N° d'ordre :

REPUBLIQUE ALGERIENNE DEMOCRATIQUE & POPULAIRE

MINISTERE DE L'ENSEIGNEMENT SUPERIEUR & DE LA RECHERCHE SCIENTIFIQUE

UNIVERSITE DJILLALI LIABES
FACULTE DES SCIENCES EXACTES
SIDI BEL ABBES

THESE DE DOCTORAT EN SCIENCES

Présentée par

Noureddine BAHRI

Spécialité : *Mathématiques*
Option : *Equations aux dérivée partielles*

Intitulée

Existence globale, stabilité et explosion en temps fini des solutions de certaines équations d'évolution non linéaire avec retard

Soutenue le : 18 /01/2022
Devant le jury composé de :

Président :	**Ali HAKEM**	Pr. Univ. Djillali Liabes SBA
Examinateurs :	**MOKHTARI Mokhtar**	M. C. A. Univ. Ibn Khaldoun Tiaret
	Mohamed KAINANE MEZADEK	M.C.A. Univ. Benbouali Chlef
	Nadia MEZOUAR	M.C.A. Univ. M. Stambouli Mascara
Directeur de thèse :	**Mama ABDELLI**	Pr. Univ. Djillali Liabes SBA
Co-Directeur de thèse :	**Abbes BENAISSA**	Pr. Univ. Djillali Liabes SBA

Année universitaire : 2021/2022

Acknowledgment

First of all, I would like thank our God fo rhis conciliation to me to achieve this point otherwise, I can not. I am deeply indebted to my supervisor, the adviser of this thesis, Prof. Mama ABDELLI, who gave permanent support and encouraged me by his help and gold advices and encouragements about the choice of my diploma theme.

I would like to thank the co-supervisor, Professor Abbas BENAISSA, who accepted me as a member with him in the laboratory of the University of Djilali Liabis in Sidi Bel Abbas, and he helped me a lot in his scientific orientations and administration.

Special thanks go to Mr.Prof. Ali HAKEM from University of Sidi Bel Abbès to have accepted to preside the jury. I would also like to thank the members of jury Mr.Prof. MOKHTARI Mokhtar M. C. A. Univ. Ibn Khaldoun Tiaret, Mohamed KAINANE MEZADEK M.C.A. Univ. Ben bouali Chlef, Nadia MEZOUAR M.C.A. Univ. M. Stambouli Mascara to have accepted inspect the contents of my thesis, and also for there advices, remarks and orientations.

I would like to say thanks to mathematical division of university Djilali Liabes of Sidi Bel Abbès for their good treatment and for informing us of all that is new.

Contents

Etude de l'existence globale et de stabilisation de certains systèmes hyperboliques

La présente thèse est consacrée à l'étude du bien-posé, du comportement asymptotique et de l'explosion en temps fini des solutions pour certaines équations d'évolution non linéaires. Ce travail se compose de cinq chapitres, sera consacré à l'étude du bien-posé, le comportement asymptotique et l'explosion en temps fini de la solution de certaines équations d'évolution avec un terme d'amortissement non linéaires, un terme de retard et un terme de source. Dans le chapitre 1, nous rappelons quelques notions utilisées dans cette thèse. Dans le chapitre 2, nous considèrons un problème de transmission onde-onde soumis à un terme d'amortissement non linéaire dans la première équation. Nous prouvons, sous certaines hypothèses appropriées, que ce système est bien posé dans des espaces de Sobolev par la méthode de Faedo-Galerkin. De plus, la stabilité générale est donnée en utilisant une combinaison des certaines propriétés des fonctions convexes avec la fonction de Lyapunov. Dans le chapitre 3, nous considèrons un système constitué de deux équations de Petrovsky couplées soumis à une dissipation non-linéaire forte et de forme générale pour chaque équation. Sous des conditions sur les termes de dissipation, nous prouvons l'existence et l'unicité d'une solution globale en se basant sur les approximations de Faedo-Galarkin. Et nous utilisons la méthode des multiplicateur pour trouver une stabilité générale. Dans le chapitre 4, nous considèrons un problème de transmission onde-onde soumis à un terme de Viscoélastique en présence d'un terme de retard pour une seul équation, nous montrons l'existence et l'unicité d'une solution globale faible et forte dans des espaces de Sobolev par la théorie des semi-groupes, ainsi que la stabilité uniforme (exponentielle) du système par la méthode de Lyapunov. Dans le chapitre 5, nous considèrons une équation d'onde d'ordre supérieur avec un amortissement non-linéaire, un terme de retard et un terme de source, nous montrons que la solution explose en temps fini où l'énergie initiale est suffisamment négative. Pour cela nous utilisons la méthode de concavité de H. Levine et nous donnons une estimation du temps d'explosion.

Mots clés:
Bien posé, système couplé, problème de transmission, décroissance exponentiel, décroissance générale, méthode de Faedo-Galerkin, théories de semi-groupe, la méthode de Lyapunov, terme viscoélastique, terme de retard, explosion en temps fini.

Study of the global existence and stabilization of some hyperbolic systems

The present thesis is devoted to the study of well-posedness and asymptotic behaviour in time of solution of some hyperbolic systems. This work consists of five chapters, will be devoted to the study of the well-posedness and asymptotic behaviour of some evolution equations with nonlinear dissipative terms, delay and other nonlinear terms. Inthis thesis is composed of 5 chapters including:

In chapter 1, we recall of some notions used in this thesis. In chapter 2, we consider the transmission problems with a nonlinear damping in the first equation. We prove, under some appropriate assumptions, that this system is well-posed using the Faedo-Galerkin schem. Furthermore, the general stability is given by using a combination of the some properties of convex functions with an appropriate Lyapunov functional. In chapter 3, we consider a coupled Petrovsky system with a nonlinear strong damping. We prove well-posedness by using Faedo-Galerkin method and establish an general decay result by introducing a multiplied method. In chapter 4, we consider a transmission problem with history and delay in the first equation. We prove well-posedness by using the semigroup theory and establish an exponential decay result by introducing a suitable Lyaponov functional.

In chapter 5, we consider a nonlinear higher-order equation with delay and source term the non-positive initial energy, it is proved that the solution blows up in the finite time.

Keywords Well-posedness, Coupled system, transmission problems, Exponential decay, general decay, Galerkin method, semigroup theory, Lyapunov method, Viscoelastic term, Delay term, blow-up.

General Introduction

In any case, if we are looking for a mathematical notion, we are often confused with an approximate science. Indeed, questions and problems evolve following the system of mathematicians knowledge and the formulation of this notion can take a long time before to have his final state.
Partial differential systems of the second order have been studied by several authors, including :
V. Komornik [32], P. A. Raviart, J. M. Thomas [66] that they treated the order two linear wave equation with the initial conditions and the boundary conditions, and they give a result of existence of a solution.
S. Guesmia [24] who is presented some problems with the bounds for the nonlinear partial differential equations. For each problem he is interested in the existence and the uniqueness of the solution.
J. L. Lions [45] who studied two partial differential equations, the first one studied in our memory and the second defined by

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + |u|^\rho u = f, \ \frac{\partial^2 u}{\partial t^2} - \Delta u + \left|u^{'}\right|^\rho u^{'} = f. \tag{1}$$

This work aims to study the existence, stability of the solutions of a nonlinear partial hyperbolic derivative systems.
For the sake of stability, we have serial methods of proof, including the following:
1) Strong stabilization : $E(t) \longrightarrow 0$ as $t \longrightarrow \infty$.
2) Logarithm stabilization : $E(t) \leq c\,(lnt)^{-\delta}\, c, \delta > 0$.
3) Polynomial stabilization :$E(t) \leq ct^{-\delta t}/c, \delta > 0$.
4) Uniform stabilization : $E(t) \leq ce^{-\delta t}/c, \delta > 0$.
More precisely we are interested in uniform and polynomial stability, and for the existence we are interested in method of Galerkin and dissipative (see [8]).
This works has been carried out exactly by many of the authors, for example (see [72], [70], [58], [18],[9], [14], [29]).
To carry out this work, we organized our research in forth important parts :
• In the chapter 1 :
We devoted to functional analysis reminders, we will evoke some reminders on the functional spaces, we are interested in the Hilbert space, distribution, theory of semi-group and the Galerkin method which contains three stages and also study the theorem of uniqueness.
• In the chapter 2 :
well-posedness and general energy decay of solution for transmission problems with weakly nonlinear dissipative

We study a nonlinear transmission problem

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t) + \mu g(u_t(x,t)) = 0, & (x,t) \in \Omega \times \mathbb{R}^+ \\ v_{tt}(x,t) - bv_{xx}(x,t) = 0, & (x,t) \in [L_1, L_2] \times \mathbb{R}^+ \end{cases} \tag{0.1}$$

where $0 < L_1 < L_2 < L_3,\ \Omega =]0, L_1[\cup]L_2, L_3[,\ a, b, \mu$ are positives constants. This system is supplemented with the following boundary and transmission conditions:

$$\begin{aligned} &u(0,t) = u(L_3,t) = 0, \\ &u(L_i,t) = v(L_i,t),\ i = 1,2, \\ &au_x(L_i,t) = bv_x(L_i,t),\ i = 1,2, \end{aligned} \tag{0.2}$$

and initial conditions

$$\begin{aligned} &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ &v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in]L_1, L_2[. \end{aligned} \tag{0.3}$$

Historical reminders:
When $g(u_t(x,t)) = u_t(x,t)$ system (0.1)-(0.3) has been investigated in [9], for $\Omega = [0, L_1]$, the authors showed the well-posedness and exponential stability of the total energy.
Ma and Oquendo [51] considered transmission problem involving two Euler-Bernoulli equations modeling the vibrations of a composite beam. By using just one boundary damping term in the boundary, they showed the global existence and decay property of the solution.
Main results of the chapter 2. We assume that the function $g : \mathbb{R} \to \mathbb{R}$ is a non-decreasing function of the class $\mathcal{C}(\mathbb{R})$ such that there exist $\varepsilon, c_2 > 0$ and a convex and increasing function $G : \mathbb{R}^+ \to \mathbb{R}^+$ of the class $\mathcal{C}^1(\mathbb{R}^+) \cap \mathcal{C}^2(]0, +\infty[)$ satisfying

$$\begin{aligned} &G(0) = 0 \textit{ and } G \textit{ is linear on } [0, \varepsilon] \textit{ or} \\ &G'(0) = 0 \textit{ and } G'' > 0 \textit{ on }]0, \varepsilon] \textit{ such that} \\ &|g(s)| \le c_2 |s| \quad \textit{if } |s| > \varepsilon \\ &s^2 + g^2(s) \le G^{-1}(sg(s)) \quad \textit{if } |s| \le \varepsilon \\ &|g'(s)| \le \tau. \end{aligned} \tag{0.4}$$

We use the Faedo-Galerkin method combined with the energy estimate method to prove the existence of global solutions and we use the Lyapunov method to study the decay of the energy.

Theorem 0.1 *Suppose that* $(u^0, v^0) \in H^2(\Omega) \times H^2(L_1, L_2) \cap H_0^1(\Omega) \times H_0^1(L_1, L_2)$, $(u^1, v^1) \in H_0^1(\Omega) \times H_0^1(L_1, L_2)$ *and that assumption (0.4) holds. Then (0.1)-(0.3) admits a unique global solution*

$$\begin{aligned} (u, v) &\in L^\infty(0, T, H^2(\Omega) \times H^2(L_1, L_2) \cap H_0^1(\Omega) \times H_0^1(L_1, L_2)), \\ (u_t, v_t) &\in L^\infty(0, T, H_0^1(\Omega) \times H_0^1(L_1, L_2)), \\ (u_{tt}, v_{tt}) &\in L^\infty(0, T, L^2(\Omega) \times L^2(L_1, L_2)). \end{aligned}$$

The energies of first and second order associated with system (0.1)-(0.3) are defined as follows:

$$E_1(t) = \frac{1}{2} \int_\Omega u_t^2(x,t)\,dx + \frac{a}{2} \int_\Omega u_x^2(x,t)\,dx,$$

$$E_2(t) = \frac{1}{2} \int_{L_1}^{L_2} v_t^2(x,t)\,dx + \frac{b}{2} \int_{L_1}^{L_2} v_x^2(x,t)\,dx.$$

The total energy is defined as

$$E(t) = E_1(t) + E_2(t).$$

Theorem 0.2 *Let (u, v) be the solution of (0.1)- (0.3). Assume that (0.4) holds and*

$$\frac{b}{a} < \frac{L_3 + L_1 - L_2}{2\left(L_2 - L_1\right)}$$

Then there exist positive constants k_1, k_2, k_3 and ε_0 such that the solution of the problem (0.1)-(0.3) satisfies

$$E(t) \leq k_3 G_1^{-1}\left(k_1 t + k_2\right), \quad \forall\, t \in \mathbb{R}_+,$$

where

$$G_1(t) = \int_t^1 \frac{1}{sG_2'(\varepsilon_0 s)} ds, \qquad G_2(t) = tG'(\varepsilon_0 t),$$

with G_1 is strictly decreasing and convex on $]0,1]$, and $\lim\limits_{t \longrightarrow 0} G_1(t) = +\infty$.

• In the chapter 3 :

well-posedness and estimates for a Petrovsky-Petrovsky system with a nonlinear dissipative term

Let us consider the following problem:

$$\begin{cases} u_{tt} + \alpha v + \Delta^2 u - g(\Delta u_t(x,t)) = 0, & in \quad \Omega \times \mathbb{R}^+, \\ v_{tt} + \alpha u + \Delta^2 v - g(\Delta v_t(x,t)) = 0, & in \quad \Omega \times \mathbb{R}^+, \\ u = \Delta u = v = \Delta v = 0 & on \quad \Gamma \times \mathbb{R}^+, \\ (u(0,x), v(0,x)) = (u^0(x), v^0(x)) & in \quad \Omega, \\ (u_t(0,x), v_t(0,x)) = (u^1(x), v^1(x)) & in \quad \Omega, \end{cases} \tag{0.5}$$

where α satisfy the following condition

$$\alpha \leq \frac{1}{2C_s}, \tag{0.6}$$

where $C_s > 0$ depending only on the geometry of Ω is the constant such that

$$\|\nabla z\|^2 \leq C_s \|\nabla \Delta z\|^2.$$

Introduce three real Hilbert spaces $\mathcal{H}$, V and W by setting

$$\mathcal{H} = H_0^1\left(\Omega\right), \quad V = \left\{ z \in H^3\left(\Omega\right) : z = \Delta z = 0 \text{ on } \Gamma \right\}$$

and

$$W = \left\{ z \in H^5\left(\Omega\right) : z = \Delta z = \Delta^2 z = 0 \text{ on } \Gamma \right\}.$$

Historical reminders:

Komornik [**70**] studied the following nonlinear Petrovsky system with a strong damping

$$\begin{cases} u_{tt}(x,t) + \Delta^2 u(x,t) - g(\Delta u_t) = 0, & x \in \Omega \times [0, +\infty[, \\ u(0,t) = \Delta u = 0, & x \in \Gamma \times [0, \infty[, \\ u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x), & x \in \Omega \times [0, +\infty[. \end{cases}$$

He used semigroup approach for sitting the well possedness and he studied the strong stability of this system by introducing a multiplier method combined with a nonlinear integral inequalities. Recently, M. Bahlil et al. [3], studied the system

$$\begin{cases} u_{tt}^1 + a(x)u^2 + \Delta^2 u^1 - g(u_t^1(x,t)) = f_1(u^1, u^2), & in \quad \Omega \times \mathbb{R}^+, \\ u_{tt}^2 + a(x)u^1 - \Delta u^2 - g(u_t^2(x,t)) = f_2(u^1, u^2), & in \quad \Omega \times \mathbb{R}^+, \\ \partial_\nu u^1 = u^1 = u^2 = 0, & on \quad \Gamma \times \mathbb{R}^+, \end{cases} \tag{0.7}$$

for g do not necessarily having a polynomial growth near the origin, by using Faedo-Galerkin method to prove the existence and uniqueness of solution and established energy decay results depending on g. A. Guesmia **[23]** studied the system (0.7) with a nonlinear damping $g(u_t^i)$. He used semigroup approach for sitting the well possedness and he showed the uniform exponential and polynomial decay of solution by introducing a multiplier method combined with a nonlinear integral inequalities given by Martinez **[51]**.

Main results of the chapter 3. Assume that $g(s)$ satisfy the following hypotheses, the function $g:\mathbb{R}\longrightarrow\mathbb{R}$ is a non-decreasing continuous function such that there exist constants $\varepsilon, c_1, c_2, \tau>0$ and a convex increasing function $G:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ of class $\mathcal{C}^1(\mathbb{R}_+)\cap\mathcal{C}^2(\mathbb{R}_+^*)$ satisfying G linear on $[0,\varepsilon]$ or ($G'(0)=0$ and $G''>0$ on $]0,\varepsilon]$, such that

$$c_1|s|\le|g(s)|\le c_2|s|,\ \text{if } |s|>\varepsilon, \tag{0.8}$$

$$|s|^2+|g(s)|^2\le G^{-1}(sg(s)),\ \text{if } |s|\le\varepsilon, \tag{0.9}$$

$$|g'(s)|\le\tau. \tag{0.10}$$

Now we define the energy associated to the solution of the system (0.5) by

$$E(t):=\frac{1}{2}\|\nabla u_t\|^2+\frac{1}{2}\|\nabla v_t\|^2+\frac{1}{2}\|\nabla\Delta u\|^2+\frac{1}{2}\|\nabla\Delta v\|^2+2\alpha\int_\Omega\nabla u.\nabla v\,dx.$$

We use the Faedo-Galerkin method combined with the energy estimate method to prove the existence of global solutions and we use some properties of convex functions to study the decay of the energy.

Theorem 0.3 *Let $(u^0,v^0)\in W$ and $(u^1,v^1)\in V$ arbitrarily, assume that (0.6) and (0.8)-(0.10) hold. Then the problem (0.5) has a unique weak solution satisfying*

$$(u,v)\in L^\infty(\mathbb{R}_+;W),$$

$$(u_t,v_t)\in L^\infty(\mathbb{R}_+;V)$$

and

$$(u_{tt},v_{tt})\in L^\infty(\mathbb{R}_+;\mathcal{H})\cap L^2(0,T,H_0^2(\Omega)).$$

Theorem 0.4 *Let $(u^0,v^0)\in W$ and $(u^1,v^1)\in V$ arbitrarily, assume that (0.6) and (0.8)-(0.10) hold. Then the global solutions of the problem (0.5) have the following asymptotic property*

$$E(t)\le\psi^{-1}\Big(h(t)+\psi(E(0))\Big),\ \forall t\ge0,$$

where $\psi(t)=\int_t^1\frac{1}{\omega\Psi(s)}\,ds$ *for* $t>0$, $h(t)=0$ *for* $0\le t\le\frac{E(0)}{\omega\Psi(E(0))}$ *and*

$$h^{-1}(t)=t+\frac{\psi(t+\psi(E(0)))}{\Psi(\psi^{-1}(t+\psi(E(0))))},\ \forall t\ge\frac{E(0)}{\Psi(E(0))},$$

$$\varphi(t)=\begin{cases} t & \text{if } G \text{ is linear on} \quad [0,\varepsilon]\\ tG'(\varepsilon_0 t) & \text{if } G'(0)=0 \text{ and } G''>0 \text{ on} \quad]0,\varepsilon],\end{cases}$$

where ω and ε are positive constants.

• In the chapter 4 :

Exponential stability of a transmission problem with history and delay

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t) + \displaystyle\int_0^{\infty} g(s)u_{xx}(x,t-s)\mathrm{d}s \\ \qquad\qquad\qquad\qquad +\mu u_t(x,t-\tau) = 0, \quad (x,t) \in \Omega \times (0,+\infty), \\ v_{tt}(x,t) - bv_{xx}(x,t) = 0, \qquad\qquad\qquad (x,t) \in (L_1, L_2) \times (0,+\infty). \end{cases} \tag{0.11}$$

Under the boundary and transmission conditions

$$\begin{array}{l} u(0,t) = u(L_3,t) = 0, \\ u(L_i,t) = v(L_i,t), \quad i = 1,2, \\ au_x(L_i,t) - \displaystyle\int_0^{\infty} g(s)u_x(L_i,t-s)\mathrm{d}s = bv_x(L_i,t), \quad i = 1,2, \end{array} \tag{0.12}$$

and the initial conditions

$$\begin{array}{l} u(x,-t) = u_0(x,t), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), \quad x \in \Omega,\ t \in (0,\tau), \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in (L_1,L_2), \end{array} \tag{0.13}$$

where $0 < L_1 < L_2 < L_3$,$\Omega =]0, L_1[\bigcup]L_2, L_3[, a, \mu, b$ are positive constants,u_0 is given history, and $\tau > 0$ is the delay.
In certain cases also it is proved by the dissipative method.
Transmission problems ([53], [55]) arise in several applications of physics and biology. We note that problem (0.11)-(0.13) is related to the wave propagation over a body which consists of two different type of materials: the elastic part and the viscoelastic part that has the past history and time delay effect.

Historical reminders:

In [72] the authors examined a system of wave equations with a linear boundary damping term with a delay:

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t) + \displaystyle\int_0^{\infty} g(s)u_{xx}(x,t-s)\mathrm{d}s \\ \qquad\qquad +\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, \quad (x,t) \in \Omega \times (0,+\infty), \\ v_{tt}(x,t) - bv_{xx}(x,t) = 0, \qquad\qquad\qquad (x,t) \in (L_1, L_2) \times (0,+\infty), \end{cases} \tag{0.14}$$

and under the assumption

$$\mu_2 \leq \mu_1 \tag{0.15}$$

they proved that the solution is exponentially stable. On the contrary, if (0.15) does not hold, they found a sequence of delays for which the corresponding solution of (0.14) will be unstable.
In [58], authors considered the equation

$$u_{tt}(x,t) - \Delta u(x,t) - \mu_1 \Delta u_t(x,t) - \mu_2 \Delta u_t(x,t-\tau) = 0,$$

and under the assumption

$$|\mu_2| \leq \mu_1,$$

they proved the well-posedness and the exponential decay of energy.

We assume that the function $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is of class C^1 satisfying:

$$g(0) > 0, \quad a - \int_0^\infty g(t)dt = a - g_0 = l > 0. \tag{0.16}$$

There exists a positive constant δ,

$$g'(s) \leq -\delta g(s) \quad \forall s \in \mathbb{R}^+. \tag{0.17}$$

Let $V := (u, v, \varphi, \psi, z, \eta^t)^T$, then V satisfies the problem

$$\begin{cases} V_t = (\mathscr{A} + \mathscr{B})V(t), \quad t > 0, \\ V(0) = V_0, \end{cases}$$

where $V_0 := (u_0(\cdot, 0), v_0, u_1, v_1, f_0(\cdot, -\tau), \eta_0)^T$. The operator $\mathscr{A}$ and $\mathscr{B}$ are linear and defined by

$$\mathscr{A}(u, v, \varphi, \psi, z, w) = l\varphi\psi u_{xx} + \int_0^{+\infty} g(s)w_{xx}(s)\mathrm{d}s - \mu\varphi - \mu z(., 1)bv_{xx} - \frac{1}{\tau}z_\rho - w_s + \varphi$$

and

$$\mathscr{B}(u, v, \varphi, \psi, z, \eta^t)^T = \mu(0, 0, \varphi, 0, 0, 0)^T,$$

where

$$X_* = \Big\{(u, v) \in H^1(\Omega) \times H^1(L_1, L_2) : u(0, t) = u(L_3, t) = 0, u(L_i, t) = v(L_i, t), \\ lu_x(L_i, t) + \textstyle\int_0^\infty g(s)\eta_x^t(L_i, s)\mathrm{d}s = bv_x(L_i, t), i = 1, 2\Big\}$$

and $L_g^2(\mathbb{R}_+, H^1(\Omega))$ denotes the Hilbert space of H^1-valued functions on $\mathbb{R}_+$.
The domain of $\mathscr{A}$ is

$$D(\mathscr{A}) = \Big\{(u, v, \varphi, \psi, z, w)^T \in \mathscr{H} : (u, v) \in \{(H^2(\Omega) \times H^2(L_1, L_2)) \cap X_*\}, \varphi \in H^1(\Omega), \\ \psi \in H^1(L_1, L_2), w \in L_g^2\left(\mathbb{R}_+, H^2(\Omega) \cap H^1(\Omega)\right), w_s \in \left(\mathbb{R}_+, H^1(\Omega)\right), \\ z_\rho \in L^2((0, 1), L^2(\Omega)), w(x, 0) = 0, z(x, 0) = \varphi(x)\Big\}.$$

Where $L_g^2(\Omega) = \left\{u : \Omega \longrightarrow \mathbb{R} measurable / \int_\Omega g\left|f(x)\right|^2 < +\infty\right\}$ with respect to the inner product

$$\langle u, v\rangle_{L_g^2} = \int_{\mathbb{R}^n} gu.v\, dx$$

and $D(\mathscr{B}) = \mathscr{H}$.
Using a semigroup theorem, we prove the existence and uniqueness of global solution under suitable assumptions on the weight of damping and the weight of distributed delay. Also we establish the exponential stability of the solution by introducing a suitable Lyapunov function.

Main results of the chapter 4.

Theorem 0.5 *Assume that (0.16) and (0.17) hold. Let* $V_0 \in \mathscr{H}$*, then there exists a unique weak solution* $V \in C(\mathbb{R}_+, \mathscr{H})$ *of problem (0.11)-(0.13). Moreover, if* $V_0 \in D(\mathscr{A})$*, then*

$$V \in C(\mathbb{R}_+, D(\mathscr{A})) \cap C^1(\mathbb{R}_+, \mathscr{H}).$$

Theorem 0.6 *Let* (u, v) *be the solution of (0.11)-(0.13). Assume that (0.16) and (0.17) hold, and that*

$$a > \frac{8(L_2 - L_1)}{L_1 + L_3 - L_2} l, \quad b > \frac{8(L_2 - L_1)}{L_1 + L_3 - L_2} l,$$

then there exist two constants $\gamma_1, \gamma_2 > 0$ *such that,*

$$E(t) \leq \gamma_2 e^{-\gamma_1 t}, \forall t \in \mathbb{R}_+.$$

• In the chapter 5 :
Blow-up of result in a nonlinear higher-order equation with delay and source term
In this chapter we are concerned with the following initial boundary value problem .

$$\begin{cases} u_{tt}(x,t) + \mathcal{A}u(x,t) + \mu_1 |u_t(x,t)|^{m-2} u_t(x,t) & \\ +\mu_2 |z(x,1,t)|^{m-2} z(x,1,t) = b|u(x,t)|^{p-2} u(x,t), & \text{in } \Omega \times]0, +\infty[, \\ D^\alpha u(x,t) = 0, \quad |\alpha| \leq k-1, & \text{on } \partial\Omega \times [0, +\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in } \Omega, \\ u_t(x, t-\tau) = f_0(x, t-\tau), & \text{in } \Omega \times]0, \tau[, \end{cases} \tag{0.18}$$

where $\mathcal{A} = (-\Delta)^k$, $k \geq 1$, $p > 1$ are real numbers, Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega$, Δ is the Laplace operator in $\mathbb{R}^n$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $|\alpha| = \Sigma_{i=1}^n \alpha_i$, $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2 ... \partial^{\alpha_n} x_n}$, $x = (x_1, x_2, ..., x_n)$, μ_1 and μ_2 are positive real numbers, $\tau > 0$ is a time delay, and the initial data (u_0, u_1, f_0) are in a suitable function space.
Main results of the chapter 5.

Lemma 0.7 *Suppose that*

$$2 \leq p < +\infty \ \ (n \leq 2k) \ \ or \ \ 2 \leq p \leq 2n/(n-2k) \ \ (n > 2k), \tag{0.19}$$

holds. Then there exists a positive constant C *depending on* Ω *only such that*

$$\|u\|_p^s \leq C(\|u\|_p^p + \|\mathcal{A}^{\frac{1}{2}} u\|_2^2),$$

for any $u \in H_0^k(\Omega)$ *and* $2 \leq s \leq p$.

Theorem 0.8 *Suppose that* $m > 1,\ p > \max\{2, m\}$ *satisfying (0.19), let* $u_0 \in H^{2k}(\Omega) \cap H_0^k(\Omega),\ u_1 \in H_0^k(\Omega)$ *and* $f_0 \in H_0^k(\Omega \times (0,1))$. *Assume further that*

$$E(0) = \frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\|\mathcal{A}^{\frac{1}{2}} u_0\|_2^2 - \frac{b}{p}\|u_0\|_p^p + \xi \int_\Omega \int_0^1 |f_0(x, -\rho\tau|^m d\rho\, dx < 0.$$

Then the solution of (0.18) blow up in finite time, i.e. there exists $T_0 < +\infty$ *such that*

$$\lim_{t \to T_0^-} (\|u_t\|_2^2 + \|\mathcal{A}^{\frac{1}{2}} u\|_2^2 + \|u\|_p^p) = \infty.$$

Chapter 1

Preliminary and Main Results

The objective of these chapters is to provide the basic tools necessary to understand the concepts that will be manipulated throughout this manuscript. We will present the definitions, as well as the directly useful properties thereafter.
All assertions in the first chapter are made without evidence (see [13], [37], [44], [71], [20]).

1 Generals Information on Analysis Functional

Partials Equations System.

Definition 1.1 *: A system of partial differential equations is a collection of several unknown functions.*

By example we have some of the partial differential **equation linear**

$$\begin{array}{ll}
\sum_{i=1}^{n} u_{x_i x_i} = 0 & \textit{Laplace Equation.} \\
u_t + \sum_{i=1}^{n} b_i u_{x_i} = 0 & \textit{Transport Equation.} \\
u_t - \Delta u = 0 & \textit{Diffusion equation (chaleur).} \\
u_{tt} - \Delta u = 0 & \textit{Andes equation.} \\
iu_t + \Delta u = 0 & \textit{Schroodinger equation.} \\
u_t + u_{xxxx} = 0 & \textit{Faisceau equation.}
\end{array}$$

And **Non-linear equation**

$$\begin{array}{ll}
\Delta u = f(u) & \textit{Poisson non-linear equation.} \\
div(|\nabla u|^{p-2}\, \nabla u) = 0 & \textit{P-Laplacian equation.} \\
u_{tt} - div\ a.(\nabla u) = 0 & \textit{Andes non-linear equation.} \\
u_t - \Delta u = f(u) & \textit{Diffusion non linear equation (chaleur).}
\end{array}$$

2 Functional Spaces

★ **Banach Space**

Definition 2.1 *:*
*Let (X, d) the metric space, we called the **Cauchy sequence** of X all sequence $(x_n)_{n\in\mathbb{N}}$ is elements of X such that*

$$\forall\epsilon > 0,\ \exists n_0 \in \mathbb{N}\ /\forall n \geq n_0,\ \forall m \geq n_0,\ d\left(x_n, x_m\right) < \epsilon.$$

Definition 2.2 *: [60]*
*Let (X, d) is a space metric, we say that this space is **complete** if and only if any continuation of Cauchy is convergent.*

Definition 2.3 *: [60]*
Let E is a vector space on $\mathbb{K}\,(\mathbb{R}$ where $\mathbb{C})$. We say that an N application of E in $\mathbb{R}^+$ denoted by $||.||_E$, is a E norm if and only if the following three conditions are met:

1. $\forall x \in E;\ ||x||_E = 0 \Longleftrightarrow x = 0$ *(Define positive).*
2. $\forall x \in E,\ \ \forall\lambda \in \mathbb{K};\ \ ||\lambda x||_E = |\lambda|\,||x||_E$ *(Homogeneity).*
3. $\forall x, y \in E;\ \ ||x + y||_E \leq ||x||_E + ||y||_E$ *(triangular inequality).*

Definition 2.4 *: [13]*
*Let E be a vector space and $||.||$ a norm on E, the pair $(E, ||.||_E)$ is called **normed space**.*

Proposition 2.5 *: [60]*
Let $(E, ||.||_E)$ a normed space, then the application defined by

$$\begin{cases} E \times E \to \mathbb{R}^+ \\ (x, y) \to ||x - y|| \end{cases}$$

is a distance on E, called distance associated with the norm $||.||$.

