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Existence globale, explosion en temps fini et le comportement asymptotique des solutions de certaines équations d'évolution non linéaire.

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Résumé

La présente thèse est consacrée à l'étude de l'existence globale, explosion en temps fini et le comportement asymptotique des solutions de certaines équations d'évolution non linéaires.
Ce travail se compose de quatre chapitres, sera consacré à l'étude du bien-posé, le comportement asymptotique et explosion en temps fini de la solution de certaines équations d'évolution avec un terme d'amortissement non linéaires, un terme de retard et un terme de source.
Dans le chapitre 1, nous rappelons quelques notions utilisées dans cette thèse.
Dans le chapitre 2, nous considérons l'équation d'onde non linéaire soumis à un amortissement, un terme de retard et un terme de source. Nous prouvons que la solution explose en temps fini si le terme de source domine le terme de dissipatif et le terme de retard $p > \max\{l+2, m\}$ sous la condition que l'énergie initiale est négative par la méthode de V. Georgiev et G. Todorova [**8**].
Dans le chapitre 3, nous considérons l'équation de Petrovsky avec un fort amortissement non linéaire et de forme générale. Nous prouvons que ce problème est bien posé en utilisant la méthode de compacité et pour la stabilité générale de la solution on introduit une méthode de Lyapunov.
Dans le chapitre 4, nous considérons un système de Petrovsky-onde couplé avec un fort amortissement non linéaire. Nous prouvons que ce système est bien posé en utilisant la méthode de compacité et pour la stabilité de solution on introduit une méthode de multiplicateur, nous trouvons la stabilité exponentille et polynomiale.

Mots clés : Bien posé, système couplé, décroissance générale, décroissance exponentielle, décroissance polynomiale, méthode Faedo-Galerkin, méthode Lyapunov, méthode multipliée, terme de source, terme de retard, explosion.

Abstract

The present thesis is devoted to the study of global existence, blow-up in finite time and the asymptotic behaviour of the solutions for some nonlinear evolution equations.
This work consists of four chapters, will be devoted to the study of the well-posedness, asymptotic behaviour and blow-up in finite time of the solution of some evolution equations with nonlinear dissipative terms, delay and source terms.
In chapter 1, we recall of some notions used in this thesis.
In chapter 2, we consider the wave equation with nonlinear source, damping and delay term. We prove that weak solutions to the systems blow up in finite time whenever the initial energy is negative and the exponent of the source terms is more dominant than the exponent of damping terms, we use the method of V. Georgiev and G. Todorova [**8**].
In chapter 3, we consider the Petrovsky equation with a nonlinear strong damping. We prove, under some appropriate assumptions, that this system is well-posed using the compactness method. Furthermore, the general stability is given by using a combination of the some properties of convex functions with an appropriate Lyapunov functional.
In chapter 4, we consider a coupled Petrovsky-wave system with a nonlinear strong damping. We prove well-posedness by using the compactness method and establish the both exponential and polynomial decay estimates by introducing a multiplied method.

Keywords : Well-posedness, coupled system, general decay, exponential decay, polynomial decay, Faedo-Galerkin method, Lyapunov method, multiplied method, source term, delay term, blow-up.

Contents

General introduction

The subject of this thesis is the study of global existence, blow-up in finite time and the asymptotic behavior of the solutions of some equations of nonlinear evolution.
The problem of stabilization consists in determining the asymptotic behavior of the energy by $E(t)$, to study its limits in order to determine if this limit is null or not and if this limit is null, to give an estimate of the decay rate of the energy to zero, they are several type of stabilization :
1. strong stabilization : $E(t) \longrightarrow 0$ as $t \longrightarrow \infty$.
2. logarithm stabilization : $E(t) \leq c\,(lnt)^{-\delta}$, $c, \delta > 0$.
3. polynomial stabilization : $E(t) \leq ct^{-\delta}$, $c, \delta > 0$.
4. uniform stabilization : $E(t) \leq ce^{-\delta t}$, $c, \delta > 0$.
In 1982, Lyapunov introduced an energy function that he used it to study the stability of some nonlinear systems without calculate explicitly their solutions. This method is known today by Lyapunov's methode and it played an important role in the stability theory of differential and ordinary equations.
In this work we prove well-posedness of the problem by using the compactness method, we establish the decay estimate by using the Lyapunov and multiplied method, and we prove that weak solutions to the systems blow up in finite time by using the method of V. Georgiev and G. Todorova [**8**].
This thesis consists of four chapters including :
Chapter 1 : Preliminaries
In this chapter, we reminder of some notions used in this thesis.
Chapter 2 : Blow-up of result in a nonlinear wave equation with delay and source term
In this chapter we consider the initial boundary value problem for a nonlinear damping and

a delay term of the form :

$$\begin{cases} |u'|^l u'' - \Delta u(x,t) - \Delta u'' + \mu_1 |u'|^{m-2}u' & \\ +\mu_2 |u'(t-\tau)|^{m-2}u'(t-\tau) = b|u|^{p-2}u, & \text{in } \Omega \times [0,+\infty[, \\ u(x,t) = 0, & \text{on } \partial\Omega \times [0,+\infty[, \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), & \text{in } \Omega, \\ u'(x,t-\tau) = f_0(x,t-\tau), & \text{in } \Omega \times [0,\tau], \end{cases} \tag{0.1}$$

with initial conditions and Dirichlet boundary conditions. Under appropriate conditions on μ_1, μ_2, we prove that there are solutions with negative initial energy that blow-up in finite time if $p > \max\{l+2, m\}$.
We suppose that

$$\max\{p, m\} \leq \frac{2(n-1)}{n-2} \quad \text{and} \quad l \leq \frac{2}{n-2} \quad \text{if} \quad n \geq 3. \tag{0.2}$$

We introduce the new variable

$$z(x,\rho,t) = u'(x,t-\rho\tau), \;\; x \in \Omega, \quad \rho \in [0,1], t > 0.$$

Therefore, the problem (0.1) is equivalent to

$$\begin{cases} |u'(x,t)|^l u''(x,t) - \Delta u(x,t) - \Delta u''(x,t) + \mu_1 |u'(x,t)|^{m-2}u'(x,t) & \\ +\mu_2 |z(x,1,t)|^{m-2} z(x,1,t) = b|u(x,t)|^{p-2}u(x,t), & \text{in } \Omega \times [0,+\infty[, \\ \tau z'(x,\rho,t) + \frac{\partial z}{\partial \rho}(x,\rho,t) = 0, & \text{in } \Omega \times [0,1] \times [0,+\infty[, \\ u(x,t) = 0, & \text{on } \partial\Omega \times [0,\infty[, \\ z(x,0,t) = u'(x,t), & \text{in } \Omega \times [0,\infty[, \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), & \text{in } \Omega, \\ z(x,\rho,0) = f_0(x,-\rho\tau), & \text{in } \Omega \times [0,1]. \end{cases}$$

We define the energy associated to the solution of the system (0.1) by

$$E(t) = \frac{1}{l+2}\|u'\|_{l+2}^{l+2} + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla u'\|_2^2 - \frac{b}{p}\|u\|_p^p + \xi \int_\Omega \int_0^1 |z(x,\rho,t)|^m \, d\rho \, dx,$$

where ξ is a positive constant such that

$$\tau \frac{\mu_2}{m}(m-1) < \xi < \tau\Big(\mu_1 - \frac{\mu_2}{m}\Big),$$

and $\mu_2 < m\mu_1$.

Theorem 0.1 *Suppose that* $m > 1$, $p > \max\{l+2, m\}$ *satisfying* (0.2)*, let* $(u_0, u_1) \in [H_0^1(\Omega) \cap H^2(\Omega)] \times H_0^1(\Omega)$ *and* $f_0 \in \mathcal{C}^1([-\tau, 0]; L^m(\Omega \times [0,1]))$*. Assume further that*

$$E(0) = \frac{1}{l+2}\|u_1\|_{l+2}^{l+2} + \frac{1}{2}\|\nabla u_0\|_2^2 + \frac{1}{2}\|\nabla u_1\|_2^2 - \frac{b}{p}\|u_0\|_p^p + \xi \int_\Omega \int_0^1 |f_0(x, -\rho\tau)|^m \, d\rho \, dx < 0.$$

Then the solution of (0.1) *blow up in finite time, i.e. there exists* $T_0 < +\infty$ *such that*

$$\lim_{t \to T_0^-} (\|u'\|_{l+2}^{l+2} + \|\nabla u\|_2^2 + \|\nabla u'\|_2^2 + \|u\|_p^p) = +\infty.$$

Chapter 3 : Well-posedness and general energy decay of solutions for a Petrovsky equation with a nonlinear strong dissipation

In this chapter we consider a nonlinear Petrovsky equation in a bounded domain with a strong dissipation

$$\begin{cases} u'' + \Delta^2 u - g(\Delta u') = 0, & \text{in } \Omega \times [0, +\infty[, \\ u(x,t) = \Delta u(x,t) = 0, & \text{on } \partial\Omega \times [0, +\infty[, \\ u(x,0) = u_0(x), \ \ u'(x,0) = u_1(x), & \text{in } \Omega. \end{cases} \tag{0.3}$$

We prove the existence and the uniqueness of the solution of this problem using the energy method combined with the compactness procedure under assumption of g. Furthermore, we study the asymptotic behaviour of solutions using a perturbed energy method.

Let us introduce three real Hilbert spaces $\mathcal{H}$, V and W by setting

$$\mathcal{H} = H_0^1(\Omega), \quad \|u\|_{\mathcal{H}}^2 = \int_\Omega |\nabla u|^2 dx,$$

and

$$V = \{u \in H^3(\Omega) | \, u = \Delta u = 0 \text{ on } \partial\Omega\}, \quad \|u\|_V^2 = \int_\Omega |\nabla \Delta u|^2 dx,$$

$$W = \{u \in H^5(\Omega) | u = \Delta u = \Delta^2 u = 0 \text{ on } \partial\Omega\}, \quad \|u\|_W^2 = \int_\Omega |\nabla \Delta^2 u|^2 \, dx.$$

We have

$$W \subset V \subset \mathcal{H} \subset V' \subset W',$$

with dense and compact imbedings.

We impose the following assumptions on g

$g : \mathbb{R} \longrightarrow \mathbb{R}$ is a non-decreasing continuous function such that there exist constants ε, c_1, c_2,

$\tau > 0$ and a convex increasing function $G : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ of class $\mathcal{C}^1(\mathbb{R}_+) \cap \mathcal{C}^2(\mathbb{R}_+^*)$ satisfying G linear on $[0, \varepsilon]$ or $G'(0) = 0$ and $G'' > 0$ on $[0, \varepsilon]$, such that

$$c_1|s| \leq |g(s)| \leq c_2|s| \quad \text{if} \quad |s| > \varepsilon, \tag{0.4}$$

$$|s|^2 + |g(s)|^2 \leq G^{-1}(sg(s)) \quad \text{if} \quad |s| \leq \varepsilon, \tag{0.5}$$

$$g'(s) \leq \tau.$$

Theorem 0.2 ***(well-posedeness and regularity)*** *Assume that* $(u_0, u_1) \in W \times V$*, then the solution of the problem* (0.3) *satisfies*

$$u' \in L^\infty(0, T; V) \ , \ \ u'' \in L^\infty(0, T; \mathcal{H}),$$

and

$$u \in L^\infty(0, T; H^4(\Omega) \cap V),$$

such that for any $T > 0$

$$u''(x,t) + \Delta^2 u(x,t) - g(\Delta u'(x,t)) = 0 \quad \textit{in} \quad L^\infty(0, T; L^2(\Omega)),$$

$$u(0) = u_0, \quad u'(0) = u_1 \quad \textit{in} \quad \Omega.$$

Now we define the energy associated with the solution of the problem (0.3) by the following formula

$$E(t) = \frac{1}{2}\|\nabla u'\|^2 + \frac{1}{2}\|\nabla \Delta u\|^2.$$

Theorem 0.3 ***(stabilization)*** *Assume that* (0.4) *and* (0.5) *hold. Then there exist positive constants* k_1, k_2, k_3 *and* ε_0 *such that the solution of the problem* (0.3) *satisfies*

$$E(t) \leq k_3 G_1^{-1}(k_1 t + k_2) \quad \forall\, t \in \mathbb{R}_+,$$

where

$$G_1(t) = \int_t^1 \frac{1}{G_2(s)}\, ds, \ \ G_2(t) = tG'(\varepsilon_0 t),$$

here G_1 *is strictly decreasing and convex on* $]0, 1]$*, with* $\lim_{t \to 0} G_1(t) = +\infty$.

Chapter 4 : Stabilization of the Petrovsky-wave nonlinear coupled system with strong damping

In this chapter we prove the well-posedness and the stabilization of the nonlinear coupled system :

$$\begin{cases} u_1'' + \Delta^2 u_1 - a(x)\Delta u_2 - g_1(\Delta u_1') = 0, & \text{in } \Omega \times [0, +\infty[, \\ u_2'' - \Delta u_2 - a(x)\Delta u_1 - g_2(\Delta u_2') = 0, & \text{in } \Omega \times [0, +\infty[, \\ \Delta u_1 = u_1 = u_2 = 0, & \text{on } \partial\Omega \times [0, +\infty[, \\ u_i(x, 0) = u_i^0(x),\ u_i'(x, 0) = u_i^1(x), & \text{in } \Omega, i = 1, 2. \end{cases} \tag{0.6}$$

The existence of global weak solutions for this problem is established by using the compactness method. Meanwhile, under suitable conditions on functions $g_i(.),\ i = 1, 2$ and $a(.)$, we estimate the energy decay rate by using the multiplier method.
Let us introduce for brevity the Hilbert spaces

$$H = L^2(\Omega) \times L^2(\Omega),$$

$$W = H_0^1(\Omega) \times H_0^1(\Omega),$$

$$H_\Delta^3(\Omega) = \{u \in H^3(\Omega) |\, u = \Delta u = 0 \text{ on } \partial\Omega\}, \quad \|u\|^2_{H_\Delta^3(\Omega)} = \int_\Omega |\nabla \Delta u|^2 dx,$$

$$V = H_\Delta^3(\Omega) \cap H^2(\Omega) \times H^2(\Omega),$$

$$\widetilde{V} = (H^4(\Omega) \cap H_\Delta^3(\Omega)) \times (H_\Delta^3(\Omega) \cap H^2(\Omega)).$$

Identifying H with its dual H' , we obtain the diagram

$$\widetilde{V} \subset V \subset W \subset H = H' \subset W' \subset V' \subset \widetilde{V}',$$

with dense and compact imbedings.
We impose the following assumptions on a and g_i.
The function $a : \Omega \to \mathbb{R}$ is a nonnegative and bounded such that

$$\begin{aligned} & a \in W^{1,\infty}(\Omega), \\ & \|a\|_\infty < \min\left\{\tfrac{1}{c'+1}, 1\right\}, \end{aligned} \tag{0.7}$$

where $c' > 0$ (depending only on the geometry of Ω) is the constant satisfies

$$\|\Delta u\| \leq c' \|\nabla \Delta u\|, \quad \forall\, u \in H_\Delta^3(\Omega).$$

Assume that $g_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$ is nondecreasing continuously differentiable function of class $\mathcal{C}^1$, and globally Lipschitz.
Suppose that $\exists\, c_j > 0,\ j = 1, 2, 3, 4,\ \exists\, \tau > 0$ such that

$$\forall\, p \geq 1 : c_1|s|^p \leq |g_i(s)| \leq c_2|s|^{\frac{1}{p}} \ \text{ if } \ |s| \leq 1, \tag{0.8}$$

$$c_3|s| \leq |g_i(s)| \leq c_4|s|, \ \text{ if } \ |s| > 1, \tag{0.9}$$

$$\forall\, s \in \mathbb{R} : g_i'(s) \leq \tau. \tag{0.10}$$

Theorem 0.4 ***(global existence)*** *Let* $(u_1^0, u_2^0) \in \widetilde{V}$ *and* $(u_1^1, u_2^1) \in V$ *arbitrarily. Assume that* (0.7)-(0.10) *hold. Then, the system* (0.6) *has a unique weak solution satisfying*

$$(u_1, u_2) \in L^\infty(\mathbb{R}_+, \widetilde{V}), \quad (u_1', u_2') \in L^\infty(\mathbb{R}_+, V),$$

and

$$(u_1'', u_2'') \in L^\infty(\mathbb{R}_+, W).$$

We define the energy associated with the solution of the problem (0.6) by the following formula

$$E(t) = \frac{1}{2}\int_\Omega |\nabla u_1'|^2 + |\nabla u_2'|^2 + |\nabla \Delta u_1|^2 + |\Delta u_2|^2\, dx + \int_\Omega a(x)\Delta u_1 \Delta u_2\, dx. \tag{0.11}$$

Theorem 0.5 ***(stabilization)*** *Let* $(u_1^0, u_2^0) \in \widetilde{V}$ *and* $(u_1^1, u_2^1) \in V$. *Assume that* (0.7)-(0.10) *hold. The energy of the unique solution of system* (0.6) *given by* (0.11) *decay estimate :*

$$E(t) \leq Ct^{-2/(p-1)} \ \ \forall\, t > 0, \ \ \textit{if} \ \ p > 1,$$

and

$$E(t) \leq C'E(0)e^{-wt} \ \ \forall\, t > 0, \ \ \textit{if} \ \ p = 1.$$

Here C *is a positive constant only depending on the initial energy* $E(0)$, *while* C' *and* w *are positive constants, independent of the initial data.*

Chapter 1

Preliminaries

In this chapter, we remind the most basic concepts, some of which we will use in the following chapters.

1 Convex functions

Let I be an interval of $\mathbb{R}$ and f a real function defined on I.

Definition 1.1 *We say that f is convex over I if and only if*

$$\forall\, u, v \in I, \forall\, t \in [0,1]; f(tu + (1-t)v) \leq t f(u) + (1-t) f(v).$$

Definition 1.2 *We say that f is concave over I if and only if $-f$ is convex.*

Proposition 1.3 *If f is twice differentiable on I, then f is convex on I if and only if $f'' \geq 0$.*

Theorem 1.4 ***(Jensen's inequality)*** *Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space such that $\mu(\Omega) = 1$ and g a μ-integrable function with values in I, f a convex function. Then*

$$f\Big(\int_{\Omega} g\, d\mu\Big) \leq \int_{\Omega} f o g\, d\mu.$$

2 Normed spaces, Hilbert spaces

Definition 2.1 *Let U be a nonempty set. We say that $(U, +, .)$ is a vector space over $\mathbb{R}$ if and only if*

1. *$(U, +)$ is a commutative group.*

2. $\forall\,\lambda \in \mathbb{R}, \forall\, u, v \in U\;:\;(u+v).\lambda = u.\lambda + v.\lambda$ *and* $\lambda.(u+v) = \lambda.u + \lambda.v.$
3. $\forall\,\lambda_1, \lambda_2 \in \mathbb{R}, \forall\, u \in U : (\lambda_1.\lambda_2).u = \lambda_1.(\lambda_2.u).$
4. $\forall\, u \in U : 1.u = u.$

Definition 2.2 *Let U and V be vector spaces over $\mathbb{R}$. A function $f : U \to V$ is said to be a linear map if and only if*

$$\forall\,\lambda_1, \lambda_2 \in \mathbb{R}, \forall\, u, v \in U : f(\lambda_1.u + \lambda_2.v) = \lambda_1.f(u) + \lambda_2.f(v).$$

Remark 2.3 *Let U and V be vector spaces over $\mathbb{R}$. We denote by $L(U, V)$ to the set of linear maps defined from U to V.*

Definition 2.4 *Let U be a vector space over $\mathbb{R}$. The vector space of linear maps from U to $\mathbb{R}$ is called the dual space, We denote it by E^*.*

Definition 2.5 *Let U be a vector space over $\mathbb{R}$, A normed vector space is a triple $(U, \mathbb{R}, \|.\|)$, where $\|.\|$ is a function called the norm, such that $\|.\|$ satisfies the following propreties*
1. $\forall\, u \in U : \|u\| \geq 0.$
2. $\forall\, u \in U : \|u\| = 0 \Leftrightarrow u = 0.$
3. $\forall\,\lambda \in \mathbb{R}, \forall\, u \in U : \|\lambda.u\| = |\lambda|\|u\|.$
4. $\forall\, u, v \in U : \|u + v\| \leq \|u\| + \|v\|.$

Definition 2.6 *Let U be a vector space over $\mathbb{R}$. U is a Banach space if and only if U is a normalized and complete space.*

Definition 2.7 *Let U be a normalized vector space of norm $\|.\|$ and V a subset of U. We say that V is bounded if and only if*

$$\exists\, M \in \mathbb{R}_+, \forall\, u \in V : \|u\| \leq M.$$

Definition 2.8 *Let U be a normalized vector space of norm $\|.\|$, (u_n) a sequence of U and u an element of U. We say that the sequence (u_n) converges to u ((u_n) strongly converges to u),and we write $\lim\limits_{n\to+\infty} u_n = u$ ($u_n \to u$), if and only if*

$$\forall\,\epsilon > 0, \exists\, n_0 \in \mathbb{N}, \forall\, n \in \mathbb{N} : n \geq n_0 \Rightarrow \|u_n - u\| < \epsilon,$$

i.e. $\lim\limits_{n\to+\infty} \|u_n - u\| = 0.$

Proposition 2.9 *Let U be a normalized vector space of norm $\|.\|$, (u_n) a sequence of U and u an element of U. We say that $u \in \overline{(u_n)}$ if and only if there exists a sub-sequence (u_{n_k}) of (u_n) such that $\lim\limits_{n\to+\infty} u_{n_k} = u$.*

Proposition 2.10 *Let U be a normalized vector space of norm $\|.\|$ and V a subset of U. V is relatively compact if and only if for any sequence (u_n) of V there exists an element u of U such that $u \in \overline{(u_n)}$.*

Definition 2.11 *Let U, V be two normalized vector spaces and $f \in L(U, V)$. We say that f is compact if and only if for any bounded set U' of U, then $f(U')$ is relatively compact.*

Definition 2.12 *Let U, V be two normalized vector spaces with norms $\|.\|_U$ and $\|.\|_V$ respectively and $f \in L(U, V)$. We say that f is continuous if and only if*

$$\exists\, M \geq 0, \forall\, u \in U : \|f(u)\|_V \leq M\|u\|_U.$$

Definition 2.13 *Let U, V be two normalized vector spaces with norms $\|.\|_U$ and $\|.\|_V$ respectively, and suppose that $U \subseteq V$. We say that U is compactly embedded in V, and we write $U \subset\subset V$, if and only if the injection $i : U \to V$ is compact and continuous.*

Definition 2.14 *Let U be a normalized vector space. The space U' is a set of the elements u of U^* such that u is continuous. And we define on U' the following norm*

$$\|u\|_{U'} := \sup_{\|x\|\neq 0} \frac{|u(x)|}{\|x\|},\ u \in U'.$$

Definition 2.15 ***(weak convergence).*** *Let U be a normalized vector space and $(u_n) \subset U, u \in U$. We say that (u_n) weakly converges to u in U, and we write $u_n \rightharpoonup u$ in U if and only if*

$$\forall\, f \in U' : \langle f, u_n\rangle \to \langle f, u\rangle,$$

$$i.e.\ \forall\, f \in U' : \lim_{n\to+\infty} |\langle f, u_n\rangle - \langle f, u\rangle| = 0.$$

Where $\langle f, u_n\rangle$ and $\langle f, u\rangle$ denote the action of f on u_n and u, i.e. $\langle f, u_n\rangle = f(u_n)$ and $\langle f, u\rangle = f(u)$.

