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Théorie des nombres: Sur les suites primitives

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Soutenue le: ../../....

Jury

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Dedicaces

To my dear parents.
To all my family.

Notations

1. $\mathbb{N}^*$ denotes the set of non-zero natural numbers.
2. We denote by $\mathcal{P}$ the set of prime numbers.
3. We denote by p_n the n-th prime number.
4. We define the degree of an integer a denoted by $\Omega(a)$, to be the number of prime factors of a counting with multiplicity.
5. We denote by $p(a)$ the last prime factor of a.
6. We denote by $\mathcal{A}$ a set of primitive integers.
7. We denote by $\deg(\mathcal{A})$ the degree of $\mathcal{A}$, it is defined as the maximum degree of its terms.
8. $\mathcal{A}_m = \{a : a \in \mathcal{A}$, the prime factors of a are $\geq p_m\}$.
9. $\mathcal{A}'_m = \{a : a \in \mathcal{A}_m,\ p_m \mid a\}$.
10. $\mathcal{A}''_m = \{a/p_m : a \in \mathcal{A}'_m\}$.
11. $\liminf\limits_{n\longrightarrow\infty}(a_n)$ indicate the limit inferior of a_n as $n \to \infty$.
12. $\limsup\limits_{n\longrightarrow\infty}(a_n)$ indicate the limit superior of a_n as $n \to \infty$.
13. $\sum_{n\leq x}$ indicate the sum of all integers lying in the interval $[1, x]$.
14. $\prod_{p\leq x}$ indecate the product of the primes numbers lying in the interval $[2, x]$.
15. $d\mathcal{A}$ indecate the asymptotic density of $\mathcal{A}$.
16. $\overline{d}\mathcal{A}$ indecate the upper asymptotic density of $\mathcal{A}$.
17. $\underline{d}\mathcal{A}$ indecate the lower asymptotic density of $\mathcal{A}$.
18. $\delta\mathcal{A}$ indecate the logarithmic density of $\mathcal{A}$.
19. $\overline{\delta}\mathcal{A}$ indecate the upper logarithmic density of $\mathcal{A}$.
20. $\underline{\delta}\mathcal{A}$ indecate the lower logarithmic density of $\mathcal{A}$.

21. For integers n and m, $m \mid n$ means "m divides n" and $m \nmid n$ means "m does not divide n".

22. $\lfloor x \rfloor$ denotes the unique integer k such that $k \leq x < k+1$ (the integer part of real x).

23. $f = \mathcal{O}(g)$ and $f \ll g$ the notations of Landau and Vinogradov means that there exists a constant $C > 0$ and a real x_0 such that for any $x \geq x_0$, we have

$$|f(x)| \leq Cg(x).$$

24. $f = o(g)$ means that $\lim_{x\to+\infty} \dfrac{|f(x)|}{g(x)} = 0$.

25. $\theta(x)$ represents Tchébichev function defined by $\theta(x) = \sum_{p\leq x} \log p$ and $\theta(x) = 0$ for $x < 0$.

26. $\varphi(n)$ represents Euler's function counts the integers $m \leq n$ such that $(m, n) = 1$

$$\varphi(n) = \sum_{m\leq n,\ (m,n)=1} 1.$$

27. $\pi(x)$ represents the function counts prime numbers less than or equal x.

Résumé

Une suite $\mathcal{A}$ d'entiers strictement positifs est dite primitive si et seulement si aucun élément de $\mathcal{A}$ ne divise les autres. Erdős a prouvé que la série $S(\mathcal{A}) = \sum_{a\in\mathcal{A}} \frac{1}{a\log a}$, où $\mathcal{A}$ est une suite primitive différente de $\{1\}$, converge. De plus, il a conjecturé que $\sum_{a\in\mathcal{A}} \frac{1}{a\log a} \leq \sum_{a\in\mathcal{P}} \frac{1}{p\log p}$, où $\mathcal{P}$ représente l'ensemble des nombres premiers. Afin de prouver cette conjecture, B. Farhi a établi la série de la forme $S(\mathcal{A}, x) = \sum_{a\in\mathcal{A}} \frac{1}{a(\log a+x)}$. Le but de cette thèse est d'introduire des résultats autour de cette dernière somme et sa relation avec la conjecture d'Erdős.

Summary

A sequence $\mathcal{A}$ of structly positive integers is said to be primitive if no term of $\mathcal{A}$ divides any other. Erdős showed that the series $S(\mathcal{A}) = \sum_{a\in\mathcal{A}} \frac{1}{a\log a}$, where $\mathcal{A}$ is a primitive sequence different from $\{1\}$, is convergent. Moreover, he conjectured that $\sum_{a\in\mathcal{A}} \frac{1}{a\log a} \leq \sum_{a\in\mathcal{P}} \frac{1}{p\log p}$ where $\mathcal{P}$ denotes the set of prime numbers. To prove this conjecture, B. Farhi established the series of the form $S(\mathcal{A}, x) = \sum_{a\in\mathcal{A}} \frac{1}{a(\log a+x)}$. The purpose of this thesis is to introduce results on this last sum and its relation with the Erdős conjecture.

Contents

Introduction

A sequence $\mathcal{A}$ of positive integers is said to be primitive if no element of $\mathcal{A}$ divides another. We can see directly that the set of primes $\mathcal{P} = (p_n)_{n \geq 1}$ is primitive. In the beginning, the research was focused on the density d of these sequences. In 1934, Bescovitch proved that for every $\epsilon > 0$ there exists a primitive sequence $\mathcal{A}$, such that the upper asymptotic density verified $\overline{d}\mathcal{A} > \frac{1}{2} - \epsilon$, see [8]. In the same subject, to check that the lower asymptotic density equal to zero ($\underline{d}\mathcal{A} = 0$), Erdős in [5], proved that if a sequence $\mathcal{A}$ is primitive different to $\{1\}$ then the series

$$S(\mathcal{A}) = \sum_{a \in \mathcal{A}} \frac{1}{a \log a}$$

is convergent, and its sum is bounded above by an absolute constant C. In 1993, Erdős and Zhang showed in [6] that $C \leq 1.84$. Years later, Clark was able to find in [2], the best bounder of C so far, he proved that $C \leq e^{\gamma} \simeq 1.78$ where γ denoted the Euler constant, but for $\mathcal{A} = \mathcal{P}$, it is well known that $S(\mathcal{P}) = 1.6366$. In 1988, Erdős conjectured if $S(\mathcal{P})$ is the maximum value of the sum $S(\mathcal{A})$ by proposing the following

Conjecture 0.1 *(Erdős): For any primitive sequence $\mathcal{A} \neq \{1\}$, we have:*

$$\sum_{a \in \mathcal{A}} \frac{1}{a \log a} \leq \sum_{p \in \mathcal{P}} \frac{1}{p \log p}.$$

In their paper [6], Erdős and Zhang showed that this conjecture is equivalent to the following one which deals with finite sums:

Conjecture 0.2 *(Erdős and Zhang): For any primitive sequence $\mathcal{A} \neq \{1\}$ and any positive integer n, we have:*

$$\sum_{a \in \mathcal{A}, a \leq n} \frac{1}{a \log a} \leq \sum_{p \in \mathcal{P}, p \leq n} \frac{1}{p \log p}.$$

In 1991, Zhang [16], proved that for each $k \geq 2$, $S(\mathbb{N}_k) < S(\mathbb{N}_1) = C$ where $\mathbb{N}_k = \{n : \Omega(n) = k\}$, $\Omega(n)$ is the number of prime factors of n counted with repetition, and in

1993, he showed that the conjecture holds for the particular case of homogenous sequences, see [17].

Recently, in 2017, still for the same objective which is to find a proof of this conjecture,B. Farhi, in [7], established the following analogue sum

$$S(\mathcal{A},\, x) = \sum_{a \in \mathcal{A}} \frac{1}{a\,(\log a + x)},$$

where x is a fixed non-negative real number and $\mathcal{A}$ is always a primitive sequence different to $\{1\}$.

In this thesis, we study precisely this series and we give some results on its relationship with the sum

$$S\,(\mathcal{P},\, x) = \sum_{p \in \mathcal{P}} \frac{1}{p\,(\log p + x)},$$

where $\mathcal{P}$ denoted the sequence of prime numbers, more exactly, we study the inequality $S(\mathcal{A},\, x) \geq S(\mathcal{P},\, x)$. As a remark, if we take $x = 0$, then we fall in the negation of Erdős conjecture which is $S(\mathcal{A},\, 0) \leq S(\mathcal{P},\, 0)$.

Our work organized into four chapters. The first is devoted to remind the main definitions and necessary theorems that we need in next chapters, especially those related to the density of sets of positive integers.

The second chapter is dedicated to two main theorems around this conjecture. So, we started by presenting the proof of Erdős conjecture for the homogeneous primitive sequences by Zhang in [17]. After that, we introduced an improved proof of his principal theorem in [16] where he proved the conjecture of Erdős for the sequences of degree less than or equal to 4. Our proof is based on drastically reducing operations, which gives us hope to raise the degree greater than 4.

In chapter three, we started by introducing improved and more precise results than that appearing in paper [9] of I. Laib and al, and we finished by generalizing the principal theorem in the same paper for any degree d.

In the last chapter, by using the primitive sequences of the form

$$\mathcal{B}_d^k = \{p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}\, |\alpha_1, \dots, \alpha_k \in \mathbb{N},\, \alpha_1 + \dots + \alpha_k = d\} \cup \{p_n\, |p_n \in \mathcal{P},\, n > k\},$$

and its properties we study the inequality $S(\mathcal{A},\, x) \geq S(\mathcal{P},\, x)$ for the largest values of x, we also used the multinomial formula.

Chapter 1

Preliminaries

In this chapter we recall some main tools used in number theory, such as Abel's summation formula and Stirling's formula then we introduce the density of a set of positive integers and its properties. More particularly, density of primitive sequences.

1.1 Abel's summation formula

Theorem 1.1 [1] *Let* $\{a(n)\}_{n\in\mathbb{N}^*}$ *be a sequence of complex numbers, define the sum*

$$A(t)=\sum_{n\leq t}a(n)$$

where $A(t)=0$ *if* $t<1$. *Assumes a continuously differentiable function* f *on the interval* $[y,\, x]$ *where* $0<y<x$, *then we have*

$$\sum_{y<n\leq x}a(n)f(n)=A(t)f(t)-A(y)f(y)-\int_y^x A(t)f'(t)\,dt.$$

Proof. Let be $m=\lfloor y\rfloor$ and $k=\lfloor x\rfloor$ then $A(y)=A(m)$, $A(k)=A(x)$ and

$$\sum_{y<n\leq x}a(n)f(n)=\sum_{n=m+1}^{k}a(n)f(n).$$

Otherwise for all $n\geq 1$,

$$\begin{aligned}a(n)-a(n-1) &= \sum_{l\leq n}a(l)-\sum_{l\leq n-1}a(l)\\ &= a(n)+\sum_{l\leq n-1}a(l)-\sum_{l\leq n-1}a(l)\\ &= a(n).\end{aligned}$$

This implies that

$$\begin{aligned}\sum_{y<n\le x} a(n) f(n) &= \sum_{n=m+1}^{k} (A(n) - A(n-1)) f(n) \\ &= \sum_{n=m+1}^{k} A(n) f(n) - A(n-1) f(n) \\ &= \sum_{n=m+1}^{k} A(n) f(n) - \sum_{n=m+1}^{k} A(n-1) f(n).\end{aligned}$$

Since

$$\sum_{n=m+1}^{k} A(n-1) f(n) = \sum_{n=m}^{k-1} A(n) f(n+1),$$

then

$$\sum_{y<n\le x} a(n) f(n) = \sum_{n=m+1}^{k} A(n) f(n) - \sum_{n=m}^{k-1} A(n) f(n+1).$$

And since

$$\begin{aligned}\sum_{n=m+1}^{k} A(n) f(n) &= \sum_{n=m+1}^{k-1} A(n) f(n) + A(k) f(k), \\ \sum_{n=m}^{k-1} A(n) f(n+1) &= A(m) f(m+1) + \sum_{n=m+1}^{k-1} A(n) f(n+1).\end{aligned}$$

Then

$$\begin{aligned}\sum_{y<n\le x} a(n) f(n) &= \sum_{n=m+1}^{k-1} A(n) f(n) + A(k) f(k) - A(m) f(m+1) - \sum_{n=m+1}^{k-1} A(n) f(n+1) \\ &= \sum_{n=m+1}^{k-1} [A(n) f(n) - A(n) f(n+1)] + A(k) f(k) - A(m) f(m+1) \\ &= \sum_{n=m+1}^{k-1} A(n) [f(n) - f(n+1)] + A(k) f(k) - A(m) f(m+1),\end{aligned}$$

but

$$\begin{aligned}\sum_{n=m+1}^{k-1} A(n) [f(n) - f(n+1)] &= \sum_{n=m+1}^{k-1} A(n) \int_{n}^{n+1} -f'(t)\, dt \\ &= -\sum_{n=m+1}^{k-1} A(n) \int_{n}^{n+1} f'(t)\, dt.\end{aligned}$$

And for $n_0 \leq n \leq t < n+1$,

$$\begin{aligned} A(t) &= \sum_{l \leq t} a(l) = a(n_0) + ... + a(n) \\ &= \sum_{n_0}^{n} a(l) = A(n). \end{aligned}$$

Hence,

$$\sum_{n=m+1}^{k-1} A(n)[f(n) - f(n+1)] = -\sum_{n=m+1}^{k-1} \int_n^{n+1} A(t) f'(t)\, dt.$$

Thus,

$$\begin{aligned} \sum_{y<n\leq x} a(n) f(n) &= -\sum_{n=m+1}^{k-1} \int_n^{n+1} A(t) f'(t)\, dt + A(k) f(k) - A(m) f(m+1) \\ &= -\int_{m+1}^{k} A(t) f'(t)\, dt + A(k) f(k) - A(m) f(m+1). \end{aligned}$$

Also, we have

$$\begin{aligned} A(k) f(k) &= A(k) f(k) + A(k) f(x) - A(k) f(x) \\ &= A(k)(f(k) - f(x)) + A(k) f(x) \\ &= A(k) \int_x^k f'(t)\, dt + A(k) f(x) \\ &= -A(k) \int_k^x f'(t)\, dt + A(k) f(x). \end{aligned}$$

For $k \leq t \leq x$, we have $A(k) = A(t)$, then $A(k) f(k) = -\int_k^x A(t) f'(t)\, dt + A(x) f(x)$. On the other hand,

$$\begin{aligned} -A(m) f(m+1) &= -A(m) f(m+1) + A(m) f(y) - A(m) f(y) \\ &= -A(m)(f(m+1) - f(y)) - A(m) f(y) \\ &= -A(m)(f(m+1) - f(y)) - A(y) f(y) \\ &= -A(m) \int_y^{m+1} f'(t)\, dt - A(y) f(y). \end{aligned}$$

