be n independent pairs, identically distributed as (X, Y). We then estimate $\psi(t, x)$ by

$$\widetilde{\psi}(t,x) = \frac{\displaystyle\sum_{i=1}^{n} k(h_x^{-1} d(x, X_i))\psi_x(Y_i - t)}{\displaystyle\sum_{i=1}^{n} k(h_x^{-1} d(x, X_i))\psi_x(Y_i - t)}, \quad \forall t \in \mathbb{R}$$

where k is a kernel function and $h_k = h_{k,n}$ is a sequence of positive real numbers which decreases to zero as n goes to infinity. A natural estimator of θ_x is a zero w.r.t.t of

$$\widetilde{\psi}(t,x) = 0 \tag{6.2}$$

Our main goal is to obtain the rate of the almost complete convergence for $\widehat{\theta}_x$.

6.3 Main results

In the following x is a fixed point in $\mathcal{F}$, $\mathcal{N}_x$ denotes a fixed neighborhood of x, and we introduce the following assumptions:
(H_1) $\mathbb{P}(X \in B(x,h)) = \phi_x(h) > 0 \, \forall h > 0$and $\lim_{h \longrightarrow 0} \phi_x(h) = 0$.
(H_2) There exist C_1 and $b > 0$ such that $\forall x_1, x_2 \in \mathcal{N}_x$, $\forall t \in \mathbb{R}$

$$|\psi(t,x_1) - \psi(t,x_2)| \leq C_1 d^b(x_1,x_2)$$

(H_3) The function ψ_x is strictly monotone, bounded, continuously differentiable, and its derivative is such that $|\psi_x^{'}| > C_2$, $\forall t \in \mathbb{R}$.
(H_4) K is a continuous function with support $[0, 1]$ such that
$0 < C_3 < K(t) < C_4 < \infty$.
(H_5) $\lim\limits_{n \longrightarrow \infty} h_K = 0$ and $\lim\limits_{n \longrightarrow \infty} \dfrac{logn}{n\phi_x(h_K)} = 0$.

Our main result is given in the following theorem.

Theorem 6.3.1 *Assume that $(H_1)-(H_5)$ are satisfied; then $\widehat{\theta}_x$ exists and is unique a.s. for all sufficientl large n, and we have*

$$\widehat{\theta}_x - \theta_x = O(h_K^b) + O\left(\sqrt{\frac{logn}{n\phi_x(h_K)}}\right) \qquad a.co \tag{6.3}$$

Proof
In what follows, we will denotes by C some strictly positive generic constant and we put $K_i = K\left(\frac{d(x,X_i)}{h_K}\right)$
Under (H_3) we have

$$\widehat{\psi}\left(\widehat{\theta}_x, x\right) = \widehat{\psi}(\theta_x, x) + (\widehat{\theta}_x - \theta_x)\widehat{\psi}^{'}(\xi_{x,n}, x)$$

for some $\xi_{x,n}$ between $\widehat{\theta}_x$ and θ_x. The condition on the derivative of ψ_x in (H_3) leads us to write

$$\exists C_2 > 0, \forall \epsilon_0 > 0, \mathbb{P}\left(|\widehat{\theta}_x - \theta_x| \geq \epsilon_0 \left(h^b + \sqrt{\frac{logn}{n\phi_x(h)}}\right)\right)$$

$$\leq \mathbb{P}\left(|\widehat{\psi}(\theta_x, x) - \psi(\theta_x, x)| \geq C_2^{-1}\epsilon_0 \left(h^b + \sqrt{\frac{logn}{n\phi_x(h)}}\right)\right)$$

Then, (6.3) is proved as soon as the following result can be checked:

$$\widehat{\psi}\left(\widehat{\theta}_x, x\right) - \psi\left(\theta_x, x\right) = O\left(h^b + \sqrt{\frac{logn}{n\phi_x(h)}}\right) a.co \tag{6.4}$$

The proof of (6.4) is based on the decomposition

$$\forall t \in \mathbb{R}, \widehat{\psi}(t,x) - \psi(t,x) = \frac{1}{\widehat{\psi}_D(x)}\left[\left(\widehat{\psi}_N(t,x) - \mathbb{E}[\widehat{\psi}_N(t,x)]\right) - \left(\psi(t,x) - \mathbb{E}[\widehat{\psi}_N(t,x)]\right)\right]$$
$$- \frac{\psi(t,x)}{\widehat{\psi}_D(x)}\left[\widehat{\psi}_D(x) - \mathbb{E}(\widehat{\psi}_D(x))\right] \qquad (3.5)$$

where

$$\widehat{\psi}_D(x) = \tfrac{1}{n\mathbb{E}[K_1]}\sum_{i=1}^{n} K_i \text{ and } \widehat{\psi}_N(t,x) = \tfrac{1}{n\mathbb{E}[K_1]}\sum_{i=1}^{n} K_i\psi_x(Y_i - t)$$

and with the fact that $\widehat{\psi}(t,x) = \frac{\psi_N(t,x)}{\psi_D(x)}$ and $\mathbb{E}[\widehat{\psi}_D(x)] = 1$.