Theorem 2.16 ***(*** [42]***page 120)*** *Let U be a normalized vector space and $(u_n) \subset U, u \in U$. If $u_n \to u$, then $u_n \rightharpoonup u$.*

Definition 2.17 ***(weak star convergence).*** *Let U be a normalized vector space and $(u_n) \subset U', u \in U'$. We say that (u_n) weakly star converges to u in U', and we write $u_n \overset{*}{\rightharpoonup} u$ in U', if and only if*

$$\forall\, x \in U : \langle u_n, x\rangle \;\to\; \langle u, x\rangle.$$

Theorem 2.18 ***(*** [42] ***page 125)*** *Let U be a normalized vector space and $(u_n) \subset U', u \in U'$. If $u_n \to u$, then $u_n \overset{*}{\rightharpoonup} u$.*

Definition 2.19 *Let U be a vector space over $\mathbb{R}$. A scalar product $(.\,,.)$ is a bilinear form on $U \times U$ with values in $\mathbb{R}$ (i.e., a map from $U \times U$ to $\mathbb{R}$ that is linear in both variables), such that*
1. $\forall\, u \in U - \{0\} : (u, u) > 0.$
2. $\forall\, u, v \in U : (u, v) = (v, u).$

Definition 2.20 *A Hilbert space is a vector space U equipped with a scalar product such that U is complete for the norm $\|.\|$, where the norm $\|.\|$ is defined as follows*

$$\forall\, u \in U : \|u\| = (u, u)^{\frac{1}{2}}.$$

Lemma 2.21 ***(Cauchy-Schwarz's inequality).*** *Let U be a Hilbert space supplied with the norm $\|.\|$, then*

$$\forall\, u, v \in U : (u, v) \leq \|u\|\|v\|.$$

3 L^p spaces

Let $1 \leq p \leq \infty$ and Ω be a nonempty set of $\mathbb{R}^n$, $n \in \mathbb{N}^*$.

Definition 3.1 *We define the space $L^p(\Omega)$ by*

$$L^p(\Omega) = \Big\{ u : \Omega \to \mathbb{R} : u \text{ is measurable and } \int_\Omega |u(x)|^p \, dx < +\infty \Big\} \text{ if } 1 \leq p < \infty,$$

and

$$L^\infty(\Omega) = \left\{ u : \Omega \to \mathbb{R} \;\middle|\; \begin{array}{l} u \text{ is measurable and there exists a constant } C \\ \text{such that } |u(x)| \leqslant C \text{ a.e in } \Omega \end{array} \right\}.$$

Proposition 3.2 *Let $u : \Omega \to \mathbb{R}$ be a measurable function. We define $\|u\|_{L^p(\Omega)}$ (or $\|u\|_p$) by*

$$\|u\|_p := \Big(\int_\Omega |u(x)|^p \, dx \Big)^{\frac{1}{p}} \text{ if } 1 \leq p < \infty,$$

and

$$\|u\|_\infty := \inf\{C\,;|u| \leqslant C \;\; a.e \; in \; \Omega\}.$$

Then for all $1 \leq p \leq \infty$, $\|.\|_p$ *is a norm on the space* $L^p(\Omega)$.

Remark 3.3 *In particularly,* $L^2(\Omega)$ *equipped with the scalar product*

$$(u, v) = \int_\Omega u(x)v(x)\,dx,$$

is a Hilbert space.

Notation 3.4 *We denote by* p' *the conjugate exponent,*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Theorem 3.5 ***(Hölder's inequality).*** *Assume that* $u \in L^p(\Omega)$ *and* $u \in L^{p'}(\Omega)$ *with* $1 \leq p \leq \infty$. *Then* $uv \in L^1(\Omega)$, *and*

$$\int_\Omega |u(x)v(x)|\,dx \leq \|u\|_p \|u\|_{p'}.$$

Proposition 3.6 *Let* $1 \leq p \leq \infty$ *and* $1 \leq q \leq \infty$. *If* $|\Omega| < +\infty$ *and* $p \leq q$ *then we have* $L^q(\Omega) \subset L^p(\Omega)$, *and*

$$\forall\, u \in L^q(\Omega) : \|u\|_p \leq |\Omega|^{\frac{1}{p}-\frac{1}{q}} \|u\|_q,$$

such that $|\Omega|$ *represents the measure of* Ω.

Definition 3.7 *The space* $L^1_{loc}(\Omega)$ *denotes the measurable functions* u *such that* $u \in L^1(K)$ *for any compact* K *included in* Ω.

Lemma 3.8 ***(Fatou's lemma).*** *Let* (u_n) *be a sequence of functions in* $L^1(\Omega)$ *that satisfy*
1. $\forall\, n \in \mathbb{N}\,, u_n(x) \geq 0$ *a.e. on* Ω.
2. $\sup_n \int_\Omega u_n(x)\,dx < +\infty$.
For almost all $x \in \Omega$ *we set* $u(x) = \lim_{n\to+\infty} \inf u_n(x) \leq +\infty$. *Then* $u \in L^1(\Omega)$, *and*

$$\int_\Omega u(x)\,dx \leq \lim_{n\to+\infty} \inf \int_\Omega u_n(x)\,dx.$$

Theorem 3.9 *Let* Ω *is a bounded open of* $\mathbb{R}^n$, $1 \leq p < \infty$ *and* $(u_n) \subset L^p(\Omega)$, $u \in L^p(\Omega)$. *Assume that* $u_n \longrightarrow u$ *strongly in* $L^p(\Omega)$. *Then there exists a subsequence* (u_{n_k}) *of* (u_n) *such* *that :*

1. $u_{n_k} \longrightarrow u$ *almost every where.*

2. $\exists\, g \in L^p(\Omega)$ *such that* $|u_{n_k}| \leq g$ *almost every where, for all* $k \in \mathbb{N}$.

Definition 3.10 *Assume that* Ω *is a bounded open of* $\mathbb{R}^n$, *and let* $\mathcal{F}$ *be a bounded part of* $L^1(\Omega)$. *We say that* $\mathcal{F}$ *is uniformly integrable over* Ω *if and only if*

$$\forall\, \epsilon > 0, \exists\, \delta > 0, \forall\, u \in \mathcal{F}, \forall\, A \subseteq \Omega : |A| < \delta \Rightarrow \int_A |u|\, dx < \epsilon.$$

Theorem 3.11 ***(Vitali's theorem).*** *Let* Ω *be an open bounded of* $\mathbb{R}^n$ *such that* $|\Omega| < +\infty$, *and* (u_n) *a sequence of functions uniformly integrable over* Ω. *If* $u_n \to u$ *pointwise a.e. on* Ω, *then* u *is integrable on* Ω, *and*

$$\lim_{n \to +\infty} \int_\Omega u_n(x)\, dx = \int_\Omega u(x)\, dx.$$

Lemma 3.12 ***(Gronwall's lemma)*** *Let* $T > 0$, $g \in L^1([0,T])$, $g \geq 0$ *a.e and* c_1, c_2 *are positives constants. Let* $\varphi \in L^1([0,T])$ $\varphi \geq 0$ *a.e such that* $g\varphi \in L^1([0,T])$ *and*

$$\varphi(t) \leq c_1 + c_2 \int_0^t g(s)\varphi(s)ds \quad a.e \ in \ [0,T].$$

Then, we have

$$\varphi(t) \leq c_1 exp\left[c_2 \int_0^t g(s)ds \right] \quad a.e \ in \ [0,T].$$

Theorem 3.13 *For all* $1 \leq p \leq \infty$, *the space* $L^p(\Omega)$ *supplied with the norm* $\|.\|_p$ *is a Banach space.*

Theorem 3.14

$$\text{For all } 1 \leq p < \infty \text{ we have } (L^p(\Omega))^* = L^{p'}(\Omega).$$

$$\text{For } p = \infty \text{ we have } (L^\infty(\Omega))^* \supset L^1(\Omega).$$

4 Sobolev spaces

Let $n \in \mathbb{N}^*$, $k, k^{'} \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$, $1 \leq p \leq \infty$ and Ω be an open of the normed vector space $\mathbb{R}^n$ supplied by the norm Euclidean.

Definition 4.1 *We denote by $C(\Omega)$ (respectively. $C^1(\Omega)$) the space of continuous functions (respectively. continuously differentiable) on Ω with real values, for $k \geq 2$, we set*

$$C^k(\Omega) = \Big\{u \in C^{k-1}(\Omega) : \frac{\partial u}{\partial x_i} \in C^{k-1}(\Omega),\ i = 1, ..., n\Big\},$$

it is the space of functions k times continuously differentiable on Ω with values in $\mathbb{R}$. We finally denote

$$C^\infty(\Omega) = \bigcap_{k \in \mathbb{N}} C^k(\Omega),$$

it is the space of functions indefinitely differentiable on Ω.

Notation 4.2 *We introduce the following notations*

$$|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n,\ |\alpha| \textit{ is called length of } \alpha.$$

$$\partial^\alpha = \Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1}\Big(\frac{\partial}{\partial x_2}\Big)^{\alpha_2} .....\Big(\frac{\partial}{\partial x_n}\Big)^{\alpha_n},\ \partial^\alpha \textit{ is called the mixed derivative operator of order } \alpha.$$

Definition 4.3 *Let $u \in C(\Omega)$. The support of u is the subset of $\mathbb{R}^n$ defined by*

$$supp\, u = \overline{\{x \in \Omega : u(x) \neq 0\}}.$$

Definition 4.4 *The space $\mathcal{D}(\Omega)$ or $C_0^\infty(\Omega)$ designates the set of elements u of $C^\infty(\Omega)$ such that the support of u is compact contained in Ω.*

Definition 4.5 *Let $u \in \mathcal{D}^{'}(\Omega)$. The distribution $\partial^\alpha u$ is called the α^{th}-weak partial derivative of the distribution u, which is defined by*

$$\langle \partial^\alpha u, \varphi\rangle = (-1)^{|\alpha|}\langle u, \partial^\alpha \varphi\rangle,\ \varphi \in \mathcal{D}(\Omega).$$

Definition 4.6 *Let $u, v \in L^1_{loc}(\Omega)$. We say that v is the α^{th}-weak partial derivative of u, written*

$$\partial^\alpha u = v,$$

provided

$$\forall\, \varphi \in \mathcal{D}(\Omega) : \int_{\Omega} u\, \partial^{\alpha} \varphi\, dx = (-1)^{|\alpha|} \int_{\Omega} v\, \varphi\, dx.$$

Definition 4.7 *The Sobolev space $W^{k,p}(\Omega)$ consists of all locally summable functions $u : \Omega \to \mathbb{R}$ such that for each multi-index α with $|\alpha| \leq k$, $\partial^{\alpha} u$ exists in the weak sense and belongs to $L^p(\Omega)$.*

Definition 4.8 *If $u \in W^{k,p}(\Omega)$, we define its norm by*

$$\|u\|_{W^{k,p}(\Omega)} := \sum_{|\alpha| \leq k} \|\partial^{\alpha} u\|_p.$$

Theorem 4.9 *For any $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, the Sobolev space $W^{k,p}(\Omega)$ is a Banach space.*

Definition 4.10 *We denote by $W_0^{k,p}(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $W^{k,p}(\Omega)$. We interpret $W_0^{k,p}(\Omega)$ as comprising those functions $u \in W^{k,p}(\Omega)$ such that*

$$\partial^{\alpha} u = 0 \text{ on } \partial\,\Omega \text{ for all } |\alpha| \leq k-1.$$

Remark 4.11 ***1.*** *If $p = 2$, we usually write $W^{k,2}(\Omega) = H^k(\Omega)$, and $W_0^{k,2}(\Omega) = H_0^k(\Omega)$.*
2. *$H^k(\Omega)$ is a Hilbert space, and we define the scalar product in $H^k(\Omega)$ by*

$$(u, v)_{H^k(\Omega)} = \sum_{|\alpha| \leq k} \int_{\Omega} \partial^{\alpha} u\, \partial^{\alpha} v\, dx,\ u, v \in H^k(\Omega).$$

3. *If $k \geq k'$, then we have $H^k(\Omega) \subset H^{k'}(\Omega)$ with continuous imbedding.*

Notation 4.12 *Let Ω be an open set of $\mathbb{R}^n$. We denote by $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$. The dual of $L^2(\Omega)$ is identified with $L^2(\Omega)$, but we do not identify $H_0^1(\Omega)$ with its dual. We have the inclusions*

$$H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega),$$

where these injections are continuous and dense.

Theorem 4.13 ***(Rellich-Kondrachov compactness theorem).*** *Let Ω be an open bounded of class $C^{0,1}$. Then we have*
1. $W^{1,p}(\Omega) \subset\subset L^q(\Omega)\ \forall q \in [1, p^*[$, *where* $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$, *if* $1 \leq p < n$.
2. $W^{1,p}(\Omega) \subset\subset L^q(\Omega)\ \forall q \in [1, +\infty[$, *if* $p = n$.
3. $W^{1,p}(\Omega) \subset\subset C(\overline{\Omega})$, *if* $p > n$.

Theorem 4.14 ***(integration by parts).*** *Let Ω be a bounded open of class $C^{0,1}$, u and v two functions of $H^1(\Omega)$. Then $uv \in W^{1,1}(\Omega)$, and we have*

$$\forall\, 1 \le i \le n : \frac{\partial(uv)}{\partial x_i} = v\frac{\partial u}{\partial x_i} + u\frac{\partial v}{\partial x_i} \quad \text{in the sense of distribution,}$$

$$\int_\Omega u(x)\frac{\partial v}{\partial x_i}(x)\,dx = -\int_\Omega v(x)\frac{\partial u}{\partial x_i}(x)\,dx + \int_{\partial\Omega} u(x)v(x)n_i(x)\,d\sigma,$$

where $n_i(x)$ is the n^{th} component of the unit vector of the outward normal $\overrightarrow{n(x)}$ at a point $x, x \in \partial\Omega$ and $d\sigma$ is the Lebesgue measure over the compact $\partial\Omega$.

Theorem 4.15 ***(Green's formula).*** *Let Ω be an open of class $C^{0,1}$ and $u \in H^2(\Omega)$, $v \in H^1(\Omega)$. Then*

$$\int_\Omega \triangle u(x)v(x)\,dx = -\int_\Omega \nabla u(x)\nabla v(x)\,dx + \int_{\partial\Omega} v(x)\frac{\partial u}{\partial n}(x)\,d\sigma,$$

where $\triangle u(x) = \displaystyle\sum_{i=1}^{n}\frac{\partial^2 u(x)}{\partial x_i^2}$ and $\nabla u(x) = \left(\frac{\partial u(x)}{\partial x_1}, \frac{\partial u(x)}{\partial x_2}, \ldots.., \frac{\partial u(x)}{\partial x_n}\right)$. If $v \in H^2(\Omega)$, then we have

$$\int_\Omega u(x)\triangle v(x)\,dx = \int_\Omega \triangle u(x)v(x)\,dx + \int_{\partial\Omega}\left\{u(x)\frac{\partial v}{\partial n}(x) - v(x)\frac{\partial u}{\partial n}(x)\right\}d\sigma.$$

5 $C^k(0,T;U)$ spaces, $L^p(0,T;U)$ spaces

Let $1 \le p \le \infty$ and U be a real Banach space supplied with the norm $\|.\|$.

Definition 5.1 *Let $u : [0,T] \to U$ be a function. We say that u is continuous on $[0,T]$, and we write $u \in C(0,T;U)$ if and only if*

$$\forall\, t_0 \in [0,T] : \lim_{t\to t_0}\|u(t) - u(t_0)\| = 0.$$

And we define the norm on $C(0,T;U)$ by

$$\|u\|_{C(0,T;U)} := \max_{0\le t\le T}\|u\|.$$

Definition 5.2 *Let $u : [0,T] \to U$ be a function. We say that u is differentiable on $[0,T]$ if and only if*

$$\forall\, t_0 \in [0,T] : \lim_{h\to 0}\frac{1}{h}\{u(t_0+h) - u(t_0)\} \text{ exists in } U.$$

And if the function $t \to \frac{\partial}{\partial t}u(t) \in U$ *is continuous, then we will say that* $u \in C^1(0,T;U)$. *More generally, for* $k \in \mathbb{N} - \{0,1\}$ *we can define*

$$C^k(0,T;U) = \{u : [0,T] \to U;\ \textit{suth that } u \in C^{k-1}(0,T;U) \textit{ and } \frac{\partial^k u}{\partial t^k} \in C(0,T;U)\}.$$

And we define the norm on $C^k(0,T;U)$ *by*

$$\|u\|_{C^k(0,T;U)} := \sum_{m \le k} \max_{0 \le t \le T} \Big\| \frac{\partial^m u}{\partial t^m} \Big\|.$$

Definition 5.3 *The space* $L^p(0,T;U)$ *consists of all measurable functions* $u : [0,T] \to U$ *with*

$$\|u\|_{L^p(0,T;U)} := \Big(\int_0^T \|u(t)\|^p dt \Big)^{\frac{1}{p}} < +\infty,$$

for $1 \le p < \infty$, *and*

$$\|u\|_{L^\infty(0,T;U)} := ess \sup_{0 \le t \le T} \|u(t)\| < +\infty.$$

Theorem 5.4 *The space* $L^p(0,T;U)$ *supplied with the norm* $\|.\|_{L^p(0,T;U)}$ *is a Banach space for any* $1 \le p \le \infty$.

Corollary 5.5 *If* U *is a Hilbert space supplied with the scalar product* $(,)$ *then* $L^2(0,T;U)$ *is a Hilbert space supplied with the scalar product*

$$((u,v)) = \int_0^T (u(t), v(t)) dt,\ u, v \in U.$$

Definition 5.6 *The vector space* $\mathcal{D}'(0,T;U)$ *(or* $\mathcal{L}(\mathcal{D}([0,T];U))$*) is defined as follows*

$$\mathcal{D}'(0,T;U) = \{u : \mathcal{D}[0,T] \to U,\ \textit{linear and continuous}\}.$$

Definition 5.7 *If* $u \in L^p(0,T;U)$, *it corresponds to a distribution denoted* u *on* $[0,T]$ *with values in* U, *by*

$$\langle u, \varphi \rangle = \int_0^T u(t)\varphi(t) dt,\ \varphi \in \mathcal{D}([0,T]).$$

Lemma 5.8 *If* $u \in L^p(0,T;U)$ *and* $\dfrac{\partial u}{\partial t} \in L^p(0,T;U)$, *then* $u \in C(0,T;U)$.

6 Existence methods

6.1 Faedo-Galerkin's approximations

We consider the Cauchy problem abstract's for a second order evolution equation in the separable Hilbert space with the inner product $\langle .,. \rangle$ and the associated norm $\|.\|$.

$$\begin{cases} u''(t) + A(t)u(t) = f(t), \quad t \in [0,T], \\ (x,0) = u_0(x), \ u'(x,0) = u_1(x), \end{cases} \tag{1.1}$$

where u and f are unknown and given function, respectively, mapping the closed interval $[0,T] \subset \mathbb{R}$ into a real separable Hilbert space H. $A(t)$ $(0 \leq t \leq T$) are linear bounded operators in H acting in the energy space $V \subset H$.
Assume that $\langle A(t)u(t), v(t) \rangle = a(t; u(t), v(t))$, for all $u, v \in V$; where $a(t;.,.)$ is a bilinear continuous in V. The problem (1.1) can be formulated as : found the solution $u(t)$ such that

$$\begin{cases} u \in C([0,T];V), u' \in C([0,T];H), \\ \langle u''(t), v \rangle + a(t; u(t), v) = \langle f, v \rangle \text{ in } D'(]0,T[), \\ u_0 \in V \ , \ u_1 \in H. \end{cases}$$

this problem can be resolved with the approximation process of Fadeo-Galerkin.

Let V_m a sub-space of V with the finite dimension d_m, and let $\{w_{jm}\}$ one basis of V_m such that .

1. $V_m \subset V(\dim V_m < \infty), \forall\, m \in \mathbb{N}$.
2. $V_m \to V$ such that, there exist a dense subspace ϑ in V and for all $\upsilon \in \vartheta$ we can get sequence $\{u_m\}_{m\in\mathbb{N}} \in V_m$ and $u_m \to u$ in V.
3. $V_m \subset V_{m+1}$ and $\overline{\cup_{m\in\mathbb{N}} V_m} = V$.