But, for $y \leq t < m+1$ we have $A(m) = A(y)$, then

$$-A(m) f(m+1) = -\int_y^{m+1} A(t) f'(t)\, dt - A(y) f(y).$$

So,

$$\begin{aligned}\sum_{y<n\leq x} a(n) f(n) &= -\int_{m+1}^{k} A(t) f'(t)\,dt - \int_{k}^{x} A(t) f'(t)\,dt + A(x) f(x) \\ &\quad -\int_{y}^{m+1} A(t) f'(t)\,dt - A(y) f(y) \\ &= A(x) f(x) - A(y) f(y) - \int_{y}^{m+1} A(t) f'(t)\,dt \\ &\quad -\int_{m+1}^{k} A(t) f'(t)\,dt - \int_{k}^{x} A(t) f'(t)\,dt \\ &= A(x) f(x) - A(y) f(y) - \int_{y}^{x} A(t) f'(t)\,dt.\end{aligned}$$

Which ends the proof. ■

Example: Writing $\pi(x)$ in terms of $\theta(x)$

Theorem 1.2 [*1*] *For $x \geq 2$, we have*

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_{2}^{x} \frac{\theta(t)}{t \log^2 t} dt.$$

Proof. We consider the caracteristique function of prime numbers $a(n)$,

$$a(n) = \begin{cases} 1 \text{ if } n = p, \\ 0 \text{ if } n \neq p. \end{cases}$$

Applied Abel's summation formula with $b(n) = a(n)\log n$, $n \in \mathbb{N}^*$ and the function $f : x \mapsto \frac{1}{\log x}$ for $x > 1$, $y = \frac{3}{2}$.
So,

$$A(n) = \sum_{n\leq x} b(n) = \sum_{p\leq x} \log p = \theta(x),$$

and $f'(x) = -\frac{1}{x^2 \log x}$.
Therefore,

$$\begin{aligned}\pi(x) &= \sum_{n\leq x} b(n) f(n) \\ &= A(x) f(x) - A\left(\frac{3}{2}\right) f\left(\frac{3}{2}\right) - \int_{\frac{3}{2}}^{x} A(t) f'(t)\,dt \\ &= \frac{\theta(x)}{\log x} - \frac{\theta\left(\frac{3}{2}\right)}{\log \frac{3}{2}} - \int_{\frac{3}{2}}^{x} -\frac{\theta(t)}{t \log^2 t} dt.\end{aligned}$$

Since for $x < 2$ we have $\theta(t) = 0$, then

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t \log^2 t} dt.$$

And the proof is finished. ■

1.2 Stirling's formula

Stirling's formula is one of the important formulas most used in the remaining three chapters, especially in approximations that contain $n!$.

Theorem 1.3 [13] *For $n \in \mathbb{N}^*$ we have*

$$n! = \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} e^{r_n}.$$

where r_n satisfies $\frac{1}{12n+1} < r_n < \frac{1}{12n}$.

Proof. Let

$$S_n = \log(n!) = \log(1) + \log(2) + ... + \log(n) = \sum_{p=1}^{n-1} \log(p+1),$$

and we put

$$\log(p+1) = A_p + B_p - \epsilon_p,$$

where

$$\begin{aligned} A_p &= \int_p^{p+1} (\log x)\, dx, \\ B_p &= \frac{1}{2}[\log(p+1) - \log p], \\ \epsilon_p &= \int_p^{p+1} (\log x)\, dx - \frac{1}{2}[\log(p+1) - \log p]. \end{aligned}$$

So,

$$S_n = \sum_{p=1}^{n-1} (A_p + B_p - \epsilon_p) = \int_1^n (\log x)\, dx + \frac{1}{2} \log n - \sum_{p=1}^{n-1} \epsilon_p.$$

Therefore

$$S_n = \left(n + \frac{1}{2}\right) \log n - n + 1 - \sum_{p=1}^{n-1} \epsilon_p,$$

where

$$\epsilon_p = \frac{2p+1}{2} \log\left(\frac{p+1}{p}\right) - 1.$$

Using the well known series

$$\log \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + ...\right) \text{ for } |x| < 1,$$

setting $x = (2p+1)^{-1}$, then $\frac{1+x}{1-x} = \frac{p+1}{p}$ and

$$\epsilon_p = \frac{1}{3(2p+1)^2} + \frac{1}{5(2p+1)^4} + \frac{1}{7(2p+1)^6} + ...,$$

then we can bound ϵ_p as follow:

$$\begin{aligned}
\epsilon_p &< \frac{1}{3(2p+1)^2}\left(1 + \frac{1}{(2p+1)^2} + \frac{1}{(2p+1)^4} + ...\right) = \frac{1}{12}\left(\frac{1}{p} - \frac{1}{p+1}\right),\\
\epsilon_p &> \frac{1}{3(2p+1)^2}\left(1 + \frac{1}{3(2p+1)^2} + \frac{1}{\left[3(2p+1)^2\right]^2} + ...\right) > \frac{1}{12}\left(\frac{1}{p+\frac{1}{12}} - \frac{1}{p+1+\frac{1}{12}}\right).
\end{aligned}$$

Now, define

$$R = \sum_{p=1}^{\infty} \epsilon_p,\ r_n = \sum_{p=n}^{\infty} \epsilon_p,$$

then

$$\frac{1}{13} < R < \frac{1}{12}.$$

So, we can write S_n on the form

$$S_n = \left(n + \frac{1}{2}\right)\log n - n + 1 - R + r_n,$$

or, setting $C = e^{1-R}$, as

$$n! = C.n^{n+\frac{1}{2}}e^{-n}e^{r_n},$$

where

$$\frac{1}{12n+1} < r_n < \frac{1}{12n}.$$

The constant C, known from the double inequality $\frac{1}{13} < R < \frac{1}{12}$ to lie between $e^{\frac{11}{12}}$ and $e^{\frac{12}{13}}$, may be shown by one of the usual methods to have the value $\sqrt{2\pi}$. This completes the proof. ■

1.3 Density

The density help us to study the manner in which a subset $\mathcal{A} \subset \mathbb{N}$ is distributed among the naturel numbers. Study of the density of primitive sequences led Erdős to lay down his conjecture: for any primitive sequence $\mathcal{A} \neq \{1\}$ we have

$$\sum_{a\in\mathcal{A}} \frac{1}{a\log a} \leq \sum_{a\in\mathcal{P}} \frac{1}{p\log p},$$

where $\mathcal{P}$ denotes the sequence of prime numbers.

Asymptotic density

Definition 1.1 [8] *Let $\mathcal{A} = \{a_i\}_{i\geq 1}$ be a sequence of positive integers. For $n \in \mathbb{N}$, we define*

$$\mathcal{A}(n) = card\{\mathcal{A} \cap [0, n]\},$$

if the limit of $\frac{\mathcal{A}(n)}{n}$ exists, then $\mathcal{A}$ is said to possess asymptotic density which defined by

$$d\mathcal{A} = \lim_{n\to\infty} \frac{\mathcal{A}(n)}{n}.$$

Remark 1.1 *If the sequence $\mathcal{A}$ is finite then $d\mathcal{A} = 0$.*

Definition 1.2 [8] *The lower asymptotic density of a sequence of positive integers $\mathcal{A}$ is defined by*

$$\underline{d}\mathcal{A} = \liminf_{n\to\infty} \frac{\mathcal{A}(n)}{n}.$$

Definition 1.3 [8] *The upper asymptotic density of a sequence of positive integers $\mathcal{A}$ is defined by*

$$\overline{d}\mathcal{A} = \limsup_{n\to\infty} \frac{\mathcal{A}(n)}{n}.$$

Example 1.1 *Let $\mathcal{A}$ be an arithmetic progression,*

$$\mathcal{A} = \{sk + r \ , \ \ k \geq 0 \ , \ 0 \leq r < s\},$$

where s and r are fixed positives integers.
We have

$$\mathcal{A}(n) = card\{\mathcal{A} \cap [0, n]\}$$

then $\mathcal{A}(n) = k$, where k satisfies the inequalities

$$sk + r \leq n < sk + r + s.$$

Therefore,

$$\frac{n-s-r}{s} < k \leq \frac{n-r}{s}$$

so,

$$\frac{n-s-r}{sn} < \frac{\mathcal{A}(n)}{n} \leq \frac{n-r}{sn}$$

hence

$$\lim_{n\to\infty} \frac{n-s-r}{sn} < \lim_{n\to\infty} \frac{\mathcal{A}(n)}{n} \leq \lim_{n\to\infty} \frac{n-r}{sn}.$$

Thus

$$d\mathcal{A} = \lim_{n\to\infty} \frac{\mathcal{A}(n)}{n} = \frac{1}{s}.$$

Theorem 1.4 *Let $\mathcal{A}$ be a sequence of positive integers, then*

1) $\underline{d}\mathcal{A} \leq \overline{d}\mathcal{A}$,

2) if $\underline{d}\mathcal{A} = \overline{d}\mathcal{A} = l$ then $d\mathcal{A}$ exists and equal to l.

Proof. 1) Let $\mathcal{A}$ be a sequence of positive integers, then we have

$$\inf\left\{\frac{\mathcal{A}(n)}{n}, \frac{\mathcal{A}(n+1)}{n+1}, ...\right\} \leq \frac{\mathcal{A}(n)}{n} \leq \sup\left\{\frac{\mathcal{A}(n)}{n}, \frac{\mathcal{A}(n+1)}{n+1}, ...\right\}.$$

then,

$$\lim_{n\to\infty}\inf_{k\geq n}\frac{\mathcal{A}(k)}{k} \leq \lim_{n\to\infty}\frac{\mathcal{A}(n)}{n} \leq \lim_{n\to\infty}\sup_{k\geq n}\frac{\mathcal{A}(k)}{k},$$

therefore,

$$\lim_{n\to\infty}\inf_{k\geq n}\frac{\mathcal{A}(k)}{k} \leq \lim_{n\to\infty}\sup_{k\geq n}\frac{\mathcal{A}(k)}{k},$$

so,

$$\underline{d}\mathcal{A} \leq \overline{d}\mathcal{A}.$$

2) If $\underline{d}\mathcal{A} = \overline{d}\mathcal{A} = l$, then for given $\epsilon > 0$ there exists n_0 such that, $\forall n \geq n_0$,

$$l - \epsilon < \inf_{k\geq n}\frac{\mathcal{A}(k)}{k} < l + \epsilon$$

and

$$l - \epsilon < \sup_{k\geq n}\frac{\mathcal{A}(k)}{k} < l + \epsilon$$

then

$$l - \epsilon < \frac{\mathcal{A}(n)}{n} < l + \epsilon.$$

Thus $d\mathcal{A} = l$. ■

Logarithmic density

Definition 1.4 [8] *Let $\mathcal{A} = \{a_i,\ i = 1,\ 2,\ ...\}$ be a sequence of positive integers, if the limit of the series $\frac{1}{\log n}\sum_{a_i\leq n}\frac{1}{a_i}$ exists, then $\mathcal{A}$ possess logarithmic density which defined by*

$$\delta\mathcal{A} = \lim_{n\to\infty}\frac{1}{\log n}\sum_{a_i\leq n}\frac{1}{a_i}.$$

Definition 1.5 [8] *The lower logarithmic density of a sequence of positive integers $\mathcal{A}$ is defined by*

$$\underline{\delta}\mathcal{A} = \liminf_{n\to\infty}\frac{1}{\log n}\sum_{a_i\leq n}\frac{1}{a_i}.$$

Definition 1.6 [8] *The upper logarithmic density of a sequence of positive integers $\mathcal{A}$ is defined by*

$$\overline{\delta}\mathcal{A} = \limsup_{n\to\infty} \frac{1}{\log n} \sum_{a_i \le n} \frac{1}{a_i}.$$

Corollay 1.1 *If a sequence $\mathcal{A}$ consists of a finite number of positive integers then*

$$\delta\mathcal{A} = 0.$$

Theorem 1.5 [8] *For any sequence of positive integers $\mathcal{A}$, we have*

$$0 \le \underline{d}\mathcal{A} \le \underline{\delta}\mathcal{A} \le \overline{\delta}\mathcal{A} \le \overline{d}\mathcal{A} \le 1.$$

Theorem 1.6 *Let $\mathcal{A} = \{a_i\}_{i\ge 1}$ be an infinite sequence of positive integers. If the series $\sum_{i=1}^{\infty} \frac{1}{a_i}$ converges then $\delta\mathcal{A} = 0$.*

Proof. Let $\{a_i, a_i, ...\}$ be an infinite sequence of positive integers and let $S = \sum_{i=1}^{\infty} \frac{1}{a_i}$. Since

$$S = \sum_{i=1}^{\infty} \frac{1}{a_i} < \infty,$$

we have

$$\sum_{a_i \le n} \frac{1}{a_i} \le \sum_{i=1}^{\infty} \frac{1}{a_i},$$

then,

$$0 \le \frac{1}{\log n} \sum_{a_i \le n} \frac{1}{a_i} \le \frac{1}{\log n} S.$$

Therefore

$$0 \le \lim_{n\to\infty} \frac{1}{\log n} \sum_{a_i \le n} \frac{1}{a_i} \le \lim_{n\to\infty} \frac{1}{\log n} S.$$

So,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{a_i \le n} \frac{1}{a_i} = 0.$$

This ends the proof. ■

Example 1.2 *Let*

$$\mathcal{A} = \left\{k^3, \quad k \ge 1\right\}.$$

Then,

$$\begin{aligned} \delta\mathcal{A} &= \lim_{n\to\infty} \frac{1}{\log n} \sum_{a_k \le n} \frac{1}{a_k} \\ &= \lim_{n\to\infty} \frac{1}{\log n} \sum_{k^3 \le n} \frac{1}{k^3}. \end{aligned}$$

It is known that the series $\sum_{k^3 \leq n} \frac{1}{k^3}$ *converges.*
Hence,

$$\delta\mathcal{A} = 0.$$

Lemma 1.1

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} = 1.$$

Proof. For n be a positive integer we have,

$$\int_1^n \frac{1}{x} dx = \log n.$$

We may then consider the sums $\sum_{k=2}^{n} \frac{1}{k}$ and $\sum_{k=1}^{n-1} \frac{1}{k}$ as being lower and upper Riemann sums respectively, of the function $x \mapsto \frac{1}{x}$ where $x \in [1,\, n]$.
Hence,

$$\sum_{k=2}^{n} \frac{1}{k} \leq \int_1^n \frac{1}{x} dx \leq \sum_{k=1}^{n-1} \frac{1}{k},$$

then

$$\sum_{k=2}^{n} \frac{1}{k} \leq \log n \leq \sum_{k=1}^{n-1} \frac{1}{k},$$

which implies

$$\frac{1}{\log n} \sum_{k=2}^{n} \frac{1}{k} \leq 1 \leq \frac{1}{\log n} \sum_{k=1}^{n-1} \frac{1}{k},$$

so,

$$\frac{1}{\log n} \sum_{k=2}^{n} \frac{1}{k} + \frac{1}{\log n} - \frac{1}{\log n} \leq 1 \leq \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k},$$

therefore

$$\frac{1}{\log n} \left(\sum_{k=2}^{n} \frac{1}{k} + 1 \right) - \frac{1}{\log n} \leq 1 \leq \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k},$$

so,

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{\log n} \leq 1 \leq \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k},$$

Thus,

$$-\frac{1}{\log n} \leq 1 - \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \leq 0.$$

Since $\frac{1}{\log n} > 0$, then we can write

$$-\frac{1}{\log n} \leq 1 - \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \leq 0 < \frac{1}{\log n}.$$

So,

$$\left|1 - \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k}\right| < \frac{1}{\log n},$$

then $\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} = 1$. ∎

1.4 Set of multiples

Definition 1.7 *Let $g(a)$ represent the greatest prime divisor of the positive integer a, and let $\mathcal{P}_a$ represent the set of naturel numbers n such that, the prime divisors of n are greater than $g(a)$.*
The set

$$a\mathcal{P}_a = \{ax,\ x \in \mathcal{P}_a\}$$

is called the set of higher multiples of a.

