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THÈSE

Préparée au Département de Mathématiques

de la Faculté Des Sciences Exactes

Présenté par

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Intitulé

Study on solutions of boundary value problems for nonlinear fractional differential equations of variable order

Pour obtenir

Le grade de **DOCTEUR**
Thèse présentée et soutenue publiquement le ....
devant le Jury composé de

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Dedicate

I dedicate this work .
To the souls of my dear father, dear mother, and my dear grandmother, may God have mercy on them.
To my wife Soumia and daughter Asmaa.
To my brothers Benlakhedar, Moussa, Islam and my sisters Kaltoum, Mabkhouta, fraiha and my father's wife Oumelkheir.

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Publications

1. **A. Benkerrouche**, D. Baleanu, M. S. Souid, A. Hakem, M. Inc, Boundary value problem for nonlinear fractional differential equations of variable order via Kuratowski MNC technique, *Advances in Difference Equations*, **2021 :365**, 1-19, (2021).

2. **A. Benkerrouche**, M.S. Souid, S. Chandok, A. Hakem, Existence and stability of a Caputo variable-order boundary value problem, *Journal of Mathematics*, **2021**, Article ID 7967880, 1-16, (2021).

3. S. Hristova, **A. Benkerrouche**, M.S. Souid, A. Hakem, Boundary value problems of Hadamard fractional differential equations of variable order, *Symmetry*, **13(5), 896**, 1-16, (2021).

4. **A. Benkerrouche**, M. S. Souid, K. Sitthithakerngkiet, A. Hakem, Implicit nonlinear fractional differential equations of variable order, *Boundary value problems* , **2021 :64**, 1-16, (2021).

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6. **A. Benkerrouche**, M.S. Souid, A. Karapinar, A. Hakem , On the boundary value problems of Hadamard fractional differential equations of variable order, *Mathematical Methods in the Applied Sciences*, (submitted).

7. **A. Benkerrouche**, M.S. Souid, F. Jarad, A. Hakem, On boundary value problems of Caputo fractional differential equation of variable order via Kuratowski MNC technique, *Advances in Continuous and Discrete Models : Theory and Applications*, (submitted).

Abstract

In this thesis, we study the existence of solutions to boundary problems for fractional differential equations of variable order with different derivatives (Riemann-Liouville, Caputo, Hadamard).

The results of this study are based on Darbo's fixed point theorem combined with the Kuratowski measure of non-compactness or the Krasnoselskii fixed point theorem.

In addition, we study the stability of the solutions obtained in the sense of Ulam-Hyers or Ulam-Hyers-Rassias.

We construct examples to illustrate the validity of the observed results.

Key words and phrases : fractional differential equations of variable order, boundary value problem, Darbo's fixed point theorem, Krasnoselskii fixed point theorem, Kuratowski measure of noncompactness, Ulam-Hyers-Rassias stability.

AMS (MOS) Subject Classifcations : 26A33, 34A08, 34A37, 34A60.

Résume

Dans cette thèse, nous étudions l'existence de solutions aux problèmes aux limites pour des équations différentielles fractionnaires d'ordre variable avec des dérivées différentes (Riemann-Liouville, Caputo, Hadamard).

Les résultats de cette étude sont basés sur le théorème du point fixe de Darbo combiné avec la mesure de non-compactité de Kuratowski ou le théorème du point fixe de Krasnoselskii.

De plus, nous étudions la stabilité des solutions obtenues au sens d'Ulam-Hyers ou d'Ulam-Hyers-Rassias.

Nous construisons des exemples pour illustrer la validité des résultats observés.

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Introduction

The primitive idea of fractional calculus is to constitute the rational numbers in the order of derivation operators with natural numbers. Although this idea seems elementary and simple, but it involves remarkable effects and outcomes which describe many physical and natural phenomena accurately. For this reason, doing research on both of the theoretical and practical aspects of boundary value problems has been attracted the focus of many mathematicians in the international academic levels [1, 7, 21, 35, 42, 47, 48, 55, 61, 62]. A main difference and novelty in this investigation is the application of the concept of variable-order operators. These versions of variable-order operators which are dependent on their power-law kernel can explain and model several hereditary aspects of various phenomena [4, 46, 52]. Generally, it is usually difficult to solve variable-order fractional boundary value problems and obtain their analytical solution, hence, some numerical methods are introduced for the approximation of solutions to different fractional boundary value problems of variable-order.

In relation to the study of the existence theory to fractional boundary value problems of variable-order, we point out some of them. In [63], Zhang studied solutions of a two-point boundary value problems with singular differential equations of variable-order. Some years later, Zhang and Hu [65] presented the existence results for approximate solutions of a variable-order fractional initial value problem on the half-axis. Recently, Hristova et al. [23] and Refice et al. [41] turned to investigation of the Hadamard fractional boundary value problems of variable-order by means of Kuratowski measure of noncompactnes method. In 2021, Bouazza et al. [19] considered a variable-order multiterm boundary value problem and derived their results by terms of fixed point methods. For other instances, refer to [23, 31, 32, 50].

The stability theory of functional equations has developed very rapidly during the past decades. In 1940, Ulam posed the problem of stability of functional equations at the University of Wisconsin. see [56]. A year later, Hayers [24] gave the first answer to the Ulam problem in the case of Banach spaces. Therefore, this type of stability came to be called the Ulam-Hyers stability. In 1978, Rassias [40] provided a generalization of the Ulam-Hyers stability.

After that, the study of these two types of stabilities, the Ulam-Hyers stability and the Ulam-Hyers-Rassias stability has grown to be one of the most essential subjects in the field of mathematical analysis and especially the stability of differential equations, see e.g. [25, 27, 28, 36, 37, 43, 59].

In the following we give an outline of our thesis organization, Consists of **5 chapters** defining the work contributed.

The **first chapter** introduce notations, definitions, and preliminary facts which are used throughout this thesis.

In Chapter 2, we deal with the existence of solutions and the stability of the obtained solution in the sense of Ulam-Hyers-Rassias for the boundary value problem (BVP for short)

$$\begin{cases} D_{0^+}^{u(t)}x(t) = f(t, x(t), I_{0^+}^{u(t)}x(t)), \;\; t \in J := [0,T] \\ x(0) = 0, \;\; x(T) = 0, \end{cases} \tag{1}$$

where $0 < T < +\infty$, $1 < u(t) \leq 2$, $f : J \times X \times X \to X$ is a continuous function and X is a real (or complex) Banach space, and $D_{0^+}^{u(t)}$, $I_{0^+}^{u(t)}$ are the Riemann-Liouville fractional derivative and integral of variable-order $u(t)$.

In Chapter 3, we will study the existence of solutions for the boundary value problem (BVP for short)

$$\begin{cases} {}^cD_{0^+}^{u(t)}x(t) = f(t, x(t), I_{0^+}^{u(t)}x(t)), \;\; t \in J \\ x(0) = 0, \;\; x(T) = 0, \end{cases} \tag{2}$$

where $1 < u(t) \leq 2$, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and ${}^cD_{0^+}^{u(t)}$ is the Caputo fractional derivative of variable-order $u(t)$ and $I_{0^+}^{u(t)}$ is the Riemann-Liouville fractional integral of variable-order $u(t)$.
Further, we study the stability of the obtained solution in the sense of Ulam-Hyers.

In Chapter 4, we shall be concerned with the boundary value problem (BVP for short) for the Hadamard fractional differential equation of variable order

$$\begin{cases} {}^HD_{1^+}^{u(t)}x(t) = f(t, x(t), \; {}^HI_{1^+}^{u(t)}x(t)), \; t \in M := [1,T] \\ x(1) = x(T) = 0, \end{cases} \tag{3}$$

where $1 < T < \infty$, $u(t) : M \to (1, 2]$ is the variable order of the fractional derivatives, $f : M \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function and ${}^HD_{1^+}^{u(t)}$ ${}^HI_{1^+}^{u(t)}$ are the left Hadamard fractional derivative and integral of variable-order $u(t)$, and we will study the stability of the obtained solution in the sense of Ulam-Hyers-Rassias.

In Chapter 5, We deal with the boundary value problem BVP

$$\begin{cases} {}^cD_{0^+}^{u(t)}x(t) = f(t, x(t), {}^cD_{0^+}^{u(t)}x(t)), \ \ t \in J \\ x(0) = 0, \ \ x(T) = 0, \end{cases} \tag{4}$$

where $u : J \to (1, 2]$, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and ${}^cD_{0^+}^{u(t)}$ is the Caputo fractional derivative of variable-order $u(t)$.
Further, we study the stability of the obtained solution in the sense of Ulam-Hyers.

Chapitre 1

Preliminary

This Chapter introduces some important fundamental definitions which are used throughout this thesis.

1.1 Notations and definitions

The symbol $C(J, X)$ represents the Banach space of continuous functions $\varkappa : J \to X$ with the norm

$$\|\varkappa\| = Sup\{\|\varkappa(t)\| : t \in J\},$$

where X is a real (or complex) Banach space.

1.2 Fractional calculus.

1.2.1 Fractional calculus of constant-order

Definition 1.1 *([29, 39]) The left Riemann-Liouville fractional integral of the function $h \in L^1([a, b], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by*

$$I_a^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s) ds,$$

where $\Gamma(.)$ is the gamma function.

Definition 1.2 *([29, 39]) The left Riemann-Liouville fractional derivative of order $\alpha > 0$ of function $h \in L^1([a, b], \mathbb{R}_+)$, is given by*

$$(D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{d}{dt}\Big)^n \int_a^t (t-s)^{n-\alpha-1} h(s) ds,$$

here $n = [\alpha] + 1$.

The following properties are some of the main ones of the fractional derivatives and integrals.

Lemma 1.1 *([29]) Let $\alpha > 0$, $a > 0$, $h \in L^1(a,b)$, $D^{\alpha}_{a^+}h \in L^1(a,b)$. Then, the differential equation*

$$D^{\alpha}_{a^+}h = 0,$$

has unique solution

$$h(t) = \omega_1(t-a)^{\alpha-1} + \omega_2(t-a)^{\alpha-2} + ... + \omega_n(t-a)^{\alpha-n},$$

where $n = [\alpha] + 1$, $\omega_\ell \in \mathbb{R}$, $\ell = 1, 2, ..., n$.

Lemma 1.2 *([29]) Let $\alpha > 0$, $a > 0$, $h \in L^1(a,b)$, $D^{\alpha}_{a^+}h \in L^1(a,b)$. Then,*

$$I^{\alpha}_{a^+}D^{\alpha}_{a^+}h(t) = h(t) + \omega_1(t-a)^{\alpha-1} + \omega_2(t-a)^{\alpha-2} + ... + \omega_n(t-a)^{\alpha-n},$$

where $n = [\alpha] + 1$, $\omega_\ell \in \mathbb{R}$, $\ell = 1, 2, ..., n$.

Lemma 1.3 *([29]) Let $\alpha > 0$, $a > 0$, $h \in L^1(a,b)$, $D^{\alpha}_{a^+}h \in L^1(a,b)$. Then,*

$$D^{\alpha}_{a^+}I^{\alpha}_{a^+}h(t) = h(t).$$

Lemma 1.4 *([29]) Let $\alpha,\ \beta > 0$, $a > 0$, $h \in L^1(a,b)$. Then,*

$$I^{\alpha}_{a^+}I^{\beta}_{a^+}h(t) = I^{\beta}_{a^+}I^{\alpha}_{a^+}h(t) = I^{\alpha+\beta}_{a^+}h(t).$$

Definition 1.3 *([29, 39]) The left Caputo fractional derivative of order $\alpha > 0$ of function $h \in L^1([a,b],\mathbb{R}_+)$, is given by*

$${}^cD^{\alpha}_{a+}h(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}h^{(n)}(s)ds,$$

where $n = [\alpha] + 1$.

Recall the following pivotal observation.

Lemma 1.5 *([29]) Let $\alpha > 0$, $a > 0$, $h \in L^1(a,b)$, ${}^cD^{\alpha}_{a^+}h \in L^1(a,b)$. Then, the differential equation*

$${}^cD^{\alpha}_{a^+}h = 0,$$

has unique solution

$$h(t) = \omega_0 + \omega_1(t-a) + \omega_2(t-a)^2 + ... + \omega_{n-1}(t-a)^{n-1},$$

where $n = [\alpha] + 1$, $\omega_\ell \in \mathbb{R}$, $\ell = 0, 1, ..., n-1$.

Lemma 1.6 *([29]) Let $\alpha > 0,\ a > 0,\ h \in L^1(a,b),\ {}^cD^\alpha_{a^+}h \in L^1(a,b)$.*

$$I^\alpha_{a^+}\ {}^cD^\alpha_{a^+}h(t) = h(t) + \omega_0 + \omega_1(t-a) + \omega_2(t-a)^2 + ... + \omega_{n-1}(t-a)^{n-1}.$$

Lemma 1.7 *([29]) Let $\alpha > 0,\ a > 0,\ h \in L^1(a,b),\ {}^cD^\alpha_{a^+}h \in L^1(a,b)$. Then,*

$${}^cD^\alpha_{a^+}I^\alpha_{a^+}h(t) = h(t).$$

Lemma 1.8 *([29]) Let $\alpha,\ \beta > 0,\ a > 0,\ h \in L^1(a,b)$.*

$$I^\alpha_{a^+}I^\beta_{a^+}h(t) = I^\beta_{a^+}I^\alpha_{a^+}h(t) = I^{\alpha+\beta}_{a^+}h(t).$$

Definition 1.4 *([29, 39]) The left Hadamard fractional integral of the function $h \in L^1([a,b],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by*

$${}^HI^\alpha_{a^+}h(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (\log\frac{t}{s})^{\alpha-1}\frac{h(s)}{s}ds,\ \ t > a.$$

Definition 1.5 *([29, 39]) The left Hadamard fractional derivative of order $\alpha > 0$ of function $h \in L^1([a,b],\mathbb{R}_+)$, is given by*

$${}^HD^\alpha_{a^+}h(t) = \frac{1}{\Gamma(n-\alpha)}(t\frac{d}{dt})^n\int_a^t (\log\frac{t}{s})^{n-\alpha-1}\frac{h(s)}{s}ds,\quad t > a,$$

where $n = [\alpha] + 1$.

The following properties are some of the main ones of the fractional derivatives and integrals.

Lemma 1.9 *([29]) Let $\alpha > 0,\ a > 1,\ h \in L^1(a,b),\ D^\alpha_{a^+}h \in L^1(a,b)$. Then, the differential equation*

$${}^HD^\alpha_{a^+}h = 0,$$

has unique solution

$$h(t) = \omega_1\left(\log\frac{t}{a}\right)^{\alpha-1} + \omega_2\left(\log\frac{t}{a}\right)^{\alpha-2} + ... + \omega_n\left(\log\frac{t}{a}\right)^{\alpha-n},$$

where $n = [\alpha] + 1$, $\omega_\ell \in \mathbb{R}$, $\ell = 1, 2, ..., n$.

Lemma 1.10 *([29]) Let $\alpha > 0,\ a > 1,\ h \in L^1(a,b),\ D^\alpha_{a^+}h \in L^1(a,b)$. Then,*

$${}^HI^\alpha_{a^+}({}^HD^\alpha_{a^+})h(t) = h(t) + \omega_1\left(\log\frac{t}{a}\right)^{\alpha-1} + \omega_2\left(\log\frac{t}{a}\right)^{\alpha-2} + ... + \omega_n\left(\log\frac{t}{a}\right)^{\alpha-n},$$

where $n = [\alpha] + 1$, $\omega_\ell \in \mathbb{R}$, $\ell = 1, 2, ..., n$.

Lemma 1.11 *([29]) Let $\alpha > 0,\ a > 1,\ h \in L^1(a,b),\ D^\alpha_{a^+}h \in L^1(a,b)$. Then,*

$${}^HD^\alpha_{a^+}({}^HI^\alpha_{a^+})h(t) = h(t).$$

Lemma 1.12 *([29]) Let $\alpha,\ \beta > 0,\ h \in L^1(a,b)$. Then,*

$${}^HI^\alpha_{a^+}({}^HI^\beta_{a^+})h(t) =^H I^\beta_{a^+}({}^HI^\alpha_{a^+})h(t) =^H I^{\alpha+\beta}_{a^+}h(t).$$

1.2.2 Fractional calculus of variable-order

Definition 1.6 *([45], [46], [57]) For $-\infty < a < b < +\infty$, we consider the mapping $u(t) : [a, b] \to (0, +\infty)$. Then, the left Riemann-Liouville fractional integral (RLFI) of variable-order $u(t)$ for function $h(t)$ is*

$$I_{a^+}^{u(t)} h(t) = \int_a^t \frac{(t-s)^{u(t)-1}}{\Gamma(u(t))} h(s) ds, \quad t > a.$$

Definition 1.7 *([45], [46], [57]) For $-\infty < a < b < +\infty$, we consider the mapping $v(t) : [a, b] \to (n-1, n)$. Then, the left Riemann-Liouville fractional derivative (RLFD) of variable-order $v(t)$ for function $h(t)$ is*

$$D_{a^+}^{v(t)} h(t) = \Big(\frac{d}{dt}\Big)^n I_{a^+}^{n-v(t)} h(t) = \Big(\frac{d}{dt}\Big)^n \int_a^t \frac{(t-s)^{n-v(t)-1}}{\Gamma(n-v(t))} h(s) ds, \quad t > a.$$

As anticipated, the case of $u(t)$ and $v(t)$ are constant, then RLFI and RLFD coincide with the standard Riemann-Liouville fractional derivative and integral, respectively see e.g. [29, 45, 46].

Remark 1.1 *([63], [65], [67]) Note that the semigroup property is not fulfilled for general functions $u(t)$, $v(t)$, i.e.,*

$$I_{a^+}^{u(t)} I_{a^+}^{v(t)} h(t) \neq I_{a^+}^{u(t)+v(t)} h(t).$$

Example 1.1 *Let*

$$u(t) = t, \quad t \in [0, 4], \quad v(t) = \begin{cases} 2, & t \in [0, 1], \\ 3, & t \in]1, 4]. \end{cases} \quad h(t) = 2, \quad t \in [0, 4].$$

$$\begin{aligned} I_{0^+}^{u(t)} I_{0^+}^{v(t)} h(t) &= \int_0^t \frac{(t-s)^{u(t)-1}}{\Gamma(u(t))} \int_0^s \frac{(s-\tau)^{v(s)-1}}{\Gamma(v(s))} h(\tau) d\tau ds \\ &= \int_0^t \frac{(t-s)^{t-1}}{\Gamma(t)} [\int_0^1 \frac{(s-\tau)}{\Gamma(2)} 2 d\tau + \int_1^s \frac{(s-\tau)^2}{\Gamma(3)} 2 d\tau] ds \\ &= \int_0^t \frac{(t-s)^{t-1}}{\Gamma(t)} [2s - 1 + \frac{(s-1)^3}{3}] ds, \end{aligned}$$

and

$$I_{0^+}^{u(t)+v(t)} h(t) = \int_0^t \frac{(t-s)^{u(t)+v(t)-1}}{\Gamma(u(t)+v(t))} h(s) ds.$$

So, we get

$$\begin{aligned} I_{0^+}^{u(t)} I_{0^+}^{v(t)} h(t)|_{t=3} &= \int_0^3 \frac{(3-s)^2}{\Gamma(3)}[2s-1+\frac{(s-1)^3}{3}]ds \\ &= \frac{21}{10}, \end{aligned}$$

$$\begin{aligned} I_{0^+}^{u(t)+v(t)} h(t)|_{t=3} &= \int_0^3 \frac{(3-s)^{u(t)+v(t)-1}}{\Gamma(u(t)+v(t))} h(s)ds \\ &= \int_0^1 \frac{(3-s)^4}{\Gamma(5)} 2ds + \int_1^3 \frac{(3-s)^5}{\Gamma(6)} 2ds \\ &= \frac{1}{12}\int_0^1 (s^4-12s^3+54s^2-108s+81)ds \\ &\quad + \frac{1}{60}\int_1^3 (-s^5+15s^4-90s^3+270s^2-405s+243)ds \\ &= \frac{665}{180}. \end{aligned}$$

Therefore, we obtain

$$I_{0^+}^{u(t)} I_{0^+}^{v(t)} h(t)|_{t=3} \neq I_{0^+}^{u(t)+v(t)} h(t)|_{t=3}.$$

Definition 1.8 *([45], [46], [57]) For $-\infty < a < b < +\infty$, we consider the mapping $v(t) : [a,b] \to (n-1,n)$. Then, the left Caputo fractional derivative (CFD) of variable-order $v(t)$ for function $h(t)$ ([46], [45], [57]) is*

$${}^cD_{a^+}^{v(t)} h(t) = \int_a^t \frac{(t-s)^{n-v(t)-1}}{\Gamma(n-v(t))} h^{(n)}(s)ds, \quad t > a.$$

As anticipated, the case of $v(t)$ is constant, then CFD coincides with the standard Caputo fractional derivative, see e.g. [29, 46, 45].

Definition 1.9 *([5], [6]) For $1 < a < b < +\infty$, we consider the mapping $u(t) : [a,b] \to (0,+\infty)$. Then, the left Hadamard fractional integral (HFI) of variable-order $u(t)$ for function $h(t)$ is*

$${}^HI_{a^+}^{u(t)} h(t) = \frac{1}{\Gamma(u(t))} \int_a^t (\log\frac{t}{s})^{u(t)-1} \frac{h(s)}{s} ds, \ \ t > a.$$

Definition 1.10 *([5], [6]) For $1 < a < b < +\infty$, we consider the mapping $v(t) : [a,b] \to (n-1,n)$. Then, the left Hadamard fractional derivative (HFD) of variable-order $v(t)$ for function $h(t)$ is*

$${}^HD_{a^+}^{v(t)} h(t) = \frac{1}{\Gamma(n-v(t))} (t\frac{d}{dt})^n \int_a^t (\log\frac{t}{s})^{n-v(t)-1} \frac{h(s)}{s} ds, \quad t > a.$$

As anticipated, the case of $u(t)$ and $v(t)$ are constant, then HFI and HFD coincide with the standard Hadamard integral and Hadamard derivative, respectively, see e.g. [29, 45, 46].

Remark 1.2 *Note the semigroup property is satisfied for standard Hadamard integral with constant orders but it is not fulfilled for the general case of variable orders* $u(t)$, $v(t)$, *i.e,* ${}^{H}I_{1^+}^{u(t)}({}^{H}I_{1^+}^{v(t)})x(t) \neq {}^{H}I_{1^+}^{u(t)+v(t)}x(t)$.

Example 1.2 *Let* $J = [1, 2]$, *and the function* $h(t) \equiv 1$ *for* $t \in J$. *Consider the following functions as order of HFI :* $v(t) \equiv 2$ *and* $u(t) = t$ *for* $t \in J$.

Then, for any $t \in J$ *we get*

$$ {}^{H}I_{1^+}^{v(t)}h(t) = \frac{1}{\Gamma(1)}\int_1^t \Big(log\frac{t}{s}\Big)^{1-1}\frac{1}{s}ds = log\ t, $$

$$ {}^{H}I_{1^+}^{u(t)}\Big({}^{H}I_{1^+}^{v(t)}h(t)\Big) = \frac{1}{\Gamma(t)}\int_1^t \Big(log\frac{t}{s}\Big)^{t-1}\frac{log\ s}{s}ds, $$

and

$$ {}^{H}I_{1^+}^{u(t)+v(t)}h(t) = \frac{1}{\Gamma(t+1)}\int_1^t \Big(log\frac{t}{s}\Big)^{t}\frac{1}{s}ds. $$

For $t = 1.5$ *we obtain*

$$ {}^{H}I_{1^+}^{u(t)}\Big({}^{H}I_{1^+}^{v(t)}h(t)\Big)|_{t=1.5} \simeq 0.027916, $$

and

$$ {}^{H}I_{1^+}^{u(t)+v(t)}h(t)|_{t=1.5} \simeq 0.0418739. $$

Therefore,the semigroup property is not satisfied for the general case of HFI of variable orders.

Lemma 1.13 *([68]) Let* $u : J \to (1, 2]$ *be a continuous function, then for*

$$ h \in C_\delta(J, \mathbb{R}) = \{h(t) \in C(J, X),\ t^\delta h(t) \in C(J, X),\ 0 \le \delta \le 1\}, $$

the variable order fractional integral $I_{0^+}^{u(t)}h(t)$ *exists for any points on* J.

Lemma 1.14 *([68]) Let* $u : J \to (1, 2]$ *be a continuous function, then*

$$ I_{0^+}^{u(t)}h(t) \in C(J, X)\ for\ h \in C(J, X). $$

Lemma 1.15 *([23]) If* $u : M \to (1, 2]$ *be a continuous function, then for*

$$ h \in C_\delta(M, \mathbb{R}) = \{h(t) \in C(M, \mathbb{R}),\ (logt)^\delta h(t) \in C(M, \mathbb{R}),\ 0 \le \delta \le 1\}, $$

the variable order fractional integral ${}^{H}I_{1^+}^{u(t)}h(t)$ *exists for any points on* M.

Proof

Taking the continuity of $\Gamma(u(t))$ into account, we shall claim that $M_u = \max_{t\in M}\left|\frac{1}{\Gamma(u(t))}\right|$ exists. We let $u^* = \max_{t\in M}|(u(t))|$. Thus, for $1 \le s \le t \le T$, we have

$$\begin{cases} \left(\log\frac{t}{s}\right)^{u(t)-1} \le 1, & \text{if} \quad 1 \le \frac{t}{s} \le e, \\ \left(\log\frac{t}{s}\right)^{u(t)-1} \le \left(log\frac{t}{s}\right)^{u^*-1}, & \text{if} \quad \frac{t}{s} > e. \end{cases}$$

Then, for $1 \le \frac{t}{s} < +\infty$, we know

$$\left(log\frac{t}{s}\right)^{u(t)-1} \le \max\left\{1, \left(log\frac{t}{s}\right)^{u^*-1}\right\} = M^*.$$

For $h \in C_\delta(M,\mathbb{R})$, by the definition (1.9), we deduce that

$$\begin{aligned} |{}^H I_{1^+}^{u(t)}h(t)| &= \frac{1}{\Gamma(u(t))}\int_1^t \left(\log\frac{t}{s}\right)^{u(t)-1}\frac{|h(s)|}{s}ds \\ &\le M_u\int_1^t \left(\log\frac{t}{s}\right)^{u(t)-1}\left(logs\right)^{-\delta}\left(logs\right)^{\delta}\frac{|h(s)|}{s}ds \\ &\le M_uM^*\int_1^t \frac{1}{s}\left(logs\right)^{-\delta}\max_{s\in M}\left(logs\right)^{\delta}|h(s)|ds \\ &\le M_uM^*\max_{s\in M}\left(logs\right)^{\delta}h^\star\int_1^t\frac{1}{s}\left(logs\right)^{-\delta}ds \\ &\le M_uM^*\max_{s\in M}\left(logs\right)^{\delta}h^\star\frac{\left(logT\right)^{1-\delta}}{1-\delta} < \infty, \end{aligned}$$

where $h^\star = \max_{t\in M}|h(t)|$. It yields that the variable order fractional integral ${}^H I_{1^+}^{u(t)}h(t)$ exists for any points on M.