We define the solution u_m of the approximate problem

$$\begin{cases} u_m(t) = \sum_{j=1}^{d_m} g_j(t) w_{jm}, \\ u_m \in C([0,T];V_m), u'_m \in C([0,T];V_m) \quad , u_m \in L^2(0,T;V_m), \\ \langle u''_m(t), w_{jm} \rangle + a(t; u_m(t), w_{jm}) = \langle f, w_{jm} \rangle, \quad 1 \leq j \leq d_m, \\ u_m(0) = \sum_{j=1}^{d_m} \xi_j(t) w_{jm} \ , \ u'_m(0) = \sum_{j=1}^{d_m} \eta_j(t) w_{jm}, \end{cases} \tag{1.2}$$

where

$$\sum_{j=1}^{d_m} \xi_j(t) w_{jm} \longrightarrow u_0 \text{ in V as } m \longrightarrow \infty,$$

$$\sum_{j=1}^{d_m} \eta_j(t) w_{jm} \longrightarrow u_1 \text{ in V as } m \longrightarrow \infty.$$

By virtue of the theory of ordinary differential equations, the system (1.2) has unique local solution which is extend to a maximal interval $[0, t_m[$ by Zorn lemma since the non-linear terms have the suitable regularity. In the next step, we obtain a priori estimates for the solution, so that can be extended outside $[0, t_m[$ to obtain one solution defined for all $t > 0$. The method is based on the three steps :

- Choose certain basis of functions in an appropriate Sobolev space, and solve the approximate problems in any finite dimensional space spanned by finite basis functions. This often turns out to be an initial value problem for nonlinear ordinary differential equations. By the well-known local existence theorem for ordinary differential equations, local existence of solution to the approximate problem follows.

- Obtain the compactness estimates for the solution of the approximate problem. It also turns out that the solution to the approximate problem globally exists.

- Further use of the obtained compactness estimates allows one to choose a subsequence of solutions of the approximate problem obtained in the second step, converging to a solution of the original problem; uniqueness of solution for the original problem has to be proved separately, but the compactness estimates obtained in the second step are still very useful for this purpose.

By the Gronwall's lemma we deduce that the solution u_m of the approximate problem (1.2) converges to the solution u of the initial problem (1.1). The uniqueness proves that u is the solution.

7 Integral inequalities

We will recall some fundamental integral inequalities introduced by A. Haraux, V. Komornik and P. Martinez to estimate the decay rate of the energy.

7.1 A result of exponential decay

The estimation of the energy decay for some dissipative problems is based on the following lemma :

Lemma 7.1 *([33]) Let $E : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a non-increasing function and assume that there is a constant $A > 0$ such that*

$$\forall\, t \geq 0, \qquad \int_t^{+\infty} E(\tau)\, d\tau \leq \frac{1}{A}\, E(t). \tag{1.3}$$

Then we have

$$\forall\, t \geq 0, \qquad E(t) \leq E(0)\, e^{1-At}. \tag{1.4}$$

Proof. The inequality (1.4) is verified for $t \leq \frac{1}{A}$, this follows from the fact that E is a decreasing function. We prove that (1.4) is verified for $t \geq \frac{1}{A}$. Introduce the function

$$h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \qquad h(t) = \int_t^{+\infty} E(\tau)\, d\tau.$$

It is non-increasing and locally absolutely continuous. Differentiating and using (1.3) we find that

$$\forall\, t \geq 0, \quad h'(t) + Ah(t) \leq 0.$$

Let

$$T_0 = \sup\{t,\ h(t) > 0\}.$$

For every $t < T_0$, we have

$$\frac{h'(t)}{h(t)} \leq -A,$$

thus

$$h(0) \leq\ e^{-At} \leq \frac{1}{A}\, E(0)\, e^{-At}, \qquad \text{for} \quad 0 \leq t < T_0.$$

Since $h(t) = 0$ if $t \geq T_0$, this inequality holds in fact for every $t \in \mathbb{R}_+$. Let $\varepsilon > 0$. As E is positive and decreasing, we deduce that

$$\forall\, t \geq \varepsilon, \quad E(t) \leq \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} E(\tau)\, d\tau \leq \frac{1}{\varepsilon}\, h(t-\varepsilon) \leq \frac{1}{A\varepsilon}\, E(0)\, e^{\varepsilon t}\, e^{-At}.$$

Choosing $\varepsilon = \frac{1}{A}$, we obtain

$$\forall\, t \geq 0, \qquad E(t) \leq E(0)\, e^{1-At}.$$

The proof of the lemma 7.1 is now completed.

7.2 A result of polynomial decay

Lemma 7.2 *([33]) Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function and assume that there are two constants $q > 0$ and $A > 0$ such that*

$$\forall\, t \geq 0, \quad \int_{t}^{+\infty} E^{q+1}(\tau)\, d\tau \leq \frac{1}{A}\, E^{q}(0) E(t). \tag{1.5}$$

Then we have :

$$\forall\, t \geq 0, \quad E(t) \leq E(0) \Big[\, \frac{1+q}{1+A\,q\,t}\, \Big]^{1/q}. \tag{1.6}$$

Remark 7.3 It is clear that the lemma 7.1 is similar to the lemma 7.2 in the case of $q = 0$.

Proof. If $E(0) = 0$, then $E \equiv 0$ and there is nothing to prove. Otherwise, replacing the function E by the function $\dfrac{E}{E(0)}$ we may assume that $E(0) = 1$. Introduce the function

$$h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \qquad h(t) = \int_{t}^{+\infty} E(\tau)\, d\tau.$$

It is non-increasing and locally absolutely continuous. Differentiating and using (1.5) we find that

$$\forall\, t \geq 0, \quad -h' \geq (Ah)^{1+q},$$

where

$$T_0 = \sup\{t,\ h(t) > 0\}.$$

Integrating in $[0, t]$ we obtain that

$$\forall\, 0 \leq t < T_0, h(t)^{-q} - h(0)^{-\sigma} \geq \sigma\omega^{1+q} t,$$

hence

$$0 \le t < T_0, \quad h(t) \le (h^{-q}(0) + qA^{1+q}\, t)^{-1/q}. \tag{1.7}$$

Since $h(t) = 0$ if $t \ge T_0$, this inequality holds in fact for every $t \in \mathbb{R}_+$. Since

$$h(0) \le \frac{1}{A} E(0)^{1+q} = \frac{1}{A},$$

by (1.5), the right-hand side of (1.7) is less than or equal to :

$$(h^{-q}(0) + qA^{1+q}\, t)^{-1/q} \le \frac{1}{A}(1 + A\, q\, t)^{-1/q}.$$

From other hand, E being nonnegative and non-increasing, we deduce from the definition of h and the above estimate that :

$$\begin{aligned}
\forall\, s \ge 0,\ E\Big[\frac{1}{A} + (q+1)s\Big]^{q+1} &\le \frac{1}{\frac{1}{A} + q + 1} \int_s^{\frac{1}{A}+(q+1)s} E(\tau)^{q+1}\, d\tau \\
&\le \frac{A}{1 + Aqs} h(s) \\
&\le \frac{A}{1 + Aqs} \frac{1}{A} (1 + Aqs)^{-\frac{1}{q}},
\end{aligned}$$

hence

$$\forall\, S \ge 0, \quad E\Big[\frac{1}{A} + (q+1)S\Big] \le \frac{1}{(1 + A\, q\, S)^{1/q}}.$$

Choosing $t = \dfrac{1}{A} + (1+q)s$ then the inequality (1.6) follows. Note that letting $q \to 0$ in this lemma we obtain (1.6).

7.3 New integral inequalities of P. Martinez

The above inequalities are verified only if the energy function is integrable. We will try to resolve this problem by introducing some weighted integral inequalities, so we can estimate the decay rate of the energy when it is slow.

Lemma 7.4 *([33]) Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing C^1 function such that*

$$\phi(0) = 0 \quad \textit{and} \quad \phi(t) \to +\infty \quad \textit{when} \quad t \to +\infty.$$

Assume that there exist $q \geq 0$ and $A > 0$ such that

$$\int_S^{+\infty} E(t)^{q+1}\phi'(t)\,dt \leq \frac{1}{A}\,E(0)^q E(S), \quad 0 \leq S < +\infty.$$

Then we have

$$\text{if } q > 0, \quad \text{then } E(t) \leq E(0)\Big[\,\frac{1+q}{1+q\,A\,\phi(t)}\,\Big]^{\frac{1}{q}}, \ \ \forall\, t \geq 0,$$

$$\text{if } q = 0, \quad \text{then } E(t) \leq E(0)\,e^{1-A\,\phi(t)}, \ \ \forall\, t \geq 0.$$

Proof.

This lemma is a generalization of the lemma 7.1. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $f(x) := E(\phi^{-1}(x))$, (we notice that ϕ^{-1} has a meaning by the hypotheses assumed on ϕ). f is non-increasing,

$f(0) = E(0)$ and if we set $x := \phi(t)$ we obtain f is non-increasing, $f(0) = E(0)$ and if we set $x := \phi(t)$ we obtain

$$\begin{aligned}\int_{\phi(S)}^{\phi(T)} f(x)^{q+1}\,dx = \int_{\phi(S)}^{\phi(T)} E(\phi^{-1}(x))^{q+1}\,dx &= \int_S^T E(t)^{q+1}\phi'(t)\,dt \\ &\leq \frac{1}{A}\,E(0)^q E(S) \\ &= \frac{1}{A}\,E(0)^q f(\phi(S)), \quad 0 \leq S < T < +\infty.\end{aligned}$$

Setting $s := \phi(S)$ and letting $T \to +\infty$, we deduce that

$$\forall s \geq 0, \quad \int_s^{+\infty} f(x)^{q+1}\,dx \leq \frac{1}{A}E(0)^q f(s).$$

Thanks to the lemma 7.4, we deduce the desired results.

Chapter 2

Blow-up of result in a nonlinear wave equation with delay and source term

1 Introduction

In this chapter we are concerned with the following initial boundary value problem .

$$\begin{cases} |u^{'}|^{l}u^{''} - \Delta u - \Delta u^{''} + \mu_1 |u^{'}|^{m-2}u^{'} & \\ +\mu_2 |u^{'}(t-\tau)|^{m-2}u^{'}(t-\tau) = b|u|^{p-2}u, & \text{in } \Omega \times [0,+\infty[, \\ u(x,t) = 0, & \text{on } \partial\Omega \times [0,+\infty[, \\ u(x,0) = u_0(x), \quad u^{'}(x,0) = u_1(x), & \text{in } \Omega, \\ u^{'}(x,t-\tau) = f_0(x,t-\tau), & \text{in } \Omega \times [0,\tau], \end{cases} \tag{2.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega$, $l > 0$, μ_1, μ_2 and b are positive real numbers, $\tau > 0$ is a time varying delay, and the initial data (u_0, u_1, f_0) are in a suitable function space.
When $l = 0$ this type of problem without delay (i.e., $\mu_2 = 0$),

$$u^{''} - \Delta u + u^{'}|u^{'}|^{m-2} = u|u|^{p-2},$$

has been extensively studied by many mathematicians. It is well known that in the further absence of the damping mechanism $u_t|u_t|^{m-2}$, the source term $u|u|^{p-2}$ causes finite-time blow-up of solutions with negative initial energy (see **[5]**, **[14]**). In contrast, in the absence of the source term, the damping term assures global existence for arbitrary initial data (see **[11]**, **[16]**). The interaction between the damping and source terms was first considered by

Levine [**23**] and [**24**] for linear damping ($m = 2$). Levine showed that solutions with negative initial energy blow up in finite time. Georgiev and Todorova [**8**] extended Levine's result to nonlinear damping ($m > 2$). In their work, the authors introduced a new method and determined relations between m and p for which there is global existence and other relations between m and p for which there is finite-time blow-up. Specifically, blow up in finite time if $p > m$ and the initial energy is sufficiently negative. Messaoudi [**35**] extended the blow-up result of [**8**] to solutions with only negative initial energy. For related results, we refer the reader to Levine and Serrin [**26**], Levine and Ro Park [**25**] , Vitillaro [**41**], Yang [**43**] and Messaoudi and Said-Houari [**37**].
In the same direction, Cavalcanti et al [**7**] have also studied the following system

$$|u^{'}|^{l}u^{''} - \Delta u - \Delta u^{''} + \int_0^t g(t-\tau)\Delta(\tau)\,d\tau - \gamma\Delta u = 0, \quad \text{in } \Omega \times [0, +\infty[,\ l > 0.$$

They proved a global existence result for $\gamma \geq 0$ and an exponential decay for $\gamma > 0$. Related to our work, we also mention the work of Wei J. Liu [**30**] in which he used the multiplier techniques to establish an exponential decay result in the higher dimensional thermo-viscoelasticity. The same method was also used in [**31**] to prove, under appropriate conditions on the coupling parameters and relaxation function, a partial exact controllability result for a linear thermo-viscoelastic model. These last results generalize earlier ones [**29**] established for thermo-elasticity.
In [**40**] Shun-Tang Wu studied a wave equation with a delay term in internal feedback

$$|u^{'}|^{l}u^{''} - \Delta u - \Delta u^{''} + \int_0^t g(t-s)\Delta u(s)\,ds + \mu_1 u^{'} + \mu_2 u^{'}(x, t-\tau) = 0.$$

They proved the local existence result by the compactness method and established the decay result by suitable Lyapunov functionals. Hao and Wei [**12**] studied a quasilinear viscoelastic problem with strong damping and source term

$$|u^{'}|^{l}u^{''} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds - \Delta u^{'} = |u|^{p-2}u.$$

They obtained a blow up result for the solution with negative initial energy and some positive initial energy if $p > l + 2$, and they proved a global existence result for any initial data if $p \leq l + 2$.
In this chapter we use some techniques from [**13**] to show that blow-up for suitably chosen initial data, any classical solution blows up in finite time.

2 Preliminaries

In this section, we present some materials needed for our main results.

Lemma 2.1 ***(Sobolev-Poincaré's inequality).*** [**1**] *Let q be a number with*

$$2 \le q < +\infty \ \ (n = 1, 2) \ \ \text{or} \ \ 2 \le q \le 2n/(n-2) \ (n \ge 3),$$

then there exists a constant $C_s = C_s(\Omega, q)$ such that

$$\|u\|_q \le C_s \|\nabla u\|_2,$$

for any $u \in H_0^1(\Omega)$.

Lemma 2.2 *Suppose that*

$$2 \le p < +\infty \ \ (n = 1, 2) \ \ \text{or} \ \ 2 \le p \le 2n/(n-2) \ (n \ge 3),$$

holds. Then there exists a positive constant C depending on Ω only such that

$$\|u\|_p^s \le C(\|u\|_p^p + \|\nabla u\|_2^2), \tag{2.2}$$

for any $u \in H_0^1(\Omega)$ and $2 \le s \le p$.

Proof. If $\|u\|_p \le 1$, then $\|u\|_p^s \le \|u\|_p^2 \le C_s \|\nabla u\|_2^2$, by Sobolev embedding the theorems.
If $\|u\|_p > 1$ then $\|u\|_p^s \le \|u\|_p^p$.
Together with the two cases, we obtain (2.2).

Lemma 2.3 *Let a, b is arbitrary real, then we have*

$$(|a| + |b|)^\alpha \le C_\alpha(|a|^\alpha + |b|^\alpha), \tag{2.3}$$

where $C_\alpha = 1$ if $0 < \alpha \le 1$, and $C_\alpha = 2^{\alpha-1}$ if $\alpha > 1$.

Proof. We set $x = \left|\frac{a}{b}\right|$, that is to proof

$$f(x) = \frac{(1+x)^\alpha}{1+x^\alpha} \le C_\alpha.$$

By taking a derivative of f, we obtain

$$f'(x) = \frac{\alpha(1+x)^{\alpha-1}(1-x^{\alpha-1})}{(1+x^\alpha)^2}.$$

If $0 < \alpha \leq 1$, then we know f is monotone decreasing on $[0,1]$ and monotone increasing on $[1,+\infty[$, and

$$\lim_{x\to 0} f(x) = 1, \quad \lim_{x\to+\infty} f(x) = 1,$$

then, we have

$$f(x) \leq 1.$$

If $\alpha > 1$, then we know f is monotone increasing on $[0,1]$ and monotone decreasing on $[1,+\infty[$. So, we have

$$f(x) \leq f(1) = 2^{\alpha-1}.$$

The proof is completed.
Now we introduce, as in Nicaise and Pignotti **[39]**, the new variable

$$z(x,\rho,t) = u^{'}(x, t-\rho\tau), \text{ in } \Omega\times[0,1]\times[0,+\infty[.$$

Then, we have

$$\tau z^{'}(x,\rho,t) + \frac{\partial z}{\partial\rho}(x,\rho,t) = 0, \text{ in } \Omega\times[0,1]\times[0,+\infty[.$$

Therefore, the problem (2.1) is equivalent to

$$\begin{cases} |u^{'}(x,t)|^l u^{''}(x,t) - \Delta u(x,t) - \Delta u^{''}(x,t) + \mu_1|u^{'}(x,t)|^{m-2}u^{'}(x,t) & \\ +\mu_2|z(x,1,t)|^{m-2}z(x,1,t) = b|u(x,t)|^{p-2}u(x,t), & \text{in } \Omega\times[0,+\infty[, \\ \tau z^{'}(x,\rho,t) + \frac{\partial z}{\partial\rho}(x,\rho,t) = 0, & \text{in } \Omega\times[0,1]\times[0,+\infty[, \\ u(x,t) = 0, & \text{on } \partial\Omega\times[0,\infty[, \\ z(x,0,t) = u^{'}(x,t), & \text{in } \Omega\times[0,\infty[, \\ u(x,0) = u_0(x), \quad u^{'}(x,0) = u_1(x), & \text{in } \Omega, \\ z(x,\rho,0) = f_0(x,-\rho\tau), & \text{in } \Omega\times[0,1]. \end{cases} \tag{2.4}$$

Theorem 2.4 *Suppose that* $m > 1$, $p > 2$, *let* $(u_0,u_1) \in [H_0^1(\Omega)\cap H^2(\Omega)]\times H_0^1(\Omega)$ *and* $f_0 \in \mathcal{C}^1([-\tau,0]; L^m(\Omega\times[0,1]))$ *satisfys the compatibility condition*

$$f_0(.,0) = u_1.$$

Assume further that

$$\max\{p,m\} \le \frac{2(n-1)}{n-2} \quad \textit{and } l \le \frac{2}{n-2} \;\; \textit{if } n \ge 3. \tag{2.5}$$

Then the problem (2.4) *has a unique local solution*

$$u \in \mathcal{C}([0,T); H_0^1(\Omega) \cap H^2(\Omega)),$$

$$u^{'} \in \mathcal{C}([0,T); H_0^1(\Omega)),$$
$$u^{''} \in \mathcal{C}([0,T); L^2(\Omega)),$$
$$z \in \mathcal{C}^1([0,T); L^2([0,1]\times\Omega)),$$

for some $T > 0$.

We define the energy associated to the solution of system (2.4) by

$$E(t) = \frac{1}{l+2}\|u^{'}\|_{l+2}^{l+2} + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla u^{'}\|_2^2 - \frac{b}{p}\|u\|_p^p + \xi\int_\Omega\int_0^1 |z(x,\rho,t)|^m \,d\rho\,dx, \tag{2.6}$$

where ξ is a positive constant such that

$$\tau\frac{\mu_2}{m}(m-1) < \xi < \tau(\mu_1 - \frac{\mu_2}{m}), \tag{2.7}$$

and $\mu_2 < m\mu_1$.

Lemma 2.5 *Let* (u,z) *be a solution of the problem* (2.4)*. Then there exists a constant* $C > 0$ *such that*

$$E'(t) \le -C\Big[\int_\Omega |z(x,1,t)|^m\,dx + \|u^{'}(x,t)\|_m^m\Big] \le 0.$$

Proof. Multiplying the first equation in (2.4) by $u^{'}$ and integrating over Ω, using integration by parts, we get

$$\frac{d}{dt}\Big[\frac{1}{l+2}\|u^{'}\|_{l+2}^{l+2} + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla u^{'}\|_2^2 - \frac{b}{p}\|u\|_p^p\Big] + \mu_1\|u^{'}\|_m^m$$

$$+\mu_2\int_\Omega |z(x,1,t)|^{m-2}z(x,1,t)u^{'}(x,t)\,dx = 0. \tag{2.8}$$

We multiply the second equation in (2.4) by $\xi|z(x,\rho,t)|^{m-2}z(x,\rho,t)$ and integrate the result over $\Omega\times[0,1]$, to obtain

$$\begin{aligned}\xi\int_\Omega\int_0^1 z'(x,\rho,t)|z(x,\rho,t)|^{m-2}z(x,\rho,t)\,d\rho\,dx &= -\frac{\xi}{\tau}\int_\Omega\int_0^1\frac{\partial z(x,\rho,t)}{\partial\rho}|z(x,\rho,t)|^{m-2}z(x,\rho,t)\,d\rho\,dx\\ &= -\frac{\xi}{\tau}\int_\Omega\int_0^1\frac{\partial z(x,\rho,t)}{\partial\rho}\ |z(x,\rho,t)|^{m-2}z(x,\rho,t)\,d\rho\,dx\\ &= -\frac{\xi}{\tau m}\int_\Omega\int_0^1\frac{\partial}{\partial\rho}\ |z(x,\rho,t)|^{m}\,d\rho\,dx\\ &= -\frac{\xi}{\tau m}\int_\Omega(|z(x,1,t)|^m-|z(x,0,t)|^m)\,dx.\end{aligned}$$

Hence

$$\xi\frac{d}{dt}\int_\Omega\int_0^1|z(x,\rho,t)|^m\,d\rho\,dx=-\frac{\xi}{\tau}\int_\Omega|z(x,1,t)|^m\,dx+\frac{\xi}{\tau}\int_\Omega|u'|^m\,dx. \tag{2.9}$$

Combining (2.8) and (2.9), we obtain

$$\begin{aligned}E'(t)=-\frac{\xi}{\tau}\int_\Omega|z(x,1,t)|^m\,dx-(\mu_1-\frac{\xi}{\tau})\|u'\|_m^m\\ -\mu_2\int_\Omega|z(x,1,t)|^{m-2}z(x,1,t)u'(x,t)\,dx,\end{aligned} \tag{2.10}$$

and using Young's inequality, we have

$$-\mu_2\int_\Omega|z(x,1,t)|^{m-2}z(x,1,t)u'\,dx\le\mu_2\Big[\frac{1}{m}\delta^m\|u'\|_m^m+\frac{m-1}{m}\frac{1}{\delta^{\frac{m}{m-1}}}\int_\Omega|z(x,1,t)|^m\,dx\Big].$$

Thus, by choosing $\delta^{-\frac{m}{m-1}}=\dfrac{m\epsilon}{m-1}$, then

$$-\mu_2\int_\Omega|z(x,1,t)|^{m-2}z(x,1,t)u'\,dx\le\mu_2\Big[\epsilon\int_\Omega|z(x,1,t)|^m\,dx\ +\frac{1}{m}(\frac{m\epsilon}{m-1})^{1-m}\|u'\|_m^m\Big],$$

with $\epsilon=\dfrac{m-1}{m}$, we have

$$-\mu_2\int_\Omega|z(x,1,t)|^{m-2}z(x,1,t)u'\,dx\le\frac{\mu_2}{m}\Big[(m-1)\int_\Omega|z(x,1,t)|^m\,dx\ +\|u'\|_m^m\Big].$$

Hence, we get from (2.10)

$$\begin{aligned}E'(t)&\le-\Big[\frac{\xi}{\tau}-\frac{\mu_2}{m}(m-1)\Big]\int_\Omega|z(x,1,t)|^m\,dx-\Big[\mu_1-\frac{\xi}{\tau}-\frac{\mu_2}{m}\Big]\|u'\|_m^m\\ &\le-C\Big[\int_\Omega|z(x,1,t)|^m\,dx+\|u'\|_m^m\Big],\end{aligned}$$

where

$$C = \min\Big\{\frac{\xi}{\tau} - \frac{\mu_2}{m}(m-1), \mu_1 - \frac{\xi}{\tau} - \frac{\mu_2}{m}\Big\},$$

which is positive by (2.7). We set

$$H(t) = -E(t). \tag{2.11}$$

Corollary 2.6 *Let the assumptions of the lemma 2.2 hold. Then we have*

$$\|u\|_p^s \leq C\Big[- H(t) - \|u'\|_{l+2}^{l+2} - \|\nabla u'\|_2^2 + \|u\|_p^p - \xi \int_\Omega \int_0^1 |z(x,\rho,t)|^m \, d\rho \, dx\Big] \quad \forall \, t \in [0,T),$$

for any $u(.,t) \in H_0^1(\Omega)$ and $2 \leq s \leq p$.