Lemma 1.2 *Let a and b be two positive integers. If $a \nmid b$ then,*

$$a\mathcal{P}_a \bigcap b\mathcal{P}_b = \phi.$$

Proof. Let a and b be two positive integers such that $p(a) \leq p(b)$ and $a\mathcal{P}_a \bigcap b\mathcal{P}_b = \phi$. Then there exists two integers n_a and n_b with $(n_a,\ a) = 1$ and $(n_b,\ b) = 1$, such that

$$an_a = bn_b,$$

and since $p(a) \leq p(b)$, we have

$$a \mid bn_b,$$

therefore

$$(n_b,\ a) = 1.$$

So, by Gauss's lemma we have $a \mid b$ which is contradictory with the fact that $a \nmid b$. ∎

Theorem 1.7 *The set $\mathcal{P}_a$ possesses asymptotic density and*

$$d\mathcal{P}_a = \frac{\varphi(m)}{m},$$

where $m = p_1 p_2 ... p_r$ and $p_1,\ p_2, ...,\ p_r$ do not exceed $p(a)$.

Proof. Since $\mathcal{P}_a = \{n : (n, m) = 1\}$, then there are exactly $\varphi(m)$ elements of $\mathcal{P}_a$ in the interval $(0, m]$, but in any intrval $((k-1)m, km]$ there are exactly $\varphi(m)$ elements of $\mathcal{P}_a$, since $\mathcal{P}_a(n) \geq \varphi(m)$ for a geven n, so there exists an integers k and t such that

$$\mathcal{P}_a(n) = k\varphi(m) + t,$$

where

$$0 \leq t < \varphi(m)$$

and

$$km < n \leq (k+1)m,$$

then

$$\frac{\mathcal{P}_a(n)}{n} = \frac{k\varphi(m) + t}{n},$$

and

$$\frac{1}{(k+1)m} \leq \frac{1}{n} \leq \frac{1}{km}$$

hence

$$\frac{k\varphi(m)}{(k+1)m} \leq \frac{k\varphi(m) + t}{(k+1)m} \leq \frac{k\varphi(m) + t}{n} \leq \frac{k\varphi(m) + t}{km},$$

since

$$\frac{k\varphi(m) + t}{km} < \frac{k\varphi(m) + \varphi(m)}{km} = \frac{(k+1)\varphi(m)}{km}.$$

Then,

$$\frac{k\varphi(m)}{(k+1)m} \leq \frac{\mathcal{P}_a(n)}{n} \leq \frac{(k+1)\varphi(m)}{km},$$

by taking the limit as $k \to \infty$ and since $km < n \leq (k+1)m$ we get

$$\lim_{n\to\infty} \frac{k\varphi(m)}{(k+1)m} = \frac{\varphi(m)}{m},$$

and

$$\lim_{n\to\infty} \frac{(k+1)\varphi(m)}{km} = (m).$$

Then,

$$\frac{\varphi(m)}{m} \leq \lim_{n\to\infty} \frac{\mathcal{P}_a(n)}{n} \leq \frac{\varphi(m)}{m}.$$

Thus, $\lim_{n\to\infty} \frac{\mathcal{P}_a(n)}{n}$ exists and equal to $\frac{\varphi(m)}{m}$. ■

1.5 Primitive sequences

Definition 1.8 *A sequence $\mathcal{A} = \{a_n\}_{n\geq 1}$ of positive integers is said to be primitive if no element of $\mathcal{A}$ divided any others.*

Particular primitive sequences

1) Let $\mathcal{A}$ be a primitive sequence, then the following sets are primitive,

$$\begin{aligned} \mathcal{A}_m &= \{a : a \in \mathcal{A}, \text{the prime factors of } a \text{ are } \geq p_m\}, \\ \mathcal{A}'_m &= \{a : a \in \mathcal{A}_m, \, p_m / a\}, \\ \mathcal{A}''_m &= \{a/p_m : a \in \mathcal{A}'_m\}. \end{aligned}$$

2) The set $\mathbb{N}_k = \{n : \Omega(n) = k\}$, where $\Omega(n)$ is the number of prime factors of n counted with repetition.

Definition 1.9 *Let $\mathcal{A}$ be a primitive sequence and let $\mathcal{A}_m$ be defined above. We call $\mathcal{A}_m$ is homogenous if for each m there is some integer s_m such that either $\mathcal{A}_m = \phi$ or $\Omega(a) = s_m$ for any $a \in \mathcal{A}_m$.*

Density of primitive sequence

Theorem 1.8 *Let $\mathcal{A}$ be an infinite primitive sequence, then for any $n \geq 1$,*

$$\mathcal{A}(2n) \leq n.$$

Proof. Suppose that $\mathcal{A}$ countained $n+1$ element that do not exted $2n$. We can write these elements under the form $a_i = 2^{\alpha_i} b_i$ where b_i is the greatest odd divisor of a_i for $i = 1, 2, ..., n+1$ and $\alpha_i \geq 0$.
Since b_i has at most n different values, then two of the integers $b_1, ..., b_{n+1}$ must be equals. So, there exists i and j such that

$$b_i = b_j, \, 1 \leq i \leq j \leq n+1,$$

this implies that

$$a_i \mid a_j \text{ or } a_j \mid a_i,$$

which contradictory with the fact that $\mathcal{A}$ is primitive sequence. ■

Theorem 1.9 *If $\mathcal{A}$ is an infinite primitive sequence, then $d\mathcal{A} \leq \frac{1}{2}$.*

Proof. From last teorem we have
1) If n is an even integer $(n = 2k)$, then

$$\frac{\mathcal{A}(2k)}{2k} \leq \frac{k}{2k} = \frac{1}{2},$$

hence

$$\underline{d}\mathcal{A} \leq \frac{1}{2}.$$

2) If n is odd $(n = 2k + 1)$ then, since $\mathcal{A}(2(k+1)) \leq k + 1$, we have

$$\mathcal{A}(n) \leq \frac{n+1}{2},$$

so,

$$\frac{\mathcal{A}(n)}{n} \leq \frac{1}{2} + \frac{1}{2n}.$$

Therefore,

$$\begin{aligned}
\underline{d}\mathcal{A} &= \limsup_{n\longrightarrow\infty} \frac{\mathcal{A}(n)}{n} \\
&\leq \limsup_{n\longrightarrow\infty} \frac{1}{2} + \frac{1}{2n} \\
&\leq \limsup_{n\longrightarrow\infty} \frac{1}{2} + \limsup_{n\longrightarrow\infty} \frac{1}{2n} = \frac{1}{2}.
\end{aligned}$$

Which ends the proof. ■

Lemma 1.3 [8] *Let p be a prime number and $x \geq 2$ a real number, then*

$$\log x < \prod_{p\leq x} \left(1 - \frac{1}{p}\right)^{-1} < M \log x,$$

where M is a positive constant.

Theorem 1.10 *(Erdős). If A is infinite primitive sequence, then the series*

$$\sum_{i=1}^{\infty} \frac{1}{a_i \log a_i}$$

converges.

Proof. For $i \geq 1$, we denote by $p(a_k)$ the greatest prime factor of $a_k \in \mathcal{A}$, we shall prove that

$$\sum_{k=1}^{\infty} \frac{1}{a_k} \prod_{p\leq p_k} \left(1 - \frac{1}{p}\right) \leq 1. \tag{4.1}$$

Let a_i, a_j be integers of $\mathcal{A}$, since for all $i \neq j$, $a_i \nmid a_j$ then by lemma 1.2, we have

$$a_i\mathcal{P}_{a_i} \bigcap a_j\mathcal{P}_{a_j} = \phi,$$

and for any $n \geq 1$, we have

$$da_n\mathcal{P}_{a_n} = \frac{1}{a_n} \prod_{p \leq p(a_n)} \left(1 - \frac{1}{p}\right),$$

then

$$\sum_{n=1}^{\infty} da_n\mathcal{P}_{a_n} = \sum_{n=1}^{\infty} \frac{1}{a_n} \prod_{p \leq p(a_n)} \left(1 - \frac{1}{p}\right),$$

since the sets $a_n\mathcal{P}_{a_n}$ are disjoints then,

$$\sum_{n=1}^{\infty} da_n\mathcal{P}_{a_n} = d\left(\bigcup_{n \geq 1} a_n\mathcal{P}_{a_n}\right),$$

then from the lemma 1.3, we have

$$\log p\left(a_n\right) \leq \prod_{p \leq p(a_n)} \left(1 - \frac{1}{p}\right)^{-1} < M \log p\left(a_n\right),$$

then

$$a_n \log p\left(a_n\right) \leq a_n \prod_{p \leq p(a_n)} \left(1 - \frac{1}{p}\right)^{-1} < M a_n \log p\left(a_n\right),$$

therefore

$$\frac{1}{M a_n \log p\left(a_n\right)} < \frac{1}{a_n} \prod_{p \leq p(a_n)} \left(1 - \frac{1}{p}\right) \leq \frac{1}{a_n \log p\left(a_n\right)},$$

hence

$$\frac{1}{a_n \log p\left(a_n\right)} < \frac{M}{a_n} \prod_{p \leq p(a_n)} \left(1 - \frac{1}{p}\right) \leq \frac{M}{a_n \log p\left(a_n\right)}.$$

So,

$$\sum_{n=1}^{\infty} \frac{1}{a_n \log p\left(a_n\right)} < M \sum_{n=1}^{\infty} \frac{1}{a_n} \prod_{p \leq p(a_n)} \left(1 - \frac{1}{p}\right).$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{a_n} \prod_{p \leq p(a_n)} \left(1 - \frac{1}{p}\right) \leq 1,$$

then, we have

$$\sum_{n=1}^{\infty} \frac{1}{a_n \log p\left(a_n\right)} < M \sum_{n=1}^{\infty} \frac{1}{a_n} \prod_{p \leq p(a_n)} \left(1 - \frac{1}{p}\right) \leq M.$$

In other hand, we have $a_n \geq p(a_n)$, then

$$a_n \log a_n \geq a_n \log p(a_n),$$

so,

$$\frac{1}{a_n \log a_n} \leq \frac{1}{a_n \log p(a_n)},$$

thus,

$$\sum_{n=1}^{\infty} \frac{1}{a_n \log a_n} \leq \sum_{n=1}^{\infty} \frac{1}{a_n \log p(a_n)} \leq M.$$

This ends the proof ■

Theorem 1.11 *For any primitive sequence $\mathcal{A}$ we have $\underline{d}\mathcal{A} = \delta\mathcal{A} = 0$.*

Proof. Let $\mathcal{A} = \{a_i\}_{i\geq 1}$ be a primitive sequence. According to theorem 1.5, it suffices to prove $\delta\mathcal{A} = 0$. For $i = 1, 2, ...$, we have $a_i < n$, then $\log a_i < \log n$, therefore

$$\frac{1}{\log n} < \frac{1}{\log a_i},$$

so,

$$\frac{1}{\log n} \sum_{1<a_i\leq n} \frac{1}{a_i} \leq \sum_{1<a_i\leq n} \frac{1}{a_i \log a_i},$$

and

$$\frac{1}{\log n} \sum_{a_i=N+1}^{n} \frac{1}{a_i} \leq \sum_{N<a_i\leq n} \frac{1}{a_i \log a_i} \leq \sum_{i=1}^{\infty} \frac{1}{a_i \log a_i},$$

then by the last theorem, the sum $\frac{1}{\log n} \sum_{a_i=N+1}^{n} \frac{1}{a_i}$ is converges.

But we have

$$\frac{1}{\log n} \sum_{1<a_i\leq n} \frac{1}{a_i} = \frac{1}{\log n} \sum_{1<a_i\leq N} \frac{1}{a_i} + \frac{1}{\log n} \sum_{N<a_i\leq n} \frac{1}{a_i}.$$

So by taking $\lim_{n\to\infty}$ we obtain $\delta\mathcal{A} = 0$,
and since $0 \leq \underline{d}\mathcal{A} \leq \delta\mathcal{A} = 0$, then

$$\underline{d}\mathcal{A} \leq \delta\mathcal{A} = 0.$$

This ends the proof. ■

Chapter 2

Erdős's conjecture for particular cases

In this chapter we will present the proof of Erdős's conjecture for homogeneous primitive sequences, and we will also investigate further the case where the primitive sequences have degree less than or equal to four.

2.1 Erdős's conjecture for the homogeneous primitive sequences

Throughout this section, we denote by $p(a)$ the last prime factor of a and $\mathcal{A}(p)$ the set of elements $a \in \mathcal{A}$ with $p(a) = p$.

Lemmas

Lemma 2.1 [6,7] *we have*

$$\begin{aligned} p_n &> n \log n \quad \textit{for} \quad n \geq 1, \\ p_n &> n(\log n + \log\log n) \quad \textit{for } n \geq 6. \end{aligned}$$

Lemma 2.2 *For any integer $m \geq 2$, we have*

$$\sum_{i>m} \frac{1}{p_i \log(i-1)} < \frac{1}{\log m}.$$

Proof. Note that for each $i \geq 3$, we have

$$\begin{aligned} \frac{1}{i \log i \log(i-1)} &< \frac{\log(i/(i-1))}{\log i \log(i-1)} \\ &= \frac{1}{\log(i-1)} - \frac{1}{\log i}. \end{aligned}$$

If we put $h(m) = \sum_{i>m} \frac{1}{p_i \log(i-1)}$, then from lemma 2.1 we have

$$\begin{aligned} h(m) &< \sum_{i>m} \frac{1}{i \log i \log(i-1)} \\ &< \frac{1}{\log(i-1)} - \frac{1}{\log i} \\ &= \frac{1}{\log m}. \end{aligned}$$

■

In the following, we define $i(a) = i$ if the largest prime factor of a is p_i.