Definition 2.6 *: [60]*
*Let $(E, ||.||_E)$ be a normed space. On dit que $(E, ||.||_E)$ is a space of **Banach** if and only if the metric space $(E, d))$ where d is a distance associated with the standard norm $||.||$ (i.e $d\,(x, y) = ||x - y||$) is a complete space.*

★ **Hilbert Spaces**

Definition 2.7 *:[13] Let E be a vector space on $\mathbb{K}$. We say that E has a scalar product if there is an application*

$$\begin{array}{lcl} h : E \times E & \to & \mathbb{K}, \\ (u, v) & \to & h(u, v) = (u, v)\ , \end{array}$$

checking the following properties:
For all u, v and $w \in E$ and $\alpha, \beta \in \mathbb{K}$,

1. $(u,v) = \overline{(u,v)}$ *(Hermitian).*
2. $(\alpha u + \beta v, w) = \alpha\,(u,w) + \beta\,(v,w)\,; \quad (u, \alpha v + \beta w,) = \overline{\alpha}\,(u,v) + \overline{\beta}\,(u,w)\,.$ *(Sesquilinear).*
3. $(u,u) \geq 0$ *et* $(u,u) = 0 \Longleftrightarrow u = 0$ *(Define positive).*

A space with a dot product is called ***Prehilbertian****.*

Definition 2.8 *:[13]*
A space of ***Hilbert***[1] ***.*** *be a vector space H equipped with a inner product (u,v) and is* ***complete*** *with the norm $(u,u)^{\frac{1}{2}}$ (i.e $|u| = (u,u)^{\frac{1}{2}}$).*

★ Sobolev Spaces

Definition 2.9 *:[13]*
Let $p \in \mathbb{R}$ where $1 \leq p < +\infty$, we define the ***Lebesgue*** [2] *space $L^p(\Omega)$ by:*

$$L^p(\Omega) = \left\{ f : \Omega \longrightarrow \mathbb{R} \text{ measurable and} \int_\Omega |f(x)|^p \, dx < +\infty \right\}.$$

$L^p(\Omega)$ is equipped with a norm

$$\|f\|_{L^p} = \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

Definition 2.10 *:[13]*
The space $L^\infty(\Omega)$ define by :

$$L^\infty(\Omega) = \left\{ u : \Omega \to \mathbb{R} \;\middle|\; \begin{array}{l} u \text{ is measurable and there is a constant } C > 0 \\ \text{such that } |u(x)| \leq C \text{ a.e on } \Omega \end{array} \right\},$$

and we denote :

$$\|u\|_\infty = \inf\{C > 0, |u| \leq C \text{ a.e on } \Omega\}.$$

Theorem 2.11 *: $L^p(\Omega)$ is equipped with a norm $\|.\|_p$ is a* ***Banach***[3] ***space****, pour tout $1 \leq p \leq \infty$.*

Remark 2.12 *A prime example of a Hilbert space in the case $p = 2$ is the collection of* ***square integrable functions*** *on Ω, and consist of all complex-valued measurable functions f that satisfy*

$$\int_\Omega |f(x)|^2 \, dx < \infty.$$

[1] David Hilbert is a German mathematician born on January 23, 1862 ?a Konigsberg in Prussia Eastern and died on February 14, 1943 in Gottingen, Germany. He is often considered one of the greatest mathematicians of the twentieth century, with the same title as Henri Poincaré. He created or developed a wide range of fundamental ideas, be it the theory of invariants, the axiomatization of the geometry or the foundations of the unctional analysis (with the Hilbert spaces).

[2] Henri Léon Lebesgue (June 28, 1875 in Beauvais - July 26, 1941 in Paris) is a mathematician French. He is recognized for his integration theory originally published in his dissertation Integral, length, area at the University of Nancy in 1902.

[3] Stefan Banach (1892 - 1945)was a Polish mathematician.

- *The resulting* $L^2(\Omega)$***-norm*** *of* f *is defined by*

$$\|f\|_{L^2(\Omega)} = \left(\int_\Omega |f(x)|^2\,dx\right)^{\frac{1}{2}}.$$

-*The space* $L^2(\Omega)$ *is naturally equipped with the following inner product*

$$\langle f,g\rangle = \int_\Omega f(x)\,\overline{g(x)}\,dx, \quad \text{whenever } f,g \in L^2(\Omega).$$

We can consider the following generalization: for all $m \in \mathbb{N}^*$ et $1 \le r < \infty$,

$$L^r(\Omega;\mathbb{R}^m) = \left\{u : \Omega \longrightarrow \mathbb{R}^m \text{ measurable such that } \int_\Omega |f|^r\,dx < +\infty\right\},$$

where $|.|$ designates any standard (for example the Euclidean norm) in $\mathbb{R}^m$.

Remark 2.13 *: We give an equivalent definition for space* $L^r(\Omega;\mathbb{R}^m)$

$$L^r(\Omega;\mathbb{R}^m) = \left\{u = (u_1,u_2,...,u_m)^T;\ u_i \in L^r(\Omega);\ \forall i = 1;2;...;m\right\}.$$

The space $\left(L^r(\Omega;\mathbb{R}^m), \|u\|_{L^r(\Omega;\mathbb{R}^m)}\right)$ *where*

$$\|u\|_{L^r(\Omega;\mathbb{R}^m)} = \left(\int_\Omega |u|^r\,dx\right)^{\frac{1}{r}},$$

is a Banach space. In the case with $r = 2$ *the space* $L^2(\Omega;\mathbb{R}^m)$ *is equipped with the following inner product*

$$(u,v)_{L^2(\Omega;\mathbb{R}^m)} = \int_\Omega (u,v)_{\mathbb{R}^m} = \sum_{i=1}^{n}\int_\Omega u_i(x).v_i(x)\,dx,$$

is a Hilbert space.

Definition 2.14 *: Let* V *is a Banach space and* $1 \le p < \infty$. *Denoted by* $L^p(0,T,V)$ *the space of functions measurable,* $u:\ ]0,T[\to V$ *such that* $\int_0^t \|u(t)\|_V^p\,dt < +\infty$.
We denote then the norm on these space by :

$$\|u\|_{L^p(0,T,V)} = \left(\int_0^T \|u(t)\|_V^p\,dt\right)^{\frac{1}{p}}. \tag{1.1}$$

Theorem 2.15 *: The space* $\left(L^p(0,T,V), \|u\|_{L^p(0,T,V)}\right)$ *is a Banach space.*

Definition 2.16 *: Let* V *is a Banach space, we define* $L^\infty(0,T,V)$ *by:*

$$L^\infty(0,T,V) = \left\{u : (0,T) \longrightarrow V \text{ measurable and } \|u\|_{L^\infty(0,T,V)} < \infty\right\},$$

where

$$\|u\|_{L^\infty(0,T,V)} = \sup_{t\in(0,T)} ess\,\|u(t)\|_V.$$

Definition 2.17 *:[63]*
We particular $p = 2$*, let* E *be a Hilbert space,* $L^2(0, T, E)$ *is the set of functions* $f : t \in [0, T] \longrightarrow E$ *if it satisfies the following:*
(1). $t \longrightarrow (f(t), g)_E$ *is a measurable function, for all* $g \in E$*,*
(2). $\int_0^T \|f(t)\|_E^2 \, dt < +\infty$*,*
with the following inner product:

$$(f, g)_{L^2(0,T,E)} = \int_0^T (f(t), g(t))_E \, dt.$$

Proposition 2.18 *:[63]*
$L^2(0, T, E)$ *is a Hilbert space.*

★ **Distributions Spaces**

Definition 2.19 *:[68]*
The ***support*** *of a measurable function* $f : \mathbb{R}^n \longrightarrow R$ (or c) *is defined to be the set of all points where* f *does not vanish.*

$$Suppf = \overline{\{x \in R^n, f(x) \neq 0\}}.$$

Definition 2.20 *:[37]*
Let Ω *be a nonempty open set in* $\mathbb{R}^n, n \geq 1$*. We denote by* $D(\Omega)$ *(or* $C_o^\infty(\Omega)$*) the reals-valued functions, infinitely differentiable in* Ω *with compact support.*
These functions of $D(\Omega)$ *are called test functions.*

Definition 2.21 *:[68]*
Let Ω *be a nonempty open set in* $\mathbb{R}^n, n \geq 1$ *. A* ***distribution*** T *on* Ω *is a map :*
$T : D(\Omega) \longrightarrow \mathbb{C}$ *such that*
(1) (Linearity) For all $\varphi, \psi \in D(\Omega)$ and all $\alpha, \beta \in \mathbb{C}$,

$$T(\alpha\varphi + \beta\psi) = \alpha T(\varphi) + \beta T(\psi).$$

(2)(Continuity) If $\varphi_n \longrightarrow \varphi$ *in* $D(\Omega)$*, then* $T(\varphi_n) \longrightarrow T(\varphi)$*.*
The set of all distributions is denoted by $D'(\Omega)$*.*

Definition 2.22 *: (Differentiation of Distributions in* $D'(\Omega)$*) [71]*
If α *is a many-indexes and* $u \in D'(\Omega)$*, the formula*

$$\left\langle \frac{\partial^\alpha u}{\partial x_i}, \varphi \right\rangle = (-1)^{|\alpha|} \left\langle u, \frac{\partial^\alpha \varphi}{\partial x_i} \right\rangle \quad \forall \varphi \in D(\Omega).$$

Definition 2.23 *: We denote by* $\mathcal{D}'(0, T, V)$ *the distributions space in* $(0, T)$ *Who lend his values in* V*, and for* $u \in L^p(0, T, V)$*, we have :*

$$u(\varphi) = \int_0^T u(t)\varphi(t) \, dt, \quad \forall \varphi \in \mathcal{D}(0, T).$$

The following result on space $L^p(0, T, V)$ *is very useful for the future.*

Lemma 2.24 *: Let $u \in L^p(0,T,V)$ and $\frac{\partial u}{\partial t} \in L^p(0,T,V)$, $(1 \leq p \leq \infty)$, then, the function $u : [0,T] \longrightarrow V$ is continuous.*

Definition 2.25 *:[15]*
Let Ω be a nonempty open set in $\mathbb{R}^n, n \geq 1$. We define the ***Sobolev***[4] *space $H^1(\Omega)$ by:*

$$H^1(\Omega) = \left\{ v : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R},\ v \in L^2(\Omega),\ \frac{\partial v}{\partial x_i} \in L^2(\Omega),\ 1 \leq i \leq n \right\}.$$

Definition 2.26 *:[15]*
• *The mapping $(.,.)_{H^1(\Omega)} : H^1(\Omega) \times H^1(\Omega) \longrightarrow \mathbb{R}$ define by:*

$$(u,v)_{H^1(\Omega)} = \int_\Omega u(x)\, v(x)\, dx + \int_\Omega \nabla u(x)\, \nabla v(x)\, dx\ \forall x \in \Omega,$$

define the inner-product in $H^1(\Omega)$.
• *$H^1(\Omega)$ is a Hilbert space with a norm:*

$$\forall v \in H^1(\Omega) : \|v\|^2_{H^1(\Omega)} = (v,v)_{H^1(\Omega)}.$$

in other words we called

$$\forall v \in H^1(\Omega) : \|v\|^2_{H^1(\Omega)} = \|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)},$$

or

$$\|\nabla v\|^2_{L^2(\Omega)} = \left\| \frac{\partial v}{\partial x_1} \right\|^2_{L^2(\Omega)} + ... + \left\| \frac{\partial v}{\partial x_n} \right\|^2_{L^2(\Omega)}.$$

Definition 2.27 *:[15]*
Let Ω be a nonempty open set in $\mathbb{R}^n, n \geq 1$, we define the Sobolev space $H^1_0(\Omega)$ by :

$$H^1_0(\Omega) = \left\{ v \in H^1(\Omega),\ \text{ such that }\ v_{|\partial\Omega} = 0 \right\}.$$

Definition 2.28 *: [37]*
Let Ω be a nonempty open set in $\mathbb{R}^n, n \geq 1$, $(n \geq 1)$ and $m \in \mathbb{N}$. We say that $u \in H^m(\Omega)$ if $u \in L^2(\Omega)$ and if all its derivatives in the sense of the distributions, up to the order m are still in $L^2(\Omega)$. i.e

$$H^m(\Omega) = \left\{ u \in L^2(\Omega), \forall\ \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \text{ where } |\alpha| \leq m, \text{ we have } D^\alpha(u) \in L^2(\Omega) \right\}.$$

Definition 2.29 *:*
Let $m \in \mathbb{N}^*$ and $p \in \mathbb{R}$ where $1 \leq p \leq \infty$. We define the Sobolev space $W^{m,p}(\Omega)$ by :

$$W^{m,p}(\Omega) = \{ u \in L^p(\Omega),\ \ \partial^\alpha u \in L^p(\Omega) \text{ for all } \alpha \in \mathbb{N}^n \text{ such that } \partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} ... \partial_n^{\alpha_n} \}.$$

[4] Specialist in differential equations applied to the physical sciences, Sobolev introduces, as early as 1934, the notion of generalized function and derivative in order to better understand phenomena where the concept of function proved insufficient in the search for solutions of equations partial derivatives. He is thus at the origin of the theory of distributions developed by his George Green (July 1793-31 May 1841), a British physicist. fellow Israel Guelfand and Frenchman Laurent Schwartz.

Theorem 2.30 *:*
$W^{m,p}(\Omega)$ equipped with the norm

$$\|u\|_{W^{m,p}(\Omega)} = \sum_{|\alpha|\leq m} \|\partial^{\alpha} u\|_{L^p}\,,\ 1 \leq p < \infty, \quad \text{for all } u \in W^{m,p}(\Omega),$$

is a Banach space.

Definition 2.31 *:*
$W_0^{m,p}(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ on $W^{m,p}(\Omega)$.

Remark 2.32 *:*
If $p = 2$, when we prefer noted $W^{m,2}(\Omega) = H^m(\Omega)$ and $W_0^{m,2}(\Omega) = H_0^m(\Omega)$. We equipped the space $H^m(\Omega)$ by the product inner

$$(u, v)_{H^m(\Omega)} = \sum_{|\alpha|\leq m} \int_{\Omega} \partial^{\alpha} u \ \partial^{\alpha} v \ dx,$$

and the norm

$$\|u\|_{H^m(\Omega)} = \left(\sum_{|\alpha|\leq m} \|\partial^{\alpha} u\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

Theorem 2.33 *:*

1. *$H^m(\Omega)$ is equipped with the following inner product $(.,.)_{H^m(\Omega)}$ is a Hilbert space.*
2. *Si $p \geq q$, $H^p(\Omega) \hookrightarrow H^q(\Omega) \hookrightarrow H^0(\Omega)$, with continuous injection.*

Since $\mathcal{D}(\Omega)$ is dense in $H_0^m(\Omega)$, we have the following :

Lemma 2.34 *:*

$$\mathcal{D}(\Omega) \hookrightarrow H_0^m(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-m}(\Omega) \hookrightarrow \mathcal{D}'(\Omega),$$

where $H^{-m}(\Omega)$ the dual of $H_0^m(\Omega)$ in a low subspace Ω.
The following results are fundamental to the study of partial differential equations.

Theorem 2.35 *:*
It is assumed that Ω is border $\partial\Omega$ is regular. Then,

i *. If $1 \leq p < n$, on a $W^{1,p} \subset L^q(\Omega)$, for each $q \in [p, p^*]$, where $p^* = \dfrac{np}{n-p}$.*

ii *. If $p = n$ on a $W^{1,p} \subset L^q(\Omega)$, for each $q \in [p, \infty)$.*

iii *. If $p > n$ on a $W^{1,p} \subset L^{\infty}(\Omega) \cap C^{0,\alpha}(\Omega)$, où $\alpha = \dfrac{p-n}{p}$.*

Remark 2.36 *:*
for all $\varphi \in H^2(\Omega)$, $\Delta\varphi \in L^2(\Omega)$ and for $\partial\Omega$ sufficiently smooth, have,

$$\|\varphi(t)\|_{H^2(\Omega)} \leq C \|\Delta\varphi(t)\|_{L^2(\Omega)}. \tag{1.2}$$

3 Some inequalities

Proposition 3.1 *: [20] (Cauchy[5] - Schwarz [6] - Schwarz inequality)*
If $f, g \in L^2(\Omega)$ the Cauchy-Schwarz inequality is:

$$|f.g|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} . \|g\|_{L^2(\Omega)} .$$

Definition 3.2 *:[13] (Hölder[7] inequality)*
Let E measurable space, $p, q > 0$ where: $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $f \in L^p(E)$ and $g \in L^q(E)$ then the product $f.g \in L^1(E)$ and the norm satisfy:

$$\|f.g\|_{L^1(E)} \leq \|f\|_{L^p(E)} . \|g\|_{L^q(E)} .$$

In other holds, for $0 < p, q < +\infty$ define by $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{r}$, if $f \in L^p(E)$ and $g \in L^q(E)$ then the product $f.g \in L^r(E)$ and

$$\|f.g\|_{L^r(E)} \leq \|f\|_{L^p(E)} . \|g\|_{L^q(E)} .$$

★ **Young[8] Inequality**

Lemma 3.3 *: For all $a, b \in \mathbb{R}_+$, we have*

$$ab \leq \frac{\delta^2}{2} a^2 + \frac{1}{2\delta^2} b^2.$$

Lemma 3.4 *: For all $a, b \in \mathbb{R}_+$, the following inequality holds*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

where, $\frac{1}{p} + \frac{1}{q} = 1$.

Definition 3.5 *:[13]*
Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ where, $p, q > 0$ et $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then

$$\|f.g\|_{L^1(E)} \leq \frac{1}{p} \|f\|^p_{L^p(E)} . \frac{1}{q} \|g\|^q_{L^q(E)} .$$

[5] Baron Cauchy (21st August 1789 in Paris - May 23rd, 1857 in Sceaux (Hauts-de-Seine)) is a French mathematician. He was one of the most prolific mathematicians, behind Euler, with close to 800 publications.

[6] Hermann Amandus Schwarz was born on January 25, 1843 in Poland and died on 30 November 1921 in Berlin. He is a famous mathematician whose work is marked by a strong inter-action between analysis and geometry.

[7] Otto Ludwig Hölder (22 December 1859 - 29 august 1937) is a mathematician German born on Stuttgart.

[8] William Henry Young (London, October 20, 1863 - Lausanne, July 7, 1942) is a English mathematician from the University of Cambridge and having worked at the University of Li-Verpool and the University of Lausanne.

We define the convolution product of two functions.

Definition 3.6 *:(Convolution)*
Let f and g two functions be locally relevant. The convolution product of the functions f and g is the function :

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(t)g(x-t)\,dt, \ \ \forall x \in \mathbb{R}.$$

Theorem 3.7 *: Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$ where $1 \leq p \leq \infty$. Then, for all $x \in \mathbb{R}^n$ the function $y \mapsto f(x-y)g(y)$ is integrable on $\mathbb{R}^n$ and we define*

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy.$$

In other $(f * g) \in L^p(\mathbb{R}^n)$ and we have :

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

Theorem 3.8 *:[13] (* ***Poincaré*** [9] ***inequity****)*
Let (a, b) an open of $\mathbb{R}$ et $u \in H_0^1(a, b)$. Then there exist the constant C (according to $b-a$) such that

$$\| u \|_{L^2(a,b)} \leq C \| u' \|_{L^2(a,b)} .$$

More than one dimension we have ,
Let Ω be an open set of $\mathbb{R}^n$ one assumes bound, convex and of border sufficiently regular. Then there is a constant $C_\Omega > 0$ such that :

$$\| u \|_{L^2(\Omega)} \leq C_\Omega \| \nabla u \|_{L^2(\Omega)},$$

for all function $u \in H_0^1(\Omega)$.

Lemma 3.9 *: Let $1 \leq p \leq r \leq q$, such that $\dfrac{1}{r} = \dfrac{\alpha}{p} + \dfrac{1-\alpha}{q}$, and $0 \leq \alpha \leq 1$. Then*

$$\|u\|_{L^r} \leq \|u\|_{L^p}^{\alpha} \|u\|_{L^q}^{1-\alpha}, \ \ \forall u \in L^p(\Omega).$$

Green[10] Formula.

Theorem 3.10 *:[62]*

Let Ω be a nonempty open set in $R^n, n \geq 1$. If u and v are in $H^1(\Omega)$, they are true

$$\int_\Omega u(x) \frac{\partial v}{\partial x_i}(x)\,dx = -\int_\Omega v(x) \frac{\partial u}{\partial x_i}(x)\,dx + \int_{\partial\Omega} u(x) v(x) \eta_i(x)\,dx,$$

where $(\eta_i)_{1 \leq i \leq n}$ is a normal unite external a $\partial\Omega$.

Corolary 3.11 *:[62]*
If $u, v \in H^1(\Omega)$ and if $\Delta u \in L^2(\Omega)$, then:

$$\int_\Omega \Delta u(x) v(x)\,dx = -\int_\Omega \nabla u(x) \nabla v(x)\,dx + \int_{\partial\Omega} \frac{\partial u}{\partial \eta}(x) v(x)\,dx.$$

[9] Henri Poincaré (April 29, 1854 in Nancy - July 17, 1912 in Paris) is a mathematician, a French physicist and philosopher. Theoretical of genius, his contributions to many fields of mathematics and physics have radically altered these two sciences.

[10] George Green (July 1793-31 may 1841), Physician Britannic.

4 Weak convergence

4.1 Weak, Weak star and strong convergence

Definition 4.1 *:(**Weak convergence in** E).*
Let $x \in E$ and let $\{x_n\} \subset E$. We say that $\{x_n\}$ weakly converges to x in E, and we write $x_n \rightharpoonup x$ in E, if

$$\langle f, x_n \rangle \longrightarrow \langle f, x \rangle \quad \text{for all } x \in E^{'}.$$

Definition 4.2 *:(**Weak Convergence in** $E^{'}$).*
Let $f \in E^{'}$ and let $\{f_n\} \subset E^{'}$. We say that $\{f_n\}$ weakly converges to f in $E^{'}$, and we write $f_n \rightharpoonup f$ in $E^{'}$, if

$$\langle f_n, x_n \rangle \longrightarrow \langle f, x \rangle \quad \text{for all } x \in E''.$$

Definition 4.3 *:(**Weak star Convergence**).*
Let $f \in E^{'}$ and let $\{f_n\} \subset E^{'}$. We say that $\{f_n\}$ weakly star converges to f in $E^{'}$, and we write $f_n \rightharpoonup^{*} f$ in $E^{'}$, if

$$\langle f_n, x_n \rangle \longrightarrow \langle f, x \rangle \quad \text{for all } x \in E.$$

Remark 4.4 *:*
As $E \subset E''$ we have $f_n <\!\rightharpoonup f$ in $E^{'}$. When E is reflexive, the last definition are the same, i.e, weak convergence in $E^{'}$ and weak star convergence coincide.

Definition 4.5 *:(**Strong Convergence**).*
Let $x \in E$ (resp. $f \in E$) and let $\{x_n\} \subset E$ (resp. $\{f_n\} \subset E^{'}$. We say that $\{x_n\}$ (resp. $\{f_n\}$) strong converge to x (resp. f), and we write $x_n \to x$ in E (resp. $f_n \to f$ in $E^{'}$), if

$$\lim_{n\to\infty} \|x_n - x\|_E = 0, \left(resp. \lim_{n\to\infty} \|f_n - f\|_{E'} = 0 \right).$$

Definition 4.6 *:(**Strong Convergence in** L^p **with** $1 \le p < \infty$).*
Let Ω an open supset of $\mathbb{R}^n$. We say that the sequence $\{x_n\}$ of L^p weakly converges to $f \in L^p(\Omega)$, if

$$\lim_n \int_\Omega f_n(x)\, g(x)\, dx = \int_\Omega f(x)\, g(x)\, dx \text{ for all } g \in L^q, \frac{1}{p} + \frac{1}{q} = 1.$$

Theorem 4.7 *(**Bolzano- Weierstrass**).*
If $dimE < \infty$ and if $\{x_n\} \subset E$ is bounded, then there existe $x \in E$ and a subsequence $\{x_{n_k}\}$ strongly converges to x.

Theorem 4.8 *(**Weak star Compactness, Banach-Alaoglu-Bourbaki**).*
Assum that E is separable and consider $\{f_n\} \subset E^{'}$. If $\{x_n\}$ is bounded, then there existe $f \in E^{'}$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$such that $\{f_{n_k}\}$ weakly star converges to f in $E^{'}$.

Theorem 4.9 *(**Weak Compactness, Kakutani-Eberlin**).*
Assum that E is reflexive and consider $\{x_n\} \subset E$. If $\{x_n\}$ is bounded, then there existe $x \in E$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$such that $\{x_{n_k}\}$ weakly star converges to x in E.

Theorem 4.10 *(**Weak Compactness** in $L^p(\Omega)$ with $1 < p < \infty$).*
Given $\{f_n\} \subset L^p(\Omega)$, If $\{f_n\}$ is bounded, then there exist $f \in L^p(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_n \rightharpoonup f$ in $L^p(\Omega)$.

Theorem 4.11 *(**Weak star Compactness, in** $L^\infty(\Omega)$).*
Given $\{x_n\} \subset L^\infty(\Omega)$ If $\{f_n\}$ is bounded, then there existe $f \in L^\infty(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_n \rightharpoonup *f$ in $L^\infty(\Omega)$.

Definition 4.12 *:[62]*
E is a Banach space, if $(f_n)_n$ is a sequence of E', then $(f_n)_n$ converge to f in the sense of weak convergence if and only if $f_n(x)$ converges to $f(x)$ for all $x \in E$.

Theorem 4.13 *: [20](**The convergence theory**)*
Suppose (f_n) is a sequence of measurable functions such that $f_n(x) \longrightarrow f(x)$ as n tends to infinity. If $|f_n(x)| \leq g(x)$, where g is integrable, then

$$\int |f_n - f| \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

and consequently

$$\int f_n \longrightarrow \int f \text{ as } n \longrightarrow \infty.$$

Corolary 4.14 *:[20]*
If $(f_n)_{n=1}^{\infty}$ converges to f in L^1, then there exists a sub-sequence $(f_{nk})_{k=1}^{\infty}$ such that

$$f_{nk}(x) \longrightarrow f(x) \text{ a.e.}$$

Theorem 4.15 *:[13] (**Density theorem**)*
The space $C_0(\Omega)$ is dense in $L^1(\Omega)$ that is to say

$$\forall f \in L^1(\Omega), \ \forall \epsilon > 0, \ \exists f_1 \in C_0(\Omega) \text{ such that } \|f - f_1\|_{L^1(\Omega)} < \epsilon.$$

Definition 4.16 *:[13]*
Let E is the Banach space and let $f \in E'$. We designate by $\varphi_f : E \longrightarrow \mathbb{R}$ the mapping define by $\varphi_f(x) = (f, x)_E$. The weak topology $\delta\left(E, E'\right)$ on E is the thinnest topology on E making continuous all mappings $(\varphi_f)_{f \in E'}$.

Theorem 4.17 *:[13] (**Dunford**[11]**-Pettis**[12])*
Let Ω be a nonempty open set in $\mathbb{R}^n$, let $U \subset L^1(\Omega)$ be a subset borne. Then U is compact so her topology $\delta(L^1(\Omega), L^\infty(\Omega))$ if only if we have

$$\forall \epsilon > 0, \forall c > 0 \text{ such that} \int_A |f| < \epsilon, \forall f \in U, \forall A \subset \Omega \text{ where } |A| < c.$$

[11]Nelson Dunford (1906-1986) is a mathematician American.
[12]Billy James Pettis (1913-1979) is a mathematician American.

4.2 Aubin Lions Lemma

Lemma 4.18 *:*
Let X_0, X and X_1 be three Banach spaces with $X_0 \subseteq X \subseteq X_1$. Assum that X_0 is compactly embedded in X and that X is continuously embedded in X_1, assum also that X_0 and X_1 are reflexives spaces. For $1 < p, q < +\infty$, let $W = \left\{ u \in L^p([0,t]; X_0) / u' \in L^q([0,t]; X_1) \right\}$. Then the embedding of W into $L^p([0,t]; X)$ is also compact.

5 Faedo-Galerkin's approximations

We consider the Cauchy problem abstract's for a second order evolution equation in the separable Hilbert space with the inner product $\langle ., . \rangle$ and the associated norm $\|.\|$.

$$(P) \qquad \begin{cases} u_{tt}(t) + A(t)u(t) = f(t), \quad t \in [0,T], \\ u(x,0) = u_0(x), \ u'(x,0) = u_1(x), \end{cases}$$

where u and f are unknown and given function, respectively, mapping the closed interval $[0,T] \subset \mathbb{R}$ into a real separable Hilbert space H. $A(t)$ $(0 \leq t \leq T)$ are linear bounded operators in H acting in the energy space $V \subset H$.
Assume that $\langle A(t)u(t), v(t) \rangle = a(t; u(t), v(t))$, for all $u, v \in V$; where $a(t; ., .)$ is a bilinear continuous in V. The problem (P) can be formulated as: Found the solution $u(t)$ such that

$$(\widetilde{P}) \qquad \begin{cases} u \in C([0,T]; V), u_t \in C([0,T]; H) \\ \langle u_{tt}(t), v \rangle + a(t; u(t), v) = \langle f, v \rangle \text{ in } D'(]0,T[) \\ u_0 \in V \ , \ u_1 \in H. \end{cases}$$

This problem can be resolved with the approximation process of Fadeo-Galerkin.
Let V^m a sub-space of V with the finite dimension d^m, and let $\{w^{jm}\}$ one basis of V^m such that.

1. $V^m \subset V(\dim V^m < \infty), \forall m \in \mathbb{N}$.
2. $V^m \to V$ such that, there exist a dense subspace ϑ in V and for all $\upsilon \in \vartheta$ we can get sequence $\{u^m\}_{m \in \mathbb{N}} \in V^m$ and $u^m \to u$ in V.
3. $V^m \subset V^{m+1}$ and $\overline{\cup_{m \in \mathbb{N}} V^m} = V$.

We define the solution u^m of the approximate problem

$$(P_m) \qquad \begin{cases} u^m(t) = \displaystyle\sum_{j=1}^{d^m} g^j(t) w^{jm}, \\ u^m \in C([0,T]; V^m), u^m_t \in C([0,T]; V^m), \quad u^m \in L^2(0,T; V^m), \\ \langle u^m_{tt}(t), w^{jm} \rangle + a(t; u^m(t), w^{jm}) = \langle f, w^{jm} \rangle, \quad 1 \leq j \leq d_m, \\ u^m(0) = \displaystyle\sum_{j=1}^{d^m} \xi^j(t) w^{jm} \ , \ u^m_t(0) = \displaystyle\sum_{j=1}^{d^m} \eta^j(t) w^{jm}, \end{cases}$$

where

$$\sum_{j=1}^{d_m} \xi^j(t) w^{jm} \longrightarrow u_0 \text{ in } V \text{ as} m \longrightarrow \infty,$$

$$\sum_{j=1}^{d^m} \eta^j(t)w^{jm} \longrightarrow u_1 \text{ in } V \text{ as} m \longrightarrow \infty.$$

By virtue of the theory of ordinary differential equations, the system (P_m) has unique local solution which is extend to a maximal interval $[0, t_m[$ by Zorn lemma since the non-linear terms have the suitable regularity. In the next step, we obtain a priori estimates for the solution, so that can be extended outside $[0, t_m[$ to obtain one solution defined for all $t > 0$. **A priori estimation and convergence.**
Using the following estimation

$$\|u^m\|^2 + \|u_t^m\|^2 \leq C \left\{ \|u^m(0)\|^2 + \|u_t^m(0)\|^2 + \int_0^T \|f(s)\|^2 ds \right\}, \quad 0 \leq t \leq T$$

and the Gronwall lemma we deduce that the solution u^m of the approximate problem (P_m) converges to the solution u of the initial problem (P). The uniqueness proves that u is the solution.

