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DEDICATIONS

“

I dedicate this modest work to my relatives, particularly

my dear parents, may Allah bless them

my brother and my sisters

Thank you.

”

- Salim Abdelkrim

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"

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Thank you.

"

- Salim Abdelkrim

PUBLICATIONS

1. **A. Salim**, M. Benchohra and J. E. Lazreg, Boundary Value Problem for Differential Equations with Generalized Hilfer-Type Fractional Derivative. (**submitted**).

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3. **A. Salim**, M. Benchohra, E. Karapinar and J. E. Lazreg, Existence and Ulam Stability for Impulsive Generalized Hilfer-Type Fractional Differential Equations, *Adv. Difference Equ.* **2020** (2020), 1-21pp.

4. **A. Salim**, M. Benchohra, J. E. Lazreg and G. N'Guérékata, Nonlinear Implicit Generalized Hilfer-Type Fractional Differential Equations with Non-Instantaneous Impulses. (**submitted**).

5. **A. Salim**, M. Benchohra, J. E. Lazreg and J. Henderson, Nonlinear Implicit Generalized Hilfer-Type Fractional Differential Equations with Non-Instantaneous Impulses in Banach Spaces. *Adv. Theory Nonlinear Anal. Appl.* **4** (2020), 332-348.

6. **A. Salim**, M. Benchohra, J.R. Graef and J. E. Lazreg, Boundary Value Problem for Fractional Order Generalized Hilfer-Type Fractional Derivative with Non-Instantaneous Impulses. *Fractal Fract.* **5** (2021), 1-21.

7. **A. Salim**, M. Benchohra, J. E. Lazreg, J. J. Nieto and Y. Zhou, Nonlocal Initial Value Problem for Hybrid Generalized Hilfer-type Fractional Implicit Differential Equations. *Nonauton. Dyn. Syst.* **8** (2021), 87-100.

8. **A. Salim**, M. Benchohra and J. E. Lazreg, Initial Value Problem for Hybrid ψ-Hilfer Fractional Implicit Differential Equations. (**submitted**).

ABSTRACT

In this thesis, we present some results on existence, uniqueness, and stability of Ulam-Hyers-Rassias for a class of initial value problem and boundary value problem for differential equations with generalized Hilfer-type fractional derivative with and without impulses (both instantaneous and non-instantaneous), We also discuss the class of initial value problem for nonlinear fractional Hybrid implicit differential equations with generalized Hilfer and ψ-Hilfer fractional derivative. The methods used are the fixed point theorems of Krasnoselskii, Dhage and Schaefer and Banach contraction principle. We also take into account the same problems, albeit in Banach Spaces, with results based on the fixed point theorems of Darbo and Mönch associated with the technique of measure of noncompactness. Further, for the justification of our results we provide various examples every chapter.

Key words and phrases : Generalized Hilfer type fractional derivative, boundary value problem, existence, measure of noncompactness, fixed point, Banach space, Ulam-Hyers-Rassias stability, ψ-Hilfer fractional derivative, implicit fractional differential equations, fractional integral, impulses, instantaneous impulses, Ulam stability, initial value problem, nonlocal problem, Gronwall lemma, non-instantaneous impulses, hybrid fractional differential equations, initial value problem, non-instantaneous impulses.

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CONTENTS

INTRODUCTION

Fractional calculus is a field in mathematical analysis which is a generalization of integer differential calculus that involves real or complex order derivatives and integrals [10–14, 25, 27, 38, 45–47]. There is a long history of this concept of fractional differential calculus. One might wonder what meaning could be attributed to the derivative of a fractional order, that is $\frac{d^n y}{dx^n}$, where n is a fraction. Indeed, in a correspondence with Leibniz, L'Hopital considered this very possibility. L'Hopital wrote to Leibniz in 1695 asking," What if n be $\frac{1}{2}$? " The study of the fractional calculus was born from this question. Leibniz responded to the question, " $d^{\frac{1}{2}}x$ will be equal to $x\sqrt{dx : x}$. This is an apparent paradox from which, one day, useful consequences will be drawn."

Over the years, many well known mathematicians have assisted in this theory. Thus, 30 September 1695 is the precise date of birth of the "fractional calculus"! Consequently, the fractional calculus has its roots in the work of Leibnitz, L'Hopital (1695), Bernoulli (1697), Euler (1730), and Lagrange (1772). Some years later, Laplace (1812), Fourier (1822), Abel (1823), Liouville (1832), Riemann (1847), Grünwald (1867), Letnikov (1868), Nekrasov (1888), Hadamard (1892), Heaviside (1892), Hardy (1915), Weyl (1917), Riesz (1922), P. Levy(1923), Davis (1924), Kober (1940), Zygmund (1945), Kuttner (1953), J. L. Lions (1959) , and Liverman (1964) and several more have developed the fundamental principle of fractional calculus.

Ross held the first fractional calculus conference at the University of New Haven in June of 1974, and edited its proceedings [88]. Thereafter, Spanier published the first monograph devoted to "Fractional Calculus" in 1974 [79]. In recent research in theoretical physics, mechanics and applied mathematics, the integrals and derivatives of non-integer order, and the fractional integrodifferential equations have seen numerous applications. Samko, Kilbas and Marichev's exceptionally detailed encyclopedic-type monograph was published in Russian in 1987 and in English in 1993 [96], (for more details see [73]). The works devoted substantially to fractional differential equations are : the book of Miller and Ross (1993) [75], of Podlubny (1999) [82], by Kilbas et *al.* (2006) [70], by Diethelm (2010) [56], by Ortigueira (2011) [80], by Abbas et *al.* (2012) [14], and by Baleanu et *al.* (2012) [37].

The origins of fixed point theory, as it is very well known, go to the system of successive approximations (or the iterative method of Picard) used to solve certain differential equations. Roughly speaking, from the process of successive approximations, Banach obtained the fixed point theorem. The fixed point theory has been immense and independent of the differential equations in the last few decades. But, lately, the outcomes of fixed points have turned out to be the instruments for the differential equation's solutions. Recently, differential fractional order equations have been shown to be an effective instrument for researching multiple phenomena in diverse fields of science and engineering, such as electrochemistry, electromagnetics, viscoelasticity, economics, etc. It is very popular in the literature to suggest a solution to fractional differential equations by adding various forms of fractional derivatives, see e.g. [7–9, 13, 14, 16, 20–22, 25, 27, 31, 38, 40, 50, 68, 69, 113]. In the other hand, there are more findings concerned with the issues of boundary value for fractional differential equations [25, 35, 41, 42, 51, 113].

In 1940, Ulam [104, 105] raised the following problem of the stability of the functional equation (of group homomorphisms): "*Under what conditions does it exist an additive mapping near an approximately additive mapping ?*"

Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given any $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

A partial answer was given by Hyers [65] in 1941, and between 1982 and 1998 Rassias [86, 87] established the Hyers-Ulam stability of linear and nonlinear mappings. Subsequently, many works have been published in order to generalize Hyers results in various directions, see for example [10, 13, 46, 47, 65, 72, 74, 84, 85, 89, 98, 105].

Many physical phenomena have short-term perturbations at some points caused by external interventions during their evolution. Adequate models for this kind of phenomena are impulsive differential equations. Two types of impulses are popular in the literature: instantaneous impulses (whose duration is negligible) and non-instantaneous impulses (these changes start impulsively and remain active on finite initially given time intervals). There are mainly two approaches for the interpretation of the solutions of impulsive fractional differential equations: one by keeping the lower bound of the fractional derivative at the fixed initial time and the other by switching the lower limit of the fractional derivative at the impulsive points. The statement of the problem depends significantly on the type of fractional derivative. Fractional derivatives have some properties similar to ordinary derivatives (such as the derivative of a constant) which lead to similar initial value problems as well as similar impulsive conditions (instantaneous and non-instantaneous). The class of problems for fractional differential equations with abrupt and instantaneous impulses is vastly studied, and different topics on the existence and qualitative properties of solutions are considered, [45, 57, 106]. In pharmacotherapy, instantaneous impulses cannot describe the dynamics of certain evolution processes. For example, when one considers the hemodynamic equilibrium of a person, the introduction of the drugs in the bloodstream and the consequent absorption for the body are a gradual and continuous process. In the literature many types of initial value problems and boundary value problems for different fractional differential equations with instantaneous and non-instantaneous impulses are studied (see, for example, [1, 3–8, 15, 24, 34, 45, 63, 71, 102, 107, 109]).

The measure of noncompactness which is one of the fundamental tools in the theory of nonlinear analysis, it was initiated by the pioneering articles of Alvàrez [32], Mönch

[76] and was developed by Bana's and Goebel [39] and many researchers in the literature. The applications of the measure of noncompactness can be seen in the wide range of applied mathematics: theory of differential equations (see [23, 81] and references therein). Recently, in [13, 32, 33, 39] the authors applied the measure of noncompactness to some classes of differential equations in Banach spaces.

Nonlocal conditions were initiated by Byszewski [52] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. The nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena. Fractional differential equations with nonlocal conditions have been discussed in [19, 26, 77] and references therein.

Hybrid fractional differential equations were also studied by several researchers. By hybrid differential equation, we mean that the terms in the equation are perturbed either linearly or quadratically or through the combination of first and second types. Perturbation taking place in form of the sum or difference of terms in an equation is called linear. On the other hand, if the equation is perturbed through the product or quotient of the terms in it, then it is called quadratic perturbation. So the study of hybrid differential equation is more general and covers several dynamic systems as particular cases. This class of equations involves the fractional derivative of an unknown function hybrid with the nonlinearity depending on it. Some recent results on hybrid differential equations can be found in a series of papers (see [28, 36, 55, 64, 112]).

In the following we give an outline of our thesis organization, which consists of seven chapters defining the contributed work.

Chapter 1: This chapter provides the notation and preliminary results, descriptions, theorems and other auxiliary results that will be needed for this study. In the first section we give some notations and definitions of the functional spaces used in this thesis. In the second section, we give the definitions of the elements from fractional calculus theory, then we present some necessary lemmata, theorems and properties. In the third section, we give some properties to the Measure of noncompactness. We finish the chapter in the last section by giving all the fixed point theorems that are used throughout the thesis.

Chapter 2: This chapter deals with some existence results for the boundary value problem of the following generalized Hilfer-type fractional differential equation:

$$\begin{cases} \left({}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}u\right)(t) = f\left(t, u(t), \left({}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}u\right)(t)\right), \quad t \in (a,b], \\ l\left({}^\rho\mathcal{J}^{1-\gamma}_{a^+}u\right)(a^+) + m\left({}^\rho\mathcal{J}^{1-\gamma}_{a^+}u\right)(b) = \phi, \end{cases}$$

where ${}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}, {}^\rho\mathcal{J}^{1-\gamma}_{a^+}$ are the generalized Hilfer-type fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$ and generalized fractional integral of order $1-\gamma, (\gamma = \alpha+\beta-\alpha\beta)$, respectively, $\phi \in E$, $0 < a < b < +\infty$, $f : (a,b] \times E \times E \to E$ is a given function where $(E, \|\cdot\|)$ is a Banach space and l, m are reals with $l + m \neq 0$. The results are based on the fixed point theorems of Darbo and Mönch associated with the technique of measure of noncompactness. Next, we prove that our problem is generalized Ulam-Hyers-Rassias stable. An example is included to show the applicability of our results.

Chapter 3: In this chapter, we give two main results. In the Section 3.2, we establish existence, uniqueness and Ulam-Hyers-Rassias results to the boundary value problem with nonlinear implicit generalized Hilfer-type fractional differential equation with impulses:

$$\begin{cases} \left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{t_k^+}u\right)(t) = f\left(t, u(t), \left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{t_k^+}u\right)(t)\right); \ t \in J_k, \ k = 0,\ldots,m, \\ \left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_k^+}u\right)(t_k^+) = \left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_{k-1}^+}u\right)(t_k^-) + L_k(u(t_k^-)); k = 1,\ldots,m, \\ c_1\left({}^{\rho}\mathcal{J}^{1-\gamma}_{a^+}u\right)(a^+) + c_2\left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_m^+}u\right)(b) = c_3, \end{cases}$$

where ${}^{\rho}\mathcal{D}^{\alpha,\beta}_{t_k^+}, {}^{\rho}\mathcal{J}^{1-\gamma}_{t_k^+}$ are the generalized Hilfer fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$ and generalized fractional integral of order $1-\gamma, (\gamma = \alpha+\beta-\alpha\beta)$ respectively, c_1, c_2, c_3 are reals with $c_1 + c_2 \neq 0$, $J_k := (t_k, t_{k+1}]; k = 0,\ldots,m$, $a = t_0 < t_1 < \ldots < t_m < t_{m+1} = b < \infty$, $u(t_k^+) = \lim\limits_{\epsilon\to 0^+} u(t_k+\epsilon)$ and $u(t_k^-) = \lim\limits_{\epsilon\to 0^-} u(t_k+\epsilon)$ represent the right and left hand limits of $u(t)$ at $t = t_k$, $f : (a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function and $L_k : \mathbb{R} \to \mathbb{R}$; $k = 1,\ldots,m$ are given continuous functions. The results are based on Banach contraction principle, Krasnoselskii's and Schaefer's fixed point theorems.

In Section 3.3, we examine the existence and the Ulam stability of the solutions to the boundary value problem with nonlinear implicit Generalized Hilfer-type fractional differential equation with instantaneous impulses:

$$\begin{cases} \left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{t_k^+}u\right)(t) = f\left(t, u(t), \left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{t_k^+}u\right)(t)\right); \ t \in J_k, \ k = 0,\cdots,m, \\ \left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_k^+}u\right)(t_k^+) = \left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_{k-1}^+}u\right)(t_k^-) + \varpi_k(u(t_k^-)); k = 1,\cdots,m, \\ c_1\left({}^{\rho}\mathcal{J}^{1-\gamma}_{a^+}u\right)(a^+) + c_2\left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_m^+}u\right)(b) = c_3, \end{cases}$$

where ${}^{\rho}\mathcal{D}^{\alpha,\beta}_{t_k^+}$, ${}^{\rho}\mathcal{J}^{1-\gamma}_{t_k^+}$ are the generalized Hilfer fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$ and generalized Hilfer fractional integral of order $1-\gamma, (\gamma = \alpha+\beta-\alpha\beta)$ respectively, c_1, c_2 are reals with $c_1 + c_2 \neq 0$, $J_k := (t_k, t_{k+1}]; k = 0,\cdots,m$, $a = t_0 < t_1 < \cdots < t_m < t_{m+1} = b < \infty$, $u(t_k^+) = \lim\limits_{\epsilon\to 0^+} u(t_k+\epsilon)$ and $u(t_k^-) = \lim\limits_{\epsilon\to 0^-} u(t_k+\epsilon)$ represent the right and left hand limits of $u(t)$ at $t = t_k$, $c_3 \in E$, $f : (a,b] \times E \times E \to E$ is a given function and $\varpi_k : E \to E$; $k = 1,\cdots,m$ are given continuous functions, where $(E, \|\cdot\|)$ is a Banach space. The results are based on fixed point theorems of Darbo and Mönch associated with the technique of measure of noncompactness. Examples are included to show the applicability of our results for each case.

Chapter 4: This chapter contain three sections. After the introduction section, in Section 4.2, we present some existence results to the initial value problem with nonlinear implicit generalized Hilfer-type fractional differential equation with non-instantaneous impulses:

$$\begin{cases} \left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}u\right)(t) = f\left(t, u(t), \left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}u\right)(t)\right); \ t \in I_k, \ k = 0,\ldots,m, \\ u(t) = g_k(t, u(t)); \ t \in \tilde{I}_k, \ k = 1,\ldots,m, \\ \left({}^{\rho}\mathcal{J}^{1-\gamma}_{a^+}u\right)(a^+) = \phi_0, \end{cases}$$

where ${}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}$, ${}^{\rho}\mathcal{J}^{1-\gamma}_{a^+}$ are the generalized Hilfer fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$ and generalized fractional integral of order $1-\gamma, (\gamma = \alpha + \beta - \alpha\beta)$ respectively, $\phi_0 \in \mathbb{R}$, $I_k := (s_k, t_{k+1}]; k = 0, \ldots, m$, $\tilde{I}_k := (t_k, s_k]; k = 1, \ldots, m$, $a = t_0 = s_0 < t_1 \leq s_1 < t_2 \leq s_2 < \ldots \leq s_{m-1} < t_m \leq s_m < t_{m+1} = b < \infty$, $u(t_k^+) = \lim\limits_{\epsilon \to 0^+} u(t_k + \epsilon)$ and $u(t_k^-) = \lim\limits_{\epsilon \to 0^-} u(t_k + \epsilon)$ represent the right and left hand limits of $u(t)$ at $t = t_k$, $f : (a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function and $g_k : \tilde{I}_k \times \mathbb{R} \to \mathbb{R}$; $k = 1, \ldots, m$, are given continuous functions such that $\left({}^{\rho}\mathcal{J}^{1-\gamma}_{s_k^+} g_k\right)(t, u(t))\,|_{t=s_k} = \phi_k \in \mathbb{R}$. The results are based on Banach contraction principle and Schaefer's fixed point theorem.

In Subsection 4.2.2, we give a generalization of the previous result to nonlocal impulsive fractional differential equations. More precisely we present some existence results for the following nonlocal problem

$$\begin{cases} \left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+} u\right)(t) = f\left(t, u(t), \left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+} u\right)(t)\right); \ t \in I_k, \ k = 0, \ldots, m, \\ u(t) = g_k(t, u(t)); \ t \in \tilde{I}_k, \ k = 1, \ldots, m, \\ \left({}^{\rho}\mathcal{J}^{1-\gamma}_{a^+} u\right)(a^+) + \xi(u) = \phi_0, \end{cases}$$

where ξ is a continuous function.

In the Section 4.3, we establish some existence results to the initial value problem of nonlinear implicit generalized Hilfer-type fractional differential equation with non-instantaneous impulses:

$$\begin{cases} \left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+} u\right)(t) = f\left(t, u(t), \left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+} u\right)(t)\right); \ t \in I_k, \ k = 0, \ldots, m, \\ u(t) = g_k(t, u(t)); \ t \in \tilde{I}_k, \ k = 1, \ldots, m, \\ \left({}^{\rho}\mathcal{J}^{1-\gamma}_{a^+} u\right)(a^+) = \phi_0, \end{cases}$$

where ${}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}$, ${}^{\rho}\mathcal{J}^{1-\gamma}_{a^+}$ are the generalized Hilfer-type fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$ and generalized fractional integral of order $1-\gamma, (\gamma = \alpha + \beta - \alpha\beta)$, respectively, $\rho > 0$, $\phi_0 \in E$, $I_k := (s_k, t_{k+1}]; k = 0, \ldots, m$, $\tilde{I}_k := (t_k, s_k]; k = 1, \ldots, m$, $a = s_0 < t_1 \leq s_1 < t_2 \leq s_2 < \ldots \leq s_{m-1} < t_m \leq s_m < t_{m+1} = b < \infty$, $u(t_k^+) = \lim\limits_{\epsilon \to 0^+} u(t_k + \epsilon)$ and $u(t_k^-) = \lim\limits_{\epsilon \to 0^-} u(t_k + \epsilon)$ represent the right and left hand limits of $u(t)$ at $t = t_k$, $f : I_k \times E \times E \to E$ is a given function and $g_k : \tilde{I}_k \times E \to E$; $k = 1, \ldots, m$ are given continuous functions such that $\left({}^{\rho}\mathcal{J}^{1-\gamma}_{s_k^+} g_k\right)(t, u(t))\,|_{t=s_k} = \phi_k \in E$, where $(E, \|\cdot\|)$ is a real Banach space. The results are based on fixed point theorems of Darbo and Mönch associated with the technique of measure of noncompactness. Examples are included to show the applicability of our results.

Chapter 5: This chapter is devoted to proving some results concerning the existence of solutions to the boundary value problem with nonlinear implicit Generalized Hilfer-type fractional differential equation with non-instantaneous impulses:

$$\begin{cases} \left({}^{\alpha}\mathcal{D}^{\vartheta,r}_{\tau_i^+}x\right)(t) = f\left(t, x(t), \left({}^{\alpha}\mathcal{D}^{\vartheta,r}_{\tau_i^+}x\right)(t)\right); \ t \in J_i, \ i = 0, \dots, m, \\ x(t) = \psi_i(t, x(t)); \ t \in \tilde{J}_i, \ i = 1, \dots, m, \\ \phi_1 \left({}^{\alpha}\mathcal{J}^{1-\xi}_{a^+}x\right)(a^+) + \phi_2\left({}^{\alpha}\mathcal{J}^{1-\xi}_{m^+}x\right)(b) = \phi_3, \end{cases}$$

where ${}^{\alpha}\mathcal{D}^{\vartheta,r}_{\tau_i^+}, {}^{\alpha}\mathcal{J}^{1-\xi}_{a^+}$ are the generalized Hilfer-type fractional derivative of order $\vartheta \in (0,1)$ and type $r \in [0,1]$ and generalized fractional integral of order $1-\xi, (\xi = \vartheta + r - \vartheta r)$ respectively, $\phi_1, \phi_2, \phi_3 \in \mathbb{R}$, $\phi_1 \neq 0$, $J_i := (\tau_i, t_{i+1}]; i = 0, \dots, m$, $\tilde{J}_i := (t_i, s_i]; i = 1, \dots, m$, $a = t_0 = \tau_0 < t_1 \leq \tau_1 < t_2 \leq \tau_2 < \dots \leq \tau_{m-1} < t_m \leq \tau_m < t_{m+1} = b < \infty$, $x(t_i^+) = \lim\limits_{\epsilon \to 0^+} x(t_i + \epsilon)$ and $x(t_i^-) = \lim\limits_{\epsilon \to 0^-} x(t_i + \epsilon)$ represent the right and left hand limits of $x(t)$ at $t = t_i$, $f : (a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function and $\psi_i : \tilde{J}_i \times \mathbb{R} \to \mathbb{R}$; $i = 1, \dots, m$ are given continuous functions such that $\left({}^{\alpha}\mathcal{J}^{1-\xi}_{\tau_i^+}\psi_i\right)(t, x(t))\,|_{t=\tau_i} = c_i \in \mathbb{R}$. The results are based on Banach contraction principle and Krasnoselskii's fixed point theorem. Further, for the justification of our results we provide two examples.

Chapter 6: In this chapter, we prove some existence results of the solutions for a class of initial value problem for nonlinear fractional hybrid implicit differential equations with Generalized Hilfer and ψ-Hilfer fractional derivatives. The results are based on fixed point theorems due to Dhage. Further, examples are provided for each section to illustrate our results. In Section 6.2, we establish existence results to the nonlocal initial value problem with nonlinear implicit hybrid Generalized Hilfer-type fractional differential equation :

$$\begin{cases} {}^{\alpha}\mathcal{D}^{\vartheta,r}_{a^+}\left(\frac{x(t)-\chi(t,x(t))}{f(t,x(t))}\right) = \varphi\left(t, x(t), {}^{\alpha}\mathcal{D}^{\vartheta,r}_{a^+}\left(\frac{x(t)-\chi(t,x(t))}{f(t,x(t))}\right)\right), \quad t \in (a,b], \\ \left({}^{\alpha}\mathcal{J}^{1-\xi}_{a^+}\left(\dfrac{x(\tau)-\chi(t,x(t))}{f(\tau,x(\tau))}\right)\right)(a^+) = \displaystyle\sum_{i=1}^{m} c_i\left(\dfrac{x(\epsilon_i)-\chi(\epsilon_i,x(\epsilon_i))}{f(\epsilon_i,x(\epsilon_i))}\right), \end{cases}$$

where ${}^{\alpha}\mathcal{D}^{\vartheta,r}_{a^+}, {}^{\alpha}\mathcal{J}^{1-\xi}_{a^+}$ are the generalized Hilfer fractional derivative of order $\vartheta \in (0,1)$ and type $r \in [0,1]$ and generalized fractional integral of order $1-\xi, (\xi = \vartheta+r-\vartheta r)$ respectively, $c_i, i = 1, \dots, m$, are real numbers, $\epsilon_i, i = 1, \dots, m$, are pre-fixed points satisfying $a < \epsilon_1 \leq \dots \leq \epsilon_m < b$, $f \in C([a,b] \times \mathbb{R}, \mathbb{R}\backslash\{0\})$, $\chi \in C([a,b] \times \mathbb{R}, \mathbb{R})$, $\varphi \in C([a,b] \times \mathbb{R}^2, \mathbb{R})$ and $\sum_{i=1}^{m} c_i \bar{\Psi}_{\xi}(\epsilon_i, a) \neq 1$, for further details see the definitions in the chapter. Then, in the Section 6.3, we consider the initial value problem with nonlinear implicit hybrid ψ-Hilfer type fractional differential equation :

$$\begin{cases} {}^{H}\mathbb{D}^{\vartheta,r;\psi}_{a^+}\left(\dfrac{x(t)}{g(t,x(t))}\right) = f\left(t, x(t), {}^{H}\mathbb{D}^{\vartheta,r;\psi}_{a^+}\left(\dfrac{x(t)}{g(t,x(t))}\right)\right), \quad t \in (a,b], \\ \left(\mathbb{J}^{1-\xi;\psi}_{a^+}\left(\dfrac{x(\tau)}{g(\tau,x(\tau))}\right)\right)(a^+) = x_0, \end{cases}$$

where ${}^{H}\mathbb{D}^{\vartheta,r;\psi}_{a^+}, \mathbb{J}^{1-\xi;\psi}_{a^+}$ are the ψ-Hilfer fractional derivative of order $\vartheta \in (0,1)$ and type $r \in [0,1]$ and ψ-Riemann-Liouville fractional integral of order $1-\xi, (\xi = \vartheta + r - \vartheta r)$ respectively, $x_0 \in \mathbb{R}$, $g \in C([a,b] \times \mathbb{R}, \mathbb{R}\backslash\{0\})$ and $f \in C([a,b] \times \mathbb{R}^2, \mathbb{R})$.

Finally we close our thesis with a conclusion and some perspectives.

Chapter 1

Preliminaries

In this chapter, we discuss the necessary mathematical tools, notations and concepts we need in the succeeding chapters. We look at some essential properties of fractional differential operators. We also review some of the basic properties of measures of non-compactness and fixed point theorems which are crucial in our results regarding fractional differential equations.

1.1 Notations and Definitions

Let $0 < a < b$ and $J = (a, b]$. Consider the following parameters α, β, γ satisfying $\gamma = \alpha + \beta - \alpha\beta, \quad 0 < \alpha, \beta, \gamma < 1$. By $C([a, b])$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|u\|_\infty = \sup\{|u(t)| : t \in [a, b]\}.$$

Consider the space $X_c^p(a, b)$, $(c \in \mathbb{R},\ 1 \le p \le \infty)$ of those complex-valued Lebesgue measurable functions f on $[a, b]$ for which $\|f\|_{X_c^p} < \infty$, where the norm is defined by

$$\|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t} \right)^{\frac{1}{p}}, \quad (1 \le p < \infty, c \in \mathbb{R}).$$

In particular, when $c = \frac{1}{p}$, the space $X_c^p(a, b)$ coincides with the $L^p(a, b)$ space: $X_{\frac{1}{p}}^p(a, b) = L^p(a, b)$.

Let $(E, \|\cdot\|)$ be a Banach space. By $C_E([a, b])$ we denote the Banach space of all continuous functions from $[a, b]$ into E with the norm

$$\|u\|_E = \sup\{\|u(t)\| : t \in [a, b]\}.$$

By $L^1([a, b])$, we denote the space of Bochner–integrable functions $f : J \longrightarrow E$ with the norm

$$\|f\|_1 = \int_a^b \|f(t)\| dt.$$

We consider the weighted spaces of continuous functions

$$C_{\gamma,\rho}(J) = \left\{ u : J \to E : \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma} u(t) \in C_E([a, b]) \right\}, 0 \le \gamma < 1,$$

and

$$C^n_{\gamma,\rho}(J) = \left\{u \in C^{n-1} : u^{(n)} \in C_{\gamma,\rho}(J)\right\}, n \in \mathbb{N},$$
$$C^0_{\gamma,\rho}(J) = C_{\gamma,\rho}(J),$$

with the norms

$$\|u\|_{C_{\gamma,\rho}} = \sup_{t\in[a,b]} \left\| \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} u(t) \right\|,$$

and

$$\|u\|_{C^n_{\gamma,\rho}} = \sum_{k=0}^{n-1} \|u^{(k)}\|_\infty + \|u^{(n)}\|_{C_{\gamma,\rho}}.$$

We define the spaces

$$C^{\alpha,\beta}_{\gamma,\rho}(J) = \left\{u \in C_{\gamma,\rho}(J),\ {}^\rho\mathcal{D}^{\alpha,\beta}_{a^+} u \in C_{\gamma,\rho}(J)\right\},$$

and

$$C^{\gamma}_{\gamma,\rho}(J) = \left\{u \in C_{\gamma,\rho}(J),\ {}^\rho\mathcal{D}^{\gamma}_{a^+} u \in C_{\gamma,\rho}(J)\right\},$$

where $\mathcal{D}^{\alpha,\beta}_{a^+}$ and $\mathcal{D}^{\gamma}_{a^+}$ are factional derivatives defined in the following sections.

1.2 Special Functions of the Fractional Calculus

1.2.1 Gamma Function

Undoubtedly, one of the basic functions of the fractional calculus is Euler's gamma function $\Gamma(z)$, which generalizes the factorial $n!$ and allows n to take also non-integer and even complex values.

Definition 1.1. *([82]) The gamma function $\Gamma(z)$ is defined by the integral :*

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} \mathrm{d}t,$$

which converges in the right half of the complex plane $\mathrm{Re}(z) > 0$.
One of the basic properties of the gamma function is that it satisfies the following functional equation:

$$\Gamma(z+1) = z\Gamma(z),$$

so, for positive integer values n, the Gamma function becomes $\Gamma(n) = (n-1)!$ and thus can be seen as an extension of the factorial function to real values.
A useful particular value of the function: $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, is used throughout many examples in this thesis.

1.2.2 Mittag-Leffler Function

The exponential function e^z plays a very important role in the theory of integer-order differential equations. Its one-parameter generalization. The function which is now denoted by :

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$$

Definition 1.2. *([82]) A two-parameter function of the Mittag-Leffler type is defined by the series expansion :*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \quad \beta > 0.$$

It follows from the definition that

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z.$$

1.3 Elements From Fractional Calculus Theory

In this section, we recall some definitions of fractional integral and fractional differential operators that include all we use throughout this thesis. We conclude it by some necessary lemmata, theorems and properties.

1.3.1 Fractional Integrals

Definition 1.3. *(**Generalized Fractional Integral** [70]) Let $\alpha \in \mathbb{R}_+$ and $g \in L^1([a,b])$. The generalized fractional integral of order α is defined by*

$$\left({}^\rho \mathcal{J}_{a^+}^{\alpha} g\right)(t) = \int_a^t s^{\rho-1} \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} \frac{g(s)}{\Gamma(\alpha)} ds, \ t > a, \rho > 0.$$

Definition 1.4. *(ψ**-Riemann-Liouville Fractional Integral** [70]).*
Let (a,b) $(-\infty \leq a < b \leq \infty)$ be a finite or infinite interval of the real line $\mathbb{R}$, $\vartheta > 0$, $c \in \mathbb{R}$ and $h \in X_c^p(a,b)$. Also let $\psi(t)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\psi'(t)$ on (a,b). The left and right-sided fractional integrals of a function h of order ϑ with respect to another function ψ on J are defined by

$$\left(\mathbb{J}_{a^+}^{\vartheta;\psi} h\right)(t) = \int_a^t \psi'(\tau) \left(\psi(t) - \psi(\tau)\right)^{\vartheta-1} \frac{h(\tau)}{\Gamma(\vartheta)} d\tau,$$

and

$$\left(\mathbb{J}_{b^-}^{\vartheta;\psi} h\right)(t) = \int_t^b \psi'(\tau) \left(\psi(\tau) - \psi(t)\right)^{\vartheta-1} \frac{h(\tau)}{\Gamma(\vartheta)} d\tau,$$

1.3.2 Fractional Derivatives

Definition 1.5. *(**Generalized Fractional Derivative** [70]) Let $\alpha \in \mathbb{R}_+ \backslash \mathbb{N}$ and $\rho > 0$. The generalized fractional derivative ${}^{\rho}\mathcal{D}^{\alpha}_{a+}$ of order α is defined by*

$$\begin{aligned} ({}^{\rho}\mathcal{D}^{\alpha}_{a+}g)(t) &= \delta^n_{\rho}({}^{\rho}\mathcal{J}^{n-\alpha}_{a+}g)(t) \\ &= \left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t s^{\rho-1}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{n-\alpha-1}\frac{g(s)}{\Gamma(n-\alpha)}ds, \ t > a, \rho > 0, \end{aligned}$$

where $n = [\alpha] + 1$ and $\delta^n_{\rho} = \left(t^{1-\rho}\dfrac{d}{dt}\right)^n$.

