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SOUNA Fethi

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Contributions aux modèles de dynamique des populations

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Devant le jury composé de :

Président :

Mr. HELAL Mohamed, Professeur à l'université de Sidi Bel Abbès

Examinateurs :

Mr. CHEKROUN Abdennasser, M.C.A à l'université de Tlemcen

Mr. HABIB Habib, M.C.A à l'université de Sidi Bel Abbès

Directeur de thèse :

Mr. LAKMECHE Abdelkader, Professeur à l'université de Sidi Bel Abbès

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Declaration

I hereby declare that the work contained in this thesis entitled "Contributions to population dynamics models " is my own work under the supervision and direction of **Pr. Abdelkader LEKMECHE**, professor at Djillali Liabès University, Department of Mathematics, Sidi Bel Abess for the award of the degree of Doctor of Philosophy in mathematics. To the best of my knowledge and belief, it contains no material previously published or written by another person.

Mai, 2021

Fethi SOUNA
Faculty of Exact Sciences
Department of Mathematics
Djillali Liabès University of S.B.A

Dedicace

*I*t *is with great emotion and immense pleasure that I dedicate this modest work:*

To my dear parents **Nacira** *and* **Omar** *who supported me throughout my studies, thank you for all your love and support, and for taking just the right amount of interest in my research.*

My dear wife **Imene**, *who encouraged me to go forward to finish my studies and gave me all her love.*

My brothers **Mohammed el Amine** *and* **Zinedine**.

To all my family members, young and old.

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To all those who love me and whom I love...

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Contents

General introduction, ecological motivation and thesis Outline

Ecological motivation

In ecology, an ecosystem is made up of different populations that interact with one another in several ways **Fig.1**. A population is a group of individuals of the same species that occupy the same patch simultaneously. Often, individuals from one population compete for the same resources, as do individuals from different populations (competition). Some populations live at the expense of others (parasitism). Certain populations help each other (mutualism, even symbiosis). All these phenomena are part of the general framework of the struggle for life. Population dynamics is a branch of mathematical ecology which aims to study these variations between living beings theoretically using simplified models. There are several interesting questions that can be asked about ecosystems, on the factors that influence the stability of an ecosystem, on the factors controlling the variability of the abundances of different components of the ecosystem, and in particular on the interactions between populations and their role to lead or not the coexistence of species in an ecosystem. To answer these questions, different methodological approaches exist throughout this last century. The first mathematical models in ecology date from the $1920s$, with the two scientific researchers **A.J. Lotka and V. Volterra**. These models are based on the representation of interactions between species through systems of ordinary differential equations. Our main objective along this thesis is to model and study certain ecological phenomena for living beings in nature through systems of ordinary differential equations or partial differential equations. The starting point is the historical model of **Lotka and Volterra**.
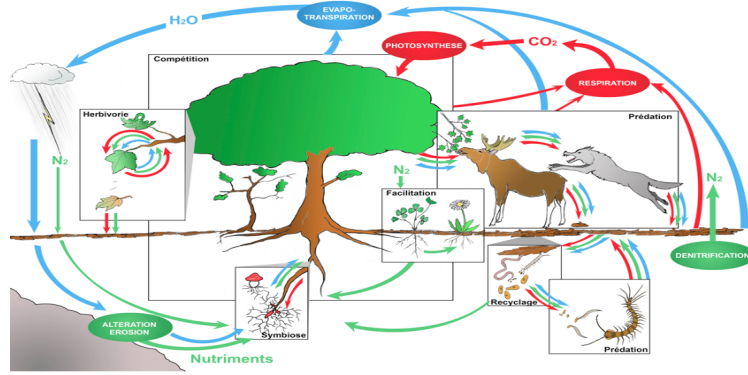
FIGURE 1: Schematic representation of interactions in ecosystems [61].

General introduction

The dynamic relationship between **predator population** and **prey population** has been considered as one of the dominant themes in modern applied mathematics due to the universal existence and the universal importance [12, 68], it is one of the reasons of the big preference by many researchers next to the huge application in different ecosystems types as aquatic, terrestrial ecosystems. The beginning was with Malthus 1798 [63] where he proposed a model of one population with an exponential increasing which called by his name **Malthus model**. Later on, in 1838, Verhulst [89] offered more realistic model which is known by the **logistic model**. The main assumption is to consider that the space that the animals lives in it has a carrying capacity where the availability of resources is the main factor in determining this food load of the space, which resolves the problem of the unboundedness of the resources generated by the bounded space offered by Malthus model [63]. It follows by the pioneering works of Lotka [56] and Volterra [90], where a system of two equations is considered for predicting **the predator–prey interaction** evolution in terms of time.

In terms of modeling, the total dynamics of a **predator-prey system** can be affected by many factors such as predator efficiency of their attack, birth, death, mature delay, so on. It is well known that **the functional response** is the crucial component to describe the relationship between the prey and the predator populations. In the last few decades, many **functional responses** appears for studying and describing a particular relationship between the species. It is the main reason for having many functional responses such as Holling $I - IV$ functional response [44, 47, 93, 103], Beddington-DeAngelis functional response [17], ratio-dependent functional response [97, 78], Crowley-Martin functional response [86, 105],

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Hassell-Varley functional response [45], so on.

Now, we introduce the prototypical **predator-prey model** with the following structure

$$\begin{cases} \frac{\partial X}{\partial t} = Xf(X) - g(X,Y), \\ \\ \frac{\partial Y}{\partial t} = -mY + \eta g(X,Y), \end{cases} \tag{0.0.1}$$

where X,Y are respectively the populations sizes of the prey and the predator at the time t, $f(X)$ is the net growth for the prey population in the absence of the predator population, $0 < \eta < 1$ is the conversion rate of prey biomass into predator biomass and m is the natural death rate of the predator which has been assumed to be constant.

In the real world, the prey population has a fast reproduction comparing with the predator population (please see [89]). It is wise to choose a reproduction functional $f(X)$ to model the crowding effect of the prey. The most known one is the logistic form :

$$f(X) = r(1 - \frac{X}{K})$$

where r is the net growth rate of the prey population, k is the carrying capacity of the prey in the absence of predator. Obviously, the functional $g(X,Y)$ represents the interaction functional. As we said, it models the quality of interaction between the prey and the predator.

In nature, an important part of savanna preys lives collectively and proceed in a huge **herd** [**?**] from one place to another, there are many examples of this behavior such as buffalos, elephants, gnus, so on. For the buffalos example, the weakest elements occupy the interior of the group and the strongest stay at the border for the purpose of defending the group and reducing the predation risk of the weakest. For the goal of protecting there own child, the strongest prey defend the group brutally. Due to the group's structure, the prey get features in their foraging efficiency and reducing predation risk. Thus, **the herd behavior** avoids the disappearance of the animals (for the reason of its big appearance), which gives better defense mechanism from predation [**?**]. The first approach in terms of modeling this phenomenon was introduced and studied by Ajraldi et *al.* [2]. The main assumption is presume that the prey population has a regular shape as circle or a square. If we assume that $X(t)$ be the density of the prey, namely number of individuals per surface unit, and the herd occupying an area S, it follows that the individuals at the outer of the packs are proportional to the border of the patch, where the herd is in a length depends on $\sqrt{S}$. Then, they are in number proportional to the square root of the density i.e. to $\sqrt{X}$, With a constant of proportionality depends on the regularity of the herd. Thus the number of the captured prey by one predator will be proportional to the number of the prey population on the boundary of the prey herd, which means that is proportional to the square root of the prey population
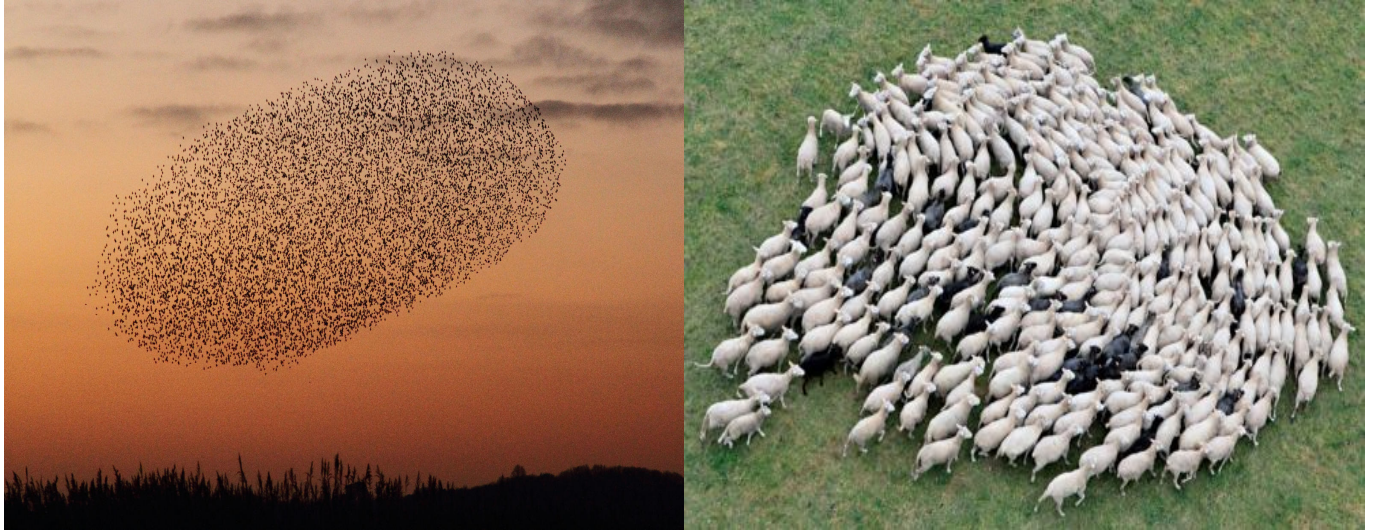
FIGURE 2: Herd behavior in wild [19].

density i.e. $\sqrt{X}$. Therfore, we leads to a new predator-prey interaction presented in [2], that can be written as

$$
\begin{cases}
\frac{\partial X}{\partial t} = rX(1 - \frac{X}{K}) - \alpha\sqrt{X}Y, \\
\\
\frac{\partial Y}{\partial t} = -mY + \eta\alpha\sqrt{X}Y,
\end{cases}
\tag{0.0.2}
$$

we keep the same notation made with the previous system, the term α is the search efficiency of predator for the prey.

Remark 1. *Noticing that any shape of the herd formed by the prey population can be approximated by a regular form such as circle or a square in $2D$.*

In share same concept of mathematical modeling in [2], some mathematical results and its ecological relevant are provided. Another pioneering work appears into the interface in the case of **predator-prey interaction** with prey grouping behavior which is presented by Braza [15]. The main result is to consider the time for a predator to handel with a prey. Holling approximation is used for building a new **functional response**. Other works offered to the wide audience for modeling other factors next to **the herd behavior** such as herd shape, predator average handling time of the prey on the bounders. We refer readers to the papers [16, 24, 27, 28, 29, 30, 31, 48, 81, 85, 87, 88, 99, 100, 101, 106], which shows the big importance of such approximation and its huge significance in the real world.

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The prey density on the outer of the packs are proportional to the border

√x

The weakest individuals

The predator interacts only with the individuals which are in the border of the pack

predator

One individual of the prey populations

The strongest individuals
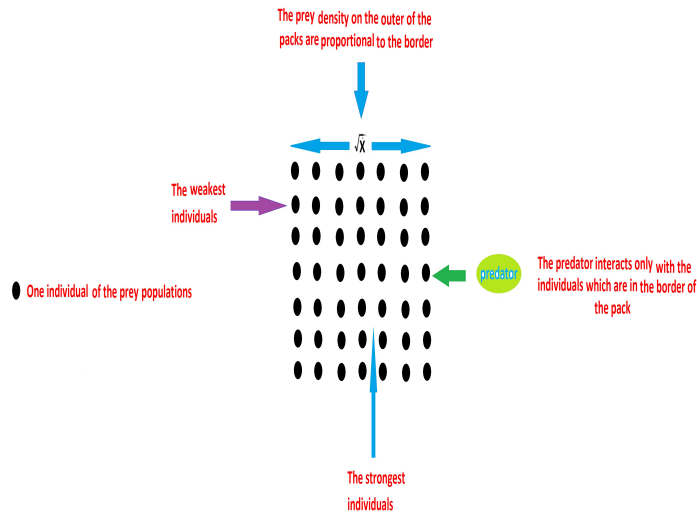
FIGURE 3: The herd structure and the mechanism of interaction between the prey and the predator population.

Thesis Outline

In this thesis, we will propose four new models of population dynamics which describes the interaction between prey and predator in nature. Our fundamental hypothesis is to suppose that the population of prey gather together in herds, on the other hand the predator shows a more individualistic behavior.

In **Chapter 1**, we scrutinize a delayed predator- prey model for the purpose of studying the impact of the strategy considered by the prey population on the evolution of the studied species. The main presumption is to assume that there exists two types of the prey with a contradictory behaviors. The first, has a social behavior and takes the advantage of living in group for defending each others, and the second one has a solitary behavior. A mathematical approach is used to study this effect, where local stability and bifurcation analysis are examined. Indeed, it has been proved that the system that we proposed has a rich dynamics such as Hopf bifurcation in both the absence and the presence of time lags. Further the stability of the periodic solution generated by the presence of the time lags are discussed using the normal form on the center of manifold theory. Some numerical simulations are provided for ensuring the obtained mathematical results.

In **Chapter 2**, we returned to dimension two and we considered a prey-predator model taking into account the shape of the herd formed by the prey. some preliminary results have been given including the boundedness of the solutions, the stability and dynamical behaviours of the equilibria of the model and

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the dynamic behavior near the origin. Then, the effect of the herd shape for the prey population on the prey and predator equilibrium densities has been analysed. After that, we investigate the global dynamics of the model where some sufficient conditions are derived to ensure the global asymptotic stability of the semi-trivial equilibrium point and coexistence equilibrium point. The analysis showed that the system might undergo Hopf bifurcation in such case a limit cycles appear for the interior equilibrium. Finally, to illustrate our theoretical results, some numerical simulations are given.

In **Chapter 3**, we deals with a diffusive predator-prey model subject to the zero flux boundary conditions with prey social behavior and the quadratic predator harvesting. First, we proved the existence of a positive solution and its bounders. The existence of the equilibrium states has been discussed. The global stability of the semi trivial constant equilibrium state is established. Concerning the non trivial equilibrium state, the local stability, Hopf bifurcation, diffusion driven instability, Turing-Hopf bifurcation are investigated. The direction and the stability of Hopf bifurcation relying on the system parameters is derived. Some numerical simulations are used to extend the analytical results and show the existence of the homogeneous and non homogeneous periodic solutions. Further the effect of the rivalry rate on the dynamical behavior of the studied species.

In **Chapter 4**, we consider a new approach of prey escaping from herd in a predator-prey model with the presence of spatial diffusion. First, the sensitivity of the equilibrium state density with respect to the escaping rate has been studied. Then, the analysis of the non diffusive system was investigated where boundedness, local, global stability, Hopf bifurcation are obtained. Besides, for the diffusive system, we proved the occurrence of Hopf bifurcation and the non existence of diffusion driven instability. Furthermore, the direction of Hopf bifurcation has been proved using the normal form on the center manifold. Some numerical simulations have been used to illustrate the obtained results.

Chapter 1

The effect of the defensive strategy taken by the prey on predator-prey interaction

This chapter is taken from publication [83].

In this chapter, we propose a model with two prey and a one predator in the presence of delay. We suppose that the two prey having two contradictory defensive strategies where, the first prey gathers in herd and the second shows a **solitary behavior**. In term of reproducing, we assume that the social population reproduces in a logistic manner and the individual population follows a Malthusian law of reproduction, which means that the solitary prey has an advantage in the reproduction on the grouped prey. We assume in this chapter that the predator consumes both types of prey, following these assumptions, we will try to choose the best strategy to defend as well as the prey preferred by the predator, which gives the efficiency of the herd for the social population.

This chapter is organized as follows: **Section 1.1** is devoted to the presentation of the mathematical model. In **Section 1.2** we analyze the stability of the equilibria for the non-delayed system. In **Section 1.3**, by considering time lags as a bifurcation parameter, we discuss the existence of Hopf bifurcation for both, the boundary and the interior equilibrium. **Section 1.4** is devoted to the analyze the stability and direction of Hopf bifurcation at the interior equilibrium using normal form on the center manifold theory. An extensive numerical simulation results have also been conducted to illustrate the theoretical results

in . A discussion section is provided for discussing and summarizing the obtained results. At last, a brief concluding remarks are provided in to close this work, where the biological significance and importance of the mathematical results are given for the conserving of the ecological species (partially and completely).

1.1 Model formulation

In nature, there are many problems for living in herds such as the big appearance of the group for the predator. In this case the predator will be able to pursuit the group, also the sharing of the resources for the interior prey herd is also a problem. The distribution of the resources between the inner group and the outer group (on the bounders) for the prey population can be considered also as a problem. In this case, the outer prey has a big availability of the resources (see [30]). In fact, this two last problems cannot be found for the prey that exhibits **solitary behavior**, where this kind of animals searches for food in a free way, the most common animals that exhibits this behavior is gazelle and giraffe. The inconvenient of this behavior is losing the safety guaranteed by living in herds, this kind of behavior is the most proffered to the predator. But in the real world the predator consumes both types of populations. For trying to resolve the query of which behavior is the most appropriate to the prey surviving, we considers a three species model with two type of prey. The first has a **social behavior** which means that the interaction happens on the outer corridor of the herd. Also to mention that this kind of population reproduces in a logistic manner for the reason of the crowding (which going to be explained better later), and the other type of the prey has a solitary behavior reproduce in a Malthusian manner (for the reason of the small number of them or its natural behavior). In fact, this means that the solitary prey has an advantage in the reproduction on the grouped prey.

Indeed, models with **time delay** are more realistic for describing the real life situations. The time lags occurs in almost every biological (resp. ecological) situation, it is responsible for regular fluctuations in population density. It describes the time spent for conversion of the prey biomass into the predator biomass (see for instance [28]), the time spent in searching and capturing the prey by a predator [27]. It is well understood that many of the processes in both natural and man-made in biology and medicine involve time lags. More precisely, in [49] it has been mentioned that an animal must take time delay to digest their food before their further activities takes place. Hence, models of species dynamics without delays can gives a wrong (or unprecise) mathematical predicting results which is not wise and it is better to avoid this kind of situations. For the subject of giving a further realistic model as possible, we consider the presence of the time delay for the prey population that exhibits **herd behavior**. As it has been mentioned previously this kind of prey has a difficulty in collecting resources from the environment.

Consequently, the time lags represents the time spends without any food that leads to the mortality of this living being. This kind of phenomenon can happens in the dry seasons for the very youngest prey, where this last has a less capability of adaptation, where the absence of resources is the main reason for the biggest prey immigrations in our planet. To highlight that the time delay attracts many researchers for the reason of its huge ecological and biological relevant, where a numerous models appears into the interface for modeling this kind of interaction. In addition to the previously mentioned examples, we cite for instance [11, 21, 23, 37, 43, 49, 59, 60, 102, 111].

In both, mathematical and ecological point of view, the delay can generates a surprising results, where a huge part of the literature papers are devoted to explore this rich dynamics. It has been confirmed by many scientists that the delay can change the dynamics of solution for a model, it is the main reason of considering it in this chapter, and the main purpose is for avoiding the misleading of the results generated by neglecting this fluctuation of time lags and providing the maximal possible precise mathematical-ecological results. In fact, it is interesting to discuss the effect of this delay on the evolution of the species for confirming that the time spending by the social prey without any food due to the crowding, can leads to the mortality. This phenomena can be modeled by the presence of the time lags in the logistic growth of the crowding prey, for more details of the time delay in logistic growth and its biological implication, we cite for instance [64]. In literature it has been mentioned that **the herd behavior** has a benefit for the population that uses it, but no one has been proved it mathematically. For the first time, we will give a comparative analysis between the two prey types and compar between the defensive strategy considered by both type of resources (prey), which is the main motivation of our chapter. In fact, we will give a preference for the solitary prey population presented by the malthusian increasing of it (exponential reproduction).

Now, we consider a new model where two types of prey are considered. The density of the first at time t is denoted by $X(t)$, this population exhibits social behavior and the one which has the time lags. For the second type of the prey its density at the time t is denoted by $Y(t)$. These two types of the resources has a contradictory behavior. In fact, these two types of prey are the subject of predation by one predator, which models the diversity of predator consumption. Based on the big motivation of our paper and ecological background we set the following model:

$$\begin{cases} \frac{d}{dt}X(t) = r_1\left(1 - \frac{X(t)}{K}\right)X(t) - \alpha_1\sqrt{X(t)}Z(t), \\ \frac{d}{dt}Y(t) = r_2Y(t) - \alpha_2Y(t)Z(t), \\ \frac{d}{dt}Z(t) = -\tilde{m}Z(t) + e\left(\alpha_1\sqrt{X(t)}Z(t) + \alpha_2Y(t)Z(t)\right), \end{cases} \quad (1.1.1)$$

with initial conditions $X(0) > 0, \quad Y(0) > 0, \quad Z(0) > 0$. In the absence of the predator, we assume that the first population of prey (with social behavior) follows a logistic growth form, where the growth of the

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second (with the solitary behavior) is exponential. All the parameters are supposed to be nonnegative with the following ecological meanings:

X	the grouped prey density	α_1	the hunting rate for the grouped prey
Y	the solitary prey density	α_2	the hunting rate for the solitary prey
Z	the predator density	e	the conversion coefficient
K	the carrying capacity of the grouped prey density	$\tilde{m}$	the natural predator mortality
r_1	the growth rate of the grouped prey		
r_2	the growth rate of the solitary prey		

Now, we introduce the effect of time lags for the system (1.1.1). In a precise way it will be put in the logistic growth of the grouped prey as it has been highlighted. The prototype model embodying this phenomenon is due to Hutchinson (1948) (see [64]) where it has been considered that in the absence of predators, the prey's growth is affected by population density only after a fixed period of time. Then, our system of equations (1.1.1) becomes

$$\begin{cases} \frac{d}{dt}X(t) = r_1\left(1 - \frac{X(t-\tau)}{K}\right)X(t) - \alpha_1\sqrt{X(t)}Z(t), \\ \frac{d}{dt}Y(t) = r_2Y(t) - \alpha_2Y(t)Z(t), \\ \frac{d}{dt}Z(t) = -\tilde{m}Z(t) + e\left(\alpha_1\sqrt{X(t)}Z(t) + \alpha_2Y(t)Z(t)\right). \end{cases} \qquad (1.1.2)$$

For the system (1.1.2) the following conditions are assumed to be hold:

$$(X(t), Y(t), Z(t)) = (X_0(t), Y_0(t), Z_0(t)), \qquad -\tau \leq t \leq 0,$$

where, $X(t), Y(t), Z(t) \in C([-\tau, 0], \mathbb{R}^3_+)$, and $X(0) > 0, \quad Y(0) > 0, \quad Z(0) > 0$.

For the subject of avoiding the heavy computation we will give a series of change of variables. First, we introduce the following one $X = u^2, \quad Y = v, \quad Z = w$ then (1.1.2) becomes:

$$\begin{cases} \frac{d}{dt}u(t) = \frac{r_1}{2}\left(1 - \frac{u^2(t-\tau)}{K}\right)u(t) - \frac{\alpha_1}{2}w(t), \\ \frac{d}{dt}v(t) = r_2v(t) - \alpha_2v(t)w(t), \\ \frac{d}{dt}w(t) = -\tilde{m}w(t) + e\left(\alpha_1u(t)w(t) + \alpha_2v(t)w(t)\right). \end{cases} \qquad (1.1.3)$$

Next, we introduce the transformations

$$x = \frac{u}{\sqrt{K}}, \quad t_{new} = \frac{r_1}{2}t_{old}, \quad z = \frac{\alpha_1}{r_1\sqrt{K}}w, \quad y = v.$$

For the goal of reducing the number of parameters for the nonlinear system (1.1.3), and after some

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straightforward calculation we reach to the following non-dimensional model.

$$\begin{cases} \frac{d}{dt}x(t) = \left(1 - x^2(t-\tau)\right)x(t) - z(t), \\ \frac{d}{dt}y(t) = y(t)\left(\alpha - \beta z(t)\right), \\ \frac{d}{dt}z(t) = z(t)\left(-m + \gamma x(t) + \delta y(t)\right), \end{cases} \tag{1.1.4}$$

where,

$$\alpha = 2\frac{r_2}{r_1}, \quad \beta = 2\frac{\alpha_2\sqrt{K}}{\alpha_1}, \quad \gamma = 2\frac{e\sqrt{K}}{r_1}, \quad \delta = 2\frac{e\alpha_2}{r_1}, \quad m = 2\frac{\tilde{m}}{r_1}.$$

1.2 Analysis of the non-delayed model $(\tau = 0)$

In this section, the qualitative stability of the equilibria is the subject of investigation in the absence of time lags, where we will use the simplified predator-prey model (1.1.4). As a first step, let us compute the equilibrium points of (1.1.4) which are the solutions of the following nonlinear system

$$\begin{cases} \left(1 - x^2\right)x - z = 0, \\ y\left(\alpha - \beta z\right) = 0, \\ z\left(-m + \gamma x + \delta y\right) = 0. \end{cases} \tag{1.2.1}$$

After solving the above system, we find the following equilibrium points

(i) The origin $E_0 := (0,0,0)$ which represents the extinction of the three population, which is not a situation that we are seeking for it.

(ii) The axial equilibrium point $E_1 := (1,0,0)$ which represent the surviving the grouped prey only without any presence of the other species.

(iii) The boundary equilibrium point $E_2 := (x_1, 0, z_1)$ where

$$x_1 = \eta, \qquad z_1 = \left(1 - \eta^2\right)\eta,$$

with positivity condition $\eta < 1$, where $\eta = \dfrac{m}{\gamma}$. This equilibrium models the surviving of the grouped prey only. If this equilibrium is stable, it means that the grouped prey has a survival advantage but the solitary prey die out.

(iv) The coexistence equilibrium point $E^* := (x^*, y^*, z^*)$ where

$$z^* = x^*(1 - (x^*)^2) = \rho, \qquad y^* = \frac{\gamma}{\delta}(\eta - x^*), \quad \text{where} \quad \rho = \frac{\alpha}{\beta} \tag{1.2.2}$$

and x^* is the positive root of the following cubic equation

$$\psi(x) \triangleq x^3 - x + \rho = 0. \tag{1.2.3}$$

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Obviously, this equation has two change of signes, using Descartes's rule, we deduce that (1.2.3) has either two or nor roots. The two roots are denoted by x_i^*, $i = 1, 2$. Hence, we obtain the two equilibria E_i^*, $i = 1, 2$. Precisely, since $\psi(0)$ is positive and $\psi'(0) < 0$, we put

$$\bar{x} = \frac{1}{\sqrt{3}},$$

where, $\bar{x}$ is the solution of $\psi'(x) = 0$, then, the Eq. (3.2.4) possess two roots if and only if $\psi(\bar{x}) < 0$, this implies that

$$\rho < \frac{2}{3\sqrt{3}}. \tag{1.2.4}$$

This last inequality is the existence condition for the interior equilibria $E_i^* := (x_i^*, y_i^*, z_i^*)$, $i = 1, 2$. From (1.2.2), the positivity requires that $x_i^* < 1$, $i = 1, 2$ and $x_i^* < \eta$, $i = 1, 2$ so, combining the two conditions gives

$$x_i^* < \min\{1, \eta\}, \quad i = 1, 2, \tag{1.2.5}$$

and

$$x_1^* < \frac{1}{\sqrt{3}} < x_2^*. \tag{1.2.6}$$

Summarizing the above discussion on the existence conditions of the interior equilibrium, we set the the following theorem

Theorem 1. *([83]) Assume that (1.2.4) and (1.2.5) holds, then the system (1.1.4) has two equilibria in the interior of $\mathbb{R}_+^2$, which are $E_1^* = (x_1^*, y_1^*, y_1^*)$ and $E_2^* = (x_2^*, y_2^*, y_2^*)$ with*

$$x_1^* < \frac{1}{\sqrt{3}} < x_2^*.$$

Now focusing on studying the local behavior of the equilibria. The Jacobian matrix associated to system (1.1.4) at an arbitrary equilibrium denoted by (x, y, z) in the absence of time delay (which means that $\tau = 0$) is accorded as follows

$$J(x, y, z) = \begin{pmatrix} 1 - 3x^2 - \lambda & 0 & -1 \\ 0 & \alpha - \beta z - \lambda & -\beta y \\ \gamma z & \delta z & -m + \gamma x + \delta y - \lambda \end{pmatrix}. \tag{1.2.7}$$

Evaluating the Jacobian matrix at the extinction equilibrium E_0, we get the eigenvalues $1, \alpha, -m$, so that this equilibrium is unstable, which shows that the three population never goes to extinction. At E_1 we get instead $-2, \alpha, -m + \gamma$ which is also unstable.

Now focusing on determining the local stability of E_2. As it has been mentioned previously, the stability of this equilibrium can determine a huge biological significant and alow us to confirm the advantage

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of using herd behavior on solitary behavior as a defensive strategy. For the mathematical analysis, it is easy to see that the Jacobian matrix (1.2.7) evaluated at E_2 has one eigenvalue that it can be computed explicitly which is given as $\lambda_1 = \alpha - \beta\eta\left(1 - \eta^2\right)$. The other two are the roots of the quadratic equation,

$$\varphi(\lambda) = \lambda^2 - \left(1 - 3\eta^2\right)\lambda + m\left(1 - \eta^2\right) = 0,$$

coming from a 2×2 minor $\tilde{J}$ of (1.2.7), where

$$\tilde{J} = \begin{pmatrix} 1 - 3\eta^2 - \lambda & -1 \\ 0 & \alpha - \beta z - \lambda \\ \gamma\eta(1 - \eta^2) & -m + \gamma\eta - \lambda \end{pmatrix}. \tag{1.2.8}$$

The local stability of E_2 impose the following conditions

$$tr(\tilde{J}) = \left(1 - 3\eta^2\right) < 0, \qquad det(\tilde{J}) = m\left(1 - \eta^2\right) > 0.$$

which leads to

$$\frac{1}{\sqrt{3}} < \eta < 1. \tag{1.2.9}$$

Then, we must combine (1.2.9) with the negativity of the first eigenvalue of the Jacobian matrix λ_1. Thus, the local stability of E_2 holds if

$$\frac{1}{\sqrt{3}} < \eta < 1, \qquad \text{and} \qquad \rho < \eta\left(1 - \eta^2\right). \tag{1.2.10}$$

Finally, we focus on the coexistence equilibrium E_i^*, $i = 1, 2$. The characteristic equation of the Jacobian matrix evaluated at E_i^*, $i = 1, 2$ can be expressed as the following cubic equation

$$P_{E_i^*}(\lambda) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0, \ i = 1, 2, \tag{1.2.11}$$

where,

$$A_1 = 3x_i^* - 1, \quad A_2 = \alpha\left(\frac{\gamma}{\beta} + m - \gamma x_i^*\right), \quad A_3 = \alpha(m - \gamma x_{1,2}^*)(3x_i^* - 1), \ i = 1, 2. \tag{1.2.12}$$

Now, by applying the Routh–Hurwitz criterion, see p. 67 of [6], the stability of E_i^*, $i = 1, 2$ is guaranteed if

$$A_1 > 0, \quad A_1 A_2 - A_3 > 0, \quad A_3 > 0.$$

From (1.2.6) we conclude that the first equilibrium E_1^* is always unstable, and the second equilibrium E_2^* is always locally asymptotically stable, when it exists.

Equilibrium point	Feasibility conditions	Stability conditions
$E_0 = (0,0,0)$	—	Always unstable
$E_1 = (1,0,0)$	—	Always unstable
$E_2 = (\eta, 0, \eta(1-\eta^2))$	$\eta < 1$	$\dfrac{1}{\sqrt{3}} < \eta < 1$ and $\rho < \eta(1-\eta^2)$
$E_i^* = (x_i^*, y_i^*, z_i^*),\ i=1,2$	$\rho < \dfrac{2}{3\sqrt{3}}$ and	from (1.2.6), we obtain
where, $x_i^*,\ i=1,2$ is solutions of	$x_i^* < \min\{1,\eta\}, 1=1,2$	E_1^* is unstable
$x^3 - x + \rho = 0$, with		
$y_i^* = \dfrac{1}{\delta}(m - \gamma x_i^*),\ i=1,2$ and $z_i^* = \rho,\ i=1,2$		and E_2^* is stable

TABLE 1.1: Dynamical behavior of the system (1.1.4) around the equilibrium points for $\tau = 0$.

1.3 Analysis of the delayed model ($\tau > 0$)

In this section, we will study the effect of the time delay on the stability of the equilibriums and shows the importance of considering it in the model construction. Indeed, we will analyze the model system (1.1.4) in the presence of time lags. We already know that time lags will not affect the number of the equilibria of the system (**??**). So, all the equilibria will exist and will be the same as in the case of the absence of time delay. For studying the local stability of equilibria we compute the Jacobian matrix evaluated at an arbitrary equilibrium denoted by (x,y,z) for the delayed system (2.2.2), where

$$J(x,y,z) = \begin{pmatrix} 1 - x^2 - 2x^2 e^{-\lambda\tau} - \lambda & 0 & -1 \\ 0 & \alpha - \beta z - \lambda & -\beta y \\ \gamma z & \delta z & -m + \gamma x + \delta y - \lambda \end{pmatrix}. \qquad (1.3.1)$$

Now, computing the characteristic equation of (1.3.1) at (x,y,z), we obtain

$$\lambda^3 + m_1\lambda^2 + m_2\lambda + m_3 + e^{-\lambda\tau}(m_4\lambda^2 + m_5\lambda + m_6) = 0, \qquad (1.3.2)$$

where

$$m_1 = -1 - \alpha + m - \gamma x - \delta y + \beta z + x^2$$
$$m_2 = \alpha - \alpha m - m + \gamma(1+\delta)x + \delta(1+\alpha)y + (\gamma + \beta m + \beta)z - \beta\gamma xz + (m-\alpha)x^2 - \gamma x^3$$
$$\qquad - \delta x^2 y + \beta x^2 z,$$
$$m_3 = \alpha m - \alpha\gamma x - \alpha\delta y - \beta mz + \beta\gamma xz - \alpha yz - \alpha m x^2 + \alpha\gamma x^3 + \alpha\delta x^2 y + \beta m x^2 z + \beta yz^2 - \beta\gamma x^3 z,$$
$$m_4 = 2x^2,$$
$$m_5 = 2(\alpha - m)x^2 - 2\beta x^2 z + 2\gamma x^3 + 2\delta x^2 y,$$
$$m_6 = -2\alpha m x^2 + 2\alpha\delta x^2 y + 2\beta m x^2 z + 2\gamma x^3 - 2\beta\gamma x^3 z.$$

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Because the equilibrium points E_0, E_1 and E_1^* in absence of the time lags, we can deduce that these equilibria stays unstable in the presence of time delay. Now we place our main attention to examine the effect of the time delay on the linear stability of E_2 and the interior equilibrium E_2^* for the delayed system (1.1.4). For insuring the stability of equilibrium point, all the eigenvalues of the characteristic equation (1.3.2) must have negative real part. Now, we put our interest on seeking for the influence of the time lags on the stability of the equilibria. In the absence of the delay the characteristic equation (1.3.2) turns into

$$\lambda^3 + (m_1 + m_4)\lambda^2 + (m_2 + m_5)\lambda + m_3 + m_6 = 0. \tag{1.3.3}$$

By standard Routh–Hurwitz criterion (see [9]), all roots of Eq. (1.3.2) will have negative real parts if the following hypothesis hold

$$(H_1): m_1 + m_4 > 0, \quad m_3 + m_6 > 0, \quad (m_1 + m_4)(m_2 + m_5) > (m_3 + m_6). \tag{1.3.4}$$

If we assume that $\lambda = 0$ to be a solution of (1.3.2) lead to, $m_3 + m_6 = 0$, which is contradictory with the second condition given in (H_1). Therefore, we can claim that $\lambda = 0$ cannot be a solution of (1.3.2). Now, for some $\tau > 0$ if we consider $\lambda = i\omega, \quad \omega > 0$ to be a solution of (1.3.2), we get

$$-i\omega^3 - m_1\omega^2 + im_2\omega + m_3 + (\cos\omega\tau - i\sin\omega\tau)(-m_4\omega^2 + im_5\omega + m_6) = 0. \tag{1.3.5}$$

separating the real and imaginary parts, we obtain the following transcendental equations

$$m_3 - m_1\omega^2 + (m_6 - -m_4\omega^2)\cos\omega\tau + m_5\omega\sin\omega\tau = 0, \tag{1.3.6}$$

$$m_2\omega - \omega^3 + m_5\omega\cos\omega\tau - (m_6 - m_4\omega^2)\sin\omega\tau = 0. \tag{1.3.7}$$

Squaring and adding (1.3.6) and (1.3.7), and by putting $\omega^2 = v$, we define a polynomial

$$\Phi(v) \triangleq v^3 + pv^2 + qv + r = 0, \tag{1.3.8}$$

where,

$$p = m_1^2 - m_4^2 - 2m_2, \quad q = m_2^2 - m_5^2 + 2m_4m_6 - 2m_1m_3, \quad r = m_3^2 - m_6^2.$$

Lemma 1. *Putting $\Delta = p^2 - 3q$ then, for the the polynomial Eq. (1.3.8), we get the following affirmations.*

(H_2) *If $r < 0$, then, Eq. (1.3.8) has at least one positive root.*

(H_3) *If $r \geq 0$ and $\Delta \leq 0$, then, Eq. (1.3.8) has no positive roots.*

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(H_4) *If $r \geq 0$ and $\Delta > 0$, then, Eq. (1.3.8) has at least one positive root if and only if $v_1^* = \dfrac{-p + \sqrt{\Delta}}{3} > 0$*
and $\Phi(v_1^*) \leq 0$.

Proof. Since $\lim\limits_{t \to +\infty} \Phi(v) = +\infty$, the Eq. (1.3.8) has at least one positive root if $r < 0$. From Eq. (1.3.8), we obtain

$$\Phi'(v) = 3v^2 + 2pv + q.$$

clearly, if $\Delta \leq 0$ then, $\Phi'(v) \geq 0$ which means that $\Phi(v)$ is monotonously increasing for $x \geq 0$. Therefore, for $r > 0$ and $\Delta \leq 0$, Eq. (1.3.8) has no positive root for $x \in [0, +\infty)$. For $r \geq 0$ and $\Delta > 0$, (1.3.8) has two real roots

$$v_1^* = \frac{-p + \sqrt{\Delta}}{3}, \qquad v_2^* = \frac{-p - \sqrt{\Delta}}{3}.$$

From $\Phi''(v_1^*) = 2\sqrt{\Delta} > 0$ and $\Phi''(v_2^*) = -2\sqrt{\Delta} < 0$, we conclude that v_1^* and v_2^* are the local minimum and the local maximum of $\Phi(v)$, respectively. So if $p(v_1^*) \leq 0$ and from $\lim\limits_{t \to +\infty} \Phi(v) = +\infty$, $\Phi(0) = r \geq 0$, we know that the sufficiency is true which means that Eq. (1.3.8) has positive roots $v_0 \in [0, v_1^*]$. This completes the proof of Lemma **Lemma 1**. $\qquad \square$

Now, we may assume that Eq. (1.3.8) has at least one positive real root $\omega_0 = \sqrt{v_0}$, that is, the characteristic equation (1.3.2) has a pair of imaginary roots of the form $\pm\omega_0$. From (1.3.6) and (1.3.7), we have

$$\cos\omega_0\tau = \frac{-(m_5\omega_0^2(m_2 - \omega_0^2) + (m_3 - m_1\omega_0^2)(m_6 - m_4\omega_0^2))}{(m_6 - m_4\omega_0^2)^2 + (m_5\omega_0^2)^2}. \tag{1.3.9}$$

Now, τ_k corresponding to the positive value of ω_0 is expressed as follows:

$$\tau_k = \frac{1}{\omega_0} \arccos\left(\frac{-(m_5\omega_0^2(m_2 - \omega_0^2) + (m_3 - m_1\omega_0^2)(m_6 - m_4\omega_0^2))}{(m_6 - m_4\omega_0^2)^2 + (m_5\omega_0^2)^2} + 2k\pi\right), \quad \text{where} \quad k = 0, 1, 2... \tag{1.3.10}$$

Lemma 2. *([83]) Assume that $\Phi'(v_0) = 3v_0^2 + 2pv_0 + q \neq 0$ and the conditions in (H_1) are satisfied. lets $\lambda(\tau) = a(\tau) \pm i\omega(\tau)$ be the root of Eq. (1.3.2) with, $a(\tau_k) = 0$ and $\omega(\tau_k) = \omega_0$ for $k = 0, 1, 2...$, where τ_k defined in (1.3.10). Then, $\pm i\omega_0$ is a simple roots and the sign of $a'(\tau_k)$ is coincident with the sign of $\Phi'(v_0)$ where $v_0 = \omega_0^2$.*

Proof. Letting $\lambda = \lambda(\tau)$ be the root of Eq. (1.3.2). Substituting $\lambda(\tau)$ into Eq. (1.3.2) and differentiating both sides of Eq. (1.3.2) with respect to the time lags τ, we get

$$\frac{d\lambda}{d\tau} = \frac{\lambda(m_4\lambda^2 + m_5\lambda + m_6)e^{-\lambda\tau}}{3\lambda^2 + 2m_1\lambda + m_2 + (-\tau(m_4\lambda^2 + m_5\lambda + m_6) + 2m_4\lambda + m_5)e^{-\lambda\tau}}.$$

Then

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(3\lambda^2 + 2m_1\lambda + m_2)e^{\lambda\tau}}{\lambda(m_4\lambda^2 + m_5\lambda + m_6)} + \frac{(2m_4\lambda + m_5)}{\lambda(m_4\lambda^2 + m_5\lambda + m_6)} - \frac{\tau}{\lambda}.$$

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From (1.3.6)-(1.3.8), we arrive at

$$
\begin{aligned}
a'(\tau_k)^{-1} &= Re\left[\frac{(3\lambda^2+2m_1\lambda+m_2)e^{\lambda\tau}}{\lambda(m_4\lambda^2+m_5\lambda+m_6)}\right]+Re\left[\frac{(2m_4\lambda+m_5)}{\lambda(m_4\lambda^2+m_5\lambda+m_6)}\right] \\
&= \frac{1}{\Pi}\left[3\omega_0^6+2(m_1^2-m_4^2-2m_2)\omega_0^4+(m_2^2-m_5^2+2m_4m_5+2m_1m_3)\omega_0^2\right] \\
&= \frac{1}{\Pi}(3\omega_0^6+2p\omega_0^4+q\omega_0^2) \\
&= \frac{v_0}{\Pi}\Phi'(v_0),
\end{aligned}
$$

where, $\Pi = m_5^2\omega_0^2+m_6^2$. Notice that $\Pi > 0$ and $v_0 > 0$, which means that

$$
sign\left[a'(\tau_k)\right] = sign\left[\Phi'(v_0)\right].
$$

The proof of **Lemma 2** is completed. $\qquad\square$

Thus, from (1.3.10) and Lemma **Lemma 2**, we can state the following theorem

Theorem 2. *([83]) If $\Phi'(v_0) \neq 0$, then system (1.1.4) undergoes Hopf bifurcation at E_2 and E_2^* for $\tau = \tau_k$, where, $k = 0, 1, 2....$*

At the equilibrium point E_2 Eq. (1.3.2) turn into

$$
\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 + e^{-\lambda\tau}(A_4\lambda^2 + A_5\lambda + A_6) = 0, \tag{1.3.11}
$$

where

$$
p = A_1^2 - A_4^2 - 2A_2, \qquad q = A_2^2 - A_5^2 + 2A_4A_6 - 2A_1A_3, \qquad r = A_3^2 - A_6^2.
$$

The expression of τ_k, $k = 0, 1, 2, ...$ for E_2 is expressed by

$$
\tau_{E_2} = \frac{1}{\omega_0}\arccos\left(\frac{-(A_5\omega_0^2(A_2-\omega_0^2)+(A_3-A_1\omega_0^2)(A_6-A_4\omega_0^2))}{(A_6-A_4\omega_0^2)^2+(A_5\omega_0^2)^2}\right) \tag{1.3.12}
$$

and

$$
\begin{aligned}
A_1 &= -1+\alpha+\eta(1-\eta^2)+\eta^2, \\
A_2 &= \alpha-\alpha m+\delta m+(1-\eta^2)(\beta\eta^3-\beta\gamma\eta^2+\eta(\gamma+\beta+\beta m))+(m-\alpha)\eta^2-\gamma\eta^3, \\
A_3 &= \alpha\gamma\eta^3-\alpha m\eta^2+(1-\eta^2)(-\beta\gamma\eta^4+\beta m\eta^3+\beta\gamma\eta^2-\beta m\eta), \\
A_4 &= 2\eta^2, \\
A_5 &= 2(\alpha-m)\eta^2-2\beta\eta^3(1-\eta^2)+2\gamma\eta^3, \\
A_6 &= -2\alpha m\eta^2+2\beta m\eta^3(1-\eta^2)+2\gamma\eta^3-2\beta\gamma\eta^4(1-\eta^2), \\
&\quad\text{with } \eta = \frac{m}{\gamma}
\end{aligned}
$$

From previous calculations, we have the following aspects on the stability of the equilibrium E_2

Theorem 3. *([83]) Suppose that (H_1) is satisfied, then we conclude that*

(1)- If $r \geq 0$ and $\Delta \leq 0$, all roots of Eq. (1.3.2) have negative real parts for all $\tau > 0$ and hence the equilibrium point E_2 of system (1.1.4) is asymptotically stable for all $\tau > 0$.