Lemma 5.1 *: [11] (**Gronwall**[13] **Inequality**)*
Let $T > 0, f \in L^1(0,T), f \geq 0$ *a.e and* $c_1, c_2 \geq 0$, *let* $\varphi \geq 0$ *almost every where such that* $f.\varphi \in L^1(0,T)$ *and*

$$\varphi(t) \leq c_1 + c_2 \int_0^T f(s)\varphi(s)\,ds, \quad \text{a.e} \quad t \in (0,T),$$

then

$$\varphi(t) \leq c_1 e^{c_2 \int_0^T f(s)ds}, \quad \text{a.e} \quad t \in (0,T).$$

6 Lax-Milgram Corollary

Definition 6.1 *:[13]*
Let E *be a Hilbert space, we called with* $a : E \times E \longrightarrow R$ *is form*
1. Bilinear *if any* $u \in E$ *the mapping*

$$\begin{array}{rccl} a: & E & \longrightarrow & \mathbb{R} \\ & v & \longrightarrow & a(u,v), \end{array}$$

is linear, and any $v \in E$ *the mapping*

$$\begin{array}{rccl} a: & E & \longrightarrow & \mathbb{R} \\ & u & \longrightarrow & a(u,v), \end{array}$$

is linear.
2. Symmetric *if*

$$a(u,v) = a(v,u), \ \forall u, v \in E.$$

[13] Thomas Hakon Gronwall (1877-1932) is a Swedish mathematician

Definition 6.2 *:[27]*
A bilinear form $a : E \times E \longrightarrow \mathbb{R}$ is said to be
(i) ***Continuous*** *if there is a constant $c > 0$ such that*

$$|a(v,u)| \leq c \|u\|_E \|v\|_E, \quad \forall u, v \in E.$$

(ii) ***Coercive*** *if there is a constant $\alpha > 0$ such that*

$$a(v,u) \geq \alpha \|v\|_E^2, \ \forall v \in E.$$

Corolary 6.3 *: [27]*
Assum that $a(u,v)$ is a continuous coercive bilinear form on $E \times E$, and L is a linear form continue on E. Then given any $\varphi \in E^*$, there exists a unique element $u \in E$ such that

$$a(u,v) = L(v), \ \forall v \in E.$$

Moreover, if a is symmetric, then u is characterized by the property

$$u \in E \text{ and } \frac{1}{2} a(u,u) - \langle \varphi, u \rangle = \min_{v \in E} \left\{ \frac{1}{2} a(v,v) - \langle \varphi, v \rangle \right\}.$$

7 Theory of Semi Group

Semi group we refer the title to [34] and [19] for more details. The semi-group theory has its origin in the study of the exponential operators. It link the operator $A : D(A) \subset X \longrightarrow X$ to the resolution of the differential equation

$$\begin{cases} u_t(t) = Au(t) \\ u(0) = u_0. \end{cases}$$

Let X the Banach space and $T(t)_{t \geq 0}$ a family of linear operators and bounded on X.

Definition 7.1 *:[71]*
Let X be a Banach space, and suppose that to every $t \geq 0$ is associated an operator $T(t)$, in such away that
i) $T(0) = I$ (operator unite on X),
ii) $T(t+s) = T(t)T(s), \ \forall t, s \geq 0$,
iii) $\lim_{t \longrightarrow 0} \|T(t)x - x\| = 0$ for every $x \in X$.
If (i) and (ii) hold, $T(t)$ is called a semi-group (or, more precisely, a one-parameter semi-group). Such semi-groups have exponential representations, provided that the mapping $t \mapsto T(t)$ satisfies some continuity assumption. The on that is chosen here, namely (iii), is easy to work with.

Definition 7.2 *:*
A linear operator A define by

$$D(A) = \left\{ x \in X, \ \lim_{t \longrightarrow 0^+} \frac{T(t)x - x}{t} exist \right\} \text{ and } Ax = \lim_{t \longrightarrow 0^+} \frac{T(t)x - x}{t},$$

is infinitesimal generator of a semi group $\{T(t)\}_{t \geq 0}$, $D(A)$ is a domain of A.

Definition 7.3 *:*
The semi-group $\{T(t)\}_{t\geq 0}$ *is called well continuous if* $\lim\limits_{t\longrightarrow 0^+} T(t)x = x$. *The semi group strongly continuous denoted* C_0 *semi-group.*

Definition 7.4 *:*
The semi-group $\{T(t)\}_{t\geq 0}$ *is called uniformity continuous if*

$$\lim_{t\longrightarrow 0^+} ||S(t) - Id|| = 0.$$

Theorem 7.5 *:*
If A *is a infinitesimal generator of a* C_0 *semi-group* $\{T(t)\}_{t\geq 0}$, *then* $D(A)$ *is dense in* X *and* A *is a closed linear operator .*

Proposition 7.6 *:*
Let $\{T(t)\}_{t\geq 0}$ *a family of* C_0 *semi-group. There exist* $w \geq 0$ *and* $M \geq 1$ *such that*
$\|T(t)\| \leq Me^{wt}$.
Si $\|T(t)\| \leq 1$, *we called a* $\{T(t)\}_{t\geq 0}$ *is a semi-group of contraction.*

Theorem 7.7 *:*
Let A *is a infinitesimal generator of a* C_0 *semi-group* $\{T(t)\}_{t\geq 0}$, *then*
i) For all $x \in X$ $t \longmapsto T(t)x$ *is continuous function in* $\mathbb{R}^+$ *to* X.
ii) For all $x \in X$, $\int_0^t T(s)\,ds \in D(A)$ and $A\int_0^t T(s)\,ds = T(t)x - x$.
iii)For all $x \in D(A)$, $T(t)x \in D(A)$ and $\frac{d}{dt}T(t)x = T(t)Ax = AT(t)x$.

Theorem 7.8 *:[60](**Lummer-Phillips, 1961**)*
Let $A : D(A) \subseteq X \longrightarrow X$ *is a linear operator such that* $\overline{D(A)} = X$. *Then the following assertions hold.*
(a) If A *is dissipative and if it exists* $\lambda_0 > 0$ *such that the rank of* $\lambda_0 - A$ *is subjective then* A *is a* ***generates infinitesimal*** *a contraction of semi-group.*
(b) If A *is a generates infinitesimal a contraction of semi-group then* $\lambda_0 - A$ *is* ***subjective*** *for all* $\lambda_0 > 0$ *and* A *is dissipative.*

Definition 7.9 *:[27]*
An unbounded linear operator $A : D(A) \subset H \longrightarrow H$ *is said to be* ***monotone***[14] *if it satisfies*

$$A(v, v) \geq 0, \ \forall v \in D(A).$$

A *it is called* ***maximal monotone*** *if, in addition,* $R(I + A) = H$ *that is to say,*

$$\forall f \in H, \exists u \in D(A) \text{ such that } u + Au = f.$$

[14] Some author say that A is accretive or that $-A$ is **dissipative**

Theorem 7.10 *:[27](**Hille-Yosida**)*
Let A be a maximal monotone operator. Then given any $u_0 \in D(A)$ there exists a unique function

$$u \in C^1([0, +\infty[; H) \cap C([0, +\infty[; D(A)),$$

satisfying

$$\begin{cases} \dfrac{du}{dt} + Au = 0, \ \ on \ [0, +\infty[, \\ u(0) = u_0. \end{cases}$$

Moreover,

$$|u(t)| \leq u_0 \ \ et \ \ \left|\frac{du}{dt}(t)\right| = |Au(t)| \leq |Au_0| \quad \forall t \geq 0.$$

Remark 7.11 *:[13]*
*The main interest of theorem of Hille-Yosida lies in the fact that we reduce the study of an "**evolution problem**" to the study of the "**stationary equation**" $u' + Au = f$ (assuming we already know that A is maximal monotone.*

8 Integral Inequalities

We will recall some fundamental integral inequalities introduced by A. Haraux, V. Komornik and P. Martinez to estimate the decay rate of the energy.

8.1 A result of exponential decay

The estimation of the energy decay for some dissipative problems is based on the following lemma:

Lemma 8.1 *([49]) Let $E : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a non-increasing function and assume that there is a constant $A > 0$ such that*

$$\forall t \geq 0, \qquad \int_t^{+\infty} E(\tau)\, d\tau \leq \frac{1}{A} E(t). \tag{1.3}$$

Then we have

$$\forall t \geq 0, \qquad E(t) \leq E(0)\, e^{1-At}. \tag{1.4}$$

Proof : The inequality (1.4) is verified for $t \leq \frac{1}{A}$, this follows from the fact that E is a decreasing function. We prove that (1.4) is verified for $t \geq \frac{1}{A}$. Introduce the function

$$h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \qquad h(t) = \int_t^{+\infty} E(\tau)\, d\tau.$$

It is non-increasing and locally absolutely continuous. Differentiating and using (1.3) we find that

$$\forall t \geq 0, \quad h'(t) + Ah(t) \leq 0.$$

Let

$$T_0 = \sup\{t, \ h(t) > 0\}. \tag{1.5}$$

For every $t < T_0$, we have

$$\frac{h'(t)}{h(t)} \leq -A,$$

thus

$$h(0) \le e^{-At} \le \frac{1}{A} E(0)\, e^{-At}, \quad \text{for} \quad 0 \le t < T_0. \tag{1.6}$$

Since $h(t) = 0$ if $t \ge T_0$, this inequality holds in fact for every $t \in \mathbb{R}_+$. Let $\varepsilon > 0$. As E is positive and decreasing, we deduce that

$$\forall t \ge \varepsilon, \quad E(t) \le \frac{1}{\varepsilon} \int_{t-\varepsilon}^{t} E(\tau)\, d\tau \le \frac{1}{\varepsilon} h(t-\varepsilon) \le \frac{1}{A\varepsilon} E(0)\, e^{\varepsilon t}\, e^{-At}.$$

Choosing $\varepsilon = \frac{1}{A}$, we obtain

$$\forall t \ge 0, \qquad E(t) \le E(0)\, e^{1-At}.$$

The proof of Lemma 8.1 is now completed. ■

8.2 A result of polynomial decay

Lemma 8.2 *([49]) Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ $(\mathbb{R}_+ = [0, +\infty))$ be a non-increasing function and assume that there are two constants $q > 0$ and $A > 0$ such that*

$$\forall t \ge 0, \quad \int_t^{+\infty} E^{q+1}(\tau)\, d\tau \le \frac{1}{A} E^q(0) E(t). \tag{1.7}$$

Then we have:

$$\forall t \ge 0, \quad E(t) \le E(0) \left(\frac{1+q}{1 + A\, q\, t} \right)^{1/q}. \tag{1.8}$$

Remark 8.3 It is clear that Lemma 8.1 is similar to Lemma 8.2 in the case of $q = 0$.

Proof : If $E(0) = 0$, then $E \equiv 0$ and there is nothing to prove. Otherwise, replacing the function E by the function $\dfrac{E}{E(0)}$ we may assume that $E(0) = 1$. Introduce the function

$$h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \qquad h(t) = \int_t^{+\infty} E(\tau)\, d\tau.$$

It is non-increasing and locally absolutely continuous. Differentiating and using (1.7) we find that

$$\forall t \ge 0, \quad -h' \ge (Ah)^{1+q},$$

where

$$T_0 = \sup\{t,\ h(t) > 0\}.$$

Integrating in $[0, t]$ we obtain that

$$\forall 0 \le t < T_0, h(t)^{-q} - h(0)^{-\sigma} \ge \sigma \omega^{1+q} t,$$

hence

$$0 \le t < T_0, \quad h(t) \le \left(h^{-q}(0) + qA^{1+q}\, t \right)^{-1/q}. \tag{1.9}$$

Since $h(t) = 0$ if $t \ge T_0$, this inequality holds in fact for every $t \in \mathbb{R}_+$. Since

$$h(0) \le \frac{1}{A} E(0)^{1+q} = \frac{1}{A},$$

by (1.7), the right-hand side of (1.9) is less than or equal to:

$$\left(h^{-q}(0)+qA^{1+q}\,t\right)^{-1/q} \le \frac{1}{A}(1+A\,q\,t)^{-1/q}. \tag{1.10}$$

From other hand, E being non negative and non-increasing, we deduce from the definition of h and the above estimate that:

$$\begin{aligned}\forall s\ge 0,\ E\left(\frac{1}{A}+(q+1)s\right)^{q+1} &\le \frac{1}{\frac{1}{A}+q+1}\int_s^{\frac{1}{A}+(q+1)s} E(\tau)^{q+1}\,d\tau \\ &\le \frac{A}{1+Aqs}h(s) \le \frac{A}{1+Aqs}\frac{1}{A}(1+Aqs)^{-\frac{1}{q}},\end{aligned}$$

hence

$$\forall S\ge 0,\quad E\left(\frac{1}{A}+(q+1)S\right) \le \frac{1}{(1+A\,q\,S)^{1/q}}.$$

Choosing $t=\dfrac{1}{A}+(1+q)s$ then the inequality (1.8) follows. Note that letting $q\to 0$ in this theorem we obtain (1.8). ■

8.3 New integral inequalities of P. Martinez

The above inequalities are verified only if the energy function is integrable. We will try to resolve this problem by introducing some weighted integral inequalities, so we can estimate the decay rate of the energy when it is slow.

Lemma 8.4 *([49]) Let $E:\mathbb{R}_+\to\mathbb{R}_+$ be a non-increasing function and $\phi:\mathbb{R}_+\to\mathbb{R}_+$ an increasing C^1 function such that*

$$\phi(0)=0 \quad \text{and} \quad \phi(t)\to+\infty \quad \text{when} \quad t\to+\infty. \tag{1.11}$$

Assume that there exist $q\ge 0$ and $A>0$ such that

$$\int_S^{+\infty} E(t)^{q+1}\phi'(t)\,dt \le \frac{1}{A}E(0)^q E(S), \quad 0\le S<+\infty, \tag{1.12}$$

then we have

$$\text{if } q>0, \quad \text{then } E(t)\le E(0)\left(\frac{1+q}{1+q\,A\,\phi(t)}\right)^{\frac{1}{q}},\ \ \forall t\ge 0,$$
$$\text{if } q=0, \quad \text{then } E(t)\le E(0)\,e^{1-A\,\phi(t)},\ \ \forall t\ge 0.$$

Proof : This Lemma is a generalization of Lemma 8.4. Let $f:\mathbb{R}_+\to\mathbb{R}_+$ be defined by $f(x):=E(\phi^{-1}(x))$, (we notice that ϕ^{-1} has a meaning by the hypotheses assumed on ϕ). f is non-increasing, $f(0)=E(0)$ and if we set $x:=\phi(t)$ we obtain f is non-increasing, $f(0)=E(0)$ and if we set $x:=\phi(t)$ we obtain

$$\begin{aligned}\int_{\phi(S)}^{\phi(T)} f(x)^{q+1}\,dx &= \int_{\phi(S)}^{\phi(T)} E\left(\phi^{-1}(x)\right)^{q+1}\,dx = \int_S^T E(t)^{q+1}\phi'(t)\,dt \\ &\le \frac{1}{A}E(0)^q E(S) \\ &= \frac{1}{A}E(0)^q f(\phi(S)), \quad 0\le S<T<+\infty.\end{aligned}$$

Setting $s := \phi(S)$ and letting $T \to +\infty$, we deduce that

$$\forall s \geq 0, \quad \int_s^{+\infty} f(x)^{q+1}\, dx \leq \frac{1}{A}\, E(0)^q f(s).$$

Thanks to Lemma 8.4, we deduce the desired results. ■

Chapter 2

Well-posedness and general energy decay of solutions for transmission problems with weakly nonlinear dissipative therms

1 Introduction

In this chapter [4], we consider a nonlinear transmission problem

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t) + \mu g(u_t(x,t)) = 0, (x,t) \in \Omega \times \mathbb{R}^+, \\ v_{tt}(x,t) - bv_{xx}(x,t) = 0, (x,t) \in [L_1, L_2] \times \mathbb{R}^+, \end{cases} \tag{2.1}$$

where $0 < L_1 < L_2 < L_3,$ $\Omega =]0, L_1[\cup]L_2, L_3[,$ a, b, μ are positives constants. This system is supplemented with the following boundary and transmission conditions:

$$\begin{aligned} & u(0,t) = u(L_3,t) = 0, \\ & u(L_i,t) = v(L_i,t), \ i = 1,2, \\ & au_x(L_i,t) = bv_x(L_i,t), \ i = 1,2, \end{aligned} \tag{2.2}$$

and initial conditions

$$\begin{aligned} & u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ & v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in]L_1, L_2[. \end{aligned} \tag{2.3}$$

When $g(u_t(x,t)) = u_t(x,t)$ system (2.1)-(2.3) has been investigated in [9], for $\Omega = [0, L_1]$, the authors showed the well-posedness and exponential stability of the total energy.
Ma and Oquendo [51] considered transmission problem involving two Euler-Bernoulli equations modeling the vibrations of a composite beam. By using just one boundary damping term in the boundary, they showed the global existence and decay property of the solution. Marzocchi et al [52] investigated a 1-D semi-linear transmission problem in classical thermoelasticity and showed that a combination of the first, second and third energies of the solution decays exponentially to zero, no matter how small the damping subdomain is. A similar result has sheen shown by Messaoudi and Said-Houari [55], where a transmission problem in thermoelasticity of type III has been

investigated. See also Marzocchi et al [53] for a multidimensional linear thermoelastic transmission problem.
To obtain global solutions of problem (2.1)-(2.3), we use the Galerkin approximation scheme (see Lions [45]) together with the energy estimate method.

To prove decay estimates, we use a perturbed energy method and some properties of convex functions. These arguments of convexity were introduced and developed by Cavalcanti et al. [14], Lasiecka and Doundykov [35], Lasiecka and Tataru [36], and used by Liu and Zuazua [46] and Alabau-Boussouira [1].

2 Preliminaries

First we recall and make use the following assumptions on the function g as:
We assume that the function $g : \mathbb{R} \to \mathbb{R}$ is a non-decreasing function of the class $\mathcal{C}(\mathbb{R})$ such that there exist $\varepsilon, c_2, \tau_0, \tau_1 > 0$ and a convex and increasing function $G : \mathbb{R}^+ \to \mathbb{R}^+$ of the class $\mathcal{C}^1(\mathbb{R}^+) \cap \mathcal{C}^2(]0, +\infty[)$ satisfying

$$\begin{array}{l} G(0) = 0 \textit{ and } G \textit{ is linear on } [0, \varepsilon] \textit{ or} \\ G'(0) = 0 \textit{ and } G'' > 0 \textit{ on }]0, \varepsilon] \textit{ such that} \\ |g(s)| \leq c_2|s| \quad \textit{if } |s| > \varepsilon, \\ s^2 + g^2(s) \leq G^{-1}(sg(s)) \quad \textit{if } |s| \leq \varepsilon, \\ \tau_0 \leq g'(s) \leq \tau_1, \end{array} \tag{2.4}$$

$$\forall s \in \mathbb{R} : sg(s) > 0.$$

We first state some Lemma which will be needed later.

Lemma 2.1 *(Sobolev-Poincaré's inequality) Let q be a number $2 \leq q \leq +\infty$ $(n = 1, 2)$ or $2 \leq q \leq 2n/(n-2)$ $(n \geq 3)$. Then there is a constant $C_s = C((0,1), q)$ such that*

$$\|u\|_q \leq C_s \|\nabla u\|_2, \quad \textit{for all} \quad u \in H_0^1(\Omega).$$

Remark 2.2 *Let us denote by ϕ^* the conjugate function of the differentiable convex function ϕ, that is to say,*

$$\phi^*(s) = \sup_{t \in \mathbb{R}_+} (st - \phi(t)).$$

*Then ϕ^* is the Legendre transform of ϕ, which is given by (see Arnold [**?** , p. 61-62])*

$$\phi^*(s) = s(\phi')^{-1}(s) - \phi\left((\phi')^{-1}(s)\right), \textit{ if } s \in \left]0, \phi'(r)\right],$$

and ϕ^* satisfies the generalized Young inequality

$$ST \leq \phi^*(S) + \phi(T), \textit{ if } S \in \left]0, \phi'(r)\right], T \in]0, r]. \tag{2.5}$$

3 Well-posedness of the problem

In this section, we prove the existence and the uniqueness of a global solution of system (2.1) -(2.3) by using the Faedo- Galerkin method.

Theorem 3.1 *Suppose that* $(u^0, v^0) \in H^2(\Omega) \times H^2(L_1, L_2) \cap H_0^1(\Omega) \times H_0^1(L_1, L_2)$, $(u^1, v^1) \in H_0^1(\Omega) \times H_0^1(L_1, L_2)$ *and that assumption (2.4) holds. Then (2.1)-(2.3) admits a unique global solution*

$$(u, v) \in L^\infty(0, T, H^2(\Omega) \times H^2(L_1, L_2) \cap H_0^1(\Omega) \times H_0^1(L_1, L_2)),$$
$$(u_t, v_t) \in L^\infty(0, T, H_0^1(\Omega) \times H_0^1(L_1, L_2)),$$
$$(u_{tt}, v_{tt}) \in L^\infty(0, T, L^2(\Omega) \times L^2(L_1, L_2)).$$

Proof :

i) *Approximate solutions:*
The main idea is to use the Galerkin Method. Let $\{\varphi^i, \psi^i\}$, $i = 1, 2, \ldots$ be a basis of $H^2(\Omega) \times H^2(L_1, L_2) \cap H_0^1(\Omega) \times H_0^1(L_1, L_2)$. Let us consider the Galerkin approximation

$$(u^m(t), v^m(t)) = \sum_{i=1}^{m} h^{im}(t) \left(\varphi^i, \psi^i\right),$$

where u^m and v^m satisfy

$$(u_{tt}^m, \varphi^i) + a(u_x^m, \varphi_x^i) + \mu(g(u_t^m), \varphi^i) + (v_{tt}^m, \psi^i) + b(v_x^m, \psi_x^i)) = 0, \tag{2.6}$$

where $i = 1, 2, \ldots$. With initial data

$$\begin{aligned} (u^m(0), v^m(0)) &= (u_0^m, v_0^m) \to (u^0, v^0) \ \text{ in } H^2(\Omega) \times H^2(L_1, L_2) \cap H_0^1(\Omega) \times H_0^1(L_1, L_2), \\ (u_t^m(0), v_t^m(0)) &= (u_1^m, v_1^m) \to (u^1, v^1) \ \text{ in } H_0^1(\Omega) \times H_0^1(L_1, L_2). \end{aligned} \tag{2.7}$$

Standard results about ordinary differential equations guarantee that there exists only one solution of this system on some interval $[0, T_m[$. The priori estimate that follow imply that in fact $T_m = +\infty$.

ii) *A priori estimate:*
The first estimate: Multiplying (2.6) by h_t^{im} and summing over i, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big\{ \int_\Omega |u_t^m|^2 dx + a\int_\Omega |u_x^m|^2 dx + \int_{L_1}^{L_2} |v_t^m|^2 dx + b\int_{L_1}^{L_2} |v_x^m|^2 dx\Big\} \\ &\quad + \mu \int_\Omega u_t^m g(u_t^m)\, dx = 0. \end{aligned}$$

Integrating in $[0, t]$, $t < t_m$ and using (2.7), we have

$$\begin{aligned} &\int_\Omega |u_t^m|^2\, dx + a\int_\Omega |u_x^m|^2\, dx + \int_{L_1}^{L_2} |v_t^m|^2\, dx + b\int_{L_1}^{L_2} |v_x^m|^2\, dx \\ &\quad + 2\textstyle\int_0^t \displaystyle\int_\Omega u_t^m(s) g(u_t^m(s))\, dx\, ds \\ &\quad \leq \int_\Omega |u_1^m|^2\, dx + a\int_\Omega |u_0^m|^2\, dx + \int_{L_1}^{L_2} |v_1^m|^2\, dx + b\int_{L_1}^{L_2} |v_0^m|^2\, dx \\ &\quad \leq C_1. \end{aligned} \tag{2.8}$$

For some C_1 independent of m.
Thus we deduce that.

$$(u^m, v^m) \text{ is bounded in } L^\infty(0,T,H_0^1(\Omega)\times H_0^1(L_1,L_2)), \tag{2.9}$$

$$(u_t^m, v_t^m) \text{ is bounded in } L^\infty(0,T,L^2(\Omega)\times L^2(L_1,L_2)), \tag{2.10}$$

$$u_t^m g(u_t^m) \text{ is bounded in } L^1(\Omega\times(0,T)). \tag{2.11}$$

The second estimate: First, we estimate $u_{tt}^m(0)$ and $v_{tt}^m(0)$ taking $t=0$ in (2.6) , we obtain

$$(u_{tt}^m(0),\varphi^i) - a(u_{xx}^m(0),\varphi^i) + \mu(g(u_t^m(0)),\varphi^i) = 0,$$

and

$$(v_{tt}^m(0),\psi^i) - b(v_{xx}^m(0),\psi^i) = 0,$$

multiplying by h_{tt}^{im} and summing over i from 1 to m,

$$(u_{tt}^m(0),u_{tt}^m(0)) - a(u_{xx}^m(0),u_{tt}^m(0)) + \mu(g(u_t^m(0)),u_{tt}^m(0)) = 0,$$

and

$$(v_{tt}^m(0),v_{tt}^m(0)) - b(v_{xx}^m(0),v_{tt}^m(0)) = 0.$$

Using Hölder's inequality and (2.7), yield

$$\begin{aligned}
&\Big(\int_\Omega |u_{tt}^m(0|^2\,dx\Big)^{\frac12} + \Big(\int_{L_1}^{L_2} |v_{tt}^m(0)|^2\,dx\Big)^{\frac12}\\
&\le a\Big(\int_\Omega |u_{xx}^m(0)|^2\,dx\Big)^{\frac12} + \mu\Big(\int_\Omega g^2(u_1^m)\,dx\Big)^{\frac12} + b(\int_{L_1}^{L_2} |v_{xx}^m(0)|^2\,dx\Big)^{\frac12}\\
&\le C_2,
\end{aligned} \tag{2.12}$$

where C_2 is a positive constant independent of m.
The third estimate: Now, differentiating (2.6) with respect to t

$$(u_{ttt}^m,\varphi^i) - a(u_{txx}^m,\varphi^i) + \mu(u_{tt}^m g'(u_t^m),\varphi^i) + (v_{ttt}^m,\psi^i) - b(v_{txx}^m,\psi^i) = 0.$$

Multiplying by h_{tt}^{mi} and summing over i from 1 to m implies

$$\frac12\frac{d}{dt}\Big[\int_\Omega |u_{tt}^m|^2\,dx + a\int_\Omega |u_{xt}^m|^2\,dx + \int_{L_1}^{L_2} |v_{tt}^m|^2\,dx + b\int_{L_1}^{L_2} |v_{xt}^m|^2\,dx\Big] + \mu\int_\Omega (u_{tt}^m)^2 g'(u_t^m)\,dx = 0.$$

Integrating it over (0, t), using (2.7),(2.4) and (2.12) and we get

$$\begin{aligned}
&\int_\Omega |u_{tt}^m(t)|^2\,dx + a\int_\Omega |u_{xt}^m(t)|^2\,dx + \int_{L_1}^{L_2} |v_{tt}^m(t)|^2\,dx\\
&+b\int_{L_1}^{L_2} |v_{xt}^m(t)|^2\,dx + 2\mu\int_0^t\int_\Omega (u_{tt}^m(s))^2 g'(u_t^m(s))\,dx\,dt\\
&= \int_\Omega |u_{tt}^m(0)|^2\,dx + a\int_\Omega |u_{xt}^m(0)|^2\,dx + \int_{L_1}^{L_2} |v_{tt}^m(0)|^2\,dx + b\int_{L_1}^{L_2} |v_{xt}^m(0)|^2\,dx\\
&\le C_3,
\end{aligned}$$

where C_3 is a positive constant independent of m. Therefore, we conclude that

$$\begin{aligned}&(u_t^m, v_t^m) \ \textit{is bounded in } L^\infty(0,T,H_0^1(\Omega)\times H_0^1(L_1,L_2)),\\&(u_{tt}^m, v_{tt}^m) \ \textit{is bounded in } L^\infty(0,T,L^2(\Omega)\times L^2(L_1,L_2)),\end{aligned} \tag{2.13}$$

we deduce that

$$(u_t^m, v_t^m) \ \textit{is bounded in } L^2(0,T,H_0^1(\Omega)\times H_0^1(L_1,L_2)).$$

Applying Rellich compactenes theorem given in **[45]**, we deduce that

$$(u_t^m, v_t^m) \ \textit{is bounded in } L^2(0,T,L^2(\Omega)\times L^2(L_1,L_2)). \tag{2.14}$$

The fourth estimate: Replacing φ^i and ψ^i by $(-u_{xx}^m\ )$and $(-v_{xx}^m)$ in (2.6), multiplying the result by h_t^{im}, summing over i from 1 to m, implies

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big[\int_\Omega |u_{tx}^m|^2\,dx + a\int_\Omega |u_{xx}^m|^2\,dx + \int_{L_1}^{L_2} |v_{tx}^m|^2\,dx + b\int_{L_1}^{L_2} |v_{xx}^m|^2\,dx\Big]\\&+\mu\int_\Omega (u_{tx}^m)^2 g'(u_t^m)\,dx = 0.\end{aligned}$$

Integrating it over $(0,t)$ and using (2.7), we have

$$\begin{aligned}&\int_\Omega |u_{tx}^m(t)|^2\,dx + a\int_\Omega |u_{xx}^m(t)|^2\,dx + \int_{L_1}^{L_2} |v_{tx}^m(t)|^2\,dx\\&+b\int_{L_1}^{L_2} |v_{xx}^m(t)|^2\,dx + \mu\int_0^t\int_\Omega (u_{tx}^m(s))^2 g'(u_t^m(s))\,dx\,ds\\&= \int_\Omega |u_{tx}^m(0)|^2\,dx + a\int_\Omega |u_{xx}^m(0)|^2\,dx + \int_{L_1}^{L_2} |v_{tx}^m(0)|^2\,dx + b\int_{L_1}^{L_2} |v_{xx}^m(0)|^2\,dx\\&\le C_4,\end{aligned}$$

where C_4 is a positive constant independent of m. we conclude that

$$\begin{aligned}&(u^m, v^m) \ \textit{is bounded in } L^\infty(0,T,H^2(\Omega)\times H^2(L_1,L_2)),\\&(u_t^m, v_t^m) \ \textit{is bounded in } L^\infty(0,T,H_0^1(\Omega)\times H_0^1(L_1,L_2)).\end{aligned} \tag{2.15}$$

ii) *Passage to the limite:*
Applying Dunford-Petti's theorem, we conclude from (2.9), (2.11) and (2.15), after replacing the sequences $\{u^m, v^m\}$ by subsequence if necessary, that

$$(u^m, v^m) \rightharpoonup (u,v)\,, \ \text{weak-star in } L^\infty(0,T;H^2(\Omega)\times H^2(L_1,L_2)\cap H_0^1(\Omega)\times H_0^1(L_1,L_2)), \tag{2.16}$$

$$(u_t^m, v_t^m) \rightharpoonup (u_t, v_t)\,, \ \text{weak-star in } L^\infty(0,T;H_0^1(\Omega)\times H_0^1(L_1,L_2)), \tag{2.17}$$

$$(u_{tt}^m, v_{tt}^m) \rightharpoonup (u_{tt}^m, v_{tt}^m)\,, \ \text{weak-star in } L^\infty(0,T;L^2(\Omega)\times L^2(L_1,L_2)), \tag{2.18}$$

$$g(u_t^m) \rightharpoonup \chi, \ \text{weak-star in } L^2(Q), \tag{2.19}$$

where $Q=(0,T)\times\Omega$. It follows at once from (2.16) and (2.18), that for each fixed $w_1 \in L^2([0,T]\times L^2(\Omega))$

$$\int_0^T\int_\Omega \Big(u_{tt}^m(x,t) - a u_{xx}^m(x,t)\Big) w_1\,dx\,dt \longrightarrow \int_0^T\int_\Omega \Big(u_{tt}(x,t) - a u_{xx}(x,t)\Big) w_1\,dx\,dt \tag{2.20}$$

and $w_2 \in L^2([0,T] \times L^2(L_1,L_2))$

$$\int_0^T \int_{L_1}^{L_2} \Big(v_{tt}^m(x,t) - bv_{xx}^m(x,t)\Big) w_2 \, dx\, dt \longrightarrow \int_0^T \int_{L_1}^{L_2} \Big(v_{tt}(x,t) - bv_{xx}(x,t)\Big) w_2 \, dx\, dt. \quad (2.21)$$

As (2.14), (2.17) and the injection of H_0^1 in L^2 is compact, we have

$$u_t^m \longrightarrow u_t, \text{ strong in } L^2(Q). \quad (2.22)$$

Therefore,

$$u_t^m \longrightarrow u_t, \text{ almost everywhere in } Q. \quad (2.23)$$

It remains to show that,

$$\int_Q g(u_t^m)\ v \, dx\, dt \longrightarrow \int_Q g(u_t)\ v\, dx\, dt,$$

Lemma 3.2 *For each $T > 0$, $g(u_t) \in L^1(Q)$, $\|g(u_t)\|_{L^1(Q)} \leq K$, where K is a constant independent of t and $g(u_t^m) \to g(u_t)$ in $L^1(Q)$.*