3 Blow-up

In this section we state and prove our main result.

Theorem 3.1 *Suppose that $m > 1$, $p > \max\{l+2, m\}$ satisfying (2.5), let $(u_0, u_1) \in [H_0^1(\Omega) \cap H^2(\Omega)] \times H_0^1(\Omega)$ and $f_0 \in \mathcal{C}^1([-\tau, 0]; L^m(\Omega \times [0,1]))$. Assume further that*

$$E(0) = \frac{1}{l+2}\|u_1\|_{l+2}^{l+2} + \frac{1}{2}\|\nabla u_0\|_2^2 + \frac{1}{2}\|\nabla u_1\|_2^2 - \frac{b}{p}\|u_0\|_p^p + \xi \int_\Omega \int_0^1 |f_0(x, -\rho\tau)|^m \, d\rho \, dx < 0.$$

Then the solution of (2.4) blows up in finite time, i.e. there exists $T_0 < +\infty$ such that

$$\lim_{t \to T_0^-} (\|u'\|_{l+2}^{l+2} + \|\nabla u\|_2^2 + \|\nabla u'\|_2^2 + \|u\|_p^p) = +\infty.$$

Proof . Assume that there exists some positive constant C such that for $t > 0$ the solution u of (2.4) satisfies

$$\|u'\|_{l+2}^{l+2} + \|\nabla u\|_2^2 + \|\nabla u'\|_2^2 + \|u\|_p^p \leq C. \tag{2.12}$$

Hence,

$$0 < H(0) < H(t) < \frac{b}{p}\|u\|_p^p. \tag{2.13}$$

We then define the function

$$G(t) = \frac{1}{l+1}\int_\Omega |u'|^l u' u \, dx + \int_\Omega \nabla u' \nabla u \, dx,$$

we have

$$G'(t) = \int_\Omega u|u^{'}|^l u''\,dx + \frac{1}{l+1}\int_\Omega |u'|^{l+2}\,dx + \int_\Omega \nabla u''\nabla u\,dx + \int_\Omega |\nabla u'|^2\,dx$$
$$= \int_\Omega u(|u'|^l u'' - \Delta u'')\,dx + \frac{1}{l+1}\int_\Omega |u'|^{l+2}\,dx + \int_\Omega |\nabla u'|^2\,dx.$$

By using the first equation of (2.4), we arrive at

$$\begin{aligned} G'(t) = &\frac{1}{l+1}\|u'\|_{l+2}^{l+2} + b\|u\|_p^p + \|\nabla u'\|_2^2 - \|\nabla u\|_2^2 \\ &-\mu_1\int_\Omega |u'|^{m-2}u'u\,dx - \mu_2\int_\Omega |z(x,1,t)|^{m-2}z(x,1,t)u\,dx. \end{aligned} \tag{2.14}$$

By Young's inequality, we obtain

$$\mu_1\int_\Omega |u'|^{m-2}u'u\,dx \le \mu_1\Big[\frac{\delta^m}{m}\|u\|_m^m + \frac{(m-1)\delta^{\frac{-m}{m-1}}}{m}\|u'\|_m^m\Big], \tag{2.15}$$

similarly, we have

$$\mu_2\int_\Omega |z(x,1,t)|^{m-2}z(x,1,t)u\,dx \le \mu_2\Big[\frac{\delta^m}{m}\|u\|_m^m + \frac{(m-1)\delta^{\frac{-m}{m-1}}}{m}\|z(x,1,t)\|_m^m\Big]. \tag{2.16}$$

We then define

$$L(t) = H^{1-\alpha}(t) + \varepsilon G(t), \tag{2.17}$$

for ε small to be chosen later and

$$0 < \alpha < \min\Big\{\frac{1}{l+2} - \frac{1}{p}, \frac{p-m}{p(m-1)}\Big\}.$$

By taking a derivative of (2.17) and using (2.14)-(2.16) we obtain

$$\begin{aligned} L'(t) &= (1-\alpha)H'(t)H^{-\alpha}(t) + \varepsilon G'(t) \\ &\ge (1-\alpha)H'(t)H^{-\alpha}(t) \\ &\quad - \varepsilon(\mu_1+\mu_2)\frac{(m-1)\delta^{\frac{-m}{m-1}}}{m}(\|u^{'}\|_m^m + \|z(x,1,t)\|_m^m) \\ &\quad + \frac{\varepsilon}{l+1}\|u^{'}\|_{l+2}^{l+2} + \varepsilon\|\nabla u^{'}\|_2^2 + b\varepsilon\|u\|_p^p - \varepsilon\|\nabla u\|_2^2 - \varepsilon(\mu_1+\mu_2)\frac{\delta^m}{m}\|u\|_m^m, \end{aligned}$$

$$L'(t) \geq \Big[\,(1-\alpha)H^{-\alpha}(t) - \,\varepsilon(\mu_1+\mu_2)\frac{(m-1)\delta^{\frac{-m}{m-1}}}{mC}\,\Big]H'(t) + \frac{\varepsilon}{l+1}\|u^{'}\|_{l+2}^{l+2}$$
$$+\,\varepsilon\|\nabla u^{'}\|_2^2 + b\varepsilon\|u\|_p^p - \varepsilon\|\nabla u\|_2^2 - \varepsilon(\mu_1+\mu_2)\frac{\delta^m}{m}\|u\|_m^m. \tag{2.18}$$

Of course (2.18) remains valid even if δ is time dependent. Therefore by taking δ so that

$$\delta^{\frac{-m}{m-1}} = kH^{-\alpha}(t),$$

for large k to be specified later, and substituting in (2.18) we arrive at

$$\begin{aligned} L'(t) \geq \Big[\,(1-\alpha) - \varepsilon(\mu_1+\mu_2)\frac{m-1}{mC}k\,\Big]H^{-\alpha}(t)H'(t) \\ + \frac{\varepsilon}{l+1}\|u^{'}\|_{l+2}^{l+2} + \,\varepsilon\|\nabla u^{'}\|_2^2 + b\varepsilon\|u\|_p^p \\ - \varepsilon(\mu_1+\mu_2)\frac{k^{1-m}}{m}H^{\alpha(m-1)}(t)\|u\|_m^m \\ - \varepsilon\|\nabla u\|_2^2. \end{aligned} \tag{2.19}$$

By exploiting (2.13) and the inequality $\|u\|_m^m \leq c\|u\|_p^m$, we obtain

$$H^{\alpha(m-1)}(t)\|u\|_m^m \leq c\Big(\frac{b}{p}\Big)^{\alpha(m-1)}\|u\|_p^{\alpha p(m-1)+m}, \tag{2.20}$$

inserting (2.20) in (2.19), using (2.6) and (2.11), we get, for $0<\beta<1$,

$$\begin{aligned} L'(t) &\geq \Big[\,(1-\alpha) - \varepsilon(\mu_1+\mu_2)\frac{m-1}{mC}k\,\Big]H^{-\alpha}(t)H'(t) + \frac{\varepsilon}{l+1}\|u^{'}\|_{l+2}^{l+2} + \varepsilon\|\nabla u^{'}\|_2^2 \\ &\quad + b\beta\varepsilon\|u\|_p^p - \varepsilon\|\nabla u\|_2^2 - \varepsilon c\Big(\frac{b}{p}\Big)^{\alpha(m-1)}(\mu_1+\mu_2)\frac{k^{1-m}}{m}\|u\|_p^{\alpha p(m-1)+m} \\ &\quad + \varepsilon(1-\beta)p\Big[\,H(t) + \frac{1}{l+2}\|u^{'}\|_{l+2}^{l+2} + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla u^{'}\|_2^2 + \xi\int_\Omega\int_0^1 |z(x,\rho,t)|^m\,d\rho\,dx\,\Big] \\ &\geq \Big[\,(1-\alpha) - \varepsilon(\mu_1+\mu_2)\frac{m-1}{mC}k\,\Big]H^{-\alpha}(t)H'(t) + \varepsilon\Big[\,\frac{1}{l+1} + \frac{p(1-\beta)}{l+2}\,\Big]\|u^{'}\|_{l+2}^{l+2} \\ &\quad + \varepsilon\Big[\,\frac{p(1-\beta)}{2} - 1\Big]\|\nabla u\|_2^2 + \varepsilon\Big[\,\frac{p(1-\beta)}{2} + 1\Big]\|\nabla u^{'}\|_2^2 + b\beta\varepsilon\|u\|_p^p + \varepsilon(1-\beta)pH(t) \\ &\quad + \varepsilon(1-\beta)p\xi\int_\Omega\int_0^1 |z(x,\rho,t)|^m\,d\rho\,dx \;- \varepsilon c\Big(\frac{b}{p}\Big)^{\alpha(m-1)}(\mu_1+\mu_2)\frac{k^{1-m}}{m}\|u\|_p^{\alpha p(m-1)+m}. \end{aligned}$$

Then we use the corollary 2.6, for $s = \alpha p(m-1) + m \leq p$, to deduce that

$$\begin{aligned} L'(t) \geq & \Big[(1-\alpha) - \varepsilon(\mu_1+\mu_2)\frac{m-1}{mC}k \Big] H^{-\alpha}(t)H'(t) + \varepsilon\Big[\frac{1}{l+1} + \frac{p(1-\beta)}{l+2} \Big] \|u^{'}\|_{l+2}^{l+2} \\ & + \varepsilon\Big[\frac{p(1-\beta)}{2} - 1 \Big] \|\nabla u\|_2^2 + \varepsilon\Big[\frac{p(1-\beta)}{2} + 1 \Big] \|\nabla u^{'}\|_2^2 + b\beta\varepsilon \|u\|_p^p \\ & + p\varepsilon(1-\beta)H(t) + p\varepsilon(1-\beta)\xi \int_\Omega \int_0^1 |z(x,\rho,t)|^m \, d\rho \, dx \\ & - \varepsilon k^{1-m} C_1 \Big[-H(t) - \|u^{'}\|_{l+2}^{l+2} - \|\nabla u^{'}\|_2^2 + \|u\|_p^p - \xi \int_\Omega \int_0^1 |z(x,\rho,t)|^m \, d\rho \, dx \Big], \end{aligned}$$

consequently, we obtain

$$\begin{aligned} L'(t) \geq & \Big[(1-\alpha) - \varepsilon(\mu_1+\mu_2)\frac{m-1}{mC}k \Big] H^{-\alpha}(t)H'(t) \\ & + \varepsilon\Big[\frac{1}{l+1} + \frac{p(1-\beta)}{l+2} + C_1 k^{1-m} \Big] \|u^{'}\|_{l+2}^{l+2} + \varepsilon\Big[\frac{p(1-\beta)}{2} + 1 + C_1 k^{1-m} \Big] \|\nabla u^{'}\|_2^2 \\ & + \varepsilon\Big[\frac{p(1-\beta)}{2} - 1 \Big] \|\nabla u\|_2^2 + \varepsilon\Big[b\beta - C_1 k^{1-m} \Big] \|u\|_p^p \\ & + \varepsilon\xi\Big[p(1-\beta) + C_1 k^{1-m} \Big] \int_\Omega \int_0^1 |z(x,\rho,t)|^m \, d\rho \, dx \\ & + \varepsilon\Big[p(1-\beta) + C_1 k^{1-m} \Big] H(t), \end{aligned} \tag{2.21}$$

where

$$C_1 = c\Big(\frac{b}{p} \Big)^{\alpha(m-1)} \frac{\mu_1+\mu_2}{m},$$

$$\frac{p(1-\beta)}{2} - 1 > 0,$$

and we choose k so large that

$$b\beta - C_1 k^{1-m} > 0.$$

Finally, we pick ε so small so that

$$(1-\alpha) - \varepsilon(\mu_1+\mu_2)\frac{m-1}{mC}k > 0,$$

and

$$L(0) = H^{1-\alpha}(0) + \frac{\varepsilon}{l+1} \int_\Omega |u_1|^l u_1 u_0 \, dx + \varepsilon \int_\Omega \nabla u_1 \nabla u_0 \, dx > 0.$$

Therefore (2.21) takes the form

$$L'(t) \geq \lambda \Big[\|u^{'}\|_{l+2}^{l+2} + \|\nabla u\|_2^2 + \|\nabla u^{'}\|_2^2 + \|u\|_p^p + H(t) + \xi \int_\Omega \int_0^1 |z(x,\rho,t)|^m \, d\rho \, dx \Big]. \quad (2.22)$$

Consequently, we have

$$L(t) \geq L(0), \;\; t \geq 0.$$

We now estimate the term $\int_\Omega |u^{'}|^l u^{'} u \, dx$ as follows

$$\Big| \int_\Omega |u^{'}|^l u^{'} u \, dx \Big| \leq \|u^{'}\|_{l+2}^{l+1} \|u\|_{l+2}$$

$$\leq C_{p,l} \|u^{'}\|_{l+2}^{l+1} \|u\|_p.$$

Using Young's inequality and (2.3), then yields

$$\Big| \int_\Omega |u^{'}|^l u^{'} u \, dx \Big|^{\frac{1}{1-\alpha}} \leq C \Big[\|u^{'}\|_{l+2}^{\frac{(l+1)\beta_1}{1-\alpha}} + \|u\|_p{}^{\frac{\beta_2}{1-\alpha}} \Big], \quad (2.23)$$

for $\dfrac{1}{\beta_1} + \dfrac{1}{\beta_2} = 1$, we take $\beta_1 = \dfrac{(l+2)(1-\alpha)}{l+1}$ which gives $\dfrac{\beta_2}{1-\alpha} = \dfrac{l+2}{1-(l+2)\alpha}$. Therefore (2.23) becomes

$$\Big| \int_\Omega |u^{'}|^l u^{'} u \, dx \Big|^{\frac{1}{1-\alpha}} \leq C \Big[\|u^{'}\|_{l+2}^{l+2} + \|u\|_p^{\frac{l+2}{1-(l+2)\alpha}} \Big]. \quad (2.24)$$

Using Cauchy-Schwarz inequality, we obtain

$$\Big| \int_\Omega \nabla u^{'} \nabla u \, dx \Big| \leq \|\nabla u\|_2 \|\nabla u^{'}\|_2.$$

Similarly, by using Young's inequality, with the conjugate exponents $2(1-\alpha)$ and $\dfrac{2(1-\alpha)}{1-2\alpha}$, we get

$$\Big| \int_\Omega \nabla u^{'} \nabla u \, dx \Big|^{\frac{1}{1-\alpha}} \leq \|\nabla u^{'}\|_2^2 + \|\nabla u\|_2^{\frac{2}{1-2\alpha}}. \quad (2.25)$$

From (2.12) and (2.13), we have

$$\|\nabla u\|_2^{\frac{2}{1-2\alpha}} \leq C^{\frac{1}{1-2\alpha}} \leq \frac{C^{\frac{1}{1-2\alpha}}}{H(0)} H(t). \quad (2.26)$$

Using (2.24)-(2.26) and the lemma 2.2, for $s = \dfrac{l+2}{1-(l+2)\alpha} \leq p$ gives

$$\Big|\int_\Omega |u^{'}|^l u^{'} u\, dx\Big|^{\frac{1}{1-\alpha}} + \Big|\int_\Omega \nabla u^{'} \nabla u\, dx\Big|^{\frac{1}{1-\alpha}} \leq C\Big[H(t) + \|u^{'}\|_{l+2}^{l+2} + \|\nabla u^{'}\|_2^2 + \|\nabla u\|_2^2 + \|u\|_p^p + \xi \int_\Omega \int_0^1 |z(x,\rho,t)|^m \, d\rho\, dx \Big].$$

Therefore, we have

$$Ł^{\frac{1}{1-\alpha}}(t) = \Big[H^{1-\alpha}(t) + \varepsilon G(t) \Big]^{\frac{1}{1-\alpha}} \leq C\Big[H(t) + \|u^{'}\|_{l+2}^{l+2} + \|\nabla u^{'}\|_2^2 + \|\nabla u\|_2^2 + \|u\|_p^p + \xi \int_\Omega \int_0^1 |z(x,\rho,t)|^m \, d\rho\, dx \Big], \quad t > 0. \tag{2.27}$$

Combining (2.22) and (2.27), we arrive at

$$L'(t) \geq \Lambda L^{\frac{1}{1-\alpha}}(t), \quad t > 0, \tag{2.28}$$

where Λ is a positive constant depending only on λ and C.
A simple integration of (2.28) over $[0, t]$ yields

$$L^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{L^{-\frac{\alpha}{1-\alpha}}(0) - \Lambda\alpha t/(1-\alpha)}, \quad t > 0.$$

Therefore, $L(t)$ blows up in time

$$T_0 \leq \frac{1-\alpha}{\Lambda\alpha L^{\frac{\alpha}{1-\alpha}}(0)}.$$

Furthermore, we have

$$\lim_{t\to T_0^-} (\|u^{'}\|_{l+2}^{l+2} + \|\nabla u\|_2^2 + \|\nabla u^{'}\|_2^2 + \|u\|_p^p) = +\infty.$$

This leads to a contradiction with (2.12). Thus, the solution of the problem (2.4) blows up in finite time. This completes the proof.

Chapter 3

Well-posedness and general energy decay of solutions for a Petrovsky equation with a nonlinear strong dissipation

1 Introduction

In this chapter, we consider the initial-boundary value problem for the nonlinear Petrovsky equation

$$\begin{cases} u'' + \Delta^2 u - g(\Delta u') = 0, & \text{in } \Omega \times [0, +\infty[, \\ u(x,t) = \Delta u(x,t) = 0, & \text{on } \partial\Omega \times [0, +\infty[, \\ u(x,0) = u_0(x), \;\; u'(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{3.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $\partial\Omega$ is a smooth boundary, (u_0, u_1) are the initial data in a suitable function space and g is real function satisfying some conditions to be specied later. In [**9**], Guesmia considered the following problem

$$\begin{cases} u'' + \Delta^2 u + q(x)u + g(u') = 0, & \text{in } \Omega \times [0, +\infty[, \\ u(x,t) = \partial_\nu u(x,t) = 0, & \text{on } \partial\Omega \times [0, +\infty[, \\ u(x,0) = u_0(x), \;\; u'(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{3.2}$$

where g is continuous, increasing, satisfying $g(0) = 0$ and $q : \Omega \longrightarrow \mathbb{R}_+$ is a bounded under suitable growth conditions on g, decay results for weak, as well as strong, solutions. Precisely, he showed that the solution decays exponentially if g behaves like a linear function, whereas

the decay is of a polynomial order otherwise. Also the system composed of the equation (3.2), with $u'|u'|^{m-2} - u|u|^{p-2}$ in the place of $q(x)u(x,t) + g(u'(x,t))$ has been treated by Messaoudi [**36**], he established an existence result and showed that the solution continues to exist globally if $m \geq p$, however, it blows up in finite time if $m < p$. Moreover, Komornik [**15**] treated the problem (3.1) for g having a polynomial growth near the origin, used semigroup method to prove the existence and uniqueness of solutions and established energy decay results depending on g.
In this chapter, we prove the global existence of the weak solutions of the problem (3.1) by using the compactness method (see Lions [**27**]). We use some technique from [**38**] to establish an explicit and general decay result, depending on g. The proof is based on the multiplier method and makes use of some properties of convex functions, the general Young inequality and Jensen's inequality. These convexity arguments were introduced and developed by Lasiecka and co-workers ([**21**],[**22**]) and used, with appropriate modifications, by Liu and Zuazua [**32**], Alabau-Boussouira [**2**] and others.