Lemma 1.16 *([23]) Let $u : M \to (1,2]$ be a continuous function, then*

$${}^H I_{1^+}^{u(t)}h(t) \in C(M,\mathbb{R}) \text{ for } h \in C(M,\mathbb{R}).$$

Proof

For $t, t_0 \in M$, $t_0 \le t$ and $h \in C(M,\mathbb{R})$, we obtain

$$\left|{}^H I_{1^+}^{u(t)}h(t) - {}^H I_{1^+}^{u(t_0)}h(t_0)\right| = \Big|\int_1^t \frac{1}{\Gamma(u(t))}\left(\log\frac{t}{s}\right)^{u(t)-1}\frac{h(s)}{s}ds$$
$$-\int_1^{t_0}\frac{1}{\Gamma(u(t_0))}\left(\log\frac{t_0}{s}\right)^{u(t_0)-1}\frac{h(s)}{s}ds\Big|$$

$$
\begin{aligned}
= & \Big|\int_0^1 \frac{1}{\Gamma(u(t))}\frac{(t-1)}{r(t-1)+1}\Big(\log\frac{t}{r(t-1)+1}\Big)^{u(t)-1}h(r(t-1)+1)dr \\
& -\int_0^1 \frac{1}{\Gamma(u(t_0))}\frac{(t_0-1)}{r(t_0-1)+1}\Big(\log\frac{t_0}{r(t_0-1)+1}\Big)^{u(t_0)-1}h(r(t_0-1)+1)dr\Big| \\
= & \Big|\int_0^1 \Big[\frac{1}{\Gamma(u(t))}\frac{(t-1)}{r(t-1)+1}\Big(\log\frac{t}{r(t-1)+1}\Big)^{u(t)-1}h(r(t-1)+1) \\
& -\frac{1}{\Gamma(u(t))}\frac{(t_0-1)}{r(t_0-1)+1}\Big(\log\frac{t}{r(t-1)+1}\Big)^{u(t)-1}h(r(t-1)+1)\Big]dr \\
& +\int_0^1 \Big[\frac{1}{\Gamma(u(t))}\frac{(t_0-1)}{r(t_0-1)+1}\Big(\log\frac{t}{r(t-1)+1}\Big)^{u(t)-1}h(r(t-1)+1) \\
& -\frac{1}{\Gamma(u(t))}\frac{(t_0-1)}{r(t_0-1)+1}\Big(\log\frac{t_0}{r(t_0-1)+1}\Big)^{u(t_0)-1}h(r(t-1)+1)\Big]dr \\
& +\int_0^1 \Big[\frac{1}{\Gamma(u(t))}\frac{(t_0-1)}{r(t_0-1)+1}\Big(\log\frac{t_0}{r(t_0-1)+1}\Big)^{u(t_0)-1}h(r(t-1)+1) \\
& -\frac{1}{\Gamma(u(t_0))}\frac{(t_0-1)}{r(t_0-1)+1}\Big(\log\frac{t_0}{r(t_0-1)+1}\Big)^{u(t_0)-1}h(r(t-1)+1)\Big]dr \\
& +\int_0^1 \Big[\frac{1}{\Gamma(u(t_0))}\frac{(t_0-1)}{r(t_0-1)+1}\Big(\log\frac{t_0}{r(t_0-1)+1}\Big)^{u(t_0)-1}h(r(t-1)+1) \\
& -\frac{1}{\Gamma(u(t_0))}\frac{(t_0-1)}{r(t_0-1)+1}\Big(\log\frac{t_0}{r(t_0-1)+1}\Big)^{u(t_0)-1}h(r(t_0-1)+1)\Big]dr\Big| \\
\leq & \ h^\star\int_0^1 \frac{1}{\Gamma(u(t))}\Big(\log\frac{t}{r(t-1)+1}\Big)^{u(t)-1}\Big|\frac{(t-1)}{r(t-1)+1}-\frac{(t_0-1)}{r(t_0-1)+1}\Big|dr \\
& +h^\star\int_0^1 \frac{1}{\Gamma(u(t))}\frac{(t_0-1)}{r(t_0-1)+1}\Big|\Big(\log\frac{t}{r(t-1)+1}\Big)^{u(t)-1}-\Big(\log\frac{t_0}{r(t_0-1)+1}\Big)^{u(t)-1}\Big|dr \\
& +h^\star\int_0^1 \frac{(t_0-1)}{r(t_0-1)+1}\Big(\log\frac{t_0}{r(t_0-1)+1}\Big)^{u(t_0)-1}\Big|\frac{1}{\Gamma(u(t))}-\frac{1}{\Gamma(u(t_0))}\Big|dr \\
& +\int_0^1 \frac{1}{\Gamma(u(t_0))}\frac{(t_0-1)}{r(t_0-1)+1}\Big(\log\frac{t_0}{r(t_0-1)+1}\Big)^{u(t_0)-1}\Big|h(r(t-1)+1)-h(r(t_0-1)+1)\Big|dr,
\end{aligned}
$$

where $h^\star = \max_{t\in M}|h(t)|$. On account of the continuity of functions $\frac{(t-1)}{r(t-1)+1}$, $\Big(\log\frac{t}{r(t-1)+1}\Big)^{u(t)-1}$, $\frac{1}{\Gamma(u(t))}$, $h(t)$, we get that the integral ${}^HI_{1^+}^{u(t)}h(t)$ is continuous at point t_0, then ${}^HI_{1^+}^{u(t)}h(t) \in C(M,\mathbb{R})$ for $h(t) \in C(M,\mathbb{R})$.

1.3 Measure of noncompactness

This subsection discusses some necessary background information about the Kuratowski measure of noncompactness (**KMNC**).

Definition 1.11 *([10]) Let X be a Banach space and Ω_X the bounded subsets of X. The (**KMNC**) is a mapping $\zeta : \Omega_X \to [0, \infty]$ which is constructed as follows :*

$$\zeta(D) = \inf \Big\{ \epsilon > 0 : \exists (D_\ell)_{\ell=1,2,\dots,n} \subset X, \ D \subseteq \bigcup_{\ell=1}^{n} D_\ell, \ diam(D_\ell) \leq \epsilon \Big\},$$

where

$$diam(D_\ell) = sup\Big\{ ||x - y|| : \ x, \ y \in D_\ell \Big\}.$$

The following properties are valid for (**KMNC**) :

Proposition 1.1 *([10, 11]). Let X be a Banach space, $D, \ D_1, \ D_2$ are bounded subsets of X, then*

1. $\zeta(D) = 0 \Longleftrightarrow D$ *is relatively compact .*
2. $\zeta(\phi) = 0$.
3. $\zeta(D) = \zeta(\overline{D}) = \zeta(convD)$.
4. $D_1 \subset D_2 \Longrightarrow \zeta(D_1) \leq \zeta(D_2)$.
5. $\zeta(D_1 + D_2) \leq \zeta(D_1) + \zeta(D_2)$.
6. $\zeta(\lambda D) = |\lambda| \zeta(D), \lambda \in \mathbb{R}$.
7. $\zeta(D_1 \cup D_2) = Max\{\zeta(D_1), \zeta(D_2)\}$.
8. $\zeta(D_1 \cap D_2) = Min\{\zeta(D_1), \zeta(D_2)\}$.
9. $\zeta(D + x_0) = \zeta(D)$ *for any* $x_0 \in X$.

Lemma 1.17 *([22]) If $U \subset C(J, X)$ is a equicontinuous and bounded set, then*

(i) the function $\zeta(U(t))$ is continuous for $t \in J$, and

$$\widehat{\zeta}(U) = \sup_{t \in J} \zeta(U(t)),$$

where

$$U(t) = \{x(t) : \ x \in U\}, \ t \in J,$$

(ii) for any $x \in U$ we have

$$\zeta\left(\int x(s)ds\right) \leq \int \zeta(x(s))ds, \ s \in J.$$

1.4 Multi-valud analysis

Definition 1.12 *([26], [64], [66]) A generalized interval is a subset I of $\mathbb{R}$ which is either an interval (i.e. a set of the form $[a_1, a_2], (a_1, a_2), [a_1, a_2)$ or $(a_1, a_2])$, a point $\{a_1\}$, or the empty set $\emptyset$.*

Definition 1.13 *([26], [64], [66]) If I is a generalized interval. A partition of I is a finite set $\mathcal{P}$ of generalized intervals contained in I, such that every x in I lies in exactly one of the generalized intervals E in $\mathcal{P}$.*

Definition 1.14 *([26], [64], [66]) Let I be a generalized interval, let $g : I \to \mathbb{R}$ be a function, and let be $\mathcal{P}$ a partition of I. g is said to be piecewise constant with respect to $\mathcal{P}$ if for every $E \in \mathcal{P}$, g is constant on E.*

1.5 Some fixed point theorems

Definition 1.15 *Let $\digamma : \Lambda \subset X \longrightarrow X$ be a bounded operator from a Banach space X into itself. The operator $\digamma$ is called a k-set contraction if there is a number $k \geq 0$ such that*

$$\mu(\digamma(S)) \leq k\mu(S),$$

for all bounded sets S in Λ. The bounded operator $\digamma$ is called condensing if $\mu(\digamma(S)) < \mu(S)$ for all bounded sets S in Λ with $\mu(\Lambda) > 0$.

Obviously, every k-set contraction for $0 \leq k < 1$ is condensing. Every compact map $\digamma$ is a k-set contraction with $k = 0$.

Theorem 1.1 *(Darbo's fixed point theorem **DFPT**) [10] Let Λ be nonempty, closed, bounded and convex subset of a Banach space X and $\digamma : \Lambda \longrightarrow \Lambda$ is a continuous operator satisfying*

$$\zeta(\digamma(S)) \leq k\zeta(S), \ for \ any \ S(\neq \emptyset) \subset \Lambda, \ k \in [0, 1),$$

i.e., $\digamma$ is $k-$ set contraction. Then, $\digamma$ has at least one fixed point in Λ.

Theorem 1.2 *(Krasnoselskii Fixed Point Theorem)([29]) Let S be a closed, bounded and convex subset of a real Banach space E and let W_1 and W_2 be operators on S satisfying the following conditions :*
(i) $W_1(S) + W_2(S) \subset S$,
(ii) W_1 is continuous on S and $W_1(S)$ is a relatively compact subset of E,
(iii) W_2 is a strict contraction on S, i.e., there exists $k \in [0, 1)$, such that

$$\|W_2(x) - W_2(y)\| \leq k\|x - y\|, \ for \ every \ x, y \in S.$$

Then, there exists $x \in S$ such that $W_1(x) + W_2(x) = x$.

1.6 The stability

Theorem 1.3 *([13], [43]) Let $\vartheta \in C(J,X)$, the BVP (1) is Ulam-Hyers-Rassias **(UHR)** stable with respect to ϑ if there exists $c_f > 0$, such that for any $\epsilon > 0$ and for every solution $z \in C(J,X)$ of the following inequality*

$$\|D_{0^+}^{u(t)}z(t) - f(t,z(t),I_{0^+}^{u(t)}z(t))\| \le \epsilon\vartheta(t),\ t \in J,$$

there exists a solution $x \in C(J,X)$ of equation (1) with

$$\|z(t) - x(t)\| \le c_f\epsilon\vartheta(t),\ t \in J.$$

Theorem 1.4 *([13], [43]) The BVP (2) is Ulam-Hyers **(UH)** stable if there exists $c_f > 0$, such that for any $\epsilon > 0$ and for every solution $z \in C(J,\mathbb{R})$ of the following inequality*

$$|^cD_{0^+}^{u(t)}z(t) - f(t,z(t),I_{0^+}^{u(t)}z(t))| \le \epsilon,\ t \in J,$$

there exists a solution $x \in C(J,\mathbb{R})$ of BVP (2) with

$$|z(t) - x(t)| \le c_f\epsilon,\ t \in J.$$

Theorem 1.5 *([13], [43]) The BVP (3) is **(UHR)** stable with respect to $\varphi \in C(M,\mathbb{R}_+)$ if there exists a real number $c_f > 0$, such that for each $\epsilon > 0$ and for each solution $z \in C(M,\mathbb{R})$ of the inequality*

$$|^HD_{1^+}^{u}z(t) - f(t,z(t),\ ^HI_{1^+}^{u(t)}z(t))| \le \epsilon\varphi(t),\ t \in M,$$

there exists a solution $x \in C(M,\mathbb{R})$ of equation (3) with

$$|z(t) - x(t)| \le c_f\epsilon\varphi(t),\ t \in M.$$

Theorem 1.6 *([13], [43]) The BVP (4) is **(UH)** stable if there exists $c_f > 0$, such that for any $\epsilon > 0$ and for every solution $z \in C(J,\mathbb{R})$ of the following inequality*

$$|^cD_{0^+}^{u(t)}z(t) - f(t,z(t),^cD_{0^+}^{u(t)}z(t))| \le \epsilon,\ t \in J,$$

there exists a solution $x \in C(J,\mathbb{R})$ of equation (4) with

$$|z(t) - x(t)| \le c_f\epsilon,\ t \in J.$$

Chapitre 2

Multiterm boundary value problem of Riemann-Liouville fractional differential equations of variable order [1]

2.1 Introduction

We deal with the existence of solutions and the stability of the obtained solution in the sense of Ulam-Hyers-Rassias for the boundary value problem (BVP for short)

$$\begin{cases} D_{0^+}^{u(t)}x(t) = f(t, x(t), I_{0^+}^{u(t)}x(t)), \ \ t \in J, \\ x(0) = 0, \ \ x(T) = 0, \end{cases} \tag{2.1}$$

where $0 < T < +\infty$, $1 < u(t) \leq 2$, $f : J \times X \times X \to X$ is a continuous function and X is a real (or complex) Banach space, and $D_{0^+}^{u(t)}$, $I_{0^+}^{u(t)}$ are the Riemann-Liouville fractional derivative and integral of variable-order $u(t)$.

2.2 Existence of solutions

Let us introduce the following assumption :

(H1) Let $n \in \mathbb{N}$ be an integer, $\mathcal{P} = \{J_1 := [0, T_1], J_2 := (T_1, T_2], J_3 := (T_2, T_3], ...J_n := (T_{n-1}, T]\}$ be a partition of the interval J, and let $u(t) : J \to (1, 2]$ be a piecewise constant function such that,

[1] **A. Benkerrouche**, D. Baleanu, M. S. Souid, A. Hakem, M. Inc, Boundary value problem for nonlinear fractional differential equations of variable order via Kuratowski MNC technique, *Advances in Difference Equations*, **2021 :365**, 1-19, (2021).

$$u(t) = \sum_{\ell=1}^{n} u_\ell I_\ell(t) = \begin{cases} u_1, & if\ t \in J_1, \\ u_2, & if\ t \in J_2, \\ . \\ . \\ . \\ u_n, & if\ t \in J_n, \end{cases}$$

where $1 < u_\ell \le 2$ are constants, and I_ℓ is the indicator of the interval $J_\ell := (T_{\ell-1}, T_\ell],\ \ell = 1, 2, ..., n,$ (with $T_0 = 0,\ T_n = T$).

Further, for a given set U of functions $u : J \to X$ let us denote by

$$U(t) = \{u(t), u \in U\},\ t \in J,$$

and

$$U(J) = \{v(t) : v \in U,\ t \in J\}.$$

We are now in a position to prove the existence of solution for the BVP (2.1) based on concept of (**MNCK**) and (**DFPT**).

For each $\ell \in \{1, 2, ..., n\}$, the symbol $E_\ell = C(J_\ell, X)$, indicated the Banach space of continuous functions $x : J_\ell \to X$ equipped with the norm

$$\|x\|_{E_\ell} = \sup_{t \in J_\ell} \|x(t)\|.$$

First, we analyze BVP defined in (2.1).
Then, for any $t \in J_\ell,\ \ell \in \{1, 2, ..., n\}$, the left Riemann-Liouville fractional derivative (RLFD) of variable order $u(t)$ for the function $x(t) \in C(J, X)$ could be presented as a sum of left Riemann-Liouville fractional derivatives of constant-orders $u_\ell,\ \ell \in \{1, 2, ..., n\}$

$$D_{0^+}^{u(t)} x(t) = \frac{d^2}{dt^2}\Big(\int_0^{T_1} \frac{(t-s)^{1-u_1}}{\Gamma(2-u_1)} x(s)ds\Big) + ... + \frac{d^2}{dt^2}\Big(\int_{T_{\ell-1}}^{t} \frac{(t-s)^{1-u_\ell}}{\Gamma(2-u_\ell)} x(s)ds\Big). \quad (2.2)$$

Thus, according to (2.2), the BVP (2.1) can be written for any $t \in J_\ell,\ \ell \in \{1, 2, ..., n\}$ in the form

$$\frac{d^2}{dt^2}\Big(\int_0^{T_1} \frac{(t-s)^{1-u_1}}{\Gamma(2-u_1)} x(s)ds\Big) + \frac{d^2}{dt^2}\Big(\int_{T_1}^{T_2} \frac{(t-s)^{1-u_2}}{\Gamma(2-u_2)} x(s)ds\Big) + ...$$

$$+ \frac{d^2}{dt^2}\Big(\int_{T_{\ell-1}}^{t} \frac{(t-s)^{1-u_\ell}}{\Gamma(2-u_\ell)} x(s)ds\Big) = f(t, x(t), I_{0^+}^{u_\ell} x(t)), \quad t \in J_\ell. \quad (2.3)$$

In what follows we shall introduce the solution to the BVP (2.1).

Definition 2.1 *BVP (2.1) has a solution, if there are functions $x_\ell, \ell = 1, 2, ..., n,$ so that, $x_\ell \in C([0, T_\ell], X)$ fulfilling equation (2.3) and $x_\ell(0) = 0 = x_\ell(T_\ell)$.*

According the observation above, BVP (2.1) can be expressed for any $t \in J_\ell,\ \ell = 1, 2, \ldots, n$ as (2.3).
For $0 \le t \le T_{\ell-1}$, we take $x(t) \equiv 0$, then (2.3) is written as

$$D^{u_\ell}_{T^+_{\ell-1}} x(t) = f(t, x(t), I^{u_\ell}_{T^+_{\ell-1}} x(t)), \quad t \in J_\ell.$$

We shall deal with following BVP

$$\begin{cases} D^{u_\ell}_{T^+_{\ell-1}} x(t) = f(t, x(t), I^{u_\ell}_{T^+_{\ell-1}} x(t)), & t \in J_\ell, \\ x(T_{\ell-1}) = 0, x(T_\ell) = 0. \end{cases} \tag{2.4}$$

For our purpose, the upcoming lemma will be a corner stone of the solution of (2.4).

Lemma 2.1 *Let $\ell \in \{1, 2, ..., n\}$ be a natural number, $f \in C(J_\ell \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and there exists a number $\delta \in (0, 1)$ such that $t^\delta f \in C(J_\ell \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.*
Then, the function $x \in E_\ell$ forms a solution of (2.4) if and only if x fulfills the integral equation

$$x(t) = -(T_\ell - T_{\ell-1})^{1-u_\ell}(t - T_{\ell-1})^{u_\ell-1} I^{u_\ell}_{T^+_{\ell-1}} f(T_\ell, x(T_\ell), I^{u_\ell}_{T^+_{\ell-1}} x(T_\ell))$$
$$+ I^{u_\ell}_{T^+_{\ell-1}} f(t, x(t), I^{u_\ell}_{T^+_{\ell-1}} x(t)). \tag{2.5}$$

Proof
We presume that $x \in E_\ell$ is solution of the BVP (2.4). Employing the operator $I^{u_\ell}_{T^+_{\ell-1}}$ to both sides of (2.4) and regarding Lemma 1.2, we find

$$x(t) = \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t - s)^{u_\ell-1} f(s, x(s), I^{u_\ell}_{T^+_{\ell-1}} x(s)) ds$$
$$+ \omega_1 (t - T_{\ell-1})^{u_\ell-1} + \omega_2 (t - T_{\ell-1})^{u_\ell-2}, \quad t \in J_\ell.$$

Due to the assumption of function f together with $x(T_{\ell-1}) = 0$, we conclude that $\omega_2 = 0$.

Let $x(t)$ satisfy $x(T_\ell) = 0$. So, we observe that

$$\omega_1 = -(T_\ell - T_{\ell-1})^{1-u_\ell} I^{u_\ell}_{T^+_{\ell-1}} f(T_\ell, x(T_\ell), I^{u_\ell}_{T^+_{\ell-1}} x(T_\ell)).$$

Then, we find

$$x(t) = -(T_\ell - T_{\ell-1})^{1-u_\ell}(t - T_{\ell-1})^{u_\ell-1} I^{u_\ell}_{T^+_{\ell-1}} f(T_\ell, x(T_\ell), I^{u_\ell}_{T^+_{\ell-1}} x(T_\ell))$$

$$+ I^{u_\ell}_{T^+_{\ell-1}} f(t, x(t), I^{u_\ell}_{T^+_{\ell-1}} x(t)), \quad t \in J_\ell.$$

Conversely, let $x \in E_\ell$ be solution of integral equation (2.5), Regarding the continuity of function $t^\delta f$ and Lemma(1.3), we deduce that x is the solution of the BVP (2.4).

Our first existence result is based on Theorem (1.1)

Theorem 2.1 *Let the conditions of Lemma (2.1) be satisfied and there exist a constants $K,\ L > 0$, such that,*

$$t^{\delta}|f(t,y_1,z_1)-f(t,y_2,z_2)| \leq K|y_1-y_2|+L|z_1-z_2|,\ \ y_i,\ z_i\ \in X,\ i=1,2,\ t\in J_\ell, \quad (2.6)$$

and the inequality

$$\frac{2(T_\ell-T_{\ell-1})^{u_\ell-1}(T_\ell^{1-\delta}-T_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\Big(K+L\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\Big)<1 \quad (2.7)$$

hold. Then, the BVP (2.4) possesses at least one solution on E_ℓ.

Proof

We construct the operator

$$W: E_\ell \to E_\ell,$$

as follow :

$$Wx(t) = -(T_\ell-T_{\ell-1})^{1-u_\ell}(t-T_{\ell-1})^{u_\ell-1}I^{u_\ell}_{T^+_{\ell-1}}f(T_\ell,x(T_\ell),I^{u_\ell}_{T^+_{\ell-1}}x(T_\ell))$$

$$+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(t-s)^{u_\ell-1}f(s,x(s),I^{u_\ell}_{T^+_{\ell-1}}x(s))ds, \quad t\in J_\ell. \quad (2.8)$$

It follows from the properties of fractional integrals and from the continuity of function $t^{\delta}f$ that the operator $W: E_\ell \to E_\ell$ defined in (2.8) is well defined.
Let

$$R_\ell \ \ \geq \ \ \frac{\frac{2f^{\star}(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)}}{1-\frac{2(T_\ell-T_{\ell-1})^{u_\ell-1}(T_\ell^{1-\delta}-T_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\Big(K+L\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\Big)},$$

with

$$f^{\star}=\sup_{t\in J_\ell}\|f(t,0,0)\|.$$

We consider the set

$$B_{R_\ell}=\{x\in E_\ell,\ \|x\|_{E_\ell}\leq R_\ell\}.$$

Clearly B_{R_ℓ} is nonempty, closed, convex and bounded.

Now, we demonstrate that W satisfies the assumption of the Theorem (1.1). We shall prove it in four phases.

STEP 1 : Claim : $W(B_{R_\ell}) \subseteq (B_{R_\ell})$.
For $x \in B_{R_\ell}$ and by (H2), we get

$$\begin{aligned}
\|Wx(t)\| &\leq \frac{(T_\ell - T_{\ell-1})^{1-u_\ell}(t-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(T_\ell - s)^{u_\ell-1}\|f(s,x(s),I^{u_\ell}_{T^+_{\ell-1}}x(s))\|ds \\
&\quad +\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(t-s)^{u_\ell-1}\|f(s,x(s),I^{u_\ell}_{T^+_{\ell-1}}x(s))\|ds \\
&\leq \frac{2}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(T_\ell - s)^{u_\ell-1}\|f(s,x(s),I^{u_\ell}_{T^+_{\ell-1}}x(s))\|ds \\
&\leq \frac{2}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(T_\ell - s)^{u_\ell-1}\|f(s,x(s),I^{u_\ell}_{T^+_{\ell-1}}x(s)) - f(s,0,0)\|ds \\
&\quad +\frac{2}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(T_\ell - s)^{u_\ell-1}\|f(s,0,0)\|ds \\
&\leq \frac{2}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(T_\ell - s)^{u_\ell-1}s^{-\delta}(K\|x(s)\| + L\|I^{u_\ell}_{T^+_{\ell-1}}x(s)\|)ds + \frac{2f^\star(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)} \\
&\leq \frac{2(T_\ell - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}(K + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)})\|x(s)\|ds + \frac{2f^\star(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)} \\
&\leq \frac{2(T_\ell - T_{\ell-1})^{u_\ell-1}(T_\ell^{1-\delta} - T_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}(K + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)})R_\ell + \frac{2f^\star(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)} \\
&\leq R_\ell,
\end{aligned}$$

which means that $W(B_{R_\ell}) \subseteq B_{R_\ell}$.