Definition 1.6. *(**Generalized Hilfer type Fractional Derivative** [78]) Let order α and type β satisfy $n-1 < \alpha < n$ and $0 \leq \beta \leq 1$, with $n \in \mathbb{N}$. The generalized Hilfer type fractional derivative to t, with $\rho > 0$ of a function g, is defined by*

$$\begin{aligned} \left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{a+}g\right)(t) &= \left({}^{\rho}\mathcal{J}^{\beta(n-\alpha)}_{a+}\left(t^{\rho-1}\frac{d}{dt}\right)^n {}^{\rho}\mathcal{J}^{(1-\beta)(n-\alpha)}_{a+}g\right)(t) \\ &= \left({}^{\rho}\mathcal{J}^{\beta(n-\alpha)}_{a+}\delta^n_{\rho}\,{}^{\rho}\mathcal{J}^{(1-\beta)(n-\alpha)}_{a+}g\right)(t). \end{aligned}$$

In this manuscript we consider the case $n = 1$ only, because $0 < \alpha < 1$.

Definition 1.7. *(ψ-**Riemann-Liouville fractional derivative** [70]).
Let $\psi'(t) \neq 0$ $(-\infty \leq a < t < b \leq \infty)$, $\vartheta > 0$ and $n \in \mathbb{N}$. The Riemann-Liouville derivatives of a function h of order ϑ with respect to another function ψ on $[a,b]$ are defined by*

$$\begin{aligned} \left(\mathbb{D}^{\vartheta;\psi}_{a+}h\right)(t) &= \delta^n(\mathbb{J}^{n-\vartheta;\psi}_{a+}h)(t) \\ &= \delta^n\int_a^t \psi'(\tau)\left(\psi(t)-\psi(\tau)\right)^{n-\vartheta-1}\frac{h(\tau)}{\Gamma(n-\vartheta)}d\tau, \end{aligned}$$

and

$$\begin{aligned} \left(\mathbb{D}^{\vartheta;\psi}_{b-}h\right)(t) &= (-1)^n\delta^n(\mathbb{J}^{n-\vartheta;\psi}_{a+}h)(t) \\ &= (-1)^n\delta^n\int_t^b \psi'(\tau)\left(\psi(\tau)-\psi(t)\right)^{n-\vartheta-1}\frac{h(\tau)}{\Gamma(n-\vartheta)}d\tau, \end{aligned}$$

where $n = [\vartheta] + 1$ and $\delta^n = \left(\dfrac{1}{\psi'(t)}\dfrac{d}{dt}\right)^n$.

Definition 1.8. *(ψ-**Hilfer Fractional Derivative**[100]) Let order ϑ and type r satisfy $n-1 < \vartheta < n$ and $0 \leq r \leq 1$, with $n \in \mathbb{N}$, let $h, \psi \in C^n([a,b], \mathbb{R})$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$. The ψ-Hilfer fractional derivatives to t of a function h, are defined by*

$$\left({}^{H}\mathbb{D}^{\vartheta,r;\psi}_{a+}h\right)(t) = \left(\mathbb{J}^{r(n-\vartheta);\psi}_{a+}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \mathbb{J}^{(1-r)(n-\vartheta);\psi}_{a+}h\right)(t)$$

and

$$\left({}^{H}\mathbb{D}^{\vartheta,r;\psi}_{b-}h\right)(t) = \left(\mathbb{J}^{r(n-\vartheta);\psi}_{b-}\left(-\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \mathbb{J}^{(1-r)(n-\vartheta);\psi}_{b-}h\right)(t).$$

In this thesis we consider the case $n = 1$ only, because $0 < \vartheta < 1$.

1.3.3 Necessary Lemma, Theorems and Properties

Theorem 1.9. *([70]) Let $\alpha > 0, \beta > 0, 1 \leq p \leq \infty, 0 < a < b < \infty$. Then, for $g \in L^1([a,b])$ we have*

$$\left({}^{\rho}\mathcal{J}^{\alpha}_{a^+}\ {}^{\rho}\mathcal{J}^{\beta}_{a^+} g\right)(t) = \left({}^{\rho}\mathcal{J}^{\alpha+\beta}_{a^+} g\right)(t).$$

Lemma 1.10. *([30]) Let $t > a$. Then, for $\alpha \geq 0$ and $\beta > 0$, we have*

$$\begin{aligned} \left[{}^{\rho}\mathcal{J}^{\alpha}_{a^+}\left(\frac{s^{\rho}-a^{\rho}}{\rho}\right)^{\beta-1}\right](t) &= \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha+\beta-1}, \\ \left[{}^{\rho}\mathcal{D}^{\alpha}_{a^+}\left(\frac{s^{\rho}-a^{\rho}}{\rho}\right)^{\alpha-1}\right](t) &= 0, \quad 0 < \alpha < 1. \end{aligned}$$

Property 1.11. *([78]) The operator ${}^{\rho}\mathcal{D}^{\alpha,\beta}_{a^+}$ can be written as*

$${}^{\rho}\mathcal{D}^{\alpha,\beta}_{a^+} = {}^{\rho}\mathcal{J}^{\beta(1-\alpha)}_{a^+}\delta_{\rho}\ {}^{\rho}\mathcal{J}^{1-\gamma}_{a^+} = {}^{\rho}\mathcal{J}^{\beta(1-\alpha)}_{a^+}\ {}^{\rho}\mathcal{D}^{\gamma}_{a^+}, \quad \gamma = \alpha+\beta-\alpha\beta.$$

Lemma 1.12. *([70, 78]) Let $\alpha > 0$, and $0 \leq \gamma < 1$. Then, ${}^{\rho}\mathcal{J}^{\alpha}_{a^+}$ is bounded from $C_{\gamma,\rho}(J)$ into $C_{\gamma,\rho}(J)$. Since ${}^{\rho}\mathcal{D}^{\alpha,\beta}_{a^+}u = {}^{\rho}\mathcal{J}^{\beta(1-\alpha)}_{a^+}\ {}^{\rho}\mathcal{D}^{\gamma}_{a^+}u$, it follows that*

$$C^{\gamma}_{1-\gamma,\rho}(J) \subset C^{\alpha,\beta}_{1-\gamma,\rho}(J) \subset C_{1-\gamma,\rho}(J).$$

Lemma 1.13. *([78]) Let $0 < a < b < \infty, \alpha > 0, 0 \leq \gamma < 1$ and $u \in C_{\gamma,\rho}(J)$. If $\alpha > 1-\gamma$, then ${}^{\rho}\mathcal{J}^{\alpha}_{a^+}u$ is continuous on J and*

$$\left({}^{\rho}\mathcal{J}^{\alpha}_{a^+}u\right)(a) = \lim_{t\to a^+}\left({}^{\rho}\mathcal{J}^{\alpha}_{a^+}u\right)(t) = 0.$$

Lemma 1.14. *([78]) Let $\alpha > 0, 0 \leq \gamma < 1$ and $g \in C_{\gamma,\rho}(J)$. Then,*

$$\left({}^{\rho}\mathcal{D}^{\alpha}_{a^+}\ {}^{\rho}\mathcal{J}^{\alpha}_{a^+}g\right)(t) = g(t), \quad \textit{for all} \quad t \in J.$$

Lemma 1.15. *([78]) Let $0 < \alpha < 1, 0 \leq \gamma < 1$. If $g \in C_{\gamma,\rho}(J)$ and ${}^{\rho}\mathcal{J}^{1-\alpha}_{a^+}g \in C^1_{\gamma,\rho}(J)$, then*

$$\left({}^{\rho}\mathcal{J}^{\alpha}_{a^+}\ {}^{\rho}\mathcal{D}^{\alpha}_{a^+}g\right)(t) = g(t) - \frac{\left({}^{\rho}\mathcal{J}^{1-\alpha}_{a^+}g\right)(a)}{\Gamma(\alpha)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha-1}, \quad \textit{for all} \quad t \in J.$$

Lemma 1.16. *([78]) Let $0 < \alpha < 1, 0 \leq \beta \leq 1$ and $\gamma = \alpha+\beta-\alpha\beta$. If $u \in C^{\gamma}_{\gamma,\rho}(J)$, then*

$${}^{\rho}\mathcal{J}^{\gamma}_{a^+}\ {}^{\rho}\mathcal{D}^{\gamma}_{a^+}u = {}^{\rho}\mathcal{J}^{\alpha}_{a^+}\ {}^{\rho}\mathcal{D}^{\alpha,\beta}_{a^+}u,$$

and

$${}^{\rho}\mathcal{D}^{\gamma}_{a^+}\ {}^{\rho}\mathcal{J}^{\alpha}_{a^+}u = {}^{\rho}\mathcal{D}^{\beta(1-\alpha)}_{a^+}u.$$

Property 1.17. *([78]) The fractional derivative ${}^{\rho}\mathcal{D}^{\alpha,\beta}_{a^+}$ interpolate the following fractional derivatives: Hilfer ($\rho \to 1$), generalized Hilfer ($\rho \to 0^+$), generalized ($\beta = 0$), Caputo–type ($\beta = 1$), Riemann–Liouville ($\beta = 0, \rho \to 1$), Hadamard ($\beta = 0, \rho \to 0^+$), Caputo ($\beta = 1, \rho \to 1$), Caputo–Hadamard ($\beta = 1, \rho \to 0^+$), Liouville ($\beta = 0, \rho \to 1, a = 0$) and Weyl ($\beta = 0, \rho \to 1, a = -\infty$).*

Property 1.18. *([100]) The operator* ${}^{H}\mathbb{D}_{a^+}^{\vartheta,r;\psi}$ *can be written as*

$$ {}^{H}\mathbb{D}_{a^+}^{\vartheta,r;\psi} = \mathbb{J}_{a^+}^{r(1-\vartheta);\psi}\ \mathbb{D}_{a^+}^{\xi;\psi}, \quad \xi = \vartheta + r - \vartheta r. $$

Lemma 1.19. *([70, 100]) Let* $\vartheta > 0, r > 0, 0 < a < b < \infty$. *Then, for* $h \in X_c^p(a,b)$ *the semigroup property is valid, i.e.*

$$ \left(\mathbb{J}_{a^+}^{\vartheta;\psi}\ \mathbb{J}_{a^+}^{r;\psi} h\right)(t) = \left(\mathbb{J}_{a^+}^{\vartheta+r;\psi} h\right)(t). $$

Lemma 1.20. *(**Gronwall's** lemma [29]) Let* u *and* w *be two integrable functions and* ζ *a continuous function, with domain* $[a,b]$. *Assume that*

- u *and* w *are nonnegative;*
- ζ *is nonnegative and nondecreasing.*

If

$$ u(t) \le w(t) + \zeta(t)\int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} u(s)ds,\ t \in [a,b], $$

then

$$ u(t) \le w(t) + \int_a^t \sum_{\tau=1}^{\infty} \frac{(\zeta(t)\Gamma(\alpha))^\tau}{\Gamma(\tau\alpha)} s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\tau\alpha-1} w(s)ds,\ t \in [a,b], $$

In addition, if w *is nondecreasing, then*

$$ u(t) \le w(t) E_\alpha\left[\zeta(t)\Gamma(\alpha)\left(\frac{t^\rho - a^\rho}{\rho}\right)^\alpha\right],\ t \in [a,b]. $$

Lemma 1.21. *(Theorem 4.1, ([78])). Let* f *be a function such that* $f \in C_{\gamma,\rho}(J)$. *Then* $u \in C_{\gamma,\rho}^{\gamma}(J)$ *is a solution of the differential equation:*

$$ \left({}^{\rho}\mathcal{D}_{a^+}^{\alpha,\beta} u\right)(t) = f(t),\ \text{for each },\ t \in J,\ 0 < \alpha < 1,\ 0 \le \beta \le 1, $$

if and only if u *satisfies the following Volterra integral equation:*

$$ u(t) = \frac{\left({}^{\rho}\mathcal{J}_{a^+}^{1-\gamma} u\right)(a^+)}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} f(s)ds, $$

where $\gamma = \alpha + \beta - \alpha\beta$.

1.4 Kuratowski Measure of Noncompactness

Now let us recall some fundamental facts of the notion of measure of noncompactness. Let Ω_X be the class of all bounded subsets of a metric space X.

Definition 1.22. *([39]) A function* $\mu : \Omega_X \to [0,\infty)$ *is said to be a measure of noncompactness on* X *if the following conditions are verified for all* $B, B_1, B_2 \in \Omega_X$.

(a) Regularity, i.e., $\mu(B) = 0$ *if and only if* B *is precompact,*

(b) invariance under closure, i.e., $\mu(B) = \mu(\overline{B})$,

(c) semi-additivity, i.e., $\mu(B_1 \cup B_2) = \max\{\mu(B_1), \mu(B_2)\}$.

Definition 1.23. *([39]) let X be a Banach space. The Kuratowski measure of noncompactness is the map $\mu : \Omega_X \longrightarrow [0, \infty)$ defined by*

$$\mu(M) = inf\{\epsilon > 0 : M \subset \bigcup_{j=1}^{m} M_j, diam(M_j) \leq \epsilon\},$$

where $M \in \Omega_X$.
The map μ satisfies the following Properties :

- $\mu(M) = 0 \Leftrightarrow \overline{M}$ *is compact (M is relatively compact).*
- $\mu(M) = \mu(\overline{M})$.
- $M_1 \subset M_2 \Rightarrow \mu(M_1) \leq \mu(M_2)$.
- $\mu(M_1 + M_2) \leq \mu(B_1) + \mu(B_2)$.
- $\mu(cM) = |c|\mu(M)$, $c \in \mathbb{R}$.
- $\mu(convM) = \mu(M)$.

1.5 Some Fixed Point Theorems

Theorem 1.24. *(**Mönch's** fixed point Theorem [76]). Let D be closed, bounded and convex subset of a Banach space X such that $0 \in D$, and let T be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv}T(V), \quad or \quad V = T(V) \cup \{0\} \Rightarrow \mu(V) = 0, \tag{1.1}$$

holds for every subset V of D, then T has a fixed point.

Theorem 1.25. *(**Darbo's** fixed point Theorem [58]). Let D be a non-empty, closed, bounded and convex subset of a Banach space X, and let T be a continuous mapping of D into itself such that for any non-empty subset C of D,*

$$\mu(T(C)) \leq k\mu(C), \tag{1.2}$$

where $0 \leq k < 1$, and μ is the Kuratowski measure of noncompactness. Then T has a fixed point in D.

Theorem 1.26. *(**Banach's** fixed point theorem [59]). Let D be a non-empty closed subset of a Banach space E, then any contraction mapping N of D into itself has a unique fixed point.*

Theorem 1.27. *(**Schaefer's** fixed point theorem [59]). Let E be a Banach space and $N : E \to E$ be a completely continuous operator. If the set*

$$D = \{u \in E : u = \lambda N u, \ for\ some\ \lambda \in (0, 1)\}$$

is bounded, then N has a fixed point.

Theorem 1.28. *(**Krasnoselskii's** fixed point theorem [59]). Let D be a closed, convex, and nonempty subset of a Banach space E, and A, B the operators such that*

1) $Ax + By \in D$ *for all* $x, y \in D$;

2) *A is compact and continuous;*

3) *B is a contraction mapping.*

Then there exists $z \in D$ such that $z = Az + Bz$.

Theorem 1.29. *(**Schauder** fixed point theorem [59]) Let X be a Banach space, D be a bounded closed convex subset of X and $T : D \to D$ be a compact and continuous map. Then T has at least one fixed point in D.*

Theorem 1.30. *(**Dhage** fixed point theorem [53]) Let Ω be a closed, convex, bounded and nonempty subset of a Banach algebra $(X, \|\cdot\|)$, and let $\mathcal{T}_1 : E \to E$ and $\mathcal{T}_2 : \Omega \to E$ be two operators such that*

1) *$\mathcal{T}_1$ is Lipschitzian with Lipschitz constant λ,*

2) *$\mathcal{T}_2$ is completely continuous,*

3) $y = \mathcal{T}_1 y \mathcal{T}_2 z \Rightarrow y \in \Omega$ *for all* $z \in \Omega$,

4) $\lambda\mathcal{M} < 1$, *where* $\mathcal{M} = \|B(\Omega)\| = \sup\{\|B(z)\| : z \in \Omega\}$.

Then the operator equation $\mathcal{T}_1 y \mathcal{T}_2 y = y$ has a solution in Ω.

Theorem 1.31. *(**Dhage** fixed point theorem with three operators [54]) Let B be a closed, convex, bounded and nonempty subset of a Banach algebra $(X, \|\cdot\|)$, and let $\mathcal{P}, \mathcal{R} : X \to X$ and $\mathcal{Q} : B \to X$ be three operators such that*

1) *$\mathcal{P}$ and $\mathcal{R}$ are Lipschitzian with Lipschitz constants η_1 and η_2, respectively,*

2) *$\mathcal{Q}$ is compact and continuous,*

3) $x = \mathcal{P}x\mathcal{Q}y + \mathcal{R}x \Rightarrow x \in B$ *for all* $y \in B$

4) $\eta_1\beta + \eta_2 < 1$, *where* $\beta = \|\mathcal{Q}(B)\| = \sup\{\|\mathcal{Q}(y)\| : y \in B\}$.

Then the operator equation $\mathcal{P}x\mathcal{Q}x + \mathcal{R}x = x$ has a solution in B.

Chapter 2

Boundary Value Problem for Differential Equations with Generalized Hilfer-Type Fractional Derivative [1]

2.1 Introduction and Motivations

This chapter deals with some existence and Ulam-Hyers-Rassias stability results for a class of boundary value problem for differential equations with generalized Hilfer type fractional derivative in Banach spaces. The results are based on the fixed point theorems of Darbo and Mönch associated with the technique of measure of noncompactness. An example is included to show the applicability of our results. The findings obtained in this chapter are studied and presented as a consequence of the following papers [7, 13, 14, 25, 25, 41, 42, 113], which are focused on linear and nonlinear initial and boundary value problems for fractional differential equations involving different kinds of fractional derivatives. As for the stability of Ulam, we are motivated by the monographs of Abbas *et al.* [7, 13], and the papers [10, 46, 47], in it, considerable attention has been given to the study of the Ulam-Hyers and Ulam-Hyers-Rassias stability of various classes of functional equations.

In this chapter we establish existence and Ulam stability results for the boundary value problem of the following generalized Hilfer type fractional differential equation:

$$\left({}^{\rho}\mathcal{D}_{a^+}^{\alpha,\beta}u\right)(t) = f\left(t, u(t), \left({}^{\rho}\mathcal{D}_{a^+}^{\alpha,\beta}u\right)(t)\right), \text{ for each }, \ t \in J, \tag{2.1}$$

$$l\left({}^{\rho}\mathcal{J}_{a^+}^{1-\gamma}u\right)(a^+) + m\left({}^{\rho}\mathcal{J}_{a^+}^{1-\gamma}u\right)(b) = \phi, \tag{2.2}$$

where ${}^{\rho}\mathcal{D}_{a^+}^{\alpha,\beta}, {}^{\rho}\mathcal{J}_{a^+}^{1-\gamma}$ are the generalized Hilfer type fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$ and generalized fractional integral of order $1-\gamma, (\gamma = \alpha+\beta-\alpha\beta)$, respectively, $\phi \in E$, $f : J \times E \times E \to E$ is a given function and l, m are reals with $l+m \neq 0$.

2.2 Existence Results

We consider the following linear fractional differential equation

$$\left({}^{\rho}\mathcal{D}_{a^+}^{\alpha,\beta}u\right)(t) = \psi(t), \quad t \in J, \tag{2.3}$$

1. **A. Salim**, M. Benchohra and J. E. Lazreg, Boundary Value Problem for Differential Equations with Generalized Hilfer Type Fractional Derivative. (**submitted**).

where $0 < \alpha < 1, 0 \leq \beta \leq 1, \rho > 0$, with the boundary condition

$$l\left({}^{\rho}\mathcal{J}_{a^+}^{1-\gamma}u\right)(a^+) + m\left({}^{\rho}\mathcal{J}_{a^+}^{1-\gamma}u\right)(b) = \phi, \tag{2.4}$$

where $\gamma = \alpha + \beta - \alpha\beta, \phi \in E$ and $l, m \in \mathbb{R}$ with $l + m \neq 0$. The following theorem shows that the problem (2.3)–(2.4) has a unique solution given by

$$\begin{aligned} u(t) &= \frac{1}{(l+m)\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1}\left[\phi - \frac{m}{\Gamma(1-\gamma+\alpha)}\int_a^b \left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-\gamma} s^{\rho-1}\psi(s)ds\right] \\ &+ \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\psi(s)ds. \end{aligned} \tag{2.5}$$

Theorem 2.1. *Let $\gamma = \alpha + \beta - \alpha\beta$, where $0 < \alpha < 1$ and $0 \leq \beta \leq 1$. If $\psi : J \to E$ is a function such that $\psi(\cdot) \in C_{\gamma,\rho}(J)$, then $u \in C_{\gamma,\rho}^{\gamma}(J)$ satisfies the problem (2.3)–(2.4) if and only if it satisfies (2.5).*

Proof: By Lemma 1.21, the solution of (2.3) can be written as

$$u(t) = \frac{\left({}^{\rho}\mathcal{J}_{a^+}^{1-\gamma}u\right)(a^+)}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\psi(s)ds. \tag{2.6}$$

Applying ${}^{\rho}\mathcal{J}_{a^+}^{1-\gamma}$ on both sides of (2.6), using Lemma 1.10 and taking $t = b$, we obtain

$$\left({}^{\rho}\mathcal{J}_{a^+}^{1-\gamma}u\right)(b) = \left({}^{\rho}\mathcal{J}_{a^+}^{1-\gamma}u\right)(a^+) + \frac{1}{\Gamma(1-\gamma+\alpha)}\int_a^b \left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-\gamma} s^{\rho-1}\psi(s)ds, \tag{2.7}$$

multiplying both sides of (2.7) by m, we get

$$m\left({}^{\rho}\mathcal{J}_{a^+}^{1-\gamma}u\right)(b) = m\left({}^{\rho}\mathcal{J}_{a^+}^{1-\gamma}u\right)(a^+) + \frac{m}{\Gamma(1-\gamma+\alpha)}\int_a^b \left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-\gamma} s^{\rho-1}\psi(s)ds.$$

Using condition (2.4), we obtain

$$m\left({}^{\rho}\mathcal{J}_{a^+}^{1-\gamma}u\right)(b) = \phi - l\left({}^{\rho}\mathcal{J}_{a^+}^{1-\gamma}u\right)(a^+).$$

Thus

$$\phi - l\left({}^{\rho}\mathcal{J}_{a^+}^{1-\gamma}u\right)(a^+) = m\left({}^{\rho}\mathcal{J}_{a^+}^{1-\gamma}u\right)(a^+) + \frac{m}{\Gamma(1-\gamma+\alpha)}\int_a^b \left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-\gamma} s^{\rho-1}\psi(s)ds,$$

which implies that

$$\left({}^{\rho}\mathcal{J}_{a^+}^{1-\gamma}u\right)(a^+) = \frac{\phi}{l+m} - \frac{m}{(l+m)\Gamma(1-\gamma+\alpha)}\int_a^b \left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-\gamma} s^{\rho-1}\psi(s)ds. \tag{2.8}$$

Substituting (2.8) into (2.6), we obtain (2.5).
Reciprocally, applying ${}^{\rho}\mathcal{J}_{a^+}^{1-\gamma}$ on both sides of (2.5), using Lemma 1.10 and Theorem 1.9, we get

$$\left({}^{\rho}\mathcal{J}_{a^+}^{1-\gamma}u\right)(t) = \frac{\phi}{l+m} - \frac{m}{(l+m)}\left({}^{\rho}\mathcal{J}_{a^+}^{1-\gamma+\alpha}\psi\right)(b) + \left({}^{\rho}\mathcal{J}_{a^+}^{1-\gamma+\alpha}\psi\right)(t). \tag{2.9}$$

Next, taking the limit $t \to a^+$ of (2.9) and using Lemma 1.13, with $1-\gamma < 1-\gamma+\alpha$, we obtain

$$\left({}^\rho\mathcal{J}_{a^+}^{1-\gamma}u\right)(a^+) = \frac{\phi}{l+m} - \frac{m}{(l+m)}\left({}^\rho\mathcal{J}_{a^+}^{1-\gamma+\alpha}\psi\right)(b). \tag{2.10}$$

Now, taking $t=b$ in (2.9), we get

$$\left({}^\rho\mathcal{J}_{a^+}^{1-\gamma}u\right)(b) = \frac{\phi}{l+m} - \frac{m}{(l+m)}\left({}^\rho\mathcal{J}_{a^+}^{1-\gamma+\alpha}\psi\right)(b) + \left({}^\rho\mathcal{J}_{a^+}^{1-\gamma+\alpha}\psi\right)(b). \tag{2.11}$$

From (2.10) and (2.11), we find that

$$\begin{aligned}
& l\left({}^\rho\mathcal{J}_{a^+}^{1-\gamma}u\right)(a^+) + m\left({}^\rho\mathcal{J}_{a^+}^{1-\gamma}u\right)(b) \\
= \; & \frac{l.\phi}{l+m} - \frac{lm}{l+m}\left({}^\rho\mathcal{J}_{a^+}^{1-\gamma+\alpha}\psi\right)(b) + \frac{m.\phi}{l+m} - \frac{m^2}{l+m}\left({}^\rho\mathcal{J}_{a^+}^{1-\gamma+\alpha}\psi\right)(b) + m\left({}^\rho\mathcal{J}_{a^+}^{1-\gamma+\alpha}\psi\right)(b) \\
= \; & \phi + \left(m - \frac{lm}{l+m} - \frac{m^2}{l+m}\right)\left({}^\rho\mathcal{J}_{a^+}^{1-\gamma+\alpha}\psi\right)(b) = \phi,
\end{aligned}$$

which shows that the boundary condition $l\left({}^\rho\mathcal{J}_{a^+}^{1-\gamma}u\right)(a^+) + m\left({}^\rho\mathcal{J}_{a^+}^{1-\gamma}u\right)(b) = \phi$, is satisfied. Next, apply operator ${}^\rho\mathcal{D}_{a^+}^{\gamma}$ on both sides of (2.5). Then, from Lemma 1.10 and Lemma 1.16 we obtain

$$({}^\rho\mathcal{D}_{a^+}^{\gamma}u)(t) = \left({}^\rho\mathcal{D}_{a^+}^{\beta(1-\alpha)}\psi\right)(t). \tag{2.12}$$

Since $u \in C_{\gamma,\rho}^{\gamma}(J)$ and by definition of $C_{\gamma,\rho}^{\gamma}(J)$, we have ${}^\rho\mathcal{D}_{a^+}^{\gamma}u \in C_{\gamma,\rho}(J)$, then (2.12) implies that

$$({}^\rho\mathcal{D}_{a^+}^{\gamma}u)(t) = \left(\delta_\rho\, {}^\rho\mathcal{J}_{a^+}^{1-\beta(1-\alpha)}\psi\right)(t) = \left({}^\rho\mathcal{D}_{a^+}^{\beta(1-\alpha)}\psi\right)(t) \in C_{\gamma,\rho}(J). \tag{2.13}$$

As $\psi(\cdot) \in C_{\gamma,\rho}(J)$ and from Lemma 1.12, it follows that

$$\left({}^\rho\mathcal{J}_{a^+}^{1-\beta(1-\alpha)}\psi\right) \in C_{\gamma,\rho}(J). \tag{2.14}$$

From (2.13), (2.14) and by the Definition of the space $C_{\gamma,\rho}^{n}(J)$, we obtain

$$\left({}^\rho\mathcal{J}_{a^+}^{1-\beta(1-\alpha)}\psi\right) \in C_{\gamma,\rho}^{1}(J).$$

Applying operator ${}^\rho\mathcal{J}_{a^+}^{\beta(1-\alpha)}$ to both sides of (2.12) and using Lemma 1.15, Lemma 1.13 and Property 1.11, we have

$$\begin{aligned}
\left({}^\rho D_{a^+}^{\alpha,\beta}u\right)(t) = {}^\rho\mathcal{J}_{a^+}^{\beta(1-\alpha)}\left({}^\rho D_{a^+}^{\gamma}u\right)(t) &= \psi(t) \\
&+ \frac{\left({}^\rho\mathcal{J}_{a^+}^{1-\beta(1-\alpha)}\psi(t)\right)(a)}{\Gamma(\beta(1-\alpha))}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta(1-\alpha)-1} \\
&= \psi(t),
\end{aligned}$$

that is, (2.3) holds. This completes the proof. ■

As a consequence of Theorem 2.1, we have the following result

Lemma 2.2. *Let $\gamma=\alpha+\beta-\alpha\beta$ where $0<\alpha<1$ and $0\leq\beta\leq 1$, let $f: J\times E\times E\to E$ be a function such that $f(\cdot,u(\cdot),v(\cdot))\in C_{\gamma,\rho}(J)$ for any $u,v\in C_{\gamma,\rho}(J)$. If $u\in C^{\gamma}_{\gamma,\rho}(J)$, then u satisfies the problem (2.1) − (2.2) if and only if u is the fixed point of the operator $\Psi: C_{\gamma,\rho}(J)\to C_{\gamma,\rho}(J)$ defined by*

$$\Psi u(t)=\frac{1}{(l+m)\Gamma(\gamma)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}\left[\phi-\frac{m}{\Gamma(1-\gamma+\alpha)}\int_a^b\left(\frac{b^\rho-s^\rho}{\rho}\right)^{\alpha-\gamma}s^{\rho-1}h(s)ds\right]$$
$$+\frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}h(s)ds, \tag{2.15}$$

where $h: J\to E$ be a function satisfying the functional equation

$$h(t)=f(t,u(t),h(t)).$$

Clearly, $h\in C_{\gamma,\rho}(J)$. Also, by Lemma 1.12, $\Psi u\in C_{\gamma,\rho}(J)$.

Lemma 2.3. *([61]) Let $D\subset C_{\gamma,\rho}(J)$ be a bounded and equicontinuous set, then*

(i) the function $t\to\mu\left(\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}D(t)\right)$ is continuous on $[a,b]$, and

$$\mu_{C_{\gamma,\rho}}(D)=\sup_{t\in[a,b]}\mu\left(\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}D(t)\right).$$

(ii) $\mu\left(\left\{\int_a^b u(s)ds: u\in D\right\}\right)\leq\int_a^b\mu(D(s))ds$, where

$$D(t)=\{u(t): u\in D\}, t\in J.$$

The following hypotheses will be used in the sequel :

(Ax_1) The function $t\mapsto f(t,u,v)$ is measurable and continuous on J for each $u,v\in E$, and the functions $u\mapsto f(t,u,v)$ and $v\mapsto f(t,u,v)$ are continuous on E for a.e. $t\in J$.

(Ax_2) There exists a continuous function $p:[a,b]\longrightarrow[0,\infty)$ such that

$$\|f(t,u,v)\|\leq p(t), \text{ for a.e. } t\in J \text{ and for each } u,v\in E.$$

(Ax_3) For each bounded set $B\subset E$ and for each $t\in J$, we have

$$\mu(f(t,B,({}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}B)))\leq\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}p(t)\mu(B),$$

where ${}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}B=\{{}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}w: w\in B\}$.

Set $p^*=\sup_{t\in[a,b]}p(t)$.

We are now in a position to state and prove our existence result for the problem (2.1)-(2.2) based on Theorem 1.24.

Theorem 2.4. *Assume* ***(Ax₁)*** – ***(Ax₃)*** *hold. If*

$$\ell := \frac{p^*}{\Gamma(\alpha+1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma+\alpha}<1, \tag{2.16}$$

then the problem (2.1)-(2.2) *has at least one solution define on* J.