(2)- If either $r < 0$ or $r \geq 0$, $\Delta > 0$, $v_1^* > 0$ and $\Phi(v_1^*) \leq 0$ hold, then $\Phi(v)$ has at least one positive root and E_2 is is asymptotically stable for all $\tau \in [0, \tau_{E_2})$.

(3)- If all conditions as stated in (2) and $\Phi'(v_0) \neq 0$ holds, the sytem (1.1.4) undergoes a Hopf bifurcation at the equilibrium point E_2 for $\tau = \tau_{E_2}$.

Similarly at equilibrium point E_2^*, the Eq. (1.3.2) turns into

$$\lambda^3 + B_1\lambda^2 + B_2\lambda + A_3 + e^{-\lambda\tau}(B_4\lambda^2 + B_5\lambda + B_6) = 0, \tag{1.3.13}$$

where

$$p = B_1^2 - B_4^2 - 2B_2, \qquad q = B_2^2 - B_5^2 + 2B_4B_6 - 2B_1B_3, \qquad r = B_3^2 - B_6^2.$$

$$\tau_{E_2^*} = \frac{1}{\omega_0} \arccos\left(\frac{-B_5\omega_0^2(B_2 - \omega_0^2) + (B_3 - B_1\omega_0^2)(B_6 - B_4\omega_0^2))}{(B_6 - B_4\omega_0^2)^2 + (B_5\omega_0^2)^2}\right) \tag{1.3.14}$$

and

$$B_1 = -1 + m - \gamma\eta + x_2^*,$$
$$B_2 = m + \gamma\eta(1 + \alpha) + \alpha\frac{\gamma}{\beta} + (m - \alpha - \gamma\eta)(x_2^*)^2 + \gamma(\delta - \alpha)x_2^*,$$
$$B_3 = -\alpha\gamma(\eta - x_2^*) - \alpha\rho y_2^* + \beta\rho(z_2^*)^2 + \alpha\gamma(\eta - x_2^*)((x_2^*)^2 - 1),$$
$$B_4 = 2(x_2^*)^2,$$
$$B_5 = 2(\gamma\eta - m)(x_2^*)^2,$$
$$B_6 = 2\alpha\gamma \ (\eta - x_2^*),$$

$$\text{with } \eta = \frac{m}{\gamma}, \qquad \rho = \frac{\alpha}{\beta}.$$

Then, for the interior equilibrium E_2^*, we set the following results

Theorem 4. *([83]) Suppose that H_1 is satisfied, then we get*

(1)- If $\Delta \leq 0$, all roots of Eq. (1.3.2) have negative real parts for all $\tau > 0$ and hence the equilibrium point E_2 of system (1.1.4) is asymptotically stable for all $\tau > 0$.

(2)- If $\Delta > 0$, $v_1^* > 0$ and $\Phi(v_1^*) \leq 0$ hold, then $\Phi(v)$ has at least a positive root and E_2^* is asymptotically stable for all $\tau \in [0, \tau_{E_2^*})$.

(3)- If all conditions as stated in (2) and $\Phi'(v_0) \neq 0$ hold, (1.1.4) undergoes a Hopf bifurcation at the equilibrium point E_2 for $\tau = \tau_{E_2^*}$.

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1.4 Stability and direction of the Hopf bifurcation Around E_2^*

In this section, we investigation the direction and the stability of the periodic solutions bifurcating from a stable equilibrium E_2^*. Following the ideas developed by **Hassard** in [43], we derive the explicit formula to determine the properties of the Hopf bifurcation at the critical value τ_k by using normal form on the center of manifold theory and center manifold reduction. Without loss of generality, we denote any one of the critical values τ_k, $k = 0, 1, 2...$ by τ^* at which the system (1.1.4) undergoes a Hopf bifurcation at E_2^*. We translate the interior equilibrium E_2^* into the origin by the translation $u_1(t) = x(t) - x_2^*$, $u_2(t) = y(t) - y_2^*$, $u_3(t) = z(t) - z_2^*$ and normalizing the delay τ by the time scaling $t \longrightarrow \frac{t}{\tau}$, then system (1.1.4) is transformed into:

$$
\begin{cases}
\dot{u}_1 = \tau a_{11} u_1 + \tau a_{12} u_1(t-1) + \tau a_{13} u_3 + \tau a_{14} u_1^2(t-1) + \tau a_{15} u_1 u_1(t-1) + \tau a_{16} u_1^2 u_1(t-1), \\
\dot{u}_2 = \tau a_{21} u_3 + \tau a_{22} u_2 u_3, \\
\dot{u}_2 = \tau a_{31} u_1 + \tau a_{32} u_2 + \tau a_{33} u_3 + \tau a_{34} u_1 u_3 + \tau a_{35} u_2 u_3.
\end{cases}
\tag{1.4.1}
$$

where, $a_{11} = 1 - (x_2^*)^2$, $a_{12} = 2(x_2^*)^2$, $a_{13} = -1$, $a_{14} = -x_2^*$, $a_{15} = -2x_2^*$, $a_{16} = -1$, $a_{21} = -\beta y_2^*$, $a_{22} = -\beta$, $a_{31} = \gamma\rho$, $a_{32} = \delta\rho$, $a_{33} = \gamma x_2^* + \delta y_2^* - m$, $a_{34} = \gamma$, $a_{35} = \delta$. Let $\tau = \tau^* + \mu$ where $\mu \in \mathbb{R}$, then $\mu = 0$ is a Hopf bifurcation value of the system (1.4.1). For simplification of notations, we rewrite (1.4.1) in $C = C([-1, 0], \mathbb{R}_+^3)$ as

$$
u'(t) = L_\mu(u_t) + F(\mu, u_t).
\tag{1.4.2}
$$

where, $u(t) = (u_1(t), u_2(t), u_3(t))^T \in \mathbb{R}^3$, $u_t(\theta) \in C$ is defined by $u_t(\theta) = u(\theta + t)$ and $L_\mu : C \longrightarrow \mathbb{R}^3$, $F : \mathbb{R} \times C \longrightarrow \mathbb{R}^3$ are given, respectively, by

$$
L_\mu \phi = (\tau^* + \mu) A_1 \phi(0) + (\tau^* + \mu) A_2 \phi(-1),
$$

where A_1 and A_2 are defined as follows:

$$
A_1 = \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & 0 & a_{21} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \qquad A_2 = \begin{bmatrix} a_{12} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\tag{1.4.3}
$$

and

$$
F(\mu, \phi) = (\tau^* + \mu) \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix},
\tag{1.4.4}
$$

with,

$$
\begin{aligned}
F_1 &= a_{14}\phi_1^2(-1) + a_{15}\phi_1(0)\phi_1(-1) + a_{16}\phi_1(0)^2\phi_1(-1), \\
F_2 &= a_{22}\phi_2(0)\phi_3(0), \\
F_3 &= a_{34}\phi_1(0)\phi_3(0) + a_{35}\phi_2(0)\phi_3(0),
\end{aligned}
$$

$\phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta))^T \in C([-1, 0], \mathbb{R})$. From the discussion in **Sec.1.3**, we know that system (1.1.4) undergoes Hopf bifurcation at E_2^* for $\mu = 0$, and the associated characteristic equation of system (1.1.4) with $\mu = 0$ has a pair of simple imaginary roots $\pm i\omega_0$. By the Riesz representation theorem [46, 72], there exists a function $\varphi(\theta, \mu)$ of bounded variation for $\theta \in [-1, 0]$, such that

$$L_\mu \phi = \int_{-1}^0 d\varphi(\theta, \mu)\phi(\theta), \qquad \forall \phi \in C.$$

In fact, we can choose

$$\varphi(\theta, \mu) = \begin{cases} A_1, & \theta = 0, \\ A_2, & \theta \in [-1, 0), \\ A_1 \xi(\tau^* + \mu), & \theta \in [-\tau^*, -1), \end{cases} \tag{1.4.5}$$

where ξ is the Dirac function. For $\phi \in C([-1, 0], \mathbb{R}_+^3)$, we set

$$B(\mu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \displaystyle\int_{-1}^0 d\varphi(\theta, \mu)\phi(\theta), & \theta = 0 \end{cases} \tag{1.4.6}$$

and

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-1, 0), \\ F(\mu, \phi), & \theta = 0. \end{cases} \tag{1.4.7}$$

Then the system (1.4.1) can be rewritten as

$$u_t' = B(\mu)u_t + R(\mu)u_t. \tag{1.4.8}$$

For $\psi \in C'([-1, 0], \mathbb{R}_+^3)$, define:

$$B^*\psi(s) = \begin{cases} \dfrac{d\psi(s)}{ds}, & s \in [-1, 0), \\ \displaystyle\int_{-1}^0 d\varphi^T(-t, 0)\psi(-t), & s = 0 \end{cases} \tag{1.4.9}$$

and bilinear inner product is given by

$$\langle \psi(s), \phi(\theta) \rangle = \overline{\psi}^T(0)\phi(0) - \int_{-1}^0 \int_{\nu=0}^\theta \overline{\psi}(\nu - \theta)d\varphi(\theta)\phi(\theta)d\nu. \tag{1.4.10}$$

We know that B^* and $B = B(0)$ are adjoint operators. By the discussion in **Sec.1.4**, we know that $i\omega_0$ are eigenvalues of $B(0)$, which are eigenvalues of B^*. Suppose that $q(\theta) = q(0)e^{i\omega_0\theta}$ is an eigenvector of $B(0)$ corresponding to the eigenvalue $i\omega_0$. Then, $B(0) = i\omega_0 q(\theta)$. For $\theta = 0$, we obtain

$$\left[i\omega_0 I - \int_{-1}^0 d\varphi(\theta)e^{i\omega_0\theta} \right] q(0) = 0,$$

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which yields to $q(0) = (1, a, b)^T$, where

$$\varepsilon_1 = \frac{a_{21}}{i\omega_0}, \tag{1.4.11}$$

$$\pi_1 = \frac{a_{31}}{-a_{32} - a_{33} + i\omega_0}. \tag{1.4.12}$$

Similarly, it can be verified that $q^s = D(1, a^*, b^*)e^{i\omega_0 s}$ is the eigenvector of B^* corresponding to $-i\omega_0$, where

$$\varepsilon_2 = -\frac{a_{21}}{i\omega_0}, \tag{1.4.13}$$

$$\pi_2 = \frac{-a_{31}}{a_{32} + a_{33} + i\omega_0}. \tag{1.4.14}$$

By (1.4.10), we get

$$
\begin{aligned}
\langle q^*(s), q(\theta)\rangle &= \overline{D}(1, \overline{\varepsilon_2}, \overline{\pi_2})(1, \varepsilon_1, \pi_1)^T - \int_{-1}^{0}\int_{\nu=0}^{\theta}(1, \overline{\varepsilon_2}, \overline{\pi_2})e^{-i\omega_0(\nu-\theta)}d\varphi(\theta)(1, \varepsilon_1, \pi_1)^T e^{i\omega_0\nu}d\nu, \\
&= 1 + \varepsilon_1\overline{\varepsilon_2} + \pi_1\overline{\pi_2} - \int_{-1}^{0}(1, \overline{\varepsilon_2}, \overline{\pi_2})\theta e^{i\omega_0\theta}d\varphi(\theta)(1, \varepsilon_1, \pi_1)^T d\nu, \\
&= 1 + \varepsilon_1\overline{\varepsilon_2} + \pi_1\overline{\pi_2} + \tau^* a_{12}e^{-i\omega_0\tau^*}.
\end{aligned}
$$

We can choose

$$\overline{D} = \frac{1}{1 + \varepsilon_1\overline{\varepsilon_2} + \pi_1\overline{\pi_2} + \tau^* a_{12}e^{-i\omega_0\tau^*}},$$

so that, $\langle q^*(s), q(\theta)\rangle = 1$, $\langle q^*(s), \overline{q}(\theta)\rangle = 0$. Following the algorithms developed in [43] and using a computation process similar to that in [65, 71], we obtain the coefficients that are crucial in determining the nature of the periodic solutions:

$$
\begin{aligned}
g_{20} &= 2\overline{D}(H_{11} + \overline{\varepsilon_2}H_{31} + \overline{\pi_2}H_{21}), \\
g_{11} &= \overline{D}(H_{12} + \overline{\varepsilon_2}H_{22} + \overline{\pi_2}H_{32}), \\
g_{02} &= 2\overline{D}(H_{13} + \overline{\varepsilon_2}H_{23} + \overline{\pi_2}H_{33}), \\
g_{21} &= 2\overline{D}(H_{14} + \overline{\varepsilon_2}H_{24} + \overline{\pi_2}H_{34}),
\end{aligned}
$$

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where,

$$H_{11} = a_{16}e^{-i\omega_0\tau^*} + a_{15}e^{-2i\omega_0\tau^*},$$

$$H_{12} = 2a_{15} + a_{16}e^{-i\omega_0\tau^*} + a_{16}e^{i\omega_0\tau^*},$$

$$H_{13} = a_{16}e^{i\omega_0\tau^*} + a_{15}e^{2i\omega_0\tau^*},$$

$$H_{14} = 2a_{17} + a_{17}e^{-2i\omega_0\tau^*} + a_{16}e^{-i\omega_0\tau^*}N_{11}^{(1)}(0) + a_{16}N_{11}^{(1)}(-1) + 2a_{15}e^{-i\omega_0\tau^*}N_{11}^{(1)}(-1) + \frac{1}{2}a_{16}e^{i\omega_0\tau^*}N_{20}^{(1)}(0),$$

$$+ \frac{a_{16}N_{20}^{(1)}(-1)}{2} + a_{15}e^{i\omega_0\tau^*}N_{20}^{(1)}(-1),$$

$$H_{21} = a_{22}\varepsilon_1,$$

$$H_{22} = a_{22}\varepsilon_1 + a_{22}\overline{\varepsilon_1},$$

$$H_{23} = a_{22}\overline{\varepsilon_1},$$

$$H_{24} = a_{22}N_{11}^{(2)}(0) + \frac{a_{22}N_{20}^{(2)}(0)}{2} + a_{22}N_{11}^{(2)}(0)\varepsilon_1 + \frac{a_{22}N_{20}^{(2)}(0)\overline{\varepsilon_1}}{2},$$

$$H_{31} = a_{33}\pi_1 + a_{34}\pi_1^2,$$

$$H_{32} = a_{33}\pi_1 + a_{33}\overline{\pi_1} + 2a_{34}\pi_1\overline{\pi_1},$$

$$H_{33} = a_{33}\overline{\pi_1} + a_{34}\overline{\pi_1}^2,$$

$$H_{34} = a_{33}N_{11}^{(3)}(0) + \frac{a_{33}N_{20}^{(3)}(0)}{2} + a_{33}N_{11}^{(3)}(0)\pi_1 + 2a_{34}N_{11}^{(3)}(0)\pi_1 + \frac{a_{33}N_{20}^{(3)}(0)\overline{\pi_1}}{2} + a_{34}N_{20}^{(3)}(0)\overline{\pi_1}$$

$$N_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau^*}q(0)e^{i\omega_0\tau^*\theta} + \frac{i\overline{g}_{20}\overline{q}(0)}{3\omega_0\tau^*}e^{-i\omega_0\tau^*\theta} + V_1e^{2i\omega_0\tau^*\theta},$$

$$N_{11}(\theta) = \frac{-ig_{11}}{\omega_0\tau^*}q(0)e^{i\omega_0\tau^*\theta} + \frac{i\overline{g}_{11}\overline{q}(0)}{3\omega_0\tau^*}e^{-i\omega_0\tau^*\theta} + V_2,$$

where V_1 and V_2 are both three-dimensional vectors, and can be determined by:

$$\begin{bmatrix} 2i\omega_0 - a_{11} - a_{12}e^{-2i\omega_0\tau^*} & -a_{13} & -a_{14} \\ -a_{21} & 2i\omega_0 & 0 \\ -a_{31} & 0 & 2i\omega_0 - a_{32} \end{bmatrix} V_1 = -2 \begin{bmatrix} H_{11} \\ H_{21} \\ H_{31} \end{bmatrix}$$

and

$$\begin{bmatrix} a_{11}+a_{12} & a_{13} & a_{14} \\ a_{21} & 0 & 0 \\ a_{31} & 0 & a_{32} \end{bmatrix} V_2 = -2 \begin{bmatrix} H_{12} \\ H_{22} \\ H_{32} \end{bmatrix}.$$

Then g_{21} can be expressed entirely by the system parameters. In fact, each g_{ij} can be determined by the parameters. Thus we can compute the following crucial quantities:

$$C_1(0) = \frac{i}{2i\omega_0\tau^*}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2},$$

$$\mu_2 = -\frac{Re\{C_1(0)\}}{Re\{\lambda'(\tau^*)\}},$$

$$\beta_2 = Re\{C_1(0)\},$$

$$T_2 = -\frac{Im\{C_1(0)\} + \mu_2 Im\{\lambda'(\tau^*)\}}{\tau^* \omega_0},$$

which determines the nature and the stability of the periodic solutions at the critical value τ^*,i.e, μ_2 determines the direction of Hopf-bifurcation. By the results of Hassard *et al.* [43], we have the following theorem.

Theorem 5. *([83]) If $\mu_2 > 0$ (respectively $\mu_2 < 0$), then the Hopf bifurcation is supercritical (respectively, subcritical) in this case the bifurcating periodic solutions exists for $\tau > \tau^*$ (respectively, $\tau < \tau^*$), β_2 determines the stability of bifurcating periodic solutions: the bifurcating periodic solutions are orbitally asymptotically stable (respectively, unstable) if $\beta_2 < 0$ (respectively, $\beta_2 > 0$), T_2 determines the period of the bifurcating periodic solutions: the period increases (respectively, decreases) if $T_2 < 0$ (respectively, $T_2 > 0$).*

1.5 Numerical simulations

For confirming the obtained mathematical results, we split our section into two subsections. The first is for verifying the dynamics presented in the absence of time lags, and the second for ensuring the results in the presence of the time delay.

1.5.1 Without delay ($\tau = 0$)

For $\alpha = 0.09$, $\beta = 9.45$, $\gamma = 0.81$, $\delta = 9.45$, $m = 0.7$, we obtain $\rho = 0.0952, \eta = 0.8642$, $E_2 = (0.8642, 0.0000, 0.2188)$, $E_2^* = (0.9952, -0.0112, 0.0952)$ and $E_1^* = (0.0095, -0.0733, 0.0952)$ the two interior equilibriums does not exists, $\rho = 0.0952 < \eta(1-\eta^2) = 0.2188$ according to the **Tab 1.1** it follows that E_2 is locally asymptotically stable. (see **Fig.1.1**)

At first we consider he values $\alpha = 3.1$, $\beta = 10.5$, $\gamma = 0.81$, $\delta = 0.9$, $m = 0.7$, we get $\rho = 0.2952 > \eta(1-\eta^2) = 0.2188$, $\eta = 0.8642$ then, $E_2 = (0.8642, 0.0000, 0.2188)$ is unstable. The first interior equilibrium $E_1^* = (0.3317, 0.4795, 0.2952)$ is unstable where the second $E_2^* = (0.7920, 0.0650, 0.2952)$ is locally asymptotically stable. (see **Fig.2.1**)

1.5.2 With delay ($\tau > 0$)

We take the following values $\alpha = 0.43$, $\beta = 19.1$, $\gamma = 0.7$, $\delta = 7.6$, $m = 0.75, \tau = 0.71$, we get $\rho = 0.0149$, $\eta = 0.5556$, $\tau_{E_2} = 0.734711$. Since, $\tau < \tau_{E_2}$, according to **Theorem 3** E_2 is locally asymptotically stable (see **Fig.1.3** and the system (1.1.4) undergoes a Hopf bifurcation at E_2 for $\tau = \tau_{E_2} = 0.734711$. (see **Fig.1.4**).
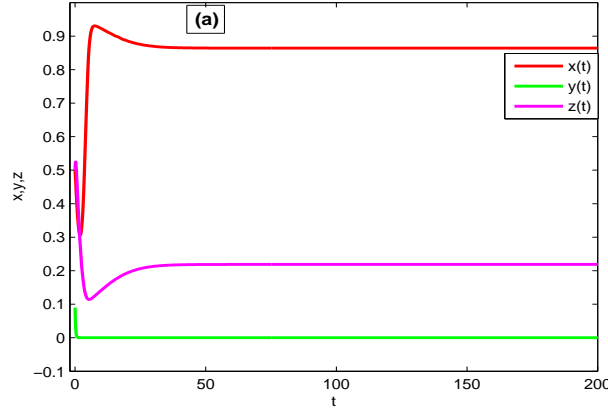
FIGURE 1.1: trajectory and phase portrait of the solution of the system (1.1.4), with $\alpha = 0.09$, $\beta = 9.45$, $\gamma = 0.81$, $\delta = 9.45$, $m = 0.7$. **(a)** time series behavior of $(x(t), y(t), z(t))$. **(b)** phase space behavior of the system (1.1.4) showing that E_2 is locally asymptotically stable for $x(0) = 0.5$, $y(0) = 0.09$ and $z(0) = 0.5$.

Finally, we take $\alpha = 3.5$, $\beta = 20.1962$, $\gamma = 0.81$, $\delta = 4.0392$, $m = 3.5$, $\tau = 0.924$, implies that $\rho = 0.1733$, $\eta = 8.7500$, $\tau_{E_2^*} = 0.920973$. $\tau > \tau_{E_2^*}$, according to **Theorem 4** E_2^* is locally asymptotically stable (see **Fig.1.5**) and the system (1.1.4) undergoes a Hopf bifurcation at $\tau_{E_2^*} = 0.921973$ as depicted in **Fig.1.6**. Further, from the above process, we can determine the stability and direction of periodic solutions bifurcating from the positive equilibrium at the critical point τ_k. For instance, for $\tau = \tau_{E_2^*} = 0.921973$, $C_1(0) = -0.565364 + 3.15609i$. From **Section 1.4**, it follows that $\mu_2 > 0$ and $\beta_2 < 0$. Therefore, the bifurcation takes place when τ crosses $\tau_{E_2^*}$ to the right $\tau > \tau_{E_2^*}$, and the corresponding periodic orbits are asymptotically stable.

1.6   Discussion

We provided in this research a new approximation of predator-prey interaction where one-predator and two-prey intermingling is considered. The first prey is assumed to have a social behavior and the other have a solitary behavior. In the absence of the predator, it has been presumed that the first population of prey reproduces in a logistic manner, and the growth of the second is assume to be in a Malthusian manner. In the absence of time lags, and from the linear stability analysis of the equilibrium points, it is concluded that for $1/\sqrt{3} < \eta < 1$ and $\rho < \eta(1 - \eta^2)$, the social prey with the predator will survive and the second population of prey will extinct this result appears clearly from **Fig.1.1** and **Fig.1.3**. In this case,
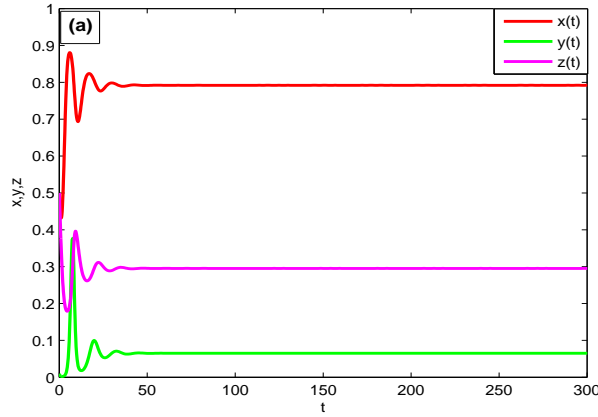
FIGURE 1.2: The local stability of E_2^*, with $\alpha = 3.1$, $\quad \beta = 10.5$, $\quad \gamma = 0.81$, $\quad \delta = 0.9$, $\quad m = 0.7$. **(a)** time series behavior of $(x(t), y(t), z(t))$. **(b)** phase space behavior of the system (1.1.4) showing that is locally asymptotically stable where E_1^* and E_2 are unstable. Here $x(0) = 0.46$, $y(0) = 0.01$ and $z(0) = 0.5$.
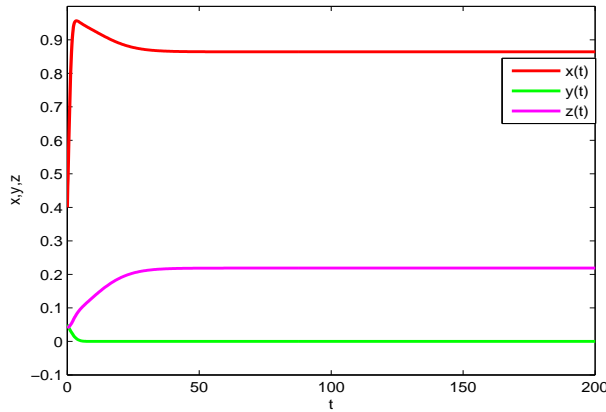

FIGURE 1.3: Numerical simulation of the system (1.1.4), with $\alpha = 0.43$, $\quad \beta = 19.1$, $\quad \gamma = 0.7$, $\quad \delta = 7.6$, $\quad m = 0.75, \tau = 0.71$, implies $\quad \tau_{E_2} = 0.734711$, $\tau < \tau_{E_2}$. Then E_2 is locally asymptotically stable. Here the initial value is $(0.44, 0.03, 0.04)$

the solitary population can not resist even though its growth is Malthusian (exponential) and goes to extinction. Therefore, it can be concluded that the predator prefers the individual prey and finds some
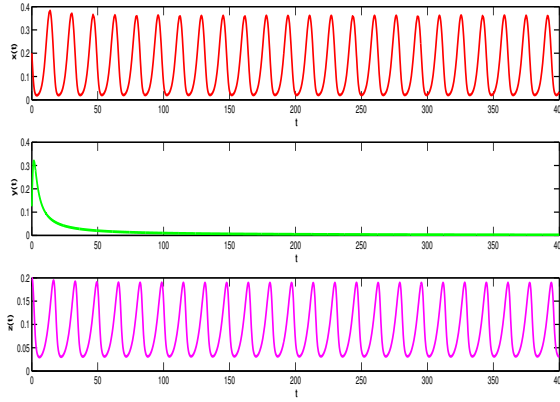
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FIGURE 1.4: The existence of periodic solutions of the system (1.1.4), with $\alpha = 0.43$, $\beta = 19.1$, $\gamma = 0.7$, $\delta = 7.6$, $m = 0.75, \tau = 0.83$, implies $\tau_{E_2} = 0.734711$, $\tau > \tau_{E_2}$. E_2 loses its stability and a Hopf-bifurcation occurs. **(a)** time series behavior of $(x(t), y(t), z(t))$, **(b)** phase space behavior of the system (1.1.4). Here the initial value is $(0.2, 0.1, 0.2)$.
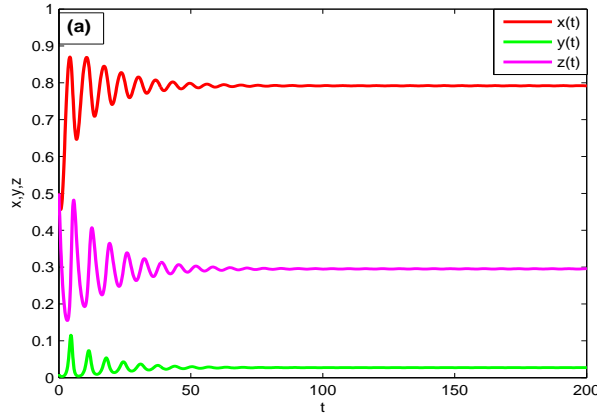
FIGURE 1.5: The local stability of E_2^*, with $\alpha = 3.1$, $\beta = 10.5$, $\gamma = 0.81$, $\delta = 9.45$, $m = 0.9, \tau = 0.89$, implies $\rho = 0.2952$, $\eta = 1.1111$, $\tau_{E_2^*} = 0.921973$. $\tau < \tau_{E_2^*}$. Then E_2^* is locally asymptotically stable. **(a)** time series behavior of $(x(t), y(t), z(t))$, **(b)** phase space behavior of the system (1.1.4). Here the initial value is $(0.5, 0.01, 0.5)$.

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FIGURE 1.6: Numerical simulation of the system (1.1.4), with $\alpha = 3.5$, $\beta = 20.1962$, $\gamma = 0.81$, $\delta = 4.0392$, $m = 3.5, \tau = 0.924$, implies $\rho = 0.1733$, $\eta = 8.7500$, $\tau_{E_2^*} = 0.920973$. $\tau > \tau_{E_2^*}$. Then E_2^* loses its stability and a Hopf bifurcation occurs. **(a)** time series behavior of $(x(t), y(t), z(t))$, **(b)** phase space behavior of the system (1.1.4). Here the initial value is $(0.1, 0.01, 0.05)$.
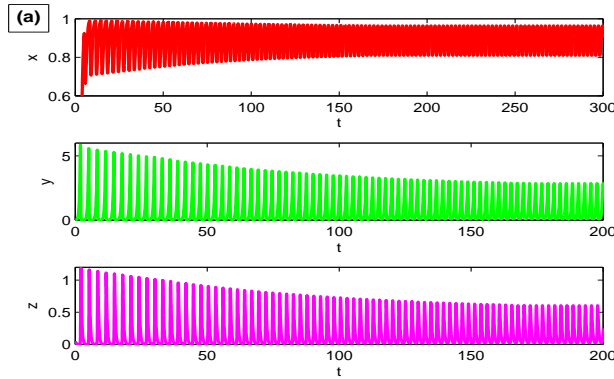
difficulty with the organized prey that lives in herds. The most interesting situation for ecologists is to conserve all species. For our model, if the equilibrium E_2^* exists then it is stable, where in this case we can avoid extinction for each species this condition is $\rho < 2/3\sqrt{3}$ and $x_2^* < \min\{1, \eta\}$. This consequence can be seen clearly from **Fig.2.1** and **Fig.1.5**, for the biological meaning, we can highlight that the solitary exists but in small densities which agrees with the real life situation. In fact, if we consider the species that is in the danger of prey extinction we can see that the most part of them have a solitary behavior which can claims the big help that the group provides for the surviving of this species (prey). It has been mentioned at **Section 1.1** that the delay can give a very good precision. Avoiding using the time lags can lead to losing the ecological precision of the ecological results. In fact, **Section 1.3** is devoted to study the misleading that can be obtained by neglecting the time lags, where a stable equilibrium in the absence of the time delay can becomes unstable, even can lead to a Hopf bifurcation. A suitable condition are provided in **Theorem 4**, where **Fig.1.4** and Fig. **Fig.1.6**. For the stability of the periodic solution generated by the presence of Hopf bifurcation on a stable equilibrium E_2^* where section **Section 1.4** is offered to show this result. In the next section we will give ecological version of our study using the affirmation used in this section and other claims which are going to be helpful in the real world.

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1.7 Concluding and remarks

In this chapter, we dealt with a comparative analysis between two contradictory defensive strategies for two type of prey. The first prey population has a social behavior. This behavior has huge benefits for this type of animals, where the predator don't dare to attack a prey on a moving herd directly. But it uses other methods to hunt the prey such as creating panicking on the herd by pursuing it for long time or attacking this moving pack from different angles. The purpose of these attacks is to divide the herd which means that the bond between the herd members is very dangerous for the predator population. In the real world, there are many predators dies for the reason of prey herd and the best example that we can consider is the buffalos, where it defends for each other brutally. Also to mention that this behavior has also other negative points such as the big feasibility, where in Africa, the predators can see the predators from miles away mostly for the prey that immigrates with a very large numbers such as gnus and buffaloes. Also in the prey herd peregrination the predator follows its prey, which makes a confusion what if the social behavior of the prey is the best strategy for the prey surviving. On the other hand, the prey with the solitary behavior has the advantage of hiding it self from a predator, and it is capable to move without being seen by predator. In general, these type of prey has a big capacity of adaptation, where the solitary behavior (small number or the big space between two solitary prey) can help in obtaining the resources very quickly at the opposite with the grouped prey (the inside prey may suffers for getting enough food mostly in the drey season). But getting enough resources is not always an issue, avoiding predation and safety is also very important for these kind of animals. This remarks make a confusion for detecting the best strategy is better for the prey population can considers.

Based on this remarks we built a model considers the intermingling of two types of preys and one predator and for the purpose of selecting the best behavior that the prey can considers. Indeed, after our mathematical analysis it is shown that the surviving of the grouped prey only without the solitary prey and the predator is not possible (which means that E_1 is always unstable). The same remark can be obtained from the extinction of the three species. This result mean that the grouped prey cannot be extinct which gives an advantage of the grouped prey on the solitary prey. Further, it has been proved that the solitary-prey-free equilibrium can be stable. This result has a huge ecological implication, for the right conditions (the condition of the stability of the solitary-prey-free equilibrium E_2) the grouping behavior is much more appropriate for the prey surviving, which confirms partially with the known idea of the advantage of using herd behavior on the solitary behavior. Furthermore, it is also proved that there exists two interior equilibriums, the first is unstable (E_1^*), and the other is stable where ever exists. For subject of conserving all the species it better to provide the right condition for avoiding any extinction (the existence condition of the interior equilibrium E_2^*). Also, to mention that we considered the time

spent without any food for the grouped prey (time delay) where we proved that neglecting this parameter can leads to unbalance between the mathematical predicting and the results on the real world, where we proved that the time delay can destabilize a stable equilibrium. By putting all together we offered a model that allows us to compar between two contradictory behaviors for the prey population where a very good anticipating results provided which agrees with the real world situation. To close, the prey social behavior is very important for the surviving of the species of it gender, in fact the most numerous population in earth are the one that exhibits this behavior we mention as example buffalos, gnu, giant honeybees, ants for the terrestrial prey animals, and sardine and tuna for the aquatic animals which shows its importance for the ecological diversity of our planet.

Chapter 2

Shape effects on herd behavior in predator-prey model with Gompertz growth function

In this chapter, we are interested in the analysis of a **predator-prey model** with **herd behavior** and the **Gompertz growth function** for the prey population. In this model we introduce a new functional response **[14]** which generalizes the interaction between the prey and the predator in the differents **herd shapes** formed by the prey (in $2D$ and $3D$ **herd shapes**). The aim of this chapter is to see the critical impact on the densities of two species when the shape of herd behavior changes.

The present chapter is organized as follows: **Section 2.1** is devoted to the formulation of the mathematical model. In **Section 2.2**, we perform some basic analysis for the model (2.2.2), including the boundedness of the solution, the local stability analysis of constant steady states. Moreover, we analyze the behaviors of the model system near the origin. **Section 2.3**, the effect of the rate of **herd shape** on the interior equilibrium has been investigated. In **Section 2.4**, we obtain the global asymptotical stability of the boundary equilibrium and the interior equilibrium. The existance of Hopf bifurcation will be shown in **Section 2.5**. To illustrate the effectiveness of our theoretical results, some numerical simulations are provided in **Section 2.6**. In next **Section** we return to the original model to project the results obtained on the simplified model and give some remarks. Finally, conclusions and comments are given in the last **Section** to close this chapter.

2.1 Presentation of the model

Recently, in [14] Venturino et *al.* introduce a new functional response which generalizes the interaction between the prey and the predator in both cases $2D$ and $3D$ **herd shapes** with a new parameter α which represents the exponent of the prey population in the interaction term with the predators. The model introduced by [14] can be written as the following forme

$$
\begin{cases}
\frac{dX(\tau)}{d\tau} = rX(\tau)\left(1 - \frac{X(\tau)}{k}\right) - aX(\tau)^\alpha Y(\tau), \\
\\
\frac{dY(\tau)}{d\tau} = -mY(\tau) + \eta aX(\tau)^\alpha Y(\tau).
\end{cases}
\tag{2.1.1}
$$

We denote by X and Y the densities of the prey and the predator, respectively. The variable τ represents time, r is the intrinsic growth rate. k is the environment carrying capacity for the prey. m represents the natural mortality rate for the predators. a stands for the predation rate of the prey population. η is the conversion rate of the prey density to a predator density.

The new parameter $0 < \alpha < 1$ represents the rate of the herd shape formed by the prey, we can easily remark that the model is widely general enough to accommodate even the case in which the prey can gather in shapes possessing a fractal geometry. For the particular case $\alpha = 1$, we don't have a herd behavior and the predator can be interact with any individual of the prey population (the classical model of Lotka and Volterra). In $2D$ space with herd behavior we obtain $\alpha = 1/2$ where the authors [2] have obtained significant results by using the Poincaré-Bendixson theorem and a technical based on the decomposition of the phase plan into regions. Moreover, for the case of $3D$ herd shape [14], when the prey makes a regular shape such as a cube, the quantity of the captured prey by one predator will be proportional to $X^{2/3}$, which gives a better explaining the value of α. We can conclude that the model (2.1.1) generalizes any form of interaction between the predator and the population of prey grouped in herds in $2D$ and $3D$.

Moreover, the mathematician **Benjamin Gompertz** established for the first time the function of Gompertz in 1850, this function can be translated into Gompertz differential equation $\frac{dy}{dx} = ry\ln(k/y))$ [36, 92], as an alternative approach, it has been proven to be a simple example to produce asymmetrical types of S-shaped curves [94], instead of the logistic function $y = \frac{k}{1-e^{a-rx}}$, (corresponding to Logistic differential equation $\frac{dy}{dx} = ry(1 - \frac{y}{k})$. Subsequently, several models have been established for biological growth by using Gompertz function. The Gompertz model is widely used to describe the evolution of a species with very rapid growth in a very short time interval as the growth of the number of bacteria [13, 51, 112], the growth of birds and animals, especially poultry [69].

Motivated by works cited above, we consider a predator–prey model (2.1.1) with Gompertz growth

as following

$$\begin{cases} \frac{dX(\tau)}{d\tau} = rX(\tau)\ln\left(\frac{k}{X(\tau)}\right) - aX(\tau)^\alpha Y(\tau), \\[2em] \frac{dY(\tau)}{d\tau} = -mY(\tau) + \eta aX(\tau)^\alpha Y(\tau). \end{cases} \tag{2.1.2}$$

2.2 Mathematical analysis and some preliminary results

Put

$$u := \frac{1}{k}X, \quad v := \frac{a}{rk^{1-\alpha}}Y, \quad t := r\tau. \tag{2.2.1}$$

Then, we obtain

$$\begin{cases} \frac{du}{dt} = f(u,v), \\[2em] \frac{dv}{dt} = g(u,v), \end{cases} \tag{2.2.2}$$

where

$$f(u,v) = \begin{cases} u\ln\left(\frac{1}{u}\right) - u^\alpha v, & \text{if} \quad u \neq 0, \\[2em] 0 & \text{if} \quad u = 0 \end{cases}$$

and

$$g(u,v) = -\delta v + \beta u^\alpha v,$$

with

$$\beta = a\frac{\eta k^\alpha}{r}, \quad \delta = \frac{m}{r}. \tag{2.2.3}$$

2.2.1 Boundedness of the solution

In the following theorem we show that solutions of system (2.2.2) are bounded.