Finally, the proof of Theorem3.1 is achieved with the following lemmas.

Lemma 6.3.1 *Under hypotheses $(H_1), (H_2), (H_4)$ and (H_5), we have*

$$\widehat{\psi}_D(x) - \mathbb{E}[\widehat{\psi}_D(x)] = O\left(\sqrt{\frac{logn}{n\phi_x(h_K)}}\right) \qquad a.co$$

Proof:
The proof of this lemma runs along the lines of that of lemma3.1 in **Ferraty et al.(2005)**.
Let $\widetilde{\delta}_i = \frac{K_i}{\mathbb{E}[K_1]}$. From (H_1) and (H_4) we deduce
$|\widetilde{\delta}_i| < \frac{C}{\phi_x(h_K)}$ and $\mathbb{E}\left[|\widetilde{\delta}_i|^2\right] < \frac{C'}{\phi_x(h_K)}$.
So we appl the Bernestein exponential inequality to get for all $\eta > 0$

$$\mathbb{P}\left(|\widehat{\psi}_D(x) - \mathbb{E}[\widehat{\psi}_D(x)]| > \eta\sqrt{\frac{logn}{n\phi_x(h_K)}}\right) \leq C' n^{-C\eta^2}$$

□

This lemma gives straightforwardly the following corollary.

Corollary 6.3.1 *Under the hypotheses of Lemma 3.3.1, we have*

$$\sum_{n\geq 1}\mathbb{P}\left(|\widehat{\psi}_D(x)| \leq 1/2\right) \leq \sum_{n\geq 1}\mathbb{P}\left(|\widehat{\psi}_D(x) - 1| > 1/2\right) < \infty$$

Lemma 6.3.2 *Under hypotheses $(H_1), (H_2), (H_4)$ and (H_5), we have for all $t \in \mathbb{R}$*

$$\psi(t,x) - \mathbb{E}[\widehat{\psi}_N(t,x)] = O(h_K^b)$$

Proof
The equidistribution of the couples (X_i, Y_i) and (H_4) imply

$$\psi(t,x) - \mathbb{E}[\widehat{\psi}_N(t,x)] = \frac{1}{\mathbb{E}[K_1]}\mathbb{E}\left[(K_1 \mathbb{I}_{B(x,h_K)}(X_1))(\psi(t,x) - \mathbb{E}[\psi_x(Y_1 - t)\ X = X_1])\right] \tag{6.5}$$

where $\mathbb{I}$ is the indicator function. Conditioning w.r.t. X_1 with the **Holder** hypothesis, and under (H_2), we prove that (H_2) allows us to write that

$$K_1 \mathbb{I}_{B(x,h_K)}(X_1)|\psi(t, X_1) - \psi(t,x)| \leq C_1 h_K^b$$

and then

$$|\psi(t,x) - \mathbb{E}[\widehat{\psi}_N(t,x)]| \leq C_1 h_K^b \square$$

Lemma 6.3.3 *Under hpotheses $(H_1), (H_3), (H_4)$ and (H_5), we have, for all $t \in \mathbb{R}$*

$$\widehat{\psi}_N(t,x) - \mathbb{E}[\widehat{\psi}_N(t,x)] = O\left(\sqrt{\frac{logn}{n\phi_x(h_K)}}\right) \qquad a.co$$

Proof:
The proof of this result is similar to the proof of Lemma3.3.1. We put

$$\Delta_i = \frac{[K_i \psi_x(Y_i - t) - \mathbb{E}[K_1 \psi_x(Y_1 - t)]]}{\mathbb{E}[K_1]}$$

Because ψ_x is bounded, we get $|\Delta_i| \leq C/\phi_x(h_K)$ and $\mathbb{E}[\Delta_i^2] \leq C^{`}/\phi_x(h_K)$, for all $i \leq n$. As in lemma 3.3.1. Bernstein's inequality is used to finish the proof. $\square$

