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Intitulée

Étude qualitative des équations différentielles d'ordre fractionnaire

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	شاءت الأقدار أن يغادرنا يوم الثلاثاء 27 جويلية 2021 البروفيسور بوعلام عتو سليماني نائب عميد كلية التكنولوجيا بالقطب الجامعي شتوان –تلمسان -الجزائر. كان عضوا في لجنة مناقشة هذه الأطروحة، رحم الله فقيدنا. رحل عنا تاركا أثرا عميقا من المحبة الصادقة في قلوب من حوله. وما علينا إلا رفع أيدينا إلى الله راجينا من المولى عز وجل أن يتغمده بواسع رحمته ومغفرته ورضوانه ويجعل مثواه الجنة رفقة الصالحين والشهداء ويرزق أهله وذويه جميل الصبر والسلوان. **"إنا لله وإنا إليه راجعون"**

Dedications

I dedicate this humble work To my Beloved Parents,

Grand Mothers, Brother and Sister,

Big Family and Friends.

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Publications

1. **A. I. N. Malti,** M. Benchohra, J. E. Lazreg, Nonlinear Boundary Implicit Differential Equations with Caputo-Exponential Fractional Order in Banach Spaces. (submitted).

2. **A. I. N. Malti,** M. Benchohra, J. R. Graef, J.E. Lazreg, Impulsive boundary problems for nonlinear implicit Caputo-exponential type fractional differential equations, *Electron. J. Qual. Theory Differ. Equ.,* **2020** (78) (2020), 1-17.

3. **A. I. N. Malti,** M. Benchohra, J. E. Lazreg, Impulsive Abstract Nonlinear Implicit Caputo–Exponential type Fractional Differential Equations. *Communications on Applied Nonlinear Analysis* (to appear).

4. **A. I. N. Malti,** M. Benchohra, J. E. Lazreg, Existence and Stability for Nonlinear implicit Caputo-Exponential type Fractional Differential Equations with Non-Instantaneous Impulses. (submitted).

5. **A. I. N. Malti,** J. E. Lazreg, M. Benchohra, and G. N'Guérékata, Existence and Stability for Nonlinear Implicit Caputo-Exponential type Fractional Differential Equations with Non-Instantaneous Impulses in Banach Spaces. (submitted).

Abstract

In this Thesis, we shall establish conditions for the existence, uniqueness of solutions and Ulam stability for various classes of initial and boundary value problems for nonlinear implicit fractional differential equations with and without impulses involving the Caputo's exponential type fractional derivative. In our study, we shall consider the Both cases of abstract and scalar. To prove the existence and uniqueness of solutions, we use some standard fixed point theorems. Several enlightening examples are also presented.

Key words and phrases :

Initial value problem, boundary value problem, Caputo's exponential type fractional derivative, implicit differential equations, exponential type fractional integral, instantaneous impulses, non-instantaneous impulses, existence, uniqueness, fixed point, Ulam stability, nonlocal conditions, Banach space, measure of noncompactness.

AMS Subject Classification : 26A33, 34A08, 34A12, 34A37, 34G20.

Résumé

Dans cette thèse, nous allons étudier l'existence, l'unicité des solutions et la stabilité d'Ulam de diverses classes de problèmes à valeur initiale et de problèmes aux limites pour les équations différentielles implicites non linéaires avec et sans impulsions en utilisant la dérivée fractionnaire de type exponentielle au sens de Caputo. Dans notre étude, Les deux cas abstrait et scalaire seront considérés. Pour prouver l'existence et l'unicité des solutions, nous utilisons certains théorèmes classiques du point fixe. Afin d'illustrer nos résultats plusieurs exemples seront présentés.

Mots clés et phrases :
Problème à valeur initiale, problème aux limites, la dérivée fractionnaire de type expenentielle au sens de Caputo, équations différentielles implicites, l'intégrale fractionnaire de type exponentielle, impulsions non-instantanées, impulsions instantanées, existence, unicité, point fixe, stabilité d'Ulam, les conditions non-locales, espace de Banach, mesure de non-compacité.

AMS Subject Classification : 26A33, 34A08, 34A12, 34A37, 34G20.

Contents

Introduction

Fractional calculus is a generalization of ordinary differentiation and integration for arbitrary non-integer order. Its history goes back to time when Marquis de L'Hopital (1661-1704) asked a question to Gottfried Wilhelm Leibniz (1646-1716), what happen to the ordinary derivative $\frac{d^n y}{dt^n}$ if $n = \frac{1}{2}$. In 30 September 1695, Leibniz replied that "...This is an apparent paradox from which, one day, useful consequences will be drawn. ..." Next, many researchers have further contributed to development of this area and we can mention the studies of L. Euler (1730), J.L. Lagrange (1772), P.S. Laplace (1812), J.B.J. Fourier (1822), N.H. Abel (1823), J. Liouville (1832), B. Riemann (1847), H.L. Greer (1859), H. Holmgren (1865), A.K. Grunwald (1867), A.V. Letnikov (1868), N.Ya. Sonin (1869), H. Laurent (1884), P.A. Nekrassov (1888), A. Krug (1890), J. Hadamard (1892), O. Heaviside (1892), S. Pincherle (1902), G.H. Hardy and J.E. Littlewood (1917), H. Weyl (1919), P. Lévy (1923), A. Marchaud (1927), H.T. Davis (1924), A. Zygmund (1935), E.R. Love (1938), A. Erdélyi (1939), H. Kober (1940), D.V. Widder (1941), M. Riesz (1949) and W. Feller (1952).

The concept of fractional calculus represents a powerful tool in applied mathematics to study a myriad of problems from different fields of science and engineering, with many break-through results found in control theory of dynamical systems, chaotic dynamics, mathematical physics, finance, biophysics, fractals, optics and signal processing, fluid flow, viscoelasticity, polymer science, rheology, physics, chemistry, biology, astrophysics, cosmology, thermodynamics, mechanics, electrochemistry, porous media, electromagnetic and bioengineering. For more details about the theory of fractional calculus, fractional differential equations and there applications, we refer to the reader the monographs of Abbas *et al.* [7], Baleanu *et al.* [19, 20, 21], Deithelm [45], Fečkan *et al.* [48], Hermann [59], Hilfer [61], Kilbas *et al.* [65], Kiryakova [66], Miller and Ross [80], Petras [87], Podlubny [89], Sahoo *et al.* [93], Samko *et al.* [96], Tarasov [99], Zhou *et al.* [107] and the papers of Abbas *et al.* [4, 5], Agarwal *et al.* [11], Benchohra *et al.* [28, 29], Diethelm and Freed [46], Gaul *et al.* [50], Glockle *et al.* [52], Metzler *et al.* [79], Mainardi [73], Oldham *et al.* [85], Salim *et al.* [94], Zhou *et al.* [106] and the reference therein.

In the past seventy years, Ulam type stability problems have been taken up by a large number of mathematicians and the study of this area has grown to be one of the most important subjects in the mathematical analysis area, since it is quite useful in many applications such as numerical analysis, optimization, biology and economics, where finding the exact solution is quite difficult. The stability of functional equations was originally raised by Ulam in 1940 in a talk given at Wisconsin University. The problem posed by Ulam was the following: "Under what conditions does there exist an additive mapping near an approximately additive mapping? "(for more details see [101]). The first answer to the question of Ulam was given by Hyers in 1941 in the case of Banach spaces (see [62]): Let E_1, E_2 be two real Banach spaces and $\epsilon > 0$. Then for every mapping $f : E_1 \to E_2$ satisfying

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon \quad \text{for all } x, y \in E_1$$

there exists a unique additive mapping $g : E_1 \to E_2$ with the property

$$\|f(x) - g(x)\| \le \epsilon \quad \text{for all } x \in E_1.$$

Thereafter, this type of stability is called the Ulam-Hyers stability. In 1978, Rassias [90] provided a remarkable generalization of the Ulam-Hyers stability of mappings by considering variables. The stability properties of all kinds of equations have attracted the attention of many mathematicians. For more details, one can see the monographs of Cădariu [43], Hyers [63] and Jung [64]. For Ulam stability of ordinary differential equations (see, [91, 92]). However, Ulam stability of fractional differential equations have been taken up by a large number of researchers, see for instance [5, 6, 31, 33, 68, 71, 94, 103] and the references therein.

The theory of impulsive differential equations was initiated in the 1960s by Vitali Davidovich Milman (1939) and Anatolii Dmitrievich Myshkis (1920-2009) [81]. This theory is an important branch of differential equations due to their application in characterizing many problems in biology, medicine, physique, engineering, etc. The reason for this applicability arises from the fact that impulsive differential problems describe the dynamics of processes in which sudden at certain moments change their state rapidly, discontinuous jumps occur. In the literature their are two popular types of impulses:

• Instantaneous impulses whose the duration of changes is relatively short and negligible. In the monographs of Benchohra *et al.* [30], Lakshmikantham *et al.* [69] and Samoilenko *et al.* [97] the authors study many classes of impulsive differential equations. This type know many contributions in fractional differential equations such as the monographs of Abbas *et al.* [8, 9], Ahmad *et al.* [13] and the papers of Benchohra *et al.* [25, 32, 37, 39], Ahmad *et al.* [15], Chang *et al.* [44], Henderson *et al.* [54, 58], Malti *et al.* [75], Wang *et al.* [102], Zhang *et al.* [105] and the references therein.

• Non-instantaneous impulses whose the impulsive action starts at some points and remain active on a finite time interval. In 2013, this class of impulses was initiated by Hernández and O'Regan [60], followed by other works as in [49, 86, 88, 98]. Recently, fractional differential equations with not-instantaneous impulses have also been discussed, see for instance the monograph of Agarwal *et al.* [12] and the papers of Abbas *et al.* [1, 2, 3], Anguraj and Kanjanadevi [17], Benchohra *et al.* [36, 38], Gautam and Dabas [51], Kumar *et al.* [67], Li and Xu.[70], Salim *et al.* [95] and the references therein.

As a motivation, this last kind of impulses is observed in pharmacotherapy. For example, one consider the hemodynamical equilibrium of a person. In the case of a decompensation (for example, high or low levels of glucose) one can prescribe some intravenous drugs (insulin). Since the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous processes, we can interpret the situation as an impulsive action which starts abruptly and stays active on a finite time interval.

In the last decades, the initial and boundary value problems for implicit differential equations involving the fractional derivative have been analyzed by many authors like in [24, 25, 26, 31, 32, 33, 34, 35, 68, 95] and the references therein.

The measure of noncompactness is one of the fundamental tools in the theory of nonlinear analysis was initiated by the pioneering articles of Alvàrez [16], Mönch [82] and the book of Bana's and Goebel [22]. Recently, in [8, 23, 27, 34, 37] the authors applied the measure of noncompactness to some classes of fractional differential equations in Banach spaces.

The nonlocal conditions were initiated by Byszewski [42] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski [40, 41], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena. Also, the fractional differential equations with nonlocal conditions have been discussed; see for instance [10, 14, 47, 57, 72, 83, 84] and the references therein.

After the above brief description of the main topics of this book, we now outline the contents of this book in more details.

Chapter 1 is devoted to the notations, definitions and some preliminary notions which are useful belong this thesis. In section 1.1, we give some notations and introduce the exponential fractional calculus. The definition of Kuratowski measure of noncompactness and their properties are presented in section 1.2. Section 1.3 consists some fixed point theorems which are used throughout this thesis.

Chapter 2 is devoted to the results obtained by Malti *et al.* [74]. In this chapter, we shall be concerned with a class of boundary value problem for nonlinear implicit fractional differential equations involving Caputo's exponential type fractional derivative given by:

$$ {}^{e}_{c}D^{\varrho}_{0}\omega(t) = f\left(t, y(t),\ {}^{e}_{c}D^{\varrho}_{0}\omega(t)\right),\ \text{for each},\ t \in J := [0,b], b > 0,\ 0 < \varrho \leq 1, $$

$$c_1\omega(0)+c_2\omega(b)=\delta,$$

where ${}_c^eD_0^{\varrho}$ is the Caputo's exponential type fractional derivative, $f : J\times E\times E\to E$ is a given function and c_1, c_2, are real constants with $c_1+c_2\neq 0$, and $\delta\in E$, where $(E,\|\cdot\|)$ is a real Banach space. Section 2.2 deals with the existence of solutions. We present two results, the first one is based on Darbo's fixed point theorem and the second one is based on Mönch's fixed point theorem. The Ulam-Hyers stability and Ulam-Hyers-Rasias stability are introduced and studied in section 2.3. In section 2.4, we give an example to show the applicability of results obtained in previous sections.

Chapter 3 is devoted to the results obtained by Malti *et al.* [75]. In this chapter, we shall be concerned with a class of impulsive boundary value problem for nonlinear implicit fractional differential equations involving Caputo's exponential type fractional derivative given by:

$$ {}_c^eD_{t_k}^{\alpha}\varpi(t)=f(t,\varpi(t),\ {}_c^eD_{t_k}^{\alpha}\varpi(t)),\quad \text{for each } t\in J_k\subseteq J,\ k=0,1,\dots,m,$$

$$\Delta\varpi|_{t=t_k}=I_k\left(\varpi\left(t_k^-\right)\right),\quad k=1,\dots,m,$$

$$c_1\varpi(a)+c_2\varpi(b)=c_3,$$

where $a=t_0<t_1<\ldots<t_m<t_{m+1}=b$, ${}_c^eD_{a^+}^{\alpha}$ denote the Caputo's exponential type fractional derivative of order α, $0<\alpha\leq 1$, $f : J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a given function and c_1, c_2, c_3 are real constants with $c_1+c_2\neq 0$, $J_k=(t_k,t_{k+1}]$, $k=1,2,\dots,m$, $J_0=[a,t_1]$, $J=[a,b]$, $\Delta\varpi|_{t=t_k}=\varpi\left(t_k^+\right)-\varpi\left(t_k^-\right)$, $\varpi\left(t_k^+\right)=\lim\limits_{h\to 0}\varpi(t_k+h)$, and $\varpi\left(t_k^-\right)=\lim\limits_{h\to 0^-}\varpi(t_k+h)$ represent the right and left limits of $\varpi(t)$ at $t=t_k$, respectively. In section 3.2, we investigates two existence results, the first one is based on Banach's contraction principle and the second one is based on Schaefer's fixed point theorem. As application, two illustrative examples are given in section 3.3.

Chapter 4 is devoted to the results obtained by Malti *et al.* [76]. In this chapter, we shall be concerned with a class of impulsive boundary value problem for nonlinear implicit fractional differential equations in Banach space involving Caputo's exponential type fractional derivative given by:

$$ {}_c^eD_{t_k}^{\alpha}y(t)=f(t,y(t),\ {}_c^eD_{t_k}^{\alpha}y(t)),\quad \text{for each } t\in J_k\subseteq J,\ k=0,1,\dots,m,$$

$$\Delta y|_{t=t_k}=I_k\left(y\left(t_k^-\right)\right),\quad k=1,\dots,m,$$

$$c_1y(a)+c_2y(b)=\sigma,$$

where $a=t_0<t_1<\ldots<t_m<t_{m+1}=b$, ${}_c^eD_{t_k}^{\alpha}$ denote the Caputo's exponential type fractional derivatives of order α, $0<\alpha\leq 1$, $(E,\|\cdot\|)$ is a real Banach space, $f : J\times E\times E\to E$

is a given function, c_1, c_2 are real constants with $c_1 + c_2 \neq 0$, and $\sigma \in E$, $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, $J_0 = [a, t_1]$, $J = [a, b]$, $\Delta y|_{t=t_k} = y\left(t_k^+\right) - y\left(t_k^-\right)$, $y\left(t_k^+\right) = \lim\limits_{h\to 0} y(t_k + h)$ and $y\left(t_k^-\right) = \lim\limits_{h\to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$. In section 4.2, we investigate the existence of solutions, we present two results, the first one is based on Darbo's fixed point theorem and the second one is based on Mönch's fixed point theorem. As application, two illustrative examples are given in section 4.3.

Chapter 5 is devoted to the results obtained by Malti *et al.* [77]. In this chapter, we shall be concerned with a class of initial value problem for nonlinear implicit fractional differential equations with non-instantaneous impulses involving Caputo's exponential type fractional derivative. The arguments of results are based on Banach's contraction principle and Schaefer's fixed point theorem. Several examples are includes to show the applicability of our results. Section 5.2 is concerned with the existence, uniqueness of solutions and Ulam-Hyers-Rassias stability for the following problem:

$$
\begin{gathered}
{}_c^eD^{\alpha}_{s_k}y(t) = f(t, y(t),\ {}_c^eD^{\alpha}_{s_k}y(t)), \quad \text{for each } t \in J_k \subseteq J,\ k = 0, 1, \ldots, m, \\
y(t) = g_k(t, y(t)), \quad \text{for each } t \in J'_k \subset J,\ k = 1, 2, \ldots, m, \\
y(a) = \delta,
\end{gathered}
$$

where ${}_c^eD^{\alpha}_{a^+}$ denote the Caputo's exponential type fractional derivatives of order $\alpha \in (0, 1]$, $\delta \in \mathbb{R}$, $J = [a, b]$, $a = t_0 = s_0 < t_1 \leq s_1 < \ldots < t_m \leq s_m < t_{m+1} = b$, $J'_k := (t_k, s_k]$, $J_k := (s_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, $J_0 := [a, t_1]$ $f : \overline{\mathcal{J}} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g_k : \overline{\mathcal{J}'} \times \mathbb{R} \to \mathbb{R}$ are a given functions such that $\overline{\mathcal{J}} = \bigcup\limits_{k=0}^{m} [s_k, t_{k+1}]$, and $\overline{\mathcal{J}'} = \bigcup\limits_{k=1}^{m} [t_k, s_k]$.

In section 5.3, we indicate and extend some generalizations to the nonlocal conditions for the results obtained in the last section with the following problem:

$$
\begin{gathered}
{}_c^eD^{\alpha}_{s_k}y\left(t\right) = f(t, x\left(t\right),\ {}_c^eD^{\alpha}_{s_k}y\left(t\right)), \quad \text{for each } t \in J_k \subseteq J,\ k = 0, 1, \ldots, m, \\
y(t) = g_k(t, y(t)), \quad \text{for each } t \in J'_k \subset J,\ k = 1, 2, \ldots, m, \\
y(a) + \widetilde{h}\left(y\right) = \delta,
\end{gathered}
$$

where α, ${}_c^eD^{\alpha}_{a^+}$, f, g_k, δ, J, J_0, J_k, J'_k, $k = 1, \ldots, m$ are defined as in section 5.2 and $\widetilde{h} : PC\left(J, \mathbb{R}\right) \to \mathbb{R}$ is a continuous functions.

Chapter 6 is devoted to the results obtained by Malti *et al.* [78]. In this chapter, we shall be concerned with a class of initial value problem for nonlinear implicit fractional differential equations with non-instantaneous impulses involving Caputo's exponential type fractional derivative in Banach Spaces. The arguments of results are based on Darbo's and Mönch's fixed point theorem combined with the technique of measure of noncompactness. Several examples are includes to show the applicability of our results. Section 6.2 is concerned with the existence, uniqueness of solutions and Ulam-Hyers-Rassias stability for the following problem:

$$ {}_{c}^{e}D_{s_k}^{\alpha}y(t) = f(t, y(t),\ {}_{c}^{e}D_{s_k}^{\alpha}y(t)), \quad \text{for each } t \in J_k \subseteq J,\ k = 0, 1, \ldots, m, $$

$$ y(t) = g_k(t, y(t)), \quad \text{for each } t \in J'_k \subset J,\ k = 1, 2, \ldots, m, $$

$$ y(a) = \delta, $$

where ${}_{c}^{e}D_{a^+}^{\alpha}$ denote the Caputo's exponential type fractional derivatives of order $\alpha \in (0,1]$, $J = [a,b]$, $a = t_0 = s_0 < t_1 \leq s_1 < \ldots < t_m \leq s_m < t_{m+1} = b$, $(E, \|\cdot\|)$ is a real Banach space, $\delta \in E$, $J'_k := (t_k, s_k]$, $J_k := (s_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, $J_0 := [a, t_1]$ $f : \overline{\mathcal{J}} \times E \times E \to E$ and $g_k : \overline{\mathcal{J}'} \times E \to E$ are a given functions such that $\overline{\mathcal{J}} = \bigcup\limits_{k=0}^{m} [s_k, t_{k+1}]$, and $\overline{\mathcal{J}'} = \bigcup\limits_{k=1}^{m} [t_k, s_k]$.

In section 6.3, we indicate and extend some generalizations to the nonlocal conditions for the results obtained in the last section with the following problem:

$$ {}_{c}^{e}D_{s_k}^{\alpha}x(t) = f(t, x(t),\ {}_{c}^{e}D_{s_k}^{\alpha}x(t)), \quad \text{for each } t \in J_k \subseteq J,\ k = 0, 1, \ldots, m, $$

$$ y(t) = g_k(t, y(t)), \quad \text{for each } t \in J'_k \subset J,\ k = 1, 2, \ldots, m, $$

$$ y(a) + \widetilde{h}(y) = \delta, $$

where α, ${}_{c}^{e}D_{a^+}^{\alpha}$, f, g_k, δ, J, J_0, J_k, J'_k, $k = 1, \ldots, m$ are defined as in section 6.2 and $\widetilde{h} : PC(J, E) \to E$ is a continuous functions.

Finally we close our thesis with a conclusion and some perspectives.

Chapter 1

Preliminaries

In this chapter, we introduce notations, definitions, lemmas, properties and fixed point theorems that will be used in the remainder of this thesis.

1.1 Notations and Definitions

Let $J := [a, b]$ such that $a < b$ and $(E, \|\cdot\|)$ be a real Banach space. By $C := C(J, E)$, we denote the Banach space of all continuous functions y from J into E with the supremum norm

$$\|y\|_\infty = \sup_{t\in J} \|y(t)\|.$$

A function $y : J \to E$ is said to be Bochner integrable if and only if $\|y(t)\|$ is Lebesgue integrable. For more details of the Bochner integral, see [104]. By $L^1(J, E)$, we denote the Banach space of functions $y : J \to E$ which are Bochner integrable with the norm

$$\|y\|_{L^1} = \int_a^b \|y(s)\|ds.$$

As usual, $AC(J)$ denote the space of absolutely continuous function from J into E. We denote by $AC_e^n(J)$ the space defined by

$$AC_e^n(J) := \left\{ y : J \to E :\ {}^eD^{n-1}y(t) \in AC(J),\ {}^eD = e^{-t}\frac{d}{dt} \right\},$$

where $n = [\alpha] + 1$ and $[\alpha]$ is the integer part of α. In particular, if $0 < \alpha \leq 1$, then $n = 1$ and $AC_e^1(J) := AC_e(J)$.

In ([65] p.99, Section 2.5) Kilbas *et al.* present the definitions and some properties of the fractional integrals and fractional derivatives of a function f with respect to another function ψ. Let $J := [a, b]$, $(-\infty \leq a < b \leq \infty)$ be a finite interval of the real line $\mathbb{R}$ and $\alpha > 0$. Also, let $\psi(t)$ be an increasing and positive monotone function on (a, b), having a continuous derivative $\psi'(t)$ on (a, b).

The fractional integrals of a function f with respect to another function ψ on $[a,b]$ are defined by

$$(I_a^\alpha f)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (\psi(t) - \psi(s))^{\alpha-1} \psi'(s) f(s) ds \text{ for } \ t > a. \tag{1.1}$$

If $a = 0$ and $b = \infty$, then

$$(I_0^\alpha f)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1} \psi'(s) f(s) ds \text{ for } \ t > 0. \tag{1.2}$$

It is well know that if $\psi(t) = t$ then all results in (1.1)-(1.2) are reduced to the *Riemann–Liouville* fractional integrals, and if $\psi(t) = ln(t)$ then the above formulas (1.1)-(1.2) are reduced to the *Hadamard* fractional integrals. In the case when $\psi(t) = e^t$ we have new kind of fractional calculus which is based on *exponential* fractional integrals defined as following.

Definition 1.1 *([84, 100]) The exponential type fractional integral of order $\alpha > 0$ of a function $h \in L^1(J, E)$ is defined by*

$$({}^eI_a^\alpha h)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \left(e^t - e^s\right)^{\alpha-1} h(s) e^s ds, \quad \textit{for each} \ \ t \in J,$$

where $\Gamma(.)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt, \quad \xi > 0.$$

Lemma 1.2 *([84, 100] Semigroup property.) Let $\alpha > 0$ and $\beta > 0$. Then, for all $t \in J$,*

$${}^eI_a^\alpha({}^eI_a^\beta h)(t) = \ {}^eI_a^\beta({}^eI_a^\alpha h)(t) = \ {}^eI_a^{\alpha+\beta} h(t).$$

Definition 1.3 *([84, 100]) Let $\alpha > 0$ and $h \in AC_e^n(J)$. The exponential derivatives of Riemann-Liouville type of order α is defined by*

$$({}^eD_a^\alpha h)(t) := \frac{1}{\Gamma(n-\alpha)} \left(e^{-t}\frac{d}{dt}\right)^n \int_a^t (e^t - e^s)^{n-\alpha-1} h(s) \frac{ds}{e^{-s}}, \quad \textit{for each } t \in J,$$

where $n = [\alpha] + 1$ and $[\alpha]$ is the integer part of α. In particular, if $\alpha = 0$, then

$$({}^eD_{(\cdot)}^0 h)(t) := h(t).$$

Definition 1.4 *([84, 100]) Let $\alpha > 0$ and $h \in AC_e^n(J)$. The Caputo's exponential type fractional derivatives of order α is defined by*

$$({}_c^eD_a^\alpha h)(t) := \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(e^t - e^s\right)^{n-\alpha-1} \left(e^{-s}\frac{d}{ds}\right)^n h(s) \frac{ds}{e^{-s}}, \quad \text{for each } t \in J,$$

where $n = [\alpha] + 1$ and $[\alpha]$ is the integer part of α. In particular, if $\alpha = 0$, then

$$\left({}_c^eD_{(\cdot)}^0 h\right)(t) := h(t),$$

Properties 1.5 *([84, 100]) If $\alpha, \beta > 0$, then*

1. ${}^eI_a^\alpha \left(e^t - e^a\right)^\beta = \dfrac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \left(e^t - e^a\right)^{\alpha+\beta}, \quad$ *for a.e.* $t \in J$.
2. ${}^eD_a^\alpha \left(e^t - e^a\right)^\beta = \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} \left(e^t - e^a\right)^{\beta-\alpha}, \quad$ *for a.e.* $t \in J$.

Lemma 1.6 *([84, 100]) Let $\alpha \geq 0$ and $n = [\alpha] + 1$. Then*

$${}_c^eD_a^\alpha f(t) = {}^eD_a^\alpha \left[f(s) - \sum_{k=0}^{n-1} \frac{{}^eD^k f(a)}{k!} (e^s - e^a)^k \right](t),$$

where ${}^eD = e^{-t}\dfrac{d}{dt}$.

Theorem 1.7 *([84, 100]) If $0 < \beta < \alpha$ and $1 \leq p < \infty$, then for $h \in L^p(J)$ we have*

$${}^eD_a^\beta({}^eI_a^\alpha h)(t) = {}^eI_a^{\alpha-\beta} h(t) \quad \text{and} \quad {}_c^eD_a^\beta({}^eI_a^\alpha h)(t) = {}^eI_a^{\alpha-\beta} h(t).$$

In addition,

$${}^eD_a^\alpha({}^eI_a^\alpha h)(t) = h(t) \quad \text{and} \quad {}_c^eD_a^\alpha({}^eI_a^\alpha h)(t) = h(t).$$

Theorem 1.8 *[84, 100] Let $\alpha \geq 0$ and $n = [\alpha] + 1$ and $h \in AC_e^n(J)$. Then we have the following formulas*

1. ${}^eI_a^\alpha({}^eD_a^\alpha h)(t) = h(t) - \displaystyle\sum_{k=1}^{n} \frac{(e^s - e^a)^{\alpha-k}}{\Gamma(\alpha-k+1)} {}^eD^{n-k}({}^eI^{n-\alpha} h)(a)$.
2. ${}^eI_a^\alpha({}_c^eD_a^\alpha h)(t) = h(t) - \displaystyle\sum_{k=0}^{n-1} \frac{(e^s - e^a)^k}{k!} {}^eD^k h(a)$.

Lemma 1.9 *Let $\alpha > 0$, and $h \in AC_e^n(J)$. Then the differential equation*

$$ {}_c^e D_a^\alpha h(t) = 0 $$

has a solutions

$$ h(t) = \eta_0 + \eta_1(e^s - e^a) + \eta_2(e^s - e^a)^2 + \ldots + \eta_{n-1}(e^s - e^a)^{n-1}, $$

$\eta_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [\alpha] + 1$.

Lemma 1.10 *Let $\alpha > 0$, and $h \in AC_e^n(J)$. Then*

$$ {}^e I_a^\alpha \left({}_c^e D_a^\alpha h \right)(t) = h(t) + \eta_0 + \eta_1(e^s - e^a) + \eta_2(e^s - e^a)^2 + \ldots + \eta_{n-1}(e^s - e^a)^{n-1}, $$

for some $\eta_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$ and $n = [\alpha] + 1$.

1.2 Measure of Noncompactness.

Now, we define the Kuratowski measure of noncompactness and give its basic properties.

Definition 1.11 *([22]) Let X be a Banach space and Ω_X be the bounded subsets of X. The Kuratowski measure of noncompactness is the map $\mu : \Omega_X \to [0, \infty)$ defined by*

$$ \mu(B) = \inf \{ \varepsilon > 0 : B \subseteq \bigcup_{i=1}^{n} B_i \ \textit{and} \ \ diam(B_i) \leqslant \varepsilon \}; \quad \textit{here}\ B \in \Omega_X, $$

where

$$ diam(B_i) = sup\{ \|x - y\| : x, y \in B_i \}. $$

Properties 1.12 *([22]) Let B_1 and B_2 be two bounded sets of the Banach space X. The Kuratowski measure of noncompactness satisfies the following properties:*

(i) $\mu(B_1) = 0 \iff \overline{B}_1$ *is compact (B_1 is relatively compact).*

(ii) $\mu\left(\overline{B}_1\right) = \mu\left(B_1\right)$.

(iii) *μ is equal to zero on every one element-set.*

(iv) $B_1 \subset B_2 \implies \mu(B_1) \leqslant \mu(B_2)$.

(v) $\mu(B_1 \cup B_2) = \max\{\mu(B_1), \mu(B_2)\}$.

(vi) $\mu(B_1 + B_2) \leq \mu(B_1) + \mu(B_2)$.

(vii) $\mu(\lambda B_2) = |\lambda| \mu(B_2)$, *where* $\lambda \in \mathbb{R}$.

(viii) $\mu(conv\ B_2) = \mu(B_2)$.

1.3 Some Fixed Point Theorems

Theorem 1.13 *([55]) (Banach's fixed point theorem). Let C be a non-empty closed subset of a Banach space X, then any contraction mapping F of C into itself has a unique fixed point.*

Theorem 1.14 *([55]) (Schaefer's fixed point theorem) Let X be a Banach space, and $F : X \to X$ completely continuous operator. If the set*

$$\varepsilon = \{y \in X : y = \lambda F y, \ \textit{for some } \lambda \in (0,1)\}$$

is bounded, then F has fixed point.

Theorem 1.15 *([55]) (Nonlinear Alternative of Leray–Schauder type). Let X be a Banach space with $C \subset X$ closed and convex. Assume U is a relatively open subset of C with $0 \in U$ and $N : \overline{U} \to C$ is a compact map. Then either,*

(i) N has a fixed point in $\overline{U}$; or

(ii) there is a point $u \in \partial U$ and $\nu \in (0,1)$ with $u = \nu N(u)$.

Theorem 1.16 *([53]) (Darbo's fixed point theorem). Let X be a Banach space and B be a bounded, closed, convex and nonempty subset of X. Suppose a continuous mapping $F : B \to B$ is such that for all closed subsets D of B,*

$$\mu(F(D)) \leq k\mu(D), \tag{1.3}$$

where $0 \leq k < 1$. Then F has a fixed point in B.

Remark 1.17 *Mappings satisfying the Darbo-condition (1.3) have subsequently been called k-set contractions.*

Theorem 1.18 *([82]) (Mönch's fixed point theorem). Let D be a bounded, closed and convex subset of a Banach space X such that $0 \in D$, and let F be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv} F(V) \ \textit{or} \ V = F(V) \cup \{0\} \Rightarrow \mu(V) = 0, \tag{1.4}$$

holds for every subset V of D, then F has a fixed point.

Chapter 2

Nonlinear Boundary Implicit Differential Equations with Caputo's Exponential Type Fractional Order in Banach Spaces

2.1 Introduction

In [26] by means of the Banach contraction principle, Benchohra and Bouriah studied the existence and Ulam stability of nonlinear fractional boundary value problem involving Caputos derivative

$$^cD^{\alpha}_{0^+}y(t) = \chi\big(t, y(t),\ ^cD^{\alpha}_{0^+}y(t)\big),\ \text{for each},\ t \in J := [0, T], T > 0,\ 0 < \alpha \leq 1,$$

$$d_1y(0) + d_2y(T) = d_3,$$

and

$$^cD^{\alpha}_{0^+}y(t) = \chi\big(t, y(t),\ ^cD^{\alpha}_{0^+}y(t)\big),\ \text{for each},\ t \in J,\ 0 < \alpha \leq 1,$$
$$y(0) + \vartheta(y) = y^*,$$

where $\chi : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\vartheta : C(J, \mathbb{R}) \to \mathbb{R}$ are a given functions and $y^*, d_1, d_2, d_3 \in \mathbb{R}$. In [24] by means of technique of measure of noncompactness and the fixed point theorems of Darbo and Mönch, the authors studied the existence of nonlinear fractional boundary value problem involving Caputo's derivative

$$^cD^{\rho}_{0^+}x(t) = \chi(t, x(t),\ ^cD^{\rho}_{0^+}x(t)),\ \text{for each},\ t \in J := [0, b], b > 0,\ 0 < \rho \leq 1,$$

$$d_1x(0) + d_2y(b) = d_3,$$

and

$$^cD^{\rho}_{0^+}x(t) = \chi\big(t, x(t),\ ^cD^{\rho}_{0^+}x(t)\big),\ \text{for each},\ t \in J, b > 0,\ 0 < \rho \leq 1,$$
$$x(0) + \vartheta(x) = x^*,$$

where $\chi : J \times E \times E \to E$, $\vartheta : C(J,E) \to E$ are a given functions and $d_1, d_2 \in \mathbb{R}$, $d_3, x^* \in E$, and $(E, \|\cdot\|)$ is a real Banach space.

This chapter is devoted to the results obtained by Malti *et al.* [74], we establish the existence of solutions for the following boundary value problem (**BVP**) of nonlinear implicit fractional differential equations (**NIFDE**) with Caputo's exponential type fractional derivative:

$$ {}^{e}_{c}D^{\varrho}_{0}\omega(t) = f\big(t, \omega(t),\ {}^{e}_{c}D^{\varrho}_{0}\omega(t)\big), \text{ for each, } t \in J := [0,b], b > 0,\ 0 < \varrho \leq 1, \tag{2.1}$$

$$ c_1\omega(0) + c_2\omega(b) = \delta, \tag{2.2}$$

where ${}^{e}_{c}D^{\varrho}_{0}$ is the Caputo's exponential type fractional derivative, $f : J \times E \times E \to E$ is a given function and c_1, c_2, are real constants with $c_1 + c_2 \neq 0$, and $\delta \in E$, where $(E, \|\cdot\|)$ is a real Banach space.

2.2 Existence Results

Let us start by defining what we mean by a solution of the problem (2.1)–(2.2).