Proof: Consider the operator $\Psi : C_{\gamma,\rho}(J)\to C_{\gamma,\rho}(J)$ defined in (2.15).
For any $u\in C_{\gamma,\rho}(J)$, and each $t\in J$ we have

$$\begin{aligned}
&\left\|\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}(\Psi u)(t)\right\|\\
\leq\ & \frac{\|\phi\|}{|l+m|\Gamma(\gamma)}+\frac{|m|}{|l+m|\Gamma(\gamma)\Gamma(1-\gamma+\alpha)}\int_a^b\left(\frac{b^\rho-s^\rho}{\rho}\right)^{\alpha-\gamma}s^{\rho-1}\|h(s)\|ds\\
&+\frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\|h(s)\|ds\\
\leq\ & \frac{\|\phi\|}{|l+m|\Gamma(\gamma)}+\frac{|m|p^*}{|l+m|\Gamma(\gamma)}\left({}^\rho\mathcal{J}_{a^+}^{1-\gamma+\alpha}(1)\right)(b)+p^*\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}\left({}^\rho\mathcal{J}_{a^+}^{\alpha}(1)\right)(t).
\end{aligned}$$

By Lemma 1.10, we have

$$\begin{aligned}
&\left\|\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}(\Psi u)(t)\right\|\\
\leq\ & \frac{\|\phi\|}{|l+m|\Gamma(\gamma)}+\frac{|m|p^*}{|l+m|\Gamma(\gamma)\Gamma(\alpha-\gamma)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma+\alpha}\\
+\ & \frac{p^*}{\Gamma(\alpha+1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma+\alpha}.
\end{aligned}$$

Hence, for any $u\in C_{\gamma,\rho}(J)$, and each $t\in J$ we get

$$\begin{aligned}
&\|(\Psi u)\|_{C_{\gamma,\rho}}\\
\leq\ & \frac{\|\phi\|}{|l+m|\Gamma(\gamma)}\\
+\ & \frac{|m|p^*}{|l+m|\Gamma(\gamma)\Gamma(\alpha-\gamma)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma+\alpha}+\frac{p^*}{\Gamma(\alpha+1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma+\alpha}\\
:=\ & R.
\end{aligned}$$

This proves that Ψ transforms the ball $B_R := B(0,R)=\{w\in C_{\gamma,\rho} : \|w\|_{C_{\gamma,\rho}}\leq R\}$ into itself . We shall show that the operator $\Psi : B_R\to B_R$ satisfies all the assumptions of Theorem 1.24. The proof will be given in several steps.

Step 1: $\Psi : B_R\to B_R$ is continuous.
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n\longrightarrow u$ in B_R . Then, for each $t\in J$, we have

$$\begin{aligned}
&\left\|\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}(\Psi u_n)(t)-\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}(\Psi u)(t)\right\|\\
\leq\ & \frac{|m|}{|l+m|\Gamma(\gamma)\Gamma(1-\gamma+\alpha)}\int_a^b\left(\frac{b^\rho-s^\rho}{\rho}\right)^{\alpha-\gamma}s^{\rho-1}\|h_n(s)-h(s)\|ds \qquad (2.17)\\
&+\frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\|h_n(s)-h(s)\|ds,
\end{aligned}$$

where $h_n, h \in C_{\gamma,\rho}(J)$ be such that

$$h_n(t) = f(t, u_n(t), h_n(t)) \ , \ h(t) = f(t, u(t), h(t)).$$

Since $u_n \longrightarrow u$ as $n \longrightarrow \infty$ and f is continuous, then by the Lebesgue dominated convergence theorem, equation (2.17) implies

$$\|\Psi u_n - \Psi u\|_{C_{\gamma,\rho}} \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Step 2: $\Psi(B_R)$ is bounded and equicontinuous.
Since $\Psi(B_R) \subset B_R$ and B_R is bounded, then $\Psi(B_R)$ is bounded.
Next, let $t_1, t_2 \in (a, b]$ such that $a < t_1 < t_2 \leq b$ and let $u \in B_R$. Thus we have

$$\begin{aligned}
&\left\| \left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma} (\Psi u)(t_2) - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{1-\gamma} (\Psi u)(t_1) \right\| \\
&\leq \left\| \frac{1}{\Gamma(\alpha)} \left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma} \int_a^{t_2} \left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} h(s) ds \right. \\
&\qquad \left. - \frac{1}{\Gamma(\alpha)} \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{1-\gamma} \int_a^{t_1} \left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} h(s) ds \right\|,
\end{aligned}$$

then,

$$\begin{aligned}
&\left\| \left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma} (\Psi u)(t_2) - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{1-\gamma} (\Psi u)(t_1) \right\| \\
\leq\ & \frac{1}{\Gamma(\alpha)} \left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma} \int_{t_1}^{t_2} \left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \|h(s)\| ds \\
&+ \int_a^{t_1} \left| \left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma} \left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha-1} - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{1-\gamma} \left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha-1} \right| \frac{s^{\rho-1}\|h(s)\|}{\Gamma(\alpha)} ds \\
\leq\ & p^* \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma} \left({}^\rho\mathcal{J}_{t_1^+}^\alpha(1)\right)(t_2) \\
&+ \frac{p^*}{\Gamma(\alpha)} \int_a^{t_1} \left| \left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma} \left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha-1} - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{1-\gamma} \left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha-1} \right| s^{\rho-1} ds.
\end{aligned}$$

By Lemma 1.10, we have

$$\begin{aligned}
&\left\| \left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma} (\Psi u)(t_2) - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{1-\gamma} (\Psi u)(t_1) \right\| \\
\leq\ & \frac{p^*}{\Gamma(\alpha+1)} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma} \left(\frac{t_2^\rho - t_1^\rho}{\rho}\right)^{\alpha} \\
&+ \frac{p^*}{\Gamma(\alpha)} \int_a^{t_1} \left| \left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma} \left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha-1} - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{1-\gamma} \left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha-1} \right| s^{\rho-1} ds.
\end{aligned}$$

As $t_1 \longrightarrow t_2$, the right side of the above inequality tends to zero. Hence, $\Psi(B_R)$ is bounded and equicontinuous.

Step 3: The implication (1.1) of Theorem 1.24 holds.
Now let D be an equicontinuous subset of B_R such that $D \subset \overline{\Psi(D)} \cup \{0\}$, therefore the function $t \longrightarrow d(t) = \mu(D(t))$ is continuous on J. By ***(Ax₃)*** and the properties of the measure μ, for each $t \in J$, we have

$$\begin{aligned}
\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} d(t) &\leq \mu\left(\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} (\Psi D)(t) \cup \{0\}\right) \\
&\leq \mu\left(\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} (\Psi D)(t)\right) \\
&\leq \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma} \\
&\times \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \frac{p(s)\mu(D(s))}{\Gamma(\alpha)} \left(\frac{s^\rho - a^\rho}{\rho}\right)^{1-\gamma} ds \\
&\leq p^* \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma} \|d\|_{C_{\gamma,\rho}} \left({}^\rho\mathcal{J}^\alpha_{a+}(1)\right)(t) \\
&\leq \frac{p^*}{\Gamma(\alpha+1)} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha} \|d\|_{C_{\gamma,\rho}}.
\end{aligned}$$

Thus

$$\|d\|_{C_{\gamma,\rho}} \leq \ell \|d\|_{C_{\gamma,\rho}}.$$

From (2.16), we get $\|d\|_{C_{\gamma,\rho}} = 0$, that is $d(t) = \mu(D(t)) = 0$, for each $t \in J$, and then $D(t)$ is relatively compact in E. In view of the Ascoli-Arzela Theorem, D is relatively compact in B_R. Applying now Theorem 1.24, we conclude that Ψ has a fixed point, which is solution of the problem (2.1)-(2.2). ■

Our next existence result for the problem (2.1)-(2.2) is based on Darbo fixed point Theorem 1.25.

Theorem 2.5. *Assume that the hypotheses* ***(Ax₁)*** – ***(Ax₃)*** *and the condition (2.16) hold. Then the problem (2.1)-(2.2) has a solution define on J.*

Proof: Consider the operator Ψ defined in (2.15). We know that $\Psi : B_R \longrightarrow B_R$ is bounded and continuous and that $\Psi(B_R)$ is equicontinuous, we need to prove that the operator Ψ is a ℓ-contraction.
Let $D \subset B_R$ and $t \in J$. Then we have

$$\begin{aligned}
&\mu\left(\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} (\Psi D)(t)\right) = \mu\left(\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} (\Psi u)(t) : u \in D\right) \\
&\leq \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma} \left\{\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \frac{p(s)\mu(D(s))}{\Gamma(\alpha)} \left(\frac{s^\rho - a^\rho}{\rho}\right)^{1-\gamma} ds : u \in D\right\} \\
&\leq p^* \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma} \mu_{C_{\gamma,\rho}}(D) \left({}^\rho\mathcal{J}^\alpha_{a+}(1)\right)(t) \\
&\leq \frac{p^*}{\Gamma(\alpha+1)} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha} \mu_{C_{\gamma,\rho}}(D).
\end{aligned}$$

Therefore

$$\mu_{C_{\gamma,\rho}}(\Psi D) \leq \frac{p^*}{\Gamma(\alpha+1)} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha} \mu_{C_{\gamma,\rho}}(D).$$

So, By (2.16), the operator Ψ is a ℓ-contraction, where

$$\ell := \frac{p^*}{\Gamma(\alpha+1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha} < 1.$$

Consequently, from Theorem 1.25 we conclude that Ψ has a fixed point $u \in B_R$, which is a solution to problem (2.1)-(2.2). ■

2.3 Ulam-Hyers-Rassias Stability

Now we are concerned with the generalized Ulam-Hyers-Rassias stability of our equation (2.1). Let $\epsilon > 0$ and $\vartheta : J \longrightarrow [0, \infty)$ be a continuous function. We consider the following inequalities :

$$\left\| \left({}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}u\right)(t) - f\left(t, u(t), \left({}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}u\right)(t)\right)\right\| \leq \epsilon; \qquad t \in J, \tag{2.18}$$

$$\left\| \left({}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}u\right)(t) - f\left(t, u(t), \left({}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}u\right)(t)\right)\right\| \leq \vartheta(t); \qquad t \in J, \tag{2.19}$$

$$\left\| \left({}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}u\right)(t) - f\left(t, u(t), \left({}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}u\right)(t)\right)\right\| \leq \epsilon\vartheta(t); \qquad t \in J. \tag{2.20}$$

Definition 2.6. *([46, 47]) Equation (2.1) is Ulam-Hyers (U-H) stable if there exists a real number $a_f > 0$ such that for each $\epsilon > 0$ and for each solution $u \in C_{\gamma,\rho}(J)$ of inequality (2.18) there exists a solution $v \in C_{\gamma,\rho}(J)$ of (2.1)–(2.2) with*

$$\|u(t) - v(t)\| \leq \epsilon a_f; \qquad t \in J.$$

Definition 2.7. *([46, 47]) Equation (2.1) is generalized Ulam-Hyers (G.U-H) stable if there exists $a_f : C([0,\infty),[0,\infty))$ with $a_f(0) = 0$ such that for each $\epsilon > 0$ and for each solution $u \in C_{\gamma,\rho}(J)$ of inequality (2.18) there exists a solution $v \in C_{\gamma,\rho}(J)$ of (2.1)–(2.2) with*

$$\|u(t) - v(t)\| \leq a_f(\epsilon); \qquad t \in J.$$

Definition 2.8. *([46, 47]) Equation (2.1) is Ulam-Hyers-Rassias (U-H-R) stable with respect to ϑ if there exists a real number $a_{f,\vartheta} > 0$ such that for each $\epsilon > 0$ and for each solution $u \in C_{\gamma,\rho}(J)$ of inequality (2.20) there exists a solution $v \in C_{\gamma,\rho}(J)$ of (2.1)–(2.2) with*

$$\|u(t) - v(t)\| \leq \epsilon a_{f,\vartheta}\vartheta(t); \qquad t \in J.$$

Definition 2.9. *([46, 47]) Equation (2.1) is generalized Ulam-Hyers-Rassias (G.U-H-R) stable with respect to ϑ if there exists a real number $a_{f,\vartheta} > 0$ such that for each solution $u \in C_{\gamma,\rho}(J)$ of inequality (2.19) there exists a solution $v \in C_{\gamma,\rho}(J)$ of (2.1)–(2.2) with*

$$\|u(t) - v(t)\| \leq a_{f,\vartheta}\vartheta(t); \qquad t \in J.$$

Remark 2.10. *It is clear that :*

1. *Definition 2.6 $\Longrightarrow$ Definition 2.7.*
2. *Definition 2.8 $\Longrightarrow$ Definition 2.9.*
3. *Definition 2.8 for $\vartheta(.) = 1 \Longrightarrow$ Definition 2.6.*

Theorem 2.11. *Assume that the hypotheses* ***(Ax₁)***, ***(Ax₂)*** *and the following hypotheses hold.*

(Ax₄) *There exists $\lambda_\vartheta > 0$ such that for each $t \in J$, we have*

$$({}^\rho\mathcal{J}_{a^+}^\alpha \vartheta)(t) \leq \lambda_\vartheta \vartheta(t).$$

(Ax₅) *There exists a continuous function $q : [a, b] \longrightarrow [0, \infty)$ such that for each $t \in J$, we have*

$$p(t) \leq q(t)\vartheta(t).$$

Then equation (2.1) is G.U-H-R stable.

Proof: Consider the operator Ψ defined in (2.15). Let u be a solution if inequality (2.19), and let us assume that v is a solution of the problem (2.1)$-$(2.2). Thus, we have

$$\begin{aligned}
\Psi v(t) &= \frac{1}{(l+m)\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1}\left[\phi - \frac{m}{\Gamma(1-\gamma+\alpha)}\int_a^b \left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-\gamma} s^{\rho-1} g(s)ds\right] \\
&+ \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g(s)ds, \ t \in (a, b],
\end{aligned}$$

where $g : J \to E$ be a function satisfying

$$g(t) = f(t, v(t), g(t)).$$

From inequality (2.19), for each $t \in (a, b]$, we have

$$\begin{aligned}
\Bigg\| u(t) &- \frac{1}{(l+m)\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1}\left[\phi - \frac{m}{\Gamma(1-\gamma+\alpha)}\int_a^b \left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-\gamma} s^{\rho-1} h(s)ds\right] \\
&- \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} h(s)ds\Bigg\| \leq ({}^\rho\mathcal{J}_{a^+}^\alpha \vartheta)(t).
\end{aligned}$$

Set $q^* = \sup\limits_{t\in[a,b]} q(t)$.

From hypotheses ***(Ax₄)*** and ***(Ax₅)***, for each $t \in J$, we get

$$\begin{aligned}
&\|u(t) - v(t)\| \\
\leq\ & \Bigg\| u(t) - \frac{1}{(l+m)\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1}\left[\phi - \frac{m}{\Gamma(1-\gamma+\alpha)}\int_a^b \left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-\gamma} s^{\rho-1} h(s)ds\right] \\
& - \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} h(s)ds\Bigg\| + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\|h(s) - g(s)\|ds \\
\leq\ & ({}^\rho\mathcal{J}_{a^+}^\alpha \vartheta)(t) + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} 2q^*\vartheta(s)ds. \\
\leq\ & \lambda_\vartheta \vartheta(t) + 2q^*({}^\rho\mathcal{J}_{a^+}^\alpha \vartheta)(t) \\
\leq\ & [1 + 2q^*]\lambda_\vartheta \vartheta(t) \\
:=\ & a_{f,\vartheta}\vartheta(t).
\end{aligned}$$

Hence, equation (2.1) is G.U-H-R stable. ■

2.4 An Example

Let

$$E = l^1 = \left\{ u = (u_1, u_2, \dots, u_n, \dots), \sum_{n=1}^{\infty} |u_n| < \infty \right\}$$

be the Banach space with the norm

$$\|u\| = \sum_{n=1}^{\infty} |u_n|.$$

Consider the following Boundary value problem fractional differential equation

$$ {}^1\mathcal{D}_{1^+}^{\frac{1}{2},0} u_n(t) = f_n\left(t, u_n(t), \left({}^1\mathcal{D}_{1^+}^{\frac{1}{2},0} u_n\right)(t)\right), \ t \in (1, e] \tag{2.21}$$

$$\left({}^1\mathcal{J}_{1^+}^{\frac{1}{2}} u_n\right)(1^+) + \left({}^1\mathcal{J}_{1^+}^{\frac{1}{2}} u_n\right)(e) = 0, \tag{2.22}$$

where

$$f_n\left(t, u_n(t), \left({}^1\mathcal{D}_{1^+}^{\frac{1}{2},0} u_n\right)(t)\right) = \frac{ct^2}{e^2}\left(sin(t-1) + u_n(t)\right), \ t \in (1, e].$$

Let

$$f = (f_1, f_2, \dots, f_n, \dots), u = (u_1, u_2, \dots, u_n, \dots) \quad c = \frac{1}{4}\Gamma\left(\frac{1}{2}\right),$$

$\gamma = \alpha = \frac{1}{2}$, $\rho = 1$ and $\beta = 0$. Clearly, the function f is continuous.
The hypothesis $(\boldsymbol{Ax_2})$ is satisfied with

$$p(t) = \frac{ct^2|sin(t-1)|}{e^2}, \ t \in (1, e].$$

A simple computation shows that the conditions of Theorem 2.4 are satisfied. Hence the problem (2.21)–(2.22) has at least one solution defined on $[1, e]$.
Also, hypothesis $(\boldsymbol{Ax_4})$ and $(\boldsymbol{Ax_5})$ are satisfied with $\vartheta(t) = e^2$, $q(t) = \dfrac{p(t)}{e^2}$ and $\lambda_\vartheta = \dfrac{4}{\sqrt{\pi}}$.
Indeed, for each $t \in (1, e]$, we get

$$\begin{aligned} ({}^\rho\mathcal{J}_{a^+}^{\alpha}\vartheta)(t) &\leq \frac{4e^2}{\sqrt{\pi}} \\ &= \lambda_\vartheta \vartheta(t). \end{aligned}$$

Consequently, Theorem 2.11 implies that equation (2.21) is G.U-H-R stable.

Chapter 3

Existence and Ulam Stability for Impulsive Generalized Hilfer-Type Fractional Differential Equations

3.1 Introduction and Motivations

The aim of this chapter is to firstly prove some existence, uniqueness and Ulam-Hyers-Rassias stability results for a class of boundary value problem for nonlinear implicit fractional differential equations with impulses and generalized Hilfer Fractional derivative. The results are based on Banach contraction principle, Krasnoselskii's and Schaefer's fixed point theorems. Secondly, the aim is to study some results concerning the existence of solutions for a class of boundary value problem for nonlinear implicit fractional differential equations with instantaneous impulses and generalized Hilfer fractional derivative in Banach spaces. The results are based on fixed point theorems of Darbo and Mönch associated with the technique of measure of noncompactness. Examples are included to show the applicability of our results for each Section. We have given and proved the results in this chapter taking into account the numerous books and articles focused on linear and nonlinear initial and boundary value problems for fractional differential equations involving different kinds of fractional derivatives [7–9, 13, 14, 25, 27, 38, 41, 42, 113]. For the stability of Ulam, we have taken into consideration the articles [10, 12, 13, 46, 47, 72, 74, 85, 89, 98]. In the literature, it is very common to propose a solution for fractional differential equations by involving different kinds of fractional derivatives, see e.g. [16, 20–22, 31, 68, 69]. The aim of the present chapter is to underline the importance of the theory of impulsive differential equations is quite important. Further, with the help of these observations, we aim to understand several phenomena that are not clarified by the non-impulsive equations. (see [42, 45, 48, 60, 97]).

The outcome of our study in this chapter is the continuation of the problem raised recently in [62], in it, Harikrishnan *et al.* investigated existence theory and different kinds of stability in the sense of Ulam, for the following boundary value problem with nonlinear generalized Hilfer type fractional differential equation with impulses:

$$\begin{cases} \left({}^{\rho}\mathcal{D}^{\alpha,\beta}u\right)(t) = f(t,u(t)); \ t \in \overline{I} := I\backslash\{t_1,\dots,t_m\}, \ I := [0,b], \\ \Delta^{\rho}\mathcal{J}^{1-\gamma}u(t)\,\big|_{t=t_k} = L_k(u(t_k^-)); k = 1,\dots,m, \\ {}^{\rho}\mathcal{J}^{1-\gamma}u(0) = u_0, \end{cases}$$

where ${}^{\rho}\mathcal{D}^{\alpha,\beta}, {}^{\rho}\mathcal{J}^{1-\gamma}$ are the generalized Hilfer fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$ and generalized fractional integral of order $1-\gamma, (\gamma = \alpha+\beta-\alpha\beta)$ respectively, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = b < \infty$, $u(t_k^+) = \lim\limits_{\epsilon\to 0^+} u(t_k+\epsilon)$ and $u(t_k^-) = \lim\limits_{\epsilon\to 0^-} u(t_k+\epsilon)$ represent the right and left hand limits of $u(t)$ at $t = t_k$, $\Delta^\rho \mathcal{J}^{1-\gamma} u(t)\big|_{t=t_k} = {}^\rho\mathcal{J}^{1-\gamma}u(t_k^+) - {}^\rho\mathcal{J}^{1-\gamma}u(t_k^-)$, $f : I \times \mathbb{R} \to \mathbb{R}$ is a given function and $L_k : \mathbb{R} \to \mathbb{R}$; $k = 1,\ldots,m$ are given continuous functions.

3.2 Boundary Value Problem for Nonlinear Generalized Hilfer-Type Fractional Differential Equations with Impulses[1]

In this section we establish the existence and uniqueness results to the boundary value problem with nonlinear implicit generalized Hilfer-type fractional differential equation with impulses:

$$\left({}^\rho\mathcal{D}^{\alpha,\beta}_{t_k^+}u\right)(t) = f\left(t, u(t), \left({}^\rho\mathcal{D}^{\alpha,\beta}_{t_k^+}u\right)(t)\right);\ t \in J_k,\ k = 0,\ldots,m, \tag{3.1}$$

$$\left({}^\rho\mathcal{J}^{1-\gamma}_{t_k^+}u\right)(t_k^+) = \left({}^\rho\mathcal{J}^{1-\gamma}_{t_{k-1}^+}u\right)(t_k^-) + L_k(u(t_k^-)); k = 1,\ldots,m, \tag{3.2}$$

$$c_1\left({}^\rho\mathcal{J}^{1-\gamma}_{a^+}u\right)(a^+) + c_2\left({}^\rho\mathcal{J}^{1-\gamma}_{t_m^+}u\right)(b) = c_3, \tag{3.3}$$

where ${}^\rho\mathcal{D}^{\alpha,\beta}_{t_k^+}, {}^\rho\mathcal{J}^{1-\gamma}_{t_k^+}$ are the generalized Hilfer fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$ and generalized fractional integral of order $1-\gamma, (\gamma = \alpha+\beta-\alpha\beta)$ respectively, c_1, c_2, c_3 are reals with $c_1 + c_2 \neq 0$, $J_k := (t_k, t_{k+1}]; k = 0,\ldots,m$, $a = t_0 < t_1 < \ldots < t_m < t_{m+1} = b < \infty$, $u(t_k^+) = \lim\limits_{\epsilon\to 0^+} u(t_k+\epsilon)$ and $u(t_k^-) = \lim\limits_{\epsilon\to 0^-} u(t_k+\epsilon)$ represent the right and left hand limits of $u(t)$ at $t = t_k$, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function and $L_k : \mathbb{R} \to \mathbb{R}$; $k = 1,\ldots,m$ are given continuous functions.

3.2.1 Existence Results

We consider the weighted spaces of continuous functions

$$C_{\gamma,\rho}(J) = \left\{u : J \to \mathbb{R} : \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} u(t) \in C([a,b])\right\}, 0 \le \gamma < 1,$$

and

$$\begin{aligned} &C^n_{\gamma,\rho}(J) = \left\{u \in C^{n-1}(J) : u^{(n)} \in C_{\gamma,\rho}(J)\right\}, n \in \mathbb{N}, \\ &C^0_{\gamma,\rho}(J) = C_{\gamma,\rho}(J), \end{aligned}$$

with the norms

$$\|u\|_{C_{\gamma,\rho}} = \sup_{t\in[a,b]} \left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} u(t)\right|,$$

1. **A. Salim**, M. Benchohra, J. E. Lazreg and G. N'Guérékata, Boundary Value Problem for Nonlinear Implicit Generalized Hilfer Type Fractional Differential Equations with impulses. *Abstract and Applied Analysis.* **2021** (2021), 17pp.

and

$$\|u\|_{C^n_{\gamma,\rho}} = \sum_{k=0}^{n-1} \|u^{(k)}\|_\infty + \|u^{(n)}\|_{C_{\gamma,\rho}}.$$

Consider the weighted Banach space

$$PC_{\gamma,\rho}(J) = \left\{ u : J \to \mathbb{R} : \left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} u(t) \in C_{\gamma,\rho}(J_k); k = 0, \ldots, m, \text{and there exist} \right.$$
$$\left. u(t_k^-) \text{ and } \left({}^\rho\mathcal{J}^{1-\gamma}_{t_k^+} u\right)(t_k^+); k = 0, \ldots, m, \text{with } u(t_k^-) = u(t_k) \right\}, 0 \le \gamma < 1.$$

and

$$PC^n_{\gamma,\rho}(J) = \left\{ u \in PC^{n-1} : u^{(n)} \in PC_{\gamma,\rho}(J) \right\}, n \in \mathbb{N},$$
$$PC^0_{\gamma,\rho}(J) = PC_{\gamma,\rho}(J),$$

with the norm

$$\|u\|_{PC_{\gamma,\rho}} = \max_{k=0,\ldots,m} \left\{ \sup_{t \in [t_k, t_{k+1}]} \left| \left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} u(t) \right| \right\}.$$

We define the space

$$PC^\gamma_{\gamma,\rho}(J) = \left\{ u \in PC_{\gamma,\rho}(J),\ {}^\rho\mathcal{D}^\gamma_{t_k^+} u \in PC_{\gamma,\rho}(J) \right\},\ k = 0, \ldots, m.$$

We consider the following linear fractional differential equation

$$\left({}^\rho\mathcal{D}^{\alpha,\beta}_{t_k^+} u\right)(t) = \psi(t), \quad t \in J_k, k = 0, \ldots, m, \tag{3.4}$$

where $0 < \alpha < 1, 0 \le \beta \le 1, \rho > 0$, with the conditions

$$\left({}^\rho\mathcal{J}^{1-\gamma}_{t_k^+} u\right)(t_k^+) = \left({}^\rho\mathcal{J}^{1-\gamma}_{t_{k-1}^+} u\right)(t_k^-) + L_k(u(t_k^-)); k = 1, \ldots, m, \tag{3.5}$$

and

$$c_1 \left({}^\rho\mathcal{J}^{1-\gamma}_{a^+} u\right)(a^+) + c_2 \left({}^\rho\mathcal{J}^{1-\gamma}_{t_m^+} u\right)(b) = c_3, \tag{3.6}$$

where $\gamma = \alpha + \beta - \alpha\beta$, $c_1, c_2, c_3 \in \mathbb{R}$ with

$$c_1 + c_2 \neq 0,\ \xi_1 = \frac{c_2}{c_1 + c_2},\ \xi_2 = \frac{c_3}{c_1 + c_2}$$

and

$$p^* = \sup\left\{ \left(\frac{t_k^\rho - t_{k-1}^\rho}{\rho}\right)^{\gamma-1} : k = 1, \ldots, m \right\},$$

such that $\psi : J \to \mathbb{R}$ be a function satisfying the functional equation

$$\psi(t) = f(t, u(t), \psi(t)).$$

The following theorem shows that the problem (3.4)–(3.6) has a unique solution given by

$$u(t)=\begin{cases}\frac{1}{\Gamma(\gamma)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}\Bigg[\xi_2-\xi_1\sum_{i=1}^{m}L_i(u(t_i^-))-\xi_1\sum_{i=1}^{m}\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{(t_{i-1})^+}\psi\right)(t_i)\\ -\xi_1\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{t_m^+}\psi\right)(b)\Bigg]+\frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\psi(s)ds \qquad if\ t\in J_0,\\ \\ \frac{1}{\Gamma(\gamma)}\left(\frac{t^\rho-t_k^\rho}{\rho}\right)^{\gamma-1}\Bigg[\xi_2-\xi_1\sum_{i=1}^{m}L_i(u(t_i^-))-\xi_1\sum_{i=1}^{m}\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{(t_{i-1})^+}\psi\right)(t_i)\\ -\xi_1\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{t_m^+}\psi\right)(b)+\sum_{i=1}^{k}L_i(u(t_i^-))+\sum_{i=1}^{k}\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{(t_{i-1})^+}\psi\right)(t_i)\Bigg]\\ +\left({}^\rho\mathcal{J}^{\alpha}_{t_k^+}\psi\right)(t) \qquad if\ t\in J_k, k=1,\dots,m.\end{cases} \tag{3.7}$$

Theorem 3.1. *Let $\gamma=\alpha+\beta-\alpha\beta$, where $0<\alpha<1$ and $0\le\beta\le1$. If $\psi:J\to\mathbb{R}$ is a function such that $\psi(\cdot)\in PC_{\gamma,\rho}(J)$, then $u\in PC^\gamma_{\gamma,\rho}(J)$ satisfies the problem (3.4)–(3.6) if and only if it satisfies (3.7).*

Proof: Assume u satisfies (3.4)–(3.6). If $t\in J_0$, then

$$\left({}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}u\right)(t)=\psi(t),$$

Lemma 1.21 implies we have the solution can be written as

$$u(t)=\frac{\left({}^\rho\mathcal{J}^{1-\gamma}_{a^+}u\right)(a^+)}{\Gamma(\gamma)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\psi(s)ds. \tag{3.8}$$

If $t\in J_1$, then Lemma 1.21 implies

$$\begin{aligned}u(t)&=\frac{\left({}^\rho\mathcal{J}^{1-\gamma}_{t_1^+}u\right)(t_1^+)}{\Gamma(\gamma)}\left(\frac{t^\rho-t_1^\rho}{\rho}\right)^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\psi(s)ds\\ &=\frac{\left({}^\rho\mathcal{J}^{1-\gamma}_{a^+}u\right)(t_1^-)+L_1(u(t_1^-))}{\Gamma(\gamma)}\left(\frac{t^\rho-t_1^\rho}{\rho}\right)^{\gamma-1}+\left({}^\rho\mathcal{J}^{\alpha}_{t_1^+}\psi\right)(t)\\ &=\frac{(t^\rho-t_1^\rho)^{\gamma-1}}{\Gamma(\gamma)\rho^{\gamma-1}}\left[\left({}^\rho\mathcal{J}^{1-\gamma}_{a^+}u\right)(a^+)+L_1(u(t_1^-))+\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{a^+}\psi\right)(t_1)\right]+\left({}^\rho\mathcal{J}^{\alpha}_{t_1^+}\psi\right)(t).\end{aligned}$$

If $t\in J_2$, then Lemma 1.21 implies

$$\begin{aligned}u(t)&=\frac{\left({}^\rho\mathcal{J}^{1-\gamma}_{t_2^+}u\right)(t_2^+)}{\Gamma(\gamma)}\left(\frac{t^\rho-t_2^\rho}{\rho}\right)^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_{t_2}^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\psi(s)ds\\ &=\frac{\left({}^\rho\mathcal{J}^{1-\gamma}_{t_1^+}u\right)(t_2^-)+L_2(u(t_2^-))}{\Gamma(\gamma)}\left(\frac{t^\rho-t_2^\rho}{\rho}\right)^{\gamma-1}+\left({}^\rho\mathcal{J}^{\alpha}_{t_2^+}\psi\right)(t)\\ &=\frac{1}{\Gamma(\gamma)}\left(\frac{t^\rho-t_2^\rho}{\rho}\right)^{\gamma-1}\Bigg[\left({}^\rho\mathcal{J}^{1-\gamma}_{a^+}u\right)(a^+)+L_1(u(t_1^-))+L_2(u(t_2^-))\\ &\quad+\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{a^+}\psi\right)(t_1)+\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{t_1^+}\psi\right)(t_2)\Bigg]+\left({}^\rho\mathcal{J}^{\alpha}_{t_2^+}\psi\right)(t).\end{aligned}$$

Repeating the process in this way, the solution $u(t)$ for $t \in J_k,\ k = 1, \ldots, m$, can be written as