Lemma 2.3 *For $m \geq 2$, $s \geq 1$, we have*

$$\sum_{p(a)>p_m,\ \Omega(a)=s} \frac{1}{a \log(i(a)-1)} \leq h(m) < \frac{1}{\log m}.$$

Proof. We proceed by induction on s.
If $s = 1$, then this is just lemma 2.2. Assume the lemma for s.
For the $s+1$ case, by lemma 2.2, we have

$$\begin{aligned} & \sum_{p(a)>p_m,\ \Omega(a)=s} \frac{1}{a \log(i(a)-1)} \\ = & \sum_{p(b)>p_m,\ \Omega(a)=s} \frac{1}{b} \sum_{j \geq i(b)} \frac{1}{p_j \log(j-1)} \\ < & \sum_{p(b)>p_m,\ \Omega(b)=s} \frac{1}{b \log(i(b)-1)} \leq h(m) < \frac{1}{\log m}. \end{aligned}$$

The proof is finished. ■

Lemma 2.4 *For $i \geq 2$ and $B \geq 2$, we have*

$$\begin{aligned} \sum_{j>i} \frac{1}{p_j \log(Bp_j)} &< \frac{\log\left(1 + \frac{\log B}{\log i}\right)}{\log B} \\ &\leq \min\left\{\frac{1}{\log i}, \frac{1}{e \log i} + \frac{1}{e \log B}\right\}, \end{aligned}$$

where $e = 2.718...$ is the base of the natural logarithms.

Proof. By lemma 2.1, we have,

$$\begin{aligned} \sum_{j>i} \frac{1}{p_j \log(Bp_j)} &< \int_i^\infty \frac{dx}{x \log x \log(Bx)} \\ &= \frac{\log(1 + \log B/\log i)}{\log B} \\ &\leq \min\left\{\frac{1}{\log i}, \frac{1}{e \log i}, \frac{1}{e \log B}\right\}, \end{aligned}$$

then

$$\sum_{j>i} \frac{1}{p_j \log(Bp_j)} \le \min\left\{\frac{1}{\log i}, \frac{1}{e\log i}, \frac{1}{e\log B}\right\}.$$

Observing that the inequality

$$\frac{\log(1+\log B/\log i)}{\log B} \le \min\left\{\frac{1}{\log i}, \frac{1}{e\log i}, \frac{1}{e\log B}\right\},$$

follows from $\log(1+x) < x$ and

$$\log x = 1 + \log\left(1 + \frac{(x-e)}{e}\right) \le \frac{x}{e},$$

for all $x > 0$. ■

Lemma 2.5 *For $m \ge 2$, $B \ge 2$, $s \ge 2$, we have*

$$\sum_{p(u)>p_m,\ \Omega(u)=s} \frac{1}{u\log(Bu)} < \left(e^{-1} + ... + e^{1-s}\right) h(m) + e^{1-s} \sum_{j>m} \frac{1}{p_j \log(Bp_j)}.$$

Proof. We proceed by induction on s.
If $s = 2$, then by lemma 2.4, we have

$$\begin{aligned}\sum_{p(u)>p_m,\ \Omega(u)=s} \frac{1}{u\log(Bu)} &= \sum_{j>m} \frac{1}{p_j} \sum_{k\ge j} \frac{1}{p_k \log(Bp_jp_k)} \\ &< e^{-1}h(m) + e^{-1} \sum_{j>m} \frac{1}{p_j \log(Bp_j)}.\end{aligned}$$

For the $s+1$ case, by lemma 2.3, lemma 2.4 and using s case, we have,

$$\begin{aligned}\sum_{p(u)>p_m,\ \Omega(u)=s+1} \frac{1}{u\log(Bu)} &= \sum_{p(u)>p_m,\ \Omega(u)=s} \frac{1}{b} \sum_{j\ge i(b)} \frac{1}{p_j \log(Bbp_j)} \\ &< \sum_{p(u)>p_m,\ \Omega(u)=s} \frac{e^{-1}}{b} \left(\frac{1}{\log(i(b)-1)} + \frac{1}{\log(Bb)}\right) \\ &\quad \left(e^{-1} + ... + e^{-s}\right) h(m) + e^{-s} \sum_{j>m} \frac{1}{p_j \log(Bp_j)}.\end{aligned}$$

And the proof is finishes. ■

Lemma 2.6 *Let a, m and s an integers such that $m \ge 5$ and $s \ge 2$ then, we have*

$$\sum_{\Omega(a)=s-1,\ p(a)>p_{m+1}} \frac{1}{a\log(ap_{m+1})} < \frac{1}{\log p_{m+1}}.$$

Proof. Put
$$w\left(s,\,m\right)=\sum_{\Omega(a)=s-1,\;p(a)>p_{m+1}}\frac{1}{a\log\left(ap_{m+1}\right)},$$
then by lemma 2.2, lemma 2.4 and lemma 2.5, we have
$$w\left(s,\,m\right)<W\left(s,\,m\right),$$
where
$$W\left(s,\,m\right)=\frac{e^{-1}+...+e^{1-S}}{\log m}+\frac{e^{1-S}}{\log p_{m+1}}.$$
Using lemma 2.1, we obtain
$$\begin{aligned}\frac{\log p_{m+1}}{\log m} &\le \frac{\log\left(m+1\right)+\log\left(\log\left(m+1\right)+\log\log\left(m+1\right)\right)}{\log m}\\ &\le \frac{\log 6+\log\left(\log 6+\log\log 6\right)}{\log 5}=1.65...<e-1.\end{aligned}$$
So, for $m\ge 5$ and $s\ge 2$,
$$W\left(s,\,m\right)-W\left(s+1,\,m\right)=e^{-S}\left(\frac{e-1}{\log p_{m+1}}-\frac{1}{\log m}\right)>0.$$
Therefore,
$$\begin{aligned}w\left(s,\,m\right) &< W\left(s,\,m\right)\le W\left(2,\,m\right)=\frac{1}{e\log m}+\frac{1}{e\log p_{m+1}}\\ &< \frac{e-1}{e\log p_{m+1}}+\frac{1}{e\log p_{m+1}}\\ &= \frac{1}{\log p_{m+1}},\end{aligned}$$
so, $w\left(s,\,m\right)<\frac{1}{\log p_{m+1}}$. ■

Lemma 2.7 *For any integer $m\le 4$, we have*
$$w\left(2,\,m\right)=\sum_{\Omega(a)=1,\;p(a)>p_{m+1}}\frac{1}{a\log\left(ap_{m+1}\right)}<\frac{1}{\log p_{m+1}}.$$

Proof. If $0\le m\le 4$ then by lemma 2.4, we have
$$w\left(2,\,m\right)<w\left(m\right)$$
where
$$w\left(m\right)=\frac{1}{p_{m+1}\log p_{m+1}^{2}}+\frac{1}{p_{m+2}\log\left(p_{m+1}p_{m+2}\right)}+\frac{1}{\log p_{m+1}}\log\left(1+\frac{\log p_{m+1}}{\log\left(m+2\right)}\right),$$

and

$$w(0) = \frac{1}{2\log 4} + \frac{1}{3\log 6} + \frac{1}{5\log 10} + \frac{1}{\log 2}\log\left(1 + \frac{\log 2}{\log 3}\right).$$

By calculation we have

m	$w(m)$	p_{m+1}	$1/\log p_{m+1}$
4	0.388	11	0.417
3	0.464	7	0.513
2	0.581	5	0.621
1	0.856	3	0.910
0			

Thus, $w(2, m) < w(m) < \frac{1}{\log p_{m+1}}$ for $0 \le m \le 4$. ■

Lemma 2.8 *Let s and m an integers such that $s \ge 3$, $2 \le m \le 4$, then*

$$w(s, m) = \sum_{\Omega(a)=s-1,\ p(a)>p_{m+1}} \frac{1}{a\log(ap_{m+1})} < \frac{1}{\log p_{m+1}}.$$

Proof. Let m be fixed integer, put

$$\gamma_s = \left(e^{-1} + \ldots + e^{2-S}\right) h(m) + e^{2-S} w(m),$$

where $w(m)$ is the upper bound of $w(s, m)$, defined in the proof of lemma 2.7. Then by lemma 2.5 we have for $s \ge 3$ that

$$w(s, m) = \left(e^{-1} + \ldots + e^{2-S}\right) h(m) + e^{2-S} w(2, m) < \gamma_s.$$

If $h(m) < (e-1) w(m)$ and $m \le 4$, then by lemma 2.2 we have

$$h(4) = \sum_{i=5}^{10} \frac{1}{p_i \log(i-1)} + h(10) < 0.6442,$$

but $h(10) < \frac{1}{\log 10}$, thus

$$\frac{h(4)}{w(4)} < 1.7 < e - 1,$$

so that the case $m = 4$ is verified.

For $m = 2$, since

$$h(2) = \frac{1}{5\log 2} + h(3) < 1.063,$$

we use the upper bound $H = 1.063$ for $h(2)$, we see that

$$\frac{H}{w(2)} > e - 1.$$

However, we then have

$$\gamma_s < \left(e^{-1} + ... + e^{2-S}\right) H + e^{2-s}\frac{H}{e-1} = \frac{H}{e-1} < 0.62 < \frac{1}{\log 5},$$

so, the case $m = 2$ is also done. ■

Lemma 2.9 [16] *We have for $s \geq 3$ and $0 \leq m \leq 4$*

$$\begin{aligned} w(s,\ 1) &< \frac{1}{\log p_2}, \\ w(s,\ 0) &< \frac{1}{\log 2}, \\ w(s,\ m) &< \frac{1}{\log p_{m+1}}. \end{aligned}$$

Erdős's conjecture and homogeneous primitive sequences

A primitive sequence $\mathcal{A}$ is called homogeneous if $\Omega(a) = c$ (constant) for all $a \in \mathcal{A}$.

Theorem 2.1 *Let $\mathcal{A}$ be a primitive sequence such that $\mathcal{A}(p)$ is homogeneous, then for $n > 1$ the inequality*

$$\sum_{a \in \mathcal{A},\ a \leq n} \frac{1}{a \log a} \leq \sum_{p\ prime,\ p \leq n} \frac{1}{p \log p}$$

is true.

Proof. According to the lemma 2.7, we have for a given prime p, if $\mathcal{B} = \mathcal{B}(p)$ is homogeneous and nonempty, then

$$\sum_{b \in \mathcal{B}} \frac{1}{a \log a} \leq \frac{1}{p \log p},$$

and this implies the theorem. ■

2.2 Erdős's conjecture for primitive sequences of degree less than or equal four

Lemmas

Lemma 2.10 *Let $n > 1$ be an interger, if we put*

$$F(n) = \log n + \log\log n - 1,$$

then we have

$$\begin{aligned} p_n &\geq nF(n) \ \text{for } n \geq 2, \ [2] & (2.1)\\ p_n &\geq n(\log(nF(n)) - \alpha) \ \text{for } n \geq 3, & (2.2)\\ p_n &\leq n(F(n) + \beta) \ \text{for } n \geq 95, & (2.3) \end{aligned}$$

where $\alpha = 1.127$ *and* $\beta = 0.305$.

Proof. Let g be the function defined on $\mathbb{N}$ by

$$n \mapsto g(n) = \frac{p_n}{n} - \log(nF(n)) \ \text{ for } n \geq 3$$

then, according to (2.1) we have $g(n) \geq h(n)$ where

$$h(n) = -1 - \log\left(1 + \frac{\log\log n - 1}{\log n}\right).$$

the stady of the real function $x \mapsto h(x)$ $(x \geq 3)$ gives us

$$h(x) \geq h(\exp(\exp 2)) > -\alpha,$$

then $g(n) > -\alpha$, which is equivalent to

$$p_n \geq n(\log(nF(n)) - \alpha) \ \text{ for } n \geq 3.$$

A computer caculation shows that for $95 \leq n < 7022$ we have

$$p_n \leq n(F(n) + \beta),$$

and on other hand, we have

$$p_n \leq n(\log n + \log\log n - 0.9385) \ \text{ for } n \geq 7022,$$

therefore the inequality (2.3) is verified for $n \geq 95$. This completes the proof. ■

Lemma 2.11 *For* $m \geq 1$ *and* $j \in \{1, 2, 3\}$, *we have*

$$\sum_{i \geq \max(m,\ j-1)} \frac{1}{p_i(k_j + \log p_i)} < \frac{1}{k_{j-1} + \log p_m}$$

where $k_0 = 0.023$, $k_1 = 0.3157$, $k_2 = 0.901$ *and* $k_3 = 2.079$.

Proof. Put $N = 95$, $C = 0.0713$,

$$\begin{array}{lll} u_1 = 0.09435, & u_2 = 0.387, & u_3 = 0.9723 \\ v_1 = 0, & v_2 = 0, & v_3 = -0.0074. \end{array}$$

It is clear that for $m \geq N$ and $j \in \{1, 2, 3\}$ we have

$$\max(m, j-1) = m,$$

and

$$\begin{aligned} C &\geq -\log(F(m)) + \log\left(1 + \frac{1}{m}\right) + \log(F(m+1) + \beta), \\ C &\leq u_j - k_{j-1}, \end{aligned} \tag{2.4}$$

and

$$v_j = \alpha - k_j + 2u_j - 1.$$

Put

$$h_j(m) = \sum_{i \geq \max(m, j-1)} \frac{1}{p_i(k_j + \log p_i)}$$

By (2.1) et (2.2) we have, for $m \geq N$ and $j \in \{1, 2, 3\}$,

$$p_i(k_j + \log p_i) > i(\log(iF(i)) - \alpha)(k_j + \log(iF(i))),$$

Since the function

$$x \mapsto \log(xF(x))$$

increases for $x > N$, we have

$$h_j(m+1) < \int_m^\infty \frac{dt}{t(\log(tF(t)) - \alpha)(\log(tF(t)) + k_j)},$$

using the change of variable $x = \log t$ give us

$$h_j(m+1) < \int_{\log m}^\infty \frac{dx}{(L(x) - \alpha)(L(x) + k_j)},$$

where

$$L(x) = \log(e^x F(e^x)),$$

then we have $1 = \frac{1}{L'(x)} \times L'(x)$ for $x > 1$, and since

$$L'(x) = \frac{1}{x} \frac{x \log x + x^2 + 1}{x + \log x - 1} > 0,$$

we also have,

$$\frac{1}{L'(x)} = 1 - \frac{1+x}{x\log x + x^2 + 1}.$$

But on other hand, for $x > \log N$,

$$(1+x)(L(x)-1) - \left(x\log x + x^2 + 1\right) = \log(x + \log x - 1) + x\log(1 + \frac{\log x - 1}{x}) - 2 > 0,$$

so,

$$\frac{1}{L'(x)} < 1 - \frac{1}{L(x)-1}$$

then

$$\frac{1}{L'(x)} < \left(1 - \frac{1}{L(x)-1}\right) \text{ for } x > \log N,$$

so,

$$\begin{aligned} h_j(m+1) &< \int_{\log m}^{\infty} \frac{\frac{1}{L'(x)} L'(x)\,dx}{(L(x)-\alpha)(L(x)+k_j)} \\ &< \int_{\log m}^{\infty} \frac{\left(1 - \frac{1}{L(x)-1}\right) L'(x)\,dx}{(L(x)-\alpha)(L(x)+k_j)}. \end{aligned}$$