STEP 2 : Claim : W is continuous.
We presume that the sequence (x_n) converges to x in E_ℓ and $t \in J_\ell$. Then,

$$\begin{aligned}
&\|(Wx_n)(t) - (Wx)(t)\| \leq \frac{(T_\ell - T_{\ell-1})^{1-u_\ell}(t-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)} \\
&\int_{T_{\ell-1}}^{T_\ell}(T_\ell - s)^{u_\ell-1}\|f(s,x_n(s),I^{u_\ell}_{T^+_{\ell-1}}x_n(s)) - f(s,x(s),I^{u_\ell}_{T^+_{\ell-1}}x(s))\|ds \\
&+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(t-s)^{u_\ell-1}\|f(s,x_n(s),I^{u_\ell}_{T^+_{\ell-1}}x_n(s)) - f(s,x(s),I^{u_\ell}_{T^+_{\ell-1}}x(s))\|ds \\
\leq\ &\frac{(T_\ell - T_{\ell-1})^{1-u_\ell}(T_\ell-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(T_\ell - s)^{u_\ell-1}\|f(s,x_n(s),I^{u_\ell}_{T^+_{\ell-1}}x_n(s)) \\
&-f(s,x(s),I^{u_\ell}_{T^+_{\ell-1}}x(s))\|ds \\
&+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(t-s)^{u_\ell-1}\|f(s,x_n(s),I^{u_\ell}_{T^+_{\ell-1}}x_n(s)) - f(s,x(s),I^{u_\ell}_{T^+_{\ell-1}}x(s))\|ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{2}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(T_\ell-s)^{u_\ell-1}\|f(s,x_n(s),I_{T_{\ell-1}^+}^{u_\ell}x_n(s))-f(s,x(s),I_{T_{\ell-1}^+}^{u_\ell}x(s))\|ds\\
&\leq \frac{2}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}(T_\ell-s)^{u_\ell-1}(K\|x_n(s)-x(s)\|+LI_{T_{\ell-1}^+}^{u_\ell}\|x_n(s)-x(s))\|)ds\\
&\leq \frac{2K}{\Gamma(u_\ell)}\|x_n-x\|_{E_\ell}\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}(T_\ell-s)^{u_\ell-1}ds\\
&\quad+\frac{2L}{\Gamma(u_\ell)}\|I_{T_{\ell-1}^+}^{u_\ell}(x_n-x)\|_{E_\ell}\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}(T_\ell-s)^{u_\ell-1}ds\\
&\leq \frac{2K}{\Gamma(u_\ell)}\|x_n-x\|_{E_\ell}\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}(T_\ell-s)^{u_\ell-1}ds\\
&\quad+\frac{2L(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)\Gamma(u_\ell+1)}\|x_n-x\|_{E_\ell}\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}(T_\ell-s)^{u_\ell-1}ds\\
&\leq \Big(\frac{2K}{\Gamma(u_\ell)}+\frac{2L(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)\Gamma(u_\ell+1)}\Big)\|x_n-x\|_{E_\ell}\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}(T_\ell-s)^{u_\ell-1}ds\\
&\leq \frac{(T_\ell-T_{\ell-1})^{u_\ell-1}({T_\ell}^{1-\delta}-{T_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\Big(2K+\frac{2L(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\Big)\|x_n-x\|_{E_\ell},
\end{aligned}
$$

i.e., we obtain

$$\|(Wx_n)-(Wx)\|_{E_\ell}\to 0\ \ as\ \ n\to\infty.$$

Ergo, the operator W is a continuous on E_ℓ.
STEP 3 : Claim : W is bounded and equicontinuous.
By Step 1, we have $W(B_{R_\ell})=\{W(x): x\in B_{R_\ell}\}\subset B_{R_\ell}$ thus for each $x\in B_{R_\ell}$ we have $\|W(x)\|_{E_\ell}\leq R_\ell$ which means that $W(B_{R_\ell})$is bounded. It remains to indicate that $W(B_{R_\ell})$ is equicontinuous.
For $t_1,t_2\in J_\ell,\ t_1<t_2$ and $x\in B_{R_\ell}$, we have

$$
\begin{aligned}
&\Big\|(Wx)(t_2)-(Wx)(t_1)\Big\| = \Big\|-\frac{(T_\ell-T_{\ell-1})^{1-u_\ell}(t_2-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\\
&\int_{T_{\ell-1}}^{T_\ell}(T_\ell-s)^{u_\ell-1}f(s,x(s),I_{T_{\ell-1}^+}^{u_\ell}x(s))ds+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_2}(t_2-s)^{u_\ell-1}f(s,x(s),I_{T_{\ell-1}^+}^{u_\ell}x(s))ds\\
&+\frac{(T_\ell-T_{\ell-1})^{1-u_\ell}(t_1-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(T_\ell-s)^{u_\ell-1}f(s,x(s),I_{T_{\ell-1}^+}^{u_\ell}x(s))ds\\
&-\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}(t_1-s)^{u_\ell-1}f(s,x(s),I_{T_{\ell-1}^+}^{u_\ell}x(s))ds\Big\|
\end{aligned}
$$

$$\leq \frac{(T_\ell - T_{\ell-1})^{1-u_\ell}}{\Gamma(u_\ell)}\Big((t_2 - T_{\ell-1})^{u_\ell-1} - (t_1 - T_{\ell-1})^{u_\ell-1}\Big)\int_{T_{\ell-1}}^{T_\ell}(T_\ell - s)^{u_\ell-1}\|f(s,x(s),I^{u_\ell}_{T^+_{\ell-1}}x(s))\|ds$$
$$+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}\Big((t_2-s)^{u_\ell-1} - (t_1-s)^{u_\ell-1}\Big)\|f(s,x(s),I^{u_\ell}_{T^+_{\ell-1}}x(s))\|ds$$
$$+\frac{1}{\Gamma(u_\ell)}\int_{t_1}^{t_2}(t_2-s)^{u_\ell-1}\|f(s,x(s),I^{u_\ell}_{T^+_{\ell-1}}x(s))\|ds$$
$$\leq \frac{(T_\ell - T_{\ell-1})^{1-u_\ell}}{\Gamma(u_\ell)}\Big((t_2 - T_{\ell-1})^{u_\ell-1} - (t_1 - T_{\ell-1})^{u_\ell-1}\Big)$$
$$\int_{T_{\ell-1}}^{T_\ell}(T_\ell - s)^{u_\ell-1}\|f(s,x(s),I^{u_\ell}_{T^+_{\ell-1}}x(s)) - f(s,0,0)\|ds$$
$$+\frac{(T_\ell - T_{\ell-1})^{1-u_\ell}}{\Gamma(u_\ell)}\Big((t_2 - T_{\ell-1})^{u_\ell-1} - (t_1 - T_{\ell-1})^{u_\ell-1}\Big)\int_{T_{\ell-1}}^{T_\ell}(T_\ell - s)^{u_\ell-1}\|f(s,0,0)\|ds$$
$$+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}\Big((t_2-s)^{u_\ell-1} - (t_1-s)^{u_\ell-1}\Big)\|f(s,x(s),I^{u_\ell}_{T^+_{\ell-1}}x(s)) - f(s,0,0)\|ds$$
$$+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}\Big((t_2-s)^{u_\ell-1} - (t_1-s)^{u_\ell-1}\Big)\|f(s,0,0)\|ds$$
$$+\frac{1}{\Gamma(u_\ell)}\int_{t_1}^{t_2}(t_2-s)^{u_\ell-1}\|f(s,x(s),I^{u_\ell}_{T^+_{\ell-1}}x(s)) - f(s,0,0)\|ds$$
$$+\frac{1}{\Gamma(u_\ell)}\int_{t_1}^{t_2}(t_2-s)^{u_\ell-1}\|f(s,0,0)\|ds$$
$$\leq \frac{(T_\ell - T_{\ell-1})^{1-u_\ell}}{\Gamma(u_\ell)}\Big((t_2 - T_{\ell-1})^{u_\ell-1} - (t_1 - T_{\ell-1})^{u_\ell-1}\Big)$$
$$\int_{T_{\ell-1}}^{T_\ell}(T_\ell - s)^{u_\ell-1}s^{-\delta}(K\|x(s)\| + L\|I^{u_\ell}_{T^+_{\ell-1}}x(s)\|)ds$$
$$+\frac{f^\star(T_\ell - T_{\ell-1})^{1-u_\ell}}{\Gamma(u_\ell)}\Big((t_2 - T_{\ell-1})^{u_\ell-1} - (t_1 - T_{\ell-1})^{u_\ell-1}\Big)\int_{T_{\ell-1}}^{T_\ell}(T_\ell - s)^{u_\ell-1}ds$$
$$+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}s^{-\delta}\Big((t_2-s)^{u_\ell-1} - (t_1-s)^{u_\ell-1}\Big)(K\|x(s)\| + L\|I^{u_\ell}_{T^+_{\ell-1}}x(s)\|)ds$$
$$+\frac{f^\star}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}\Big((t_2-s)^{u_\ell-1} - (t_1-s)^{u_\ell-1}\Big)ds$$
$$+\frac{1}{\Gamma(u_\ell)}\int_{t_1}^{t_2}s^{-\delta}(t_2-s)^{u_\ell-1}(K\|x(s)\| + L\|I^{u_\ell}_{T^+_{\ell-1}}x(s)\|)ds + \frac{f^\star}{\Gamma(u_\ell)}\int_{t_1}^{t_2}(t_2-s)^{u_\ell-1}ds$$
$$\leq \frac{1}{\Gamma(u_\ell)}\Big((t_2 - T_{\ell-1})^{u_\ell-1} - (t_1 - T_{\ell-1})^{u_\ell-1}\Big)(K\|x\|_{E_\ell} + L\|I^{u_\ell}_{T^+_{\ell-1}}x\|_{E_\ell})\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}ds$$
$$+\frac{f^\star(T_\ell - T_{\ell-1})}{\Gamma(u_\ell+1)}\Big((t_2 - T_{\ell-1})^{u_\ell-1} - (t_1 - T_{\ell-1})^{u_\ell-1}\Big)$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(u_\ell)}(K\|x\|_{E_\ell}+L\|I^{u_\ell}_{T^+_{\ell-1}}x\|_{E_\ell})\int_{T_{\ell-1}}^{t_1} s^{-\delta}((t_2-t_1)^{u_\ell-1})ds\\
&+\frac{f^\star}{\Gamma(u_\ell)}\Big(\frac{(t_2-T_{\ell-1})^{u_\ell}}{u_\ell}-\frac{(t_2-t_1)^{u_\ell}}{u_\ell}-\frac{(t_1-T_{\ell-1})^{u_\ell}}{u_\ell}\Big)\\
&+\frac{(t_2-t_1)^{u_\ell-1}}{\Gamma(u_\ell)}(K\|x\|_{E_\ell}+L\|I^{u_\ell}_{T^+_{\ell-1}}x\|_{E_\ell})\int_{t_1}^{t_2} s^{-\delta}ds+\frac{f^\star}{\Gamma(u_\ell)}\frac{(t_2-t_1)^{u_\ell}}{u_\ell}\\
\leq\ & \frac{{T_\ell}^{1-\delta}-{T_{\ell-1}}^{1-\delta}}{(1-\delta)\Gamma(u_\ell)}\Big((t_2-T_{\ell-1})^{u_\ell-1}-(t_1-T_{\ell-1})^{u_\ell-1}\Big)\Big(K\|x\|_{E_\ell}+\frac{L(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\|x\|_{E_\ell}\Big)\\
&+\frac{f^\star(T_\ell-T_{\ell-1})}{\Gamma(u_\ell+1)}\Big((t_2-T_{\ell-1})^{u_\ell-1}-(t_1-T_{\ell-1})^{u_\ell-1}\Big)\\
&+\Big(\frac{({t_1}^{1-\delta}-{T_{\ell-1}}^{1-\delta})(t_2-t_1)^{u_\ell-1}}{(1-\delta)\Gamma(u_\ell)}\Big)(K\|x\|_{E_\ell}+\frac{L(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\|x\|_{E_\ell})\\
&+\frac{f^\star}{\Gamma(u_\ell+1)}\Big((t_2-T_{\ell-1})^{u_\ell}-(t_2-t_1)^{u_\ell}-(t_1-T_{\ell-1})^{u_\ell}\Big)\\
&+\frac{({t_2}^{1-\delta}-{t_1}^{1-\delta})(t_2-t_1)^{u_\ell-1}}{(1-\delta)\Gamma(u_\ell)}(K\|x\|_{E_\ell}+\frac{L(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\|x\|_{E_\ell})+\frac{f^\star(t_2-t_1)^{u_\ell}}{\Gamma(u_\ell+1)}\\
\leq\ & \Big(\frac{{T_\ell}^{1-\delta}-{T_{\ell-1}}^{1-\delta}}{(1-\delta)\Gamma(u_\ell)}(K+\frac{L(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)})\|x\|_{E_\ell}+\frac{f^\star(T_\ell-T_{\ell-1})}{\Gamma(u_\ell+1)}\Big)\\
&\Big((t_2-T_{\ell-1})^{u_\ell-1}-(t_1-T_{\ell-1})^{u_\ell-1}\Big)\\
&+\Big(\frac{{t_2}^{1-\delta}-{T_{\ell-1}}^{1-\delta}}{(1-\delta)\Gamma(u_\ell)}(K+\frac{L(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)})\|x\|_{E_\ell}\Big)(t_2-t_1)^{u_\ell-1}\\
&+\frac{f^\star}{\Gamma(u_\ell+1)}\Big((t_2-T_{\ell-1})^{u_\ell}-(t_1-T_{\ell-1})^{u_\ell}\Big).
\end{aligned}
$$

Hence $\|(Wx)(t_2)-(Wx)(t_1)\|_{E_\ell}\to 0$ as $|t_2-t_1|\to 0$. It implies that $T(B_{R_\ell})$ is equicontinuous.

Remark 2.1 *According to remark of ([12]) page 20 we can easily show that the inequality (2.6) is equivalent to the following inequality*

$$\zeta(t^\delta\|f(t,B_1,B_2)\|)\leq K\zeta(B_1)+L\zeta(B_2),$$

for any bounded sets $B_1,B_2\subset X$ *and for each* $t\in J_\ell$.

STEP 4 : Claim : W is k-*set contraction.*
For $U\in B_{R_\ell}$, $t\in J_\ell$, we get,

$$\zeta(W(U)(t)) \;=\; \zeta((Wx)(t),x\in U)$$

$$\leq \Big(\frac{(T_\ell - T_{\ell-1})^{1-u_\ell}(t-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(T_\ell - s)^{u_\ell-1}\zeta f(s,x(s),I^{u_\ell}_{T^+_{\ell-1}}x(s))ds$$
$$+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(t-s)^{u_\ell-1}\zeta f(s,x(s),I^{u_\ell}_{T^+_{\ell-1}}x(s))ds, x\in U\Big).$$

Then Remark 2.1 implies that, for each $s \in J_\ell$,

$$\zeta(W(U)(t)) \leq \Big(\frac{(T_\ell - T_{\ell-1})^{1-u_\ell}(t-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}$$
$$\int_{T_{\ell-1}}^{T_\ell}(T_\ell - s)^{u_\ell-1}\Big[K\widehat{\zeta}(U)\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}ds + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\widehat{\zeta}(U)\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}ds\Big]$$
$$+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(t-s)^{u_\ell-1}\Big[K\widehat{\zeta}(U)\int_{T_{\ell-1}}^{t}s^{-\delta}ds + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\widehat{\zeta}(U)\int_{T_{\ell-1}}^{t}s^{-\delta}ds\Big], x\in U\Big).$$
$$\leq \Big(\frac{(t-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}\Big[K\widehat{\zeta}(U)\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}ds + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\widehat{\zeta}(U)\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}ds\Big]$$
$$+\frac{(t-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}\Big[K\widehat{\zeta}(U)\int_{T_{\ell-1}}^{t}s^{-\delta}ds + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\widehat{\zeta}(U)\int_{T_{\ell-1}}^{t}s^{-\delta}ds\Big], x\in U\Big).$$
$$\leq \frac{[({T_\ell}^{1-\delta} - {T_{\ell-1}}^{1-\delta}) + (t^{1-\delta} - {T_{\ell-1}}^{1-\delta})](t-T_{\ell-1})^{u_\ell-1}}{(1-\delta)\Gamma(u_\ell)}\Big(K + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\Big)\widehat{\zeta}(U)$$
$$\leq \frac{2({T_\ell}^{1-\delta} - {T_{\ell-1}}^{1-\delta})(T_\ell - T_{\ell-1})^{u_\ell-1}}{(1-\delta)\Gamma(u_\ell)}\Big(K + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\Big)\widehat{\zeta}(U).$$

Therefore

$$\widehat{\zeta}(WU) \leq \tfrac{2({T_\ell}^{1-\delta} - {T_{\ell-1}}^{1-\delta})(T_\ell - T_{\ell-1})^{u_\ell-1}}{(1-\delta)\Gamma(u_\ell)}\Big(K + L\tfrac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\Big)\widehat{\zeta}(U).$$

Consequently by (2.7), we deduce that W forms a set contraction. Hence, by the Theorem (1.1), the BVP (2.4) has at least solution $\widetilde{x_\ell}$ in B_{R_ℓ}. Since $B_{R_\ell} \subset E_\ell$, the claim of Theorem (2.1) is proved.

Now, we will prove the existence result for the BVP (2.1).

Introduce the following assumption :

(H2) Let $f \in C(J\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and there exists a number $\delta\in(0,1)$ such that $t^\delta f \in C(J\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and there exist a constants $K,\ L>0$, such that,
$t^\delta\|f(t,y_1,z_1) - f(t,y_2,z_2)\| \leq K\|y_1-y_2\| + L\|z_1-z_2\|$, for any $y_1,\ y_2,\ z_1,\ z_2\in\mathbb{R}$ and $t\in J$.

Theorem 2.2 *Let the conditions (H1), (H2) and inequality* (2.7) *be satisfied for all* $\ell \in 1,2,...,n$. *Then, the BVP (2.1) possesses at least one solution in* $C(J,X)$.

Proof For $\ell \in 1, 2, ..., n$, according to Theorem (2.1), the BVP (2.4) possesses at least one solution $\widetilde{x_\ell}$ in E_ℓ. For $\ell \in 1, 2, ..., n$, we define the function

$$x_\ell = \begin{cases} 0, & t \in [0, T_{\ell-1}], \\ \widetilde{x}_\ell, & t \in J_\ell. \end{cases}$$

Thus, the function $x_\ell \in C([0, T_\ell], X)$ solves the integral equation (2.3) for $t \in J_\ell$, with $x_\ell(0) = 0,\ x_\ell(T_\ell) = \widetilde{x}_\ell(T_\ell) = 0$.
Then, the function

$$x(t) = \begin{cases} x_1(t), \quad t \in J_1, \\ x_2(t) = \begin{cases} 0, & t \in J_1, \\ \widetilde{x}_2, & t \in J_2, \end{cases} \\ . \\ . \\ . \\ . \\ x_n(t) = \begin{cases} 0, & t \in [0, T_{\ell-1}], \\ \widetilde{x}_\ell, & t \in J_\ell. \end{cases} \end{cases}$$

forms a solution of BVP (2.1) in $C(J, X)$.

2.3 Ulam-Hyers-Rassias Stability

Theorem 2.3 *Assume that conditions (H1), (H2) and the inequality (2.7) hold and*

(H3) *Let $\vartheta \in C(J_\ell, X)$ is a increasing function and there exists $\lambda_\vartheta > 0$ such that*

$$I^{u_\ell}_{T_{\ell-1}^+}\vartheta(t) \leq \lambda_{\vartheta(t)}\vartheta(t),\ for\ any\ t \in J_\ell.$$

Then, the BVP (2.1) *is* ***(UHR)*** *stable with respect to ϑ.*

Proof

Let $\epsilon > 0$ an arbitrary number and the function $z(t)$ from $z \in C(J_\ell, X)$ satisfy the following inequality

$$\|D^{u_\ell}_{T_{\ell-1}^+}z(t) - f(t, z(t), I^{u_\ell}_{T_{\ell-1}^+}z(t))\| \leq \epsilon\vartheta(t), t \in J_\ell. \tag{2.9}$$

For any $\ell \in \{1, 2, ..., n\}$ we define the functions $z_1(t) \equiv z(t), t \in [0, T_1]$ and for $\ell = 2, 3, ..., n$:

$$z_\ell(t) = \begin{cases} 0, & t \in [0, T_{\ell-1}], \\ z(t), & t \in J_\ell. \end{cases}$$

Then, for $t \in J_\ell$, $\ell \in \{1,2,...,n\}$ and taking the (RLFI) $I^{u_\ell}_{T^+_{\ell-1}}$ of both sides of the inequality (2.9), we obtain

$$\begin{aligned}&\Big\| z_\ell(t) + \tfrac{(T_\ell-T_{\ell-1})^{1-u_\ell}(t-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} f(s, z_\ell(s), I^{u_\ell}_{T^+_{\ell-1}} z_\ell(s))ds \\ &-\tfrac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t-s)^{u_\ell-1} f(s, z_\ell(s), I^{u_\ell}_{T^+_{\ell-1}} z_\ell(s))ds \Big\| \\ &\leq \epsilon \int_{T_{\ell-1}}^{t} \tfrac{(t-s)^{u(i)-1}}{\Gamma(u(i))} \vartheta(s)ds \\ &\leq \epsilon \lambda_{\vartheta(t)} \vartheta(t).\end{aligned}$$

According to Theorem (2.2), BVP (2.1) has a solution $x \in C(J,X)$ defined by $x(t) = x_\ell(t)$ for $t \in J_\ell,\ \ell = 1,2,...,n$, where

$$x_\ell = \begin{cases} 0, & t \in [0, T_{\ell-1}], \\ \widetilde{x}_\ell, & t \in J_\ell, \end{cases} \tag{2.10}$$

and $\widetilde{x}_\ell \in E_\ell$ is a solution of (2.4).
According to Lemma 2.1, the integral equation

$$\begin{aligned}\widetilde{x}_\ell(t) \quad = \quad & -\frac{(T_\ell-T_{\ell-1})^{1-u_\ell}(t-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} f(s, \widetilde{x}_\ell(s), I^{u_\ell}_{T^+_{\ell-1}} \widetilde{x}_\ell(s))ds \\ & +\frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t-s)^{u_\ell-1} f(s, \widetilde{x}_\ell(s), I^{u_\ell}_{T^+_{\ell-1}} \widetilde{x}_\ell(s))ds,\end{aligned} \tag{2.11}$$

holds.
Let $t \in J_\ell,\ \ell = 1,2,...,n$. Then, by (2.10) and (2.11) we get

$$\begin{aligned}&\|z(t) - x(t)\| = \|z(t) - x_\ell(t)\| = \|z_\ell(t) - \widetilde{x}_\ell(t)\| \\ = \quad & \|z_\ell(t) + \frac{(T_\ell-T_{\ell-1})^{1-u_\ell}(t-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} f(s, \widetilde{x}_\ell(s), I^{u_\ell}_{T^+_{\ell-1}} \widetilde{x}_\ell(s))ds \\ & -\frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t-s)^{u_\ell-1} f(s, \widetilde{x}_\ell(s), I^{u_\ell}_{T^+_{\ell-1}} \widetilde{x}_\ell(s))ds\| \\ \leq \quad & \|z_\ell(t) + \frac{(T_\ell-T_{\ell-1})^{1-u_\ell}(t-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} f(s, z_\ell(s), I^{u_\ell}_{T^+_{\ell-1}} z_\ell(s))ds \\ & -\frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t-s)^{u_\ell-1} f(s, z_\ell(s), I^{u_\ell}_{T^+_{\ell-1}} z_\ell(s))ds\| \\ & +\frac{(T_\ell-T_{\ell-1})^{1-u_\ell}(t-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} \\ & \|f(s, z_\ell(s), I^{u_\ell}_{T^+_{\ell-1}} z_\ell) - f(s, \widetilde{x}_\ell(s), I^{u_\ell}_{T^+_{\ell-1}} \widetilde{x}_\ell)\|ds \\ & +\frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t-s)^{u_\ell-1} \|f(s, z_\ell(s), I^{u_\ell}_{T^+_{\ell-1}} z_\ell) - f(s, \widetilde{x}_\ell(s), I^{u_\ell}_{T^+_{\ell-1}} \widetilde{x}_\ell)\|ds\end{aligned}$$

$$\begin{aligned}
&\leq \lambda_{\vartheta(t)}\epsilon\vartheta(t)+\frac{(T_\ell-T_{\ell-1})^{1-u_\ell}(t-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\\
&\quad\int_{T_{\ell-1}}^{T_\ell}(T_\ell-s)^{u_\ell-1}s^{-\delta}(K\|z_\ell(s)-\widetilde{x}_\ell(s)\|+LI^{u_\ell}_{T^+_{\ell-1}}\|z_\ell(s)-\widetilde{x}_\ell(s)\|)ds\\
&\quad+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(t-s)^{u_\ell-1}s^{-\delta}(K\|z_\ell(s)-\widetilde{x}_\ell(s)\|+LI^{u_\ell}_{T^+_{\ell-1}}\|z_\ell(s)-\widetilde{x}_\ell(s)\|)ds\\
&\leq \lambda_{\vartheta(t)}\epsilon\vartheta(t)+\frac{(T_\ell-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}(K\|z_\ell-\widetilde{x}_\ell\|_{E_\ell}+LI^{u_\ell}_{T^+_{\ell-1}}\|z_\ell-\widetilde{x}_\ell\|_{E_\ell})\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}ds\\
&\quad+\frac{(T_\ell-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}(K\|z_\ell-\widetilde{x}_\ell\|_{E_\ell}+LI^{u_\ell}_{T^+_{\ell-1}}\|z_\ell-\widetilde{x}_\ell\|_{E_\ell})\int_{T_{\ell-1}}^{t}s^{-\delta}ds\\
&\leq \lambda_{\vartheta(t)}\epsilon\vartheta(t)+\frac{(T_\ell-T_{\ell-1})^{u_\ell-1}({T_\ell}^{1-\delta}-{T_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}(K\|z_\ell-\widetilde{x}_\ell\|_{E_\ell}+L\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\|z_\ell-\widetilde{x}_\ell\|_{E_\ell})\\
&\quad+\frac{(T_\ell-T_{\ell-1})^{u_\ell-1}(t^{1-\delta}-{T_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}(K\|z_\ell-\widetilde{x}_\ell\|_{E_\ell}+L\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\|z_\ell-\widetilde{x}_\ell\|_{E_\ell})\\
&\leq \lambda_{\vartheta(t)}\epsilon\vartheta(t)+\frac{2(T_\ell-T_{\ell-1})^{u_\ell-1}({T_\ell}^{1-\delta}-{T_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}(K+L\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)})\|z_\ell-\widetilde{x}_\ell\|_{E_\ell}\\
&\leq \lambda_{\vartheta(t)}\epsilon\vartheta(t)+\mu\|z-x\|,
\end{aligned}$$

where

$$\mu=\max_{\ell=1,2,\ldots,n}\frac{2(T_\ell-T_{\ell-1})^{u_\ell-1}({T_\ell}^{1-\delta}-{T_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\Big(K+L\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\Big).$$

Then,

$$\|z-x\|(1-\mu)\leq\lambda_{\vartheta(t)}\epsilon\vartheta(t).$$

We obtain, for each $t\in J$

$$|z(t)-x(t)|\leq\|z-x\|\leq\frac{\lambda_{\vartheta(t)}\vartheta(t)}{1-\mu}\epsilon:=c_{f_1}\epsilon.$$

Therefore, by Theorem (1.3), the BVP (2.1) is **(UHR)** stable with respect to ϑ.

2.4 Example

In this example, we deal with the fractional boundary value problem,

$$\begin{cases} D_{0^+}^{u(t)}x(t) = \dfrac{t^{-\frac{1}{3}}e^{-t}}{\left(e^{e^{\frac{t^2}{1+t}}} + 4e^{2t} + 1\right)\left(1 + |x(t)| + |I_0^{u(t)}x(t)|\right)}, \quad t \in J := [0, 2], \\ x(0) = 0, \ \ x(2) = 0, \end{cases} \tag{2.12}$$

where

$$u(t) = \begin{cases} \frac{3}{2}, & t \in J_1 := [0, 1], \\ \frac{9}{5}, & t \in J_2 :=]1, 2]. \end{cases} \tag{2.13}$$

Denote

$$f_1(t, y, z) = \frac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}} + 4e^{2t} + 1)(1 + y + z)}, \ (t, y, z) \in [0, 2] \times [0, +\infty) \times [0, +\infty).$$

Then, we have

$$\begin{aligned} t^{\frac{1}{3}}|f_1(t, y_1, z_1) - f_1(t, y_2, z_2)| &= \left| \frac{e^{-t}}{(e^{e^{\frac{t^2}{1+t}}} + 4e^{2t} + 1)} \left(\frac{1}{1 + y_1 + z_1} - \frac{1}{1 + y_2 + z_2} \right) \right| \\ &\le \frac{e^{-t}(|y_1 - y_2| + |z_1 - z_2|)}{(e^{e^{\frac{t^2}{1+t}}} + 4e^{2t} + 1)(1 + y_1 + z_1)(1 + y_2 + z_2)} \\ &\le \frac{e^{-t}}{(e^{e^{\frac{t^2}{1+t}}} + 4e^{2t} + 1)}(|y_1 - y_2| + |z_1 - z_2|) \\ &\le \frac{1}{(e+5)}|y_1 - y_2| + \frac{1}{(e+5)}|z_1 - z_2|. \end{aligned}$$

Ergo, (H2) holds with $\delta = \frac{1}{3}$, $K = L = \frac{1}{e+5}$.