Definition 2.1 *A function $\omega \in AC_e(J,E)$ is said to be a solution of the problem (2.1)–(2.2) is ω satisfied equation (2.1) on J and conditions (2.2).*

For the existence of solutions or the problem (2.1) – (2.2), we need the following auxiliary lemmas:

Lemma 2.2 *Let $0 < \varrho \leq 1$ and $\xi : J \to E$ be a continuous function. Then the linear fractional boundary value problem*

$$ {}^{e}_{c}D^{\varrho}_{0}\omega(t) = \xi(t), \text{ for each, } t \in J,\ 0 < \varrho \leq 1, \tag{2.3}$$

$$ c_1\omega(0) + c_2\omega(b) = \delta, \tag{2.4}$$

where c_1, c_2, are real constants with $c_1 + c_2 \neq 0$, and $\delta \in E$ has a unique solution given by

$$\begin{aligned} \omega(t) &= \frac{1}{\Gamma(\varrho)}\int_0^t (e^t - e^s)^{\varrho-1} e^s \xi(s) ds \\ &\quad - \frac{1}{(c_1+c_2)}\left[\frac{c_2}{\Gamma(\varrho)}\int_0^b (e^b - e^s)^{\varrho-1} e^s \xi(s) ds - \delta\right]. \end{aligned}$$

Proof. By integrating the formula (2.3), we get

$$ \omega(t) = \omega_0 + \frac{1}{\Gamma(\varrho)}\int_0^t (e^t - e^s)^{\varrho-1} e^s \xi(s) ds. \tag{2.5}$$

By (2.5), we get $c_1\omega(0) = c_1\omega_0$, and

$$c_2\omega(b) = c_2\omega_0 + \frac{c_2}{\Gamma(\varrho)} \int_0^b (e^b - e^s)^{\varrho-1} e^s \xi(s) ds.$$

Then by condition (2.4), we deduce

$$\omega_0 = -\frac{1}{(c_1 + c_2)} \left[\frac{c_2}{\Gamma(\varrho)} \int_0^b (e^b - e^s)^{\varrho-1} e^s \xi(s) ds - \delta \right].$$

Replacing in (2.5), we get

$$\omega(t) = \frac{1}{\Gamma(\varrho)} \int_0^t (e^t - e^s)^{\varrho-1} e^s \xi(s) ds - \frac{1}{(c_1 + c_2)} \left[\frac{c_2}{\Gamma(\varrho)} \int_0^b (e^b - e^s)^{\varrho-1} e^s \xi(s) ds - \delta \right].$$

Lemma 2.3 *Let a function $f(t,u,v) : J \times E \times E \to E$ be continuous. Then the problem (2.1)–(2.2) is equivalent to the problem:*

$$\omega(t) = \Psi + {}^{e}I_0^{\varrho}\vartheta(t) \tag{2.6}$$

where $\vartheta \in C(J,E)$ satisfies the functional equation:

$$\Psi = \frac{1}{(c_1 + c_2)} \left[\delta - \frac{c_2}{\Gamma(\varrho)} \int_0^b (e^b - e^s)^{\varrho-1} e^s \vartheta(s) ds \right]$$

and

$$\vartheta(t) = f\Big(t, \Psi + {}^{e}I_0^{\varrho}\vartheta(t), \vartheta(t)\Big).$$

Proof. Let ω be a solution of (2.6). Then $\omega(0) = \Psi$ and

$$\omega(b) = \Psi + \frac{1}{\Gamma(\varrho)} \int_0^b (e^b - e^s)^{\varrho-1} e^s \vartheta(s) ds.$$

So,

$$\begin{aligned}
c_1\omega(0) + c_2\omega(b) &= c_1\Psi + \left[c_2\Psi + \frac{c_2}{\Gamma(\varrho)} \int_0^b (e^b - e^s)^{\varrho-1} e^s \vartheta(s) ds \right] \\
&= (c_1 + c_2)\Psi + \frac{c_2}{\Gamma(\varrho)} \int_0^b (e^b - e^s)^{\varrho-1} e^s \vartheta(s) ds \\
&= \frac{(c_1 + c_2)}{(c_1 + c_2)} \left[\delta - \frac{c_2}{\Gamma(\varrho)} \int_0^b (e^b - e^s)^{\varrho-1} e^s \vartheta(s) ds \right] \\
&\quad + \frac{c_2}{\Gamma(\varrho)} \int_0^b (e^b - e^s)^{\varrho-1} e^s \vartheta(s) ds \\
&= \delta.
\end{aligned}$$

Then

$$c_1\omega(0) + c_2\omega(b) = \delta.$$

On the other hand, we have

$$\begin{aligned} {}^{e}_{c}D^{\varrho}_{0}\omega(t) &= {}^{e}_{c}D^{\varrho}_{0}\Big(\Psi + {}^{e}I^{\varrho}_{0}\vartheta(t)\Big) = \vartheta(t) \\ &= f\Big(t, y(t), {}^{e}_{c}D^{\varrho}_{0}\omega(t)\Big). \end{aligned}$$

Thus, ω is a solution of the problem (2.1)–(2.2).

Lemma 2.4 *([56]) If $V \subset C(J,E)$ is a bounded and equicontinuous set, then*

(i) the function $t \to \mu(V(t))$ is continuous on J, and

$$\mu_c(V) = \sup_{a\leq t\leq b} \mu(V(t)).$$

(ii) $\mu\left(\int_a^b \omega(s)ds : \omega \in V\right) \leq \int_a^b \mu(V(s))ds,$

where μ is the Kuratowski measure of noncompactness and

$$V(s) = \{\omega(s) : \omega \in V\},\ s \in J.$$

The following hypotheses will be used in the sequel:

(K_1) The function $f : J \times E \times E \to E$ is continuous.

(K_2) There exist constants $\ell_1 > 0$ and $0 < \ell_2 < 1$ such that

$$\|f(t,u,v) - f(t,\bar{u},\bar{v})\| \leq \ell_1\|u-\bar{u}\| + \ell_2\|v-\bar{v}\|, \quad \text{for any } u,v,\bar{u},\bar{v} \in E,\ t \in J.$$

Remark 2.5 *([18]) Conditions (K_2) is equivalent to the inequality*

$$\mu\big(f(t,B_1,B_2)\big) \leq \ell_1\mu(B_1) + \ell_2\mu(B_2),$$

for any bounded sets $B_1, B_2 \subseteq E$ and for each $t \in J$.

Now, we are in a position to state and prove our existence result for the problem (2.1)-(2.2) based on Darbo's fixed point theorem.
Set

$$\phi = \frac{\ell_1}{1-\ell_2},\ \Theta = \frac{(e^b-1)^{\varrho}}{\Gamma(\varrho+1)}\left(1 + \frac{|c_2|}{|c_1+c_2|}\right) \quad \text{and} \quad \overline{f} = \sup_{t\in J}\|f(t,0,0)\|.$$

Theorem 2.6 *Assume* (K_1) *and* (K_2) *holds. If*

$$\phi\Theta < 1, \tag{2.7}$$

then BVP (2.1)-(2.2) has at least one solution on J.

Proof. Transform the problem (2.1)-(2.2) into a fixed point problem. Define the operator $\Lambda : C(J,E) \to C(J,E)$ by

$$\begin{aligned} \Lambda(\omega)(t) &= \frac{\delta}{c_1+c_2} + \frac{1}{\Gamma(\varrho)}\int_0^t (e^t-e^s)^{\varrho-1}e^s\vartheta(s)ds \\ & -\frac{c_2}{(c_1+c_2)\Gamma(\varrho)}\int_0^b (e^b-e^s)^{\varrho-1}e^s\vartheta(s)ds, \end{aligned} \tag{2.8}$$

where $\vartheta \in C(J,E)$ such that

$$\vartheta(t) = f(t,\omega(t),\vartheta(t)).$$

Claim 1: Λ *is continuous.*
Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C(J,E)$. Then, for each $t \in J$:

$$\begin{aligned} \|\Lambda(u_n)(t)-\Lambda(u)(t)\| &\leq \frac{1}{\Gamma(\varrho)}\int_0^t (e^t-e^s)^{\varrho-1}e^s\|\vartheta_n(s)-\vartheta(s)\|ds \\ & +\frac{|c_2|}{|c_1+c_2|\Gamma(\varrho)}\int_0^b (e^b-e^s)^{\varrho-1}e^s\|\vartheta_n(s)-\vartheta(s)\|ds, \end{aligned} \tag{2.9}$$

where $\vartheta_n, \vartheta \in C(J,E)$ such that

$$\vartheta_n(t) = f(t,u_n(t),\vartheta_n(t)),$$

and

$$\vartheta(t) = f\big(t,u(t),\vartheta(t)\big).$$

By condition (K_2), we have

$$\begin{aligned} \|\vartheta_n(t)-\vartheta(t)\| &= \|f(t,u_n(t),\vartheta_n(t))-f(t,u(t),\vartheta(t))\| \\ &\leq \ell_1\|u_n(t)-u(t)\| + \ell_2\|\vartheta_n(t)-\vartheta(t)\|. \end{aligned}$$

Then

$$\|\vartheta_n(t)-\vartheta(t)\| \leq \phi\|u_n(t)-u(t)\|.$$

Since $u_n \to u$, then we get $\vartheta_n(t) \to \vartheta(t)$ as $n \to \infty$ for each $t \in J$, and let $\eta > 0$ be such that, for each $t \in J$, we have $\|\vartheta_n(t)\| \leq \eta$ and $\|\vartheta(t)\| \leq \eta$, then, we have

$$\begin{aligned} (e^t-e^s)^{\varrho-1}e^s\|\vartheta_n(s)-\vartheta(s)\| &\leq (e^t-e^s)^{\varrho-1}e^s\big[\|\vartheta_n(s)\|+\|\vartheta(s)\|\big] \\ &\leq 2\eta(e^t-e^s)^{\varrho-1}e^s. \end{aligned}$$

For each $t \in J$, the function $s \to 2\eta(e^t - e^s)^{\varrho-1}e^s$ is integrable on $[0, t]$, then the Lebesgue dominated convergence theorem and (2.9) imply that

$$\|\Lambda(u_n)(t) - \Lambda(u)(t)\| \to 0 \ as \ n \to \infty,$$

and hence

$$\|\Lambda(u_n) - \Lambda(u)\|_\infty \to 0 \text{ as } n \to \infty.$$

Consequently, Λ is continuous.
Let R be a constant such that

$$R \geq \left[\frac{|\delta|}{|c_1 + c_2|} + \frac{\overline{f}}{1 - \ell_2}\Theta\right]\left[1 - \Theta\phi\right]^{-1}, \tag{2.10}$$

and define

$$B_R = \{u \in C\,(J, E) : \|u\|_\infty \leq R\}.$$

It is clear that B_R is a bounded, closed and convex subset of $C\,(J, E)$.

Claim 2: $\Lambda(B_R) \subset B_R$. Let $u \in B_R$ we show that $\Lambda u \in B_R$. We have, for each $t \in J$

$$\begin{aligned}\|\Lambda u(t)\| \ &\leq \ \frac{|\delta|}{|c_1 + c_2|} + \frac{1}{\Gamma(\varrho)}\int_0^t (e^t - e^s)^{\varrho-1}e^s\|\vartheta(s)\|ds \\ &+\frac{|c_2|}{|c_1 + c_2|\Gamma(\varrho)}\int_0^b (e^b - e^s)^{\varrho-1}e^s\|\vartheta(s)\|ds.\end{aligned} \tag{2.11}$$

By condition (K_2), for each $t \in J$, we have that

$$\begin{aligned}\|\vartheta(t)\| &= \|f(t, u(t), \vartheta(t))\| \\ &\leq \|f(t, u(t), \vartheta(t)) - f(t, 0, 0)\| + \|f(t, 0, 0)\| \\ &\leq \ell_1\|u(t)| + \ell_2\|\vartheta(t)\| + \|f(t, 0, 0)\| \\ &\leq \ell_1 R + \ell_2\|\vartheta(t)\| + \overline{f}.\end{aligned}$$

Then

$$\begin{aligned}\|\vartheta(t)\| &\leq \frac{\ell_1 R + \overline{f}}{1 - \ell_2} \\ &= \phi R + \frac{\overline{f}}{1 - \ell_2} := \widetilde{M}.\end{aligned}$$

Thus, (2.10) and (2.11) implies that

$$\begin{aligned}
\|\Lambda u(t)\| &\leq \frac{|\delta|}{|c_1+c_2|} + \left(\phi R + \frac{\overline{f}}{1-\ell_2}\right)\frac{(e^b-1)^\varrho}{\Gamma(\varrho+1)} \\
&+ \left(\phi R + \frac{\overline{f}}{1-\ell_2}\right)\frac{|c_2|(e^b-1)^\varrho}{|c_1+c_2|\Gamma(\varrho+1)} \\
&\leq \frac{|\delta|}{|c_1+c_2|} + \frac{\phi(e^b-1)^\varrho}{\Gamma(\varrho+1)}\left[1+\frac{|c_2|}{|c_1+c_2|}\right]R \\
&+ \left(\frac{\overline{f}}{1-\ell_2}\right)\frac{(e^b-1)^\varrho}{\Gamma(\varrho+1)}\left[1+\frac{|c_2|}{|c_1+c_2|}\right] \\
&\leq \frac{|\delta|}{|c_1+c_2|} + \phi\Theta R + \left(\frac{\overline{f}}{1-\ell_2}\right)\Theta \\
&\leq R.
\end{aligned}$$

Then

$$\|\Lambda u\|_\infty \leq R.$$

Thus $\Lambda(B_R) \subset B_R$.

Claim 3: *$\Lambda(B_R)$ is bounded and equicontinuous.*
Let $\tau_1, \tau_2 \in J,\ \tau_1 < \tau_2$, and let $u \in B_R$. Then

$$\begin{aligned}
\|\Lambda(u)(\tau_2) - \Lambda(u)(\tau_1)\| &= \left\| \frac{1}{\Gamma(\varrho)}\int_0^{\tau_1}\left[(e^{\tau_2}-e^s)^{\varrho-1} - (e^{\tau_1}-e^s)^{\varrho-1}\right]e^s\vartheta(s)ds \right. \\
&\left. + \frac{1}{\Gamma(\varrho)}\int_{\tau_1}^{\tau_2}(e^{\tau_2}-e^s)^{\varrho-1}e^s\vartheta(s)ds \right\| \\
&\leq \frac{1}{\Gamma(\varrho)}\int_0^{\tau_1}\left|(e^{\tau_2}-e^s)^{\varrho-1} - (e^{\tau_1}-e^s)^{\varrho-1}\right|e^s\|\vartheta(s)\|ds \\
&+ \frac{1}{\Gamma(\varrho)}\int_{\tau_1}^{\tau_2}\left|(e^{\tau_2}-e^s)^{\varrho-1}\right|e^s\|\vartheta(s)\|ds \\
&\leq \frac{\widetilde{M}}{\Gamma(\varrho+1)}\Big[(e^{\tau_1}-1)^\varrho - (e^{\tau_2}-1)^\varrho + 2(e^{\tau_2}-e^{\tau_1})^\varrho\Big].
\end{aligned}$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero.

Claim 4: *The operator $\Lambda : B_R \to B_R$ is a contraction.*
Let $V \subset B_R$ and $t \in J$, then we have

$$\begin{aligned}\mu(\Lambda(V)(t)) &= \mu\Big(\{(\Lambda y)(t), y \in V\}\Big)\\ &\leq \frac{1}{\Gamma(\varrho)}\Big\{\int_0^t (e^t - e^s)^{\varrho-1}e^s\mu(\vartheta(s))ds, y \in V\Big\}.\end{aligned}$$

Then for each $s \in J$, the Remark 2.5 implies that

$$\begin{aligned}\mu\Big(\{\vartheta(s), y \in V\}\Big) &= \mu\Big(\{f(s, y(s), \vartheta(s)), y \in V\}\Big)\\ &\leq \ell_1\mu\Big(\{y(s), y \in V\}\Big) + \ell_2\mu\Big(\{\vartheta(s), y \in V\}\Big).\end{aligned}$$

Thus,

$$\begin{aligned}\mu\Big(\{\vartheta(s), y \in V\}\Big) &\leq \frac{\ell_1}{1-\ell_2}\mu\Big(\{y(s), y \in V\}\Big)\\ &\leq \phi\, \mu\Big(\{y(s), y \in V\}\Big).\end{aligned}$$

Then

$$\begin{aligned}\mu\Big(\Lambda(V)(t)\Big) &\leq \frac{\phi}{\Gamma(\varrho)}\left\{\int_0^t (e^t - e^s)^{\varrho-1}e^s\left\{\mu\left(y(s)\right)\right\}ds,\ y \in V\right\}\\ &\leq \frac{\phi\mu_c(V)}{\Gamma(\varrho)}\int_0^t (e^t - e^s)^{\varrho-1}e^s ds\\ &\leq \frac{\phi(e^b-1)^\varrho}{\Gamma(\varrho+1)}\mu_c(V).\end{aligned}$$

Therefore

$$\mu_c(\Lambda V) \leq \frac{\phi(e^b-1)^\varrho}{\Gamma(\varrho+1)}\mu_c(V).$$

So, by (2.7), the operator Λ is a contraction. As a consequence of Theorem 1.16, we deduce that Λ has a fixed point, which is solution to the problem (2.1) $-$ (2.2). This completes the proof.

Our next existence result is based on Mönch's fixed point theorem.

Theorem 2.7 *Assume (K_1), (K_2) and (2.7) holds. Then the BVP (2.1)-(2.2) has at least one solution on J.*

Proof. Consider the operator Λ defined in (2.8). We shall show that Λ satisfies the assumption of Mönch's fixed point theorem. We know that $\Lambda : B_R \to B_R$ is bounded and continuous, we need to prove that there the implication

$$V = \overline{conv}\Lambda(V) \quad \text{or} \quad V = \Lambda(V) \cup \{0\} \Rightarrow \mu(V) = 0$$

holds for every equicontinuous subset V of B_R. Now let V be a subset of B_R such that $V \subset \overline{conv}(\Lambda(V) \cup \{0\})$. V is bounded and equicontinuous and therefore the function $t \to v(t) = \mu(V(t))$ is continuous on J. By Lemma 2.4, Remark 2.5 and the properties of the measure μ, we have for each $t \in J$

$$\begin{aligned}
v(t) &\leq \mu(\Lambda(V)(t) \cup \{0\}) \\
&\leq \mu(\Lambda(V)(t)) \\
&\leq \mu\{(\Lambda y)(t), y \in V\} \\
&\leq \frac{\phi}{\Gamma(\varrho)} \int_0^t (e^t - e^s)^{\varrho-1} e^s \mu(\{y(s),\ y \in V\}) ds \\
&\leq \frac{\phi}{\Gamma(\varrho)} \int_0^t (e^t - e^s)^{\varrho-1} e^s v(s) ds \\
&\leq \frac{\phi \left(e^b - 1\right)^{\varrho}}{\Gamma(\varrho+1)} \|v\|_\infty \\
&\leq \phi\ \Theta\ \|v\|_\infty .
\end{aligned}$$

Therefore,

$$\|v\|_\infty \leq \phi\ \Theta\ \|v\|_\infty .$$

From (2.7), we get $v(t) = 0$ for each $t \in J$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzelà theorem, V is relatively compact in B_R. Applying now Theorem 1.18 we conclude that Λ has a fixed point $y \in B_R$, which is solution to the problem $(2.1) - (2.2)$. This completes the proof.

2.3 Ulam Stability

In this section, we are concerned with Ulam-Hyers (**U-H**) stability and Ulam-Hyers-Rasias (**U-H-R**) stability. So, we adopt the definitions in Rus [91] to our problem (2.1)-(2.2).

Definition 2.8 *The problem (2.1)-(2.2) is U-H stable if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $\varpi \in AC_e(J, E)$ of the inequality*

$$\|{}^e_c D^\varrho_0 \varpi(t) - f(t, \varpi(t),\ {}^e_c D^\varrho_0 \varpi(t))\| \leq \epsilon,\ \ t \in J, \tag{2.12}$$

there exists a solution $\omega \in AC_e(J, E)$ of equation (2.1) with

$$\|\varpi(t) - \omega(t)\|_E \leq c_f \epsilon,\ \ t \in J.$$

Definition 2.9 *Equation (2.1) is U-H-R stable with respect to $\varphi \in C(J, \mathbb{R}_+)$ if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $\varpi \in AC_e(J, E)$ of the inequality*

$$\|{}^e_c D^\varrho_0 \varpi(t) - f(t, \varpi(t),\ {}^e_c D^\varrho_0 \varpi(t))\| \leq \epsilon\varphi(t),\ t \in J, \tag{2.13}$$

there exists a solution $\omega \in AC_e(J, E)$ of problem (2.1)-(2.2) with

$$\|\varpi(t) - \omega(t)\|_E \leq c_f \epsilon \varphi(t),\ t \in J.$$

Remark 2.10 *A function $\varpi \in AC_e(J, E)$ is a solution of the inequality (2.12) if and only if there exists a function $g \in C(J, E)$ (which depend on ϖ) such that*

(i) $\|g(t)\| \leq \epsilon$, $\forall t \in J$.

(ii) ${}^e_c D^\varrho_0 \varpi(t) = f(t, \varpi(t),\ {}^e_c D^\varrho_0 \varpi(t)) + g(t)$, $t \in J$.

Theorem 2.11 *Assume that the assumptions (K1), (K2) and (2.7) hold, then the problem (2.1)-(2.2) is U-H stable.*

Proof. Let $\epsilon > 0$ and ϖ be a solution of the inequality:

$$\|{}^e_c D^\varrho_0 \varpi(t) - f(t, \varpi(t),\ {}^e_c D^\varrho_0 \varpi(t))\| \leq \epsilon,\ t \in J. \tag{2.14}$$

Let us denote by ω the unique solution for the following problem

$$\begin{gathered} {}^e_c D^\varrho_0 \omega(t) = f(t, \omega(t),\ {}^e_c D^\varrho_0 \omega(t)),\ \text{for each } t \in J,\ 0 < \varrho \leq 1, \\ \omega(0) = \varpi(0), \quad \omega(b) = \varpi(b). \end{gathered}$$

By using Lemma 2.3, we have

$$\begin{aligned} \omega(t) &= \Psi_\omega + {}^e I^\varrho_0 \vartheta_\omega(t) \\ &= \Psi_\omega + \frac{1}{\Gamma(\varrho)} \int_0^t (e^t - e^s)^{\varrho-1} e^s \vartheta_\omega(s) ds, \end{aligned}$$

where $\vartheta_\omega \in C(J, E)$ such that

$$\vartheta_\omega(t) = f(t, \Psi_\omega + {}^e I^\varrho_0 \vartheta_\omega(t), \vartheta_\omega(t))$$

and

$$\Psi_\omega = \frac{1}{(c_1 + c_2)} \left[\delta - \frac{c_2}{\Gamma(\varrho)} \int_0^b (e^b - e^s)^{\varrho-1} e^s \vartheta_\omega(s) ds \right].$$

Note that $\omega(0) = \varpi(0)$ and $\omega(b) = \varpi(b)$ implies $\Psi_\omega = \Psi_\varpi$.
Indeed, by (K_2), we have, for each $t \in J$

$$\begin{aligned} \|\vartheta_\varpi(t) - \vartheta_\omega(t)\| &= \|f(t, \varpi(t), \vartheta_\varpi(t)) - f(t, \omega(t), \vartheta_\omega(t))\| \\ &\leq \ell_1 \|\varpi(t) - \omega(t)\| + \ell_2 \|\vartheta_\varpi(t) - \vartheta_\omega(t)\|. \end{aligned}$$

Then

$$\|\vartheta_{\varpi}(t)-\vartheta_{\omega}(t)\|\leq\phi\|\varpi(t)-\omega(t)\|. \tag{2.15}$$

Thus,

$$\begin{aligned}
\|\Psi_{\omega}-\Psi_{\varpi}\| &\leq \frac{|\,c_2\,|}{|\,c_1+c_1\,|\,\Gamma(\varrho)}\int_0^b (e^b-e^s)^{\varrho-1}e^s\,\|\vartheta_{\omega}(s)-\vartheta_{\varpi}(s)\|\,ds\\
&\leq \frac{|\,c_2\,|\,\phi}{|\,c_1+c_2\,|\,\Gamma(\varrho)}\int_0^b (e^b-e^s)^{\varrho-1}e^s\,\|\omega(s)-\varpi(s)\|\,ds\\
&= \frac{|\,c_2\,|\,\phi}{|\,c_1+c_2\,|}\;{}^{e}I_0^{\varrho}\,\|\omega(b)-\varpi(b)\|\\
&= 0.
\end{aligned}$$

By integration of the formula (2.14), we obtain

$$\begin{aligned}
\left\|\varpi(t)-\Psi_{\varpi}-\frac{1}{\Gamma(\varrho)}\int_0^t (e^t-e^s)^{\varrho-1}e^s\vartheta_{\varpi}(s)ds\right\| &\leq \frac{\epsilon(e^t-1)^{\varrho}}{\Gamma(\varrho+1)}\\
&\leq \frac{\epsilon(e^b-1)^{\varrho}}{\Gamma(\varrho+1)},
\end{aligned}$$

where $\vartheta_{\varpi}\in C(J,E)$ such that

$$\vartheta_{\varpi}(t)=f(t,\Psi_{\varpi}+\;{}^{e}I_0^{\varrho}\vartheta_{\varpi}(t),\vartheta_{\varpi}(t)).$$

We have, for each $t\in J$

$$\begin{aligned}
\|\varpi(t)-\omega(t)\| &= \left\|\varpi(t)-\Psi_{\omega}-\frac{1}{\Gamma(\varrho)}\int_0^t (e^t-e^s)^{\varrho-1}e^s\vartheta_{\omega}(s)ds\right\|\\
&= \left\|\varpi(t)-\Psi_{\varpi}-\frac{1}{\Gamma(\varrho)}\int_0^t (e^t-e^s)^{\varrho-1}e^s\vartheta_{\varpi}(s)ds\right.\\
&\quad \left.+\frac{1}{\Gamma(\varrho)}\int_0^t (e^t-e^s)^{\varrho-1}e^s\left(\vartheta_{\varpi}(s)-\vartheta_{\omega}(s)\right)ds\right\|\\
&\leq \left\|\varpi(t)-\Psi_{\varpi}-\frac{1}{\Gamma(\varrho)}\int_0^t (e^t-e^s)^{\varrho-1}e^s\vartheta_{\omega}(s)ds\right\|\\
&\quad +\frac{1}{\Gamma(\varrho)}\int_0^t (e^t-e^s)^{\varrho-1}e^s\,\|\vartheta_{\varpi}(s)-\vartheta_{\omega}(s)\|\,ds.
\end{aligned}$$

Using (2.15), we obtain

$$\|\varpi(t)-\omega(t)\| \leq \frac{\epsilon(e^b-1)^\varrho}{\Gamma(\varrho+1)} + \frac{\phi}{\Gamma(\varrho)}\int_0^t (e^t-e^s)^{\varrho-1}e^s\,\|\varpi(s)-\omega(s)\|\,ds.$$

Then

$$\begin{aligned}\|\varpi(t)-\omega(t)\| &\leq \frac{\epsilon(e^b-1)^\varrho}{\Gamma(\varrho+1)} + \frac{\phi\,\|\varpi-\omega\|_E}{\Gamma(\varrho)}\int_0^t (e^t-e^s)^{\varrho-1}e^s ds \\ &\leq \frac{\epsilon(e^b-1)^\varrho}{\Gamma(\varrho+1)} + \frac{\phi\parallel \varpi-\omega\parallel_E (e^b-1)^\varrho}{\Gamma(\varrho+1)}.\end{aligned}$$

So

$$\|\varpi-\omega\|_E \leq \frac{\epsilon(e^b-1)^\varrho}{\Gamma(\varrho+1)} + \frac{\phi\parallel \varpi-\omega\parallel_E (e^b-1)^\varrho}{\Gamma(\varrho+1)}.$$

Thus,

$$\|\varpi-\omega\|_E\left[1-\frac{\phi(e^b-1)^\varrho}{\Gamma(\varrho+1)}\right] \leq \frac{\epsilon(e^b-1)^\varrho}{\Gamma(\varrho+1)}.$$

By (2.7), we obtain

$$\|\varpi-\omega\|_E \leq \frac{(e^b-1)^\varrho}{\Gamma(\varrho+1)}\left[1-\frac{\phi(e^b-1)^\varrho}{\Gamma(\varrho+1)}\right]^{-1}\epsilon := c_f\epsilon.$$

Therefore, the problem (2.1)-(2.2) is U-H stable. This completes the proof.

Theorem 2.12 *Assume (K_1), (K_2), (2.7) and*

(K_3) *There exists an increasing function $\varphi \in C(J, \mathbb{R}_+)$ and there exists $\lambda_\varphi > 0$ such that for each $t \in J$, we have*

$$^{e}I_0^\varrho\varphi(t) \leq \lambda_\varphi\varphi(t)$$

are satisfied. Then the problem (2.1)-(2.2) is U-H-R stable with respect to φ.

Proof. Let ϖ be a solution of the following inequality

$$\|{}_c^eD_0^\varrho\varpi(t) - f(t,\varpi(t),\ {}_c^eD_0^\varrho\varpi(t))\| \leq \epsilon\varphi(t),\ t\in J. \tag{2.16}$$

Let us denote by ω the unique solution of the problem

$${}_c^eD_0^\varrho\omega(t) = f(t,\omega(t),\ {}_c^eD_0^\varrho\omega(t)),\ \text{for each } t\in J,\ 0<\varrho\leq 1,$$

$$\omega(0)=\varpi(0),\ \omega(b)=\varpi(b).$$

By using Lemma 2.3, we have

$$\omega(t) = \Psi_\omega + \frac{1}{\Gamma(\varrho)}\int_0^t (e^t-e^s)^{\varrho-1}e^s\vartheta_\omega(s)ds,$$

where $\vartheta_\omega \in C(J,E)$ such that

$$\vartheta_\omega(t) = f(t, \Psi_\omega + {}^eI_0^\varrho \vartheta_\omega(t), \vartheta_\omega(t))$$

and

$$\Psi_\omega = \frac{1}{(c_1+c_2)}\left[\delta - \frac{c_2}{\Gamma(\varrho)}\int_0^b (e^b-e^s)^{\varrho-1}e^s\vartheta_\omega(s)ds\right].$$

By integration of the formula (2.16), we obtain

$$\begin{aligned}\left\|\varpi(t) - \Psi_\varpi - \frac{1}{\Gamma(\varrho)}\int_0^t (e^t-e^s)^{\varrho-1}e^s\vartheta_\varpi(s)ds\right\| &\leq \frac{\epsilon}{\Gamma(\varrho)}\int_0^b (e^b-e^s)^{\varrho-1}e^s\varphi(s)ds \\ &\leq \epsilon\lambda_\varphi\varphi(t). \qquad (2.17)\end{aligned}$$

We have, for each $t \in J$

$$\begin{aligned}\|\varpi(t)-\omega(t)\| &= \left\|\varpi(t) - \Psi_\omega - \frac{1}{\Gamma(\varrho)}\int_0^t (e^t-e^s)^{\varrho-1}e^s\vartheta_\omega(s)ds\right\| \\ &= \left\|\varpi(t) - \Psi_\varpi - \frac{1}{\Gamma(\varrho)}\int_0^t (e^t-e^s)^{\varrho-1}e^s\vartheta_\varpi(s)ds\right. \\ &\quad \left. + \frac{1}{\Gamma(\varrho)}\int_0^t (e^t-e^s)^{\varrho-1}e^s\left(\vartheta_\varpi(s)-\vartheta_\omega(s)\right)ds\right\| \\ &\leq \left\|\varpi(t) - \Psi_\varpi - \frac{1}{\Gamma(\varrho)}\int_0^t (e^t-e^s)^{\varrho-1}e^s\vartheta_\varpi(s)ds\right\| \\ &\quad + \frac{1}{\Gamma(\varrho)}\int_0^t (e^t-e^s)^{\varrho-1}e^s\left\|\vartheta_\varpi(s)-\vartheta_\omega(s)\right\|ds.\end{aligned}$$

Using (2.15) and (2.17), we obtain

$$\|\varpi(t)-\omega(t)\| \leq \epsilon\lambda_\varphi\varphi(t) + \frac{\phi}{\Gamma(\varrho)}\int_0^t (e^t-e^s)^{\varrho-1}e^s\left\|\varpi(s)-\omega(s)\right\|ds.$$

Then

$$\begin{aligned}\|\varpi(t)-\omega(t)\| &\leq \epsilon\lambda_\varphi\varphi(t) + \frac{\phi \parallel \varpi-\omega \parallel_E}{\Gamma(\varrho)}\int_0^t (e^t-e^s)^{\varrho-1}e^s ds \\ &\leq \epsilon\lambda_\varphi\varphi(t) + \frac{\phi \parallel \varpi-\omega \parallel_E (e^b-1)^\varrho}{\Gamma(\varrho+1)}.\end{aligned}$$

So

$$\parallel \varpi-\omega \parallel_E \;\leq\; \epsilon\lambda_\varphi\varphi(t) + \frac{\phi \parallel \varpi-\omega \parallel_E (e^b-1)^\varrho}{\Gamma(\varrho+1)}.$$

Thus,

$$||\varpi - \omega||_E \left[1 - \frac{\phi(e^b - 1)^\varrho}{\Gamma(\varrho + 1)}\right] \leq \epsilon \lambda_\varphi \varphi(t).$$

From the condition (2.7), it follows that

$$||\varpi - \omega||_E \leq \epsilon \lambda_\varphi \varphi(t) \left[1 - \frac{\phi(e^b - 1)^\varrho}{\Gamma(\varrho + 1)}\right]^{-1}.$$

Then, for each $t \in J$

$$\|\varpi(t) - \omega(t)\|_E \leq \lambda_\varphi \epsilon \varphi(t) \left[1 - \frac{\phi(e^b - 1)^\varrho}{\Gamma(\varrho + 1)}\right]^{-1} := c_f \epsilon \varphi(t).$$

Therefore, the problem (2.1)-(2.2) is U-H-R stable with respect to φ. This completes the proof.

Remark 2.13 *Our results for the boundary value problem (2.1)-(2.2) remain true for the following cases:*

- *Initial value problem:* $c_1 = 1, c_2 = 0$ *and* δ *arbitrary.*
- *Terminal value problem:* $c_1 = 0, c_2 = 1$ *and* δ *arbitrary.*
- *Anti-periodic problem:* $c_1 = c_2 \neq 0$ *and* $\delta = 0$.

However, our results are not applicable for the periodic problem, i.e. for $c_1 = 1,\ c_2 = -1,$ *and* $\delta = 0$.

2.4 An Example

In this section, we will give an example to illustrate our main results. Let

$$E = l^1 = \left\{\omega = (\omega_1, \omega_2, \ldots, \omega_n, \ldots) : \sum_{n=1}^{\infty} |\omega_n| < \infty\right\}$$

be the Banach space with the norm

$$\|\omega\|_E = \sum_{n=1}^{\infty} |\omega_n|.$$

Consider the following boundary value problem for the nonlinear implicit fractional differential equation:

$$ {}_{c}^{e}D_{0}^{\frac{1}{2}}\omega_n(t) = \frac{(3+|\omega_n(t)|+|^{c}D^{\frac{1}{2}}\omega_n(t)|)}{3e^{t+200}(1+|\omega_n(t)|+|^{c}D^{\frac{1}{2}}\omega_n(t)|)}, \text{ for each, } t\in[0,1], \tag{2.18} $$

$$ \omega_n(0)+\omega_n(1)=1. \tag{2.19} $$

where $J=[0,1]$, $b=1$, $c_1=c_2=\delta=1$, $\omega=(\omega_1,\omega_2,\ldots,\omega_n,\ldots)$, $f=(f_1,f_2,\ldots,f_n,\ldots)$, ${}_{c}^{e}D_{0}^{\frac{1}{2}}\omega=\left({}_{c}^{e}D_{0}^{\frac{1}{2}}\omega_1,\ {}_{c}^{e}D_{0}^{\frac{1}{2}}\omega_2,\ldots,\ {}_{c}^{e}D_{0}^{\frac{1}{2}}\omega_n,\ldots\right)$ and

$$ f(t,u,v)=\frac{(3+||u||+||v||)}{3e^{t+200}(1+||u||+||v||)}, \quad t\in[0,1],\ u,v\in E. $$

For any $u,v,\bar{u},\bar{v}\in E$ and $t\in[0,1]$, we can show that

$$ ||f(t,u,v)-f(t,\bar{u},\bar{v})||\le\frac{2}{3e^{200}}\big(\|u-\bar{u}\|_E+\|v-\bar{v}\|_E\big). $$

Thus, for $\ell_1=\ell_2=\dfrac{2}{3e^{200}}$, we have

$$ \Theta\phi=\frac{(e^b-1)^{\varrho}}{\Gamma(\varrho+1)}\left(1+\frac{|\mu|}{|\lambda+\mu|}\right)\frac{\ell_1}{1-\ell_2}=\sqrt{\frac{e-1}{\pi}}\times\frac{6}{3e^{200}-2}\approx 2.047\times 10^{-87}<1, $$

Hence, from Theorem 2.6. The boundary value problem (2.18)-(2.19) has at least one solution on J. Also, from Theorem 2.11, this BVP is U-H stable.