By setting $y = L(x)$ and $y_m = L(\log m)$, we obtain

$$h_j(m+1) < \int_{y_m}^{\infty} \frac{(y-2)\,dy}{(y-1)(y-\alpha)(y+k_j)}.$$

Now, for $j \in \{1,\ 2,\ 3\}$ we put

$$g_j(m) = \frac{1}{\log p_m + k_{j-1}},$$

then according to (2.3) and (2.4) we have

$$\begin{aligned} g_j(m+1) &\geq \frac{1}{k_{j-1} + \log((m+1)(F(m+1)+\beta))} \\ &> \frac{1}{\log(mF(m)) + u_j} \\ &= \int_{y_m}^{\infty} \frac{dy}{(y+u_j)^2}. \end{aligned}$$

For $j \in \{1,\ 2,\ 3\}$, we put

$$\Delta_j = (y-2)(y+u_j)^2 - (y-1)(y-\alpha)(y+k_j),$$

then, for $y > y_m$ and $j \in \{1, 2, 3\}$

$$\Delta_j = v_j y^2 + \left(u_j^2 - 4u_j - \alpha + k_j + \alpha k_j\right) y - \left(2u_j^2 + \alpha k_j\right) < 0.$$

So, for $y > y_m$ and $j \in \{1, 2, 3\}$ we have

$$\frac{(y-2)}{(y-1)(y-\alpha)(y+k_j)} < \frac{1}{(y+u_j)^2},$$

thus

$$h_j(m+1) < \frac{dt}{\log m + \log(\log m + \log\log m - 1) + u_j},$$

according to (2.3) and (2.4) we have

$$\begin{aligned} & \log p_{m+1} + k_{j-1} - \log m - \log(\log m + \log\log m - 1) - u_j \\ = \ & \log\left(1 + \frac{1}{m}\right) + \log\left(\frac{\log(m+1) + \log\log(m+1) - 0.7}{\log m + \log\log m - 1}\right) - u_j + k_{j-1} \\ \leq \ & C - -u_j + k_{j-1} \leq 0, \end{aligned}$$

then

$$g_j(m+1) \geq \frac{1}{\log m + \log(\log m + \log\log m - 1) + u_j}.$$

Thus, for $m \geq N$ and $j \in \{1, 2, 3\}$ we have $h_j(m+1) < g_j(m+1)$ i.e.

$$h_j(m) < g_j(m) \text{ for } m \geq N+1.$$

And for $1 \leq m \leq N$, a computer calculation gives

$$\begin{aligned} h_j(m) &= \sum_{i \geq m}^{N} \frac{1}{p_i(k_j + \log p_i)} + h_j(N+1) \\ &< \sum_{i \geq m}^{N} \frac{1}{p_i(k_j + \log p_i)} + \frac{1}{\log N + \log(\log N + \log\log N - 1) + u_j} \\ &< g_j(m). \end{aligned}$$

Which ends the proof. ■

Lemma 2.12 *Let* $m \geq 1$ *be fixed and let* $B = B_m$ *be primitive sequence with* $\deg(B) \leq 3$. *For* $1 \leq t \leq 4 - \deg(B)$, *we have*

$$\sum_{b \in B} \frac{1}{b(t\log p_m + \log b)} < \frac{1}{k_{t-1} + \log p_m} \text{ where } p_1^{4-t} \notin B_1, \quad (2.5)$$

$$\sum_{b \in B} \frac{1}{b(t\log p_m + \log b)} < \frac{1}{k_0 + \log p_m} \text{ where } p_1^3 \notin B_1. \quad (2.6)$$

Proof. For $m \geq 1$ and $1 \leq t \leq 4 - \deg(B)$, put

$$g_t(B) = \sum_{b \in B} \frac{1}{b(t \log p_m + \log b)} \text{ where } (g_t(\phi) = 0).$$

By induction on $\deg(B)$, if $\deg(B) = 1$ and $1 \leq t \leq 3$ we have $t \log p_m \geq t \log 2 > k_t$ and $p_1 \notin B_1$ when $t = 3$, so by lemma 2.11 we get

$$\begin{aligned} g_t(B) &= \sum_{b \in B} \frac{1}{b(t \log p_m + \log b)} \\ &< \sum_{i \geq \max(m,\, t-1)} \frac{1}{p_i(k_t + \log p_i)} \\ &< \frac{1}{k_{t-1} + \log p_m}. \end{aligned}$$

If $\deg(B) = s > 1$ and $1 \leq t \leq 4 - s$, we know that $B = \bigcup_{i \geq m} B'_i$ is disjoint, so

$$g_t(B) = \sum_{i \geq m} g_t(B'_i) \text{ where } p_1^{4-t} \notin B'_1.$$

We distinguish the folowing two cases:
1) If $\deg(B'_i) \leq 1$ then

$$g_t(B'_i) < \frac{1}{p_i(k_t + \log p_i)}. \tag{2.7}$$

2) If $\deg(B'_i) > 1$ then

$$\begin{aligned} g_t(B'_i) &= \sum_{b \in B''_i} \frac{1}{p_i b((t+1) \log p_i + \log b)} \\ &= \frac{1}{p_i} g_t(B''_i) \text{ where } p_1^{3-t} \notin B''_1, \end{aligned}$$

and since

$$\deg(B''_i) < s \text{ and } t + 1 \leq 4 - \deg(B''_i),$$

we have

$$g_{t+1}(B''_i) < \frac{1}{k_t + \log p_i} \text{ where } p_1^{4-(t+1)} \notin B''_1,$$

thus

$$g_t(B'_i) \leq \frac{1}{p_i(k_t + \log p_i)}. \tag{2.8}$$

So, from (2.7) et (2.8) and lemma 2.11 we obtain

$$g_t(B) < \frac{1}{k_{t-1} + \log p_m} \text{ where } p_1^{4-t} \notin B_1,$$

then for $t = 1$ we get the inequality (2.6), and the proof is finished. ■

Theorem of Zhang

Theorem 2.2 *For any primitive sequence $\mathcal{A}$ whose the number of the prime factors of its terms counted with multiplicity is at most 4, we have*

$$\sum_{a\in\mathcal{A},a\leq n}\frac{1}{a\log a}\leq\sum_{a\in\mathcal{P},a\leq n}\frac{1}{p\log p}\ \textit{for } n>1.$$

Proof. Let n be fixed and $A=\{a: a\in\mathcal{A},\ a\leq n\}$ a subsequence of $\mathcal{A}$ with $\deg(\mathcal{A})\leq 4$. Put $\pi(n)=m$, the number of primes $\leq n$ then $A=\bigcup_{1\leq i\leq m}A_i'$ is disjoint and

$$f(A)=\sum_{1\leq i\leq m}f(A_i').$$

Let $1\leq i\leq m$, we distinguish the following two cases:
1^{st} case: we suppose that $p_1^4\notin A$, i.e. , $p_1^3\notin A_1''$.
If $\deg(A_i')\leq 1$ then

$$f(A_i')\leq\frac{1}{p_i\log p_i},$$

and if $\deg(A_i')>1$ then

$$f(A_i')=\frac{1}{p_i}\sum_{b\in A_i''}\frac{1}{b(\log p_i+\log b)}\ \text{where } p_1^3\notin A_1'',$$

and

$$\deg(A_i'')\leq\deg(A_i')-1\leq 3.$$

So, according to (2.6) we get

$$\sum_{b\in A_i''}\frac{1}{b(\log p_i+\log b)}<\frac{1}{k_0+\log p_i}<\frac{1}{\log p_i}\ \text{where } p_1^3\notin A_1'',$$

therefore

$$f(A_i')\leq\frac{1}{p_i\log p_i}\ \text{for } 1\leq i\leq m. \tag{2.9}$$

2^{nd} case: if $p_1^4\in A$, since A is a primitive sequence then $p_1\notin A_1'$, so,

$$\deg\left(A_1'-\{p_1^4\}\right)\neq 1,$$

i.e.,

$$f\left(A_1'-\{p_1^4\}\right)<\frac{1}{p_1(k_0+\log p_1)},$$

thus

$$\begin{aligned} f\left(A_1'\right) &= f\left(\left\{p_1^4\right\}\right) + f\left(A_1' - \left\{p_1^4\right\}\right) \\ &= \frac{1}{p_1^4 \log p_1^4} + \frac{1}{p_1\left(k_0 + \log p_1\right)} \\ &< \frac{1}{p_1 \log p_1}, \end{aligned}$$

and from (2.9), we have

$$f\left(A_i'\right) \leq \frac{1}{p_i \log p_i} \text{ for } 2 \leq i \leq m,$$

then

$$f\left(A_i'\right) \leq \frac{1}{p_i \log p_i} \text{ for } 1 \leq i \leq m, \tag{2.10}$$

thus, by (2.9) and (2.10) we get

$$f(A) = \sum_{1 \leq i \leq m} f\left(A_i'\right) \leq \sum_{1 \leq i \leq m} \frac{1}{p_i \log p_i}.$$

This completes the proof. ■

Chapter 3

Principal results on the sum $S(\mathcal{A},\, x)$ and its relationship with Erdős's conjecture

In this chapter, we discuss the results obtained in [7], In particular, we improuved the value of x such that $S\left(\mathcal{A},\, x\right) \geq S\left(\mathcal{P},\, x\right)$. The authors in this paper studies only the case where primitive sequence have degree $d = 2$, we gives a generalization of this result for any degree d.

3.1 Some results on primitive sequences of the form $\mathcal{B}_d^k$

Lemmas

Lemma 3.1 [15] *For any real $x > 1$, we have*

$$\sum_{p\in\mathcal{P},\ p\leq x} \frac{1}{p} > \log\log x.$$

Lemma 3.2 *For any integer $n > 1$, we have*

$$2.5n^n e^{-n}\sqrt{n} < n! \leq n^{n-1}.$$

Proof. For $n = 2$, the inequality is verifed. For $n > 2$, we can use the inequality

$$n^n e^{-n}\sqrt{2\pi n} e^{\frac{1}{12n+1}} < n! \leq n^n e^{-n}\sqrt{2\pi n} e^{\frac{1}{12n}}.$$

See[13]. ∎

Lemma 3.3 *Let the real number $\epsilon > 1$ and the integer $n > 1$, then we have*

$$\inf_{n>1,\ \epsilon>1}\left(\frac{nn!e^{\epsilon n}}{\epsilon^{n-1}n^{n-1}-n!}\right)=4e^{3}.$$

Proof. For $n \geq 2$, we define for $\epsilon > 1$ the sequence $t_n(\epsilon)$ by

$$t_n(\epsilon)=\frac{nn!e^{\epsilon n}}{\epsilon^{n-1}n^{n-1}-n!}.$$

According to lemma 3.2**,** for $n \geq 2$, $\epsilon > 1$

$$\frac{\epsilon^{n-1}n^{n-1}-n!}{nn!}<\frac{\epsilon^{n-1}e^{n}-2.5n\sqrt{n}}{2.5n^{2}\sqrt{n}},$$

then

$$t_n(\epsilon)>\frac{2.5n^{2}\sqrt{n}e^{\epsilon n}}{\epsilon^{n-1}e^{n}-2.5n\sqrt{n}}.$$

Since, for $\epsilon > 1$ the real function

$$x\mapsto f_{\epsilon}(x)=\frac{2.5x^{2}\sqrt{x}e^{\epsilon x}}{e^{x+(x-1)\log\epsilon}-2.5x\sqrt{x}},$$

is increases on $[4, \infty)$, then for $n \geq 4$, we have

$$t_n(\epsilon)>f_{\epsilon}(x)\geq f_{\epsilon}(4),$$

and since $t_3(\epsilon) < f_{\epsilon}(4)$ for $\epsilon > 1$, we have

$$\begin{aligned}\inf_{n>1,\ \epsilon>1}\left(\frac{nn!e^{\epsilon n}}{\epsilon^{n-1}n^{n-1}-n!}\right) &= \inf_{n>1,\ \epsilon>1}\{t_3(\epsilon),\, t_2(\epsilon)\}\\ &= t_2\left(\frac{3}{2}\right)=4e^{3}.\end{aligned}$$

The proof is achieved. ■

Lemma 3.4 *For any integer $k \geq 1$ and any integer $d \geq 2$, we define*

$$\mathcal{A}_d^k=\{p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_k^{\alpha_k}:\alpha_1,\ldots,\alpha_k\in\mathbb{N},\ \alpha_1+\ldots+\alpha_k=d\}$$

then we have the disjoint union

$$\mathcal{A}_d^{k+1}=\mathcal{A}_d^{k}\bigcup\left\{ap_{_{k+1}}:a\in\mathcal{A}_{d-1}^{k+1}\right\}.$$

Proof. Let $y \in \mathcal{A}_d^{k+1}$ such that $p_{k+1} \mid y$. Then, $y = ap_{k+1}$ where $a \in \mathcal{A}_{d-1}^{k+1}$, and

$$\mathcal{A}_d^{k+1}=\left\{y\in\mathcal{A}_d^{k+1}\mid p_{k+1}\nmid y\right\}\bigcup\left\{y\in\mathcal{A}_d^{k+1}\mid p_{k+1}\mid y\right\},$$

therefore

$$\mathcal{A}_d^{k+1}=\mathcal{A}_d^{k}\bigcup\left\{ap_{k+1}\mid a\in\mathcal{A}_{d-1}^{k+1}\right\},$$

which is disjoint union. ■

Lemma 3.5 *Let $k_0 = 13674662$, then for any real number $x > 0$ and for any $k \geq k_0$ the sequence $S(\mathcal{B}_2^k,\, x)$ is strictly increases.*

Proof. For any integers $k \geq 1$, $d \geq 2$, the multinomial formula give us

$$\sum_{a \in \mathcal{A}_d^k} \frac{1}{a} \geq \frac{1}{d!} \left(\sum_{n=1}^{k} \frac{1}{p_n} \right)^d . \tag{3.1}$$

According to lemma 3.4, we have

$$\begin{aligned} \mathcal{B}_2^{k+1} &= \mathcal{A}_2^{k+1} \bigcup \mathcal{A}^{k+1} \\ &= \mathcal{A}_2^{k} \bigcup \left\{ a p_{k+1} \mid a \in \mathcal{A}_1^{k+1} \right\} \bigcup \mathcal{A}^{k+1}, \end{aligned}$$

then

$$S(\mathcal{B}_2^{k+1},\, x) = S(\mathcal{B}_2^{k},\, x) + E,$$

where

$$E = \frac{1}{p_{k+1}} \left(S(\mathcal{A}_1^{k+1},\ \log p_{k+1} + x) - \frac{1}{\log p_{k+1} + x} \right).$$