2 Notation and preliminaries

We begin by introducing some notation that will be used throughout this work.
Let us introduce three real Hilbert spaces $\mathcal{H}$, V and W by setting

$$\mathcal{H} = H_0^1(\Omega), \quad \|u\|_{\mathcal{H}}^2 = \int_\Omega |\nabla u|^2 dx,$$

and

$$V = \{u \in H^3(\Omega) |\, u = \Delta u = 0 \text{ on } \partial\Omega\}, \quad \|u\|_V^2 = \int_\Omega |\nabla \Delta u|^2 dx,$$

$$W = \{u \in H^5(\Omega) |\, u = \Delta u = \Delta^2 u = 0 \text{ on } \partial\Omega\}, \quad \|u\|_W^2 = \int_\Omega |\nabla \Delta^2 u|^2\, dx.$$

We have

$$W \subset V \subset \mathcal{H} \subset V' \subset W',$$

with dense and compact imbedings.
If $u \in L^2(\Omega)$, we denote by $\|u\|_{L^2(\Omega)} = \|u\|$.
We impose the following assumptions on g
$g : \mathbb{R} \longrightarrow \mathbb{R}$ is a non-decreasing continuous function such that there exist constants ε, c_1, c_2, $\tau > 0$ and a convex increasing function $G : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+^*)$ satisfying

G linear on $[0,\varepsilon]$ or $G'(0)=0$ and $G''>0$ on $[0,\varepsilon]$, such that

$$c_1|s| \leq |g(s)| \leq c_2|s| \quad \text{if} \quad |s|>\varepsilon, \tag{3.3}$$

$$|s|^2+|g(s)|^2 \leq G^{-1}(sg(s)) \quad \text{if} \quad |s|\leq\varepsilon, \tag{3.4}$$

$$g'(s) \leq \tau. \tag{3.5}$$

Remark 2.1 *Let us denote by ϕ^* the conjugate function of the differentiable convex function ϕ, i.e.,*

$$\phi^*(s)=\sup_{t\in\mathbb{R}_+}(st-\phi(t)).$$

Then ϕ^* is the Legendre transform of ϕ, which is given by (see **[3]** *p. 61 − 62)*

$$\phi^*(s)=s(\phi^{'})^{-1}(s)-\phi((\phi^{'})^{-1}(s)) \;\; \textit{if} \;\; s\in]0,\phi^{'}(r)],$$

and ϕ^* satisfies the generalized Young inequality

$$ST\leq\phi^*(S)+\phi(T) \;\; \textit{if} \;\; S\in]0,\phi^{'}(r)],\, T\in]0,r]. \tag{3.6}$$

Lemma 2.2 *For all $u\in H_0^1(\Omega)\cap H^2(\Omega)$, we have*

$$\|\nabla u\| \leqslant c\|\Delta u\|_{H^{-1}(\Omega)} \leqslant c\|\Delta u\|. \tag{3.7}$$

3 Well-posedeness and regularity

Theorem 3.1 *Assume that $(u_0,u_1)\in W\times V$, then the solution of the problem (3.1) satisfies*

$$u'\in L^\infty(0,T;V)\;,\;\; u''\in L^\infty(0,T;\mathcal{H}),$$

and

$$u\in L^\infty(0,T;H^4(\Omega)\cap V),$$

such that for any $T>0$

$$u''+\Delta^2u-g(\Delta u')=0 \quad \textit{in} \quad L^\infty(0,T;L^2(\Omega)),$$

$$u(0)=u_0,\; u'(0)=u_1 \quad \textit{in} \quad \Omega.$$

Proof.

The step 1 : Approximate solutions
We will construct solutions approximated by the Faedo-Galekin. Let $T > 0$ be fixed and let $\{w_j\}, j \in \mathbb{N}$ be a basis of $\mathcal{H}$, V and W, i.e. the space generated by $\mathcal{B}_k = \{w_1, w_2, \ldots, w_k\}$ is dense in $\mathcal{H}$, V and W.
We construct approximate solutions u_k, $k = 1, 2, 3, \ldots$, in the form

$$u_k(t) = \sum_{j=1}^{k} c_{j,k}(t) w_j(x),$$

where c_{jk} is determined by the ordinary differential equations.
For any v in $\mathcal{B}_k$, $u_k(t)$ satisfies the approximate equation

$$\int_\Omega (u_k''(t) + \Delta^2 u_k(t) - g(\Delta u_k'(t)))v \, dx = 0, \tag{3.8}$$

with initial conditions

$$u_k(0) = u_k^0 = \sum_{j=1}^{k} \langle u_0, w_j \rangle w_j \to u_0 \quad \text{in } W \text{ as } k \to +\infty, \tag{3.9}$$

$$u_k'(0) = u_k^1 = \sum_{j=1}^{k} \langle u_1, w_j \rangle w_j \to u_1 \quad \text{in } V \text{ as } k \to +\infty. \tag{3.10}$$

The standard theory of ODE guarantees that the system (3.8)-(3.10) has an unique local solution which is extended to a maximal interval in $[0, t_k)$ (with $0 < t_k < T$) by Zorn lemma.
In the next step, we obtain a priori estimates for the solution of the system (3.8)-(3.10), so that it can be extended outside $[0, t_k)$ to obtain one solution defined for all $T > 0$, using a standard compactness argument for the limiting procedure.

The step 2 : A priori estimates
The first estimate : Setting $v = -2\Delta u_k'$ in (3.8), we have

$$\frac{d}{dt}\Big[\, \|\nabla u_k'\|^2 + \|\nabla \Delta u_k\|^2 \,\Big] + 2\int_\Omega \Delta u_k' g(\Delta u_k') \, dx = 0.$$

Integrating in $[0,t]$, $t < t_k$ and using (3.9) and (3.10), we obtain

$$\begin{aligned} &\|\nabla u_k'(t)\|^2 + \|\nabla \Delta u_k(t)\|^2 + 2\int_0^t \int_\Omega \Delta u_k'(s) g(\Delta u_k'(s))\, dx\, ds \\ &\le c(\|\nabla u_k^1\|^2 + \|\nabla \Delta u_k^0\|^2) \le C_1, \end{aligned} \tag{3.11}$$

and C_1 is a positive constant depending only on $\|u_1\|_V$ and $\|u_0\|_W$. Estimate (3.11) yields that

$$u_k \text{ is bounded in } L^\infty(0,T;V), \tag{3.12}$$

$$u_k' \text{ is bounded in } L^\infty(0,T;\mathcal{H}), \tag{3.13}$$

$$\Delta u_k' g(\Delta u_k') \text{ is bounded in } L^1(\Omega \times (0,T)). \tag{3.14}$$

From (3.3), (3.4) and (3.14), it follows that

$$g(\Delta u_k') \text{ is bounded in } L^2(\Omega \times (0,T)). \tag{3.15}$$

The second estimate : First, we estimate $u_k''(0)$. Differentiating (3.8) with respect to x, taking $v = \nabla u_k''(t)$ and choosing $t = 0$, we obtain that

$$\|\nabla u_k''(0)\|^2 + \Big[\nabla u_k''(0), \nabla \Delta^2 u_k^0 - \nabla(g(\Delta u_k^1))\Big] = 0.$$

Using Cauchy-Schwarz inequality and (3.5), we have

$$\begin{aligned} \|\nabla u_k''(0)\| &\le \|\nabla \Delta^2 u_k^0\| + \|\nabla \Delta u_k^1 g'(\Delta u_k^1)\| \\ &\le \|\nabla \Delta^2 u_k^0\| + \tau \|\nabla \Delta u_k^1\|. \end{aligned} \tag{3.16}$$

By (3.9) and (3.10) yields

$$u_k''(0) \quad \text{is bounded in} \quad \mathcal{H}. \tag{3.17}$$

The third estimate : Differentiating (3.8) with respect to t get

$$\int_\Omega (u_k'''(t) + \Delta^2 u_k')v\ dx - \int_\Omega \Delta u_k'' g'(\Delta u_k')v\, dx = 0.$$

Taking $v = -2\Delta u_k''$, applying the Green formula, we obtain

$$\frac{d}{dt}\Big[\, \|\nabla u_k''\|^2 + \|\nabla \Delta u_k'\|^2 \,\Big] + 2\int_\Omega (\Delta u_k'')^2 g'(\Delta u_k')\,dx = 0,$$

by integrating it over $[0,t]$, we get

$$\|\nabla u_k''(t)\|^2 + \|\nabla \Delta u_k'(t)\|^2 + 2\int_0^t \int_\Omega (\Delta u_k''(s))^2 g'(\Delta u_k'(s)\,dx\,ds$$

$$= \|\nabla u_k''(0)\|^2 + \|\nabla \Delta u_k^1\|^2.$$

By (3.10) and (3.17), we deduce that

$$u_k'' \text{ is bounded in } \; L^\infty(0,T;\mathcal{H}). \tag{3.18}$$

The fourth estimate : Setting $v = 2\Delta^2 u_k'$ in (3.8), we have

$$2\int_\Omega u_k''\Delta^2 u_k'\,dx + \frac{d}{dt}\|\Delta^2 u_k\|^2 - 2\int_\Omega g(\Delta u_k')\Delta^2 u_k'\,dx = 0.$$

Therefore by using the Green's formula, we have

$$\begin{aligned}\frac{d}{dt}\|\Delta^2 u_k\|^2 &= -2\int_\Omega \Delta u_k''\Delta u_k'\,dx - 2\int_\Omega g'(\Delta u_k')(\nabla\Delta u_k')^2\,dx\\ &= -\frac{d}{dt}\|\Delta u_k'\|^2 - 2\int_\Omega g'(\Delta u_k')(\nabla\Delta u_k')^2\,dx.\end{aligned}$$

Integrating it over $[0,t]$, we arrive at

$$\|\Delta^2 u_k(t)\|^2 + \|\Delta u_k'(t)\|^2 + 2\int_0^t \int_\Omega g'(\Delta u_k')(\nabla\Delta u_k')^2\,dx\,ds = \|\Delta^2 u_k^0\|^2 + \|\Delta u_k^1\|^2.$$

By using $g' \geq 0$, (3.9) and (3.10), we deduce that

$$\|\Delta^2 u_k(t)\|^2 + \|\Delta u_k'(t)\|^2 \leq c'(\|\Delta^2 u_0\|^2 + \|\Delta u_1\|^2),$$

then

$$\Delta^2 u_k \text{ is bounded in } \; L^\infty(0,T;L^2(\Omega)), \tag{3.19}$$

hence

$$u_k \text{ is bounded in } \; L^\infty(0,T;H^4(\Omega)). \tag{3.20}$$

From (3.12) and (3.20) we deduce that

$$u_k \text{ is bounded in } \; L^\infty(0,T; V \cap H^4(\Omega)). \tag{3.21}$$

The step 3 : Passage to the limit

Applying Dunford-Pettis theorem we conclude from (3.13), (3.15), (3.18) and (3.21), there is a sub-sequence u_{k_l} that we note again u_k, which verifies

$$u_k \rightharpoonup u \;\text{ weak-star in } L^\infty(0,T; V \cap H^4(\Omega)), \tag{3.22}$$

$$u_k' \rightharpoonup u' \;\text{ weak-star in } L^\infty(0,T; V), \tag{3.23}$$

$$u_k'' \rightharpoonup u'' \;\text{ weak-star in } L^\infty(0,T; \mathcal{H}), \tag{3.24}$$

$$g(\Delta u_k') \rightharpoonup \phi \;\text{ weak-star in } L^2(\mathcal{A}), \tag{3.25}$$

where $\mathcal{A} = \Omega \times [0,T]$. It follows at once from (3.22) and (3.24), that for each fixed $v \in L^2(0,T; L^2(\Omega))$

$$\int_0^T \int_\Omega (u_k''(x,t) + \Delta^2 u_k(x,t)) v \, dx \, dt \longrightarrow \int_0^T \int_\Omega (u''(x,t) + \Delta^2 u(x,t)) v \, dx \, dt.$$

It remains to show that

$$\int_0^T \int_\Omega g(\Delta u_k') \; v \, dx \, dt \longrightarrow \int_0^T \int_\Omega g(\Delta u') \; v \, dx \, dt.$$

From (3.13) and (3.18) we deduce that u_k' is bounded in $L^2(0,T;\mathcal{H})$, u_k'' is bounded in $L^2(0,T;\mathcal{H})$, and $\|u_k''\| \le \|\nabla u_k''\|$, we obtain u_k'' is bounded in $L^2(0,T; L^2(\Omega))$. Consequently u_k' is bounded in $H^1(\mathcal{A})$. And the injection of $H^1(\mathcal{A})$ in $L^2(\mathcal{A})$ is compact (see **[28]** Theorem 16. 1, Chap 1), then

$$u_k' \longrightarrow u' \;\text{ strongly in } L^2(\mathcal{A}),$$

therefore

$$u_k' \longrightarrow u' \;\text{ almost every where in } \mathcal{A}. \tag{3.26}$$

Lemma 3.2 *For each $T > 0$, we have $g(\Delta u') \in L^1(\mathcal{A})$,*

$$\|g(\Delta u')\|_{L^1(\mathcal{A})} \le K,$$

where K is a constant independent of t, and

$$g(\Delta u_k') \to g(\Delta u') \ \textit{in} \ L^1(\mathcal{A}).$$

Proof. We claim that

$$g(\Delta u') \in L^1(\mathcal{A}).$$

Indeed, since g is continuous, we deduce from (3.26)

$$g(\Delta u_k') \longrightarrow g(\Delta u') \ \text{ almost every where in } \ \mathcal{A},$$

$$\Delta u_k' g(\Delta u_k') \longrightarrow \Delta u' g(\Delta u') \ \text{ almost every where in } \ \mathcal{A}.$$

Hence, by (3.14) and Fatou's lemma, we have

$$\int_0^T \int_\Omega \Delta u'(x,t) g(\Delta u'(x,t))\, dx\, dt \leq K_1 \ \text{ for } \ T > 0. \tag{3.27}$$

Now, we can estimate $\int_0^T \int_\Omega |\Delta g(u'(x,t))|\, dx\, dt$. By Cauchy-Schwarz inequality, we have

$$\int_0^T \int_\Omega |\Delta g(u'(x,t))|\, dx\, dt \leq c|\mathcal{A}|^{1/2} \Big[\int_0^T \int_\Omega |\Delta g(u'(x,t))|^2\, dx\, dt \Big]^{1/2}.$$

Using (3.3), (3.4) and (3.27), we obtain

$$\begin{aligned}
\int_0^T \int_\Omega |\Delta g(u'(x,t))|^2\, dx\, dt &\leq \int_0^T \int_{|\Delta u'| > \varepsilon} \Delta u' g(\Delta u')\, dx\, dt + \int_0^T \int_{|\Delta u'| \leq \varepsilon} G^{-1}(\Delta u' g(\Delta u'))\, dx\, dt \\
&\leq c \int_0^T \int_\Omega \Delta u' g(\Delta u')\, dx\, dt + cG^{-1} \Big[\int_{\mathcal{A}} \Delta u' g(\Delta u')\, dx\, dt \Big] \\
&\leq c \int_0^T \int_\Omega \Delta u' g(\Delta u')\, dx\, dt + c' G^*(1) + c'' \int_\Omega \Delta u' g(\Delta u')\, dx\, dt \\
&\leq cK_1 + c' G^*(1) \quad \text{for } T > 0.
\end{aligned}$$

Then

$$\int_0^T \int_\Omega |\Delta g(u'(x,t))|\, dx\, dt \leq K \quad \text{for } T > 0.$$

Let $E \subset \Omega \times [0,T]$, and set

$$E_1 = \Big\{ (x,t) \in E : |g(\Delta u_k'(x,t))| \leq \frac{1}{\sqrt{|E|}} \Big\},$$

$$E_2 = E\backslash E_1,$$

where $|E|$ is the measure of E.
If $M(r) = \inf\{|s| : s \in \mathbb{R} \text{ and } |g(s)| \geq r\}$, then

$$\int_E |g(\Delta u_k')|\, dx\, dt \leq c\sqrt{|E|} + \Big[\, \frac{M}{\sqrt{|E|}} \,\Big]^{-1} \int_{E_2} \Delta u_k' g(\Delta u_k')\, dx\, dt.$$

By applying (3.14), we deduce that

$$\sup_k \int_E |g(\Delta u_k')|\, dx\, dt \longrightarrow 0 \quad \text{when } |E| \longrightarrow 0.$$

From Vitali's convergence theorem, we deduce that

$$g(\Delta u_k') \to g(\Delta u') \ \text{ in } L^1(\mathcal{A}).$$

This completes the proof.
Then (3.25) implies that

$$g(\Delta u_k') \rightharpoonup g(\Delta u') \ \text{ weak-star in } L^2([0,T] \times \Omega).$$

We deduce, for all $v \in L^2(0,T; L^2(\Omega))$, that

$$\int_0^T \int_\Omega g(\Delta u_k') v\, dx\, dt \longrightarrow \int_0^T \int_\Omega g(\Delta u') v\, dx\, dt.$$

Finally we have shown that, for all $v \in L^2(0,T; L^2(\Omega))$:

$$\int_0^T \int_\Omega (u''(x,t) + \Delta^2 u(x,t) - g(\Delta u')) v\, dx\, dt = 0.$$

Therefore, u is a solution for the problem (3.1).

The step 4 : Proof of uniqueness

Let u_1, u_2 be two solutions of the problem (3.1) with the same initial data. Let us denote it is straightforward to see that $z = u_1 - u_2$ satisfies

$$\|\nabla z'\|^2 + \|\nabla \Delta z\|^2 + 2\int_0^t \int_\Omega (\Delta u_1'(x,s) - \Delta u_2'(x,s))(g(\Delta u_1'(x,s)) - g(\Delta u_2'(x,s)))\, dx\, ds = 0.$$

Using the monotonicity of g, we have

$$2\int_0^t \int_\Omega (\Delta u_1'(x,s) - \Delta u_2'(x,s))(g(\Delta u_1'(x,s)) - g(\Delta u_2'(x,s)))\,dx\,ds \geq 0,$$

we conclude that

$$\|\nabla z'\|^2 + \|\nabla \Delta z\|^2 = 0,$$

which implies $z = 0$. This finishes the proof of theorem (3.1).

4 Asymptotic behavior

Now we define the energy associated with the solution of the problem (3.1) by the following formula

$$E(t) = \frac{1}{2}\|\nabla u'\|^2 + \frac{1}{2}\|\nabla \Delta u\|^2.$$

Lemma 4.1 *Let u be a solution to the problem* (3.1)*. Then E is a non-increasing function.*

Proof. Multiplying the first equation in (3.1) by $-\Delta u'$ and integrating over Ω, we get

$$\frac{d}{dt}\Big[\frac{1}{2}\|\nabla u'\|^2 + \frac{1}{2}\|\nabla \Delta u\|^2\Big] = -\int_\Omega \Delta u' g(\Delta u')\,dx \leq 0. \tag{3.28}$$

Lemma 4.2 *We define the following functional F by*

$$F(t) = ME(t) - \int_\Omega \Delta u u'\,dx, \tag{3.29}$$

where $M > 0$ will be determined later. Then there are positive constants λ_1, λ_2 such that

$$\lambda_1 E(t) \leq F(t) \leq \lambda_2 E(t) \quad \forall\, t \in \mathbb{R}_+. \tag{3.30}$$

Proof. Using the obvious estimates

$$\|u'\| \leq c_3\|\nabla u'\|, \tag{3.31}$$

and

$$\|\Delta u\| \leq c_4\|\nabla \Delta u\|. \tag{3.32}$$

By Cauchy-Schwarz's inequality and (3.31) and (3.32), we obtain

$$-\int_\Omega \Delta u.u'\,dx \le \frac{c_3^2}{2}\|\nabla\Delta u\|^2 + \frac{c_4^2}{2}\|\nabla u'\|^2 \le \max(c_3^2, c_4^2)E(t),$$

hence

$$(M - \max(c_3^2, c_4^2))E(t) \le F(t) \le (M + \max(c_3^2, c_4^2))E(t),$$

choosing $M > \max(c_3^2, c_4^2)$, we obtain (3.30), where $\lambda_1 = M - \max(c_3^2, c_4^2)$ and $\lambda_2 = M + \max(c_3^2, c_4^2)$.

Lemma 4.3 *We define the following functional L by*

$$L(t) = F(t) + \lambda E(t),$$

where λ will be chosen later. Then there are positive constants μ_1, μ_2 such that

$$\mu_1 E(t) \le L(t) \le \mu_2 E(t) \quad \forall\, t \in \mathbb{R}_+. \tag{3.33}$$

It is aisy to see (3.33) hold from the lemma (4.2) with $\mu_1 = \lambda_1 + \lambda$ and $\mu_2 = \lambda_2 + \lambda$.

Theorem 4.4 *Assume that* (3.3) *and* (3.4) *hold. Then there are positive constants k_1, k_2, k_3 and ε_0 such that the solution of the problem* (3.1) *satisfies*

$$E(t) \le k_3 G_1^{-1}(k_1 t + k_2) \quad \forall\, t \in \mathbb{R}_+, \tag{3.34}$$

where

$$G_1(t) = \int_t^1 \frac{1}{G_2(s)}\,ds, \;\; G_2(t) = tG'(\varepsilon_0 t), \tag{3.35}$$

here G_1 is strictly decreasing and convex on $]0,1]$, with $\lim_{t\to 0} G_1(t) = +\infty$.

Proof. Let $\varepsilon_1 \in]0,\varepsilon]$, we define two sets Ω_1, Ω_2 such that

$$\Omega_1 = \{x \in \Omega : |\Delta u'| \le \varepsilon_1\}, \;\; \Omega_2 = \{x \in \Omega : |\Delta u'| > \varepsilon_1\}.$$

Differentiating (3.29) with respect to t, using (3.7), (3.28), and the first equation of the problem (3.1), we get

$$\begin{aligned} F'(t) &= ME'(t) + \int_\Omega |\nabla u'|^2\,dx - \int_\Omega |\nabla\Delta u|^2\,dx - \int_\Omega \Delta u g(\Delta u')\,dx \\ &\le -2E(t) + \int_\Omega 2|\nabla u'|^2 + |\Delta u g(\Delta u')|\,dx \\ &\le -2E(t) + 2\int_\Omega |\nabla u'|^2 + |\Delta u g(\Delta u')|\,dx \\ &\le -2E(t) + C\int_\Omega |\Delta u'|^2 + |\Delta u g(\Delta u')|\,dx. \end{aligned} \tag{3.36}$$

Using Young's inequality, (3.32) to obtain

$$\begin{aligned} \int_{\Omega_1} |\Delta u g(\Delta u')|\,dx &\le c_4\delta\|\nabla\Delta u\|^2 + C_\delta\int_{\Omega_1} |g(\Delta u')|^2\,dx \\ &\le c\delta E(t) + C_\delta\int_{\Omega_1} |g(\Delta u')|^2\,dx. \end{aligned} \tag{3.37}$$

By Cauchy-Schwarz's inequality and (3.32), we have

$$\begin{aligned} \int_{\Omega_2} |\Delta u g(\Delta u')|\,dx &\le \Big[\int_{\Omega_2} |\Delta u|^2\,dx\Big]^{\frac12}\Big[\int_{\Omega_2} |g(\Delta u')|^2\,dx\Big]^{\frac12} \\ &\le c_4\|\nabla\Delta u\|^{\frac12}\Big[\int_{\Omega_2} |g(\Delta u')|^2\,dx\Big]^{\frac12}. \end{aligned}$$

Then, we use Young's inequality and (3.3), for any $\delta > 0$, we have

$$\begin{aligned} \int_{\Omega_2} |\Delta u'|^2 + |\Delta u g(\Delta u')|\,dx &\le \int_{\Omega_2} |\Delta u' g(\Delta u')|\,dx + c_4\|\nabla\Delta u\|^{\frac12}\Big[\int_{\Omega_2} |\Delta u' g(\Delta u')|\,dx\Big]^{\frac12} \\ &\le -cE'(t) + c_4E^{\frac12}(t)(-E'(t))^{\frac12} \\ &\le -cE'(t) + c_4\delta E(t) + C_\delta(-E'(t)) \\ &\le c_4\delta E(t) - (c + C_\delta)E'(t). \end{aligned} \tag{3.38}$$

By (3.36) - (3.38), and the function $L = F + \lambda E$ satisfies

$$L'(t) \le (-2+cC\delta+c_4C\delta)E(t)+(\lambda-cC-C_\delta C)E'(t)+C\int_{\Omega_1} |\Delta u'|^2\,dx+CC_\delta\int_{\Omega_1} |g(\Delta u')|^2\,dx,$$

for $\delta < \frac{2}{cC+c_4C}$ and $\lambda \geq cC + C_\delta C$, we have

$$L'(t) \leq -mE(t) + c\int_{\Omega_1} |\Delta u'|^2 + |g(\Delta u')|^2\, dx, \tag{3.39}$$

where $m = 2 - cC\delta - c_4 C\delta$.