By (2.13), according to (2.4) we consider two auxiliary BVP for Riemann-Liouville fractional differential equations of constant order

$$\begin{cases} D_{0^+}^{\frac{3}{2}}x(t) = \dfrac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}} + 4e^{2t} + 1)(1 + |x(t)| + |I_0^{\frac{3}{2}}x(t)|)}, \quad t \in J_1, \\ x(0) = 0, \ \ x(1) = 0, \end{cases} \tag{2.14}$$

and

$$\begin{cases} D_{1^+}^{\frac{9}{5}}x(t) = \dfrac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}} + 4e^{2t} + 1)(1 + |x(t)| + |I_0^{\frac{9}{5}}x(t)|)}, \quad t \in J_2, \\ x(1) = 0, \ \ x(2) = 0. \end{cases} \tag{2.15}$$

Next, we prove that the condition (2.7) is fulfilled for $\ell = 1$. Indeed,

$$\frac{2({T_1}^{1-\delta} - {T_0}^{1-\delta})(T_1 - T_0)^{u_1-1}}{(1-\delta)\Gamma(u_1)}\Big(K+\frac{L(T_1-T_0)^{u_1}}{\Gamma(u_1+1)}\Big) = \frac{2}{\frac{2}{3}(e+5)\Gamma(\frac{3}{2})}\Big(1+\frac{1}{\Gamma(\frac{5}{2})}\Big) \simeq 0.7685 < 1.$$

Let $\vartheta(t) = t^{\frac{1}{2}}$. Then, we get

$$\begin{aligned} I_{0^+}^{u_1}\vartheta(t) &= \frac{1}{\Gamma(\frac{3}{2})}\int_0^t (t-s)^{\frac{1}{2}} s^{\frac{1}{2}} ds \\ &\leq \frac{1}{\Gamma(\frac{3}{2})}\int_0^t (t-s)^{\frac{1}{2}} ds \\ &\leq \frac{2}{3\Gamma(\frac{3}{2})}\vartheta(t) := \lambda_{\vartheta(t)}\vartheta(t), \end{aligned}$$

where $\lambda_{\vartheta(t)} = \frac{2}{3\Gamma(\frac{3}{2})}$. Thus, (H3) is satisfied.

Accordingly the condition (2.7) is achieved. By Theorem (2.1), the BVP (2.14) has a solution $\widetilde{x}_1 \in E_1$.

We prove that the condition (2.7) is fulfilled for $\ell = 2$. Indeed,

$$\frac{2({T_2}^{1-\delta} - {T_1}^{1-\delta})(T_2 - T_1)^{u_2-1}}{(1-\delta)\Gamma(u_2)}\Big(K+\frac{L(T_2-T_1)^{u_2}}{\Gamma(u_2+1)}\Big) = \frac{2(2^{\frac{2}{3}}-1)}{\frac{2}{3}\Gamma(\frac{9}{5})}\frac{1}{e+5}\Big(1+\frac{1}{\Gamma(\frac{14}{5})}\Big) \simeq 0.3913 < 1.$$

Accordingly the condition (2.7) is achieved.

We have

$$\begin{aligned} I_{1^+}^{u_2}\vartheta(t) &= \frac{1}{\Gamma(\frac{9}{5})}\int_1^t (t-s)^{\frac{4}{5}} s^{\frac{1}{2}} ds \\ &\leq \frac{1}{\Gamma(\frac{9}{5})}\int_1^t (t-s)^{\frac{4}{5}} ds \\ &\leq \frac{5}{9\Gamma(\frac{9}{5})}\vartheta(t) := \lambda_{\vartheta(t)}\vartheta(t), \end{aligned}$$

where $\lambda_{\vartheta(t)} = \frac{5}{9\Gamma(\frac{9}{5})}$. Thus, (H3) is fulfilled.

According to Theorem 2.1, the BVP (2.15) possesses a solution $\widetilde{x}_2 \in E_2$.

As a result, by Theorem (2.2), the BVP (2.12) has a solution

$$x(t) = \begin{cases} \widetilde{x}_1(t), & t \in J_1, \\ x_2(t), & t \in J_2, \end{cases}$$

where

$$x_2(t) = \begin{cases} 0, & t \in J_1, \\ \widetilde{x}_2(t), & t \in J_2. \end{cases}$$

According to Theorem (2.3), the BVP (2.12) is **(UHR)** stable with respect to ϑ.

Chapitre 3

Multiterm boundary value problem of Caputo fractional differential equations of variable order [2]

3.1 Introduction

We will study the existence of solutions for the boundary value problem (BVP for short)

$$\begin{cases} {}^cD_{0^+}^{u(t)}x(t) = f(t, x(t), I_{0^+}^{u(t)}x(t)), \;\; t \in J := [0, T], \\ x(0) = 0, \;\; x(T) = 0, \end{cases} \tag{3.1}$$

where $1 < u(t) \leq 2$, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and ${}^cD_{0^+}^{u(t)}$ is the Caputo fractional derivative of variable-order $u(t)$ and $I_{0^+}^{u(t)}$ is the Riemann-Liouville fractional integral of variable-order $u(t)$.
Further, we study the stability of the obtained solution in the sense of Ulam-Hyers.

3.2 Existence of solutions

Let us introduce the following assumptions :

(H1) Let $n \in \mathbb{N}$ be an integer, $\mathcal{P} = \{J_1 := [0, T_1], J_2 := (T_1, T_2], J_3 := (T_2, T_3], ...J_n := (T_{n-1}, T]\}$ be a partition of the interval J, and let $u(t) : J \to (1, 2]$ be a piecewise constant function with respect to $\mathcal{P}$, i.e.,

[2] **A. Benkerrouche**, M.S. Souid, S. Chandok, A. Hakem, Existence and stability of a Caputo variable-order boundary value problem, *Journal of Mathematics*, **2021**, Article ID 7967880, 1-16, (2021).

$$u(t)=\sum_{\ell=1}^{n}u_\ell I_\ell(t)=\begin{cases} u_1, & if\ t\in J_1,\\ u_2, & if\ t\in J_2,\\ \cdot & \\ \cdot & \\ \cdot & \\ u_n, & if\ t\in J_n,\end{cases}$$

where $1<u_\ell\leq 2$ are constants, and I_ℓ is the indicator of the interval $J_\ell := (T_{\ell-1},T_\ell],\ \ell = 1,2,...,n,$ (with $T_0=0,\ T_n=T$) such that

$$I_\ell(t)=\begin{cases} 1, & for\ t\in J_\ell,\\ 0, & for\ elsewhere.\end{cases}$$

Further, for a given set U of functions $u:J\to X$ let us denote by

$$U(t)=\{u(t),u\in U\},\ t\in J,$$

and

$$U(J)=\{v(t):v\in U,\ t\in J\}.$$

For each $\ell\in\{1,2,...,n\}$, the symbol $E_\ell=C(J_\ell,\mathbb{R})$, indicated the Banach space of continuous functions $x:J_\ell\to\mathbb{R}$ equipped with the norm

$$\|x\|_{E_\ell}=\sup_{t\in J_\ell}|x(t)|.$$

Then, for any $t\in J_\ell,\ \ell=1,2,...,n$ the left Caputo fractional derivative (CFD) of variable order $u(t)$ for function $x(t)\in C(J,\mathbb{R})$, could be presented as a sum of left Caputo fractional derivatives of constant-orders $u_\ell,\ \ell=1,2,...,n$

$$^cD^{u(t)}_{0^+}x(t)=\int_0^{T_1}\frac{(t-s)^{1-u_1}}{\Gamma(2-u_1)}x^{(2)}(s)ds+...+\int_{T_{\ell-1}}^{t}\frac{(t-s)^{1-u_\ell}}{\Gamma(2-u_\ell)}x^{(2)}(s)ds. \tag{3.2}$$

Thus, according to (3.2), BVP(3.1) can be written for any $t\in J_\ell,\ \ell=1,2,...,n$ in the form

$$\int_0^{T_1}\frac{(t-s)^{1-u_1}}{\Gamma(2-u_1)}x^{(2)}(s)ds+\int_{T_1}^{T_2}\frac{(t-s)^{1-u_2}}{\Gamma(2-u_2)}x^{(2)}(s)ds+...$$
$$+\int_{T_{\ell-1}}^{t}\frac{(t-s)^{1-u_\ell}}{\Gamma(2-u_\ell)}x^{(2)}(s)ds=f(t,x(t),I^{u_\ell}_{0^+}x(t)),\quad t\in J_\ell. \tag{3.3}$$

In what follows we shall introduce the solution to the BVP (3.1).

Definition 3.1 *BVP (3.1) has a solution, if there are functions $x_\ell,\ell=1,2,...,n,$ so that, $x_\ell\in C([0,T_\ell],\mathbb{R})$ fulfilling equation (3.3) and $x_\ell(0)=0=x_\ell(T_\ell)$.*

Let the function $x \in C(J, \mathbb{R})$ be such that $x(t) \equiv 0$ on $t \in [0, T_{\ell-1}]$ and it solves integral equation (3.3). Then (3.3) is reduced to

$$^{c}D^{u_\ell}_{T^+_{\ell-1}} x(t) = f(t, x(t), I^{u_\ell}_{T^+_{\ell-1}} x(t)), \quad t \in J_\ell.$$

We shall deal with following BVP

$$\begin{cases} ^{c}D^{u_\ell}_{T^+_{\ell-1}} x(t) = f(t, x(t), I^{u_\ell}_{T^+_{\ell-1}} x(t)), \quad t \in J_\ell, \\ x(T_{\ell-1}) = 0, x(T_\ell) = 0. \end{cases} \tag{3.4}$$

For our purpose, the upcoming lemma will be a corner stone of the solution of (3.4).

Lemma 3.1 *Let $\ell \in \{1, 2, ..., n\}$ be a natural number, $f \in C(J_\ell \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and there exists a number $\delta \in (0, 1)$ such that $t^\delta f \in C(J_\ell \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.*
Then, the function $x \in E_\ell$ is a solution of BVP (3.4) if and only if x solves the integral equation

$$\begin{aligned} x(t) = &-(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} f(T_\ell, x(T_\ell), I^{u_\ell}_{T^+_{\ell-1}} x(T_\ell)) \\ &+ I^{u_\ell}_{T^+_{\ell-1}} f(t, x(t), I^{u_\ell}_{T^+_{\ell-1}} x(t)). \end{aligned} \tag{3.5}$$

Proof

We presume that $x \in E_\ell$ is solution of BVP (3.4). Employing the operator $I^{u_\ell}_{T^+_{\ell-1}}$ to both sides of (3.4) and regarding Lemma 1.6, we find

$$x(t) = \omega_1 + \omega_2(t - T_{\ell-1}) + \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t-s)^{u_\ell - 1} f(s, x(s), I^{u_\ell}_{T^+_{\ell-1}} x(s)) ds, \quad t \in J_\ell.$$

By $x(T_{\ell-1}) = 0$, we get $\omega_1 = 0$.

Let $x(t)$ satisfy $x(T_\ell) = 0$. So, we observe that

$$\omega_2 = -(T_\ell - T_{\ell-1})^{-1} I^{u_\ell}_{T^+_{\ell-1}} f(T_\ell, x(T_\ell), I^{u_\ell}_{T^+_{\ell-1}} x(T_\ell)).$$

Then, we find

$$\begin{aligned} x(t) = &-(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} f(T_\ell, x(T_\ell), I^{u_\ell}_{T^+_{\ell-1}} x(T_\ell)) \\ &+ I^{u_\ell}_{T^+_{\ell-1}} f(t, x(t), I^{u_\ell}_{T^+_{\ell-1}} x(t)), \quad t \in J_\ell. \end{aligned}$$

Conversely, let $x \in E_\ell$ be solution of integral equation (3.5). Regarding the continuity of function $t^\delta f$ and Lemma(1.7), we deduce that x is the solution of BVP (3.4).

We are now in a position to prove the existence of solution for the BVP (3.4) based on concept of (**MNCK**) and (**DFPT**).

Theorem 3.1 *Let the conditions of Lemma 3.1 be satisfied and there exist a constants* $K,\ L > 0$, *such that,*

$$t^{\delta}|f(t,y_1,z_1)-f(t,y_2,z_2)| \leq K|y_1-y_2|+L|z_1-z_2|,\ \ y_i,\ z_i \in \mathbb{R},\ i=1,2,\ t\in J_\ell, \tag{3.6}$$

and the inequality

$$\frac{2(T_\ell - T_{\ell-1})^{u_\ell-1}(T_\ell^{1-\delta} - T_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\Big(K + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\Big) < 1, \tag{3.7}$$

holds.

Then, BVP (3.4) possesses at least one solution in E_ℓ.

Proof

We construct the operator

$$W : E_\ell \to E_\ell,$$

as follow :

$$Wx(t) = -(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} f(T_\ell, x(T_\ell), I^{u_\ell}_{T^+_{\ell-1}} x(T_\ell))$$

$$+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t} (t-s)^{u_\ell-1} f(s, x(s), I^{u_\ell}_{T^+_{\ell-1}} x(s))ds, \quad t\in J_\ell. \tag{3.8}$$

It follows from the properties of fractional integrals and from the continuity of function $t^{\delta}f$ that the operator $W : E_\ell \to E_\ell$ defined in (3.8) is well defined.
Let

$$R_\ell \ \ \geq \ \ \frac{\frac{2f^\star(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)}}{1-\frac{2(T_\ell-T_{\ell-1})^{u_\ell-1}(T_\ell^{1-\delta}-T_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\Big(K+L\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\Big)},$$

with

$$f^\star = \sup_{t\in J_\ell}|f(t,0,0)|.$$

We consider the set

$$B_{R_\ell} = \{x \in E_\ell,\ \|x\|_{E_\ell} \leq R_\ell\}.$$

Clearly B_{R_ℓ} is nonempty, closed, convex and bounded.

Now, we demonstrate that W satisfies the assumption of the Theorem (1.1). We shall prove it in four phases.

STEP 1 : Claim : $W(B_{R_\ell}) \subseteq (B_{R_\ell})$.
For $x \in B_{R_\ell}$ and by (H2), we get

$$\begin{aligned}
|Wx(t)| \leq\; & \frac{(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1})}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} |f(s, x(s), I^{u_\ell}_{T^+_{\ell-1}} x(s))| ds \\
& + \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t - s)^{u_\ell - 1} |f(s, x(s), I^{u_\ell}_{T^+_{\ell-1}} x(s))| ds \\
\leq\; & \frac{2}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} |f(s, x(s), I^{u_\ell}_{T^+_{\ell-1}} x(s))| ds \\
\leq\; & \frac{2}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} |f(s, x(s), I^{u_\ell}_{T^+_{\ell-1}} x(s)) - f(s, 0, 0)| ds \\
& + \frac{2}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} |f(s, 0, 0)| ds \\
\leq\; & \frac{2}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} s^{-\delta} (K|x(s)| + L|I^{u_\ell}_{T^+_{\ell-1}} x(s)|) ds + \frac{2f^\star (T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)} \\
\leq\; & \frac{2(T_\ell - T_{\ell-1})^{u_\ell - 1}}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} s^{-\delta} (K + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)}) |x(s)| ds + \frac{2f^\star (T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)} \\
\leq\; & \frac{2(T_\ell - T_{\ell-1})^{u_\ell - 1}(T_\ell^{1-\delta} - T_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)} (K + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)}) R_\ell + \frac{2f^\star (T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)} \\
\leq\; & R_\ell,
\end{aligned}$$

which means that $W(B_{R_\ell}) \subseteq B_{R_\ell}$.
STEP 2 : Claim : W is continuous.
We presume that the sequence (x_n) converges to x in E_ℓ and $t \in J_\ell$. Then,

$$\begin{aligned}
& |(Wx_n)(t) - (Wx)(t)| \leq \frac{(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1})}{\Gamma(u_\ell)} \\
& \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} |f(s, x_n(s), I^{u_\ell}_{T^+_{\ell-1}} x_n(s)) - f(s, x(s), I^{u_\ell}_{T^+_{\ell-1}} x(s))| ds \\
& + \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t - s)^{u_\ell - 1} |f(s, x_n(s), I^{u_\ell}_{T^+_{\ell-1}} x_n(s)) - f(s, x(s), I^{u_\ell}_{T^+_{\ell-1}} x(s))| ds \\
\leq\; & \frac{2}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} |f(s, x_n(s), I^{u_\ell}_{T^+_{\ell-1}} x_n(s)) - f(s, x(s), I^{u_\ell}_{T^+_{\ell-1}} x(s))| ds \\
\leq\; & \frac{2}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} s^{-\delta} (T_\ell - s)^{u_\ell - 1} (K|x_n(s) - x(s)| + LI^{u_\ell}_{T^+_{\ell-1}} |x_n(s) - x(s))|) ds \\
\leq\; & \frac{2K}{\Gamma(u_\ell)} \|x_n - x\|_{E_\ell} \int_{T_{\ell-1}}^{T_\ell} s^{-\delta} (T_\ell - s)^{u_\ell - 1} ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2L}{\Gamma(u_\ell)}\|I^{u_\ell}_{T^+_{\ell-1}}(x_n-x)\|_{E_\ell}\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}(T_\ell-s)^{u_\ell-1}ds\\
\le\ & \frac{2K}{\Gamma(u_\ell)}\|x_n-x\|_{E_\ell}\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}(T_\ell-s)^{u_\ell-1}ds\\
&+\frac{2L(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)\Gamma(u_\ell+1)}\|x_n-x\|_{E_\ell}\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}(T_\ell-s)^{u_\ell-1}ds\\
\le\ & \Big(\frac{2K}{\Gamma(u_\ell)}+\frac{2L(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)\Gamma(u_\ell+1)}\Big)\|x_n-x\|_{E_\ell}\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}(T_\ell-s)^{u_\ell-1}ds\\
\le\ & \frac{2(T_\ell-T_{\ell-1})^{u_\ell-1}({T_\ell}^{1-\delta}-{T_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\Big(K+L\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\Big)\|x_n-x\|_{E_\ell},
\end{aligned}
$$

i.e., we obtain

$$\|(Wx_n)-(Wx)\|_{E_\ell}\to 0 \quad as \quad n\to\infty.$$

Ergo, the operator W is a continuous on E_ℓ.

Step 3 : Claim : W is bounded and equicontinuous.
By Step 1, we have $W(B_{R_\ell})=\{W(x):x\in B_{R_\ell}\}\subset B_{R_\ell}$ thus for each $x\in B_{R_\ell}$ we have $\|W(x)\|_{E_\ell}\le R_\ell$ which means that $W(B_{R_\ell})$is bounded. It remains to indicate that $W(B_{R_\ell})$ is equicontinuous.
For $t_1,t_2\in J_\ell,\ t_1<t_2$ and $x\in B_{R_\ell}$, we have

$$
\begin{aligned}
&\Big|(Wx)(t_2)-(Wx)(t_1)\Big| = \Big| -\frac{(T_\ell-T_{\ell-1})^{-1}(t_2-T_{\ell-1})}{\Gamma(u_\ell)}\\
&\int_{T_{\ell-1}}^{T_\ell}(T_\ell-s)^{u_\ell-1}f(s,x(s),I^{u_\ell}_{T^+_{\ell-1}}x(s))ds\\
&+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_2}(t_2-s)^{u_\ell-1}f(s,x(s),I^{u_\ell}_{T^+_{\ell-1}}x(s))ds\\
&+\frac{(T_\ell-T_{\ell-1})^{-1}(t_1-T_{\ell-1})}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(T_\ell-s)^{u_\ell-1}f(s,x(s),I^{u_\ell}_{T^+_{\ell-1}}x(s))ds\\
&-\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}(t_1-s)^{u_\ell-1}f(s,x(s),I^{u_\ell}_{T^+_{\ell-1}}x(s))ds\Big|\\
\le\ & \frac{(T_\ell-T_{\ell-1})^{-1}}{\Gamma(u_\ell)}\Big((t_2-T_{\ell-1})-(t_1-T_{\ell-1})\Big)\int_{T_{\ell-1}}^{T_\ell}(T_\ell-s)^{u_\ell-1}|f(s,x(s),I^{u_\ell}_{T^+_{\ell-1}}x(s))|ds\\
&+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}\Big((t_2-s)^{u_\ell-1}-(t_1-s)^{u_\ell-1}\Big)|f(s,x(s),I^{u_\ell}_{T^+_{\ell-1}}x(s))|ds\\
&+\frac{1}{\Gamma(u_\ell)}\int_{t_1}^{t_2}(t_2-s)^{u_\ell-1}|f(s,x(s),I^{u_\ell}_{T^+_{\ell-1}}x(s))|ds
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \frac{(T_\ell - T_{\ell-1})^{-1}}{\Gamma(u_\ell)}\Big((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\Big) \\
& \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} |f(s, x(s), I^{u_\ell}_{T^+_{\ell-1}} x(s)) - f(s,0,0)| ds \\
& + \frac{(T_\ell - T_{\ell-1})^{-1}}{\Gamma(u_\ell)}\Big((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\Big) \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} |f(s,0,0)| ds \\
& + \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t_1} \Big((t_2 - s)^{u_\ell - 1} - (t_1 - s)^{u_\ell - 1}\Big) |f(s, x(s), I^{u_\ell}_{T^+_{\ell-1}} x(s)) - f(s,0,0)| ds \\
& + \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t_1} \Big((t_2 - s)^{u_\ell - 1} - (t_1 - s)^{u_\ell - 1}\Big) |f(s,0,0)| ds \\
& + \frac{1}{\Gamma(u_\ell)} \int_{t_1}^{t_2} (t_2 - s)^{u_\ell - 1} |f(s, x(s), I^{u_\ell}_{T^+_{\ell-1}} x(s)) - f(s,0,0)| ds \\
& + \frac{1}{\Gamma(u_\ell)} \int_{t_1}^{t_2} (t_2 - s)^{u_\ell - 1} |f(s,0,0)| ds. \\
\leq\ & \frac{(T_\ell - T_{\ell-1})^{-1}}{\Gamma(u_\ell)}\Big((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\Big) \\
& \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} s^{-\delta} (K|x(s)| + L|I^{u_\ell}_{T^+_{\ell-1}} x(s)|) ds \\
& + \frac{f^\star (T_\ell - T_{\ell-1})^{-1}}{\Gamma(u_\ell)}\Big((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\Big) \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} ds \\
& + \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t_1} s^{-\delta} \Big((t_2 - s)^{u_\ell - 1} - (t_1 - s)^{u_\ell - 1}\Big) (K|x(s)| + L|I^{u_\ell}_{T^+_{\ell-1}} x(s)|) ds \\
& + \frac{f^\star}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t_1} \Big((t_2 - s)^{u_\ell - 1} - (t_1 - s)^{u_\ell - 1}\Big) ds \\
& + \frac{1}{\Gamma(u_\ell)} \int_{t_1}^{t_2} s^{-\delta} (t_2 - s)^{u_\ell - 1} (K|x(s)| + L|I^{u_\ell}_{T^+_{\ell-1}} x(s)|) ds \\
& + \frac{f^\star}{\Gamma(u_\ell)} \int_{t_1}^{t_2} (t_2 - s)^{u_\ell - 1} ds \\
\leq\ & \frac{(T_\ell - T_{\ell-1})^{u_\ell - 2}}{\Gamma(u_\ell)}\Big((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\Big) \\
& (K\|x\|_{E_\ell} + L\|I^{u_\ell}_{T^+_{\ell-1}} x\|_{E_\ell}) \int_{T_{\ell-1}}^{T_\ell} s^{-\delta} ds \\
& + \frac{f^\star (T_\ell - T_{\ell-1})^{u_\ell - 1}}{\Gamma(u_\ell + 1)}\Big((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\Big) \\
& + \frac{1}{\Gamma(u_\ell)} (K\|x\|_{E_\ell} + L\|I^{u_\ell}_{T^+_{\ell-1}} x\|_{E_\ell}) \int_{T_{\ell-1}}^{t_1} s^{-\delta} ((t_2 - t_1)^{u_\ell - 1}) ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{f^\star}{\Gamma(u_\ell)}\Big(\frac{(t_2-T_{\ell-1})^{u_\ell}}{u_\ell}-\frac{(t_2-t_1)^{u_\ell}}{u_\ell}-\frac{(t_1-T_{\ell-1})^{u_\ell}}{u_\ell}\Big)\\
&+\frac{(t_2-t_1)^{u_\ell-1}}{\Gamma(u_\ell)}(K\|x\|_{E_\ell}+L\|I^{u_\ell}_{T^+_{\ell-1}}x\|_{E_\ell})\int_{t_1}^{t_2}s^{-\delta}ds\\
&+\frac{f^\star}{\Gamma(u_\ell)}\frac{(t_2-t_1)^{u_\ell}}{u_\ell}\\
\leq\ &\frac{(T_\ell-T_{\ell-1})^{u_\ell-2}({T_\ell}^{1-\delta}-{T_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\Big((t_2-T_{\ell-1})-(t_1-T_{\ell-1})\Big)\\
&(K\|x\|_{E_\ell}+L\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\|x\|_{E_\ell})\\
&+\frac{f^\star(T_\ell-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell+1)}\Big((t_2-T_{\ell-1})-(t_1-T_{\ell-1})\Big)\\
&+\Big(\frac{({t_1}^{1-\delta}-{T_{\ell-1}}^{1-\delta})(t_2-t_1)^{u_\ell-1}}{(1-\delta)\Gamma(u_\ell)}\Big)(K\|x\|_{E_\ell}+L\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\|x\|_{E_\ell})\\
&+\frac{f^\star}{\Gamma(u_\ell+1)}\Big((t_2-T_{\ell-1})^{u_\ell}-(t_2-t_1)^{u_\ell}-(t_1-T_{\ell-1})^{u_\ell}\Big)\\
&+\frac{({t_2}^{1-\delta}-{t_1}^{1-\delta})(t_2-t_1)^{u_\ell-1}}{(1-\delta)\Gamma(u_\ell)}(K\|x\|_{E_\ell}+L\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\|x\|_{E_\ell})+\frac{f^\star(t_2-t_1)^{u_\ell}}{\Gamma(u_\ell+1)}\\
\leq\ &\Big(\frac{(T_\ell-T_{\ell-1})^{u_\ell-2}({T_\ell}^{1-\delta}-{T_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\\
&(K+L\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)})\|x\|_{E_\ell}+\frac{f^\star(T_\ell-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell+1)}\Big)\Big((t_2-T_{\ell-1})-(t_1-T_{\ell-1})\Big)\\
&+\Big(\frac{{t_2}^{1-\delta}-{T_{\ell-1}}^{1-\delta}}{(1-\delta)\Gamma(u_\ell)}(K+L\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)})\|x\|_{E_\ell}\Big)(t_2-t_1)^{u_\ell-1}\\
&+\frac{f^\star}{\Gamma(u_\ell+1)}\Big((t_2-T_{\ell-1})^{u_\ell}-(t_1-T_{\ell-1})^{u_\ell}\Big).
\end{aligned}
$$

Hence $\|(Wx)(t_2)-(Wx)(t_1)\|_{E_\ell}\to 0$ as $|t_2-t_1|\to 0$. It implies that $T(B_{R_\ell})$ is equicontinuous.

Remark 3.1 *According to remark of ([12]) page 20 we can easily show that the inequality (3.6) is equivalent to the following inequality*

$$\zeta(t^\delta|f(t,B_1,B_2)|)\leq K\zeta(B_1)+L\zeta(B_2),$$

for any bounded sets $B_1,B_2\subset X$ and for each $t\in J_\ell$.