Since p_{k+1} is the largest element in $\mathcal{A}_1^{k+1}$, we have

$$\begin{aligned} S(\mathcal{A}_1^{k+1},\ \log p_{k+1} + x) &= \sum_{a \in \mathcal{A}_1^{k+1}} \frac{1}{a\,(\log a + \log p_{k+1} + x)} \\ &\geq \frac{1}{2 \log p_{k+1} + x} \sum_{a \in \mathcal{A}_1^{k+1}} \frac{1}{a}. \end{aligned}$$

and according to (3.1) and lemma 3.1 we obtain for $k \geq k_0$,

$$\begin{aligned} \sum_{a \in \mathcal{A}_1^{k+1}} \frac{1}{a} &\geq \sum_{n=1}^{k+1} \frac{1}{p_n} \\ &\geq \log\log p_{k+1} > 2. \end{aligned}$$

therefore

$$\begin{aligned} S(\mathcal{A}_1^{k+1},\ \log p_{k+1} + x) - \frac{1}{\log p_{k+1} + x} &> \frac{2}{2 \log p_{k+1} + x} - \frac{1}{\log p_{k+1} + x} \\ &= \frac{x}{(2 \log p_{k+1} + x)\,(\log p_{k+1} + x)} \\ &> 0. \end{aligned}$$

then

$$S(\mathcal{B}_2^{k+1},\, x) - S(\mathcal{B}_2^{k+1},\, x) > 0.$$

This ends the proof. ■

Improved result over B_d^k

Theorem 3.1 *Let $k_0 = 13674662$ and $x_0 = 80.4$. Then for any primitive sequence*

$$\mathcal{B}_2^k = \{p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k};\ \alpha_1, \ldots, \alpha_k \in \mathbb{N},\ \alpha_1 + \ldots + \alpha_k = 2,\ k \geq k_0\} \bigcup \{p_n \in \mathcal{P} \mid n > k\},$$

we have

$$S(\mathcal{B}_2^k,\ x) > S(\mathcal{P},\ x) \text{ for } x \geq x_0.$$

Proof. For any natural numbers $k \geq 1$ and $d \geq 2$, p_k^d is the greatest element of the primitive sequence $\mathcal{A}_d^k$, then $\log a \leq d \log p_k$ for any $a \in \mathcal{A}_d^k$. So, for any $x > 0$ we have

$$\begin{aligned}
\sum_{a \in \mathcal{B}_d^k} \frac{1}{a(\log a + x)} &= \sum_{a \in \mathcal{A}_d^k \cup \mathcal{A}^k} \frac{1}{a(\log a + x)} \\
&= \sum_{a \in \mathcal{A}_d^k} \frac{1}{a(\log a + x)} + \sum_{a \in \mathcal{A}^k} \frac{1}{a(\log a + x)} \\
&\geq \frac{1}{d \log p_k + x} \sum_{a \in \mathcal{A}_d^k} \frac{1}{a} + \sum_{n > k} \frac{1}{p_n(\log p_n + x)}.
\end{aligned}$$

from (3.1) and lemma 2.1, we have

$$\sum_{a \in \mathcal{A}_d^k} \frac{1}{a} > \frac{(\log \log p_k)^{d-1}}{d!} \sum_{n=1}^{k} \frac{1}{p_n},$$

then

$$\begin{aligned}
\sum_{a \in \mathcal{B}_d^k} \frac{1}{a(\log a + x)} &\geq \frac{x (\log \log p_k)^{d-1}}{d!\,(d \log p_k + x)} \sum_{n=1}^{k} \frac{1}{x p_n} + \sum_{n > k} \frac{1}{p_n(\log p_n + x)} \\
&> \frac{x (\log \log p_k)^{d-1}}{d!\,(d \log p_k + x)} \sum_{n=1}^{k} \frac{1}{p_n(\log p_n + x)} + \sum_{n > k} \frac{1}{p_n(\log p_n + x)}.
\end{aligned}$$

To obtain the inequality of theorem, we must choose d, k and x such that

$$\frac{x (\log \log p_k)^{d-1}}{d!\,(d \log p_k + x)} > 1,$$

it is clear that for $d \geq 2$ and $k > 1$, the function

$$x \mapsto h(x) = \frac{x (\log \log p_k)^{d-1}}{d!\,(d \log p_k + x)}$$

increases for $x > 0$.
Let x_0 be the smalest value such that the last inequality is verifed, then

$$\frac{(\log\log p_k)^{d-1} - d!}{dd!\log p_k} > \frac{1}{x_0}. \tag{3.2}$$

Since $x_0 > 0$, we must choose k such that $(\log\log p_k)^{d-1} - d!x > 0$, then according to lemma 3.2, it must be

$$\log\log p_k > d,$$

then there exist $\epsilon > 1$ such that $\log\log p_k = \epsilon d$. Therefore, (3.2) equivalent to

$$\frac{dd!e^{\epsilon d}}{\epsilon^{d-1}d^{d-1} - d!} < x_0,$$

so we must choose d and ϵ such that the number

$$\frac{dd!e^{\epsilon d}}{\epsilon^{d-1}d^{d-1} - d!}$$

is the smalest possible. According to lemma 3.3, we obtain $d = 2, \epsilon = 1.481$ and $x_0 > 4e^3$, then we must find an integer k_0 so that $\log\log p_{k_0}$ be in the neighborhood of 2.962, a computer calculation give us

$$(p_{k_0},\, k_0) = (249910007,\, 13674662)\,.$$

Then if we take $k_0 = 13674662$ and $d = 2$, we obtain $\mathcal{B}_2^{k_0}$ and $x_0 = 80.4$. So, according to lemma 3.5 we have

$$S(\mathcal{B}_2^k,\, x) > S(\mathcal{P},\, x) \text{ for } k > k_0,\, x \geq x_0.$$

The proof is achieved. ■

3.2 Generalized result on $S(\mathcal{B}_2^k,\, x)$ concerning primitive sequences of the form $\mathcal{B}_d^k$

Lemmas

Lemma 3.6 [4] *For $x \geq 3275$ there exists a prime number p such that*

$$x < p \leq x\left(1 + \frac{1}{2\ln^2 x}\right).$$

Lemma 3.7 *For any integer $n > 1$, we have*

$$n! \leq n^n e^{1-n}\sqrt{n}, \tag{3.3}$$
$$n! \leq 2(n+1)^{n-2}, \tag{3.4}$$
$$n! < n^{n-2} \quad (n \geq 5). \tag{3.5}$$

Proof. For $n = 2$, the inequality (3.3) is verifed. For $n > 2$, it is comes from the inequality [13]

$$n^n e^{-n}\sqrt{2\pi n}e^{\frac{1}{12n+1}} < n! < n^n e^{-n}\sqrt{2\pi n}e^{\frac{1}{12n}},$$

and we can prove (3.4) and (3.5) by induction. ■

Lemma 3.8 *Let $n \geq 2$ be an integer and x be a reel number such that $x \geq n-1$. The function*

$$x \mapsto f_n(x) = \frac{nn!e^x}{x^{n-1} - n!}$$

reaches its minimum in x_n where $x_n \in \left]n-1, n+1\right]$, moreover $x_2 = 2$, $x_3 = \sqrt{7}+1$, $x_4 \simeq 4.298$ and $x_n < n$ for $n \geq 5$.

Proof. Let $n \geq 2$ be an integer and let f_n be the function defined on the interval $I = \left]n-1, +\infty\right[$ by

$$f_n(x) = \frac{nn!e^x}{x^{n-1} - n!},$$

f is differentiable on I and

$$f_n'(x) = \frac{nn!e^x\left(x^{n-1} - (n-1)x^{n-2} - n!\right)}{\left(x^{n-1} - n!\right)^2}.$$

Put for $x > n-1$,

$$g_n(x) = x^{n-1} - (n-1)x^{n-2} - n!,$$

then

$$g_n'(x) = (n-1)x^{n-3}(x-(n-2)) > 0 \text{ on } I,$$

hence g_n increases on I. On the other hand, since g_n is continuous then by lemma 3.7, we have

$$\lim_{x \to n-1} g_n(x) = -n! < 0,$$
$$g_n(n) = n^{n-2} - n! > 0 \text{ for } n \geq 5,$$
$$g_n(n+1) = 2(n+1)^{n-2} - n! \geq 0,$$

therefore, there exists only one root $x_n \in]n-1,\, n+1]$ and for $n \geq 5$, $x_n \in]n-1,\, n]$ such that $f_n^{'}(x_n) = 0$. Since $g_n(x) < 0$ for $x < x_n$ and $g_n(x) > 0$ for $x > x_n$ then f_n strictly decreases on $]n-1,\, x_n]$ and strictly increases on $[x_n,\, +\infty[$, so we have

$$f_n(x) \geq f_n(x_n) \text{ where } x_n \in]n-1,\, n+1].$$

It is clear that for $n = 2, 3, 4$ the equality

$$x^{n-1} - (n-1)\, x^{n-2} - n! = 0$$

gives $x_2 = 2$, $x_3 = \sqrt{7} + 1$, $x_4 \simeq 4.298$. ■

Lemma 3.9 *For any integer $d \geq 2$, there exists a prime p such that*

$$e^{e^{x_d}} < p \leq e^{e^{d+1}}, \tag{3.6}$$

moreover

$$\max\left\{p : p \in \left]e^{e^{x_d}},\, e^{e^{d+1}}\right[\right\} > e^{e^d},$$

where $(x_d)_{d\geq 2}$ is the sequence defined in lemma 3.8.

Proof. The inequality (3.6) is easy to verify for $d = 2, 3, 4$. By lemma 3.8, we have, for $d \geq 5$

$$d - 1 \leq x_d \leq d\ , \tag{3.7}$$

therefore $e^{e^{x_d}} > 3275$, then from lemma 3.6 there exists a prime p such that

$$e^{e^{x_d}} < p \leq e^{e^{x_d}}\left(1 + \frac{1}{2e^{2x_d}}\right).$$

from (3.7), we get $4 \leq x_d \leq d$, then $1 + \frac{1}{2e^{2x_d}} < 2$ and $e^{e^{x_d}} < e^{e^d}$, thus

$$e^{e^{x_d}}\left(1 + \frac{1}{2e^{2x_d}}\right) < 2e^{e^d} < e^{e^{d+1}}.$$

Since

$$4e^{e^d} < \left(e^{e^d}\right)^2 < e^{e^{d+1}},$$

then according to the Bertrand's postilate there exists a prime number in $\left[2e^{e^d},\, 4e^{e^d}\right]$, thus, the greatest prime number in $\left[e^{e^{x_d}},\, e^{e^{d+1}}\right]$ is greater than e^{e^d}, which finishes the proof. ■

Lemma 3.10 *Let $d \geq 2$ and let k_0 be the integer such that $p_{k_0} \geq \exp(\exp d)$. For any real number $x > 0$ the sequence $\left(\mathcal{S}(\mathcal{B}_d^k,\, x)\right)_{k\geq k_0}$ is strictly increases.*

Proof. For any integer $k \geq 1$ and any integer $d \geq 2$, the multinomial formula ensures that

$$\begin{aligned}\sum_{a\in\mathcal{A}_d^k}\frac{1}{a} &= \sum_{\alpha_1+...+\alpha_k=d}\frac{1}{p_1^{\alpha_1}p_2^{\alpha_2}...p_k^{\alpha_k}}\\ &\geq \sum_{\alpha_1+...+\alpha_k=d}\frac{(1/p_1)^{\alpha_1}}{(\alpha_1)!}...\frac{(1/p_k)^{\alpha_k}}{(\alpha_k)!}\\ &= \frac{1}{d!}\left(\sum_{n=1}^{k}\frac{1}{p_n}\right)^d\end{aligned}$$

therefore

$$\sum_{a\in\mathcal{A}_d^k}\frac{1}{a} \geq \frac{1}{d!}\left(\sum_{n=1}^{k}\frac{1}{p_n}\right)^d. \tag{3.8}$$

Put $\mathcal{A}^k = \{p_n\,|p_n \in \mathcal{P},\, n > k\}$, then from lemma 3.4 we have

$$\begin{aligned}\mathcal{B}_d^{k+1} &= \mathcal{A}_d^{k+1}\bigcup\mathcal{A}^{k+1}\\ &= \mathcal{A}_d^{k}\bigcup\left\{ap_{k+1}|a\in\mathcal{A}_{d-1}^{k+1}\right\}\bigcup\mathcal{A}^{k+1}\end{aligned}$$

so,

$$S\left(\mathcal{B}_d^{k+1},\, x\right) = S\left(\mathcal{B}_d^{k},\, x\right) + E$$

where

$$E = \frac{1}{p_{k+1}}\left(S\left(\mathcal{A}_{d-1}^{k+1},\ \log p_{k+1} + x\right) - \frac{1}{\log p_{k+1} + x}\right).$$

Since p_{k+1}^{d-1} is the greatest element of $\mathcal{A}_{d-1}^{k+1}$, we have

$$\begin{aligned}S\left(\mathcal{A}_{d-1}^{k+1},\ \log p_{k+1} + x\right) &= \sum_{a\in\mathcal{A}_{d-1}^{k+1}}\frac{1}{a(\log a + \log p_{k+1} + x)}\\ &\geq \sum_{a\in\mathcal{A}_{d-1}^{k+1}}\frac{1}{a((d-1)\log p_{k+1} + \log p_{k+1} + x)}\\ &\geq \frac{1}{d\log p_{k+1} + x}\sum_{a\in\mathcal{A}_{d-1}^{k+1}}\frac{1}{a},\end{aligned}$$

and by lemma 3.1 we obtain

$$\begin{aligned}
\sum_{a\in\mathcal{A}_{d-1}^{k+1}} \frac{1}{a} &\geq \frac{1}{(d-1)!}\left(\sum_{n=1}^{k+1}\frac{1}{p_n}\right)^{d-1} \\
&\geq \frac{1}{(d-1)!}\left(\log\log p_{k+1}\right)^{d-1} \\
&\geq \frac{d^{d-1}}{(d-1)!} \\
&\geq \frac{d^{d-1}}{d!}d \text{ for } k\geq k_0,
\end{aligned}$$

according to lemma 3.7 we have $d!\leq d^{d-1}$ then

$$\sum_{a\in\mathcal{A}_{d-1}^{k+1}} \frac{1}{a}\geq d \text{ for } k\geq k_0,$$

which implies

$$\begin{aligned}
S\left(\mathcal{A}_{d-1}^{k+1},\ \log p_{_{k+1}}+x\right)-\frac{1}{\log p_{_{k+1}}+x} &> \frac{d}{d\log p_{_{k+1}}+x}-\frac{1}{\log p_{_{k+1}}+x} \\
&= \frac{dx-x}{\left(d\log p_{_{k+1}}+x\right)\left(\log p_{_{k+1}}+x\right)}>0
\end{aligned}$$

thus

$$S\left(B_d^{k+1},\, x\right)-S\left(B_d^{k},\, x\right)>0.$$

The proof is finished. ■

Generalized result on $S(B_2^k,\, x)$

Theorem 3.2 *Let $d\geq 2$ be an integer, $x_0=\frac{dd!e^{d+1}}{(d+1)^{d-1}-d!}$ and let k_0 be the greatest integer such that $p_{k_0}\leq e^{e^{d+1}}$.Then for any $k\geq k_0$ and any primitive sequence of the form*

$$\mathcal{B}_d^k=\{p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}\,|\alpha_1,\dots,\alpha_k\in\mathbb{N},\ \alpha_1+\dots+\alpha_k=d\}\bigcup\{p_n\,|p_n\in\mathcal{P},\ n>k\}$$

we have, for $x\geq x_0$

$$S(\mathcal{B}_d^k,\, x)>S(\mathcal{P},\, x).$$

Proof. For any integer $k\geq 1$ and any integer $d\geq 2$, we have

$$\begin{aligned}
\sum_{a\in\mathcal{B}_d^k}\frac{1}{a(\log a+x)} &= \sum_{a\in\mathcal{A}_d^k\cup\mathcal{A}^k}\frac{1}{a(\log a+x)} \\
&= \sum_{a\in\mathcal{A}_d^k}\frac{1}{a(\log a+x)}+\sum_{a\in\mathcal{A}^k}\frac{1}{a(\log a+x)} \\
&\geq \frac{1}{d\log p_k+x}\sum_{a\in\mathcal{A}_d^k}\frac{1}{a}+\sum_{n>k}\frac{1}{p_n(\log p_n+x)}.
\end{aligned}$$