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\gamma)}\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{\gamma-1}\left[\left({}^\rho\mathcal{J}_{a^+}^{1-\gamma}u\right)(a^+) + \sum_{i=1}^{k} L_i(u(t_i^-)) + \sum_{i=1}^{k}\left({}^\rho\mathcal{J}_{(t_{i-1})^+}^{1-\gamma+\alpha}\psi\right)(t_i)\right] \\ &\quad + \left({}^\rho\mathcal{J}_{t_k^+}^{\alpha}\psi\right)(t). \end{aligned} \tag{3.9}$$

Applying ${}^\rho\mathcal{J}_{t_m^+}^{1-\gamma}$ on both sides of (3.9), using Lemma 1.10 and taking $t = b$, we obtain

$$\begin{aligned} \left({}^\rho\mathcal{J}_{t_m^+}^{1-\gamma}u\right)(b) &= \left({}^\rho\mathcal{J}_{a^+}^{1-\gamma}u\right)(a^+) + \sum_{i=1}^{m} L_i(u(t_i^-)) + \sum_{i=1}^{m}\left({}^\rho\mathcal{J}_{(t_{i-1})^+}^{1-\gamma+\alpha}\psi\right)(t_i) \\ &\quad + \left({}^\rho\mathcal{J}_{(t_m)^+}^{1-\gamma+\alpha}\psi\right)(b). \end{aligned} \tag{3.10}$$

Multiplying both sides of (3.10) by c_2 and using condition (3.6), we obtain

$$\begin{aligned} c_3 - c_1\left({}^\rho\mathcal{J}_{a^+}^{1-\gamma}u\right)(a^+) &= c_2\left({}^\rho\mathcal{J}_{a^+}^{1-\gamma}u\right)(a^+) + c_2\sum_{i=1}^{m} L_i(u(t_i^-)) + c_2\sum_{i=1}^{m}\left({}^\rho\mathcal{J}_{(t_{i-1})^+}^{1-\gamma+\alpha}\psi\right)(t_i) \\ &\quad + c_2\left({}^\rho\mathcal{J}_{(t_m)^+}^{1-\gamma+\alpha}\psi\right)(b), \end{aligned}$$

which implies that

$$\begin{aligned} \left({}^\rho\mathcal{J}_{a^+}^{1-\gamma}u\right)(a^+) &= \xi_2 - \xi_1\sum_{i=1}^{m} L_i(u(t_i^-)) - \xi_1\sum_{i=1}^{m}\left({}^\rho\mathcal{J}_{(t_{i-1})^+}^{1-\gamma+\alpha}\psi\right)(t_i) \\ &\quad - \xi_1\left({}^\rho\mathcal{J}_{(t_m)^+}^{1-\gamma+\alpha}\psi\right)(b). \end{aligned} \tag{3.11}$$

Substituting (3.11) into (3.9) and (3.8) we obtain (3.7).
Reciprocally, applying ${}^\rho\mathcal{J}_{t_k^+}^{1-\gamma}$ on both sides of (3.7) and using Lemma 1.10 and Theorem 1.9, we get

$$\left({}^\rho\mathcal{J}_{t_k^+}^{1-\gamma}u\right)(t) = \begin{cases} \xi_2 - \xi_1\sum_{i=1}^{m} L_i(u(t_i^-)) - \xi_1\sum_{i=1}^{m}\left({}^\rho\mathcal{J}_{(t_{i-1})^+}^{1-\gamma+\alpha}\psi\right)(t_i) \\ -\xi_1\left({}^\rho\mathcal{J}_{(t_m)^+}^{1-\gamma+\alpha}\psi\right)(b) + \left({}^\rho\mathcal{J}_{a^+}^{1-\gamma+\alpha}\psi\right)(t) \qquad if\ t \in J_0, \\ \\ \xi_2 - \xi_1\sum_{i=1}^{m} L_i(u(t_i^-)) - \xi_1\sum_{i=1}^{m}\left({}^\rho\mathcal{J}_{(t_{i-1})^+}^{1-\gamma+\alpha}\psi\right)(t_i) \\ -\xi_1\left({}^\rho\mathcal{J}_{t_m^+}^{1-\gamma+\alpha}\psi\right)(b) + \sum_{i=1}^{k} L_i(u(t_i^-)) + \sum_{i=1}^{k}\left({}^\rho\mathcal{J}_{(t_{i-1})^+}^{1-\gamma+\alpha}\psi\right)(t_i) \\ + \left({}^\rho\mathcal{J}_{t_k^+}^{1-\gamma+\alpha}\psi\right)(t) \qquad if\ t \in J_k, k = 1, \ldots, m. \end{cases} \tag{3.12}$$

Next, taking the limit $t \to a^+$ of (3.12) and using Lemma 1.13, with $1-\gamma < 1-\gamma+\alpha$, we obtain

$$\begin{aligned} \left({}^\rho\mathcal{J}_{a^+}^{1-\gamma}u\right)(a^+) &= \xi_2 - \xi_1\sum_{i=1}^{m} L_i(u(t_i^-)) - \xi_1\sum_{i=1}^{m}\left({}^\rho\mathcal{J}_{(t_{i-1})^+}^{1-\gamma+\alpha}\psi\right)(t_i) \\ &\quad - \xi_1\left({}^\rho\mathcal{J}_{t_m^+}^{1-\gamma+\alpha}\psi\right)(b). \end{aligned} \tag{3.13}$$

Now, taking $t=b$ in (3.12), we get

$$\begin{aligned}\left({}^\rho\mathcal{J}^{1-\gamma}_{t_m^+}u\right)(b) &= \xi_2+(1-\xi_1)\left(\sum_{i=1}^m L_i(u(t_i^-))+\sum_{i=1}^m\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{(t_{i-1})^+}\psi\right)(t_i)\right.\\ &\left.\quad+\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{t_m^+}\psi\right)(b)\right).\end{aligned}\tag{3.14}$$

From (3.13) and (3.14), we find that

$$c_1\left({}^\rho\mathcal{J}^{1-\gamma}_{a^+}u\right)(a^+)+c_2\left({}^\rho\mathcal{J}^{1-\gamma}_{t_m^+}u\right)(b)=c_3,$$

which shows that the boundary condition $c_1\left({}^\rho\mathcal{J}^{1-\gamma}_{a^+}u\right)(a^+)+c_2\left({}^\rho\mathcal{J}^{1-\gamma}_{t_m^+}u\right)(b)=c_3$, is satisfied.
Next, apply operator ${}^\rho\mathcal{D}^{\gamma}_{t_k^+}$ on both sides of (3.7), where $k=0,\dots,m$. Then, from Lemma 1.10 and Lemma 1.16 we obtain

$$({}^\rho\mathcal{D}^{\gamma}_{t_k^+}u)(t)=\left({}^\rho\mathcal{D}^{\beta(1-\alpha)}_{t_k^+}\psi\right)(t).\tag{3.15}$$

Since $u\in C^{\gamma}_{\gamma,\rho}(J_k)$ and by definition of $C^{\gamma}_{\gamma,\rho}(J_k)$, we have ${}^\rho\mathcal{D}^{\gamma}_{t_k^+}u\in C_{\gamma,\rho}(J_k)$, then (3.15) implies that

$$({}^\rho\mathcal{D}^{\gamma}_{t_k^+}u)(t)=\left(\delta_\rho\ {}^\rho\mathcal{J}^{1-\beta(1-\alpha)}_{t_k^+}\psi\right)(t)=\left({}^\rho\mathcal{D}^{\beta(1-\alpha)}_{t_k^+}\psi\right)(t)\in C_{\gamma,\rho}(J_k).\tag{3.16}$$

As $\psi(\cdot)\in C_{\gamma,\rho}(J_k)$ and from Lemma 1.12, follows

$$\left({}^\rho\mathcal{J}^{1-\beta(1-\alpha)}_{t_k^+}\psi\right)\in C_{\gamma,\rho}(J_k).\tag{3.17}$$

From (3.16), (3.17) and by the definition of the space $C^n_{\gamma,\rho}(J_k)$, we obtain

$$\left({}^\rho\mathcal{J}^{1-\beta(1-\alpha)}_{t_k^+}\psi\right)\in C^1_{\gamma,\rho}(J_k).$$

Applying operator ${}^\rho\mathcal{J}^{\beta(1-\alpha)}_{t_k^+}$ on both sides of (3.15) and using Lemma 1.15, Lemma 1.13 and Property 1.11, we have

$$\begin{aligned}\left({}^\rho\mathcal{D}^{\alpha,\beta}_{t_k^+}u\right)(t)={}^\rho\mathcal{J}^{\beta(1-\alpha)}_{t_k^+}\left({}^\rho\mathcal{D}^{\gamma}_{t_k^+}u\right)(t) &= \psi(t)-\frac{\left({}^\rho\mathcal{J}^{1-\beta(1-\alpha)}_{t_k^+}\psi\right)(t_k)}{\Gamma(\beta(1-\alpha))}\left(\frac{t^\rho-t_k^\rho}{\rho}\right)^{\beta(1-\alpha)-1}\\ &= \psi(t),\end{aligned}$$

that is, (3.4) holds.
Also, we can easily show that

$$\left({}^\rho\mathcal{J}^{1-\gamma}_{t_k^+}u\right)(t_k^+)=\left({}^\rho\mathcal{J}^{1-\gamma}_{t_{k-1}^+}u\right)(t_k^-)+L_k(u(t_k^-));k=1,\dots,m.$$

This completes the proof. ∎

As a consequence of Theorem 3.1, we have the following result

Lemma 3.2. *Let $\gamma = \alpha+\beta-\alpha\beta$ where $0<\alpha<1$ and $0\leq\beta\leq 1$, let $f : J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a function such that $f(\cdot,u(\cdot),w(\cdot))\in PC_{\gamma,\rho}(J)$ for any $u,w\in PC_{\gamma,\rho}(J)$.*
If $u\in PC^{\gamma}_{\gamma,\rho}(J)$, then u satisfies the problem (3.1)-(3.3) if and only if u is the fixed point of the operator $\Psi : PC_{\gamma,\rho}(J)\to PC_{\gamma,\rho}(J)$ defined by

$$\begin{aligned}\Psi u(t) &= \frac{1}{\Gamma(\gamma)}\left(\frac{t^\rho-t_k^\rho}{\rho}\right)^{\gamma-1}\Bigg[\xi_2-\xi_1\sum_{i=1}^{m}L_i(u(t_i^-))-\xi_1\sum_{i=1}^{m}\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{(t_{i-1})^+}h\right)(t_i)\\ &\quad -\xi_1\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{t_m^+}h\right)(b)+\sum_{a<t_k<t}L_k(u(t_k^-))+\sum_{a<t_k<t}\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{(t_{k-1})^+}h\right)(t_k)\Bigg]\\ &\quad +\left({}^\rho\mathcal{J}^{\alpha}_{t_k^+}h\right)(t)\qquad t\in J_k,\ k=0,\dots,m,\end{aligned}\tag{3.18}$$

where $h : J\to\mathbb{R}$ be a function satisfying the functional equation

$$h(t)=f(t,u(t),h(t)).$$

Assume that the function $f : J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous and satisfies the conditions:

(Ax_1) The function $f : J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be such that

$$f(\cdot,u(\cdot),w(\cdot))\in PC^{\beta(1-\alpha)}_{\gamma,\rho}(J)\ \text{ for any } u,w\in PC_{\gamma,\rho}(J).$$

(Ax_2) There exist constants $K>0$ and $0<M<1$ such that

$$|f(t,u,w)-f(t,\bar{u},\bar{w})|\leq K|u-\bar{u}|+M|w-\bar{w}|$$

for any $u,w,\bar{u},\bar{w}\in\mathbb{R}$ and $t\in J$.

(Ax_3) There exists a constant $l^*>0$ such that

$$|L_k(u)-L_k(\bar{u})|\leq l^*|u-\bar{u}|$$

for any $u,\bar{u}\in\mathbb{R}$ and $k=1,\dots,m$.

(Ax_4) There exist functions $p_1,p_2,p_3\in C([a,b],\mathbb{R}_+)$ with

$$p_1^*=\sup_{t\in[a,b]}p_1(t),\ \ p_2^*=\sup_{t\in[a,b]}p_2(t),\ \ p_3^*=\sup_{t\in[a,b]}p_3(t)<1$$

such that

$$|f(t,u,w)|\leq p_1(t)+p_2(t)|u|+p_3(t)|w|\ \text{ for } t\in J \text{ and } u,w\in\mathbb{R}.$$

(Ax_5) The functions $L_k:\mathbb{R}\longrightarrow\mathbb{R}$ are continuous and there exist constants $\Phi_1,\Phi_2>0$ such that

$$|L_k(u)|\leq\Phi_1|u|+\Phi_2\ \text{ for each } u\in\mathbb{R}, k=1,\dots,m.$$

We are now in a position to state and prove our existence result for problem (3.1)-(3.3) based on Banach's fixed point.

Theorem 3.3. *Assume* ***(Ax₁)*** – ***(Ax₃)*** *hold. If*

$$\begin{aligned} L \;:=\; & \Big(|\xi_1|+1\Big)\left(\frac{ml^*p^*}{\Gamma(\gamma)}+\frac{mK}{(1-M)\Gamma(1+\alpha)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^\alpha\right) \\ & +\frac{K}{(1-M)}\left(\frac{|\xi_1|}{\Gamma(1+\alpha)}+\frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\right)\left(\frac{b^\rho-a^\rho}{\rho}\right)^\alpha<1, \end{aligned} \tag{3.19}$$

then the problem (3.1)-(3.3) *has a unique solution in* $PC^\gamma_{\gamma,\rho}(J)$.

Proof: The proof will be given in two steps.
Step 1: We show that the operator Ψ defined in (3.18) has a unique fixed point u^* in $PC_{\gamma,\rho}(J)$. Let $u,w\in PC_{\gamma,\rho}(J)$ and $t\in J$, then we have

$$\begin{aligned} & |\Psi u(t)-\Psi w(t)| \\ \leq\; & \frac{(t^\rho-t_k^\rho)^{\gamma-1}}{\Gamma(\gamma)\rho^{\gamma-1}}\Bigg[|\xi_1|\sum_{i=1}^m|L_i(u(t_i^-))-L_i(w(t_i^-))|+|\xi_1|\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{t_m^+}|h(s)-g(s)|\right)(b) \\ & +|\xi_1|\sum_{i=1}^m\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{(t_{i-1})^+}|h(s)-g(s)|\right)(t_i)+\sum_{a<t_k<t}|L_k(u(t_k^-))-L_k(w(t_k^-))| \\ & +\sum_{a<t_k<t}\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{(t_{k-1})^+}|h(s)-g(s)|\right)(t_k)\Bigg]+\left({}^\rho\mathcal{J}^{\alpha}_{t_k^+}|h(s)-g(s)|\right)(t), \end{aligned}$$

where $h,g\in PC_{\gamma,\rho}(J)$ such that

$$\begin{aligned} h(t)&=f(t,u(t),h(t)),\\ g(t)&=f(t,w(t),g(t)). \end{aligned}$$

By ***(Ax₂)***, we have

$$\begin{aligned} |h(t)-g(t)| &= |f(t,u(t),h(t))-f(t,w(t),g(t))| \\ &\leq K|u(t)-w(t)|+M|h(t)-g(t)|. \end{aligned}$$

Then,

$$|h(t)-g(t)|\leq\frac{K}{1-M}|u(t)-w(t)|.$$

Therefore, for each $t\in J$

$$\begin{aligned} & |\Psi u(t)-\Psi w(t)| \\ \leq\; & \frac{(t^\rho-t_k^\rho)^{\gamma-1}}{\Gamma(\gamma)\rho^{\gamma-1}}\Bigg[|\xi_1|\sum_{i=1}^m l^*|u(t_i)-w(t_i)|+\frac{|\xi_1|K}{1-M}\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{t_m^+}|u(s)-w(s)|\right)(b) \\ & +\frac{|\xi_1|K}{1-M}\sum_{i=1}^m\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{(t_{i-1})^+}|u(s)-w(s)|\right)(t_i)+\sum_{i=1}^m l^*|u(t_i)-w(t_i)| \\ & +\frac{K}{1-M}\sum_{i=1}^m\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{(t_{i-1})^+}|u(s)-w(s)|\right)(t_i)\Bigg]+\frac{K}{1-M}\left({}^\rho\mathcal{J}^{\alpha}_{t_k^+}|u(s)-w(s)|\right)(t). \end{aligned}$$

Thus

$$\begin{aligned}
&|\Psi u(t)-\Psi w(t)|\\
\leq\ &\frac{(t^\rho-t_k^\rho)^{\gamma-1}}{\Gamma(\gamma)\rho^{\gamma-1}}\Bigg[|\xi_1|ml^*p^*+\frac{|\xi_1|K}{1-M}\left({}^\rho\mathcal{J}_{t_m^+}^{1-\gamma+\alpha}\left(\frac{s^\rho-t_m^\rho}{\rho}\right)^{\gamma-1}\right)(b)\\
&+\frac{mK|\xi_1|}{1-M}\left({}^\rho\mathcal{J}_{(t_{k-1})^+}^{1-\gamma+\alpha}\left(\frac{s^\rho-t_{k-1}^\rho}{\rho}\right)^{\gamma-1}\right)(t_k)\\
&+ml^*p^*+\frac{mK}{1-M}\left({}^\rho\mathcal{J}_{(t_{k-1})^+}^{1-\gamma+\alpha}\left(\frac{s^\rho-t_{k-1}^\rho}{\rho}\right)^{\gamma-1}\right)(t_k)\Bigg]\|u-w\|_{PC_{\gamma,\rho}}\\
&+\frac{K}{1-M}\|u-w\|_{PC_{\gamma,\rho}}\left({}^\rho\mathcal{J}_{t_k^+}^{\alpha}\left(\frac{s^\rho-t_k^\rho}{\rho}\right)^{\gamma-1}\right)(t).
\end{aligned}$$

By Lemma 1.10, we have

$$\begin{aligned}
&|\Psi u(t)-\Psi w(t)|\\
\leq\ &\frac{1}{\Gamma(\gamma)}\left(\frac{t^\rho-t_k^\rho}{\rho}\right)^{\gamma-1}\|u-w\|_{PC_{\gamma,\rho}}\Bigg[|\xi_1|ml^*p^*\\
&+\frac{|\xi_1|K\Gamma(\gamma)}{(1-M)\Gamma(1+\alpha)}\left(\frac{b^\rho-t_m^\rho}{\rho}\right)^{\alpha}+\frac{mK|\xi_1|\Gamma(\gamma)}{(1-M)\Gamma(1+\alpha)}\left(\frac{t_k^\rho-t_{k-1}^\rho}{\rho}\right)^{\alpha}\\
&+ml^*p^*+\frac{mK\Gamma(\gamma)}{(1-M)\Gamma(1+\alpha)}\left(\frac{t_k^\rho-t_{k-1}^\rho}{\rho}\right)^{\alpha}\Bigg]\\
&+\frac{K\Gamma(\gamma)}{(1-M)\Gamma(\gamma+\alpha)}\|u-w\|_{PC_{\gamma,\rho}}\left(\frac{t^\rho-t_k^\rho}{\rho}\right)^{\alpha+\gamma-1},
\end{aligned}$$

hence

$$\begin{aligned}
&\left|\left(\frac{t^\rho-t_k^\rho}{\rho}\right)^{1-\gamma}(\Psi u(t)-\Psi w(t))\right|\\
\leq\ &\Bigg[(|\xi_1|+1)\left(\frac{ml^*p^*}{\Gamma(\gamma)}+\frac{mK}{(1-M)\Gamma(1+\alpha)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha}\right)\\
&+\frac{K}{(1-M)}\left(\frac{|\xi_1|}{\Gamma(1+\alpha)}+\frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\right)\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha}\Bigg]\|u-w\|_{PC_{\gamma,\rho}},
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|\Psi u-\Psi w\|_{PC_{\gamma,\rho}}\\
\leq\ &\Bigg[(|\xi_1|+1)\left(\frac{ml^*p^*}{\Gamma(\gamma)}+\frac{mK}{(1-M)\Gamma(1+\alpha)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha}\right)\\
&+\frac{K}{(1-M)}\left(\frac{|\xi_1|}{\Gamma(1+\alpha)}+\frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\right)\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha}\Bigg]\|u-w\|_{PC_{\gamma,\rho}}.
\end{aligned}$$

By (3.19), the operator Ψ is a contraction. Hence, by Theorem 1.26, Ψ has a unique fixed point $u^*\in PC_{\gamma,\rho}(J)$.

Step 2: We show that such a fixed point $u^*\in PC_{\gamma,\rho}(J)$ is actually in $PC_{\gamma,\rho}^{\gamma}(J)$.

Since u^* is the unique fixed point of operator Ψ in $PC_{\gamma,\rho}(J)$, then for each $t \in J_k$, with $k = 0, \dots, m$, we have

$$
\begin{aligned}
u^*(t) &= \frac{1}{\Gamma(\gamma)}\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{\gamma-1}\left[\xi_2 - \xi_1\sum_{i=1}^m L_i(u(t_i^-)) - \xi_1\sum_{i=1}^m \left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{(t_{i-1})^+}h\right)(t_i)\right.\\
&- \left.\xi_1\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{t_m^+}h\right)(b) + \sum_{a<t_k<t} L_k(u(t_k^-)) + \sum_{a<t_k<t}\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{(t_{k-1})^+}h\right)(t_k)\right] + \left({}^\rho\mathcal{J}^\alpha_{t_k^+}h\right)(t),
\end{aligned}
$$

where $h \in PC_{\gamma,\rho}(J)$ such that

$$h(t) = f(t, u^*(t), h(t)).$$

Applying ${}^\rho\mathcal{D}^\gamma_{t_k^+}$ to both sides and by Lemma 1.10 and Lemma 1.16, we have

$$
\begin{aligned}
{}^\rho\mathcal{D}^\gamma_{t_k^+}u^*(t) &= \left({}^\rho\mathcal{D}^\gamma_{t_k^+}\,{}^\rho\mathcal{J}^\alpha_{t_k^+}f(s,u^*(s),h(s))\right)(t)\\
&= \left({}^\rho\mathcal{D}^{\beta(1-\alpha)}_{t_k^+}f(s,u^*(s),h(s))\right)(t).
\end{aligned}
$$

Since $\gamma \geq \alpha$, by ***(Ax₁)***, the right hand side is in $PC_{\gamma,\rho}(J)$ and thus ${}^\rho\mathcal{D}^\gamma_{t_k^+}u^* \in PC_{\gamma,\rho}(J)$ which implies that $u^* \in PC^\gamma_{\gamma,\rho}(J)$. As a consequence of Steps 1 and 2 together with Theorem 3.3, we can conclude that the problem (3.1)-(3.3) has a unique solution in $PC^\gamma_{\gamma,\rho}(J)$. ■

Our second result is based on Schaefer's fixed point theorem.

Theorem 3.4. *Assume that the hypothesis* ***(Ax₁)***, ***(Ax₄)*** *and* ***(Ax₅)*** *hold. If*

$$(|\xi_1|+1)\left(\frac{m\Phi_1 p^*}{\Gamma(\gamma)} + \frac{mp_2^*(b^\rho-a^\rho)^\alpha}{(1-p_3^*)\Gamma(1+\alpha)\rho^\alpha}\right) + \left(\frac{|\xi_1|}{\Gamma(1+\alpha)} + \frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\right)\left(\frac{p_2^*(b^\rho-a^\rho)^\alpha}{(1-p_3^*)\rho^\alpha}\right) < 1, \tag{3.20}$$

then the problem (3.1)-(3.3) *has at least one solution in* $PC^\gamma_{\gamma,\rho}(J)$.

Proof: We shall use Schaefer's fixed point theorem to prove in several steps that the operator Ψ defined in (3.18) has a fixed point.
Step 1: Ψ is continuous.
Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $PC_{\gamma,\rho}(J)$.
Then for each $t \in J$ we have,

$$
\begin{aligned}
&\left|((\Psi u_n)(t) - (\Psi u)(t))\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma}\right|\\
\leq\ & \frac{1}{\Gamma(\gamma)}\left[|\xi_1|\sum_{i=1}^m |L_i(u_n(t_i^-)) - L_i(u(t_i^-))| + |\xi_1|\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{t_m^+}|h_n(s)-h(s)|\right)(b)\right.\\
&+|\xi_1|\sum_{i=1}^m\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{(t_{i-1})^+}|h_n(s)-h(s)|\right)(t_i) + \sum_{a<t_k<t}|L_k(u_n(t_k^-)) - L_k(u(t_k^-))|\\
&+\left.\sum_{a<t_k<t}\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{(t_{k-1})^+}|h_n(s)-h(s)|\right)(t_k)\right] + \left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma}\left({}^\rho\mathcal{J}^\alpha_{t_k^+}|h_n(s)-h(s)|\right)(t),
\end{aligned}
$$

where $h_n, h \in PC_{\gamma,\rho}(J)$ such that

$$\begin{aligned} h_n(t) &= f(t, u_n(t), h_n(t)), \\ h(t) &= f(t, u(t), h(t)). \end{aligned}$$

Since $u_n \to u$, then we get $h_n(t) \to h(t)$ as $n \to \infty$ for each $t \in J$, and since f and L_k are continuous, then we have

$$\|\Psi u_n - \Psi u\|_{PC_{\gamma,\rho}} \to 0 \text{ as } n \to \infty.$$

Step 2: We show that Ψ is the mapping of two bounded sets in $PC_{\gamma,\rho}(J)$.
For $\eta > 0$, there exists a positive constant r such that $B_\eta = \{u \in PC_{\gamma,\rho}(J) : \|u\|_{PC_{\gamma,\rho}} \leq \eta\}$, we have $\|\Psi(u)\|_{PC_{\gamma,\rho}} \leq r$.
By ***(Ax₄)*** and from (3.18), We have for each $t \in J_k, k = 0, \ldots, m$,

$$\begin{aligned} \left|\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} h(t)\right| &= \left|\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} f(t, u(t), h(t))\right| \\ &\leq \left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} \left(p_1(t) + p_2(t)|u(t)| + p_3(t)|h(t)|\right). \end{aligned}$$

Which implies that

$$\|h\|_{PC_{\gamma,\rho}} \leq p_1^* \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma} + p_2^*\eta + p_3^*\|h\|_{PC_{\gamma,\rho}}.$$

Then

$$\|h\|_{PC_{\gamma,\rho}} \leq \frac{p_1^* \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma} + p_2^*\eta}{1 - p_3^*} := \Lambda.$$

Thus (3.18) implies

$$\begin{aligned} &\left|\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} (\Psi u)(t)\right| \\ \leq\ & \frac{1}{\Gamma(\gamma)} \Bigg[|\xi_2| + |\xi_1| \sum_{i=1}^{m} |L_i(u(t_i^-))| + |\xi_1| \sum_{i=1}^{m} \left({}^\rho\mathcal{J}_{(t_{i-1})^+}^{1-\gamma+\alpha}|h(s)|\right)(t_i) \\ &+ |\xi_1| \left({}^\rho\mathcal{J}_{t_m^+}^{1-\gamma+\alpha}|h(s)|\right)(b) + \sum_{a<t_k<t} |L_k(u(t_k^-))| + \sum_{a<t_k<t} \left({}^\rho\mathcal{J}_{(t_{k-1})^+}^{1-\gamma+\alpha}|h(s)|\right)(t_k) \Bigg] \\ &+ \left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} \left({}^\rho\mathcal{J}_{t_k^+}^{\alpha}|h(s)|\right)(t). \end{aligned}$$

Then

$$\begin{aligned}
&\left|\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma}(\Psi u)(t)\right| \\
\leq\ & \frac{1}{\Gamma(\gamma)}\Bigg[|\xi_2| + |\xi_1| m(\Phi_1 p^*\eta + \Phi_2) + |\xi_1| m\Lambda\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{(t_{k-1})^+}\left(\frac{s^\rho - t_{k-1}^\rho}{\rho}\right)^{\gamma-1}\right)(t_k) \\
&+|\xi_1|\Lambda\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{t_m^+}\left(\frac{s^\rho - t_m^\rho}{\rho}\right)^{\gamma-1}\right)(b) + m(\Phi_1 p^*\eta + \Phi_2) \\
&+m\Lambda\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{(t_{k-1})^+}\left(\frac{s^\rho - t_{k-1}^\rho}{\rho}\right)^{\gamma-1}\right)(t_k)\Bigg] + \Lambda\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma}\left({}^\rho\mathcal{J}^{\alpha}_{t_k^+}\left(\frac{s^\rho - t_k^\rho}{\rho}\right)^{\gamma-1}\right)(t).
\end{aligned}$$

By Lemma 1.10, we have

$$\begin{aligned}
\|\Psi u\|_{PC_{\gamma,\rho}} \leq\ & (|\xi_1|+1)\left(\frac{m(\Phi_1 p^*\eta + \Phi_2)}{\Gamma(\gamma)} + \frac{m\Lambda}{\Gamma(1+\alpha)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha}\right) \\
&+\Lambda\left(\frac{|\xi_1|}{\Gamma(1+\alpha)} + \frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\right)\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha} + \frac{|\xi_2|}{\Gamma(\gamma)} \\
:=\ & r.
\end{aligned}$$

Step 3: Ψ maps bounded sets into equicontinuous sets of $PC_{\gamma,\rho}$.
Let $\epsilon_1, \epsilon_2 \in J$, $\epsilon_1 < \epsilon_2$, B_η be a bounded set of $PC_{\gamma,\rho}$ as in Step 2, and let $u \in B_\eta$. Then

$$\begin{aligned}
&\left|\left(\frac{\epsilon_1^\rho - t_k^\rho}{\rho}\right)^{1-\gamma}(\Psi u)(\epsilon_1) - \left(\frac{\epsilon_2^\rho - t_k^\rho}{\rho}\right)^{1-\gamma}(\Psi u)(\epsilon_2)\right| \\
\leq\ & \frac{1}{\Gamma(\gamma)}\left[\sum_{\epsilon_1 < t_k < \epsilon_2}|L_k(u(t_k^-))| + \sum_{\epsilon_1 < t_k < \epsilon_2}\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{(t_{k-1})^+}|h(s)|\right)(t_k)\right] \\
&+\frac{\Lambda\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\left|\left(\frac{\epsilon_1^\rho - t_k^\rho}{\rho}\right)^{\alpha} - \left(\frac{\epsilon_2^\rho - t_k^\rho}{\rho}\right)^{\alpha}\right|.
\end{aligned}$$

As $\epsilon_1 \to \epsilon_2$, the right-hand side of the above inequality tends to zero. From step 1 to 3 with Arzela-Ascoli theorem, we conclude that $\Psi : PC_{\gamma,\rho} \to PC_{\gamma,\rho}$ is continuous and completely continuous.
Step 4: A priori bound. Now it remains to show that the set

$$G = \{u \in PC_{\gamma,\rho} : u = \lambda^*\Psi(u) \text{ for some } 0 < \lambda^* < 1\}$$

is bounded. Let $u \in G$, then $u = \lambda^*\Psi(u)$ for some $0 < \lambda^* < 1$.
By **(Ax_4)**, we have for each $t \in J$,

$$\begin{aligned}
\left|\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma}h(t)\right| &= \left|\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma}f(t, u(t), h(t))\right| \\
&\leq \left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma}\left(p_1(t) + p_2(t)|u(t)| + p_3(t)|h(t)|\right),
\end{aligned}$$

which implies that

$$\|h\|_{PC_{\gamma,\rho}} \leq p_1^*\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma} + p_2^*\|u\|_{PC_{\gamma,\rho}} + p_3^*\|h\|_{PC_{\gamma,\rho}},$$

then

$$\|h\|_{PC_{\gamma,\rho}} \leq \frac{p_1^*\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma}+p_2^*\|u\|_{PC_{\gamma,\rho}}}{1-p_3^*}.$$

This implies, by (3.18), ***(Ax₅)*** and by letting the estimation of Step 2, that for each $t\in J$ we have

$$\begin{aligned}
\|u\|_{PC_{\gamma,\rho}} &\leq (|\xi_1|+1)\left(\frac{m(\Phi_1 p^*\|u\|_{PC_{\gamma,\rho}}+\Phi_2)}{\Gamma(\gamma)}+\frac{mp_1^*\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1+\alpha-\gamma}+mp_2^*\|u\|_{PC_{\gamma,\rho}}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha}}{(1-p_3^*)\Gamma(1+\alpha)}\right)\\
&\quad+\frac{p_1^*\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1+\alpha-\gamma}+p_2^*\|u\|_{PC_{\gamma,\rho}}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha}}{(1-p_3^*)}\left(\frac{|\xi_1|}{\Gamma(1+\alpha)}+\frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\right)+\frac{|\xi_2|}{\Gamma(\gamma)},\\
&\leq \Bigg[(|\xi_1|+1)\left(\frac{m\Phi_1 p^*}{\Gamma(\gamma)}+\frac{mp_2^*\left(b^\rho-a^\rho\right)^{\alpha}}{(1-p_3^*)\Gamma(1+\alpha)\rho^\alpha}\right)\\
&\quad+\left(\frac{|\xi_1|}{\Gamma(1+\alpha)}+\frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\right)\left(\frac{p_2^*\left(b^\rho-a^\rho\right)^{\alpha}}{(1-p_3^*)\rho^\alpha}\right)\Bigg]\|u\|_{PC_{\gamma,\rho}}\\
&\quad+\frac{|\xi_2|}{\Gamma(\gamma)}+(|\xi_1|+1)\left(\frac{m\Phi_2}{\Gamma(\gamma)}+\frac{mp_1^*\left(b^\rho-a^\rho\right)^{1+\alpha-\gamma}}{(1-p_3^*)\Gamma(1+\alpha)\rho^{1+\alpha-\gamma}}\right)\\
&\quad+\left(\frac{|\xi_1|}{\Gamma(1+\alpha)}+\frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\right)\left(\frac{p_1^*\left(b^\rho-a^\rho\right)^{1+\alpha-\gamma}}{(1-p_3^*)\rho^{1+\alpha-\gamma}}\right).
\end{aligned}$$

By (3.20), we have

$$\begin{aligned}
&\|u\|_{PC_{\gamma,\rho}}\\
&\leq \frac{\frac{|\xi_2|}{\Gamma(\gamma)}+\left[(|\xi_1|+1)\left(\frac{m\Phi_2}{\Gamma(\gamma)}+\frac{mp_1^*(b^\rho-a^\rho)^{1+\alpha-\gamma}}{(1-p_3^*)\Gamma(1+\alpha)\rho^{1+\alpha-\gamma}}\right)+\left(\frac{|\xi_1|}{\Gamma(1+\alpha)}+\frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\right)\left(\frac{p_1^*(b^\rho-a^\rho)^{1+\alpha-\gamma}}{(1-p_3^*)\rho^{1+\alpha-\gamma}}\right)\right]}{1-\left[(|\xi_1|+1)\left(\frac{m\Phi_1p^*}{\Gamma(\gamma)}+\frac{mp_2^*(b^\rho-a^\rho)^{\alpha}}{(1-p_3^*)\Gamma(1+\alpha)\rho^{\alpha}}\right)+\left(\frac{|\xi_1|}{\Gamma(1+\alpha)}+\frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\right)\left(\frac{p_2^*(b^\rho-a^\rho)^{\alpha}}{(1-p_3^*)\rho^{\alpha}}\right)\right]}\\
&:= R.
\end{aligned}$$

As consequence of Theorem 1.27, and using Step 2 of the last result, we deduce that Ψ has a fixed point which is a solution of the problem (3.1)-(3.3). ■

Our third result is based on Krasnoselskii fixed point theorem.