Proof : We claim that

$$g(u_t) \in L^1(Q).$$

Indeed, since g is continuous, we deduce from (2.23)

$$g(u_t^m) \longrightarrow g(u_t) \quad \text{almost everywhere in} \quad Q. \quad (2.24)$$

$$u_t^m g(u_t^m) \longrightarrow u_t g(u_t) \quad \text{almost everywhere in} \quad Q.$$

Hence, by (2.10) and Fatou's Lemma, we have

$$\int_Q u_t(x,t) g(u_t(x,t))\, dx\, dt \leq K_1, \quad \text{for } T > 0. \quad (2.25)$$

Now, we can estimate $\int_Q |g(u_t(x,t))|\, dx\, dt$. By Cauchy-Schwarz inequality, we have

$$\int_0^T \int_\Omega |g(u_t(x,t))|\, dx\, dt \leq c|Q|^{1/2}\Big(\int_0^T \int_\Omega |g(u_t(x,t))|^2\, dx\, dt\Big)^{1/2}.$$

Using (2.4) and (2.25), we obtain

$$\begin{aligned}
\int_0^T \int_\Omega |g(u_t(x,t))|^2\, dx\, dt \quad &\leq \int_0^T \int_{|u_t|>\varepsilon} u_t g(u_t)\, dx\, dt + \int_0^T \int_{|u_t|\leq\varepsilon} G^{-1}(u_t g(u_t))\, dx\, dt \\
&\leq c\int_0^T \int_\Omega u_t g(u_t)\, dx\, dt + cG^{-1}\Big(\int_Q u_t g(u_t)\, dx\, dt\Big) \\
&\leq c\int_0^T \int_\Omega u_t g(u_t)\, dx\, dt + c'G^*(1) + c''\int_0^T \int_\Omega u_t g(u_t)\, dx\, dt \\
&\leq cK_1 + c'G^*(1), \quad \text{for} \quad T > 0.
\end{aligned}$$

Then,

$$\int_Q |g(u_t(x,t))|\, dx\, d \leq K, \quad \text{for } \ T > 0.$$

Let $B \subset \Omega \times [0,T]$ and set

$$B_1 = \Big\{(x,t) \in B : |g(u_t^m(x,t))| \leq \frac{1}{\sqrt{|B|}}\Big\}, \quad B_2 = B \backslash B_1,$$

where $|B|$ is the measure of B. If $M(r) = \inf\{|s| : s \in \mathbb{R} \text{ and } |g(s)| \geq r\}$

$$\int_B |g(u_t^m)|\,dx\,dt \leq c\sqrt{|B|} + \Big(M\Big(\frac{1}{\sqrt{|B|}}\Big)\Big)^{-1} \int_{B_2} |u_t^m g(u_t^m)|\,dx\,dt.$$

By applying (2.10) we deduce that

$$\sup_m \int_B g(u_t^m)\ dx\ dt \longrightarrow 0,\ \text{when } |E| \longrightarrow 0.$$

From Vitali's convergence theorem we deduce that

$$g(u_t^m) \to g(u_t) \quad \text{in } L^1(Q).$$

This completes the proof. ∎

Then (2.19) implies that

$$g(u_t^m) \rightharpoonup g(u_t),\ \text{weak-star in } L^2(Q).$$

We deduce, for all $w_1 \in L^2\Big([0,T] \times L^2(\Omega)\Big)$, that

$$\int_0^T \int_\Omega g(u_t^m) w_1\,dx\,dt \longrightarrow \int_0^T \int_\Omega g(u_t) w_1\,dx\,dt.$$

Finally we have shown that, for all $w_1 \in L^2\Big([0,T] \times L^2(\Omega)\Big)$:

$$\int_0^T \int_\Omega \Big(u_{tt}(x,t) - a u_{xx}(x,t) - \mu g(u_t)\Big) w_1\,dx\,dt = 0.$$

iv) *Proof of uniqueness:*

Let u_1, u_2 be two solutions of $(2.1)_1$ and v_1, v_2 be two solutions of $(2.1)_2$ with the same initial data. Let us denote it is straightforward to see that $z = u_1 - u_2$ and $w = v_1 - v_2$ satisfies

$$\begin{aligned} &\int_\Omega z_t^2(x,t)\,dx + a\int_\Omega z_x^2(x,t)\,dx + \int_{L_1}^{L_2} w_t^2(x,t)\,dx \\ &+ b\int_{L_1}^{L_2} w_x^2(x,t)\,dx + \mu\int_0^t \int_\Omega (g(u_1^{'}t(s) - g(u_2^{'}t(s))w_t\,dx\,ds = 0. \end{aligned} \tag{2.26}$$

Using the monotonicity of g hence we conclude that

$$\int_\Omega z_t^2(x,t)\,dx + a\int_\Omega z_x^2(x,t)\,dx + \int_{L_1}^{L_2} w_t^2(x,t)\,dx + b\int_{L_1}^{L_2} w_x^2(x,t)\,dx \leq 0, \tag{2.27}$$

which implies $z = 0$ and $w = 0$. This finishes the proof of Theorem (3.1).

∎

4 Asymptotic behavior

In this section, we state and prove our stability result for the energy of the solution of system (2.1)-(2.3), using the multiplier technique.
The energies of first and second order associated with system (2.1)-(2.3) are defined as follows:

$$E_1(t) = \frac{1}{2}\int_\Omega u_t^2(x,t)\,dx + \frac{a}{2}\int_\Omega u_x^2(x,t)\,dx, \tag{2.28}$$

$$E_2(t) = \frac{1}{2}\int_{L_1}^{L_2} v_t^2(x,t)\,dx + \frac{b}{2}\int_{L_1}^{L_2} v_x^2(x,t)\,dx. \tag{2.29}$$

The total energy is defined as

$$E\,(t) = E_1\,(t) + E_2\,(t)\,. \tag{2.30}$$

Our decay result reads as follows.

Theorem 4.1 *Let (u, v) be the solution of (2.1)- (2.3). Assume that (2.4) holds and*

$$\frac{b}{a} < \frac{L_3 + L_1 - L_2}{2\,(L_2 - L_1)}. \tag{2.31}$$

Then there exist positive constants k_1, k_2, k_3 and ε_0 such that the solution of the problem (2.1)-(2.3) satisfies

$$E(t) \le k_3 G_1^{-1}\,(k_1 t + k_2)\,, \quad \forall\; t \in \mathbb{R}_+, \tag{2.32}$$

where

$$G_1(t) = \int_t^1 \frac{1}{sG_2'(\varepsilon_0 s)}ds, \quad G_2(t) = tG'(\varepsilon_0 t), \tag{2.33}$$

with G_1 is strictly decreasing and convex on $]0,1]$, and $\lim\limits_{t\longrightarrow 0} G_1(t) = +\infty$.

For the proof of theorem 4.1 we use the following Lemmas.

Lemma 4.2 *The total energy $E(t)$ satisfies*

$$E^{'}(t) = -\mu\int_\Omega u_t(x,t)g(u_t(x,t))\,dx \le 0. \tag{2.34}$$

Proof : Multiplying equation (2.1) by u_t and integrating in Ω, we have

$$\int_\Omega u_t\,(x,t)\,u_{tt}(x,t)\,dx - a\int_\Omega u_t(x,t)u_{xx}(x,t)\,dx = -\mu\int_\Omega u_t(x,t)g(u_t(x,t))\,dx,$$

which integrated by parts leads to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega [u_t^2(x,t) + au_x^2(x,t)]\,dx = \;& -\mu\int_\Omega u_t(x,t)g(u_t(x,t))\,dx \\ & -a\,(u_x\,(L_1,t)\,u_t\,(L_1,t) - u_x\,(0,t)\,u_t\,(0,t)) \\ & -a\,(u_x\,(L_3,t)\,u_t\,(L_3,t) - u_x\,(L_2,t)\,u_t\,(L_2,t))\,.\end{aligned} \tag{2.35}$$

Multiplying equation (2.1) by v_t and performing an integration in (L_1, L_2), we get

$$\int_{L_1}^{L_2} v_t(x,t)v_{tt}(x,t)\,dx - b\int_{L_1}^{L_2} v_t(x,t)v_{xx}(x,t)\,dx = 0.$$

After integrating by parts we arrive at

$$\frac{1}{2}\frac{d}{dt}\int_{L_1}^{L_2}[v_t^2(x,t)+bv_x^2(x,t)]\,dx=-b\left(v_t\left(L_2,t\right)\;v_x\left(L_2,t\right)-v_t\left(L_1,t\right)\;v_x\left(L_1,t\right)\right). \tag{2.36}$$

Adding (2.35) with (2.36) and using the transmission conditions (2.2) we conclude

$$\frac{d}{dt}E(t)=-\mu\int_\Omega u_t(x,t)g(u_t(x,t))\,dx.$$

■

Lemma 4.3 *Let (u,v) be the solution of (2.1)-(2.3). Then the functional*

$$J\left(t\right)=\int_\Omega u\left(x,t\right)u_t\left(x,t\right)dx+\int_{L_1}^{L_2}v\left(x,t\right)v_t\left(x,t\right)dx,$$

satisfies, for any $\delta>0$, the estimate

$$\begin{aligned}\frac{d}{dt}J\left(t\right)\;&\le\int_\Omega u_t^2(x,t)\,dx+\int_{L_1}^{L_2}v_t^2\left(x,t\right)\,dx-\left(a-\delta C_s\right)\int_\Omega u_x^2(x,t)\,dx\\&+b\int_{L_1}^{L_2}v_x^2(x,t)\,dx+C(\delta)\mu^2\int_\Omega g^2(u_t(x,t))\,dx.\end{aligned} \tag{2.37}$$

Proof : Taking the derivative of $J\left(t\right)$ with respect to t and using (2.1) we find that

$$\begin{aligned}\frac{d}{dt}J(t)\;&=\int_\Omega u_t^2(x,t)\,dx+\int_{L_1}^{L_2}v_t^2(x,t)\,dx-a\int_\Omega u_x^2(x,t)dx\\&-b\int_{L_1}^{L_2}v_x^2\left(x,t\right)dx-\mu\int_\Omega u\left(x,t\right)g\left(u_t\left(x,t\right)\right)dx+\left[auu_x\right]_{\partial\Omega}+\left[bvv_x\right]_{L_1}^{L_2}.\end{aligned} \tag{2.38}$$

Using the boundary conditions (2.2) we have

$$\begin{aligned}\left[auu_x\right]_{\partial\Omega}+\left[bvv_x\right]_{L_1}^{L_2}\;&=\;a\left\{u\left(L_1,t\right)u_x\left(L_1,t\right)-u\left(0,t\right)u_x\left(0,t\right)\right\}\\&\quad+a\left\{u\left(L_3,t\right)u_x\left(L_3,t\right)-u\left(L_2,t\right)u_x\left(L_2,t\right)\right\}\\&\quad+b\left\{v\left(L_2,t\right)v_x\left(L_2,t\right)-v\left(L_1,t\right)v_x\left(L_1,t\right)\right\}\\&=\;0.\end{aligned}$$

Applying Young's and Poincaré's inequalities, we have

$$\mu\int_\Omega u\left(x,t\right)g\left(u_t\left(x,t\right)\right)dx\le\delta C_s\int_\Omega u_x^2\left(x,t\right)\,dx+C(\delta)\mu^2\int_\Omega g^2\left(u_t\left(x,t\right)\right)\,dx,$$

where δ is a positive constant. We arrive at

$$\begin{aligned}\frac{d}{dt}J\left(t\right)\;&\le\int_\Omega u_t^2\left(x,t\right)dx+\int_{L_1}^{L_2}v_t^2\left(x,t\right)dx-\left(a-\delta C_s\right)\int_\Omega u_x^2\left(x,t\right)dx\\&-b\int_{L_1}^{L_2}v_x^2\left(x,t\right)dx+C(\delta)\mu^2\int_\Omega g^2\left(u_t\left(x,t\right)\right)dx\\&\le\int_\Omega u_t^2\left(x,t\right)dx+\int_{L_1}^{L_2}v_t^2\left(x,t\right)dx-\left(a-\delta C_s\right)\int_\Omega u_x^2\left(x,t\right)dx\\&+b\int_{L_1}^{L_2}v_x^2\left(x,t\right)dx+C(\delta)\mu^2\int_\Omega g^2\left(u_t\left(x,t\right)\right)dx.\end{aligned}$$

The proof of Lemma 4.3 is now completed. ■

Now, inspired by [52], we introduce the functional

$$q(x)=\begin{cases} x-\frac{L_1}{2} & x\in[0,L_1],\\ x-\frac{L_2+L_3}{2}, & x\in[L_2,L_3]\\ \frac{L_2-L_3-L_1}{2(L_2-L_1)}(x-L_1)+\frac{L_1}{2}, & x\in[L_1,L_2]. \end{cases} \tag{2.39}$$

Lemma 4.4 *Let u be the solution of (2.1). Then the functional*

$$J_1(t)=-\int_\Omega q(x)u_x(x,t)u_t(x,t)\,dx,$$

satisfies, the estimate

$$\begin{aligned} \frac{d}{dt}J_1(t)\le\ & \frac{1}{2}\int_\Omega u_t^2(x,t)\,dx+\left(\frac{a}{2}+\delta_1\right)\int_\Omega u_x^2(x,t)\,dx\\ & +C(\delta_1)\mu^2\int_\Omega g^2\left(u_t(x,t)\right)\,dx-\frac{a}{4}\left[\left(L_3-L_2\right)u_x^2\left(L_2,t\right)+L_1u_x^2\left(L_1,t\right)\right]. \end{aligned} \tag{2.40}$$

Proof : Taking the derivative of $J_1(x)$ with respect to t and using (2.1), we obtain

$$\begin{aligned} \frac{d}{dt}J_1(t)=\ & -\int_\Omega q(x)u_{xt}(x,t)u_t(x,t)\,dx-a\int_\Omega q(x)u_x(x,t)u_{xx}(x,t)\,dx\\ & +\mu\int_\Omega q(x)u_x(x,t)g(u_t(x,t))\,dx. \end{aligned}$$

Integrating by parts, we have

$$-\int_\Omega q(x)u_{xt}(x,t)u_t(x,t)\,dx=-\frac{1}{2}\left[q(x)u_t^2(x,t)\right]_{\partial\Omega}+\frac{1}{2}\int_\Omega q_x(x)u_t^2\left(x,t\right)\,dx. \tag{2.41}$$

On the other hand, then

$$-a\int_\Omega q(x)u_x(x,t)u_{xx}(x,t)\,dx=-\frac{a}{2}\left[q(x)u_x^2(x,t)\right]_{\partial\Omega}+\frac{a}{2}\int_\Omega q_x(x)u_x^2(x,t)\,dx. \tag{2.42}$$

By using the boundary conditions (2.2) we have

$$\frac{1}{2}\left[q(x)u_t^2(x,t)\right]_{\partial\Omega}=\frac{L_1}{4}u_t^2(L_1,t)+\frac{L_3-L_2}{2}u_t^2(L_2,t)\ge 0. \tag{2.43}$$

Also, we have

$$\begin{aligned} -\frac{a}{2}\left[q(x)u_x^2(x,t)\right]_{\partial\Omega} & =-\frac{aL_1}{4}[u_x^2(L_1,t)-u_x^2(0,t)]-\frac{a(L_2-L_3)}{4}[u_x^2(L_3,t)-u_x^2(L_2,t)]\\ & =-\frac{aL_1}{4}u_x^2(L_1,t)-\frac{a(L_3-L_2)}{4}u_x^2(L_2,t), \end{aligned} \tag{2.44}$$

using the Young inequality as, we obtain

$$\mu\int_\Omega q(x)u_x(x,t)g(u_t(x,t))\,dx\le\delta_1\int_\Omega u_x^2(x,t)\,dx+C(\delta_1)\mu^2\int_\Omega g^2(u_t(x,t))\,dx. \tag{2.45}$$

Thus (2.40) follows from (2.41)-(2.45). ■

Lemma 4.5 *Let v be the solution of (2.1). Then the functional*

$$J_2(t) = -\int_{L_1}^{L_2} q(x)v_x(x,t)v_t(x,t)\,dx,$$

satisfies, the estimate

$$\begin{aligned}\frac{d}{dt}J_2(t) \leq \;& \frac{L_2-L_3-L_1}{4(L_2-L_1)}\left(\int_{L_1}^{L_2} v_t^2(x,t)\,dx + \int_{L_1}^{L_2} b v_x^2(x,t)\,dx\right)\\ &+\frac{b}{4}\left[(L_3-L_2)\,v_x^2(L_2,t) + L_1 v_x^2(L_1,t)\right].\end{aligned} \tag{2.46}$$

Proof : By the same method, taking the derivative of J_2 with respect to t and using (2.1), we obtain

$$\begin{aligned}\frac{d}{dt}J_2(t) \;&= -\int_{L_1}^{L_2} q(x)v_{xt}(x,t)v_t(x,t)\,dx - \int_{L_1}^{L_2} q(x)v_x(x,t)v_{tt}(x,t)\,dx\\ &-\int_{L_1}^{L_2} q(x)v_{xt}(x,t)v_t(x,t)\,dx - b\int_{L_1}^{L_2} q(x)v_x(x,t)v_{xx}(x,t)\,dx.\end{aligned} \tag{2.47}$$

Integrating by parts, we have

$$\begin{aligned}-\int_{L_1}^{L_2} q(x)v_{xt}(x,t)v_t(x,t)\,dx = \;& -\frac{1}{2}\left[q(x)v_t^2(x,t)\right]_{L_1}^{L_2} + \frac{1}{2}\int_{L_1}^{L_2} q_x(x)v_t^2(x,t)\;dx\\ = \;& -\frac{L_2-L_3}{4}v_t^2(L_2,t) + \frac{L_1}{4}v_t^2(L_1,t)\\ &+\frac{L_2-L_3-L_1}{4(L_2-L_1)}\int_{L_1}^{L_2} v_t^2(x,t)\;dx,\end{aligned} \tag{2.48}$$

and

$$\begin{aligned}-b\int_{L_1}^{L_2} q(x)v_x(x,t)v_{xx}(x,t)\,dx = \;& -\frac{b}{2}\left[q(x)v_x^2(x,t)\right]_{L_1}^{L_2} + \frac{b}{2}\int_{L_1}^{L_2} q_x(x)v_x^2(x,t)\;dx\\ = \;& -b\frac{L_2-L_3}{4}v_x^2(L_2,t) + b\frac{L_1}{4}v_x^2(L_1,t)\\ &+b\frac{L_2-L_3-L_1}{4(L_2-L_1)}\int_{L_1}^{L_2} v_x^2(x,t)\;dx.\end{aligned} \tag{2.49}$$

Estimate (2.46) follows by substituting (2.48) and (2.49) into (2.47). ■

Next, we define a Lyapunov functional $\mathcal{L}$ and show that it is equivalent to the energy functional E.

Lemma 4.6 *For N sufficiently large, the functional defined by*

$$\mathcal{L}(t) := NE(t) + \gamma J(t) + \gamma_1 J_1(t) + \gamma_2 J_2(t), \tag{2.50}$$

where N, γ, γ_1 and γ_2 are positive real numbers to be chosen appropriately later, satisfies

$$\beta_1 E(t) \leq \mathcal{L}(t) \leq \beta_2 E(t), \quad \forall\; \beta_1, \beta_2 > 0. \tag{2.51}$$

Proof : Let $L(t)=\gamma J(t)+\gamma_1 J_1(t)+\gamma_2 J_2(t)$

$$\begin{aligned}|L(t)|\leq\ & \gamma\int_\Omega |u\,(x,t)\,u_t\,(x,t)\,|\,dx+\gamma\int_{L_1}^{L_2}|v\,(x,t)\,v_t\,(x,t)\,|\,dx\\ & +\gamma_1\int_\Omega |q(x)u_x(x,t)u_t(x,t)|\,dx+\gamma_2\int_{L_1}^{L_2}|q(x)v_x(x,t)v_t(x,t)|\,dx.\end{aligned}$$

Exploiting Young's, Poincaré inequalities and (2.30), we obtain

$$\begin{aligned}|L(t)|\leq\ & \frac{C_s}{2}\int_\Omega u_x^2\,(x,t)\ dx+\frac{1}{2}\int_\Omega u_t^2\,(x,t)\ dx+\frac{C_s}{2}\int_{L_1}^{L_2}v_x^2\,(x,t)\ dx+\frac{1}{2}\int_{L_1}^{L_2}v_t^2\,(x,t)\,|\,dx\\ & +\frac{c_1}{2}\int_\Omega u_x^2(x,t)+\frac{c_1}{2}\int_\Omega u_t^2(x,t)\,dx+\frac{c_2}{2}\int_{L_1}^{L_2}v_x^2(x,t)+\frac{c_2}{2}\int_{L_1}^{L_2}v_t^2(x,t)\,dx\\ & \leq cE(t).\end{aligned}$$

Consequently, $|L(t)-NE(t)|\leq cE(t)$, which yields

$$(N-c)E(t)\leq\mathcal{L}(t)\leq(N+c)E(t).$$

Choosing N large enough, we obtain estimate (2.51). ■

Lemma 4.7 *Let (u,v) be a solution of (2.1)-(2.3). Then $\mathcal{L}(t)$ satisfies the following estimate, along the solution and for some positive constants $m,\ c>0$*

$$\frac{d}{dt}\mathcal{L}(t)\leq -mE(t)+c\int_\Omega[u_t^2(x,t)+g^2(u_t(x,t))]\,dx. \tag{2.52}$$

Proof : Taking the derivative of (2.50) with respect to t and making use of (2.34), (2.37), (2.40) and (2.46), we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t)\leq\ & (\gamma+\frac{\gamma_1}{2})\int_\Omega u_t^2(x,t)\,dx-\left[\gamma(a-\delta C_s)-\gamma_1\left(\frac{a}{2}+\delta_1\right)\right]\int_\Omega u_x^2(x,t)\,dx\\ & +\left[\gamma_2\frac{L_2-L_3-L_1}{4\,(L_2-L_1)}+\gamma\right]\int_{L_1}^{L_2}v_t^2\,(x,t)\ dx+b\left[\gamma_2\frac{L_2-L_3-L_1}{4\,(L_2-L_1)}+\gamma\right]\int_{L_1}^{L_2}v_x^2\,(x,t)\ dx\\ & -\frac{a}{4}\left[\gamma_1-\gamma_2\frac{b}{a}\right]\left[(L_3-L_2)\,v_x^2\,(L_2,t)+L_1v_x^2\,(L_1,t)\right]\\ & +\mu^2\left[\gamma C(\delta)+\gamma_1 C(\delta_1)\right]\int_\Omega g^2(u_t(x,t))\,dx.\end{aligned}\tag{2.53}$$

At this point, we choose our constants in (2.53), carefully, such that all the coefficients in (2.53) will be negative. Indeed, under the assumption (2.31), we can always find γ, γ_1 and γ_2 such that

$$\gamma_2\frac{L_2-L_3-L_1}{4\,(L_2-L_1)}+\gamma<0,\quad \gamma_1>\gamma_2\frac{b}{a},\quad \gamma>\frac{\gamma_1}{2},$$

we may δ and δ_1 small enough such that $\gamma\delta C_s+\gamma_1\delta_1<a(\gamma-\frac{\gamma_1}{2})$.
Then

$$\frac{d}{dt}\mathcal{L}(t)\leq -mE(t)+c\int_\Omega[u_t^2(x,t)+g^2(u_t(x,t))]\,dx.$$

This completes the proof. ■

Proof : of Theorem 4.1. As in Komornik [30], we consider the following partition of Ω,

$$\Omega_1 = \{x \in \Omega : |u_t| > \varepsilon\}, \quad \Omega_2 = \{x \in \Omega : |u_t| \leq \varepsilon\}.$$

Case 1. G is linear on $[0, \varepsilon]$, then we deduce that

$$\mathcal{L}'(t) \leq -mE(t) + c\int_\Omega u_t(x,t)g(u_t(x,t))\,dx \leq -mE(t) - cE'(t).$$

Consequently, we arrive at

$$(\mathcal{L}(t) + cE(t))' \leq -mE(t).$$

Recalling that

$$\mathcal{L}(t) + cE(t) \sim E(t),$$

we obtain

$$E(t) \leq c'e^{-c''t}.$$

Case 2. G is nonlinear on $[0, \varepsilon]$ In this case, we define

$$I\left(t\right) = \frac{1}{|\Omega_1|}\int_{\Omega_1} u_t\left(x,t\right) g\left(u_t\left(x,t\right)\right) dx,$$

and exploit Jensen's inequality and the concavity of G^{-1} to obtain

$$G^{-1}\left(I\left(t\right)\right) \geq c\int_{\Omega_1} G^{-1}(u_t g(u_t))\,dx,$$

by using this inequality and (2.4) , we obtain

$$\begin{aligned} \int_{\Omega_1}[u_t^2(x,t) + g^2(u_t(x,t)]dx &\leq \int_{\Omega_1} G^{-1}(u_t g(u_t))\,dx \\ &\leq cG^{-1}(I(t)), \end{aligned} \tag{2.54}$$

using (2.52) and (2.54), we have

$$\mathcal{L}'(t) \leq -mE(t) + cG^{-1}(I(t)). \tag{2.55}$$

We define F_0 by

$$F_0(t) = H'\Big(\frac{E(t)}{E(0)}\Big)\mathcal{L}(t) + c_0E(t).$$

Then, we see easily that, for $a_1,\ a_2 > 0$

$$a_1F_0(t) \leq E(t) \leq a_2F_0(t). \tag{2.56}$$

By recalling that $E' \leq 0,\ G' > 0,\ G'' > 0$ on $(0; \varepsilon]$ and making use of (2.30) and (2.55), we obtain

$$\begin{aligned} F_0'(t) &= \frac{E'(t)}{E(0)}G''\Big(\frac{E(t)}{E(0)}\Big)\mathcal{L}(t) + G'\Big(\frac{E(t)}{E(0)}\Big)\mathcal{L}'(t) + c_0E'(t) \\ &\leq -mE(t)G'\Big(\frac{E(t)}{E(0)}\Big) + cG'\Big(\frac{E(t)}{E(0)}\Big)G^{-1}(I(t)) + c_0E'(t). \end{aligned} \tag{2.57}$$

Let G^* be the convex conjugate of G in the sense of Young

$$G^*(s) = s(G')^{-1}(s) - G[(G')^{-1}(s)],$$

and G satisfies the generalized Young's inequality

$$AB \leq G^*(A) + G(B),$$

with $A = G'\Big(\frac{E(t)}{E(0)}\Big)$ and $B = G^{-1}(I(t))$

$$\begin{aligned} F_0'(t) &\leq -mE(t)G'\Big(\frac{E(t)}{E(0)}\Big) + cG^*\Big(G'\Big(\frac{E(t)}{E(0)}\Big)\Big) + cI(t) + c_0E'(t) \\ &\leq -mE(t)G'\Big(\frac{E(t)}{E(0)}\Big) + c\frac{E(t)}{E(0)}G'\Big(\frac{E(t)}{E(0)}\Big) - cE'(t) + c_0E'(t). \end{aligned} \tag{2.58}$$

Choosing $c_0 > c$, we obtain

$$F_0'(t) \leq -k\frac{E(t)}{E(0)}G'\Big(\frac{E(t)}{E(0)}\Big) = -kG_1\Big(\frac{E(t)}{E(0)}\Big), \tag{2.59}$$

where $G_2(t) = tG'(t)$. Since

$$G_2'(t) = G'(t) + tG''(t),$$

and G is convex on $(0, \varepsilon]$ we find that $G_2'(t) > 0$ and $G_2(t) > 0$ on $(0, 1]$. By setting $F(t) = \frac{a_1F_0(t)}{E(0)}$ (a_1 is given in (2.56)), we easily see that, by (2.56), we have

$$F(t) \sim E(t). \tag{2.60}$$

Using (2.59), we arrive at $F'(t) \leq -k_1G_2(F(t))$.
By recalling (2.33), we deduce $G_2(t) = -1/G_1'(t)$.

$$F'(t) \leq \frac{k}{G_1'(F(t))} \quad \textit{which gives} \quad [G_1(F(t))]' = F'(t)G_1'(t) \leq k_1.$$

A simple integration leads to

$$G_1(F(t)) \leq k_1t + G_1(F(0)).$$

Consequently,

$$F(t) \leq G_1^{-1}(k_1t + k_2). \tag{2.61}$$

Using (2.60) and (2.61) we obtain (2.32). The proof is complete. ■

Chapter 3

Well-posedness and decay estimates for a Petrovsky-Petrovsky system

1 Introduction and statement of main results

Let us consider the following problem:

$$\begin{cases} u_{tt} + \alpha v + \Delta^2 u - g(\Delta u_t(x,t)) = 0, & in \quad \Omega \times \mathbb{R}^+, \\ v_{tt} + \alpha u + \Delta^2 v - g(\Delta v_t(x,t)) = 0, & in \quad \Omega \times \mathbb{R}^+, \\ u = \Delta u = v = \Delta v = 0, & on \quad \Gamma \times \mathbb{R}^+, \\ (u(0,x), v(0,x)) = (u^0(x), v^0(x)), & in \quad \Omega, \\ (u_t(0,x), v_t(0,x)) = (u^1(x), v^1(x)), & in \quad \Omega. \end{cases} \tag{3.1}$$

The problem of stabilization of weakly coupled systems has also been studied by several authors. Under certain conditions imposed on the subset where the damping term is effective, Kapitonov [31] showed uniform stabilization of the solutions of a pair of hyperbolic systems coupled in velocities. In [2], the authors developed an approach to prove that, for $\alpha \in \mathbb{R}^+$ with α small enough,

$$\begin{cases} u_{tt} - \Delta u + \alpha v + u_t = 0, & in \quad \Omega \times \mathbb{R}^+, \\ v_{tt} - \Delta v + \alpha u = 0, & in \quad \Omega \times \mathbb{R}^+, \end{cases} \tag{3.2}$$

is not exponentially stable and the asymptotic behavior of solutions is at least of polynomial type $\frac{1}{t^m}$ with decay rate m depending on the smoothness of initial data.

In [10], Beniani consider the Petrovsky-Petrovsky system, that is,

$$\begin{cases} u_{tt} + \phi(x)\left(\Delta^2 u - \int_{-\infty}^{t} \mu(t-s)\Delta^2 u(s)ds\right) + \alpha v = 0, & \mathbb{R}^n \times \mathbb{R}^+ \\ v_{tt} + \phi(x)\Delta^2 v + \alpha u = 0, & \mathbb{R}^n \times \mathbb{R}^+ \\ u = v = \Delta u = \Delta v = 0, & \Gamma \times \mathbb{R}^+ \\ (u_0, v_0) \in \mathcal{D}^{2,2}(\mathbb{R}^n), & (u_1, v_1) \in L^2_g(\mathbb{R}^n), \end{cases} \tag{3.3}$$

the authors proved, under suitable conditions, that the system is polynomial stable. Komornik [70] considered the problem

$$\begin{cases} u_{tt} + \Delta^2 u - g(\Delta u_t(s)) = 0, & \Omega \times \mathbb{R}^+, \\ u = \Delta u = 0, & \Gamma \times \mathbb{R}^+, \\ u(0) = u_0, \quad u'(0) = u_1, & \Omega. \end{cases} \tag{3.4}$$

and used semi-group to obtain the well posedness and also showed the both exponential and polynomial decay of energy by introducing a multiplier method combined with a nonlinear integral inequalities given by Martinez [49].

The paper is organized as follows. In Section 1, we prove the global existence and uniqueness of solutions of (3.1). In Section 3, we prove the stability results.

The plan of this paper is as follows. In section 1, we present some notations and assumptions needed for our work, and then establish the well-posedness and the stability result of our problem. In section 2, we use the Faedo-Galerkin to prove the global existence and uniqueness of solutions (3.1). In the last section, we prove the stability result by the multiplier method and some properties of convex functions. These arguments of convexity were introduced and developed by Lasiecka et al. [36] and Alabau-Boussouira, [1].