Theorem 6. *All solutions of the system (2.2.2) that start in $\mathbb{R}_+^2$ are bounded.*

Proof. Let $(u(t), v(t))$ be any solution of the system $(2.2.2)$, we put $w(t) = \beta u(t) + v(t)$, then

$$\frac{dw}{dt} + \delta w = \beta u \left(\ln\left(\frac{1}{u}\right) + \delta \right) = \psi(u). \tag{2.2.4}$$

The function $\psi(u)$ reaches the maximum value $\beta e^{1-\delta} > 0$ at $u = e^{1-\delta}$, so we leads to

$$\frac{dw}{dt} + \delta w = \beta u \left(\ln\left(\frac{1}{u}\right) + \delta \right) = \psi(u) \leq \beta e^{1-\delta}. \tag{2.2.5}$$

Applying the theory of differential inequality [9], we obtain

$$0 \leq w(t) \leq \frac{\beta e^{1-\delta}}{\delta}(1 - e^{-\delta t}) + w(0)e^{-\delta t} \leq \frac{\beta e^{1-\delta}}{\delta} + w(0), \quad \forall t \geq 0. \tag{2.2.6}$$

This completes the proof of **Theorem 6**. $\qquad\square$

2.2.2 Equilibrium points and their stability

In this subsection, we find the possible equilibia of $(2.2.2)$. We see that this system has three nonnegative equilibria:

(i) the trivial equilibrium point $E_0 := (0,0)$,

(ii) the boundary equilibrium point $E_1 := (1,0)$,

(ii) the interior equilibrium point $E^* := (u^*, v^*)$, exists if and only if $u^* < 1$ (i.e. $\beta > \delta$), where

$$u^* = \left(\frac{\delta}{\beta}\right)^{\frac{1}{\alpha}}, \qquad v^* = (u^*)^{1-\alpha} \ln\left(\frac{1}{u^*}\right). \tag{2.2.7}$$

Now, we study the local stability analysis of the system $(2.2.2)$. The variational matrix for $(2.2.2)$ is

$$J(u,v) = \begin{pmatrix} \ln\left(\dfrac{1}{u}\right) - 1 - \alpha u^{\alpha-1} v & -u^\alpha \\ \\ \beta \alpha u^{\alpha-1} v & \beta u^{\alpha-1} - \delta \end{pmatrix}. \tag{2.2.8}$$

The eigenvalues of the variational matrix evaluated at the boundary equilibrium point E_1 are $\lambda_1 = -1$ and $\lambda_2 = \beta - \delta$. Hence, E_1 is locally asymptotically stable if $\beta < \delta$ and unstable if $\beta > \delta$.

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Condition	E_1	E^*	Bifurcation
$0 < \beta < \delta$	Asymptotically stable	Infeasible	
$\beta e^{-\alpha/1-\alpha} < \delta < \beta$	unstable	Asymptotically stable	
$\delta = \delta_H = \beta e^{-\alpha/1-\alpha}$	unstable		Hopf

TABLE 2.1: Dynamical behavior of the system (2.2.2) around E_1 and E^*.

Next, we investigate the coexistence equilibrium E^*. The Jacobian evaluated at E^* is given by

$$
J(E^*) = \begin{pmatrix} (1-\alpha)\left[\ln\left(\dfrac{1}{u^*}\right) - \dfrac{1}{1-\alpha}\right] & -(u^*)^\alpha \\[2em] \beta\alpha(u^*)^{\alpha-1}v^* & 0 \end{pmatrix}. \tag{2.2.9}
$$

The characteristic equation of $J(E^*)$ is given by $\lambda^2 - T\lambda + D = 0$, where

$$
T = tr(J(E^*)) = (1-\alpha)\left[\ln\left(\frac{1}{u^*}\right) - \frac{1}{1-\alpha}\right] = (1-\alpha)\left[\left(\frac{1}{\alpha}\right)\ln\left(\frac{\beta}{\delta}\right) - \frac{1}{1-\alpha}\right]
$$

and

$$
D = det(J(E^*)) = \beta\alpha(u^*)^{2\alpha-1}v^* = \beta\alpha(u^*)^\alpha \ln\left(\frac{1}{u^*}\right) > 0.
$$

As $D > 0$, the real parts of the eigenvalues have the same sign and therfore the local stability of E^* depends only on the sign of $T = tr(J(E^*))$. That is, E^* is locally asymptotically stable for $T < 0$ and unstable for $T > 0$. Put $C_H = e^{-1/1-\alpha}$, then, the interior equilibrium E^* is locally asymptotically stable for $u^* > C_H$ i.e $\beta e^{-\alpha/1-\alpha} < \delta < \beta$, and unstable when $\beta e^{-\alpha/1-\alpha} > \delta$. By the Poincare-Andronov-Hopf bifurcation theorem [**43**], we know that (2.2.2) undergoes Hopf bifurcation at E^* for $u^* = C_H$. Then $u^* = C_H$ i.e $\delta = \delta_H = \beta e^{-\alpha/1-\alpha}$ is the Hopf bifurcation critical value.

2.2.3 Stability analysis of E_0

Here, we investigate the linear stability of E_0. Due to the singularity of matrix $J(u,v)$, the stability of E_0 cannot be determined by simply evaluating (2.2.8) at $u = 0$ and $v = 0$, since $u^{1-\alpha}v$ is indeterminate. Basically, system (2.2.2) is not linearizable at E_0 because of the terms of the α power of prey, where $\alpha \in (0,1)$. In [**2**] for the particular case $\alpha = 1/2$ with logistic growth for prey, the authors remove the singularity by rescaling the variable $\sqrt{u} = x$, the eigenvalues of such transformed system being $1/2$ and

$-\delta$, we may conclude that, the trivial equilibrium E_0 has an unstable manifold. After analyzing the local stability near the origin of our proposed model, the system behavior shows a mixed stability of the steady state at the origin given by the following novel theorem

Theorem 7. *Suppose that $0 < \alpha < 1$, let $u_0 = u(0) \geq 0$ and $v_0 = v(0) \geq 0$ be the initial conditions for the system (2.2.2) and let E be the part of the system's phase plane in the interior of the positive quadrant of the (u,v) plane defined by*

$$D_0 = \left\{ (u,v) : u > e^{-1}, v > \tilde{v}(u) \right\}, \qquad where$$

$$\tilde{v}(u) = \frac{\delta + 1 - \alpha}{1 - \alpha} u^{1-\alpha}. \tag{2.2.10}$$

Then, for any $(u_0, v_0) \in D_0$ there exists some suitable time t^* for which the prey population goes to extinction. Thus the system's trajectory reaches the axis v at a finite time.

Proof. From the first equation of system (2.2.2). Since u and v are non-negative, we have

$$\frac{dv}{dt} = -\delta v + \beta u^\alpha v \geq -\delta v. \tag{2.2.11}$$

Applying the comparison principle for ordinary differential equations [40], and let $\tilde{v}$ be the solution of the following initial value problem

$$\begin{cases} \frac{d\tilde{v}}{dt} = -\delta \tilde{v}, \\ \\ \tilde{v}(0) = v_0. \end{cases} \tag{2.2.12}$$

Then, we obtain

$$v(t) \geq \tilde{v}(t) = v_0 e^{-\delta t}, \qquad \text{for all} \quad t > 0. \tag{2.2.13}$$

Consider the second equation, from $u > e^{-1}$ and (2.2.13) , we get

$$\begin{align} \frac{du}{dt} &= u \ln\left(\frac{1}{u}\right) u^\alpha - v \tag{2.2.14} \\ &\leq u - u^\alpha v \tag{2.2.15} \\ &\leq u - u^\alpha v_0 e^{-\delta t}. \tag{2.2.16} \end{align}$$

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Similarly, by the comparison principle [**40**], if $\tilde{u}$ is the solution of the initial value problem

$$\begin{cases} \frac{d\tilde{u}}{dt} = \tilde{u} - \tilde{u}^\alpha v_0 e^{-\delta t}, \\ \\ \tilde{u}(0) = u_0. \end{cases} \tag{2.2.17}$$

It follows that

$$u(t) \leq \tilde{u}(t), \qquad \text{for all} \quad t > 0. \tag{2.2.18}$$

In order to solve (2.2.17), we define a new variable $w(t)$ as follows

$$\tilde{u} = w(t)e^t, \qquad \text{with} \quad w(0) = \tilde{u}(0) = u_0. \tag{2.2.19}$$

Substituting (2.2.19) into (2.2.17). We lead to the following equation for w:

$$\frac{d}{dt}w(t) = -w(t)^\alpha v_0 e^{-\lambda t}. \tag{2.2.20}$$

where $\lambda = \delta + 1 - \alpha$.

Solving the above equations, we obtain,

$$w(t)^{1-\alpha} = w(0)^{1-\alpha} - \frac{v_0(1-\alpha)}{\lambda}(1 - e^{-\lambda t}). \tag{2.2.21}$$

Obviously, the right-hand side of Eq.(2.2.21) monotonously decreases as function of t. In addition, we know that $w(0) > 0$, this implies that for some $\tilde{t}$, $w(\tilde{t}) = 0$ if and only if

$$w(0)^{1-\alpha} < \frac{v_0(1-\alpha)}{\lambda}. \tag{2.2.22}$$

Now, we come back to the relation (2.2.19) between w and $\tilde{u}$, It is obvious to see that $w(\tilde{t}) = 0$ which implies that $\tilde{u}(\tilde{t}) = 0$. Note that $w(0) = u_0$ and from (2.2.22) we lead to $w(\tilde{t}) = 0$ for any u_0 and v_0 satisfying the following condition

$$v_0 > \frac{\lambda}{1-\alpha}u_0^{1-\alpha} = \frac{\delta + 1 - \alpha}{1 - \alpha}u_0^{1-\alpha}. \tag{2.2.23}$$

From (2.2.21), we obtain the explicit expression of $\tilde{t}$ as follows

$$\tilde{t} = -\frac{1}{\delta + 1 - \alpha}\ln\left[1 - \frac{\delta + 1 - \alpha}{v_0(1-\alpha)}u_0^{1-\alpha}\right]. \tag{2.2.24}$$

Finally, we know from (2.2.18) that $\tilde{u}$ is an upper bound for $u(t)$ then, if $\tilde{u}(\tilde{t}) = 0$ we obtain $0 \leq u(\tilde{t}) \leq 0$, which means that $u(t)$ becomes zero for some $t^* \leq \tilde{t}$. Inequality (2.2.23) gives a sufficient condition for the prey species extinction. This completes the proof of **Theorem 7**. $\qquad\square$
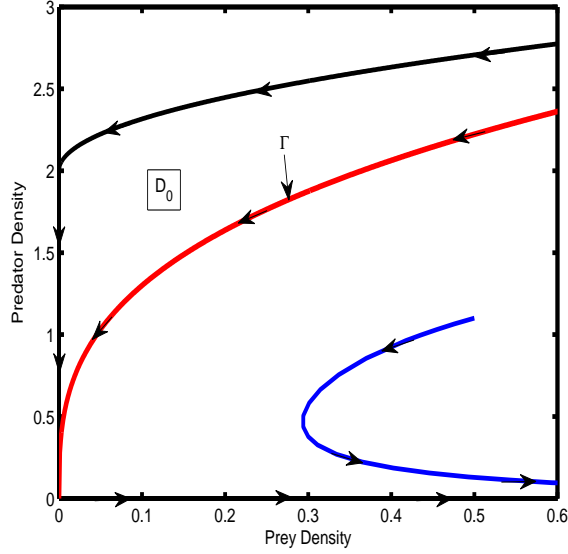
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FIGURE 2.1: The phase plane of (2.2.2) with parameters $\alpha = 2/3$, $\beta = 0.5$, $\delta = 0.6$. The thick red curve is the separatrix $v = 2.8u^{1/3}$, it shows the boundary of the extinction domain D_0. Please see (2.2.10). For any initial conditions above this curve, the trajectories (shown in black color with $v_0 > 2.8u_0^{1/3}$) hit the vertical axis, which means species extinction. For initial conditions below the curve, the trajectories (shown in blue color with $v_0 < 2.8u_0^{1/3}$) approach an attractor in the interior of the phase plane (either the stable coexistence steady state or the stable limit cycle, depending on equation parameters).

Note that inequality (2.2.23) gives only a sufficient condition of prey extinction but not a necessary one. Therefore, the actual extinction domain in the (u,v) phase plane is somewhat larger than the domain D_0 defined by (2.2.10); the extinction may happen as well for some $v_0 < \tilde{v}(u_0)$. Now we define the separatrix curve separating the two different trajectories as fllows :

$$(\Gamma): \qquad v = \frac{\delta + 1 - \alpha}{1 - \alpha} u^{1-\alpha}, \tag{2.2.25}$$

From (2.2.25) and **Fig.2.1**, the singular dynamics of model (2.2.2) near the origin is given in the following lemma

Lemma 3.

(i) For any initial value (u_0, v_0) such that, $v_0 = \dfrac{\delta + 1 - \alpha}{1 - \alpha} u_0^{1-\alpha}$ the orbit of model (2.2.2) goes into the

*origin equilibrium along the parabola $v = \dfrac{\delta + 1 - \alpha}{1 - \alpha} u^{1-\alpha}$, see **Fig.2.1**.*

*(ii) For any initial value (u_0, v_0) such that, $v_0 > \dfrac{\delta + 1 - \alpha}{1 - \alpha} u_0^{1-\alpha}$ the orbit of model (2.2.2) terminates at $u = 0$ and some positive value of v, after which v decreases to zero, see **Fig.2.1**.*

*(iii) For any initial value (u_0, v_0) such that, $v_0 < \dfrac{\delta + 1 - \alpha}{1 - \alpha} u_0^{1-\alpha}$ the orbit of model (2.2.2) gives saddle behavior, see **Fig.2.1**.*

Lemma 3, together with **Theorem 7** lead to the following theorem for the stability of the trivial equilibrium E_0,

Theorem 8. *The equilibrium point E_0 is globally asymptotically stable in the domain D_0 and unstable below the separatrix Γ (please see **Fig.2.1**).*

Lemma 4. *Below the separatrix Γ, E_0 has the same behavior as a saddle point (please see **Fig.2.1**).*

2.3 The impact of the prey herd shape

The form of the herd for the prey plays an effective role in maintaining the cohesion and toughness of the herd against predators. In this section, we will study the impact of prey herd shape on the equilibrium densities of both the prey and the predator populations. Let's start with the prey population, the differentiation of the density equilibrium of the prey population with respect to α gives

$$\frac{du^*}{d\alpha} = \frac{1}{\alpha^2} \ln\left(\frac{\beta}{\alpha}\right) e^{-\frac{1}{\alpha} \ln\left(\frac{\beta}{\alpha}\right)} > 0.$$

Obviously, the prey equilibrium density is monotonously increases as α increases because $\beta > \delta$ (The feasibility condition of E^*), means that prey herd shape has a positive effect on the prey equilibrium density **Fig.2.2**. This result give the importance of the social behavior for the prey population, see **Fig.2.2**.

Now focusing on studying the impact of the defense mechanism on the predator density equilibrium state. Noticing that

$$v^*(\alpha) = (u^*)^{1-\alpha} \ln\left(\frac{1}{u^*}\right) \qquad \text{with} \qquad u^* = \left(\frac{\delta}{\beta}\right)^{\frac{1}{\alpha}}. \tag{2.3.1}$$
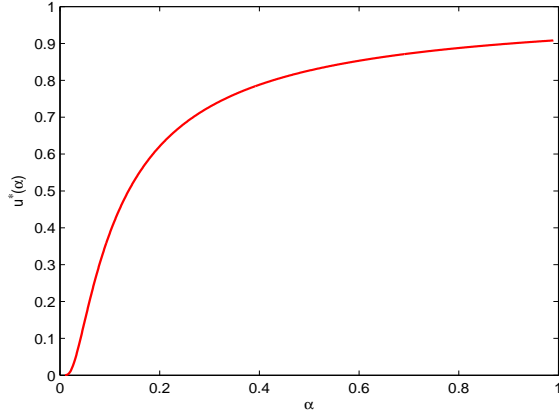
45

FIGURE 2.2: The effect of the prey herd shape on the prey density equilibrium state for the values $\beta = 1.1, \delta = 1$ where $\alpha \in [0.01, 0.99]$.

After differentiating the equilibrium state of the predatory density $(2.3.1)$ with respect to α, we obtain

$$\frac{dv^*}{d\alpha} = \frac{1}{\alpha^2} \ln\left(\frac{\beta}{\delta}\right) \left(\frac{\beta}{\delta}\right)^{\frac{\alpha-1}{\alpha}} \left[-1 + \frac{1}{\alpha} \ln\left(\frac{\beta}{\delta}\right)\right]. \tag{2.3.2}$$

Lemma 5. *Assume that $\beta > \delta$, then we have*

(i) If $0 < \delta < \beta e^{-1}$, the predator equilibrium density is monotonously increases for $\alpha \in (0,1)$ (see ***Fig.2.3****).*

(ii) If $\delta > \beta e^{-1}$, the predator equilibrium density is monotonously increases for $\alpha \in \left]0, \ln\left(\frac{\beta}{\delta}\right)\right[$ and monotonously decreases for $\alpha \in \left]\ln\left(\frac{\beta}{\delta}\right), 1\right[$ (see ***Fig.2.3****).*

Proof. It is not dffcult to verify **Lemme 5**. From $(2.3.2)$, we know that $\frac{dv^*}{d\alpha} = 0$ for $\alpha = \ln\left(\frac{\beta}{\delta}\right)$ where, $\frac{dv^*}{d\alpha} < 0$ for $\alpha > \ln\left(\frac{\beta}{\delta}\right)$ and $\frac{dv^*}{d\alpha} > 0$ for $\alpha < \ln\left(\frac{\beta}{\delta}\right)$. Then, if we take $0 < \delta < \beta e^{-1}$ we deduce that the predator equilibrium density is monotonously increases over the interval $(0,1)$ which completes the

rst part of the proof. On the other hand if $\delta > \beta e^{-1}$ means that the function v^* is monotonously increases for $\alpha \in \left]0, \ln\left(\frac{\beta}{\delta}\right)\right[$ and monotonously decreases for $\alpha \in \left]\ln\left(\frac{\beta}{\delta}\right), 1\right[$. The proof is completed. $\square$
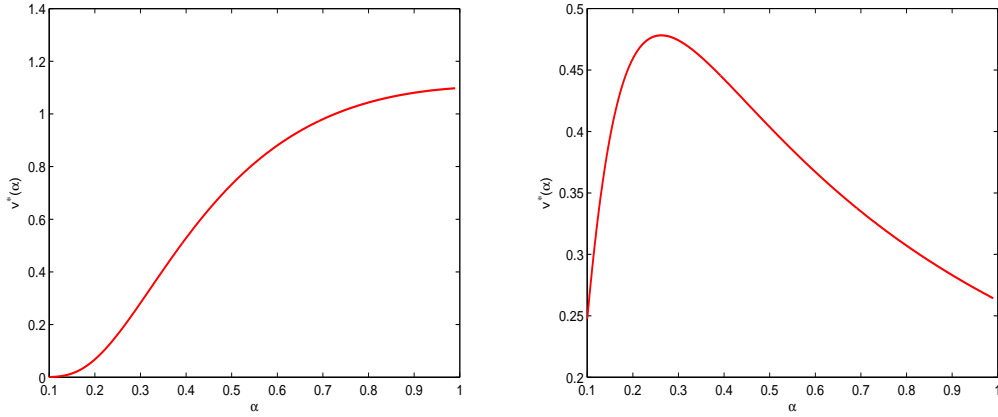
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FIGURE 2.3: The effect of the prey herd shape on the predator density equilibrium state. Left: the values $\beta = 3, \delta = 1$, then $\delta < \beta e^{-1} = 1.1036$. Right: the values $\beta = 1.3, \delta = 1$, then $\delta > \beta e^{-1} = 0.4782$. Here $\alpha \in [0.1; 0.99]$.

clearly the prey herd shape has a remarkable influence on the predator equilibrium density, this importance can be seen when the shape of the herd changes. Obviously the predator prefers the form where there exists more prey on the border. From **Lemme 5** we also see that the predator mortality rate plays an important role in the predator density, if it is less than βe^{-1} the prey herd shape has a positive impact on the predator density equilibrium for any forms of herds ($\alpha \in (0,1)$). But if $\delta > \beta e^{-1}$, the herd shape has a positive impact on the predator density equilibrium for $\alpha \in \left] 0, \ln\left(\dfrac{\beta}{\delta}\right) \right[$ and a negative impact for $\alpha \in \left] \ln\left(\dfrac{\beta}{\delta}\right), 1 \right[$.

2.4   Global asymptotic stability

From **Section 2.2**, we know that system (2.2.2) always has the equilibria E_0 and E_1 for all parameter values. Further (2.2.2) has a unique coexistence equilibrium $E^* = (u^*, v^*)$ if $\beta > \delta$, where

$$u^* = \left(\frac{\delta}{\beta}\right)^{\frac{1}{\alpha}}, \qquad v^* = (u^*)^{1-\alpha} \ln\left(\frac{1}{u^*}\right).$$

Now, we study the global asymptotic stability of the equilibrium points E_1 and E^*.

Theorem 9. *Assume that $\beta < \delta$, then the predator-extinction equilibrium point E_1 of system (2.2.2) is globally asymptotically stable.*

Proof. From **Sec.2.2**, we khow that the boundary equilibrium E_1 is locally asymptotically stable if $\beta < \delta$. The eigenvalues of the jacobian matrix evaluated at this equilibrium are $\lambda_1 = -1$ and $\lambda_2 = \beta - \delta$.

Since $u' < u \ln \left(\dfrac{1}{u} \right) < u(1-u)$, we have $\lim\limits_{t \to +\infty} \sup u(x) \leq 1$.

So, for any $\epsilon > 0$ there exist $T_\epsilon > 0$ such that

$$u(t) \leq 1 + \epsilon, \qquad \text{for } t > T_\epsilon.$$

From the second equation of (2.2.2), we have

$$v' = v(-\delta + \beta u^\alpha).$$

And by choosing $0 < \epsilon < \left(\dfrac{\delta}{\beta} \right)^{\frac{1}{\alpha}} - 1$, we can obtain

$$v' \leq v(-\delta + \beta(1+\epsilon)^\alpha), \quad t > T_\epsilon.$$

Applying a comparaison theorem on differential inequalities, we obtain

$$v(t) \leq v(0) e^{(-\delta + \beta(1+\epsilon)^\alpha)t} \tag{2.4.1}$$

with

$$-\delta + \beta(1+\epsilon)^\alpha < 0, \qquad \text{for} \quad t > T. \tag{2.4.2}$$

Now, from (2.4.1) and (2.4.2), for $t \to +\infty$ we get $v(t) \to 0$ and for a sufficiently large time the first equation of system (2.2.2) becomes

$$u' \leq u \ln \left(\frac{1}{u} \right),$$

thus, we get $u(t) \leq \dfrac{1}{1 - e^{u_0 - t}} \longrightarrow 1$ for $t \longrightarrow +\infty$. This completes the proof of **Theorem 9** $\qquad \square$

In the following, we try to show the global stability of the interior equilibrium E^*. But before announcing the theorem we must claim that system (2.2.2) has no nontrivial periodic orbits in $\mathbb{R}_+^2$, clearly when $u^* > 1$, we can affirm that model (2.2.2) has no nontrivial periodic orbits due to the fact that when $u^* > 1$ model (2.2.2) has no positive equilibrium. Therfore, we must assume $C_H < u^* < 1$ in the rest of this section. So, we have the following theorem

Theorem 10. *The system (2.2.2) has no nontrivial periodic orbits in $\mathbb{R}_+^2$ when $C_H < u^* < 1$.*

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Proof. Suppose that $C_H < u^* < 1$ and there exists $\Lambda(t) = (u(t), v(t))$ called the nontrivial periodic orbits of (2.2.2) with period $T > 0$ in $\mathbb{R}_+^2$. Then, we have

$$\int_0^T \left(\ln\left(\frac{1}{u}\right) - u^{\alpha-1}v \right) dt = \int_0^T \frac{u'(t)}{u(t)} dt = \ln(u(T)) - \ln(u(0)) = 0, \tag{2.4.3}$$

$$\int_0^T (-\delta + \beta u^\alpha) \, dt = \int_0^T \frac{v'(t)}{v(t)} dt = \ln(v(T)) - \ln(v(0)) = 0, \tag{2.4.4}$$

and

$$
\begin{aligned}
tr(J_\Lambda) &= \ln\left(\frac{1}{u}\right) - 1 - \alpha u^{\alpha-1}v + \beta u^{\alpha-1} - \delta \\
&= \frac{u'}{u} + \frac{v'}{v} - 1 + (1-\alpha)\left(\left(\ln\left(\frac{1}{u}\right) - \frac{u'}{u}\right) \right) \\
&= \alpha\frac{u'}{u} + \frac{v'}{v} + tr(J_{E^*}) + (1-\alpha)\left(\ln\left(\frac{1}{u}\right) - \ln\left(\frac{1}{u^*}\right) \right).
\end{aligned}
$$

Then, we obtain

$$\int_0^T tr(J_\Lambda) \, dt = tr(J_{E^*})T + (1-\alpha)\int_0^T \left(\ln\left(\frac{1}{u}\right) - \ln\left(\frac{1}{u^*}\right) \right) dt. \tag{2.4.5}$$

Setting

$$\psi_1(u) = \frac{\ln\left(\frac{1}{u}\right) - \ln\left(\frac{1}{u^*}\right)}{(u^*)^\alpha - u^\alpha}, \tag{2.4.6}$$

then, we have

$$\psi'_1(u) = \frac{\frac{1}{u}\left[u^\alpha - (u^*)^\alpha - \alpha u^\alpha \left(\ln\left(\frac{1}{u}\right) - \ln\left(\frac{1}{u^*}\right) \right) \right]}{((u^*)^\alpha - u^\alpha)^2}. \tag{2.4.7}$$

It is easy to prove that $\psi'_1(u) > 0$ for $u \neq u^*$. Defining the functional

$$\psi_2(u) = u^\alpha - (u^*)^\alpha - \alpha u^\alpha \left(\ln\left(\frac{1}{u}\right) - \ln\left(\frac{1}{u^*}\right) \right), \tag{2.4.8}$$

with

$$\psi'_2(u) = \alpha^2 u^{\alpha-1} \left(\ln(u) - \ln(u^*) \right), \tag{2.4.9}$$

which implies that $\psi_2(u) > \min \psi_2(u) = \psi_2(u^*) = 0$ for $u \neq u^*$. Then, we lead to the conclusion $\psi'_1(u) > 0$.

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It follows from the Green's theorem and

$$\left(\ln\left(\frac{1}{u}\right) - \ln\left(\frac{1}{u^*}\right) \right) = ((u^*)^\alpha - u^\alpha)\psi_1(u) = -\frac{v'}{\beta v}\psi_1(u), \qquad (2.4.10)$$

that

$$\int_0^T \left(\ln\left(\frac{1}{u}\right) - \ln\left(\frac{1}{u^*}\right) \right) \mathrm{d}t = -\frac{1}{\beta}\int_0^T \frac{v'}{\beta v}\psi_1(u) = -\frac{1}{\beta}\int\int_\Omega \frac{\psi'_1(u)}{v}\,\mathrm{d}u\mathrm{d}v < 0, \qquad \text{over} \quad \Omega, \quad (2.4.11)$$

where Ω is the bounded region enclosed by Λ. From the condition $C_H < u^* < 1$ we know that $T = tr(J(E_c)) < 0$ and using (2.4.5) and (2.4.11), lead to

$$\int_0^T tr(J_\Lambda)\,\mathrm{d}t < 0, \qquad (2.4.12)$$

The divergency criterion [40] affirms that all the periodic solutions are orbitally stable, which is contradictory with the stability of E^*. Therefore, we conclude that (2.2.2) has no nontrivial periodic orbits in $\mathbb{R}_+^2$ for $C_H < u^* < 1$. Which completes the proof. $\qquad \square$

Since E^* is the only equilibrium locally asymptotically stable for $C_H < u^* < 1$ then, the system (2.2.2) has no closed orbits in $\mathbb{R}_+^2$ .Hence, the interior equilibrium point E^* must be globally asymptotically stable.

In summary we have the following theorem

Theorem 11. *Below the separatrix* Γ, *the interior equilibrium point* E^* *of the model system (2.2.2) is globally asymptotically stable when* $C_H < u^* < 1$. *see* **Fig.2.5**.

2.5 Hopf bifurcation

In this section, we show the critical case of $tr(J(E^*)) = 0$ in which the system (2.2.2) undergoes a Hopf bifurcation at the equilibrium E^*. The coexistence equilibrium state E^* is globally asymptotically stable if $\beta e^{-\alpha/1-\alpha} < \delta < \beta$, while, the coexistence equilibrium state E^* is unstable if $\beta e^{-\alpha/1-\alpha} > \delta$. Hopf bifurcation is characterised by the stability change of an equilibrium point with small amplitude limit cycle behaviour around the equilibrium point, as some parameter μ (called bifurcation parameter) of the system passes through a critical value μ^*. By choosing the parameter δ as the bifurcation parameter, we have the following result

Theorem 12. *The system (2.2.2) undergoes Hopf bifurcation with respect to the parameter* δ *around the coexistence equilibrium point* $E^* = (u^*, v^*)$ *if* $\delta = \delta_H = \beta e^{-\alpha/1-\alpha}$.

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Proof. We know that, if $T = tr(J(E^*)) = 0$, then both the eigenvalues will be purely imaginary provided $D = det(J(E^*)) > 0$. Therefore, from the implicit function theorem a Hopf bifurcation occurs where a periodic orbit is created as the stability of the equilibrium point E^* changes. Using the above two conditions it is found that the critical value of the Hopf bifurcation parameter is $\delta_H = \beta e^{-\alpha/1-\alpha}$. The critical δ_H value depends on the parameter α where $0 < \alpha < 1$.

It is clear that the given conditions [6]

(i) $T = tr(J(E^*)) = 0$,

(ii) $D = det(J(E^*)) > 0$ and

(iii) $\dfrac{d}{\delta} tr(J(E^*)) \neq 0$ at $\delta = \delta_H = \beta e^{-\alpha/1-\alpha}$

implie the occurrence of Hopf bifurcation for system (2.2.2) around E^*.
where $\dfrac{d}{d\delta} tr(J(E^*))|_{\delta=\delta_H} = -\dfrac{1-\alpha}{\alpha\beta e^{-\alpha/1-\alpha}} < 0$ is the transversality condition, the proof is complete. $\square$

2.6 Numerical results

In this section, some numerical simulations are provided to illustrate the theoretical findings which are established in the previous section of this work. In the following, we take the parameters of system (2.2.2) as given in **Tab. 2.2**.

Case	α	β	δ	Figure
1	2/3	0.7	2	**Fig. 2.4**
2	2/3	0.23	0.1	**Fig. 2.5**
3	2/9	0.127	0.05	**Fig. 2.6**

TABLE 2.2: Parameter values used in the simulations of system (2.2.2).

Case 1 in **Tab. 2.2**. For this set of parameter values, $\alpha = 2/3$ $\beta = 0.7$ $\delta = 2$ where the interior equilibrium is infeasable ($\bar{v} = -2.6618 < 0$). It follows from **Theorem 9** that the boundary equilibrium $E_1 = (1,0)$ of the system is globally asymptotically stable. The simulation results can be seen in **Fig.2.4** with different initial values.

Case 2 in **Tab. 2.2**. For this set of parameter values, $\alpha = 2/3$ $\beta = 0.23$ $\delta = 0.1$, we have $\bar{u} = 0.2838 > C_H = 0.0498$. So, from **Theorem 10**, we khow that system (2.2.2) has no limit cycle. Therfore,
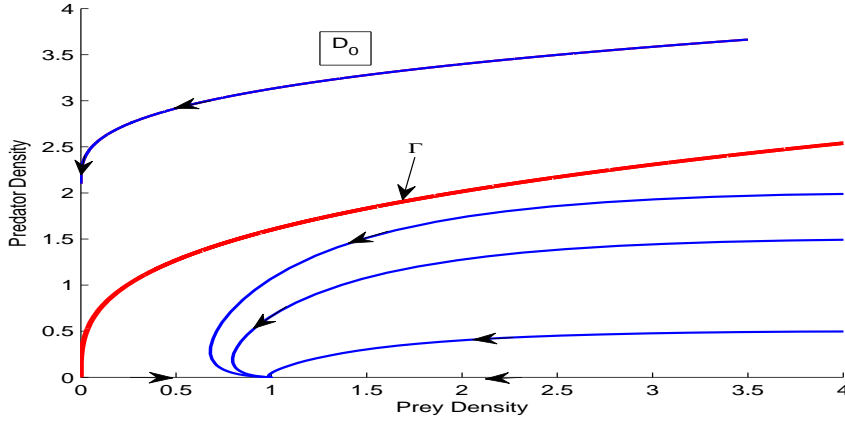
FIGURE 2.4: Global asymptotic stability of the predator-extinction equilibrium point E_1 for different initial values with parameter values given as Case 1 in **Tab. 2.2**.

by **Theorem 11** the interior equilibrium $E^* = (0.2838, 0.4703)$ is globally asymptotically stable. The corresponding phase plane plots of the system are depicted in **Fig.2.5** with different initial values. Clearly it is a stable spiral converging to the equilibrium.

Case 3 in **Table 2.2**. We choose the parameters of the system (2.2.2) as $\alpha = 2/9 \quad \beta = 0.127 \quad \delta = 0.05$ with the initial condition $(u(0), v(0)) = (0.1, 0.1)$ then the interior equilibrium $E^* = (0.2470, 0.4725)$ and $C_H = 0.2765 > \bar{u} = 0.2470$, by **Theorem 12** there exists a limit cycle near the interior equilibrium $E^* = (0.2470, 0.4725)$. **Fig.2.6** shows the graphical presentation of the prey–predator system with the specified values of the parameters.

2.7 Discussion for the original model (2.1.2)

After doing the qualitative study of the model (2.2.2), it is necessary to project the results obtained on (2.1.2) to have a more general idea of the asymptotic behavior on our proposed model.

From previous **Sections**, we know that the boundary equilibrium states E_1 of (2.2.2) is globally asymptotically stable if and only if $\beta < \delta$ where the interior equilibium E^* doest not exixst. For the original model (2.1.2), this result remains valid when the boundary equilibrium becomes $P_1 = (K, 0)$, is globaly asymptotically stable for $m > a\eta K^\alpha$ (using (2.2.3). Clearly, this result is logically obvious, when the mortality rate m is greater than a certain value $a\eta K^\alpha$ we have predator extinction.
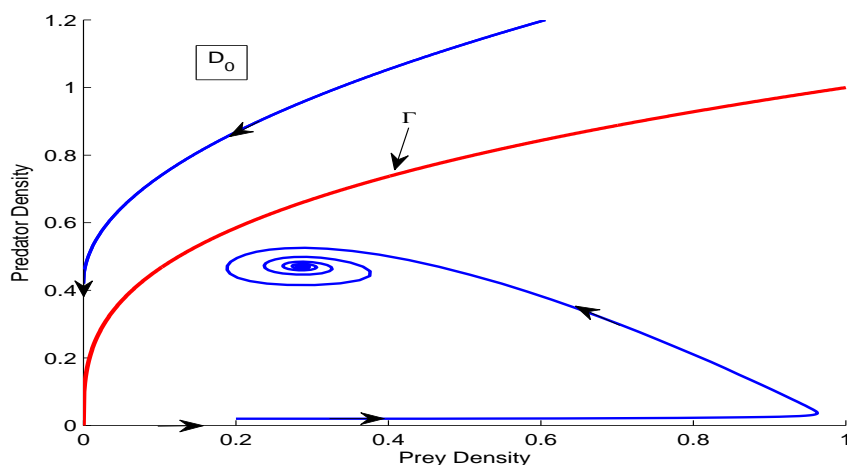
FIGURE 2.5: Global asymptotic stability of the interior equilibrium point E^* with parameter values given as Case 2 in **Tab. 2.2** and $(u_0, v_0) = (0.2, 0.02)$.
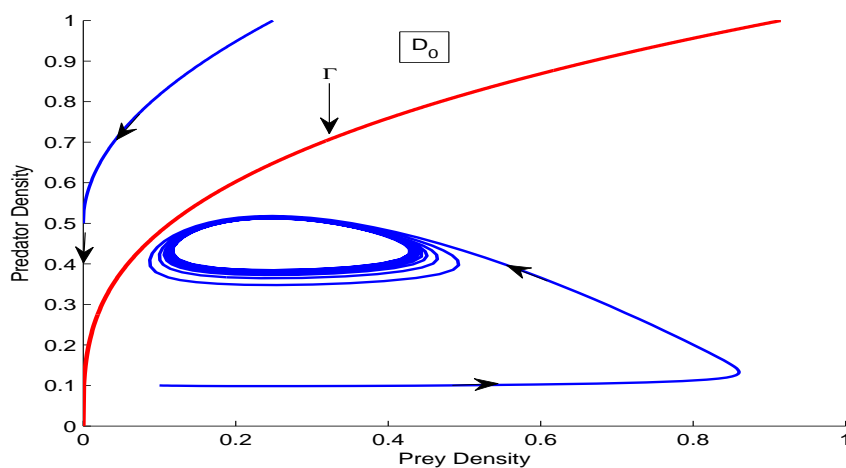

FIGURE 2.6: Phase plane of the system $(2.2.2)$ for $\alpha = 2/9$ $\beta = 0.127$ and $\delta = 0.1$ the positive interior equilibrium point E^* of system $(2.2.2)$ looses its stability and a Hopf-bifurcation occurs. Here the initial condition is $(u(0), v(0)) = (0.1, 0.1)$.

From **Theorem 11** we know that the interior equilibrium E^* is globally asymptotically stable for $C_H < u^* < 1$ where $C_H = e^{-1/1-\alpha}$ and for the original model $(2.1.2)$ we have the following theorem

Theorem 13. *Assume that $a\eta K^\alpha e^{-\alpha/1-\alpha} < m < a\eta K^\alpha$, then the interior equilibrium $P^* = (X^*, Y^*)$ is globally asymptotically stable where,*

$$X^* = K\left(\frac{m}{a\eta K^\alpha}\right)^{\frac{1}{\alpha}}, \ Y^* = r\frac{K^{2-2\alpha}}{a}(X^*)^{1-\alpha}\ln(\frac{1}{KX^*}).$$

From **Section 2.4**, the system $(2.2.2)$ undergoes Hopf befurcation at the coexistence equilibrium for $u^* = C_H$, then for the original system $(2.1.2)$, the occurrence of Hopf befurcation arising when the parameter m crosses the critical value aeK^α please see **Fig.2.7**.

Now, we put

$$m = a\eta K^\alpha = f_1(\alpha), \ m = a\eta K^\alpha e^{-\alpha/1-\alpha} = f_2(\alpha).$$

Then, we have the qualitative behavior of the model $(2.1.2)$ in the $m - \alpha$ parameter plane for $a = 0.5, \eta = 0.4, K = 10$ and $r = 0.1$. It is observed from **Fig.2.7** that the predator will extinct when the mortality rate m is greater than the curve with the equation $f_1(\alpha)$. When the parameter m becomes between the two curves $f_1(\alpha)$ and $f_2(\alpha)$, the boundary equilibrium E_1 loses their stability when the interior becomes globally asymptotically stable. In the region where the parameter m is less than the curve $f_2(\alpha)$(the green region), we known that the two equilibria points become unstable and the system transits to an oscillatory regime.

2.8   Conclusions and comments

The prey-predator systems have been studied extensively in theoretical ecology. However, only a little attention has been paid on one of the most realistic phenomenon that many species of preys exhibits a herd behavior against unexpected attacks of predator. In this paper, a new approximation of herd behavior with Gompertz growth function have been investigated. Depending on the physiological characteristics of the prey and the environmental characteristics, the herd shape changes from one species to another, whereas this change has a direct influence on the interaction between the two species. Clearly, more prey on the herd boundary implies a good solidarity of the group, therefore the prey population keeps as many individuals as possible over time which means that the herd shape have a positively influences on the prey population if the parameter α increases, this is easily seen in **Fig.2.2**. For predator from the study in the **Section 2.3**, it can be seen that the predator mortality rate along with the prey herd shape both affect the predator level for the interior equilibrium **Fig.2.3**. It is obvious that whatever the physiology of a species of predator they prefer a form of prey where there is more chance for hunting, this is incarnated when the border of the herd possesses the large amount of prey among the density total. In the real world,
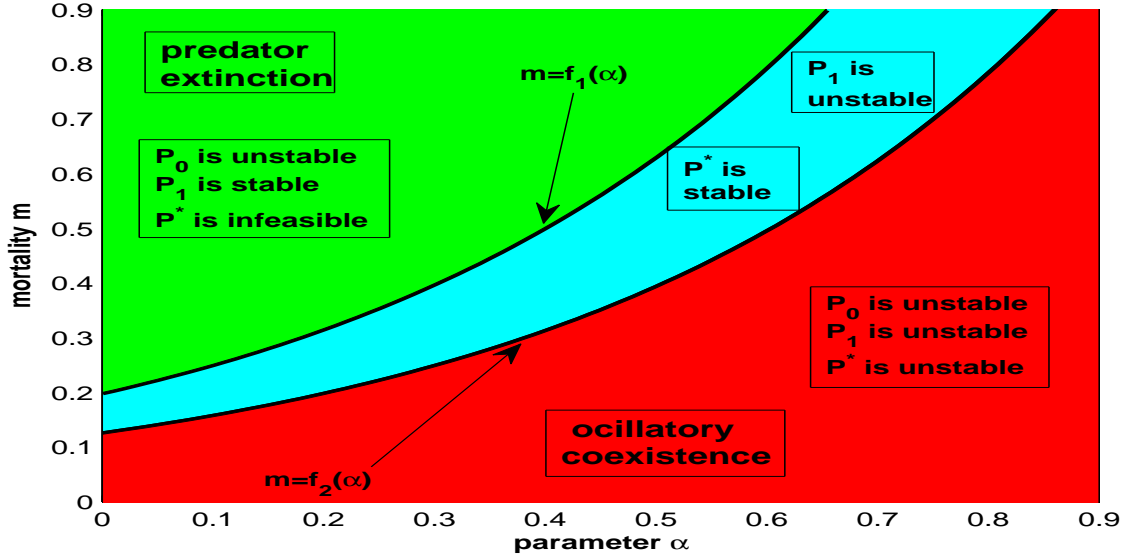
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FIGURE 2.7: Qualitative behaviour of the model (2.1.2) in the $m - \alpha$ phase plane is presented. Results are presented for parameter $\alpha \in [0, 0.9]$ and mortality rate m with $a = 0.5, \eta = 0.4, K = 10$ and $r = 0.1$. It shows that parameter space is divided into three regions (oscillatory coexistence region, stable coexistence region and predator extinction region).

several prey population reproduces in a quick way in a small time interval in the nature, a very good example on that, the zebra and buffalo in the savana, the evolution of the prey population is influenced by the initial density all of this explains the reason for injecting Gompetz's growth into the model from a modeling point of view. From the analysis of the model (2.2.2) in the **Section 2.2**, we can ensure that. The model we have considered is biologically well behaved, as it is bounded and remains infeasible range forever. From **Theorem 10** and boundedness of the solutions, we khow that the system (2.2.2), for $u^* \geq 1$ has no positive equilibrium point and the boundary equilibrium point E_1 is a local attractor; for $C_H < u^* < 1$ the unique positive equilibrium point is a local attractor and boundary equilibrium point is unstable. Finally for $C_H > u^*$ the limit cycle is a local attractor and the positive equilibrium and boundary equilibrium are unstable. Fanally for $C_H > u^*$ the limit cycle is a local attractor and the positive equilibrium and boundary equilibrium are unstable. In **Section 2.2**, from **Theorem 7**, we khow that the inequality (2.2.23) gives the condition for the prey species extinction which prove the existence of the extinction domain D_0. See **Fig.2.1**. Thus we conclude that the positive invariant set of model

(2.2.2) is divided into two parts by a separatrix in which can be defined as $v = \frac{\delta+1-\alpha}{1-\alpha} u^{1-\alpha}$ near the origin equilibrium, with one part as the extinction domain in which orbits terminate at positive vertical axis and then decrease to zero along the vertical axis, and the other part as the attraction domain of the corresponding attractor (boundary equilibrium point, positive equilibrium point or limit cycle) in which orbits converge to the attractor. Based on these result we obtained the global dynamics of the model (2.2.2) as well as the original model (2.1.2). It is necessary to make clear that model (2.2.2) represents a transition to the qualitative study of model (2.1.2) (a kind of auxiliary model). One can see that the dynamics of the predator prey model with prey herd shape is more plentiful than the traditional models and makes much more sense ecologically. The standard models usually predict the origin equilibrium point is a saddle point, i.e. the prey species will recover no matter how small it is relative to the predator species. In contrast, for model (2.2.2) as well the origin model (2.1.2) solution behavior near the origin equilibrium is singular. If the prey population is suitably smaller than the predator population in wild, then the prey species first goes extinct, causing the predator species to follow suit (see **Fig.2.1**). This makes perfect ecological sense. The result in the present chapter may help us to better understanding the interaction of predator with prey in a real ecosystem, and could help ecologists find the right factors to insure the continuation of living for species in nature.