Chapter 3

Impulsive Boundary Value Problem for Nonlinear Implicit Caputo's Exponential Type Fractional Differential Equations

3.1 Introduction

This chapter is devoted to the results obtained by Malti *et al.* [75], we establish the existence and uniqueness results to the following boundary value problem (**BVP**) for nonlinear implicit fractional differential equations (**NIFDE**) with impulses and Caputo's exponential type fractional derivative:

$$
{}_{c}^{e}D_{t_k}^{\alpha}\varpi(t) = f(t, \varpi(t),\ {}_{c}^{e}D_{t_k}^{\alpha}\varpi(t)), \quad \text{for each } t \in J_k \subseteq J,\ k = 0, 1, \ldots, m, \tag{3.1}
$$

$$
\Delta\varpi|_{t=t_k} = I_k\left(\varpi\left(t_k^-\right)\right), \quad k = 1, \ldots, m, \tag{3.2}
$$

$$
c_1\varpi(a) + c_2\varpi(b) = c_3, \tag{3.3}
$$

where $a = t_0 < t_1 < \ldots < t_m < t_{m+1} = b$, ${}_{c}^{e}D_{a^+}^{\alpha}$ denote the Caputo's exponential type fractional derivative of order α, $0 < \alpha \leq 1$, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function and c_1, c_2, c_3 are real constants with $c_1 + c_2 \neq 0$, $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, $J_0 = [a, t_1]$, $J = [a, b]$, $\Delta\varpi|_{t=t_k} = \varpi\left(t_k^+\right) - \varpi\left(t_k^-\right)$, $\varpi\left(t_k^+\right) = \lim\limits_{h\to 0} \varpi(t_k + h)$, and $\varpi\left(t_k^-\right) = \lim\limits_{h\to 0^-} \varpi(t_k + h)$ represent the right and left limits of $\varpi(t)$ at $t = t_k$, respectively.

3.2 Existence Results

The following notations which are useful on this chapter. Consider the set of function

$$PC(J,\mathbb{R}) = \Big\{\varpi : J \to \mathbb{R},\ \varpi \in C\left(J_k,\mathbb{R}\right),\ k=0,\ldots,m \text{ and there exist } \varpi(t_k^+) \text{ and } \varpi(t_k^-), k=1,\ldots,m \text{ with } \varpi(t_k^-)=\varpi(t_k)\Big\}.$$

This set is the Banach space with the norm

$$\|\varpi\|_{PC} = \max_{k=0,1,\ldots,m}\Big\{\sup_{t\in J_k}|\varpi(t)|\Big\}.$$

Now, we defining what we mean by a solution of the problem (3.1)-(3.3).

Definition 3.1 *A function $\varpi \in PC\left(J,\mathbb{R}\right) \cap \Big(\cup_{k=0}^{m} AC_e\left(J_k,\mathbb{R}\right)\Big)$ is said a solution of (3.1)-(3.3) if ϖ satisfies the equation ${}_c^eD_{a^+}^{\alpha}\varpi(t) = f(t,\varpi(t),\ {}_c^eD_{a^+}^{\alpha}\varpi(t))$, on J_k and the conditions*

$$\Delta\varpi|_{t=t_k} = I_k\left(\varpi\left(t_k^-\right)\right), \quad \textit{for } k=1,\ldots,m,$$
$$c_1\varpi(a)+c_2\varpi(b)=c_3,$$

To prove the existence of solutions to (3.1)-(3.3), we need the following auxiliary lemmas.

Lemma 3.2 *Let $0<\alpha\leq 1$ and let $\varphi : J \to \mathbb{R}$ be continuous. A function ϖ is a solution of the integral equation*

$$\varpi(t) = \begin{cases} \dfrac{-1}{c_1+c_2}\left[c_2\displaystyle\sum_{i=1}^{m} I_i\left(\varpi\left(t_i^-\right)\right) + c_2\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\left(e^{t_i}-e^s\right)^{\alpha-1}\frac{\varphi(s)}{\Gamma(\alpha)}e^s ds \right. \\ \left. +c_2\displaystyle\int_{t_m}^{b}\left(e^{b}-e^s\right)^{\alpha-1}\frac{\varphi(s)}{\Gamma(\alpha)}e^s ds - c_3\right] + \int_a^t\left(e^{t}-e^s\right)^{\alpha-1}\frac{\varphi(s)}{\Gamma(\alpha)}e^s ds, \ \textit{if } t\in[a,t_1], \\ \\ \dfrac{-1}{c_1+c_2}\left[c_2\displaystyle\sum_{i=1}^{m} I_i\left(\varpi\left(t_i^-\right)\right) + c_2\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\left(e^{t_i}-e^s\right)^{\alpha-1}\frac{\varphi(s)}{\Gamma(\alpha)}e^s ds \right. \\ \left. +c_2\displaystyle\int_{t_m}^{b}\left(e^{b}-e^s\right)^{\alpha-1}\frac{\varphi(s)}{\Gamma(\alpha)}e^s ds - c_3\right] + \sum_{i=1}^{k} I_i\left(\varpi\left(t_i^-\right)\right) \\ +\displaystyle\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(e^{t_i}-e^s\right)^{\alpha-1}\frac{\varphi(s)}{\Gamma(\alpha)}e^s ds + \int_{t_k}^t\left(e^{t}-e^s\right)^{\alpha-1}\frac{\varphi(s)}{\Gamma(\alpha)}e^s ds, \ \textit{if } t\in(t_k,t_{k+1}], \end{cases} \tag{3.4}$$

where $k=1,\ldots,m$, if and only if, ϖ is a solution of the fractional BVP

$${}_c^eD_{t_k}^{\alpha}\varpi(t) = \varphi(t), \quad t\in J_k, \tag{3.5}$$

$$\Delta\varpi|_{t=t_k} = I_k\left(\varpi\left(t_k^-\right)\right), \quad for\ k = 1, \ldots, m, \tag{3.6}$$

$$c_1\varpi(a) + c_2\varpi(b) = c_3. \tag{3.7}$$

Proof: Assume that ϖ satisfies (3.5)-(3.7). If $t \in [a, t_1]$, then

$${}_{c}^{e}D_a^{\alpha}\varpi(t) = \varphi(t).$$

By Lemma 1.10, we get

$$\varpi(t) = \eta_0 + {}^{e}I_a^{\alpha}\varphi(t) = \eta_0 + \frac{1}{\Gamma(\alpha)}\int_a^t \left(e^t - e^s\right)^{\alpha-1}\varphi(s)e^s ds.$$

If $t \in (t_1, t_2]$, then by Lemma 1.10 we get

$$\begin{aligned}
\varpi(t) &= \varpi\left(t_1^+\right) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t \left(e^t - e^s\right)^{\alpha-1}\varphi(s)e^s ds \\
&= \Delta\varpi|_{t=t_1} + \varpi\left(t_1^-\right) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t \left(e^t - e^s\right)^{\alpha-1}\varphi(s)e^s ds \\
&= I_1\left(\varpi\left(t_1^-\right)\right) + \left[\eta_0 + \frac{1}{\Gamma(\alpha)}\int_a^{t_1} \left(e^{t_1} - e^s\right)^{\alpha-1}\varphi(s)e^s ds\right] \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t \left(e^t - e^s\right)^{\alpha-1}\varphi(s)e^s ds \\
&= \eta_0 + I_1\left(\varpi\left(t_1^-\right)\right) + \frac{1}{\Gamma(\alpha)}\int_a^{t_1} \left(e^{t_1} - e^s\right)^{\alpha-1}\varphi(s)e^s ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t \left(e^t - e^s\right)^{\alpha-1}\varphi(s)e^s ds.
\end{aligned}$$

If $t \in (t_2, t_3]$, then by Lemma 1.10 we get

$$\begin{aligned}
\varpi(t) &= \varpi\left(t_2^+\right) + \frac{1}{\Gamma(\alpha)}\int_{t_2}^t \left(e^t - e^s\right)^{\alpha-1}\varphi(s)e^s ds \\
&= \Delta\varpi|_{t=t_2} + \varpi\left(t_2^-\right) + \frac{1}{\Gamma(\alpha)}\int_{t_2}^t \left(e^t - e^s\right)^{\alpha-1}\varphi(s)e^s ds \\
&= I_2\left(\varpi\left(t_2^-\right)\right) + \left[\eta_0 + I_1\left(\varpi\left(t_1^-\right)\right) + \frac{1}{\Gamma(\alpha)}\int_a^{t_1} \left(e^{t_1} - e^s\right)^{\alpha-1}\varphi(s)e^s ds\right. \\
&\quad \left. + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} \left(e^{t_2} - e^s\right)^{\alpha-1}\varphi(s)e^s ds\right] + \frac{1}{\Gamma(\alpha)}\int_{t_2}^t \left(e^t - e^s\right)^{\alpha-1}\varphi(s)e^s ds \\
&= \eta_0 + \left[I_1\left(\varpi\left(t_1^-\right)\right) + I_2\left(\varpi\left(t_2^-\right)\right)\right] + \left[\frac{1}{\Gamma(\alpha)}\int_a^{t_1} \left(e^{t_1} - e^s\right)^{\alpha-1}\varphi(s)e^s ds\right. \\
&\quad \left. + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} \left(e^{t_2} - e^s\right)^{\alpha-1}\varphi(s)e^s ds\right] + \frac{1}{\Gamma(\alpha)}\int_{t_2}^t \left(e^t - e^s\right)^{\alpha-1}\varphi(s)e^s ds.
\end{aligned}$$

Repeating the process in this ways, the solution $\varpi(t)$ for $t \in (t_k, t_{k+1}]$ where $k = 1, \ldots, m$, can be written as

$$\begin{aligned}\varpi(t) &= \eta_0 + \sum_{i=1}^{k} I_i\left(\varpi\left(t_i^-\right)\right) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(e^{t_i} - e^s\right)^{\alpha-1} \varphi(s) e^s ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(e^{t} - e^s\right)^{\alpha-1} \varphi(s) e^s ds.\end{aligned}$$

It clear that

$$\varpi(a) = \eta_0$$

and

$$\begin{aligned}\varpi(b) &= \eta_0 + \sum_{i=1}^{m} I_i\left(\varpi\left(t_i^-\right)\right) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \left(e^{t_i} - e^s\right)^{\alpha-1} \varphi(s) e^s ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_m}^{b} \left(e^{b} - e^s\right)^{\alpha-1} \varphi(s) e^s ds.\end{aligned}.$$

Hence, by applying the boundary conditions $c_1\varpi(a) + c_2\varpi(b) = c_3$, we get

$$\begin{aligned}c_3 &= \eta_0(c_1 + c_2) + c_2 \sum_{i=1}^{m} I_i\left(\varpi\left(t_i^-\right)\right) + \frac{c_2}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \left(e^{t_i} - e^s\right)^{\alpha-1} \varphi(s) e^s ds \\ &\quad + \frac{c_2}{\Gamma(\alpha)} \int_{t_m}^{b} \left(e^{b} - e^s\right)^{\alpha-1} \varphi(s) e^s ds.\end{aligned}$$

Then

$$\begin{aligned}\eta_0 &= \frac{-1}{c_1 + c_2} \left[c_2 \sum_{i=1}^{m} I_i\left(\varpi\left(t_i^-\right)\right) + \frac{c_2}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \left(e^{t_i} - e^s\right)^{\alpha-1} \varphi(s) e^s ds \right. \\ &\quad \left. + \frac{c_2}{\Gamma(\alpha)} \int_{t_m}^{b} \left(e^{b} - e^s\right)^{\alpha-1} \varphi(s) e^s ds - c_3 \right].\end{aligned}$$

Thus, if $t \in (t_k, t_{k+1}]$, where $k = 1, \ldots, m$, then

$$\begin{aligned}\varpi(t) &= \frac{-1}{c_1 + c_2} \left[c_2 \sum_{i=1}^{m} I_i\left(\varpi\left(t_i^-\right)\right) + \frac{c_2}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \left(e^{t_i} - e^s\right)^{\alpha-1} \varphi(s) e^s ds \right. \\ &\quad \left. + \frac{c_2}{\Gamma(\alpha)} \int_{t_m}^{b} \left(e^{b} - e^s\right)^{\alpha-1} \varphi(s) e^s ds - c_3 \right] + \sum_{i=1}^{k} I_i\left(\varpi\left(t_i^-\right)\right) \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(e^{t_i} - e^s\right)^{\alpha-1} \varphi(s) e^s ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(e^{t} - e^s\right)^{\alpha-1} \varphi(s) e^s ds.\end{aligned}$$

Conversely, assume that ϖ satisfies the impulsive fractional integral equation (3.4). If $t \in [a, t_1]$ then $c_1\varpi(a) + c_2\varpi(b) = c_3$ and using the fact that ${}_c^eD_a^\alpha$ is the left inverse of ${}^eI_a^\alpha$ we get

$${}_c^eD_a^\alpha \varpi(t) = \varphi(t), \quad \text{for each } t \in [a, t_1].$$

If $t \in (t_k, t_{k+1}]$, where $k = 1, \ldots, m$. Then, by using the fact that ${}^{e}_{c}D^{\alpha}_{t_k} C = 0$, where C is a constant and ${}^{e}_{c}D^{\alpha}_{t_k}$ is the left inverse of ${}^{e}I^{\alpha}_{t_k}$, we get

$$ {}^{e}_{c}D^{\alpha}_{t_k} \varpi(t) = \varphi(t), \quad \text{for each } t \in (t_k, t_{k+1}]. $$

Also, we can easily show that

$$ \Delta \varpi|_{t=t_k} = I_k\left(\varpi\left(t_k^-\right)\right), \quad k = 1, \ldots, m. $$

Now, we pass to state and proof our first existence result for the problem (3.1)-(3.3) based on the Banach contraction principle.

The following hypotheses will be used in the sequel:

(H1) The function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H2) There exist constants $k_1 > 0$ and $0 < k_2 < 1$ such that

$$ |f(t, \varpi_1, \omega_1) - f(t, \varpi_2, \omega_2)| \le k_1 |\varpi_1 - \varpi_2| + k_2 |\omega_1 - \omega_2|, $$

for any $\varpi_1, \varpi_2, \omega_1, \omega_2 \in \mathbb{R}$ and $t \in J$.

(H3) There exists a constant $\xi > 0$ such that

$$ |I_k(\varpi_1) - I_k(\varpi_2)| \le \xi |\varpi_1 - \varpi_2|, $$

for each $\varpi_1, \varpi_2 \in \mathbb{R}$ and $k = 1, 2, \ldots, m$.

Set

$$ \gamma = \frac{k_1}{1 - k_2}, \quad \mu_1 = \frac{|c_2|}{|c_1 + c_2|} + 1 \quad \text{and} \quad \mu_2 = \frac{\gamma\,(m+1)\left(e^b - e^a\right)^{\alpha}}{\Gamma\left(\alpha + 1\right)}. $$

Theorem 3.3 *Assume that (H1)-(H3) are satisfied. If*

$$ \mu_1\left(m\xi + \mu_2\right) < 1, \tag{3.8} $$

then the boundary value problem (3.1)-(3.3) has a unique solution on J.

Proof. Transform the problem (3.1)-(3.3) into a fixed point problem, consider the operator $\Theta : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$, defined by

$$ \begin{aligned} \Theta(\varpi)(t) \;=\; & \frac{-1}{c_1 + c_2}\left[c_2 \sum_{i=1}^{m} I_i\left(\varpi\left(t_i^-\right)\right) + \frac{c_2}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \left(e^{t_i} - e^s\right)^{\alpha-1} \varphi(s) e^s ds \right. \\ & \left. + \frac{c_2}{\Gamma(\alpha)} \int_{t_m}^{b} \left(e^b - e^s\right)^{\alpha-1} \varphi(s) e^s ds - c_3 \right] + \sum_{a<t_k<t} I_k\left(\varpi\left(t_k^-\right)\right) \\ & + \frac{1}{\Gamma(\alpha)} \sum_{a<t_k<t} \int_{t_{k-1}}^{t_k} \left(e^{t_k} - e^s\right)^{\alpha-1} \varphi(s) e^s ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(e^t - e^s\right)^{\alpha-1} \varphi(s) e^s ds, \end{aligned} \tag{3.9} $$

where $\varphi \in C(J, \mathbb{R})$ satisfies

$$\varphi(t) = f(t, \varpi(t), \varphi(t)).$$

It clear that, the fact of finding solutions for problem (3.1)-(3.3) is to find the fixed points of the operator Θ. Now, for $\varpi_1, \varpi_2 \in PC(J, \mathbb{R})$ and for each $t \in J$, we have

$$\begin{aligned}
|\Theta(\varpi_1)(t) - \Theta(\varpi_2)(t)| \leq\ & \frac{|c_2|}{|c_1 + c_2|}\left[\sum_{i=1}^{m} \left|I_i\left(\varpi_1\left(t_i^-\right)\right) - I_i\left(\varpi_2\left(t_i^-\right)\right)\right|\right. \\
& + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \left(e^{t_i} - e^s\right)^{\alpha-1} e^s |\varphi_1(s) - \varphi_2(s)| ds \\
& \left. + \frac{1}{\Gamma(\alpha)} \int_{t_m}^{b} \left(e^b - e^s\right)^{\alpha-1} e^s |\varphi_1(s) - \varphi_2(s)| ds\right] \\
& + \sum_{a<t_k<t} \left|I_k\left(\varpi_1\left(t_k^-\right)\right) - I_k\left(\varpi_2\left(t_k^-\right)\right)\right| \\
& + \frac{1}{\Gamma(\alpha)} \sum_{a<t_k<t} \int_{t_{k-1}}^{t_k} \left(e^{t_k} - e^s\right)^{\alpha-1} e^s |\varphi_1(s) - \varphi_2(s)| ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(e^t - e^s\right)^{\alpha-1} e^s |\varphi_1(s) - \varphi_2(s)| ds,
\end{aligned}$$

where $\varphi_1, \varphi_2 \in C(J, \mathbb{R})$ are such that

$$\varphi_1(t) = f(t, \varpi_1(t), \varphi_1(t)),$$

and

$$\varphi_2(t) = f(t, \varpi_2(t), \varphi_2(t)).$$

By (H2) we have

$$\begin{aligned}
|\varphi_1(s) - \varphi_2(s)| &= |f(t, \varpi_1(t), \varphi_1(t)) - f(t, \varpi_2(t), \varphi_2(t))| \\
&\leq k_1|\varpi_1(t) - \varpi_2(t)| + k_2|\varphi_1(t) - \varphi_2(t)|.
\end{aligned}$$

Then

$$|\varphi_1(s) - \varphi_2(s)| \leq \gamma\ |\varpi_1(s) - \varpi_2(s)|. \tag{3.10}$$

Hence, for each $t \in J$,

$$
\begin{aligned}
|\Theta(\varpi_1)(t) - \Theta(\varpi_2)(t)| \;&\le\; \frac{|c_2|}{|c_1+c_2|}\left[\sum_{k=1}^{m} \xi \left|\varpi_1\left(t_k^-\right) - \varpi_2\left(t_k^-\right)\right|\right. \\
&+\frac{\gamma}{\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\left(e^{t_k}-e^s\right)^{\alpha-1} e^s \left|\varpi_1(s)-\varpi_2(s)\right| ds \\
&\left.+\frac{\gamma}{\Gamma(\alpha)}\int_{t_m}^{b}\left(e^{b}-e^s\right)^{\alpha-1} e^s \left|\varpi_1(s)-\varpi_2(s)\right| ds\right] \\
&+\sum_{i=1}^{m} \xi \left|\varpi_1\left(t_i^-\right) - \varpi_2\left(t_i^-\right)\right| \\
&+\frac{\gamma}{\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\left(e^{t_k}-e^s\right)^{\alpha-1} e^s \left|\varpi_1(s)-\varpi_2(s)\right| ds \\
&+\frac{\gamma}{\Gamma(\alpha)}\int_{t_k}^{t}\left(e^{t}-e^s\right)^{\alpha-1} e^s \left|\varpi_1(s)-\varpi_2(s)\right| ds \\
&\le\; \frac{|c_2|}{|c_1+c_2|}\left[m\xi + \frac{\gamma m\left(e^b-e^a\right)^{\alpha}}{\Gamma(\alpha+1)} + \frac{\gamma\left(e^b-e^a\right)^{\alpha}}{\Gamma(\alpha+1)}\right]\left\|\varpi_1-\varpi_2\right\|_{PC} \\
&+\left[m\xi + \frac{\gamma m\left(e^b-e^a\right)^{\alpha}}{\Gamma(\alpha+1)} + \frac{\gamma\left(e^b-e^a\right)^{\alpha}}{\Gamma(\alpha+1)}\right]\left\|\varpi_1-\varpi_2\right\|_{PC} \\
&=\; \left(\frac{|c_2|}{|c_1+c_2|}+1\right)\left[m\xi + \frac{\gamma\left(m+1\right)\left(e^b-e^a\right)^{\alpha}}{\Gamma(\alpha+1)}\right]\left\|\varpi_1-\varpi_2\right\|_{PC}.
\end{aligned}
$$

Thus,

$$
\left\|\Theta(\varpi_1) - \Theta(\varpi_2)\right\|_{PC} \;\le\; \mu_1\left(m\xi + \mu_2\right)\left\|\varpi_1-\varpi_2\right\|_{PC}.
$$

By (3.8), the operator Θ is a contraction. Hence, by Banach's contraction principle, we deduce that Θ has a unique fixed point which is a unique solution of (3.1)-(3.3). This completes the proof.

The second existence result is based on Schaefer's fixed point theorem. Set

$$
\widetilde{f} = \sup_{t\in J}\left|f\left(t,0,0\right)\right| \qquad \text{and} \qquad \widetilde{I} = \max_{k=1,\ldots,m}\left|I_k\left(0\right)\right|.
$$

Theorem 3.4 *Assume that (H1)-(H3) and (3.8) are satisfied. Then the problem (3.1)-(3.3) has at least one solution on J.*

Proof. We shall use Schaefer's fixed point theorem to prove that Θ, defined by (3.9), has at least one fixed point on J. The proof will be given in several steps.

Step 1: Θ *is continuous.*
Let $\{v_n\}$ be a sequence such that $v_n \to v$ in $PC(J, \mathbb{R})$. Then, for each $t \in J$

$$\begin{aligned}
|\Theta(v_n)(t) - \Theta(v)(t)| \leq \; & \frac{|c_2|}{|c_1 + c_2|} \left[\sum_{i=1}^{m} \left| I_i \left(v_n \left(t_i^- \right) \right) - I_i \left(v \left(t_i^- \right) \right) \right| \right. \\
& + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \left(e^{t_i} - e^s \right)^{\alpha-1} e^s |\varphi_n(s) - \varphi(s)| ds \\
& \left. + \frac{1}{\Gamma(\alpha)} \int_{t_m}^{b} \left(e^b - e^s \right)^{\alpha-1} e^s |\varphi_n(s) - \varphi(s)| ds \right] \\
& + \sum_{a < t_k < t} \left| I_k \left(v_n \left(t_k^- \right) \right) - I_k \left(v \left(t_k^- \right) \right) \right| \\
& + \frac{1}{\Gamma(\alpha)} \sum_{a < t_k < t} \int_{t_{k-1}}^{t_k} \left(e^{t_k} - e^s \right)^{\alpha-1} e^s |\varphi_n(s) - \varphi(s)| ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(e^t - e^s \right)^{\alpha-1} e^s |\varphi_n(s) - \varphi(s)| ds,
\end{aligned} \tag{3.11}$$

where $\varphi_n, \varphi \in C(J, E)$ such that

$$\varphi_n(t) = f\left(t, v_n(t), \varphi_n(t)\right)$$

and

$$\varphi(t) = f\left(t, v(t), \varphi(t)\right).$$

By (H2), we have

$$\begin{aligned}
|\varphi_n(t) - \varphi(t)| &= |f\left(t, v_n(t), \varphi_n(t)\right) - f\left(t, v(t), \varphi(t)\right)| \\
&\leq k_1 |v_n(t) - v(t)| + k_2 |\varphi_n(t) - \varphi(t)|.
\end{aligned}$$

Then

$$|\varphi_n(t) - \varphi(t)| \leq \gamma \; |v_n(t) - v(t)|.$$

Since $v_n \to v$ then we get $\varphi_n(t) \to \varphi(t)$ as $n \to \infty$ for each $t \in J$. And let $\delta > 0$ be such that, for each $t \in J$, we have $|\varphi_n(t)| \leq \delta$ and $|\varphi(t)| \leq \delta$. Then, we have

$$\begin{aligned}
(e^t - e^s)^{\alpha-1} e^s |\varphi_n(s) - \varphi(s)| &\leq (e^t - e^s)^{\alpha-1} e^s [|\varphi_n(s)| + |\varphi(s)|] \\
&\leq 2\delta (e^t - e^s)^{\alpha-1} e^s
\end{aligned}$$

and
$$\begin{aligned}(e^{t_k}-e^s)^{\alpha-1}e^s|\varphi_n(s)-\varphi(s)| &\leq (e^{t_k}-e^s)^{\alpha-1}e^s[|\varphi_n(s)|+|\varphi(s)|]\\ &\leq 2\delta(e^{t_k}-e^s)^{\alpha-1}e^s.\end{aligned}$$
For each $t\in J$, the functions $s\to 2\delta(e^t-e^s)^{\alpha-1}e^s$ and $s\to 2\delta(e^{t_k}-e^s)^{\alpha-1}e^s$ are integrable on $[a,t]$. Then, the Lebesgue dominated convergence theorem and (3.11) imply that
$$|\Theta(v_n)(t)-\Theta(v)(t)|\to 0 \ as \ n\to\infty.$$
Thus,
$$\|\Theta(u_n)-\Theta(u)\|_{PC}\to 0 \text{ as } n\to\infty.$$
Consequently, Θ is continuous.

Step 2 : Θ *maps bounded sets into bounded sets in* $PC(J,\mathbb{R})$.
Indeed, it is enough to show that for any $\overline{\delta}>0$, there exists a positive constant $\overline{\ell}$ such that for any $v\in B_{\overline{\delta}}$, with $B_{\overline{\delta}}=\left\{v\in PC(J,\mathbb{R}):\|v\|_{PC}\leq\overline{\delta}\right\}$, we have $\|\Theta(v)\|_{PC}\leq\overline{\ell}$. We have that, for each $t\in J$,
$$\begin{aligned}|\Theta(v)(t)| \leq\ & \frac{|c_2|}{|c_1+c_2|}\left[\sum_{i=1}^m\left|I_i\left(v\left(t_i^-\right)\right)\right|+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^m\int_{t_{i-1}}^{t_i}\left(e^{t_i}-e^s\right)^{\alpha-1}e^s\left|\varphi(s)\right|ds\right.\\ &\left.+\frac{1}{\Gamma(\alpha)}\int_{t_m}^{b}\left(e^{b}-e^s\right)^{\alpha-1}e^s\left|\varphi(s)\right|ds\right]+\frac{|c_3|}{|c_1+c_2|}+\sum_{a<t_k<t}\left|I_k\left(v\left(t_k^-\right)\right)\right|\\ &+\frac{1}{\Gamma(\alpha)}\sum_{a<t_k<t}\int_{t_{k-1}}^{t_k}\left(e^{t_k}-e^s\right)^{\alpha-1}e^s\left|\varphi(s)\right|ds\\ &+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(e^{t}-e^s\right)^{\alpha-1}e^s\left|\varphi(s)\right|ds,\end{aligned} \tag{3.12}$$
where $\varphi\in C(J,\mathbb{R})$ such that
$$\varphi(t)=f\left(t,v(t),\varphi(t)\right).$$
By (H2), we have for each $t\in J$
$$\begin{aligned}|\varphi(t)| &= |f\left(t,v(t),\varphi(t)\right)-f\left(t,0,0\right)+f\left(t,0,0\right)|\\ &\leq |f\left(t,v(t),\varphi(t)\right)-f\left(t,0,0\right)|+|f\left(t,0,0\right)|\\ &\leq k_1\left|v\right|+k_2\left|\varphi(t)\right|+\widetilde{f}.\end{aligned}$$
Thus,
$$|\varphi(t)| \ \leq \ \gamma\left|v\right|+\frac{\widetilde{f}}{1-k_2}. \tag{3.13}$$

Analogously, by using (H3), we obtain

$$|I_k(v)| \leq \xi |v| + \widetilde{I}. \tag{3.14}$$

From this and using (3.12), for any $v \in B_{\overline{\delta}}$, we have

$$\begin{aligned}
|\Theta(v)(t)| &\leq \frac{|c_2|}{|c_1+c_2|}\left[m\left(\xi|v|+\widetilde{I}\right)+m\left(\gamma|v|+\frac{\widetilde{f}}{1-k_2}\right)\frac{\left(e^b-e^a\right)^\alpha}{\Gamma(\alpha+1)}\right.\\
&\quad\left.+\left(\gamma|v|+\frac{\widetilde{f}}{1-k_2}\right)\frac{\left(e^b-e^a\right)^\alpha}{\Gamma(\alpha+1)}\right]+\frac{|c_3|}{|c_1+c_2|}+m\left(\xi|v|+\widetilde{I}\right)\\
&\quad+m\left(\gamma|v|+\frac{\widetilde{f}}{1-k_2}\right)\frac{\left(e^b-e^a\right)^\alpha}{\Gamma(\alpha+1)}+\left(\gamma|v|+\frac{\widetilde{f}}{1-k_2}\right)\frac{\left(e^b-e^a\right)^\alpha}{\Gamma(\alpha+1)}\\
&= \left(\frac{|c_2|}{|c_1+c_2|}+1\right)\left[m\left(\xi|v|+\widetilde{I}\right)+\left(\gamma|v|+\frac{\widetilde{f}}{1-k_2}\right)\frac{(m+1)\left(e^b-e^a\right)^\alpha}{\Gamma(\alpha+1)}\right]\\
&\quad+\frac{|c_3|}{|c_1+c_2|}\\
&\leq \left(\frac{|c_2|}{|c_1+c_2|}+1\right)\left[m\left(\xi\overline{\delta}+\widetilde{I}\right)+\left(\gamma\overline{\delta}+\frac{\widetilde{f}}{1-k_2}\right)\frac{(m+1)\left(e^b-e^a\right)^\alpha}{\Gamma(\alpha+1)}\right]\\
&\quad+\frac{|c_3|}{|c_1+c_2|}\\
&= \mu_1\left[m\left(\xi\overline{\delta}+\widetilde{I}\right)+\left(\overline{\delta}+\frac{\widetilde{f}}{k_1}\right)\mu_2\right]+\frac{|c_3|}{|c_1+c_2|}\\
&:= \overline{\ell},
\end{aligned}$$

which implies that $\|\Theta(v)\|_{PC} \leq \overline{\ell}$.

Step 3 : Θ *maps bounded sets into equicontinuous sets of* $PC(J,\mathbb{R})$.
Let $\tau_1,\tau_2 \in J$, $\tau_1 < \tau_2$, $B_{\overline{\delta}}$ be a bounded set of $PC(J,\mathbb{R})$ as in Step 2, and let $v \in B_{\overline{\delta}}$. Then we have

$$\begin{aligned}
|\Theta(v)(\tau_2)-\Theta(v)(\tau_1)| &\leq \frac{1}{\Gamma(\alpha)}\int_a^{\tau_1}\left|\left[(e^{\tau_2}-e^s)^{\alpha-1}-(e^{\tau_1}-e^s)^{\alpha-1}\right]e^s\right||\varphi(s)|\,ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\left|(e^{\tau_2}-e^s)^{\alpha-1}e^s\right||\varphi(s)|\,ds+\sum_{\tau_1<t_k<\tau_2}\left|I_k\left(v\left(t_k^-\right)\right)\right|\\
&\quad+\frac{1}{\Gamma(\alpha)}\sum_{\tau_1<t_k<\tau_2}\int_{t_{k-1}}^{t_k}\left|\left(e^{t_k}-e^s\right)^{\alpha-1}e^s\right||\varphi(s)|\,ds
\end{aligned}$$

Hence

$$
\begin{aligned}
|\Theta(v)(\tau_2)-\Theta(v)(\tau_1)| \leq\ & \left(\gamma|v|+\frac{\widetilde{f}}{(1-k_2)}\right)\frac{1}{\Gamma(\alpha+1)}\left[(e^{\tau_1}-e^a)^\alpha-(e^{\tau_2}-e^a)^\alpha\right.\\
& \left.+2\left(e^{\tau_2}-e^{\tau_1}\right)^\alpha\right]+(\tau_2-\tau_1)\left[\left(\xi|v|+\widetilde{I}\right)\right.\\
& \left.+\left(\gamma|v|+\frac{\widetilde{f}}{(1-k_2)}\right)\frac{\left(e^b-e^a\right)^\alpha}{\Gamma(\alpha+1)}\right]\\
\leq\ & \left(\gamma\overline{\delta}+\frac{\widetilde{f}}{(1-k_2)}\right)\frac{1}{\Gamma(\alpha+1)}\left[(e^{\tau_1}-e^a)^\alpha-(e^{\tau_2}-e^a)^\alpha\right.\\
& \left.+2\left(e^{\tau_2}-e^{\tau_1}\right)^\alpha\right]+(\tau_2-\tau_1)\left[\left(\xi\overline{\delta}+\widetilde{I}\right)\right.\\
& \left.+\left(\gamma\overline{\delta}+\frac{\widetilde{f}}{(1-k_2)}\right)\frac{\left(e^b-e^a\right)^\alpha}{\Gamma(\alpha+1)}\right].
\end{aligned}
$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero. As a consequence of the steps 1 to 3 together with the Ascoli-Arzelà theorem, we deduce that $\Theta : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ is completely continuous.

Step 4: *A priori bounds.* Now it remain to show that the set

$$\varepsilon = \{v \in PC(J,\mathbb{R}) : v = \lambda\Theta(v),\ \text{for some } \lambda \in (0,1)\}$$

is bounded. Let $v \in \varepsilon$, then $v = \lambda\Theta(v)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$ we have

$$
\begin{aligned}
v(t) =\ & \frac{-\lambda}{c_1+c_2}\left[c_2\sum_{i=1}^{m} I_i\left(v\left(t_i^-\right)\right)+\frac{c_2}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\left(e^{t_i}-e^s\right)^{\alpha-1}\varphi(s)e^s ds\right.\\
& \left.+\frac{c_2}{\Gamma(\alpha)}\int_{t_m}^{b}\left(e^b-e^s\right)^{\alpha-1}\varphi(s)e^s ds-c_3\right]+\lambda\sum_{a<t_k<t} I_k\left(v\left(t_k^-\right)\right)\\
& +\frac{\lambda}{\Gamma(\alpha)}\sum_{a<t_k<t}\int_{t_{k-1}}^{t_k}\left(e^{t_k}-e^s\right)^{\alpha-1}\varphi(s)e^s ds+\frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t}\left(e^t-e^s\right)^{\alpha-1}\varphi(s)e^s ds.
\end{aligned}
$$

By using (3.13) and (3.14), for each $t \in J$, we obtain

$$
\begin{aligned}
|v(t)| &\leq \frac{|c_2|}{|c_1+c_2|}\left[m\left(\xi|v|+\widetilde{I}\right)+m\left(\gamma|v|+\frac{\widetilde{f}}{1-k_2}\right)\frac{\left(e^b-e^a\right)^\alpha}{\Gamma(\alpha+1)}\right.\\
&\quad\left.+\left(\gamma|v|+\frac{\widetilde{f}}{1-k_2}\right)\frac{\left(e^b-e^a\right)^\alpha}{\Gamma(\alpha+1)}\right]+\frac{|c_3|}{|c_1+c_2|}+m\left(\xi|v|+\widetilde{I}\right)\\
&\quad m\left(\gamma|v|+\frac{\widetilde{f}}{1-k_2}\right)\frac{\left(e^b-e^a\right)^\alpha}{\Gamma(\alpha+1)}+\left(\gamma|v|+\frac{\widetilde{f}}{1-k_2}\right)\frac{\left(e^b-e^a\right)^\alpha}{\Gamma(\alpha+1)}\\
&= \left(\frac{|c_2|}{|c_1+c_2|}+1\right)\left[m\left(\xi|v|+\widetilde{I}\right)+\left(\gamma|v|+\frac{\widetilde{f}}{1-k_2}\right)\frac{(m+1)\left(e^b-e^a\right)^\alpha}{\Gamma(\alpha+1)}\right]\\
&\quad+\frac{|c_3|}{|c_1+c_2|}\\
&\leq \left(\frac{|c_2|}{|c_1+c_2|}+1\right)\left(m\xi+\frac{\gamma(m+1)\left(e^b-e^a\right)^\alpha}{\Gamma(\alpha+1)}\right)|v|\\
&\quad+\left(\frac{|c_2|}{|c_1+c_2|}+1\right)\left(m\widetilde{I}+\frac{\widetilde{f}(m+1)\left(e^b-e^a\right)^\alpha}{(1-k_2)\Gamma(\alpha+1)}\right)+\frac{|c_3|}{|c_1+c_2|}.\\
&\leq \mu_1(m\xi+\mu_2)|v|+\mu_1\left(m\widetilde{I}+\frac{\widetilde{f}\mu_2}{k_1}\right)+\frac{|c_3|}{|c_1+c_2|}.
\end{aligned}
$$

Thus,

$$
\left[1-\mu_1(m\xi+\mu_2)\right]\|v\|_{PC} \leq \mu_1\left(m\widetilde{I}+\frac{\widetilde{f}\mu_2}{k_1}\right)+\frac{|c_3|}{|c_1+c_2|}.
$$

By using the condition (3.8), it follows that

$$
\|v\|_{PC} \leq \left[\mu_1\left(m\widetilde{I}+\frac{\widetilde{f}\mu_2}{k_1}\right)+\frac{|c_3|}{|c_1+c_2|}\right]\left[1-\mu_1(m\xi+\mu_2)\right]^{-1}:=\overline{M}.
$$

This shows that the set ε is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that Θ has at least one fixed point which is solution of (3.1)-(3.3). This completes the proof.