First, assume that α and $g(s)$ satisfy the following hypotheses

(A1) α satisfy the following condition

$$\alpha \leq \frac{1}{2C_s}, \tag{3.5}$$

where $C_s > 0$ depending only on the geometry of Ω is the constant such that

$$\|\nabla z\|^2 \leq C_s \|\nabla \Delta z\|^2.$$

(A2) The function $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a non-decreasing continuous function such that there exist constants $\varepsilon, c_1, c_2, \tau > 0$ and a convex increasing function $G : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ of class $\mathcal{C}^1(\mathbb{R}_+) \cap \mathcal{C}^2(\mathbb{R}_+^*)$ satisfying G linear on $[0, \varepsilon]$ or ($G'(0) = 0$ and $G'' > 0$ on $]0, \varepsilon]$, such that

$$c_1 |s| \leq |g(s)| \leq c_2 |s|, \text{ if } |s| > \varepsilon, \tag{3.6}$$

$$|s|^2 + |g(s)|^2 \leq G^{-1}(sg(s)), \text{ if } |s| \leq \varepsilon, \tag{3.7}$$

$$|g'(s)| \leq \tau. \tag{3.8}$$

Introduce three real Hilbert spaces $\mathcal{H}$, V and W by setting

$$\mathcal{H} = H_0^1(\Omega), \qquad \|z\|_{\mathcal{H}}^2 = \int_\Omega |\nabla z|^2 \, dx,$$

$$V = \left\{ z \in H^3(\Omega) : z = \Delta z = 0 \text{ on } \Gamma \right\}, \qquad \|z\|_V^2 = \int_\Omega |\nabla \Delta z|^2 \, dx,$$

and

$$W = \left\{ z \in H^5(\Omega) : z = \Delta z = \Delta^2 z = 0 \text{ on } \Gamma \right\}, \qquad \|z\|_W^2 = \int_\Omega \left| \nabla \Delta^2 z \right|^2 dx.$$

Identifying $\mathcal{H}$ with its dual $\mathcal{H}'$ we have

$$W \subset V \subset \mathcal{H} = \mathcal{H}' \subset V' \subset W',$$

with dense and compact imbedding. Now we define the energy associated to the solution of the system (3.1) by

$$E\left(t\right) := \frac{1}{2}\left\|\nabla u_t\right\|^2 + \frac{1}{2}\left\|\nabla v_t\right\|^2 + \frac{1}{2}\left\|\nabla \Delta u\right\|^2 + \frac{1}{2}\left\|\nabla \Delta v\right\|^2 + 2\alpha \int\limits_{\Omega} \nabla u.\nabla v \, dx. \tag{3.9}$$

$$2\alpha \int\limits_{\Omega} \nabla u.\nabla v \, dx \geq -\alpha C_s \int\limits_{\Omega} (|\nabla \Delta u|^2 + |\nabla \Delta v|^2) \, dx.$$

Hence

$$\begin{aligned} E\left(t\right) &\geq \frac{1}{2}\left\|\nabla u_t\right\|^2 + \frac{1}{2}\left\|\nabla v_t\right\|^2 + \Big(\frac{1}{2} - \alpha C_s\Big)\left\|\nabla \Delta u\right\|^2 + \Big(\frac{1}{2} - \alpha C_s\Big)\left\|\nabla \Delta v\right\|^2 \\ &\geq 0. \end{aligned} \tag{3.10}$$

Note that E is the natural energy for system (3.1), given the structure of the damping term. The energy E is a non-increasing function of the time variable t and we have for almost every $t \geq 0$,

$$E^{'}\left(t\right) = -\int\limits_{\Omega} (\Delta u_t g(\Delta u_t) + \Delta v_t g(\Delta v_t)) \, dx \leq 0. \tag{3.11}$$

Theorem 1.1 *Let $(u^0, v^0) \in W$ and $(u^1, v^1) \in V$ arbitrarily, assume that (3.5)-(3.8) hold. Then the problem (3.1) has a unique weak solution satisfying*

$$(u, v) \in L^{\infty}(\mathbb{R}_+; W),$$

$$(u_t, v_t) \in L^{\infty}(\mathbb{R}_+; V)$$

and

$$(u_{tt}, v_{tt}) \in L^{\infty}(\mathbb{R}_+; \mathcal{H}) \cap L^2(0, T, H_0^2(\Omega)).$$

Theorem 1.2 *Let $(u^0, v^0) \in W$ and $(u^1, v^1) \in V$ arbitrarily, assume that (3.5)-(3.8) hold. Then the global solutions of the problem (3.1) have the following asymptotic property*

$$E(t) \leq \psi^{-1}\Big(h(t) + \psi(E(0))\Big), \ \forall t \geq 0, \tag{3.12}$$

where $\psi(t) = \int_t^1 \frac{1}{\omega \Psi(s)} \, ds$ *for* $t > 0$, $h(t) = 0$ *for* $0 \leq t \leq \frac{E(0)}{\omega \Psi(E(0))}$ *and*

$$h^{-1}(t) = t + \frac{\psi(t + \psi(E(0)))}{\Psi(\psi^{-1}(t + \psi(E(0))))}, \ \forall t \geq \frac{E(0)}{\Psi(E(0))},$$

$$\varphi(t) = \begin{cases} t & \text{if } G \text{ is linear on } [0, \varepsilon] \\ tG'(\varepsilon_0 t) & \text{if } G'(0) = 0 \text{ and } G'' > 0 \text{ on }]0, \varepsilon], \end{cases} \tag{3.13}$$

where ω and ε are positive constants.

2 Some technical lemmas

Lemma 2.1 *For all $u \in H_0^1(\Omega) \cap H^2(\Omega)$, we have*

$$\|\nabla u\| \leq c\,\|\Delta u\|_{H^{-1}(\Omega)} \leq c\|\Delta u\|, \tag{3.14}$$

where $H^{-1}(\Omega) = \left(H_0^1(\Omega)\right)'$.

Remark 2.2 *Let us denote by ϕ^* the conjugate function of the differentiable convex function ϕ, i.e.,*

$$\phi^* = \sup_{s \in \mathbb{R}^+} (st - \phi(t)).$$

*Then ϕ^* is the Legendre transform of ϕ, which is given by (see Arnold [**?** , p. 61-62])*

$$\phi^*(s) = s(\phi')^{-1}(s) - \phi\left((\phi')^{-1}(s)\right), \text{ if } s \in \left]0, \phi'(r)\right],$$

and ϕ^* satisfies the generalized Young inequality

$$ST \leq \phi^*(S) + \phi(T), \text{ if } S \in \left]0, \phi'(r)\right], T \in]0, r]. \tag{3.15}$$

Lemma 2.3 *[7] Let $E : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a non-increasing function and $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a convex and increasing function such that $\psi(0) = 0$ assume that*

$$\int_S^T \psi(E(t)) \leq E(S), \quad 0 \leq S < T. \tag{3.16}$$

Then E satisfies the following estimate:

$$E(t) \leq \psi^{-1}\left(h(t) + \psi(E(0))\right), \quad \forall t \geq 0. \tag{3.17}$$

Where $\psi(t) = \int_t^1 \frac{1}{\psi(s)} ds$ for $t > 0$, $h(t) = 0$ for $0 \leq t \leq \frac{E(0)}{\psi(E(0))}$ and

$$h^{-1}(t) = t + \frac{\psi^{-1}(t + \psi(E(0))}{\psi\left(\psi^{-1}(t + \psi(E(0)))\right)}, \quad \forall t \geq \frac{E(0)}{\psi(E(0))}.$$

Proof : of theorem 1.1

i) *Approximate solutions:* We will use the Faedo-Galerkin method to prove the existence of a global solution.
Let $T > 0$ be fixed and let $\{w_j\}, j \in \mathbb{N}$ be a basis of $\mathcal{H}$, V and W, i.e. the space generated by $\mathcal{B}_k = \{w_1, w_2, \ldots, w_k\}$ is dense in $\mathcal{H}$, V and W.
We construct approximate solutions (u^k, v^k), $k = 1, 2, 3, \ldots$, in the form

$$u^k(t, x) = \sum_{j=1}^k c_{jk}(t) w_j(x), \quad v^k(t, x) = \sum_{i=0}^k h_{jk}(t) w_j(x),$$

where c_{jk} and h_{jk} is determined by the ordinary differential equations.

For any w in $\mathcal{B}_k$, $(u^k(t), v^k(t))$ satisfies the approximate equation satisfies the approximate equation

$$\begin{cases} \int_\Omega (u^k_{tt}(t) + \alpha v^k + \Delta^2 u^k - g(\Delta u^k_t))w\,dx = 0, \\ \int_\Omega (v^k_{tt}(t) + \alpha u^k + \Delta^2 v^k - g(\Delta v^k_t))w\,dx = 0, \end{cases} \tag{3.18}$$

with initial conditions

$$u^k(0) = u^k_0 = \sum_{j=1}^{k} \langle u_0, w_j\rangle w_j \to u^0, \quad \text{in } W \text{ as } k \to +\infty, \tag{3.19}$$

$$v^k(0) = v^k_0 = \sum_{j=1}^{k} \langle v_0, w_j\rangle w_j \to v^0, \quad \text{in } W \text{ as } k \to +\infty, \tag{3.20}$$

$$u^k_t(0) = u^k_1 = \sum_{j=1}^{k} \langle u_1, w_j\rangle w_j \to u^1, \quad \text{in } V \text{ as } k \to +\infty, \tag{3.21}$$

$$v^k_t(0) = v^k_1 = \sum_{j=1}^{k} \langle v_1, w_j\rangle w_j \to v^1, \quad \text{in } V \text{ as } k \to +\infty, \tag{3.22}$$

$$-\Delta^2 u^k_0 - \alpha v^k_0 + g(\Delta u^k_1) \longrightarrow -\Delta^2 u^0 - \alpha v^0 + g(\Delta u^1), \quad \text{in } \mathcal{H} \quad \text{as } k \to +\infty, \tag{3.23}$$

$$-\Delta^2 v^k_0 - \alpha u^k_0 + g(\Delta v^k_1) \longrightarrow -\Delta^2 v^0 - \alpha u^0 + g(\Delta v^1), \quad \text{in } \mathcal{H} \quad \text{as} \quad k \to +\infty. \tag{3.24}$$

The standard theory of ODE guarantees that the system (3.18)-(3.24) has an unique solution in $[0, t_k)$, with $0 < t_k < T$, by Zorn lemma since the nonlinear terms in (3.18) are locally Lipschitz continuous. Note that $(u^k(t), v^k(t))$ the class $\mathcal{C}^2$.
In the next step, we obtain a priori estimates for the solution of the system (3.18)-(3.24), so that it can be extended outside $[0, t_k)$ to obtain one solution defined for all $T > 0$, using a standard compactness argument for the limiting procedure.

ii) *A priori estimates:*
The first estimate: Setting $w = -2\Delta u^k_t$ in the first equation and $w = -2\Delta v^k_t$ in the second equation in (3.18), add the resulting equations, we have

$$\begin{aligned} &\frac{d}{dt}\Big[\|\nabla u^k_t(t)\|^2 + \|\nabla v^k_t(t)\|^2 + \|\nabla\Delta u^k(t)\|^2 + \|\nabla\Delta v^k(t)\|^2 + 2\alpha \int_\Omega \nabla u^k \nabla v^k\,dx\Big] \\ &+2\int_\Omega \Delta u^k_t g(\Delta u^k_t)\,dx + 2\int_\Omega \Delta v^k_t g(\Delta v^k_t)\,dx = 0. \end{aligned}$$

Integrating it over $(0, t)$ and using Hölder's inequality, we get

$$\begin{aligned} &\|\nabla u^k_t(t)\|^2 + \|\nabla v^k_t(t)\|^2 + \|\nabla\Delta u^k(t)\|^2 + \|\nabla\Delta v^k(t)\|^2 + 2\alpha \int_\Omega \nabla u^k \nabla v^k\,dx \\ &+2\int_0^t \int_\Omega \Delta u^k_t(s) g(\Delta u^k_t(s))\,dx\,ds + 2\int_0^t \int_\Omega \Delta v^k_t(s) g(\Delta v^k_t(s))\,dx\,ds \\ &\le \|\nabla u^k_1(t)\|^2 + \|\nabla v^k_1(t)\|^2 + \|\nabla\Delta u^k_0(t)\|^2 + \|\nabla\Delta v^k_0(t)\|^2 + 2\alpha \int_\Omega \nabla u^k_0 \nabla v^k_0\,dx \\ &\le \|\nabla u^k_1(t)\|^2 + \|\nabla v^k_1(t)\|^2 + \|\nabla\Delta u^k_0(t)\|^2 + \|\nabla\Delta v^k_0(t)\|^2 + \alpha\|\nabla u^k_0\|^2 + \alpha\|\nabla v^k_0\|^2, \end{aligned}$$

using (3.19)-(3.22), we obtain

$$\begin{aligned}&\|\nabla u_t^k(t)\|^2+\|\nabla v_t^k(t)\|^2+(1-2\alpha C_s)\|\nabla\Delta u^k(t)\|^2+(1-2\alpha C_s)\|\nabla\Delta v^k(t)\|^2\\&+2\int_0^t\int_\Omega \Delta u_t^k(s)g(\Delta u_t^k(s))\,dx\,ds+2\int_0^t\int_\Omega \Delta v_t^k(s)g(\Delta v_t^k(s))\,dx\,ds\\&\le C_1\end{aligned}\tag{3.25}$$

and C_1 is a positive constant depending only on $\|u_0\|_V,\ \|v_0\|_V,\ \|u_1\|_{\mathcal{H}}$ and $\|v_1\|_{\mathcal{H}}$.
This estimate imply that the solution (u^k,v^k) exists globally in $[0,+\infty)$. Estimate (3.25) yields that

$$\Big(u^k,v^k\Big)\ \text{are bounded in } L^\infty(0,T;V),\tag{3.26}$$

$$\Big(u_t^k,v_t^k\Big)\ \text{are bounded in } L^\infty(0,T;\mathcal{H}),\tag{3.27}$$

$$\Big(\Delta u_t^k g(\Delta u_t^k),\Delta v_t^k g(\Delta v_t^k)\Big)\ \text{are bounded in } L^1(\Omega\times(0,T)).\tag{3.28}$$

We show that

$$\Big(g(\Delta u_t^k),g(\Delta v_t^k)\Big)\ \text{are bounded in } L^2(\Omega\times(0,T)).\tag{3.29}$$

As in Komornik [30], we consider the following partition of Ω,

$$\Omega_1=\{x\in\Omega:|\Delta u_t^k|>\varepsilon\},\quad \Omega_2=\{x\in\Omega:|\Delta u_t^k|\le\varepsilon\}.$$

Using (3.6) and (3.28), we have

$$\begin{aligned}\int_0^T\int_{\Omega_1}|g(\Delta u_t^k(s))|^2\,dx\,ds &\le c_2\int_0^T\int_{\Omega_1}\Delta u_t^k(s)g(\Delta u_t^k(s))\,dx\,ds\\&\le C,\end{aligned}$$

exploit Jensen's inequality and the concavity of G^{-1}, we obtain

$$\begin{aligned}\int_{\Omega_2}|g(\Delta u_t^k(t))|^2\,dx &\le \int_{\Omega_2}G^{-1}(\Delta u_t^k(t)g(\Delta u_t^k(t)))\,dx\,ds\\&\le G^{-1}\Big(\frac{1}{|\Omega_2|}\int_{\Omega_2}\Delta u_t^k(t)g(\Delta u_t^k(t))\,dx\Big),\end{aligned}$$

using Remark 2.2, we have

$$\begin{aligned}\int_0^T\int_{\Omega_2}|g(\Delta u_t^k(s))|^2\,dx\,dt &\le G^*(1)+\frac{1}{|\Omega_2|}\int_0^T\int_{\Omega_2}\Delta u_t^k(s)g(\Delta u_t^k(s))\,dx\,dt\\&\le C.\end{aligned}$$

The second estimate.
First, we estimate $\|u_{tt}^k(0)\|$ and $\|v_{tt}^k(0)\|$. Differentiating (3.18) with respect to x, setting $w=\nabla u_{tt}^k(t)$ in the first equation and $w=\nabla v_{tt}^k(t)$ in the second equation in (3.18), add the resulting equations and choosing $t=0$, we obtain that

$$\|\nabla u_{tt}^k(0)\|^2+\Big(\nabla u_{tt}^k(0),\nabla\Delta^2u_0^k-\nabla\Big(g(\Delta u_1^k)\Big)+\alpha\nabla v_0^k\Big)=0.$$

and

$$\|\nabla v_{tt}^k(0)\|^2+\Big(\nabla v_{tt}^k(0),\nabla\Delta^2v_0^k-\nabla\Big(g(\Delta v_1^k)\Big)+\alpha\nabla u_0^k\Big)=0.$$

Using Cauchy-Schwarz inequality, we have

$$\|\nabla u_{tt}^k(0)\| \quad \leq \quad \Big(\int_\Omega |\nabla(\Delta^2 u_0^k + g(\Delta u_1^k) + \alpha v_0^k)|^2\,dx\Big)^{\frac{1}{2}}$$

and

$$\|\nabla v_{tt}^k(0)\| \quad \leq \quad \Big(\int_\Omega |\nabla(\Delta^2 v_0^k + g(\Delta v_1^k) + \alpha u_0^k)|^2\,dx\Big)^{\frac{1}{2}}.$$

By (3.23) and (3.24) yields

$$\Big(u_{tt}^k(0), v_{tt}^k(0)\Big) \quad \text{are bounded in} \quad \mathcal{H}. \tag{3.30}$$

The Third estimate.
Differentiating (3.18) with respect to t, we get

$$\begin{cases} \displaystyle\int_\Omega (u_{ttt}^k(t) + \alpha v_t^k + \Delta^2 u_t^k - \Delta u_{tt} g'(\Delta u_t))w\,dx = 0, \\ \displaystyle\int_\Omega (v_{ttt}^k(t) + \alpha u_t^k + \Delta^2 v_t^k - \Delta v_{tt} g'(\Delta v_t))w\,dx = 0, \end{cases} \tag{3.31}$$

taking $w = -2\Delta u_{tt}^k$ in the first equation and $w = -2\Delta v_{tt}^k$ in the second equation in (3.31), add the resulting equations and applying the Green formula, we have

$$\begin{aligned} &\frac{d}{dt}\Big[\|\nabla u_{tt}^k\|^2 + \|\nabla\Delta u_t^k\|^2 + \|\nabla v_{tt}^k\|^2 + \|\nabla\Delta v_t^k\|^2 + 2\alpha\int_\Omega \nabla u_t^k.\nabla v_t^k\,dx\Big] \\ &+2\int_\Omega |\Delta u_{tt}^k|^2 g'(\Delta u_t^k)\,dx + 2\int_\Omega |\Delta v_{tt}^k|^2 g'(\Delta v_t^k)\,dx = 0. \end{aligned}$$

Integrating the last equality over $(0,t)$ and using Hölder's inequality, we get

$$\begin{aligned} &\|\nabla u_{tt}^k(t)\|^2 + \|\nabla v_{tt}^k(t)\|^2 + \|\nabla\Delta u_t^k(t)\|^2 + \|\nabla\Delta v_t^k(t)\|^2 + 2\alpha\int_\Omega \nabla u_t^k \nabla v_t^k\,dx \\ &+2\int_0^t\int_\Omega |\Delta u_{tt}^k(s)|^2 g'(\Delta u_t^k(s))\,dx\,ds + 2\int_0^t\int_\Omega |\Delta v_{tt}^k(s)|^2 g'(\Delta v_t^k(s))\,dx\,ds \\ &\leq \|\nabla u_{tt}^k(0)\|^2 + \|\nabla v_{tt}^k(0)\|^2 + \|\nabla\Delta u_0^k(t)\|^2 + \|\nabla\Delta v_0^k(t)\|^2 + 2\alpha\int_\Omega \nabla u_1^k \nabla v_1^k\,dx \\ &\leq \|\nabla u_{tt}^k(0)\|^2 + \|\nabla v_{tt}^k(0)\|^2 + \|\nabla\Delta u_1^k(t)\|^2 + \|\nabla\Delta v_1^k(t)\|^2 + \alpha\|\nabla u_1^k\|^2 + \alpha\|\nabla v_1^k\|^2, \end{aligned} \tag{3.32}$$

using (3.21), (3.22) and (3.30), we have

$$\begin{aligned} &\|\nabla u_{tt}^k(t)\|^2 + \|\nabla v_{tt}^k(t)\|^2 + (1-2\alpha C_s)\|\nabla\Delta u_t^k(t)\|^2 + (1-2\alpha C_s)\|\nabla\Delta v_t^k(t)\|^2 \\ &+c\int_0^t\int_\Omega |\Delta u_{tt}^k(s)|^2\,dx\,ds + c'\int_0^t\int_\Omega |\Delta v_{tt}^k(s)|^2\,dx\,ds \\ &\leq C_2, \end{aligned}$$

for all $t \in \mathbb{R}^+$, therefore, we conclude that

$$\Big(u_t^k, v_t^k\Big) \quad \text{are bounded in} \quad L^\infty(0,T;V) \tag{3.33}$$

and

$$\Big(u_{tt}^k, v_{tt}^k\Big) \quad \text{are bounded in} \quad L^\infty(0,T;\mathcal{H}) \cap L^2(0,T,H_0^2(\Omega)), \tag{3.34}$$

we deduce that

$$\left(u_t^k, v_t^k\right) \quad \text{are bounded in} \quad L^2(0,T;V). \tag{3.35}$$

Applying Rellich compactenes theorem given in **[45]**, we deduce that

$$\left(u_t^k, v_t^k\right) \quad \text{are precompact in} \quad L^2(0,T;L^2(\Omega)). \tag{3.36}$$

The fourth estimate.

Differentiating (3.18) with respect to x, taking $w = \nabla\Delta^2 u_t^k$ in the first equation and $w = \nabla\Delta^2 v_t^k$ in the second equation in (3.18), add the resulting equations, we obtain that

$$\|\nabla\Delta^2 u^k\|^2 = \int_\Omega \nabla\Delta^2 u^k(-\nabla u_{tt}^k - \alpha\nabla v^k + \nabla\Delta u_t^k g'(\Delta u_t^k))\,dx$$

and

$$\|\nabla\Delta^2 v^k\|^2 = \int_\Omega \nabla\Delta^2 v^k(-\nabla v_{tt}^k - \alpha\nabla u^k + \nabla\Delta v_t^k g'(\Delta v_t^k))\,dx.$$

Using Cauchy-Schwarz inequality, we have

$$\|\nabla\Delta^2 u^k\| \le 2\Big(\int_\Omega \{|\nabla u_{tt}^k|^2 + \alpha^2|\nabla v^k|^2 + |\nabla\Delta u_t^k g'(\Delta u_t^k)|^2\}\,dx\Big)^{\frac{1}{2}}$$

and

$$\|\nabla\Delta^2 v^k\| \le 2\Big(\int_\Omega \{|\nabla v_{tt}^k|^2 + \alpha^2|\nabla u^k|^2 + |\nabla\Delta v_t^k g'(\Delta v_t^k)|^2\}\,dx\Big)^{\frac{1}{2}}.$$

Using (3.8), (3.33) and (3.34), we obtain

$$\|\nabla\Delta^2 u^k\| + \|\nabla\Delta^2 v^k\| \le C_3,$$

for some C_3 independent of k, then

$$\left(u^k, v^k\right) \quad \text{are bounded in} \quad L^\infty(0,T;W). \tag{3.37}$$

iii) *Passage to the limit.* Applying Dunford-Petit theorem we conclude from (3.26), (3.29), (3.33), (3.34) and (3.37) that there exists a subsequence u^m of u^k and a functions $\{u,v\}$ such that

$$(u^m, v^m) \rightharpoonup (u,v) \ \text{ weak-star in } L^\infty(0,T;W), \tag{3.38}$$

$$(u_t^m, v_t^m) \rightharpoonup (u_t, v_t)\,, \ \text{ weak-star in } L^\infty(0,T;V), \tag{3.39}$$

$$(u_{tt}^m, v_{tt}^m) \rightharpoonup (u_{tt}, v_{tt})\,, \ \text{ weak-star in } L^\infty(0,T;\mathcal{H}) \cap L^2(0,T,H_0^2(\Omega)), \tag{3.40}$$

$$(u_t^m, v_t^m) \longrightarrow (u_t, v_t)\,, \ \text{almost everywhere in } \mathcal{A}, \tag{3.41}$$

$$(g(\Delta(u_t^m), g(\Delta v_t^m)) \rightharpoonup (\phi_1, \phi_2)\,, \ \text{ weak-star in } L^2(\mathcal{A}), \tag{3.42}$$

where $\mathcal{A} = \Omega \times [0,T]$. It follows at once from (3.38) and (3.40), that for each fixed $w \in L^2([0,T] \times L^2(\Omega))$,

$$\int_0^T \int_\Omega \Big(\Big(u_{tt}^m + \Delta^2 u^m + \alpha v^m\Big)(x,t)\Big) w\,dx\,dt \longrightarrow \int_0^T \int_\Omega \Big(\Big(u_{tt} + \Delta^2 u + \alpha v\Big)(x,t)\Big) w\,dx\,dt$$

and

$$\int_0^T\int_\Omega\Big(\Big(v_{tt}^m+\Delta^2u^m+\alpha u^m\Big)(x,t)\Big)w\,dx\,dt\longrightarrow\int_0^T\int_\Omega\Big(\Big(v_{tt}+\Delta^2v+\alpha u\Big)(x,t)\Big)w\,dx\,dt.$$

As (u_t^m,v_t^m) are bounded in $L^\infty(0,T;V)$ and the injection of V in $\mathcal{H}$ is compact, we have

$$(u_t^m,v_t^m)\longrightarrow(u_t,v_t)\,,\text{ strong in } L^2(0,T;\mathcal{H}). \tag{3.43}$$

It remains to show that

$$\int_0^T\int_\Omega g(\Delta u_t^m)\ w\,dx\,dt\longrightarrow\int_0^T\int_\Omega g(\Delta u_t)\ w\,dx\,dt$$

and

$$\int_0^T\int_\Omega g(\Delta v_t^m)\ w\,dx\,dt\longrightarrow\int_0^T\int_\Omega g(\Delta v_t)\ w\,dx\,dt.$$

Lemma 2.4 *For each $T>0$, $(g(\Delta u_t),g(\Delta v_t))\in L^1(\mathcal{A})$, $\|g(\Delta u_t)\|_{L^1(\mathcal{A})}\le K$ and $\|g(\Delta v_t)\|_{L^1(\mathcal{A})}\le K$, where K is a constant independent of t and*

$$(g(\Delta(u_t^m),g(\Delta v_t^m))\to(g(\Delta u_t),g(\Delta v_t))\,,\quad in\ \ L^1(\mathcal{A}).$$

Proof : We claim that

$$(g(\Delta u_t),g(\Delta v_t))\in L^1(\mathcal{A}).$$

Indeed, since g is continuous, we deduce from (3.41)

$$(g(\Delta(u_t^m),g(\Delta v_t^m))\longrightarrow(g(\Delta u_t),g(\Delta v_t))\quad\text{almost everywhere in }\ \mathcal{A}. \tag{3.44}$$

$$(\Delta u_t^m g(\Delta u_t^m),\Delta v_t^m g(\Delta v_t^m)\,,\longrightarrow(\Delta u_t g(\Delta u_t),\Delta v_t g(\Delta v_t))\quad\text{almost everywhere in }\ \mathcal{A}.$$

Hence, by (3.28) and Fatou's Lemma, we have

$$\int_0^T\int_\Omega\Delta u_t(x,t)g(\Delta u_t(x,t))\,dx\,dt\le K_1,\quad\text{for },\ T>0 \tag{3.45}$$

and

$$\int_0^T\int_\Omega\Delta v_t(x,t)g(\Delta v_t(x,t))\,dx\,dt\le K_1,\quad\text{for },\ T>0. \tag{3.46}$$

Now, we can estimate $\int_0^T\int_\Omega|g(\Delta u_t(x,t))|\,dx\,dt$. By Cauchy-Schwarz inequality, we have

$$\int_0^T\int_\Omega|g(\Delta u_t(x,t))|\,dx\,dt\le c|\mathcal{A}|^{1/2}\Big(\int_0^T\int_\Omega|g(\Delta u_t(x,t))|^2\,dx\,dt\Big)^{1/2}.$$

Using (3.6), (3.7) and (3.45), we obtain

$$\begin{aligned}\int_0^T\int_\Omega|g(\Delta u_t)|^2\,dx\,dt &\le \int_0^T\int_{|\Delta u_t|>\varepsilon}\Delta u_t g(\Delta u_t)\,dx\,dt+\int_0^T\int_{|\Delta u_t|\le\varepsilon}G^{-1}(\Delta u_t g(\Delta u_t))\,dx\,dt\\ &\le c\int_0^T\int_\Omega\Delta u_t g(\Delta u_t)\,dx\,dt+cG^{-1}\Big(\int_{\mathcal{A}}\Delta u_t g(\Delta u_t)\,dx\,dt\Big)\\ &\le c\int_0^T\int_\Omega\Delta u_t g(\Delta u_t)\,dx\,dt+c'G^*(1)+c''\int_\Omega\Delta u_t g(\Delta u_t)\,dx\,dt\\ &\le cK_1+c'G^*(1),\quad\text{for }\ T>0.\end{aligned}$$

Then
$$\int_0^T \int_{\mathcal{A}} |g(\Delta u_t(x,t))|\, dx\, d \leq K, \qquad \text{for } T>0.$$
Let $B \subset \Omega \times [0,T]$ and set
$$B_1 = \Big\{(x,t) \in B : |g(\Delta u_t^m(x,t))| \leq \frac{1}{\sqrt{|B|}}\Big\}, \quad B_2 = B \backslash B_1,$$
where $|B|$ is the measure of E. If $M(r) = \inf\{|s| : s \in \mathbb{R} \quad \text{and} \quad |g(s)| \geq r\}$
$$\int_B |g(\Delta u_t^m)|\, dx\, dt \leq c\sqrt{|B|} + \Big(M\Big(\frac{1}{\sqrt{|B|}}\Big)\Big)^{-1} \int_{B_2} |\Delta u_t^m g(\Delta u_t^m))|\, dx\, dt.$$
By applying (3.28) we deduce that
$$\sup_k \int_B g(\Delta u_t^m)\ dx\ dt \longrightarrow 0, \text{ when } |B| \longrightarrow 0.$$
From Vitali's convergence theorem we deduce that
$$g(\Delta u_t^m) \to g(\Delta u_t) \qquad in\ \ L^1(\mathcal{A}).$$
Similarly, we have
$$g(\Delta v_t^m) \to g(\Delta v_t) \qquad in\ \ L^1(\mathcal{A}).$$
This completes the proof. ■

Then (3.42) implies that
$$\{g(\Delta u_t^m), g(\Delta v_t^m)\} \rightharpoonup (g(\Delta u_t), g(\Delta v_t)), \text{ weak-star in } L^2([0,T] \times \Omega).$$
We deduce, for all $w \in L^2([0,T] \times L^2(\Omega)$, that
$$\int_0^T \int_\Omega g(\Delta u_t^m) w\, dx\, dt \longrightarrow \int_0^T \int_\Omega g(\Delta u_t) w\, dx\, dt$$
and
$$\int_0^T \int_\Omega g(\Delta v_t^m) w\, dx\, dt \longrightarrow \int_0^T \int_\Omega g(\Delta v_t) w\, dx\, dt.$$
Finally, we have shown that, for all $w \in L^2([0,T] \times L^2(\Omega))$
$$\begin{cases} \int_\Omega (u_{tt}(t) + \alpha v + \Delta^2 u - g(\Delta u_t)) w\, dx = 0, \\ \int_\Omega (v_{tt}(t) + \alpha u + \Delta^2 v - g(\Delta v_t)) w\, dx = 0. \end{cases}$$
■

Proof : of theorem 1.2

From now on, S and T denote two real numbers such that $1 \leq S < T < \infty$. We write E instead of $E(t)$.