Chapter 3

Spatiotemporal patterns in a diffusive predator-prey model with protection zone and predator harvesting

This chapter is taken from publication [82].

We always stay in the same topic as the two previous chapters, namely a predator-prey model, with **protection zone** (**herd behavior**) for the prey population and in the presence of predator harvesting. Except that in this chapter we take into account the spatial distribution of two species to represent the diffusion in nature subject to the zero flux boundary conditions. The main objective of this chapter is to study the impact of **predator harvesting** on the two densities of populations.

We organize this chapter in the following form : in **Section 3.1** we propose a diffusive predator-prey model with **protection zone** and **quadratic predator harvesting**. **Section 3.2**, the existence of a positive solution and some priori estimates for the system are proved. **Section 3.3**, is devoted to study the existence of the equilibrium states for the system (3.1.3). **Section 3.4**, the global stability of the semi trivial equilibrium state $(1,0)$ has been successfully established under a suitable condition on the model parameters. **Section 3.5**, the spatiotemporal dynamics of the system (3.1.3) near the unique non trivial equilibrium state (u^*, v^*) is investigated, where the existence of **Hopf bifurcation** is discussed. On

the other hand, the presence of **Turing driven instability**, **Turing-Hopf bifurcation** have been also discussed. For the stability of the **homogeneous** and **nonhomogeneous periodic solutions** generated by the presence of **Hopf bifurcation** are studied using the **normal form** on the center of manifold. The obtained results were checked using numerical simulations. A discussion **Section** ends this chapter.

3.1   The mathematical model

The herd behavior for the prey population can affect the availability of the prey to the predator, and using the fact that the most part of the predator population put their focus on hunting the weakest one (the weakest prey located in inside of the pack) which can make perturbation on the predator population, where will push the predator to fight with each other for a small quantity of resources, this behavior is known by **harvesting**. Also, it can be seen on the competition between the predators for the hunting zones of for reproduction. The most famous predator that can exhibit this behavior is the lions, where the solitary males enter into hunting zone of other lions and fight their leader for the exclusivity of reproduction. The harvest has a strong influence on the dynamic evolution of a population, this phenomenon is common in the fields of forestry, fishing and wildlife management. It is well know that these predators in a huge decreasing due to the human intervention and disappearance of many zones of living for these living being. The interest of this paper is to study the effect of the herd behavior and the predator harvesting in the existence of the considered species. During the few last years, several forms of harvest have been proposed and studied, mainly consisting of constant harvest [57, 107, 109], proportional harvesting [52] and nonlinear harvesting [39, 105]. Here, we deals with a model with quadratic harvesting which can be modeled by $(\tilde{\beta}\gamma V^2)$. This case of predator has been widely investigated, we cite for instance the papers [1, 3, 4, 5, 96, 97]. Based on the best knowledge there is no results on its effect on the interaction predator-prey in the presence of herd behavior where in this case the predator has two difficulties, the hunting from the herd, which is dangerous and protecting the zone of hunting from a predator of it gender. Further, in [79], it has been given a comparative analysis between the linear mortality and quadratic one, where two models have been investigated, but in the real world, the predator can exhibits the two types of mortality, where the previous example (lions example) is the simplest one that can hold the types of mortality. The harvesting can hold for the smaller lions and the bigger males, and the natural mortality can insured for the females, where it has been noticed that in the savanna, the males never attacks their females (also in [100] it has been considered that the predator exhibits quadratic mortality only). Based

on the above discussion we formulate the following system

$$
\begin{cases}
\dfrac{\partial U}{\partial \tau} = rU(1 - \frac{U}{K}) - a\sqrt{U}V, \\[2mm]
\dfrac{\partial V}{\partial \tau} = \tilde{\beta}V(-\tilde{m} + a\sqrt{U} - \gamma V),
\end{cases}
\tag{3.1.1}
$$

where, U, V denote the prey and predator population, respectively, at any time t. a is the search effeciency of predator for the prey, r represents the intrinsic growth rate of the prey populations, K is the carrying capacity of the prey in the absence of predator. the term $\tilde{\beta}\tilde{m}$ is the natual death rate of the predator and the new parameter $\tilde{\beta}\gamma$ which represents the harvesting rate for the predator population.

Besides, the predator and the prey are in different spatial locations, there exist many reasons for moving such as search for resources, currents and turbulent diffusion. The movement of the prey and predator can be modeled by the presence of **spatial diffusion** on the system (3.1.1). The ecological reaction diffusion systems becomes the one of the most dominant themes in mathematical biology we give as example the papers [74, 103, 75, 76, 77, 79, 98]. We assume that the two considered populations are always in movement thus each one of those populations will follow a trajectory (patch). The length of this path we denoted by x. This point of view is a strong simplification of the general case, where it has been used widely we cite for instance the papers [78, 84, 85]. Besides, we will assume that the two populations are in isolated patches, which can be elaborated using the zero flux boundary condition. Naturally, the presence of spatial diffusion leads to the following predator-prey model

$$
\begin{cases}
\dfrac{\partial U}{\partial \tau} = rU(1 - \frac{U}{K}) - a\sqrt{U}V + d_1 \Delta U, & x \in \Omega,\ \tau > 0, \\[2mm]
\dfrac{\partial V}{\partial \tau} = \tilde{\beta}V(-\tilde{m} + a\sqrt{U} - \gamma V) + d_2 \Delta V, & x \in \Omega,\ \tau > 0, \\[2mm]
\dfrac{\partial U}{\partial \vec{n}} = \dfrac{\partial V}{\partial \vec{n}} = 0, & x \in \partial\Omega,\ \tau > 0, \\[2mm]
U(x,0) = U_0(x) \geq 0, \quad V(x,0) = V_0(x) \geq 0, & x \in \Omega,
\end{cases}
\tag{3.1.2}
$$

where Ω is a bounded domain in $\mathbb{R}^n, n \geq 1$ with a smooth boundary, d_1 and d_2 are the diffusion rates for the prey and the predator. The vector $\vec{n}$ represents the unit outward unit normal vector of the boundary $\partial\Omega$. The homogeneous zero flux boundary conditions describes isolated patches of the two populations. For reducing the number of parameters we set the following change of variables

$$
u = \frac{U}{k}, \quad v = \frac{a}{r\sqrt{k}}V, \quad t = r\tau, \quad \beta = \frac{a\tilde{\beta}\sqrt{k}}{r}, \quad c = \gamma\frac{r}{a}, \quad \eta = \frac{\tilde{m}}{a\sqrt{k}}.
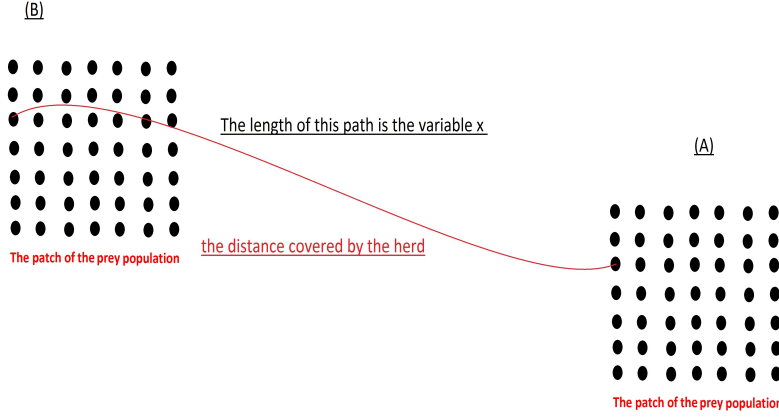$$

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(B)

The length of this path is the variable x

(A)

The patch of the prey population

the distance covered by the herd

The patch of the prey population

FIGURE 3.1: The manner in which the population of prey diffuse from a point (A) to another point (B).

Then the system (3.1.2) becomes

$$
\begin{cases}
\frac{\partial u}{\partial t} = u(1-u) - \sqrt{u}\,v + d_1 \Delta u, & x \in \Omega,\ t > 0, \\[2ex]
\frac{\partial v}{\partial t} = \beta v(-\eta + \sqrt{u} - cv) + d_2 \Delta v, & x \in \Omega,\ t > 0, \\[2ex]
\frac{\partial u}{\partial \overrightarrow{n}} = \frac{\partial v}{\partial \overrightarrow{n}} = 0, & x \in \partial\Omega,\ t > 0, \\[2ex]
u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0, & x \in \Omega,
\end{cases}
\tag{3.1.3}
$$

The most interesting about the presence of predator harvesting and the **spatial diffusion** on predator prey model with herd behavior is its effect on the existence of species (more precisely the harvesting). For the reason of the novelty of the prey herd behavior it is interesting to study the effect of the variable c on the spatiotemporal behavior of solution of the system (3.1.3). In wild world, the predator harvesting affect hugely on the existence of species, where for the small competition between the predators (which means that the predator are compatible) this can affect negatively the prey population and positively the predator, and the opposite for the bigger competitions.

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3.2 Existence of a positive solution, a priori bound of solution and some estimations of solution

In this section, we study the existence and uniqueness of positive solution of the system (3.1.3). Further, a priori bound of solution is given. The following theorem summarizes the obtained results

Theorem 14. ([82]) Assume that $u_0(x) \geq 0$, $v_0(x) \geq 0$ and $\Omega \in \mathbb{R}^m$ with a smooth boundary bounded domain in $\mathbb{R}^m$, then

(i) The system (3.1.3) has a unique solution $(u(x,t), v(x,t))$ satisfying

$$0 < u(x,t) \leq u^*(t), \quad 0 < v(x,t) \leq v^*(t) \quad for \ t > 0, \quad x \in \Omega.$$

where $(u^*(t), v^*(t))$ is the unique solution of the ordinary differential equation

$$\begin{cases} u_t = u(1-u), \\ v_t = \beta v(-\eta + \sqrt{u} - cv), \\ u(0) = u_0^* = \sup_{x \in \Omega} u_0(x) \ , \quad v(0) = v_0^* = \sup_{x \in \Omega} v_0(x). \end{cases} \tag{3.2.1}$$

(ii) Moreover, we have $\lim_{t \to +\infty} \sup u(x,t) \leq 1$, $\lim_{t \to +\infty} \sup \int_\Omega v(x,t)\,dx \leq \dfrac{1+\beta\eta}{\eta}|\Omega|$.

Proof. (i) We put $h(u,v) = u(1-u) - \sqrt{u}v$, $\quad k(u,v) = \beta v(-\eta + \sqrt{u} - cv)$, then, we have $h_u = -\sqrt{u} \leq 0$, $\quad k_u = -\beta v \geq 0$ for $(u,v) \in \mathbb{R}_+^2$ and from [70] we can conclude that the functionals h and k are a mixed quasi monotone functionals in $\mathbb{R}$. Now we let,

$(u_1, v_1) = (0,0)$ satisfying

$$\begin{cases} \frac{\partial u_1}{\partial t} - d_1 \Delta u_1 - h(u_1, v_1) = 0 \leq 0, \\ \frac{\partial v_1}{\partial t} - d_2 \Delta v_1 - k(u_1, v_1) = 0 \leq 0, \end{cases} \tag{3.2.2}$$

and $(u_2, v_2) = (u^*(t), v^*(t))$ satisfying the system

$$\begin{cases} \frac{\partial u_2}{\partial t} - d_1 \Delta u_2 - h(u_2, v_2) = 0 \geq 0, \\ \frac{\partial v_2}{\partial t} - d_2 \Delta v_2 - k(u_2, v_2) = 0 \geq 0, \end{cases} \tag{3.2.3}$$

where $0 \leq u_0(x) \leq u_0^*$, $0 \leq v_0(x) \leq v_0^*$. $(u_1(x,t), v_1(x,t))$ and $(u_2(x,t), v_2(x,t))$ called the lower and the upper solution of the system (3.1.3), according to the **Theorem 8.3.1** in [70], the system (3.1.3) has a unique global solution $(u(x,t), v(x,t))$ satisfying the condition

$0 \leq u(x,t) \leq u^*(t)$, $0 \leq v(x,t) \leq v^*(t)$ for $t > 0$ and $x \in \Omega$.

The strong maximum principle implies that $u(x,t) > 0$, $v(x,t) > 0$ for $t > 0$ and $x \in \Omega$.

(ii) Now, lets prove the second part of the **Theorem 14**, we know that $u(x,t) \leq u^*(t)$ and $v(x,t) \leq v^*(t)$

for all $t > 0$ where $u^*(t)$ is the unique solution of the equation

$$\frac{du}{dt} = u(1-u), \qquad u(0) = u_0^* > 0.$$

It is easy to verify that $u^*(t) \to 1$ as $t \to +\infty$. So, for any $\varepsilon > 0$ there exist $\delta_0 > 0$ such that

$u(x,t) \leq 1 + \varepsilon$, for $t > \delta_0$, $x \in \overline{\Omega}$, which leads to $\lim\limits_{t \to +\infty} \sup u(x,t) \leq 1$.

Now let

$$\sigma(t) = \int_\Omega u(x,t)\,\mathrm{d}x, \qquad \theta(t) = \int_\Omega v(x,t)\,\mathrm{d}x, \tag{3.2.4}$$

and

$$w(t) = \beta\sigma(t) + \theta(t). \tag{3.2.5}$$

Then we have

$$\begin{aligned}
\frac{d\sigma}{dt} &= \int_\Omega d_1\Delta u\,\mathrm{d}x + \int_\Omega [u(1-u) - \sqrt{u}v]\,\mathrm{d}x, \\
\frac{d\theta}{dt} &= \int_\Omega d_2\Delta v\,\mathrm{d}x + \int_\Omega [\beta v(-\eta + \sqrt{u} - cv)]\,\mathrm{d}x.
\end{aligned} \tag{3.2.6}$$

Now, making use of the boundary conditions

$$\begin{aligned}
\frac{dw}{dt} &= \beta\frac{d\sigma}{dt} + \frac{d\theta}{dt} \\
&= \beta\int_\Omega u(1-u)\,\mathrm{d}x - \beta\eta\theta(t) - c\beta\int_\Omega v^2\,\mathrm{d}x \\
&= \beta\int_\Omega u(1-u)\,\mathrm{d}x - \beta\eta(\beta\sigma(t) + \theta(t)) + \eta\beta^2\sigma(t) - c\beta\int_\Omega v^2\,\mathrm{d}x \\
&\leq -\beta\eta w + \beta(m\beta + 1)\sigma.
\end{aligned}$$

From $\lim\limits_{t \to +\infty} \sup u(x,t) \leq 1$ we have $\lim\limits_{t \to +\infty} \sup \sigma(t) \leq |\Omega|$. Thus, for sufficiently small $\varepsilon > 0$, there exists $T_0 > 0$ such that

$$\frac{dw}{dt} \leq -\eta\beta w + \beta(\eta\beta + 1)(1 + \varepsilon)\,|\Omega|, \qquad t > T_0, \tag{3.2.7}$$

we consider the equation

$$\frac{dw}{dt} = -\eta\beta w + \beta(\eta\beta + 1)(1 + \varepsilon)\,|\Omega|, \tag{3.2.8}$$

By a straightforward calculation, we obtain

$$\lim_{t \to +\infty} w(t) = \frac{\eta\beta + 1}{\eta}(1 + \varepsilon)\,|\Omega|. \tag{3.2.9}$$

Using the comparison principle and (3.2.7) we can obtain for $T_1 > T_0$

$$\int_\Omega v(x,t)\,\mathrm{d}x = \theta(t) < w(t) \leq \frac{\eta\beta + 1}{\eta}(1 + \varepsilon)\,|\Omega| + \varepsilon, \qquad t > T_1, \tag{3.2.10}$$

which leads to

$$\lim_{t \to +\infty} \sup \int_\Omega v(x,t)\,\mathrm{d}x \leq \frac{1 + \beta\eta}{\eta}\,|\Omega|. \tag{3.2.11}$$

This completes the proof of **Theorem 14**. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.3 Existence and uniqueness of positive equilibrium state for system (3.1.3)

In this section, we shall prove the existence and uniqueness of the positive equilibrium state for system (3.1.3).

Obviously the investigated system has two boundary equilibria state $E_0 = (0,0)$, $E_1 = (1,0)$. Now we focus on proving the existence of positive equilibrium state E^*. Then we have the following result.

Theorem 15. ([82]) Assume that $0 < \eta < 1$. The system (3.1.3) has a unique positive equilibrium state $E^* = (u^*, v^*)$ with $0 < u^* < 1$. Further when $\eta \geq 1$ the positive homogeneous steady state does not exists.

Proof. It is easy to verify that the positive equilibrium state E^* is the solution of the following system

$$\begin{cases} u(1-u) - u^{\frac{1}{2}}v = 0, \\ -\eta + u^{\frac{1}{2}} - cv = 0, \end{cases} \tag{3.3.1}$$

from the first equation of the system (3.3.1) we have

$$v = u^{\frac{1}{2}}(1-u) \tag{3.3.2}$$

in the first quadrant of the $u - v$-phase plane. since $v^* > 0$, we have $0 < u^* < 1$. By replacing (3.3.2) in (3.3.1) the positive equilibrium state is the solution of the equation of the variable u

$$\Phi(u) = 0, \tag{3.3.3}$$

where $\Phi(u) = -\eta + (1-c)u^{\frac{1}{2}} + cu^{\frac{3}{2}}$ for $u \in (0,1)$.

Now, we put $x = u^{\frac{1}{2}}$, thus, (3.3.3) becomes

$$\Phi(x) = -\eta + (1-c)x + cx^3 = 0, \tag{3.3.4}$$

and we have

$$\Phi(0) = -\eta, \qquad \Phi(1) = 1 - \eta.$$

Clearly, if $0 \leq \eta < 1$ and for any value of the parameter c the equation (3.3.4) has always a unique change of sign, then it is clear from Descarte's rule of sign that the above equation possesses a unique positive root $u^* \in (0,1)$ then, we conclude that the system (3.1.3) has a unique positive equilibrium states. Which completes the proof. $\square$

In the next of this paper, we confine system (3.1.3) to the spatial domain $\Omega = (0, L\pi)$ with $L \in \mathbb{R}^+$. This assumption has been put due to biological meaning. If we assume that the prey and the predator are in movement, then each one of the considered populations will follow a trajectory, the length of this trajectory will be considered as our variable x, and $L\pi$ can be considered as the maximal distance can be attained by a prey or predator, Δu becomes the one dimensional Laplace operator $\Delta u = u_{xx}$.

Defining the real-valued Sobolev space subject to the homogeneous Neumann boundary condition as follows

$$X = \left\{ U = (u, v)^T \in H^2(0, L\pi) \times H^2(0, L\pi) : (u_x, v_x)|_{x=0, L\pi} = 0 \right\}. \tag{3.3.5}$$

For $U_1 = (u_1, v_1)^T, U_2 = (u_2, v_2)^T \in X$, defining the usual inner product

$$< U_1, U_2 >= \int_0^{L\pi} (u_1 u_2 + v_1 v_2) \, dx.$$

Then the complexification of X is

$$X_{\mathbb{C}} = X \oplus iX = \{ U_1 + iU_2 : U_1, U_2 \in X \},$$

with the complex-valued inner product defined by

$$< U_1, U_2 >= \int_0^{L\pi} (\overline{u_1} u_2 + \overline{v_1} v_2) \, dx, \qquad U_i = (u_i, v_i)^T \in X_{\mathbb{C}}, \quad i = 1, 2.$$

3.4 Global stability of the semi trivial equilibrium state E_1

In this section, we analyze the global stability of the semi trivial equilibrium state $E_1 = (1, 0)$ where the positive equilibrium state does not exists (means that $\eta > 1$), We have the following theorem

Theorem 16. *([82]) For $\eta > 1$, the semi-trivial equilibrium state $E_1 = (1, 0)$ of the system (3.1.3) is globally asymptotically stable.*

Proof. Letting

$$f(u, v) = u(1 - u) - \sqrt{u}v, \qquad g(u, v) = \beta v(-m + \sqrt{u} - cv). \tag{3.4.1}$$

Defining the Jacobian matrix of the system (3.1.3) in the absence of diffusion as follows

$$J|_{(u,v)} = \begin{pmatrix} A(u, v) & B(u, v) \\ C(u, v) & D(u, v) \end{pmatrix}, \tag{3.4.2}$$
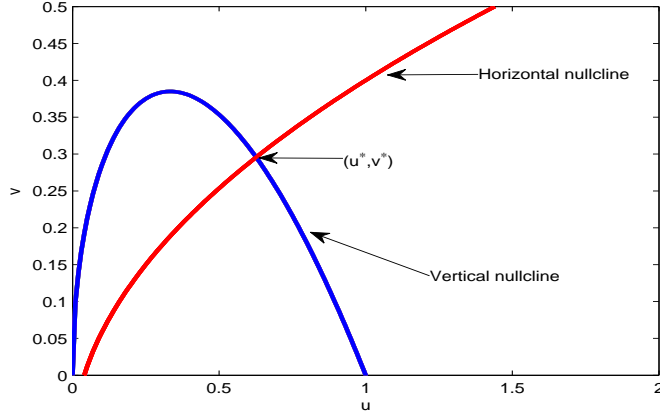
FIGURE 3.2: Graph of null-cline curves associated with model **(3.1.3)** shows the existence of the interior equilibrium (u^*, v^*). Here,we set the parameter values as $\eta = 0.2, c = 2,$.

with the domain $D_L = X_{\mathbb{C}}$, where

$$A(u,v) = 1 - 2u - \frac{v}{2\sqrt{u}}, \quad B(u,v) = -\sqrt{u}, \quad C(u,v) = \beta\frac{v}{2\sqrt{u}}, \quad D(u,v) = \beta(-\eta + \sqrt{u} - 2cv). \quad (3.4.3)$$

The linearized operator of the steady state system of (3.1.3) evaluated at the semi trivial equilibrium state $E_1 = (1,0)$ is given by

$$U_t = (D\Delta + J|_{(1,0)})U, \quad (3.4.4)$$

where $D = diag(d_1, d_2)$,and

$$J|_{(1,0)} = \begin{pmatrix} -1 & -1 \\ 0 & \beta(-m+1) \end{pmatrix}. \quad (3.4.5)$$

It is well know that the eigenvalue problem

$$-\varphi'' = \lambda\varphi, \quad x \in (0, L\pi); \quad \varphi'(0) = \varphi'(L\pi) = 0, \quad (3.4.6)$$

has eigenvalue $\lambda_n = \left(\frac{n}{L}\right)^2$ $(n = 0,1,2,...)$, with the corresponding eigenfunction $\varphi_n(x) = \cos\left(\frac{n}{L}x\right)$ in the Sobolev space X. In terms of the boundary condition

$$u_x(0,t) = u_x(L\pi,t) = v_x(0,t) = v_x(L\pi,t) = 0. \quad (3.4.7)$$

Setting

$$U = \sum_{n=0}^{+\infty} \begin{pmatrix} a_n \\ b_n \end{pmatrix} \cos\left(\frac{n\pi}{L}x\right)e^{\lambda_n t}, \quad (3.4.8)$$

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and then substituting it into (3.4.4). Then from a straightforward calculates , the eigenvalue of the matrix $-\left(\frac{n}{L}\right)^2 D + J|_{(1,0)}$ are:

$$\lambda_1 = -1 - \left(\frac{n}{L}\right)^2 d_1 < 0, \qquad \lambda_2 = \beta(-\eta+1) - \left(\frac{n}{L}\right)^2 d_2, \tag{3.4.9}$$

for $\eta > 1$ $\lambda_2 < 0$ for any $n \geq 0$ which means that the equilibrium states E_1 is locally asymptotically stable. Now, it is remain to prove the global attraction of the boundary equilibrium when $m > 1$, we use the result of **Theorem 2.1**, page 188 in [70]. From **Theorem 14** we proved that $(u^*(t), v^*(t))$ and $(0,0)$ are the upper and the lower solution of the system (2.1.2), respectively, we know that $\lim\limits_{t \to +\infty} \sup u(x,t) \leq 1$ and $u^*(t) \to 1$, so for an $\varepsilon > 0$ there exist $T_\varepsilon > 0$ such that

$$u^*(t) \leq 1 + \varepsilon, \qquad t > T_\varepsilon. \tag{3.4.10}$$

and there exist $M > 0$ for which $v(x,t) < M$ (using the estimations in **Theorem 2.1** in [70], then for ε sufficiently small there exists $T_1 > T$ such that $u(x,t) > \varepsilon$. Further we define the constant upper and the lower solution of the system (3.1.3) as

$$(u_1, v_1) = (1+\varepsilon, M), \qquad (u_2, v_2) = (\varepsilon, 0), \tag{3.4.11}$$

such that

$$\begin{cases} \frac{\partial u_1}{\partial t} - d_1 \Delta u_1 - h(u_1, v_1) = \varepsilon(1+\varepsilon) + \sqrt{1+\varepsilon}M \geq 0, \\ \frac{\partial v_1}{\partial t} - d_2 \Delta v_1 - k(u_1, v_1) = -M\beta(-\eta + \sqrt{1+\varepsilon} - cM) \geq -M\beta(-\eta + \sqrt{\varepsilon} - cM) \geq 0, \end{cases}$$

$$\begin{cases} \frac{\partial u_2}{\partial t} - d_1 \Delta u_2 - h(u_2, v_2) = -\varepsilon(1-\varepsilon) \leq 0, \\ \frac{\partial v_2}{\partial t} - d_2 \Delta v_2 - k(u_2, v_2) = 0 \leq 0, \end{cases}$$

where ε and M are positive constants and ε is sufficiently small. We define also the monotone sequences for coupled parabolic equation defined in [70] by $(\overline{u}^{(k)}, \overline{v}^{(k)})$ and $(\underline{u}^{(k)}, \underline{v}^{(k)})$ for $k = 1, 2, \dots$ as follows

$$\begin{cases} \overline{u}^{(k)} = \overline{u}^{(k-1)} + \dfrac{1}{c_1} F(\overline{u}^{(k-1)}, \underline{v}^{(k-1)}), \\ \overline{v}^{(k)} = \overline{v}^{(k-1)} + \dfrac{1}{c_2} G(\overline{u}^{(k-1)}, \overline{v}^{(k-1)}), \end{cases} \tag{3.4.12}$$

and

$$\begin{cases} \underline{u}^{(k)} = \underline{u}^{(k-1)} + \dfrac{1}{c_1} F(\underline{u}^{(k-1)}, \overline{v}^{(k-1)}), \\ \underline{v}^{(k)} = \underline{v}^{(k-1)} + \dfrac{1}{c_2} G(\underline{u}^{(k-1)}, \underline{v}^{(k-1)}), \end{cases} \tag{3.4.13}$$

where $\overline{u}^{(0)} = 1 + \varepsilon, \quad \overline{v}^{(0)}) = M$ and $\underline{u}^{(0)} = \varepsilon, \quad \underline{v}^{(0)} = 0$.

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Using **lemma 2.1** (page 199) of [70] we have

$$\begin{cases} \overline{u}^{(k)} \to \overline{u}, \qquad \underline{u}^{(k)} \to \underline{u} \\ \overline{v}^{(k)} \to \overline{v}, \qquad \underline{v}^{(k)} \to \underline{v}, \end{cases} \tag{3.4.14}$$

verifying

$$\varepsilon \le \underline{u}^{(k)} \le \underline{u}^{(k+1)} \le \underline{u} \le \overline{u} \le \overline{u}^{(k+1)} \le \overline{u}^{(k)} \le 1+\varepsilon,$$
$$0 \le \underline{v}^{(k)} \le \underline{v}^{(k+1)} \le \underline{v} \le \overline{v} \le \overline{v}^{(k+1)} \le \overline{v}^{(k)} \le M, \tag{3.4.15}$$

which means that $(\overline{u}^{(k)}, \overline{v}^{(k)})$ is increasing and $(\underline{u}^{(k)}, \underline{v}^{(k)})$ is decreasing. From (3.4.12), (3.4.13) and [70] $\overline{u}, \overline{v}$ and $\underline{u}, \underline{v}$ verify

$$F(\overline{u}, \underline{v}) = 0, \ F(\underline{u}, \overline{v}) = 0,$$
$$G(\overline{u}, \overline{v}) = 0, \ \ G(\underline{u}, \underline{v}) = 0, \tag{3.4.16}$$

with

$$F(\overline{u}, \underline{v}) = \overline{u}(1-\overline{u}) - \sqrt{\overline{u}}\underline{v} = 0, \qquad F(\underline{u}, \overline{v}) = \underline{u}(1-\underline{u}) - \sqrt{\underline{u}}\overline{v} = 0,$$
$$G(\overline{u}, \overline{v}) = \beta\overline{v}(-m + \sqrt{\overline{u}} - c\overline{v}) = 0, \qquad G(\underline{u}, \underline{v}) = \beta\underline{v}(-m + \sqrt{\underline{u}} - c\underline{v}) = 0, \tag{3.4.17}$$

Since, $\underline{v}^{(0)} = 0$ then $\underline{u}^{(k)} = 0$, $k = 0,1,...$, leads to $\underline{v}^{(k)} = 0$, therefore we have $\underline{v} = 0$ and from $F(\overline{u}, \underline{v}) = 0$ we obtain $\overline{u} = 1$ and it follows that $\overline{v} = 0$ and $\underline{u} = 1$, then from **Theorem 2.2** (page 189) of Pao [70] and the arbitrary largeness of M we have

$$(u,v) \to (1,0) \quad as \ t \to +\infty$$

which leads to the global attraction of the semi trivial equilibrium state E_1. Combining this result with the local stability of $(1,0)$, we can deduce that the semi trivial equilibrium state E_1 is globally asymptotically stable. Which completes the proof. □

3.5 Bifurcation analysis, Turing instability

In this context, we shall analyze the dynamics of the system (3.1.3) near the non trivial equilibrium state E^*, the existence of Hopf bifurcation, and Turing instability at the non trivial equilibrium state E^*. Throughout the rest part of this paper, we assume that $0 < \eta < 1$ and we choose β as a bifurcation parameter. Obviously u^* is independent on the parameter β, then system (3.1.3) can be rewritten in the following forme

$$\frac{\partial U}{\partial t} = D\Delta U + J(\beta)U + H(U), \tag{3.5.1}$$

where

$$J|_{(u^*,v^*)} = \begin{pmatrix} A(\beta) & B(\beta) \\ C(\beta) & D(\beta) \end{pmatrix}, \qquad D\Delta = diag\left(d_1\frac{\partial^2}{\partial x^2}, d_2\frac{\partial^2}{\partial y^2}\right), \tag{3.5.2}$$

$$H(U) = \begin{pmatrix} u(1-u) - \sqrt{u}v - A(\beta)u - B(\beta)v \\ \beta v(-m + \sqrt{u} - cv) - C(\beta)u - D(\beta)v \end{pmatrix}. \tag{3.5.3}$$

and

$$A(\beta) = 1 - 2u^* - \frac{v^*}{2\sqrt{u^*}}, \quad B(\beta) = -\sqrt{u^*}, \quad C(\beta) = \beta \frac{v^*}{2\sqrt{u^*}}, \quad D(\beta) = -\beta v^*. \tag{3.5.4}$$

Using Eq. (3.1.1), we obtain

$$A(\beta) = \frac{1}{2} - \frac{3}{2}u^*, \quad B(\beta) = -\sqrt{u^*}, \quad C(\beta) = \frac{1}{2}\beta(1-u^*), \quad D(\beta) = -\beta c\sqrt{u^*}(1-u^*). \tag{3.5.5}$$

The linearized system of system (3.5.1) at E^* is given by

$$\frac{\partial U}{\partial t} = \left(D\Delta + J|_{(u^*, v^*)} \right) U, \tag{3.5.6}$$

it is easy to prove that the solution of (3.5.6) is a nontrivial solution of a linear problem (3.5.6) if and only if there is a $n \in \mathbb{N}$ for which λ satisfies

$$\det \left(\lambda I - \left(\frac{n}{L} \right)^2 D - J|_{(u^*, v^*)} \right) = 0 \tag{3.5.7}$$

where I is 2×2 identity matrix, and $D = diag(d_1, d_1)$. From a straightforward computation, we obtain the characteristic equation of (3.5.1) as follows

$$\lambda^2 - T_n(\beta)\lambda + D_n(\beta) = 0, \tag{3.5.8}$$

with

$$T_n(\beta) = \frac{1}{2}(1 - 3u^*) - \beta c\sqrt{u^*}(1-u^*) - (d_1 + d_2)\left(\frac{n}{L} \right)^2, \tag{3.5.9}$$

$$D_n(\beta) = d_1 d_2 \left(\frac{n}{L} \right)^4 + \left(\frac{n}{L} \right)^2 \left[d_1 \beta c\sqrt{u^*}(1-u^*) - \frac{1}{2}d_2(1-3u^*) \right] + \frac{1}{2}\beta\sqrt{u^*}(1-u^*)(1-c(1-3u^*)). \tag{3.5.10}$$

3.5.1 Hopf bifurcation:

In this subsection, we show the existence of time-periodic orbits with a careful Hopf bifurcation and give the eventual Hopf bifurcation points. For having Hopf bifurcation it is necessary to put the condition $0 < u^* < (3)^{-1}$. From [43], it is known that $T_n(\beta) = 0$, $D_n(\beta) > 0$ and $\frac{\partial}{\partial \beta}\lambda(\beta)|_{\beta=\beta_H} \neq 0$ are necessary conditions for Hopf bifurcation to occur. First, we shall prove the non existence of Hopf bifurcation for some value of the model parameters. The following theorem illustrate the obtained results.

Lemma 6. *([82]) If $3^{-1} < u^* < 1$, the system (3.1.3) has no Hopf bifurcation and the non trivial constant steady state E^* is always stable for any value of β (with diffusion and without diffusion).*

Proof. It is easy to check that for $3^{-1} < u^* < 1$ we have, $T_n(\beta) < 0$ for any value of β and consequently there is no Hopf bifurcation. So, the non trivial constant steady state E^* is always stable for any value of β The proof is completed. $\qquad\square$

Now, discussing the positivity of $D_0(\beta)$ then we resume the obtained results in the following lemma

Lemma 7. *([82]) Under the condition $0 < \eta < 1$ we have*
(i): For $0 < u^* < \dfrac{1}{3}\left(1 - \dfrac{1}{c}\right)$ then $D_0(\beta) < 0$ with $c > 1$.
(ii): For $\dfrac{1}{3}\left(1 - \dfrac{1}{c}\right) < u^* < 1$ then $D_0(\beta) > 0$.

Proof. Recalling that from (3.5.10) we have

$$D_0(\beta) = \frac{1}{2}\beta\sqrt{u^*}(1 - u^*)(1 - c(1 - 3u^*)). \tag{3.5.11}$$

Obviously the positivity of $D_0(\beta)$ depends only on $1 - c(1 - 3u^*)$. So the results obtained in **Lemme 7** can be easily deduced. Which completes the proof. $\qquad\square$

For the existence of the eventual Hopf bifurcation we draw the following lemma.

Lemma 8. *([82]) Putting*

$$N^* = \left[\frac{(1 - 3u^*)L^2}{2(d_1 + d_2)}\right]. \tag{3.5.12}$$

where [.] stands for the integer part function. The system (3.1.3) undergo Hopf bifurcation at $\beta = \beta_n (0 \le n \le N^*)$ where β_n is decreasing sequence in n $(n = 0, 1, ..., N^*)$ and

$$\beta_n = \frac{(1 - 3u^*) - 2(d_1 + d_2)\left(\dfrac{n}{L}\right)^2}{2c\sqrt{u^*}(1 - u^*)}.$$

Proof. Noticing that Hopf bifurcation occurs if and only if $T_n(\beta) = 0$, Solving this equation in β we obtain

$$\beta_n = \frac{(1 - 3u^*) - 2(d_1 + d_2)\left(\dfrac{n}{L}\right)^2}{2c\sqrt{u^*}(1 - u^*)}. \tag{3.5.13}$$

Next, we prove that β_n is a decreasing sequence. So, a straight forward calculation gives

$$\beta_{n+1} - \beta_n = \frac{-(d_1 + d_2)(2n + 1)}{cL^2\sqrt{u^*}(1 - u^*)} < 0. \tag{3.5.14}$$

Thus, β_n is strictly decreasing sequence. For the ecological meaning of β_n, we must choose some value of the positive integer n such that $\beta_n \geq 0$ which means that

$$\frac{1}{2}(1 - 3u^*) > (d_1 + d_2)\left(\frac{n}{L}\right)^2, \tag{3.5.15}$$

which leads to $n < N^*$ where N^* is defined in (3.5.12) and the bifurcation points are $\beta_0, \beta_1, ..., \beta_N$. $\qquad \square$

Let $\lambda(\beta) = \alpha(\beta) \pm i\omega(\beta)$ be the solution of the characteristic equation (3.5.8) satisfying

$$\alpha(\beta_n) = 0, \quad \omega(\beta_n) = \sqrt{D(\beta_n)},$$

then, we have the following transversality condition

$$\alpha'(\beta_n) = -c\sqrt{u^*}(1 - u^*) < 0. \tag{3.5.16}$$

Combining the transversality condition with **Lemme 8** then, the bifurcation points and their order is given by the following theorem

Theorem 17. *If there exist N_1 $(1 \leq N_1 \leq N^*)$ a critical values denoted by $j_0, ..., j_{N_1}$ such that $j_0 = 0 < j_1 < ... < j_{N_1} < N^*$ and $D_{j\xi}(\beta(j\xi)) > 0, \xi = 0...N_1$ we have the following estimation:*

$$\beta_{jN_1} < ... < \beta_{j\xi} < \beta_{j\xi - 1} < ... < \beta_{j1} < \beta_{j0}. \tag{3.5.17}$$

Now discussing the dynamics introduced by the presence of Hopf bifurcation, which is given by the following theorem.

Theorem 18. *([82]) Assume that $0 < \eta < 1$ then we have the following results*

(i) If $(3)^{-1} < u^* < 1$ holds then the non trivial equilibrium state $E^* = (u^*, v^*)$ is always stable for any value of β.

(ii) If $\frac{1}{3}\left(1 - \frac{1}{c}\right) < u^* < (3)^{-1}$ holds then we have

1. *for $n = 0$ the non trivial equilibrium state $E^* = (u^*, v^*)$ is asymptotically stable for $\beta > \beta_0$ and unstable for $\beta < \beta_0$.*

2. *The system (3.1.3) undergoes Hopf bifurcation near the interior equilibrium $E^* = (u^*, v^*)$ when $\beta = \beta_n, (n < N_1)$. A family of homogenous periodic solutions occurs for $n = 0$, and a family of nonhomogeneous periodic solution occurs for $n = 1, 2, ..., N_1$.*

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3.5.2 Turing driven instability:

In this subsection, the main objective is to discuss the existence/nonexistence of the diffusion-driven instability and give a sufficient condition for Turing driven instability. Turing concluded that the reaction-diffusion model may exhibit spatial patterns under the following two conditions:

(i) The equilibrium is linearly stable in the absence of diffusion $(D_0(\beta) > 0$ and $T_0(\beta) < 0)$;

(ii) The equilibrium state becomes linearly unstable in the presence of diffusion$(D_n(\beta) < 0$ for some value of $n)$.

Such as instability called by diffusion-driven instability. To discuss the Turing instability, we must assume that $D_0(\beta) > 0$, and $T_0(\beta) < 0$ which is ensured if

$$\frac{1}{3}\left(1 - \frac{1}{c}\right) < u^* < (3)^{-1}, \text{ and } \beta > \beta_0.$$

Then, we rewrite the system (3.5.10) as the following form:

$$D\left(\left(\frac{n}{l}\right)^2\right) = d_1 d_2 \left(\left(\frac{n}{l}\right)^2\right)^2 + \left(\frac{n}{l}\right)^2 \left[d_1 \beta c \sqrt{u^*}(1 - u^*) - \frac{1}{2}d_2(1 - 3u^*)\right] + \frac{1}{2}\beta(1 - u^*)(1 - c(1 - 3u^*)).$$
(3.5.18)

Now discussing the non existence of diffusion driven instability, the following lemma illustrate the obtained results

Lemma 9. *([82]) Assume that*

$$d_2 \leq \frac{d_1 \beta}{\beta_0} \tag{3.5.19}$$

then the system (3.1.3) has no diffusion driven instability

It is easy to see that under the condition (3.5.19) we have $D\left(\left(\frac{n}{l}\right)^2\right) > 0$ then we can not have Turing driven instability. Now assuming that $d_2 > \frac{d_1 \beta}{\beta_0}$ then the minimum of the functional $D\left(\left(\frac{n}{l}\right)^2\right)$ occurs when

$$\left(\frac{n}{l}\right)^2 = \left(\frac{n}{l}\right)^2_{cr}, \tag{3.5.20}$$

where

$$\left(\frac{n}{l}\right)^2_{cr} = \frac{-d_1 \beta c \sqrt{u^*}(1 - u^*) + \frac{1}{2}d_2(1 - 3u^*)}{2d_1 d_2}, \tag{3.5.21}$$

Remark 2. *When $(\sqrt{3})^{-1} < u^* < 1$, we have $\left(\frac{n}{l}\right)^2_{cr} < 0$ for any value of β, but when $0 < u^* < (\sqrt{3})^{-1}$ we obtain $\left(\frac{n}{l}\right)^2_{cr} > 0$ for $\beta < \beta^*$ with*

$$\beta^* = \frac{d_2(1 - 3u^*)}{2d_1 c \sqrt{u^*}(1 - u^*)}. \tag{3.5.22}$$

We know that the sufficient condition for Turing instability is that $D(\left(\frac{n}{l}\right)^2_{cr}) < 0$, which is equivalent to

$$d_1 \beta c \sqrt{u^*}(1 - u^*) > 2\sqrt{d_1 d_2}\sqrt{\frac{1}{2}\beta(1 - u^*)(1 - c(1 - 3u^*))} + \frac{1}{2}d_2(1 - 3u^*) \quad \text{for} \quad u^* < (\sqrt{3})^{-1}. \quad (3.5.23)$$

which is a sufficient condition for the occurrence of Turing driven instability.