STEP 4 : Claim : W is k-*set contraction*.
For $U \in B_{R_\ell}$, $t \in J_\ell$, we get,

$$\begin{aligned}\zeta(W(U)(t)) &= \zeta((Wx)(t), x \in U) \\ &\leq \Big(\frac{(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1})}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} \zeta f(s, x(s), I_{T_{\ell-1}^+}^{u_\ell} x(s)) ds \\ &\quad + \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t-s)^{u_\ell - 1} \zeta f(s, x(s), I_{T_{\ell-1}^+}^{u_\ell} x(s)) ds, x \in U\Big).\end{aligned}$$

Then Remark 3.1 implies that, for each $s \in J_\ell$,

$$\begin{aligned}\zeta(W(U)(t)) &\leq \Big(\frac{(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1})}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} \Big[K\widehat{\zeta}(U) \int_{T_{\ell-1}}^{T_\ell} s^{-\delta} ds \\ &\quad + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)} \widehat{\zeta}(U) \int_{T_{\ell-1}}^{T_\ell} s^{-\delta} ds\Big] \\ &\quad + \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t-s)^{u_\ell - 1} \Big[K\widehat{\zeta}(U) \int_{T_{\ell-1}}^{t} s^{-\delta} ds \\ &\quad + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)} \widehat{\zeta}(U) \int_{T_{\ell-1}}^{t} s^{-\delta} ds\Big], x \in U\Big) \\ &\leq \Big(\frac{(T_\ell - T_{\ell-1})^{u_\ell - 2}(t - T_{\ell-1})}{\Gamma(u_\ell)} \\ &\quad \int_{T_{\ell-1}}^{T_\ell} \Big[K\widehat{\zeta}(U) \int_{T_{\ell-1}}^{T_\ell} s^{-\delta} ds + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)} \widehat{\zeta}(U) \int_{T_{\ell-1}}^{T_\ell} s^{-\delta} ds\Big] \\ &\quad + \frac{(t - T_{\ell-1})^{u_\ell - 1}}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} \Big[K\widehat{\zeta}(U) \int_{T_{\ell-1}}^{t} s^{-\delta} ds \\ &\quad + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)} \widehat{\zeta}(U) \int_{T_{\ell-1}}^{t} s^{-\delta} ds\Big], x \in U\Big) \\ &\leq \frac{({T_\ell}^{1-\delta} - {T_{\ell-1}}^{1-\delta})(T_\ell - T_{\ell-1})^{u_\ell - 2}(t - T_{\ell-1})}{(1-\delta)\Gamma(u_\ell)} \Big(K + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)}\Big) \widehat{\zeta}(U) \\ &\quad + \frac{(t^{1-\delta} - {T_{\ell-1}}^{1-\delta})(t - T_{\ell-1})^{u_\ell - 1}}{(1-\delta)\Gamma(u_\ell)} \Big(K + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)}\Big) \widehat{\zeta}(U) \\ &\leq \frac{2({T_\ell}^{1-\delta} - {T_{\ell-1}}^{1-\delta})(T_\ell - T_{\ell-1})^{u_\ell - 1}}{(1-\delta)\Gamma(u_\ell)} \Big(K + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)}\Big) \widehat{\zeta}(U).\end{aligned}$$

Thus,

$$\widehat{\zeta}(WU) \leq \frac{2({T_\ell}^{1-\delta} - {T_{\ell-1}}^{1-\delta})(T_\ell - T_{\ell-1})^{u_\ell - 1}}{(1-\delta)\Gamma(u_\ell)} \Big(K + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)}\Big) \widehat{\zeta}(U).$$

Therefore, all conditions of Theorem 1.1 are fulfilled and thus, the BVP (3.4) has at least solution $\widetilde{x}_\ell \in B_{R_\ell}$. Since $B_{R_\ell} \subset E_\ell$, the claim of Theorem 3.1 is proved.

Now, we will prove the existence result for BVP (3.1).

Introduce the following assumption :

(H2) Let $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and there exists a number $\delta \in (0,1)$ such that $t^\delta f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and there exist a constants $K, \ L > 0$, such that, $t^\delta |f(t, y_1, z_1) - f(t, y_2, z_2)| \leq K|y_1 - y_2| + L|z_1 - z_2|, \ y_1, \ y_2, \ z_1, \ z_2 \in \mathbb{R}, \ t \in J.$

Theorem 3.2 *Let the conditions (H1), (H2) and inequality (3.7) be satisfied for all* $\ell \in \{1, 2, ..., n\}$.
Then, the problem (3.1) possesses at least one solution in $C(J, \mathbb{R})$.

Proof
According to Theorem 3.1, the BVP (3.4) has a solution $\widetilde{x_\ell} \in E_\ell$, $\ell \in \{1, 2, ..., n\}$
For any $\ell \in \{1, 2, ..., n\}$ we define the function

$$x_\ell = \begin{cases} 0, & t \in [0, T_{\ell-1}], \\ \widetilde{x}_\ell, & t \in J_\ell. \end{cases}$$

Thus, the function $x_\ell \in C([0, T_\ell], \mathbb{R})$ solves the integral equation (3.3) for $t \in J_\ell$ with $x_\ell(0) = 0, \ x_\ell(T_\ell) = \widetilde{x}_\ell(T_\ell) = 0$.

Then, the function

$$x(t) = \begin{cases} x_1(t), & t \in J_1, \\ x_2(t) = \begin{cases} 0, & t \in J_1, \\ \widetilde{x}_2, & t \in J_2, \end{cases} \\ . \\ . \\ . \\ x_n(t) = \begin{cases} 0, & t \in [0, T_{\ell-1}], \\ \widetilde{x}_\ell, & t \in J_\ell, \end{cases} \end{cases} \tag{3.9}$$

is a solution of the BVP (3.1) in $C(J, \mathbb{R})$.

3.3 Ulam-Hyers stability

Theorem 3.3 *Let the conditions (H1), (H2) and inequality (3.7) be satisfied. Then, BVP* (3.1) *is **(UH)** stable.*

Proof Let $\epsilon > 0$ an arbitrary number and the function $z(t)$ from $z \in C(J_\ell, \mathbb{R})$ satisfy the following inequality

$$|{}^c D_{0^+}^{u(t)} z(t) - f(t, z(t), I_{0^+}^{u(t)} z(t))| \leq \epsilon, \ t \in J. \tag{3.10}$$

For any $\ell \in \{1,2,...,n\}$ we define the functions $z_1(t) \equiv z(t), t \in [0,T_1]$ and for $\ell = 2,3,...,n$:

$$z_\ell(t) = \begin{cases} 0, & t \in [0,T_{\ell-1}], \\ z(t), & t \in J_\ell. \end{cases}$$

For any $\ell \in \{1,2,...,n\}$ according to equality (3.2) for $t \in J_\ell$ we get

$${}^cD_{0^+}^{u(t)} z_\ell(t) = \int_{T_{\ell-1}}^{t} \frac{(t-s)^{1-u_\ell}}{\Gamma(2-u_\ell)} z^{(2)}(s)ds.$$

Taking the (RLFI) $I_{T_{\ell-1}^+}^{u_\ell}$ of both sides of the inequality (3.10), we obtain

$$\begin{aligned}
&\Big| z_\ell(t) + \frac{(T_\ell - T_{\ell-1})^{-1}(t-T_{\ell-1})}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} f(s, z_\ell(s), I_{T_{\ell-1}^+}^{u_\ell} z_\ell(s))ds \\
&- \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t-s)^{u_\ell-1} f(s, z_\ell(s), I_{T_{\ell-1}^+}^{u_\ell} z_\ell(s))ds \Big| \\
&\leq \epsilon \int_{T_{\ell-1}}^{t} \frac{(t-s)^{u_\ell-1}}{\Gamma(u_\ell)} ds \\
&\leq \epsilon \frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}.
\end{aligned}$$

According to Theorem 3.2, BVP (3.1) has a solution $x \in C(J,\mathbb{R})$ defined by $x(t) = x_\ell(t)$ for $t \in J_\ell,\ \ell = 1,2,...,n$, where

$$x_\ell = \begin{cases} 0, & t \in [0,T_{\ell-1}], \\ \widetilde{x}_\ell, & t \in J_\ell, \end{cases} \tag{3.11}$$

and $\widetilde{x}_\ell \in E_\ell$ is a solution of (3.4). According to Lemma (3.1) the integral equation

$$\begin{aligned}
\widetilde{x}_\ell(t) = &-\frac{(T_\ell - T_{\ell-1})^{-1}(t-T_{\ell-1})}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} f(s, \widetilde{x}_\ell(s), I_{T_{\ell-1}^+}^{u_\ell} \widetilde{x}_\ell(s))ds \\
&+ \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t-s)^{u_\ell-1} f(s, \widetilde{x}_\ell(s), I_{T_{\ell-1}^+}^{u_\ell} \widetilde{x}_\ell(s))ds,
\end{aligned} \tag{3.12}$$

holds.

Let $t \in J_\ell,\ \ell = 1,2,...,n$. Then by (3.11) and (3.12) we get

$$\begin{aligned}
&|z(t) - x(t)| = |z(t) - x_\ell(t)| = |z_\ell(t) - \widetilde{x}_\ell(t)| \\
&= \Big| z_\ell(t) + \frac{(T_\ell - T_{\ell-1})^{-1}(t-T_{\ell-1})}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} f(s, \widetilde{x}_\ell(s), I_{T_{\ell-1}^+}^{u_\ell} \widetilde{x}_\ell(s))ds \\
&- \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t-s)^{u_\ell-1} f(s, \widetilde{x}_\ell(s), I_{T_{\ell-1}^+}^{u_\ell} \widetilde{x}_\ell(s))ds \Big|
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \Big|z_\ell(t)+\frac{(T_\ell-T_{\ell-1})^{-1}(t-T_{\ell-1})}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(T_\ell-s)^{u_\ell-1}f(s,z_\ell(s),I^{u_\ell}_{T^+_{\ell-1}}z_\ell(s))ds\\
&-\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(t-s)^{u_\ell-1}f(s,z_\ell(s),I^{u_\ell}_{T^+_{\ell-1}}z_\ell(s))ds\Big|\\
&+\frac{(T_\ell-T_{\ell-1})^{-1}(t-T_{\ell-1})}{\Gamma(u_\ell)}\\
&\int_{T_{\ell-1}}^{T_\ell}(T_\ell-s)^{u_\ell-1}|f(s,z_\ell(s),I^{u_\ell}_{T^+_{\ell-1}}z_\ell)-f(s,\widetilde{x}_\ell(s),I^{u_\ell}_{T^+_{\ell-1}}\widetilde{x}_\ell)|ds\\
&+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(t-s)^{u_\ell-1}|f(s,z_\ell(s),I^{u_\ell}_{T^+_{\ell-1}}z_\ell)-f(s,\widetilde{x}_\ell(s),I^{u_\ell}_{T^+_{\ell-1}}\widetilde{x}_\ell)|ds\\
\leq\ & \epsilon\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}+\frac{(T_\ell-T_{\ell-1})^{-1}(t-T_{\ell-1})}{\Gamma(u_\ell)}\\
&\int_{T_{\ell-1}}^{T_\ell}(T_\ell-s)^{u_\ell-1}s^{-\delta}(K|z_\ell(s)-\widetilde{x}_\ell(s)|+LI^{u_\ell}_{T^+_{\ell-1}}|z_\ell(s)-\widetilde{x}_\ell(s)|)ds\\
&+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(t-s)^{u_\ell-1}s^{-\delta}(K|z_\ell(s)-\widetilde{x}_\ell(s)|+LI^{u_\ell}_{T^+_{\ell-1}}|z_\ell(s)-\widetilde{x}_\ell(s)|)ds\\
\leq\ & \epsilon\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}+\frac{(T_\ell-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\\
&\Big(K\|z_\ell-\widetilde{x}_\ell\|_{E_\ell}+L\|I^{u_\ell}_{T^+_{\ell-1}}(z_\ell-\widetilde{x}_\ell)\|_{E_\ell}\Big)\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}ds\\
&+\frac{(T_\ell-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}(K\|z_\ell-\widetilde{x}_\ell\|_{E_\ell}+L\|I^{u_\ell}_{T^+_{\ell-1}}(z_\ell-\widetilde{x}_\ell)\|_{E_\ell})\int_{T_{\ell-1}}^{t}s^{-\delta}ds\\
\leq\ & \epsilon\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}+\frac{(T_\ell-T_{\ell-1})^{u_\ell-1}({T_\ell}^{1-\delta}-{T_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\\
&\Big(K\|z_\ell-\widetilde{x}_\ell\|_{E_\ell}+L\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\|z_\ell-\widetilde{x}_\ell\|_{E_\ell}\Big)\\
&+\frac{(T_\ell-T_{\ell-1})^{u_\ell-1}(t^{1-\delta}-{T_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}(K\|z_\ell-\widetilde{x}_\ell\|_{E_\ell}+L\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\|z_\ell-\widetilde{x}_\ell\|_{E_\ell})\\
\leq\ & \epsilon\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}+\frac{2(T_\ell-T_{\ell-1})^{u_\ell-1}({T_\ell}^{1-\delta}-{T_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\\
&\Big(K+L\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\Big)\|z_\ell-\widetilde{x}_\ell\|_{E_\ell}\\
\leq\ & \epsilon\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}+\mu\|z-x\|,
\end{aligned}
$$

where

$$\mu = \max_{\ell=1,2,\ldots,n} \frac{2(T_\ell - T_{\ell-1})^{u_\ell - 1}({T_\ell}^{1-\delta} - {T_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}(K + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)}).$$

Then,

$$\|z - x\|(1-\mu) \leq \frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)}\epsilon.$$

We obtain, for each $t \in J$

$$|z(t) - x(t)| \leq \|z - x\| \leq \frac{(T_\ell - T_{\ell-1})^{u_\ell}}{(1-\mu)\Gamma(u_\ell + 1)}\epsilon := c_f\epsilon.$$

Therefore, by Theorem (1.4), the BVP (3.1) is **(UH)** stable.

Chapitre 4

Multiterm boundary value problem of Hadamard fractional differential equations of variable order [3]

4.1 Introduction

We shall be concerned with the boundary value problem (BVP for short) for the Hadamard fractional differential equation of variable order

$$\begin{cases} {}^{H}D_{1^+}^{u(t)}x(t) = f(t, x(t), \; {}^{H}I_{1^+}^{u(t)}x(t)), \; t \in M := [1, T], \\ x(1) = x(T) = 0, \end{cases} \tag{4.1}$$

where $1 < T < \infty$, $u(t) : M \to (1, 2]$ is the variable order of the fractional derivatives, $f : M \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function and ${}^{H}D_{1^+}^{u(t)}$ ${}^{H}I_{1^+}^{u(t)}$ are the left Hadamard fractional derivative and integral of variable-order $u(t)$, and we will study the stability of the obtained solution in the sense of Ulam-Hyers-Rassias.

4.2 Existence of solutions

Let us introduce the following assumptions :

(H1) Let $n \in \mathbb{N}$ be an integer, $\mathcal{P} = \{J_1 := [1, T_1], J_2 := (T_1, T_2], J_3 := (T_2, T_3], ...J_n := (T_{n-1}, T]\}$ be a partition of the interval M, and let $u(t) : M \to (1, 2]$ be a piecewise constant function with respect to $\mathcal{P}$, i.e.,

---

[3] S. Hristova, **A. Benkerrouche**, M.S. Souid, A. Hakem, Boundary value problems of Hadamard fractional differential equations of variable order. *Symmetry*, **13(5), 896**, 1-16, (2021).

$$u(t)=\sum_{\ell=1}^{n}u_\ell I_\ell(t)=\begin{cases} u_1, & if\ t\in J_1,\\ u_2, & if\ t\in J_2,\\ . & \\ . & \\ . & \\ u_n, & if\ t\in J_n,\end{cases}$$

where $1<u_\ell\leq 2$ are constants, and I_ℓ is the indicator of the interval $J_\ell:=(T_{\ell-1},T_\ell],\ \ell=1,2,...,n$, (here $T_0=1,\ T_n=T$) such that

$$I_\ell(t)=\begin{cases} 1, & for\ t\in J_\ell,\\ 0, & for\ elsewhere.\end{cases}$$

Further, for a given set U of functions $u:M\to X$ let us denote by

$$U(t)=\{u(t),u\in U\},\ t\in M,$$

and

$$U(M)=\{v(t):v\in U,\ t\in M\}.$$

We are now in a position to prove the existence of solution for the BVP (4.1) based on concept of (**MNCK**) and (**DFPT**).

For each $\ell\in\{1,2,...,n\}$, the symbol $E_\ell=C(J_\ell,X)$, indicated the Banach space of continuous functions $x:J_\ell\to X$ equipped with the norm

$$\|x\|_{E_\ell}=\sup_{t\in J_\ell}\|x(t)\|.$$

Then, for any $t\in J_\ell,\ \ell=1,2,...,n$ the left Hadamard fractional derivative (HFD) of variable order $u(t)$ for function $x(t)\in C(M,\mathbb{R})$, could be presented as a sum of left Hadamard fractional derivatives of constant-orders $u_\ell,\ \ell=1,2,...,n$

$$\begin{aligned}{}^HD_{0^+}^{u(t)}x(t)=&\Big(t\frac{d}{dt}\Big)^2\Big(\frac{1}{\Gamma(2-u_1)}\int_1^{T_1}\Big(\log\frac{t}{s}\Big)^{1-u_1}\frac{x(s)}{s}ds\Big)+\\ &\Big(t\frac{d}{dt}\Big)^2\Big(\frac{1}{\Gamma(2-u_2)}\int_{T_1}^{T_2}\Big(\log\frac{t}{s}\Big)^{1-u_2}\frac{x(s)}{s}ds\Big)+\\ &...+\Big(t\frac{d}{dt}\Big)^2\Big(\frac{1}{\Gamma(2-u_\ell)}\int_{T_{\ell-1}}^{t}\Big(\log\frac{t}{s}\Big)^{1-u_\ell}\frac{x(s)}{s}ds\Big).\end{aligned}\tag{4.2}$$

Thus, according to (4.2), BVP(4.1) can be written for any $t\in J_\ell,\ \ell=1,2,...,n$ in the form

$$\begin{aligned}&\Big(t\frac{d}{dt}\Big)^2\Big(\frac{1}{\Gamma(2-u_1)}\int_1^{T_1}\Big(\log\frac{t}{s}\Big)^{1-u_1}\frac{x(s)}{s}ds\Big)+\Big(t\frac{d}{dt}\Big)^2\Big(\frac{1}{\Gamma(2-u_2)}\int_{T_1}^{T_2}\Big(\log\frac{t}{s}\Big)^{1-u_2}\frac{x(s)}{s}ds\Big)+\\ &...+(t\frac{d}{dt})^2\Big(\frac{1}{\Gamma(2-u_\ell)}\int_{T_{\ell-1}}^{t}(\log\frac{t}{s})^{1-u_\ell}\frac{x(s)}{s}ds\Big)=f(t,x(t),{}^HI_{1^+}^{u(t)}x(t)),\quad t\in J_\ell.\end{aligned}\tag{4.3}$$

In what follows we shall introduce the solution to the BVP (4.1).

Definition 4.1 *BVP (4.1) has a solution, if there are functions x_ℓ, $\ell = 1, 2, ..., n$, so that, $x_\ell \in C([1, T_\ell], X)$ fulfilling equation (4.3) and $x_\ell(1) = 0 = x_\ell(T_\ell)$.*

According the observation above, BVP (4.1) can be expressed as in (4.2), with $J_\ell, \ell \in \{1, 2, ..., n\}$ as (4.3).
For $1 \leq t \leq T_{\ell-1}$, we take $x(t) \equiv 0$, then (4.3) is written as

$$ {}^H D^{u_\ell}_{T^+_{\ell-1}} x(t) = f(t, x(t), {}^H I^{u_\ell}_{T^+_{\ell-1}} x(t)), \quad t \in J_\ell. $$

We shall deal with following BVP

$$ \begin{cases} {}^H D^{u_\ell}_{T^+_{\ell-1}} x(t) = f(t, x(t), {}^H I^{u_\ell}_{T^+_{\ell-1}} x(t)), \quad t \in J_\ell, \\ x(T_{\ell-1}) = 0, x(T_\ell) = 0. \end{cases} \tag{4.4} $$

For our purpose, the upcoming lemma will be a corner stone of the solution of (4.4).

Lemma 4.1 *Let $\ell \in \{1, 2 \ldots, n\}$ be a natural number , $f \in C(J_\ell \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and there exists a number $\delta \in (0, 1)$ such that $(\log t)^\delta f \in C(J_\ell \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.*

Then, the function $x \in E_\ell$ forms a solution of (4.4) if and only if x fulfills the integral equation

$$ x(t) = -\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{1-u_\ell}\Big(log\frac{t}{T_{\ell-1}}\Big)^{u_\ell-1} \; {}^H I^{u_\ell}_{T^+_{\ell-1}} f(T_\ell, x(T_\ell), {}^H I^{u_\ell}_{T^+_{\ell-1}} x(T_\ell)) + {}^H I^{u_\ell}_{T^+_{\ell-1}} f(t, x(t), {}^H I^{u_\ell}_{T^+_{\ell-1}} x(t)). \tag{4.5} $$

Proof

We presume that $x \in E_\ell$ is solution of BVP (4.4). Employing the operator ${}^H I^{u_\ell}_{T^+_{\ell-1}}$ to both sides of (4.4) and regarding Lemma 1.10, we find

$$ x(t) = \omega_1\Big(log\frac{t}{T_{\ell-1}}\Big)^{u_\ell-1} + \omega_2\Big(log\frac{t}{T_{\ell-1}}\Big)^{u_\ell-2} + {}^H I^{u_\ell}_{T^+_{\ell-1}} f(t, x(t), {}^H I^{u_\ell}_{T^+_{\ell-1}} x(t)), \quad t \in J_\ell. $$

Due to the assumption of function f together with $x(T_{\ell-1}) = 0$, we conclude that $\omega_2 = 0$.

Let $x(t)$ satisfy $x(T_\ell) = 0$. So, we observe that

$$ \omega_1 = -\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{1-u_\ell} \; {}^H I^{u_\ell}_{T^+_{\ell-1}} f(T_\ell, x(T_\ell), {}^H I^{u_\ell}_{T^+_{\ell-1}} x(T_\ell)). $$

Then, we find

$$ x(t) = -\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{1-u_\ell}\Big(log\frac{t}{T_{\ell-1}}\Big)^{u_\ell-1} \; {}^H I^{u_\ell}_{T^+_{\ell-1}} f(T_\ell, x(T_\ell), {}^H I^{u_\ell}_{T^+_{\ell-1}} x(T_\ell)) $$

$$+^H I^{u_\ell}_{T^+_{\ell-1}} f(t, x(t),^H I^{u_\ell}_{T^+_{\ell-1}} x(t)), \quad t \in J_\ell.$$

Conversely, let $x \in E_\ell$ be a solution of integral equation (4.5). Regarding the continuity of function $(logt)^\delta f$ and Lemma 1.11, we deduce that x is the solution of BVP (4.4).

Our first existence result is based on Theorem 1.1.

Theorem 4.1 *Let the conditions of Lemma 4.1 be satisfied and there exist constants* $K,\ L > 0$ *such that*

$$(\log t)^\delta |f(t,x_1,y_1) - f(t,x_2,y_2)| \le K|x_1 - x_2| + L|y_1 - y_2|,\ for\ any\ x_i,\ y_i \in \mathbb{R},\ i = 1,\ 2,$$

$$and\ t \in J_\ell, \tag{4.6}$$

and the inequality

$$\frac{(\log T_\ell)^{1-\delta} - (\log T_{\ell-1})^{1-\delta}}{(1-\delta)\Gamma(u_\ell)} (\log \frac{T_\ell}{T_{\ell-1}})^{u_\ell - 1} \Big(K + \frac{L}{\Gamma(u_\ell + 1)} (\log \frac{T_\ell}{T_{\ell-1}})^{u_\ell} \Big) < \frac{1}{2}, \tag{4.7}$$

holds. Then, the BVP (4.4) possesses at least one solution on E_ℓ.

Proof

We construct the operator

$$W : E_\ell \to E_\ell,$$

as follow

$$Wx(t) = -\frac{1}{\Gamma(u_\ell)} \Big(log \frac{T_\ell}{T_{\ell-1}} \Big)^{1-u_\ell} \Big(log \frac{t}{T_{\ell-1}} \Big)^{u_\ell - 1} \int_{T_{\ell-1}}^{T_\ell} \Big(\log \frac{T_\ell}{s} \Big)^{u_\ell - 1} \frac{f(s, x(s),^H I^{u_\ell}_{T^+_{\ell-1}} x(s))}{s} ds$$

$$+\frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} \Big(\log \frac{t}{s} \Big)^{u_\ell - 1} \frac{f(s, x(s),^H I^{u_\ell}_{T^+_{\ell-1}} x(s))}{s} ds. \tag{4.8}$$

It follows from the properties of fractional integrals and from the continuity of function $(logt)^\delta f$ that the operator $W : E_\ell \to E_\ell$ defined in (4.8) is well defined.
Let

$$R_\ell \ge \frac{\frac{2f^*}{\Gamma(u_\ell+1)} (\log \frac{T_\ell}{T_{\ell-1}})^{u_\ell}}{1 - \frac{2[(\log T_\ell)^{1-\delta} - (\log T_{\ell-1})^{1-\delta}]}{(1-\delta)\Gamma(u_\ell)} (\log \frac{T_\ell}{T_{\ell-1}})^{u_\ell - 1} \Big(K + \frac{L}{\Gamma(u_\ell+1)} (\log \frac{T_\ell}{T_{\ell-1}})^{u_\ell} \Big)},$$

with

$$f^\star = \sup_{t \in J_\ell} |f(t,0,0)|.$$

We consider the set

$$B_{R_\ell} = \{x \in E_\ell,\ \|x\|_{E_\ell} \le R_\ell\}.$$

Clearly B_{R_ℓ} is nonempty, closed, convex and bounded.

Now, we demonstrate that W satisfies the assumption of the Theorem 1.1. We shall prove it in four phases.

Step 1 : Claim : $W(B_{R_\ell}) \subseteq (B_{R_\ell})$.