The final existence result for (3.1)-(3.3) is based on Nonlinear alternative of Leray-Schauder type.

Theorem 3.5 *Assume that (H1)-(H3) and (3.8) are satisfied. Then the problem (3.1)-(3.3) has at least one solution on J.*

Proof. We shall show that the operator $\Theta : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ defined by (3.9) is continuous and completely continuous. Obviously according to the steps 1 to 3 in the proof of Theorem 3.4, we conclude that f is continuous and completely continuous. Now we show that there exists an open set $U \subseteq PC(J,\mathbb{R})$ with $v \neq \rho\Theta(v)$, for $\rho \in (0,1)$ and $v \in \partial U$. Let $v \in PC(J,\mathbb{R})$ and $v = \rho\Theta(v)$ for some $0 < \rho < 1$. Thus for each $t \in J$, we have that

$$\begin{aligned} v(t) &= \frac{-\rho}{c_1+c_2}\left[c_2\sum_{i=1}^{m} I_i\left(v\left(t_i^-\right)\right) + \frac{c_2}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\left(e^{t_i}-e^s\right)^{\alpha-1}\varphi(s)e^s ds \right. \\ &\left. +\frac{c_2}{\Gamma(\alpha)}\int_{t_m}^{b}\left(e^b-e^s\right)^{\alpha-1}\varphi(s)e^s ds - c_3\right] + \rho\sum_{a<t_k<t} I_k\left(v\left(t_k^-\right)\right) \\ &+\frac{\rho}{\Gamma(\alpha)}\sum_{a<t_k<t}\int_{t_{k-1}}^{t_k}\left(e^{t_k}-e^s\right)^{\alpha-1}\varphi(s)e^s ds + \frac{\rho}{\Gamma(\alpha)}\int_{t_k}^{t}\left(e^t-e^s\right)^{\alpha-1}\varphi(s)e^s ds. \end{aligned}$$

As in Theorem 3.4, we obtain

$$\|v\|_{PC} \leq \left[\mu_1\left(m\widetilde{I} + \frac{\widetilde{f}\mu_2}{k_1}\right) + \frac{|c_3|}{|c_1+c_2|}\right]\left[1-\mu_1\left(m\xi+\mu_2\right)\right]^{-1} := \overline{M}.$$

Hence, for $U = \{v \in PC(J,\mathbb{R}) : \|v\|_{PC} < \overline{M}+1\}$, there is no $v \in \partial U$ such that $v = \rho\Theta(v)$, for $\rho \in (0,1)$. As a consequence of Leray–Schauder's theorem, we deduce that Θ has a fixed point v in $\overline{U}$ which is a solution of problem (3.1)-(3.3). This completes the proof.

Remark 3.6 *Our results for the boundary value problem (3.1)-(3.3) remain true for the following cases:*

- *Initial value problem:* $c_1 = 1,\ c_2 = 0$ *and* c_3 *arbitrary.*
- *Terminal value problem:* $c_1 = 0,\ c_2 = 1$ *and* c_3 *arbitrary.*
- *Anti-periodic problem:* $c_1 = c_2 \neq 0$ *and* $c_3 = 0$.

However, our results are not applicable for the periodic problem, i.e. for $c_1 = 1,\ c_2 = -1,$ *and* $c_3 = 0$.

3.3 Examples

In this section, we will give two examples to illustrate our main results.

Example 1. Consider the following impulsive boundary value problem for nonlinear implicit fractional differential equation:

$$ {}_c^eD_{t_k}^{\frac{1}{2}}\varpi(t) = \frac{e^{-\sqrt{t+9}}\sin t}{7\left(t^2+1\right)\left(\sqrt{3}+|\varpi(t)|+\left|{}_c^eD_{t_k}^{\frac{1}{2}}\varpi(t)\right|\right)}, \quad \text{for each } t\in J_0\cup J_1, \tag{3.15}$$

$$ \Delta\varpi|_{t=\frac{\pi}{2}} = \frac{\left|\varpi(\frac{\pi}{2}^-)\right|}{19+\left|\varpi(\frac{\pi}{2}^-)\right|}, \tag{3.16}$$

$$ \varpi(0)+\varpi(\pi) = 13, \tag{3.17}$$

where $J_0=\left[0,\frac{\pi}{2}\right]$, $J_1=(\frac{\pi}{2},\pi]$, $m=1$, $\alpha=\frac{1}{2}$, $a=0$, $b=\pi$, $c_1=c_2=1$, $c_3=13$,

$$ f\left(t,\varpi,\omega\right) = \frac{e^{-\sqrt{t+9}}\sin t}{7\left(t^2+1\right)\left(\sqrt{3}+|\varpi|+|\omega|\right)}$$

and

$$ I_1(\varpi) = \frac{|\varpi|}{19+|\varpi|}.$$

Now, for each $t\in[0,\pi]$ and for any $\varpi_1,\varpi_2,\omega_1,\omega_2\in\mathbb{R}$, we can show that

$$ \left|f\left(t,\varpi_1,\omega_1\right)-f\left(t,\varpi_2,\omega_2\right)\right| \le \frac{1}{21e^3}\left(|\varpi_1-\varpi_2|+|\omega_1-\omega_2|\right)$$

and

$$ \left|I_1(\varpi_1)-I_1(\varpi_2)\right| \le \frac{1}{19}\left|\varpi_1-\varpi_2\right|.$$

Thus, for $k_1=k_2=\dfrac{1}{21e^3}$ and $\xi=\dfrac{1}{19}$ we have that

$$\begin{aligned} \mu_1\left(m\xi+\mu_2\right) &= \left(\frac{|c_2|}{|c_1+c_2|}+1\right)\left[m\xi+\frac{k_1\left(m+1\right)\left(e^b-e^a\right)^{\alpha}}{\left(1-k_2\right)\Gamma(\alpha+1)}\right] \\ &= \frac{3}{2}\left\{\frac{1}{19}+\frac{2\sqrt{e^{\pi}-1}}{21e^3}\left[\left(1-\frac{1}{21e^3}\right)\Gamma\left(\frac{3}{2}\right)\right]^{-1}\right\} \\ &= \frac{3}{2}\left[\frac{1}{19}+\frac{4\sqrt{e^{\pi}-1}}{\left(21e^3-1\right)\sqrt{\pi}}\right] \\ &\approx 0.1168003443 \\ &< 1. \end{aligned}$$

Hence, all of assumptions (H1)-(H3) and the condition (3.8) are satisfied. As a consequence of Theorem 3.3 the impulsive problem (3.15)-(3.17) has a unique solution on $[0,\pi]$.

Example 2. Consider the following impulsive boundary value problem for nonlinear implicit fractional differential equation:

$$ {}_{c}^{e}D_{t_k}^{\frac{1}{2}}\varpi(t) = \frac{e^{-\sqrt{t+16}}\left(2+|\varpi(t)|+\left|{}_{c}^{e}D_{t_k}^{\frac{1}{2}}\varpi(t)\right|\right)}{179\left(t^2+1\right)\left(1+|\varpi(t)|+\left|{}_{c}^{e}D_{t_k}^{\frac{1}{2}}\varpi(t)\right|\right)}, \quad \text{for each } t \in J_0 \cup J_1, \tag{3.18} $$

$$ \Delta\varpi|_{t=\frac{1}{4}} = \frac{5\left|\varpi(\frac{1}{4}^-)\right|}{20+\left|\varpi(\frac{1}{4}^-)\right|}, \tag{3.19} $$

$$ \varpi(0) = -\varpi(1), \tag{3.20} $$

where $J_0 = \left[0, \frac{1}{4}\right]$, $J_1 = \left(\frac{1}{4}, 1\right]$, $m = 1$, $\alpha = \frac{1}{2}$, $a = 0$, $b = 1$, $c_1 = c_2 = 1$, $c_3 = 0$,

$$ f(t,\varpi,\omega) = \frac{e^{-\sqrt{t+16}}\left(2+|\varpi|+|\omega|\right)}{179\left(t^2+1\right)\left(1+|\varpi|+|\omega|\right)}, \quad \text{for each } t \in J_0 \cup J_1, $$

and

$$ I_1(\varpi) = \frac{5|\varpi|}{20+|\varpi|}. $$

Now, for each $t \in [0,1]$ and for any $\varpi_1, \varpi_2, \omega_1, \omega_2 \in \mathbb{R}$, we can show that

$$ |f(t,\varpi_1,\omega_1) - f(t,\varpi_2,\omega_2)| \le \frac{1}{179e^4}\left(|\varpi_1-\varpi_2| + |\omega_1-\omega_2|\right) $$

and

$$ |I_1(\varpi_1) - I_1(\varpi_2)| \le \frac{1}{4}|\varpi_1-\varpi_2|. $$

Thus, for $k_1 = k_2 = \dfrac{1}{179e^4}$ and $\xi = \dfrac{1}{4}$ we have that

$$ \begin{aligned} \mu_1(m\xi+\mu_2) &= \left(\frac{|c_2|}{|c_1+c_2|}+1\right)\left[m\xi + \frac{k_1(m+1)\left(e^b-e^a\right)^\alpha}{(1-k_2)\Gamma(\alpha+1)}\right] \\ &= \frac{3}{2}\left\{\frac{1}{4} + \frac{2\sqrt{e-1}}{179e^4}\left[\left(1-\frac{1}{179e^4}\right)\Gamma\left(\frac{3}{2}\right)\right]^{-1}\right\} \\ &= \frac{3}{2}\left[\frac{1}{4} + \frac{4\sqrt{e-1}}{\left(179e^4-1\right)\sqrt{\pi}}\right] \\ &= \frac{3}{8} + \frac{4\sqrt{e-1}}{\left(179e^4-1\right)\sqrt{\pi}} \\ &\approx 0.3753057 \\ &< 1. \end{aligned} $$

Hence, all of assumptions (H1), (H2), (H3) and the condition (3.8) are satisfied. As a consequence of Theorem 3.4 or Theorem 3.5 the problem (3.18)-(3.20) has at least one solution on $[0, 1]$.

Chapter 4

Impulsive Abstract Nonlinear Implicit Caputo's Exponential Type Fractional Differential Equations

4.1 Introduction

This chapter is devoted to the results obtained by Malti *et al.* [76], we establish the existence results of solutions for a class of impulsive boundary value problem (**BVP**) of the following nonlinear implicit fractional differential equations (**NIFD**) involving Caputo's exponential type fractional derivative:

$$ {}_{c}^{e}D_{t_k}^{\alpha} y(t) = f(t, y(t), {}_{c}^{e}D_{t_k}^{\alpha} y(t)), \quad \text{for each } t \in J_k \subseteq J, \; k = 0, 1, \ldots, m, \tag{4.1}$$

$$ \Delta y|_{t=t_k} = I_k\left(y\left(t_k^-\right)\right), \quad k = 1, \ldots, m, \tag{4.2}$$

$$ c_1 y(a) + c_2 y(b) = \sigma, \tag{4.3}$$

where $a = t_0 < t_1 < \ldots < t_m < t_{m+1} = b$, ${}_{c}^{e}D_{t_k}^{\alpha}$ denote the Caputo's exponential type fractional derivatives of order α, $0 < \alpha \leq 1$, $(E, \|\cdot\|)$ is a real Banach space, $f : J\times E\times E \to E$ is a given function, c_1, c_2 are real constants with $c_1 + c_2 \neq 0$, and $\sigma \in E$, $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, $J_0 = [a, t_1]$, $J = [a, b]$, $\Delta y|_{t=t_k} = y\left(t_k^+\right) - y\left(t_k^-\right)$, $y\left(t_k^+\right) = \lim\limits_{h\to 0} y(t_k + h)$ and $y\left(t_k^-\right) = \lim\limits_{h\to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$.

4.2 Existence Results

The following notations which are useful on this chapter. Consider the set of function

$$PC(J, E) = \Big\{y : J \to E, \; y \in C\left(J_k, E\right), \; k = 0, \ldots, m \text{ and there exist}$$
$$y(t_k^+) \text{ and } y(t_k^-), k = 1, \ldots, m \text{ with } y(t_k^-) = y(t_k)\Big\}.$$

This set is the Banach space with the norm

$$\|y\|_{PC} = \max_{k=0,1,\dots,m} \Big\{ \sup_{t\in J_k} \|y(t)\| \Big\}.$$

Lemma 4.1 *([56]) If $V \subset PC(J,E)$ is a bounded and equicontinuous set, then*

(i) the function $t \to \mu(V(t))$ is continuous on J, and

$$\mu_{PC}(V) = \max_{k=0,1,\dots,m} \Big\{ \sup_{t\in J_k} \mu(V(t)) \Big\}.$$

(ii) $\mu\left(\int_a^b y(s)ds : y \in V\right) \leq \int_a^b \mu(V(s))ds,$

where μ is the Kuratowski measure of noncompactness and

$$V(s) = \{y(s) : y \in V\}, \ s \in J.$$

Now, we define what we mean by a solution of the problem (4.1)-(4.3).

Definition 4.2 *A function $y \in PC(J,E) \cap \Big(\cup_{k=0}^m AC_e(J_k,E) \Big)$ is said a solution of (4.1)-(4.3) if y satisfies the equation ${}_c^eD_{a+}^{\alpha} y(t) = f(t, y(t), {}_c^eD_{a+}^{\alpha} y(t))$, on J_k and the conditions*

$$\Delta y|_{t=t_k} = I_k\left(y\left(t_k^-\right)\right), \quad \text{for } k = 1,\dots,m,$$
$$c_1 y(a) + c_2 y(b) = \sigma,$$

To Prove the existence of solutions to (4.1)-(4.3), we need the following auxiliary lemmas.

Lemma 4.3 *Let $0 < \alpha \leq 1$ and let $\varphi : J \to E$ be continuous. A function y is a solution of the integral equation*

$$y(t) = \begin{cases} \dfrac{-1}{c_1+c_2}\Bigg[c_2 \displaystyle\sum_{i=1}^m I_i\left(y\left(t_i^-\right)\right) + c_2 \sum_{i=1}^m \int_{t_{i-1}}^{t_i} \left(e^{t_i}-e^s\right)^{\alpha-1} \frac{\varphi(s)}{\Gamma(\alpha)} e^s ds \\ \quad + c_2 \displaystyle\int_{t_m}^b \left(e^b - e^s\right)^{\alpha-1} \frac{\varphi(s)}{\Gamma(\alpha)} e^s ds - \sigma \Bigg] + \int_a^t \left(e^t-e^s\right)^{\alpha-1} \frac{\varphi(s)}{\Gamma(\alpha)} e^s ds, \ \text{if } t\in[a,t_1], \\ \\ \dfrac{-1}{c_1+c_2}\Bigg[c_2 \displaystyle\sum_{i=1}^m I_i\left(y\left(t_i^-\right)\right) + c_2 \sum_{i=1}^m \int_{t_{i-1}}^{t_i} \left(e^{t_i}-e^s\right)^{\alpha-1} \frac{\varphi(s)}{\Gamma(\alpha)} e^s ds \\ \quad + c_2 \displaystyle\int_{t_m}^b \left(e^b - e^s\right)^{\alpha-1} \frac{\varphi(s)}{\Gamma(\alpha)} e^s ds - \sigma \Bigg] + \sum_{i=1}^k I_i\left(y\left(t_i^-\right)\right) \\ \quad + \displaystyle\sum_{i=1}^k \int_{t_{i-1}}^{t_i} \left(e^{t_i}-e^s\right)^{\alpha-1} \frac{\varphi(s)}{\Gamma(\alpha)} e^s ds + \int_{t_k}^t \left(e^t-e^s\right)^{\alpha-1} \frac{\varphi(s)}{\Gamma(\alpha)} e^s ds, \ \text{if } t\in(t_k,t_{k+1}], \end{cases} \tag{4.4}$$

where $k = 1, \ldots, m$, *if and only if* y *is a solution of the fractional BVP*

$$ {}_c^eD_{t_k}^{\alpha} y(t) = \varphi(t), \quad t \in J_k, \tag{4.5}$$

$$\Delta y|_{t=t_k} = I_k\left(y\left(t_k^-\right)\right), \quad \textit{for } k = 1, \ldots, m, \tag{4.6}$$

$$c_1 y(a) + c_2 y(b) = \sigma. \tag{4.7}$$

Proof: Assume that y satisfies (4.5)-(4.7). If $t \in [a, t_1]$, then

$$ {}_c^eD_a^{\alpha} y(t) = \varphi(t).$$

By Lemma 1.10, we get

$$y(t) = \eta_0 + {}^eI_a^{\alpha}\varphi(t) = \eta_0 + \frac{1}{\Gamma(\alpha)} \int_a^t \left(e^t - e^s\right)^{\alpha-1} \varphi(s) e^s ds.$$

If $t \in (t_1, t_2]$, then by Lemma 1.10 we get

$$\begin{aligned}
y(t) &= y\left(t_1^+\right) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \left(e^t - e^s\right)^{\alpha-1} \varphi(s) e^s ds \\
&= \Delta y|_{t=t_1} + y\left(t_1^-\right) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \left(e^t - e^s\right)^{\alpha-1} \varphi(s) e^s ds \\
&= I_1\left(y\left(t_1^-\right)\right) + \left[\eta_0 + \frac{1}{\Gamma(\alpha)} \int_a^{t_1} \left(e^{t_1} - e^s\right)^{\alpha-1} \varphi(s) e^s ds\right] \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \left(e^t - e^s\right)^{\alpha-1} \varphi(s) e^s ds \\
&= \eta_0 + I_1\left(y\left(t_1^-\right)\right) + \frac{1}{\Gamma(\alpha)} \int_a^{t_1} \left(e^{t_1} - e^s\right)^{\alpha-1} \varphi(s) e^s ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \left(e^t - e^s\right)^{\alpha-1} \varphi(s) e^s ds.
\end{aligned}$$

If $t \in (t_2, t_3]$, then by Lemma 1.10 we get

$$\begin{aligned}
y(t) &= y\left(t_2^+\right) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} \left(e^t - e^s\right)^{\alpha-1} \varphi(s) e^s ds \\
&= \Delta y|_{t=t_2} + y\left(t_2^-\right) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} \left(e^t - e^s\right)^{\alpha-1} \varphi(s) e^s ds \\
&= I_2\left(y\left(t_2^-\right)\right) + \left[\eta_0 + I_1\left(y\left(t_1^-\right)\right) + \frac{1}{\Gamma(\alpha)} \int_{a}^{t_1} \left(e^{t_1} - e^s\right)^{\alpha-1} \varphi(s) e^s ds \right. \\
&\quad \left. + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(e^{t_2} - e^s\right)^{\alpha-1} \varphi(s) e^s ds \right] + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} \left(e^t - e^s\right)^{\alpha-1} \varphi(s) e^s ds \\
&= \eta_0 + \left[I_1\left(y\left(t_1^-\right)\right) + I_2\left(y\left(t_2^-\right)\right)\right] + \left[\frac{1}{\Gamma(\alpha)} \int_{a}^{t_1} \left(e^{t_1} - e^s\right)^{\alpha-1} \varphi(s) e^s ds \right. \\
&\quad \left. + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(e^{t_2} - e^s\right)^{\alpha-1} \varphi(s) e^s ds \right] + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} \left(e^t - e^s\right)^{\alpha-1} \varphi(s) e^s ds.
\end{aligned}$$

Repeating the process in this ways, the solution $y(t)$ for $t \in (t_k, t_{k+1}]$ where $k = 1, \ldots, m$, can be written as

$$\begin{aligned}
y(t) &= \eta_0 + \sum_{i=1}^{k} I_i\left(y\left(t_i^-\right)\right) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(e^{t_i} - e^s\right)^{\alpha-1} \varphi(s) e^s ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(e^t - e^s\right)^{\alpha-1} \varphi(s) e^s ds.
\end{aligned} \tag{4.8}$$

It clear that

$$y(a) = \eta_0$$

and

$$\begin{aligned}
y(b) &= \eta_0 + \sum_{i=1}^{m} I_i\left(y\left(t_i^-\right)\right) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \left(e^{t_i} - e^s\right)^{\alpha-1} \varphi(s) e^s ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_m}^{b} \left(e^b - e^s\right)^{\alpha-1} \varphi(s) e^s ds.
\end{aligned}$$

Hence, by applying the boundary conditions $c_1 y(a) + c_2 y(b) = \sigma$, we get

$$\begin{aligned}
\sigma &= \eta_0(c_1 + c_2) + c_2 \sum_{i=1}^{m} I_i\left(y\left(t_i^-\right)\right) + \frac{c_2}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \left(e^{t_i} - e^s\right)^{\alpha-1} \varphi(s) e^s ds \\
&\quad + \frac{c_2}{\Gamma(\alpha)} \int_{t_m}^{b} \left(e^b - e^s\right)^{\alpha-1} \varphi(s) e^s ds.
\end{aligned}$$

Then

$$\begin{aligned}\eta_0 &= \frac{-1}{c_1+c_2}\left[c_2\sum_{i=1}^{m} I_i\left(y\left(t_i^-\right)\right)+\frac{c_2}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\left(e^{t_i}-e^s\right)^{\alpha-1}\varphi(s)e^s ds\right.\\ &\left.+\frac{c_2}{\Gamma(\alpha)}\int_{t_m}^{b}\left(e^{b}-e^s\right)^{\alpha-1}\varphi(s)e^s ds-\sigma\right].\end{aligned}$$

Thus, if $t\in(t_k,t_{k+1}]$, where $k=1,\ldots,m$, then

$$\begin{aligned}y(t) &= \frac{-1}{c_1+c_2}\left[c_2\sum_{i=1}^{m} I_i\left(y\left(t_i^-\right)\right)+\frac{c_2}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\left(e^{t_i}-e^s\right)^{\alpha-1}\varphi(s)e^s ds\right.\\ &\left.+\frac{c_2}{\Gamma(\alpha)}\int_{t_m}^{b}\left(e^{b}-e^s\right)^{\alpha-1}\varphi(s)e^s ds-\sigma\right]+\sum_{i=1}^{k} I_i\left(y\left(t_i^-\right)\right)\\ &+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(e^{t_i}-e^s\right)^{\alpha-1}\varphi(s)e^s ds+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(e^{t}-e^s\right)^{\alpha-1}\varphi(s)e^s ds.\end{aligned}$$

Conversely, assume that y satisfies the impulsive fractional integral equation (4.4).
If $t\in[a,t_1]$ then $c_1y(a)+c_2y(b)=\sigma$ and using the fact that ${}_c^eD_a^\alpha$ is the left inverse of ${}^eI_a^\alpha$, we get

$$ {}_c^eD_a^\alpha y(t)=\varphi(t),\quad \text{for each } t\in[a,t_1].$$

If $t\in(t_k,t_{k+1}]$, where $k=1,\ldots,m$. Then, by using the fact that ${}_c^eD_{t_k}^\alpha C=0$, where C is a constant and ${}_c^eD_{t_k}^\alpha$ is the left inverse of ${}^eI_{t_k}^\alpha$, we get

$$ {}_c^eD_{t_k}^\alpha y(t)=\varphi(t),\quad \text{for each } t\in(t_k,t_{k+1}].$$

Also, we can easily show that

$$\Delta y|_{t=t_k}=I_k\left(y\left(t_k^-\right)\right),\quad k=1,\ldots,m.$$

Now, we pass to state and proof our first existence result for the problem (4.1)-(4.3) based on concept of measure of noncompactness and Darbo's fixed point theorem.

The following hypotheses will be used in the sequel:

(H1) The function $f:J\times E\times E\to E$ is continuous.

(H2) There exist constants $k_1>0$ and $0<k_2<1$ such that

$$\|f(t,y_1,z_1)-f(t,y_2,z_2)\|\le k_1\|y_1-y_2\|+k_2\|z_1-z_2\|,$$

for any $y_1,y_2,z_1,z_2\in E$ and for each $t\in J$.

(H3) There exists a constant $\xi > 0$ such that

$$\|I_k(y_1) - I_k(y_2)\| \leq \xi \|y_1 - y_2\|,$$

for each $y_1, y_2 \in E$ and $k = 1, 2, \ldots, m$.

Remark 4.4 *[18] Conditions (H2) and (H3) are respectively equivalent to the inequalities*

$$\mu(f(t, B_1, B_2)) \leq k_1 \mu(B_1) + k_2 \mu(B_2)$$

and

$$\mu(I_k(B_2)) \leq \xi \mu(B_2),$$

for any bounded sets $B_1, B_2 \subseteq E$, for each $t \in J$, $k = 1, \ldots, m$ and μ is a Kuratowski measure of noncompactness in E.

Set

$$\gamma = \frac{k_1}{1 - k_2}, \quad \varepsilon = \frac{|c_2|}{|c_1 + c_2|} + 1 \quad \text{and} \quad \rho = \frac{\gamma(m+1)\left(e^b - e^a\right)^\alpha}{\Gamma(\alpha + 1)}.$$

Theorem 4.5 *Assume that (H1)-(H3) are satisfied. If*

$$\varepsilon(m\xi + \rho) < 1, \tag{4.9}$$

then the impulsive boundary value problem (4.1)-(4.3) has at least one solution on J.

Proof. Transform the problem (4.1)-(4.3) into a fixed point problem, consider the operator $F : PC(J, E) \to PC(J, E)$, defined by

$$\begin{aligned} F(y)(t) = \; & \frac{-1}{c_1 + c_2}\left[c_2 \sum_{i=1}^{m} I_i\left(y\left(t_i^-\right)\right) + \frac{c_2}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \left(e^{t_i} - e^s\right)^{\alpha-1} \varphi(s) e^s ds\right. \\ & \left. + \frac{c_2}{\Gamma(\alpha)} \int_{t_m}^{b} \left(e^b - e^s\right)^{\alpha-1} \varphi(s) e^s ds - \sigma\right] + \sum_{a < t_k < t} I_k\left(y\left(t_k^-\right)\right) \\ & + \frac{1}{\Gamma(\alpha)} \sum_{a < t_k < t} \int_{t_{k-1}}^{t_k} \left(e^{t_k} - e^s\right)^{\alpha-1} \varphi(s) e^s ds) \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(e^t - e^s\right)^{\alpha-1} \varphi(s) e^s ds, \end{aligned} \tag{4.10}$$

where $\varphi \in C(J, E)$ such that

$$\varphi(t) = f(t, y(t), \varphi(t)).$$

It clear that, the fact of finding solutions for problem (4.1)-(4.3) is to find the fixed points of the operator equation $F(y) = y$. Now, we shall use Darbo's fixed point theorem to prove that F, defined by (4.10), has at least one fixed point on J. The proof will be given in several steps.

Step 1: *F is continuous.*
Let $\{u_n\}$ be a sequence such that $u_n \to y$ in $PC(J,E)$. Then, for each $t \in J$

$$\begin{aligned} \|F(u_n)(t) - F(u)(t)\| \leq & \frac{|c_2|}{|c_1+c_2|}\left[\sum_{i=1}^{m}\left\|I_i\left(u_n\left(t_i^-\right)\right) - I_i\left(u\left(t_i^-\right)\right)\right\|\right. \\ & + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\left(e^{t_i}-e^s\right)^{\alpha-1} e^s \|\varphi_n(s)-\varphi(s)\|\, ds \\ & \left. + \frac{1}{\Gamma(\alpha)}\int_{t_m}^{b}\left(e^{b}-e^s\right)^{\alpha-1} e^s \|\varphi_n(s)-\varphi(s)\|\, ds\right] \\ & + \sum_{a<t_k<t}\left\|I_k\left(u_n\left(t_k^-\right)\right) - I_k\left(u\left(t_k^-\right)\right)\right\| \\ & + \frac{1}{\Gamma(\alpha)}\sum_{a<t_k<t}\int_{t_{k-1}}^{t_k}\left(e^{t_k}-e^s\right)^{\alpha-1} e^s \|\varphi_n(s)-\varphi(s)\|\, ds \\ & + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(e^{t}-e^s\right)^{\alpha-1} e^s \|\varphi_n(s)-\varphi(s)\|\, ds, \end{aligned} \tag{4.11}$$

where $\varphi_n, \varphi \in C(J,E)$ such that

$$\varphi_n(t) = f\left(t, u_n(t), \varphi_n(t)\right)$$

and

$$\varphi(t) = f\left(t, u(t), \varphi(t)\right).$$

By (H2), we have

$$\begin{aligned} \|\varphi_n(t) - \varphi(t)\| &= \|f\left(t, u_n(t), \varphi_n(t)\right) - f\left(t, u(t), \varphi(t)\right)\| \\ &\leq k_1\|u_n(t) - u(t)\| + k_2\|\varphi_n(t) - \varphi(t)\|. \end{aligned}$$

Then

$$\|\varphi_n(t) - \varphi(t)\| \leq \gamma\ \|u_n(t) - u(t)\|.$$

Since $u_n \to u$, then we get $\varphi_n(t) \to \varphi(t)$ as $n \to \infty$ for each $t \in J$. And let $\delta > 0$ be such that, for each $t \in J$, we have $\|\varphi_n(t)\| \leq \delta$ and $\|\varphi(t)\| \leq \delta$. Then, we have

$$\begin{aligned} (e^t - e^s)^{\alpha-1}e^s\|\varphi_n(s) - \varphi(s)\| &\leq (e^t - e^s)^{\alpha-1}e^s\left[\|\varphi_n(s)\| + \|\varphi(s)\|\right] \\ &\leq 2\delta(e^t - e^s)^{\alpha-1}e^s \end{aligned}$$

and

$$\begin{aligned}(e^{t_k}-e^s)^{\alpha-1}e^s\|\varphi_n(s)-\varphi(s)\| &\leq (e^{t_k}-e^s)^{\alpha-1}e^s\left[\|\varphi_n(s)\|+\|\varphi(s)\|\right]\\ &\leq 2\delta(e^{t_k}-e^s)^{\alpha-1}e^s.\end{aligned}$$

For each $t\in J$, the functions $s\to 2\delta(e^t-e^s)^{\alpha-1}e^s$ and $s\to 2\delta(e^{t_k}-e^s)^{\alpha-1}e^s$ are integrable on $[a,t]$. Then, by Lebesgue dominated convergence theorem, (4.11) imply that

$$\|F(v_n)(t)-F(v)(t)\|\to 0\ as\ n\to\infty.$$

Thus,

$$\|F(u_n)-F(u)\|_{PC}\to 0 \text{ as } n\to\infty.$$

Consequently, F is continuous.

Let $\widetilde{f}=\sup_{t\in J}\|f(t,0,0)\|$, $\widetilde{I}=\max_{k=1,\dots,m}\|I_k(0)\|$ and consider the ball $B_R=\{y\in PC(J,E):\|y\|_{PC}\leq R\}$, such that

$$R\geq\left[\varepsilon m\widetilde{I}+\frac{\widetilde{f}}{k_1}\varepsilon\rho+\frac{|c_3|}{|c_1+c_2|}\right]\left[1-\varepsilon\rho-\varepsilon m\xi\right]^{-1}.$$

Step 2: *F maps B_R into itself.*

For any $y\in B_R$ and each $t\in J$, we have

$$\begin{aligned}\|F(y)(t)\| \leq\ & \frac{|c_2|}{|c_1+c_2|}\left[\sum_{i=1}^{m}\left\|I_i\left(y\left(t_i^-\right)\right)\right\|+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\left(e^{t_i}-e^s\right)^{\alpha-1}e^s\|\varphi(s)\|\,ds\right.\\ &\left.+\frac{1}{\Gamma(\alpha)}\int_{t_m}^{b}\left(e^b-e^s\right)^{\alpha-1}e^s\|\varphi(s)\|\,ds\right]+\frac{\|c_3\|}{|c_1+c_2|}+\sum_{a<t_k<t}\left\|I_k\left(y\left(t_k^-\right)\right)\right\|\\ &+\frac{1}{\Gamma(\alpha)}\sum_{a<t_k<t}\int_{t_{k-1}}^{t_k}\left(e^{t_k}-e^s\right)^{\alpha-1}e^s\|\varphi(s)\|\,ds\\ &+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(e^t-e^s\right)^{\alpha-1}e^s\|\varphi(s)\|\,ds,\end{aligned}\tag{4.12}$$

where $\varphi\in C(J,E)$ such that

$$\varphi(t)=f\left(t,y(t),\varphi(t)\right).$$

By (H2), we have for each $t \in J$

$$\begin{aligned}\|\varphi(t)\| &= \|f(t,y(t),\varphi(t)) - f(t,0,0) + f(t,0,0)\| \\ &\leq \|f(t,y(t),\varphi(t)) - f(t,0,0)\| + \|f(t,0,0)\| \\ &\leq k_1 \|y\| + k_2 \|\varphi(t)\| + \widetilde{f} \\ &\leq k_1 R + k_2 \|\varphi(t)\| + \widetilde{f}.\end{aligned}$$

Thus,

$$\|\varphi(t)\| \leq \gamma R + \frac{\widetilde{f}}{1-k_2} := \overline{M}.$$

Analogous to the recent calculus and by using (H3), we obtain

$$\|I_k(y)\| \leq \xi R + \widetilde{I} := \overline{N}.$$

From this and using the formula (4.12), it follows that

$$\begin{aligned}\|F(y)(t)\| &\leq \frac{|c_2|}{|c_1+c_2|}\Bigg[m\left(\xi R + \widetilde{I}\right) + m\left(\gamma R + \frac{\widetilde{f}}{1-k_2}\right)\frac{\left(e^b - e^a\right)^\alpha}{\Gamma(\alpha+1)} \\ &\quad + \left(\gamma R + \frac{\widetilde{f}}{1-k_2}\right)\frac{\left(e^b - e^a\right)^\alpha}{\Gamma(\alpha+1)}\Bigg] + \frac{\|c_3\|}{|c_1+c_2|} + m\left(\xi R + \widetilde{I}\right) \\ &\quad + m\left(\gamma R + \frac{\widetilde{f}}{1-k_2}\right)\frac{\left(e^b - e^a\right)^\alpha}{\Gamma(\alpha+1)} + \left(\gamma R + \frac{\widetilde{f}}{1-k_2}\right)\frac{\left(e^b - e^a\right)^\alpha}{\Gamma(\alpha+1)} \\ &= \left(\frac{|c_2|}{|c_1+c_2|} + 1\right)\left[m\left(\xi R + \widetilde{I}\right) + \left(\gamma R + \frac{\widetilde{f}}{1-k_2}\right)\frac{(m+1)\left(e^b - e^a\right)^\alpha}{\Gamma(\alpha+1)}\right] \\ &\quad + \frac{\|c_3\|}{|c_1+c_2|} \\ &= \varepsilon\left[m\left(\xi R + \widetilde{I}\right) + \rho R + \frac{\widetilde{f}}{k_1}\rho\right] + \frac{\|c_3\|}{|c_1+c_2|} \\ &= \varepsilon m \xi R + \varepsilon m \widetilde{I} + \varepsilon \rho R + \frac{\widetilde{f}}{k_1}\varepsilon\rho + \frac{\|c_3\|}{|c_1+c_2|} \\ &\leq R.\end{aligned}$$

Which implies $\|Fy\|_{PC} \leq R$. Therefore, $F : B_R \to B_R$.