The case 1 : G is linear on $[0,\varepsilon]$, using (3.4) and (3.39), we deduce that

$$\begin{aligned} L^{'}(t) &\leq -mE(t) + c\int_{\Omega_1} G^{-1}(\Delta u' g(\Delta u'))\, dx \\ &\leq -mE(t) + c\int_{\Omega_1} \Delta u' g(\Delta u')\, dx \\ &\leq -mE(t) - cE'(t), \end{aligned}$$

we deduce that

$$(L(t) + cE(t))^{'} \leq -mE(t).$$

From (3.33), we have

$$L(t) \sim E(t),$$

then

$$L(t) + cE(t) \sim E(t),$$

we obtain

$$E(t) \leq E(0)e^{-mt},$$

thus, we have

$$E(t) \leq E(0)G_1^{-1}(mt).$$

The case 2 : G is nonlinear, we define the following functional I by

$$I(t) = \frac{1}{|\Omega_1|}\int_{\Omega_1} \Delta u' g(\Delta u')\, dx.$$

From Jensen's inequality and the concavity of G^{-1}, we conclude that

$$G^{-1}(I(t)) \geq c\int_{\Omega_1} G^{-1}(\Delta u' g(\Delta u')\, dx.$$

By using this inequality and (3.4), we obtain

$$\int_{\Omega_1} |\Delta u'|^2 + |g(\Delta u')|^2 \, dx \leq \int_{\Omega_1} G^{-1}(\Delta u' g(\Delta u')) \, dx,$$

implies

$$\int_{\Omega_1} |\Delta u'|^2 + |g(\Delta u')|^2 \, dx \leq cG^{-1}(I(t)), \tag{3.40}$$

using (3.39) and (3.40), we obtain

$$L'(t) \leq -mE(t) + cG^{-1}(I(t)). \tag{3.41}$$

For $\varepsilon_0 < \varepsilon$ and $c_0 > 0$, we define H_1 by

$$H_1(t) = G^{'}\Big[\varepsilon_0 \frac{E(t)}{E(0)}\Big] L(t) + c_0 E(t).$$

Since $L(t)$ is equivalent to $E(t)$ (see (3.33)), there exists positive constants α_1, α_2, such that

$$\alpha_1 H_1(t) \leq E(t) \leq \alpha_2 H_1(t) \quad \forall\, t \in \mathbb{R}_+. \tag{3.42}$$

By recalling that $E^{'} \leq 0$, $G^{'} > 0$, and $G^{''} > 0$ on $]0, \varepsilon]$, making use of (3.41), we obtain

$$\begin{aligned} H_1^{'}(t) &= \varepsilon_0 \frac{E'(t)}{E(0)} G''\Big[\varepsilon_0 \frac{E(t)}{E(0)}\Big] L(t) + G'\Big[\varepsilon_0 \frac{E(t)}{E(0)}\Big] L'(t) + c_0 E^{'}(t) \\ &\leq -mE(t) G'\Big[\varepsilon_0 \frac{E(t)}{E(0)}\Big] + cG'\Big[\varepsilon_0 \frac{E(t)}{E(0)}\Big] G^{-1}(I(t)) + c_0 E^{'}(t). \end{aligned}$$

Using (3.6) with $S = G^{'}\Big[\varepsilon_0 \dfrac{E(t)}{E(0)}\Big]$ and $T = G^{-1}(I(t))$, and the lemma 4.1 , we deduce that

$$\begin{aligned} H_1^{'}(t) &\leq -mE(t) G^{'}\Big[\varepsilon_0 \tfrac{E(t)}{E(0)}\Big] + cG^{*}\Big[G^{'}\Big[\varepsilon_0 \tfrac{E(t)}{E(0)}\Big]\Big] + cI(t) + c_0 E^{'}(t) \\ &\leq -mE(t) G^{'}\Big[\varepsilon_0 \tfrac{E(t)}{E(0)}\Big] + c\varepsilon_0 E(0) \tfrac{E(t)}{E(0)} G^{'}\Big[\varepsilon_0 \tfrac{E(t)}{E(0)}\Big] - cE'(t) + c_0 E^{'}(t). \end{aligned}$$

Choosing $c_0 > c$ and ε_0 small enough, we obtain

$$\begin{aligned} H_1^{'}(t) &\leq -k \frac{E(t)}{E(0)} G^{'}\Big[\varepsilon_0 \frac{E(t)}{E(0)}\Big] \\ &\leq -kG_2\Big[\frac{E(t)}{E(0)}\Big]. \end{aligned} \tag{3.43}$$

Since

$$G_2^{'}(t) = G^{'}(\varepsilon_0 t) + \varepsilon_0 t G^{''}(\varepsilon_0 t),$$

we find that $G_2'(t) > 0$ and $G_2(t) > 0$ on $]0, 1]$. By setting

$$H(t) = \frac{\alpha_1}{E(0)} H_1(t),$$

(α_1 is given in (3.42)) we easily see that, by (3.42), we have

$$H(t) \sim E(t). \tag{3.44}$$

Using (3.43), we arrive at

$$H^{'}(t) \leq -k_1 G_2(H(t)).$$

By recalling (3.35), we deduce that $G_2(t) = -1/G_1'(t)$, hence

$$H^{'}(t) \leq \frac{k_1}{G_1'(H(t))},$$

which gives

$$(G_1(H(t)))^{'} = H'(t) G_1'(H(t)) \geq k_1,$$

by integrating over $[0, t]$, we obtain

$$G_1(H(t)) \geq k_1 t + G_1(H(0)),$$

consequently

$$H(t) \leq G_1^{-1}\left(k_1 t + k_2\right). \tag{3.45}$$

Combining (3.44) and (3.45), we obtain (3.34).

5 Examples

Example 5.1 *Let g given by $g(s) = s^p(-\ln s)^q$, where $p \geq 1$ and $q \in \mathbb{R}$ on $[0, \epsilon]$, and the function G is defined in the neighborhood of 0 by*

$$G(s) = c s^{\frac{p+1}{2}} (-\ln \sqrt{s})^q,$$

we have

$$G'(s) = cs^{\frac{p-1}{2}}(-\ln\sqrt{s})^{q-1}\Big[\frac{p+1}{2}(-\ln\sqrt{s}) - \frac{q}{2}\Big],$$

thus

$$G_1(t) = \frac{1}{c}\int_t^1 \frac{1}{s^{\frac{p+1}{2}}(-\ln\sqrt{s})^{q-1}(\frac{p+1}{2}(-\ln\sqrt{s}) - \frac{q}{2})}\,ds.$$

Making the following changement of variable : $z = \frac{1}{\sqrt{s}}$*, we obtain*

$$G_1(t) = \frac{2}{c}\int_1^{\frac{1}{\sqrt{t}}} \frac{z^{p-2}}{(\ln z)^{q-1}(\frac{p+1}{2}\ln z - \frac{q}{2})}\,dz.$$

We have three cases :

The case 1 : *If* $p = 1$, $q = 1$, *we have*

$$G_1(t) = \frac{2}{c}\ln(-\ln\sqrt{et}),$$

we deduce that

$$G_1^{-1}(t) = \frac{1}{e}e^{-2e^{\frac{c}{2}t}},$$

then

$$E(t) \leq \frac{k_3}{e}e^{-2e^{\frac{c}{2}(k_1 t + k_2)}}.$$

The case 2 : *If* $p = 1$, $q < 1$, *we have*

$$\begin{aligned} G_1(t) &= \tfrac{2}{c}\int_1^{\frac{1}{\sqrt{t}}} \frac{1}{z(\ln z)^{q-1}(\ln z - \frac{q}{2})}\,dz \\ &\sim \tfrac{2}{c}\int_1^{\frac{1}{\sqrt{t}}} \frac{1}{z(\ln z)^{q}}\,dz \\ &\sim \tfrac{2^q}{c(1-q)}(-\ln t)^{1-q}, \text{ as } t \to 0, \end{aligned}$$

we deduce that

$$G_1^{-1}(t) \sim e^{-kt^{\frac{1}{1-q}}}, \text{ as } t \to +\infty,$$

then

$$E(t) \leq k_3 e^{-k(k_1 t + k_2)^{\frac{1}{1-q}}},$$

where $k = (\frac{c(1-q)}{2^q})^{\frac{1}{1-q}}$.

The case 3 : *If $p>1$, using the lemma 6.1 (i) (see* **[6]***), we obtain in the neighborhood of* 0

$$G_1(t)\sim \frac{1}{\frac{p^2-1}{2^{q+1}}(-\ln t)^q t^{\frac{p-1}{2}}},.$$

Applying the lemma 6.2 (see **[6]***), we obtain in the neighborhood of $+\infty$*

$$G_1^{-1}(t)\sim (\frac{p-1}{2})^{\frac{2q}{p-1}} t^{\frac{-2}{p-1}} (\ln t)^{\frac{-2q}{p-1}},$$

then

$$E(t)\leq k_3(\frac{p-1}{2})^{\frac{2q}{p-1}}(k_1t+k_2)^{\frac{-2}{p-1}}(\ln(k_1t+k_2))^{\frac{-2q}{p-1}}.$$

Example 5.2 *Let g given by $g(s)=e^{-(-\ln s)^{\gamma}}$, where $1<\gamma<2$, and*

$$G(t)=c\sqrt{s}e^{-(-\ln\sqrt{s})^{\gamma}}.$$

We have

$$G'(s)=\frac{c}{2\sqrt{s}}e^{-(-\ln\sqrt{s})^{\gamma}}(1+\gamma(-\ln\sqrt{s})^{\gamma-1}),$$

thus

$$G_1(t)=\frac{2}{c}\int_t^1 \frac{e^{(-\ln\sqrt{s})^{\gamma}}}{\sqrt{s}(1+\gamma(-\ln\sqrt{s})^{\gamma-1})}\,ds.$$

Making the following changement of variable : $z=\frac{1}{\sqrt{s}}$, we obtain

$$G_1(t)=\frac{4}{c}\int_1^{\frac{1}{\sqrt{t}}} \frac{e^{(\ln z)^{\gamma}}}{z^2(1+\gamma(\ln z)^{\gamma-1})}\,dz.$$

Using the lemma 6.1 (see **[6]***), we obtain in the neighborhood of* 0

$$G_1(t)\sim c\frac{\sqrt{t}e^{(-\ln\sqrt{t})^{\gamma}}}{(-\ln\sqrt{t})^{2(\gamma-1)}},$$

and applying the lemma 6.2 (see **[6]***), we obtain in the neighborhood of $+\infty$*

$$G_1^{-1}(t)\sim \exp\Big(-2\Big[\ln t+\Big[\ln t+\ln^{\frac{1}{\gamma}}(t)+2\frac{\gamma-1}{\gamma}\ln\ln t\Big]^{\frac{1}{\gamma}}+2\frac{\gamma-1}{\gamma}\ln\ln t\Big]^{\frac{1}{\gamma}}\Big),$$

then

$$E(t)\leq k_3\exp\Big(-2\Big[\ln(k_1t+k_2)+\Big[\ln(k_1t+k_2)+\ln^{\frac{1}{\gamma}}(k_1t+k_2)$$
$$+2\frac{\gamma-1}{\gamma}\ln\ln(k_1t+k_2)\Big]^{\frac{1}{\gamma}}+2\frac{\gamma-1}{\gamma}\ln\ln(k_1t+k_2)\Big]^{\frac{1}{\gamma}}\Big).$$

Chapter 4

Stabilization of the Petrovsky-wave nonlinear coupled system with strong damping

1 Introduction

In this chapter we investigate the existence and decay properties of solutions for the initial boundary value problem of system of Petrovsky-wave of the type

$$\begin{cases} u_1'' + \Delta^2 u_1 - a(x)\Delta u_2 - g_1(\Delta u_1') = 0, & \text{in } \Omega \times [0, +\infty[, \\ u_2'' - \Delta u_2 - a(x)\Delta u_1 - g_2(\Delta u_2') = 0, & \text{in } \Omega \times [0, +\infty[, \\ \Delta u_1 = u_1 = u_2 = 0, & \text{on } \partial\Omega \times [0, +\infty[, \\ u_i(x,0) = u_i^0(x),\ u_i'(x,0) = u_i^1(x), & \text{in } \Omega, i = 1, 2, \end{cases} \tag{4.1}$$

here Ω is a bounded domain of $\mathbb{R}^n$ with regular boundary $\partial\Omega$.
When $a(x) = 0$, the Petrovsky equation was treated by Komornik **[15]**, he used semigroup approach for sitting the well-posedness and he studied the strong stability by introducing a multiplier method combined with a nonlinear integral inequalities. Recently, Bahlil and Baowei **[4]**, studied the system

$$\begin{cases} u_1'' + a(x)u_2 + \Delta^2 u_1 - g_1(u_1'(x,t)) = f_1(u_1, u_2), & \text{in } \Omega \times [0, +\infty[, \\ u_2'' + a(x)u_1 - \Delta u_2 - g_2(u_2'(x,t)) = f_2(u_1, u_2), & \text{in } \Omega \times [0, +\infty[, \\ \partial_\nu u_1 = u_1 = v = u_2 = 0, & \text{on } \partial\Omega \times [0, +\infty[, \end{cases} \tag{4.2}$$

for g_i $(i = 1, 2)$ do not necessarily having a polynomial growth near the origin, by using compactness method to prove the existence and uniqueness of solution and established energy decay results

depending on g_i. Guesmia [**10**] consider the problem (4.2) without source terms f_1 and f_2. He deal with global existence and uniform decay of solutions.
In this chapter, we prove the global existence of weak solutions of the problem (4.1) by using the compactness method (see Lions [**27**]). We use some technique from [**4**] to establish an explicit and general decay result, depending on g_i. The proof is based on the multiplier method and makes use of some properties of convex functions, the general Young inequality. These convexity arguments were introduced and developed by Lasiecka and co-workers ([**21**],[**22**]) and used, with appropriate modifications, by Liu and Zuazua [**32**], Alabau-Boussouira [**2**] and others.

2 Preliminaries and assumptions

Let us introduce for brevity the Hilbert spaces

$$H = L^2(\Omega) \times L^2(\Omega),$$

$$W = H_0^1(\Omega) \times H_0^1(\Omega),$$

$$H_\Delta^3(\Omega) = \{u \in H^3(\Omega) |\, u = \Delta u = 0 \text{ on } \partial\Omega\}, \quad \|u\|^2_{H_\Delta^3(\Omega)} = \int_\Omega |\nabla \Delta u|^2 dx,$$

$$V = H_\Delta^3(\Omega) \cap H^2(\Omega) \times H^2(\Omega),$$

$$\widetilde{V} = (H^4(\Omega) \cap H_\Delta^3(\Omega)) \times (H_\Delta^3(\Omega) \cap H^2(\Omega)).$$

Identifying H with its dual H' , we obtain the diagram

$$\widetilde{V} \subset V \subset W \subset H = H' \subset W' \subset V' \subset \widetilde{V}',$$

with dense and compact imbedings. If $u \in L^2(\Omega)$, we denote by $\|u\|_{L^2(\Omega)} = \|u\|$.
We impose the following assumptions on a and g_i.
The function $a : \Omega \to \mathbb{R}$ is a nonnegative such that

$$\begin{aligned} & a \in W^{1,\infty}(\Omega), \\ & \|a\|_\infty < \min\left\{\tfrac{1}{c'+1}, 1\right\}, \end{aligned} \tag{4.3}$$

where $c' > 0$ (depending only on the geometry of Ω) is the constant satisfies

$$\|\Delta u\| \leq c' \|\nabla \Delta u\|, \quad \forall\, u \in H_\Delta^3(\Omega),$$

$$\|\nabla u\| \leq c^{'} \|\Delta u\|, \quad \forall\, u \in H_0^2(\Omega).$$

Assume that $g_i : \mathbb{R} \to \mathbb{R}$, $i = 1,2$ is nondecreasing continuously differentiable function of class $\mathcal{C}^1$, and globally Lipschitz.
Suppose that $\exists\, c_j > 0,\ j = 1,2,3,4,\ \exists\, \tau > 0$ such that

$$\forall\, p \geq 1 : c_1|s|^p \leq |g_i(s)| \leq c_2|s|^{\frac{1}{p}} \quad \text{if} \quad |s| \leq 1, \tag{4.4}$$

$$c_3|s| \leq |g_i(s)| \leq c_4|s|, \quad \text{if} \quad |s| > 1, \tag{4.5}$$

$$\forall\, s \in \mathbb{R} : g_i'(s) \leq \tau. \tag{4.6}$$

3 Global existence

Theorem 3.1 *Let* $(u_1^0, u_2^0) \in \widetilde{V}$ *and* $(u_1^1, u_2^1) \in V$ *arbitrarily. Assume that* (4.3)-(4.6) *hold. Then, the system* (4.1) *has a unique weak solution satisfying*

$$(u_1, u_2) \in L^\infty(\mathbb{R}_+, \widetilde{V}), \quad (u_1', u_2') \in L^\infty(\mathbb{R}_+, V),$$

and

$$(u_1'', u_2'') \in L^\infty(\mathbb{R}_+, W).$$

Proof.

The step 1 : Approximate solutions
We will construct solutions approximated by the Faedo-Galekin. Let $T > 0$ be fixed and denote by V_k the space generated by $\{w_{i,1}, w_{i,2}, ..., w_{i,k}\}$, where the set $\{w_{i,k},\ i = 1,2,\ k \in \mathbb{N}\}$ is a basis of $\widetilde{V}$.
We construct approximate solution $u_{i,k},\ k = 1,2,3, ...,\ i = 1,2$ in the form

$$u_{i,k}(x,t) = \sum_{j=1}^{k} c_{j,k}(t) w_{i,j}(x),\ i = 1,2,$$

where $c_{j,k}$ $(j = 1,2,...,k)$ are determined by the following ordinary differential equations

$$\begin{cases} (u_{1,k}'' + \Delta^2 u_{1,k} - a(x)\Delta u_{2,k} - g_1(\Delta u_{1,k}'), w_{1,j}) = 0, & \text{for all } w_{1,j} \in V_k, \\ (u_{2,k}'' - \Delta u_{2,k} - a(x)\Delta u_{1,k} - g_2(\Delta u_{2,k}'), w_{2,j}) = 0, & \text{for all } w_{2,j} \in V_k, \\ u_{i,k}(0) = u_{i,k}^0,\ u_{i,k}'(0) = u_{i,k}^1, & \text{in} \quad \Omega,\ i = 1,2, \end{cases} \tag{4.7}$$

with initial conditions

$$u_{1,k}(0) = u_{1,k}^0 = \sum_{j=1}^{k} \langle u_1^0, w_{1,j} \rangle w_{1,j} \to u_1^0, \quad \text{in } H^4(\Omega) \cap H_\Delta^3(\Omega) \text{ as } k \to +\infty,$$

$$u_{2,k}(0)=u^0_{2,k}=\sum_{j=1}^{k}\langle u^0_2,w_{2,j}\rangle w_{2,j}\to u^0_2,\quad \text{in } H^3_\Delta(\Omega)\cap H^2(\Omega) \text{ as } k\to+\infty,$$

$$u'_{1,k}(0)=u^1_{1,k}=\sum_{j=1}^{k}\langle u^1_1,w_{1,j}\rangle w_{1,j}\to u^1_1,\quad \text{in } H^3_\Delta(\Omega)\cap H^2(\Omega) \text{ as } k\to+\infty, \tag{4.8}$$

$$u'_{2,k}(0)=u^1_{2,k}=\sum_{j=1}^{k}\langle u^1_2,w_{2,j}\rangle w_{2,j}\to u^1_2,\quad \text{in } H^2(\Omega) \text{ as } k\to+\infty, \tag{4.9}$$

$$-\Delta^2 u^0_{1,k}+a(x)\Delta u^0_{2,k}+g_1(\Delta u^1_{1,k})\longrightarrow -\Delta^2 u^0_1+a(x)\Delta u^0_2+g_1(\Delta u^1_1),\ \text{in } H^1_0(\Omega) \text{ as } k\to+\infty, \tag{4.10}$$

$$\Delta u^0_{2,k}+a(x)\Delta u^0_{1,k}+g_2(\Delta u^1_{2,k})\longrightarrow \Delta u^0_2+a(x)\Delta u^0_1+g_2(\Delta u^1_2),\qquad \text{in } H^1_0(\Omega) \text{ as } k\to+\infty, \tag{4.11}$$