For $x \in B_{R_\ell}$ and by the conditions of function f , we obtain

$$\begin{aligned}
\|Wx(t)\| &\leq \frac{1}{\Gamma(u_\ell)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{1-u_\ell}\Big(log\frac{t}{T_{\ell-1}}\Big)^{u_\ell-1} \\
&\quad \int_{T_{\ell-1}}^{T_\ell}\Big(\log\frac{T_\ell}{s}\Big)^{u_\ell-1}\|f(s,x(s),{}^H I^{u_\ell}_{T^+_{\ell-1}}x(s))\|\frac{ds}{s} \\
&\quad +\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}\Big(\log\frac{t}{s}\Big)^{u_\ell-1}\|f(s,x(s),{}^H I^{u_\ell}_{T^+_{\ell-1}}x(s))\|\frac{ds}{s} \\
&\leq \frac{2}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}\Big(\log\frac{T_\ell}{s}\Big)^{u_\ell-1}\|f(s,x(s),{}^H I^{u_\ell}_{T^+_{\ell-1}}x(s))\|\frac{ds}{s} \\
&\leq \frac{2}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}\Big(\log\frac{T_\ell}{s}\Big)^{u_\ell-1}\|f(s,x(s),{}^H I^{u_\ell}_{T^+_{\ell-1}}x(s))-f(s,0,0)\|\frac{ds}{s} \\
&\quad +\frac{2}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}\Big(\log\frac{T_\ell}{s}\Big)^{u_\ell-1}\|f(s,0,0)\|\frac{ds}{s} \\
&\leq \frac{2}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}\Big(\log\frac{T_\ell}{s}\Big)^{u_\ell-1}(\log s)^{-\delta}\Big(K\|x(s)\|+L\|{}^H I^{u_\ell}_{T^+_{\ell-1}}x(s)\|\Big)\frac{ds}{s} \\
&\quad +\frac{2f^\star}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}\Big(\log\frac{T_\ell}{s}\Big)^{u_\ell-1}\frac{ds}{s} \\
&\leq \frac{2}{\Gamma(u_\ell)}\Big(\log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell-1}\Big(K\|x\|_{E_\ell}+L\|{}^H I^{u_\ell}_{T^+_{\ell-1}}x\|_{E_\ell}\Big)\int_{T_{\ell-1}}^{T_\ell}(\log s)^{-\delta}\frac{ds}{s} \\
&\quad +\frac{2f^\star}{\Gamma(u_\ell+1)}\Big(\log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell} \\
&\leq \frac{2}{\Gamma(u_\ell)}\Big(\log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell-1}\Big(K+\frac{L}{\Gamma(u_\ell+1)}\Big(\log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell}\Big)\|x\|_{E_\ell}\int_{T_{\ell-1}}^{T_\ell}(\log s)^{-\delta}\frac{ds}{s} \\
&\quad +\frac{2f^\star}{\Gamma(u_\ell+1)}\Big(\log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell} \\
&\leq \frac{2\Big[(\log T_\ell)^{1-\delta}-(\log T_{\ell-1})^{1-\delta}\Big]}{(1-\delta)\Gamma(u_\ell)}\Big(\log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell-1}\Big(K+\frac{L}{\Gamma(u_\ell+1)}(\log\frac{T_\ell}{T_{\ell-1}})^{u_\ell}\Big)R_\ell \\
&\quad +\frac{2f^\star}{\Gamma(u_\ell+1)}\Big(\log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell} \\
&\leq R_\ell,
\end{aligned}$$

which means that $W(B_{R_\ell}) \subseteq (B_{R_\ell})$.

Step 2 : Claim : W is continuous.
We presume that the sequence (x_n) converges to x in E_ℓ. Then,

$$
\begin{aligned}
&\|(Wx_n)(t)-(Wx)(t)\| \le \frac{1}{\Gamma(u_\ell)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{1-u_\ell}\Big(log\frac{t}{T_{\ell-1}}\Big)^{u_\ell-1}\\
&\int_{T_{\ell-1}}^{T_\ell}\Big(\log\frac{T_\ell}{s}\Big)^{u_\ell-1}\|f(s,x_n(s),{}^H I^{u_\ell}_{T^+_{\ell-1}}x_n(s))-f(s,x(s),{}^H I^{u_\ell}_{T^+_{\ell-1}}x(s))\|\frac{ds}{s}\\
&+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}\Big(\log\frac{t}{s}\Big)^{u_\ell-1}\|f(s,x_n(s),{}^H I^{u_\ell}_{T^+_{\ell-1}}x_n(s))-f(s,x(s),{}^H I^{u_\ell}_{T^+_{\ell-1}}x(s))\|\frac{ds}{s}\\
\le\;& \frac{2}{\Gamma(u_\ell)}\Big(log\frac{t}{T_{\ell-1}}\Big)^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}\|f(s,x_n(s),{}^H I^{u_\ell}_{T^+_{\ell-1}}x_n(s))-f(s,x(s),{}^H I^{u_\ell}_{T^+_{\ell-1}}x(s))\|\frac{ds}{s}\\
\le\;& \frac{2}{\Gamma(u_\ell)}(log\frac{t}{T_{\ell-1}})^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}(logs)^{-\delta}\Big(K\|x_n(s)-x(s)\|+L\ {}^H I^{u_\ell}_{T^+_{\ell-1}}\|x_n(s)-x(s)\|\Big)\frac{ds}{s}\\
\le\;& \frac{2}{\Gamma(u_\ell)}\Big(K\|x_n-x\|_{E_\ell}+L\ \|{}^H I^{u_\ell}_{T^+_{\ell-1}}(x_n-x)\|_{E_\ell}\Big)\Big(\log\frac{t}{T_{\ell-1}}\Big)^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}(logs)^{-\delta}\frac{ds}{s}\\
\le\;& \frac{2\Big[(\log T_\ell)^{1-\delta}-(\log T_{\ell-1})^{1-\delta}\Big]}{(1-\delta)\Gamma(u_\ell)}\Big(\log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell-1}\Big(K+\frac{L}{\Gamma(u_\ell+1)}(\log\frac{T_\ell}{T_{\ell-1}})^{u_\ell}\Big)\|x_n-x\|_{E_\ell},
\end{aligned}
$$

i.e., we obtain

$$\|(Wx_n)-(Wx)\|_{E_\ell}\to 0 \quad as \quad n\to\infty.$$

Ergo, the operator W is continuous on E_ℓ.

Step 3 : Claim : W is is bounded and equicontinuous.
By Step 1, we have $W(B_{R_\ell})=\{W(x): x\in B_{R_\ell}\}\subset B_{R_\ell}$ thus for each $x\in B_{R_\ell}$ we have $\|W(x)\|_{E_\ell}\le R_\ell$ which means that $W(B_{R_\ell})$is bounded. It remains to indicate that $W(B_{R_\ell})$ is equicontinuous.
For $t_1,t_2\in J_\ell,\ t_1<t_2$ and $x\in B_{R_\ell}$, we have

$$
\begin{aligned}
&\|(Wx)(t_2)-(Wx)(t_1)\|\\
=\;& \Big\|-\frac{1}{\Gamma(u_\ell)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{1-u_\ell}\Big(log\frac{t_2}{T_{\ell-1}}\Big)^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}\Big(\log\frac{T_\ell}{s}\Big)^{u_\ell-1}f(s,x(s),{}^H I^{u_\ell}_{T^+_{\ell-1}}x(s))\frac{ds}{s}\\
&+\frac{1}{\Gamma(u_\ell)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{1-u_\ell}\Big(log\frac{t_1}{T_{\ell-1}}\Big)^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}\Big(\log\frac{T_\ell}{s}\Big)^{u_\ell-1}f(s,x(s),{}^H I^{u_\ell}_{T^+_{\ell-1}}x(s))\frac{ds}{s}\\
&+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_2}\Big(\log\frac{t_2}{s}\Big)^{u_\ell-1}f(s,x(s),{}^H I^{u_\ell}_{T^+_{\ell-1}}x(s))\frac{ds}{s}\\
&-\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}\Big(\log\frac{t_1}{s}\Big)^{u_\ell-1}f(s,x(s),{}^H I^{u_\ell}_{T^+_{\ell-1}}x(s))\frac{ds}{s}\Big\|
\end{aligned}
$$

$$\begin{aligned}
\leq\ & \frac{1}{\Gamma(u_\ell)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{1-u_\ell}\Big(\Big(log\frac{t_2}{T_{\ell-1}}\Big)^{u_\ell-1}-\Big(log\frac{t_1}{T_{\ell-1}}\Big)^{u_\ell-1}\Big)\\
& \int_{T_{\ell-1}}^{T_\ell}\Big(\log\frac{T_\ell}{s}\Big)^{u_\ell-1}\|f(s,x(s),{}^H I^{u_\ell}_{T^+_{\ell-1}}x(s))\|\frac{ds}{s}\\
& +\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}\Big(\Big(\log\frac{t_2}{s}\Big)^{u_\ell-1}-\Big(\log\frac{t_1}{s}\Big)^{u_\ell-1}\Big)\|f(s,x(s),{}^H I^{u_\ell}_{T^+_{\ell-1}}x(s))\|\frac{ds}{s}\\
& +\frac{1}{\Gamma(u_\ell)}\int_{t_1}^{t_2}\Big(\log\frac{t_2}{s}\Big)^{u_\ell-1}\|f(s,x(s),{}^H I^{u_\ell}_{T^+_{\ell-1}}x(s))\|\frac{ds}{s}\\
\leq\ & \frac{1}{\Gamma(u_\ell)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{1-u_\ell}\Big(\Big(log\frac{t_2}{T_{\ell-1}}\Big)^{u_\ell-1}-\Big(log\frac{t_1}{T_{\ell-1}}\Big)^{u_\ell-1}\Big)\\
& \int_{T_{\ell-1}}^{T_\ell}\Big(\log\frac{T_\ell}{s}\Big)^{u_\ell-1}\|f(s,x(s),{}^H I^{u_\ell}_{T^+_{\ell-1}}x(s))-f(s,0,0)\|\frac{ds}{s}\\
& +\frac{1}{\Gamma(u_\ell)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{1-u_\ell}\Big(\Big(log\frac{t_2}{T_{\ell-1}}\Big)^{u_\ell-1}-\Big(log\frac{t_1}{T_{\ell-1}}\Big)^{u_\ell-1}\Big)\\
& \int_{T_{\ell-1}}^{T_\ell}\Big(\log\frac{T_\ell}{s}\Big)^{u_\ell-1}\|f(s,0,0)\|\frac{ds}{s}\\
& +\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}\Big(\log\frac{t_2}{t_1}\Big)^{u_\ell-1}\|f(s,x(s),{}^H I^{u_\ell}_{T^+_{\ell-1}}x(s))-f(s,0,0)\|\frac{ds}{s}\\
& +\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}\Big(\log\frac{t_2}{t_1}\Big)^{u_\ell-1}\|f(s,0,0)\|\frac{ds}{s}\\
& +\frac{1}{\Gamma(u_\ell)}\int_{t_1}^{t_2}\Big(\log\frac{t_2}{s}\Big)^{u_\ell-1}\|f(s,x(s),{}^H I^{u_\ell}_{T^+_{\ell-1}}x(s))-f(s,0,0)\|\frac{ds}{s}\\
& +\frac{1}{\Gamma(u_\ell)}\int_{t_1}^{t_2}\Big(\log\frac{t_2}{s}\Big)^{u_\ell-1}\|f(s,0,0)\|\frac{ds}{s}\\
\leq\ & \frac{1}{\Gamma(u_\ell)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{1-u_\ell}\Big(\Big(log\frac{t_2}{T_{\ell-1}}\Big)^{u_\ell-1}-\Big(log\frac{t_1}{T_{\ell-1}}\Big)^{u_\ell-1}\Big)\\
& \int_{T_{\ell-1}}^{T_\ell}\Big(\log\frac{T_\ell}{s}\Big)^{u_\ell-1}(logs)^{-\delta}\Big(K\|x(s)\|+L\ {}^H I^{u_\ell}_{T^+_{\ell-1}}\|x(s)\|\Big)\frac{ds}{s}\\
& +\frac{f^\star}{\Gamma(u_\ell)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{1-u_\ell}\Big(\Big(log\frac{t_2}{T_{\ell-1}}\Big)^{u_\ell-1}-\Big(log\frac{t_1}{T_{\ell-1}}\Big)^{u_\ell-1}\Big)\int_{T_{\ell-1}}^{T_\ell}\Big(\log\frac{T_\ell}{s}\Big)^{u_\ell-1}\frac{ds}{s}\\
& +\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}\Big(\log\frac{t_2}{t_1}\Big)^{u_\ell-1}(logs)^{-\delta}\Big(K\|x(s)\|+L\ {}^H I^{u_\ell}_{T^+_{\ell-1}}\|x(s)\|\Big)\frac{ds}{s}\\
& +\frac{f^\star}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}\Big(\log\frac{t_2}{t_1}\Big)^{u_\ell-1}\frac{ds}{s}\\
& +\frac{1}{\Gamma(u_\ell)}\int_{t_1}^{t_2}\Big(\log\frac{t_2}{s}\Big)^{u_\ell-1}(logs)^{-\delta}\Big(K\|x(s)\|+L\ {}^H I^{u_\ell}_{T^+_{\ell-1}}\|x(s)\|\Big)\frac{ds}{s}\\
& +\frac{f^\star}{\Gamma(u_\ell)}\int_{t_1}^{t_2}\Big(\log\frac{t_2}{s}\Big)^{u_\ell-1}\frac{ds}{s}
\end{aligned}$$

$$\leq \frac{1}{\Gamma(u_\ell)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{1-u_\ell}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell-1}\Big(\Big(log\frac{t_2}{T_{\ell-1}}\Big)^{u_\ell-1}-\Big(log\frac{t_1}{T_{\ell-1}}\Big)^{u_\ell-1}\Big)$$
$$\Big(K\|x\|_{E_\ell}+L\|^H I^{u_\ell}_{T^+_{\ell-1}}x\|_{E_\ell}\Big)\int_{T_{\ell-1}}^{T_\ell}(logs)^{-\delta}\frac{ds}{s}$$
$$+\frac{f^\star}{\Gamma(u_\ell+1)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{1-u_\ell}\Big(\Big(log\frac{t_2}{T_{\ell-1}}\Big)^{u_\ell-1}-\Big(log\frac{t_1}{T_{\ell-1}}\Big)^{u_\ell-1}\Big)\Big(\log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell}$$
$$+\frac{(logt_2)^{1-\delta}-(logt_1)^{1-\delta}}{(1-\delta)\Gamma(u_\ell)}\Big(\log\frac{t_2}{t_1}\Big)^{u_\ell-1}\Big(K\|x\|_{E_\ell}+L\|^H I^{u_\ell}_{T^+_{\ell-1}}x\|_{E_\ell}\Big)$$
$$+\frac{f^\star}{\Gamma(u_\ell)}\Big(\log\frac{t_2}{t_1}\Big)^{u_\ell-1}\Big(logt_1-logT_{\ell-1}\Big)$$
$$+\frac{(logt_2)^{1-\delta}-(logt_1)^{1-\delta}}{(1-\delta)\Gamma(u_\ell)}\Big(\log\frac{t_2}{t_1}\Big)^{u_\ell-1}\Big(K\|x\|_{E_\ell}+L\|^H I^{u_\ell}_{T^+_{\ell-1}}x\|_{E_\ell}\Big)$$
$$+\frac{f^\star}{\Gamma(u_\ell+1)}\Big(\log\frac{t_2}{t_1}\Big)^{u_\ell}$$
$$\leq \frac{(logT_\ell)^{1-\delta}-(logT_{\ell-1})^{1-\delta}}{(1-\delta)\Gamma(u_\ell)}\Big(K+\frac{L}{\Gamma(u_\ell+1)}\Big(\log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell}\Big)\|x\|_{E_\ell}$$
$$\Big(\Big(log\frac{t_2}{T_{\ell-1}}\Big)^{u_\ell-1}-\Big(log\frac{t_1}{T_{\ell-1}}\Big)^{u_\ell-1}\Big)$$
$$+\frac{f^\star}{\Gamma(u_\ell+1)}\Big(log\frac{T_\ell}{T_{\ell-1}})\Big(\Big(log\frac{t_2}{T_{\ell-1}}\Big)^{u_\ell-1}-\Big(log\frac{t_1}{T_{\ell-1}}\Big)^{u_\ell-1}\Big)$$
$$+\frac{2((logt_2)^{1-\delta}-(logt_1)^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\Big(\log\frac{t_2}{t_1}\Big)^{u_\ell-1}\Big(K+\frac{L}{\Gamma(u_\ell+1)}\Big(\log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell}\Big)\|x\|_{E_\ell}$$
$$+\frac{f^\star}{\Gamma(u_\ell)}\Big(\log\frac{t_2}{t_1}\Big)^{u_\ell-1}\Big(logt_1-logT_{\ell-1}\Big)+\frac{f^\star}{\Gamma(u_\ell+1)}\Big(\log\frac{t_2}{t_1}\Big)^{u_\ell}$$
$$\leq \Big[\frac{(logT_\ell)^{1-\delta}-(logT_{\ell-1})^{1-\delta}}{(1-\delta)\Gamma(u_\ell)}\Big(K+\frac{L}{\Gamma(u_\ell+1)}\Big(\log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell}\Big)\|x\|_{E_\ell}$$
$$+\frac{f^\star}{\Gamma(u_\ell+1)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)\Big]\Big(\Big(log\frac{t_2}{T_{\ell-1}}\Big)^{u_\ell-1}-\Big(log\frac{t_1}{T_{\ell-1}}\Big)^{u_\ell-1}\Big)$$
$$+\Big[\frac{2\Big((logt_2)^{1-\delta}-(logt_1)^{1-\delta}\Big)}{(1-\delta)\Gamma(u_\ell)}\Big(K+\frac{L}{\Gamma(u_\ell+1)}\Big(\log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell}\Big)\|x\|_{E_\ell}$$
$$+\frac{f^\star}{\Gamma(u_\ell)}\Big(logt_1-logT_{\ell-1}\Big)\Big]\Big(\log\frac{t_2}{t_1}\Big)^{u_\ell-1}+\frac{f^\star}{\Gamma(u_\ell+1)}\Big(\log\frac{t_2}{t_1}\Big)^{u_\ell}.$$

Hence $\|(Wx)(t_2)-(Wx)(t_1)\|\to 0$ as $|t_2-t_1|\to 0$. It implies that $W(B_{R_\ell})$ is equicontinuous.

Remark 4.1 *According to remark of ([12]) page 20 we can easily show that the inequality (4.6) is equivalent to the following inequality*

$$\zeta(t^\delta|f(t,B_1,B_2)|)\leq K\zeta(B_1)+L\zeta(B_2),$$

for any bounded sets $B_1, B_2 \subset X$ *and for each* $t \in J_\ell$.

Step 4 : W is k-*set contraction*.
For $U \in B_{R_\ell}$, $t \in J_\ell$, we get,

$$\begin{aligned}
&\zeta(W(U)(t)) = \zeta((Wy)(t), y \in U)\\
&\leq \Big[\frac{1}{\Gamma(u_\ell)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{1-u_\ell}\Big(log\frac{t}{T_{\ell-1}}\Big)^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}\Big(log\frac{T_\ell}{s}\Big)^{u_\ell-1}\zeta f(s,x(s),{}^H I^{u_\ell}_{T^+_{\ell-1}}x(s))ds\\
&+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}\Big(log\frac{t}{s}\Big)^{u_\ell-1}\zeta f(s,x(s),{}^H I^{u_\ell}_{T^+_{\ell-1}}x(s))ds,\ y \in U\Big].
\end{aligned}$$

Then Remark 4.1 implies that, for each $s \in J_\ell$,

$$\begin{aligned}
&\zeta(W(U)(t)) \leq \Big[\frac{1}{\Gamma(u_\ell)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{1-u_\ell}\Big(log\frac{t}{T_{\ell-1}}\Big)^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}\Big(log\frac{T_\ell}{s}\Big)^{u_\ell-1}\\
&\Big(K\widehat{\zeta}(U)s^{-\delta} + \frac{L}{\Gamma(u_\ell+1)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell}\ \widehat{\zeta}(U)s^{-\delta}\Big)ds\\
&+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}\Big(log\frac{t}{s}\Big)^{u_\ell-1}\Big(K\widehat{\zeta}(U)s^{-\delta} + \frac{L}{\Gamma(u_\ell+1)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell}\ \widehat{\zeta}(U)s^{-\delta}\Big)ds,\ y \in U\Big].\\
\leq\ &\Big[\frac{1}{\Gamma(u_\ell)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}\Big(K\widehat{\zeta}(U)s^{-\delta} + \frac{L}{\Gamma(u_\ell+1)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell}\ \widehat{\zeta}(U)s^{-\delta}\Big)ds\\
&+\frac{1}{\Gamma(u_\ell)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell-1}\int_{T_{\ell-1}}^{t}\Big(K\widehat{\zeta}(U)s^{-\delta} + \frac{L}{\Gamma(u_\ell+1)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell}\ \widehat{\zeta}(U)s^{-\delta}\Big)ds,\ y \in U\Big].\\
\leq\ &\frac{(logT_\ell)^{1-\delta} - (logT_{\ell-1})^{1-\delta}}{(1-\delta)\Gamma(u_\ell)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell-1}\Big(K + \frac{L}{\Gamma(u_\ell+1)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell}\ \Big)\widehat{\zeta}(U)\\
&+\frac{(logt)^{1-\delta} - (logT_{\ell-1})^{1-\delta}}{(1-\delta)\Gamma(u_\ell)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell-1}\Big(K + \frac{L}{\Gamma(u_\ell+1)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell}\ \Big)\widehat{\zeta}(U)\\
\leq\ &\frac{2[(logT_\ell)^{1-\delta} - (logT_{\ell-1})^{1-\delta}]}{(1-\delta)\Gamma(u_\ell)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell-1}\Big(K + \frac{L}{\Gamma(u_\ell+1)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell}\Big)\widehat{\zeta}(U).
\end{aligned}$$

Thus,

$$\widehat{\zeta}(WU) \leq \frac{2[(logT_\ell)^{1-\delta} - (logT_{\ell-1})^{1-\delta}]}{(1-\delta)\Gamma(u_\ell)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell-1}\Big(K+\frac{L}{\Gamma(u_\ell+1)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell}\Big)\widehat{\zeta}(U).$$

From inequality (4.7), it follows W is a k-set contraction.

Therefore, all conditions of Theorem 1.1 are fulfilled and thus, there exists $\widetilde{x_\ell} \in B_{R_\ell}$, such that $W(\widetilde{x_\ell}) = \widetilde{x_\ell}$, which is a solution of the BVP (4.4). Since $B_{R_\ell} \subset E_\ell$ the claim of Theorem 4.1 is proved.

Now, we will prove the existence result for BVP (4.1).

Introduce the following assumption :

(H2) Let $f \in C(M \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and there exists a number $\delta \in (0,1)$ such that $(logt)^{\delta} f \in C(M \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and there exist constants $K,\ L > 0$ such that

$$(logt)^{\delta}|f(t,x_1,y_1) - f(t,x_2,y_2)| \leq K|x_1 - x_2| + L|y_1 - y_2|,$$

for any $x_1,\ x_2,\ y_1,\ y_2 \in \mathbb{R}$ and $t \in M$.

Theorem 4.2 *Let the conditions (H1), (H2) and inequality (4.7) be satisfied for all $\ell \in \{1, 2, \ldots, n\}$.*

Then, the BVP (4.1) possesses at least one solution in $C(M, \mathbb{R})$.

Proof

For any $\ell \in \{1, 2, \ldots, n\}$ according to Theorem 4.1 the BVP (4.4) possesses at least one solution $\widetilde{x}_\ell \in E_\ell$.

For any $\ell \in \{1, 2, \ldots, n\}$ we define the function

$$x_\ell = \begin{cases} 0, & t \in [1, T_{\ell-1}], \\ \widetilde{x}_\ell, & t \in J_\ell. \end{cases}$$

Thus, the function $x_\ell \in C([1, T_\ell], \mathbb{R})$ solves the integral equation (4.3) for $t \in J_\ell,\ \ \ell \in \{1, 2, \ldots, n\}$ and $x_\ell(1) = 0,\ x_\ell(T_\ell) = \widetilde{x}_\ell(T_\ell) = 0$.

Then, the function,

$$x(t) = \begin{cases} x_1(t), & t \in J_1, \\ x_2(t) = \begin{cases} 0, & t \in J_1, \\ \widetilde{x}_2, & t \in J_2, \end{cases} \\ . \\ . \\ . \\ . \\ x_n(t) = \begin{cases} 0, & t \in [1, T_{\ell-1}], \\ \widetilde{x}_\ell, & t \in J_\ell, \end{cases} \end{cases}$$

forms a solution of BVP (4.1).

4.3 Ulam-Hyers-Rassias stability

Theorem 4.3 *Assume (H1), (H2), (4.7), and*

(H3) *The function $\varphi \in C(J_\ell, \mathbb{R}_+)$ is increasing and there exists $\lambda_\varphi > 0$ such that, for each $t \in J_\ell$, we have*

$${}^{H}I^{u_\ell}_{T_{\ell-1}^+}\varphi(t) \leq \lambda_{\varphi(t)}\varphi(t),$$

then, the BVP (4.1) is ***UHR*** *stable with respect to φ.*

Proof

Let $\epsilon > 0$ be an arbitrary number and the function $z(t)$ from $C(M, \mathbb{R})$ satisfy the following inequality

$$\|{}^H D^{u_\ell}_{T_{\ell-1}{}^+} z(t) - f(t, z(t), {}^H I^{u_\ell}_{T^+_{\ell-1}} z(t))\| \leq \epsilon\varphi(t), t \in J_\ell. \tag{4.9}$$

For any $\ell \in \{1, 2, \dots, n\}$ we define the functions $z_1(t) \equiv z(t),\ t \in [1, T_1]$ and for $\ell = 2, 3, \dots, n$:

$$z_\ell(t) = \begin{cases} 0, & t \in [1, T_{\ell-1}], \\ z(t), & t \in J_\ell. \end{cases}$$

For any $\ell \in \{1, 2, \dots, n\}$ according to equality (4.2) for $t \in J_\ell$ we get

$${}^H D^{u(t)}_{1^+} z_\ell(t) = \frac{t^2}{\Gamma(2 - u(t))} \frac{d^2}{dt^2} \int_{T_{\ell-1}}^t \Big(log\frac{t}{s}\Big)^{1-u_\ell} \frac{z(s)}{s} ds.$$

we take the (HFI) ${}^H I^{u_\ell}_{T_{\ell-1}{}^+}$ of both sides of the inequality (4.9), we obtain

$$\begin{aligned}
&\|z(t) + \frac{1}{\Gamma(u_\ell)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{1-u_\ell}\Big(log\frac{t}{T_{\ell-1}}\Big)^{u_\ell-1} \int_{T_{\ell-1}}^{T_\ell} \Big(log\frac{T_\ell}{s}\Big)^{u_\ell-1} \frac{f(s, z(s), {}^H I^{u_\ell}_{T^+_{\ell-1}} z(s))}{s} ds \\
&- \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^t \Big(log\frac{t}{s}\Big)^{u_\ell-1} \frac{f(s, z(s), {}^H I^{u_\ell}_{T^+_{\ell-1}} z(s))}{s} ds\| \\
&\leq \epsilon \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^t \frac{1}{s}\Big(log\frac{t}{s}\Big)^{u_\ell-1} \varphi(s) ds \\
&\leq \epsilon\lambda_{\varphi(t)}\varphi(t).
\end{aligned}$$

According to Theorem 4.2, BVP (4.1) has a solution $x \in C(M, \mathbb{R})$ defined by $x(t) = x_\ell(t)$ for $t \in J_\ell,\ \ell = 1, 2, \dots, n$, where

$$x_\ell = \begin{cases} 0, & t \in [1, T_{\ell-1}], \\ \widetilde{x}_\ell, & t \in J_\ell, \end{cases} \tag{4.10}$$

and $\widetilde{x}_\ell \in E_\ell$ is a solution of (4.4). According to Lemma 4.1 the integral equation

$$\widetilde{x}_\ell(t) = -\frac{1}{\Gamma(u_\ell)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{1-u_\ell}\Big(log\frac{t}{T_{\ell-1}}\Big)^{u_\ell-1} \int_{T_{\ell-1}}^{T_\ell} \Big(log\frac{T_\ell}{s}\Big)^{u_\ell-1} \frac{f(s, x(s), {}^H I^{u_\ell}_{T^+_{\ell-1}} x(s))}{s} ds$$

$$+\frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^t \Big(log\frac{t}{s}\Big)^{u_\ell-1} \frac{f(s, x(s), {}^H I^{u_\ell}_{T^+_{\ell-1}} x(s))}{s} ds, \tag{4.11}$$

holds.