Step 3: *$F(B_R)$ is bounded.*

This is clear since from the previous step we know that $F(B_R) \subset B_R$ and B_R is bounded. Thus, for each $v \in B_R$, we have $\|F(y)\|_{PC} \leq R$. Then, $F(B_R)$ is bounded.

Step 4: *$F(B_R)$ is equicontinuous.*

Let $\tau_1, \tau_2 \in J$, such that $\tau_1 < \tau_2$ and let $y \in B_R$. Then.

$$\begin{aligned}
\|F(y)(\tau_2) - F(y)(\tau_1)\| \leq & \frac{1}{\Gamma(\alpha)} \int_a^{\tau_1} \left| \left[(e^{\tau_2} - e^s)^{\alpha-1} - (e^{\tau_1} - e^s)^{\alpha-1} \right] e^s \right| \|\varphi(s)\| \, ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{\tau_2} \left| (e^{\tau_2} - e^s)^{\alpha-1} e^s \right| \|\varphi(s)\| \, ds + \sum_{\tau_1 < \tau_k < \tau_2} \left\| I_k \left(v \left(\tau_k^- \right) \right) \right\| \\
& + \frac{1}{\Gamma(\alpha)} \sum_{\tau_1 < t_k < \tau_2} \int_{t_{k-1}}^{t_k} \left| \left(e^{t_k} - e^s \right)^{\alpha-1} e^s \right| \|\varphi(s)\| \, ds
\end{aligned}$$

Hence

$$\begin{aligned}
\|F(y)(\tau_2) - F(y)(\tau_1)\| \leq & \frac{\overline{M}}{\Gamma(\alpha+1)} \left[(e^{\tau_1} - e^a)^\alpha - (e^{\tau_2} - e^a)^\alpha + 2 (e^{\tau_2} - e^{\tau_1})^\alpha \right] \\
& + (\tau_2 - \tau_1) \left[\overline{N} + \frac{\overline{M}}{\Gamma(\alpha+1)} \left(e^b - e^a \right)^\alpha \right].
\end{aligned}$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero.

Before the next step, by using (4.8) in the proof of Lemma 4.3. For each $t \in J$ and $k = 1, \ldots, m$, we can write the operator F as

$$\begin{aligned}
F(y)(t) = & \eta_0 + \sum_{a < t_k < t} I_k \left(y \left(t_k^- \right) \right) + \frac{1}{\Gamma(\alpha)} \sum_{a < t_k < t} \int_{t_{k-1}}^{t_k} \left(e^{t_k} - e^s \right)^{\alpha-1} \varphi(s) e^s ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(e^t - e^s \right)^{\alpha-1} \varphi(s) e^s ds,
\end{aligned}$$

where $\varphi \in C(J, E)$ such that

$$\varphi(t) = f\left(t, y(t), \varphi(t)\right).$$

Step 5: *The operator $F : B_R \to B_R$ is a strict set contraction.*
Let $V \subset B_R$. For each $t \in J$, we have

$$\begin{aligned}
\mu\left(F\left(V\right)\left(t\right)\right) &= \mu\left(\left(Fy\right)\left(t\right),\ y \in V\right) \\
&\leq \mu\left(\eta_0\right) + \sum_{a<t_k<t} \left\{\mu\left(I_k\left(y\left(t_k^-\right)\right)\right),\ y \in V\right\} \\
&\quad + \frac{1}{\Gamma(\alpha)} \sum_{a<t_k<t} \left\{\int_{t_{k-1}}^{t_k} \left(e^{t_k} - e^s\right)^{\alpha-1} e^s \mu\left(\varphi(s)\right) ds,\ y \in V\right\} \\
&\quad + \frac{1}{\Gamma(\alpha)} \left\{\int_{t_k}^{t} \left(e^t - e^s\right)^{\alpha-1} e^s \mu\left(\varphi(s)\right) ds,\ y \in V\right\}.
\end{aligned}$$

By the Remark 4.4 and the Properties 1.12, for each $s \in J$, we get

$$\begin{aligned}
\mu\left(\left\{\varphi\left(s\right),\ y \in V\right\}\right) &= \mu\left(\left\{f\left(s, y(s), \varphi(s)\right),\ y \in V\right\}\right) \\
&\leq k_1 \mu\left(\left\{y\left(s\right),\ \in V\right\}\right) + k_2\left(\left\{\varphi\left(s\right),\ y \in V\right\}\right).
\end{aligned}$$

Thus,

$$\mu\left(\left\{\varphi\left(s\right),\ y \in V\right\}\right) \leq \gamma\, \mu\left(\left\{y\left(s\right),\ y \in V\right\}\right). \tag{4.13}$$

Also, for each $t \in J$ and $k = 1, \ldots, m$, we get

$$\mu\left(\left\{\left(I_k\left(y\left(t_k^-\right)\right)\right),\ y \in V\right\}\right) \leq \xi\, \mu\left(\left\{y\left(t\right),\ y \in V\right\}\right). \tag{4.14}$$

Hence,

$$\begin{aligned}
\mu\left(F\left(V\right)\left(t\right)\right) &\leq m\xi\, \mu\left(\left\{y\left(t\right),\ y \in V\right\}\right) \\
&\quad + \frac{m\gamma}{\Gamma(\alpha)} \left\{\int_a^b \left(e^b - e^s\right)^{\alpha-1} e^s \mu\left(y(s)\right) ds,\ y \in V\right\} \\
&\quad + \frac{\gamma}{\Gamma(\alpha)} \left\{\int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \mu\left(y(s)\right) ds,\ y \in V\right\} \\
&\leq \left[m\xi + \frac{\gamma\left(m+1\right)\left(e^b - e^a\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}\right] \mu_{PC}\left(V\right).
\end{aligned}$$

Therefore,

$$\mu_{PC}\left(FV\right) \leq \left(m\xi + \rho\right) \mu_{PC}\left(V\right).$$

Since $\varepsilon > 1$. So, from (4.9), it clear that $(m\xi + \rho) < 1$, which implies that the operator F is a λ-set contraction, where

$$\lambda = \left(m\xi + \rho\right).$$

As a consequence of steps 1 to 5 together with Theorem 1.16, we deduce that the operator F has at least one fixed point in B_R. This shows that the impulsive boundary value problem (4.1)-(4.3) has at least one solution on J. This completes the proof.

The second existence result for the impulsive boundary value problem (4.1)-(4.3) is based on the concept of measure of noncompactness and Mönch's fixed point theorem.

Theorem 4.6 *Assume that (H1)-(H3) and the condition (4.9) hold. Then the impulsive boundary value problem (4.1)-(4.3) has at least one solution on J.*

Proof. Consider the operator F as defined in (4.10). We shall show that F satisfies the assumption of Mönch's fixed point theorem. We know that $F : B_R \to B_R$ is bounded and continuous, we need to prove that the implication

$$V = \overline{conv}F(V), \quad or \quad V = F(V) \cup \{0\} \Rightarrow \mu(V) = 0$$

holds for every subset V of B_R.
Now let V be a subset of B_R such that $V \subset \overline{conv}(F(V) \cup \{0\})$. Then V is bounded and equicontinuous and therefore the function $t \to v(t) = \mu(V(t))$ is continuous on J.
By using the Remark 4.4, Lemma 4.1 and Properties 1.12, for each $t \in J$, we have

$$\begin{aligned}
v(t) &= \mu(V(t)) \\
&= \mu\left(F(V) \cup \{0\}\right) \\
&\leq \mu\left(\left(FV\right)(t)\right) \\
&\leq \mu\left(y\left(a\right)\right) + \sum_{a<t_k<t} \left\{\mu\left(I_k\left(y\left(t_k^-\right)\right)\right),\ y \in V\right\} \\
&\quad + \frac{1}{\Gamma(\alpha)} \sum_{a<t_k<t} \left\{\int_{t_{k-1}}^{t_k} \left(e^{t_k} - e^s\right)^{\alpha-1} e^s \mu\left(\varphi(s)\right) ds,\ y \in V\right\} \\
&\quad + \frac{1}{\Gamma(\alpha)} \left\{\int_{t_k}^{t} \left(e^t - e^s\right)^{\alpha-1} e^s \mu\left(\varphi(s)\right) ds,\ y \in V\right\}.
\end{aligned}$$

By using (4.13) and (4.14), we obtain that

$$\begin{aligned}
v(t) &\leq m\xi\ \mu\left(\{y\left(t\right),\ y \in V\}\right) \\
&\quad + \frac{m\gamma}{\Gamma(\alpha)} \left\{\int_a^b \left(e^b - e^s\right)^{\alpha-1} e^s \mu\left(y(s)\right) ds,\ y \in V\right\} \\
&\quad + \frac{\gamma}{\Gamma(\alpha)} \left\{\int_a^b \left(e^b - e^s\right)^{\alpha-1} e^s \mu\left(y(s)\right) ds,\ y \in V\right\}
\end{aligned}$$

Then

$$\begin{aligned} v(t) &\leq m\xi v(t) + \frac{\gamma\,(m+1)}{\Gamma(\alpha)} \int_a^b \left(e^b - e^s\right)^{\alpha-1} e^s v(s)ds, \\ &\leq \left[m\xi + \frac{\gamma\,(m+1)\left(e^b - e^a\right)^{\alpha}}{\Gamma\,(\alpha+1)}\right] \|v\|_{PC}. \end{aligned}$$

Therefore,

$$\|v\|_{PC} \leq (m\xi + \rho)\,\|v\|_{PC}.$$

Since $\varepsilon > 1$. So, from (4.9), it clear that $(m\xi + \rho) < 1$. Which implies that $v(t) = 0$, for each $t \in J$. Hence, $V(t)$ is relatively compact in E. In view of the Ascoli-Arzelà theorem, V is relatively compact in B_R. Therefore, by applying Theorem 1.18, we deduce that the operator F has at least one fixed point in B_R. This shows that the impulsive boundary value problem (4.1)-(4.3) has at least one solution on J. This completes the proof.

Remark 4.7 *Our results for the boundary value problem (4.1)-(4.3) remain true for the following cases:*

- *Initial value problem:* $c_1 = 1,\ c_2 = 0$ *and* c_3 *arbitrary.*
- *Terminal value problem:* $c_1 = 0,\ c_2 = 1$ *and* c_3 *arbitrary.*
- *Anti-periodic problem:* $c_1 = c_2 \neq 0$ *and* $c_3 = 0$.

However, our results are not applicable for the periodic problem, i.e. for $c_1 = 1,\ c_2 = -1$, *and* $c_3 = 0$.

4.3 Examples

In this section, we will give two examples to illustrate our main results. Let

$$E = l^1 = \left\{ y = (y_1, y_2, \ldots, y_n, \ldots) : \sum_{n=1}^{\infty} |y_n| < \infty \right\}$$

be the Banach space with the norm

$$\|y\|_E = \sum_{n=1}^{\infty} |y_n|.$$

Example 1. Consider the following impulsive boundary value problem for nonlinear implicit fractional differential equation:

$${}^e_cD^{\frac{1}{2}}_{t_k} y_n(t) = \frac{e^{-\sqrt{t+25}} \sin t}{13\,(t^4+1)\left(1 + |y_n(t)| + \left|{}^e_cD^{\frac{1}{2}}_{t_k} y_n(t)\right|\right)}, \quad \text{for each } t \in J_0 \cup J_1, \tag{4.15}$$

$$\Delta y_n|_{t=\frac{\pi}{2}} = \frac{\left|y_n(\frac{\pi}{2}^-)\right|}{89+\left|y_n(\frac{\pi}{2}^-)\right|}, \tag{4.16}$$

$$2y_n(0)+y_n(\pi) = 283, \tag{4.17}$$

where $J_0 = \left[0, \frac{\pi}{2}\right]$, $J_1 = (\frac{\pi}{2}, \pi]$, $m = 1$, $\alpha = \frac{1}{2}$, $a = 0$, $b = \pi$, $c_1 = 2$, $c_2 = 1$, $\sigma = 283$, $y = (y_1, y_2, \ldots, y_n, \ldots)$, $f = (f_1, f_2, \ldots, f_n, \ldots)$, ${}_c^eD_{t_k}^{\frac{1}{2}}y = \left({}_c^eD_{t_k}^{\frac{1}{2}}y_1, {}_c^eD_{t_k}^{\frac{1}{2}}y_2, \ldots, {}_c^eD_{t_k}^{\frac{1}{2}}y_n, \ldots\right)$,

$$f(t,y,z) = \frac{e^{-\sqrt{t+25}}\sin t}{13\left(t^4+1\right)\left(1+\|y\|_E+\|z\|_E\right)}$$

and

$$I_1(y) = \frac{\|y\|_E}{89+\|y\|_E}.$$

Now, for each $t \in [0,\pi]$ and for any $y_1, y_2, z_1, z_2 \in E$, we can show that

$$\|f(t,y_1,z_1) - f(t,y_2,z_2)\|_E \le \frac{1}{13e^5}\left(\|y_1-y_2\|_E + \|z_1-z_2\|_E\right)$$

and

$$\|I_1(y_1) - I_1(y_2)\|_E \le \frac{1}{89}\|y_1-y_2\|_E.$$

Thus, for $k_1 = k_2 = \dfrac{1}{13e^5}$ and $\xi = \dfrac{1}{89}$ we have that

$$\begin{aligned}
\varepsilon(m\xi+\rho) &= \left(\frac{|c_2|}{|c_1+c_2|}+1\right)\left[m\xi + \frac{k_1(m+1)\left(e^b-e^a\right)^\alpha}{(1-k_2)\,\Gamma(\alpha+1)}\right] \\
&= \frac{5}{3}\left\{\frac{1}{89} + \frac{2\sqrt{e^\pi-1}}{13e^5}\left[\left(1-\frac{1}{13e^5}\right)\Gamma\left(\frac{3}{2}\right)\right]^{-1}\right\} \\
&= \frac{5}{3}\left[\frac{1}{89} + \frac{4\sqrt{e^\pi-1}}{\left(13e^5-1\right)\sqrt{\Pi}}\right] \\
&\approx 0.02790439619 \\
&< 1.
\end{aligned}$$

Hence, from Theorem 4.5. The impulsive nonlinear fractional boundary value problem (4.15)-(4.17) has at least one solution on $[0,\pi]$.

Example 2. Consider the following impulsive boundary value problem for nonlinear implicit fractional differential equation:

$${}_c^eD_{t_k}^{\frac{1}{2}}y_n(t) = \frac{e^{-\sqrt{t+36}}\left(2+|y_n(t)|+\left|{}_c^eD_{t_k}^{\frac{1}{2}}y_n(t)\right|\right)}{181\left(t^6+1\right)\left(1+|y_n(t)|+\left|{}_c^eD_{t_k}^{\frac{1}{2}}y_n(t)\right|\right)}, \quad \text{for each } t \in J_0 \cup J_1, \tag{4.18}$$

$$\Delta y_n|_{t=\frac{3}{4}} = \frac{10\left|y_n(\frac{3}{4}^-)\right|}{20+\left|y_n(\frac{3}{4}^-)\right|}, \tag{4.19}$$

$$y_n(0) = -y_n(1), \tag{4.20}$$

where $J_0 = \left[0, \frac{3}{4}\right]$, $J_1 = (\frac{3}{4}, 1]$, $m = 1$, $\alpha = \frac{1}{2}$, $a = 0$, $b = 1$, $c_1 = c_2 = 1$, $\sigma = 0$, $y = (y_1, y_2, \dots, y_n, \dots)$, $f = (f_1, f_2, \dots, f_n, \dots)$, ${}_c^eD_{t_k}^{\frac{1}{2}}y = \left({}_c^eD_{t_k}^{\frac{1}{2}}y_1, {}_c^eD_{t_k}^{\frac{1}{2}}y_2, \dots, {}_c^eD_{t_k}^{\frac{1}{2}}y_n, \dots\right)$,

$$f(t, y, z) = \frac{e^{-\sqrt{t+36}}\left(2+\|y\|_E+\|z\|_E\right)}{181\left(t^6+1\right)\left(1+\|y\|_E+\|z\|_E\right)}$$

and

$$I_1(y) = \frac{10\|y\|_E}{20+\|y\|_E}.$$

Now, for each $t \in [0, 1]$ and for any $y_1, y_2, z_1, z_2 \in E$, we can show that

$$\|f(t, y_1, z_1) - f(t, y_2, z_2)\|_E \;\leq\; \frac{1}{181e^6}\left(\|y_1-y_2\|_E+\|z_1-z_2\|_E\right)$$

and

$$\|I_1(y_1) - I_1(y_2)\|_E \;\leq\; \frac{1}{2}\|y_1-y_2\|_E.$$

Thus, for $k_1 = k_2 = \dfrac{1}{181e^6}$ and $\xi = \dfrac{1}{2}$ we have that

$$\begin{aligned}
\varepsilon(m\xi+\rho) &= \left(\frac{|c_2|}{|c_1+c_2|}+1\right)\left[m\xi+\frac{k_1(m+1)\left(e^b-e^a\right)^\alpha}{(1-k_2)\Gamma(\alpha+1)}\right] \\
&= \frac{3}{2}\left\{\frac{1}{2}+\frac{2\sqrt{e-1}}{181e^6}\left[\left(1-\frac{1}{181e^6}\right)\Gamma\left(\frac{3}{2}\right)\right]^{-1}\right\} \\
&= \frac{3}{2}\left[\frac{1}{2}+\frac{4\sqrt{e-1}}{\left(181e^6-1\right)\sqrt{\Pi}}\right] \\
&\approx 0.7500607693 \\
&< 1.
\end{aligned}$$

Hence, from Theorem 4.6. The impulsive nonlinear fractional boundary value problem (4.18)-(4.20) has at least one solution on $[0, 1]$.

Chapter 5

Initial Value Problem for Nonlinear Implicit Caputo's Exponential Type Fractional Differential Equations with Non-Instantaneous Impulses

5.1 Introduction

This chapter is devoted to the results obtained by Malti *et al.* [77], we establish first the existence of solutions and Ulam-Hyers-Rassias (**U-H-R**) stability for a class of initial value problem (**IVP**) for nonlinear implicit Caputo's exponential type fractional differential equations with non-instantaneous impulses. At the end, we give some generalization of our results to the nonlocal cases.

5.2 Existence and Stability Results for the IVP

In this section, we study the existence, uniqueness of solutions and Ulam-Hyers-Rassias (**U-H-R**) stability for a class of initial value problem (**IVP**) for the following nonlinear implicit Caputo's exponential type fractional differential equations with non-instantaneous impulses:

$$ {}^{e}_{c}D^{\alpha}_{s_k}y(t) = f\left(t, y(t),\ {}^{e}_{c}D^{\alpha}_{s_k}y(t)\right), \quad \text{for each } t \in J_k \subseteq J,\ k = 0, 1, \ldots, m, \tag{5.1} $$

$$ y(t) = g_k(t, y(t)), \quad \text{for each } t \in J'_k \subset J,\ k = 1, 2, \ldots, m, \tag{5.2} $$

$$ y(a) = \delta, \tag{5.3} $$

where ${}^{e}_{c}D^{\alpha}_{a^+}$ denote the Caputo's exponential type fractional derivatives of order $\alpha \in (0, 1]$, $\delta \in \mathbb{R}$, $J = [a, b]$, $a = t_0 = s_0 < t_1 \leq s_1 < \ldots < t_m \leq s_m < t_{m+1} = b$, $J'_k := (t_k, s_k]$, $J_k := (s_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, $J_0 := [a, t_1]$, $f : \overline{\mathcal{J}} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g_k : \overline{\mathcal{J}'} \times \mathbb{R} \to \mathbb{R}$

are a given functions such that $\overline{\mathcal{J}} = \bigcup\limits_{k=0}^{m} [s_k, t_{k+1}]$, and $\overline{\mathcal{J}'} = \bigcup\limits_{k=1}^{m} [t_k, s_k]$. Here, the impulses start abruptly at the points $t_k,\ k = 1, \ldots, m$ and their action continues on the intervals $J'_k,\ k = 1, \ldots, m$. The function y takes an impulses at $t_k,\ k = 1, \ldots, m$ and it follows different rules in two consecutive intervals J_k and J'_k. At the point $s_k,\ k = 1, \ldots, m$, the function y is continuous. The intervals $J'_k,\ k = 1, \ldots, m$ are called intervals of non-instantaneous impulses for the problem (5.1)-(5.3) and the functions $g_k(t, y(t))$, $k = 1, \ldots, m$ are called non-instantaneous impulsive functions for the problem (5.1)-(5.3). Also, the problem (5.1)-(5.3) can be reduced to an impulsive fractional differential equations when $t_k = s_k, k = 1, \ldots, m$.

5.2.1 Existence of Solutions

The following notations which are useful on this chapter. Consider the set of functions

$$PC(J, \mathbb{R}) = \Big\{ y : J \to \mathbb{R},\ y \in C(J_0, \mathbb{R}) \text{ and } y \in C(J'_k \cup J_k, \mathbb{R}),\ k = 1, 2, ..., m$$
$$\text{and there exist } y(t_k^-), y(t_k^+) \text{ for every } k = 1, \ldots, m \text{ with } y(t_k^-) = y(t_k) \Big\}.$$

This set is the Banach space with the norm

$$\|y\|_{PC} = \max\left\{ \sup_{t \in \bigcup\limits_{k=1}^{m} J'_k} |y(t)|,\ \sup_{t \in \bigcup\limits_{k=0}^{m} J_k} |y(t)| \right\}.$$

Now, we define what we mean by a solution of the IVP (5.1)-(5.3).

Definition 5.1 *A function $y \in PC(J, \mathbb{R}) \cap \left(\bigcup\limits_{k=0}^{m} AC_e(J_k, \mathbb{R}) \right)$ is said a solution of (5.1)-(5.3) if y satisfies the condition $y(a) = \delta$, the equations ${}^e_cD^{\alpha}_{s_k} y(t) = f(t, y(t), {}^e_cD^{\alpha}_{s_k} y(t))$ on $J_k,\ k = 0, 1, \ldots, m$ and $y(t) = g_k(t, y(t))$ on $J'_k,\ k = 1, 2, \ldots, m$.*

To Prove the existence of solutions of the IVP (5.1)-(5.3), we need the following auxiliary lemma.

Lemma 5.2 *Let $\alpha \in (0, 1]$ and let $\varphi : \overline{\mathcal{J}} \to \mathbb{R}$ be continuous. A function y is a solution of the fractional integral equation*

$$y(t) = \begin{cases} \delta + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t \left(e^t - e^s\right)^{\alpha - 1} e^s \varphi(s) ds, & \text{if } t \in [a, t_1] \\ g_k(t, y(t)), & \text{if } t \in J'_k = (t_k, s_k] \\ g_k(s_k, y(s_k)) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{s_k}^t \left(e^t - e^s\right)^{\alpha - 1} e^s \varphi(s) ds, & \text{if } t \in J_k = (s_k, t_{k+1}]. \end{cases} \tag{5.4}$$

if and only if y is a solution of the fractional IVP with non-instantaneous impulses

$$ {}_c^eD^{\alpha}_{s_k}y(t) = \varphi\left(t\right), \quad \textit{for each } t \in J_k \subseteq J,\ k = 0, 1, \ldots, m, \tag{5.5}$$

$$ y(t) = g_k(t, y(t)), \quad \textit{for each } t \in J'_k \subset J,\ k = 1, 2, \ldots, m, \tag{5.6}$$

$$ y(a) = \delta. \tag{5.7}$$

Proof. Assume that y satisfies (5.5)-(5.7). If $t \in [a, t_1]$, then

$$ {}_c^eD^{\alpha}_{a}y(t) = \varphi(t).$$

By Lemma 1.10, we get

$$ y(t) = \eta_0 + {}^eI^{\alpha}_a\varphi(t).$$

Since $y\left(a\right) = \delta$, then $\eta_0 = \delta$ and

$$ y\left(t\right) = \delta + \frac{1}{\Gamma(\alpha)}\int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s\varphi(s)ds.$$

If $t \in J'_1 := (t_1, s_1]$, we have

$$ y(t) = g_1(t, y(t)).$$

If $t \in J_1 := (s_1, t_2]$, then by Lemma 1.10 we get

$$\begin{aligned} y(t) &= y\left(s_1\right) + \frac{1}{\Gamma(\alpha)}\int_{s_1}^t \left(e^t - e^s\right)^{\alpha-1} e^s\varphi(s)ds \\ &= g_1(s_1) + \frac{1}{\Gamma(\alpha)}\int_{s_1}^t \left(e^t - e^s\right)^{\alpha-1} e^s\varphi(s)ds. \end{aligned}$$

If $t \in J'_2 := (t_2, s_2]$, we have

$$ y(t) = g_2(t, y(t)).$$

If $t \in J_2 := (s_2, t_3]$, then by Lemma 1.10 we get

$$\begin{aligned} y(t) &= y\left(s_2, y(s_2)\right) + \frac{1}{\Gamma(\alpha)}\int_{s_2}^t \left(e^t - e^s\right)^{\alpha-1} e^s\varphi(s)ds \\ &= g_2(s_2, y(s_2)) + \frac{1}{\Gamma(\alpha)}\int_{s_2}^t \left(e^t - e^s\right)^{\alpha-1} e^s\varphi(s)ds. \end{aligned}$$

If $t \in J'_k := (t_k, s_k]$, we have

$$ y(t) = g_k(t, y(t)).$$

If $t \in J_k := (s_k, t_{k+1}]$, then by Lemma 1.10 we get

$$\begin{aligned} y(t) &= y\left(s_k, y(s_k)\right) + \frac{1}{\Gamma(\alpha)}\int_{s_k}^t \left(e^t - e^s\right)^{\alpha-1} e^s\varphi(s)ds \\ &= g_k(s_k, y(s_k)) + \frac{1}{\Gamma(\alpha)}\int_{s_k}^t \left(e^t - e^s\right)^{\alpha-1} e^s\varphi(s)ds. \end{aligned}$$

By the same ways, for every non-zero integer $k \leq m$, the solution $y(\cdot)$ can be written as

$$y(t) = \begin{cases} g_k(t, y(t)), & \text{if } t \in J'_k = (t_k, s_k] \\ g_k(s_k, y(s_k)) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{s_k}^{t} \left(e^t - e^s\right)^{\alpha-1} e^s \varphi(s) ds, & \text{if } t \in J_k = (s_k, t_{k+1}]. \end{cases}$$

Conversely, assume that y satisfies the equation (5.4). If $t \in [a, t_1]$ then $y(a) = \delta$ and using the fact that ${}^e_c D^\alpha_a$ is the left inverse of ${}^e I^\alpha_a$ we get

$${}^e_c D^\alpha_a y(t) = \varphi(t), \quad \text{for each } t \in [a, t_1].$$

If $t \in (s_k, t_{k+1}]$, where $k = 1, \ldots, m$. Then, by using the fact that ${}^e_c D^\alpha_{s_k} C = 0$, where C is a constant and ${}^e_c D^\alpha_{s_k}$ is the left inverse of ${}^e I^\alpha_{s_k}$, we get

$${}^e_c D^\alpha_{s_k} y(t) = \varphi(t), \quad \text{for each } t \in (s_k, t_{k+1}].$$

Obviously, it easy to see that

$$y(t) = g_k(t, y(t)), \quad \text{for each } t \in (t_k, s_k], \; k = 1, 2, \ldots, m.$$

The following hypotheses will be used in the sequel:

(H1) The function $f : \overline{\mathcal{J}} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H2) There exist constants $k_1 > 0$ and $0 < k_2 < 1$ such that

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq k_1 |y_1 - y_2| + k_2 |z_1 - z_2| \quad \text{for } t \in \bigcup_{k=0}^{m} J_k \text{ and } y_1, y_2, z_1, z_2 \in \mathbb{R}.$$

(H3) The function $g_k : \overline{\mathcal{J}'} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist constants $\xi_k > 0$, $k = 1, \ldots, m$ such that

$$|g_k(t, y_1) - g_k(t, y_2)| \leq \xi_k |y_1 - y_2| \quad \text{for } t \in J'_k, \; k = 1, \ldots, m \text{ and } y_1, y_2 \in \mathbb{R}.$$

(H4) There exist the functions $p, q_1, q_2 \in C\left(\overline{\mathcal{J}}, \mathbb{R}_+\right)$ such that

$$|f(t, y, z)| \leq p(t) + q_1(t)|y| + q_2(t)|z| \quad \text{for } t \in \bigcup_{k=0}^{m} J_k \text{ and } y, z \in \mathbb{R}.$$

(H5) The function g_k is continuous and there exist $p_{g_k} \in C\left(\overline{\mathcal{J}'}, \mathbb{R}_+\right)$ such that

$$|g_k(t, y)| \leq p_{g_k}(t)(|y| + 1) \quad \text{for } t \in J'_k, \; k = 1, \ldots, m \text{ and } y \in \mathbb{R}.$$

Now, we pass to state and proof our first existence result for the problem (5.1)-(5.3) based on Banach's fixed point theorem.
Set

$$\gamma = \frac{k_1}{1 - k_2}, \quad \rho = \frac{\left(e^b - e^a\right)^\alpha}{\Gamma(\alpha + 1)}, \quad \xi := \max_{k=1,\ldots,m} \{\xi_k\} \text{ and } p^*_g = \max_{k=1,\ldots,m} \left\{ \sup_{t \in [t_k, s_k]} p_{g_k}(t) \right\} < 1.$$

Theorem 5.3 *Assume that (H1)-(H3) are satisfied. If*

$$\xi + \gamma\rho < 1, \tag{5.8}$$

then the IVP (5.1)-(5.3) has a unique solution on J.

Proof. Transform the problem (5.1)-(5.3) into a fixed point problem, consider the operator $\Xi : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ defined by

$$\Xi(y)(t) = \begin{cases} \delta + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi(s) ds, & \text{if } t \in [a, t_1] \\ g_k(t, y(t)), & \text{if } t \in J_k' = (t_k, s_k] \\ g_k(s_k, y(s_k)) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{s_k}^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi(s) ds, & \text{if } t \in J_k = (s_k, t_{k+1}], \end{cases} \tag{5.9}$$

where $\varphi \in C(\overline{\mathcal{J}}, \mathbb{R})$ such that

$$\varphi(t) = f(t, y(t), \varphi(t)).$$

It is clear that, the fact of finding solutions for problem (5.1)-(5.3) is to find fixed points of the operator Ξ. Let $y_1, y_2 \in PC(J, \mathbb{R})$ for each $t \in J_k$, we have

$$\begin{aligned} \left|\Xi(y_1)(t) - \Xi(y_2)(t)\right| &\leq \left|g_k(s_k, y_1(s_k)) - g_k(s_k, y_2(s_k))\right| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{s_k}^t \left(e^t - e^s\right)^{\alpha-1} e^s \left|\varphi_1(s) - \varphi_2(s)\right| ds, \end{aligned}$$

where $\varphi_1, \varphi_2 \in C(\overline{\mathcal{J}}, \mathbb{R})$, such that

$$\varphi_1(t) = f(t, y_1(t), \varphi_1(t)) \quad \text{and} \quad \varphi_2(t) = f(t, y_2(t), \varphi_2(t)).$$

By (H2), we have

$$\begin{aligned} \left|\varphi_1(t) - \varphi_2(t)\right| &= \left|f(t, y_1(t), \varphi_1(t)) - f(t, y_2(t), \varphi_2(t))\right| \\ &\leq k_1 \left|y_1(t) - y_2(t)\right|| + k_2 \left|\varphi_1(t) - \varphi_2(t)\right|. \end{aligned}$$

Then

$$\left|\varphi_1(t) - \varphi_2(t)\right| \leq \gamma \ \left|y_1(t) - y_2(t)\right|.$$

By (H3), we have

$$\left|g_k(s_k, y_1(s_k)) - g_k(s_k, y_2(s_k))\right| \leq \xi_k \left|y_1(s_k) - y_2(s_k)\right| \leq \xi \left|y_1(s_k) - y_2(s_k)\right|.$$

Hence, for each $t \in J_k$

$$\begin{aligned} |\Xi(y_1)(t) - \Xi(y_2)(t)| &\leq \xi |y_1(s_k) - y_2(s_k)| \\ &\quad + \frac{\gamma}{\Gamma(\alpha)} \int_{s_k}^{t} \left(e^t - e^s\right)^{\alpha-1} e^s |y_1(t) - y_2(t)| \, ds \\ &\leq \left(\xi + \frac{\gamma \left(e^b - e^a\right)^{\alpha}}{\Gamma(\alpha+1)} \right) \|y_1 - y_2\|_{PC}. \end{aligned}$$

Thus,

$$\|\Xi(y_1) - \Xi(y_2)\|_{PC} \leq (\xi + \gamma\rho) \|y_1 - y_2\|_{PC}.$$

Analogous to the recent calculus, for each $t \in [a, t_1]$, we get

$$|\Xi(y_1)(t) - \Xi(y_2)(t)| \leq \gamma\rho |y_1(t) - y_2(t)|.$$

Then

$$\|\Xi(y_1) - \Xi(y_2)\|_{PC} \leq \gamma\rho \|y_1 - y_2\|_{PC}.$$

Also, for each $t \in J'_k$, we obtain

$$|\Xi(y_1)(t) - \Xi(y_2)(t)| \leq \xi |y_1(t) - y_2(t)|.$$

Then

$$\|\Xi(y_1) - \Xi(y_2)\|_{PC} \leq \xi \|y_1 - y_2\|_{PC}.$$

Therefore, it easy to see that for each $t \in J$, we have

$$|\Xi(y_1)(t) - \Xi(y_2)(t)| \leq (\xi + \gamma\rho) |y_1(t) - y_2(t)|.$$

Then

$$\|\Xi(y_1) - \Xi(y_2)\|_{PC} \leq (\xi + \gamma\rho) \|y_1 - y_2\|_{PC}.$$

From (5.8) it follows that the operator Ξ is a contraction. Hence, by Banach's contraction principle, we deduce that Ξ has a unique fixed point which is a unique solution of the fractional IVP (5.1)-(5.3) on J. This completes the proof.

The second existence result is based on Schaefer's fixed point theorem.
Set

$$p^* = \sup_{t \in \overline{J}} p(t), \quad q_1^* = \sup_{t \in \overline{J}} q_1(t), \quad q_2^* = \sup_{t \in \overline{J}} q_2(t) < 1.$$

Theorem 5.4 *Assume that (H1) and (H4)-(H5) hold. If*

$$p_g^* + \frac{q_1^* \rho}{1 - q_2^*} < 1, \tag{5.10}$$

then the problem (5.1)-(5.3) has at least one solution on J.

Proof. We shall use Schaefer's fixed point theorem to prove that Ξ defined by (5.9) has at least one fixed point on J. The proof will be given in several steps.

Step 1: Ξ *is continuous.*
Let $\{v_n\}$ be a sequence such that $v_n \to v$ in $PC(J, \mathbb{R})$.
Case 1. For each $t \in J_k$, we obtain

$$\begin{aligned} |\Xi(v_n)(t) - \Xi(v)(t)| &\leq |g_k(s_k, v_n(s_k)) - g_k(s_k, v(s_k))| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{s_k}^{t} \left(e^t - e^s\right)^{\alpha-1} e^s |\varphi_n(s) - \varphi(s)| \, ds, \end{aligned}$$

where $\varphi_n, \varphi \in C(\overline{\mathcal{J}}, \mathbb{R})$, such that

$$\varphi_n(t) = f(t, v_n(t), \varphi_n(t)) \quad \text{and} \quad \varphi(t) = f(t, v(t), \varphi(t)).$$

Case 2. For each $t \in J'_k$, we have

$$|\Xi(v_n)(t) - \Xi(v)(t)| \leq |g_k(t, v_n(t)) - g_k(t, v(t))|.$$

Case 3. For each $t \in [a, t_1]$, we get

$$|\Xi(v_n)(t) - \Xi(v)(t)| \leq \frac{1}{\Gamma(\alpha)} \int_{a}^{t} \left(e^t - e^s\right)^{\alpha-1} e^s |\varphi_n(s) - \varphi(s)| \, ds,$$

where $\varphi_n, \varphi \in C(\overline{\mathcal{J}}, \mathbb{R})$, such that

$$\varphi_n(t) = f(t, v_n(t), \varphi_n(t)) \quad \text{and} \quad \varphi(t) = f(t, v(t), \varphi(t)).$$

Since $v_n \to v$ as $n \to \infty$ and f, g_k are continuous, then by Lebesgue dominated convergence theorem, we have

$$|\varphi_n(t) - \varphi(t)| \to \infty \quad \text{as} \quad n \to \infty \quad \text{and} \quad |g_k(t, v_n(t)) - g_k(t, v(t))| \to \infty \quad \text{as} \quad n \to \infty,$$

which leads to $\|\Xi(v_n)(t) - \Xi(v)(t)\|_{PC} \to \infty$ as $n \to \infty$. Therefore Ξ is continuous.