The step 2 : A priori estimates. We are going to use some a priori estimates to show that $t_k=\infty$. Then, we will show that the sequence of solutions to (4.7) converges to a solution of (4.1) with the claimed smoothness.
Choosing $w_{i,j}=-2\Delta u'_{i,k}$, $i=1,2$ in (4.7), we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla u'_{1,k}|^2+|\nabla u'_{2,k}|^2+|\nabla\Delta u_{1,k}|^2+|\Delta u_{2,k}|^2+2a(x)\Delta u_{1,k}\Delta u_{2,k}\,dx\\ +2\int_\Omega \Delta u'_{1,k}g_1(\Delta u'_{1,k})\,dx+2\int_\Omega \Delta u'_{2,k}g_2(\Delta u'_{2,k})\,dx=0,\end{aligned} \tag{4.12}$$

and choosing $w_{i,j}=2\Delta^2 u'_{i,k}$, $i=1,2$ in (4.7), implies

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\Delta u'_{1,k}|^2+|\Delta u'_{2,k}|^2+|\Delta^2 u_{1,k}|^2+|\nabla\Delta u_{2,k}|^2+2a(x)\nabla\Delta u_{1,k}\nabla\Delta u_{2,k}\,dx\\ +2\int_\Omega \nabla a(x)\Delta u_{2,k}\nabla\Delta u'_{1,k}\,dx+2\int_\Omega \nabla a(x)\Delta u'_{1,k}\nabla\Delta u'_{2,k}\,dx\\ +2\int_\Omega |\nabla\Delta u'_{1,k}|^2 g'_1(\Delta u'_{1,k})\,dx+2\int_\Omega |\nabla\Delta u'_{2,k}|^2 g'_2(\Delta u'_{2,k})\,dx=0.\end{aligned} \tag{4.13}$$

Summing (4.12) and (4.13), we obtain

$$\begin{aligned}&\frac{d}{dt}\int_\Omega |\Delta u'_{1,k}|^2+|\nabla u'_{1,k}|^2+|\Delta^2 u_{1,k}|^2+|\nabla\Delta u_{1,k}|^2\,dx\\ &+\frac{d}{dt}\int_\Omega |\Delta u'_{2,k}|^2+|\nabla u'_{2,k}|^2+|\Delta u_{2,k}|^2+|\nabla\Delta u_{2,k}|^2\,dx\\ &+2\frac{d}{dt}\int_\Omega a(x)\Delta u_{1,k}\Delta u_{2,k}+a(x)\nabla\Delta u_{1,k}\nabla\Delta u_{2,k}\,dx\\ &+2\int_\Omega \Delta u'_{1,k}g_1(\Delta u'_{1,k})\,dx+2\int_\Omega \Delta u'_{2,k}g_2(\Delta u'_{2,k})\,dx\end{aligned} \tag{4.14}$$

$$+2\int_\Omega \nabla a(x)\Delta u_{2,k}\nabla\Delta u_{1,k}^{'}\,dx+2\int_\Omega \nabla a(x)\Delta u_{1,k}\nabla\Delta u_{2,k}^{'}\,dx$$

$$+2\int_\Omega |\nabla\Delta u_{1,k}^{'}|^2 g_1'(\Delta u_{1,k}^{'})\,dx+2\int_\Omega |\nabla\Delta u_{2,k}^{'}|^2 g_2'(\Delta u_{2,k}^{'})\,dx=0.$$

Using Cauchy-Schwarz's inequality and Sobolev embedding, we have

$$\begin{aligned}2\int_\Omega a(x)\Delta u_{2,k}\Delta u_{1,k}\,dx &\le 2\int_\Omega |a(x)||\Delta u_{2,k}||\Delta u_{1,k}|\,dx\\ &\le c'\|a\|_\infty\int_\Omega |\nabla\Delta u_{1,k}(x,t)|^2\,dx\\ &\quad+c'\|a\|_\infty\int_\Omega |\Delta u_{2,k}(x,t)|^2\,dx,\end{aligned}\tag{4.15}$$

and

$$2\int_\Omega a(x)\nabla\Delta u_{1,k}\nabla\Delta u_{2,k}\,dx\le \|a\|_\infty\int_\Omega |\nabla\Delta u_{1,k}|^2\,dx+\|a\|_\infty\int_\Omega |\nabla\Delta u_{2,k}|^2\,dx.\tag{4.16}$$

By Cauchy-Schwarz's inequality, Sobolev embedding and the condition (4.4), we get

$$\begin{aligned}2\int_\Omega \nabla a(x)\Delta u_{2,k}\nabla\Delta u_{1,k}^{'}\,dx &\le 2\int_\Omega |\nabla a(x)||\Delta u_{2,k}||\nabla\Delta u_{1,k}^{'}|\,dx\\ &\le 2\int_\Omega |\nabla a(x)||\Delta u_{2,k}||\nabla\Delta_{1,k}^{'}|\frac{\sqrt{g_1'(\Delta u_{1,k}^{'})}}{\sqrt{\tau}}\,dx\\ &\le \|\nabla a\|_\infty\int_\Omega |\nabla\Delta u_{1,k}^{'}|^2 g_1'(\Delta u_{1,k}^{'})\,dx\\ &\quad+\frac{\|\nabla a\|_\infty}{\tau}\int_\Omega |\Delta u_{2,k}|^2\,dx.\end{aligned}\tag{4.17}$$

Similarly, we have

$$\begin{aligned}2\int_\Omega \nabla a(x)\Delta u_{1,k}\nabla\Delta u_{2,k}^{'}\,dx &\le \int_\Omega |\nabla\Delta u_{2,k}^{'}|^2 g_2'(\Delta u_{2,k}^{'})\,dx+\frac{\|\nabla a\|_\infty}{\tau}\int_\Omega |\Delta u_{1,k}|^2\,dx\\ &\le \|\nabla a\|_\infty\int_\Omega |\nabla\Delta u_{2,k}^{'}|^2 g_2'(\Delta u_{2,k}^{'})\,dx\\ &\quad+\frac{c'\|\nabla a\|_\infty}{\tau}\int_\Omega |\nabla\Delta u_{1,k}|^2\,dx.\end{aligned}\tag{4.18}$$

Reporting (4.15)-(4.18), into (4.14) and integrating over $[0,t]$, we find

$$\begin{aligned}&F_k(t)+2\int_0^t\int_\Omega \Delta u_{1,k}^{'}(s)g_1(\Delta u_{1,k}^{'}(s))\,dx\,dt+2\int_0^t\int_\Omega \Delta u_{2,k}^{'}(s)g_2(\Delta u_{2,k}^{'}(s))\,dx\,dt\\ &\quad+\int_0^t\int_\Omega |\nabla\Delta u_{1,k}^{'}(s)|^2 g_1'(\Delta u_{1,k}^{'}(s))\,dx\,dt+\int_0^t\int_\Omega |\nabla\Delta u_{2,k}^{'}(s)|^2 g_2'(\Delta u_{2,k}^{'}(s))\,dx\,dt\\ &\quad\le K+C_1\int_0^t F_k(s)\,ds,\quad \forall\, t\in[0,t_k),\end{aligned}$$

where

$$F_k(t) = \int_\Omega |\Delta u'_{1,k}(t)|^2 + |\Delta u'_{2,k}(t)|^2 + |\nabla u'_{1,k}(t)|^2 + |\nabla u'_{2,k}(t)|^2 + |\Delta^2 u_{1,k}(t)|^2\,dx$$

$$+ (1 - c'\|a\|_\infty - \|a\|_\infty) \int_\Omega |\nabla\Delta u_{1,k}(t)|^2\,dx + (1 - c'\|a\|_\infty) \int_\Omega |\Delta u_{2,k}(t)|^2\,dx$$

$$+ (1 - \|a\|_\infty) \int_\Omega |\nabla\Delta u_{2,k}(t)|^2\,dx,$$

and C_1 is a positive constant depending only on $\|a\|_\infty$, $\|\nabla a\|_\infty$ and τ.
So that, thanks to the monotonicity condition on the function g_i and using Gronwall's lemma, we conclude that

$$u_{1,k} \quad \text{is bounded in} \quad L^\infty(0,T; H^4(\Omega) \cap H^3_\Delta(\Omega)), \tag{4.19}$$

$$u_{2,k} \quad \text{is bounded in} \quad L^\infty(0,T; H^3_\Delta(\Omega) \cap H^2(\Omega)), \tag{4.20}$$

$$u'_{1,k} \quad \text{is bounded in} \quad L^\infty(0,T; H^2(\Omega) \cap H^1_0(\Omega)), \tag{4.21}$$

$$u'_{2,k} \quad \text{is bounded in} \quad L^\infty(0,T; H^2(\Omega) \cap H^1_0(\Omega)), \tag{4.22}$$

$$\Delta u'_{i,k} g_i(\Delta u'_{i,k}),\ i = 1,2 \quad \text{is bounded in} \quad L^1(\mathcal{A}),$$

and from (4.4), (4.5) we deduce that

$$g_i(\Delta u'_{i,k}),\ i = 1,2 \quad \text{is bounded in} \quad L^2(\mathcal{A}), \tag{4.23}$$

where $\mathcal{A} = \Omega \times (0,T)$.
We assume first $t < T$ and let $0 < \xi < T - t$. Set

$$u^\xi_{i,k}(x,t) = u_{i,k}(x, t+\xi),\ \ i = 1,2,$$

$$U^\xi_k = u_{1,k}(x,t+\xi) - u_{1,k}(x,t),$$

and

$$D^\xi_k = u_{2,k}(x,t+\xi) - u_{2,k}(x,t).$$

Then, U^ξ_k solves the differential equation

$$((U^\xi_k)'' + \Delta^2 U^\xi_k - a(x)\Delta D^\xi_k - (g_1(\Delta (u^\xi_{1,k})') - g_1(\Delta u'_{1,k})), w_{1,j}) = 0, \quad \forall\, w_{1,j} \in V_k, \tag{4.24}$$

and D^ξ_k solves

$$((D^\xi_k)'' - \Delta D^\xi_k - a(x)\Delta U^\xi_k - (g_2(\Delta (u^\xi_{2,k})') - g_2(\Delta u'_{2,k})), w_{2,j}) = 0, \quad \forall\, w_{2,j} \in V_k. \tag{4.25}$$

Choosing $w_{1,j} = -\Delta(U_k^\xi)'$ in (4.24) and $w_{2,j} = \Delta(D_k^\xi)'$ in (4.25), and using the fact that g_i is nondecreasing, we find

$$\frac{d}{dt}\int_\Omega |\nabla(U_k^\xi)'(x,t)|^2 + |\nabla(D_k^\xi)'(x,t)|^2 + |\nabla\Delta U_k^\xi(x,t)|^2 + |\Delta D_k^\xi(x,t)|^2\,dx$$
$$+2\frac{d}{dt}\int_\Omega a(x)\Delta D_k^\xi(x,t)\Delta U_k^\xi(x,t)\,dx \le 0, \quad \forall\, t \ge 0,$$

integrating in $[0,t]$, to get

$$\int_\Omega |\nabla(U_k^\xi)'(t)|^2 + |\nabla(D_k^\xi)'(t)|^2\,dx + (1 - c'\|a\|_\infty)\int_\Omega |\nabla\Delta U_k^\xi|^2 + |\Delta D_k^\xi(t)|^2\,dx$$
$$\le C_2\int_\Omega |\nabla(U_k^\xi)'(0)|^2 + |\nabla(D_k^\xi)'(0)|^2 + |\nabla\Delta U_k^\xi(0)|^2 + |\Delta D_k^\xi(0)|^2\,dx,$$

and C_2 is a positive constant depending only on $\|a\|_\infty$ and c'.
Dividing by ξ^2, and letting $\xi \to 0$, we find

$$\int_\Omega |\nabla u''_{1,k}(t)|^2 + |\nabla u''_{2,k}(t)|^2 + |\nabla\Delta u'_{1,k}(t)|^2 + |\Delta u'_{2,k}(t)|^2\,dx$$
$$\le C'_2\int_\Omega |\nabla u''_{1,k}(0)|^2 + |\nabla u''_{2,k}(0)|^2 + |\nabla\Delta u^1_{1,k}|^2 + |\Delta u^1_{2,k}|\,dx.$$

We estimate $\|\nabla u''_{i,k}(0)\|$. Choosing $v = -\Delta u''_{i,k}$, $i = 1,2$ and $t = 0$ in (4.7), we obtain that

$$\|\nabla u''_{1,k}(0)\|^2 = \int_\Omega \nabla u''_{1,k}(0)\nabla(-\Delta^2 u^0_{1,k} - a(x)u^0_{2,k} + g_1(\Delta u^1_{1,k}))\,dx,$$

and

$$\|\nabla u''_{2,k}(0)\|^2 = \int_\Omega \nabla u''_{2,k}(0)\nabla(\Delta u^0_{2,k} - a(x)u^0_{1,k} + g_2(\Delta u^1_{2,k}))\,dx.$$

Using Cauchy-Schwarz's inequality, we have

$$\|\nabla u''_{1,k}(0)\| \le \Big[\int_\Omega |\nabla(-\Delta^2 u^0_{1,k} - a(x)u^0_{2,k} + g_1(\Delta u^1_{1,k}))|^2\,dx\Big]^{\frac{1}{2}},$$

and

$$\|\nabla u''_{2,k}(0)\| \le \Big[\int_\Omega |\nabla(\Delta u^0_{2,k} - a(x)u^0_{1,k} + g_2(\Delta u^1_{2,k}))|^2\,dx\Big]^{\frac{1}{2}}.$$

By (4.10) and (4.11) yields

$$(u''_{1,k}(0), u''_{2,k}(0)) \quad \text{is bounded in } \; W. \tag{4.26}$$

By (4.8), (4.9) and (4.26), we deduce that

$$\int_\Omega |\nabla u''_{1,k}(t)|^2 + |\nabla u''_{2,k}(t)|^2 + |\nabla\Delta u'_{1,k}(t)|^2 + |\Delta u'_{2,k}(t)|^2\,dx \le C_3, \quad \forall\, t \ge 0,$$

where C_3 is a positive constant independent of $k \in \mathbb{N}$. Therefore, we conclude that

$$u_{1,k}^{'} \quad \text{is bounded in} \quad L^\infty(0,T;H^3_\Delta(\Omega)), \tag{4.27}$$

$$u_{2,k}^{'} \quad \text{is bounded in} \quad L^\infty(0,T;H^2(\Omega)), \tag{4.28}$$

$$u_{1,k}^{''} \quad \text{is bounded in} \quad L^\infty(0,T;H^1_0(\Omega)), \tag{4.29}$$

$$u_{2,k}^{''} \quad \text{is bounded in} \quad L^\infty(0,T;H^1_0(\Omega)). \tag{4.30}$$

The step 3 : Passage to the limit

Applying Dunford-Pettis and Banach-Alaoglu-Bourbaki theorems, we conclude from (4.19)-(4.23) and (4.27)-(4.30) that there exists a subsequence $\{u_{i,m}\}$ of $\{u_{i,k}\}$, $i = 1, 2$ such that

$$(u_{1,m}, u_{2,m}) \rightharpoonup (u_1, u_2) \quad \text{weak-star in} \quad L^\infty(0,T;\widetilde{V}), \tag{4.31}$$

$$(u_{1,m}^{'}, u_{2,m}^{'}) \rightharpoonup (u_1', u_2') \quad \text{weak-star in} \quad L^\infty(0,T;V),$$

$$(u_{1,m}^{''}, u_{2,m}^{''}) \rightharpoonup (u_1'', u_2'') \quad \text{weak-star in} \quad L^\infty(0,T;W), \tag{4.32}$$

$$g_i(\Delta u_{i,m}^{'}) \rightharpoonup \chi_i,\ i = 1, 2 \quad \text{weak-star in} \quad L^2(\mathcal{A}). \tag{4.33}$$

As $(u_{1,m}, u_{2,m})$ is bounded in $L^\infty(0,T;\widetilde{V})$ (by (4.19), (4.20)), then $(u_{1,m}, u_{2,m})$ is bounded in $L^2(0,T;\widetilde{V})$ and the injection of $\widetilde{V}$ in H is compact, we have

$$(u_{1,m}, u_{2,m}) \longrightarrow (u_1, u_2) \quad \text{strongly in} \quad L^2(0,T;H). \tag{4.34}$$

In the other hand, using (4.31), (4.32) and (4.34), we have

$$\int_0^T \int_\Omega \Big[u_{1,m}^{''}(x,t) + \Delta^2 u_{1,m}(x,t) - a(x)\Delta u_{2,m}(x,t) \Big] w\, dx\, dt \longrightarrow$$
$$\int_0^T \int_\Omega \Big[u_1''(x,t) + \Delta^2 u_1(x,t) - a(x)\Delta u_2(x,t) \Big] w\, dx\, dt, \text{ as } m \to +\infty$$

and

$$\int_0^T \int_\Omega \Big[u_{2,m}^{''}(x,t) - \Delta u_{2,m}(x,t) - a(x)\Delta u_{1,m}(x,t) \Big] w\, dx\, dt \longrightarrow$$
$$\int_0^T \int_\Omega \Big[u_2''(x,t) - \Delta u_2(x,t) - a(x)\Delta u_1(x,t) \Big] w\, dx\, dt, \text{ as } m \to +\infty$$

for all $w \in L^2(0,T;L^2(\Omega))$.

It remains to show the convergence

$$\int_0^T \int_\Omega g_i(\Delta u_{i,m}^{'})\, w\, dx\, dt \longrightarrow \int_0^T \int_\Omega g_i(\Delta u_i')\, w\, dx\, dt, \text{ as } m \to +\infty,\ i = 1, 2.$$

From (4.21), (4.22) and (4.29), (4.30), we have $(u^{'}_{1,k}, u^{'}_{2,k})$ is bounded in $L^{\infty}(0,T;W)$, $(u^{''}_{1,k}, u^{''}_{2,k})$ is bounded in $L^{\infty}(0,T;W)$ and $W \subset H$, then $(u^{''}_{1,k}, u^{''}_{2,k})$ is bounded in $L^{\infty}(0,T;H)$, therefore $(u^{'}_{1,k}, u^{'}_{2,k})$ is bounded in $L^{2}(0,T;W)$ and $(u^{''}_{1,k}, u^{''}_{2,k})$ is bounded in $L^{2}(0,T;H)$, hence $(u^{'}_{1,k}, u^{'}_{2,k})$ is bounded in $(H^1(\mathcal{A}))^2$, and the injection of $H^1(\mathcal{A})$ in $L^2(\mathcal{A})$ is compact, we find that

$$(u^{'}_{1,k}, u^{'}_{2,k}) \to (u^{'}_{1}, u^{'}_{2}) \text{ strongly in } \quad (L^2(\mathcal{A}))^2,$$

consequently, we have

$$(u^{'}_{1,m}, u^{'}_{2,m}) \longrightarrow (u'_1, u'_2) \qquad \text{almost every where in} \qquad \mathcal{A}^2, \tag{4.35}$$

Lemma 3.2 *For each $T > 0$, we have $g_i(\Delta u'_i) \in L^1(\mathcal{A})$, and*

$$\|g_i(\Delta u'_i)\|_{L^1(\mathcal{A})} \leq K,\ i = 1, 2,$$

where K is a constant independent of t, and

$$g_i(\Delta u^{'}_{i,m}) \to g_i(\Delta u'_i),\ i = 1, 2 \text{ in } L^1(\mathcal{A}).$$

Proof. We claim that

$$g_i(\Delta u') \in L^1(\mathcal{A}),\ i = 1, 2.$$

Indeed, since g_i is continuous, we deduce from (4.35)

$$g_i(\Delta u^{'}_{i,m}) \longrightarrow g_i(\Delta u'_i),\ i = 1, 2 \quad \text{almost every where in} \quad \mathcal{A}.$$

$$\Delta u^{'}_{i,m} g_i(\Delta u^{'}_{i,m}) \longrightarrow \Delta u'_i g_i(\Delta u'_i),\ i = 1, 2 \quad \text{almost every where in} \quad \mathcal{A}.$$

Hence, by (4.23) and Fatou's lemma, we have

$$\int_0^T \int_\Omega \Delta u'_i(x,t) g_i(\Delta u'_i(x,t))\, dx\, dt \leq K_1,\ i = 1, 2 \quad \text{for } T > 0.$$

Now, we can estimate $\int_0^T \int_\Omega |g_i(\Delta u'_i(x,t))|\, dx\, dt$, $i = 1, 2$. By Cauchy-Schwarz's inequality.

If $|\Delta u'_i| > 1$, by using (4.5), we have

$$\begin{aligned} \int_0^T \int_\Omega |g_i(\Delta u'_i(x,t))|\, dx\, dt &\leq c|\mathcal{A}|^{1/2} \Big[\int_0^T \int_\Omega |g_i(\Delta u'_i(x,t))|^2\, dx\, dt \Big]^{1/2} \\ &\leq c|\mathcal{A}|^{1/2} \Big[\int_0^T \int_\Omega \Delta u'_i g_i(\Delta u'_i(x,t))\, dx\, dt \Big]^{1/2} \leq K_2,\ i = 1, 2. \end{aligned}$$

If $|\Delta u_i'| \leq 1$, by using (4.4), we have

$$\int_0^T \int_\Omega |g_i(\Delta u_i'(x,t))|\,dx\,dt \leq c|\mathcal{A}|^{1/2}\Big[\int_0^T \int_\Omega |g_i(\Delta u_i'(x,t))|^2\,dx\,dt\Big]^{1/2}$$
$$\leq c|\mathcal{A}|^{1/2}\Big[\int_0^T \int_\Omega \Delta u_i^{'} g_i(\Delta u_i^{'})\,dx\,dt\Big]^{1/2}$$
$$\leq K_3,\ i=1,2 \quad \text{for } T>0.$$

Then

$$\int_0^T \int_{\mathcal{A}} |g_i(\Delta u_i'(x,t))|\,dx\,dt \leq K,\ i=1,2 \quad \text{for } T>0.$$

Let $E \subset \Omega \times [0,T]$ and set

$$E_1 = \Big\{(x,t) \in E : |g_i(\Delta u_{i,m}^{'}(x,t))| \leq \frac{1}{\sqrt{|E|}},\ i=1,2\Big\},$$

$$E_2 = E \backslash E_1,$$

where $|E|$ is the measure of E. If $M(r) = \inf\{|s| : s \in \mathbb{R} \text{ and } |g_i(s)| \geq r,\ i=1,2\}$, we have

$$\int_E |g_i(\Delta u_{i,m}^{'})|\,dx\,dt \leq c\sqrt{|E|} + \Big[\frac{M}{\sqrt{|E|}}\Big]^{-1} \int_{E_2} \Delta u_{i,m}^{'} g_i(\Delta u_{i,m}^{'})\,dx\,dt,\ i=1,2.$$

By applying (4.23) we deduce that

$$\sup_m \int_E |g_i(\Delta u_{i,m}^{'})|\ dx\ dt \longrightarrow 0,\ i=1,2,\ \text{when } |E| \longrightarrow 0.$$

From Vitali's convergence theorem, we deduce that

$$g_i(\Delta u_{i,m}^{'}) \to g_i(\Delta u_i') \quad \text{in } L^1(\mathcal{A}),\ i=1,2.$$

This completes the proof.
Then (4.33) implies that

$$g_i(\Delta u_{i,m}^{'}) \rightharpoonup g_i(\Delta u_i'),\ i=1,2 \text{ weak-star in } L^2([0,T]\times\Omega).$$

We deduce, for all $w \in L^2(0,T;L^2(\Omega))$, that

$$\int_0^T \int_\Omega g_i(\Delta u_{i,m}^{'}) w\,dx\,dt \longrightarrow \int_0^T \int_\Omega g_i(\Delta u_i') w\,dx\,dt,\ i=1,2.$$

Finally we have shown that, for all $w \in L^2(0,T;L^2(\Omega))$:

$$\int_0^T \int_\Omega \Big[u_1''(x,t) + \Delta^2 u_1(x,t) - a(x)\Delta u_2(x,t) - g_1(\Delta u_1'(x,t))\Big] w \, dx \, dt = 0,$$

and

$$\int_0^T \int_\Omega \Big[u_2''(x,t) - \Delta u_2(x,t) - a(x)\Delta u_1(x,t) - g_2(\Delta u_2'(x,t))\Big] w \, dx \, dt = 0.$$

Therefore, (u_1, u_2) is a solution for the problem (4.1).