Multiplying first equation of (3.1) by $-\frac{\varphi(E)}{E}\Delta u$, where $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ is convex, increasing and of class $\mathcal{C}^1$ on $]0,+\infty[$ such that $\varphi(0)=0$ and we integrate by parts, we have, for all $0\le S\le T$

$$\begin{aligned}
0 &= -\int_S^T \frac{\varphi(E)}{E}\int_\Omega \Delta u\Big(u_{tt}+\Delta^2 u+\alpha v-g(\Delta u_t)\Big)\,dx\,dt\\
&= -\Big[\frac{\varphi(E)}{E}\int_\Omega u_t\Delta u\,dx\Big]_S^T+\int_S^T\Big(\frac{\varphi(E)}{E}\Big)'\int_\Omega u_t\Delta u\,dx\,dt\\
&\quad -2\int_S^T\frac{\varphi(E)}{E}\int_\Omega|\nabla u_t|^2\,dx\,dt+\int_S^T\frac{\varphi(E)}{E}\int_\Omega\Big(|\nabla u_t|^2+|\nabla\Delta u|^2+\alpha\nabla u\nabla v\Big)\,dx\,dt\\
&\quad +\int_S^T\frac{\varphi(E)}{E}\int_\Omega\Delta u g(\Delta u_t)\,dx\,dt.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
0 &= -\int_S^T \frac{\varphi(E)}{E}\int_\Omega \Delta v\Big(v_{tt}+\Delta^2 v+\alpha u-g(\Delta v_t)\Big)\,dx\,dt\\
&= -\Big[\frac{\varphi(E)}{E}\int_\Omega v_t\Delta v\,dx\Big]_S^T+\int_S^T\Big(\frac{\varphi(E)}{E}\Big)'\int_\Omega v_t\Delta v\,dx\,dt\\
&\quad -2\int_S^T\frac{\varphi(E)}{E}\int_\Omega|\nabla v_t|^2\,dx\,dt+\int_S^T\frac{\varphi(E)}{E}\int_\Omega\Big(|\nabla v_t|^2+|\nabla\Delta v|^2+\alpha\nabla u\nabla v\Big)\,dx\,dt\\
&\quad +\int_S^T\frac{\varphi(E)}{E}\int_\Omega\Delta v g(\Delta v_t)\,dx\,dt.
\end{aligned}$$

Taking their sum, we obtain that

$$\begin{aligned}
&\int_S^T\frac{\varphi(E)}{E}\int_\Omega\Big(|\nabla u_t|^2+|\nabla v_t|^2+|\nabla\Delta u|^2+|\nabla\Delta v|^2+2\alpha\nabla u\nabla v)\,dx\,dt\\
&=\Big[\frac{\varphi(E)}{E}\int_\Omega(u_t\Delta u+v_t\Delta v)\,dx\Big]_S^T-\int_S^T\Big(\frac{\varphi(E)}{E}\Big)'\int_\Omega(u_t\Delta u+v_t\Delta v)\,dx\,dt\\
&+2\int_S^T\frac{\varphi(E)}{E}\int_\Omega(|\nabla u_t|^2+|\nabla v_t|^2)\,dx\,dt\\
&-\int_S^T\frac{\varphi(E)}{E}\int_\Omega(\Delta u g(\Delta u_t)+\Delta v g(\Delta v_t))\,dx\,dt.
\end{aligned}$$

Using the definition of the energy, we have

$$\begin{aligned}
2\int_S^T\varphi(E)dt &= \left[\frac{\varphi(E)}{E}\int_\Omega(u_t\Delta u+v_t\Delta v)\,dx\right]_S^T-\int_S^T\Big(\frac{\varphi(E)}{E}\Big)'\int_\Omega(u_t\Delta u+v_t\Delta v)\,dx\,dt\\
&\quad+2\int_S^T\frac{\varphi(E)}{E}\int_\Omega(|\nabla u_t|^2+|\nabla v_t|^2)\,dx\,dt\\
&\quad+\int_S^T\frac{\varphi(E)}{E}\int_\Omega(\Delta u g(\Delta u_t)+\Delta v g(\Delta v_t))\,dx\,dt.
\end{aligned}\tag{3.47}$$

Now, we estimate the terms of the right-hand side of (3.47) in order to apply the results of Lemma 2.1.
Since E is non-increasing, we have

$$\begin{aligned}&\Big[\frac{\varphi(E)}{E}\int_\Omega (u_t\Delta u+v_t\Delta v)\,dx\Big]_S^T\\&=\frac{\varphi(E(T))}{E(T)}\int_\Omega u_t(T)\Delta u(T)+v_t(T)\Delta v(T)\,dx-\frac{\varphi(E(S))}{E(S)}\int_\Omega u_t(S)\Delta u(S)+v_t(S)\Delta v(S)\,dx\\&\leq C_s\varphi(E(S)).\end{aligned}$$

In the other hand, we have φ is convex, increasing and of class $\mathcal{C}^1$ on $]0,+\infty[$ such that $\varphi(0)=0$. Then we deduce that

$$\begin{aligned}\int_S^T\Big(\frac{\varphi(E)}{E}\Big)'\int_\Omega (u_t\Delta u+v_t\Delta v)\,dx\,dt &\leq C_s\int_S^T\Big|\Big(\frac{\varphi(E)}{E}\Big)'\Big|E\,dt\\&\leq C_s\varphi(E(S)).\end{aligned}$$

Using these estimates, we conclude from (3.47) that

$$\begin{aligned}\int_S^T\varphi(E)\,dt &\leq C\varphi(E(S))+\int_S^T\frac{\varphi(E)}{E}\int_\Omega |\nabla u_t|^2+|\nabla v_t|^2\,dx\,dt\\&\quad+\int_S^T\frac{\varphi(E)}{E}\int_\Omega |\Delta u|.|g(\Delta u_t(x,t))|+|\Delta v|.|g(\Delta v_t(x,t))|\,dx\,dt.\end{aligned}\tag{3.48}$$

We distinguish two cases
Case 1. G is linear on $[0,\epsilon]$. By using Sobolev embedding, we have

$$\begin{aligned}\int_S^T\frac{\varphi(E)}{E}\int_{\Omega_1}|\nabla u_t|^2+|\nabla v_t|^2\,dx\,dt &\leq C_s\int_S^T\frac{\varphi(E)}{E}\int_\Omega|\Delta u_t|^2+|\Delta v_t|^2\,dx\,dt\\&\leq C_s\int_S^T\frac{\varphi(E)}{E}\int_\Omega \Delta u_t g(\Delta u_t)+\Delta v_t g(\Delta v_t)\,dx\,dt\\&\leq -C_s\int_S^T\frac{\varphi(E)}{E}E'(t)\,dt\\&\leq C\varphi(E(S))\end{aligned}$$

and

$$\begin{aligned}\int_S^T\frac{\varphi(E)}{E}\int_{\Omega_2}|\nabla u_t|^2+|\nabla v_t|^2\,dx\,dt &\leq C_s\int_S^T\frac{\varphi(E)}{E}\int_\Omega|\Delta u_t|^2+|g(\Delta u_t)|^2+|\Delta v_t|^2+g(\Delta v_t)|^2\,dx\,dt\\&\leq C_s\int_S^T\frac{\varphi(E)}{E}\int_\Omega \Delta u_t g(\Delta u_t)+\Delta v_t g(\Delta v_t)\,dx\,dt\\&\leq C\varphi(E(S)).\end{aligned}$$

Exploiting Young's and Poincaré's inequalities, we obtain

$$\begin{aligned}&\int_S^T\frac{\varphi(E)}{E}\int_{\Omega_1}|\Delta u|.|g(\Delta u_t)|\,dx\,dt\\&\leq\varepsilon\int_S^T\frac{\varphi(E)}{E}\int_{\Omega_1}|\Delta u|^2\,dx\,dt+C(\varepsilon)\int_S^T\frac{\varphi(E)}{E}\int_{\Omega_1}|g(\Delta u_t)|^2\,dx\,dt\\&\leq\varepsilon C_s\int_S^T\frac{\varphi(E)}{E}\int_\Omega|\nabla\Delta u|^2\,dx\,dt+C(\varepsilon)\int_S^T\frac{\varphi(E)}{E}\int_\Omega\Delta u_t g(\Delta u_t)\,dx\,dt\\&\leq\varepsilon C_s\int_S^T\varphi(E)\,dt+C(\varepsilon)\int_S^T\frac{\varphi(E)}{E}\int_\Omega\Delta u_t g(\Delta u_t)\,dx\,dt.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\int_S^T \frac{\varphi(E)}{E}\int_{\Omega_1} |\Delta v|.|g(\Delta v_t)|\,dx\,dt\\ &\le \varepsilon C_s\int_S^T \varphi(E)\,dt + C(\varepsilon)\int_S^T \frac{\varphi(E)}{E}\int_\Omega \Delta v_t g(\Delta v_t)\,dx\,dt.\end{aligned}$$

Taking their sum, we obtain that

$$\begin{aligned}&\int_S^T \frac{\varphi(E)}{E}\int_{\Omega_1} |\Delta u|.|g(\Delta u_t)| + |\Delta v|.|g(\Delta v_t)|\,dx\,dt\\ &\le \varepsilon C_s\int_S^T \varphi(E)\,dt + C(\varepsilon)\int_S^T \frac{\varphi(E)}{E}\int_\Omega \Delta u_t g(\Delta u_t) + \Delta v_t g(\Delta v_t)\,dx\,dt\\ &\le \varepsilon C_s\int_S^T \varphi(E)\,dt + C(\varepsilon)\varphi(E(S)).\end{aligned}\tag{3.49}$$

Similarly, we have

$$\begin{aligned}&\int_S^T \frac{\varphi(E)}{E}\int_{\Omega_2} |\Delta u|.|g(\Delta u_t)| + |\Delta v|.|g(\Delta v_t)|\,dx\,dt\\ &\le \varepsilon C_s\int_S^T \varphi(E)\,dt + C(\varepsilon)\int_S^T \frac{\varphi(E)}{E}\int_\Omega \Delta u_t g(\Delta u_t) + \Delta v_t g(\Delta v_t)\,dx\,dt\\ &\le \varepsilon C_s\int_S^T \varphi(E)\,dt + C(\varepsilon)\varphi(E(S)).\end{aligned}\tag{3.50}$$

By these estimates, we find

$$\int_S^T \varphi(E(t))\,dt \le c\varphi(E(S)).$$

Using Lemma 2.2 (cf. Guesmia [25]) for E in the particular case where $\varphi(s) = s$ we deduce from (3.17) that

$$E(t) \le ce^{-wt}.$$

Case 2. $G'(0) = 0,\ \ G'' > 0$ **on** $]0, \epsilon]$

$$\int_S^T \frac{\varphi(E)}{E}\int_{\Omega_1} |\nabla u_t|^2\,dx\,dt \le C\varphi(E(S)),$$

$$\begin{aligned}\int_S^T \frac{\varphi(E)}{E}\int_{\Omega_2} |\nabla u_t|^2 + |\nabla v_t|^2\,dx\,dt &\le C_s\int_S^T \frac{\varphi(E)}{E}\int_\Omega (|\Delta u_t|^2 + |g(\Delta u_t)|^2 + |\Delta v_t|^2 + |g(\Delta v_t)|^2)\,dx\,dt\\ &\le C_s\int_S^T \frac{\varphi(E)}{E}\int_\Omega G^{-1}\Big(\Delta u_t g(\Delta u_t) + \Delta v_t g(\Delta v_t)\Big)\,dx\,dt\\ &\le C_s\int_S^T \frac{\varphi(E)}{E}|\Omega|G^{-1}\Big(\frac{1}{|\Omega|}\int_\Omega \Delta u_t g(\Delta u_t) + \Delta v_t g(\Delta v_t)\Big)\,dx\,dt.\end{aligned}$$

Using remark 2.2, we obtain

$$\begin{aligned}&\int_S^T \frac{\varphi(E)}{E}\int_\Omega (|\Delta u_t|^2 + |g(\Delta u_t)|^2 + |\Delta v_t|^2 + |g(\Delta v_t)|^2)\,dx\,dt\\ &\le c\int_S^T G^*(\frac{\varphi(s)}{s})\,dt + c\int_S^T\int_\Omega \Delta u_t g(\Delta u_t) + \Delta v_t g(\Delta v_t)\,dx\,dt.\end{aligned}\tag{3.51}$$

Choosing $\varphi(s) = sG'(\epsilon s)$ and using remark 2.2, we obtain

$$G^*(\frac{\varphi(s)}{s}) = s\epsilon G'(\epsilon s) = \epsilon s G'(\epsilon s) - G(\epsilon s) \leq \epsilon\varphi(s). \tag{3.52}$$

Making use of (3.51) and (3.52), we have

$$\begin{aligned}
&\int_S^T \frac{\varphi(E)}{E}\int_\Omega \Big(|\Delta u_t|^2 + |g(\Delta u_t)|^2 + |\Delta v_t|^2 + |g(\Delta v_t)|^2\Big)\,dx\,dt \\
&\leq c\epsilon\int_S^T \varphi(E)\,dt + cE(S).
\end{aligned}$$

$$\begin{aligned}
&\int_S^T \frac{\varphi(E)}{E}\int_{\Omega_1} |\Delta u|.|g(\Delta u_t)|\,dx\,dt \\
&\leq \varepsilon\int_S^T \frac{\varphi(E)}{E}\int_\Omega |\Delta u|^2\,dx\,dt + C(\varepsilon)\int_S^T \frac{\varphi(E)}{E}\int_\Omega |g(\Delta u_t)|^2\,dx\,dt \\
&\leq \varepsilon C_s\int_S^T \frac{\varphi(E)}{E}\int_\Omega |\nabla\Delta u|^2\,dx\,dt + C(\varepsilon)\int_S^T \frac{\varphi(E)}{E}\int_\Omega \Delta u_t g(\Delta u_t)\,dx\,dt \\
&\leq \varepsilon C_s\int_S^T \varphi(E)\,dt + C(\varepsilon)\int_S^T \frac{\varphi(E)}{E}\int_\Omega \Delta u_t g(\Delta u_t)\,dx\,dt.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_S^T \frac{\varphi(E)}{E}\int_{\Omega_1} |\Delta v|.|g(\Delta v_t)|\,dx\,dt \\
&\leq \varepsilon C_s\int_S^T \varphi(E)\,dt + C(\varepsilon)\int_S^T \frac{\varphi(E)}{E}\int_\Omega \Delta v_t g(\Delta v_t)\,dx\,dt.
\end{aligned}$$

Taking their sum, we obtain that

$$\begin{aligned}
&\int_S^T \frac{\varphi(E)}{E}\int_{\Omega_1} |\Delta u|.|g(\Delta u_t)| + |\Delta v|.|g(\Delta v_t)|\,dx\,dt \\
&\leq \varepsilon C_s\int_S^T \varphi(E)\,dt + C(\varepsilon)\int_S^T \frac{\varphi(E)}{E}\int_\Omega \Delta u_t g(\Delta u_t) + \Delta v_t g(\Delta v_t)\,dx\,dt \\
&\leq \varepsilon C_s\int_S^T \varphi(E)\,dt + C(\varepsilon)\varphi(E(S))
\end{aligned}$$

$$\begin{aligned}
&\int_S^T \frac{\varphi(E)}{E}\int_{\Omega_2} |\Delta u|.|g(\Delta u_t)|\,dx\,dt \\
&\leq \varepsilon\int_S^T \frac{\varphi(E)}{E}\int_\Omega |\Delta u|^2\,dx\,dt + C(\varepsilon)\int_S^T \frac{\varphi(E)}{E}\int_\Omega |g(\Delta(u_t)|^2\,dx\,dt \\
&\leq \varepsilon C_s\int_S^T \frac{\varphi(E)}{E}\int_\Omega |\nabla\Delta u|^2\,dx\,dt + C(\varepsilon)\int_S^T \frac{\varphi(E)}{E}\int_\Omega (|\Delta u_t|^2 + |g(\Delta(u_t))|^2)\,dx\,dt \\
&\leq \varepsilon C_s\int_S^T \frac{\varphi(E)}{E}\int_\Omega |\nabla\Delta u|^2\,dx\,dt + C(\varepsilon)\int_S^T \frac{\varphi(E)}{E}|\Omega|G^{-1}\Big(\frac{1}{|\Omega|}\int_\Omega \Delta u_t g(\Delta u_t)\Big)\,dx\,dt \\
&\leq \varepsilon C\int_S^T \varphi(E)\,dt + C(\varepsilon)\int_S^T \frac{\varphi(E)}{E}|\Omega|G^{-1}\Big(\frac{1}{|\Omega|}\int_\Omega \Delta u_t g(\Delta u_t)\Big)\,dx\,dt.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\int_S^T \frac{\varphi(E)}{E}\int_{\Omega_2}|\Delta v|.|g(\Delta v_t)|\,dx\,dt\\&\leq \varepsilon C\int_S^T \varphi(E)\,dt + C(\varepsilon)\int_S^T \frac{\varphi(E)}{E}|\Omega|G^{-1}\Big(\frac{1}{|\Omega|}\int_\Omega \Delta v_t g(\Delta v_t)\Big)\,dx\,dt.\end{aligned}$$

Taking their sum, we obtain that

$$\begin{aligned}&\int_S^T \frac{\varphi(E)}{E}\int_{\Omega_2}|\Delta u|.|g(\Delta u_t)| + |\Delta v|.|g(\Delta v_t)|\,dx\,dt\\&\leq \varepsilon C\int_S^T \varphi(E)\,dt + C(\varepsilon)\varphi(E(S)).\end{aligned}$$

Then, choosing $\varepsilon > 0$ small enough and using (3.48), we obtain in both cases

$$\begin{aligned}\int_S^T \varphi(E(t))\,dt &\leq c(E(S) + \varphi(E(S)))\\&\leq c\Big(1 + \tfrac{\varphi(E(S))}{E(S)}\Big)E(S)\\&\leq cE(S),\ \ \forall S \geq 0.\end{aligned}$$

Using Lemma 2.1 in the particular case where $\Psi(s) = \omega\varphi(s)$ we deduce from (3.17) our estimate (3.12). The proof of Theorem 1.2 is now complete. ∎

Chapter 4

Exponential stability of a transmission problem with history and delay

1 Introduction

In this chapter we study the following transmission system with a past history and a delay term

$$
\begin{cases} u_{tt}(x,t) - au_{xx}(x,t) + \displaystyle\int_0^\infty g(s)u_{xx}(x,t-s)\mathrm{d}s \\ \qquad\qquad\qquad\qquad +\mu u_t(x,t-\tau) = 0, \quad (x,t) \in \Omega \times (0,+\infty), \\ v_{tt}(x,t) - bv_{xx}(x,t) = 0, \qquad\qquad\qquad (x,t) \in (L_1,L_2) \times (0,+\infty), \end{cases} \tag{4.1}
$$

Under the boundary and transmission conditions

$$
\begin{array}{l} u(0,t) = u(L_3,t) = 0, \\ u(L_i,t) = v(L_i,t), \quad i = 1,2, \\ au_x(L_i,t) - \displaystyle\int_0^\infty g(s)u_x(L_i,t-s)\mathrm{d}s = bv_x(L_i,t), \quad i = 1,2, \end{array} \tag{4.2}
$$

and the initial conditions

$$
\begin{array}{l} u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), \quad x \in \Omega, \ t \in (0,\tau), \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in (L_1,L_2), \end{array} \tag{4.3}
$$

where $0 < L_1 < L_2 < L_3$,$\Omega =]0, L_1[\bigcup]L_2, L_3[, a, \mu, b$ are positives constants, u_0 is given history, and $\tau > 0$ is the delay.
In certain cases also it is proved by the dissipative method.
Transmission problems ([53], [55]) arise in several applications of physics and biology. We note that problem (4.1)-(4.2) is related to the wave propagation over a body which consists of two different type of materials: the elastic part and the viscoelastic part that has the past history and time delay effect.

For wave equations with various dissipation, many results concerning stabilization of solutions have been proved. Recently, wave equations with viscoelastic damping have been investigated by many authors, see [12], [16], [17], [21], [42], [43], [67] and the references therein. It is showed that the dissipation produced by the viscoelastic part can produce the decay of the solution. For example, A. Guesmia [24] studied the equation

$$u_{tt} - Au + \int_0^{\infty} g(t)Au(t-s)ds + \mu u_t(t-\tau) = 0, \quad \text{in } \Omega \times (0,\infty),$$

and under the condition:

$$\exists \delta > 0, \quad g'(s) \leq -\delta g(s) \quad \forall\, s \in \mathbb{R}^+,$$

the authors showed the exponential decay.
Messaoudi [59] investigated the following viscoelastic equation:

$$u_{tt} - \Delta u + \int_0^{t} g(t)\Delta u(t-s)ds = 0, \quad \textit{in } \Omega \times (0,\infty),$$

in a bounded domain, and established a more general decay result, from which the usual exponential and polynomial decay rates are only special cases.
In [72] the authors examined a system of wave equations with a linear boundary damping term with a delay:

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t) + \displaystyle\int_0^{\infty} g(s)u_{xx}(x,t-s)\mathrm{d}s & \\ \qquad\qquad +\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, & (x,t) \in \Omega \times (0,+\infty), \\ v_{tt}(x,t) - bv_{xx}(x,t) = 0, & (x,t) \in (L_1,L_2) \times (0,+\infty), \end{cases} \tag{4.4}$$

and under the assumption

$$\mu_2 \leq \mu_1 \tag{4.5}$$

they proved that the solution is exponentially stable. On the contrary, if (4.5) does not hold, they found a sequence of delays for which the corresponding solution of (4.4) will be unstable.
In [58], authors considered the equation

$$u_{tt}(x,t) - \Delta u(x,t) - \mu_1 \Delta u_t(x,t) - \mu_2 \Delta u_t(x,t-\tau) = 0,$$

and under the assumption

$$|\mu_2| \leq \mu_1, \tag{4.6}$$

they proved the well-posedness and the exponential decay of energy.
Recently, in [73] Yadav and Jiwari considered Burgers'-Fisher equation:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + au\frac{\partial u}{\partial x} + bu(1-u) = 0, \quad (x,t) \in (0,T) \times \Omega,$$

the authors proved existence and uniqueness of solution. Furthermore, they also presented finite element analysis and approximation. The paper is organized as follows. The well-posedness of the problem is analyzed in Section 2 using the semigroup theory. In Section 3, we prove the exponential decay of the energy when time goes to infinity.

2 Preliminaries

We assume that the function g satisfies the following:
We assume that the function $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is of class C^1 satisfying:

$$g(0) > 0, \quad a - \int_0^\infty g(t)dt = a - g_0 = l > 0. \tag{4.7}$$

There exists a positive constant δ,

$$g'(s) \leq -\delta g(s) \quad \forall s \in \mathbb{R}^+, \tag{4.8}$$

As in [61], we introduce the variable

$$z(x, \rho, t) = u_t(x, t - \tau\rho), \quad (x, \rho, t) \in \Omega \times (0, 1) \times (0, \infty).$$

Then

$$\tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0, \quad (x, \rho, t) \in \Omega \times (0, 1) \times (0, \infty).$$

Following the ideal in [18], we set

$$\eta^t(x, s) = u(x, t) - u(x, t - s), \quad (x, t, s) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+. \tag{4.9}$$

Then

$$\eta^t_t(x, s) + \eta^t_s(x, s) = u_t(x, t), \quad (x, t, s) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+.$$

Thus, system (4.1) becomes

$$\begin{cases} u_{tt}(x,t) - l u_{xx}(x,t) - \displaystyle\int_0^\infty g(s)\eta^t_{xx}(x,s)\mathrm{d}s & \\ \qquad\qquad + \mu z(x,1,t) = 0, & (x,t) \in \Omega \times (0,+\infty), \\ v_{tt}(x,t) - b v_{xx}(x,t) = 0, & (x,t) \in (L_1, L_2) \times (0,+\infty), \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & (x,\rho,t) \in \Omega \times (0,1) \times (0,+\infty), \\ \eta^t_t(x,s) + \eta^t_s(x,s) = u_t(x,t), & (x,s,t) \in \Omega \times (0,+\infty) \times (0,+\infty), \end{cases} \tag{4.10}$$

the boundary and transmission conditions (4.2) become

$$\begin{array}{l} u(0,t) = u(L_3,t) = 0, \\ u(L_i,t) = v(L_i,t), \quad i = 1,2,\ t \in (0,+\infty), \\ l u_x(L_i,t) + \displaystyle\int_0^\infty g(s)\eta^t_x(L_i,s)\mathrm{d}s = b v_x(L_i,t), \quad i = 1,2,\ t \in (0,+\infty), \end{array} \tag{4.11}$$

and the initial conditions (4.3) become

$$\begin{array}{ll} u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & x \in (L_1, L_2), \\ z(x,0,t) = u_t(x,t), & (x,t) \in \Omega \times (0,+\infty), \\ \quad z(x,\rho,0) = f_0(x,-\rho\tau), & (x,t) \in \Omega \times (0,1). \end{array} \tag{4.12}$$

It is clear that

$$\begin{array}{ll} \eta^t(x,0) = 0, & \textit{for all } x > 0, \\ \eta^t(0,s) = \eta^t(L_3,s) = 0, & \textit{for all } s > 0, \\ \eta^0(x,s) = \eta_0(s), & \textit{for all } s > 0. \end{array} \tag{4.13}$$

3 Well-posedness of the problem

Let $V := (u, v, \varphi, \psi, z, \eta^t)^T$, then V satisfies the problem

$$\begin{cases} V_t = (\mathscr{A} + \mathscr{B})V(t), & t > 0, \\ V(0) = V_0, \end{cases} \tag{4.14}$$

where $V_0 := \Big(u_0(\cdot, 0), v_0, u_1, v_1, f_0(\cdot, -\tau), \eta_0\Big)^T$. The operator $\mathscr{A}$ and $\mathscr{B}$ are linear and defined by

$$\mathscr{A}\begin{pmatrix} u \\ v \\ \varphi \\ \psi \\ z \end{pmatrix} = \begin{pmatrix} \varphi \\ \psi \\ lu_{xx} + \int_0^{+\infty} g(s)w_{xx}(s)\mathrm{d}s - \mu\varphi - \mu z(.,1) \\ bv_{xx} \\ -\frac{1}{\tau}z_\rho \\ -w_s + \varphi \end{pmatrix} \tag{4.15}$$

and $\mathscr{B}(u, v, \varphi, \psi, z, \eta^t)^T = \mu(0, 0, \varphi, 0, 0, 0)^T$ where

$$X_* = \Big\{(u, v) \in H^1(\Omega) \times H^1(L_1, L_2) : u(0,t) = u(L_3,t) = 0, u(L_i,t) = v(L_i,t), \\ lu_x(L_i,t) + \int_0^\infty g(s)\eta_x^t(L_i,s)\mathrm{d}s = bv_x(L_i,t), i = 1,2\Big\}$$

and $L_g^2(\mathbb{R}_+, H^1(\Omega))$ denotes the Hilbert space of H^1-valued functions on $\mathbb{R}_+$, endowed with the inner product

$$(\phi, \vartheta)_{L_g^2(\mathbb{R}_+, H^1(\Omega))} = \int_\Omega \int_0^{+\infty} g(s)\phi_x(s)\vartheta_x(s)ds\,dx.$$

Let

$$V = (u, v, \varphi, \psi, z, w)^T, \quad \bar{V} = (\bar{u}, \bar{v}, \bar{\varphi}, \bar{\psi}, \bar{z}, \bar{w})^T.$$

We define the inner product in the energy space $\mathscr{H}$,

$$\begin{aligned} \langle V, \bar{V}\rangle_{\mathscr{H}} &= \int_\Omega \varphi\bar{\varphi}\mathrm{d}x + \int_{L_1}^{L_2} \psi\bar{\psi}\,dx + \int_\Omega lu_x\bar{u}_x\mathrm{d}x + \int_{L_1}^{L_2} bv_x\bar{v}_x\,dx \\ &+ \int_\Omega \int_0^{+\infty} g(s)w_x(s)\bar{w}_x(s)\,ds\,dx + \tau\mu \int_\Omega \int_0^1 z\bar{z}\,d\rho\,dx. \end{aligned}$$

The domain of $\mathscr{A}$ is

$$D(\mathscr{A}) = \Big\{(u, v, \varphi, \psi, z, w)^T \in \mathscr{H} : (u, v) \in \{(H^2(\Omega) \times H^2(L_1, L_2)) \cap X_*\}, \varphi \in H^1(\Omega), \\ \psi \in H^1(L_1, L_2), w \in L_g^2\left(\mathbb{R}_+, H^2(\Omega) \cap H^1(\Omega)\right), w_s \in \left(\mathbb{R}_+, H^1(\Omega)\right), z_\rho \in L^2((0,1), L^2(\Omega)), \\ w(x,0) = 0, z(x,0) = \varphi(x)\Big\}$$

and $D(\mathscr{B}) = \mathscr{H}$ The well-posedness of problem (4.10)-(4.11) is ensured by the following theorem.