Let $t \in J_\ell$ where $\ell \in \{1, 2, \dots, n\}$. Then by (4.10) and (4.11) we get

$$
\begin{aligned}
& \|z(t) - x(t)\| = \|z(t) - x_\ell(t)\| = \|z_\ell(t) - \widetilde{x}_\ell(t)\| \\
= \;& \|z_\ell(t) + \frac{1}{\Gamma(u_\ell)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{1-u_\ell}\Big(log\frac{t}{T_{\ell-1}}\Big)^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}\Big(log\frac{T_\ell}{s}\Big)^{u_\ell-1} f(s, \widetilde{x}_\ell(s), {}^H I^{u_\ell}_{T^+_{\ell-1}}\widetilde{x}_\ell(s))\frac{ds}{s} \\
& -\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}\Big(log\frac{t}{s}\Big)^{u_\ell-1} f(s, \widetilde{x}_\ell(s), {}^H I^{u_\ell}_{T^+_{\ell-1}}\widetilde{x}_\ell(s))\frac{ds}{s}\| \\
\le \;& \|z_\ell(t) + \frac{1}{\Gamma(u_\ell)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{1-u_\ell}\Big(log\frac{t}{T_{\ell-1}}\Big)^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}\Big(log\frac{T_\ell}{s}\Big)^{u_\ell-1} f(s, z_\ell(s), {}^H I^{u_\ell}_{T^+_{\ell-1}} z_\ell(s))\frac{ds}{s} \\
& -\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}\Big(log\frac{t}{s}\Big)^{u_\ell-1} f(s, z_\ell(s), {}^H I^{u_\ell}_{T^+_{\ell-1}} z_\ell(s))\frac{ds}{s}\| \\
& +\frac{1}{\Gamma(u_\ell)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{1-u_\ell}\Big(log\frac{t}{T_{\ell-1}}\Big)^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}\Big(log\frac{T_\ell}{s}\Big)^{u_\ell-1} \\
& \|f(s, z_\ell(s), {}^H I^{u_\ell}_{T^+_{\ell-1}} z_\ell(s)) - f(s, \widetilde{x}_\ell(s), {}^H I^{u_\ell}_{T^+_{\ell-1}}\widetilde{x}_\ell(s))\|\frac{ds}{s} \\
& +\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}\Big(log\frac{t}{s}\Big)^{u_\ell-1}\|f(s, z_\ell(s), {}^H I^{u_\ell}_{T^+_{\ell-1}} z_\ell(s)) - f(s, \widetilde{x}_\ell(s), {}^H I^{u_\ell}_{T^+_{\ell-1}}\widetilde{x}_\ell(s))\|\frac{ds}{s} \\
\le \;& \lambda_{\varphi(t)}\epsilon\varphi(t) + \frac{1}{\Gamma(u_\ell)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{1-u_\ell}\Big(log\frac{t}{T_{\ell-1}}\Big)^{u_\ell-1} \\
& \int_{T_{\ell-1}}^{T_\ell}\Big(log\frac{T_\ell}{s}\Big)^{u_\ell-1}(\log s)^{-\delta}\Big(K\|z_\ell(s) - \widetilde{x}_\ell(s)\| + L\ \|{}^H I^{u_\ell}_{T^+_{\ell-1}}(z_\ell(s) - \widetilde{x}_\ell(s))\|\Big)\frac{ds}{s} \\
& +\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}\Big(log\frac{t}{s}\Big)^{u_\ell-1}(\log s)^{-\delta}\Big(K\|z_\ell(s) - \widetilde{x}_\ell(s)\| + L\ \|{}^H I^{u_\ell}_{T^+_{\ell-1}}(z_\ell(s) - \widetilde{x}_\ell(s))\|\Big)\frac{ds}{s} \\
\le \;& \lambda_{\varphi(t)}\epsilon\varphi(t) + \frac{1}{\Gamma(u_\ell)}\Big(log\frac{t}{T_{\ell-1}}\Big)^{u_\ell-1} \\
& \int_{T_{\ell-1}}^{T_\ell}(\log s)^{-\delta}\Big(K\|z_\ell(s) - \widetilde{x}_\ell(s)\| + L\ \|{}^H I^{u_\ell}_{T^+_{\ell-1}}(z_\ell(s) - \widetilde{x}_\ell(s))\|\Big)\frac{ds}{s} \\
& +\frac{1}{\Gamma(u_\ell)}\Big(log\frac{t}{T_{\ell-1}}\Big)^{u_\ell-1}\int_{T_{\ell-1}}^{t}(\log s)^{-\delta}\Big(K\|z_\ell(s) - \widetilde{x}_\ell(s)\| + L\ \|{}^H I^{u_\ell}_{T^+_{\ell-1}}(z_\ell(s) - \widetilde{x}_\ell(s))\|\Big)\frac{ds}{s} \\
\le \;& \lambda_{\varphi(t)}\epsilon\varphi(t) + \frac{(\log T_\ell)^{1-\delta} - (\log T_{\ell-1})^{1-\delta}}{(1-\delta)\Gamma(u_\ell)}\Big(log\frac{t}{T_{\ell-1}}\Big)^{u_\ell-1} \\
& \Big(K\|z_\ell - \widetilde{x}_\ell\|_{E_\ell} + L\ \|{}^H I^{u_\ell}_{T^+_{\ell-1}}(z_\ell - \widetilde{x}_\ell)\|_{E_\ell}\Big) \\
& +\frac{(\log t)^{1-\delta} - (\log T_{\ell-1})^{1-\delta}}{(1-\delta)\Gamma(u_\ell)}\Big(log\frac{t}{T_{\ell-1}}\Big)^{u_\ell-1}\Big(K\|z_\ell - \widetilde{x}_\ell\|_{E_\ell} + L\ \|{}^H I^{u_\ell}_{T^+_{\ell-1}}(z_\ell - \widetilde{x}_\ell)\|_{E_\ell}\Big) \\
\le \;& \lambda_{\varphi(t)}\epsilon\varphi(t) + \frac{(\log T_\ell)^{1-\delta} - (\log T_{\ell-1})^{1-\delta}}{(1-\delta)\Gamma(u_\ell)}\Big(log\frac{t}{T_{\ell-1}}\Big)^{u_\ell-1} \\
& \Big(K\|z_\ell - \widetilde{x}_\ell\|_{E_\ell} + \frac{L}{\Gamma(u_\ell+1)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell}\|z_\ell - \widetilde{x}_\ell\|_{E_\ell}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(\log t)^{1-\delta}-(\log T_{\ell-1})^{1-\delta}}{(1-\delta)\Gamma(u_\ell)}\Big(log\frac{t}{T_{\ell-1}}\Big)^{u_\ell-1}\Big(K\|z_\ell-\widetilde{x}_\ell\|_{E_\ell}+\frac{L}{\Gamma(u_\ell+1)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell}\|z_\ell-\widetilde{x}_\ell\|_{E_\ell}\Big)\\
\leq\ & \lambda_{\varphi(t)}\epsilon\varphi(t)+\frac{2\Big[(\log T_\ell)^{1-\delta}-(\log T_{\ell-1})^{1-\delta}\Big]}{(1-\delta)\Gamma(u_\ell)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell-1}\\
&\Big(K+\frac{L}{\Gamma(u_\ell+1)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell}\Big)\|z_\ell-\widetilde{x}_\ell\|_{E_\ell}\\
\leq\ & \lambda_{\varphi(t)}\epsilon\varphi(t)+\mu\|z-x\|,
\end{aligned}
$$

where

$$
\mu=\max_{\ell=1,2,\ldots,n}\frac{2\Big[(\log T_\ell)^{1-\delta}-(\log T_{\ell-1})^{1-\delta}\Big]}{(1-\delta)\Gamma(u_\ell)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell-1}\Big(K+\frac{L}{\Gamma(u_\ell+1)}\Big(log\frac{T_\ell}{T_{\ell-1}}\Big)^{u_\ell}\Big).
$$

Then,

$$
\|z-x\|(1-\mu)\leq\lambda_{\varphi(t)}\epsilon\varphi(t).
$$

We obtain, for each $t\in M$

$$
|z(t)-x(t)|\leq\|z-x\|\leq\frac{\lambda_{\varphi(t)}\varphi(t)}{(1-\mu)}\epsilon:=c_f\epsilon.
$$

Then, by Theorem (1.5), BVP (4.1) is (**UHR**) stable with respect to φ.

4.4 Example

In this example, we deal with the following BVP

$$
\begin{cases}
{}^HD_{1^+}^{u(t)}x(t)=\dfrac{7}{5\sqrt{\pi}}(logt)^{u(t)}+\dfrac{(logt)^{-\frac{1}{3}}}{t+3}x(t)+\dfrac{\sqrt{\pi}logt}{t^2+1}\ {}^HI_{1^+}^{u(t)}x(t),\quad t\in M:=[1,e],\\
x(1)=0,\ \ x(e)=0,
\end{cases}
\tag{4.12}
$$

where

$$
u(t)=\begin{cases}1.3, & t\in J_1:=[1,2],\\ 1.7, & t\in J_2:=]2,e].\end{cases}
\tag{4.13}
$$

Denote

$$
f(t,x,y)=\frac{7}{5\sqrt{\pi}}(logt)^{u(t)}+\frac{(logt)^{-\frac{1}{3}}}{t+3}x(t)+\frac{\sqrt{\pi}logt}{t^2+1}y(t),\ (t,x,y)\in[1,e]\times[0,+\infty)\times[0,+\infty).
$$

Then, we have

$$
\begin{aligned}
&(logt)^{\frac{1}{3}}|f(t,x_1,y_1)-f(t,x_2,y_2)|\\
= &\ \Big|\frac{1}{t+3}x_1(t)+\frac{\sqrt{\pi}(logt)^{\frac{4}{3}}}{t^2+1}y_1(t)-\frac{1}{t+3}x_2(t)-\frac{\sqrt{\pi}(logt)^{\frac{4}{3}}}{t^2+1}y_2(t)\Big|\\
\leq &\ \frac{1}{t+3}|x_1(t)-x_2(t)|+\frac{\sqrt{\pi}(logt)^{\frac{4}{3}}}{t^2+1}|y_1(t)-y_2(t)|\\
\leq &\ \frac{1}{4}|x_1(t)-x_2(t)|+\frac{\sqrt{\pi}}{2}|y_1(t)-y_2(t)|.
\end{aligned}
$$

Ergo, (H2) holds with $\delta=\frac{1}{3}$, $K=\frac{1}{4}, L=\frac{\sqrt{\pi}}{2}$.

By (4.13), according to (4.4) we consider two auxiliary BVP for Hadamard fractional differential equations of constant order

$$
\begin{cases}
{}^H D_{1^+}^{1.3}x(t)=\frac{7}{5\sqrt{\pi}}(logt)^{1.3}+\frac{(logt)^{-\frac{1}{3}}}{t+3}x(t)+\frac{\sqrt{\pi}logt}{t^2+1}\ {}^H I_{1^+}^{1.3}x(t), \quad t\in J_1,\\
x(1)=0,\ \ x(2)=0,
\end{cases}
\tag{4.14}
$$

and

$$
\begin{cases}
{}^H D_{2^+}^{1.7}x(t)=\frac{7}{5\sqrt{\pi}}(logt)^{1.7}+\frac{(logt)^{-\frac{1}{3}}}{t+3}x(t)+\frac{\sqrt{\pi}logt}{t^2+1}\ {}^H I_{1^+}^{1.7}x(t), \quad t\in J_2,\\
x(2)=0,\ \ x(e)=0.
\end{cases}
\tag{4.15}
$$

Next, we prove that the inequality (4.7) is fulfilled for $\ell=1$. Indeed,

$$
\begin{aligned}
&\frac{(\log T_1)^{1-\delta}-(\log T_0)^{1-\delta}}{(1-\delta)\Gamma(u_1)}\Big(log\frac{T_1}{T_0}\Big)^{u_1-1}\Big(K+\frac{L}{\Gamma(u_1+1)}\Big(log\frac{T_1}{T_0}\Big)^{u_1}\Big)\\
= &\ \frac{(\log 2)^{\frac{2}{3}}}{(\frac{2}{3})\Gamma(1.3)}(\log 2)^{0.3}\Big(\frac{1}{4}+\frac{\frac{\sqrt{\pi}}{2}}{\Gamma(2.3)}(\log 2)^{1.3}\Big)\\
\simeq &\ 0.3809<\frac{1}{2}.
\end{aligned}
$$

Let $\varphi(t)=(logt)^{\frac{1}{2}}$. Then, we get

$$
\begin{aligned}
{}^H I_{1^+}^{u_1}\varphi(t) &= \frac{1}{\Gamma(1.3)}\int_1^t\Big(log\frac{t}{s}\big)^{1.3-1}\frac{(logs)^{\frac{1}{2}}}{s}ds\\
&\leq \frac{1}{\Gamma(1.3)}\int_1^t\Big(log\frac{t}{s}\Big)^{0.3}\frac{ds}{s}\\
&\leq \frac{0.75}{\Gamma(2.3)}(logt)^{\frac{1}{2}}:=\lambda_{\varphi(t)}\varphi(t),
\end{aligned}
$$

where $\lambda_\varphi=\frac{0.75}{\Gamma(2.3)}$. Thus, condition (H3) is satisfied.

By Theorem 4.1, the BVP (4.14) has a solution $\widetilde{x}_1 \in E_1$.
We prove that the inequality (4.7) is fulfilled for $\ell = 2$. Indeed,

$$\begin{aligned}
&\frac{(\log T_2)^{1-\delta} - (\log T_1)^{1-\delta}}{(1-\delta)\Gamma(u_2)} \Big(log\frac{T_2}{T_1}\Big)^{u_2-1} \Big(K + \frac{L}{\Gamma(u_2+1)} \Big(log\frac{T_2}{T_1}\Big)^{u_2}\Big) \\
= \ & \frac{1-(\log 2)^{\frac{2}{3}}}{(\frac{2}{3})\Gamma(1.7)} \Big(log\frac{e}{2}\Big)^{0.7} \Big(\frac{1}{4} + \frac{\frac{\sqrt{\pi}}{2}}{\Gamma(2.7)} (\log\frac{e}{2})^{1.7}\Big) \\
\simeq \ & 0.0242 < \frac{1}{2}.
\end{aligned}$$

Accordingly the inequality (4.7) is achieved. We get

$$\begin{aligned}
{}^H I_{2^+}^{u_2} \varphi(t) &= \frac{1}{\Gamma(1.7)} \int_2^t \Big(log\frac{t}{s}\Big)^{1.7-1} \frac{(logs)^{\frac{1}{2}}}{s} ds \\
&\leq \frac{1}{\Gamma(1.7)} \int_2^t \Big(log\frac{t}{s}\Big)^{0.7} \frac{ds}{s} \\
&\leq \frac{1}{\Gamma(2.7)} (logt)^{\frac{1}{2}} := \lambda_{\varphi(t)} \varphi(t),
\end{aligned}$$

where $\lambda_\varphi = \frac{1}{\Gamma(2.7)}$. Thus condition (H3) is fulfilled.
According to Theorem 4.1, the BVP (4.15) possesses a solution $\widetilde{x}_2 \in E_2$.
Thus, by Theorem 4.2, the BVP (4.12) has a solution

$$x(t) = \begin{cases} \widetilde{x}_1(t), & t \in J_1, \\ x_2(t), & t \in J_2, \end{cases}$$

where

$$x_2(t) = \begin{cases} 0, & t \in J_1 \\ \widetilde{x}_2(t), & t \in J_2. \end{cases}$$

According to Theorem (4.3) BVP (4.12) is **UHR** stable with respect to φ.

Chapitre 5

Implicit boundary value problem of Caputo fractional differential equations of variable order [(4)]

5.1 Introduction

We deal with the boundary value problem BVP

$$\begin{cases} {}^cD_{0^+}^{u(t)}x(t) = f(t, x(t), {}^cD_{0^+}^{u(t)}x(t)), \;\; t \in J := [0, T], \\ x(0) = 0, \;\; x(T) = 0, \end{cases} \tag{5.1}$$

where $u : J \to (1, 2]$, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and ${}^cD_{0^+}^{u(t)}$ is the Caputo fractional derivative of variable-order $u(t)$. Further, we study the stability of the obtained solution of (5.1) in the sense of Ulam-Hyers **(UH)**.

5.2 Existence of solutions

Let us introduce the following assumption :

(H1) Let $n \in N$ be an integer, $\mathcal{P} = \{J_1 := [0, T_1], J_2 := (T_1, T_2], J_3 := (T_2, T_3], ...J_n := (T_{n-1}, T]\}$ be a partition of the interval J, and let $u(t) : J \to (1, 2]$ be a piecewise constant function with respect to $\mathcal{P}$, i.e.,

[(4)] **A. Benkerrouche**, M. S. Souid, K. Sitthithakerngkiet, A. Hakem, Implicit nonlinear fractional differential equations of variable order, *Boundary value problems* , **2021 :64**, 1-16, (2021).

$$u(t)=\sum_{\ell=1}^{n}u_\ell I_\ell(t)=\begin{cases} u_1, & if\ t\in J_1,\\ u_2, & if\ t\in J_2,\\ \cdot \\ \cdot \\ \cdot \\ u_n, & if\ t\in J_n,\end{cases}$$

where $1<u_\ell\leq 2$ are constants, and I_ℓ is the indicator of the interval $J_\ell:=(T_{\ell-1},T_\ell],\ \ell=1,2,...,n,$ (with $T_0=0,\ T_n=T$) such that

$$I_\ell(t)=\begin{cases} 1, & for\ t\in J_\ell,\\ 0, & for\ elsewhere.\end{cases}$$

For each $\ell\in\{1,2,...,n\}$, the symbol $E_\ell=C(J_\ell,\mathbb{R})$, indicated the Banach space of continuous functions $x:J_\ell\to\mathbb{R}$ equipped with the norm

$$\|x\|_{E_\ell}=\sup_{t\in J_\ell}|x(t)|.$$

Then, for any $t\in J_\ell,\ \ell=1,2,...,n$ the left caputo fractional derivative of variable order $u(t)$ for function $x(t)\in C(J,\mathbb{R})$, could be presented as a sum of left caputo fractional derivatives of constant-orders $u_\ell,\ \ell=1,2,...,n$

$$^cD_{0^+}^{u(t)}x(t)=\int_0^{T_1}\frac{(t-s)^{1-u_1}}{\Gamma(2-u_1)}x^{(2)}(s)ds+...+\int_{T_{\ell-1}}^{t}\frac{(t-s)^{1-u_\ell}}{\Gamma(2-u_\ell)}x^{(2)}(s)ds. \tag{5.2}$$

Thus, according to (5.2), BVP(5.1) can be written for any $t\in J_\ell,\ \ell=1,2,...,n$ in the form

$$\int_0^{T_1}\frac{(t-s)^{1-u_1}}{\Gamma(2-u_1)}x^{(2)}(s)ds+...+\int_{T_{\ell-1}}^{t}\frac{(t-s)^{1-u_\ell}}{\Gamma(2-u_\ell)}x^{(2)}(s)ds=f(t,x(t),^cD_{0^+}^{u(t)}x(t)). \tag{5.3}$$

In what follows we shall introduce the solution to the BVP (5.1).

Definition 5.1 *BVP (5.1) has a solution, if there are functions $x_\ell,\ell=1,2,...,n,$ so that, $x_\ell\in C([0,T_\ell],\mathbb{R})$ fulfilling equation (5.3) and $x_\ell(0)=0=x_\ell(T_\ell)$.*

Let the function $x\in C(J,\mathbb{R})$ be such that $x(t)\equiv 0$ on $t\in[0,T_{\ell-1}]$ and it solves integral equation (5.3). Then (5.3) is reduced to

$$^cD_{T_{\ell-1}^+}^{u_\ell}x(t)=f(t,x(t),^cD_{T_{\ell-1}^+}^{u_\ell}x(t)),\quad t\in J_\ell.$$

We shall deal with following BVP

$$\begin{cases} ^cD_{T_{\ell-1}^+}^{u_\ell}x(t)=f(t,x(t),^cD_{T_{\ell-1}^+}^{u_\ell}x(t)), & t\in J_\ell,\\ x(T_{\ell-1})=0,x(T_\ell)=0.\end{cases} \tag{5.4}$$

For our purpose, the upcoming lemma will be a corner stone of the solution of BVP (5.4).

Lemma 5.1 *Let $\ell \in \{1, 2, ..., n\}$ be a natural number, $f \in C(J_\ell \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and there exists a number $\delta \in (0, 1)$ such that $t^\delta f \in C(J_\ell \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.*
Then, the function $x \in E_\ell$ is a solution of BVP (5.4) if and only if x solves the integral equation

$$x(t) = -(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} y(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}} y(t), \tag{5.5}$$

where

$$y(t) = f\Big(t, -(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} y(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}} y(t), y(t)\Big), \quad t \in J_\ell.$$

Proof
We presume that $x \in E_\ell$ is solution of BVP (5.4) and we take ${}^cD^{u_\ell}_{T^+_{\ell-1}} x(t) = y(t)$. Employing the operator $I^{u_\ell}_{T^+_{\ell-1}}$ to both sides of (5.4) and regarding Lemma 1.2, we find

$$x(t) = \omega_1 + \omega_2(t - T_{\ell-1}) + I^{u_\ell}_{T^+_{\ell-1}} y(t), \quad t \in J_\ell.$$

By $x(T_{\ell-1}) = 0$, we get $\omega_1 = 0$.

Let $x(t)$ satisfy $x(T_\ell) = 0$. So, we observe that

$$\omega_2 = -(T_\ell - T_{\ell-1})^{-1} I^{u_\ell}_{T^+_{\ell-1}} y(T_\ell).$$

Then, we find

$$x(t) = -(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} y(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}} y(t),$$

where

$$y(t) = f\Big(t, -(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} y(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}} y(t), y(t)\Big), \quad t \in J_\ell.$$

Conversely, let $x \in E_\ell$ be solution of integral equation (5.5). Regarding the continuity of function $t^\delta f$ and Lemma 1.3, we deduce that x is the solution of BVP (5.4).

We will prove the existence result for the BVP (5.4). This result is based on Theorem 1.2.

Theorem 5.1 *Let the conditions of Lemma 5.1 be satisfied and there exist a constants $K,\ L > 0$, such that,*
$t^\delta |f(t, y_1, z_1) - f(t, y_2, z_2)| \le K|y_1 - y_2| + L|z_1 - z_2|$, *for any $y_i,\ z_i \in \mathbb{R}$, $i = 1, 2$, $t \in J_\ell$, and the inequality*

$$\frac{2(T_\ell - T_{\ell-1})^{u_\ell - 1}(T_\ell^{1-\delta} - T_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)} \Big(2K \frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)} + L\Big) < 1, \tag{5.6}$$

holds.
Then, BVP (5.4) possesses at least one solution in E_ℓ.

Proof

We construct the operators

$$W_1, W_2 : E_\ell \to E_\ell,$$

as follow :

$$W_1 y(t) = -(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} y(T_\ell), \quad W_2 y(t) = I^{u_\ell}_{T^+_{\ell-1}} y(t), \tag{5.7}$$

where

$$y(t) \ = \ f\Big(t, -(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} y(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}} y(t), y(t)\Big), \quad t \in J_\ell.$$

It follows from the properties of fractional integrals and from the continuity of function $t^\delta f$ that the operator $W_1, W_2 : E_\ell \to E_\ell$ defined in (5.7) are well defined. Let

$$R_\ell \ \geq \ \frac{\frac{2f^\star (T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}}{1 - \frac{2(T_\ell - T_{\ell-1})^{u_\ell - 1}(T_\ell^{1-\delta} - T_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}(2K\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)} + L)},$$

where

$$f^\star = \sup_{t \in J_\ell} |f(t, 0, 0)|.$$

We consider the set

$$B_{R_\ell} = \{y \in E_\ell, \ \|y\|_{E_\ell} \leq R_\ell\}.$$

Clearly B_{R_ℓ} is nonempty, closed, convex and bounded.

Now, we demonstrate that W_1, W_2 satisfies the assumption of the Theorem 1.2. We shall prove it in four phases.

STEP 1 : Claim : $W_1(B_{R_\ell}) + W_2(B_{R_\ell}) \subseteq (B_{R_\ell})$.

For $y \in B_{R_\ell}$, we have

$$\begin{aligned}
&|(W_1 y)(t) + (W_2 y)(t)| \leq \frac{(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1})}{\Gamma(u_\ell)} \\
&\int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1}\Big|f\Big(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} y(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}} y(s), y(s)\Big)\Big| ds \\
&+ \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t - s)^{u_\ell - 1}\Big|f\Big(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} y(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}} y(s), y(s)\Big)\Big| ds \\
\leq \ & \frac{2}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1}\Big|f\Big(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} y(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}} y(s), y(s)\Big)\Big| ds
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{2}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(T_\ell-s)^{u_\ell-1}\Big|f\Big(s,-(T_\ell-T_{\ell-1})^{-1}(s-T_{\ell-1})I^{u_\ell}_{T^+_{\ell-1}}y(T_\ell)\\
&\quad +I^{u_\ell}_{T^+_{\ell-1}}y(s),y(s)\Big)-f(s,0,0)\Big|ds+\frac{2}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(T_\ell-s)^{u_\ell-1}|f(s,0,0)|ds\\
&\le \frac{2}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(T_\ell-s)^{u_\ell-1}s^{-\delta}\Big(K|-(T_\ell-T_{\ell-1})^{-1}(s-T_{\ell-1})I^{u_\ell}_{T^+_{\ell-1}}y(T_\ell)\\
&\quad +I^{u_\ell}_{T^+_{\ell-1}}y(s)|+L|y(s)|\Big)ds+\frac{2f^\star(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\\
&\le \frac{2(T_\ell-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}\Big(K|I^{u_\ell}_{T^+_{\ell-1}}y(T_\ell)+I^{u_\ell}_{T^+_{\ell-1}}y(s)|+L|y(s)|\Big)ds\\
&\quad +\frac{2f^\star(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\\
&\le \frac{2(T_\ell-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\Big(2K\|I^{u_\ell}_{T^+_{\ell-1}}y\|_{E_\ell}+L\|y\|_{E_\ell}\Big)\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}ds\\
&\quad +\frac{2f^\star(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\\
&\le \frac{2(T_\ell-T_{\ell-1})^{u_\ell-1}(T_\ell^{1-\delta}-T_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\Big(2K\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}+L\Big)R_\ell\\
&\quad +\frac{2f^\star(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\\
&\le R_\ell,
\end{aligned}
$$

which means that $W_1(B_{R_\ell})+W_2(B_{R_\ell})\subseteq B_{R_\ell}$.