The step 4 : Proof of uniqueness

To prove the uniqueness of the problem (4.1), we need of the following lemma

Lemma 3.3 *If* (u_1, u_2) *is a solution of the problem* (4.1)*, then we have*

$$\int_\Omega a(x)\Delta u_1 \Delta u_2 \, dx \geq -\frac{c'}{2}\|a\|_\infty \int_\Omega |\nabla \Delta u_1|^2 + |\Delta u_2|^2 \, dx. \tag{4.36}$$

Proof. Using Cauchy-schwartz's inequality and Cauchy's inequality, Sobolev embedding and the condition (4.3), we get

$$\begin{aligned}
\int_\Omega a(x)\Delta u_1 \Delta u_2 \, dx &\geq -\frac{1}{2}\|a\|_\infty \int_\Omega \frac{1}{c'}|\Delta u_1|^2 + c'|\Delta u_2|^2 \, dx \\
&\geq -\frac{1}{2}\|a\|_\infty \int_\Omega \frac{c'^2}{c'}|\nabla \Delta u_1|^2 + c'|\Delta u_2|^2 \, dx \\
&\geq -\frac{c'}{2}\|a\|_\infty \int_\Omega |\nabla \Delta u_1|^2 + |\Delta u_2|^2 \, dx.
\end{aligned}$$

Let (u_1, u_2), $(\widetilde{u}_1, \widetilde{u}_2)$ be two solutions of the problem (4.1) with the same initial data. Let us denote it is straightforward to see that $z_1 = u_1 - \widetilde{u}_1$, $z_2 = u_2 - \widetilde{u}_2$ satisfy

$$\|\nabla z_1'\|^2 + \|\nabla z_2'\|^2 + \|\nabla \Delta z_1\|^2 + \|\Delta z_2\|^2 + 2\int_\Omega a(x)\Delta z_1 \Delta z_2 \, dx +$$

$$2\int_0^t \int_\Omega \Delta z_1'(g_1(\Delta u_1') - g_1(\Delta \widetilde{u}_1')) \, dx \, dt + 2\int_0^t \int_\Omega \Delta z_2'(g_2(\Delta u_2') - g_2(\Delta \widetilde{u}_2')) \, dx \, dt = 0,$$

from (4.36), we have

$$\begin{aligned}
&\|\nabla z_1'\|^2 + \|\nabla z_2'\|^2 + (1 - c'\|a\|_\infty)(\|\nabla \Delta z_1\|^2 + \|\Delta z_2\|^2) \\
&+ 2\int_0^t \int_\Omega \Delta z_1'(g_1(\Delta u_1') - g_1(\Delta \widetilde{u}_1')) \, dx \, dt \\
&+ 2\int_0^t \int_\Omega \Delta z_2'(g_2(\Delta u_2') - g_2(\Delta \widetilde{u}_2')) \, dx \, dt \leq 0,
\end{aligned}$$

we have g_1, g_2 are increasing, we conclude that $2\int_0^t\int_\Omega \Delta z_1^{'}(g_1(\Delta u_1^{'}) - g_1(\Delta \widetilde{u}_1^{'}))\,dx\,dt \geq 0$, $2\int_0^t\int_\Omega \Delta z_2^{'}(g_2(\Delta u_2^{'}) - g_2(\Delta \widetilde{u}_2^{'}))\,dx\,dt \geq 0$. And $1 - c^{'}\|a\|_\infty > 0$ (see (4.3), therefore

$$\|\nabla z_1^{'}\|^2 + \|\nabla z_2^{'}\|^2 + \|\nabla \Delta z_1\|^2 + \|\Delta z_2\|^2 = 0,$$

which implies $z_1 = z_2 = 0$. This finishes the proof of theorem (3.1).

4 Energy estimates

In this section, we prove our stability result for the energy of the solution of system (4.1), using the multiplier technique.

Lemma 4.1 *We define the energy associated with the solution of the problem* (4.1) *by the following formula*

$$E(t) = \frac{1}{2}\int_\Omega |\nabla u_1'|^2 + |\nabla u_2'|^2 + |\nabla \Delta u_1|^2 + |\Delta u_2|^2\,dx + \int_\Omega a(x)\Delta u_1 \Delta u_2\,dx. \tag{4.37}$$

Then E is a non-increasing function, and it is a nonnegative function.

Proof. Multiplying the first equation in (4.1) by $-\Delta u_1'$ and the second equation by $-\Delta u_2'$, and integrating over Ω and using integration by parts and the monotonicity of g_i, $i = 1, 2$, we obtain

$$\frac{1}{2}\frac{d}{dt}\Big[\int_\Omega |\nabla u_1'|^2 + |\nabla u_2'|^2 + |\nabla \Delta u_1|^2 + |\Delta u_2|^2\,dx + 2\int_\Omega a(x)\Delta u_1 \Delta u_2\,dx\Big]$$

$$= -\int_\Omega \Delta u_1' g_1(\Delta u_1') + \Delta u_2' g_2(\Delta u_2')\,dx \leq 0.$$

And using (4.36), we obtain then

$$E(t) \geq \frac{1}{2}\int_\Omega |\nabla u_1'|^2 + |\nabla u_2'|^2 + (1 - c'\|a\|_\infty)(|\nabla \Delta u_1|^2 + |\Delta u_2|^2)\,dx \geq 0.$$

Theorem 4.2 *Let $(u_1^0, u_2^0) \in \widetilde{V}$ and $(u_1^1, u_2^1) \in V$. Assume that* (4.3)-(4.6) *hold. The energy of the unique solution of the system* (4.1), *given by* (4.37) *decay estimate :*

$$E(t) \leq Ct^{-2/(p-1)} \quad \forall\, t > 0, \quad \textit{if} \;\; p > 1,$$

and

$$E(t) \leq C'E(0)e^{-wt} \quad \forall\, t > 0, \;\; \textit{if} \;\; p = 1.$$

Here C is a positive constant only depending on the initial energy $E(0)$, while C' and w are positive constants, independent of the initial data.

Proof. Multiplying the first equation of (4.1) by $-E^{\mu}\Delta u_1$, we obtain

$$
\begin{aligned}
0 &= \int_S^T -E^{\mu}\int_\Omega \Delta u_1(u_1''+\Delta^2 u_1 - a(x)\Delta u_2 + g_1(\Delta u_1'))\,dx\,dt\\
&= -\Big[E^{\mu}\int_\Omega u_1'\Delta u_1\,dx\Big]_S^T + \mu\int_S^T E'E^{\mu-1}\int_\Omega \Delta u_1 u_1'\,dx\,dt\\
&\quad -2\int_S^T E^{\mu}\int_\Omega |\nabla u_1'|^2\,dx\,dt + \int_S^T E^{\mu}\int_\Omega |\nabla u_1'|^2+|\nabla\Delta u_1|^2\,dx\,dt\\
&\quad + \int_S^T E^{\mu}\int_\Omega a(x)\Delta u_1\Delta u_2\,dx\,dt + \int_S^T E^{\mu}\int_\Omega \Delta u_1.g_1(\Delta u_1')\,dx\,dt.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
0 &= \int_S^T -E^{\mu}\int_\Omega \Delta u_2(u_2''+\Delta u_2 - a(x)\Delta u_1 + g_2(\Delta u_2'))\,dx\,dt\\
&= -\Big[E^{\mu}\int_\Omega u_2'\Delta u_2\,dx\Big]_S^T + \mu\int_S^T E'E^{\mu-1}\int_\Omega \Delta u_2 u_2'\,dx\,dt\\
&\quad -2\int_S^T E^{\mu}\int_\Omega |\nabla u_2'|^2\,dx\,dt + \int_S^T E^{\mu}\int_\Omega |\nabla u_2'|^2+|\Delta u_2|^2\,dx\,dt\\
&\quad + \int_S^T E^{\mu}\int_\Omega a(x)\Delta u_2\Delta u_1\,dx\,dt + \int_S^T E^{\mu}\int_\Omega \Delta u_2.g_2(\Delta u_2')\,dx\,dt.
\end{aligned}
$$

Taking their sum, we obtain

$$
\begin{aligned}
\int_S^T E^{\mu+1}\,dt \le \Big[&E^{\mu}\int_\Omega u_1'\Delta u_1 + u_2'\Delta u_2\,dx\Big]_S^T\\
&-\mu\int_S^T E'E^{\mu-1}\int_\Omega \Delta u_1 u_1' + \Delta u_2 u_2'\,dx\,dt\\
&+2\int_S^T E^{\mu}\int_\Omega |\nabla u_1'|^2+|\nabla u_2'|^2\,dx\,dt\\
&-\int_S^T E^{\mu}\int_\Omega \Delta u_1.g_1(\Delta u_1') + \Delta u_2.g_2(\Delta u_2')\,dx\,dt.
\end{aligned}
\tag{4.38}
$$

Since E is non-increasing, we find that

$$
\Big[E^{\mu}\int_\Omega u_1'\Delta u_1 + u_2'\Delta u_2\,dx\Big]_S^T \le cE^{\mu+1}(S),
$$

$$
\mu\Big|\int_S^T E'E^{\mu-1}\int_\Omega \Delta u_1 u_1' + \Delta u_2 u_2'\,dx\,dt\Big| \le cE^{\mu+1}(S).
$$

Using these estimates, we conclude from (4.38) that

$$\int_S^T E^{\mu+1}\,dt \le CE^{\mu+1}(S) + 2\int_S^T E^{\mu}\int_\Omega |\nabla u_1'|^2 + |\nabla u_2'|^2\,dx\,dt$$
$$+\int_S^T E^{\mu}\int_\Omega |\Delta u_1|.|g_1(\Delta u_1')| + |\Delta u_2|.|g_2(\Delta u_2')|\,dx\,dt. \tag{4.39}$$

Now, we estimate the terms of the right-hand side of (4.39).
As in Komornik [**15**], we consider the following partition of Ω,

$$\Omega^+ = \{x\in\Omega : |\Delta u_i'| > 1\},\quad \Omega^- = \{x\in\Omega : |\Delta u_i'| \le 1\}.$$

By using Sobolev embedding and Young's inequality, we obtain

$$\begin{aligned}
&\int_S^T E^{\mu}\int_{\Omega^+} |\Delta u_1|.|g_1(\Delta u_1')|\,dx\,dt + \int_S^T E^{\mu}\int_{\Omega^+} |\nabla u_1'|^2\,dx\,dt\\
&\le \varepsilon \textstyle\int_S^T E^{\mu}\displaystyle\int_{\Omega^+} |\Delta u_1|^2\,dx\,dt + C(\varepsilon)\int_S^T E^{\mu}\int_{\Omega^+} |g_1(\Delta u_1')|^2\,dx\,dt + c\int_S^T E^{\mu}\int_{\Omega^+} |\Delta u_1'|^2\,dx\,dt\\
&\le \varepsilon c'\int_S^T E^{\mu}\int_\Omega |\nabla\Delta u_1|^2\,dx\,dt + \Big[C(\varepsilon)c_2 + \frac{c}{c_1}\Big]\int_S^T E^{\mu}\int_\Omega \Delta u_1' g_1(\Delta u_1')\,dx\,dt\\
&\qquad\qquad \le \varepsilon C\int_S^T E^{\mu+1}\,dt + C_1(\varepsilon)\int_S^T E^{\mu}E'\,dt\\
&\qquad\qquad \le \varepsilon C\int_S^T E^{\mu+1}\,dt + C_1(\varepsilon,\mu)E^{\mu+1}(S).
\end{aligned} \tag{4.40}$$

Similarly, we have

$$\int_S^T E^{\mu}\int_{\Omega^+} |\Delta u_2|.|g_2(\Delta u_2')|\,dx\,dt + \int_S^T E^{\mu}\int_{\Omega^+} |\nabla u_2'|^2\,dx\,dt$$
$$\le \varepsilon C\int_S^T E^{\mu+1}\,dt + C_2(\varepsilon,\mu)E^{\mu+1}(S). \tag{4.41}$$

Summing (4.40) and (4.41), we obtain

$$\begin{aligned}
&\int_S^T E^{\mu}\int_{\Omega^+} |\Delta u_1|.|g_1(\Delta u_1')| + |\Delta u_2|.|g_2(\Delta u_2')|\,dx\,dt\\
&\qquad + \int_S^T E^{\mu}\int_{\Omega^+} |\nabla u_1'|^2 + |\nabla u_2'|^2\,dx\,dt\\
&\qquad \le \varepsilon C\int_S^T E^{\mu+1}\,dt + C(\varepsilon,\mu)E^{\mu+1}(S).
\end{aligned} \tag{4.42}$$

By using Sobolev embedding and Young's inequality, we obtain

$$\begin{aligned}&\int_S^T E^\mu \int_{\Omega^-} |\Delta u_1|.|g_1(\Delta u_1')| + |\nabla u_1'|^2 \,dx\,dt \\ &\le \varepsilon' c' \int_S^T E^\mu \int_\Omega |\nabla \Delta u_1|^2 \,dx\,dt + C(\varepsilon') \int_S^T E^\mu \int_\Omega (|\Delta u_1'|^2 + |g_1(\Delta u_1')|^2) \,dx\,dt \\ &\le \varepsilon' c' \int_S^T E^{\mu+1} \,dt + C(\varepsilon') \int_S^T E^\mu \int_\Omega (\Delta u_1' g_1(\Delta u_1'))^{\frac{2}{p+1}} \,dx\,dt \\ &\le \varepsilon' C \int_S^T E^{\mu+1} \,dt + C(\varepsilon', p) \int_S^T E^\mu \Big[\int_\Omega \Delta u_1' g_1(\Delta u_1') \,dx \Big]^{\frac{2}{p+1}} dt.\end{aligned} \tag{4.43}$$

Similarly, we have

$$\begin{aligned}&\int_S^T E^\mu \int_{\Omega^-} |\Delta u_2|.|g_2(\Delta u_2')| + |\nabla u_2'|^2 \,dx\,dt \\ &\le \varepsilon' C \int_S^T E^{\mu+1} \,dt + C(\varepsilon', p) \int_S^T E^\mu \Big[\int_\Omega \Delta u_2' g_2(\Delta u_2') \,dx \Big]^{\frac{2}{p+1}} dt.\end{aligned} \tag{4.44}$$

Summing (4.43) and (4.44), we obtain

$$\begin{aligned}&\int_S^T E^\mu \int_{\Omega^-} |\Delta u_1|.|g_1(\Delta u_1')| + |\Delta u_2|.|g_2(\Delta u_2')| \,dx\,dt \\ &\qquad + \int_S^T E^\mu \int_{\Omega^-} |\nabla u_1'|^2 + |\nabla u_2'|^2 \,dx\,dt \\ &\qquad \le \varepsilon_0 C \int_S^T E^{\mu+1} \,dt + C(\varepsilon_0, p) \int_S^T E^\mu (-E')^{\frac{2}{p+1}} \,dt \\ &\qquad \le \varepsilon_0 C \int_S^T E^{\mu+1} \,dt + \varepsilon_1 \int_S^T E^{\mu\frac{p+1}{p-1}} \,dt + C(\varepsilon_1, p) E(S).\end{aligned} \tag{4.45}$$

Reporting (4.42) and (4.45) in (4.39), we find

$$\int_\Omega E^{\mu+1} \,dt \le C E(S) + C' E^{\mu+1}(S) + \varepsilon_0 C \int_S^T E^{\mu+1} \,dt + \varepsilon_1 \int_S^T E^{\mu\frac{p+1}{p-1}} \,dt,$$

we choose μ such that

$$\mu \frac{p+1}{p-1} = \mu + 1.$$

Thus we find

$$\mu = \frac{p-1}{2}.$$

Choosing ε_0 and ε_1 small enough, we obtain

$$\int_\Omega E^{\mu+1} \,dt \le C' E(S) + C' E^\mu(0) E(S),$$

where C' is a positive constant independent of $E(0)$.

We may thus complete the proof by applying lemmas 7.1 and 7.2.

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الملخص (بالعربية) :

هذه الأطروحة مخصصة لدراسة الوجود الكلي و الانفجار في الزمن المحدد و السلوك المقارب لحلول بعض المعادلات الزمنية الغير خطية. تتكون هذه الأطروحة من أربعة فصول مخصصة لدراسة الطرح الجيد، و السلوك المقارب، و الانفجار في الزمن المحدد لحلول بعض المعادلات الزمنية مع عبارة الإخماد الغير خطية و بحد التأخر و حد المنبع. في الفصل الأول نذكر ببعض المفاهيم الأساسية التي ستستخدم في هذه الأطروحة. في الفصل الثاني نعتبر معادلة الأمواج الغير خطية الخاضعة للتخميد وحد التأخر وحد المنبع. نبرهن أن الحل ينفجر في زمن محدد إذا كان حد المنبع يتحكم في عبارة التبديد و حد التأخير max{2+l,m}<p وذالك باستعمال طريقة V. Georgiev و G.Todorova ، تحت الشرط الذي تكون فيه الطاقة الابتدائية سالبة. في الفصل الثالث نعتبر معادلة Petrovsky الغير خطية ومعممة الشكل و بتخميد قوي. نبرهن أن هذه المسألة مطروحة جيدا و ذالك بإستعمال طريقة الإكتناز. أما الإستقرار العام للحل نستعمل طريقة Lyapunov. في الفصل الرابع نعتبر جملة الامواج المزدوجة الغير خطية لـ Petrovsky بتخميد قوي. نبرهن أن هذه المسألة مطروحة جيدا و ذالك بإستعمال طريقة الإكتناز. أما استقرار الحل نستعمل الطريقة المضاعفة فنجد أن إستقرارالحل يخضع لتابع اسي ولتابع كثير حدود.

الكلمات المفتاحية : الطرح الجيد، الجملة المزدوجة، التناقص العام، التناقص الأسي، كثير حدود الاضمحلال، طريقة Faedo -Galerkin ، طريقة Lyapunov، طريقة مضاعفة، حد المنبع، حد التأخر، انفجار.

Résumé (en Français) :

La présente thèse est consacrée à l’étude de l’existence globale, explosion en temps fini et le comportement asymptotique des solutions de certaines équations d’évolution non linéaires. Ce travail se compose de quatre chapitres, sera consacré à l’étude du bien-posé, le comportement asymptotique et explosion en temps fini de la solution de certaines équations d’évolution avec un terme d’amortissement non linéaires, un terme de retard et un terme de source. Dans le chapitre 1, nous rappelons quelques notions utilisées dans cette thèse. Dans le chapitre 2, nous considérons l’équation d’onde non linéaire soumis à un amortissement, un terme de retard et un terme de source. Nous prouvons que la solution explose en temps fini si le terme de source domine le terme de dissipatif et le terme de retard p > max{l + 2, m} sous la condition que l’énergie initiale est négative par la méthode de V. Georgiev et G. Todorova. Dans le chapitre 3, nous considérons l’équation de Petrovsky avec un fort amortissement non linéaire et de forme générale. Nous prouvons que ce problème est bien posé en utilisant la méthode de compacité, et pour la stabilité générale de la solution introduisant une méthode de Lyapunov. Dans le chapitre 4, nous considérons un système Petrovsky-onde couplé avec un fort amortissement non linéaire. Nous prouvons la bien posé en utilisant la méthode de compacité, et pour la stabilité de solution introduisant une méthode de multiplicateur, nous trouvons la stabilité exponentielle et polynomiale.

Les mots clés : Bien posé, système couplé, décroissance générale, décroissance exponentielle, polynomiale décroissance, méthode Faedo-Galerkin, méthode Lyapunov, méthode multipliée, terme de source, terme de retard, explosion.

Abstract (en Anglais) :

The present thesis is devoted to the study of global existence, blow-up in finite time and the asymptotic behaviour of the solutions for some nonlinear evolution equations.. This work consists of four chapters, will be devoted to the study of the well-posedness, asymptotic behaviour and blow-up in finite time of the solution of some evolution equations with nonlinear dissipative terms, delay and source terms. In chapter 1, we recall of some notions used in this thesis. In chapter 2, we consider the wave equation with nonlinear source, damping and delay term. We prove that weak solutions to the systems blow up in finite time whenever the initial energy is negative and the exponent of the source terms is more dominant than the exponent of damping terms, we use the method of V. Georgiev and G. Todorova. In chapter 3, we consider the Petrovsky equation with a nonlinear strong damping. We prove, under some appropriate assumptions, that this system is well-posed using the compactness method. Furthermore, the general stability is given by using a combination of the some properties of convex functions with an appropriate Lyapunov functional. In chapter 4, we consider a coupled Petrovsky-wave system with a nonlinear strong damping. We prove well-posedness by using the compactness method, and establish the both exponentialand polynomial decay estimates by introducing a multiplied method.

Keywords : Well-posedness, coupled system, general decay, exponential decay, polynomial decay, Faedo-Galerkin method, Lyapunov method, multiplied method, source term, delay term, blow-up.