Theorem 3.5. *Assume that* ***(Ax₁)***, ***(Ax₄)*** *and* ***(Ax₅)*** *hold. If*

$$(|\xi_1|+1)\left(\frac{m\Phi_1 p^*}{\Gamma(\gamma)}+\frac{mp_2^*\left(b^\rho-a^\rho\right)^{\alpha}}{(1-p_3^*)\Gamma(1+\alpha)\rho^\alpha}\right)+\frac{p_2^*|\xi_1|\left(b^\rho-a^\rho\right)^{\alpha}}{(1-p_3^*)\Gamma(1+\alpha)\rho^\alpha}<1, \tag{3.21}$$

then the problem (3.1)-(3.3) *has at least one solution in* $PC^{\gamma}_{\gamma,\rho}(J)$.

Proof: Consider the set

$$B_\eta=\{u\in PC_{\gamma,\rho}(J):\|u\|_{PC_{\gamma,\rho}}\leq\eta\},$$

where

$$\eta\geq\frac{(|\xi_1|+1)\left(\frac{m\Phi_2}{\Gamma(\gamma)}+\frac{m\Lambda}{\Gamma(1+\alpha)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha}\right)+\Lambda\left(\frac{|\xi_1|}{\Gamma(1+\alpha)}+\frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\right)\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha}+\frac{|\xi_2|}{\Gamma(\gamma)}}{1-(|\xi_1|+1)\frac{m\Phi_1p^*}{\Gamma(\gamma)}}.$$

We define the operators Q_1 and Q_2 on B_η by

$$\begin{aligned} Q_1u(t) &= \frac{1}{\Gamma(\gamma)}\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{\gamma-1}\left[\xi_2 - \xi_1\sum_{i=1}^{m}L_i(u(t_i^-)) - \xi_1\sum_{i=1}^{m}\left({}^\rho\mathcal{J}_{(t_{i-1})^+}^{1-\gamma+\alpha}h\right)(t_i)\right. \\ & \left. -\xi_1\left({}^\rho\mathcal{J}_{t_m^+}^{1-\gamma+\alpha}h\right)(b) + \sum_{a<t_k<t}L_k(u(t_k^-)) + \sum_{a<t_k<t}\left({}^\rho\mathcal{J}_{(t_{k-1})^+}^{1-\gamma+\alpha}h\right)(t_k)\right], \end{aligned} \tag{3.22}$$

$$Q_2u(t) = \left({}^\rho\mathcal{J}_{t_k^+}^{\alpha}h\right)(t), \tag{3.23}$$

where $k = 0,\dots,m$ and $h : J \to \mathbb{R}$ be a function satisfying the functional equation

$$h(t) = f(t, u(t), h(t)).$$

Then the fractional integral equation (3.18) can be written as operator equation

$$\Psi u(t) = Q_1u(t) + Q_2u(t), \quad u \in PC_{\gamma,\rho}(J).$$

The proof will be given in several steps.

Step 1: We prove that $Q_1u + Q_2w \in B_\eta$ for any $u, z \in B_\eta$.
Same as Step 2 of the last result, by ***(Ax₄)***, ***(Ax₅)*** and Lemma 1.10, for each $t \in J$ we have

$$\begin{aligned} \|Q_1u + Q_2w\|_{PC_{\gamma,\rho}} &\le \|Q_1u\|_{PC_{\gamma,\rho}} + \|Q_2w\|_{PC_{\gamma,\rho}} \\ &\le (|\xi_1|+1)\left(\frac{m(\Phi_1p^*\eta + \Phi_2)}{\Gamma(\gamma)} + \frac{m\Lambda}{\Gamma(1+\alpha)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^\alpha\right) \\ &\quad +\Lambda\left(\frac{|\xi_1|}{\Gamma(1+\alpha)} + \frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\right)\left(\frac{b^\rho - a^\rho}{\rho}\right)^\alpha + \frac{|\xi_2|}{\Gamma(\gamma)}. \end{aligned}$$

Since

$$\eta \ge \frac{(|\xi_1|+1)\left(\frac{m\Phi_2}{\Gamma(\gamma)} + \frac{m\Lambda}{\Gamma(1+\alpha)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^\alpha\right) + \Lambda\left(\frac{|\xi_1|}{\Gamma(1+\alpha)} + \frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\right)\left(\frac{b^\rho-a^\rho}{\rho}\right)^\alpha + \frac{|\xi_2|}{\Gamma(\gamma)}}{1 - (|\xi_1|+1)\frac{m\Phi_1p^*}{\Gamma(\gamma)}}.$$

we have

$$\|Q_1y + Q_2z\|_{PC_{\gamma,\rho}} \le \eta,$$

which infers that $Q_1u + Q_2w \in B_\eta$.

Step 2: Q_1 is a contraction.
Let $u, w \in PC_{\gamma,\rho}(J)$ and $t \in J$.
By ***(Ax₄)***, we have

$$\begin{aligned} |h(t) - g(t)| &= |f(t, u(t), h(t)) - f(t, w(t), g(t))| \\ &\le p_2(t)|u(t) - w(t)| + p_3(t)|h(t) - g(t)|. \end{aligned}$$

Then,

$$|h(t) - g(t)| \le \frac{p_2(t)}{1 - p_3(t)}|u(t) - w(t)| \le \frac{p_2^*}{1 - p_3^*}|u(t) - w(t)|.$$

where $p_1^* = \sup_{t\in[a,b]} p_1(t)$, $p_2^* = \sup_{t\in[a,b]} p_2(t)$ and $h, g \in C([a,b], \mathbb{R})$ such that

$$h(t) = f(t, u(t), h(t)),$$
$$g(t) = f(t, w(t), g(t)).$$

Then by $(\boldsymbol{Ax_5})$ and using the estimation in Step 1 of the first result, we have

$$\begin{aligned}
&|Q_1 y(t) - Q_1 z(t)| \\
\leq\;& \frac{1}{\Gamma(\gamma)}\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{\gamma-1}\|u-w\|_{PC_{\gamma,\rho}}\Bigg[|\xi_1| m\Phi_1 p^* \\
&+\frac{p_2^*|\xi_1|\Gamma(\gamma)}{(1-p_3^*)\Gamma(1+\alpha)}\left(\frac{b^\rho - t_m^\rho}{\rho}\right)^\alpha + \frac{mp_2^*|\xi_1|\Gamma(\gamma)}{(1-p_3^*)\Gamma(1+\alpha)}\left(\frac{t_k^\rho - t_{k-1}^\rho}{\rho}\right)^\alpha \\
&+m\Phi_1 p^* + \frac{mp_2^*\Gamma(\gamma)}{(1-p_3^*)\Gamma(1+\alpha)}\left(\frac{t_k^\rho - t_{k-1}^\rho}{\rho}\right)^\alpha\Bigg],
\end{aligned}$$

hence

$$\begin{aligned}
\|Q_1 u - Q_1 w\|_{PC_{\gamma,\rho}} \;\leq\;& \Bigg[(|\xi_1|+1)\left(\frac{m\Phi_1 p^*}{\Gamma(\gamma)} + \frac{mp_2^*}{(1-p_3^*)\Gamma(1+\alpha)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^\alpha\right) \\
&+\frac{p_2^*|\xi_1|}{(1-p_3^*)\Gamma(1+\alpha)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^\alpha\Bigg]\|u-w\|_{PC_{\gamma,\rho}}.
\end{aligned}$$

By (3.21), the operator Q_1 is a contraction.

Step 3: Q_2 is continuous and compact.
The continuity of Q_2 follows from the continuity of f. Next we prove that Q_2 is uniformly bounded on B_η. Let any $w \in B_\eta$. By using the estimation in Step 2 of the last result, (3.23) implies

$$\begin{aligned}
\left|\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma}(Q_2 z)(t)\right| \;&\leq\; \left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma}\left({}^\rho\mathcal{J}_{t_k^+}^\alpha |g(s)|\right)(t), \\
&\leq\; \Lambda\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma}\left({}^\rho\mathcal{J}_{t_k^+}^\alpha\left(\frac{s^\rho - t_k^\rho}{\rho}\right)^{\gamma-1}\right)(t),
\end{aligned}$$

where $k = 0, \ldots, m$ and $g : J \to \mathbb{R}$ be a function satisfying the functional equation

$$g(t) = f(t, w(t), g(t)).$$

By Lemma 1.10, we have

$$\|Q_2 z\|_{PC_{\gamma,\rho}} \leq \frac{\Lambda\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^\alpha.$$

This means that Q_2 is uniformly bounded on B_η. Next, we show that $Q_2 B_\eta$ is equicontinuous. Let any $w \in B_\eta$ and $a < \epsilon_1 < \epsilon_2 \leq b$. Then

$$\begin{aligned}
&\left|\left(\frac{\epsilon_1^\rho - t_k^\rho}{\rho}\right)^{1-\gamma}(Q_2 z)(\epsilon_1) - \left(\frac{\epsilon_2^\rho - t_k^\rho}{\rho}\right)^{1-\gamma}(Q_2 z)(\epsilon_2)\right| \\
\leq\;& \frac{\Lambda\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\left|\left(\frac{\epsilon_1^\rho - t_k^\rho}{\rho}\right)^\alpha - \left(\frac{\epsilon_2^\rho - t_k^\rho}{\rho}\right)^\alpha\right|.
\end{aligned}$$

Note that

$$\left|\left(\frac{\epsilon_1^{\rho}-t_k^{\rho}}{\rho}\right)^{1-\gamma}(Q_2z)(\epsilon_1)-\left(\frac{\epsilon_2^{\rho}-t_k^{\rho}}{\rho}\right)^{1-\gamma}(Q_2z)(\epsilon_2)\right|\to 0 \quad \text{as} \quad \epsilon_1\to\epsilon_2.$$

This shows that Q_2B_η is equicontinuous on J. Therefore Q_2B_η is relatively compact. By PC_γ type Arzela-Ascoli Theorem Q_2 is compact.
As a consequence of Theorem 1.28, we deduce that Ψ has at least a fixed point $u^*\in PC_{\gamma,\rho}(J)$ and by the same way of the proof of Theorem 3.3, we can easily show that $u^*\in PC^{\gamma}_{\gamma,\rho}(J)$. Using Lemma 3.2, we conclude that the problem (3.1)-(3.3) has at least one solution in the space $PC^{\gamma}_{\gamma,\rho}(J)$. ■

3.2.2 Ulam-Hyers-Rassias stability

Now we are concerned with the Ulam-Hyers-Rassias Stability of our problem (3.1)-(3.3). Let $u\in PC_{\gamma,\rho}(J)$, $\epsilon>0$, $\tau>0$ and $\vartheta: J\longrightarrow[0,\infty)$ be a continuous function. We consider the following inequality :

$$\begin{cases}\left|\left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{t_k^+}u\right)(t)-f\left(t,u(t),\left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{t_k^+}u\right)(t)\right)\right|\le\epsilon\vartheta(t), t\in J_k, k=0,\ldots,m,\\[2ex] \left|\left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_k^+}u\right)(t_k^+)-\left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_{k-1}^+}u\right)(t_k^-)-L_k(u(t_k^-))\right|\le\epsilon\tau, k=1,\ldots,m.\end{cases}\tag{3.24}$$

Definition 3.6. *([108]) Problem (3.1)-(3.3) is Ulam-Hyers-Rassias (U-H-R) stable with respect to (ϑ,τ) if there exists a real number $a_{f,m,\vartheta}>0$ such that for each $\epsilon>0$ and for each solution $u\in PC_{\gamma,\rho}(J)$ of inequality (3.24) there exists a solution $w\in PC_{\gamma,\rho}(J)$ of (3.1)-(3.3) with*

$$|u(t)-w(t)|\le\epsilon a_{f,m,\vartheta}(\vartheta(t)+\tau),\qquad t\in(a,b].$$

Remark 3.7. *([108]) A function $u\in PC_{\gamma,\rho}(J)$ is a solution of inequality (3.24) if and only if there exist $\sigma\in PC_{\gamma,\rho}(J)$ and a sequence $\sigma_k, k=0,\ldots,m$ such that*

1. $|\sigma(t)|\le\epsilon\vartheta(t)$ *and* $|\sigma_k|\le\epsilon\tau$, $t\in J_k, k=1,\ldots,m$;
2. $\left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{t_k^+}u\right)(t)=f\left(t,u(t),\left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{t_k^+}u\right)(t)\right)+\sigma(t)$, $t\in J_k, k=0,\ldots,m$;
3. $\left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_k^+}u\right)(t_k^+)=\left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_{k-1}^+}u\right)(t_k^-)+L_k(u(t_k^-))+\sigma_k$, $k=1,\ldots,m$.

Theorem 3.8. *Assume that in addition to **(Ax1)** – **(Ax3)** and (3.19), the following hypothesis holds:*

(Ax6) *There exist a nondecreasing function $\vartheta\in PC_{\gamma,\rho}(J)$ and $\lambda_\vartheta,\tilde{\lambda}_\vartheta>0$ such that for each $t\in(a,b]$, we have*

$$({}^{\rho}\mathcal{J}^{\alpha}_{a^+}\vartheta)(t)\le\lambda_\vartheta\vartheta(t),$$

and

$$({}^{\rho}\mathcal{J}^{1-\gamma}_{a^+}\vartheta)(t)\le\tilde{\lambda}_\vartheta\vartheta(t).$$

Then equation (3.1) is U-H-R stable with respect to (ϑ,τ).

Proof: Consider the operator Ψ defined in (3.18). Let $u \in PC_{\gamma,\rho}(J)$ be a solution if inequality (3.24), and let us assume that w is the unique solution of the problem

$$\begin{cases} \left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{t_k^+}w\right)(t) = f\left(t, w(t), \left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{t_k^+}w\right)(t)\right); \ t \in J_k, \ k = 0, \dots, m, \\ \left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_k^+}w\right)(t_k^+) = \left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_{k-1}^+}w\right)(t_k^-) + L_k(w(t_k^-)); k = 1, \dots, m, \\ c_1\left({}^{\rho}\mathcal{J}^{1-\gamma}_{a^+}w\right)(a^+) + c_2\left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_m^+}w\right)(b) = c_3, \\ \left({}^{\rho}\mathcal{J}^{1-\gamma}_{a^+}w\right)(a^+) = \left({}^{\rho}\mathcal{J}^{1-\gamma}_{a^+}u\right)(a^+). \end{cases}$$

By Lemma 3.2, we obtain for each $t \in J$

$$\begin{aligned} w(t) &= \frac{1}{\Gamma(\gamma)}\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{\gamma-1}\left[\left({}^{\rho}\mathcal{J}^{1-\gamma}_{a^+}w\right)(a^+) + \sum_{a<t_k<t} L_k(w(t_k^-)) + \sum_{a<t_k<t}\left({}^{\rho}\mathcal{J}^{1-\gamma+\alpha}_{(t_{k-1})^+}h\right)(t_k)\right] \\ &\quad + \left({}^{\rho}\mathcal{J}^{\alpha}_{t_k^+}h\right)(t) \qquad t \in J_k, k = 0, \dots, m, \end{aligned}$$

where $h : J \to \mathbb{R}$ be a function satisfying the functional equation

$$h(t) = f(t, w(t), h(t)).$$

Since u is a solution of the inequality (3.24), by Remark 3.7, we have

$$\begin{cases} \left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{t_k^+}u\right)(t) = f\left(t, u(t), \left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{t_k^+}u\right)(t)\right) + \sigma(t), t \in J_k, k = 0, \dots, m; \\ \left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_k^+}u\right)(t_k^+) = \left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_{k-1}^+}u\right)(t_k^-) + L_k(u(t_k^-)) + \sigma_k, k = 1, \dots, m. \end{cases} \tag{3.25}$$

Clearly, the solution of (3.25) is given by

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\gamma)}\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{\gamma-1}\left[\left({}^{\rho}\mathcal{J}^{1-\gamma}_{a^+}u\right)(a^+) + \sum_{a<t_k<t} L_k(u(t_k^-)) + \sum_{a<t_k<t}\sigma_k \right. \\ &\quad \left. + \sum_{a<t_k<t}\left({}^{\rho}\mathcal{J}^{1-\gamma+\alpha}_{(t_{k-1})^+}g\right)(t_k) + \sum_{a<t_k<t}\left({}^{\rho}\mathcal{J}^{1-\gamma+\alpha}_{(t_{k-1})^+}\sigma\right)(t_k)\right] \\ &\quad + \left({}^{\rho}\mathcal{J}^{\alpha}_{t_k^+}g\right)(t) + \left({}^{\rho}\mathcal{J}^{\alpha}_{t_k^+}\sigma\right)(t) \qquad t \in J_k, k = 0, \dots, m, \end{aligned}$$

where $g : J \to \mathbb{R}$ be a function satisfying the functional equation

$$g(t) = f(t, u(t), g(t)).$$

Hence, for each $t \in J$, we have

$$\begin{aligned} |u(t) - w(t)| &\leq \frac{1}{\Gamma(\gamma)}\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{\gamma-1}\left[\sum_{k=1}^{m}|L_k(u(t_k^-)) - L_k(w(t_k^-))| + \sum_{k=1}^{m}|\sigma_k| \right. \\ &\quad \left. + \sum_{k=1}^{m}\left({}^{\rho}\mathcal{J}^{1-\gamma+\alpha}_{(t_{k-1})^+}|g(s) - h(s)|\right)(t_k) + \sum_{k=1}^{m}\left({}^{\rho}\mathcal{J}^{1-\gamma+\alpha}_{(t_{k-1})^+}|\sigma(s)|\right)(t_k)\right] \\ &\quad + \left({}^{\rho}\mathcal{J}^{\alpha}_{t_k^+}|g(s) - h(s)|\right)(t) + \left({}^{\rho}\mathcal{J}^{\alpha}_{t_k^+}|\sigma(s)|\right)(t). \end{aligned}$$

Thus,

$$\begin{aligned}\|u-w\|_{PC_{\gamma,\rho}} &\leq \frac{1}{\Gamma(\gamma)}\Bigg[m\epsilon\tau+(m\tilde{\lambda}_{\vartheta}+1)\epsilon\lambda_{\vartheta}\vartheta(t)+\sum_{k=1}^{m} l^{*}|u(t_k^{-})-w(t_k^{-})| \\ &+\sum_{k=1}^{m}\left({}^{\rho}\mathcal{J}^{1-\gamma+\alpha}_{(t_{k-1})^{+}}|g(s)-h(s)|\right)(t_k)\Bigg]+\left(\frac{t^{\rho}-t_k^{\rho}}{\rho}\right)^{1-\gamma}\left({}^{\rho}\mathcal{J}^{\alpha}_{t_k^{+}}|g(s)-h(s)|\right)(t).\end{aligned}$$

By $(\boldsymbol{Ax_2})$ and Lemma 1.10, for $t\in J$, we have

$$\begin{aligned}&\|u-w\|_{PC_{\gamma,\rho}} \\ \leq\ &\frac{1}{\Gamma(\gamma)}\left[m\epsilon\tau+(m\tilde{\lambda}_{\vartheta}+1)\epsilon\lambda_{\vartheta}\vartheta(t)+ml^{*}p^{*}\|u-w\|_{PC_{\gamma,\rho}}\right] \\ +\ &\left[\frac{mK}{(1-M)\Gamma(1+\alpha)}\left(\frac{t_k^{\rho}-t_{k-1}^{\rho}}{\rho}\right)^{\alpha}+\frac{K\Gamma(\gamma)}{(1-M)\Gamma(\gamma+\alpha)}\left(\frac{t^{\rho}-t_k^{\rho}}{\rho}\right)^{\alpha}\right]\|u-w\|_{PC_{\gamma,\rho}}.\end{aligned}$$

Thus,

$$\begin{aligned}\|u-w\|_{PC_{\gamma,\rho}} &\leq \frac{1}{\Gamma(\gamma)}\left(m\epsilon\tau+(m\tilde{\lambda}_{\vartheta}+1)\epsilon\lambda_{\vartheta}\vartheta(t)\right) \\ &+\left[\frac{ml^{*}p^{*}}{\Gamma(\gamma)}+\frac{K}{1-M}\left(\frac{m}{\Gamma(1+\alpha)}+\frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\right)\left(\frac{b^{\rho}-a^{\rho}}{\rho}\right)^{\alpha}\right]\|u-w\|_{PC_{\gamma,\rho}}.\end{aligned}$$

Then by 3.19, we have

$$\|u-w\|_{PC_{\gamma,\rho}}\leq a_{\vartheta}\epsilon(\tau+\vartheta(t)),$$

where

$$a_{\vartheta}=\frac{1}{\Gamma(\gamma)}(m+(m\tilde{\lambda}_{\vartheta}+1)\lambda_{\vartheta})\left[1-\frac{ml^{*}p^{*}}{\Gamma(\gamma)}+\frac{K}{1-M}\left(\frac{m}{\Gamma(1+\alpha)}+\frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\right)\left(\frac{b^{\rho}-a^{\rho}}{\rho}\right)^{\alpha}\right]^{-1}.$$

Hence, equation (3.1) is U-H-R stable with respect to (ϑ,τ). ■

3.2.3 Examples

Example 3.9. *Consider the following impulsive boundary value problem of generalized Hilfer Fractional differential equation*

$$\left({}^{\frac{1}{2}}\mathcal{D}^{\frac{1}{2},0}_{t_k^{+}}u\right)(t)=\frac{1}{97e^{t+2}\left(1+|u(t)|+\left|{}^{\frac{1}{2}}\mathcal{D}^{\frac{1}{2},0}_{t_k^{+}}u(t)\right|\right)}+\frac{\ln(e+\sqrt{t})}{e^{2}\sqrt{t-1}},\ \textit{for each } t\in J_0\cup J_1, \tag{3.26}$$

$$\left({}^{\frac{1}{2}}\mathcal{J}^{\frac{1}{2}}_{e^{+}}u\right)(e^{+})-\left({}^{\frac{1}{2}}\mathcal{J}^{\frac{1}{2}}_{1^{+}}u\right)(e^{-})=\frac{|u(e^{-})|}{3+|u(e^{-})|}, \tag{3.27}$$

$$3\left({}^{\frac{1}{2}}\mathcal{J}^{\frac{1}{2}}_{1^{+}}u\right)(1^{+})-2\left({}^{\frac{1}{2}}\mathcal{J}^{\frac{1}{2}}_{e^{+}}u\right)(3)=0, \tag{3.28}$$

where $J_0 = (1, e]$, $J_1 = (e, 3]$, $t_0 = 1$ *and* $t_1 = e$.
Set

$$f(t, u, w) = \frac{1}{97e^{t+2}(1 + |u| + |w|)} + \frac{\ln(e + \sqrt{t})}{e^2\sqrt{t-1}}, \ t \in (1, 3], \ u, w \in \mathbb{R}.$$

We have

$$PC^{\beta(1-\alpha)}_{\gamma,\rho}([1, 3]) = PC^{0}_{\frac{1}{2},\frac{1}{2}}([1, 3]) = \left\{ g : (1, 3] \to \mathbb{R} : \sqrt{2}\left(\sqrt{t} - \sqrt{t_k}\right)^{\frac{1}{2}} g \in PC([1, 3]) \right\},$$

with $\gamma = \alpha = \frac{1}{2}$, $\rho = \frac{1}{2}$, $\beta = 0$, *and* $k \in \{0, 1\}$. *Clearly, the continuous function* $f \in PC^{0}_{\frac{1}{2},\frac{1}{2}}([1, 3])$. *Hence the condition* ***(Ax₁)*** *is satisfied.*
For each $u, \bar{u}, w, \bar{w} \in \mathbb{R}$ *and* $t \in (1, 3]$:

$$\begin{aligned} |f(t, u, w) - f(t, \bar{u}, \bar{w})| &\leq \frac{1}{97e^{t+2}}(|u - \bar{u}| + |w - \bar{w}|) \\ &\leq \frac{1}{97e^3}\left(|u - \bar{u}| + |w - \bar{w}|\right). \end{aligned}$$

Hence condition ***(Ax₂)*** *is satisfied with* $K = M = \dfrac{1}{97e^3}$.
And let

$$L_1(u) = \frac{u}{3 + u}, u \in [0, \infty).$$

Let $u, w \in [0, \infty)$. *Then we have*

$$|L_1(u) - L_1(w)| = |\frac{u}{3 + u} - \frac{w}{3 + w}| = \frac{3|u - w|}{(3 + u)(3 + w)} \leq \frac{1}{3}|u - w|,$$

and so the condition ***(Ax₃)*** *is satisfied and* $l^* = \frac{1}{3}$.
A simple computation shows that the condition (3.19) of Theorem 3.3 is satisfied, for

$$\begin{aligned} L &= \frac{1}{\sqrt{2\pi(\sqrt{e} - 1)}} + \frac{3\sqrt{2}(\sqrt{3} - 1)^{\frac{1}{2}}}{(97e^3 - 1)\Gamma(\frac{3}{2})} + \frac{\sqrt{2}(\sqrt{3} - 1)^{\frac{1}{2}}}{(97e^3 - 1)}\left(\frac{2}{\Gamma(\frac{3}{2})} + \sqrt{\pi}\right) \\ &\approx 0.52720987569 < 1. \end{aligned}$$

Then the problem (3.26)-(3.28) *has a unique solution in* $PC^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}([1, 3])$.
Also, hypothesis ***(Ax₆)*** *is satisfied with*

$$\vartheta(t) = e^5, \ \tau = 1 \ \textit{and} \ \lambda_\vartheta = \tilde{\lambda}_\vartheta = \frac{2}{\Gamma(\frac{3}{2})}.$$

Indeed, for each $t \in J_0 \cup J_1$, *we get*

$$\begin{aligned} ({}^\rho\mathcal{J}^{\frac{1}{2}}_{1^+}\vartheta)(t) &\leq \frac{2e^5}{\Gamma(\frac{3}{2})} \\ &= \lambda_\vartheta\vartheta(t) = \tilde{\lambda}_\vartheta\vartheta(t). \end{aligned}$$

Consequently, Theorem 3.8 implies that equation (3.26) is U-H-R stable.