Using (3.8) and lemma 3.1, we get

$$\sum_{a\in\mathcal{A}_d^k}\frac{1}{a}>\frac{(\log\log p_k)^{d-1}}{d!}\sum_{n=1}^{k}\frac{1}{p_n},$$

therefore

$$\begin{aligned}\sum_{a\in\mathcal{B}_d^k}\frac{1}{a(\log a+x)} &\geq \frac{(\log\log p_k)^{d-1}}{d!(d\log p_k+x)}\sum_{n=1}^{k}\frac{1}{p_n}+\sum_{n>k}\frac{1}{p_n(\log p_n+x)}\\ &> \frac{x\,(\log\log p_k)^{d-1}}{d!(d\log p_k+x)}\sum_{n=1}^{k}\frac{1}{xp_n}+\sum_{n>k}\frac{1}{p_n(\log p_n+x)}\\ &> \frac{x\,(\log\log p_k)^{d-1}}{d!(d\log p_k+x)}\sum_{n=1}^{k}\frac{1}{p_n(\log p_n+x)}+\sum_{n>k}\frac{1}{p_n(\log p_n+x)}.\end{aligned}$$

To obtain the inequality required in theorem, we must choose k and x so that

$$\frac{x\,(\log\log p_k)^{d-1}}{d!(d\log p_k+x)}>1. \tag{3.9}$$

Since for $d\geq 2$, $k>1$, the function

$$x\mapsto h_{k,d}(x)=\frac{x\,(\log\log p_k)^{d-1}}{d!(d\log p_k+x)},$$

is strictly increases for $x>0$, let x_0 the smallest value for which the inequality (3.9) is verified, that is

$$\frac{(\log\log p_k)^{d-1}-d!}{dd!\log p_k}>\frac{1}{x_0}. \tag{3.10}$$

Since $x_0>0$, we need to find k such that

$$(\log\log p_k)^{d-1}-d!>0,$$

then by lemma 3.9, we just take $\log\log p_k>d$, and if we put $\log\log p_k=z$, (3.10) becomes

$$\frac{dd!e^z}{z^{d-1}-d!}<x_0.$$

Now, we must choose z so that, the number $\frac{dd!e^z}{z^{d-1}-d!}$ is the smallest possible.
According to lemma 3.8, the function

$$x\mapsto f_d(x)=\frac{dd!e^x}{x^{d-1}-d!}$$

reaches its minimum x_d in

$$]d-1,\,d+1],$$

so we can take $z \in]x_d,\, d+1[$ and $x_0 = \frac{dd!e^{d+1}}{(d+1)^{d-1}-d!}$.
From lemma 3.9, there exists a prime number p_k such that

$$x_d < \log\log p_k < d+1.$$

Now, we can choose

$$p_{k_0} = \max\left\{p_k : \log\log p_k \in]x_d,\, d+1[\right\} \text{ and } z = \log\log p_{k_0},$$

then we obtain, for $x \geq x_0$

$$S(\mathcal{B}_d^{k_0},\, x) > S(\mathcal{P},\, x).$$

Finally, by lemma 3.9, we have

$$\exp\left(\exp d\right) \leq p_{k_0} \leq \exp\left(\exp\left(d+1\right)\right),$$

and from lemma 3.10, we get for $k \geq k_0$ and $x \geq x_0$,

$$S(\mathcal{B}_d^{k},\, x) > S(\mathcal{P},\, x).$$

Which ends the proof. ■

Remark 3.1 *If we take $d = 2$, then we get , for $k \geq 27775592$ and $x \geq 80.4$*

$$S(\mathcal{B}_2^{k},\, x) > S(\mathcal{B}_1^{k},\, x).$$

Since for x is sufficiently large, we have $S(\mathcal{B}_d^{k},\, x) > S(\mathcal{P},\, x)$, so we can conjecture that: for any $d \geq 1$ there exists k_0 such that

$$S(\mathcal{B}_{d+1}^{k},\, x) > S(\mathcal{B}_d^{k},\, x), \;\; k \geq k_0,\, x > 0.$$

Chapter 4

Study the sum $S(\mathcal{A},\, x)$ for largest values of x

As explained in the introduction, the main objective of this chapter is to study the sum $S(\mathcal{A},\, x)$ for largest values of x, in this work we use the primitive sequences of the form

$$\mathcal{B}_d^k = \{p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}\,|\alpha_1,\dots,\alpha_k \in \mathbb{N},\, \alpha_1+\dots+\alpha_k = d\} \bigcup \{p_n\,|p_n \in \mathcal{P},\, n>k\}\,.$$

Lemmas

Lemma 4.1 [4] *For $k \geq 463$,*

$$p_{k+1} \leq p_k\left(1+\frac{1}{2\log^2 p_k}\right).$$

Lemma 4.2 *For any real number $x>0$ and any integer $k \geq 2$ the following holds*

$$\sum_{n>k}\frac{1}{p_n(\log p_n + x)} \leq \begin{cases} \frac{\log(1+\frac{x}{\log k})}{x} & \textit{if } x \neq 0 \\ \frac{1}{\log k} & \textit{if } x = 0. \end{cases}$$

Proof. Let $x>0$ be a real number and $k \geq 2$ be an integer. By lemma 2.1 and since the function

$$t \mapsto \frac{dt}{t\log t(\log t + x)}$$

decreases on $[1,+\infty)$, we obtain then

$$\begin{aligned}\sum_{n>k}\frac{1}{p_n(\log p_n + x)} &\leq \sum_{n>k}\frac{1}{n\log n(\log n+\log\log n + x)} \\ &\leq \sum_{n>k}\frac{1}{n\log n(\log n + x)} \\ &\leq \int_k^{+\infty}\frac{dt}{t\log t(\log t + x)}.\end{aligned}$$

Put $u = \log t$, so,
if $x \neq 0$, we have

$$\begin{aligned}
\int_{k}^{+\infty} \frac{dt}{t\log t(\log t + x)} &= \int_{\log k}^{+\infty} \frac{du}{u(u+x)} \\
&= \frac{1}{x}\int_{\log k}^{+\infty} \left(\frac{1}{u} - \frac{1}{u+x}\right) du \\
&= \frac{\log(1+\frac{x}{\log k})}{x},
\end{aligned}$$

if $x = 0$, we have

$$\begin{aligned}
\int_{k}^{+\infty} \frac{dt}{t\log t(\log t + x)} &= \int_{\log k}^{+\infty} \frac{du}{u^2} \\
&= \frac{1}{\log k}.
\end{aligned}$$

This ends the proof. ■

Lemma 4.3 *For any integer $n \neq 0$, we have:*

$$n! \leq n^n e^{1-n}\sqrt{n}.$$

Proof. For $n = 1$, the inequality is verified.
For $n \geq 2$, we use the inequality [8]

$$n! \leq n^n e^{-n}\sqrt{2\pi n}e^{1/12n}.$$

We have

$$\begin{aligned}
n! &\leq n^n e^{-n}\sqrt{2\pi n}e^{1/12n} \\
&\leq n^n e^{1-n}\sqrt{n}\sqrt{2\pi}e^{1/12n-1},
\end{aligned}$$

and since the function $x \mapsto \sqrt{2\pi}e^{1/12x-1}$ decreases on $[2, +\infty)$, then

$$\sqrt{2\pi}e^{1/12n-1} < 1,$$

therefore

$$n! \leq n^n e^{1-n}\sqrt{n}.$$

Which ends the proof. ■

Lemma 4.4 *For any* $\lambda \geq 1$ *and any* $n > 0$, *we have*

$$\frac{71.383\lambda^{2n} + 20.978}{64.975 \ln\left(\lambda^{2n} + 2\right) + 20.978} > \lambda^n.$$

Proof. For any λ, we have:

$$\begin{aligned}
& \frac{71.383\lambda^{2n} + 20.978}{64.975 \ln\left(\lambda^{2n} + 2\right) + 20.978} - \lambda^n \\
= & \frac{71.383\lambda^{2n} + 20.978 - 64.975\lambda^n \ln\left(\lambda^{2n} + 2\right) - 20.978\lambda^n}{64.975 \ln\left(\lambda^{2n} + 2\right) + 20.978} \\
= & \frac{1}{64.975 \ln\left(\lambda^{2n} + 2\right) + 20.978}\left(\lambda^n(71.383\lambda^n - 64.975 \ln\left(\lambda^{2n} + 2\right) - 20.978) + 20.978\right).
\end{aligned}$$

Put $w_n(\lambda) = 71.383\lambda^n - 64.975 \ln\left(\lambda^{2n} + 2\right) - 20.978$, then

$$\begin{aligned}
\frac{d}{d\lambda} w_n(\lambda) = & \frac{n}{\lambda\left(\lambda^{2n} + 2\right)}\left(1.4277 \times 10^2\lambda^n - 1.2995 \times 10^2\lambda^{2n} + 71.383.\lambda^{3n}\right) \\
= & \frac{n\lambda^n}{\lambda\left(\lambda^{2n} + 2\right)}\left(1.4277 \times 10^2 - 1.2995 \times 10^2\lambda^n + 71.383.\lambda^{2n}\right)
\end{aligned}$$

and since $1.4277 \times 10^2 - 1.2995 \times 10^2 x + 71.383.x^2 > 0$, then

$$\frac{d}{d\lambda} w_n(\lambda) > 0.$$

Hence the function w_n increases for $\lambda \geq 1$, therefore

$$71.383\lambda^n - 64.975 \ln\left(\lambda^{2n} + 2\right) - 20.978 \geq -20.977,$$

so,

$$\left(\lambda^n(71.383\lambda^n - 64.975 \ln\left(\lambda^{2n} + 2\right) - 20.978) + 20.978\right) \geq 6.6654 \times 10^{-4} > 0.$$

Thus

$$\frac{71.383\lambda^{2n} + 20.978}{64.975 \ln\left(\lambda^{2n} + 2\right) + 20.978} - \lambda^n > 0,$$

and the proof is achieved. ■

Lemma 4.5 *Let the sequence* $(u_n)_{n\geq 2}$ *where*

$$u_n = \frac{n^{n-1} - n!}{nn!}.$$

(u_n) *increases on* $[2,\infty)$.

Proof. We have.

$$\begin{aligned}
u_{n+1}-u_n &= \frac{(n+1)^n-(n+1)!}{(n+1)\,(n+1)!}-\frac{n^{n-1}-n!}{nn!}\\
&= \frac{(n+1)^n}{(n+1)\,(n+1)!}-\frac{(n+1)^n}{(n+1)\,(n+1)!}-\frac{n^{n-2}}{n!}+\frac{1}{n}\\
&= \frac{(n+1)^{n-1}}{(n+1)!}-\frac{1}{(n+1)}-\frac{n^{n-2}}{n!}+\frac{1}{n}\\
&\geq \frac{(n+1)^{n-2}}{n!}-\frac{n^{n-2}}{n!}+\frac{1}{n}-\frac{1}{(n+1)}\geq 0.
\end{aligned}$$

The proof finished. ■

4.1 Study of $S(\mathcal{A},x)$ for largest values of x

Theorem 4.1 *Let $\lambda \geq 1$ and $t>0$, then for any $x \geq 1656\lambda^{2t}\left(\log(\lambda^{2t}+2)\right)^{3/2}$, there exists a primitive sequence $\mathcal{A}$ such that*

$$S(\mathcal{A},\, x) \geq \lambda^t S(\mathcal{P},\, x).$$

Proof. Let $\lambda \geq 1$ and let $t>0$. To prove this theorem, we need the parameters α, c and β which satisfy :

$$c\alpha \geq e^{\beta}+\log 1.008,\ 0<\alpha\leq\frac{5}{12} \tag{C1}$$

$$\beta \geq 1.950 \tag{C2}$$

those parameters will be chosen later, the real c is chosen to be the smallest possible value so that; for any $x \geq c\lambda^{2t}\left(\log(\lambda^{2t}+2)\right)^{3/2}$, there exists a primitive sequence $\mathcal{A}\neq\{1\}$ such that

$$\sum_{a\in\mathcal{A}}\frac{1}{a(\log a+x)}>\lambda^t\sum_{p\in\mathcal{P}}\frac{1}{p(\log p+x)}.$$

Let p_k be the largest prime satisfying $p_k\leq e^{\alpha x}$, then according to lemmas 4.1 and 2.1, we obtain

$$p_k \leq e^{\alpha x}<p_{k+1}<1.008p_k, \tag{4.1}$$

Assume that

$$d=\left\lfloor \beta+\log\lambda^{2t}+\frac{3}{2}\log\log\left(\lambda^{2t}+2\right)\right\rfloor,$$

then from (C1) and (C2), we have

$$x\geq\frac{1}{\alpha}\left(e^d+\log 1.008\right),$$

and from lemma 3.1 and (4.1), we obtain

$$\begin{aligned}\sum_{n=1}^{k}\frac{1}{p_n} &> \log\log p_k\\ &> \log\log\frac{p_{k+1}}{1.008}\\ &> \log\log\frac{e^{\alpha x}}{1.008}\\ &\geq d.\end{aligned}$$