3.5.3 Turing-Hopf bifurcation

In this subsection, our main focus is on studying the existence of Turing-Hopf bifurcation, and deducing the spatiotemporal dynamics near this point. This type of bifurcation can be obtained using two bifurcation parameters and occurs if there exists two integers $n_H \neq n_T$ such that the system undergoes Hopf bifurcation for $n = n_H$ (means that $T_{n_H}(\beta_{TH}, d_{2TH}) = 0$ and $D_{n_H}(\beta_{TH}, d_{2TH}) > 0$) and Turing bifurcation $n = n_T$ (means that $D_{n_T}(\beta_{TH}, d_{2TH}) = 0$ and $T_{n_T}(\beta_{TH}, d_{2TH}) \neq 0$). Now, we assume that $\frac{c-1}{3c} < u^* < \frac{1}{3}$ (for having $T_0(\beta, d_2) = 0$ and $D_0 > 0$) and taking β, d_2 as bifurcation parameters. If we choose $n_H = 0$, then $T_0(\beta) = 0$ equivalent to

$$\beta = \beta_H(d_2) = \frac{1 - 3u^*}{2c\sqrt{u^*}(1 - u^*)} = \beta_{TH}, \quad (3.5.24)$$

where, β_{TH} represents the line of Hopf bifurcation in $d_2 - \beta$ plan. On the other hand, we look for the existence of a positive integer $n_T \neq 0$ such that $D_{n_T}(\beta, d_2) = 0$. Solving $D_n = 0$ in β we obtain

$$\beta = \beta_T(d_2, n) = \frac{\frac{1}{2}\left(\frac{n}{L}\right)^2(1 - 3u^*) - d_1\left(\frac{n}{L}\right)^4}{d_1\left(\frac{n}{L}\right)^2 c\sqrt{u^*}(1 - u^*) + \frac{1}{2}\sqrt{u^*}(1 - u^*)(1 - c(1 - 3u^*))}d_2, \quad (3.5.25)$$

For simplicity we put

$$A_1 = \frac{1}{2L^2}(1 - 3u^*) > 0,$$
$$A_2 = \frac{d_1 c\sqrt{u^*}(1 - u^*)}{L^2} > 0,$$
$$A_3 = \frac{1}{2}\sqrt{u^*}(1 - u^*)(1 - c(1 - 3u^*)) > 0,$$

then (3.5.25) becomes

$$\beta = \beta_T(d_2, n) = \frac{n^2 A_1 - \frac{d_1}{L^4}n^4}{n^2 A_2 + A_3}d_2, \quad (3.5.26)$$

Obviously, for having the intersection between Hopf bifurcation line defined by (3.5.24) and Turing bifurcation defined by (3.5.26) in $d_2 - \beta$ plan we must have $\frac{n^2 A_1 - \frac{d_1}{L^4}n^4}{n^2 A_2 + A_3} > 0$. Defining the following functional

$$f(x) = \frac{x^2 A_1 - \frac{d_1}{L^4}x^4}{x^2 A_2 + A_3}, \quad \text{for } x \in [1, n^*],$$

where $n^* = \max\{n \in \mathbb{N} - \{0\}/A_1 - \frac{d_1}{l^4}n^2 > 0\}$. Now setting

$$\tilde{x} = \sqrt{\frac{-3d_1 A_3 + \sqrt{9d_1^2 A_3^2 + 8l^4 A_1 A_2 A_3 d_1}}{2d_1 A_2}}, \quad (3.5.27)$$

then we have

$$f'(x) = \begin{cases} > 0, & \text{for } x < \tilde{x}, \\ < 0, & \text{for } x > \tilde{x}, \end{cases}$$

In order to chose the maximum positive value of the functional f, we put

$$n_T = \begin{cases} 1, & \text{if } \tilde{x} \leq 1, \\ [\tilde{x}], & \text{if } f([\tilde{x}]+1) \leq f([\tilde{x}]), 1 \leq \tilde{x} \leq n^*, \\ [\tilde{x}]+1, & \text{if } f([\tilde{x}]) \leq f([\tilde{x}]+1), 1 \leq \tilde{x} \leq n^*, \\ n^*, & \text{if } \tilde{x} \geq n^*. \end{cases}$$

Obviously, we proved that n_T verify

$$f(n_T) = \max_{1 \leq n \leq n^*} f(n).$$

Thus, the line of Hopf bifurcation H_0 intersects Turing bifurcation curve T_{n_T} at the point

$$(d_{2TH}, \beta_{TH}) = \left(\frac{1-3u^*}{2c\sqrt{u^*(1-u^*)}f(n_T)}, \frac{1-3u^*}{2c\sqrt{u^*(1-u^*)}} \right),$$

which called the Turing-Hopf bifurcation point. Now, we will focus on verifying the transversality condition. Fixing d_2, and taking β as parameter we denote by $\lambda(\beta)$ the roots of the characteristic equation. we have

$$\frac{dRe(\lambda(\beta))}{d\beta}\bigg|_{H_0} = \frac{dRe(\lambda(\beta))}{d\beta}\bigg|_{T_{n_T}} = -c\sqrt{u^*}(1-u^*) < 0,$$

then we have the following results

Theorem 19. *([82]) Assume that $0 < \eta < 1$ and $\frac{1}{3} - \frac{1}{3c} < u^* < \frac{1}{3}$ then we have **(i)** The Hopf bifurcation curve H_0 intersects with Turing bifurcation curve T_{n_T} at $d_2 - \beta$ plan and the Turing-Hopf bifurcation occurs a the intersection point*

$$(d_{2TH}, \beta_{TH}) = \left(\frac{1-3u^*}{2c\sqrt{u^*(1-u^*)}f(n_T)}, \frac{1-3u^*}{2c\sqrt{u^*(1-u^*)}} \right).$$

(ii) *for $(d_2, \beta) = (d_{2TH}, \beta_{TH})$ the characteristic equation (3.5.8) has a pair of purely imaginary roots for $n = 0$ and a simple zero root for $n = n_T$.*

3.6 Normal form on the center of manifold for Hopf bifurcation:

In this context, we shall study the direction and stability of Hopf bifurcation by using the normal form of Hopf bifurcation on the center manifold. From **Theorem 18**, we know that the system (3.1.3) undergoes Hopf bifurcation near the interior equilibrium state $E^* = (u^*, v^*)$ when the parameter β across the curve $d_2 = d_{2H}(\beta, n)$. Thanks to the paper [43] we will compute the normal form on the center manifold

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associated to the Hopf bifurcation and we mainly focus on the existence of spatially homogeneous and nonhomogeneous periodic solutions bifurcating from the Hopf bifurcation of the reaction-diffusion system at the interior equilibrium state $E^* = (u^*, v^*)$. To continue our discussion into the frame work of the Hopf bifurcation theorem, we translate (3.1.3) into the following system by the transition $\tilde{u} = u - u^*$ and $\tilde{v} = v - v^*$ and $\mu = u^*$ and $v^* = v_\mu^* = \sqrt{\mu}(1 - \mu)$. For convenience, we drop the tilde. Thus, the reaction-diffusion system (3.1.3) becomes

$$
\begin{cases}
\frac{\partial u}{\partial t} = (u + u^*)(1 - (u + u^*)) - \sqrt{u + u^*}(v + v^*) + d_1 \Delta u, \\[2mm]
\frac{\partial v}{\partial t} = \beta(v + v^*)(-m + \sqrt{u + u^*} - c(v + v^*)) + d_2 \Delta v,
\end{cases}
\tag{3.6.1}
$$

for $x \in \Omega = (0, l\pi)$ and $t \in (0, \infty)$. We rewrite system (3.6.1) as

$$
\dot{U}(t) = LU + F(U, \mu), \quad \text{for any} \quad U = (u, v)^T \in X,
\tag{3.6.2}
$$

such that

$$
L = D\Delta + J(\mu) = \begin{pmatrix} A(\mu) + d_1 \frac{\partial^2}{\partial x^2} & B(\mu) \\[2mm] C(\mu) & d_2 \frac{\partial^2}{\partial x^2} + D(\mu) \end{pmatrix},
$$

and

$$
F(U, \mu) = \begin{pmatrix} F_1(U, \mu) \\[2mm] F_2(U, \mu) \end{pmatrix},
$$

$$
= \begin{pmatrix} (u + u^*)(1 - u - u^*) - \sqrt{u + u^*}(v + v^*) - a_{11}(\mu)u - a_{12}(\mu)v \\[2mm] \beta(v + v^*)(-m + \sqrt{u + u^*} - c(v + v^*) - a_{21}(\mu)u - a_{22}(\mu)v \end{pmatrix}
$$

and

$$
A(\mu) = \frac{1}{2} - \frac{3}{2}\mu, \quad B(\mu) = -\sqrt{\mu}, \quad C(\mu) = \frac{1}{2}\beta(1 - \mu), \quad D(\mu) = -\beta c \sqrt{\mu}(1 - \mu).
\tag{3.6.3}
$$

Let $< ., . >$ be the complex-valued L^2 inner product on Hilbert space $X_\mathbb{C}$, defined by

$$
< U_1, U_2 > = \int_0^{l\pi} (\overline{u_1}u_2 + \overline{v_1}v_2) \, dx, \qquad U_j = (u_j, v_{ji})^T \in X_\mathbb{C}, \quad j = 1, 2.
$$

Then, we define the adjoint operator of the operator L^* as

$$
L^* = D\Delta + J(\mu) = \begin{pmatrix} A(\mu) + d_1 \frac{\partial^2}{\partial x^2} & C(\mu) \\[2mm] B(\mu) & d_2 \frac{\partial^2}{\partial x^2} + D(\mu) \end{pmatrix},
$$

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then $L_n^*(\mu) = -n^2 D + J^*(\mu)$ where $J^*(\mu)$ is the adjoint matrix of $J(\mu)$. After computation, we obtain the eigenfunctions of $L_n(\mu)$ and $L_n^*(\mu)$ corresponding to the eigenvalue $i\hat{\omega}$ and $-i\hat{\omega}$ on $X_{\mathbb{C}}$ as follows

$$q = \begin{pmatrix} a_n \\ b_n \end{pmatrix} \cos(\frac{n}{l}x) \quad \text{and} \quad q^* = \begin{pmatrix} a_n^* \\ b_n^* \end{pmatrix} \cos(\frac{n}{l}x)$$

satisfying $< q^*, q >= 1$ and $< q^*, \bar{q} >= 0$ and also $L_n^*(\mu)q^* = -i\hat{\omega}q^*$, where

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{i\omega - (A - d_1(\frac{n}{l})^2}{C} \end{pmatrix} n = 0, 1, ..N_1 \text{ and } \mu = 0.$$

And for $\mu = 0$ we have

$$\begin{pmatrix} a_n^* \\ b_n^* \end{pmatrix} = \begin{cases} (\frac{1}{l\pi} \frac{BC}{BC + (i\hat{\omega} + A)^2}, \frac{1}{l\pi} \frac{-B(i\hat{\omega} + A)}{BC + (i\hat{\omega} + A)^2})^T & \text{if } n = 0 \\ (\frac{2}{l\pi} \frac{BC}{BC + (i\hat{\omega} + A - d_1(\frac{n}{l})^2)^2}, \frac{2}{l\pi} \frac{-BC}{BC + (i\hat{\omega} + A - d_1(\frac{n}{l})^2)^2})^T & \text{if } n \neq 0 \end{cases}.$$

We decompose $X = X^{\mathbb{C}} \oplus X^{\mathbb{S}}$ with the center subspace $X^{\mathbb{C}} := \{zq + \bar{z}\bar{q} : z \in \mathbb{C}\}$ and the stable subspace $X^{\mathbb{S}} := \{U \in X :< q^*, U >= 0\}$. For any $U = (u, v) \in X$, there exists $z \in \mathbb{C}$ and $W = (w_1, w_2) \in X^{\mathbb{S}}$ such that

$$\begin{pmatrix} u \\ v \end{pmatrix} = zq + \bar{z}\bar{q} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{cases} u = a_n z \cos(nx) + \bar{a}_n \bar{z} \cos(nx) + w_1 \\ v = b_n z \cos(nx) + \bar{b}_n \bar{z} \cos(nx) + w_2 \end{cases}. \tag{3.6.4}$$

Now we will follow the work in [43] so we have

$$\begin{cases} \dfrac{dz}{dt} = i\hat{\omega}z + < q^*, F_n(U) >, \\[3mm] \dfrac{dW}{dt} = L(\mu)w + H(z, \bar{z}, W), \end{cases} \tag{3.6.5}$$

where

$$\begin{cases} H(z, \bar{z}, w) = F_n(U) - < q^*, F_n(U) > q - < \bar{q}^*, F_n(U) > \bar{q}, \\ F_n(U) = \frac{1}{2}Q_{UU} + \frac{1}{6}C_{UUU} + o(|U|^4). \end{cases} \tag{3.6.6}$$

where Q_{UU} and C_{UUU} represent the second and the third order of the studied system (3.1.3), where

$$\begin{cases} Q_{qq} = \begin{pmatrix} C_n \\ D_n \end{pmatrix} \cos^2(\frac{n}{l}x), \\[3mm] Q_{\bar{q}\bar{q}} = \bar{Q}_{qq}, \\[3mm] Q_{q\bar{q}} = \begin{pmatrix} E_n \\ F_n \end{pmatrix} \cos^2(\frac{n}{l}x), \\[3mm] Q_{q\bar{q}} = \begin{pmatrix} G_n \\ H_n \end{pmatrix} \cos^2(\frac{n}{l}x), \end{cases} \tag{3.6.7}$$

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and

$$
\begin{cases}
C_n = F_{1uu}a_n^2 + 2F_{1uv}a_nb_n + F_{1vv}b_n^2, \\
D_n = F_{2uu}a_n^2 + 2F_{2uv}a_nb_n + F_{2vv}b_n^2, \\
E_n = F_{1uu}a_n\bar{a}_n + F_{1uv}(a_n\bar{b}_n + \bar{a}_nb_n) + F_{1vv}b_n\bar{b}_n, \\
F_n = F_{2uu}a_n\bar{a}_n + F_{2uv}(a_n\bar{b}_n + \bar{a}_nb_n) + F_{2vv}b_n\bar{b}_n, \\
G_n = F_{1uuu}a_n^2\bar{a}_n + F_{1uuv}(a_n^2\bar{b}_n + 2a_n\bar{a}_nb_n) + F_{1uvv}(\bar{a}_nb_n^2 + 2a_nb_n\bar{b}_n) + F_{1vvv}b_n^2\bar{b}_n, \\
H_n = F_{2uuu}a_n^2\bar{a}_n + F_{2uuv}(a_n^2\bar{b}_n + 2a_n\bar{a}_nb_n) + F_{2uvv}(\bar{a}_nb_n^2 + 2a_nb_n\bar{b}_n) + F_{2PPP}b_n^2\bar{b}_n.
\end{cases}
\tag{3.6.8}
$$

The normal form of system (3.1.3) restricted to the center manifold is given by

$$
\frac{dz}{dt} = i\hat{\omega}z + <q^*, F_n(U)> = i\hat{\omega}z + \sum_{2 \le i+j \le 3} \frac{g_{ij}}{i!j!}z^i z^j + o(|z|^4),
\tag{3.6.9}
$$

where

$$
\begin{cases}
g_{20} = <q^*, Q_{qq}>, \\
g_{11} = <q^*, Q_{q\bar{q}}>, \\
g_{02} = <q^*, Q_{\bar{q}\bar{q}}>, \\
g_{21} = 2<q^*, Q_{W_{11}q}> + <q^*, Q_{W_{20}\bar{q}}> + <q^*, C_{qq\bar{q}}>.
\end{cases}
\tag{3.6.10}
$$

For the spatially homogeneous Hopf bifurcation, that is, $n = 0$, we have

$$
\begin{cases}
g_{20} = l\pi(\bar{a}_0^* C_0 + \bar{b}_0^* D_0), \\
g_{11} = l\pi(\bar{a}_0^* E_0 + \bar{b}_0^* F_0), \\
g_{02} = l\pi(\bar{a}_0^* \bar{C}_0 + \bar{b}_0^* \bar{D}_0), \\
g_{21} = 2<q^*, Q_{w_{11}q}> + <q^*, Q_{w_{20}\bar{q}}> + <q^*, C_{qq\bar{q}}>,
\end{cases}
$$

where

$$
<q^*, Q_{w_{11}q}> = l\pi[\bar{a}_0^*(F_{1uu}w_{11}^{(1)}a_0 + F_{1uv}(w_{11}^{(1)}b_0 + w_{11}^{(2)}a_0) + F_{1vv}w_{11}^{(2)}b_0)
$$
$$
+ \bar{b}_0^*(F_{2uu}w_{11}^{(1)}a_0 + F_{2uv}(w_{11}^{(1)}b_0 + w_{11}^{(2)}a_0) + F_{2vv}w_{11}^{(2)}b_0)].
$$

$$
<q^*, Q_{w_{20}\bar{q}}> = l\pi[\bar{a}_0^*(F_{1uu}w_{20}^{(1)}\bar{a}_0 + F_{1uv}(w_{20}^{(1)}\bar{b}_0 + w_{20}^{(2)}\bar{a}_0) + F_{1vv}w_{20}^{(2)}\bar{b}_0)
$$
$$
+ \bar{b}_0^*(F_{2uu}w_{20}^{(1)}\bar{a}_0 + F_{2uv}(w_{20}^{(1)}\bar{b}_0 + w_{20}^{(2)}\bar{a}_0) + F_{2vv}w_{20}^{(2)}\bar{b}_0).
$$

$$
<q^*, C_{qq\bar{q}}> = l\pi(\bar{a}_0^* g_0 + \bar{b}_0^* h_0),
$$

and

$$
w_{11} = -[L_0(\mu)]^{-1}H_{11} = \begin{pmatrix} w_{11}^{(1)} \\ w_{11}^{(2)} \end{pmatrix} = \begin{pmatrix} -\frac{H_{11}^{(2)}}{C} \\ -\frac{-H_{11}^{(1)}C + H_{11}^{(2)}A}{BC} \end{pmatrix}.
$$

$$
w_{20} = [2i\hat{\omega}diag(1,1) - L_0(\mu)]^{-1}H_{20} = \begin{pmatrix} w_{20}^{(1)} \\ w_{20}^{(2)} \end{pmatrix} = \begin{pmatrix} \frac{H_{20}^{(1)}2i\hat{\omega} + H_{20}^{(2)}B}{(2i\hat{\omega}-A)2i\hat{\omega}-BC} \\ -\frac{H_{20}^{(1)}C + H_{20}^{(2)}(2i\hat{\omega}-A)}{(2i\hat{\omega}-A)2i\hat{\omega}-BC} \end{pmatrix},
$$

where

$$H_{11} = Q_{q\bar{q}} - <q^*, Q_{q\bar{q}}> q - <\bar{q}^*, Q_{q\bar{q}}> \bar{q} = \begin{pmatrix} H_{11}^{(1)} \\ H_{11}^{(2)} \end{pmatrix},$$

$$= \begin{pmatrix} E_0 - l\pi a_0(\bar{a}_0^* E_0 + \bar{b}_0^* F_0) - l\pi\bar{a}_0(a_0^* E_0 + b_0^* F_0) \\ F_0 - l\pi b_0(\bar{a}_0^* E_0 + \bar{b}_0^* F_0) - l\pi\bar{b}_0(a_0^* E_0 + b_0^* F_0) \end{pmatrix}.$$

$$H_{20} = Q_{qq} - <q^*, Q_{qq}> q - <\bar{q}^*, Q_{qq}> \bar{q} = \begin{pmatrix} H_{20}^{(1)} \\ H_{20}^{(2)} \end{pmatrix},$$

$$= \begin{pmatrix} C_0 - \pi a_0(\bar{a}_0^* C_0 + \bar{b}_0^* D_0) - \pi\bar{a}_0(a_0^* C_0 + b_0^* D_0) \\ D_0 - \pi b_0(\bar{a}_0^* C_0 + \bar{b}_0^* D_0) - \pi\bar{b}_0(a_0^* C_0 + b_0^* D_0) \end{pmatrix}.$$

After some straightforward calculation and manipulation we obtain

$$<q^*, Q_{q\bar{q}}> = <q^*, Q_{qq}> = <\bar{q}^*, Q_{qq}> = <\bar{q}^*, Q_{\bar{q}\bar{q}}> = 0,$$

then we obtain

$$g_{20} = g_{11} = g_{02} = 0,$$

$$g_{21} = 2<q^*, Q_{w_{11}q}> + <q^*, Q_{w_{20}\bar{q}}> + <q^*, C_{qq\bar{q}}>,$$

where

$$w_{11} = -\tfrac{1}{2}[L_n(\mu)]^{-1}[\cos\tfrac{2n}{l}x + 1]\begin{pmatrix} E_n \\ F_n \end{pmatrix},$$

$$= w_{11n}\cos\tfrac{2n}{l}x + w_{11n},$$

and

$$w_{20} = -\tfrac{1}{2}[2i\hat{\omega}diag(1,1) - L_n(\mu)]^{-1}[\cos\tfrac{2n}{l}x + 1]\begin{pmatrix} C_n \\ D_n \end{pmatrix},$$

$$= w_{20n}\cos\tfrac{2n}{l}x + w_{20n},$$

with

$$<q^*, Q_{w_{11}q}> = \frac{3l\pi}{4}[\bar{a}_n^*(F_{1uu}w_{11n}^{(1)}a_n + F_{1uv}(w_{11n}^{(1)}b_n + w_{11}^{(2)}a_n) + F_{1vv}w_{11n}^{(2)}b_n)$$
$$+\bar{b}_n^*(F_{2uu}w_{11n}^{(1)}a_n + F_{2uv}(w_{11n}^{(1)}b_n + w_{11n}^{(2)}a_n) + F_{2vv}w_{11n}^{(2)}b_n)],$$

and

$$<q^*, Q_{w_{20}\bar{q}}> = \frac{3l\pi}{4}[\bar{a}_0^*(F_{1uu}w_{20n}^{(1)}\bar{a}_0 + F_{1uv}(w_{20n}^{(1)}\bar{b}_0 + w_{20}^{(2)}\bar{a}_0) + F_{1vv}w_{20n}^{(2)}\bar{b}_0)$$
$$+\bar{b}_0^*(F_{2uu}w_{20n}^{(1)}\bar{a}_0 + F_{2uv}(w_{20n}^{(1)}\bar{b}_0 + w_{20}^{(2)}\bar{a}_0) + F_{2vv}w_{20n}^{(2)}\bar{b}_0),$$

$$<q^*, C_{qq\bar{q}}> = \frac{3l\pi}{8}(\bar{a}_0^* G_0 + \bar{b}_0^* H_0),$$

where

$$w_{20n} = \tfrac{1}{2}[2i\hat{\omega}diag(1,1) - L_n(\mu)]^{-1}\begin{pmatrix} C_n \\ D_n \end{pmatrix} = \begin{pmatrix} w_{20n}^{(1)} \\ w_{20n}^{(2)} \end{pmatrix},$$

$$= \frac{1}{2\bar{D}_{n\hat{\omega}}}\begin{pmatrix} C_n(2i\hat{\omega} + d_2(\tfrac{n}{l})^2) + D_n B \\ D_n(2i\hat{\omega} - A + d_1(\tfrac{n}{l})^2) + E_n C \end{pmatrix},$$

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such that

$$
\begin{aligned}
\tilde{D}_{n\hat{\omega}} &= \det(2i\hat{\omega}\,diag(1,1) - L_n(\mu)) \\
&= (2i\hat{\omega} + d_2(\frac{n}{l})^2)(2i\hat{\omega} - A + d_1(\frac{n}{l})^2) - BC,
\end{aligned}
$$

and

$$
\begin{aligned}
w_{11n} &= -\frac{1}{2}[L_n(\mu)]^{-1}\begin{pmatrix} E_n \\ F_n \end{pmatrix} = \begin{pmatrix} w_{11n}^{(1)} \\ w_{11n}^{(2)} \end{pmatrix} \\
&= -\frac{1}{2D_n}\begin{pmatrix} -E_n d_2(\frac{n}{l})^2 + f_n B \\ f_n(A - d_1(\frac{n}{l})^2) - E_n C \end{pmatrix},
\end{aligned}
$$

where D_n is defined in (3.5.10). Using the results in [43] the direction and stability of Hopf bifurcation can be determined by the following values

$$
\mu_2 = -\frac{Re(c_1(0))}{\beta'(0)},
$$

$$
v_2 = 2Re(c_1(0)),
$$

with

$$
c_1(0) = \frac{i}{2\hat{\omega}}\left(g_{20}g_{11} - 2g_{11}\bar{g}_{11} - \frac{1}{3}g_{02}\bar{g}_{02}\right) + \frac{g_{21}}{2}, \tag{3.6.11}
$$

which can be completely determined by the parameters of system (3.1.3) evaluated at the bifurcation point. The direction of Hopf bifurcation is determined by μ_2 where the Hopf bifurcation is supercritical (resp subcritical) if $\mu_2 > 0$ (resp $\mu_2 < 0$). Further, the bifurcating periodic solutions exists for $\mu > 0$ (resp $\mu < 0$) and the stability of the periodic solutions determined by v_2, where the bifurcating periodic solution are asymptotically stable (resp unstable) if $v_2 < 0$ (resp $v_2 > 0$).

3.7 Discussion and conclusion

In this paper, the dynamical behavior of a diffusive predator-prey model (3.1.3) with protection zone and quadratic predator harvesting subject to the zero flux boundary conditions has been investigated. First, we showed the existence of a positive solution and it bounders. Then, we discussed the existence of the equilibrium states and the effects of the predator harvesting on the prey density equilibrium, by figure 3.3 where it has been noticed that the predator harvesting affect positively the density equilibrium of the prey (means the value of u^* increases when c increases). In the next section we proved the global stability of the semi trivial equilibrium state $(1,0)$ in the case $\eta > 1$ it has been justified using a numerical simulation figure **Fig.3.7** where it has been noticed that for multi value of the initial conditions we have $(u(t,x), v(x,t)) \to (1,0)$ as $t \to +\infty$.

By choosing β as bifurcation parameter, we studied the existence of Hopf bifurcation by analyzing the characteristic equation, where we obtained that for $u^* > (3)^{-1}$ we cannot have Hopf bifurcation, its means that the interior equilibrium state is locally stable. This result has been carried out using numerical simulation figure **Fig.3.8** (A)-(D) where for figures (A)-(B) we used $c = 1$ which means that $u^* = 0.36 > (3)^{-1}$ (see figure 3.3), and $c = 1.5$ which means that $u^* = 0.52 > \frac{1}{3}$ (see figure **Fig.3.3** for figures (C)-(D). Besides, for (E)-(F) in figure **Fig.3.8** we took $c = 0.5$, means that $u^* = 0.13 < (3)^{-1}$ and the critical value of Hopf bifurcation $\beta_0 = 1.9449$ leads to deduce that the interior equilibrium state is always unstable. In this figure we used $\beta = 0.56 < \beta_0$ using **Theorem 18** we have the existence of a homogeneous periodic solutions. On the other hand, we have $\beta_1 = 0.4782$, and for $\beta < \beta_1$ we have the existence of a non homogeneous periodic solutions, and we took a multi value of the parameter β for justify this result. In figure **Fig.3.9** we used $\beta = 0.33$ for (A)-(B), and $\beta = 0.31$ for (C)-(D), $\beta = 0.28$ for (E)-(F), and $\beta = 0.24$ for (E)-(F).

On the other hand, we give also a sufficient condition for having Turing driven instability and justified using a numerical simulation (**Fig.3.4**). For the existence of Turing-Hopf bifurcation a two parameters β and d_2 has been used for proving the possibility of having the intersection between Hopf bifurcation curve H_0 defined by $\beta = \beta_0$ at the $d_2 - \beta$ plan and Turing bifurcation curve defined by $\beta = \beta_T(d_2, n_T)$. The most interesting about calculation the Turing-Hopf bifurcation point is to determine the regions of stability an instability of the interior equilibrium state, where the regions D_3, D_4 are the instability regions, and D_1 is the stable region, D_2 is Turing driven instability region.

For the biological meaning of the obtained results, the harvesting affect positively the prey population and negatively the predator population, where in the real world the increased harvesting (competition) will give the opportunity for the prey to escape which means it will survive and reproduce. Also the predator harvesting plays an important component in the evolution of species in the presence of herd behavior where it can be seen clearly in **Fig.3.5** and figure **Fig.3.7** where a small change in the value of predator harvesting rate can lead to huge change in population patterns. In other word, the presence of the predator harvesting is a very important component in model construction and give an important behavior on a mathematical point of view.
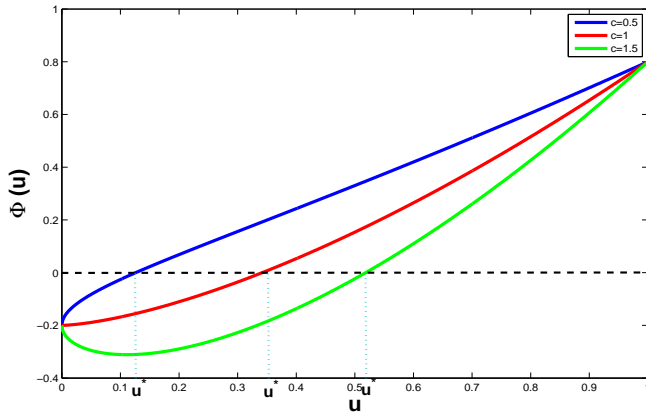
FIGURE 3.3: The effect of the harvesting rate c on the prey constant steady state for $\eta = 0.2$ and different values of the variable c.
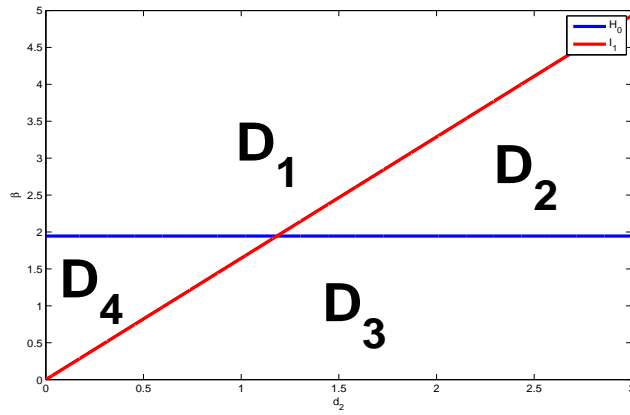

FIGURE 3.4: the existence of Turing-Hopf bifurcation for the values $\eta = 0.2 < 1$; $d_1 = 0.1$; $(u^*, v^*) = (0.13, 0.3137)$; $c = 0.5$ and $n_T = 1$.
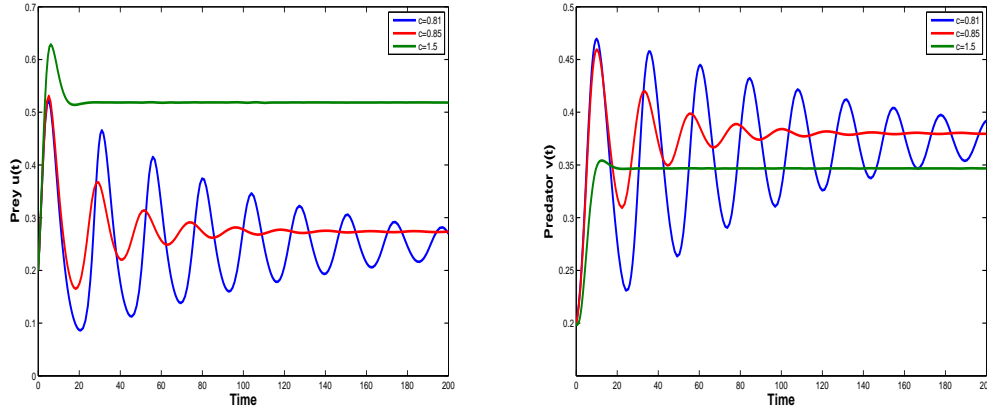
FIGURE 3.5: The sensitivity of the system ($3.1.3$) with respect to c in the absence of spatial diffusion for a fixed value $\eta = 0.2$; $\beta = 0.5$ and different value of the predator harvesting c, and the same initial conditions $u(0) = 0.2$, $v(0) = 0.2$.
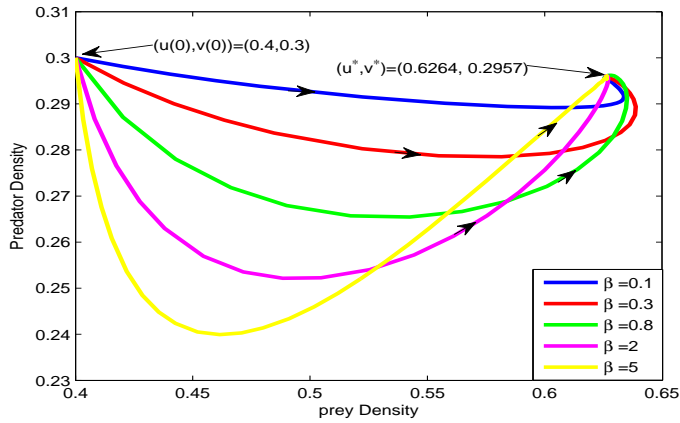
FIGURE 3.6: Asymptotic stability of E^* in the absence of spatial diffusion for a fixed value $\eta = 0.2$, $c = 2$ and different value of the parameter β.
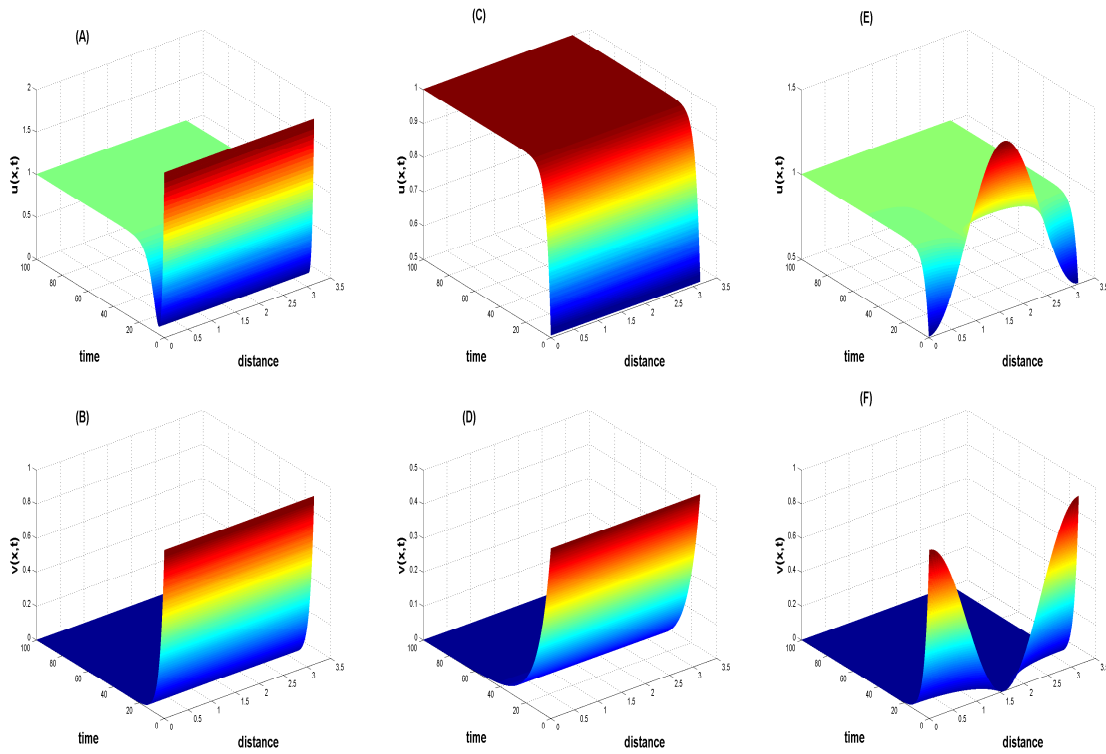
FIGURE 3.7: The global stability of the semi trivial equilibrium state of the system (3.1.3) for a fixed value $\eta = 1.2 > 1$; $\beta = 0.5$; $c = 0.5$; $d_1 = 0.1$; $d_2 = 0.2$ and different initial conditions.
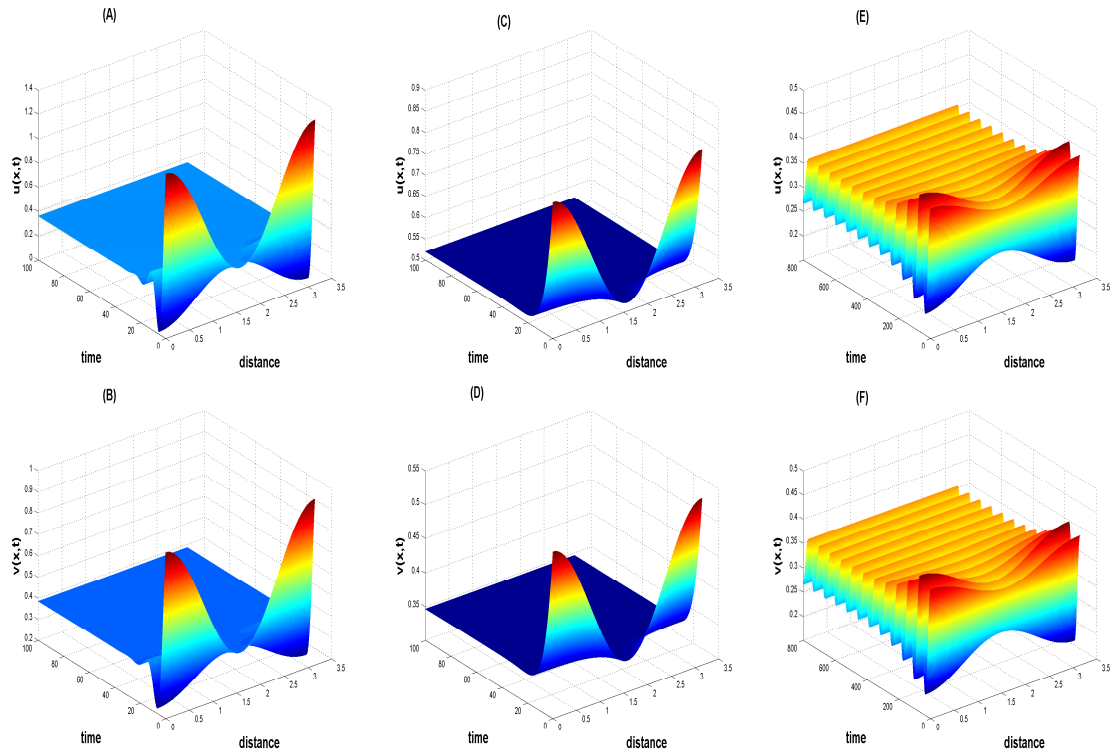
FIGURE 3.8: The sensibility of the system (3.1.3) with respect to c for a fixed value $\eta = 0.2 < 1$; $d_1 = 0.08$; $d_2 = 0.15$; $\beta = 0.56$ with a multi values of c. $(u^*, v^*) = (0.36, 0.384)$ $\beta = 0.5$; $c = 0.5$; $d_1 = 0.1$; $d_2 = 0.2$ and different initial conditions.
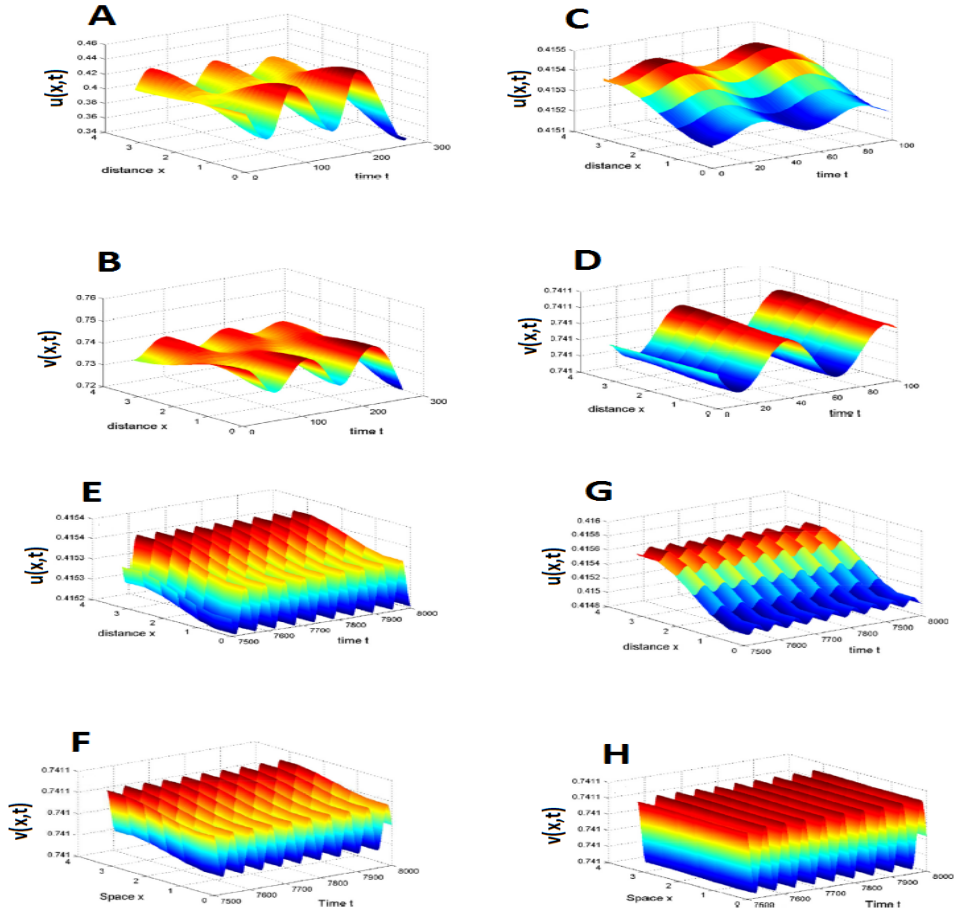
FIGURE 3.9: Existence of the non-homogeneous periodic solutions for system (3.1.3) with a fixed value $\eta = 0.235 <$ 1; $d_1 = 0.075$; $d_2 = 0.17$ with a multi values of β and different initial conditions where $(u^*, v^*) = (0.4136, 0.7412)$.