Step 2 : Ξ *maps bounded sets into bounded sets in* $PC(J, \mathbb{R})$.
Indeed, it is enough to show that for any $\varrho > 0$ there exists a positive constant ℓ such that for any $v \in B_\varrho = \{v \in PC(J, \mathbb{R}) : \|v\|_{PC} \leq \varrho\}$, we have $\|\Xi(v)\|_{PC} \leq \ell$.
Case 1. For each $t \in [a, t_1]$, we have

$$|\Xi(v)(t)| \leq |\delta| + \frac{1}{\Gamma(\alpha)} \int_{a}^{t} \left(e^t - e^s\right)^{\alpha-1} e^s |\varphi(s)| \, ds,$$

where $\varphi \in C(\overline{\mathcal{J}}, \mathbb{R})$ such that

$$\varphi(t) = f(t, v(t), \varphi(t)).$$

By using (H4), we get

$$\begin{aligned}|\varphi(t)| &= |f(t, v(t), \varphi(t))| \\ &\leq p(t) + q_1(t)|v(t)| + q_2(t)|\varphi(t)| \\ &\leq p(t) + q_1(t)\varrho + q_2(t)|\varphi(t)| \\ &\leq p^* + q_1^*\varrho + q_2^*|\varphi(t)|.\end{aligned}$$

Then

$$|\varphi(t)| \leq \frac{p^* + q_1^*\varrho}{1 - q_2^*} := \mathfrak{M}.$$

Thus,

$$\|\Xi(v)\|_{PC} \leq |\delta| + \frac{\mathfrak{M}\left(\mathfrak{e}^{\mathfrak{b}} - \mathfrak{e}^{\mathfrak{a}}\right)^{\alpha}}{\Gamma(\alpha+1)} := \ell_1.$$

Case 2. For each $t \in J_k'$, we have

$$|\Xi(v)(t)| \leq |g_k(t, y(t))|.$$

By (H5), we know that

$$|g_k(t, y(t))| \leq p_{g_k}(t)(|y(t)| + 1).$$

Thus,

$$\|\Xi(v)(t)\|_{PC} \leq p_g^*(\varrho + 1) := \ell_2.$$

Case 3. Similarly to the recent calculus for each $t \in J_k$, we get

$$\|\Xi(v)(t)\|_{PC} \leq p_g^*(\varrho + 1) + \frac{\mathfrak{M}\left(\mathfrak{e}^{\mathfrak{b}} - \mathfrak{e}^{\mathfrak{a}}\right)^{\alpha}}{\Gamma(\alpha+1)} := \ell_3.$$

Now, if $\ell = \max\{\ell_1, \ell_2, \ell_3\}$, then we have

$$\|\Xi v\|_{PC} \leq \ell.$$

This shows that Ξ maps bounded sets into bounded sets in $PC(J, \mathrm{I\!R})$.

Step 3 : Ξ *maps bounded sets into equicontinuous sets of* $PC(J, \mathrm{I\!R})$.
Case 1. For $\tau_1, \tau_2 \in [a, t_1]$, $\tau_1 < \tau_2$ and $v \in B_\varrho$, we have

$$\begin{aligned}|\Xi(v)(\tau_2) - \Xi(v)(\tau_1)| &\leq \frac{1}{\Gamma(\alpha)}\int_a^{\tau_1}\left|\left[(e^{\tau_2} - e^s)^{\alpha-1} - (e^{\tau_1} - e^s)^{\alpha-1}\right]e^s\right||\varphi(s)|\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\left|(e^{\tau_2} - e^s)^{\alpha-1}e^s\right||\varphi(s)|\,ds \\ &\leq \frac{M}{\Gamma(\alpha+1)}\left[(e^{\tau_1} - e^a)^{\alpha} - (e^{\tau_2} - e^a)^{\alpha} + 2(e^{\tau_2} - e^{\tau_1})^{\alpha}\right].\end{aligned}$$

Case 2. For $\tau_1, \tau_2 \in J'_k$, $\tau_1 < \tau_2$ and $v \in B_\varrho$, we obtain

$$|\Xi(v)(\tau_2) - \Xi(v)(\tau_1)| \leq |g_k(\tau_2, y(\tau_2)) - g_k(\tau_1, y(\tau_1))|.$$

Case 3. For $\tau_1, \tau_2 \in J_k$, $\tau_1 < \tau_2$ and $v \in B_\varrho$, we get

$$\begin{aligned}|\Xi(v)(\tau_2) - \Xi(v)(\tau_1)| &\leq \frac{1}{\Gamma(\alpha)}\int_{s_k}^{\tau_1} \left|\left[(e^{\tau_2} - e^s)^{\alpha-1} - (e^{\tau_1} - e^s)^{\alpha-1}\right] e^s\right| |\varphi(s)|\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2} \left|(e^{\tau_2} - e^s)^{\alpha-1} e^s\right| |\varphi(s)|\, ds \\ &\leq \frac{M}{\Gamma(\alpha+1)}\left[(e^{\tau_1} - e^{s_k})^\alpha - (e^{\tau_2} - e^{s_k})^\alpha + 2(e^{\tau_2} - e^{\tau_1})^\alpha\right].\end{aligned}$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero. As a consequence of the steps 1 to 3 together with the Ascoli-Arzelà theorem, we deduce that Ξ is completely continuous.

Step 4: *A priori bounds.* We will show that the set

$$\varepsilon = \{u \in PC(J, \mathbb{R}) : u = \lambda\Xi(v), \text{ for some } \lambda \in (0,1)\}$$

is bounded. Let $u \in \varepsilon$, then $u = \lambda\Xi(u)$ for some $0 < \lambda < 1$.
Case 1. For each $t \in [a, t_1]$, we have

$$u(t) = \delta\lambda + \frac{\lambda}{\Gamma(\alpha)}\int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi(s) ds.$$

Taking the absolute values of both sides and using $0 < \lambda < 1$, we get

$$|u(t)| \leq |\delta| + \frac{1}{\Gamma(\alpha)}\int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s |\varphi(s)|\, ds.$$

By (H4), we have

$$\begin{aligned}|\varphi(t)| &= |f(t, u(t), \varphi(t))| \\ &\leq p(t) + q_1(t)|u(t)| + q_2(t)|\varphi(t)| \\ &\leq p^* + q_1^*|u(t)| + q_2^*|\varphi(t)| \\ &\leq \frac{1}{1-q_2^*}\left(p^* + q_1^*|u(t)|\right) \\ &\leq \frac{p^* + q_1^*\|u\|_{PC}}{1-q_2^*}.\end{aligned}$$

This implies that

$$|u(t)| \leq |\delta| + \left(\frac{p^* + q_1^*\|u\|_{PC}}{1-q_2^*}\right)\rho.$$

Thus,

$$\left(1-\frac{q_1^*\rho}{1-q_2^*}\right)\|u\|_{PC} \;\leq\; |\delta|+\frac{p^*\rho}{1-q_2^*}.$$

From the condition (5.10), it easy to see that $\dfrac{q_1^*\rho}{1-q_2^*}<1$. So,

$$\|u\|_{PC} \;\leq\; \left[|\delta|+\frac{p^*\rho}{1-q_2^*}\right]\left[1-\frac{q_1^*\rho}{1-q_2^*}\right]^{-1}:=\zeta_1.$$

Case 2. For each $t\in J_k'$, we have

$$|u\,(t)| \;\leq\; |g_k(t,u\,(t))|\,.$$

By (H5), we obtain

$$\begin{aligned}|u\,(t)| &\leq p_{g_k}(t)\big(|u\,(t)|+1\big)\\ &\leq p_g^*\big(\|u\|_{PC}+1\big).\end{aligned}$$

From the condition (5.10), it follows that

$$\|u\|_{PC} \;\leq\; \frac{p_g^*}{1-p_g^*}:=\zeta_2.$$

Case 3. Analogous for each $t\in J_k$, we get

$$|u(t)| \;\leq\; \left[p_g^*+\frac{p^*\rho}{1-q_2^*}\right]\left[1-\left(p_g^*+\frac{q_1^*\rho}{1-q_2^*}\right)\right]^{-1}.$$

Then

$$\|u\|_{PC} \;\leq\; \left[p_g^*+\frac{p^*\rho}{1-q_2^*}\right]\left[1-\left(p_g^*+\frac{q_1^*\rho}{1-q_2^*}\right)\right]^{-1}:=\zeta_3.$$

Hence, $\|u\|_{PC}\leq\zeta$ with $\zeta=\max\{\zeta_1,\zeta_2,\zeta_3\}$. Therefore, the set ε is bounded. Finally, as a consequence of Schaefer's fixed point theorem, we deduce that Ξ has at least one fixed point which is solution of the IVP (5.1)-(5.3) on J. This completes the proof.

5.2.2 Ulam–Hyers–Rassias Stability

In this subsection, we study U-H-R stability of the IVP (5.1)-(5.3). For this, we introduce the concepts of U-H-R stability for the problem (5.1)-(5.3). Let $x\in PC(J,\mathbb{R})$, $\epsilon>0$, $\omega>0$, and $\psi\in C\left(J,\mathbb{R}_+\right)$ be a nondecreasing function on every J_k, $k=0,1,\ldots,m$. We consider the following inequalities:

$$\begin{cases}\left|{}^{e}_{c}D^{\alpha}_{s_k}x\,(t)-f(t,x\,(t)\,,\,{}^{e}_{c}D^{\alpha}_{s_k}x\,(t))\right|\leq\epsilon\psi\,(t)\,, & t\in J_k,\ k=0,1,\ldots,m\\ |x\,(t)-g_k(t,x\,(t))|\leq\epsilon\omega, & t\in J_k',\ k=1,2,\ldots,m.\end{cases}\tag{5.11}$$

Definition 5.5 *The problem (5.1)-(5.3) is U-H-R stable with respect to* (ψ, ω) *if there exists a real number* $\nu_{f,\psi} > 0$ *such that for each* $\epsilon > 0$ *and for each solution* $x \in PC(J, \mathbb{R})$ *of the inequality (5.11) there exists a solution* $y \in PC(J, \mathbb{R})$ *of the problem (5.1)-(5.3) with*

$$\|x(t) - y(t)\|_{PC} \leq \epsilon \nu_{f,\psi} (\psi(t) + \omega) \quad \text{for all } t \in J.$$

Remark 5.6 *A function* $x \in PC(J, \mathbb{R})$ *is a solution of the inequality (5.11) if and only if there is* $\Phi \in \bigcap_{k=0}^{m} C(J_k, \mathbb{R})$ *and a sequence* Φ_k, $k = 1, 2 \ldots, m$ *(which depend on* x*) such that*

(i) $|\Phi(t)| \leq \epsilon \psi(t)$, *for* $t \in J_k$, $k = 0, \ldots, m$ *and* $|\Phi_k| \leq \epsilon \omega$, *for* $k = 1, \ldots, m$;

(ii) ${}^e_c D^\alpha_{s_k} x(t) = f(t, x(t), {}^e_c D^\alpha_{s_k} x(t)) + \Phi(t)$, *for* $t \in J_k$, $k = 0, \ldots, m$;

(iii) $x(t) = g_k(t, x(t)) + \Phi_k$, *for* $t \in J'_k$, $k = 1, \ldots, m$.

Now, we need the following assumption in the sequel:

(H6) There exists $\psi \in C(J, \mathbb{R}_+)$ which is nondecreasing on every J_k, $k = 0, 1, \ldots, m$, and $\beta_\psi > 0$ such that

$${}^e I^\alpha_{s_k} \psi(t) \leq \beta_\psi \psi(t), \quad \text{for each } t \in J_k, \ k = 0, 1, \ldots, m.$$

Theorem 5.7 *Assume that (H1)-(H3), (H6) and (5.8) are satisfied. Then the IVP (5.1)-(5.3) is U-H-R stable with respect to* (ψ, ω).

Proof. Let $x \in PC(J, \mathbb{R})$ be a solution of the inequality (5.11). Denote by y the unique solution of the non-instantaneous impulsive Cauchy problem

$$\begin{cases} {}^e_c D^\alpha_{s_k} y(t) = f(t, y(t), {}^e_c D^\alpha_{s_k} y(t)), & t \in J_k, \ k = 0, 1, \ldots, m \\ y(t) = g_k(t, y(t)), & t \in J'_k, \ k = 1, \ldots, m \\ y(a) = x(a) = \delta. \end{cases}$$

From Lemma 5.2, we know that

$$y(t) = \begin{cases} \delta + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi_y(s) ds, & \text{if } t \in [a, t_1] \\ g_k(t, y(t)), & \text{if } t \in J'_k, \ k = 1, \ldots, m \\ g_k(s_k, y(s_k)) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{s_k}^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi_y(s) ds, & \text{if } t \in J_k, \ k = 1, \ldots, m, \end{cases}$$

where $\varphi_y \in C(\overline{\mathcal{J}}, \mathbb{R})$ such that

$$\varphi_y(t) = f(t, y(t), \varphi_y(t)).$$

Since x is a solution of the inequality (5.11) and from Remark 5.6, we have

$$\begin{cases} {}_c^eD_{s_k}^{\alpha} x(t) = f(t, x(t), {}_c^eD_{s_k}^{\alpha} x(t)) + \Phi(t), & t \in J_k, \ k = 0, 1, \ldots, m \\ x(t) = g_k(t, x(t)) + \Phi_k, & t \in J_k', \ k = 1, \ldots, m. \end{cases} \tag{5.12}$$

Clearly, the solution of (5.12) is given by

$$x(t) = \begin{cases} \delta + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi_x(s) ds \\ + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \Phi(s) ds, \quad \text{if } t \in [a, t_1] \\ g_k(t, x(t)) + \Phi_k, \quad \text{if } t \in J_k', \ k = 1, \ldots, m \\ g_k(s_k, x(s_k)) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{s_k}^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi_x(s) ds \\ + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{s_k}^t \left(e^t - e^s\right)^{\alpha-1} e^s \Phi(s) ds, \quad \text{if } t \in J_k, \ k = 1, \ldots, m, \end{cases}$$

where $\varphi_x \in C(\overline{\mathcal{J}}, \mathbb{R})$ such that

$$\varphi_x(t) = f(t, x(t), \varphi_x(t)).$$

Now, by using (H2), (H3), (H6) and the previous computations, we shall find $\nu_{f,\psi}$.
Case 1. For each $t \in [a, t_1]$, we have

$$\begin{aligned} |x(t) - y(t)| &= \left| \frac{1}{\Gamma(\alpha)} \int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi_x(s) ds + \frac{1}{\Gamma(\alpha)} \int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \Phi(s) ds \right. \\ &\left. - \frac{1}{\Gamma(\alpha)} \int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi_y(s) ds \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \left|\varphi_x(s) - \varphi_y(s)\right| ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \left|\Phi(s)\right| ds \\ &\leq \frac{\gamma}{\Gamma(\alpha)} \int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \left|x(s) - y(s)\right| ds + \frac{\epsilon}{\Gamma(\alpha)} \int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \psi(t) ds \\ &\leq \frac{\gamma \left(e^b - e^a\right)^{\alpha}}{\Gamma(\alpha+1)} \|x - y\|_{PC} + \epsilon \beta_\psi \psi(t). \end{aligned}$$

Then,

$$\|x-y\|_{PC} \leq \gamma\rho\|x-y\|_{PC}+\epsilon\beta_{\psi}\psi(t).$$

From (5.8), it clear that $\gamma\rho<1$. So,

$$\|x-y\|_{PC} \leq \frac{\epsilon\beta_{\psi}\psi(t)}{1-\gamma\rho}. \tag{5.13}$$

Case 2. For each $t\in(t_k,s_k]$, we have

$$\begin{aligned}|x(t)-y(t)| &= |g_k(t,x(t))+\Phi_k-g_k(t,y(t))|\\ &\leq |g_k(t,x(t))-g_k(t,y(t))|+|\Phi_k|\\ &\leq \xi_k|x(t)-y(t)|+\epsilon\omega.\end{aligned}$$

Then,

$$\|x-y\|_{PC} \leq \xi\|x-y\|_{PC}+\epsilon\omega.$$

From (5.8), it clear that $\xi<1$. So,

$$\|x(t)-y(t)\|_{PC} \leq \frac{\epsilon\omega}{1-\xi}. \tag{5.14}$$

Case 3. For each $t\in(s_k,t_{k+1}]$, we have

$$\begin{aligned}|x(t)-y(t)| &= \Bigg|g_k(s_k,x(s_k))+\frac{1}{\Gamma(\alpha)}\int_{s_k}^{t}\left(e^t-e^s\right)^{\alpha-1}e^s\varphi_x(s)ds\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{s_k}^{t}\left(e^t-e^s\right)^{\alpha-1}e^s\Phi(s)ds-g_k(s_k,y(s_k))\\ &\quad-\frac{1}{\Gamma(\alpha)}\int_{s_k}^{t}\left(e^t-e^s\right)^{\alpha-1}e^s\varphi_y(s)ds\Bigg|\\ &\leq |g_k(s_k,x(s_k))-g_k(s_k,y(s_k))|+\frac{1}{\Gamma(\alpha)}\int_{s_k}^{t}\left(e^t-e^s\right)^{\alpha-1}e^s|\Phi(s)|ds\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{s_k}^{t}\left(e^t-e^s\right)^{\alpha-1}e^s|\varphi_x(s)-\varphi_y(s)|ds\\ &\leq \xi\|x-y\|_{PC}+\epsilon\beta_{\psi}\psi(t)+\frac{\gamma\left(e^b-e^a\right)^{\alpha}}{\Gamma(\alpha+1)}\|x-y\|_{PC}.\end{aligned}$$

Then,

$$\|x-y\|_{PC} \leq (\xi+\gamma\rho)\|x-y\|_{PC}+\epsilon\beta_{\psi}\psi(t).$$

From (5.8), we get

$$\|x-y\|_{PC} \leq \frac{\epsilon\beta_{\psi}\psi(t)}{1-(\xi+\gamma\rho)}. \tag{5.15}$$

Finally, From (5.13), (5.14) and (5.15), we have

$$\|x-y\|_{PC} \leq \frac{\epsilon\omega}{1-\xi}+\frac{\epsilon\beta_\psi\psi(t)}{1-(\xi+\gamma\rho)}, \quad \text{for } t\in J.$$

Consequently,

$$\|x-y\|_{PC} \leq \epsilon\nu_{f,\psi}\left[\psi(t)+\omega\right], \quad \text{for } t\in J,$$

with

$$\nu_{f,\psi} := \max\left\{\frac{1}{1-\xi}, \frac{\beta_\psi}{1-(\xi+\gamma\rho)}\right\},$$

which implies that the IVP (5.1)-(5.3) is U-H-R stable with respect to (ψ,ω). This completes the proof.

5.2.3 Examples

In this subsection, we will give two examples to illustrate the above results.
Example 1. Consider the following initial value problem :

$$ {}^e_cD^{\frac{1}{2}}_{s_k}y(t) = \frac{e^{-\sqrt{t^2+24}}}{181t^2\left(1+|y(t)|+\left|{}^e_cD^{\frac{1}{2}}_{s_k}y(t)\right|\right)}, \quad \text{for each } t\in J_0\cup J_1, \tag{5.16}$$

$$y(t) = \frac{e^{-t^2}}{13(t^2-1)}\left(\frac{|y(t)|}{|y(t)|+1}\right), \quad \text{for each } t\in J_1', \tag{5.17}$$

$$y(1) = 769, \tag{5.18}$$

where $\alpha=\frac{1}{2}$, $J_0=\left[1,\sqrt{2}\right]$, $J_1'=(\sqrt{2},\sqrt{3}]$, $J_1=(\sqrt{3},2]$, $J=[1,2]$, $\overline{\mathcal{J}}=\left[1,\sqrt{2}\right]\cup\left[\sqrt{3},2\right]$, $\overline{\mathcal{J}'}=\left[\sqrt{2},\sqrt{3}\right]$, $\delta=769$.
Set

$$f(t,y,z) = \frac{e^{-\sqrt{t^2+24}}}{181t^2(1+|y|+|z|)}, \quad \text{for } (t,y,z)\in\overline{\mathcal{J}}\times\mathbb{R}\times\mathbb{R},$$

and

$$g_1(t,y) = \frac{e^{-t^2}}{13(t^2-1)}\left(\frac{|y|}{|y|+1}\right), \quad \text{for } (t,y)\in\overline{\mathcal{J}'}\times\mathbb{R}.$$

Now, for $t\in\bigcup\limits_{k=0}^{1}J_k$ and $y_1,y_2,z_1,z_2\in\mathbb{R}$, we can show that

$$|f(t,y_1,z_1)-f(t,y_2,z_2)| \leq \frac{1}{181e^5}\left(|y_1-y_2|+|z_1-z_2|\right).$$

Also,

$$|g_1(t,y_1)-g_1(t,y_2)| \leq \frac{1}{13e^2}|y_1-y_2|, \quad \text{for } t\in J_1' \text{ and } y_1,y_2\in\mathbb{R}.$$

Thus, for $k_1 = k_2 = \dfrac{1}{181e^5}$ and $\xi = \dfrac{1}{13e^2}$, we have that

$$\begin{aligned}
\xi + \gamma\rho &= \xi + \frac{k_1 \left(e^b - e^a\right)^{\alpha}}{(1-k_2)\,\Gamma(\alpha+1)} \\
&= \frac{1}{13e^2} + \frac{\sqrt{e^2-e}}{181e^5}\left[\left(1-\frac{1}{181e^5}\right)\Gamma\left(\frac{3}{2}\right)\right]^{-1} \\
&= \frac{1}{13e^2} + \frac{2\sqrt{e^2-e}}{(181e^5-1)\sqrt{\pi}} \\
&\approx 0.01050119152 \\
&< 1.
\end{aligned}$$

Hence, all of assumptions (H1)-(H3) and the condition (5.8) are satisfied. As a consequence of Theorem 5.3, the initial value problem (5.16)-(5.18) has a unique solution on $[1,2]$. On the other hand, with the choice of $w = 1$ and

$$\psi(t) = \begin{cases} e^t - e, & \text{if } t \in \left[1, \sqrt{2}\right] \\ 0, & \text{if } t \in \left(\sqrt{2}, \sqrt{3}\right] \\ e^t - e^{\sqrt{3}}, & \text{if } t \in \left(\sqrt{3}, 2\right]. \end{cases}$$

We find that

$$\begin{aligned}
{}^{e}I_{1}^{\frac{1}{2}}\psi(t) &= \frac{4}{3\sqrt{\pi}}\sqrt{e^t-e}\left(e^t-e\right) \\
&\leq \frac{4\sqrt{e^{\sqrt{2}}}}{3\sqrt{\pi}}\psi(t)
\end{aligned}$$

and

$$\begin{aligned}
{}^{e}I_{\sqrt{3}}^{\frac{1}{2}}\psi(t) &= \frac{4}{3\sqrt{\pi}}\sqrt{e^t-e^{\sqrt{3}}}\left(e^t-e^{\sqrt{3}}\right) \\
&\leq \frac{4e}{3\sqrt{\pi}}\psi(t).
\end{aligned}$$

Thus, (H6) is satisfied with $\beta_{\psi} = \dfrac{4e}{3\sqrt{\pi}}$. Therefore, from Theorem 5.7, the IVP (5.16)-(5.18) is U-H-R stable with respect to (ψ, ω).

Example 2. Consider the following initial value problem :

$$ {}_{c}^{e}D_{s_k}^{\frac{1}{2}}y(t) = \frac{e^{-7t}}{47\left(t^2+1\right)}\left[2\cos(t) + \frac{3\,|y(t)|}{1+|y(t)|} + \frac{5\left|{}_{c}^{e}D_{s_k}^{\frac{1}{2}}y(t)\right|}{1+\left|{}_{c}^{e}D_{s_k}^{\frac{1}{2}}y(t)\right|}\right], \text{ for } t \in J_k,\ k = 0,\ldots,9, \tag{5.19}$$

$$y(t)=\frac{|y(t)|+1}{43+e^{\frac{10t}{\pi}}},\quad \text{for } t\in J'_k,\ k=1,\dots,9, \tag{5.20}$$

$$y(0)=563, \tag{5.21}$$

where $\alpha=\frac{1}{2}$, $t_k=\frac{k\pi}{10}$, $s_k=\frac{k\pi}{10}+\frac{\pi}{17}$, $J'_k=(t_k,s_k]$, $J_k=(s_k,t_{k+1}]$, $k=1,..,9$, $J_0=[0,\frac{\pi}{10}]$, $J=[0,\pi]$, $\overline{\mathcal{J}}=\bigcup_{k=0}^{9}[s_k,t_{k+1}]$, $\overline{\mathcal{J}'}=\bigcup_{k=1}^{9}[t_k,s_k]$, $\delta=563$.
Set

$$f(t,y,z)=\frac{e^{-7t}}{47(t^2+1)}\left[2\cos(t)+\frac{3|y|}{1+|y|}+\frac{5|z|}{1+|z|}\right],\quad \text{for } (t,y,z)\in\overline{\mathcal{J}}\times\mathbb{R}\times\mathbb{R}$$

and

$$g_k(t,y)=\frac{|y|+1}{43+e^{\frac{10t}{\pi}}},\quad \text{for } (t,y)\in\overline{\mathcal{J}'}\times\mathbb{R}.$$

Choosing

$$p(t)=\frac{2e^{-7t}\cos(t)}{47(t^2+1)},\qquad q_1(t)=\frac{3e^{-7t}}{47(t^2+1)},$$

$$q_2(t)=\frac{5e^{-7t}}{47(t^2+1)}\quad\text{and}\quad p_{g_k}(t)=\frac{1}{43+e^{\frac{10t}{\pi}}}.$$

Hence (H4) and (H5) are satisfied with $p^*=\frac{2}{47}$, $q_1^*=\frac{3}{47}$, $q_2^*=\frac{5}{47}$ and $p_g^*=\frac{1}{43+e}$. Also, we can show that

$$\begin{aligned}
p_g^*+\frac{q_1^*\rho}{1-q_2^*} &= p_g^*+\frac{q_1^*\left(e^b-e^a\right)^\alpha}{(1-q_2^*)\,\Gamma(\alpha+1)}\\
&= \frac{1}{43+e}+\frac{3\sqrt{e-1}}{47}\left[\left(1-\frac{5}{47}\right)\Gamma\left(\frac{3}{2}\right)\right]^{-1}\\
&= \frac{1}{43+e}+\frac{6\sqrt{e-1}}{42\sqrt{\pi}}\\
&\approx 0.1275242364\\
&< 1.
\end{aligned}$$

Hence, all of assumptions (H1), (H4), (H5) and the condition (5.10) are satisfied. As a consequence of Theorem 5.4, the IVP (5.19)-(5.21) has at least one solution on $[0,\pi]$.

5.3 Existence and Stability Results for the Nonlocal IVP

In this section, we present some results of existence, uniqueness of solutions and U-H-R stability for a class of nonlocal IVP :

$$ {}_{c}^{e}D_{s_k}^{\alpha} y\left(t\right) = f(t, x\left(t\right), {}_{c}^{e}D_{s_k}^{\alpha} y\left(t\right)), \quad \text{for each } t \in J_k \subseteq J,\ k = 0, 1, \ldots, m, \tag{5.22} $$

$$ y(t) = g_k(t, y(t)), \quad \text{for each } t \in J_k' \subset J,\ k = 1, 2, \ldots, m, \tag{5.23} $$

$$ y(a) + \widetilde{h}\left(y\right) = \delta, \tag{5.24} $$

where $\alpha,\ {}_{c}^{e}D_{a+}^{\alpha},\ f,\ g_k,\ \delta,\ J,\ J_0,\ J_k,\ J_k',\ k = 1, \ldots, m$ are defined as in (5.1)-(5.3) and $\widetilde{h} : PC\left(J, \mathbb{R}\right) \to \mathbb{R}$ is a continuous functions. In [42], Byszewski *et al.* gives a theorem about the existence and uniqueness of a solution of a nonlocal Cauchy problem for an ordinary differential equation. The nonlocal condition (5.24) can be more useful than the standard initial condition (5.3) to describe some motion of physical phenomena with better effect.

5.3.1 Existence of Solutions

In this subsection, let us start by defining what we mean by a solution of the problem (5.22)-(5.24).

Definition 5.8 *A function* $y \in PC\left(J, \mathbb{R}\right) \cap \left(\bigcup_{k=0}^{m} AC_e\left(J_k, \mathbb{R}\right) \right)$ *is said a solution of (5.22)-(5.24) if* y *satisfies the condition (5.24), the equations (5.22) and (5.23).*

Lemma 5.9 *The nonlocal IVP (5.22)-(5.24) is equivalent to the following integral equation*

$$ y(t) = \begin{cases} \delta - \widetilde{h}\left(y\right) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi(s) ds, & \text{if } t \in [a, t_1] \\ g_k(t, y(t)), & \text{if } t \in J_k',\ k = 1, \ldots, m \\ g_k(s_k, y(s_k)) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{s_k}^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi(s) ds, & \text{if } t \in J_k,\ k = 1, \ldots, m, \end{cases} $$

where $\varphi \in C(\overline{\mathcal{J}}, \mathbb{R})$ *such that*

$$ \varphi(t) = f(t, y(t), \varphi(t)). $$

Proof. The proof is taken just with repeating the same process applied in the proof of Lemma 5.2. We state the following assumption:

(H7) There exists a constant $\xi_{\widetilde{h}} > 0$ such that

$$\left|\widetilde{h}\left(y_1\right)-\widetilde{h}\left(y_2\right)\right| \leq \xi_{\widetilde{h}}\|y_1-y_2\|, \quad \text{for any } y_1, y_2 \in PC\left(J, \mathbb{R}\right).$$

Theorem 5.10 *Assume that (H1)-(H3) and (H7) hold. If*

$$\xi_{\widetilde{h}} + \xi + \gamma\rho < 1, \tag{5.25}$$

then the nonlocal problem (5.22)-(5.24) has a unique solution on J.

Proof. Transform the problem (5.22)-(5.24) into a fixed point problem, consider the operator $\widetilde{\Xi} : PC\left(J, \mathbb{R}\right) \to PC\left(J, \mathbb{R}\right)$ as

$$\widetilde{\Xi}\left(y\right)\left(t\right) = \begin{cases} \delta - \widetilde{h}\left(y\right) + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi(s) ds, & \text{if } t \in [a, t_1] \\ g_k\left(t, y\left(t\right)\right), & \text{if } t \in J_k', \ k = 1, \ldots, m \\ g_k\left(s_k, y\left(s_k\right)\right) + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{s_k}^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi(s) ds, & \text{if } t \in J_k, \ k = 1, \ldots, m, \end{cases}$$

where $\varphi \in C(\overline{\mathcal{J}}, \mathbb{R})$ such that

$$\varphi(t) = f(t, y(t), \varphi(t)).$$

We can easily show that $\widetilde{\Xi}$ is a contraction simply by following the computations as it is done in the proof of Theorem 5.3.

5.3.2 Ulam–Hyers–Rassias Stability

In this subsection, we present an result in U-H-R stability for the nonlocal IVP (5.22)-(5.24).

Theorem 5.11 *Assume that (H1)-(H3), (H6)-(H7) and (5.25) are satisfied. Then the nonlocal IVP (5.22)-(5.24) is U-H-R stable with respect to* (ψ, ω).

Proof. To prove that the problem (5.22)-(5.24) is U-H-R stable with respect to (ψ, ω), we follow the computations as it is done in the proof of theorem 5.7.

5.3.3 An Example

Consider the following nonlocal IVP :

$${}_c^eD_{s_k}^{\frac{1}{2}} y(t) = \frac{e^{-t}}{\left(22 + e^t\right)\left(1 + |y(t)| + \left|{}_c^eD_{s_k}^{\frac{1}{2}} y(t)\right|\right)}, \quad \text{for each } t \in J_0 \cup J_1, \tag{5.26}$$

$$y\left(t\right)=\frac{|y(t)|+1}{(13+e^{t})},\quad t\in J_{1}^{\prime}, \tag{5.27}$$

$$y\left(0\right)+\frac{1}{15}\int_{0}^{3}\frac{y(t)}{|y(t)|+1}dt=3, \tag{5.28}$$

where $\alpha=\frac{1}{2}$, $J_0=[0,1]$, $J_1'=(1,2]$, $J_1=(2,3]$, $J=[0,3]$, $\overline{\mathcal{J}}=[0,1]\cup[2,3]$, $\overline{\mathcal{J}'}=[1,2]$, $\delta=3$. Set

$$f\left(t,y,z\right)=\frac{e^{-t}}{\left(22+e^{t}\right)\left(1+|y|+|z|\right)},\quad \text{for } (t,y,z)\in\overline{\mathcal{J}}\times\mathrm{I\!R}\times\mathrm{I\!R},$$

$$g_{1}\left(t,y\right)=\frac{|y|+1}{(13+e^{t})},\quad \text{for } (t,y)\in\overline{\mathcal{J}'}\times\mathrm{I\!R}$$

and

$$\widetilde{h}\left(y\right)=\frac{1}{15}\int_{0}^{3}\frac{y(t)}{|y(t)|+1}dt,\quad \text{for } y\in PC([0,3]\,,\mathrm{I\!R}).$$

Since,

$$\left|f\left(t,y_{1},z_{1}\right)-f\left(t,y_{2},z_{2}\right)\right| \leq \frac{1}{23}\left(|y_{1}-y_{2}|+|z_{1}-z_{2}|\right),\quad \text{for } (t,y,z)\in\overline{\mathcal{J}}\times\mathrm{I\!R}\times\mathrm{I\!R},$$

$$\left|g_{1}\left(t,y_{1}\right)-g_{1}\left(t,y_{2}\right)\right| \leq \frac{1}{13+e}\left|y_{1}-y_{2}\right|,\quad \text{for } (t,y)\in\overline{\mathcal{J}'}\times\mathrm{I\!R}$$

and

$$\left|\widetilde{h}(y_{1})-\widetilde{h}(y_{2})\right| \leq \frac{1}{5}\left|y_{1}-y_{2}\right|,\quad \text{for } y\in PC\left([0,3]\,,\mathrm{I\!R}\right).$$

Then for $k_1=k_2=\dfrac{1}{23}$, $\xi=\dfrac{1}{13+e}$ and $\xi_{\widetilde{h}}=\dfrac{1}{5}$, we have

$$\begin{aligned}
\xi_{\widetilde{h}}+\xi+\gamma\rho &= \xi_{\widetilde{h}}+\xi+\frac{k_{1}\left(e^{b}-e^{a}\right)^{\alpha}}{\left(1-k_{2}\right)\Gamma(\alpha+1)}\\
&= \frac{1}{5}+\frac{1}{13+e}+\frac{\sqrt{e^{3}-1}}{23}\left[\left(1-\frac{1}{23}\right)\Gamma\left(\frac{3}{2}\right)\right]^{-1}\\
&= \frac{1}{5}+\frac{1}{13+e}+\frac{2\sqrt{e^{3}-1}}{22\sqrt{\pi}}\\
&\approx 0.4876906265\\
&< 1.
\end{aligned}$$

Therefore, all of assumptions (H1), (H2), (H3), (H7) and the condition (5.25) are satisfied. By Theorem 5.10, we deduce that the nonlocal IVP (5.26)-(5.28) has a unique solution on $[0,3]$. Also, form Theorem 5.11, it follows that the problem (5.26)-(5.28) is U-H-R stable with respect to (ψ,ω).