Example 3.10. *Consider the following impulsive initial value problem of generalized Hilfer Fractional differential equation*

$$\left({}^{1}\mathcal{D}^{\frac{1}{2},0}_{t_k^+}u\right)(t)=\frac{3+|u(t)|+|{}^{1}\mathcal{D}^{\frac{1}{2},0}_{t_k^+}u(t)|}{53e^{-t+4}(1+|u(t)|+|{}^{1}\mathcal{D}^{\frac{1}{2},0}_{t_k^+}u(t)|)},\ \text{for each } t\in J_0\cup J_1, \tag{3.29}$$

$$\left({}^{1}\mathcal{J}^{\frac{1}{2}}_{e^+}u\right)(e^+)-\left({}^{1}\mathcal{J}^{\frac{1}{2}}_{1^+}u\right)(e^-)=\frac{|u(e^-)|}{2+|u(e^-)|}, \tag{3.30}$$

$$\left({}^{1}\mathcal{J}^{\frac{1}{2}}_{1^+}u\right)(1^+)=0, \tag{3.31}$$

where $J_0=(1,e]$, $J_1=(e,3]$, $t_0=1$ *and* $t_1=e$.
Set

$$f(t,u,w)=\frac{3+|u|+|w|}{53e^{-t+4}(1+|u|+|w|)},\ t\in(1,3],\ u,w\in\mathbb{R}.$$

We have

$$PC^{\beta(1-\alpha)}_{\gamma,\rho}([1,3])=PC^{0}_{\frac{1}{2},1}([1,3])=\left\{g:(1,3]\to\mathbb{R}:(\sqrt{t-t_k})g\in PC([1,3])\right\},$$

with $\gamma=\alpha=\frac{1}{2}$, $\rho=1$, $\beta=0$ *and* $k\in\{0,1\}$.
Clearly, the continuous function $f\in PC^{0}_{\frac{1}{2},1}([1,3])$. *Hence the condition* $(\boldsymbol{Ax_1})$ *is satisfied. For each* $u,w\in\mathbb{R}$ *and* $t\in(1,3]$:

$$|f(t,u,w)|\ \le\ \frac{1}{53e^{-t+4}}(3+|u|+|w|).$$

Hence condition $(\boldsymbol{Ax_4})$ *is satisfied with*

$$p_1(t)=\frac{3}{53e^{-t+4}},\ p_2(t)=p_3(t)=\frac{1}{53e^{-t+4}},$$

and

$$p_1^*=\frac{3}{53e},\ p_2^*=p_3^*=\frac{1}{53e}.$$

And let

$$L_1(u)=\frac{u}{2+u},u\in[0,\infty).$$

Then we have

$$|L_1(u)|\le\frac{1}{2}|u|+2,$$

and so the condition $(\boldsymbol{Ax_5})$ *is satisfied with* $\Phi_1=\dfrac{1}{2}$ *and* $\Phi_2=2$.
The condition (3.20) *of Theorem 3.4 is satisfied, for*

$$\begin{aligned}&(|\xi_1|+1)\left(\tfrac{m\Phi_1p^*}{\Gamma(\gamma)}+\tfrac{mp_2^*(b^\rho-a^\rho)^\alpha}{(1-p_3^*)\Gamma(1+\alpha)\rho^\alpha}\right)+\left(\tfrac{|\xi_1|}{\Gamma(1+\alpha)}+\tfrac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\right)\left(\tfrac{p_2^*(b^\rho-a^\rho)^\alpha}{(1-p_3^*)\rho^\alpha}\right)\\ =&\left(\frac{1}{2\sqrt{2\pi}}+\frac{\sqrt{2}}{(53e-1)\Gamma(\frac{3}{2})}\right)+\frac{\sqrt{2\pi}}{53e-1}\\ \approx&\ 0.22814541069\le 1.\end{aligned}$$

Then the problem (3.29)-(3.31) *has at least one solution in* $PC^{\frac{1}{2}}_{\frac{1}{2},1}([1,3])$. *Also, hypothesis* $(\boldsymbol{Ax_6})$ *is satisfied with*

$$\vartheta(t) = t-1,\ \tau = 1 \text{ and } \lambda_\vartheta = \tilde{\lambda}_\vartheta = \frac{\sqrt{2}\Gamma(2)}{\Gamma(\frac{5}{2})}.$$

Indeed, for each $t \in J_0 \cup J_1$, we get

$$\begin{aligned}({}^\rho\mathcal{J}_{1^+}^{\frac{1}{2}}\vartheta)(t) &\leq \frac{\sqrt{2}\Gamma(2)}{\Gamma(\frac{5}{2})}(t-1)\\ &= \lambda_\vartheta\vartheta(t) = \tilde{\lambda}_\vartheta\vartheta(t).\end{aligned}$$

Consequently, by a simple change of the constants l^*, K and M from hypothesis ***(Ax1)*** *and* ***(Ax2)*** *to Φ_1, p_2^* and p_3^* from* ***(Ax4)*** *and* ***(Ax5)****, Theorem 3.8 implies that equation (3.29) is G.U-H-R stable.*

Example 3.11. *Consider the following impulsive anti–Periodic boundary value problem of generalized Hilfer Fractional differential equation*

$$\left({}^1\mathcal{D}_{t_k^+}^{\frac{1}{2},0}u\right)(t) = \frac{e^2 + |u(t)| + |{}^1\mathcal{D}_{t_k^+}^{\frac{1}{2},0}u(t)|}{77e^{-t+2}(1+|u(t)| + |{}^1\mathcal{D}_{t_k^+}^{\frac{1}{2},0}u(t)|)},\ \text{for each } t \in J_k, k = 0,\dots,4, \tag{3.32}$$

$$\left({}^1\mathcal{J}_{t_k^+}^{\frac{1}{2}}u\right)(t_k^+) - \left({}^1\mathcal{J}_{t_{(k-1)}^+}^{\frac{1}{2}}u\right)(t_k^-) = \frac{|u(t_k^-)|}{10k + |u(t_k^-)|}, k = 1,\dots,4, \tag{3.33}$$

$$\left({}^1\mathcal{J}_{1^+}^{\frac{1}{2}}u\right)(1^+) = -\left({}^1\mathcal{J}_{\frac{9}{5}^+}^{\frac{1}{2}}u\right)(2), \tag{3.34}$$

where $J_k = (t_k, t_{k+1}]$, $t_k = 1 + \dfrac{k}{5}$ for $k = 0,\dots,4$, $m = 4$, $a = t_0 = 1$ and $b = t_5 = 2$. Set

$$f(t,u,w) = \frac{e^2 + |u| + |w|}{77e^{-t+2}(1+|u|+|w|)},\ t \in (1,2],\ u,w \in \mathbb{R}.$$

We have

$$PC_{\gamma,\rho}^{\beta(1-\alpha)}([1,2]) = PC_{\frac{1}{2},1}^{0}([1,2]) = \left\{g : (1,2] \to \mathbb{R} : (\sqrt{t-t_k})g \in PC([1,2])\right\},$$

with $\gamma = \alpha = \frac{1}{2}$, $\rho = 1$, $\beta = 0$ and $k = 0,\dots,4$.
Clearly, the continuous function $f \in PC_{\frac{1}{2},1}^{0}([1,2])$. Hence the condition ***(Ax1)*** *is satisfied.*
For each $u,w \in \mathbb{R}$ and $t \in (1,2]$:

$$|f(t,u,w)| \leq \frac{1}{77e^{-t+2}}(e^2 + |u| + |w|).$$

Hence condition ***(Ax4)*** *is satisfied with*

$$p_1(t) = \frac{e^2}{77e^{-t+2}},\ p_2(t) = p_3(t) = \frac{1}{77e^{-t+2}},$$

and

$$p_1^* = \frac{e^2}{77},\ p_2^* = p_3^* = \frac{1}{77}.$$

And let

$$L_k(u) = \frac{u}{10k+u}, k = 1, \dots, 4, u \in [0, \infty).$$

Then we have

$$|L_k(u)| \leq \frac{1}{10}|u| + 1, k = 1, \dots, 4,$$

and so the condition $(\boldsymbol{Ax_5})$ *is satisfied with* $\Phi_1 = \dfrac{1}{10}$ *and* $\Phi_2 = 1$.
The condition (3.21) of Theorem 3.5 is satisfied, for

$$(|\xi_1|+1)\left(\tfrac{m\Phi_1 p^*}{\Gamma(\gamma)} + \tfrac{mp_2^*(b^\rho - a^\rho)^\alpha}{(1-p_3^*)\Gamma(1+\alpha)\rho^\alpha}\right) + \tfrac{p_2^*|\xi_1|(b^\rho-a^\rho)^\alpha}{(1-p_3^*)\Gamma(1+\alpha)\rho^\alpha} = \frac{3\sqrt{5}}{5\sqrt{\pi}} + \frac{125}{1463\Gamma(\frac{3}{2})} < 1.$$

Then the problem (3.32)-(3.34) *has at least one solution in* $PC^{\frac{1}{2}}_{\frac{1}{2},1}([1,2])$. *Also, hypothesis* $(\boldsymbol{Ax_6})$ *is satisfied with*

$$\vartheta(t) = (1-t)^2,\ \tau = 1 \text{ and } \lambda_\vartheta = \tilde{\lambda}_\vartheta = \frac{\Gamma(3)}{\Gamma(\frac{7}{2})}.$$

Indeed, for each $t \in J_k, k = 0, \dots, 4$, *we get*

$$\begin{aligned}({}^\rho\mathcal{J}^{\frac{1}{2}}_{1^+}\vartheta)(t) &\leq \frac{\Gamma(3)}{\Gamma(\frac{7}{2})}(t-1)^2\\ &= \lambda_\vartheta\vartheta(t) = \tilde{\lambda}_\vartheta\vartheta(t).\end{aligned}$$

Same as Example 3.10, Theorem 3.8 implies that equation (3.32) is U-H-R stable.

3.3 Existence and Ulam Stability for Impulsive Generalized Hilfer-Type Fractional Differential Equations in Banach Spaces [2]

Motivated by the works mentioned in the Introduction of the current chapter, in this section we discuss the existence results to the boundary value problem with nonlinear implicit generalized Hilfer-type fractional differential equation with instantaneous impulses:

$$\left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{t_k^+}u\right)(t)=f\left(t,u(t),\left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{t_k^+}u\right)(t)\right);\ t\in J_k,\ k=0,\cdots,m, \tag{3.35}$$

$$\left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_k^+}u\right)(t_k^+)=\left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_{k-1}^+}u\right)(t_k^-)+\varpi_k(u(t_k^-));k=1,\cdots,m, \tag{3.36}$$

$$c_1\left({}^{\rho}\mathcal{J}^{1-\gamma}_{a^+}u\right)(a^+)+c_2\left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_m^+}u\right)(b)=c_3, \tag{3.37}$$

where ${}^{\rho}\mathcal{D}^{\alpha,\beta}_{t_k^+}$, ${}^{\rho}\mathcal{J}^{1-\gamma}_{t_k^+}$ are the generalized Hilfer fractional derivative of order $\alpha\in(0,1)$ and type $\beta\in[0,1]$ and generalized Hilfer fractional integral of order $1-\gamma,(\gamma=\alpha+\beta-\alpha\beta)$ respectively, c_1,c_2 are reals with $c_1+c_2\neq 0$, $J_k:=(t_k,t_{k+1}];k=0,\cdots,m$, $a=t_0<t_1<\cdots<t_m<t_{m+1}=b<\infty$, $u(t_k^+)=\lim_{\epsilon\to 0^+}u(t_k+\epsilon)$ and $u(t_k^-)=\lim_{\epsilon\to 0^-}u(t_k+\epsilon)$ represent the right and left hand limits of $u(t)$ at $t=t_k$, $c_3\in E$, $f:J\times E\times E\to E$ is a given function and $\varpi_k:E\to E$; $k=1,\cdots,m$ are given continuous functions.

3.3.1 Existence Results

Consider the weighted Banach space

$$PC_{\gamma,\rho}(J)=\left\{u:J\to E:\left(\frac{t^\rho-t_k^\rho}{\rho}\right)^{1-\gamma}u(t)\in C_E(J_k);k=0,\cdots,m,\text{and there exist } u(t_k^-)\right.$$
$$\left.\text{and }\left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_k^+}u\right)(t_k^+);k=0,\cdots,m,\text{with } u(t_k^-)=u(t_k)\right\},0\leq\gamma<1.$$

and

$$PC^n_{\gamma,\rho}(J)=\left\{u\in PC^{n-1}:u^{(n)}\in PC_{\gamma,\rho}(J)\right\},n\in\mathbb{N},$$
$$PC^0_{\gamma,\rho}(J)=PC_{\gamma,\rho}(J),$$

with the norm

$$\|u\|_{PC_{\gamma,\rho}}=\max_{k=0,\ldots,m}\left\{\sup_{t\in[t_k,t_{k+1}]}\left\|\left(\frac{t^\rho-t_k^\rho}{\rho}\right)^{1-\gamma}u(t)\right\|\right\}.$$

We define the space

$$PC^\gamma_{\gamma,\rho}(J)=\left\{u\in PC_{\gamma,\rho}(J),\ {}^{\rho}\mathcal{D}^{\gamma}_{t_k^+}u\in PC_{\gamma,\rho}(J)\right\},\ k=0,\ldots,m.$$

2. **A. Salim**, M. Benchohra, E. Karapinar and J. E. Lazreg, Existence and Ulam stability for impulsive generalized Hilfer-type fractional differential equations, *Adv. Difference Equ.* **2020** (2020), 1-21pp.

Lemma 3.12. *([61]) Let $D \subset PC_{\gamma,\rho}(J)$ be a bounded and equicontinuous set, then (i) the function $t \to \mu(D(t))$ is continuous on J, and*

$$\mu_{PC_{\gamma,\rho}}(D) = \sup_{t\in[a,b]} \mu\left(\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} D(t)\right),$$

(ii) $\mu\left(\int_a^b u(s)ds : u \in D\right) \leq \int_a^b \mu(D(s))ds$, where

$$D(t) = \{u(t) : t \in D\}, t \in J.$$

By following the same results from the first section, we have the following result

Lemma 3.13. *Let $\gamma = \alpha+\beta-\alpha\beta$ where $0 < \alpha < 1$ and $0 \leq \beta \leq 1$, let $f : J \times E \times E \to E$ be a function such that $f(\cdot, u(\cdot), w(\cdot)) \in PC_{\gamma,\rho}(J)$ for any $u, w \in PC_{\gamma,\rho}(J)$.*
If $u \in PC^{\gamma}_{\gamma,\rho}(J)$, then u satisfies the problem (3.35) − (3.37) if and only if u is the fixed point of the operator $\Psi : PC_{\gamma,\rho}(J) \to PC_{\gamma,\rho}(J)$ defined by

$$\begin{aligned} \Psi u(t) &= \frac{1}{\Gamma(\gamma)}\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{\gamma-1}\Bigg[\xi_2 - \xi_1\sum_{i=1}^{m}\varpi_i(u(t_i^-)) - \xi_1\sum_{i=1}^{m}\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{(t_{i-1})^+}h\right)(t_i) \\ &\quad -\xi_1\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{t_m^+}h\right)(b) + \sum_{a<t_k<t}\varpi_k(u(t_k^-)) + \sum_{a<t_k<t}\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{(t_{k-1})^+}h\right)(t_k)\Bigg] \\ &\quad + \left({}^\rho\mathcal{J}^{\alpha}_{t_k^+}h\right)(t) \qquad t \in J_k, k = 0, \cdots, m, \end{aligned} \tag{3.38}$$

where $h : J \to \mathrm{I\!R}$ be a function satisfying the functional equation

$$h(t) = f(t, u(t), h(t)).$$

The following hypotheses will be used in the sequel :

(Ax_1) The function $t \mapsto f(t, u, w)$ is measurable and continuous on J for each $u, w \in E$, and the functions $u \mapsto f(t, u, w)$ and $w \mapsto f(t, u, w)$ are continuous on E for a.e. $t \in J$, and

$$f(\cdot, u(\cdot), w(\cdot)) \in PC^{\beta(1-\alpha)}_{\gamma,\rho} \text{ for any } u, w \in PC_{\gamma,\rho}(J).$$

(Ax_2) There exists a continuous function $p : [a, b] \longrightarrow [0, \infty)$ such that

$$\|f(t, u, w)\| \leq p(t), \text{ for a.e. } t \in J \text{ and for each } u, w \in E.$$

(Ax_3) For each bounded set $B \subset E$ and for each $t \in (a, b]$, we have

$$\mu(f(t, B, ({}^\rho D^{\alpha,\beta}_{a^+}B))) \leq \left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} p(t)\mu(B),$$

where ${}^\rho D^{\alpha,\beta}_{a^+}B = \{{}^\rho D^{\alpha,\beta}_{a^+}w : w \in B\}$ and $k = 1, \cdots, m$.

(Ax_4) The functions $\varpi_k : E \longrightarrow E$ are continuous and there exists $\eta^* > 0$ such that

$$\|\varpi_k(u)\| \leq \eta^*\|u\| \text{ for each } u \in E, k = 1, \cdots, m.$$

(Ax_5) For each bounded set $B \subset E$ and for each $t \in J$, we have

$$\mu(\varpi_k(B)) \leq \eta^* \left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} \mu(B), k = 1, \cdots, m.$$

We are now in a position to state and prove our existence result for the problem (3.35)–(3.37) based on Mönch's fixed point theorem.

Theorem 3.14. *Assume* **(Ax_1)** – **(Ax_5)** *hold. If*

$$\mathfrak{L} := \frac{m\eta^*}{\Gamma(\gamma)} + p^* \left(\frac{1}{\Gamma(\alpha+1)} + \frac{m}{\Gamma(\gamma)\Gamma(2-\gamma+\alpha)}\right) \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha} < 1, \tag{3.39}$$

where $p^* = \sup_{t\in[a,b]} p(t)$, *then the problem* (3.35)-(3.37) *has at least one solution in* $PC^\gamma_{\gamma,\rho}(J)$.

Proof: Consider the operator $\Psi : PC_{\gamma,\rho}(J) \to PC_{\gamma,\rho}(J)$ defined in (3.38) and the ball $B_R := B(0, R) = \{w \in PC_{\gamma,\rho}(J) : \|w\|_{PC_{\gamma,\rho}} \leq R\}$.
For any $u \in B_R$, and each $t \in J$ we have

$$\begin{aligned}
&\left\|\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} (\Psi u)(t)\right\| \\
\leq\ & \frac{1}{\Gamma(\gamma)}\Bigg[\|\xi_2\| + |\xi_1| \sum_{i=1}^{m} \|\varpi_i(u(t_i^-))\| + |\xi_1| \sum_{i=1}^{m} \left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{(t_{i-1})^+}\|h(s)\|\right)(t_i) \\
& + |\xi_1| \left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{t_m^+}\|h\|\right)(b) + \sum_{a<t_k<t} \|\varpi_k(u(t_k^-))\| + \sum_{a<t_k<t} \left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{(t_{k-1})^+}\|h(s)\|\right)(t_k)\Bigg] \\
& + \left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} \left({}^\rho\mathcal{J}^{\alpha}_{t_k^+}\|h(s)\|\right)(t) \\
\leq\ & \frac{\|\xi_2\|}{\Gamma(\gamma)} + \frac{|\xi_1|+1}{\Gamma(\gamma)}\left(ml^*R + mp^*\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{(t_{i-1})^+}(1)\right)(t_i)\right) + \frac{|\xi_1|p^*}{\Gamma(\gamma)}\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{t_m^+}(1)\right)(b) \\
& + p^*\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma}\left({}^\rho\mathcal{J}^{\alpha}_{t_k^+}(1)\right)(t).
\end{aligned}$$

By Lemma 1.10, we have

$$\begin{aligned}
&\left\|\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} (\Psi u)(t)\right\| \\
\leq\ & \frac{\|\xi_2\|}{\Gamma(\gamma)} + \frac{|\xi_1|+1}{\Gamma(\gamma)}\left(ml^*R + \frac{mp^*}{\Gamma(2-\gamma+\alpha)}\left(\frac{t_i^\rho - t_{i-1}^\rho}{\rho}\right)^{1-\gamma+\alpha}\right) \\
& + \frac{|\xi_1|p^*}{\Gamma(\gamma)\Gamma(2-\gamma+\alpha)}\left(\frac{b^\rho - t_m^\rho}{\rho}\right)^{1-\gamma+\alpha} + \frac{p^*}{\Gamma(\alpha+1)}\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma+\alpha}.
\end{aligned}$$

Hence, for any $u \in PC_{\gamma,\rho}(J)$, and each $t \in (a, b]$ we get

$$\begin{aligned}
\|(\Psi u)\|_{PC_{\gamma,\rho}} \leq\ & \frac{\|\xi_2\|}{\Gamma(\gamma)} + \frac{|\xi_1|+1}{\Gamma(\gamma)}\left(ml^*R + \frac{mp^*}{\Gamma(2-\gamma+\alpha)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha}\right) \\
& + \left(\frac{|\xi_1|p^*}{\Gamma(\gamma)\Gamma(2-\gamma+\alpha)} + \frac{p^*}{\Gamma(\alpha+1)}\right)\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha}. \\
\leq\ & R.
\end{aligned}$$

This proves that Ψ transforms the ball B_R into itself. We shall show that the operator $\Psi : B_R \to B_R$ satisfies all the assumptions of Theoreme 1.24. The proof will be given in several steps.

Step 1: $\Psi : B_R \to B_R$ is continuous.
Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $PC_{\gamma,\rho}(J)$.
Then for each $t \in (a, b]$ we have ,

$$\begin{aligned}
&\left\| ((\Psi u_n)(t) - (\Psi u)(t)) \left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} \right\| \\
\leq\ & \frac{1}{\Gamma(\gamma)} \Bigg[|\xi_1| \sum_{i=1}^{m} \|\varpi_i(u_n(t_i^-)) - \varpi_i(u(t_i^-))\| + |\xi_1| \left({}^\rho\mathcal{J}_{t_m^+}^{1-\gamma+\alpha} \|h_n(s) - h(s)\| \right)(b) \\
&+ |\xi_1| \sum_{i=1}^{m} \left({}^\rho\mathcal{J}_{(t_{i-1})^+}^{1-\gamma+\alpha} \|h_n(s) - h(s)\| \right)(t_i) + \sum_{a<t_k<t} \|\varpi_k(u_n(t_k^-)) - \varpi_k(u(t_k^-))\| \\
&+ \sum_{a<t_k<t} \left({}^\rho\mathcal{J}_{(t_{k-1})^+}^{1-\gamma+\alpha} \|h_n(s) - h(s)\| \right)(t_k) \Bigg] + \left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} \left({}^\rho\mathcal{J}_{t_k^+}^{\alpha} \|h_n(s) - h(s)\| \right)(t),
\end{aligned}$$

where $h_n, h \in PC_{\gamma,\rho}$ such that

$$\begin{aligned}
h_n(t) &= f(t, u_n(t), h_n(t)), \\
h(t) &= f(t, u(t), h(t)).
\end{aligned}$$

Since $u_n \to u$, then we get $h_n(t) \to h(t)$ as $n \to \infty$ for each $t \in J$, and by Lebesgue dominated convergence theorem we have

$$\|\Psi u_n - \Psi u\|_{PC_{\gamma,\rho}} \to 0 \text{ as } n \to \infty.$$

Step 2: $\Psi(B_R)$ is bounded and equicontinuous.
Since $\Psi(B_R) \subset B_R$ and B_R is bounded, then $\Psi(B_R)$ is bounded.
Next, let $\epsilon_1, \epsilon_2 \in J$, $\epsilon_1 < \epsilon_2$, and let $u \in B_R$. Then

$$\begin{aligned}
&\left\| \left(\frac{\epsilon_1^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} (\Psi u)(\epsilon_1) - \left(\frac{\epsilon_2^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} (\Psi u)(\epsilon_2) \right\| \\
\leq\ & \frac{1}{\Gamma(\gamma)} \left[\sum_{\epsilon_1<t_k<\epsilon_2} \|\varpi_k(u(t_k^-))\| + \sum_{\epsilon_1<t_k<\epsilon_2} \left({}^\rho\mathcal{J}_{(t_{k-1})^+}^{1-\gamma+\alpha} \|h(s)\| \right)(t_k) \right] \\
&+ \frac{p^*}{\Gamma(\alpha+1)} \left| \left(\frac{\epsilon_1^\rho - t_k^\rho}{\rho}\right)^{1-\gamma+\alpha} - \left(\frac{\epsilon_2^\rho - t_k^\rho}{\rho}\right)^{1-\gamma+\alpha} \right|.
\end{aligned}$$

As $\epsilon_1 \to \epsilon_2$, the right-hand side of the above inequality tends to zero. Hence, $\Psi(B_R)$ is bounded and equicontinuous.

Step 3: The implication (1.1) of Theorem 1.24 holds.

Now let D be an equicontinuous subset of B_R such that $D \subset \overline{\Psi(D)} \cup \{0\}$, therefore the function $t \longrightarrow d(t) = \mu(D(t))$ are continuous on J. By $(\boldsymbol{Ax_3})$, $(\boldsymbol{Ax_5})$ and the properties of the measure μ, for each $t \in J$, we have

$$\begin{aligned}
\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} d(t) &\leq \mu\left(\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} (\Psi D)(t) \cup \{0\}\right) \\
&\leq \mu\left(\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} (\Psi D)(t)\right) \\
&\leq \frac{1}{\Gamma(\gamma)}\left[\sum_{a<t_k<t} \eta^* \left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} \mu(D(t))\right. \\
&\quad \left. + \sum_{a<t_k<t}\left({}^\rho\mathcal{J}_{(t_{k-1})^+}^{1-\gamma+\alpha}\left(\frac{s^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} p(s)\mu(D(s))\right)(t_k)\right] \\
&\quad + \left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma}\left({}^\rho\mathcal{J}_{t_k^+}^{\alpha}\left(\frac{s^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} p(s)\mu(D(s))\right)(t) \\
&\leq \frac{m\eta^*}{\Gamma(\gamma)}\|d\|_{PC_{\gamma,\rho}} + p^*\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma}\left({}^\rho\mathcal{J}_{a^+}^{\alpha}\left(\frac{s^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} d(s)\right)(t) \\
&\quad + \frac{mp^*}{\Gamma(\gamma)}\left({}^\rho\mathcal{J}_{a^+}^{1-\gamma+\alpha}\left(\frac{s^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} d(s)\right)(t) \\
&\leq \left[\frac{m\eta^*}{\Gamma(\gamma)} + \frac{p^*}{\Gamma(\alpha+1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha}\right. \\
&\quad \left. + \frac{mp^*}{\Gamma(\gamma)\Gamma(2-\gamma+\alpha)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha}\right]\|d\|_{PC_{\gamma,\rho}}.
\end{aligned}$$

Thus

$$\|d\|_{PC_{\gamma,\rho}} \leq \mathfrak{L}\|d\|_{PC_{\gamma,\rho}}.$$

From (3.39), we get $\|d\|_{PC_{\gamma,\rho}} = 0$, that is $d(t) = \mu(D(t)) = 0$, for each $t \in J_k, k = 0, \cdots, m$, and then $D(t)$ is relatively compact in E. In view of the Ascoli-Arzela Theorem, D is relatively compact in B_R. Applying now Theorem 1.24, we conclude that Ψ has a fixed point $u^* \in PC_{\gamma,\rho}(J)$, which is solution of the problem (3.35)-(3.37).

Step 4: We show that such a fixed point $u^* \in PC_{\gamma,\rho}(J)$ is actually in $PC^\gamma_{\gamma,\rho}(J)$.
Since u^* is the unique fixed point of operator Ψ in $PC_{\gamma,\rho}(J)$, then for each $t \in J_k$, with $k = 0, \cdots, m$, we have

$$\begin{aligned}
u^*(t) &= \frac{1}{\Gamma(\gamma)}\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{\gamma-1}\left[\xi_2 - \xi_1\sum_{i=1}^{m}\varpi_i(u(t_i^-)) - \xi_1\sum_{i=1}^{m}\left({}^\rho\mathcal{J}_{(t_{i-1})^+}^{1-\gamma+\alpha}h\right)(t_i)\right. \\
&- \left.\xi_1\left({}^\rho\mathcal{J}_{t_m^+}^{1-\gamma+\alpha}h\right)(b) + \sum_{a<t_k<t}\varpi_k(u(t_k^-)) + \sum_{a<t_k<t}\left({}^\rho\mathcal{J}_{(t_{k-1})^+}^{1-\gamma+\alpha}h\right)(t_k)\right] + \left({}^\rho\mathcal{J}_{t_k^+}^{\alpha}h\right)(t),
\end{aligned}$$

where $h \in PC_{\gamma,\rho}$ such that

$$h(t) = f(t, u^*(t), h(t)).$$

Applying ${}^{\rho}\mathcal{D}^{\gamma}_{t_k^+}$ to both sides and by Lemma 1.10 and Lemma 1.16, we have

$$\begin{aligned} {}^{\rho}\mathcal{D}^{\gamma}_{t_k^+}u^*(t) &= \left({}^{\rho}\mathcal{D}^{\gamma}_{t_k^+}\ {}^{\rho}\mathcal{J}^{\alpha}_{t_k^+}f(s,u^*(s),h(s))\right)(t) \\ &= \left({}^{\rho}\mathcal{D}^{\beta(1-\alpha)}_{t_k^+}f(s,u^*(s),h(s))\right)(t). \end{aligned}$$

Since $\gamma \geq \alpha$, by $(\boldsymbol{Ax_1})$, the right hand side is in $PC_{\gamma,\rho}(J)$ and thus ${}^{\rho}\mathcal{D}^{\gamma}_{t_k^+}u^* \in PC_{\gamma,\rho}(J)$ which implies that $u^* \in PC^{\gamma}_{\gamma,\rho}(J)$. As a consequence of Steps 1 to 4 together with Theorem 3.14, we can conclude that the problem $(3.35)-(3.37)$ has at least one solution in $PC^{\gamma}_{\gamma,\rho}(J)$. ■

Our second existence result for the problem (3.35)-(3.37) is based on Darbo's fixed point Theorem.

Theorem 3.15. *Assume $(\boldsymbol{Ax_1})$ – $(\boldsymbol{Ax_5})$ and (3.39) hold. then the problem (3.35)-(3.37) has at least one solution in $PC^{\gamma}_{\gamma,\rho}(J)$.*

Proof: Consider the operator Ψ defined in (3.38). We know that $\Psi : B_R \longrightarrow B_R$ is bounded and continuous and that $\Psi(B_R)$ is equicontinuous, we need to prove that the operator Ψ is a $\mathfrak{L}$-contraction.
Let $D \subset B_R$ and $t \in J$. Then we have

$$\begin{aligned} &\mu\left(\left(\frac{t^{\rho}-t_k^{\rho}}{\rho}\right)^{1-\gamma}(\Psi D)(t)\right) = \mu\left(\left(\frac{t^{\rho}-t_k^{\rho}}{\rho}\right)^{1-\gamma}(\Psi u)(t) : u \in D\right) \\ \leq\ & \frac{1}{\Gamma(\gamma)}\left[\sum_{a<t_k<t}\eta^*\mu\left(\left\{\left(\frac{t^{\rho}-t_k^{\rho}}{\rho}\right)^{1-\gamma}u(t), u \in D\right\}\right)\right. \\ &+ \left.\sum_{a<t_k<t}\left\{\left({}^{\rho}\mathcal{J}^{1-\gamma+\alpha}_{(t_{k-1})^+}p^*\mu\left(\left(\frac{s^{\rho}-t_k^{\rho}}{\rho}\right)^{1-\gamma}u(s)\right)\right)(t_k), u \in D\right\}\right] \\ &+ \left(\frac{b^{\rho}-a^{\rho}}{\rho}\right)^{1-\gamma}\left\{\left({}^{\rho}\mathcal{J}^{\alpha}_{t_k^+}p^*\mu\left(\left(\frac{s^{\rho}-t_k^{\rho}}{\rho}\right)^{1-\gamma}u(s)\right)\right)(t), u \in D\right\}. \end{aligned}$$

By Lemma 1.10, we have

$$\mu_{PC_{\gamma,\rho}}(\Psi D) \leq \left[\frac{m\eta^*}{\Gamma(\gamma)} + \left(\frac{p^*}{\Gamma(\alpha+1)} + \frac{mp^*}{\Gamma(\gamma)\Gamma(2-\gamma+\alpha)}\right)\left(\frac{b^{\rho}-a^{\rho}}{\rho}\right)^{1-\gamma+\alpha}\right]\mu_{PC_{\gamma,\rho}}(D).$$

Therefore

$$\mu_{PC_{\gamma,\rho}}(\Psi D) \leq \mathfrak{L}\mu_{PC_{\gamma,\rho}}(D).$$

So, By (3.39), the operator Ψ is a $\mathfrak{L}$-contraction.
As consequence of Theorem 1.25 and using Step 4 of the last result, we deduce that Ψ has a fixed point which is a solution of the problem (3.35)-(3.37). ■

3.3.2 Ulam Type Stability

Now, we consider the Ulam stability for problem (3.35)-(3.37). Let $u \in PC_{\gamma,\rho}(J)$, $\epsilon > 0$, $\tau > 0$ and $\vartheta : J \longrightarrow [0, \infty)$ be a continuous function. We consider the following inequality :

$$\begin{cases} \left\| \left({}^\rho\mathcal{D}^{\alpha,\beta}_{t_k^+}u\right)(t) - f\left(t, u(t), \left({}^\rho\mathcal{D}^{\alpha,\beta}_{t_k^+}u\right)(t)\right) \right\| \leq \epsilon\vartheta(t), t \in J_k, k = 0, \dots, m, \\ \left\| \left({}^\rho\mathcal{J}^{1-\gamma}_{t_k^+}u\right)(t_k^+) - \left({}^\rho\mathcal{J}^{1-\gamma}_{t_{k-1}^+}u\right)(t_k^-) - \varpi_k(u(t_k^-)) \right\| \leq \epsilon\tau, k = 1, \dots, m. \end{cases} \tag{3.40}$$

Definition 3.16. *([108]) Problem (3.35)–(3.37) is Ulam-Hyers-Rassias (U-H-R) stable with respect to (ϑ, τ) if there exists a real number $a_{f,m,\vartheta} > 0$ such that for each $\epsilon > 0$ and for each solution $u \in PC_{\gamma,\rho}(J)$ of inequality (3.40) there exists a solution $w \in PC_{\gamma,\rho}(J)$ of (3.35)–(3.37) with*

$$\|u(t) - w(t)\| \leq \epsilon a_{f,m,\vartheta}(\vartheta(t) + \tau), \qquad t \in J.$$

Remark 3.17. *([108]) A function $u \in PC_{\gamma,\rho}(J)$ is a solution of inequality (3.40) if and only if there exist $\sigma \in PC_{\gamma,\rho}(J)$ and a sequence $\sigma_k, k = 0, \dots, m$ such that*

1. $\|\sigma(t)\| \leq \epsilon\vartheta(t)$ *and* $\|\sigma_k\| \leq \epsilon\tau$, $t \in J_k, k = 1, \dots, m$;
2. $\left({}^\rho\mathcal{D}^{\alpha,\beta}_{t_k^+}u\right)(t) = f\left(t, u(t), \left({}^\rho\mathcal{D}^{\alpha,\beta}_{t_k^+}u\right)(t)\right) + \sigma(t)$, $t \in J_k, k = 0, \dots, m$;
3. $\left({}^\rho\mathcal{J}^{1-\gamma}_{t_k^+}u\right)(t_k^+) = \left({}^\rho\mathcal{J}^{1-\gamma}_{t_{k-1}^+}u\right)(t_k^-) + \varpi_k(u(t_k^-)) + \sigma_k$, $k = 1, \dots, m$.