Chapter 4

Mathematical analysis of a diffusive predator-prey model with herd behavior and prey escaping

This chapter is taken from publication [81].

In this chapter, we deals with a new approach of a predator-prey model with social behavior prey escaping. For any species of prey, it is almost impossible to preserve the regrouping of the herd during the attacks by the predators, that obliges some prey to escape from the herd due to the panic. Here, we propose and study a new system which describing the escape of the prey from their herd.

The rest of the paper is organized as follows. In **Section 4.2**, we focus on studying the effect of the escaping rate on the positive equilibrium state (u^*, v^*). In the next section the existence of a priori bound of the solution, the global stability of the semi trivial equilibrium $(k, 0)$, and the occurrence of **Hopf bifurcation** for the system (4.1.1) have been proved. In **Section 4.4** the analysis of the diffusive system has been successfully studied where the local stability and the occurrence of **Hopf bifurcation** have been shown. Furthermore, the stability of the **homogeneous** and **nonhomogeneous periodic solutions** have been established using the **normal form** on the center of manifold. An extend numerical simulations have been carried out to insure the theoretical results. A discussion **Section** ends the chapter.

4.1 Mathematical modelling of the model

In the real world, during the predator hunting of the prey, it is almost impossible for the prey population to keep the pack all together, due to the panic of some of them. The attack of the predator will push the prey pack to split into two groups. The first will stay in the group, and the second will leave the pack and go in any direction. In the present paper, we will consider that there is a constant proportional density of the prey population denoted by $0 < P < 1$ which will abandon the prey pack during the hunting for the reason of panic, which can be called also by escaping rate, and the other $1 - P$ stays in the pack. For modeling this behavior, there are two ways for the predator to consume the prey. The first, is to hunt on the boundary of the prey pack which means $\sqrt{(1-P)u}$ (is the density of the prey on the outer bound of the pack) or hunt the **escaping prey** which means Pu. According to the above discussion, we propose the following model

$$\begin{cases} \frac{du}{dt} = ru(1 - \frac{u}{k}) - a_1\sqrt{(1-P)u}v - a_2Puv, \\ \frac{dv}{dt} = -mv + ea_1\sqrt{(1-P)u}v + ea_2Puv, \end{cases} \tag{4.1.1}$$

where u, v represent the social prey and predator population, respectively, at any time. a_1 and a_2 stands for the maximum values at which per capita reduction rate of the prey population in pack and solitary prey can attained, respectively. For $a_1 = a_2$ represents the non selective hunting of the prey by a predator, and for $a_1 \neq a_2$ shows the predator preference of one prey on another (if $a_1 > a_2$ shows the predator preference of the solitary prey on the prey in the pack).

Remark 3. *It is easy to see that for $P = 0$ (there is no escaping), the system (4.1.1) becomes the system (0.0.2). Besides, if $P = 1$ (all the prey escape, which means that there is no herd behavior) then the system (4.1.1) becomes the classical model of Lotka and volterra which has mentioned in the introduction. Furthermore, we proved the existence of a new functional response which describes the interaction predator-prey in the presence of **herd behavior** and **prey escaping** from the herd. The proposed functional response is $h(u,v) = a_1\sqrt{(1-P)u}v + a_2Puv$.*

As mentioned in **chapter.3**, the prey and the predator are always in movement, which can be modeled by the presence of self-dispersal. The **spatial diffusion** has been widely studied in literature, see [54, 77, 78, 84, 85, 100, 101, 104, 106, 108]. In the considered chapter, the study of the system (4.1.1) and the effect of the spatial diffusion on the system (4.1.1) has been investigated. The system (4.1.1) becomes in the presence of **spatial diffusion**:

$$\begin{cases} u_t - du_{xx} = ru(1 - \frac{u}{k}) - a_1\sqrt{(1-P)u}v - a_2Puv, \\ v_t - \tilde{d}v_{xx} = -mv + ea_1\sqrt{(1-P)u}v + ea_2Puv, \\ u_x(0,t) = u_x(l\pi,t) = v_x(0,t) = v_x(l\pi,t) = 0 \qquad \forall t \geq 0, \\ u(x,0) = \phi(x) \geq 0 \qquad v(x,0) = \psi(x) \geq 0 \qquad x \in (0,l\pi), \end{cases} \tag{4.1.2}$$
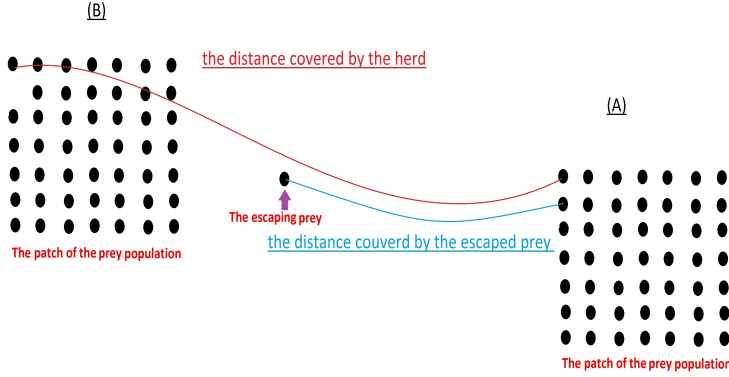
FIGURE 4.1: The manner in which the individual escaping from the herd when the pack of the prey moving from point (A) to point (B).

where x is the location of the prey or the predator at the time t, and $l\pi$ is the domain size, $d, \tilde{d}$ are the diffusion rates for the prey and the predator, respectively. The homogeneous Neumann boundary conditions represents that the prey and the predator move with a distance between 0 and $l\pi$. For more examples see [26, 27, 58, 73].

It is easy to check that the homogeneous steady states of the system (4.1.2) are $(0,0)$, $(k,0)$ and (u^*, v^*) where

$$
\begin{cases}
u^* = \left(\dfrac{ea_1\sqrt{(1-P)} + \sqrt{\left(ea_1\sqrt{(1-P)} \right)^2 + 4mea_2P}}{2ea_2P} \right)^2, \\
v^* = \dfrac{r\sqrt{u^*}}{a_1\sqrt{(1-P)} + a_2P\sqrt{u^*}} \left(1 - \dfrac{u^*}{k} \right) > 0.
\end{cases}
\quad \text{exists if and only if } k > u^*. \qquad (4.1.3)
$$

4.2   Sensitivity analysis

In this section, we will study the impact of the prey escaping on the equilibrium densities of both the prey and the predator populations. First, we differentiate the density equilibrium of the prey population
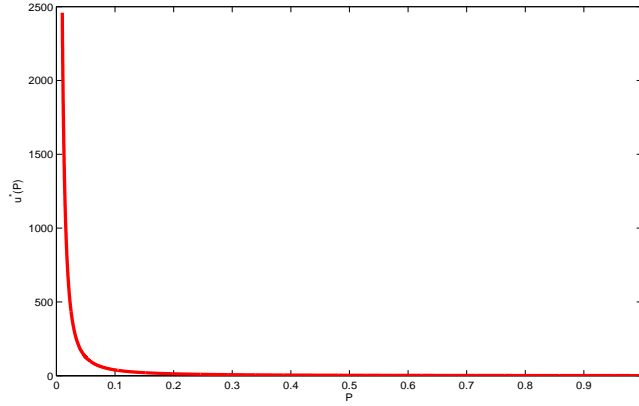
FIGURE 4.2: The negative effect of the prey escaping on the prey density equilibrium state for the values $a_1 = 1$; $a_2 = 2.1$; $e = 0.44$; $m = 1.01$; $r = 0.2$, $k = 8000$.

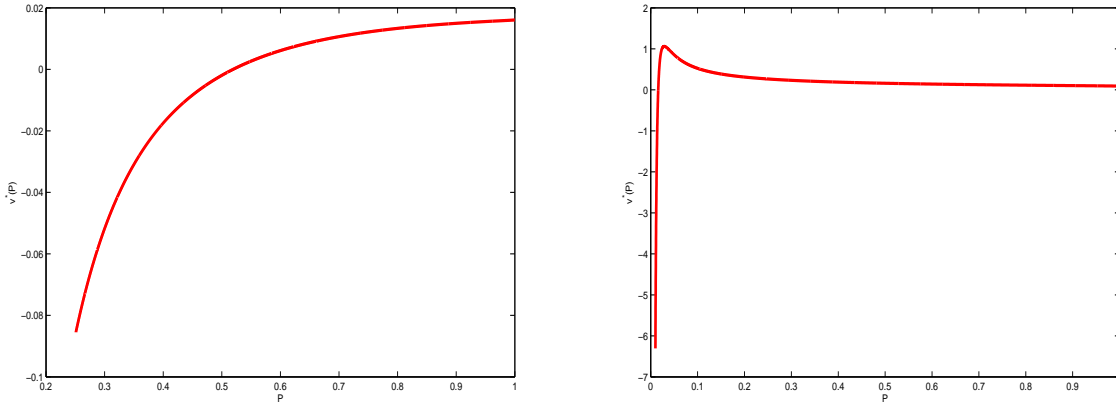FIGURE 4.3: The effect of the prey escaping on the predator density equilibrium state for the values
(i) (Left hand figure) the values $a_1 = 1.9$; $a_2 = 0.1$; $e = 0.0044$; $m = 2.01$; $r = 0.2$, $k = 10000 < \frac{2m}{ea_2} = 91364$.
(ii) (Right hand figure) the values $a_1 = 1$; $a_2 = 2.1$; $e = 0.44$; $m = 1.01$; $r = 0.2$, $k = 8000 > \frac{2m}{ea_2} = 2.1861$.

with respect to P we obtain

$$\frac{du^*}{dP} = \left(\frac{ea_1\sqrt{(1-P)} + \sqrt{\left(ea_1\sqrt{(1-P)}\right)^2 + 4mea_2P}}{2ea_2P} \right) \left(\frac{a_1(P-2)}{a_2P^2\sqrt{1-P}} - \frac{4ma_2P + ea_1^2(1-P) + ea_1^2}{\sqrt{e}a_2P^2\sqrt{ea_1^2 + P(4ma_2 - ea_1^2)}} \right) < 0. \quad (4.2.1)$$

Obviously, the prey equilibrium density is decreasing with respect to the escaping rate, means that the prey escaping has a negative effect on the prey equilibrium density. Which shows the importance
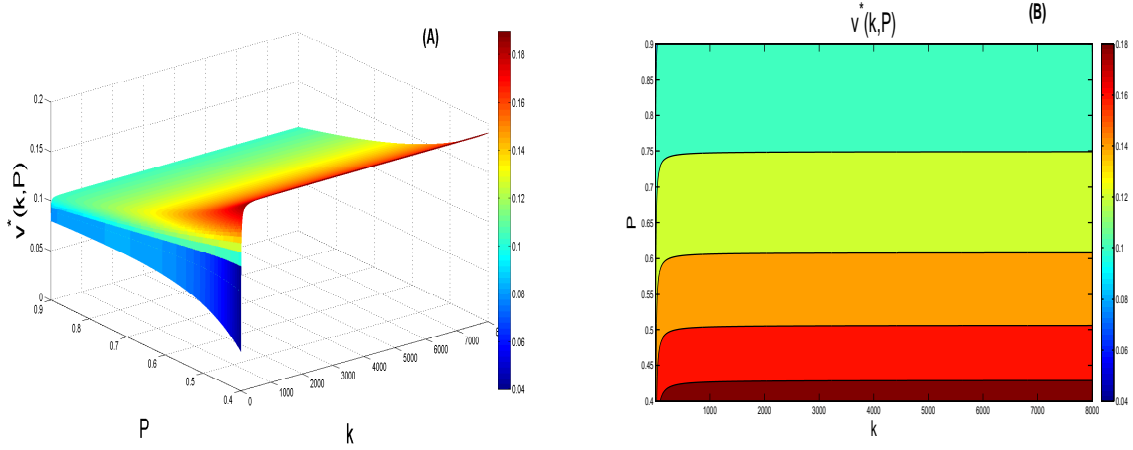
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FIGURE 4.4: The impact of both the prey escaping and the carrying capacity of the space for the values $a_1 = 1$; $a_2 = 2.1$; $e = 0.44$; $m = 1.01$; $r = 0.2$.

of the social behavior for the prey population. **Fig.4.2** shows the effect of the variable P on the prey density equilibrium state.

Now focusing on studying the impact of the escaping rate P on the predator density equilibrium state. The predator density equilibrium can be written as follows

$$v^* = \frac{eru^*}{m}(1 - \frac{u^*}{k}). \tag{4.2.2}$$

By a differentiation of the predator density equilibrium state (4.2.2) with respect to the escaping rate P we obtain

$$\frac{dv^*}{dP} = \frac{er}{m}\frac{du^*}{dP}(1 - \frac{2u^*}{k}),$$

$$= \frac{2er}{km}\frac{du^*}{dP}(\frac{k}{2} - u^*). \tag{4.2.3}$$

In order to study the positivity of $\frac{k}{2} - u^*$ we draw the following lemma

Lemma 10. *([81])*

(i) If $k \leq \frac{2m}{ea_2}$ then $\frac{k}{2} - u^* < 0$ for any $0 < P < 1$.

(ii) if $k > \frac{2m}{ea_2}$ then there exists $0 < P_{cr} < 1$ such that $\frac{k}{2} - u^* < 0$ for $0 < P < P_{cr}$ and $\frac{k}{2} - u^* > 0$ for $P_{cr} < P < 1$.
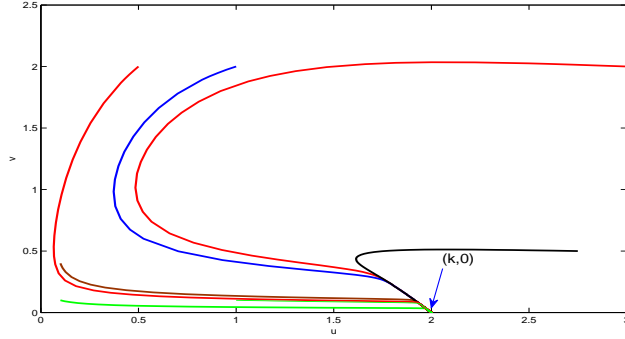
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FIGURE 4.5: Phase portraits of the system $(4.1.1)$ when (u^*, v^*) does not exists and $(k,0)$ is globally asymptotically stable for $k = 2 < u^* = 3.5871$.

Proof. (i) It is not difficult to verify that $\lim_{P \to 0^+} u^*(P) = +\infty$ and $\lim_{P \to 1} u^*(P) = \frac{m}{ea_2}$ and using the fact that $\frac{du^*}{dP} < 0$ then for $\frac{k}{2} \leq \frac{m}{ea_2} = \min_{p \in]0,1]} u^*(P)$ we deduce that $\frac{k}{2} - u^* < 0$ which completes the first part of the proof.

(ii) for $\frac{k}{2} > \frac{m}{ea_2} = \min_{p \in]0,1]} u^*(P)$ then the curve of the functional $u^*(P)$, $0 < P < 1$ intersect the line $\frac{k}{2}$ at $0 < P_{cr} < 1$ then we have $\frac{k}{2} - u^* < 0$ for $0 < P < P_{cr}$ and $\frac{k}{2} - u^* > 0$ for $P_{cr} < P < 1$. The proof is completed. □

Using **Lemme 10** together with the fact that $\frac{du^*}{dP} < 0$, we draw the following results

(i)If $k \leq \frac{2m}{ea_2}$ then $\frac{dv^*}{dP} > 0$ and based on the ecological meaning we deduce that the prey escaping has a positive impact on the predator density equilibrium.

(ii)If $k > \frac{2m}{ea_2}$ then there exists $0 < P_{cr} < 1$ such that

$$\frac{dv^*}{dP} = \begin{cases} > 0 & \text{for } 0 < P < P_{cr}, \\ < 0 & \text{for } P_{cr} < P < 1, \end{cases}$$

which means that the prey escaping has a positive impact on the predator density equilibrium for $P < P_{cr}$ and a negative impact for $P_{cr} < P < 1$ (see **Fig.4.3**).

4.3 Stability, bifurcation analysis of the non spatial system

This section is devoted to study the solution behavior for the model $(4.1.1)$ where a priori bound of solution, global stability of the boundary equilibrium $(k,0)$ has been investigated. Further, the existence of Hopf bifurcation has been shown.

In order to show the existence of the bounds of the system (4.1.1) we set the following theorem

Theorem 20. *([81]) Let $(u(t), v(t))$ be the solution of the system (4.1.1) then*

$$\limsup_{t \to +\infty} u(t) \le k,$$
$$\limsup_{t \to +\infty} v(t) \le \frac{m+r}{m} ek.$$

Proof. Obviously, for the system (4.1.1) the positive invariant set is the first quadrant $\mathbb{R}^2_+$, since $u = 0$, $v = 0$ are its solutions. From the first equation of the system (4.1.1) we have

$$u'(t) \le ru(t)(1 - \frac{u(t)}{k}).$$

Let $\tilde{u}(t)$ be the solution of the following initial value problem

$$\begin{cases} \dfrac{d\tilde{u}(t)}{dt} = r\tilde{u}(t)(1 - \dfrac{u(t)}{k}), \\ \tilde{u}(0) = u(0), \end{cases} \qquad (4.3.1)$$

using the standard comparison principle, we have $u(t) \le \tilde{u}(t)$ for all $t \in [0, +\infty)$. Thus

$$\limsup_{t \to +\infty} u(t) \le \limsup_{t \to +\infty} \tilde{u}(t) = k.$$

Now we put

$$w(t) = eu(t) + v(t).$$

then

$$\dot{w}(t) = e\dot{u}(t) + \dot{v}(t),$$
$$= -mv(t) + ru(t)(1 - \frac{u(t)}{k}),$$
$$= -m(eu(t) + v(t)) + emu(t) + ru(t)(1 - \frac{u(t)}{k}),$$

leads to

$$\frac{dw(t)}{dt} \le -mw(t) + eu(t)(m+r).$$

From $\limsup\limits_{t \to +\infty} u(t) \le k$ we can deduce that there exists $T > 0$ such that for $t > T$, $u(t) < k$. Then for $t > T$ we have

$$\frac{dw(t)}{dt} \le -mw(t) + ek(m+r), \qquad (4.3.2)$$

by a similar argument we have

$$\limsup_{t \to +\infty} v(t) \le \frac{m+r}{m} ek,$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$
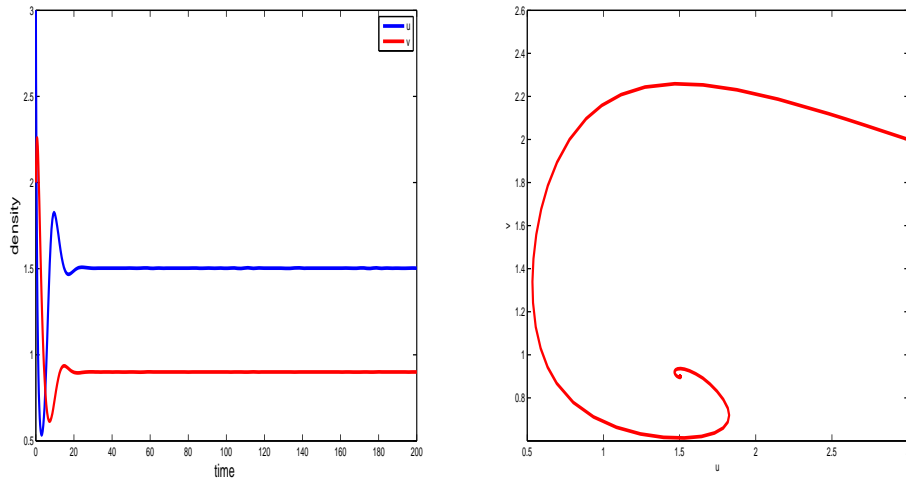
FIGURE 4.6: Trajectory and phase portraits of the system (4.1.1) when $(u^*, v^*) = (1.55, 0.89)$ is locally asymptotically stable and $k = 2 < k^* = 17.0855$ for the values $P = 0.5$; $a_1 = 0.2$; $a_2 = 1.1$; $e = 0.5$; $m = 0.5$; $r = 1.2$.
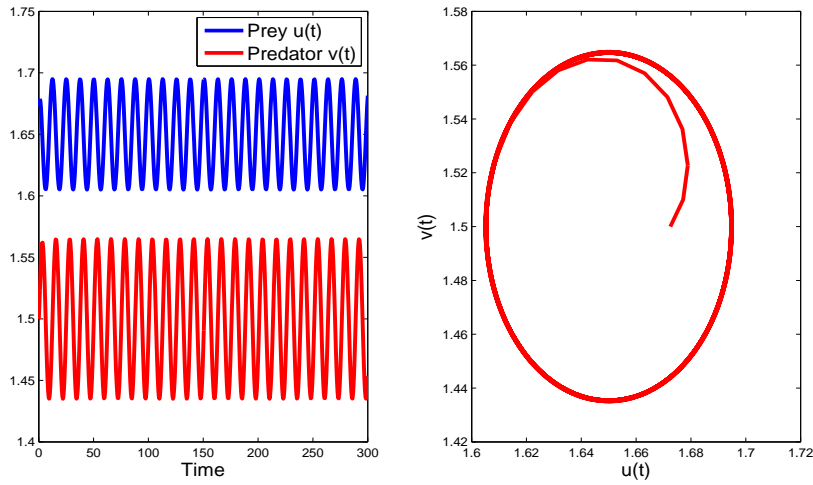

FIGURE 4.7: Trajectory and phase portraits of the system (4.1.1) when $(u^*, v^*) = (1.51, 1.65)$ is unstable and $k = 18 > k^* = 17.0855$ for the values $P = 0.5$; $a_1 = 0.2$; $a_2 = 1.1$; $e = 0.5$; $m = 0.5$; $r = 1.2$.

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Now, focusing on proving the global stability of the semi trivial equilibrium state $(k,0)$. The following theorem summarize the obtained results

Theorem 21. *([81]) Assume that $k < u^*$ then the semi trivial equilibrium is globally asymptotically stable.*

Proof. The eigenvalues of Jacobian matrix of the system(4.1.1) on $(k,0)$ are $\lambda_1 = -r < 0, \lambda_2 = -m + eb_1\sqrt{k} + eb_2k < 0$(for $k < u^*$) which means that $(k,0)$ is locally asymptotically stable, it remain to prove that $(k,0)$ is globally attractive. From Theorem (4.1.1) we have $\limsup\limits_{t\to+\infty} u(t) \le k$. We know that from the second equation that

$$v_t = v(-m + ea_1\sqrt{(1-P)u} + ea_2Pu).$$

It is well known that there exists $T > 0$ such that for $t > T$, there exists $\varepsilon > 0$ such that $u(t) \le k + \varepsilon$. Choosing $0 < \varepsilon < u^* - k$. Thus, we can obtain

$$v_t \le v(-m + ea_1\sqrt{(1-P)(k+\varepsilon)} + ea_2P(k+\varepsilon)),$$

where

$$-m + ea_1\sqrt{(1-P)(k+\varepsilon)} + ea_2P(k+\varepsilon) < 0 ,$$

which leads to

$$v(t) \le v(0)exp[(-m + eb_1\sqrt{k+\varepsilon} + ea_2P(k+\varepsilon))t].$$

Putting $t \to +\infty$ we obtain that $v(t) \to 0$ for $t \to +\infty$. By replacing this result in the first equation we obtain $u(t) \to k$ for $t \to +\infty$. The proof is completed. $\square$

Now, let us prove the existence of Hopf bifurcation. Firstly, we consider the carrying capacity k as bifurcation parameter. we calculate the Jacobian matrix of the system (4.1.1) at the positive equilibrium (u^*,v^*) and it is given by

$$J(u^*,v^*) = \begin{pmatrix} a_{11}(k) & a_{12} \\ a_{21}(k) & 0 \end{pmatrix},$$

where

$$\begin{aligned}
a_{11}(k) &= r(1 - \frac{2u^*}{k}) - \frac{v^*}{2\sqrt{u^*}}(a_1\sqrt{1-P} + 2a_2P\sqrt{u^*}), \\
&= \frac{ra_1\sqrt{1-P}}{2(a_1\sqrt{1-P}+a_2P\sqrt{u^*})} - \frac{ru^*}{k}\left(\frac{2a_1\sqrt{1-P}+a_2P\sqrt{u^*}}{2a_1\sqrt{1-P}+a_2P\sqrt{u^*}}\right), \\
a_{21}(k) &= ev^*\left(\frac{a_1\sqrt{1-P}}{2\sqrt{u^*}} + a_2P\right), \\
&= \frac{re}{2}\left(1 + \frac{a_2P\sqrt{u^*}}{a_1\sqrt{(1-P)}+a_2P\sqrt{u^*}}\right)(1 - \frac{u^*}{k}) > 0, \\
a_{12} &= -\left(a_1\sqrt{1-P}\sqrt{u^*} + a_2Pu^*\right) = -\frac{m}{e} < 0.
\end{aligned} \qquad (4.3.3)$$

The characteristic equation is given by

$$\lambda^2 - T_0(k)\lambda + D_0 = 0, \tag{4.3.4}$$

where

$$T_0(k) = \frac{ra_1\sqrt{1-P}}{2(a_1\sqrt{1-P}+a_2P\sqrt{u^*})} - \frac{ru^*}{k}\left(\frac{2a_1\sqrt{1-P}+a_2P\sqrt{u^*}}{2(a_1\sqrt{1-P}+a_2P\sqrt{u^*})}\right),$$
$$D_0(k) = \frac{rm}{2}\left(1+\frac{a_2P\sqrt{u^*}}{a_1\sqrt{(1-P)}+a_2P\sqrt{u^*}}\right)(1-\frac{u^*}{k}) > 0. \tag{4.3.5}$$

Obviously, $D_0(k) > 0$ which means that the system (4.1.1) undergoes a Hopf bifurcation if $T_0(k) = 0$ and it is equivalent to

$$k = k^* = \frac{u^*(2a_1\sqrt{1-P}+a_2P\sqrt{u^*})}{a_1\sqrt{1-P}}. \tag{4.3.6}$$

Remark 4. *It is easy to verify that $k^* > u^*$ which means that this bifurcation point exists.*

Letting $\lambda(k) = \beta(k) \pm \omega(k)$ be the roots of the characteristic equation (4.3.4) satisfying $\beta(k^*) = 0, \omega(k^*) = \sqrt{D_0(k^*)}$. Then we have

$$\beta'(k^*) = \frac{1}{2}T_0'(k^*) = \frac{1}{2}\frac{ru^*}{k^{*2}}\left(\frac{2a_1\sqrt{1-P}+a_2P\sqrt{u^*}}{2(a_1\sqrt{1-P}+a_2P\sqrt{u^*})}\right) > 0,$$

which together with the fact that the characteristic equation (4.3.4) has a pair of purely imaginary roots $\pm i\sqrt{D_0(k^*)}$ at k^*, implies that the system (4.1.1) undergoes Hopf bifurcation at $k = k^*$.

Thus, we deduce that the interior equilibrium (u^*, v^*) is locally asymptotically stable for $k < k^*$ and instable for $k > k^*$ and the system (4.1.1) undergoes Hopf bifurcation at $k = k^*$.

4.4 Stability, Hopf bifurcation for the diffusive system

In this section, the stability of the positive equilibrium (u^*, v^*) has been studied. Further, the existence of the Hopf bifurcation for the system (4.1.2), non existence of diffusion driven instability has been successfully proved. Throughout the rest part of the paper, the condition $k > u^*$ has been assumed to be verified.

4.4.1 Characteristic equation

Consider the following problem

$$\begin{cases} -\psi'' = \kappa\psi & x \in (0, l\pi), \\ \psi'(0) = \psi^{l\pi}(0) = 0 \end{cases} \tag{4.4.1}$$

the eigenvalue of the problem (4.4.1) are $\kappa_n = \left(\dfrac{n}{l}\right)^2$, $n = 0,1,2,...$, and the corresponding eigenfunction are

$$\psi_n(x) = \begin{cases} \sqrt{\frac{1}{\pi}}, & n = 0, \\ \sqrt{\frac{2}{\pi}}\cos\frac{n}{l}x, & n = 1,2,3,..., \end{cases}$$

and $\{\psi_n(x)\}_1^\infty$ describes the orthogonal basis of $L^2(0,l\pi)$. Now letting

$$\chi = \left\{ U = (u,v)^T \in W^{2,2}(0,l\pi) \big/ u_x = v_x = 0 \text{ at } x = 0, l\pi \right\}. \tag{4.4.2}$$

The space χ is Banach space, and $\Omega = L^2(0,l\pi) \times L^2(0,l\pi)$ is Hilbert space with the inner product $(U_1, U_2) = \int\limits_0^{l\pi}(u_1 u_2 + v_1 v_2)dx$, where $U_1 = (u_1, v_1)^T$ and $U_2 = (u_2, v_2)^T$. Defining the following mapping $F : (0,\infty) \times \chi \to \Omega$ by

$$F(k,U) = \begin{pmatrix} du_{xx} + ru(1 - \frac{u}{k}) - a_1\sqrt{(1-P)}uv - a_2 Puv \\ \tilde{d}v_{xx} - my + ea_1\sqrt{(1-P)}uv + ea_2 Puv \end{pmatrix},$$

where $U = (u,v)^T$. Then for any $(u,v)^T \in \chi$,

$$U = (u,v)^T \text{ is solution of } (4.1.2) \iff F(k,U) = 0.$$

At the homogeneous steady state (u^*, v^*) the Frechet derivative of $F(k,U)$ for U is given as follows

$$L(k) = \operatorname{diag}(d\Delta, \tilde{d}\Delta) + J(u^*, v^*), \tag{4.4.3}$$

$$= \begin{pmatrix} d\Delta u + \frac{ra_1\sqrt{1-P}}{2(a_1\sqrt{1-P}+a_2 P\sqrt{u^*})} - \frac{ru^*}{k}\left(\frac{2a_1\sqrt{1-P}+a_2 P\sqrt{u^*}}{2(a_1\sqrt{1-P}+a_2 P\sqrt{u^*})}\right) & -\frac{m}{e} \\ \frac{re}{2}\left(1 + \frac{a_2 P\sqrt{u^*}}{a_1\sqrt{(1-P)}+a_2 P\sqrt{u^*}}\right)\left(1 - \frac{u^*}{k}\right) & \tilde{d}\Delta \end{pmatrix}. \tag{4.4.4}$$

Letting $u = \sum_{n=0}^\infty a_n\psi_n$ and $v = \sum_{n=0}^\infty b_n\psi_n$. Then the characteristic equation becomes

$$\sum_{n=0}^\infty (\operatorname{diag}(-d\kappa_n, -\tilde{d}\kappa_n) + J(E^*))\begin{pmatrix} a_n \\ b_n \end{pmatrix}\psi_n = 0. \tag{4.4.5}$$

Letting $\left|\operatorname{diag}(-d\kappa_n, -\tilde{d}\kappa_n) + J(E^*)\right| = 0$, n=0,1,2,..., we obtain

$$\lambda^2 - T_n(k)\lambda + D_n(k) = 0, \tag{4.4.6}$$

with

$$T_n(k) = \frac{ra_1\sqrt{1-P}}{2(a_1\sqrt{1-P}+a_2 P\sqrt{u^*})} - \frac{ru^*}{k}\left(\frac{2a_1\sqrt{1-P}+a_2 P\sqrt{u^*}}{2(a_1\sqrt{1-P}+a_2 P\sqrt{u^*})}\right) - (d + \tilde{d})(\frac{n}{l})^2,$$
$$D_n(k) = d\tilde{d}(\frac{n}{l})^4 - \tilde{d}(\frac{n}{l})^2\left[\frac{ra_1\sqrt{1-P}}{2(a_1\sqrt{1-P}+a_2 P\sqrt{u^*})} - \frac{ru^*}{k}\left(\frac{2a_1\sqrt{1-P}+a_2 P\sqrt{u^*}}{2(a_1\sqrt{1-P}+a_2 P\sqrt{u^*})}\right)\right] \tag{4.4.7}$$
$$+ \frac{rm}{2}\left(1 + \frac{a_2 P\sqrt{u^*}}{a_1\sqrt{(1-P)}+a_2 P\sqrt{u^*}}\right)\left(1 - \frac{u^*}{k}\right).$$

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4.4.2 The existence of Hopf bifurcation

Recall that a Hopf bifurcation occurs if and only if $T_n(k) = 0$ and $D_n(k) > 0$. Obviously $D_0(k) > 0$ and $\lim_{n \to +\infty} D_n(k) = +\infty$. The critical value of the bifurcation parameter k must be the solution of the following equation of the variable k

$$\frac{ra_1\sqrt{1-P}}{2(a_1\sqrt{1-P}+a_2P\sqrt{u^*})} - \frac{ru^*}{k}\left(\frac{2a_1\sqrt{1-P}+a_2P\sqrt{u^*}}{2(a_1\sqrt{1-P}+a_2P\sqrt{u^*})}\right) - (d+\tilde{d})(\frac{n}{l})^2 = 0, \tag{4.4.8}$$

which is equivalent to

$$k = k(n),$$

where

$$k(n) = \frac{ru^*(2a_1\sqrt{1-P}+a_2P\sqrt{u^*})}{rb_1 - 2(d+\tilde{d})(\frac{n}{l})^2(a_1\sqrt{1-P}+a_2P\sqrt{u^*})}. \tag{4.4.9}$$

Then we have the following results:

Lemma 11. *([81]) Putting*

$$N_1 = \max\{n \in \mathbb{N}/ \ ra_1\sqrt{1-P} - 2(d+\tilde{d})(\frac{n}{l})^2(a_1\sqrt{1-P}+a_2P\sqrt{u^*}) > 0\}. \tag{4.4.10}$$

Hopf bifurcation occurs for the system (4.1.2) at $k = k(n)$ and for $n \leq N_1$ (where $k(n)$ is defined in (4.4.9)) and $k(n)$ verifies the following estimation

$$u^* < k(0) < k(1) < .. < k(n) < k(n+1) < ... < k(N_1). \tag{4.4.11}$$

Proof. Hopf bifurcation occurs if and only if $T_n(k) = 0$, which equivalent to

$$T_0(k) = (d+\tilde{d})(\frac{n}{l})^2, \tag{4.4.12}$$

and it is easy to see that $\lim_{k \to u^*} T_0(k) = -r < 0$, and

$$T_0'(k) = \frac{r}{2k^2}\left(\frac{2a_1\sqrt{1-P}+a_2P\sqrt{u^*}}{a_1\sqrt{1-P}+a_2P\sqrt{u^*}}\right) > 0, \tag{4.4.13}$$

and

$$\lim_{k \to +\infty} T_0(k) = \frac{ra_1\sqrt{1-P}}{2(a_1\sqrt{1-P}+a_2P\sqrt{u^*})} > 0. \tag{4.4.14}$$

From (4.4.12) and (4.4.13), $T_0(k)$ is strictly increasing with respect to the carrying capacity k and the solution of the equation $T_0(k) = 0$ is
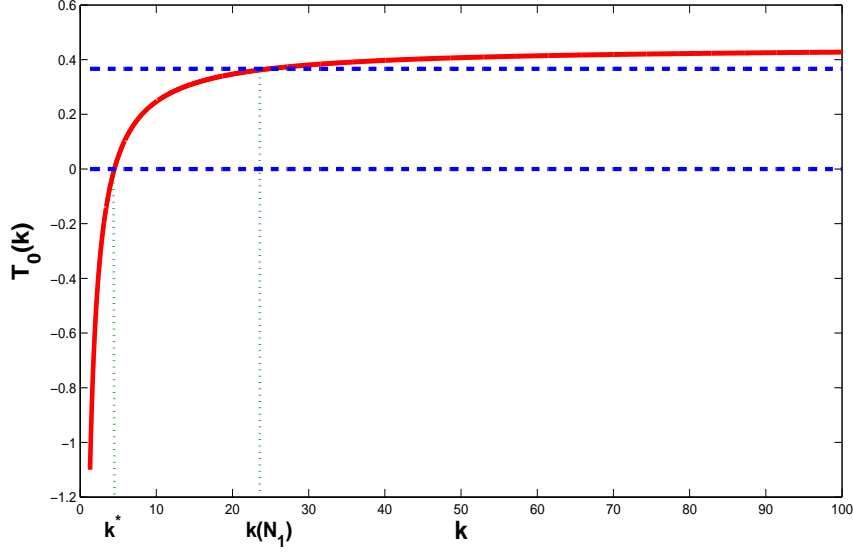
$$k = k^* > u^*, \tag{4.4.15}$$

FIGURE 4.8: The existence and the order of Hopf bifurcation points the solution of the equation (4.4.12) where $a = 3.5$; $P = 0.7$; $l = 5$; $e = 0.5$; $m = 0.5$; $r = 2.2$; $d = 0.02$; $\widetilde{d} = 0.1$ which means $u^* = 1.3005$; $N_1 = 6$.

where k^* is defined by (4.3.6). The equation (4.4.12) possesses a solution if and only if the constant integer n verifies

$$(d + \tilde{d})(\frac{n}{l})^2 < \frac{ra_1\sqrt{1-P}}{2(a_1\sqrt{1-P} + a_2 P\sqrt{u^*})}, \tag{4.4.16}$$

which equivalent to $n < N_1$.

The function $(d + \tilde{d})(\frac{n}{l})^2$ is strictly increasing with respect to n, which leads to the estimation obtained in (4.4.11) (see **Fig.4.8**). The proof is completed. $\qquad\square$

Now we put $\lambda(k) = \beta(k) \pm i\omega(k)$ as the solution of the characteristic equation with $\beta(k(n)) = 0$ and $\omega(k(n)) = \sqrt{D(k(n))}$, then

$$\beta'(k(n)) = \frac{r}{2k^2(n)} \left(\frac{2a_1\sqrt{1-P} + a_2 P\sqrt{u^*}}{2(a_1\sqrt{1-P} + a_2 P\sqrt{u^*})} \right) > 0. \tag{4.4.17}$$

Under the result (4.4.17) the bifurcation points and their order is given by the following theorem

Theorem 22. ([81]) *If there exists $N^* \leq N_1$ a critical value such that $i_0 = 0 < i_1 < ... < i_{N^*} \leq N_1$ and $D_{i\xi}(k(i\xi)) > 0$, $\xi = 0...N^*$ we have the following estimation:*

$$u^* < k(0) < k(1) < .. < k(n) < k(n+1) < ... < k(i_{N^*}). \tag{4.4.18}$$

In order to the above analysis we have the following theorem

Theorem 23. *([81]) Assume that $k > u^*$ holds, then we have the following results*

(i) If $n = 0$ the equilibrium (u^*, v^*) is asymptotically stable for $0 < k < k^* := k(0)$ and unstable for $k > k^*$

.

(ii) The system (4.1.2) has a Hopf bifurcation near (u^*, v^*) for $k = k(n)$ where a spatially homogenous periodic solution appears for n=0, and spatially nonhomogeneous periodic solution for $n = 1, 2, ..., i_{N^*}$.

Remark 5. *For Theorem 22 we have $T_n = 0$ for $k = k_n, n = 0, 1, ..., i_{N^*}$ and it is well known that $D_0 > 0$ for $k > u^*$. Thus there exist an integer denoted by N^* such that $D_n > 0$ for $n = 0, 1, ..., N^*$. Taking $i_{N^*} = min\{N_1, N^*\}$. Thus we have:*

(i) $T_n < 0$ for $n = 0, 1, ..., i_{N^*}$.

(ii) $D_n > 0$ for $n = 0, 1, ..., i_{N^*}$.

(iii) The transversality condition (4.4.17).

Thus the system undergoes Hopf bifurcation at $k = k(n)$ for $n = 0, 1, ..., i_{N^*}$.

4.4.3 Non existence of Turing instability:

In this subsection, our main focus is on proving that the system (4.1.2) can not undergo diffusion driven instability. Note that Turing driven instability occurs when the equilibrium point is linearly stable in the absence of spatial diffusion and in the presence of this last becomes unstable, which means that we will assume that $u^* < k < k^*$.