STEP 2 : Claim : W_1 is continuous.
We presume that the sequence (y_n) converges to y in E_ℓ and $t\in J_\ell$. Then,

$$
\begin{aligned}
&|(W_1y_n)(t)-(W_1y)(t)|\le \frac{(T_\ell-T_{\ell-1})^{-1}(t-T_{\ell-1})}{\Gamma(u_\ell)}\\
&\int_{T_{\ell-1}}^{T_\ell}(T_\ell-s)^{u_\ell-1}\Big|f\Big(s,-(T_\ell-T_{\ell-1})^{-1}(s-T_{\ell-1})I^{u_\ell}_{T^+_{\ell-1}}y_n(T_\ell)+I^{u_\ell}_{T^+_{\ell-1}}y_n(s),y_n(s)\Big)\\
&-f\Big(s,-(T_\ell-T_{\ell-1})^{-1}(s-T_{\ell-1})I^{u_\ell}_{T^+_{\ell-1}}y(T_\ell)+I^{u_\ell}_{T^+_{\ell-1}}y(s),y(s)\Big)\Big|ds\\
\le\;& \frac{(T_\ell-T_{\ell-1})^{-1}(t-T_{\ell-1})}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(T_\ell-s)^{u_\ell-1}s^{-\delta}\\
&\Big(K|-(T_\ell-T_{\ell-1})^{-1}(s-T_{\ell-1})I^{u_\ell}_{T^+_{\ell-1}}(y_n(T_\ell)-y(T_\ell))\\
&+I^{u_\ell}_{T^+_{\ell-1}}(y_n(s)-y(s))|+L|(y_n(s)-y(s))|\Big)ds
\end{aligned}
$$

$$\leq \frac{(T_\ell - T_{\ell-1})^{u_\ell - 1}}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} s^{-\delta}\Big(K|I^{u_\ell}_{T^+_{\ell-1}}(y_n(T_\ell) - y(T_\ell)) + I^{u_\ell}_{T^+_{\ell-1}}(y_n(s) - y(s))|$$
$$+L|(y_n(s) - y(s))|\Big)ds$$
$$\leq \frac{(T_\ell - T_{\ell-1})^{u_\ell - 1}}{\Gamma(u_\ell)}\Big(2K\|I^{u_\ell}_{T^+_{\ell-1}}(y_n - y)\|_{E_\ell} + L\|y_n - y\|_{E_\ell}\Big)\int_{T_{\ell-1}}^{T_\ell} s^{-\delta}ds$$
$$\leq \frac{(T_\ell - T_{\ell-1})^{u_\ell - 1}(T_\ell^{1-\delta} - T_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\Big(2K\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)} + L\Big)\|y_n - y\|_{E_\ell},$$

i.e., we obtain

$$\|(W_1 y_n) - (W_1 y)\|_{E_\ell} \to 0 \;\; as \;\; n \to \infty.$$

Ergo, the operator W_1 is a continuous on E_ℓ.

STEP 3 : W_1 is compact

Now, we will show that $W_1(B_{R_\ell})$ is relatively compact, meaning that W_1 is compact. Clearly $W_1(B_{R_\ell})$ is uniformly bounded because by Step 1, we have $W_1(B_{R_\ell}) = \{W_1(y) : y \in B_{R_\ell}\} \subset W_1(B_{R_\ell}) + W_2(B_{R_\ell}) \subseteq (B_{R_\ell})$ thus for each $y \in B_{R_\ell}$ we have $\|W_1(y)\|_{E_\ell} \leq R_\ell$ which means that $W_1(B_{R_\ell})$is bounded. It remains to indicate that $W_1(B_{R_\ell})$ is equicontinuous.

For $t_1, t_2 \in J_\ell,\; t_1 < t_2$ and $y \in B_{R_\ell}$, we have

$$|(W_1 y)(t_2) - (W_1 y)(t_1)|$$
$$= \Big| - \frac{(T_\ell - T_{\ell-1})^{-1}(t_2 - T_{\ell-1})}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} f\Big(s, -(T_\ell - T_{\ell-1})^{-1}$$
$$(s - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} y(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}} y(s), y(s)\Big)ds + \frac{(T_\ell - T_{\ell-1})^{-1}(t_1 - T_{\ell-1})}{\Gamma(u_\ell)}$$
$$\int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} f\Big(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} y(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}} y(s), y(s)\Big)ds\Big|$$
$$\leq \frac{(T_\ell - T_{\ell-1})^{-1}}{\Gamma(u_\ell)}\Big((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\Big)$$
$$\int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1}\Big| f\Big(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} y(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}} y(s), y(s)\Big)\Big|ds$$
$$\int_{T_{\ell-1}}^{T_\ell} \Big| f\Big(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} y(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}} y(s), y(s)\Big) - f(s, 0, 0)\Big|ds$$
$$\leq \frac{(T_\ell - T_{\ell-1})^{u_\ell - 2}}{\Gamma(u_\ell)}\Big((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\Big)$$
$$+\frac{(T_\ell - T_{\ell-1})^{u_\ell - 2}}{\Gamma(u_\ell)}\Big((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\Big)\int_{T_{\ell-1}}^{T_\ell} |f(s, 0, 0)|ds$$

$$\begin{aligned}
&\leq \frac{(T_\ell - T_{\ell-1})^{u_\ell-2}}{\Gamma(u_\ell)}\Big((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\Big) \\
&\quad \int_{T_{\ell-1}}^{T_\ell} s^{-\delta}\Big(K| - (T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} y(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}} y(s)| + L|y(s)|)\Big)ds \\
&\quad + \frac{f^\star (T_\ell - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\Big((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\Big) \\
&\leq \frac{(T_\ell - T_{\ell-1})^{u_\ell-2}}{\Gamma(u_\ell)}\Big((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\Big) \\
&\quad \int_{T_{\ell-1}}^{T_\ell} s^{-\delta}\Big(K|I^{u_\ell}_{T^+_{\ell-1}} y(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}} y(s)| + L|y(s)|)\Big)ds \\
&\quad + \frac{f^\star (T_\ell - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\Big((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\Big) \\
&\leq \frac{(T_\ell - T_{\ell-1})^{u_\ell-2}}{\Gamma(u_\ell)}\Big((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\Big)\Big(2K\|I^{u_\ell}_{T^+_{\ell-1}} y\|_{E_\ell} \\
&\quad + L\|y\|_{E_\ell}\Big)\int_{T_{\ell-1}}^{T_\ell} s^{-\delta}ds + \frac{f^\star (T_\ell - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\Big((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\Big) \\
&\leq \frac{(T_\ell - T_{\ell-1})^{u_\ell-2}(T_\ell^{1-\delta} - T_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\Big((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\Big) \\
&\quad \Big(2K\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)} + L\Big)\|y\|_{E_\ell} \\
&\quad + \frac{f^\star (T_\ell - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\Big((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\Big) \\
&\leq \Big[\frac{(T_\ell - T_{\ell-1})^{u_\ell-2}(T_\ell^{1-\delta} - T_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\Big(2K\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)} + L\Big)\|y\|_{E_\ell} \\
&\quad + \frac{f^\star (T_\ell - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\Big]\Big((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\Big).
\end{aligned}$$

Hence $\|(W_1y)(t_2) - (W_1y)(t_1)\|_{E_\ell} \to 0$ as $|t_2 - t_1| \to 0$. It implies that $W_1(B_{R_\ell})$ is equicontinuous.

STEP 4 : W_2 is a strict contraction.
For $x(t), y(t) \in E_\ell$, we obtain that

$$\begin{aligned}
&|(W_2x)(t) - (W_2y)(t)| \\
= &\Big|\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(t-s)^{u_\ell-1} f\Big(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} x(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}} x(s), x(s)\Big)ds \\
&- \frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(t-s)^{u_\ell-1} f\Big(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} y(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}} y(s), y(s)\Big)ds\Big|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(t-s)^{u_\ell-1}\Big|f\Big(s,-(T_\ell-T_{\ell-1})^{-1}(s-T_{\ell-1})I_{T_{\ell-1}^+}^{u_\ell}x(T_\ell)+I_{T_{\ell-1}^+}^{u_\ell}x(s),x(s)\Big)\\
&\quad -f\Big(s,-(T_\ell-T_{\ell-1})^{-1}(s-T_{\ell-1})I_{T_{\ell-1}^+}^{u_\ell}y(T_\ell)+I_{T_{\ell-1}^+}^{u_\ell}y(s),y(s)\Big)\Big|ds\\
&\leq \frac{(T_\ell-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}s^{-\delta}\Big(K\Big|(T_\ell-T_{\ell-1})^{-1}(s-T_{\ell-1})(I_{T_{\ell-1}^+}^{u_\ell}(x-y)(T_\ell))\\
&\quad +(I_{T_{\ell-1}^+}^{u_\ell}(x-y)(s))\Big|+L\Big|(x-y)(s)\Big|\Big)ds\\
&\leq \frac{(T_\ell-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}s^{-\delta}\Big(K\Big|(I_{T_{\ell-1}^+}^{u_\ell}(x-y)(T_\ell)+I_{T_{\ell-1}^+}^{u_\ell}(x-y)(s)\Big|+L\Big|(x-y)(s)\Big|\Big)ds\\
&\leq \frac{(T_\ell-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\Big(2K\|(I_{T_{\ell-1}^+}^{u_\ell}(x-y)\|_{E_\ell}+L\|x-y\|_{E_\ell}\Big)\int_{T_{\ell-1}}^{t}s^{-\delta}ds\\
&\leq \frac{(T_\ell-T_{\ell-1})^{u_\ell-1}(T_\ell^{1-\delta}-T_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\Big(2K\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}+L\Big)\|x-y\|_{E_\ell}.
\end{aligned}
$$

Consequently by (5.6), the operator W_2 is strict contraction.
Therefore, all conditions of Theorem 1.2 are fulfilled and thus, there exists $\widetilde{x_\ell}\in B_{R_\ell}$, such that $W_1\widetilde{x_\ell}+W_2\widetilde{x_\ell}=\widetilde{x_\ell}$, which is a solution of the BVP (5.4). Since $B_{R_\ell}\subset E_\ell$, the claim of Theorem 5.1 is proved.

Now, we will prove the existence result for BVP (5.1).
Introduce the following assumption :

(H2) Let $f\in C(J\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and there exists a number $\delta\in(0,1)$ such that $t^\delta f\in C(J\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and there exist a constants $K,\ L>0$, such that,
$t^\delta|f(t,y_1,z_1)-f(t,y_2,z_2)|\leq K|y_1-y_2|+L|z_1-z_2|$, for any $y_1,\ y_2,\ z_1,\ z_2\in\mathbb{R}$ and $t\in J$.

Theorem 5.2 *Let the conditions (H1), (H2) and inequality (5.6) be satisfied for all* $\ell\in\{1,2,...,n\}$.
Then, the problem (5.1) possesses at least one solution in $C(J,\mathbb{R})$.

Proof For any $\ell\in\{1,2,...,n\}$ according to Theorem 5.1 the BVP (5.4) possesses at least one solution $\widetilde{x_\ell}\in E_\ell$.
For any $\ell\in\{1,2,...,n\}$ we define the function

$$x_\ell=\begin{cases}0, & t\in[0,T_{\ell-1}],\\ \widetilde{x_\ell}, & t\in J_\ell.\end{cases}$$

Thus, the function $x_\ell\in C([0,T_\ell],\mathbb{R})$ solves the integral equation (5.3) for $t\in J_\ell$ with $x_\ell(0)=0,\ x_\ell(T_\ell)=\widetilde{x_\ell}(T_\ell)=0$.

Then, the function

$$x(t) = \begin{cases} x_1(t), \quad t \in J_1, \\ x_2(t) = \begin{cases} 0, \quad t \in J_1, \\ \widetilde{x}_2, \quad t \in J_2, \end{cases} \\ . \\ . \\ . \\ x_n(t) = \begin{cases} 0, \quad t \in [0, T_{\ell-1}], \\ \widetilde{x}_\ell, \quad t \in J_\ell, \end{cases} \end{cases} \tag{5.8}$$

is a solution of the BVP (5.1) in $C(J, \mathbb{R})$.

5.3 Ulam-Hyers stability

Theorem 5.3 *Let the conditions (H1), (H2) and inequality (5.6) be satisfied. Then, BVP* (5.1) *is* ***(UH)*** *stable.*

Proof Let $\epsilon > 0$ an arbitrary number and the function $z(t)$ from $z \in C(J_\ell, \mathbb{R})$ satisfy the following inequality

$$|{}^cD_{0^+}^{u(t)} z(t) - f(t, z(t), {}^cD_{0^+}^{u(t)} z(t))| \le \epsilon, \ t \in J. \tag{5.9}$$

For any $\ell \in \{1, 2, ..., n\}$ we define the functions $z_1(t) \equiv z(t), t \in [0, T_1]$ and for $\ell = 2, 3, ..., n$:

$$z_\ell(t) = \begin{cases} 0, \quad t \in [0, T_{\ell-1}], \\ z(t), \quad t \in J_\ell. \end{cases}$$

For any $\ell \in \{1, 2, ..., n\}$ according to equality (5.2) for $t \in J_\ell$ we get

$${}^cD_{T_{\ell-1}^+}^{u(t)} z_\ell(t) = \int_{T_{\ell-1}}^t \frac{(t-s)^{1-u_\ell}}{\Gamma(2-u_\ell)} z^{(2)}(s) ds.$$

Taking the (RLFI) $I_{T_{\ell-1}^+}^{u_\ell}$ of both sides of the inequality (5.9),we obtain

$$\begin{aligned} &\Big| z_\ell(t) + \frac{(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1})}{\Gamma(u_\ell)} \\ &\int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} f\Big(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I_{T_{\ell-1}^+}^{u_\ell} z_\ell(T_\ell) + I_{T_{\ell-1}^+}^{u_\ell} z_\ell(s), z_\ell(s)\Big) ds \\ &- \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^t (t-s)^{u_\ell - 1} f\Big(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I_{T_{\ell-1}^+}^{u_\ell} z_\ell(T_\ell) + I_{T_{\ell-1}^+}^{u_\ell} z_\ell(s), z_\ell(s)\Big) ds \Big| \\ &\le \epsilon \int_{T_{\ell-1}}^t \frac{(t-s)^{u_\ell - 1}}{\Gamma(u_\ell)} ds \\ &\le \epsilon \frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)}. \end{aligned}$$

According to Theorem 5.2, BVP (5.1) has a solution $x \in C(J, \mathbb{R})$ defined by $x(t) = x_\ell(t)$ for $t \in J_\ell,\ \ell = 1, 2, ..., n$, where

$$x_\ell = \begin{cases} 0, & t \in [0, T_{\ell-1}], \\ \widetilde{x}_\ell, & t \in J_\ell, \end{cases} \tag{5.10}$$

and $\widetilde{x}_\ell \in E_\ell$ is a solution of (5.4). According to Lemma 5.1 the integral equation

$$\begin{aligned} \widetilde{x}_\ell(t) &= -\frac{(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1})}{\Gamma(u_\ell)} \\ \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} f\Big(s, &- (T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} \widetilde{x}_\ell(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}} \widetilde{x}_\ell(s), \widetilde{x}_\ell(s)\Big) ds \\ + \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t &- s)^{u_\ell - 1} f\Big(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} \widetilde{x}_\ell(T_\ell) \\ & + I^{u_\ell}_{T^+_{\ell-1}} \widetilde{x}_\ell(s), \widetilde{x}_\ell(s)\Big) ds, \end{aligned} \tag{5.11}$$

holds.

Let $t \in J_\ell,\ \ell = 1, 2, ..., n$. Then by Eq (5.10) and (5.11) we get

$$\begin{aligned} & |z(t) - x(t)| = |z(t) - x_\ell(t)| = |z_\ell(t) - \widetilde{x}_\ell(t)| \\ = & \Big| z_\ell(t) + \frac{(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1})}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} f\Big(s, -(T_\ell - T_{\ell-1})^{-1} \\ & (s - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} \widetilde{x}_\ell(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}} \widetilde{x}_\ell(s), \widetilde{x}_\ell(s)\Big) ds \\ & - \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t - s)^{u_\ell - 1} f\Big(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} \widetilde{x}_\ell(T_\ell) \\ & + I^{u_\ell}_{T^+_{\ell-1}} \widetilde{x}_\ell(s), \widetilde{x}_\ell(s)\Big) ds \Big| \\ \leq & \Big| z_\ell(t) + \frac{(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1})}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} f\Big(s, -(T_\ell - T_{\ell-1})^{-1} \\ & (s - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} z_\ell(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}} z_\ell(s), z_\ell(s)\Big) ds \\ & - \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t - s)^{u_\ell - 1} f\Big(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} z_\ell(T_\ell) \\ & + I^{u_\ell}_{T^+_{\ell-1}} z_\ell(s), z_\ell(s)\Big) ds \Big| \end{aligned}$$

$$
\begin{aligned}
&+\frac{(T_\ell - T_{\ell-1})^{-1}(t-T_{\ell-1})}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(T_\ell - s)^{u_\ell-1}\Big|f\Big(s, -(T_\ell - T_{\ell-1})^{-1}(s-T_{\ell-1})I^{u_\ell}_{T^+_{\ell-1}}z_\ell(T_\ell)\\
&+I^{u_\ell}_{T^+_{\ell-1}}z_\ell(s), z_\ell(s)\Big)ds - f\Big(s, -(T_\ell - T_{\ell-1})^{-1}(s-T_{\ell-1})I^{u_\ell}_{T^+_{\ell-1}}\widetilde{x}_\ell(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}}\widetilde{x}_\ell(s), \widetilde{x}_\ell(s)\Big)\Big|ds\\
&+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(t-s)^{u_\ell-1}\Big|f\Big(s, -(T_\ell - T_{\ell-1})^{-1}(s-T_{\ell-1})I^{u_\ell}_{T^+_{\ell-1}}z_\ell(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}}z_\ell(s), z_\ell(s)\Big)ds\\
&-f\Big(s, -(T_\ell - T_{\ell-1})^{-1}(s-T_{\ell-1})I^{u_\ell}_{T^+_{\ell-1}}\widetilde{x}_\ell(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}}\widetilde{x}_\ell(s), \widetilde{x}_\ell(s)\Big)\Big|ds\\
\leq\;& \epsilon\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)} + \frac{(T_\ell - T_{\ell-1})^{-1}(t-T_{\ell-1})}{\Gamma(u_\ell)}\\
&\int_{T_{\ell-1}}^{T_\ell}(T_\ell - s)^{u_\ell-1}s^{-\delta}\Big(K\Big|(T_\ell - T_{\ell-1})^{-1}(s-T_{\ell-1})(I^{u_\ell}_{T^+_{\ell-1}}(z_\ell(T_\ell) - \widetilde{x}_\ell(T_\ell)))\\
&+(I^{u_\ell}_{T^+_{\ell-1}}(z_\ell(s) - \widetilde{x}_\ell(s)))\Big| + L\Big|(z_\ell(s) - \widetilde{x}_\ell(s))\Big|\Big)ds\\
&+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(t-s)^{u_\ell-1}s^{-\delta}\Big(K\Big|(T_\ell - T_{\ell-1})^{-1}(s-T_{\ell-1})(I^{u_\ell}_{T^+_{\ell-1}}(z_\ell(T_\ell) - \widetilde{x}_\ell(T_\ell)))\\
&+(I^{u_\ell}_{T^+_{\ell-1}}(z_\ell(s) - \widetilde{x}_\ell(s)))\Big| + L\Big|(z_\ell(s) - \widetilde{x}_\ell(s))\Big|\Big)ds\\
\leq\;& \epsilon\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)} + \frac{2(T_\ell - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}\Big(K\Big|(I^{u_\ell}_{T^+_{\ell-1}}(z_\ell(T_\ell) - \widetilde{x}_\ell(T_\ell)))\\
&+(I^{u_\ell}_{T^+_{\ell-1}}(z_\ell(s) - \widetilde{x}_\ell(s)))\Big| + L\Big|(z_\ell(s) - \widetilde{x}_\ell(s))\Big|\Big)ds\\
\leq\;& \epsilon\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)} + \frac{2(T_\ell - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\\
&\Big(2K\|I^{u_\ell}_{T^+_{\ell-1}}(z_\ell - \widetilde{x}_\ell)\|_{E_\ell} + L\|z_\ell - \widetilde{x}_\ell\|_{E_\ell}\Big)\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}ds\\
\leq\;& \epsilon\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)} + \frac{2(T_\ell - T_{\ell-1})^{u_\ell-1}(T_\ell^{1-\delta} - T_{\ell-1}{}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\\
&\Big(2K\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\|z_\ell - \widetilde{x}_\ell\|_{E_\ell} + L\|z_\ell - \widetilde{x}_\ell\|_{E_\ell}\Big)\\
\leq\;& \epsilon\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)} + \frac{2(T_\ell - T_{\ell-1})^{u_\ell-1}(T_\ell^{1-\delta} - T_{\ell-1}{}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\\
&\Big(2K\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)} + L\Big)\|z_\ell - \widetilde{x}_\ell\|_{E_\ell}\\
\leq\;& \epsilon\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)} + \mu\|z-x\|,
\end{aligned}
$$

where

$$\mu = \max_{\ell=1,2,\ldots,n} \frac{2(T_\ell - T_{\ell-1})^{u_\ell-1}({T_\ell}^{1-\delta} - {T_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\Big(2K\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)} + L\Big).$$

Then,

$$\|z - x\|(1-\mu) \le \frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\epsilon.$$

We obtain, for each $t \in J$

$$|z(t) - x(t)| \le \|z - x\| \le \frac{(T_\ell - T_{\ell-1})^{u_\ell}}{(1-\mu)\Gamma(u_\ell+1)}\epsilon := c_f\epsilon.$$

Therefore, by Theorem (5.3), the BVP (5.1) is **(UH)** stable.

5.4 Example

Let us consider the following fractional boundary value problem,

$$\begin{cases} {}^cD_{0^+}^{u(t)}x(t) = \dfrac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}} + 4e^{2t} + 1)(1 + |x(t)| + |{}^cD_{0^+}^{u(t)}x(t)|)}, & t \in J := [0,2], \\ x(0) = 0, \ \ x(2) = 0, \end{cases} \tag{5.12}$$

where

$$u(t) = \begin{cases} \frac{3}{2}, & t \in J_1 := [0,1], \\ \\ \frac{9}{5}, & t \in J_2 :=]1,2]. \end{cases} \tag{5.13}$$

Let

$$f_1(t,y,z) = \frac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}} + 4e^{2t} + 1)(1+y+z)}, \ (t,y,z) \in [0,2] \times [0,+\infty) \times [0,+\infty).$$

Then, we have

$$\begin{aligned} t^{\frac{1}{3}}|f_1(t,y_1,z_1) - f_1(t,y_2,z_2)| &= \left|\frac{e^{-t}}{(e^{e^{\frac{t^2}{1+t}}} + 4e^{2t} + 1)}\left(\frac{1}{1+y_1+z_1} - \frac{1}{1+y_2+z_2}\right)\right| \\ &\le \frac{e^{-t}(|y_1 - y_2| + |z_1 - z_2|)}{(e^{e^{\frac{t^2}{1+t}}} + 4e^{2t} + 1)(1+y_1+z_1)(1+y_2+z_2)} \\ &\le \frac{e^{-t}}{(e^{e^{\frac{t^2}{1+t}}} + 4e^{2t} + 1)}(|y_1 - y_2| + |z_1 - z_2|) \\ &\le \frac{1}{(e+5)}|y_1 - y_2| + \frac{1}{(e+5)}|z_1 - z_2|. \end{aligned}$$

Hence the condition (H2) holds with $\delta = \frac{1}{3}$ and $K = L = \frac{1}{e+5}$.
By (5.13), according to (5.4) we consider two auxiliary BVP for Caputo fractional differential equations of constant order

$$\begin{cases} {}^cD_{0^+}^{\frac{3}{2}} x(t) = \dfrac{t^{-\frac{1}{3}} e^{-t}}{(e^{e^{\frac{t^2}{1+t}}} + 4e^{2t} + 1)(1 + |x(t)| + |{}^cD^{\frac{3}{2}} x(t)|)}, & t \in J_1, \\ x(0) = 0, \ \ x(1) = 0, \end{cases} \tag{5.14}$$

and

$$\begin{cases} {}^cD_{1^+}^{\frac{9}{5}} x(t) = \dfrac{t^{-\frac{1}{3}} e^{-t}}{(e^{e^{\frac{t^2}{1+t}}} + 4e^{2t} + 1)(1 + |x(t)| + |{}^cD^{\frac{9}{5}} x(t)|)}, & t \in J_2, \\ x(1) = 0, \ \ x(2) = 0. \end{cases} \tag{5.15}$$

Next, we prove that the condition (5.6) is fulfilled for $\ell = 1$. Indeed,

$$\frac{2({T_1}^{1-\delta} - {T_0}^{1-\delta})(T_1 - T_0)^{u_1 - 1}}{(1-\delta)\Gamma(u_1)} \Big(\frac{2K(T_1 - T_0)^{u_1}}{\Gamma(u_1 + 1)} + L \Big) = \frac{1}{\frac{2}{3}(e+5)\Gamma(\frac{3}{2})} \Big(\frac{2}{\Gamma(\frac{5}{2})} + 1 \Big) \simeq 0.3664 < 1.$$

Accordingly the condition (5.6) is achieved.
By Theorem 5.1, the problem (5.14) has a solution $\widetilde{x}_1 \in E_1$.
We prove that the condition (5.6) is fulfilled for $\ell = 2$. Indeed,

$$\frac{2({T_2}^{1-\delta} - {T_1}^{1-\delta})(T_2 - T_1)^{u_2 - 1}}{(1-\delta)\Gamma(u_2)} \Big(\frac{2K(T_2 - T_1)^{u_2}}{\Gamma(u_2 + 1)} + L \Big) = \frac{2^{\frac{2}{3}} - 1}{\frac{2}{3}\Gamma(\frac{9}{5})} \frac{1}{e+5} \Big(\frac{2}{\Gamma(\frac{14}{5})} + 1 \Big) \simeq 0.2682 < 1.$$

Thus, the condition (5.6) is satisfied.
According to Theorem 5.1, the BVP (5.15) possesses a solution $\widetilde{x}_2 \in E_2$.
Then, by Theorem 5.2, the BVP (5.12) has a solution

$$x(t) = \begin{cases} \widetilde{x}_1(t), & t \in J_1, \\ x_2(t), & t \in J_2, \end{cases}$$

where

$$x_2(t) = \begin{cases} 0, & t \in J_1, \\ \widetilde{x}_2(t), & t \in J_2. \end{cases}$$

According to Theorem 5.3, BVP (5.12) is **(UH)** stable.

Conclusion

In this thesis, we introduced an abstract variable-order boundary value problems of different derivatives (Riemann-Liouville, Caputo, Hadamard), where the function $u(t) : J \to (1, 2]$ stands for the variable order of the given systems. First, we reviewed some important specifications of different variable-order operators and we showed that the semi-group property is not valid for variable-order integrals.

Then by defining a partition based on the generalized intervals, we introduced a piecewise constant function $u(t)$ and in every chapter converted the given variable-order boundary value problem to an equivalent standard boundary value problem of the fractional constant order.

The results in this study are established with the help of the Darbo's fixed point theorem combined with Kuratowski measure of noncompactness in the first three chapters and Krasnoselsky's fixed point theorem in the last chapter.

The Ulam-Hyers or Ulam-Hyers-Rassias stability of its possible solutions was checked.

A numerical examples is given at the end of the chapters to support and validate the potentiality of all our obtained results.

As a result of our investigation into this fascinating special research subject, all of our findings are unique and noteworthy.

Furthermore, all of the findings in this thesis have a lot of potential for use in a variety of transdisciplinary science applications. With the support of our original findings in this research study, we may be able to do further research on this open research subject.

To put it another way, the proposed BVP could be extended to more sophisticated real mathematical fractional models in the future.

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