Chapter 6

Initial Value Problem for Nonlinear Implicit Caputo's Exponential Type Fractional Differential Equations with Non-Instantaneous Impulses in Banach Spaces

6.1 Introduction

In [38], Benchohra and Slimane studied the existence of solutions for the following fractional initial value problem with non-instantaneous impulses in Banach spaces involving the Caputo fractional derivative

$$^{c}D^{r}y(t) = f(t, y(t)), \quad \text{for a.e. } t \in (s_k, t_{k+1}],\ k = 0, \ldots, m,\ 0 < r \leq 1,$$

$$y(t) = g_k(t, y(t)), \quad t \in (t_k, s_k],\ k = 1, \ldots, m,$$

$$y(0) = y_0,$$

where $J = [0, T]$ $0 = s_0 < t_1 < s_1 < \ldots < t_m < s_m < t_{m+1} = T$, $^{c}D^{r}$ is the Caputo fractional derivative , $(E, \|\cdot\|)$ is a real Banach space, $y_0 \in E$, $f : J \times E \to E$ and $g_k : (t_k, s_k] \times E \times E$, $k = 1, ..., m$ are given functions.

This chapter is devoted to the results obtained by Malti *et al.* [78], at the beginning, we establish the existence of solutions and Ulam-Hyers-Rassias (**U-H-R**) stability for a class of initial value problem (**IVP**) for nonlinear implicit Caputo's exponential type fractional differential equations with non-instantaneous impulses in abstract space. At the end, we give some generalization of our results to the nonlocal cases.

6.2 Existence and Stability Results for the IVP in Banach Space

In this section, we study the existence of solutions and Ulam-Hyers-Rassias stability for an initial value problem of nonlinear implicit Caputo's exponential type fractional differential equations with non-instantaneous impulses given by

$$ {}_c^eD_{s_k}^{\alpha} y(t) = f(t, y(t), \ {}_c^eD_{s_k}^{\alpha} y(t)), \quad \text{for each } t \in J_k \subseteq J, \ k = 0, 1, \ldots, m, \tag{6.1}$$

$$ y(t) = g_k(t, y(t)), \quad \text{for each } t \in J_k' \subset J, \ k = 1, 2, \ldots, m, \tag{6.2}$$

$$ y(a) = \delta, \tag{6.3}$$

where $J = [a, b]$, $a = t_0 = s_0 < t_1 \leq s_1 < \ldots < t_m \leq s_m < t_{m+1} = b$, ${}_c^eD_{a^+}^{\alpha}$ denote the Caputo's exponential type fractional derivatives of order $\alpha \in (0, 1]$, $(E, \|\cdot\|)$ is a real Banach space, $\delta \in E$, $J_k' := (t_k, s_k]$, $J_k := (s_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, $J_0 := [a, t_1]$, $f : \overline{\mathcal{J}} \times E \times E \to E$ and $g_k : \overline{\mathcal{J}'} \times E \to E$ are a given functions such that $\overline{\mathcal{J}} = \overset{m}{\underset{k=0}{\cup}} [s_k, t_{k+1}]$, and $\overline{\mathcal{J}'} = \overset{m}{\underset{k=1}{\cup}} [t_k, s_k]$. Here, the impulses start abruptly at the points t_k, $k = 1, \ldots, m$ and their action continues on the intervals J_k', $k = 1, \ldots, m$. The function y takes an impulses at t_k, $k = 1, \ldots, m$ and it follows different rules in two consecutive intervals J_k and J_k'. At the point s_k, $k = 1, \ldots, m$, the function y is continuous. The intervals J_k', $k = 1, \ldots, m$ are called intervals of non-instantaneous impulses for the problem (6.1)-(6.3) and the functions $g_k(t, y(t))$, $k = 1, \ldots, m$ are called non-instantaneous impulsive functions for the problem (6.1)-(6.3). Also, the problem (6.1)-(6.3) can be reduced to an impulsive fractional differential equations when $t_k = s_k, k = 1, \ldots, m$.

6.2.1 Existence of Solutions

The following notations which are useful on this chapter. Consider the following set of functions

$$ PC(J, E) = \Big\{ y : J \to E, \ y \in C(J_0, E) \text{ and } y \in C(J_k' \cup J_k, E), \ k = 1, 2, \ldots, m $$
$$ \text{and there exist } y(t_k^-), y(t_k^+) \text{ for every } k = 1, \ldots, m \text{ with } y(t_k^-) = y(t_k) \Big\}. $$

This set is the Banach space equipped with the norm

$$ \|y\|_{PC} = \max \left\{ \sup_{t \in \overset{m}{\underset{k=1}{\cup}} J_k'} \|y(t)\|, \ \sup_{t \in \overset{m}{\underset{k=0}{\cup}} J_k} \|y(t)\| \right\}. $$

Lemma 6.1 *([56]) If $V \subset PC(J,E)$ is a bounded and equicontinuous set, then*

(i) the function $t \to \mu(V(t))$ is continuous on J, and

$$\mu_{PC}(V) = \max\left\{ \sup_{t\in\bigcup_{k=1}^{m} J'_k} \mu(V(t)),\ \sup_{t\in\bigcup_{k=0}^{m} J_k} \mu(V(t)) \right\}.$$

(ii) $\mu\left\{ \int_a^b y(s)ds : y \in V \right\} \le \int_a^b \mu(V(s))ds$,

where μ is the Kuratowski measure of noncompactness and

$$V(s) = \{y(s) : y \in V\},\ s \in J.$$

Now, we define what we mean by a solution of the IVP (6.1)-(6.3).

Definition 6.2 *A function $y \in PC(J,E) \cap \left(\bigcup_{k=0}^{m} AC_e(J_k,E) \right)$ is said a solution of (6.1)-(6.3) if y satisfies the condition $y(a)=\delta$, the equations ${}_c^eD_{s_k}^{\alpha} y(t) = f(t, y(t),\ {}_c^eD_{s_k}^{\alpha} y(t))$ on J_k, $k = 0,1,\dots,m$ and $y(t) = g_k(t,y(t))$ on J'_k, $k=1,2,\dots,m$.*

To prove the existence of solutions of the IVP (6.1)-(6.3), we need the following auxiliary lemma.

Lemma 6.3 *Let $\alpha \in (0,1]$ and let $\varphi : \overline{\mathcal{J}} \to E$ be continuous. A function y is a solution of the fractional integral equation*

$$y(t) = \begin{cases} \delta + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi(s)ds, & \text{if } t \in (a,t_1] \\ g_k(t,y(t)), & \text{if } t \in J'_k = (t_k, s_k] \\ g_k(s_k, y(s_k)) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{s_k}^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi(s)ds, & \text{if } t \in J_k = (s_k, t_{k+1}]. \end{cases} \tag{6.4}$$

if and only if y is a solution of the fractional IVP with non-instantaneous impulses

$${}_c^eD_{s_k}^{\alpha} y(t) = \varphi(t), \quad \text{for each } t \in J_k \subseteq J,\ k = 0,1,\dots,m, \tag{6.5}$$

$$y(t) = g_k(t,y(t)), \quad \text{for each } t \in J'_k \subset J,\ k=1,2,\dots,m, \tag{6.6}$$

$$y(a) = \delta. \tag{6.7}$$

Proof. Assume that y satisfies (6.5)-(6.7). If $t \in [a,t_1]$, then

$${}_c^eD_a^{\alpha} y(t) = \varphi(t).$$

By Lemma 1.10, we get

$$y(t) = \eta_0 + {}^{e}I_a^{\alpha}\varphi(t).$$

Since $y\,(a) = \delta$, then $\eta_0 = \delta$ and

$$y\,(t) = \delta + \frac{1}{\Gamma(\alpha)} \int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi(s) ds.$$

If $t \in J_1' := (t_1, s_1]$, we have

$$y(t) = g_1(t, y(t)).$$

If $t \in J_1 := (s_1, t_2]$, then by Lemma 1.10 we get

$$\begin{aligned} y(t) &= y\,(s_1) + \frac{1}{\Gamma(\alpha)} \int_{s_1}^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi(s) ds \\ &= g_1(s_1) + \frac{1}{\Gamma(\alpha)} \int_{s_1}^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi(s) ds. \end{aligned}$$

If $t \in J_2' := (t_2, s_2]$, we have

$$y(t) = g_2(t, y(t)).$$

If $t \in J_2 := (s_2, t_3]$, then by Lemma 1.10 we get

$$\begin{aligned} y(t) &= y\,(s_2, y(s_2)) + \frac{1}{\Gamma(\alpha)} \int_{s_2}^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi(s) ds \\ &= g_2(s_2, y(s_2)) + \frac{1}{\Gamma(\alpha)} \int_{s_2}^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi(s) ds. \end{aligned}$$

If $t \in J_k' := (t_k, s_k]$, we have

$$y(t) = g_k(t, y(t)).$$

If $t \in J_k := (s_k, t_{k+1}]$, then by Lemma 1.10 we get

$$\begin{aligned} y(t) &= y\,(s_k, y(s_k)) + \frac{1}{\Gamma(\alpha)} \int_{s_k}^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi(s) ds \\ &= g_k(s_k, y(s_k)) + \frac{1}{\Gamma(\alpha)} \int_{s_k}^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi(s) ds. \end{aligned}$$

By the same ways, for every non-zero integer $k \leq m$, the solution $y(t)$ can be written as

$$y(t) = \begin{cases} g_k(t, y(t)), & \text{if } t \in J_k' = (t_k, s_k] \\ g_k(s_k, y(s_k)) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{s_k}^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi(s) ds, & \text{if } t \in J_k = (s_k, t_{k+1}]. \end{cases}$$

Conversely, assume that y satisfies the equation (6.4). If $t \in [a, t_1]$ then $y(a) = \delta$ and using the fact that ${}_c^e D_a^\alpha$ is the left inverse of ${}^e I_a^\alpha$ we get

$$ {}_c^e D_a^\alpha y(t) = \varphi(t), \quad \text{for each } t \in [a, t_1]. $$

If $t \in (s_k, t_{k+1}]$, where $k = 1, \dots, m$. Then, by using the fact that ${}_c^e D_{s_k}^\alpha C = 0$, where C is a constant and ${}_c^e D_{s_k}^\alpha$ is the left inverse of ${}^e I_{s_k}^\alpha$, we get

$$ {}_c^e D_{s_k}^\alpha y(t) = \varphi(t), \quad \text{for each } t \in (s_k, t_{k+1}]. $$

Obviously, it easy to see that

$$ y(t) = g_k(t, y(t)), \quad \text{for each } t \in (t_k, s_k],\ k = 1, 2, \dots, m. $$

The following hypotheses will be used in the sequel:

(H_{01}) The function $t \to f(t, y, z)$ is measurable on $\overline{\mathcal{J}}$ for each $y, z \in E$, and the functions $y \to f(t, y, z)$ and $z \to f(t, y, z)$ are continuous on E for a.e. $t \in \overline{\mathcal{J}}$.

(H_{02}) There exist constants $k_1 > 0$ and $0 < k_2 < 1$ such that

$$ \|f(t, y_1, z_1) - f(t, y_2, z_2)\| \leq k_1\|y_1 - y_2\| + k_2\|z_1 - z_2\|, \text{ for } t \in \bigcup_{k=0}^{m} J_k,\ y_1, y_2, z_1, z_2 \in E. $$

(H_{03}) The function $g_k : \overline{\mathcal{J}'} \times E \to E$ is continuous and there exist constants $\xi_k > 0$, $k = 1, \dots, m$ such that

$$ \|g_k(t, y_1) - g_k(t, y_2)\| \leq \xi_k\|y_1 - y_2\|, \quad \text{for } t \in J_k',\ k = 1, \dots, m \text{ and } y_1, y_2 \in E. $$

Remark 6.4 *[18] Conditions of (H_{02}) and (H_{03}) are respectively equivalent to the inequalities*

$$ \mu\left(f\left(t, B_1, B_2\right)\right) \leq k_1 \mu\left(B_1\right) + k_2 \mu\left(B_2\right), \quad \textit{for } (t, B_1, B_2) \in \bigcup_{k=0}^{m} J_k \times E \times E $$

and

$$ \mu\left(g_k\left(t, B\right)\right) \leq \xi_k \mu\left(B\right), \quad \textit{for } (t, B) \in J_k' \times E,\ k = 1, \dots, m. $$

Now, we pass to state and proof our first existence result for the problem (6.1)-(6.3) based on concept of measure of noncompactness and Darbo's fixed point theorem.

Set

$$ \gamma := \frac{k_1}{1 - k_2},\ \rho := \frac{\left(e^b - e^a\right)^\alpha}{\Gamma(\alpha + 1)},\ f^* := \sup_{t \in \overline{\mathcal{J}}} \left\|f\left(t, 0, 0\right)\right\|,\ g^* := \max_{k=1,\dots,m} \left\{ \sup_{t \in [t_k, s_k]} \left\|g_k\left(t, 0\right)\right\| \right\}, $$

$\xi := \max\limits_{k=1,\dots,m} \{\xi_k\}$ and $\sigma := \max\limits_{k=1,\dots,m} \left\|g_k\left(s_k, y\left(s_k\right)\right)\right\|$.

Theorem 6.5 *Assume that* $(H_{01})-(H_{03})$ *are satisfied. If*

$$\lambda_1 := \max\{\gamma\rho,\ \xi\} < 1, \tag{6.8}$$

then the IVP (6.1)-(6.3) has a solution defined on J.

Proof. Transform the problem (6.1)-(6.3) into a fixed point problem, consider the operator $\Lambda : PC(J,E) \to PC(J,E)$ defined by

$$\Lambda(y)(t) = \begin{cases} \delta + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi(s)ds, & \text{if } t \in [a, t_1] \\ g_k(t, y(t)), & \text{if } t \in J'_k = (t_k, s_k] \\ g_k(s_k, y(s_k)) + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{s_k}^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi(s)ds, & \text{if } t \in J_k = (s_k, t_{k+1}], \end{cases} \tag{6.9}$$

where $\varphi \in C(\overline{\mathcal{J}}, E)$ such that

$$\varphi(t) = f(t, y(t), \varphi(t)).$$

It clear that, the fact of finding solutions for problem (6.1)-(6.3) is to find the fixed points of the operator equation $\Lambda(y) = y$. Now, we shall use Darbo's fixed point theorem to prove that Λ defined by (6.9) has a fixed point on J. The proof will be given in several steps.

Step 1: Λ *is continuous.*
Let $\{v_n\}$ be a sequence such that $v_n \to v$ in $PC(J,E)$.
Case 1. For each $t \in [a, t_1]$, we obtain

$$\|\Lambda(v_n)(t) - \Lambda(v)(t)\| \leq \frac{1}{\Gamma(\alpha)}\int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \|\varphi_n(s) - \varphi(s)\| ds,$$

where $\varphi_n, \varphi \in C(\overline{\mathcal{J}}, E)$, such that

$$\varphi_n(t) = f(t, v_n(t), \varphi_n(t)) \quad \text{and} \quad \varphi(t) = f(t, v(t), \varphi(t)).$$

Case 2. For each $t \in J'_k$, we have

$$\|\Lambda(v_n)(t) - \Lambda(v)(t)\| \leq \|g_k(t, v_n(t)) - g_k(t, v(t))\|.$$

Case 3. For each $t \in J_k$, we get

$$\begin{aligned}\|\Lambda(v_n)(t) - \Lambda(v)(t)\| \leq\ & \|g_k(s_k, v_n(s_k)) - g_k(s_k, v(s_k))\| \\ & + \frac{1}{\Gamma(\alpha)}\int_{s_k}^t \left(e^t - e^s\right)^{\alpha-1} e^s \|\varphi_n(s) - \varphi(s)\| ds,\end{aligned}$$

where $\varphi_n, \varphi \in C(\overline{\mathcal{J}}, E)$, such that

$$\varphi_n(t) = f(t, v_n(t), \varphi_n(t)) \quad \text{and} \quad \varphi(t) = f(t, v(t), \varphi(t)).$$

Since $v_n \to v$ as $n \to \infty$ and f, g_k are continuous, then by Lebesgue dominated convergence theorem, we have

$$\|\varphi_n(t) - \varphi(t)\| \to 0 \text{ as } n \to \infty \quad \text{and} \quad \|g_k(t, v_n(t)) - g_k(t, v(t))\| \to 0 \text{ as } n \to \infty,$$

which leads to $\|\Lambda(v_n)(t) - \Lambda(v)(t)\|_{PC} \to \infty$ as $n \to \infty$. Therefore Λ is continuous. Before the next step, we consider the ball $B_R = \left\{y \in PC(J, E) : \|y\|_{PC} \le R\right\}$ where

$$\max\left\{\frac{\|\delta\| + \dfrac{f^*\rho}{1-k_2}}{1-\gamma\rho}, \ \frac{g^*}{1-\xi}, \ \frac{\sigma + \dfrac{f^*\rho}{1-k_2}}{1-\gamma\rho}\right\} \le R.$$

Step 2: *Prove that for any $y \in B_R$, F maps B_R into itself.*
Case 1. For each $t \in [a, t_1]$, we obtain

$$\|\Lambda(y)(t)\| \le \|\delta\| + \frac{1}{\Gamma(\alpha)} \int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \|\varphi(s)\| ds.$$

where $\varphi \in C(\overline{\mathcal{J}}, E)$ such that

$$\varphi(s) = f(s, y(s), \varphi(s)).$$

By (H_{02}), we have

$$\begin{aligned}
\|\varphi(t)\| &= \|f(t, y(t), \varphi(t)) - f(t, 0, 0) + f(t, 0, 0)\| \\
&\le \|f(t, y(t), \varphi(t)) - f(t, 0, 0)\| + \|f(t, 0, 0)\| \\
&\le k_1 \|y\| + k_2 \|\varphi(t)\| + f^* \\
&\le k_1 \|y\|_{PC} + k_2 \|\varphi(t)\| + f^* \\
&\le k_1 R + k_2 \|\varphi(t)\| + f^*.
\end{aligned}$$

Then

$$\|\varphi(t)\| \le \gamma R + \frac{f^*}{1-k_2} := M.$$

Next, we have

$$\begin{aligned}
\|\Lambda(y)(t)\| &\le \|\delta\| + \left(\gamma R + \frac{f^*}{1-k_2}\right) \frac{1}{\Gamma(\alpha)} \int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s ds \\
&\le \|\delta\| + \left(\gamma R + \frac{f^*}{1-k_2}\right) \frac{\left(e^b - e^a\right)^{\alpha}}{\Gamma(\alpha+1)} \\
&= \|\delta\| + \left(\gamma R + \frac{f^*}{1-k_2}\right) \rho.
\end{aligned}$$

Thus,

$$\|Fy\|_{PC} \leq R, \quad \text{for each } t \in [a, t_1].$$

Case 2. For each $t \in J_k'$, we have

$$\begin{aligned}
\|\Lambda(y)(t)\| &= \|g_k(t, y(t))\| \\
&= \|g_k(t, y(t)) - g_k(t, 0) + g_k(t, 0)\| \\
&\leq \|g_k(t, y(t)) - g_k(t, 0)\| + \|g_k(t, 0)\| \\
&\leq \xi_k \|y\| + g^* \\
&\leq \xi \|y\|_{PC} + g^* \\
&\leq \xi R + g^*.
\end{aligned}$$

Thus,

$$\|\Lambda y\|_{PC} \leq R, \quad \text{for each } t \in J_k'.$$

Case 3. For each $t \in J_k$, we have

$$\begin{aligned}
\|\Lambda(y)(t)\| &\leq \|g_k(s_k, y(s_k))\| + \frac{1}{\Gamma(\alpha)} \int_{s_k}^{t} \left(e^t - e^s\right)^{\alpha-1} e^s \|\varphi(s)\| \, ds \\
&\leq \sigma + \frac{1}{\Gamma(\alpha)} \int_{s_k}^{t} \left(e^t - e^s\right)^{\alpha-1} e^s \|\varphi(s)\| \, ds.
\end{aligned}$$

Similarly to the case 1, we get

$$\|\Lambda(y)(t)\| \leq \sigma + \left(\gamma R + \frac{f^*}{1-k_2}\right)\rho.$$

Thus,

$$\|\Lambda y\|_{PC} \leq R, \quad \text{for each } t \in J_k.$$

Hence $\|\Lambda y\|_{PC} \leq R$, for each $t \in J$. This implies that Λ transforms the ball B_R into itself.

Step 3:$\Lambda(B_R)$ *is bounded.*
Since $\Lambda(B_R) \subseteq B_R$ and B_R is bounded, then $\Lambda(B_R)$ is bounded.

Step 4: $\Lambda(B_R)$ *is equicontinuous.*
Case 1. For $\tau_1, \tau_2 \in [a, t_1]$, $\tau_1 < \tau_2$ and $y \in B_R$, we have

$$\begin{aligned}
\|\Lambda(v)(\tau_2) - F(v)(\tau_1)\| &\leq \frac{1}{\Gamma(\alpha)} \int_{a}^{\tau_1} \left\| \left[(e^{\tau_2} - e^s)^{\alpha-1} - (e^{\tau_1} - e^s)^{\alpha-1}\right] e^s \right\| \|\varphi(s)\| \, ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} \left\| (e^{\tau_2} - e^s)^{\alpha-1} e^s \right\| \|\varphi(s)\| \, ds \\
&\leq \frac{M}{\Gamma(\alpha+1)} \left[(e^{\tau_1} - e^a)^\alpha - (e^{\tau_2} - e^a)^\alpha + 2(e^{\tau_2} - e^{\tau_1})^\alpha\right].
\end{aligned}$$

Case 2. For $\tau_1, \tau_2 \in J'_k,\ \tau_1 < \tau_2$ and $y \in B_R$, we obtain

$$\|\Lambda(v)(\tau_2) - F(v)(\tau_1)\| \le \|g_k(\tau_2, y(\tau_2)) - g_k(\tau_1, y(\tau_1))\|.$$

Case 3. For $\tau_1, \tau_2 \in J_k,\ \tau_1 < \tau_2$ and $y \in B_R$, we get

$$\begin{aligned}
\|\Lambda(v)(\tau_2) - F(v)(\tau_1)\| &\le \frac{1}{\Gamma(\alpha)} \int_{s_k}^{\tau_1} \left\| \left[(e^{\tau_2} - e^s)^{\alpha-1} - (e^{\tau_1} - e^s)^{\alpha-1} \right] e^s \right\| \|\varphi(s)\| \, ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} \left\| (e^{\tau_2} - e^s)^{\alpha-1} e^s \right\| \|\varphi(s)\| \, ds \\
&\le \frac{M}{\Gamma(\alpha+1)} \left[(e^{\tau_1} - e^{s_k})^\alpha - (e^{\tau_2} - e^{s_k})^\alpha + 2 (e^{\tau_2} - e^{\tau_1})^\alpha \right].
\end{aligned}$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero.

Step 5: *The operator $\Lambda : B_R \to B_R$ is a strict set contraction.*
Case 1. For $t \in [a, t_1]$ and $V \subset B_R$, one has

$$\begin{aligned}
\mu(\Lambda(V)(t)) &= \mu(\{(\Lambda y)(t),\ y \in V\}) \\
&\le \mu(\delta) + \frac{1}{\Gamma(\alpha)} \left\{ \int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \mu(\varphi(s)) \, ds, y \in V \right\}.
\end{aligned}$$

By the Remark 6.4, for each $s \in \overset{m}{\underset{k=0}{\cup}} J_k$, we have

$$\begin{aligned}
\mu(\{\varphi(s),\ y \in V\}) &= \mu(\{f(s, y(s), \varphi(s)),\ y \in V\}) \\
&\le k_1 \mu(\{y(s),\ y \in V\}) + k_2(\{\varphi(s),\ y \in V\}).
\end{aligned}$$

Then

$$\mu(\{\varphi(s),\ y \in V\}) \le \gamma\, \mu(\{y(s),\ y \in V\}).$$

It follows that

$$\begin{aligned}
\mu(\Lambda(V)(t)) &\le \frac{\gamma}{\Gamma(\alpha)} \left\{ \int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \mu(y(s)) \, ds, y \in V \right\} \\
&\le \frac{\gamma \left(e^b - e^a\right)^\alpha}{\Gamma(\alpha+1)} \mu_{PC}(V) \\
&\le \gamma\rho\, \mu_{PC}(V).
\end{aligned}$$

Case 2. For $t \in J'_k$, and $V \subset B_R$, we have

$$\begin{aligned}
\mu(\Lambda(V)(t)) &= \mu(\{(\Lambda y)(t),\ y \in V\}) \\
&\le \mu(\{g_k(t, y(t)),\ y \in V\}) \\
&\le \xi_k\, \mu(\{y(t),\ y \in V\}) \\
&\le \xi\, \mu_{PC}(V).
\end{aligned}$$

Case 3. For $t \in J_k$, and $V \subset B_R$, one has

$$\begin{aligned}
\mu\left(\Lambda\left(V\right)\left(t\right)\right) &= \mu\left(\left\{\left(\Lambda y\right)\left(t\right),\ y \in V\right\}\right) \\
&\leq \mu\left(g_k(s_k, y\left(s_k\right))\right) + \frac{1}{\Gamma(\alpha)}\left\{\int_{s_k}^{t}\left(e^t - e^s\right)^{\alpha-1} e^s \mu\left(\varphi(s)\right) ds,\ y \in V\right\} \\
&\leq \frac{\gamma}{\Gamma(\alpha)}\left\{\int_{s_k}^{t}\left(e^t - e^s\right)^{\alpha-1} e^s \mu\left(y(s)\right) ds,\ y \in V\right\} \\
&\leq \frac{\gamma\left(e^b - e^a\right)^{\alpha}}{\Gamma(\alpha+1)}\ \mu_{PC}\left(V\right) \\
&\leq \gamma\rho\ \mu_{PC}\left(V\right).
\end{aligned}$$

Thus, for each $t \in J$, we have

$$\mu_{PC}\left(\Lambda V\right) \ \leq \ \lambda_1\, \mu_{PC}\left(V\right).$$

Hence by (6.8), the operator Λ is a contraction. As a consequence of steps 1 to 5 together with Theorem 1.16, we deduce that the operator Λ has a fixed point which is solution of the IVP (6.1)-(6.3). This completes the proof.

The following hypotheses will be used in the sequel:

(H_{04}) There exists a continuous function $p : \overline{\mathcal{J}'} \to [0, \infty)$ such that

$$\left\|f\left(t, y, z\right)\right\| \leq \frac{p\left(t\right)}{1 + \|y\| + \|z\|}, \quad \text{for a.e. } t \in \bigcup_{k=0}^{m} J_k \text{ and } y, z \in E.$$

(H_{05}) For each bounded set $B \subset E$ and for each $t \in \bigcup_{k=0}^{m} J_k$, we have

$$\mu\Big(f\left(t, B,\ {}^{e}_{c}D^{\alpha}_{s_k} B\right)\Big) \leq p\left(t\right)\mu\left(B\right),$$

where ${}^{e}_{c}D^{\alpha}_{s_k} B = \left\{{}^{e}_{c}D^{\alpha}_{s_k} y : y \in B\right\}.$

(H_{06}) The function $g_k : \overline{\mathcal{J}'} \times E \to E$ is continuous and there exists $q_k \in C\left(\left[t_k, s_k\right], \mathbb{R}_+\right)$, $k = 1, \ldots, m$ such that

$$\left\|g_k\left(t, y\right)\right\| \leq \frac{q_k\left(t\right)}{1 + \|y\|}, \quad \text{for a.e. } t \in J'_k,\ k = 1, \ldots, m \text{ and each } y \in E.$$

(H_{07}) For each bounded set $B \subset E$ and for each $t \in J_k$, $k = 1, \ldots, m$, we have

$$\mu\Big(g_k\left(t, B\right)\Big) \leq q_k\left(t\right)\mu\left(B\right),\ k = 1, \ldots, m.$$

The second existence result for the IVP (6.1)-(6.3) is based on the concept of measure of noncompactness and Mönch's fixed point theorem.
Set

$$p^* := \sup_{t\in\overline{\mathcal{J}}} p\,(t) \quad \text{and} \quad q^* := \max_{k=1,\ldots,m}\left\{\sup_{t\in[t_k,s_k]} q_k\,(t)\right\}.$$

Theorem 6.6 *Assume that* (H_{01}) *and* $(H_{04})-(H_{07})$ *are satisfied. If*

$$\lambda_2 := \max\{q^*,\ p^*\rho\} < 1, \tag{6.10}$$

then the IVP (6.1)-(6.3) has at least one solution defined on J.

Proof. We shall use Mönch's fixed point theorem to prove that Λ defined by (6.9) has at least one fixed point on J. The proof will be given in several steps.

Step 1: Λ *is continuous.*
Let $\{v_n\}$ be a sequence such that $v_n \to v$ in $PC(J,E)$.
Case 1. For each $t\in[a,t_1]$, we obtain

$$\|\Lambda\,(v_n)\,(t)-\Lambda\,(v)\,(t)\| \;\leq\; \frac{1}{\Gamma(\alpha)}\int_a^t \left(e^t-e^s\right)^{\alpha-1} e^s\,\|\varphi_n\,(s)-\varphi\,(s)\|\,ds,$$

where $\varphi_n,\varphi\in C(\overline{\mathcal{J}},E)$, such that

$$\varphi_n\,(t)=f\,(t,v_n\,(t)\,,\varphi_n\,(t)) \quad \text{and} \quad \varphi\,(t)=f\,(t,v\,(t)\,,\varphi\,(t))\,.$$

Case 2. For each $t\in J'_k$, we have

$$\|\Lambda\,(v_n)\,(t)-\Lambda\,(v)\,(t)\| \;\leq\; \|g_k(t,v_n\,(t))-g_k(t,v\,(t))\|\,.$$

Case 3. For each $t\in J_k$, we get

$$\begin{aligned}\|\Lambda\,(v_n)\,(t)-\Lambda\,(v)\,(t)\| \;\leq\;& \|g_k(s_k,v_n\,(s_k))-g_k(s_k,v\,(s_k))\|\\ &+\frac{1}{\Gamma(\alpha)}\int_{s_k}^t \left(e^t-e^s\right)^{\alpha-1} e^s\,\|\varphi_n\,(s)-\varphi\,(s)\|\,ds,\end{aligned}$$

where $\varphi_n,\varphi\in C(\overline{\mathcal{J}},E)$, such that

$$\varphi_n\,(t)=f\,(t,v_n\,(t)\,,\varphi_n\,(t)) \quad \text{and} \quad \varphi\,(t)=f\,(t,v\,(t)\,,\varphi\,(t))\,.$$

Since $v_n \to v$ as $n\to\infty$ and f,g_k are continuous, then by Lebesgue dominated convergence theorem, we have

$$\|\varphi_n\,(t)-\varphi\,(t)\|\to 0 \text{ as } n\to\infty \quad \text{and} \quad \|g_k(t,v_n\,(t))-g_k(t,v\,(t))\|\to 0 \text{ as } n\to\infty,$$

which leads to $\|\Lambda(v_n)(t) - \Lambda(v)(t)\|_{PC} \to \infty$ as $n \to \infty$. Therefore Λ is continuous. Before the next step, we consider the ball $B_{R_2} = \left\{y \in PC(J,E) : \|y\|_{PC} \le R_2\right\}$ where

$$R_2 \ge \max\left\{\|\delta\| + p^*\rho,\ q^*,\ \sigma + p^*\rho\right\}.$$

Step 2: *Prove that for any $y \in B_{R_2}$, Λ maps B_{R_2} into itself.*
Case 1. For each $t \in [a, t_1]$, we obtain

$$\begin{aligned}
\|\Lambda(y)(t)\| &\le \|\delta\| + \frac{1}{\Gamma(\alpha)}\int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \|\varphi(s)\| \, ds \\
&\le \|\delta\| + \frac{1}{\Gamma(\alpha)}\int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s p(s) \, ds \\
&\le \|\delta\| + \frac{p^*}{\Gamma(\alpha)}\int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s ds \\
&\le \|\delta\| + \frac{p^*\left(e^b - e^a\right)^{\alpha}}{\Gamma(\alpha+1)} = \|\delta\| + p^*\rho.
\end{aligned}$$

Case 2. For each $t \in J_k'$, we have

$$\|\Lambda(y)(t)\| = \|g_k(t, y(t))\| \le q_k(t) \le q^*.$$

Case 3. For each $t \in J_k$, we have

$$\begin{aligned}
\|\Lambda(y)(t)\| &\le \|g_k(s_k, y(s_k))\| + \frac{1}{\Gamma(\alpha)}\int_{s_k}^t \left(e^t - e^s\right)^{\alpha-1} e^s \|\varphi(s)\| \, ds \\
&\le \sigma + \frac{1}{\Gamma(\alpha)}\int_{s_k}^t \left(e^t - e^s\right)^{\alpha-1} e^s \|\varphi(s)\| \, ds.
\end{aligned}$$

Similarly to the case 1, we get

$$\|\Lambda(y)(t)\| \le \sigma + p^*\rho.$$

Hence $\|\Lambda y\|_{PC} \le R_2$, for each $t \in J$. This implies that Λ transforms the ball B_{R_2} into itself.

Step 3: $\Lambda(B_{R_2})$ *is bounded.*
Since $\Lambda(B_{R_2}) \subseteq B_{R_2}$ and B_{R_2} is bounded, then $\Lambda(B_{R_2})$ is bounded.

Step 4: $\Lambda(B_{R_2})$ *is equicontinuous.*
Case 1. For $\tau_1, \tau_2 \in [a, t_1]$, $\tau_1 < \tau_2$ and $y \in B_{R_2}$, we have

$$\begin{aligned}
\|\Lambda(v)(\tau_2) - \Lambda(v)(\tau_1)\| &\le \frac{1}{\Gamma(\alpha)}\int_a^{\tau_1} \left\|\left[(e^{\tau_2} - e^s)^{\alpha-1} - (e^{\tau_1} - e^s)^{\alpha-1}\right] e^s\right\| \|\varphi(s)\| \, ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2} \left\|(e^{\tau_2} - e^s)^{\alpha-1} e^s\right\| \|\varphi(s)\| \, ds \\
&\le \frac{p^*}{\Gamma(\alpha+1)}\left[(e^{\tau_1} - e^a)^{\alpha} - (e^{\tau_2} - e^a)^{\alpha} + 2\left(e^{\tau_2} - e^{\tau_1}\right)^{\alpha}\right].
\end{aligned}$$

Case 2. For $\tau_1, \tau_2 \in J'_k,\ \tau_1 < \tau_2$ and $y \in B_{R_2}$, we obtain

$$\|\Lambda(v)(\tau_2) - \Lambda(v)(\tau_1)\| \ \leq \ \|g_k(\tau_2, y(\tau_2)) - g_k(\tau_1, y(\tau_1))\| .$$

Case 3. For $\tau_1, \tau_2 \in J_k,\ \tau_1 < \tau_2$ and $y \in B_{R_2}$, we get

$$\begin{aligned}\|\Lambda(v)(\tau_2) - \Lambda(v)(\tau_1)\| &\leq \frac{1}{\Gamma(\alpha)} \int_{s_k}^{\tau_1} \left\| \left[(e^{\tau_2} - e^s)^{\alpha-1} - (e^{\tau_1} - e^s)^{\alpha-1} \right] e^s \right\| \|\varphi(s)\| \, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} \left\| (e^{\tau_2} - e^s)^{\alpha-1} e^s \right\| \|\varphi(s)\| \, ds \\ &\leq \frac{p^*}{\Gamma(\alpha+1)} \left[(e^{\tau_1} - e^{s_k})^\alpha - (e^{\tau_2} - e^{s_k})^\alpha + 2 (e^{\tau_2} - e^{\tau_1})^\alpha \right].\end{aligned}$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero. Hence, $\Lambda(B_{R_2})$ is equicontinuous.