Theorem 3.18. *Assume that in addition to **(Ax₁)** – **(Ax₅)** and (3.39), the following hypothesis hold.*

(Ax₆) *There exist a nondecreasing function $\vartheta \in PC_{\gamma,\rho}(J)$ and $\lambda_\vartheta > 0$ such that for each $t \in J$, we have*

$$({}^\rho\mathcal{J}^{\alpha}_{a^+}\vartheta)(t) \leq \lambda_\vartheta\vartheta(t).$$

(Ax₇) *There exists a continuous function $\chi : [a, b] \longrightarrow [0, \infty)$ such that for each $t \in J_k; k = 0, \dots, m$, we have*

$$p(t) \leq \chi(t)\vartheta(t).$$

Then equation (3.35) is U-H-R stable with respect to (ϑ, τ).

Set $\chi^* = \sup_{t\in[a,b]} \chi(t)$.

Proof: Consider the operator Ψ defined in (3.38). Let $u \in PC_{\gamma,\rho}(J)$ be a solution if inequality (3.40), and let us assume that w is the unique solution of the problem

$$\begin{cases} \left({}^\rho\mathcal{D}^{\alpha,\beta}_{t_k^+}w\right)(t) = f\left(t, w(t), \left({}^\rho\mathcal{D}^{\alpha,\beta}_{t_k^+}w\right)(t)\right); \ t \in J_k, \ k = 0, \dots, m, \\ \left({}^\rho\mathcal{J}^{1-\gamma}_{t_k^+}w\right)(t_k^+) = \left({}^\rho\mathcal{J}^{1-\gamma}_{t_{k-1}^+}w\right)(t_k^-) + \varpi_k(w(t_k^-)); k = 1, \dots, m, \\ c_1\left({}^\rho\mathcal{J}^{1-\gamma}_{a^+}w\right)(a^+) + c_2\left({}^\rho\mathcal{J}^{1-\gamma}_{t_m^+}w\right)(b) = c_3, \\ \left({}^\rho\mathcal{J}^{1-\gamma}_{t_k^+}w\right)(t_k^+) = \left({}^\rho\mathcal{J}^{1-\gamma}_{t_k^+}u\right)(t_k^+); \ k = 0, \dots, m. \end{cases}$$

By Lemma 1.21, we obtain for each $t \in (a,b]$

$$w(t) = \frac{\left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_k^+} w\right)(t_k^+)}{\Gamma(\gamma)} \left(\frac{t^{\rho} - t_k^{\rho}}{\rho}\right)^{\gamma-1} + \left({}^{\rho}\mathcal{J}^{\alpha}_{t_k^+} h\right)(t) \qquad t \in J_k, k = 0, \ldots, m,$$

where $h : (a,b] \to E$ be a function satisfing the functional equation

$$h(t) = f(t, w(t), h(t)).$$

Since u is a solution of the inequality (3.40), by Remark 3.17, we have

$$\begin{cases} \left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{t_k^+} u\right)(t) = f\left(t, u(t), \left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{t_k^+} u\right)(t)\right) + \sigma(t), t \in J_k, k = 0, \ldots, m; \\ \left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_k^+} u\right)(t_k^+) = \left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_{k-1}^+} u\right)(t_k^-) + \varpi_k(u(t_k^-)) + \sigma_k, k = 1, \ldots, m. \end{cases} \tag{3.41}$$

Clearly, the solution of (3.41) is given by

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\gamma)} \left(\frac{t^{\rho} - t_k^{\rho}}{\rho}\right)^{\gamma-1} \Bigg[\left({}^{\rho}\mathcal{J}^{1-\gamma}_{a^+} u\right)(a^+) + \sum_{a<t_k<t} \varpi_k(u(t_k^-)) + \sum_{a<t_k<t} \sigma_k \\ &\quad + \sum_{a<t_k<t} \left({}^{\rho}\mathcal{J}^{1-\gamma+\alpha}_{(t_{k-1})^+} g\right)(t_k) + \sum_{a<t_k<t} \left({}^{\rho}\mathcal{J}^{1-\gamma+\alpha}_{(t_{k-1})^+} \sigma\right)(t_k)\Bigg] \\ &\quad + \left({}^{\rho}\mathcal{J}^{\alpha}_{t_k^+} g\right)(t) + \left({}^{\rho}\mathcal{J}^{\alpha}_{t_k^+} \sigma\right)(t) \qquad t \in J_k, k = 0, \ldots, m, \end{aligned}$$

where $g : (a,b] \to E$ be a function satisfing the functional equation

$$g(t) = f(t, u(t), g(t)).$$

We have for each $t \in J_k, k = 0, \ldots, m$,

$$\begin{aligned} \left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_k^+} u\right)(t_k^+) &= \left({}^{\rho}\mathcal{J}^{1-\gamma}_{a^+} u\right)(a^+) + \sum_{a<t_k<t} \varpi_k(u(t_k^-)) + \sum_{a<t_k<t} \sigma_k \\ &\quad + \sum_{a<t_k<t} \left({}^{\rho}\mathcal{J}^{1-\gamma+\alpha}_{(t_{k-1})^+} g\right)(t_k) + \sum_{a<t_k<t} \left({}^{\rho}\mathcal{J}^{1-\gamma+\alpha}_{(t_{k-1})^+} \sigma\right)(t_k). \end{aligned}$$

Hence, for each $t \in J$, we have

$$\|u(t) - w(t)\| \leq \left({}^{\rho}\mathcal{J}^{\alpha}_{t_k^+} |g(s) - h(s)|\right)(t) + \left({}^{\rho}\mathcal{J}^{\alpha}_{t_k^+} |\sigma(s)|\right)(t).$$

Thus,

$$\begin{aligned} \|u(t) - w(t)\| &\leq \left({}^{\rho}\mathcal{J}^{\alpha}_{a^+} \|g(s) - h(s)\|\right)(t) + \left({}^{\rho}\mathcal{J}^{\alpha}_{a^+} \|\sigma(s)\|\right) \\ &\leq \epsilon\lambda_{\vartheta}\vartheta(t) + \int_a^t s^{\rho-1} \left(\frac{t^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1} \frac{2\chi(t)\vartheta(t)}{\Gamma(\gamma)} ds \\ &\leq \epsilon\lambda_{\vartheta}\vartheta(t) + 2\chi^* \left({}^{\rho}\mathcal{J}^{\alpha}_{a^+} \vartheta\right)(t) \\ &\leq (\epsilon + 2\chi^*)\lambda_{\vartheta}\vartheta(t) \\ &\leq (1 + \frac{2\chi^*}{\epsilon})\lambda_{\vartheta}\epsilon(\tau + \vartheta(t)) \\ &\leq a_{\vartheta}\epsilon(\tau + \vartheta(t)), \end{aligned}$$

where $a_{\vartheta} = (1 + \frac{2\chi^*}{\epsilon})\lambda_{\vartheta}$. Hence, equation (3.35) is U-H-R stable with respect to (ϑ, τ). ■

3.3.3 Examples

Let

$$E = l^1 = \left\{ u = (u_1, u_2, \cdots, u_n, \cdots), \sum_{n=1}^{\infty} |u_n| < \infty \right\}$$

be the Banach space with the norm

$$\|u\| = \sum_{n=1}^{\infty} |u_n|.$$

Example 3.19. *Consider the following impulsive boundary value problem of generalized Hilfer fractional differential equation*

$$\left({}^1\mathcal{D}^{\frac{1}{2},0}_{t_k^+} u_n\right)(t) = \frac{3t^2-20}{213e^{-t+3}(1+|u_n(t)|+|{}^1\mathcal{D}^{\frac{1}{2},0}_{t_k^+} u_n(t)|)}, \quad t \in J_k, k = 0, \cdots, 9, \tag{3.42}$$

$$\left({}^1\mathcal{J}^{\frac{1}{2}}_{t_k^+} u_n\right)(t_k^+) - \left({}^1\mathcal{J}^{\frac{1}{2}}_{t_{(k-1)}^+} u_n\right)(t_k^-) = \frac{|u_n(t_k^-)|}{10(k+3)+|u_n(t_k^-)|}, k = 1, \cdots, 9, \tag{3.43}$$

$$\left({}^1\mathcal{J}^{\frac{1}{2}}_{1^+} u_n\right)(1^+) + 2\left({}^1\mathcal{J}^{\frac{1}{2}}_{\frac{9}{5}^+} u_n\right)(3) = 0, \tag{3.44}$$

where $J_k = (t_k, t_{k+1}]$, $t_k = 1 + \frac{k}{5}$ for $k = 0, \cdots, 9$, $m = 9$, $a = t_0 = 1$, and $b = t_{10} = 3$. Set

$$f(t,u,w) = \frac{3t^2-20}{213e^{-t+3}(1+\|u\|+\|w\|)}, \quad t \in (1,3], \ u, w \in E.$$

We have

$$PC^{\beta(1-\alpha)}_{\gamma,\rho}([1,3]) = PC^{0}_{\frac{1}{2},1}([1,3]) = \left\{ g : (1,3] \to \mathbb{R} : (\sqrt{t-t_k})g \in PC([1,3]) \right\},$$

with $\gamma = \alpha = \frac{1}{2}$, $\rho = 1$, $\beta = 0$ and $k = 0, \cdots, 9$. Clearly, the continuous function $f \in PC^0_{\frac{1}{2},1}([1,2])$.
*Hence the condition **(Ax_1)** is satisfied.*
For each $u, w \in E$ and $t \in (1,3]$:

$$\|f(t,u,w)\| \leq \frac{3t^2-20}{213e^{-t+3}}.$$

*Hence condition **(Ax_2)** is satisfied with $p^* = \frac{7}{213}$.*
And let

$$\varpi_k(u) = \frac{\|u\|}{10(k+3)+\|u\|}, k = 1, \cdots, 9, u \in E.$$

Let $u \in E$. Then we have

$$\|\varpi_k(u)\| \leq \frac{1}{40}\|u\|, k = 1, \cdots, 9,$$

and so the condition $(\boldsymbol{Ax_4})$ *is satisfied with* $\eta^* = \dfrac{1}{40}$.
The condition (3.39) *of Theorem* 3.14 *is satisfied, for*

$$\begin{aligned}\mathfrak{L} &:= \frac{m\eta^*}{\Gamma(\gamma)} + \left(\frac{p^*}{\Gamma(\alpha+1)} + \frac{mp^*}{\Gamma(\gamma)\Gamma(2-\gamma+\alpha)}\right)\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha} \\ &= \frac{9}{40\sqrt{\pi}} + 2\left(\frac{17}{213\sqrt{\pi}} + \frac{63}{213\Gamma(2)\sqrt{\pi}}\right) \\ &\approx 0.55074703829 < 1.\end{aligned}$$

Then the problem (3.42)–(3.44) *has at least one solution in* $PC^{\frac{1}{2}}_{\frac{1}{2},1}([1,3])$.

Example 3.20. *Let the following impulsive anti–periodic boundary value problem*

$$\left({}^{\frac{1}{2}}\mathcal{D}^{\frac{1}{2},0}_{t_k^+} u_n\right)(t) = \frac{(3t^3+5e^{-3})|u_n(t)|}{144e^{-t+e}(1+\|u(t)\|+\|{}^{\frac{1}{2}}\mathcal{D}^{\frac{1}{2},0}_{t_k^+} u(t)\|)}, \text{ for each } t \in J_0 \cup J_1, \tag{3.45}$$

$$\left({}^{\frac{1}{2}}\mathcal{J}^{\frac{1}{2}}_{2^+} u_n\right)(2^+) - \left({}^{\frac{1}{2}}\mathcal{J}^{\frac{1}{2}}_{1^+} u_n\right)(2^-) = \frac{|u_n(2^-)|}{77e^{-t+4}+2}, \tag{3.46}$$

$$\left({}^{\frac{1}{2}}\mathcal{J}^{\frac{1}{2}}_{1^+} u\right)(1^+) = -\left({}^{\frac{1}{2}}\mathcal{J}^{\frac{1}{2}}_{2^+} u\right)(e), \tag{3.47}$$

where $J_0 = (1,2]$, $J_1 = (2,e]$, $t_1 = 2$, $m = 1$, $a = t_0 = 1$ *and* $b = t_2 = e$.
Set

$$f(t,u,w) = \frac{(3t^3+5e^{-3})\|u\|}{144e^{-t+e}(1+\|u\|+\|w\|)}, \quad t \in (1,e], \ u,w \in E.$$

We have

$$PC^{\beta(1-\alpha)}_{\gamma,\rho}([1,2]) = PC^{0}_{\frac{1}{2},\frac{1}{2}}([1,e]) = \left\{ g : (1,e] \to E : \sqrt{2}(\sqrt{t}-\sqrt{t_k})^{\frac{1}{2}} g \in C([1,e]) \right\},$$

with $\gamma = \alpha = \frac{1}{2}$, $\rho = \frac{1}{2}$, $\beta = 0$ *and* $k \in \{0,1\}$. *Clearly, the continuous function* $f \in PC^{0}_{\frac{1}{2},\frac{1}{2}}([1,e])$.
Hence the condition $(\boldsymbol{Ax_1})$ *is satisfied.*
For each $u,w \in E$ *and* $t \in (1,e]$:

$$\|f(t,u,w)\| \le \frac{(3t^3+5e^{-3})}{144e^{-t+e}}.$$

Hence condition $(\boldsymbol{Ax_2})$ *is satisfied with*

$$p(t) = \frac{(3t^3+5e^{-3})}{144e^{-t+e}},$$

and

$$p^* = \frac{(3e^3+5e^{-3})}{144}.$$

And let

$$\varpi_1(u) = \frac{\|u\|}{77e^{-t+4}+2}, u \in E.$$

Let $u \in E$. Then we have

$$\|\varpi_k(u)\| \leq \frac{1}{77e^{-t+4}+2}\|u\|,$$

*and so the condition **(Ax_4)** is satisfied with* $\eta^* = \dfrac{1}{77e^{4-e}+2}$.
The condition (3.39) of Theorem 3.14 is satisfied, for

$$\begin{aligned}
\mathfrak{L} &:= \frac{m\eta^*}{\Gamma(\gamma)} + \left(\frac{p^*}{\Gamma(\alpha+1)} + \frac{mp^*}{\Gamma(\gamma)\Gamma(2-\gamma+\alpha)}\right)\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha} \\
&= \frac{1}{(77e^{4-e}+2)\sqrt{\pi}} + (2\sqrt{e}-2)\left(\frac{6e^3+10e^{-3}}{144\sqrt{\pi}} + \frac{3e^3+5e^{-3}}{144\sqrt{\pi}\Gamma(2)}\right) \\
&\approx 0.92473323802 < 1.
\end{aligned}$$

Then the problem (3.45)–(3.47) has at least one solution in $PC^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}([1,e])$. *Also, hypothesis* **($Ax_6$)** *is satisfied with* $\tau = 1$, $\vartheta(t) = e^3$ *and* $\lambda_\vartheta = 3$. *Indeed, for each* $t \in (1,e]$, *we get*

$$({}^{\frac{1}{2}}\mathcal{J}^{\frac{1}{2}}_{1^+}\vartheta)(t) \leq \frac{2e^3}{\Gamma(\frac{3}{2})} \leq \lambda_\vartheta \vartheta(t).$$

Let the function $\chi : [1,e] \longrightarrow [0,\infty)$ *defined by :*

$$\chi(t) = \frac{(3e^{-3}t^3 + 5e^{-6})}{144e^{-t+e}},$$

then, for each $t \in (1,e]$, *we have*

$$p(t) = \chi(t)\vartheta(t),$$

with $\chi^* = p^* e^{-3}$. *Hence, the condition* **(Ax_7)** *is satisfied. Consequently, Theorem 3.18 implies that equation (3.45) is U-H-R stable.*

Chapter 4

Nonlinear Implicit Generalized Hilfer-Type Fractional Differential Equations with Non-Instantaneous Impulses

4.1 Introduction and Motivations

In the present chapter, we prove some results concerning the existence of solutions for a class of initial value problems for nonlinear implicit fractional differential equations with non-instantaneous impulses and generalized Hilfer fractional derivative. The results are based on Banach contraction principle and Schaefer's fixed point theorem. Then, we study the same problem in Banach spaces with results based on fixed point theorems of Darbo and Mönch associated with the technique of measure of noncompactness. Examples are included to show the applicability of our results. There are numerous books and articles focused on linear and nonlinear problems for fractional differential equations involving different kinds of fractional derivatives. One can refer to [7, 10–14, 25, 27, 38, 45–47] for instance and references therein. The class of problems for fractional differential equations with abrupt and instantaneous impulses is vastly studied, and different topics on the existence and qualitative properties of solutions are considered, see the papers [45,57,106]. Fractional differential equations with not instantaneous impulses have been developed in the last years; see the books [7, 102], the papers [1–6, 8, 15, 24, 34, 63, 71, 91, 92, 107, 109], and the references therein.

4.2 Initial Value Problem for Nonlinear Implicit Generalized Hilfer-Type Fractional Differential Equations [1]

In this section, we establish existence results to the initial value problem with nonlinear implicit generalized Hilfer-type fractional differential equation with non-instantaneous impulses:

$$\left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}u\right)(t) = f\left(t, u(t), \left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}u\right)(t)\right); \ t \in I_k, \ k = 0, \dots, m, \tag{4.1}$$

$$u(t) = g_k(t, u(t)); \ t \in \tilde{I}_k, \ k = 1, \dots, m, \tag{4.2}$$

$$\left({}^{\rho}\mathcal{J}^{1-\gamma}_{a^+}u\right)(a^+) = \phi_0, \tag{4.3}$$

where ${}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}$, ${}^{\rho}\mathcal{J}^{1-\gamma}_{a^+}$ are the generalized Hilfer fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$ and generalized fractional integral of order $1-\gamma, (\gamma = \alpha + \beta - \alpha\beta)$ respectively, $\phi_0 \in \mathbb{R}$, $I_k := (s_k, t_{k+1}]; k = 0, \dots, m$, $\tilde{I}_k := (t_k, s_k]; k = 1, \dots, m$, $a = t_0 = s_0 < t_1 \leq s_1 < t_2 \leq s_2 < \dots \leq s_{m-1} < t_m \leq s_m < t_{m+1} = b < \infty$, $u(t_k^+) = \lim\limits_{\epsilon \to 0^+} u(t_k + \epsilon)$ and $u(t_k^-) = \lim\limits_{\epsilon \to 0^-} u(t_k + \epsilon)$ represent the right and left hand limits of $u(t)$ at $t = t_k$, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function and $g_k : \tilde{I}_k \times \mathbb{R} \to \mathbb{R}$; $k = 1, \dots, m$ are given continuous functions such that $\left({}^{\rho}\mathcal{J}^{1-\gamma}_{s_k^+}g_k\right)(t, u(t))\,|_{t=s_k} = \phi_k \in \mathbb{R}$.

4.2.1 Existence Results

Consider the weighted Banach space

$$C_{\gamma,\rho}(I_k) = \left\{ u : I_k \to \mathbb{R} : t \to \left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} u(t) \in C([s_k, t_{k+1}], \mathbb{R}) \right\},$$

where $0 \leq \gamma < 1$, $k = 0, \dots, m$, and

$$C^n_{\gamma,\rho}(I_k) = \left\{ u \in C^{n-1}(I_k) : u^{(n)} \in C_{\gamma,\rho}(I_k) \right\}, n \in \mathbb{N},$$
$$C^0_{\gamma,\rho}(I_k) = C_{\gamma,\rho}(I_k).$$

Also consider the Banach space

$$PC_{\gamma,\rho}(J) = \Bigg\{ u : J \to \mathbb{R} : u \in C_{\gamma,\rho}(I_k); k = 0, \dots, m, \text{ and } u \in C(\tilde{I}_k, \mathbb{R}); k = 1, \dots, m,$$
$$\text{and there exist } u(t_k^-), u(t_k^+), u(s_k^-), \text{ and } u(s_k^+) \text{ with } u(t_k^-) = u(t_k) \Bigg\},$$

and

$$PC^n_{\gamma,\rho}(J) = \left\{ u \in PC^{n-1}(J) : u^{(n)} \in PC_{\gamma,\rho}(J) \right\}, n \in \mathbb{N},$$
$$PC^0_{\gamma,\rho}(J) = PC_{\gamma,\rho}(J).$$

1. **A. Salim**, M. Benchohra, J. E. Lazreg and G. N'Guérékata, Nonlinear implicit generalized Hilfer-type fractional differential equations with non-instantaneous impulses. (**submitted**).

with the norm

$$\|u\|_{PC_{\gamma,\rho}} = \max\left\{ \max_{k=0,\dots,m} \left\{ \sup_{t\in[s_k,t_{k+1}]} \left| \left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} u(t) \right| \right\}, \max_{k=1,\dots,m} \left\{ \sup_{t\in[t_k,s_k]} |u(t)| \right\} \right\},$$

We define the space,

$$PC^{\gamma}_{\gamma,\rho}(J) = \Big\{ u : J \to \mathbb{R} : u \in C^{\gamma}_{\gamma,\rho}(I_k); k = 0,\dots,m, \text{ and } u \in C(\tilde{I}_k, \mathbb{R}); k = 1,\dots,m,$$

$$\text{and there exist } u(t_k^-), u(t_k^+), u(s_k^-), \text{ and } u(s_k^+) \text{ with } u(t_k^-) = u(t_k) \Big\}.$$

We consider the following linear fractional differential equation

$$\left({}^\rho\mathcal{D}^{\alpha,\beta}_{s_k^+} u\right)(t) = \psi(t),\ t \in I_k,\ k = 0,\dots,m, \tag{4.4}$$

where $0 < \alpha < 1, 0 \le \beta \le 1, \rho > 0$, with the conditions

$$u(t) = g_k(t, u(t));\ t \in \tilde{I}_k,\ k = 1,\dots,m, \tag{4.5}$$

and

$$\left({}^\rho\mathcal{J}^{1-\gamma}_{a^+} u\right)(a^+) = \phi_0, \tag{4.6}$$

where $\gamma = \alpha + \beta - \alpha\beta$, $\phi_0 \in \mathbb{R}$ and $\phi^* = max\{|\phi_k| : k = 0,\dots,m\}$. The following theorem shows that the problem (4.4)–(4.6) has a unique solution given by

$$u(t) = \begin{cases} \dfrac{\phi_k}{\Gamma(\gamma)}\left(\dfrac{t^\rho - s_k^\rho}{\rho}\right)^{\gamma-1} + \left({}^\rho\mathcal{J}^{\alpha}_{s_k^+}\psi\right)(t) & if\ t \in I_k,\ k = 0,\dots,m, \\ u(t) = g_k(t, u(t)) \quad if\ t \in \tilde{I}_k,\ k = 1,\dots,m. \end{cases} \tag{4.7}$$

Theorem 4.1. *Let $\gamma = \alpha + \beta - \alpha\beta$, where $0 < \alpha < 1$ and $0 \le \beta \le 1$. If $\psi : I_k \to \mathbb{R}; k = 0,\dots,m$, is a function such that $\psi(\cdot) \in C_{\gamma,\rho}(I_k)$, then $u \in PC^{\gamma}_{\gamma,\rho}(J)$ satisfies the problem (4.4)–(4.6) if and only if it satisfies (4.7).*

Proof: Assume u satisfies (4.4)–(4.6). If $t \in I_0$, then

$$\left({}^\rho\mathcal{D}^{\alpha,\beta}_{a^+} u\right)(t) = \psi(t),$$

Lemma 1.21 implies we have the solution can be written as

$$u(t) = \frac{\left({}^\rho\mathcal{J}^{1-\gamma}_{a^+} u\right)(a)}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\psi(s)ds.$$

If $t \in \tilde{I}_1$, then we have $u(t) = g_1(t, u(t))$.
If $t \in I_1$, then Lemma 1.21 implies

$$\begin{aligned} u(t) &= \frac{\left({}^\rho\mathcal{J}^{1-\gamma}_{s_1^+} u\right)(s_1)}{\Gamma(\gamma)}\left(\frac{t^\rho - s_1^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_{s_1}^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\psi(s)ds \\ &= \frac{\phi_1}{\Gamma(\gamma)}\left(\frac{t^\rho - s_1^\rho}{\rho}\right)^{\gamma-1} + \left({}^\rho\mathcal{J}^{\alpha}_{s_1^+}\psi\right)(t). \end{aligned}$$

If $t \in \tilde{I}_2$, then we have $u(t) = g_2(t, u(t))$.
If $t \in I_2$, then Lemma 1.21 implies

$$\begin{aligned} u(t) &= \frac{\left({}^\rho \mathcal{J}_{s_2^+}^{1-\gamma} u\right)(s_2)}{\Gamma(\gamma)} \left(\frac{t^\rho - s_2^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_{s_2}^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\psi(s)ds \\ &= \frac{\phi_2}{\Gamma(\gamma)} \left(\frac{t^\rho - s_2^\rho}{\rho}\right)^{\gamma-1} + \left({}^\rho \mathcal{J}_{s_2^+}^{\alpha} \psi\right)(t). \end{aligned}$$

Repeating the process in this way, the solution $u(t)$ for $t \in J$, can be written as

$$u(t) = \begin{cases} \dfrac{\phi_k}{\Gamma(\gamma)} \left(\dfrac{t^\rho - s_k^\rho}{\rho}\right)^{\gamma-1} + \left({}^\rho \mathcal{J}_{s_k^+}^{\alpha} \psi\right)(t) & if\ t \in I_k, k = 0, \dots, m, \\ u(t) = g_k(t, u(t)) \quad if\ t \in \tilde{I}_k,\ k = 1, \dots, m. \end{cases}$$

Reciprocally, for $t \in I_0$, applying ${}^\rho \mathcal{J}_{a^+}^{1-\gamma}$ on both sides of (4.7) and using Lemma 1.10 and Theorem 1.9, we get

$$\left({}^\rho \mathcal{J}_{a^+}^{1-\gamma} u\right)(t) = \phi_0 + \left({}^\rho \mathcal{J}_{a^+}^{1-\gamma+\alpha} \psi\right)(t). \tag{4.8}$$

Next, taking the limit $t \to a^+$ of (4.8) and using Lemma 1.13, with $1-\gamma < 1-\gamma+\alpha$, we obtain

$$\left({}^\rho \mathcal{J}_{a^+}^{1-\gamma} u\right)(a^+) = \phi_0. \tag{4.9}$$

which shows that the initial condition $\left({}^\rho \mathcal{J}_{a^+}^{1-\gamma} u\right)(a^+) = \phi_0$, is satisfied. Next, for $t \in I_k; k = 0, \dots, m$, apply operator ${}^\rho \mathcal{D}_{s_k^+}^{\gamma}$ on both sides of (4.7). Then, from Lemma 1.10 and Lemma 1.16 we obtain

$$({}^\rho \mathcal{D}_{s_k^+}^{\gamma} u)(t) = \left({}^\rho \mathcal{D}_{s_k^+}^{\beta(1-\alpha)} \psi\right)(t). \tag{4.10}$$

Since $u \in C_{\gamma,\rho}^{\gamma}(I_k)$ and by definition of $C_{\gamma,\rho}^{\gamma}(I_k)$, we have ${}^\rho \mathcal{D}_{s_k^+}^{\gamma} u \in C_{\gamma,\rho}(I_k)$, then (4.10) implies that

$$({}^\rho \mathcal{D}_{s_k^+}^{\gamma} u)(t) = \left(\delta_\rho\ {}^\rho \mathcal{J}_{s_k^+}^{1-\beta(1-\alpha)} \psi\right)(t) = \left({}^\rho \mathcal{D}_{s_k^+}^{\beta(1-\alpha)} \psi\right)(t) \in C_{\gamma,\rho}(I_k). \tag{4.11}$$

As $\psi(\cdot) \in C_{\gamma,\rho}(I_k)$ and from Lemma 1.12, follows

$$\left({}^\rho \mathcal{J}_{s_k^+}^{1-\beta(1-\alpha)} \psi\right) \in C_{\gamma,\rho}(I_k), k = 0, \dots, m. \tag{4.12}$$

From (4.11), (4.12) and by the definition of the space $C_{\gamma,\rho}^{n}(I_k)$, we obtain

$$\left({}^\rho \mathcal{J}_{s_k^+}^{1-\beta(1-\alpha)} \psi\right) \in C_{\gamma,\rho}^{1}(I_k), k = 0, \dots, m.$$

Applying operator ${}^\rho \mathcal{J}_{s_k^+}^{\beta(1-\alpha)}$ on both sides of (4.10) and using Lemma 1.15, Lemma 1.13 and Property 1.11, we have

$$\begin{aligned} \left({}^\rho \mathcal{D}_{s_k^+}^{\alpha,\beta} u\right)(t) = {}^\rho \mathcal{J}_{s_k^+}^{\beta(1-\alpha)} \left({}^\rho \mathcal{D}_{s_k^+}^{\gamma} u\right)(t) &= \psi(t) - \frac{\left({}^\rho \mathcal{J}_{s_k^+}^{1-\beta(1-\alpha)} \psi\right)(s_k)}{\Gamma(\beta(1-\alpha))} \left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\beta(1-\alpha)-1} \\ &= \psi(t), \end{aligned}$$

that is, (4.4) holds.
Also, we can easily show that

$$u(t) = g_k(t, u(t_k^-)); \ t \in \tilde{I}_k, \ k = 1, \dots, m.$$

This completes the proof. ■

As a consequence of Theorem 4.1, we have the following result

Lemma 4.2. *Let $\gamma = \alpha + \beta - \alpha\beta$ where $0 < \alpha < 1$ and $0 \leq \beta \leq 1$, and $k = 0, \dots, m$, let $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, be a function such that $f(\cdot, u(\cdot), w(\cdot)) \in C_{\gamma,\rho}(I_k)$, for any $u, w \in PC_{\gamma,\rho}(J)$. If $u \in PC^{\gamma}_{\gamma,\rho}(J)$, then u satisfies the problem (4.1)-(4.3) if and only if u is the fixed point of the operator $\Psi : PC_{\gamma,\rho}(J) \to PC_{\gamma,\rho}(J)$ defined by*

$$\Psi u(t) = \begin{cases} \dfrac{\phi_k}{\Gamma(\gamma)} \left(\dfrac{t^\rho - s_k^\rho}{\rho} \right)^{\gamma - 1} + \left({}^\rho \mathcal{J}^\alpha_{s_k^+} h \right)(t) & if \ t \in I_k, \ k = 0, \dots, m, \\ \\ g_k(t, u(t)) \qquad if \ t \in \tilde{I}_k, \ k = 1, \dots, m. \end{cases} \tag{4.13}$$

where $h \in C_{\gamma,\rho}(I_k)$, $k = 0, \dots, m$ be a function satisfying the functional equation

$$h(t) = f(t, u(t), h(t)).$$

Also, by Lemma 1.12, $\Psi u \in PC_{\gamma,\rho}(J)$.

The following hypotheses will be used in the sequel :

(Ax_1) The function $f : I_k \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous on $I_k; k = 0, \dots, m$, and

$$f(\cdot, u(\cdot), w(\cdot)) \in C^{\beta(1-\alpha)}_{\gamma,\rho}(I_k), k = 0, \dots, m, \quad \text{for any } u, w \in PC_{\gamma,\rho}(J).$$

(Ax_2) There exist constants $\mathfrak{M}_1 > 0$ and $0 < \mathfrak{M}_2 < 1$ such that

$$|f(t, u, w) - f(t, \bar{u}, \bar{w})| \leq \mathfrak{M}_1 |u - \bar{u}| + \mathfrak{M}_2 |w - \bar{w}|$$

for any $u, w, \bar{u}, \bar{w} \in \mathbb{R}$ and $t \in I_k$, $k = 0, \dots, m$.

(Ax_3) The functions g_k are continuous and there exists a constant $l^* > 0$ such that $|g_k(u) - g_k(\bar{u})| \leq l^* |u - \bar{u}|$ for any $u, \bar{u} \in \mathbb{R}$ and $k = 1, \dots, m$.