Lemma 6.3.4 *Under hypotheses of Theorem3.1, $\widehat{\theta}$ exists and is unique a.s for all sufficiently large n.*

Proof:
We prove this lemma by means of arguments similar to those used for Theorem1 in **Collomb and Hardle (1986)** .Indeed for all $\epsilon > 0$, the strict monotonocity of ψ_x implies

$$\psi(\theta_x - \epsilon, x) < \psi(\theta_x, x) < \psi(\theta_x + \epsilon, x)$$

Lemma 3.3.1, 3.3.2, 3.3.3 and Corollary3.3.1 show that

$$\widehat{\psi}(\theta_x, x) - \psi(\theta_x, x) = O\left(h_K^b + \sqrt{\frac{logn}{n\phi_x(h_K)}}\right) \qquad a.co$$

for all fixed real t. So, for sufficiently large n

$$\widehat{\psi}(\theta_x - \epsilon, x) \leq 0 \leq \widehat{\psi}(\theta_x + \epsilon, x) \qquad a.co$$

Since ψ_x and K are continuous functions, then $\widehat{\psi}(t, x)$ is continuous function of t; then there exists a $t_0 = \widehat{\theta}_x \in [\theta_x - \epsilon; \theta_x + \epsilon]$ such that $\widehat{\psi}(\widehat{\theta}_x, x) = 0$. Finally, the unicity of $\widehat{\theta}_x$ is a direct consequence of the strict monotonocity of ψ_x and the positivity of K $\square$

Comment:

(1) Remarks on the functional variable: The concentration hypothesis (H_1) is less restrictive than the strict positivity of the explanatory variable's density X which is usually assumed in most of the previous works in the finite-dimensional case (see Collomb and Hardle (1986) and Laib and Ould-Said (2000)). Moreover, it is checked for a great class of continuous time processes (see for instance Bogachev (1999) for a Gaussian measure and Li and Shao (2001) for a general Gaussian process).

(2) Remarks on the nonparametric model: The functional character of our model is well exploited in this work. Indeed, hypothesis (H_2) is a regularity condition which characterizes the functional space.

(3) Remarks on the robustness properties: In this work, we consider a family of ψ-functions indexed by x , in order to cover most of the $M-$ estimate classes. It is also worth noting that we keep the same conditions on the function ψ_x (assumption (H_3)) as were given by Collomb and Hardle (1986) in the multivariate case. Furthermore, the boundeness assumption on ψ_x is made for the simplicity of the proof. It can be dropped while using truncation methods as to those used in Laib and Ould-Said (2000).

(4) Remarks on the convergence rate: The expression for the convergence rate (3) is identical to those of Ferraty and Vieu (2006) and Collomb and Hardle (1986) for the regression model in the functional and the multivariate cases respectively. Thus, by considering the same arguments as Ferraty et al. (2005), we obtain the almost convergence rate $O((logn)^{-b/2})$ for the estimator $\widehat{\theta}_x$ for continuous-time stochastic process having a probability measure which is absolutely continuous with respect to the Wiener measure, under suitable bandwidth choise $(h_K \longrightarrow \eta(logn)^{-1/2})$ and for the L^∞ metric.

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الملخص:

كانت التفاوتات في الاحتمال الأسي أدوات مهمة في الاحتمالات والإحصاءات .في هذه الأطروحة ، أثبتنا وجود تفاوت احتمالي جديد لتوزيعات المتغيرات العشوائية المعتمدة السلبية الخطية ، وحصلنا على نتيجة تتعامل مع التقارب الكامل المشروط لعمليات الانحدار الذاتي من الدرجة الأولى مع ابتكارات F-LNQD الموزعة بشكل متماثل.

Abstract :

The exponential probability inequalities have been important tools in probability and statistics. In this thisis , we prove a new tail probability inequality for the distributions of sums of conditionally linearly negative quadrant dependent random variables, and obtain a result dealing with conditionally complete convergence of first-order autoregressive processes with identically distributed *F-LNQD* innovations.

Résumé :

Les inégalités de probabilité exponentielles ont été des outils importants en probabilité et en statistique.
Dans cette thèse, nous prouvons une nouvelle inégalité de probabilité pour les distributions de variables aléatoires dépendantes linéairement négatives, et obtenons un résultat traitant de la convergence conditionnellement complète des processus autorégressifs du premier ordre avec des innovations F-LNQD identiquement distribuées.