Step 4: *The implication (1.4) holds.*
Now let V be a subset of B_{R_2} such that $V \subset \overline{conv}(F(V) \cup \{0\})$. V is bounded and equicontinuous and therefore the function $t \to v(t) = \mu(V(t))$ is continuous on J.
By using the Lemma 6.1 and Properties 1.12, for each $t \in J$, we have

$$v(t) = \mu(V(t)) \ \leq \mu\big(\Lambda(V) \cup \{0\}\big) \ \leq \mu\left((\Lambda V)(t)\right).$$

Case 1. If $t \in [a, t_1]$, we obtain

$$\begin{aligned} v(t) &\leq \mu(\delta) + \frac{1}{\Gamma(\alpha)} \int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s p(s)\, \mu(V(s))\, ds \\ &\leq \frac{p^*}{\Gamma(\alpha)} \int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s v(s) ds \\ &\leq p^* \rho \ \|v\|_\infty . \end{aligned}$$

Case 2. If $t \in J'_k$, we have

$$v(t) \ \leq \mu\left(g_k(t, V(t))\right) \ \leq q_k(t)\, \mu(V(t)) \ \leq q^* v(t) \ \leq q^* \|v\|_\infty .$$

Case 3. If $t \in J_k$, one has

$$\begin{aligned} v(t) &\leq \mu\left(g_k(s_k, y(s_k))\right) + \frac{1}{\Gamma(\alpha)} \int_{s_k}^t \left(e^t - e^s\right)^{\alpha-1} e^s p(s)\, \mu(V(s))\, ds \\ &\leq \frac{p^*}{\Gamma(\alpha)} \int_{s_k}^t \left(e^t - e^s\right)^{\alpha-1} e^s v(s) ds \\ &\leq p^* \rho \ \|v\|_{PC} . \end{aligned}$$

Consequently, for each $t \in J$, we have

$$\|v\|_{PC} \ \leq \ \lambda_2 \|v\|_{PC} .$$

From (6.10), we get $\|v\|_\infty = 0$, that is $v(t) = \mu(V(t)) = 0$, for each $t \in J$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzelà theorem, V is relatively compact in B_{R_2}. Applying now Theorem 1.18, we deduce that the operator Λ has at least one fixed point in B_{R_2}. This shows that the IVP (6.1)-(6.3) has at least one solution on J. This completes the proof.

6.2.2 Ulam-Hyers-Rassias Stability

In this subsection, we study U-H-R stability of the IVP (6.1)-(6.3). For this, we start by introducing the concepts of U-H-R stability for the problem (6.1)-(6.3). Let $x \in PC(J,E)$, $\epsilon > 0$, $\omega > 0$, and $\psi \in C(J, \mathbb{R}_+)$ be a nondecreasing function on every J_k, $k = 0, 1, \ldots, m$. We consider the following inequalities:

$$\begin{cases} \left\| {}^e_cD^\alpha_{s_k} x(t) - f(t, x(t), {}^e_cD^\alpha_{s_k} x(t)) \right\| \leq \epsilon\psi(t), & t \in J_k, \ k = 0, 1, \ldots, m \\ \|x(t) - g_k(t, x(t))\| \leq \epsilon\omega, & t \in J'_k, \ k = 1, 2, \ldots, m. \end{cases} \tag{6.11}$$

Definition 6.7 *The problem (6.1)-(6.3) is U-H-R stable with respect to (ψ, ω) if there exists a real number $\nu_{f,\psi} > 0$ such that for each $\epsilon > 0$ and for each solution $x \in PC(J,E)$ of the inequality (6.11) there exists a solution $y \in PC(J,E)$ of the problem (6.1)-(6.3) with*

$$\|x(t) - y(t)\|_{PC} \leq \epsilon\nu_{f,\psi}(\psi(t) + \omega) \quad \text{for all } t \in J.$$

Remark 6.8 *A function $x \in PC(J,E)$ is a solution of the inequality (6.11) if and only if there is $\Phi \in \bigcap_{k=0}^{m} C(J_k, E)$ and a sequence Φ_k, $k = 1, 2 \ldots, m$ (which depend on x) such that*

(i) $\|\Phi(t)\| \leq \epsilon\psi(t)$, *for* $t \in J_k$, $k = 0, \ldots, m$ *and* $\|\Phi_k\| \leq \epsilon\omega$, *for* $k = 1, \ldots, m$;

(ii) ${}^e_cD^\alpha_{s_k} x(t) = f(t, x(t), {}^e_cD^\alpha_{s_k} x(t)) + \Phi(t)$, *for* $t \in J_k$, $k = 0, \ldots, m$;

(iii) $x(t) = g_k(t, x(t)) + \Phi_k$, *for* $t \in J'_k$, $k = 1, \ldots, m$.

To discuss stability, we need the following additional assumption:

(H_{08}) Let $\psi \in C(J, \mathbb{R}_+)$ be a nondecreasing function on every J_k, $k = 0, 1, \ldots, m$. There exists $\beta_\psi > 0$ such that

$${}^eI^\alpha_{s_k}\psi(t) \leq \beta_\psi\psi(t), \quad \text{for each } t \in J_k, \ k = 0, \ldots, m.$$

Theorem 6.9 *Assume that $(H_{01}) - (H_{03})$ and (H_{08}) hold. If*

$$\xi + \gamma\rho < 1, \tag{6.12}$$

then IVP (6.1)-(6.3) is U-H-R stable with respect to (ψ, ω).

Proof. Let $x \in PC(J, E)$ be a solution of the inequality (6.11). Denote by y the unique solution of the non-instantaneous impulsive Cauchy problem

$$\begin{cases} {}_c^eD_{s_k}^{\alpha} y(t) = f(t, y(t), {}_c^eD_{s_k}^{\alpha} y(t)), \quad t \in J_k, \; k = 0, 1, \ldots, m \\ y(t) = g_k(t, y(t)), \quad t \in J_k', \; k = 1, \ldots, m \\ y(a) = x(a) = \delta. \end{cases}$$

From Lemma 6.3, we know that

$$y(t) = \begin{cases} \delta + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi_y(s) ds, & \text{if } t \in [a, t_1] \\ g_k(t, y(t)), & \text{if } t \in J_k', \; k = 1, \ldots, m \\ g_k(s_k, y(s_k)) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{s_k}^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi_y(s) ds, & \text{if } t \in J_k, \; k = 1, \ldots, m, \end{cases}$$

where $\varphi_y \in C(\overline{\mathcal{J}}, E)$ such that

$$\varphi_y(t) = f(t, y(t), \varphi_y(t)).$$

Since x is a solution of the inequality (6.11) and from Remark 6.8, we have

$$\begin{cases} {}_c^eD_{s_k}^{\alpha} x(t) = f(t, x(t), {}_c^eD_{s_k}^{\alpha} x(t)) + \Phi(t), \quad t \in J_k, \; k = 0, 1, \ldots, m \\ x(t) = g_k(t, x(t)) + \Phi_k, \quad t \in J_k', \; k = 1, \ldots, m. \end{cases} \tag{6.13}$$

Clearly, the solution of (6.13) is given by

$$x(t) = \begin{cases} \delta + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi_x(s) ds \\ + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \Phi(s) ds, \quad \text{if } t \in [a, t_1] \\ g_k(t, x(t)) + \Phi_k, \quad \text{if } t \in J_k', \; k = 1, \ldots, m \\ g_k(s_k, x(s_k)) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{s_k}^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi_x(s) ds \\ + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{s_k}^t \left(e^t - e^s\right)^{\alpha-1} e^s \Phi(s) ds, \quad \text{if } t \in J_k, \; k = 1, \ldots, m, \end{cases}$$

where $\varphi_x \in C(\overline{\mathcal{J}}, E)$ such that

$$\varphi_x(t) = f(t, x(t), \varphi_x(t)).$$

Now, by using the hypothesis (H_{02}), (H_{03}), (H_{08}) and the previous computations, we shall find $\nu_{f,\psi}$.
Case 1. For each $t \in [a, t_1]$, we have

$$\begin{aligned}
&\|x(t) - y(t)\| \\
&= \left\| \frac{1}{\Gamma(\alpha)} \int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi_x(s) ds + \frac{1}{\Gamma(\alpha)} \int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \Phi(s) ds \right. \\
&\left. - \frac{1}{\Gamma(\alpha)} \int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi_y(s) ds \right\| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \|\varphi_x(s) - \varphi_y(s)\| ds + \frac{1}{\Gamma(\alpha)} \int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \|\Phi(s)\| ds \\
&\leq \frac{\gamma}{\Gamma(\alpha)} \int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \|x(s) - y(s)\| ds + \frac{\epsilon}{\Gamma(\alpha)} \int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \psi(t) ds \\
&\leq \frac{\gamma \left(e^b - e^a\right)^{\alpha}}{\Gamma(\alpha+1)} \|x - y\|_{PC} + \epsilon \beta_\psi \psi(t).
\end{aligned}$$

Then,

$$\|x - y\|_{PC} \leq \gamma\rho \|x - y\|_{PC} + \epsilon \beta_\psi \psi(t).$$

From (6.12), it clear that $\gamma\rho < 1$. So

$$\|x - y\|_{PC} \leq \frac{\epsilon \beta_\psi \psi(t)}{1 - \gamma\rho}. \tag{6.14}$$

Case 2. For each $t \in (t_k, s_k]$, we have

$$\begin{aligned}
\|x(t) - y(t)\| &= \|g_k(t, x(t)) + \Phi_k - g_k(t, y(t))\| \\
&\leq \|g_k(t, x(t)) - g_k(t, y(t))\| + \|\Phi_k\| \\
&\leq \xi \|x(t) - y(t)\| + \epsilon\omega.
\end{aligned}$$

Then,

$$\|x - y\|_{PC} \leq \xi \|x - y\|_{PC} + \epsilon\omega.$$

From (6.12), it clear that $\xi < 1$. So

$$\|x - y\|_{PC} \leq \frac{\epsilon\omega}{1 - \xi}. \tag{6.15}$$

Case 3. For each $t \in (s_k, t_{k+1}]$, we have

$$\begin{aligned}
&\|x(t) - y(t)\| \\
&= \left\| g_k(s_k, x(s_k)) + \frac{1}{\Gamma(\alpha)} \int_{s_k}^{t} \left(e^t - e^s\right)^{\alpha-1} e^s \varphi_x(s) ds + \frac{1}{\Gamma(\alpha)} \int_{s_k}^{t} \left(e^t - e^s\right)^{\alpha-1} e^s \Phi(s) ds \right. \\
&\quad \left. - g_k(s_k, y(s_k)) - \frac{1}{\Gamma(\alpha)} \int_{s_k}^{t} \left(e^t - e^s\right)^{\alpha-1} e^s \varphi_y(s) ds \right\| \\
&\leq \|g_k(s_k, x(s_k)) - g_k(s_k, y(s_k))\| + \frac{1}{\Gamma(\alpha)} \int_{s_k}^{t} \left(e^t - e^s\right)^{\alpha-1} e^s \|\varphi_x(s) - \varphi_y(s)\| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{s_k}^{t} \left(e^t - e^s\right)^{\alpha-1} e^s \|\Phi(s)\| ds \\
&\leq \xi \|x - y\|_{PC} + \frac{\gamma \left(e^b - e^a\right)^{\alpha}}{\Gamma(\alpha+1)} \|x - y\|_{PC} + \epsilon \beta_{\psi} \psi(t).
\end{aligned}$$

Then

$$\|x - y\|_{PC} \leq (\xi + \gamma\rho) \|x - y\|_{PC} + \epsilon \beta_{\psi} \psi(t).$$

From (6.12), we get

$$\|x - y\|_{PC} \leq \frac{\epsilon \beta_{\psi} \psi(t)}{1 - (\xi + \gamma\rho)}. \tag{6.16}$$

Finally, From (6.14), (6.15) and (6.16), we have

$$\|x - y\|_{PC} \leq \frac{\epsilon\omega}{1 - \xi} + \frac{\epsilon \beta_{\psi} \psi(t)}{1 - (\xi + \gamma\rho)}, \quad \text{for } t \in J.$$

Consequently

$$\|x - y\|_{PC} \leq \epsilon \nu_{f,\psi} [\psi(t) + \omega], \quad \text{for } t \in J$$

with

$$\nu_{f,\psi} := \max\left\{ \frac{1}{1 - \xi}, \frac{\beta_{\psi}}{1 - (\xi + \gamma\rho)} \right\}.$$

Which implies that the IVP (6.1)-(6.3) is U-H-R stable with respect to (ψ, ω).

6.2.3 Examples

In this subsection, we will give two examples to illustrate the above results. Let

$$E = l^1 = \left\{ y = (y_1, y_2, \ldots, y_n, \ldots) : \sum_{n=1}^{\infty} |y_n| < \infty \right\}$$

be the Banach space with the norm $\|y\|_E = \sum_{n=1}^{\infty} |y_n|$.

Example 1. Consider the following initial value problem for the nonlinear implicit Caputo's exponential type fractional differential equation with non-instantaneous impulses:

$${}_{c}^{e}D_{s_k}^{\frac{1}{2}} y_n(t) = f_n\left(t, y_n(t), {}_{c}^{e}D_{s_k}^{\frac{1}{2}} y_n(t)\right), \quad \text{for each } t \in J_k,\ k = 0, 1, \tag{6.17}$$

$$y_n(t) = g_{1_n}(t, y_n(t)), \quad \text{for each } t \in J_1', \tag{6.18}$$

$$y_n(1) = 487, \tag{6.19}$$

where $\alpha = \frac{1}{2}$, $\delta = 487$, $J_0 = [1, \ln(2)]$, $J_1' = (\ln(2), \ln(3)]$, $J_1 = (\ln(3), \ln(5)]$, $J = [1, \ln(5)]$, $\overline{\mathcal{J}} = [1, \ln(2)] \cup [\ln(3), \ln(5)]$, $\overline{\mathcal{J}'} = [\ln(2), \ln(3)]$, $y = (y_1, y_2, \ldots, y_n, \ldots)$, ${}_c^eD_{s_k}^{\frac{1}{2}} y = \left({}_c^eD_{s_k}^{\frac{1}{2}} y_1, {}_c^eD_{s_k}^{\frac{1}{2}} y_2, \ldots, {}_c^eD_{s_k}^{\frac{1}{2}} y_n, \ldots\right)$, $f = (f_1, f_2, \ldots, f_n, \ldots)$, $g_1 = (g_{1_1}, g_{1_2}, \ldots, g_{1_n}, \ldots)$,

$$f_n\left(t, y_n(t), {}_c^eD_{s_k}^{\frac{1}{2}} y_n(t)\right) = \frac{\sqrt{\pi} e^{-2t+1}}{11(2t-1)} \left[\frac{|y_n(t)|}{1 + |y_n(t)|} + \frac{\left|{}_c^eD_{s_k}^{\frac{1}{2}} y_n(t)\right|}{1 + \left|{}_c^eD_{s_k}^{\frac{1}{2}} y_n(t)\right|}\right], \quad \forall t \in \overline{\mathcal{J}}$$

and

$$g_{1_n}(t, y_n(t)) = \frac{e^{-t}|y_n(t)|}{5(1 + |y_n(t)|)}, \quad \forall t \in \overline{\mathcal{J}'}.$$

Now, for $t \in \bigcup_{k=0}^{1} J_k$ and $y, \overline{y}, z, \overline{z} \in E$, we can show that

$$\|f(t, y, z) - f(t, \overline{y}, \overline{z})\|_E \leq \frac{\sqrt{\pi}}{11e}\left(\|y - \overline{y}\|_E + \|z - \overline{z}\|_E\right).$$

Also,

$$\|g_1(t, y) - g_1(t, \overline{y})\|_E \leq \frac{1}{10}\|y - \overline{y}\|_E, \quad \text{for } t \in J_1' \text{ and } y, \overline{y} \in E.$$

Thus, for $k_1 = k_2 = \dfrac{\sqrt{\pi}}{11e}$ and $\xi = \frac{1}{10}$, we have that

$$\begin{aligned}
\lambda_1 &= \max\{\gamma\rho,\ \xi\} \\
&= \max\left\{\frac{k_1\left(e^b - e^a\right)^\alpha}{(1 - k_2)\Gamma(\alpha + 1)},\ \xi\right\} \\
&= \max\left\{\frac{\sqrt{\pi}\sqrt{5 - e}}{11e}\left[\left(1 - \frac{\sqrt{\pi}}{11e}\right)\Gamma\left(\frac{3}{2}\right)\right]^{-1},\ \frac{1}{10}\right\} \\
&= \max\left\{\frac{2\sqrt{5 - e}}{(11e - \sqrt{\pi})},\ \frac{1}{10}\right\} \\
&\approx 0.1074019495 \\
&< 1.
\end{aligned}$$

Hence, all of assumptions $(H_{01}) - (H_{03})$ and the condition (6.8) are satisfied. As a consequence of Theorem 6.5, the problem (6.17)-(6.19) has a solution on $[1, \ln(5)]$. On the other hand, with the choice of $w = 1$ and

$$\psi(t) = \begin{cases} e^t - e, & \text{if } t \in [1, \ln(2)] \\ 0, & \text{if } t \in (\ln(2), \ln(3)] \\ e^t - 3, & \text{if } t \in (\ln(3), \ln(5)]. \end{cases}$$

We find that

$$\begin{aligned} {}^{e}I_1^{\frac{1}{2}}\psi(t) &= \frac{4}{3\sqrt{\pi}}\sqrt{e^t - e}\left(e^t - e\right) \\ &\leq \frac{4\sqrt{2-e}}{3\sqrt{\pi}}\psi(t) \end{aligned} \quad \text{and} \quad \begin{aligned} {}^{e}I_{\ln(3)}^{\frac{1}{2}}\psi(t) &= \frac{4}{3\sqrt{\pi}}\sqrt{e^t - 3}\left(e^t - 3\right) \\ &\leq \frac{4\sqrt{2}}{3\sqrt{\pi}}\psi(t). \end{aligned}$$

Thus, (H_{08}) is satisfied with $\beta_\psi = \frac{4\sqrt{2}}{3\sqrt{\pi}}$. In addition, $(\xi + \gamma\rho) \approx 0.2074019495 < 1$. Therefore, From Theorem 6.9, the IVP (6.17)-(6.19) is U-H-R stable with respect to (ψ, ω).

Example 2. Consider the following initial value problem for the nonlinear Caputo's exponential type implicit fractional differential equation with non-instantaneous impulses:

$$ {}_{c}^{e}D_{s_k}^{\frac{1}{2}}y_n(t) = f_n\left(t, y_n(t), {}_{c}^{e}D_{s_k}^{\frac{1}{2}}y_n(t)\right), \text{ for } t \in J_k,\ k = 0, \ldots, 9, \tag{6.20}$$

$$y_n(t) = g_{k_n}(t, y_n(t)), \quad \text{for } t \in J'_k,\ k = 1, \ldots, 9, \tag{6.21}$$

$$y_n(0) = 523, \tag{6.22}$$

where $\alpha = \frac{1}{2}$, $t_k = \frac{k\pi}{5}$, $s_k = \frac{k\pi}{5} + \frac{\pi}{8}$, $J'_k = (t_k, s_k]$, $J_k = (s_k, t_{k+1}]$, $k = 1, .., 9$, $J_0 = \left[0, \frac{\pi}{5}\right]$, $J = [0, 2\pi]$, $\overline{\mathcal{J}} = \bigcup\limits_{k=0}^{9}[s_k, t_{k+1}]$, $\overline{\mathcal{J}'} = \bigcup\limits_{k=1}^{9}[t_k, s_k]$, $\delta = 523$, $y = (y_1, y_2, \ldots, y_n, \ldots)$, ${}_{c}^{e}D_{s_k}^{\frac{1}{2}}y = \left({}_{c}^{e}D_{s_k}^{\frac{1}{2}}y_1, {}_{c}^{e}D_{s_k}^{\frac{1}{2}}y_2, \ldots, {}_{c}^{e}D_{s_k}^{\frac{1}{2}}y_n, \ldots\right)$, $f = (f_1, f_2, \ldots, f_n, \ldots)$, $g_k = (g_{k_1}, g_{k_2}, \ldots, g_{k_n}, \ldots)$,

$$f_n\left(t, y_n(t), {}_{c}^{e}D_{s_k}^{\frac{1}{2}}y_n(t)\right) = \frac{\sqrt{\pi}e^{-\sqrt{t^2+16}}\cos(t)}{(t^2+1)\left(1 + |y_n(t)| + \left|{}_{c}^{e}D_{s_k}^{\frac{1}{2}}y_n(t)\right|\right)}, \quad \forall t \in \overline{\mathcal{J}}$$

and

$$g_{k_n}(t, y_n(t)) = \frac{e^{\frac{5t}{\pi}}}{(2+k)(1+|y_n(t)|)}, \quad \forall t \in \overline{\mathcal{J}'}.$$

Choosing $p(t) = \dfrac{\sqrt{\pi} e^{-\sqrt{t^2+16}} \cos(t)}{(t^2+1)}$ and $q_k(t) = \dfrac{e^{\frac{5t}{\pi}}}{(2+k)}$, $k = 1, \ldots, 9$. The hypotheses $(H_{04}) - (H_{07})$ are satisfied with $p^* = \dfrac{\sqrt{\pi}}{e^4}$ and $q^* = \dfrac{1}{3e}$. Also, we can show that

$$
\begin{aligned}
\max\{q^*,\ p^*\rho\} &= \max\left\{q^*,\ \frac{p^*\left(e^b - e^a\right)^\alpha}{\Gamma(\alpha+1)}\right\} \\
&= \max\left\{\frac{1}{3e},\ \sqrt{\pi}\sqrt{e^{2\pi}-1}\left[e^4\Gamma\left(\frac{3}{2}\right)\right]^{-1}\right\} \\
&= \max\left\{\frac{1}{3e},\ \frac{2\sqrt{e^{2\pi}-1}}{e^4}\right\} \\
&\approx 0.8468812794 \\
&< 1.
\end{aligned}
$$

Hence, all of assumptions (H_{01}), $(H_{04}) - (H_{07})$ and the condition (6.10) are satisfied. As a consequence of Theorem 6.6, the IVP (6.20)-(6.22) has at least one solution on $[0, 2\pi]$.

6.3 Existence and Stability Results for the Nonlocal IVP in Banach Space

In this section, we present some results of existence of solutions and U-H-R stability for a class of nonlocal IVP

$$ {}_c^e D_{s_k}^\alpha x(t) = f(t, x(t),\ {}_c^e D_{s_k}^\alpha x(t)), \quad \text{for each } t \in J_k \subseteq J,\ k = 0, 1, \ldots, m, \tag{6.23} $$

$$ y(t) = g_k(t, y(t)), \quad \text{for each } t \in J'_k \subset J,\ k = 1, 2, \ldots, m, \tag{6.24} $$

$$ y(a) + \widetilde{h}(y) = \delta, \tag{6.25} $$

where α, ${}_c^e D_{a^+}^\alpha$, f, g_k, δ, J, J_0, J_k, J'_k, $k = 1, \ldots, m$ are defined as in (6.1)-(6.3) and $\widetilde{h} : PC(J, E) \to E$ is a continuous functions. In [42], Byszewski *et al.* gives a theorem about the existence and uniqueness of a solution of a nonlocal Cauchy problem for an ordinary differential equation. The nonlocal condition (6.25) can be more useful than the standard initial condition (6.3) to describe some motion of physical phenomena with better effect.

6.3.1 Existence of Solutions

In this subsection, let us start by defining what we mean by a solution of the problem (6.23)-(6.25).

Definition 6.10 *A function* $y \in PC(J,E) \cap \left(\bigcup\limits_{k=0}^{m} AC_e(J_k,E) \right)$ *is said a solution of (6.23)-(6.25) if* y *satisfies the condition* $y(a) - \widetilde{h}(y) = \delta$, *the equations* ${}_c^eD_{s_k}^{\alpha} y(t) = f(t, y(t), {}_c^eD_{s_k}^{\alpha} y(t))$ *on* J_k, $k = 0,1,\ldots,m$ *and* $y(t) = g_k(t, y(t))$ *on* J'_k, $k = 1,2,\ldots,m$.

Lemma 6.11 *The nonlocal IVP (6.23)-(6.25) is equivalent to the following integral equation*

$$y(t) = \begin{cases} \delta - \widetilde{h}(y) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi(s)ds, & \text{if } t \in [a,t_1] \\ g_k(t,y(t)), & \text{if } t \in J'_k,\ k = 1,\ldots,m \\ g_k(s_k, y(s_k)) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{s_k}^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi(s)ds, & \text{if } t \in J_k,\ k = 1,\ldots,m, \end{cases}$$

where $\varphi \in C(\overline{\mathcal{J}}, E)$ *such that*

$$\varphi(t) = f(t, y(t), \varphi(t)).$$

Proof. The proof is taken just with Repeating the same process applied in the proof of Lemma 6.3. Now, we state the following assumptions:

(H_{09}) There exists a constant $\xi_{\widetilde{h}} > 0$ such that

$$\left\| \widetilde{h}(y_1) - \widetilde{h}(y_2) \right\| \le \xi_{\widetilde{h}} \|y_1 - y_2\|, \quad \text{for any } y_1, y_2 \in PC(J,E).$$

(H_{10}) There exists a nondecreasing function $p_{\widetilde{h}} \in C(J, \mathbb{R}_+)$ such that

$$\left\| \widetilde{h}(y) \right\| \le \frac{p_{\widetilde{h}}(t)}{1 + \|y\|}, \quad \text{for any } y \in PC(J,E).$$

(H_{11}) For each bounded set $B \subset PC(J,E)$, we have

$$\mu\left(\widetilde{h}(B)\right) \le p_{\widetilde{h}}(t) \sup_{t \in J} \mu\left(B(t)\right).$$

Remark 6.12 *[18] Conditions of* (H_{09}) *is equivalent to the inequality*

$$\mu\left(\widetilde{h}(B)\right) \le \xi_k \sup_{t \in J} \mu\left(B(t)\right), \quad \textit{for any bounded set } B \subset PC(J,E).$$

Set

$$p_{\widetilde{h}}^* := \sup_{t \in J} p_{\widetilde{h}}(t).$$

Theorem 6.13 *Assume that* $(H_{01}) - (H_{03})$ *and* (H_{09}) *hold. If*

$$\max\left\{\xi_{\widetilde{h}} + \gamma\rho,\ \xi\right\} < 1, \tag{6.26}$$

then the nonlocal problem (6.23)-(6.25) has a solution defined on J.

Proof. Transform the problem (6.23)-(6.25) into a fixed point problem, consider the operator $\widetilde{\Lambda} : PC(J,E) \to PC(J,E)$ as

$$\widetilde{\Lambda}(y)(t) = \begin{cases} \delta - \widetilde{h}(y) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi(s) ds, & \text{if } t \in [a, t_1] \\ g_k(t, y(t)), & \text{if } t \in J_k',\ k = 1, \dots, m \\ g_k(s_k, y(s_k)) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{s_k}^t \left(e^t - e^s\right)^{\alpha-1} e^s \varphi(s) ds, & \text{if } t \in J_k,\ k = 1, \dots, m, \end{cases}$$

where $\varphi \in C(\overline{\mathcal{J}}, E)$ such that

$$\varphi(t) = f(t, y(t), \varphi(t)).$$

We can easily show that $\widetilde{\Lambda}$ satisfies the assumptions of Darbo's fixed point theorem. By following the computations as it is done in the proof of Theorem 6.5.

Theorem 6.14 *Assume that* $(H_{01}),\ (H_{04}) - (H_{07}),\ (H_{10})$ *and* (H_{11}) *hold. If*

$$\max\left\{q^*,\ p^*_{\widetilde{h}} + p^*\rho\right\} < 1,$$

then the nonlocal problem (6.23)-(6.25) has at least one solution defined on J.

Proof. The proof is taken just by following the computations as it is done in the proof of theorem 6.6.

6.3.2 Ulam-Hyers-Rassias Stability

In this subsection, we present an result in U-H-R stability for the nonlocal IVP (6.23)-(6.25).

Theorem 6.15 *Assume that* $(H_{01}) - (H_{03}),\ (H_{08})$ *and* (H_{09}) *hold. If*

$$\xi_{\widetilde{h}} + \gamma\rho + \xi < 1,$$

then the nonlocal IVP (6.23)-(6.25) is U-H-R stable with respect to (ψ, ω).

Proof. To prove that the problem (6.23)-(6.25) is U-H-R stable with respect to (ψ, ω), we follow the computations as it is done in the proof of theorem 6.9.

6.3.3 An Example

Consider the following nonlocal IVP

$$ {}_c^eD_{s_k}^{\frac{1}{2}}y_n(t) = f_n\left(t, y_n(t), {}_c^eD_{s_k}^{\frac{1}{2}}y_n(t)\right), \qquad \text{for each } t\in J_k,\ k=0,1, \tag{6.27} $$

$$ y_n(t) = g_{1_n}(t, y_n(t)), \qquad \text{for each } t\in J_1', \tag{6.28} $$

$$ y_n(0)+\widetilde{h}_n(y_n) = 191, \tag{6.29} $$

where $\alpha=\frac{1}{2}$, $J_0=[0,1]$, $J_1'=(1,2]$, $J_1=(2,3]$, $J=[0,3]$, $\overline{\mathcal{J}}=[0,1]\cup[2,3]$, $\overline{\mathcal{J}'}=[1,2]$, $\delta=191$, $y=(y_1,y_2,\ldots,y_n,\ldots)$, $f=(f_1,f_2,\ldots,f_n,\ldots)$, $g_1=(g_{1_1},g_{1_2},\ldots,g_{1_n},\ldots)$, ${}_c^eD_{s_k}^{\frac{1}{2}}y=\left({}_c^eD_{s_k}^{\frac{1}{2}}y_1, {}_c^eD_{s_k}^{\frac{1}{2}}y_2,\ldots, {}_c^eD_{s_k}^{\frac{1}{2}}y_n,\ldots\right)$, $\widetilde{h}=\left(\widetilde{h}_1,\widetilde{h}_2,\ldots,\widetilde{h}_n,\ldots\right)$,

$$ f_n\left(t, y_n(t), {}_c^eD_{s_k}^{\frac{1}{2}}y_n(t)\right) = \frac{e^{-t-1}\left(2+|y_n(t)|+\left|{}_c^eD_{s_k}^{\frac{1}{2}}y_n(t)\right|\right)}{(3+e^t)\left(1+|y_n(t)|+\left|{}_c^eD_{s_k}^{\frac{1}{2}}y_n(t)\right|\right)}, \quad \forall t\in\overline{\mathcal{J}}, $$

$$ g_{1_n}(t, y_n(t)) = \frac{e^{-\sqrt{t^2+8}}\left(2+|y_n(t)|\right)}{3t\left(1+|y_n(t)|\right)}, \quad \forall t\in\overline{\mathcal{J}'}, $$

$$ \widetilde{h}_n(y_n) = \sum_{i=1}^{\ell} c_i y_n(\tau_i), $$

with $0<\tau_1<\tau_2<\ldots<\tau_\ell<3$, $c_1,c_2,\ldots,c_\ell>0$ and $\sum_{i=1}^{\ell} c_i<\frac{1}{5}$. Since,

$$ \|f(t,y,z)-f(t,\overline{y},\overline{z})\|_E \le \frac{1}{4e}\left(\|y-\overline{y}\|_E+\|z-\overline{z}\|_E\right), \quad \text{for } t\in\bigcup_{k=0}^{1}J_k,\ y,\overline{y},z,\overline{z}\in E, $$

$$ \|g_1(t,y)-g_1(t,\overline{y})\|_E \ \le\ \frac{1}{3e^3}\|y-\overline{y}\|_E, \quad \text{for } t\in J_1',\ y,\overline{y}\in E, $$

and

$$ \left\|\widetilde{h}(y)-\widetilde{h}(\overline{y})\right\|_E \ \le\ \frac{1}{5}\|y-\overline{y}\|_E, \quad \text{for } y,\overline{y}\in PC([0,3],E). $$

Then for $k_1 = k_2 = \dfrac{1}{4e}$, $\xi = \dfrac{1}{3e^3}$ and $\xi_{\widetilde{h}} = \frac{1}{5}$, we have

$$\begin{aligned}
\max\left\{\xi_{\widetilde{h}} + \gamma\rho,\ \xi\right\} &= \max\left\{\xi_{\widetilde{h}} + \frac{k_1\left(e^b - e^a\right)^\alpha}{(1-k_2)\,\Gamma(\alpha+1)},\ \xi\right\} \\
&= \max\left\{\frac{1}{5} + \frac{\sqrt{e^3-1}}{4e}\left[\left(1-\frac{1}{4e}\right)\Gamma(\frac{3}{2})\right]^{-1},\ \frac{1}{3e^3}\right\} \\
&= \max\left\{\frac{1}{5} + \frac{2\sqrt{e^3-1}}{(4e-1)\sqrt{\pi}},\ \frac{1}{3e^3}\right\} \\
&\approx 0.6992895922 \\
&< 1.
\end{aligned}$$

Therefore, all of assumptions $(H_{01}) - (H_{03})$, (H_{09}) and the condition (6.26) are satisfied. By Theorem 6.13, we deduce that the nonlocal IVP (6.27)-(6.29) has a solution defined on $[0, 3]$. Also, form Theorem 6.15, it follows that the problem (6.27)-(6.29) is U-H-R stable with respect to (ψ, ω).

Conclusion and Perspective

In this thesis, we have presented some results in the existence, uniqueness of solutions and Ulam stability for an nonlinear implicit fractional differential equations involving Caputo's exponential type fractional derivative of order $\alpha \in (0,1]$. We have discussed Several various classes: boundary value problems in Banach spaces, boundary value problems with instantaneous impulses, abstract boundary value problems with instantaneous impulses, initial value problems with not-instantaneous impulses for the both case of with and without nonlocal conditions and initial value problems with not-instantaneous impulses in Banach spaces for the both case of with and without nonlocal conditions. Our approach for the classes of problems in Banach spaces is based on Darbo and Mönch fixed points theorems combined with the technique of measure of noncompactness and for the remaind classes of problems we have used Banach's contraction principle and Schaefer's fixed point theorem.

For the prespective, we project to study the case of nonlinear fractional differential inclusions with and without impulses and the case of hybrid nonlinear coupled system implicit generalized fractional pantograph equations. Also, We plan to extend our results with ψ-Caputo fractional derivative.

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الملخص

في هذه الأطروحة، سنضع شروطًا لدراسة وجود ووحدانية الحلول وكذا الاستقرار بمفهوم إيلام(Ulam) لمختلف فئات من المشاكل عند القيمة الابتدائية وعند الحدود للمعادلات التفاضلية الضمنية الغير الخطية بوجود وغياب النبضات. موظفين المشتقة الكسرية الأسية لكابوتو(Caputo). في دراستنا، تم التطرق إلى الحالتين المجردة والسلمية. لإثبات وجود ووحدانية الحلول، نستخدم بعض نظريات النقطة الثابتة. كما يتم تقديم العديد من الأمثلة التوضيحية.

الكلمات المفتاحية:

مشكلة القيمة الابتدائية، مشكلة القيمة عند الحدود، المشتقة الكسرية الأسية لكابوتو، المعادلات التفاضلية الكسرية الضمنية، التكامل الكسري الأسي، النبضات غير الآنية، النبضات اللحظية، الوجود، الوحدانية، النقطة الثابتة، الاستقرار بمفهوم إيلام، الظروف الغير المحلية، فضاء باناخ، مقياس عدم الاكتناز.

Résumé

Dans cette thèse, nous avons établi des conditions d'existence, d'unicité des solutions et de la stabilité d'Ulam de diverses classes de problèmes à valeur initiale et de problèmes aux limites pour les équations différentielles implicites non linéaires avec et sans impulsions en utilisant la dérivée fractionnaire de type exponentielle au sens de Caputo. Dans notre étude, Les deux cas abstrait et scalaire ont été considérés. Pour prouver l'existence et l'unicité des solutions, nous utilisons certains théorèmes classiques du point fixe. Afin d'illustrer nos résultats plusieurs exemples ont été présentés.

Mots clés et phrases :

Problème à valeur initiale, problème aux limites, la dérivée fractionnaire de type exponentielle au sens de Caputo, équations différentielles implicites, l'intégrale fractionnaire de type exponentielle, impulsions non-instantanées, impulsions instantanées, existence, unicité, point fixe, stabilité d'Ulam, les conditions non-locales, espace de Banach, mesure de non-compacité.

Abstract

In this Thesis, we shall establish conditions for the existence, uniqueness of solutions and Ulam stability for various classes of initial and boundary value problems for nonlinear implicit fractional differential equations with and without impulses involving Caputo's exponential type fractional derivative. In our study, the Both cases of abstract and scalar have been considered. To prove the existence and uniqueness of solutions, we use some standard fixed point theorems. several enlightening examples are also presented.

Key words and phrases:

Initial value problem, boundary value problem, Caputo's exponential type fractional derivative, implicit differential equations, exponential type fractional integral, non-instantaneous impulses, instantaneous impulses, existence, uniqueness, fixed point, Ulam stability, nonlocal conditions, Banach space, measure of noncompactness.

AMS Subject Classification: 26A33, 34A08, 34A12, 34A37, 34G20.