Lemma 12. *([81]) The system (4.1.2) has no Turing instability at (u^*, v^*).*

Proof. Obviously $D_0(k) > 0$, then for having the existence of Turing instability we must prove $D_n(k) < 0$ which can be rewritten as follows

$$D_n(k) = D((\frac{n}{l})^2) = d\tilde{d}((\frac{n}{l})^2)^2 - \tilde{d}(\frac{n}{l})^2 \left[\frac{ra_1\sqrt{1-P}}{2(a_1\sqrt{1-P}+a_2P\sqrt{u^*})} - \frac{ru^*}{k} \left(\frac{2a_1\sqrt{1-P}+a_2P\sqrt{u^*}}{2(a_1\sqrt{1-P}+a_2P\sqrt{u^*})} \right) \right] + D_0(k).$$

(4.4.19)

Thus, using the fact that $u^* < k < k^*$, we obtain that

$$\frac{ra_1\sqrt{1-P}}{2(a_1\sqrt{1-P}+a_2P\sqrt{u^*})} - \frac{ru^*}{k} \left(\frac{2a_1\sqrt{1-P}+a_2P\sqrt{u^*}}{2(a_1\sqrt{1-P}+a_2P\sqrt{u^*})} \right) < 0,$$

which means $D_n(k) > 0$ for any integer n the we deduce the non existence of Turing driven instability. The proof is completed. □

4.4.4 Direction and stability of Hopf bifurcation

In this section our aim is to study the stability of the homogeneous and the non homogeneous periodic solution using the normal form on the center manifold at the Hopf bifurcation points. Before that we will make a change of variable by putting $\tilde{u} = u - u^*$ and $\tilde{v} = v - v^*$ where $v^* = v_\mu^* = \dfrac{r\sqrt{u^*}}{a_1\sqrt{1-P}+a_2P\sqrt{u^*}}(1-\dfrac{u^*}{\tilde{k}+\mu})$ and $\mu = k - k(n)$ where $\mu = 0$ is equivalent to and $k = k(n)$, $n = 1,2,...,i_{N^*}$ and dropping tilde for convenience, the system (4.1.2) becomes:

$$\begin{cases} u_t - du_{xx} = r(u+u^*)(1-\frac{(u+u^*)}{k(n)+\mu}) - a_1\sqrt{(1-P)(u+u^*)}v - a_2P(u+u^*)(v+v^*), \\ v_t - \tilde{d}v_{xx} = -my + ea_1\sqrt{(1-P)(u+u^*)}(v+v^*) + ea_2P(u+u^*)(v+v^*), \end{cases} \quad x \in [0,l\pi]; t > 0$$

(4.4.20)

and it is equivalent to

$$\dot{U}(t) = LU + F(U,\mu),$$

(4.4.21)

where

$$L = D\Delta + J(E^*) = \begin{pmatrix} a_{11}(\mu) + d\frac{\partial^2}{\partial x^2} & a_{12} \\ a_{21}(\mu) & \tilde{d}\frac{\partial^2}{\partial x^2} \end{pmatrix},$$

and

$$F(U,R^*) = \begin{pmatrix} F_1(U,\mu) \\ F_2(U,\mu) \end{pmatrix},$$

$$= \begin{pmatrix} r(u+u^*)(1-\frac{(u+u^*)}{k(n)+\mu}) - a_1\sqrt{(1-P)(u+u^*)}v - a_2P(u+u^*)(v+v^*) - a_{11}(\mu)u - a_{12}v \\ -my + ea_1\sqrt{(1-P)(u+u^*)}(v+v^*) + ea_2(u+u^*)(v+v^*) - a_{21}(\mu)u \end{pmatrix},$$

where $a_{11}(\mu), a_{12}$ and $a_{21}(\mu)$ are defined in (4.3.3). Now we define the following complex space of χ by the decomposition:

$$\chi_\mathbb{C} = \chi \oplus i\chi,$$

and the inner product

$$<U_1,U_2> = \int_0^\pi \bar{u}_1 u_2 + \bar{v}_1 v_2 dx \quad \text{where } U_1 = (u_1,v_1)^T, U_2 = (u_2,v_2)^T \in \chi_\mathbb{C}.$$

The adjoint operator L^* is given by

$$L^* = D\Delta + J^*(\mu) = \begin{pmatrix} a_{11}(\mu) + d\frac{\partial^2}{\partial x^2} & a_{21}(\mu) \\ a_{12} & \tilde{d}\frac{\partial^2}{\partial x^2} \end{pmatrix}.$$

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Then $L_n^*(\mu) = -n^2 D + J^*(\mu)$ where $J^*(\mu)$ is the adjoint matrix of $J(\mu)$. Now we define the eigenfunction of the $L_n(\mu)$ and $L_n^*(\mu)$ corresponding the eigenvalue $i\omega$ and $-i\omega$ by

$$q = \begin{pmatrix} a_n \\ b_n \end{pmatrix}\cos(\frac{n}{l}x) \quad \text{and} \quad q^* = \begin{pmatrix} a_n^* \\ b_n^* \end{pmatrix}\cos(\frac{n}{l}x),$$

such that $< q^*, q >= 1$ and $< q^*, \bar{q} >= 0$ and also $L_n^*(\mu)q^* = -i\omega q^*$. Where

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{i\omega - (a_{11} - d(\frac{n}{l})^2}{a_{21}} \end{pmatrix} n = 0, 1, .. i_{N^*} \text{ and } \mu = 0.$$

And for $\mu = 0$ we have

$$\begin{pmatrix} a_n^* \\ b_n^* \end{pmatrix} = \begin{cases} (\frac{1}{l\pi}\frac{a_{21}a_{12}}{a_{21}a_{12}+(i\omega+a_{11})^2}, \frac{1}{l\pi}\frac{-a_{12}(i\omega+a_{11})}{a_{21}a_{12}+(i\omega+a_{11})^2})^T & \text{if } n = 0, \\ (\frac{2}{l\pi}\frac{a_{21}a_{12}}{a_{21}a_{12}+(i\omega+a_{11}-d(\frac{n}{l})^2)^2}, \frac{2}{l\pi}\frac{-a_{21}a_{12}}{a_{21}a_{12}+(i\omega+a_{11}-d(\frac{n}{l})^2)^2})^T & \text{if } n \neq 0, \end{cases}.$$

and use the decomposition of our space

$$\chi = \chi^{\mathbb{C}} \oplus \chi^s,$$

where $\chi^{\mathbb{C}} := \{zq + \bar{z}\bar{q}/z \in \mathbb{C}\}$ and $\chi^s := \{U \in \chi/ < q^*, U >= 0\}$ for any $U = (R, P)^T \in \chi$ there exists $z \in \mathbb{C}$ ($u \in \chi^{\mathbb{C}}$ means there exist $z \in \mathbb{C}$ such that $u = zq + \bar{z}\bar{q}$) and $w = (w_1, w_2)^T \in \chi^s$ such that

$$U = zq + \bar{z}\bar{q} + w = \begin{cases} R = a_n z\cos(nx) + \bar{a}_n\bar{z}\cos(nx) + w_1, \\ P = b_n z\cos(nx) + \bar{b}_n\bar{z}\cos(nx) + w_1. \end{cases} \tag{4.4.22}$$

Now, we will follow the steps given in [43] so we have

$$\begin{cases} \frac{dz}{dt} = i\omega z + < q^*, F_n(U) >, \\ \frac{dw}{dt} = L(\mu)w + H(z, \bar{z}, w), \end{cases} \tag{4.4.23}$$

where

$$\begin{cases} H(z, \bar{z}, w) = F_n(U) - < q^*, F_n(U) > q - < \bar{q}^*, F_n(U) > \bar{q}, \\ F_n(U) = \frac{1}{2}Q_{UU} + \frac{1}{6}C_{UUU} + o(|U|^4), \end{cases} \tag{4.4.24}$$

where Q_{UU} and C_{UUU} are the second and the third order of the system (4.4.7) then we have

$$\begin{cases} Q_{qq} = \begin{pmatrix} c_n \\ d_n \end{pmatrix}\cos^2(\frac{n}{l}x), \\ Q_{\bar{q}\bar{q}} = \bar{Q}_{qq}, \\ Q_{q\bar{q}} = \begin{pmatrix} e_n \\ f_n \end{pmatrix}\cos^2(\frac{n}{l}x), \\ Q_{q\bar{q}} = \begin{pmatrix} g_n \\ h_n \end{pmatrix}\cos^2(\frac{n}{l}x), \end{cases} \tag{4.4.25}$$

where the partial derivative evaluated at origin

$$
\begin{cases}
c_n = F_{1RR}a_n^2 + 2F_{1RP}a_n b_n + F_{1PP}b_n^2, \\
d_n = F_{2RR}a_n^2 + 2F_{2RP}a_n b_n + F_{2PP}b_n^2, \\
e_n = F_{1RR}a_n\bar{a}_n + F_{1RP}(a_n\bar{b}_n + \bar{a}_n b_n) + F_{1PP}b_n\bar{b}_n, \\
f_n = F_{2RR}a_n\bar{a}_n + F_{2RP}(a_n\bar{b}_n + \bar{a}_n b_n) + F_{2PP}b_n\bar{b}_n, \\
g_n = F_{1RRR}a_n^2\bar{a}_n + F_{1RRP}(a_n^2\bar{b}_n + 2a_n\bar{a}_n b_n) + F_{1RPP}(\bar{a}_n b_n^2 + 2a_n b_n\bar{b}_n) + F_{1PPP}b_n^2\bar{b}_n, \\
h_n = F_{2RRR}a_n^2\bar{a}_n + F_{2RRP}(a_n^2\bar{b}_n + 2a_n\bar{a}_n b_n) + F_{2RPP}(\bar{a}_n b_n^2 + 2a_n b_n\bar{b}_n) + F_{2PPP}b_n^2\bar{b}_n.
\end{cases}
\tag{4.4.26}
$$

Then, the reaction-diffusion system restricted to the center of manifold is given by

$$
\frac{dz}{dt} = i\omega z + < q^*, F_n(U) >= i\omega z + \sum_{2 \le i+j \le 3} \frac{g_{ij}}{i!j!} z^i z^j + o(|z|^4),
\tag{4.4.27}
$$

where

$$
\begin{cases}
g_{20} =< q^*, Q_{qq} >, \\
g_{11} =< q^*, Q_{q\bar{q}} >, \\
g_{02} =< q^*, Q_{\bar{q}\bar{q}} >, \\
g_{21} = 2 < q^*, Q_{w_{11}q} > + < q^*, Q_{w_{20}\bar{q}} > + < q^*, C_{qq\bar{q}} > .
\end{cases}
\tag{4.4.28}
$$

Then we have two cases:

(i) If $n = 0$ (for the spatially homogenous Hopf bifurcation)

$$
\begin{cases}
g_{20} = l\pi(\bar{a}_0^* c_0 + \bar{b}_0^* d_0), \\
g_{11} = l\pi(\bar{a}_0^* e_0 + \bar{b}_0^* f_0), \\
g_{02} = l\pi(\bar{a}_0^* \bar{c}_0 + \bar{b}_0^* \bar{d}_0), \\
g_{21} = 2 < q^*, Q_{w_{11}q} > + < q^*, Q_{w_{20}\bar{q}} > + < q^*, C_{qq\bar{q}} > .
\end{cases}
$$

where

$$
\begin{aligned}
< q^*, Q_{w_{11}q} >= l\pi[\bar{a}_0^*(F_{1RR}w_{11}^{(1)}a_0 + F_{1RP}(w_{11}^{(1)}b_0 + w_{11}^{(2)}a_0) + F_{1PP}w_{11}^{(2)}b_0) \\
+ \bar{b}_0^*(F_{2RR}w_{11}^{(1)}a_0 + F_{2RP}(w_{11}^{(1)}b_0 + w_{11}^{(2)}a_0) + F_{2PP}w_{11}^{(2)}b_0)].
\end{aligned}
$$

$$
\begin{aligned}
< q^*, Q_{w_{20}\bar{q}} >= l\pi[\bar{a}_0^*(F_{1RR}w_{20}^{(1)}\bar{a}_0 + F_{1RP}(w_{20}^{(1)}\bar{b}_0 + w_{20}^{(2)}\bar{a}_0) + F_{1PP}w_{20}^{(2)}\bar{b}_0) \\
+ \bar{b}_0^*(F_{2RR}w_{20}^{(1)}\bar{a}_0 + F_{2RP}(w_{20}^{(1)}\bar{b}_0 + w_{20}^{(2)}\bar{a}_0) + F_{2PP}w_{20}^{(2)}\bar{b}_0).
\end{aligned}
$$

$$
< q^*, C_{qq\bar{q}} >= l\pi(\bar{a}_0^* g_0 + \bar{b}_0^* h_0).
$$

And

$$
w_{11} = -[L_0(\mu)]^{-1}H_{11} = \begin{pmatrix} w_{11}^{(1)} \\ w_{11}^{(2)} \end{pmatrix} = \begin{pmatrix} -\frac{H_{11}^{(2)}}{a_{21}} \\ -\frac{-H_{11}^{(1)}a_{21} + H_{11}^{(2)}a_{11}}{a_{21}a_{12}} \end{pmatrix}.
$$

$$w_{20} = [2i\omega diag(1,1) - L_0(\mu)]^{-1}H_{20} = \begin{pmatrix} w_{20}^{(1)} \\ w_{20}^{(2)} \end{pmatrix} = \begin{pmatrix} \frac{H_{20}^{(1)}2i\omega + H_{20}^{(2)}a_{12}}{(2i\omega - a_{11})2i\omega - a_{21}a_{12}} \\ -\frac{H_{20}^{(1)}a_{21} + H_{20}^{(2)}(2i\omega - a_{11})}{(2i\omega - a_{11})2i\omega - a_{21}a_{12}} \end{pmatrix}.$$

where

$$H_{11} = Q_{q\bar{q}} - <q^*, Q_{q\bar{q}}>q - <\bar{q}^*, Q_{q\bar{q}}>\bar{q} = \begin{pmatrix} H_{11}^{(1)} \\ H_{11}^{(2)} \end{pmatrix},$$

$$= \begin{pmatrix} e_0 - l\pi a_0(\bar{a}_0^* e_0 + \bar{b}_0^* f_0) - l\pi\bar{a}_0(a_0^* e_0 + b_0^* f_0) \\ f_0 - l\pi b_0(\bar{a}_0^* e_0 + \bar{b}_0^* f_0) - l\pi\bar{b}_0(a_0^* e_0 + b_0^* f_0) \end{pmatrix}.$$

$$H_{20} = Q_{qq} - <q^*, Q_{qq}>q - <\bar{q}^*, Q_{qq}>\bar{q} = \begin{pmatrix} H_{20}^{(1)} \\ H_{20}^{(2)} \end{pmatrix},$$

$$= \begin{pmatrix} c_0 - \pi a_0(\bar{a}_0^* c_0 + \bar{b}_0^* d_0) - \pi\bar{a}_0(a_0^* c_0 + b_0^* d_0) \\ d_0 - \pi b_0(\bar{a}_0^* c_0 + \bar{b}_0^* d_0) - \pi\bar{b}_0(a_0^* c_0 + b_0^* d_0) \end{pmatrix}.$$

(ii) If $n = 1, 2, ..., i_{N^*}$ (for the spatially inhomogeneous Hopf bifurcation)

Not that

$$\int_0^{l\pi} \cos^3 \frac{n}{l}x \, dx = 0.$$

By straight forward calculation we have

$$<q^*, Q_{q\bar{q}}> = <q^*, Q_{qq}> = <\bar{q}^*, Q_{qq}> = <\bar{q}^*, Q_{\bar{q}\bar{q}}> = 0.$$

Then we have

$$g_{20} = g_{11} = g_{02} = 0,$$

$$g_{21} = 2 <q^*, Q_{w_{11}q}> + <q^*, Q_{w_{20}\bar{q}}> + <q^*, C_{qq\bar{q}}>.$$

(4.4.29)

where

$$w_{11} = -\frac{1}{2}[L_n(\mu)]^{-1}[\cos \frac{2n}{l}x + 1]\begin{pmatrix} e_n \\ f_n \end{pmatrix},$$

$$= w_{11n}\cos \frac{2n}{l}x + w_{11n}.$$

and

$$w_{20} = -\frac{1}{2}[2i\omega diag(1,1) - L_n(\mu)]^{-1}[\cos \frac{2n}{l}x + 1]\begin{pmatrix} c_n \\ d_n \end{pmatrix},$$

$$= w_{20n}\cos \frac{2n}{l}x + w_{20n}.$$

Note that $\int_0^{l\pi} \cos \frac{2n}{l}x \cos^2 \frac{n}{l}x \, dx = \frac{\pi}{4}$ and $\int_0^{l\pi} \cos^2 \frac{n}{l}x \, dx = \frac{\pi}{2}$ then we have

$$<q^*, Q_{w_{11}q}> = \frac{3l\pi}{4}[\bar{a}_n^*(F_{1RR}w_{11n}^{(1)}a_n + F_{1RP}(w_{11n}^{(1)}b_n + w_{11}^{(2)}a_n) + F_{1PP}w_{11n}^{(2)}b_n)$$

$$+\bar{b}_n^*(F_{2RR}w_{11n}^{(1)}a_n + F_{2RP}(w_{11n}^{(1)}b_n + w_{11n}^{(2)}a_n) + F_{2PP}w_{11n}^{(2)}b_n)],$$

(4.4.30)

$$<q^*, Q_{w_{20}\bar{q}}> = \frac{3l\pi}{4}[\bar{a}_0^*(F_{1RR}w_{20n}^{(1)}\bar{a}_0 + F_{1RP}(w_{20n}^{(1)}\bar{b}_0 + w_{20}^{(2)}\bar{a}_0) + F_{1PP}w_{20n}^{(2)}\bar{b}_0)$$

$$+\bar{b}_0^*(F_{2RR}w_{20n}^{(1)}\bar{a}_0 + F_{2RP}(w_{20n}^{(1)}\bar{b}_0 + w_{20}^{(2)}\bar{a}_0) + F_{2PP}w_{20n}^{(2)}\bar{b}_0),$$

(4.4.31)

$$< q^*, C_{qq\bar{q}} >= \frac{3l\pi}{8}(\bar{a}_0^* g_0 + \bar{b}_0^* h_0),$$ (4.4.32)

where

$$w_{20n} = \frac{1}{2}[2i\omega \, diag(1,1) - L_n(\mu)]^{-1}\begin{pmatrix} c_n \\ d_n \end{pmatrix} = \begin{pmatrix} w_{20n}^{(1)} \\ w_{20n}^{(2)} \end{pmatrix},$$
$$= \frac{1}{2\tilde{D}_{n\omega}}\begin{pmatrix} c_n(2i\omega + \tilde{d}(\frac{n}{l})^2) + d_n a_{12} \\ d_n(2i\omega - a_{11} + d(\frac{n}{l})^2) + e_n a_{21} \end{pmatrix}.$$

Such that

$$\tilde{D}_{n\omega} = \det(2i\omega \, diag(1,1) - L_n(\mu)),$$
$$= (2i\omega + \tilde{d}(\tfrac{n}{l})^2)(2i\omega - a_{11} + d(\tfrac{n}{l})^2) - a_{12}a_{21},$$

and

$$w_{11n} = -\frac{1}{2}[L_n(\mu)]^{-1}\begin{pmatrix} e_n \\ f_n \end{pmatrix} = \begin{pmatrix} w_{11n}^{(1)} \\ w_{11n}^{(2)} \end{pmatrix},$$
$$= -\frac{1}{2D_n}\begin{pmatrix} -e_n\tilde{d}(\frac{n}{l})^2 + f_n a_{12} \\ f_n(a_{11} - d(\frac{n}{l})^2) - e_n a_{21} \end{pmatrix},$$

where D_n is defined in (4.4.7). Using the results in **Hassard** [43] the direction and stability of Hopf bifurcation can be determined by the following values

$$\mu_2 = -\frac{Re(c_1(0))}{\beta'(0)},$$

$$v_2 = 2Re(c_1(0)),$$

with

$$c_1(0) = \frac{i}{2\omega}\left(g_{20}g_{11} - 2g_{11}\bar{g}_{11} - \frac{1}{3}g_{02}\bar{g}_{02}\right) + \frac{g_{21}}{2}.$$ (4.4.33)

The direction of Hopf bifurcation is determined by μ_2 where the Hopf bifurcation is supercritical (resp. subcritical) if $\mu_2 > 0$ (resp. $\mu_2 < 0$). Further, the bifurcating periodic solutions exists for $\mu > 0$ (resp. $\mu < 0$) and the stability of the periodic solutions determined by v_2, where the bifurcating periodic solution are asymptotically stable (resp. unstable) if $v_2 < 0$ (resp. $v_2 > 0$)

4.5 Numerical simulation

In this section we shall discuss the relationship between the theoretical results and the graphical representation, which is given in the following form

Fig.4.2: Represents the negative effect of the prey escaping rate P on the prey density equilibrium, which has been shown in (4.2.1). This result shows that the escaping never serves the prey population.

Fig.4.3: represents the impact of the prey escaping rate P on the predator density equilibrium obtained in **Section 4.2** where it varies between a positive and negative impact which depends only on
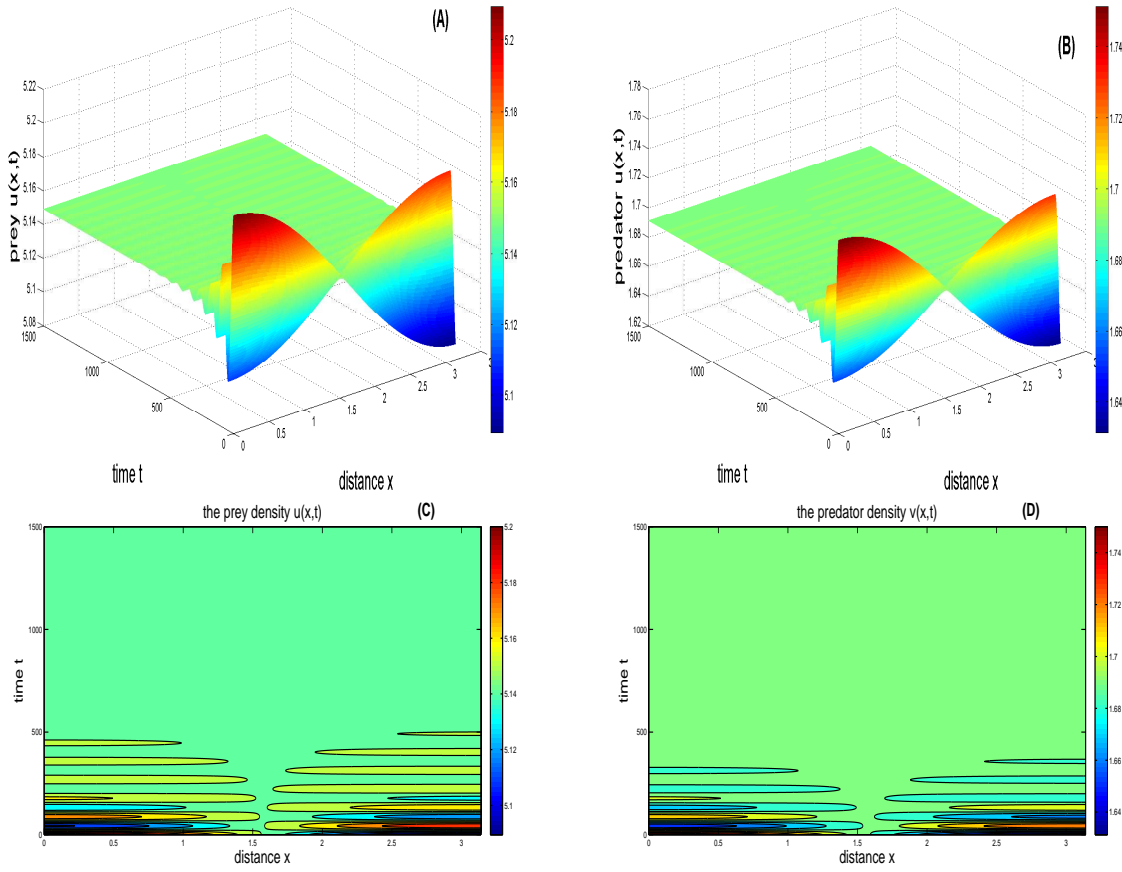
FIGURE 4.9: Numerical simulation of the system (4.1.2) when the equilibrium $(u^*, v^*) = (5.1494, 1.6639)$ is locally asymptotically stable for $k = 32$ and $d = 0.02$; $\widetilde{d} = 0.01$; $a_1 = 0.5$; $a_2 = 0.7$; $P = 0.5$; $l = 1$ $e = 0.5$; $m = 0.5$; $r = 1.2$ with the initial condition $u(0,x) = u^* + 0.06cosx$ and $v(0,x) = v^* + 0.06cosx$.

two critical values $k_{cr} := \frac{2m}{ea_2}$ and P_{cr}. This results shows that the prey escaping can serve (or not) the predator population.

Fig.4.4: Shows the impact of both the prey escaping rate and the carrying capacity of the environment for the prey population on the predator density equilibrium, which generalizes the obtained figure in **Fig.4.3**. This results can be seen easily from **Section 4.2**. Means that both the carrying capacity of the environment for the prey population and the prey escaping from the pack affect hugely the predator population density equilibrium.

Fig.4.5: Shows the global stability of the semi trivial equilibrium $(k,0)$ for $u^* > 2$ for multi values

FIGURE 4.10: Numerical simulation of the system (4.1.2) when the equilibrium $(u^*, v^*) = (5.1494, 1.6827)$ is unstable and the existence of a homogeneous periodic solution for the values $k = 34$ and $d = 0.02$; $\widetilde{d} = 0.01$; $a_1 = 0.5$; $a_2 = 0.7$; $P = 0.5$; $l = 1$ $e = 0.5$; $m = 0.5$; $r = 1.2$ with the initial condition $u(0, x) = u^* - 0.1 + 0.03 cos x$ and $v(0, x) = v^* - 0.06 + 0.35 cos x$.

of the initial conditions. This result has been used to illustrate the results of **Theorem 21**, also shows that the predator population extinct for the smaller capacities of the environment for the prey.

Fig.4.6: Represents the local stability of the interior equilibrium (u^*, v^*) for $k < k^*$ determined by studying quality of solution of the characteristic equation (4.3.4).

Fig.4.7: represents that the interior equilibrium is unstable for $k > k^*$ and the existence of a stable periodic orbit. This result has been obtained due to the qualitative analysis of the characteristic equation (4.3.4).
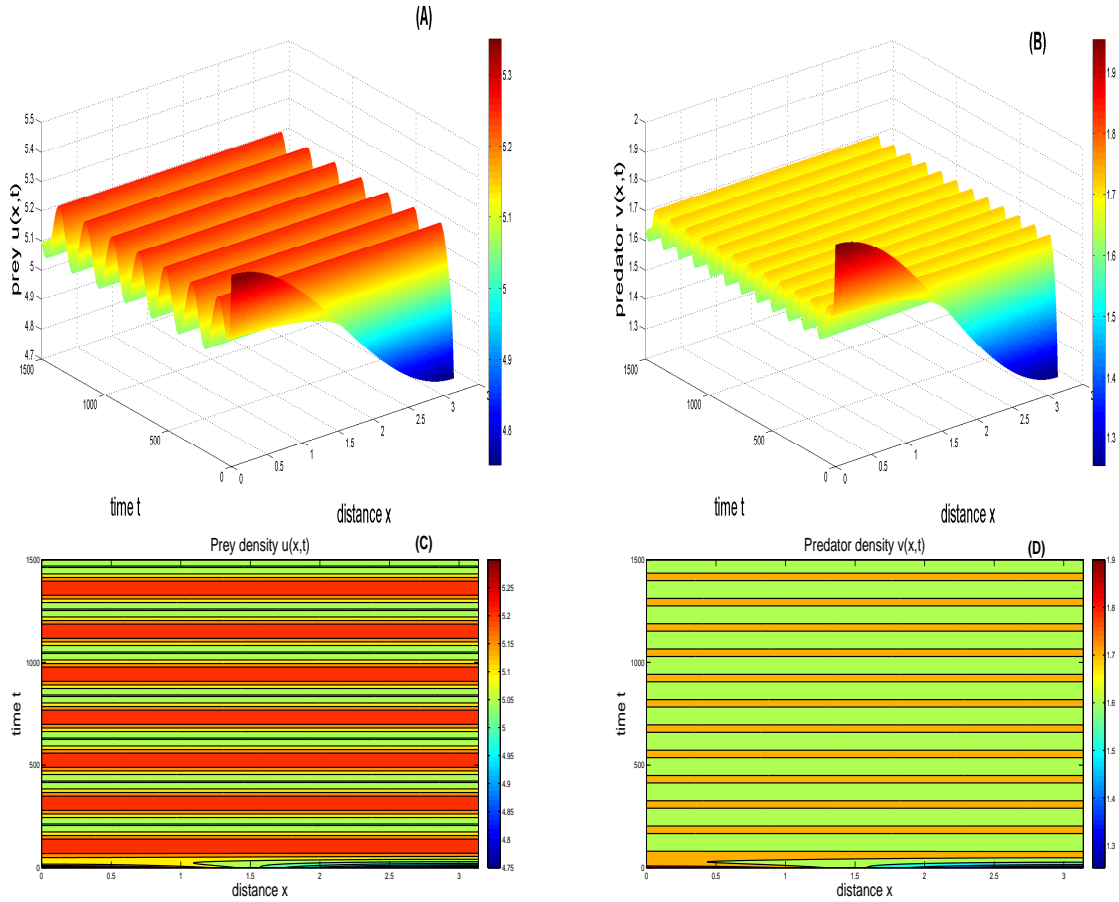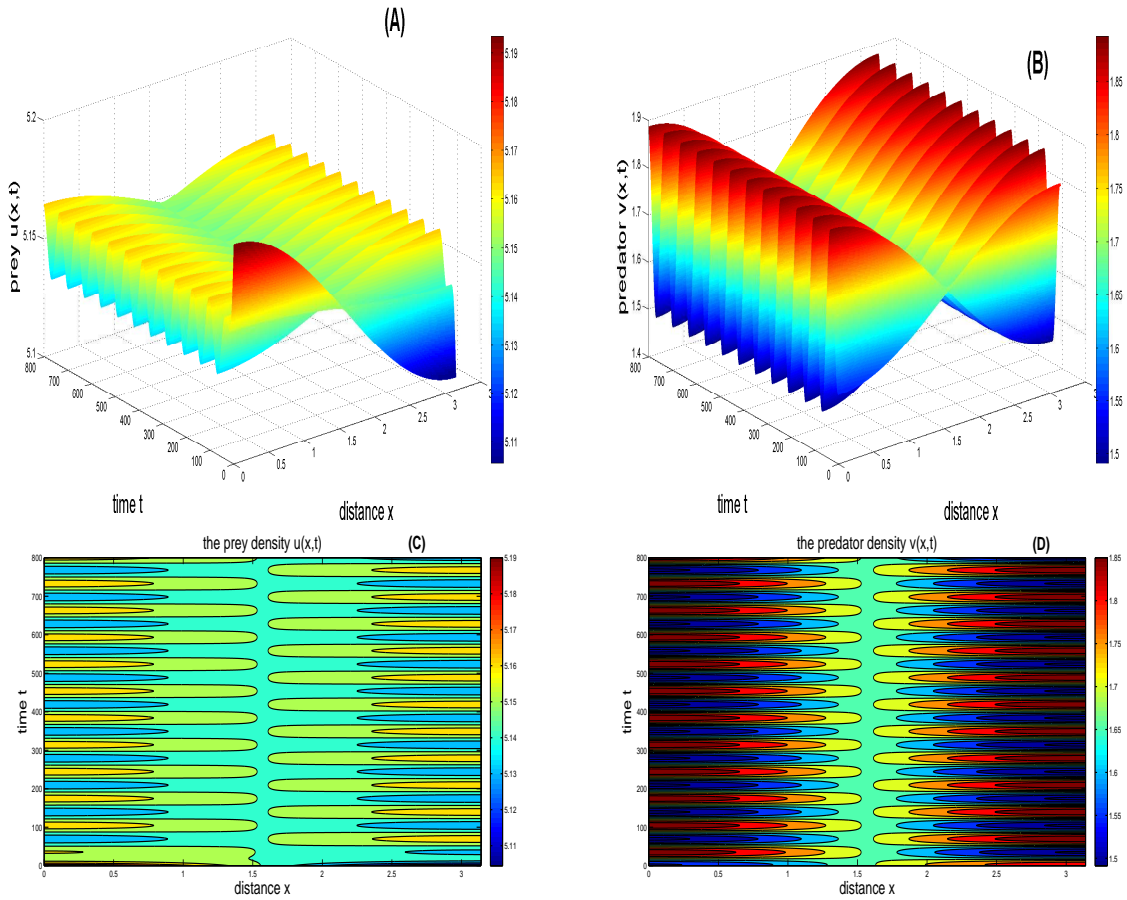
FIGURE 4.11: Numerical simulation of the system (4.1.2) when the equilibrium state $(u^*, v^*) = (5.1494, 1.6912)$ is unstable and the existence of a non homogeneous periodic solution for the values $k = 44$ and $d = 0.02$; $\widetilde{d} = 0.01$; $a_1 = 0.5$; $a_2 = 0.7$; $P = 0.5$; $l = 1$ $e = 0.5$; $m = 0.5$; $r = 1.2$ with the initial data $u(0,x) = u^* - 0.015 + 0.06 cos x$ and $v(0,x) = v^* - 0.2 + 0.06 cos 0.04 x$.

Fig.4.8: Shows the existence and the order of Hopf bifurcation points which insures the obtained results in **Theorem 22**. Further, shows the maximal value of the integer n in which the system undergoes Hopf bifurcation (in this case we have $N_1 = 6$). Also, means that we have seven Hopf bifurcation points.

Now lets investigate with the graphical representation of solutions of the system (4.1.2), the following

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FIGURE 4.12: Numerical simulation of the effect of the prey escaping rate P on the Hopf bifurcation value shown in (4.3.6), the values $e = 0.44$; $m = 1.01$; $r = 0.2$; $k = 8000$ are used, where for (A) we take $a_1 = 1$; $a_2 = 2.1$; and for (B) we take $a_1 = 2.5$ and $a_2 = 2.1$.

numerical schema is used

$$
\begin{cases}
u_i^{n+1} = u_i^n - \Delta t \frac{d}{\Delta x^2}(2u_i^n - u_{i-1}^n - u_{i+1}^n) + f(u_i^n, v_i^n), i = 1, 2, ..., M, n = 1, 2, ..., N, \\[2mm]
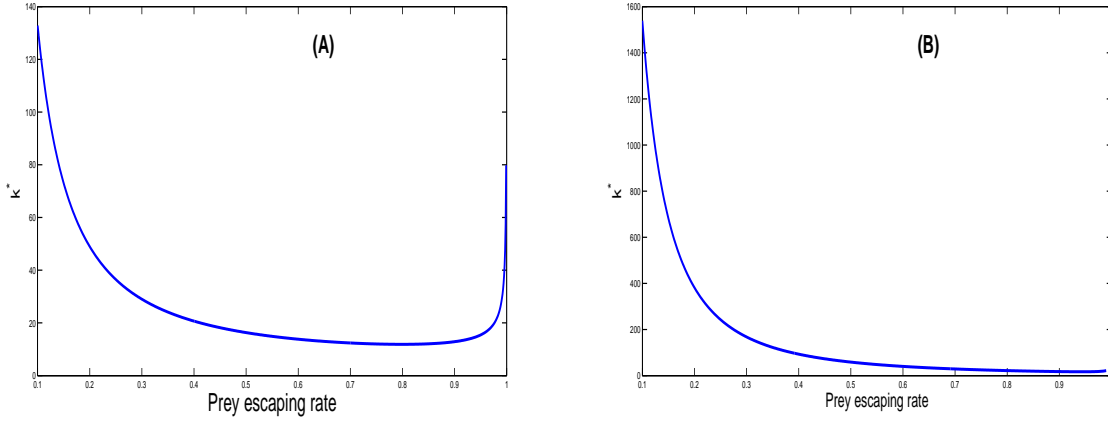v_i^{n+1} = v_i^n - \Delta t \frac{\widetilde{d}}{\Delta x^2}(2v_i^n - v_{i-1}^n - v_{i+1}^n) + g(u_i^n, v_i^n), i = 1, 2, ..., M, n = 1, 2, ..., N, \\[2mm]
u_1^n = u_0^n, v_1^n = v_0^n, u_M^n = u_{M-1}^n, v_M^n = v_{M-1}^n, n = 1, 2, ..., N, \\[2mm]
u_i^0 = \phi_i, v_i^0 = \psi_i, i = 1, 2, ..., M,
\end{cases}
\tag{4.5.1}
$$

where $\Delta x = 0.005$ and $\Delta t = 0.05$ and $\phi_i = \phi(i \times \Delta x)$, $\psi_i = \psi(i \times \Delta x)$, $M = \frac{l\pi}{\Delta x}$, $N = \frac{T}{\Delta t}$ and

$$
f(u_i^n, v_i^n) = r u_i^n (1 - \frac{u_i^n}{k}) - a_1 \sqrt{(1-P)u_i^n} v_i^n - a_2 P u_i^n v_i^n,
$$

$$
g(u_i^n, v_i^n) = -m v_i^n + e a_1 \sqrt{(1-P)u_i^n} v_i^n + e a_2 P u_i^n v_i^n.
$$

Fig.4.9: Represents the local stability of the interior equilibrium state (u^*, v^*) for $k = 32 < k(0) = 33.4344$ this result has been obtained in **Theorem 23**, where (C) and (D) represents the projection of the surfaces (A) and (B) on the plan t-x, respectively.

Fig.4.10: Represents the existence of a homogeneous periodic orbits for $k = 34 > k(0) = 33.4344$ (means that n=0) shown in **Theorem 23**, where (C) and (D) represents the projection of the surfaces (A) and (B) on the plan t-x, respectively.

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Fig.4.11: Represents the existence of a non homogeneous periodic orbits for $k > k(1) = 43.3963$ (means that n=1) shown in **Theorem 23**, where (C) and (D) represents the projection of the surfaces (A) and (B) on the plan t-x, respectively.

Now shall we discuss the direction of Hopf bifurcation using a numerical simulation. Taking the following values $d = 0.02$; $\widetilde{d} = 0.01$; $a_1 = 0.5$; $a_2 = 0.7$; $P = 0.5$; $l = 1$; $e = 0.5$; $m = 0.5$; $r = 1.5$. Thus we have:

(i) The set of all bifurcation points is $\Lambda = \{k(0), k(1), k(2)\}$ where $k(0) = 22.1027$, $k(1) = 25.4397$, $k(2) = 46.5027$.

(ii) For $k = k(1)$ we have $a_{11} = 0.0324$, $a_{21} = 0.1280$, $a_{12} = -1$ and $\omega = 0.3576$, $a_1 = 1$, $b_1 = -0.0012 + 2.7939i$, $a_1^* = -0.1677 - 0.0051i$, $b_1^* = 0.1677 + 0.0051i$. Thus by a straight forward calculation, and by replacing those results in (4.4.29) (see also (4.4.30), (4.4.31), (4.4.32)) we obtained that $Re\{c_1(0)\} \approx -0.8205 < 0$. Then the the periodic solutions are stables at $k = k(1)$.

4.6 Discussion

In this paper, a study of a new approach of escaping prey from a pack with a predator-prey interaction in the presence of spatial diffusion and subject to the homogeneous Neumann boundary conditions. The impact of the prey escaping on the positive equilibrium state (u^*, v^*) has been successfully studied in the second section, where the negative effect on the prey density equilibrium has been obtained (see **Fig.4.2**). This results shows that this specific behavior of the prey population has always a negative impact on the evolution of the prey density. On the other hand, the effect of the variable P on the predator density equilibrium has been divided between a positive and negative effect, where it depends on two parameters the carrying capacity of the space of the prey k and a critical value of the prey escaping rate denoted by P_{cr}. Also those results were justified using numerical simulations (**Fig.4.3**). Because of the dependence of the predator density equilibrium (see **Lemme 10**) on two parameters P and k we presented the **Fig.4.4** which shows the effect of those two parameters on the density equilibrium of the predator. Besides, in **Fig.4.12** the impact of the variable P on the value of the Hopf bifurcation shown in (4.3.6 which prove the huge sensitivity of the Hopf bifurcation value to the escaping prey rate. Mathematically, this sensitivity can lead to a huge variation in the solution behavioral of the considered systems (4.1.1) and (4.1.2).

In section 3, our main focus has been on the study of the non spatial system (4.1.1). Firstly, the existence of the upper bound of solution has been successfully established. This upper bound has been used to prove the global stability of the semi trivial equilibrium state $(k, 0)$ in the absence of the interior equilibrium state (u^*, v^*) (which means if $k < u^*$) the result has been also justified by a numerical

simulation (**Fig.4.5**). Besides, the local stability of the positive equilibrium state (u^*, v^*) has been also determined, where a critical value of the carrying capacity of the space for the prey denoted by k^* such that for $k < k^*$ the interior equilibrium is locally stable and unstable for $k > k^*$. Further, we obtained that the system $(4.1.1)$ undergoes a Hopf bifurcation at $k = k^*$. This result has been checked out using numerical simulations (**Fig.4.6** and **Fig.4.7**).

In the next section, the analysis of the diffusive system has been investigated. The characteristic equation has been calculated for the interior equilibrium state (u^*, v^*). By analyzing the characteristic equation, it has been proved that the diffusive system undergoes a Hopf bifurcation at multi value $k = k(n)$ (where $n < N_1$) and the sequence $k(n)$ is increasing in n (see **Fig.4.8**). Note that for $n = 0$ a homogeneous periodic solution appears (see **Fig.4.10**) and non homogeneous periodic solution appears for $n = 1, 2, ..., j_{N^*}$ (see **Fig.4.11**) (which appears when the interior equilibrium state (u^*, v^*) is unstable).For determining the stability of the periodic solution the normal form on the center manifold theory has been used (see [18]). An important quantity denoted by v_2 shows the stability of the homogeneous and non homogeneous periodic solutions has been determined, where for $v_2 < 0$ the periodic solutions are unstable and stable for $v_2 > 0$.

Based on the biological meaning of those results, the main remark is the escaping rate never serves the prey population. In the real world, the objective of the attack of the predator on the prey pack is to divide this herd, where the main focus will be on hunting the escaping prey. This behavior can be seen widely in the real world, where the most hunted prey are the youngest ones. In other word, the prey must stay organized, and the pack is the easiest and effective way for prey to survive in the wild. Besides, the global stability of the semi trivial equilibrium $(k, 0)$ for $k < u^*$ means that for the smaller species the predator cannot satisfies it need (the prey is no more available to the predator), which leads to the extinction of the predator population. On the other hand, after the analysis of the system there exists a critical value of the carrying capacity of the space for the prey population such that for the small spaces (means that the carrying capacity is lower then this critical value) we can conserve the two populations at a constant density in the long term, and for large spaces (means that the carrying capacity is bigger then this critical value) we conserve the two population but with a periodic patterns.

Chapter 5

Appendix

We present in this appendix some necessary notions as well as the mathematical tools that we used for the good understanding of previous chapters.

All of the following concepts are adapted from [6, 33, 42, 43, 70, 91]

Theorem 24. *(**Existence and uniqueness of solution** [42]) let I be an open interval of $\mathbb{R}$ and let*

$$x : I \longrightarrow \mathbb{R} : \quad t \longrightarrow x(t) \tag{5.0.1}$$

be a real-valued differentiable function of a real variable t. We will use the notation $\dot{x}$ to denote the derivative dx/dt, and refer to t as time or independent variable. Also let

$$f : \mathbb{R} \longrightarrow \mathbb{R} : \quad x \longrightarrow f(x) \tag{5.0.2}$$

be a given real-valued, where x is an unknow function of t and f is a given function of x. Eq. (5.0.2) is called a scalar autonomous differential equation because the function f does not depend on t.