Theorem 3.1 *Assume that (4.7 and (4.8) hold. Let $V_0 \in \mathscr{H}$, then there exists a unique weak solution $V \in C(\mathbb{R}_+, \mathscr{H})$ of problem (4.14). Moreover, if $V_0 \in D(\mathscr{A})$, then*

$$V \in C(\mathbb{R}_+, D(\mathscr{A})) \cap C^1(\mathbb{R}_+, \mathscr{H}).$$

Proof : We use the semi group approach. So, first, we prove that the operator $\mathscr{A}$ is dissipative. In fact, for $(u, v, \varphi, \psi, z, w)^T \in D(\mathscr{A})$, where $\varphi(L_i) = \psi(L_i), i = 1, 2$, we have

$$\begin{aligned}\langle \mathscr{A}V, V\rangle_{\mathscr{H}} &= \int_\Omega lu_{xx}\varphi\, dx + \int_\Omega \Big(\int_0^{+\infty} g(s)w_{xx}(s)\, ds - \mu\varphi - \mu z(.,1)\Big)\varphi\, dx \\ &+ \int_\Omega lu_x\varphi_x \mathrm{d}x + \int_{L_1}^{L_2} bv_x\psi_x\, dx + \int_{L_1}^{L_2} bv_{xx}\psi\, dx \\ &+ \int_\Omega\int_0^{+\infty} g(s)w_x(s)(-w_s+\varphi)_x\, ds\, dx \\ &- \mu\int_\Omega\int_0^1 zz_\rho(x,\rho)\, d\rho\, dx.\end{aligned} \tag{4.16}$$

For the last term of the right side of (4.16), we obtain

$$\begin{aligned}\mu\int_\Omega\int_0^1 zz_\rho(x,\rho)\, d\rho\, dx &= \mu\int_\Omega\int_0^1 \frac{1}{2}\frac{\partial}{\partial\rho}z^2(x,\rho)\, d\rho\, dx \\ &= \frac{\mu}{2}\int_\Omega (z^2(x,1) - z^2(x,0))\, dx.\end{aligned}$$

Noticing that $z(x,0,t) = \varphi(x,t)$, $w(x,0) = 0$ and $\varphi(L_i) = \psi(L_i), i = 1, 2$, we obtain

$$\begin{aligned}\langle \mathscr{A}V, V\rangle_{\mathscr{H}} &= \Big[lu_x\varphi + \int_0^{+\infty} g(s)w_x(s)\, ds\varphi\Big]_{\partial\Omega} + [bv_x\psi]_{L1}^{L2} \\ &+ \int_\Omega (-\mu\varphi - \mu z(.,1))\varphi \mathrm{d}x - \frac{1}{2}\int_\Omega \Big[g(s)|w_x(x,s)|^2\Big]_0^{+\infty} dx \\ &+ \frac{1}{2}\int_\Omega\int_0^{+\infty} g'(s)|w_x(x,s)|^2\, ds\, dx - \frac{\mu}{2}\int_\Omega (z^2(x,1) - \varphi^2(x)\, dx,\end{aligned}$$

where we have used that

$$\begin{aligned}&[lu_x\varphi + \int_0^{+\infty} g(s)w_x(s)\, ds\varphi]_{\partial\Omega} \\ &= \Big(lu_x(L_1,t) + \int_0^{+\infty} g(s)w_x(L_1,s)\, ds\Big)\varphi(L_1,t) \\ &\quad - \Big(lu_x(L_2,t) + \int_0^{+\infty} g(s)w_x(L_2,s)\, ds\Big)\varphi(L_2,t) \\ &= -[bv_x\psi]_{L1}^{L2}.\end{aligned}$$

Using Young's inequality, we have

$$\langle \mathscr{A}V, V\rangle_{\mathscr{H}} \le \frac{1}{2}\int_\Omega\int_0^{+\infty} g'(s)|w_x(x,s)|^2\, ds\, dx.$$

Consequently, taking (4.8) into account, we conclude that

$$\langle \mathscr{A}V, V\rangle_{\mathscr{H}} \le 0,$$

that is, $\mathscr{A}$ is dissipative. Next, we prove that $-\mathscr{A}$ is maximal. Actually, let $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T \in \mathscr{H}$, we prove that there exists $V = (u, v, \varphi, \psi, z, w)^T \in D(\mathscr{A})$ satisfying

$$(\lambda I - \mathscr{A})V = F, \tag{4.17}$$

which is equivalent to

$$\begin{array}{l}\lambda u-\varphi=f_1,\\ \lambda v-\psi=f_2,\\ \lambda\varphi-lu_{xx}-\displaystyle\int_0^\infty g(s)w_{xx}(s)\mathrm{d}s+\mu\varphi+\mu z(.,t)=f_3,\\ \lambda\psi-bv_{xx}=f_4,\\ \lambda z+\frac{1}{\tau}z_\rho=f_5,\\ \lambda w+w_s-\varphi=f_6.\end{array}\tag{4.18}$$

Assume that with the suitable regularity we have found u and v, then

$$\begin{array}{l}\varphi=\lambda u-f_1,\\ \psi=\lambda v-f_2.\end{array}\tag{4.19}$$

So we have $\varphi\in H^1(\Omega)$ and $\psi\in H^1(L_1,L_2)$. Moreover, we can find z with

$$z(x,0)=\varphi(x),\qquad \textit{for } x\in\Omega.$$

Using the equation in (4.18), we obtain

$$z(x,\rho)=\varphi(x)e^{-\lambda\rho\tau}+\tau e^{-\lambda\rho\tau}\int_0^\rho f_5(x,\sigma)e^{\lambda\sigma\tau}\mathrm{d}\sigma.$$

From (4.19), we obtain

$$z(x,\rho)=\lambda ue^{-\lambda\rho\tau}-f_1e^{-\lambda\rho\tau}+\tau e^{-\lambda\rho\tau}\int_0^\rho f_5(x,\sigma)e^{\lambda\sigma\tau}\mathrm{d}\sigma.\tag{4.20}$$

It is easy to see that the last equation in (4.18) with $w(x,0)=0$ has a unique solution

$$\begin{aligned}w(x,s)&=\Big(\int_0^s e^{\lambda y}(f_6(x,y)+\varphi(x))\,dy\Big)e^{-\lambda s}\\ &=\Big(\int_0^s e^{\lambda y}(f_6(x,y)+\lambda u(x)-f_1(x))\,dy\Big)e^{-\lambda s}.\end{aligned}\tag{4.21}$$

By using (4.18)-(4.21), the functions u and v satisfy

$$\begin{array}{l}\left(\lambda^2+\mu\lambda+\mu\lambda e^{-\lambda\tau}\right)u-\tilde{l}u_{xx}=\tilde{f},\\ \lambda^2v-bv_{xx}=f_4+\lambda f_2,\end{array}\tag{4.22}$$

where

$$\tilde{l}=l+\lambda\int_0^\infty g(s)e^{-\lambda s}\Big(\int_0^s e^{\lambda y}\mathrm{d}y\Big)\,ds$$

and

$$\begin{aligned}\tilde{f}\ &=\int_0^\infty g(s)e^{-\lambda s}\Big(\int_0^s e^{\lambda y}(f_6(x,y)-f_1(x,y))_{xx}\mathrm{d}y\Big)\,ds\\ &\quad-\mu\tau e^{-\lambda\tau}\int_0^1 f_5(x,\sigma)e^{\lambda\sigma\tau}\,d\sigma+\left(\lambda+\mu+\mu e^{-\lambda\tau}\right)f_1+f_3.\end{aligned}$$

We just need to prove that (4.22) has a solution $(u, v) \in X_*$ and replace in (4.19), (4.20) and (4.21) to get $V = (u, v, \varphi, \psi, z, w)^T \in D(\mathscr{A})$ satisfying (4.17).
Consequently, problem (4.22) is equivalent to the problem

$$\Phi((u, v), (\omega_1, \omega_2)) = l(\omega_1, \omega_2), \tag{4.23}$$

where the bilinear form $\Phi : (X_*, X_*) \to \mathbb{R}$ and the linear form $l : X_* \to \mathbb{R}$ are defined by

$$\begin{aligned}\Phi((u, v), (\omega_1, \omega_2)) &= \int_\Omega [\left(\lambda^2 + \mu\lambda + \mu\lambda e^{-\lambda\tau}\right) u\omega_1 + \tilde{l}u_x(\omega)_x]\mathrm{d}x - [\tilde{l}u_x\omega_1]_{\partial\Omega} \\ &+ \int_{L_1}^{L_2} \left(\lambda^2 v\omega_2 + bv_x(\omega_2)_x\right) \mathrm{d}x - [bv_x\omega_2]_{L_1}^{L_2}\end{aligned}$$

and

$$l(\omega_1, \omega_2) = \int_\Omega \tilde{f}\omega_1 \mathrm{d}x + \int_{L_1}^{L_2} (f_4 + \lambda f_2)\omega_2 \, dx.$$

Using the properties of the space X_*, it is easy to see that Φ is continuous and coercive, and l is continuous. Applying the Lax[1]- Milligram [2] theorem, we infer that for all $(\omega_1, \omega_2) \in X_*$, problem (4.23) has a unique solution $(u, v) \in X_*$. It follows from (4.22) that $(u, v) \in \{(H^2(\Omega) \times H^2(L_1, L_2)) \cap X_*\}$. Thence, the operator $\lambda I - \mathscr{A}$ is surjective for any $\lambda > 0$. That mean $\mathscr{A}$ is maximal monotone operator. Then, using Lummer-Phillips theorem [65], we deduce that $\mathscr{A}$ is an infinitesimal generator of a linear C_0-semi group on $\mathscr{H}$.
On the other hand, it is clear that the linear operator $\mathscr{B}$ is Lipschitz continuous. Finally, also $\mathscr{A} + \mathscr{B}$ is an infinitesimal generator of a linear C_0-semi group on $\mathscr{H}$. Consequently (4.14) is well-posed in the sense of Theorem 3.1(see [65]). ■

4 Exponential stability

In this section, we consider a decay result of problem (4.1)-(4.3). In fact using the energy method to produce a suitable Lyapunov functional

Theorem 4.1 *Let (u, v) be the solution of (4.1)-(4.3). Assume that (4.7)-(4.8 hold, and that*

$$a > \frac{8(L_2 - L_1)}{L_1 + L_3 - L_2} l, \quad b > \frac{8(L_2 - L_1)}{L_1 + L_3 - L_2} l, \tag{4.24}$$

then there exist two constants $\gamma_1, \gamma_2 > 0$ such that,

$$E(t) \le \gamma_2 e^{-\gamma_1 t}, \forall t \in \mathbb{R}_+. \tag{4.25}$$

For the proof of Theorem 4.1, we need some lemmas.

For a solution of (4.1)-(4.3), we define the energy

$$\begin{aligned}E(t) &= \frac{1}{2}\int_\Omega [u_t^2(x,t) + l u_x^2(x,t)]\, dx + \frac{1}{2}\int_{L_1}^{L_2} [v_t^2(x,t) + b v_x^2(x,t)]\, dx \\ &+ \frac{1}{2}\int_\Omega \int_0^\infty g(s)|\eta_x^t(x,s)|^2 \, ds\, dx + \frac{\tau\mu}{2}\int_\Omega \int_0^1 z^2(x,\rho,t)\, d\rho\, dx.\end{aligned} \tag{4.26}$$

[1]Peter Lax is a mathematician Hungarian born on 1926 in Budapest. the 2005 Abel prize was awarded to him.

[2]R. James Milgram is a mathematician American born on 1939. He is currently a professor in Stanford University

Lemma 4.2 *Let (u, v, η, z) be the solution of (4.10) and (4.11). Then we have the inequality*

$$\frac{d}{dt}E(t) \leq \mu \int_\Omega u_t^2(x,t)\,dx + \frac{1}{2}\int_\Omega \int_0^\infty g'(s)|\eta_x^t(x,s)|^2\,ds\,dx. \tag{4.27}$$

Proof : We have

$$\begin{aligned}
\tfrac{d}{dt}E(t) &= \int_\Omega \Big(u_t u_{tt} + l u_x u_{xt} + \int_0^\infty g(s)\eta_x^t(x,s)\eta_{xt}^t\,ds\Big)\,dx \\
&\quad + \int_{L_1}^{L_2}(v_t v_{tt} + b v_x v_{xt})\,dx + \tau|\mu| \int_\Omega \int_0^1 z_t(x,\rho,t) z(x,\rho,t)\,d\rho\,dx \\
&= \Big[\Big(l u_x + \int_0^\infty g(s)\eta_x^t(x,s)\,ds\Big)u_t\Big]_{\partial\Omega} - [b v_x v_t]_{L_1}^{L_2} \\
&\quad - \int_\Omega \int_0^\infty g(s)\eta_x^t(x,s)\eta_{xs}^t(x,s)\mathrm{d}s\mathrm{d}x \\
&\quad -\mu \int_\Omega u_t z(x,1,t)dx + \frac{\mu}{2}\int_\Omega u_t^2(x,t)\mathrm{d}x - \frac{\mu}{2}\int_\Omega z^2(x,1,t)\,dx \\
&= \frac{1}{2}\int_\Omega \int_0^\infty g'(s)|\eta_x^t(x,s)|^2\,ds\,dx - \mu\int_\Omega u_t z(x,1,t)\,dx + \frac{\mu}{2}\int_\Omega u_t^2(x,t)\,dx \\
&\quad -\frac{\mu}{2}\int_\Omega z^2(x,1,t)\,dx,
\end{aligned} \tag{4.28}$$

where we have used that

$$\begin{aligned}
\Big[\Big(l u_x + \int_0^\infty g(s)\eta_x^t(x,s)\,ds\Big)u_t\Big]_{\partial\Omega} &= \Big(l u_x(L_1,t) + \int_0^\infty g(s)\eta_x^t(L_1,s)\,ds\Big)u_t(L_1,t) \\
&\quad -\Big(l u_x(L_2,t) + \int_0^\infty g(s)\eta_x^t(L_2,s)\,ds\Big)u_t(L_2,t) \\
&= -[b v_x v_t]_{L_1}^{L_2},
\end{aligned}$$

$$\Big[\frac{1}{2}\int_\Omega g(s)|\eta_x^t(x,s)|^2\mathrm{d}x\Big]_0^\infty = 0,$$

and

$$\frac{\tau\mu}{2}\frac{d}{dt}\int_\Omega \int_0^1 z^2(x,\rho,t)\,d\rho\,dx = -\frac{\mu}{2\tau}\int_\Omega (z^2(x,1) - z^2(x,0))\,dx. \tag{4.29}$$

Young's inequality gives us

$$\frac{d}{dt}E(t) \leq \mu \int_\Omega u_t^2(x,t)\,dx + \frac{1}{2}\int_\Omega \int_0^\infty g'(s)|\eta_x^t(x,s)|^2\,ds\,dx.$$

■

Lemma 4.3 *Let (u, u_t, v, v_t) be the solution of (4.1)-(4.3). Then the functional*

$$\mathscr{D}(t) = \int_\Omega u u_t\,dx + \int_{L_1}^{L_2} v v_t\,dx,$$

satisfies, for any $\varepsilon > 0$, the estimate

$$\begin{aligned}
\frac{d}{dt}\mathscr{D}(t) &\leq \int_\Omega u_t^2\,dx + \int_{L_1}^{L_2} v_t^2\,dx + (L\varepsilon + \varepsilon - l)\int_\Omega u_x^2\,dx - \int_{L_1}^{L_2} b v_x^2\,dx \\
&\quad + \frac{g_0}{4\varepsilon}\int_\Omega \int_0^\infty g(s)|\eta_x^t(x,s)|^2\,ds\,dx + \frac{\mu^2}{4\varepsilon}\int_\Omega z^2(x,1,t)\,dx.
\end{aligned} \tag{4.30}$$

Proof : Taking the derivative of $\mathscr{D}(t)$ with respect to t and using (4.10), we have

$$\begin{aligned}\frac{d}{dt}\mathscr{D}(t) &= \int_\Omega u_t^2 \mathrm{d}x - l\int_\Omega u_x^2\,dx - \mu\int_\Omega z(x,1,t)u\,dx + [bv_x v]_{L_1}^{L_2} + \int_{L_1}^{L_2} v_t^2\,dx \\ &+\Big[\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\,ds\Big)u\Big]_{\partial\Omega} \\ &-\int_\Omega u_x(x,t)\int_0^\infty g(s)\eta_x^t(x,s)\,ds\,dx - \int_{L_1}^{L_2} bv_x^2\,dx \\ &= \int_\Omega u_t^2 \mathrm{d}x - l\int_\Omega u_x^2\,dx - \mu\int_\Omega z(x,1,t)u\mathrm{d}x + \int_{L_1}^{L_2} v_t^2\,dx \\ &-\int_{L_1}^{L_2} bv_x^2\,dx - \int_\Omega u_x(x,t)\int_0^\infty g(s)\eta_x^t(x,s)\,ds\,dx,\end{aligned} \tag{4.31}$$

where we used that

$$\begin{aligned}\Big[\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\,ds\Big)u\Big]_{\partial\Omega} &= \Big(lu_x(L_1,t) + \int_0^\infty g(s)\eta_x^t(L_1,s)\,ds\Big)u(L_1,t) \\ &-\Big(lu_x(L_2,t) + \int_0^\infty g(s)\eta_x^t(L_2,s)\,ds\Big)u(L_2,t) \\ &= -[bv_x v_t]_{L_1}^{L_2}.\end{aligned}$$

By the boundary conditions (4.2), we have

$$\begin{aligned}u^2(x,t) &= \Big(\int_0^x u_x(x,t)\mathrm{d}x\Big)^2 \\ &\le L_1\int_0^{L_1} u_x^2(x,t)\,dx, \quad x\in[0,L_1], \\ &u^2(x,t) \le (L_3-L_2)\int_{L_2}^{L_3} u_x^2(x,t)\,dx, \quad x\in[L_2,L_3],\end{aligned}$$

which implies

$$\int_\Omega u^2(x,t)\,dx \le L\int_\Omega u_x^2\,dx, \quad x\in\Omega, \tag{4.32}$$

where $L=\max\{L_1, L_3-L_2\}$. By making use of Young's inequality and (4.32), for any $\varepsilon>0$, we obtain

$$\mu\int_\Omega z(x,1,t)u\,dx \le \frac{\mu^2}{4\varepsilon}\int_\Omega z^2(x,1,t)\,dx + L\varepsilon\int_\Omega u_x^2\,dx. \tag{4.33}$$

Young's and Hölder's inequalities and (A2) imply that

$$\int_\Omega u_x(x,t)\int_0^\infty g(s)\eta_x^t(x,s)\,ds\,dx \le \varepsilon\int_\Omega u_x^2(x,t)\mathrm{d}x + \frac{g_0}{4\varepsilon}\int_\Omega\int_0^\infty g(s)|\eta_x^t(x,s)|^2\,ds\,dx. \tag{4.34}$$

Inserting the estimates (4.33) and (4.34) into (4.31), then (4.30) is fulfilled. ∎

Next, enlightened by [52], we introduce the functional

$$q(x) = \begin{cases} x-\frac{L_1}{2}, & x\in[0,L_1], \\ \frac{L_1}{2} - \frac{L_1+L_3-L_2}{2(L_2-L_1)}(x-L_1), & x\in(L_1,L_2), \\ x-\frac{L_2+L_3}{2}, & x\in[L_2,L_3]. \end{cases}$$

It is easy to see that $q(x)$ is bounded: $|q(x)| \le M$, where $M = \max\{\frac{L_1}{2}, \frac{L_3-L_2}{2}\}$.
We define the functionals

$$\mathscr{F}_1(t) = -\int_\Omega q(x)u_t\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\,ds\Big)\,dx, \quad \mathscr{F}_2(t) = -\int_{L_1}^{L_2} q(x)v_x v_t\,dx,$$

then we have the following results.

Lemma 4.4 *The functional $\mathscr{F}_1(t)$ and $\mathscr{F}_2(t)$ satisfy*

$$\begin{aligned}\frac{d}{dt}\mathscr{F}_1(t) \;&\le \Big(\frac{l+g_0}{2} + \varepsilon_1 M^2\Big)\int_\Omega u_t^2\,dx + \Big(l^2 + l^2\varepsilon_1\Big)\int_\Omega u_x^2\,dx \\ &+\frac{M^2\mu^2}{4\varepsilon_1}\int_\Omega z^2(x,1,t)\,dx + (g_0 + g_0\varepsilon_1)\int_\Omega\int_0^\infty g(s)|\eta_x^t(x,s)|^2\,ds\,dx \\ &-\frac{g(0)}{4\varepsilon_1}\int_\Omega\int_0^\infty g'(s)|\eta_x^t(x,s)|^2\,ds\,dx - \Big[\frac{l+g_0}{2}q(x)u_t^2\Big]_{\partial\Omega} \\ &-\Big[\frac{q(x)}{2}\Big(lu_x(x,t) + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)^2\Big]_{\partial\Omega}\end{aligned} \tag{4.35}$$

and

$$\begin{aligned}\frac{d}{dt}\mathscr{F}_2(t) \;&\le -\frac{L_1+L_3-L_2}{4(L_2-L_1)}\Big(\int_{L_1}^{L_2} v_t^2\mathrm{d}x + \int_{L_1}^{L_2} bv_x^2\mathrm{d}x\Big) + \frac{L_1}{4}v_t^2(L_1) \\ &+\frac{L_3-L_2}{4}v_t^2(L_2) + \frac{b}{4}\Big((L_3-L_2)v_x^2(L_2,t) + L_1 v_x^2(L_1,t)\Big).\end{aligned} \tag{4.36}$$

Proof : Taking the derivative of $\mathscr{F}_1(t)$ with respect to t and using (4.10), we obtain

$$\begin{aligned}\frac{d}{dt}\mathscr{F}_1(t) \;&= -\int_\Omega q(x)u_{tt}\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)\,dx \\ &\quad -\int_\Omega q(x)u_t\Big(lu_{xt} + \int_0^\infty g(s)\eta_{xt}^t(x,s)\mathrm{d}s\Big)\,dx \\ &= -\int_\Omega q(x)\Big(lu_{xx} + \int_0^\infty g(s)\eta_{xx}^t(x,s)\,ds\Big)\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)\,dx \\ &\quad +\mu\int_\Omega q(x)z(x,1,t)\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)\,dx \\ &\quad -\int_\Omega q(x)u_t\Big(lu_{xt} + \int_0^\infty g(s)\eta_{xt}^t(x,s)\,ds\Big)\,dx.\end{aligned} \tag{4.37}$$

We pay attention to

$$\begin{aligned}-\int_\Omega q(x)\Big(lu_{xx} + \int_0^\infty g(s)\eta_{xx}^t(x,s)\,ds\Big)\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\,ds\Big)\,dx \\ = \frac{1}{2}\int_\Omega q'(x)\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\,ds\Big)^2\,dx \\ -\Big[\frac{q(x)}{2}\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\,ds\Big)^2\Big]_{\partial\Omega}.\end{aligned} \tag{4.38}$$

The last term in (4.37) can be treated as follows

$$
\begin{aligned}
&-\int_\Omega q(x)u_t\Big(lu_{xt}+\int_0^\infty g(s)\eta^t_{xt}(x,s)\,ds\Big)\,dx\\
&\qquad= -l\int_\Omega q(x)u_tu_{xt}\,dx-\int_\Omega q(x)u_t\int_0^\infty g(s)\eta^t_{xt}(x,s)\,ds\,dx\\
&\qquad= \Big[-\frac{l}{2}q(x)u_t^2\Big]_{\partial\Omega}+\frac{l}{2}\int_\Omega q'(x)u_t^2\,dx-\int_\Omega q(x)u_t\int_0^\infty g(s)\,\Big(u_t-\eta^t_s\Big)_x\,ds\,dx\\
&\qquad= \Big[-\tfrac{l}{2}q(x)u_t^2\Big]_{\partial\Omega}+\frac{l}{2}\int_\Omega q'(x)u_t^2\,dx-g_0\int_\Omega q(x)u_tu_{tx}\,dx\\
&\qquad+\int_\Omega q(x)u_t\int_0^\infty g(s)\eta^t_{sx}(x,s)\,ds\,dx\\
&\qquad= \Big[-\frac{l+g_0}{2}q(x)u_t^2\Big]_{\partial\Omega}+\frac{l+g_0}{2}\int_\Omega q'(x)u_t^2\,dx-\int_\Omega q(x)u_t\int_0^\infty g'(s)\eta^t_x\,ds\,dx,
\end{aligned}
\tag{4.39}
$$

where we used that

$$
-\Big[\int_\Omega q(x)u_tg(s)\eta^t_x(x,s)\,dx\Big]_0^\infty=0.
$$

Inserting (4.38) and (4.39) in (4.37), we arrive at

$$
\begin{aligned}
\tfrac{d}{dt}\mathscr{F}_1(t)\quad &= -\Big[\tfrac{q(x)}{2}\Big(lu_x+\int_0^\infty g(s)\eta^t_x(x,s)\,ds\Big)^2\Big]_{\partial\Omega}-\Big[\frac{l+g_0}{2}q(x)u_t^2\Big]_{\partial\Omega}\\
&+\tfrac{1}{2}\int_\Omega q'(x)\Big(lu_x+\int_0^\infty g(s)\eta^t_x(x,s)\,ds\Big)^2\,dx\\
&+\mu\int_\Omega q(x)z(x,1,t)\Big(lu_x+\int_0^\infty g(s)\eta^t_x(x,s)\,ds\Big)\,dx\\
&+\frac{l+g_0}{2}\int_\Omega q'(x)u_t^2\,dx-\int_\Omega q(x)u_t\int_0^\infty g'(s)\eta^t_x\,ds\,dx.
\end{aligned}
\tag{4.40}
$$

Using Malinowski and Cauchy-Schwarz inequalities, we have

$$
\frac{1}{2}\int_\Omega\Big(lu_x+\int_0^\infty g(s)\eta^t_x(x,s)\mathrm{d}s\Big)^2\,dx\quad\le l^2\int_\Omega u_x^2\,dx+g_0\int_\Omega\int_0^\infty g(s)|\eta^t_x(x,s)|^2\,ds\,dx.
\tag{4.41}
$$

Young's inequality gives us that for any $\varepsilon_1>0$,

$$
\begin{aligned}
\Big|\mu\int_\Omega q(x)z(x,1,t)\Big(lu_x+\int_0^\infty g(s)\eta^t_x(x,s)\mathrm{d}s\Big)\mathrm{d}x\Big|\quad &\le\frac{M^2\mu^2}{4\varepsilon_1}\int_\Omega z^2(x,1,t)\,dx+l^2\varepsilon_1\int_\Omega u_x^2(x,t)\,dx\\
&+g_0\varepsilon_1\int_\Omega\int_0^\infty g(s)|\eta^t_x(x,s)|^2\,ds\,dx.
\end{aligned}
\tag{4.42}
$$

It is clear that

$$
\Big|\int_\Omega q(x)u_t\int_0^\infty g'(s)\eta^t_x\,ds\mathrm{d}x\Big|\le\varepsilon_1M^2\int_\Omega u_t^2\,dx-\frac{g(0)}{4\varepsilon_1}\int_\Omega\int_0^\infty g'(s)|\eta^t_x(x,s)|^2\,ds\,dx.
\tag{4.43}
$$

Inserting (4.41)-(4.43) into (4.40), we obtain (4.35).

By the same method, taking the derivative of $\mathscr{F}_1(t)$ with respect to t, we obtain

$$\begin{aligned}\frac{d}{dt}\mathscr{F}_2(t) &= -\int_{L_1}^{L_2} q(x)v_{xt}v_t\,dx - \int_{L_1}^{L_2} q(x)v_x v_{tt}\,dx \\ &= \Big[-\frac{1}{2}q(x)v_t^2\Big]_{L_1}^{L_2} + \frac{1}{2}\int_{L_1}^{L_2} q'(x)v_t^2\,dx + \frac{1}{2}\int_{L_1}^{L_2} bq'(x)v_x^2\,dx \\ &\quad + \Big[-\frac{b}{2}q(x)v_x^2\Big]_{L_1}^{L_2} \\ &\le -\frac{L_1+L_3-L_2}{4(L_2-L_1)}\Big(\int_{L_1}^{L_2} v_t^2\,dx + \int_{L_1}^{L_2} bv_x^2\,dx\Big) + \frac{L_1}{4}v_t^2(L_1) \\ &\quad + \frac{L_3-L_2}{4}v_t^2(L_2) + \frac{b}{4}\Big((L_3-L_2)v_x^2(L_2,t) + L_1 v_x^2(L_1,t)\Big).\end{aligned}$$

Thus, the proof of Lemma 4.4 is complete. ∎

We define the functional

$$\mathscr{F}_3(t) = \tau\int_\Omega\int_0^1 e^{-\tau\rho}z^2(x,\rho,t)\,d\rho\,dx,$$

then we have the following estimate.

Lemma 4.5 *The functional $\mathscr{F}_3(t)$ satisfies*

$$\frac{d}{dt}\mathscr{F}_3(t) \le -c_2\Big(\int_\Omega z^2(x,1,t)\,dx + \tau\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx\Big) + \int_\Omega u_t^2(x,t)\,dx.$$

Proof :

$$\begin{aligned}\frac{d}{dt}\mathscr{F}_3(t) &= 2\tau\int_0^1\int_\Omega e^{-\tau\rho}z_t(x,\rho,t)z(x,\rho,t)\,d\rho\,dx \\ &= -2\int_0^1\int_\Omega e^{-\tau\rho}z_\rho(x,\rho,t)z(x,\rho,t)\,d\rho\,dx \\ &= -\int_0^1\int_\Omega e^{-\tau\rho}\frac{\partial}{\partial\rho}\Big(z^2(x,\rho,t)\Big)\,d\rho\,dx \\ &= -\tau\int_0^1\int_\Omega e^{-\tau\rho}z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x + \int_\Omega u_t^2(x,t)\mathrm{d}x - e^{-\tau}\int_\Omega z^2(x,1,t)\,dx \\ &\le -e^{-\tau}\Big(\tau\int_0^1\int_\Omega z^2(x,\rho,t)\,d\rho\,dx + \int_\Omega z^2(x,1,t)\mathrm{d}x\Big) + \int_\Omega u_t^2(x,t)\,dx.\end{aligned}$$

∎

We define the functional

$$\mathscr{F}_4(t) = -\int_\Omega u_t\int_0^\infty g(s)(u(t)-u(t-s))\,ds\,dx,$$

then we have the following estimate.

Lemma 4.6 *The functional $\mathscr{F}_4(t)$ satisfies*

$$\begin{aligned}\frac{d}{dt}\mathscr{F}_4(t) &\le -(g_0-\delta_2)\int_\Omega u_t^2\,dx + \delta_2 l^2\int_\Omega u_x^2\,dx + \delta_2\mu\int_\Omega z^2(x,1,t)\,dx \\ &\quad + \Big(g_0 + \frac{g_0}{4\delta_2} + \frac{\mu g_0 L^2}{2\delta_2}\Big)\int_\Omega\int_0^\infty g(s)|\eta_x^t(x,s)|^2\,ds\,dx \\ &\quad - \frac{g(0)L^2}{\delta_2}\int_\Omega\int_0^\infty g'(s)|\eta_x^t(x,s)|^2\,ds\,dx.\end{aligned} \tag{4.44}$$

Proof : Taking the derivative of $\mathscr{F}_4(t)$ with respect to t and using (4.10), we have

$$\begin{aligned}\frac{d}{dt}\mathscr{F}_4(t) &= -\int_\Omega \Big(lu_{xx} + \int_0^\infty g(s)\eta^t_{xx}(x,s)\,ds - \mu z(x,1,t)\Big)\\ &\quad\times \int_0^\infty g(s)(u(t)-u(t-s))\,ds\,dx - \int_\Omega u_t\int_0^\infty g(s)(u_t(t)-u_t(t-s))\,ds\,dx\\ &= \int_\Omega lu_x\int_0^\infty g(s)(u_x(t)-u_x(t-s))\,ds\,dx - g_0\int_\Omega u_t^2\mathrm{d}x \\ &\quad+\int_\Omega u_t\int_0^\infty g(s)\eta^t_s(s)\,ds\,dx + \int_\Omega\Big(\int_0^\infty g(s)(u_x(t)-u_x(t-s))\,dxs\Big)^2\mathrm{d}x\\ &\quad+\int_\Omega \mu z(x,1,t)\int_0^\infty g(s)(u(t)-u(t-s))\,ds\,dx.\end{aligned} \tag{4.45}$$

Using Young's inequality and (4.32), we obtain for any $\delta_2>0$,

$$\int_\Omega lu_x\int_0^\infty g(s)(u_x(t)-u_x(t-s))\,ds\,dx \le \delta_2 l^2\int_\Omega u_x^2\,dx + \frac{g_0}{4\delta_2}\int_\Omega\int_0^\infty g(s)|\eta^t_x(x,s)|^2ds\,dx, \tag{4.46}$$

$$\int_\Omega \mu z(x,1,t)\int_0^\infty g(s)(u(t)-u(t-s))\,ds\,dx \le \delta_2\mu\int_\Omega z^2(x,1,t)\,dx + \frac{\mu g_0 L}{4\delta_2}\int_\Omega\int_0^\infty g(s)|\eta^t_x(x,s)|^2\,ds\,dx.$$

We notice that

$$\begin{aligned}\int_\Omega\Big(\int_0^\infty g(s)(u_x(t)-u_x(t-s))\mathrm{d}s\Big)^2dx &= \int_\Omega\Big(\int_0^\infty \sqrt{g(s)}\sqrt{g(s)}(u_x(t)-u_x(t-s))\,ds\Big)^2dx\\ &\le \int_\Omega\int_0^\infty g(s)\,ds\Big(\int_0^\infty g(s)|\eta^t_x(x,s)|^2\,ds\Big)\,dx\\ &\le g_0\int_\Omega\int_0^\infty g(s)|\eta^t_x(x,s)|^2\,ds\,dx\end{aligned} \tag{4.47}$$

and

$$\begin{aligned}\int_\Omega u_t\int_0^\infty g(s)\eta^t_s(s)\,ds\,dx &= -\int_\Omega u_t\int_0^\infty g'(s)\eta^t(s)\,ds\,dx\\ &\le \delta_2\int_\Omega u_t^2\,dx - \frac{g(0)L^2}{4\delta_2}\int_\Omega\int_0^\infty g'(s)|\eta^t_x(x,s)|^2\,ds\,dx.\end{aligned} \tag{4.48}$$

Inserting the estimates (4.46)-(4.48) into (4.45), we obtain (4.44). The proof is complete. ∎

Proof : (**of theorem 4.1**) We define the Lyapunov functional

$$\mathscr{L}(t) = N_1E(t) + N_2\mathscr{D}(t) + \mathscr{F}_1(t) + N_4\mathscr{F}_2(t) + N_5\mathscr{F}_3(t) + N_6\mathscr{F}_4(t), \tag{4.49}$$

where N_1, N_2, N_4, N_5 and N_6 are positive constants that will be fixed later.
Taking the derivative of (4.49) with respect to t and taking advantage of the above lemmas,

we have

$$\begin{aligned}
\frac{d}{dt}\mathscr{L}(t) \le & -\Big[N_6(g_0-\delta_2) - N_2 - \Big(\frac{l+g_0}{2} + \varepsilon_1 M^2\Big) - N_5 - N_1\mu\Big]\int_\Omega u_t^2\,dx \\
& -\Big[N_5c_2 - \frac{N_2\mu^2}{4\varepsilon} - \frac{M^2\mu^2}{4\varepsilon_1} - N_6\delta_2\mu\Big]\int_\Omega z^2(x,1,t)\,dx \\
& -\Big[N_2(l-L\varepsilon-\varepsilon) - (l^2+l^2\varepsilon_1) - N_6\delta_2 l^2\Big]\int_\Omega u_x^2\,dx \\
& -\Big[\frac{b(L_1+L_3-L_2)}{4(L_2-L_1)}N_4 + N_2 b\Big]\int_{L_1}^{L_2} v_x^2\,dx \\
& -\Big[\frac{L_1+L_3-L_2}{4(L_2-L_1)}N_4 - N_2\Big]\int_{L_1}^{L_2} v_t^2\,dx \\
& -(b-N_4)\frac{b}{4}\Big((L_3-L_2)v_x^2(L_2,t) + L_1 v_x^2(L_1,t)\Big) \\
& -(a-N_4)\Big[\frac{L_1}{4}v_t^2(L_1,t) + \frac{L_3-L_2}{4}v_t^2(L_2,t)\Big] \\
& +c(N_2,N_6)\int_\Omega\int_0^\infty g(s)|\eta_x^t(x,s)|^2\,ds\,dx \\
& +\Big(\frac{N_1}{2} - \frac{g(0)}{4\varepsilon_1} - \frac{N_6 g(0)L}{4\delta_2}\Big)\int_\Omega\int_0^\infty g'(s)|\eta_x^t(x,s)|^2\,ds\,dx.
\end{aligned} \tag{4.50}$$

At this moment, we wish all coefficients except the last two in (4.50) will be negative. We want to choose N_2 and N_4 to ensure that

$$\begin{aligned}
& a - N_4 \ge 0, \quad b - N_4 \ge 0, \\
& \frac{L_1+L_3-L_2}{4(L_2-L_1)}N_4 - N_2 > 0.
\end{aligned}$$

For this purpose, since $\frac{8l(L_2-L_1)}{L_1+L_3-L_2} < \min\{a,b\}$ we first choose N_4 satisfying

$$\frac{8l(L_2-L_1)}{L_1+L_3-L_2} < N_4 \le \min\{a,b\}.$$

Once N_4 is fixed, we pick N_2 satisfying

$$2l < N_2 < \frac{L_1+L_3-L_2}{4(L_2-L_1)}N_4.$$

Then we take ε and ε_1 small enough, and $\delta_2 < \frac{1}{2N_6}$ we have

$$N_2(l-L\varepsilon-\varepsilon) - 2l^2\varepsilon_1 > \frac{3}{2}l^2.$$

Once ε and ε_1 are fixed, we take N_5 satisfying

$$N_5 > \max\Big\{\frac{2N_2\mu^2}{\varepsilon c_2}, \frac{2M^2\mu^2}{\varepsilon_1 c_2}\Big\},$$

and $\delta_2 < \dfrac{N_5 c_2}{8N_6\mu}$ such that

$$N_5c_2 - \frac{N_2\mu^2}{4\varepsilon} - \frac{M^2\mu^2}{4\varepsilon_1} > \frac{3}{8}N_5c_2.$$

Further, we take $\delta_2 < \frac{g_0}{2}$ we choose N_6 satisfying

$$N_6 > \frac{2N_2}{g_0} + \frac{l+g_0}{g_0} + \frac{2\varepsilon_1 M^2}{g_0} + \frac{2N_5}{g_0} + \frac{2N_1\mu}{g_0}.$$

Then we have

$$N_6 > \max\Big\{\frac{2N_2}{g_0}, \frac{l+g_0}{g_0} + \frac{2\varepsilon_1 M^2}{g_0}, \frac{2N_5}{g_0}, \frac{2N_1\mu}{g_0}\Big\}.$$

Then, we pick δ_2 satisfying

$$\delta_2 < \min\Big\{\frac{g_0}{2}, \frac{N_5 c_2}{8N_6\mu}, \frac{1}{2N_6}\Big\},$$

$$N_5 c_2 - \frac{N_2\mu^2}{4\varepsilon} - \frac{M^2\mu^2}{4\varepsilon_1} - N_6\delta_2\mu \geq 0.$$

Once

$$N_2(l - L^2\varepsilon - \varepsilon) - (l^2 + l^2\varepsilon_1) - N_6\delta_2 l^2 \geq 0.$$

Finally, choosing N_1 large enough such that the first and the last coefficients in (4.50) is positive.