So,

$$\sum_{n=1}^{k}\frac{1}{p_n} \geq d. \tag{4.2}$$

Now, we define the following sets of positive integers :

$$\begin{aligned}\mathcal{P}^k &= \{p_n\,|p_n \in \mathcal{P},\, p_n > p_k\},\\ \mathcal{A} &= \mathcal{A}_d^k \bigcup \mathcal{P}^k.\end{aligned}$$

It is clear that $\mathcal{A}_d^k \bigcap \mathcal{P}^k = \varnothing$ and the sets $\mathcal{A}_d^k$, $\mathcal{P}^k$ and $\mathcal{A}$ are primitive sequences. Then, according to the multinomial formula and (4.2), we have

$$\begin{aligned}\sum_{a\in\mathcal{A}_d^k}\frac{1}{a} &= \sum_{\alpha_1+\ldots+\alpha_k=d}\frac{1}{p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_k^{\alpha_k}}\\ &\geq \sum_{\alpha_1+\ldots+\alpha_k=d}\frac{(1/p_1)^{\alpha_1}}{(\alpha_1)!}\ldots\frac{(1/p_k)^{\alpha_k}}{(\alpha_k)!}\\ &= \frac{1}{d!}(\sum_{n=1}^{k}\frac{1}{p_n})^d\\ &> \frac{d^{d-1}}{d!}\sum_{n=1}^{k}\frac{1}{p_n}.\end{aligned}$$

So,

$$\sum_{a\in\mathcal{A}_d^k}\frac{1}{a} > \frac{d^{d-1}}{d!}\sum_{n=1}^{k}\frac{1}{p_n}. \tag{4.3}$$

Since

$$x \geq c\lambda^{2t}\left(\log(\lambda^{2t}+2)\right)^{3/2}$$

then, from $(C1)$ and $(C2)$, we obtain

$$e^{\alpha x} \geq 3303 \geq p_{464}.$$

Hence, according to (4.1) we have

$$p_{464} \leq p_k \leq e^{\alpha x} < p_{k+1} < 1.008 p_k.$$

By using lemma 2.1, we get

$$\begin{aligned} \log p_k &\leq \alpha x \\ &\leq \log p_k + \log 1.008 \\ &\leq \log\left(k\left(\log k + \log\log k\right)\right) + \log 1.008. \end{aligned}$$

Now, since the function

$$t \mapsto \frac{\log\left(t\left(\log t + \log\log t\right)\right) + \log 1.008}{\log t}$$

decreases on $[464, +\infty)$, then we have

$$\frac{\log\left(t\left(\log t + \log\log t\right)\right) + \log 1.008}{\log t} \leq \frac{\log\left(464\left(\log 464 + \log\log 464\right)\right) + \log 1.008}{\log 464} \simeq 1.339$$

that is,

$$\alpha x \leq 1.339 \log k. \tag{4.4}$$

By using (4.4) and lemma 4.3, we find

$$\sum_{n>k} \frac{1}{p_n(\log p_n + x)} \leq \frac{\log(1 + \frac{x}{\log k})}{x} < \frac{\log(1 + \frac{1.339}{\alpha})}{x},$$

therefore,

$$\frac{1}{x} > \frac{1}{\log(1 + \frac{1.339}{\alpha})} \sum_{n>k} \frac{1}{p_n(\log p_n + x)}. \tag{4.5}$$

On other hand, according to (4.1) and (4.2), we have for $x \neq 0$

$$\begin{aligned} \sum_{n=1}^{k} \frac{1}{p_n(\log p_n + x)} &\geq \sum_{n=1}^{k} \frac{1}{p_n(\alpha x + x)} \\ &\geq \frac{1}{(\alpha + 1)\, x} \sum_{n=1}^{k} \frac{1}{p_n} \\ &\geq \frac{d}{(\alpha + 1)\, x}, \end{aligned}$$

and from (4.5) we obtain

$$\sum_{n=1}^{k} \frac{1}{p_n(\log p_n + x)} \geq \frac{d}{(\alpha + 1) \log(1 + \frac{1.339}{\alpha})} \sum_{n>k} \frac{1}{p_n(\log p_n + x)}.$$

Put $h(\alpha) = (\alpha+1)\log(1+\frac{1.339}{\alpha})$, then we have

$$\sum_{n=1}^{k} \frac{1}{p_n(\log p_n + x)} \geq \frac{d}{h(\alpha)} \left(\sum_{n=1}^{+\infty} \frac{1}{p_n(\log p_n + x)} - \sum_{n=1}^{k} \frac{1}{p_n(\log p_n + x)} \right),$$

therfore,

$$\begin{aligned} \left(1+\frac{d}{h(\alpha)}\right) \sum_{n=1}^{k} \frac{1}{p_n(\log p_n + x)} &\geq \frac{d}{h(\alpha)} \sum_{n=1}^{+\infty} \frac{1}{p_n(\log p_n + x)} \\ &\geq \frac{d}{d+h(\alpha)} \sum_{n=1}^{+\infty} \frac{1}{p_n(\log p_n + x)}. \end{aligned}$$

Thus,

$$\sum_{n=1}^{k} \frac{1}{p_n(\log p_n + x)} \geq \frac{d}{d+h(\alpha)} \sum_{n=1}^{+\infty} \frac{1}{p_n(\log p_n + x)}. \tag{4.6}$$

Since p_k^d is the largest element in $\mathcal{A}_d^k$, then according to (4.1), we have for any $a \in \mathcal{A}_d^k$

$$\log a \leq d \log p_k \leq d\alpha x,$$

hence, from (4.3), we obtain

$$\begin{aligned} \sum_{a\in\mathcal{A}} \frac{1}{a(\log a + x)} &= \sum_{a\in\mathcal{A}_d^k\cup\mathcal{P}^k} \frac{1}{a(\log a + x)} \\ &= \sum_{a\in\mathcal{A}_d^k} \frac{1}{a(\log a + x)} + \sum_{a\in\mathcal{P}^k} \frac{1}{a(\log a + x)} \\ &\geq \frac{1}{(d\alpha x + x)} \sum_{a\in\mathcal{A}_d^k} \frac{1}{a} + \sum_{n>k} \frac{1}{p_n(\log p_n + x)} \\ &> \frac{d^{d-1}}{d!x(d\alpha+1)} \sum_{n=1}^{k} \frac{1}{p_n} + \sum_{n>k} \frac{1}{p_n(\log p_n + x)} \\ &> \frac{d^{d-1}}{d!(d\alpha+1)} \sum_{n=1}^{k} \frac{1}{p_n(\log p_n + x)} + \sum_{n>k} \frac{1}{p_n(\log p_n + x)} \\ &= \left(\frac{d^{d-1}}{d!(d\alpha+1)} - 1\right) \sum_{n=1}^{k} \frac{1}{p_n(\log p_n + x)} + \sum_{n=1}^{+\infty} \frac{1}{p_n(\log p_n + x)}. \end{aligned}$$

According to $(C1)$, and lemma 4.5, we have, for $d \geq 4$,

$$\left(\frac{d^{d-1}}{d!(d\alpha+1)} - 1\right) > 0.$$

By using this last inequality and (4.6), we obtain

$$\left(\frac{d^{d-1}}{d!(d\alpha+1)}-1\right)\sum_{n=1}^{k}\frac{1}{p_n(\log p_n+x)} \geq \left(\frac{d^{d-1}}{d!(d\alpha+1)}-1\right)\frac{d}{d+h\left(\alpha\right)}\sum_{n=1}^{+\infty}\frac{1}{p_n(\log p_n+x)}.$$

Therefore,

$$\begin{aligned}\sum_{a\in\mathcal{A}}\frac{1}{a(\log a+x)} &> \left(\left(\frac{d^{d-1}}{d!(d\alpha+1)}-1\right)\frac{d}{d+h\left(\alpha\right)}+1\right)\sum_{n=1}^{+\infty}\frac{1}{p_n(\log p_n+x)}\\ &= \frac{d^d+d!(d\alpha+1)h\left(\alpha\right)}{d!(d\alpha+1)\left(d+h\left(\alpha\right)\right)}\sum_{n=1}^{+\infty}\frac{1}{p_n(\log p_n+x)},\end{aligned}$$

by applying lemma 4.3, we get

$$\begin{aligned}\frac{d^d+d!(d\alpha+1)h\left(\alpha\right)}{d!(d\alpha+1)\left(d+h\left(\alpha\right)\right)} &> \frac{d!e^{d-1}+d!\sqrt{d}(d\alpha+1)h\left(\alpha\right)}{\sqrt{d}d!(d\alpha+1)\left(d+h\left(\alpha\right)\right)}\\ &> \frac{e^{d-1}+\sqrt{d}(d\alpha+1)h\left(\alpha\right)}{\sqrt{d}(d\alpha+1)\left(d+h\left(\alpha\right)\right)}.\end{aligned}$$

So,

$$\sum_{a\in\mathcal{A}}\frac{1}{a(\log a+x)} > \left(\frac{e^{d-1}+\sqrt{d}(d\alpha+1)h\left(\alpha\right)}{\sqrt{d}(d\alpha+1)\left(d+h\left(\alpha\right)\right)}\right)\sum_{n=1}^{+\infty}\frac{1}{p_n(\log p_n+x)}. \tag{4.7}$$

It follows from the expression of d, that

$$d > \beta-1+\log\lambda^{2t}+\frac{3}{2}\log\log(\lambda^{2t}+2),$$

then

$$e^{d-1} > e^{\beta-2}\lambda^{2t}\left(\log(\lambda^{2t}+2)\right)^{3/2}.$$

And since

$$\begin{aligned}\log\lambda^{2t} &< \log\left(\lambda^{2t}+2\right),\\ \log\log\left(\lambda^{2t}+2\right) &\leq \log\left(\lambda^{2t}+2\right)-1\end{aligned}$$

and $\beta \geq 1.950$, we have

$$d < (\beta+1)\log\left(\lambda^{2t}+2\right),$$

then

$$d\alpha+1 < ((\beta+1)\alpha+1)\log\left(\lambda^{2t}+2\right),$$

and

$$d < (\beta+1)\log\left(\lambda^{2t}+2\right).$$

So, the formula (4.7) becomes

$$\sum_{a\in\mathcal{A}} \frac{1}{a(\log a + x)} > j_{\alpha,\beta}(\lambda) \sum_{n=1}^{+\infty} \frac{1}{p_n(\log p_n + x)}, \tag{4.8}$$

where

$$j_{\alpha,\beta}(\lambda) = \frac{e^{\beta-2}\lambda^{2t} + \sqrt{\beta+1}((\beta+1)\alpha+1)h(\alpha)}{\sqrt{\beta+1}((\beta+1)\alpha+1)\left((\beta+1)\log\left(\lambda^{2t}+2\right) + h(\alpha)\right)}.$$

Now, we must choose α and β so that, for any $\lambda \geq 1$ and any $t > 0$, $j_{\alpha,\beta}(\lambda) \geq 1$ and $\frac{e^{\beta}+\log 1.008}{\alpha}$ be the smallest possible. That is, for any $\lambda \geq 1$ and for any $t > 0$

$$\frac{e^{\beta-2}}{\sqrt{\beta+1}\,(\beta+1)\,((\beta+1)\alpha+1)} \geq \frac{\log\left(\lambda^{2t}+2\right)}{\lambda^{2t}}.$$

Since, for any $t > 0$ the function

$$\lambda \mapsto \frac{\log\left(\lambda^{2t}+2\right)}{\lambda^{2t}}$$

decreases on $[1, +\infty)$, then

$$\frac{e^{\beta-2}}{\sqrt{\beta+1}\,(\beta+1)\,((\beta+1)\alpha+1)} \geq \log 3.$$

Hence,

$$\frac{e^{\beta-2} - (\beta+1)^{\frac{3}{2}}\log 3}{(\beta+1)^{\frac{5}{2}}\log 3} \geq \alpha$$

and

$$\frac{e^{\beta}+\log 1.008}{\alpha} \geq \frac{\left(e^{\beta}+\log 1.008\right)(\beta+1)^{\frac{5}{2}}\log 3}{e^{\beta-2} - (\beta+1)^{\frac{3}{2}}\log 3}.$$

Finally, we will choose β so that the quantity

$$\frac{\left(e^{\beta}+\log 1.008\right)(\beta+1)^{\frac{5}{2}}\log 3}{e^{\beta-2} - (\beta+1)^{\frac{3}{2}}\log 3}$$

is also the smallest possible. A computer calculation gives $\beta \simeq 6.264$, $\alpha \simeq 0.317$ and $c \simeq 1655.234$. By replacing α and β in the formula of $j_{\alpha,\beta}$ we get

$$j_{\alpha,\beta}(\lambda) = \frac{71.094\lambda^{2t} + 19.381}{64.659\ln\left(\lambda^{2t}+2\right) + 19.381},$$

and (4.8) becomes

$$\sum_{a\in\mathcal{A}} \frac{1}{a(\log a + x)} > \frac{71.094\lambda^{2t} + 19.381}{64.659\ln\left(\lambda^{2t}+2\right) + 19.381} \sum_{n=1}^{+\infty} \frac{1}{p_n(\log p_n + x)}.$$

By lemma 4.3, for every $\lambda \geq 1$ and every $t > 0$, we have

$$\frac{71.094\lambda^{2t} + 19.381}{64.659 \ln\left(\lambda^{2t} + 2\right) + 19.381} > \lambda^{t},$$

which leads to the inequality of theorem. Thus, for $\lambda \geq 1$, $t > 0$ and for any

$$x \geq 1656.3\lambda^{2t} \left(\log(\lambda^{2t} + 2)\right)^{3/2},$$

since

$$d = \left\lfloor 6.264 + \log \lambda^{2t} + \frac{3}{2} \log\log\left(\lambda^{2t} + 2\right) \right\rfloor$$

and k is the greatest integer such that $p_k \leq e^{0.317x}$, the sequence $\mathcal{A}$ is well defined. This ends the proof. This ends the proof. ■

Conclusion

The content of this thesis is focused on the Erdos's conjecture, so on the inequality

$$\sum_{a\in\mathcal{A}} \frac{1}{a\log a} \leq \sum_{a\in\mathcal{P}} \frac{1}{p\log p}$$

where $\mathcal{A}$ is a primitive sequence diferent to $\{1\}$ and $\mathcal{P}$ reprente the set of prime numbers. We took two paths in our work:

1) In chapter 2, by using a new estimations of n-th prime number, we simplified the proof of Zhang's theorem in which he proved the conjecture of Erdős for the primitive sequences of degree less or equal four. The first perspective is to extend Zhang's theorem to sequences of higher degree.

2) In chapitre 3 and 4, we study the inequality

$$\sum_{p\in\mathcal{A}} \frac{1}{a\,(\log a + x)} \leq \sum_{p\in\mathcal{P}} \frac{1}{p\,(\log p + x)}$$

where x is a positive real number, we proved that for $x \geq 80.4$ this last inequality is false, so the second perspective is to improve the value of x.

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