We say that a function x is a solution of Eq. (5.0.2) on the interval I if $\dot{x}(t) = f(x(t))$ for all $t \in I$. We will often be interested in a specific solution of Eq. (5.0.2) which at some initiale time $t_0 \in I$ has the value x_0. Thus we will study x satisfying

$$\dot{x} = f(x), \qquad x(t_0) = x_0. \tag{5.0.3}$$

(i) *If $f \in C^0(\mathbb{R}, \mathbb{R})$, then, for any $x_0 \in \mathbb{R}$, there is an interval(possible infinite) $I_{x_0} = (\alpha_{x_0}, \beta_{x_0})$ containing $t_0 = 0$ and a solution $\phi(t, x_0)$ of the initial-value problem (5.0.3), defined for all $t \in I_{x_0}$, satisfying the initial condition $\phi(0, x_0) = x_0$. Also if α_{x_0} is finite, then*

$$\lim_{t \to \alpha_{x_0}^+} \phi(t, x_0) = +\infty, \tag{5.0.4}$$

or, if β_{x_0} is finite, then

$$\lim_{t \to \beta_{x_0}^-} \phi(t, x_0) = +\infty, \tag{5.0.5}$$

(ii) If, in addition, $f \in C^1(\mathbb{R}, \mathbb{R})$, then $\phi(t, x_0)$ is unique in I_{x_0} and $\phi(t, x_0)$ is continuous in $(t, x_0$ toghere with its first partial derivatives that is $\phi(t, x_0)$ is a C^1 function.

A planar autonomous system

let I be an open interval of the real line $\mathbb{R}$ and

$$x_i : I \longrightarrow \mathbb{R}; \qquad t \longrightarrow x_i(t) \qquad \text{for} i = 1, 2 \tag{5.0.6}$$

be two C^1 functions of a real variable t.Also, let

$$f_i : \mathbb{R}^2 \longrightarrow \mathbb{R}; \quad (x_1, x_2) \longrightarrow f_i(x_1, x_2) \qquad i = 1, 2 \tag{5.0.7}$$

be two given real-valued functions in two variables. We obtain a pair of simultaneous differential equations of the form

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2), \\ \dot{x}_2 = f_2(x_1, x_2). \end{cases} \tag{5.0.8}$$

Before announcing the definitions, wi will be convenient to use boldface letters to denote vectors quantities. For instance, if we let $X = (x_1, x_2), \quad \dot{X} = (\dot{x}_1, \dot{x}_2)$, and $f = (f_1, f_2)$, then Eq. (5.0.8) can be written as

$$\dot{X} = f(X). \tag{5.0.9}$$

Definition 1. *(**Stability of an equilibrium** [42]) An equilibrium point $\bar{X}$ of a planar autonomous system $\dot{X} = f(X)$ is said to be **stable** if, for any given $\epsilon > 0$, there is a $\delta > 0$ (depending only in ϵ) such that, for every X^0 for which $\|X^0 - \bar{X}\| < \delta$, the solution $\phi(t, X^0)$ of $\dot{X} = f(X)$ through X^0 at $t = 0$ satisfies the enequality $\|\phi(t, X^0) - \bar{X}\| < \epsilon$ for all $t \geq 0$. The equilibrium $\bar{X}$ is said to be unstable if it is not stable, that is, there is an $\eta > 0$ such that, for any $\delta > 0$, there is an X^0 with $\|X^0 - \bar{X}\| < \delta$ and $t_{X^0} > 0$ such that $\|\phi(t_{X^0}, X^0) - \bar{X}\| = \eta$.*

Definition 2. *(**Asymptotic stability** [42]) An equilibrium point $\bar{X}$ is said to be **symptotically stable** if it is stable in addition, there is an $r > 0$ such that $\|\phi(t, X^0) - \bar{X}\| \longrightarrow 0$ as $t \longrightarrow +\infty$ for all X^0 satisfying $\|X^0 - \bar{X}\| < r$*

Now, let

$$
Df(X) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1}(X) & \dfrac{\partial f_1}{\partial x_2}(X) \\[2em] \dfrac{\partial f_2}{\partial x_1}(X) & \dfrac{\partial f_2}{\partial x_2}(X) \end{pmatrix}
\tag{5.0.10}
$$

be the Jacobian matrix of f at the point X.

Definition 3. *If $\bar{X}$ is an equilibrium point of $\dot{X} = f(X)$, then the linear differential equation*

$$
\dot{X} = Df(\bar{X})X
$$

is called the linear variationel equation or the linearization of the vector field f at the equilirium point $\bar{X}$.

Theorem 25. *([42]) Let f be a C^1 function. If all the eigenvalues of the Jacobian matrix $Df(\bar{X})$ have negative real parts, then the equilibrium point $\bar{X}$ of the differential equation $\dot{X} = f(X)$ is **symptotically stable**.*

Theorem 26. *(**Poincaré-Bendixson** [91]) Let $\mathcal{M}$ be a positively invariant region for the vector field containing a finite number of fixed points. Let $p \in \mathcal{M}$, and consider $\omega(p)$. Then one of the following possibilities holds.*

(i) $\omega(p)$ is a fixed point;

(ii) $\omega(p)$ is a closed orbit;

(iii) $\omega(p)$ consists of a finite number of fixed points $p_1, ..., p_n$ and orbits γ with $\omega(\gamma) = p_i$.

(Note: $\omega(\gamma)$ is the ω-limit set of every point on γ.) defined as

$$
\omega(p) = \cup_{t \geq 0} \overline{\{x_1(s,p), x_2(s,p), s \geq t\}}
$$

Theorem 27. *(**Poincaré-Andronov-Hopf Bifurcation** [91]) We consider the normal form associated with the differential system (5.0.8)*

$$
\begin{cases} r_t = d\mu r + a r^3 \\ \theta_t = w + c\mu + b r^3 \end{cases}
\tag{5.0.11}
$$

whre, a, b, c, d, w are constant parameters. For μ sufficiently small we have the following cases

***case 1:** $\mathbf{d > 0, a > 0}$. In this case the origin is an unstable fixed point for $\mu > 0$ and an asymptotically stable fixed point for $\mu < 0$, with an unstable periodic orbit for $\mu < 0$ (note: the reader should realize that if the origin is stable for $\mu < 0$, then the periodic orbit should be unstable).*

case 2: **d > 0, a < 0**. *In this case the origin is an asymptotically stable fixed point for $\mu < 0$ and an unstable fixed point for $\mu > 0$, with an asymptotically stable periodic orbit for $\mu > 0$.*

case 3: **d < 0, a > 0**. *In this case the origin is an unstable fixed point for $\mu < 0$ and an asymptotically stable fixed point for $\mu > 0$, with an unstable periodic orbit for $\mu > 0$.*

case 4: **d < 0, a < 0**. *In this case the origin is an asymptotically stable fixed point for $\mu < 0$ and an unstable fixed point for $\mu > 0$, with an asymptotically stable periodic orbit for $\mu < 0$.*

With, $d = \dfrac{d}{d\mu}(Re\lambda(\mu))|_{\mu=0}$, and μ is the critical value of bifurcation.

An n-dimensional autonomous system

Now, we will consider dynamics systems in dimension greater than two. The general form of an n-dimensional system is as follows:

$$\dot{x}_i = f_i(x_1, x_2, ..., x_n), \tag{5.0.12}$$

with, $i \in [1, n]$. If $(x_1^*, x_2^*, ...x_n^*)$ is an equilibrium point for (5.0.12), then, we have

$$f_i(x_1^*, x_2^*, ...x_n^*) = 0, \quad \text{for} \quad i \in [1, n].$$

the linear system associated with (5.0.12) near the equilibrium point is written in the following form

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}x_j, \tag{5.0.13}$$

where, $i \in [1, n]$ and $A = [a_{ij}]_{1 \leq i \leq n, 1 \leq j \leq n}$ is a square matrix of dimension n with constant coefficients. We assume that $detA \neq 0$, which implies that the origin is the only equilibrium for system (5.0.13). The matrix A has n eigenvalues which are solutions of the characteristic equation $det(A - \lambda I) = 0$, which is a polynomial of degree n that we write in the following form:

$$P_A(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ... + a_{n-1}\lambda + a_n = 0 \tag{5.0.14}$$

Theorem 28. *(Routh-Hurwitz stability criterion [6]) Let the characteristic equation defined in (5.0.14) and the linear system (5.0.13). We consider the following n determinants:*

$$H_1 = a_1$$

$$H_2 = \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix},$$

$$H_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix},$$

$$H_k = \begin{vmatrix} a_1 & a_3 & a_5 & . & . & . \\ 1 & a_2 & a_4 & . & . & . \\ 0 & a_1 & a_3 & . & . & . \\ 0 & 1 & a_2 & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & a_k \end{vmatrix},$$

such as, $k \in [1,n]$. In the case of n dimension, all a_j with $j > n$ are taken equal to zeros. We have the following result:

The equilibrium point $(x_1^*, x_2^*, ... x_n^*)$ is asymptotically stable $\iff \forall k \in [1,n], \; H_k > 0$.

The spatial diffusion system

We consider the following system:

$$\begin{cases} u_{i_t} - \Delta u_i = f_i(U), \\ \frac{\partial u_i}{\partial \vec{n}} = 0 \quad (t > 0, \; x \in \partial \Omega), \quad i = 1,2, \\ u_i(0,x) = \eta_i(0,x), \end{cases} \tag{5.0.15}$$

with $f_i : \mathbb{R}^2 \to \mathbb{R}; \quad (i = 1,2)$ and $U = (u_1, u_2)^T$.

Definition 4. (*quasi-monotone [70]*) Let $f_i (i = 1,2)$ be two Lipschitzian functions.

$$\left| f_i(U) - f_i(U') \right| \le k_i \left(\left| u - u' \right| + \left| v - v' \right| \right), \qquad i = 1,2,$$

where, $U = (u,v)^T, U' = (u',v')^T$ and let $\chi \subset \mathbb{R}^2$ such that $f_{1v} \le 0$ and $f_{2u} \ge 0$ in χ, so f_i are a mixed quasi-monotone functional in χ.

Theorem 29. (*[70]*) Let $\hat{c} = (\hat{c}_1, \hat{c}_2)^T, \tilde{c} = (\tilde{c}_1, \tilde{c}_2)^T$ be two constant vectors in $\mathbb{R}^2$ such that $\tilde{c}_1 \le \hat{c}_1$, and $\tilde{c}_2 \le \hat{c}_2$ which satisfies the condition $f_i(\hat{c}) \le 0, \; f_i(\tilde{c}) \ge 0$ then, the system (5.0.15) has a unique global solution $U(t,x)$, satisfyings $\hat{c} \le U(t,x) \le \tilde{c}, \; \forall t > 0, x \in \chi$ for any $\hat{c} \le \eta_i(t,x) \le \tilde{c}, \; i = 1,2$.

Definition 5. (*Lower and upper solution [70]*) We say that $\hat{U}$ is a lower solution (resp $\tilde{U}$ is a upper solution) of problem (5.0.15) if
$\forall x \in \chi, \; \hat{U}_t - d\Delta\hat{U} \le f(\hat{U})$ (resp $\tilde{U}_t - d\Delta\tilde{U} \ge f(\tilde{U})$).

Lemma 13. (*[70]*) Let $\{\overline{C}^{(n)}\}, \{\underline{C}^{(n)}\}$ be two sequences satisfies

$$\overline{C}_i^{(n)} = \overline{C}_i^{(n-1)} + \frac{1}{k_i} f_i(\overline{C}_i^{(n-1)}, \underline{C}_i^{(n-1)}), \qquad i = 1,2, \quad n \in \mathbb{N}^* \tag{5.0.16}$$

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and

$$\underline{C_i}^{(n)} = \underline{C_i}^{(n-1)} + \frac{1}{k_i} f_i(\underline{C_i}^{(n-1)}, \overline{C_i}^{(n-1)}), \qquad i = 1, 2, \quad n \in \mathbb{N}^*, \tag{5.0.17}$$

with $\overline{C}^{(0)} = \tilde{C}, \underline{C}^{(0)} = \hat{C}$. *So, the sequences satisfies the following monotone property.*

$$\hat{C}_i \leq \underline{C_i}^{(n)} \leq \underline{C_i}^{(n+1)} \leq \overline{C_i}^{(n+1)} \leq \overline{C_i}^{(n)} \leq \tilde{C}_i, \qquad i = 1, 2, \quad n \in \mathbb{N}^*.$$

Theorem 30. *([70]) Assume that the condition of Theorem 2.2 [70] is hold and let $\overline{C}, \underline{C}$ are a limits of(5.0.16) and (5.0.17), so, for any initial function $\eta_i(0, x) \in (\hat{C}_i, \tilde{C}_i)$, the solution $U(t, x)$ satisfy $\underline{C_i} \leq u_i(t, x) \leq \underline{C_i}$, $i = 1, 2$,*
If we have $C^* = \overline{C} = \underline{C}$ then, C^* is the unique solution of the system (5.0.15) in $(\hat{C}_i, \tilde{C}_i)$ and

$$\lim_{t \to +\infty} U(t, x) = C^*.$$

Now, we write the system (5.0.15) in the following form

$$\frac{d}{dt} U = d\Delta U + L(U) + f(U) \quad t > 0, \tag{5.0.18}$$

with $Dom(\Delta) \subset \chi$, $L(C, \chi))$ i.e. $L : C \to \chi$ is a bounded linear operator and $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a $C^k (k \geq 2)$ function such that, $f(0) = 0$ and $D(f(0)) = 0$ and under the following assumptions

(H_1): $d\Delta$ generate a C_0 semi group $\{T(t)\}_{t \geq 0}$ in χ with $|T(t)| \leq Me^{wt}$ (for some $M \geq 1, w \in \mathbb{R}$);

(H_2): The eigenfunctions $\{\mu_n\}_{n=1}^\infty$ of $d\Delta$ corresponding to the eigenvalues $\{f_n\}_{n=1}^\infty$ form an orthonormal basis in χ, and for all $x \in Dom(\Delta)$, there exists a unique expression $x = \sum_{n=1}^\infty x_n$ with $\Delta x = \sum_{n=1}^\infty \mu_n x_n$ where $x_n = P_n x$ and $P_n : \chi \longrightarrow \chi_n = \{\alpha f_n, \ \alpha \in \mathbb{C}\}$ is a projection operator;

(H_3): $L : C \longrightarrow \chi$ defined by $L\varphi = \int_{-a}^{0} d\eta(s)ds, \ \varphi \in C$, for a function $\eta : [-a, 0] \longrightarrow B(\chi, \chi)$ of bounded variation, and L satisfies $L(P_n\varphi) \in \chi_n$ for all $n \geq 1$ and $\varphi \in C$, where $P_n\varphi \in C$ is given by $(P_n\varphi)(s) = P_n\varphi(s)$ for all $s \in [-a, 0]$. Moreover, $L\left(\sum_{n=1}^\infty P_n\varphi\right) = \sum_{n=1}^\infty L(P_n\varphi)$ if $\sum_{n=1}^\infty P_n\varphi \in C$.

Definition 6. *(forme normale [33]) The system*

$$\begin{cases} \dot{\bar{z}} = B\bar{z} + \sum_{j \geq 2} \frac{1}{j!} g_j^1(\bar{z}, \bar{y}), \\ \dot{\bar{y}} = A\bar{y} + \sum_{j \geq 2} \frac{1}{j!} g_j^2(\bar{z}, \bar{y}), \end{cases} \tag{5.0.19}$$

with $g_j = (g_j^1, g_j^2)^T, j \geq 2$ *and*

$$\begin{cases} g_j^1 = \bar{f}_j^1(z, y) - [DU_j^1(z)Bz - BU_j^1(z)], \\ g_j^2 = \bar{f}_j^2(z, y) - [DU_j^2(z)Bz - A_1 U_j^2(z)], \end{cases}$$

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is called a normal form, on the center of the manifold associated to Λ if g_j^1, g_j^2 are defined by $g_j = \bar{f}_j - M_j U_j$ and 5.0.19 is equivalent to the following normal form

$$\dot{\bar{z}} = B\bar{z} + \sum_{j \geq 2} \frac{1}{j!} g_j^1(\bar{z}, 0), \tag{5.0.20}$$

i.e. (5.0.20) is also a Λ normal form, on the center of the manifold associated to Λ with

$$\Lambda := \{\lambda \in \sigma(A); Re\lambda = 0\} \neq \emptyset$$

.

Theorem 31. ([33]) Assume that (H_1)-(H_3) are hold and $\Lambda := \{\lambda \in \sigma(A); Re\lambda = 0\} \neq \emptyset$, then, there is a normal form of variables $(z, y) = (\bar{z}, \bar{y}) + o(|\bar{z}|^2)$ as the normal form is given by the ordinary differential equation

$$\dot{\bar{z}} = B\bar{z} + \sum_{j \geq 2} \frac{1}{j!} g_j^1(\bar{z}, 0).$$

Definition 7. (*Turing instability*) Consider the system

$$\begin{cases} u_t = d_1 u_{xx} + f(u, v), \\ v_t = d_2 v_{xx} + g(u, v), \end{cases} \quad t > 0, x \in \Omega, \tag{5.0.21}$$

with the initial conditions and the associated Neumann boundary conditions

$$\begin{cases} \frac{\partial u}{\partial \vec{n}} = \frac{\partial v}{\partial \vec{n}} = 0, \quad \forall t > 0, \ x \in \partial\Omega, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), \quad x \in \Omega. \end{cases}$$

We assume that the sytem (5.0.21) has a steady state denoted by $E^* = (u^*, v^*)$ i.e. $f(u^*, v^*) = 0$ and $g(u^*, v^*) = 0$ with the associated characteristic equation

$$\lambda^2 - T_n \lambda + D_n = 0, \qquad n \in \mathbb{N}.$$

We talk about *Turing instability* when the steady state E^* is locally asymptotically stable in the absence of diffusion i.e. ($n = 0$, $D_0 > 0$ and $T_0 < 0$), but, in the presence of diffusion and for some values of d_1 and d_2, there is a frequency ($n \neq 0$) for which E^* loses its stability (in other words, diffusion does not always have a stabilizing effect for the system (5.0.21).

Theorem 32. (*Turing-Hopf bifurcation*) The *Turing-Hopf bifurcation* is a particular case of Hopf bifurcation, is a Hopf bifurcation with two parameters. We consider the characteristic equation around the equilibrium state E^*

$$\lambda^2 - T_n(\beta)\lambda + D_n(\beta) = 0$$

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where, β is the critical parameter of the bifurcation. We know that the Hopf bifurcation occurs if $T_n(\beta) = 0$ et $D_n(\beta) > 0$ and the Turing instability occurs when $D_n(\beta) < 0$ for some n where, $D_0(\beta) > 0$. Then, if there are two positive reals k^*, d_1^* such that $T_n(\beta^*, d_1^*) = 0$ and $D_n(\beta^*, d_1^*) = 0$ for some values of n then, (β^*, d_1^*) is called the critical point of **Turing-Hopf bifurcation**.

Definition 8. *(Invariant Manifold [91]) An invariant set $S \in \mathbb{R}^n$ is said to be a $C^r (r \geq 1)$ invariant manifold if S has the structure of a C^r differentiable manifold. Similarly, a positively (resp., negatively) invariant set $S \subset \mathbb{R}^n$ is said to be a $C^r (r \geq 1)$ positively (resp., negatively) invariant manifold if S has the structure of a C^r differentiable manifold.*

Definition 9. *(Homogeneous periodic solutions (resp non-homogeneous) [43]) We say that the system (5.0.15) has a periodic homogeneous solution $U(x,t)$ (resp non-homogeneous), if it satisfies the following conditions: $\exists\ T > 0$, such that $U(x, t+T) = U(x,t)$ and $\dfrac{\partial U}{\partial x}(x,t) = 0$, respectivement $U(x, t+T) = U(x,t)$ and $\dfrac{\partial U}{\partial x}(x,t) \neq 0$).*

Prospect and Future Directions

For the past century, the ecosystem models becomes one of the most important tools to understand the interaction between living beings in nature. The predator-prey model is one of the most used model in ecosystems. It has been considered as the dominant theme in mathematical ecology for a long time. The big part of the recent works in mathematic ecology are devoted to study the interaction between the predator and the prey to ensure the conservation of the species and keep the environment equilibrium. Among the factors which affect on the living beings of an ecosystem is the way in which animals live, in addition, their behavior and their systematic defense approach to defend themselves against predators. Through this thesis, we highlighted one of the most important behaviors among prey in the wild, **"the herd behavior"**. We have analyzed a new mathematical models describing the interaction between prey and predator, the main hypothesis is that the prey has a social behavior which gather together in herd. The main objective is to analyze the asymptotic behavior, theoretically and numerically of some differential systems (EDO and EDP) with herd behavior. To our knowledge, such as models have never been studied in the literature. For our models, we have more particularly interested in studying the existence, the local stability and the global stability of the equilibrium points. In addition, we have analyzed the occurrence of certain types of bifurcations such as Hopf bifurcation, Turing-Hopf bifurcation and Turing driven instability.

The study carried out during this thesis on ordinary differential equation systems (with delay and without delay) and reaction-diffusion systems allowed us to establish the bases for future developments. Much remains to be discovered, we are convinced that the study of these models will open up new interesting perspectives from both a mathematical and an application point of view.

In the real world, birth and death rates, carrying capacity, predation coefficients and other parameters involved in the systems exhibit a large random fluctuation. For the deterministic models this is negligible, so, it is interesting to consider the influence of **environmental noise** for these models by introducing **stochastic noise** to reveal the impact of environmental variability on the population dynamics in ecology.

In this context we address the following question: do the results obtained in this thesis remain true

when we introduce the environmental noise on the two species? We intend to bring back some answers for these kinds of questions.

This manuscript presents the work carried out in the biomathematics research laboratory, Faculty of Exact Sciences, Department of Mathematics, University of Djillali Liabès, Sidi Bel Abbes.

Finally, we hope that this thesis can help to further educate students about the interest of this theme.

Bibliography

[1] P.A. Abrams, L.R. Ginzburg. The nature of predation: prey dependent, ratio-dependent or neither?. Trends in Ecology and Evolution, 15(8):337–341,2000.

[2] V. Ajraldi, M. Pittavino, and E. Venturino. Modeling herd behavior in population systems. Nonlinear Analysis: Real World Applications, 12(1):2319–2338,2011.

[3] H.R. Akcakaya, R. Arditi, and L.R. Ginzburg. Ratio-dependent prediction: An abstraction that works. Ecology, 76(3):995–1004,1995.

[4] R. Arditi, L.R. Ginzburg, Coupling in predator-prey dynamics: ratio-dependence. Journal of theoretical biology, 139(3):311-326,1989.

[5] T.M. Asfaw. Dynamics of Generalized Time Dependent Predator Prey Model with Nonlinear Harvesting. Journal of Math. Analysis, 3(30):1473-1485,2009.

[6] P. Auger, C. Lett, and J.C. Poggiale. Modélisation mathématique en écologie, cours et exercices corrigés. Dunod, 2010.

[7] N.J.T. Bailey. The Mathematical Theory of Infectious Disease and Its Applications. Griffin, London, 1975.

[8] M. Baurmanna, T. Gross, and U. Feudel. Instabilities in spatially extended predator–prey systems: spatiotemporal patterns in the neighborhood of Turing–Hopf bifurcations. Journal of Theoretical Biology. Biol, 245(2):220–229,2007.

[9] G. Birkhoff, G.C. Rota. Ordinary Differential Equations. 3rd ed, Wiley. New York, 1978.

[10] S.P. Bera, A. Maiti, and G.P. Samanta . Dynamics of a food chain model with herd behaviour of the prey. Modeling Earth Systems and Environment, 131(2):2363-6203,2016.

[11] E. Beretta, Y. Kuang. Global analysis in some delayed ratio-dependent predator–prey systems. Nonlinear Analysis: Theory, Methods & Applications, 32(3):381–408,1998.

[12] A.A. Berryman. The origins and evolution of predator–prey theory. Ecology 73(5):1530–1535,1992.

[13] R.L. Buchanan. Predictive food microbiology. Trends in Food Science & Technology, 4(1):6–11,1993.

[14] I.M. Bulai, E. Venturino. Shape effects on herd behavior in ecological interacting population models. Mathematics and Computers in Simulation, 141:40-55,2017.

[15] P.A. Braza. Predator–prey dynamics with square root functional responses. Nonlinear Analysis: Real World Applications, 13(4):1837–1843,2012.

[16] E. Cagliero, E. Venturino. Ecoepidemics with infected prey in herd defense: the harmless and toxic cases. International Journal of Computer Mathematics, 93(1):108-127,2014.

[17] R.S. Cantrell, C. Cosner. On the dynamics of predator–prey models with the Beddington–DeAngelis functional response. Journal of Mathematical Analysis & Applications, 257(1):206–222,2001.

[18] J. Carr. Applications of Center Manifold Theory. New York: SpringerVerlag, 1981.

[19] C. Contreras, Á. Contreras. Herding in financial markets: Why do financial investors herd?. La revista Análisis Financiero,

[20] J. Chattopadhyay, S. Chatterjee, and E. Venturino. Patchy agglomeration as a transition from monospecies to recurrent plankton blooms. Journal of theoretical biology, 253(2):289–295,2008.

[21] R. Chinnathambi, F.A. Rihan. Stability of fractional-order prey–predator system with time-delay and Monod–Haldane functional response. Nonlinear Dynamics, 92(4):1637–1648,2018.

[22] C. Cosner, D.L. DeAngelis. Effects of spatial grouping on the functional response of predators. Theoretical population biology, 56(1):65–75,1999.

[23] J.M. Cushing. Periodic time-dependent predator–prey system. SIAM Journal on Applied Mathematics, 32(1):82–95,1977.

[24] S. Djilali. Herd behavior in a predator-prey model with spatial diffusion: bifurcation analysis and Turing instability. Journal of Applied Mathematics & Computing, 58(1-2):125-149,2017.

[25] S. Djilali. Herd behavior in a predator-prey model with spatial diffusion: bifurcation analysis and Turing instability. Journal of Applied Mathematics and Computing, 58(1-2) :125-149, 2017.

[26] S. Djilali, T.M. Touaoula, S.E.H. Miri. A Heroin epidemic model: very general non linear incidence, treat-age, and global stability. Acta Applicandae Mathematicae, 152(1):171-194,2017.

[27] S. Djilali. Impact of prey herd shape on the predator-prey interaction. Chaos Solitons & Fractals, 120:139-148,2019.

[28] S. Djilali. Effect of herd shape in a diffusive predator-prey model with time delay. Journal of Applied Analysis & Computation, 9(2):638-654,2019.

[29] S. Djilali, S. Bentout. Spatiotemporal patterns in a diffusive predator-prey model with prey social behavior. Acta Applicandae Mathematicae, 169:125–143, 2019.

[30] S. Djilali. Pattern formation of a diffusive predator-prey model with herd behavior and nonlocal prey competition. Mathematical Methods in the Applied Sciences, 43(5):2233-2250,2019.

[31] B. Ghanabri, S. Djilali. Mathematical and numerical analysis of a three-species on predator-prey model with herd behavior and time fractional-order derivative. Mathematical Methods in the Applied sciences, 43(4):1736-1752,2019.

[32] O. Diekmann, J.A.P. Heesterbeek. Mathematical epidemiology of infectious diseases: model building, analysis and interpretation. Princeton university. USA, 1994.

[33] T. Faria. Normal forms and Hopf bifurcation for partial differential equations with delay. Transactions of the American Mathematical Society, 352(5):2217-2238,2000.

[34] H.I. Freedman. Deterministic Mathematical Models in Population Ecology. Marcel Dekker. New York 1980.

[35] G. Gimmelli, B.W. Kooi, and E. Venturino. Ecoepidemic models with prey group defense and feeding saturation. Ecological Complexity, 22:50-58,2015.

[36] B. Gompertz. On the nature of the function expressive of the law of human mortality and on a new method of determining the value of life contingencies. Philosophical Transactions of the Royal Society of London, 1825.

[37] K. Gopalsamy. Stability and Oscillation in Delay Differential Equation of Population Dynamics. Kluwer Academic. Dordrecht, 1992.

[38] R.P. Gupta, P. Chandra. Bifurcation analysis of modified leslie-gower preda- tor-prey model with michaelis-menten type prey harvesting. Journal of Mathematical Analysis & Applications, 398(1):278–95,2013.

[39] R.P. Gupta, M. Banerjee and P. Chandra. Bifurcation analysis and control of Leslie–Gower predator-prey model with Michaelis-Menten type prey-harvesting. Differential Equations & Dynamical systems, 20(3):339–66,2012.

[40] H. Halil. Nonlinear Systems, 3rd ed. Prentice Hall. New Jersey, 2002.

[41] J. Hale. Ordinary Differential Equations. Krieger. Malabar, 1980.

[42] J. Hale, H. KoÇak. Dynamics and bifurcations. Springer. New York. Berlin Heidelberg, 1996.

[43] B. Hassard, N. Kazarinoff and Y. Wan. Theory and application of Hopf bifurcation. Cambridge. Cambridge University Press, 1981.

[44] Y. Huang, F. Chen, and L. Zhong. Stability analysis of a prey-predator model with Holling type III response function incorporating a prey refuge. Applied Mathematics & Computation, 182(1):672-683,2006.

[45] S.B. Hsu, T.W. Hwang and Y. Kuang. Global dynamics of a predator-prey model with Hassell-Varley type functional response. Discrete and Continuous Dynamical Systems, 10(4):857-871,2008.

[46] R. Kent Goodrich. A Riesz Representation Theorem. American Mathematical Society, 24(3):629-636,1970.

[47] W. Ko, K. Ryu. Qualitative analysis of a predator-prey model with Holling type II functional response incorporating a prey refuge. Journal of Differential Equations, 231(2):534-550, 2006.

[48] B.W. Kooi, E. Venturino. Ecoepidemic predator–prey model with feeding satiation, prey herd behavior and abandoned infected prey. Mathematical Biosciences, 274:58-72,2016.

[49] Y. Kuang. Delay Differential Equation with Applications in Population Dynamics. Academic Press. New York, 1993.

[50] Y. Kuang, E. Beretta. Global qualitative analysis of a ratio-dependent predator-prey system. Journal of mathematical biology, 36(4):389-406,1998.

[51] AK. Laird. Dynamics of tumor growth. British Journal of Cancer, 18(3):490–502,1964.

[52] P. Lenzini, J. Rebaza. Nonconstant predator harvesting on ratio-dependent predator-prey models. Applied Mathematical Sciences, 4(13–16):791–803,2010 .

[53] Q. Liu, D. Jiang, T. Hayat and A. Alsaedi, Stationary distribution and extinction of a stochastic predator–prey model with herd behavior. Journal of the Franklin Institute, 355(16):8177-8193, 2018.

[54] C. Li. A diffusive Holling-Tanner prey-predator model with free boundary. International Journal of Biomathematics, 11(5):1850066,2018.

[55] X. LI, Y. LV and Y. PEI. Social behavior of group defense in a predator–prey system with delay, Journal of Biological Systems, 26(3):399-419,2018.

[56] A.J. Lotka. Relation between birth rates and death rates. Science, 26(653) :21–22, 1907.

[57] J. Luo, Y. Zhao. Stability and bifurcation analysis in a predator-prey system with constant harvesting and prey group defense. International Journal of Bifurcation & Chaos, 27(11):1750179,2017.

[58] Z. Ma, S. Wang. A delay-induced predator-prey model with Holling type interaction functional response and habitat complexity. Nonlinear. Dynamics, 93(3):1519-1544,2018.

[59] N. MacDonald. Time delay in prey–predator models. Mathematical Biosciences, 33(3-4):227-234,1977.

[60] A.P. Maiti, B. Dubey and J. Tushar. A delayed prey–predator model with Crowley–Martin-type functional response including prey refuge. Mathematical Methods In The Applied Sciences, 40(16):5792-5809,2017.

[61] N. Mouquet, I. Gounand and D. Gravel, Biodiversité et fonctionnement des écosystèmes, Société Française d'écologie et d'évolution (SFE). 2010.

[62] I. Martina, E. Venturino. Shape effects on herd behavior in ecological interacting population models. Mathematics & Computers in Simulation, 141:40-55,2017.

[63] T.R. Malthus. An Essay on the Principle of Population, and a Summary View of the Principle of Populations. Penguin. Harmondsworth, 1798.

[64] R.M. May. Time delay versus stability in population models with two and three trophic levels. Ecology, 54(2):315–325,1973.

[65] X.Y. Meng, H.F. Huo and H. Xiang. Hopf bifurcation in a three-species system with delays. Journal of Applied Mathematics & Computing, 35(1-2):635–661,2011.

[66] Q. Meng, L. Yang. Steady state in a cross-diffusion predator-prey model with the Beddington-DeAngelis functional response. Nonlinear Analysis: Real World Applications, 45:401-413,2019.

[67] A. Mousaoui, S. Bassaid and E.H.A. Dads. The impact of water level fluctuations on a delayed prey-predator model. Nonlinear Analysis: Real World Applications, 21:170-184,2015.

[68] J.D. Murray. Mathematical Biology. Springer Verlag. New York, 1989.

[69] S.N. Nahashon, S.E. Aggrey, N.A. Adefope. A. Amenyenu and D. Wright. Growth Characteristics of Pearl Grey Guinea Fowl as predicted by the Richards, Gompertz and logistic models. Poultry Science, 85(2):359-363,2006.

[70] C. Pao. Nonlinear Parabolic and Elliptic Equations. Plenum Press. New York, 1992.

[71] S. Ruan, J. Wei. On the zeros of third degree exponential polynomial with applications to a delayed model for the control of testosteron. Mathematical Medicine and Biology, 18(1):41–52,2001.

[72] W. Rudin. Analyse réelle et complexe. Dunod. Paris, 1998.

[73] K. Ryu, W. Ko, and M. Haque. Bifurcation analysis in a predator-prey system with a functional response increasing in both predator and prey densities. Nonlinear Dynamics, 94(3):1639-1656,2018.

[74] Y. Song, X. Tang. Stability, Steady-State Bifurcations, and Turing Patterns in a Predator–Prey Model with Herd Behavior and Prey-taxis. Studies in Applied Mathematics, 139(3):371-404, 2017.

[75] Y.Song, X.Zou. Bifurcation analysis of a diffusive ratio-dependent predator-prey model. Nonlinear Dynamics, 78(1):49-70,2014.

[76] Y. Song, X. Zou. Spatiotemporal dynamics in a diffusive ratio-dependent predator-prey model near a Hopf-Turing bifurcation point. Computers & Mathematics with Applications, 67(20):1978-1997,2014.

[77] Y. Song, X. Tang. Stability, steady-state bifurcation, and Turing patterns in predator-prey model with herd behavior and prey-taxis. Studies in Applied Mathematics, 139(3):371-404,2017.

[78] Y. Song, X. Zou. Spatiotemporal dynamics in a diffusive ratio-dependent predator-prey model near a Hopf-Turing bifurcation point. Computers & Mathematics with Applications, 67(10):1978-1997,2014.

[79] T.Singh, S.Banerjee. Spatiotemporal Model of a Predator–Prey System with Herd Behavior and Quadratic Mortality. International Journal of Bifurcation & Chaos, 29(04):1950049,2019.

[80] H.B. Shi, W.T. Li, and G.L .In, Positive steady states of a diffusive predator-prey system with modified Holling-Tanner functional response. Nonlinear Analysis: Real World Applications, 11(5):3711-3721,2010.

[81] F. Souna, S. Djilali and F. Charif. Mathematical analysis of a diffusive predator-prey model with herd behavior and prey escaping. Mathematical modeling of natural phenomenon, 15:23-24,2020.

[82] F. Souna, A. Lakmeche and S. Djilali. Spatiotemporal patterns in a diffusive predator-prey model with protection zone and predator harvesting. Chaos, Solitons & Fractals, 140:110180,2020.

[83] F. Souna, A. Lakmeche and S. Djilali. The effect of the defensive strategy taken by the prey on predator–prey interaction. Journal of Applied Mathematics and Computing, 64:665–690,2020.

[84] X. Tang, Y. Song. Bifurcation analysis and Turing instability in a diffusive predator prey model with herd behavior and hyperbolic mortality. Chaos, Solitons & Fractals, 81:303-314,2015.

[85] X. Tang, Y. Song. Turing-hopf bifurcation analysis of a predator-prey model with herd behavior and cross-diffusion. nonlinear Dynamics, 86(1):73-89,2016.

[86] R.K. Upadhyay, R.K. Naji. Dynamics of a three species food chain model with Crowley–Martin type functional response. Chaos, Solitons & Fractals, 42(3):1337–1346,2009.

[87] E. Venturino. A mimimal model for ecoepidemics with group defense. Journal of Biological Systems, 19(4):763–785,2011.

[88] E. Venturino, S. Petrovskii. Spatiotemporal behavior of a prey-predator system with a group defense for prey. Ecological Complexity, 14:37-47,2013.

[89] P.F. Verhulst. Notice sur la loi que la population suit dans son accroissement. Correspondance Mathématique et Physique, 10(2):113–121,1938.

[90] V. Volterra. Sui tentativi di applicazione della matematiche alle scienze biologiche esociali. Giornale degli economisti, 23(12):436–458,1901.

[91] S. Wiggins. Introduction To Applied Nonlinear Dynamical Systems And Chaos. Second Edition. Springer. New York Berlin Heidelberg, 2003.

[92] CP. Windson. The Gompertz curve as a growth curve. Proceedings of the National Academy of Sciences of the United States of America, 18(1):1–8,1932.

[93] Q. Wang, B. Dai and Y. Chen. Multiple periodic solutions of an impulsive predator–prey model with Holling-type IV functional response. Mathematical & Computer Modelling, 49(9-10):1829-1836,2009.

[94] S. Wright. Book review. Journal of the American Statistical Association, 21(5) :494–494, 1926.

[95] J. Wu. Theory and applications of partial functional differential equations. Springer-Verlag. New York, 1996.

[96] D. Xiao, L. Jennings. Bifurcations of a ratio-dependent predator–prey system with constant rate harvesting. SIAM Journal on Applied Mathematics, 65(3):737–753,2005.

[97] D.Xiao, W. Li and M. Han. Dynamics in a ratio-dependent predator-prey model with predator harvesting. Journal of Mathematical Analysis and Applications, 324(1):14-29,2006.

[98] Z. Xie. Turing instability in a coupled predator-prey model with different Holling type functional responses. Discrete & Continuous Dynamical Systems-S, 4(6):1621-1628,2011.

[99] C. Xu, S. Yuan and T. Zhang. Global dynamics of a predator-prey model with defense mechanism for prey. Applied Mathematics Letters, 62:42-48,2016.

[100] Z. Xu, Y. Song. Bifurcation analysis of a diffusive predator-prey system with a herd behavior and quadratic mortality. Mathematical Methods in the Applied Sciences, 38(14):2994-3006,2015.

[101] W. Yang. Analysis on existence of bifurcation solutions for a predator-prey model with herd behavior. Applied Mathematical Modelling, 53:433-446,2018.

[102] R. Yang, J. Ma. Analysis of a diffusive predator-prey system with anti-predator behaviour and maturation delay. Chaos, Solitons & Fractals, 109:128-139,2018.

[103] R. Yang, J. Wei. Stability and bifurcation analysis of a diffusive prey-predator system in Holling type III with a prey refuge. Nonlinear Dynamics, 79(1):631-646,2015.

[104] R. Yang, M. Liu and C. Zhang. A diffusive predator-prey system with additional food and intra-specific competition among predators. International Journal of Biomathematics, 11(3):1850060,2018.

[105] F. Zhang, Y. Li. Stability and Hopf bifurcation of a delayed-diffusive predator-prey model with hyperbolic mortality and nonlinear prey harvesting. Nonlinear Dynamics, 88(2):1397-412,2017.

[106] F. Zhang, Y. Li, and C. Li, Hopf Bifurcation in a Delayed Diffusive Leslie–Gower Predator–Prey Model with Herd Behavior. International Journal of Bifurcation and Chaos, 29(4):1950055, 2019.

[107] L. Zhang, W. Wang and Y. Xue, Spatiotemporal complexity of a predator-prey system with constant harvest rate. Chaos, Solitons & Fractals, 41(1):38-46,2009.

[108] J. Zhang, W. Li, and X. Yan. Hopf bifurcation and Turing instability in spatial homogeneous and inhomogeneous predator-prey models. Applied Mathematics and Computation, 218(5):1883-1893,2011.

[109] C. Zhu, K. Lan. Phase portraits, Hopf bifurcations and limit cycles of Leslie–Gower predator-prey systems with harvesting rates, Discrete and Continuous Dynamial Systems Series B, 14(1):289–306,2010.

[110] H. Zhu, X. Zhang. Dynamics and patterns of a diffusive prey-predator system with a group defence for prey. Discrete Dynamics in Nature & Society, 2018:page 9,2018.

[111] J. Zhao, JP. Tian and J. Wei J. Minimal model of plankton systems revisited with spatial diffusion and maturation delay. Bulletin of Mathematical Biology, 78(3):381–412,2016.

[112] M.H. Zwietering, I. Jongenburger and F.M. Rombouts, and K. van 't Riet. Modeling of the bacterial growth Curve. Applied & Environmental Micriobiol. 56(6):1975–81,1990.

<u>**Abstract**</u>

This thesis is devoted to the mathematical study of certain ecological phenomena from the great savanna (in Africa). It focuses on the mathematical modeling of various prey-predator models where the prey exhibits a social behavior. Our main topic in this thesis is to investigate the impact of the social behavior on the relationships between species in nature.
Key words: Herd behavior; Ecological models; Ordinary differential equations with delay and without delay; periodic solutions; Diffusion of species; Hopf bifurcation; Turing driven instability; Turing-Hopf bifurcation.

<u>**Résumé**</u>

Cette thèse est consacrée à l'étude mathématique de certains phénomènes écologiques issue de la grande savanna (en Afrique). Elle se porte sur la modélisation mathématique de divers modèles proie-prédateur où la population de proie présente un comportement social. Notre objectif principal dans cette thèse est d'étudier l'impact du comportement social sur les relations entre les espèces dans la nature.
Mots clés: Comportement du troupeau; Modèles écologiques; Equations différentielles ordinaires à retard et sans retard; solutions périodiques; La diffusion des espèces; La bifurcation de Hopf; L'instabilité du Turing; La bifurcation de Turing-Hopf.

<u>**ملخص**</u>

هذه الأطروحة مخصصة للدراسة الرياضية لظواهر بيئية معينة من السافانا الكبرى (في أفريقيا). تركز هذه الأطروحة على النمذجة الرياضية لعدة نماذج الفريسة و المفترسة أين يُظهر سكان الفرائس سلوكًا اجتماعيًا. موضوعنا الرئيسي في هذه الأطروحة هو دراسة تأثير السلوك الاجتماعي على العلاقات بين الأنواع في الطبيعة.
الكلمات الرئيسية : السلوك القطيعي ، النماذج الإيكولوجية ، معادلات تفاضلية عادية مع تأخير وبدون تأخير ، حلول دورية ، إنتشار الكائنات ، تشعّب هوبف ، عدم استقرار تورينج ، تشعّب تورينج هوبف.