From the above, we deduce that there exist two positive constants α_1 and α_2 such that (4.50) becomes

$$\begin{aligned}\frac{d}{dt}\mathscr{L}(t) &\leq -\alpha_1 E(t) + \alpha_2 \int_\Omega \int_0^\infty g(s)|\eta_x^t(x,s)|^2\, ds\, dx \\ &\leq -\alpha_1 E(t) - \tfrac{\alpha_2}{\delta} \int_\Omega \int_0^\infty g'(s)|\eta_x^t(x,s)|^2\, ds\, dx \\ &\leq -\alpha_1 E(t) - \alpha_3 E'(t).\end{aligned} \tag{4.51}$$

That is

$$(\mathscr{L}(t) + \alpha_3 E(t))' \leq -\alpha_1 E(t), \tag{4.52}$$

where $\alpha_3 > 0$. Denote $\mathcal{E}(t) = \mathscr{L}(t) + \alpha_3 E(t)$, then it is easy to see that

$$\mathcal{E}(t) \sim E(t),$$

i.e., there exist two positive constants β_1, β_2 such that

$$\beta_1\ E(t) \leq \mathcal{E}(t) \leq \beta_2 E(t), \quad \forall t \geq 0. \tag{4.53}$$

Combining (4.52) and (4.53), we deduce that there exists $\gamma_1 > 0$ for which the estimate

$$\frac{d\mathcal{E}(t)}{dt} \leq -\gamma_1 \mathcal{E}(t), \quad \forall t \geq 0, \tag{4.54}$$

since

$$\mathcal{E}(t)(t) \leq \mathcal{E}(0)e^{-\gamma_1 t}, \quad \forall t \geq 0. \tag{4.55}$$

Consequently, using (4.55) and (4.53), we find

$$E(t) \leq \frac{1}{\beta_1}\mathcal{E}(t) \leq \frac{1}{\beta_1}\mathcal{E}(0)e^{-\gamma_1 t}, \quad \forall t \geq 0. \tag{4.56}$$

Thus, the proof of Theorem 4.1 is complete. ∎

Chapter 5

Blow-up of result in a nonlinear higher-order equation with delay and source term

1 Introduction

In this chapter we are concerned with the following initial boundary value problem

$$\begin{cases} u_{tt}(x,t) + \mathcal{A}u(x,t) + \mu_1 |u_t(x,t)|^{m-2}u_t(x,t) & \\ +\mu_2 |z(x,1,t)|^{m-2}z(x,1,t) = b|u(x,t)|^{p-2}u(x,t), & \text{in } \Omega\times]0,+\infty[, \\ D^\alpha u(x,t) = 0, \ \ |\alpha| \le k-1, & \text{on } \partial\Omega \times [0,+\infty[, \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), & \text{in } \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), & \text{in } \Omega\times]0,\tau[, \end{cases} \tag{5.1}$$

where $\mathcal{A} = (-\Delta)^k$, $k \ge 1$, $p > 1$ are real numbers, Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega$, Δ is the Laplace operator in $\mathbb{R}^n$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $|\alpha| = \Sigma_{i=1}^n \alpha_i$, $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial^{\alpha_1}x_1\partial^{\alpha_2}x_2...\partial^{\alpha_n}x_n}$, $x = (x_1, x_2, ..., x_n)$, b, μ_1 and μ_2 are positives real numbers, $\tau > 0$ is a time delay, and the initial data (u_0, u_1, f_0) are in a suitable function space.
Without delay (i.e., $\mu_2 = 0$),

$$u_{tt} - \Delta u + u_t|u_t|^{m-2} = u|u|^{p-2},$$

has been extensively studied by many mathematicians. It is well known that in the further absence of the damping mechanism $u_t|u_t|^{m-2}$, the source term $u|u|^{p-2}$ causes finite-time blow-up of solutions with negative initial energy. In contrast, in the absence of the source term, the damping term assures global existence for arbitrary initial data (see **[26]**, **[33]**). The interaction between the damping and source terms was first considered by Levine **[40]** and **[41]** for linear damping ($m = 2$). Levine showed that solutions with negative initial energy blow up in finite time. Georgiev and Todorova **[22]** extended Levine's result to nonlinear damping ($m > 2$). In their work, the authors introduced a new method and determined relations between m and p for which there is global existence and other relations between m and p for which there is finite-time blow-up. Specifically, blow up in finite time if $p > m$ and the initial energy is sufficiently negative. Messaoudi **[56]** extended the

blow-up result of [**22**] to solutions with only negative initial energy. For related results, we refer the reader to Levine and Serrin [**38**], Levine and Ro Park [**39**] , Vitillaro [**69**], Yang [**74**] and Messaoudi and Said-Houari [**57**].
In this paper we use some techniques from [**28**] to show that blow-up for suitably chosen initial data, any classical solution blows up in finite time. This paper is organized as follows. In section 2, we establish some preliminary results. Section 3 is devoted to the blow-up result.

2 Preliminary results

In this section, we present some materials needed for our main results.

Lemma 2.1 *Let q be a real number with $2 \leq q < +\infty$ if $n \leq 2k$ and $2 \leq q \leq \dfrac{2n}{n-2k}$ if $n > 2s$. Then there is a constant C_* depending on Ω and q such that*

$$\|u\|_q \leq C_* \|\mathcal{A}^{\frac{1}{2}} u\|_2, \quad \forall u \in H_0^k(\Omega).$$

Lemma 2.2 *Suppose that*

$$2 \leq p < +\infty \ \ (n \leq 2k) \ \ or \ \ 2 \leq p \leq 2n/(n-2k) \ \ (n > 2k), \tag{5.2}$$

holds. Then there exists a positive constant C depending on Ω only such that

$$\|u\|_p^s \leq C(\|u\|_p^p + \|\mathcal{A}^{\frac{1}{2}} u\|_2^2), \tag{5.3}$$

for any $u \in H_0^k(\Omega)$ and $2 \leq s \leq p$.

Proof : If $\|u\|_p \leq 1$ then $\|u\|_p^s \leq \|u\|_p^2 \leq C_*^2 \|\mathcal{A}^{\frac{1}{2}} u\|_2^2$ by Sobolev embedding the theorems. If $\|u\|_p > 1$ then $\|u\|_p^s \leq \|u\|_p^p$.
Together with the two cases, we obtain (5.3). ∎

Now we introduce, as in Nicaise and Pignotti [**64**], the new variable

$$z(x,\rho,t) = u_t(x, t-\rho\tau), \ \ x \in \Omega, \quad \rho \in (0,1), t > 0. \tag{5.4}$$

Then, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \ \text{ in } \Omega \times (0,1) \times (0,+\infty). \tag{5.5}$$

Therefore, problem (5.1) is equivalent to

$$\begin{cases} u_{tt}(x,t) + \mathcal{A}u(x,t) + \mu_1 |u_t(x,t)|^{m-2} u_t(x,t) & \\ +\mu_2 |z(x,1,t)|^{m-2} z(x,1,t) = b|u(x,t)|^{p-2} u(x,t), & \text{in } \Omega \times]0,+\infty[, \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & \text{in } \Omega \times]0,1[\times]0,+\infty[\\ D^\alpha u(x,t) = 0, \ \ |\alpha| \leq \sigma - 1, & \text{on } \partial\Omega \times [0,+\infty[, \\ z(x,0,t) = u_t(x,t), & \text{in } \Omega \times [0,\infty[\\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in } \Omega \\ z(x,\rho,0) = f_0(x, t-\rho\tau), & \text{in } \Omega \times]0,1[. \end{cases} \tag{5.6}$$

Theorem 2.3 *(Local existence) Suppose that $m > 1$, $p > 2$ and $k \geq 1$, let $u_0 \in H^{2k}(\Omega) \cap H_0^k(\Omega)$, $u_1 \in H_0^k(\Omega)$ and $f_0 \in H_0^k(\Omega \times (0,1))$ satisfy the compatibility condition*

$$f_0(.,0) = u_1.$$

Assume further that

$$\max\{p, m\} \leq \frac{2(n-1)}{n-2} \quad \text{and } l \leq \frac{2}{n-2}, \quad \text{if } n \geq 3,$$

and

$$p \text{ satisfies } 2 \leq p < +\infty \text{ if } n \leq 2k \text{ and } 2 \leq p \leq \frac{2(n-k)}{n-2k} \quad \text{if } n > 2k.$$

Then problem (5.6) has a unique local solution

$$u \in \mathcal{C}([0,T]; H_0^k(\Omega)),$$

$$u_t \in \mathcal{C}([0,T]; H_0^k(\Omega)) \cap L^m((0,T) \times \Omega),$$

$$u_{tt} \in L^2([0,T];\ L^2(\Omega)),$$

$$z \in \mathcal{C}([0,T]; L^2(\Omega \times (0,1))),$$

for some $T > 0$.

We define the energy associated to the solution of system (5.6) by

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\mathcal{A}^{\frac{1}{2}}u\|_2^2 - \frac{b}{p}\|u\|_p^p + \xi \int_\Omega \int_0^1 |z(x,\rho,t)|^m \, d\rho \, dx, \tag{5.7}$$

where ξ is a positive constant such that

$$\tau \frac{\mu_2}{m}(m-1) < \xi < \tau\Big(\mu_1 - \frac{\mu_2}{m}\Big), \tag{5.8}$$

and $\mu_2 < m\mu_1$.

Lemma 2.4 *Let (u,z) be a solution of the problem (5.6). Then, there exists $C > 0$ such that*

$$E'(t) \leq -C\Big[\int_\Omega |z(x,1,t)|^m \, dx + \|u_t(x,t)\|_m^m\Big] \leq 0. \tag{5.9}$$

Proof : Multiplying the first equation in (5.6) by u_t and integrating over Ω, using integration by parts, we get

$$\begin{aligned} \frac{d}{dt}\Big(\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\mathcal{A}^{\frac{1}{2}}u\|_2^2 \quad &- \frac{b}{p}\|u\|_p^p\Big) + \mu_1\|u_t\|_m^m \\ &+\mu_2 \int_\Omega |z(x,1,t)|^{m-2} z(x,1,t) u_t(x,t) \, dx = 0. \end{aligned} \tag{5.10}$$

We multiply the second equation in (5.6) by $\xi|z(x,\rho,t)|^{m-2}z(x,\rho,t)$ and integrate the result over $\Omega\times(0,1)$, to obtain

$$\begin{aligned}
&\xi\int_\Omega\int_0^1 z_t(x,\rho,t)|z(x,\rho,t)|^{m-2}z(x,\rho,t)\,d\rho\,dx\\
&\qquad=-\frac{\xi}{\tau}\int_\Omega\int_0^1 z_\rho(x,\rho,t)|z(x,\rho,t)|^{m-2}z(x,\rho,t)\,d\rho\,dx\\
&\qquad=-\frac{\xi}{\tau}\int_\Omega\int_0^1 \frac{\partial z(x,\rho,t)}{\partial\rho}\,|z(x,\rho,t)|^{m-2}z(x,\rho,t)\,d\rho\,dx\\
&\qquad=-\frac{\xi}{\tau m}\int_\Omega\int_0^1 \frac{\partial}{\partial\rho}\,|z(x,\rho,t)|^{m}\,d\rho\,dx\\
&\qquad=-\frac{\xi}{\tau m}\int_\Omega(|z(x,1,t)|^m-|z(x,0,t)|^m)\,dx.
\end{aligned}\tag{5.11}$$

Hence

$$\xi\frac{d}{dt}\int_\Omega\int_0^1|z(x,\rho,t)|^m\,d\rho\,dx=-\frac{\xi}{\tau}\int_\Omega|z(x,1,t)|^m dx+\frac{\xi}{\tau}\int_\Omega|u_t(x,t)|^m\,dx.\tag{5.12}$$

Combining (5.10) and (5.12), we obtain

$$E'(t)=-\frac{\xi}{\tau}\int_\Omega|z(x,1,t)|^m\,dx-(\mu_1-\frac{\xi}{\tau})\|u_t(x,t)\|_m^m-\mu_2\int_\Omega|z(x,1,t)|^{m-2}z(x,1,t)u_t(x,t)\,dx,\tag{5.13}$$

and using Young's inequality, we have

$$-\mu_2\int_\Omega|z(x,1,t)|^{m-2}z(x,1,t)u_t(x,t)\,dx\le\mu_2\Big[\frac{1}{m}\delta^m\|u_t(x,t)\|_m^m+\frac{m-1}{m}\frac{1}{\delta^{\frac{m}{m-1}}}\int_\Omega|z(x,1,t)|^m\,dx\Big].$$

Thus, by choosing $\delta^{-\dfrac{m}{m-1}}=\dfrac{m\epsilon}{m-1}$, then

$$-\mu_2\int_\Omega|z(x,1,t)|^{m-2}z(x,1,t)u_t(x,t)\,dx\le\mu_2\Big[\epsilon\int_\Omega|z(x,1,t)|^m\,dx\;+\frac{1}{m}\Big(\frac{m\epsilon}{m-1}\Big)^{1-m}\|u_t(x,t)\|_m^m\Big],$$

with $\epsilon=\dfrac{m-1}{m}$, we have

$$-\mu_2\int_\Omega|z(x,1,t)|^{m-2}z(x,1,t)u_t(x,t)\,dx\le\frac{\mu_2}{m}\Big[(m-1)\int_\Omega|z(x,1,t)|^m\,dx\;+\|u_t(x,t)\|_m^m\Big].\tag{5.14}$$

Hence, we get from (5.13)

$$\begin{aligned}
E'(t)\;&\le-\Big(\frac{\xi}{\tau}-\frac{\mu_2}{m}(m-1)\Big)\int_\Omega|z(x,1,t)|^m\,dx-\Big(\mu_1-\frac{\xi}{\tau}-\frac{\mu_2}{m}\Big)\|u_t(x,t)\|_m^m\\
&\le-C\Big(\int_\Omega|z(x,1,t)|^m\,dx+\|u_t(x,t)\|_m^m\Big),
\end{aligned}\tag{5.15}$$

where

$$C=\min\Big\{\frac{\xi}{\tau}-\frac{\mu_2}{m}(m-1),\mu_1-\frac{\xi}{\tau}-\frac{\mu_2}{m}\Big\},$$

which is positive by (5.8). This completes the proof of the Lemma. ∎

We set

$$H(t)=-E(t).\tag{5.16}$$

3 Blow-up

In this section we state and prove our main result.

Theorem 3.1 *Suppose that $m > 1,\ p > \max\{2, m\}$ satisfying (5.2), let $u_0 \in H^{2k}(\Omega) \cap H_0^k(\Omega),\ u_1 \in H_0^k(\Omega)$ and $f_0 \in H_0^k(\Omega \times (0,1))$. Assume further that*

$$E(0) = \frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\|\mathcal{A}^{\frac{1}{2}} u_0\|_2^2 - \frac{b}{p}\|u_0\|_p^p + \xi \int_\Omega \int_0^1 |f_0(x, -\rho\tau|^m \, d\rho \, dx < 0.$$

Then the solution of (5.6) blow up in finite time, i.e. there exists $T_0 < +\infty$ such that

$$\lim_{t \to T_0^-} (\|u_t\|_2^2 + \|\mathcal{A}^{\frac{1}{2}} u\|_2^2 + \|u\|_p^p) = \infty.$$

The proof of Theorem 3.1 relies on the following

Corolary 3.2 *Let the assumptions of the Lemma 2.2 hold. Then we have the following*

$$\|u\|_p^s \leq C\Big(-H(t) - \|u_t\|_2^2 - \|\mathcal{A}^{\frac{1}{2}} u\|_2^2 + \|u\|_p^p - \xi \int_\Omega \int_0^1 |z(x,\rho,t)|^m \, d\rho \, dx\Big), \quad \text{for all} \quad t \in [0,T), \tag{5.17}$$

for any $u(.,t) \in H_0^1(\Omega)$ and $2 \leq s \leq p$.

Proof : Assume that there exists some positive constant C such that for $t > 0$ the solution $u(t)$ of (5.6) satisfies

$$\|u_t\|_2^2 + \|\mathcal{A}^{\frac{1}{2}} u\|_2^2 + \|u\|_p^p \leq C. \tag{5.18}$$

Hence,

$$0 < H(0) < H(t) < \frac{b}{p}\|u\|_p^p. \tag{5.19}$$

We then define the function

$$G(t) = \int_\Omega u_t u \, dx,$$

with

$$G'(t) = \int_\Omega u u_{tt} dx + \int_\Omega |u_t|^2 dx. \tag{5.20}$$

By using the equation of (5.6), we arrive at

$$\begin{aligned} G'(t) \quad &= \|u_t\|_2^2 + b\|u\|_p^p - \|\mathcal{A}^{\frac{1}{2}} u\|_2^2 - \mu_1 \int_\Omega |u_t(x,t)|^{m-2} u_t(x,t) u(x,t) \, dx \\ &-\mu_2 \int_\Omega |z(x,1,t)|^{m-2} z(x,1,t) u(x,t) \, dx. \end{aligned} \tag{5.21}$$

By Young's inequality, we obtain

$$\mu_1 \int_\Omega |u_t(x,t)|^{m-2} u_t(x,t) u(x,t) \, dx \leq \mu_1 \Big[\frac{\delta^m}{m}\|u\|_m^m + \frac{(m-1)\delta^{\frac{-m}{m-1}}}{m}\|u_t\|_m^m\Big], \tag{5.22}$$

similarly, we have

$$\mu_2 \int_\Omega |z(x,1,t)|^{m-2} z(x,1,t) u(x,t) \, dx \leq \mu_2 \Big[\frac{\delta^m}{m}\|u\|_m^m + \frac{(m-1)\delta^{\frac{-m}{m-1}}}{m}\|z(x,1,t)\|_m^m\Big]. \tag{5.23}$$

We then define

$$L(t) = H^{1-\alpha}(t) + \epsilon G(t), \tag{5.24}$$

for ϵ small to be chosen later and

$$0 < \alpha < \min\Big\{\frac{1}{l+2} - \frac{1}{p}, \frac{p-m}{p(m-1)}\Big\}.$$

By taking a derivative of (5.24) and using (5.21) , (5.22) and (5.23) we obtain

$$\begin{aligned} L'(t) &= (1-\alpha)H'(t)H^{-\alpha}(t) + \epsilon G'(t) \\ &\geq (1-\alpha)H'(t)H^{-\alpha}(t) - \epsilon(\mu_1+\mu_2)\frac{(m-1)\delta^{\frac{-m}{m-1}}}{m}(\|u_t\|_m^m + \|z(x,1,t)\|_m^m) \\ &+\epsilon\|u_t\|_2^2 + b\epsilon\|u\|_p^p - \epsilon\|\mathcal{A}^{\frac{1}{2}}u\|_2^2 - \epsilon(\mu_1+\mu_2)\tfrac{\delta^m}{m}\|u\|_m^m \\ &\geq \Big[(1-\alpha)H^{-\alpha}(t) - \epsilon(\mu_1+\mu_2)\frac{(m-1)\delta^{\frac{-m}{m-1}}}{mC}\Big]H'(t) + \frac{\epsilon}{l+1}\|u_t\|_{l+2}^{l+2} + b\epsilon\|u\|_p^p \\ &-\epsilon\|\mathcal{A}^{\frac{1}{2}}u\|_2^2 - \epsilon(\mu_1+\mu_2)\frac{\delta^m}{m}\|u\|_m^m. \end{aligned} \tag{5.25}$$

Of course (5.25) remains valid even if δ is time dependent. Therefore by taking δ so that

$$\delta^{\frac{-m}{m-1}} = kH^{-\alpha}(t),$$

for large k to be specified later, and substituting in (5.25) we arrive at

$$\begin{aligned} L'(t) &\geq \Big[(1-\alpha) - \epsilon(\mu_1+\mu_2)\tfrac{m-1}{mC}k\Big]H^{-\alpha}(t)H'(t) + \tfrac{\epsilon}{l+1}\|u_t\|_{l+2}^{l+2} + b\epsilon\|u\|_p^p \\ &-\epsilon\|\mathcal{A}^{\frac{1}{2}}u\|_2^2 - \epsilon(\mu_1+\mu_2)\frac{k^{1-m}}{m}H^{\alpha(m-1)}(t)\|u\|_m^m. \end{aligned} \tag{5.26}$$

By exploiting (5.19) and the inequality $\|u\|_m^m \leq c\|u\|_p^m$, we obtain

$$H^{\alpha(m-1)}(t)\,\|u\|_m^m \leq c\Big(\frac{b}{p}\Big)^{\alpha(m-1)}\|u\|_p^{\alpha p(m-1)+m}, \tag{5.27}$$

inserting (5.27) in (5.26), using (5.7) and (5.16), we get, for $0 < \beta < 1$,

$$\begin{aligned} L'(t) &\geq \Big[(1-\alpha) - \epsilon(\mu_1+\mu_2)\frac{m-1}{mC}k\Big]H^{-\alpha}(t)H'(t) + \epsilon\|u_t\|_2^2 \\ &+b\beta\epsilon\|u\|_p^p - \epsilon\|\mathcal{A}^{\frac{1}{2}}u\|_2^2 - \epsilon c\Big(\frac{b}{p}\Big)^{\alpha(m-1)}(\mu_1+\mu_2)\frac{k^{1-m}}{m}\|u\|_p^{\alpha p(m-1)+m} \\ &+\epsilon(1-\beta)p\Big[H(t) + \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\mathcal{A}^{\frac{1}{2}}u\|_2^2 + \xi\int_\Omega\int_0^1 |z(x,\rho,t)|^m\,d\rho\,dx\Big] \\ &\geq \Big[(1-\alpha) - \epsilon(\mu_1+\mu_2)\tfrac{m-1}{mC}k\Big]H^{-\alpha}(t)H'(t) + \epsilon\Big(1 + \frac{p(1-\beta)}{2}\Big)\|u_t\|_2^2 \\ &+\epsilon\Big(\tfrac{p(1-\beta)}{2} - 1\Big)\|\mathcal{A}^{\frac{1}{2}}u\|_2^2 + b\beta\epsilon\|u\|_p^p + \epsilon(1-\beta)pH(t) \\ &+\epsilon(1-\beta)p\xi\int_\Omega\int_0^1 |z(x,\rho,t)|^m\,d\rho\,dx - \epsilon c\Big(\frac{b}{p}\Big)^{\alpha(m-1)}(\mu_1+\mu_2)\frac{k^{1-m}}{m}\|u\|_p^{\alpha p(m-1)+m}. \end{aligned} \tag{5.28}$$

Then we use Corollary 3.2, for $s=\alpha p(m-1)+m\leq p$, to deduce that

$$\begin{aligned} L'(t) &\geq \Big[(1-\alpha)-\varepsilon(\mu_1+\mu_2)\tfrac{m-1}{mC}k\Big]H^{-\alpha}(t)H'(t)+\varepsilon\Big(\tfrac{1}{l+1}+\tfrac{p(1-\beta)}{l+2}\Big)\|u_t\|_{l+2}^{l+2} \\ &+\epsilon\Big(\tfrac{p(1-\beta)}{2}-1\Big)\|\mathcal{A}^{\frac{1}{2}}u\|_2^2+b\beta\epsilon\|u\|_p^p \\ &+p\epsilon(1-\beta)H(t)+p\epsilon(1-\beta)\xi\int_\Omega\int_0^1|z(x,\rho,t)|^m d\rho\,dx \\ &-\epsilon k^{1-m}C_1\Big\{-H(t)-\|u_t\|_2^2+\|u\|_p^p-\xi\int_\Omega\int_0^1|z(x,\rho,t)|^m\,d\rho\,dx\Big\}. \end{aligned} \tag{5.29}$$

Consequently, we obtain

$$\begin{aligned} L'(t) &\geq \Big[(1-\alpha)-\varepsilon(\mu_1+\mu_2)\tfrac{m-1}{mC}k\Big]H^{-\alpha}(t)H'(t) \\ &+\varepsilon\Big(1+\tfrac{p(1-\beta)}{l+2}+C_1k^{1-m}\Big)\|u_t\|_2^2 \\ &+\epsilon\Big(\tfrac{p(1-\beta)}{2}-1\Big)\|\mathcal{A}^{\frac{1}{2}}u\|_2^2+\epsilon\Big(b\beta-C_1k^{1-m}\Big)\|u\|_p^p \\ &+\epsilon\xi\Big(p(1-\beta)+C_1k^{1-m}\Big)\int_\Omega\int_0^1|z(x,\rho,t)|^m d\rho\,dx \\ &+\epsilon\Big(p(1-\beta)+C_1k^{1-m}\Big)H(t), \end{aligned} \tag{5.30}$$

where

$$C_1=c\Big(\frac{b}{p}\Big)^{\alpha(m-1)}\frac{\mu_1+\mu_2}{m},$$

$$\frac{p(1-\beta)}{2}-1>0,$$

and we choose k so large that

$$b\beta-C_1k^{1-m}>0.$$

Finally, we pick ε so small so that

$$(1-\alpha)-\epsilon(\mu_1+\mu_2)\frac{m-1}{mC}k>0,$$

and

$$L(0)=H^{1-\alpha}(0)+\epsilon\int_\Omega u_1u_0dx>0.$$

Therefore (5.30) takes the form

$$L'(t)\geq\lambda\Big[\|u_t\|_2^2+\|\mathcal{A}^{\frac{1}{2}}u\|_2^2+\|u\|_p^p+H(t)+\xi\int_\Omega\int_0^1|z(x,\rho,t)|^m d\rho dx\Big]. \tag{5.31}$$

Consequently, we have

$$L(t)\geq L(0),\ \ t\geq 0.$$

We now estimate

$$\Big|\int_\Omega u_tudx\Big|\leq\|u_t\|_2\|u\|_2\leq C\|u_t\|_2\|u\|_p, \tag{5.32}$$

which implies

$$\Big|\int_\Omega u_tudx\Big|^{\frac{1}{1-\alpha}}\leq C\|u_t\|_2^{\frac{1}{1-\alpha}}\|u\|_p^{\frac{1}{1-\alpha}}. \tag{5.33}$$

Using Young's inequality then yields

$$\Big|\int_\Omega u_t u\,dx\Big|^{\frac{1}{1-\alpha}} \quad \le C\Big[\|u_t\|_2^{\frac{\beta_1}{1-\alpha}} + \|u\|_p{}^{\frac{\beta_2}{1-\alpha}}\Big], \tag{5.34}$$

for $\frac{1}{\beta_1}+\frac{1}{\beta_2}=1$, we take $\beta_1 = 2(1-\alpha)$ which gives $\frac{\beta_2}{1-\alpha} = \frac{2}{1-2\alpha}$.
Therefore, (5.34) becomes

$$\Big|\int_\Omega u_t u\,dx\Big|^{\frac{1}{1-\alpha}} \le C\Big[\|u_t\|_2^2 + \|u\|_p^s\Big]. \tag{5.35}$$

Using (5.35) and Corollary 3.2, for $s = \frac{2}{1-2\alpha} \le p$ gives

$$\Big|\int_\Omega u_t u\,dx\Big|^{\frac{1}{1-\alpha}} \le C\Big(H(t) + \|u_t\|_{l+2}^{l+2} + \|\mathcal{A}^{\frac{1}{2}}u\|_2^2 + \|u\|_p^p + \xi \int_\Omega\int_0^1 |z(x,\rho,t)|^m d\rho\,dx\Big). \tag{5.36}$$

Therefore, we have

$$\begin{aligned} L^{\frac{1}{1-\alpha}}(t) &= \Big(H^{1-\alpha}(t) + \epsilon G(t)\Big)^{\frac{1}{1-\alpha}} \\ &\le C\Big(H(t) + \|u_t\|_{l+2}^{l+2} + \|\mathcal{A}^{\frac{1}{2}}u\|_2^2 + \|u\|_p^p + \xi \int_\Omega\int_0^1 |z(x,\rho,t)|^m d\rho\,dx\Big), \quad t>0. \end{aligned} \tag{5.37}$$

Combining (5.31) and (5.37), we arrive at

$$L'(t) \ge \Lambda L^{\frac{1}{1-\alpha}}(t),\ t>0, \tag{5.38}$$

where Λ is a positive constant depending only on λ and C.
A simple integration of (5.38) over $(0,t)$ yields

$$L^{\frac{\alpha}{1-\alpha}}(t) \ge \frac{1}{L^{-\frac{\alpha}{1-\alpha}}(0) - \Lambda\alpha t/(1-\alpha)},\ t>0.$$

Therefore, $L(t)$ blows up in time

$$T_0 \le \frac{1-\alpha}{\Lambda\alpha L^{\frac{\alpha}{1-\alpha}}(0)}.$$

Furthermore, we have

$$\lim_{t\to T_0^-}(\|u_t\|_{l+2}^{l+2} + \|\nabla u\|_2^2 + \|\nabla u_t\|_2^2 + \|u\|_p^p) = \infty.$$

This leads to a contradiction with (5.18). Thus, the solution of problem (5.6) blows up in finite time. This completes the proof. ∎

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