(Ax_4) There exist functions $p_1, p_2, p_3 \in C([a, b], \mathbb{R}_+)$ such that

$$|f(t, u, w)| \leq p_1(t) + p_2(t)|u| + p_3(t)|w| \ \text{ for } t \in I_k, k = 0, \dots, m, \text{ and } u, w \in \mathbb{R}.$$

(Ax_5) The functions g_k are continuous and there exist constants $\Phi_1, \Phi_2 > 0$ such that

$$|g_k(t, u)| \leq \Phi_1 |u| + \Phi_2 \ \text{ for each } u \in \mathbb{R}, t \in J'_k, \quad k = 1, \dots, m,$$

We are now in a position to state and prove our existence result for the problem (4.1)-(4.3) based on based on Banach's fixed point theorem. Set $\Upsilon = \frac{K}{1-M}$.

Theorem 4.3. *Assume* **(Ax_1)** – **(Ax_3)** *hold. If*

$$L \quad := \quad l^* + \frac{\Upsilon\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^\alpha < 1, \tag{4.14}$$

then the problem (4.1)-(4.3) has a unique solution in $PC_{\gamma,\rho}(J)$.

Proof: The proof will be given in two steps.
Step 1: We show that the operator Ψ defined in (4.13) has a unique fixed point u^* in $PC_{\gamma,\rho}(J)$. Let $u, w \in PC_{\gamma,\rho}(J)$ and $t \in J$.
For $t \in I_k, k = 0, \ldots, m$, we have

$$|\Psi u(t) - \Psi w(t)| \leq \left({}^\rho\mathcal{J}^\alpha_{s_k^+}|h(s) - g(s)|\right)(t),$$

where $h, g \in C_{\gamma,\rho}(I_k); k = 0, \ldots, m$, such that

$$\begin{aligned} h(t) &= f(t, u(t), h(t)), \\ g(t) &= f(t, w(t), g(t)). \end{aligned}$$

By **(Ax_2)**, we have

$$\begin{aligned} |h(t) - g(t)| &= |f(t, u(t), h(t)) - f(t, w(t), g(t))| \\ &\leq \mathfrak{M}_1|u(t) - w(t)| + \mathfrak{M}_2|h(t) - g(t)|. \end{aligned}$$

Then,

$$|h(t) - g(t)| \leq \Upsilon|u(t) - w(t)|.$$

Therefore, for each $t \in I_k, k = 0, \ldots, m$,

$$|\Psi u(t) - \Psi w(t)| \leq \Upsilon\left({}^\rho\mathcal{J}^\alpha_{s_k^+}|u(s) - w(s)|\right)(t).$$

Thus

$$|\Psi u(t) - \Psi w(t)| \leq \left[\Upsilon\left({}^\rho\mathcal{J}^\alpha_{s_k^+}\left(\frac{s^\rho - s_k^\rho}{\rho}\right)^{\gamma-1}\right)(t)\right]\|u - w\|_{PC_{\gamma,\rho}}.$$

By Lemma 1.10, we have

$$|\Psi u(t) - \Psi w(t)| \leq \left[\frac{\Upsilon\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\alpha+\gamma-1}\right]\|u - w\|_{PC_{\gamma,\rho}}.$$

Hence

$$\begin{aligned} \left|(\Psi u(t) - \Psi w(t))\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\right| &\leq \left[\frac{\Upsilon\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\alpha}\right]\|u - w\|_{PC_{\gamma,\rho}} \\ &\leq \left[l^* + \frac{\Upsilon\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha}\right]\|u - w\|_{PC_{\gamma,\rho}}. \end{aligned}$$

For $t \in \tilde{I}_k, k = 1, \ldots, m$, we have

$$\begin{aligned} |\Psi u(t) - \Psi w(t)| &\leq |(g_k(t, u(t)) - g_k(t, w(t)))| \\ &\leq l^*\|u - w\|_{PC_{\gamma,\rho}} \\ &\leq \left[l^* + \frac{\Upsilon\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha}\right]\|u - w\|_{PC_{\gamma,\rho}}. \end{aligned}$$

Then, for each $t \in J$, we have

$$\|\Psi u - \Psi w\|_{PC_{\gamma,\rho}} \leq \left[l^* + \frac{\Upsilon\Gamma(\gamma)}{\Gamma(\gamma+\alpha)} \left(\frac{b^\rho - a^\rho}{\rho} \right)^\alpha \right] \|u - w\|_{PC_{\gamma,\rho}}.$$

By (4.14), the operator Ψ is a contraction. Hence, by Theorem 1.26, Ψ has a unique fixed point $u^* \in PC_{\gamma,\rho}(J)$.

Step 2: We show that such a fixed point $u^* \in PC_{\gamma,\rho}(J)$ is actually in $PC^\gamma_{\gamma,\rho}(J)$. Since u^* is the unique fixed point of operator Ψ in $PC_{\gamma,\rho}(J)$, then for each $t \in J$, we have

$$\Psi u^*(t) = \begin{cases} \dfrac{\phi_k}{\Gamma(\gamma)} \left(\dfrac{t^\rho - s_k^\rho}{\rho} \right)^{\gamma-1} + \left({}^\rho\mathcal{J}^\alpha_{s_k^+} h \right)(t) & if\ t \in I_k, k = 0, \ldots, m, \\ g_k(t, u^*(t)) & if\ t \in \tilde{I}_k, k = 1, \ldots, m. \end{cases}$$

where $h \in C_{\gamma,\rho}(I_k); k = 0, \ldots, m$, such that

$$h(t) = f(t, u^*(t), h(t)).$$

Applying ${}^\rho\mathcal{D}^\gamma_{s_k^+}$ to both sides and by Lemma 1.10 and Lemma 1.16, we have

$$\begin{aligned} {}^\rho\mathcal{D}^\gamma_{s_k^+} u^*(t) &= \left({}^\rho\mathcal{D}^\gamma_{s_k^+} \, {}^\rho\mathcal{J}^\alpha_{s_k^+} f(s, u^*(s), h(s)) \right)(t) \\ &= \left({}^\rho\mathcal{D}^{\beta(1-\alpha)}_{s_k^+} f(s, u^*(s), h(s)) \right)(t). \end{aligned}$$

Since $\gamma \geq \alpha$, by **(*Ax₁*)**, the right hand side is in $C_{\gamma,\rho}(I_k)$ and thus ${}^\rho\mathcal{D}^\gamma_{s_k^+} u^* \in C_{\gamma,\rho}(I_k)$. And since $g_k \in C(\tilde{I}_k, \mathbb{R}); k = 1, \ldots, m$, then $u^* \in PC^\gamma_{\gamma,\rho}(J)$. As a consequence of Steps 1 and 2 together with Theorem 4.3, we can conclude that the problem (4.1)-(4.3) has a unique solution in $PC_{\gamma,\rho}(J)$. ■

Our second result is based on Schaefer's fixed point theorem. Set

$$p_1^* = \sup_{t\in[a,b]} p_1(t), \; p_2^* = \sup_{t\in[a,b]} p_2(t), \; p_3^* = \sup_{t\in[a,b]} p_3(t) < 1.$$

Theorem 4.4. *Assume* **(*Ax₁*)**, **(*Ax₄*)** *and* **(*Ax₅*)** *hold. If*

$$max \left\{ \Phi_1, \left(\frac{p_2^*\Gamma(\gamma)}{(1-p_3^*)\Gamma(\gamma+\alpha)} \right) \left(\frac{b^\rho - a^\rho}{\rho} \right)^\alpha \right\} < 1, \tag{4.15}$$

then the problem (4.1)-(4.3) has at least one solution in $PC_{\gamma,\rho}(J)$.

Proof: We shall use Schaefer's fixed point theorem to prove in several steps that the operator Ψ defined in (4.13) has a fixed point.
Step 1: Ψ is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $PC_{\gamma,\rho}(J)$. Then for each $t \in I_k, k = 0, \ldots, m$, we have,

$$\left| ((\Psi u_n)(t) - (\Psi u)(t)) \left(\frac{t^\rho - s_k^\rho}{\rho} \right)^{1-\gamma} \right| \leq \left(\frac{t^\rho - s_k^\rho}{\rho} \right)^{1-\gamma} \left({}^\rho\mathcal{J}^\alpha_{s_k^+} |h_n(s) - h(s)| \right)(t),$$

where $h_n, h \in C_{\gamma,\rho}(I_k)$, such that

$$h_n(t) = f(t, u_n(t), h_n(t)),$$
$$h(t) = f(t, u(t), h(t)).$$

For each $t \in \tilde{I}_k, k = 1, \ldots, m$, we have,

$$|\Psi u_n(t) - \Psi u(t)| \leq |(g_k(t, u_n(t)) - g_k(t, u(t)))|.$$

Since $u_n \to u$, then we get $h_n(t) \to h(t)$ as $n \to \infty$ for each $t \in J$, and since f and g_k are continuous, then we have

$$\|\Psi u_n - \Psi u\|_{PC_{\gamma,\rho}} \to 0 \text{ as } n \to \infty.$$

Step 2: We show that Ψ is the mapping of two bounded sets in $PC_{\gamma,\rho}(J)$.
For $\eta > 0$, there exists a positive constant r such that $B_\eta = \{u \in PC_{\gamma,\rho}(J) : \|u\|_{PC_{\gamma,\rho}} \leq \eta\}$, we have $\|\Psi(u)\|_{PC_{\gamma,\rho}} \leq r$.
By **(Ax_4)** and from (4.13), We have for each $t \in I_k, k = 0, \ldots, m$,

$$\begin{aligned}\left|\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} h(t)\right| &= \left|\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} f(t, u(t), h(t))\right| \\ &\leq \left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} \left(p_1(t) + p_2(t)|u(t)| + p_3(t)|h(t)|\right),\end{aligned}$$

which implies that

$$\left|\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} h(t)\right| \leq p_1^* \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma} + p_2^*\eta + p_3^* \left|\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} h(t)\right|.$$

Then

$$\left|\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} h(t)\right| \leq \frac{p_1^* \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma} + p_2^*\eta}{1 - p_3^*} := \Lambda.$$

Thus, for $t \in I_k, k = 0, \ldots, m$, (4.13) implies

$$\left|\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} (\Psi u)(t)\right| \leq \frac{|\phi_k|}{\Gamma(\gamma)} + \left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} \left({}^\rho\mathcal{J}_{s_k^+}^\alpha |h(s)|\right)(t).$$

By Lemma 1.10, for $t \in I_k, k = 0, \ldots, m$, we have

$$\begin{aligned}\left|\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} (\Psi u)(t)\right| &\leq \frac{\phi^*}{\Gamma(\gamma)} + \Lambda \left(\frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\right) \left(\frac{b^\rho - a^\rho}{\rho}\right)^\alpha \\ &:= r_1.\end{aligned}$$

And for each $t \in \tilde{I}_k, k = 1, \ldots, m$, we have,

$$|\Psi u(t)|_{PC_{\gamma,\rho}} \leq |g_k(t, u(t))| \leq \Phi_1\eta + \Phi_2 := r_2.$$

Thus, for each $t \in J$ we have,

$$\|\Psi u\|_{PC_{\gamma,\rho}} \leq max\{r_1, r_2\} := r.$$

Step 3: Ψ maps bounded sets into equicontinuous sets of $PC_{\gamma,\rho}(J)$.
Let $\epsilon_1, \epsilon_2 \in J$, $\epsilon_1 < \epsilon_2$, B_η be a bounded set of $PC_{\gamma,\rho}(J)$ as in Step 2, and let $u \in B_\eta$. Then for each $t \in I_k, k = 0, \dots, m$, and by Lemma 1.10, we have

$$\begin{aligned}
&\left|\left(\frac{\epsilon_1^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}(\Psi u)(\epsilon_1) - \left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}(\Psi u)(\epsilon_2)\right| \\
&\leq \left|\left(\frac{\epsilon_1^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\left({}^\rho\mathcal{J}^\alpha_{s_k^+}h(\tau)\right)(\epsilon_1) - \left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\left({}^\rho\mathcal{J}^\alpha_{s_k^+}h(\tau)\right)(\epsilon_2)\right| \\
&\leq \left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\left({}^\rho\mathcal{J}^\alpha_{\epsilon_1^+}|h(\tau)|\right)(\epsilon_2) + \frac{1}{\Gamma(\alpha)}\int_{s_k}^{\epsilon_1}\left|\tau^{\rho-1}H(\tau)h(\tau)\right|d\tau,
\end{aligned}$$

where $H(\tau) = \left[\left(\frac{\epsilon_1^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\left(\frac{\epsilon_1^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} - \left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\left(\frac{\epsilon_2^\rho - \tau^\rho}{\rho}\right)^{\alpha-1}\right]$.
Then by Lemma 1.10, we have

$$\begin{aligned}
&\left|\left(\frac{\epsilon_1^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}(\Psi u)(\epsilon_1) - \left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}(\Psi u)(\epsilon_2)\right| \\
\leq& \frac{\Lambda\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\left(\frac{\epsilon_2^\rho - \epsilon_1^\rho}{\rho}\right)^{\alpha+\gamma-1} + \Lambda\int_{s_k}^{\epsilon_1}\left|H(\tau)\frac{\tau^{\rho-1}}{\Gamma(\alpha)}\right|\left(\frac{\tau^\rho - s_k^\rho}{\rho}\right)^{\gamma-1}d\tau.
\end{aligned}$$

And for each $t \in \tilde{I}_k, k = 1, \dots, m$, we have

$$|(\Psi u)(\epsilon_1) - (\Psi u)(\epsilon_2)| \leq |g_k(\epsilon_1, u(\epsilon_1)) - g_k(\epsilon_2, u(\epsilon_2))|.$$

As $\epsilon_1 \to \epsilon_2$, the right-hand side of the above inequality tends to zero. From step 1 to 3 with Arzela-Ascoli theorem, we conclude that $\Psi : PC_{\gamma,\rho} \to PC_{\gamma,\rho}$ is continuous and completely continuous.
Step 4: A priori bound. Now it remains to show that the set

$$G = \{u \in PC_{\gamma,\rho} : u = \lambda^*\Psi(u) \text{ for some } 0 < \lambda^* < 1\}$$

is bounded. Let $u \in G$, then $u = \lambda^*\Psi(u)$ for some $0 < \lambda^* < 1$.
By ***(Ax₄)***, we have for each $t \in I_k, k = 0, \dots, m$,

$$\begin{aligned}
\left|\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}h(t)\right| &= \left|\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}f(t, u(t), h(t))\right| \\
&\leq \left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\left(p_1(t) + p_2(t)|u(t)| + p_3(t)|h(t)|\right),
\end{aligned}$$

which implies that

$$\left|\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}h(t)\right| \leq p_1^*\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma} + p_2^*\|u\|_{PC_{\gamma,\rho}} + p_3^*\left|\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}h(t)\right|,$$

then

$$\left|\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} h(t)\right| \leq \frac{p_1^*\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma} + p_2^*\|u\|_{PC_{\gamma,\rho}}}{1-p_3^*}.$$

This implies, by (4.13), $(\boldsymbol{Ax_5})$ and by letting the estimation of Step 2, that for each $t \in I_k, k = 0, \ldots, m$, we have

$$\left|\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} u(t)\right| \leq \frac{|\phi_k|}{\Gamma(\gamma)} + \frac{p_1^*\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma} + p_2^*\|u\|_{PC_{\gamma,\rho}}}{1-p_3^*}\left(\frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\right)\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha},$$

thus

$$\begin{aligned}\left|\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} u(t)\right| \leq\ & \frac{\phi^*}{\Gamma(\gamma)} + \left(\frac{p_1^*\Gamma(\gamma)}{(1-p_3^*)\Gamma(\gamma+\alpha)}\right)\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha} \\ & + \left(\frac{p_2^*\Gamma(\gamma)}{(1-p_3^*)\Gamma(\gamma+\alpha)}\right)\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha}\|u\|_{PC_{\gamma,\rho}}.\end{aligned}$$

And for each $t \in \tilde{I}_k, k = 1, \ldots, m$, we have,

$$|u(t)| \leq |g_k(t, u(t))| \leq \Phi_1\|u\|_{PC_{\gamma,\rho}} + \Phi_2,$$

Then, for each $t \in J$ we have,

$$\|u\|_{PC_{\gamma,\rho}} \leq \chi_1 + \chi_2\|u\|_{PC_{\gamma,\rho}},$$

where

$$\chi_1 = \max\left\{\Phi_2, \frac{\phi^*}{\Gamma(\gamma)} + \left(\frac{p_1^*\Gamma(\gamma)}{(1-p_3^*)\Gamma(\gamma+\alpha)}\right)\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha}\right\},$$

and

$$\chi_2 = \max\left\{\Phi_1, \left(\frac{p_2^*\Gamma(\gamma)}{(1-p_3^*)\Gamma(\gamma+\alpha)}\right)\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha}\right\}.$$

Then by (4.15), we have

$$\|u\|_{PC_{\gamma,\rho}} \leq \frac{\chi_1}{1-\chi_2} := R.$$

As consequence of Theorem 1.27, and using Step 2 of the last result, we deduce that Ψ has a fixed point which is a solution of the problem (4.1)-(4.3). ■

4.2.2 Nonlocal Impulsive Differential Equations

This section is concerned with a generalization of the results presented in the previous section to nonlocal impulsive fractional differential equations. More precisely we shall present some existence results for the following nonlocal problem

$$\left({}^\rho\mathcal{D}_{s_k^+}^{\alpha,\beta} u\right)(t) = f\left(t, u(t), \left({}^\rho\mathcal{D}_{s_k^+}^{\alpha,\beta} u\right)(t)\right);\ t \in I_k,\ k = 0, \ldots, m, \tag{4.16}$$

$$u(t) = g_k(t, u(t));\ t \in \tilde{I}_k,\ k = 1, \ldots, m, \tag{4.17}$$

$$\left({}^{\rho}\mathcal{J}_{a^+}^{1-\gamma}u\right)(a^+) + \xi(u) = \phi_0, \tag{4.18}$$

where ${}^{\rho}\mathcal{D}_{s_k^+}^{\alpha,\beta}, {}^{\rho}\mathcal{J}_{a^+}^{1-\gamma}$ are the generalized Hilfer fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$ and generalized Hilfer fractional integral of order $1-\gamma, (\gamma = \alpha + \beta - \alpha\beta)$ respectively, $\phi_0 \in \mathbb{R}$ and I_k, $\tilde{I}_k$, f, g_k are as in Section 3, $\xi : PC_{\gamma,\rho}(J) \mapsto \mathbb{R}$ is a continuous function. Nonlocal conditions were initiated by Byszewski [52] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. The nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena. The following hypothesis will be used in the sequel.

(Ax6) There exist constants $K^* > 0$ such that

$$|\xi(u) - \xi(\bar{u})| \leq K^*|u(t) - \bar{u}(t)|$$

for any $u, \bar{u} \in PC_{\gamma,\rho}(J)$.

Theorem 4.5. *Assume* ***(Ax1)*** *–* ***(Ax3)***, ***(Ax6)*** *hold. If*

$$l^* + K^* + \frac{\Upsilon\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^\alpha < 1, \tag{4.19}$$

then the nonlocal problem (4.16)–(4.18) has a unique solution in $PC_{\gamma,\rho}(J)$.

Proof: We transform the problem (4.16)–(4.18) into a fixed point problem. Consider the operator $\tilde{\Psi} : PC_{\gamma,\rho}(J) \longrightarrow PC_{\gamma,\rho}(J)$ defined by

$$\tilde{\Psi}u(t) = \begin{cases} \dfrac{\phi_0 - \xi(u)}{\Gamma(\gamma)}\left(\dfrac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \left({}^{\rho}\mathcal{J}_{a^+}^{\alpha}h\right)(t) & if\ t \in I_0, \\ \dfrac{\phi_k}{\Gamma(\gamma)}\left(\dfrac{t^\rho - s_k^\rho}{\rho}\right)^{\gamma-1} + \left({}^{\rho}\mathcal{J}_{s_k^+}^{\alpha}h\right)(t) & if\ t \in I_k, k = 1,\dots,m, \\ g_k(t,u(t)) \quad if\ t \in \tilde{I}_k, k = 1,\dots,m. \end{cases} \tag{4.20}$$

where $h \in C_{\gamma,\rho}(I_k)$, $k = 0,\dots,m$ be a function satisfying the functional equation

$$h(t) = f(t, u(t), h(t)).$$

Clearly, the fixed points of the operator $\tilde{\Psi}$ are solutions of the problem (4.16)–(4.18). We can easily show that $\tilde{\Psi}$ is a contraction and its fixed points are in $PC_{\gamma,\rho}^{\gamma}(J)$. ■

Theorem 4.6. *Assume* ***(Ax1)***, ***(Ax4)*** *–* ***(Ax6)*** *hold. If*

$$max\left\{\Phi_1, \left(\frac{p_2^*\Gamma(\gamma)}{(1-p_3^*)\Gamma(\gamma+\alpha)}\right)\left(\frac{b^\rho - a^\rho}{\rho}\right)^\alpha\right\} < 1, \tag{4.21}$$

then the nonlocal problem (4.16)–(4.18) has at least one solution in $PC_{\gamma,\rho}(J)$.

4.2.3 Ulam-Hyers-Rassias stability

First, we consider the Ulam Stability for problem (4.1)–(4.3). Let $u \in PC_{\gamma,\rho}(J)$, $\epsilon > 0$, $\tau > 0$ and $\vartheta : J \longrightarrow [0, \infty)$ be a continuous function. We consider the following inequalities :

$$\begin{cases} \left| \left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+} u \right)(t) - f\left(t, u(t), \left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+} u \right)(t) \right) \right| \leq \epsilon\vartheta(t), \ t \in I_k, \ k = 0, \ldots, m, \\ |u(t) - g_k(t, u(t))| \leq \epsilon\tau, \ t \in \tilde{I}_k, \ k = 1, \ldots, m. \end{cases} \tag{4.22}$$

Definition 4.7. *([108, 110]) Problem (4.1)–(4.3) is Ulam-Hyers-Rassias (U-H-R) stable with respect to (ϑ, τ) if there exists a real number $a_{f,\vartheta} > 0$ such that for each $\epsilon > 0$ and for each solution $u \in PC_{\gamma,\rho}(J)$ of inequality (4.22) there exists a solution $w \in PC_{\gamma,\rho}(J)$ of (4.1)–(4.3) with*

$$|u(t) - w(t)| \leq \epsilon a_{f,\vartheta}(\vartheta(t) + \tau), \qquad t \in J.$$

Remark 4.8. *([108, 110]) A function $u \in PC_{\gamma,\rho}(J)$ is a solution of inequality (4.22) if and only if there exist $\sigma \in PC_{\gamma,\rho}(J)$ and a sequence $\sigma_k, k = 0, \ldots, m$ such that*

1. $|\sigma(t)| \leq \epsilon\vartheta(t)$, $t \in I_k, k = 0, \ldots, m$; *and* $|\sigma_k| \leq \epsilon\tau$, $\ t \in \tilde{I}_k, \ k = 1, \ldots, m$,
2. $\left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+} u \right)(t) = f\left(t, u(t), \left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+} u \right)(t) \right) + \sigma(t), \ \ t \in I_k, \ k = 0, \ldots, m,$
3. $u(t) = g_k(t, u(t)) + \sigma_k$, $t \in \tilde{I}_k, \ k = 1, \ldots, m$.

Theorem 4.9. *Assume that in addition to **(Ax1)** – **(Ax3)** and (4.14), the following hypothesis holds.*

(Ax7) *There exist a nondecreasing function $\vartheta : J \longrightarrow [0, \infty)$ and $\lambda_\vartheta > 0$ such that for each $t \in I_k; k = 0, \ldots, m$, we have*

$$({}^{\rho}\mathcal{J}^{\alpha}_{s_k^+}\vartheta)(t) \leq \lambda_\vartheta \vartheta(t).$$

Then problem (4.1)–(4.3) is U-H-R stable with respect to (ϑ, τ).

Proof: Consider the operator Ψ defined in (4.13). Let $u \in PC_{\gamma,\rho}(J)$ be a solution if inequality (4.22), and let us assume that w is the unique solution of the problem

$$\begin{cases} \left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+} w \right)(t) = f\left(t, w(t), \left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+} w \right)(t) \right); \ t \in I_k, \ k = 0, \ldots, m, \\ u(t) = g_k(y, w(t_k^-)); \ t \in \tilde{I}_k, \ k = 1, \ldots, m, \\ \left({}^{\rho}\mathcal{J}^{1-\gamma}_{s_k^+} w \right)(s_k^+) = \left({}^{\rho}\mathcal{J}^{1-\gamma}_{s_k^+} u \right)(s_k^+) = \phi_k, \ k = 0, \ldots, m. \end{cases}$$

By Lemma 4.2, we obtain for each $t \in (a, b]$

$$w(t) = \begin{cases} \dfrac{\phi_k}{\Gamma(\gamma)} \left(\dfrac{t^\rho - s_k^\rho}{\rho} \right)^{\gamma-1} + \left({}^{\rho}\mathcal{J}^{\alpha}_{s_k^+} h \right)(t) & if \ t \in I_k, k = 0, \ldots, m, \\ g_k(t, w(t)) & if \ t \in \tilde{I}_k, k = 1, \ldots, m, \end{cases}$$

where $h \in C_{\gamma,\rho}(I_k); k = 0, \ldots, m$, be a function satisfying the functional equation

$$h(t) = f(t, w(t), h(t)).$$

Since u is a solution of the inequality (4.22), by Remark 4.8, we have

$$\begin{cases} \left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+} u\right)(t) = f\left(t, u(t), \left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+} u\right)(t)\right) + \sigma(t), t \in I_k, k = 0, \ldots, m; \\ u(t) = g_k(t, u(t)) + \sigma_k, t \in \tilde{I}_k, k = 1, \ldots, m. \end{cases} \tag{4.23}$$

Clearly, the solution of (4.23) is given by

$$u(t) = \begin{cases} \frac{\phi_k}{\Gamma(\gamma)}\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\gamma-1} + \left({}^{\rho}\mathcal{J}^{\alpha}_{s_k^+} g\right)(t) + \left({}^{\rho}\mathcal{J}^{\alpha}_{s_k^+} \sigma\right)(t) & if\ t \in I_k, k = 1, \ldots, m, \\ g_k(t, u(t)) + \sigma_k & if\ t \in \tilde{I}_k, k = 1, \ldots, m, \end{cases}$$

where $g : I_k \to \mathbb{R}, k = 0, \ldots, m$, be a function satisfying the functional equation

$$g(t) = f(t, u(t), g(t)).$$

Hence, for each $t \in I_k$,$k = 0, \ldots, m$, we have

$$\begin{aligned} |u(t) - w(t)| &\leq \left({}^{\rho}\mathcal{J}^{\alpha}_{s_k^+}|g(s) - h(s)|\right)(t) + \left({}^{\rho}\mathcal{J}^{\alpha}_{s_k^+}|\sigma(s)|\right) \\ &\leq \epsilon\lambda_\vartheta\vartheta(t) + \Upsilon\int_{s_k}^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\frac{|u(s) - w(s)|}{\Gamma(\alpha)}ds. \end{aligned}$$

We apply Lemma 1.20 to obtain

$$\begin{aligned} |u(t) - w(t)| &\leq \epsilon\lambda_\vartheta\vartheta(t) + \int_{s_k}^t \sum_{\tau=1}^{\infty}\frac{(\Upsilon)^\tau}{\Gamma(\tau\alpha)} s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\tau\alpha-1}(\epsilon\lambda_\vartheta\vartheta(s))ds \\ &\leq \epsilon\lambda_\vartheta\vartheta(t)E_\alpha\left[\Upsilon\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^\alpha\right] \\ &\leq \epsilon\lambda_\vartheta\vartheta(t)E_\alpha\left[\Upsilon\left(\frac{b^\rho - a^\rho}{\rho}\right)^\alpha\right]. \end{aligned}$$

And for each $t \in \tilde{I}_k$,$k = 1, \ldots, m$, we have

$$\begin{aligned} |u(t) - w(t)| &\leq |g_k(t, u(t)) - g_k(t, w(t))| + |\sigma_k| \\ &\leq l^*|u(t) - w(t)| + \epsilon\tau, \end{aligned}$$

then by 4.14, we have

$$|u(t) - w(t)| \leq \frac{\epsilon\tau}{1 - l^*}.$$

Then for each $t \in J$, we have

$$|u(t) - w(t)| \leq a_\vartheta\epsilon(\tau + \vartheta(t)),$$

where

$$a_\vartheta = \frac{1}{1 - l^*} + \lambda_\vartheta E_\alpha\left[\Upsilon\left(\frac{b^\rho - a^\rho}{\rho}\right)^\alpha\right].$$

Hence, problem (4.1)–(4.3) is U-H-R stable with respect to (ϑ, τ). Now we are concerned with the Ulam-Hyers-Rassias stability of our problem (4.16)−(4.18). ■

Theorem 4.10. *Assume that* **(Ax_1)**, **(Ax_4)** – **(Ax_7)**, *and* (4.21) *hold. Then the problem* (4.16)–(4.18) *is U-H-R stable with respect to* (ϑ, τ).

4.2.4 Examples

Example 4.11. *Consider the following impulsive Cauchy problem*

$$\left({}^{\frac{1}{2}}\mathcal{D}^{\frac{1}{2},0}_{s_k^+}u\right)(t) = \frac{e^{-t}}{79e^{t+3}(1+|u(t)|+|{}^{\frac{1}{2}}\mathcal{D}^{\frac{1}{2},0}_{s_k^+}u(t)|}, \text{ for each } t \in I_0 \cup I_1, \tag{4.24}$$

$$u(t) = \frac{|u(t)|}{e^t + 2|u(t)|}, \text{ for each } t \in \tilde{I}_1, \tag{4.25}$$

$$\left({}^{\frac{1}{2}}\mathcal{J}^{\frac{1}{2}}_{1^+}u\right)(1^+) = 0, \tag{4.26}$$

where

$$I_0 = (1,2],\ I_1 = (e,3],\ \tilde{I}_1 = (2,e],\ s_0 = 1,\ t_1 = 2,\ \text{and } s_1 = e.$$

Set

$$f(t,u,w) = \frac{e^{-t}}{79e^{t+3}(1+|u|+|w|)},\ t \in I_0 \cup I_1,\ u, w \in \mathbb{R}.$$

We have

$$C^{\beta(1-\alpha)}_{\gamma,\rho}((1,2]) = C^{0}_{\frac{1}{2},\frac{1}{2}}((1,2]) = \left\{v : (1,2] \to \mathbb{R} : \sqrt{2}\left(\sqrt{t}-1\right)^{\frac{1}{2}} v \in C([1,2],\mathbb{R})\right\},$$

and

$$C^{\beta(1-\alpha)}_{\gamma,\rho}((e,3]) = C^{0}_{\frac{1}{2},\frac{1}{2}}((e,3]) = \left\{v : (e,3] \to \mathbb{R} : \sqrt{2}\left(\sqrt{t}-\sqrt{2}\right)^{\frac{1}{2}} v \in C([e,3],\mathbb{R})\right\},$$

with

$$\gamma = \alpha = \frac{1}{2}\quad \rho = \frac{1}{2},\quad \beta = 0, \text{and}\quad k \in \{0,1\}.$$

Clearly, the continuous function $f \in C^{0}_{\frac{1}{2},\frac{1}{2}}((1,2]) \cap C^{0}_{\frac{1}{2},\frac{1}{2}}((e,3])$.
Hence the condition **(Ax_1)** *is satisfied.*
For each $u, \bar{u}, w, \bar{w} \in \mathbb{R}$ *and* $t \in I_0 \cup I_1$, *we have*

$$\begin{aligned} |f(t,u,w) - f(t,\bar{u},\bar{w})| &\le \frac{e^{-t}}{79e^{t+3}}(|u-\bar{u}| + |w-\bar{w}|) \\ &\le \frac{1}{79e^5}\left(|u-\bar{u}| + |w-\bar{w}|\right). \end{aligned}$$

Hence condition **(Ax_2)** *is satisfied with* $\mathfrak{M}_1 = \mathfrak{M}_2 = \dfrac{1}{79e^5}$.
And let

$$g_1(u) = \frac{u}{e^t + 2u}, u \in [0,\infty).$$

Let $u, w \in [0,\infty)$. *Then we have*

$$|g_1(u) - g_1(w)| = |\frac{u}{e^t+2u} - \frac{w}{e^t+2w}| = \frac{e^t|u-w|}{(e^t+2u)(e^t+2w)} \le \frac{1}{e}|u-w|,$$

and so the condition **(Ax_3)** *is satisfied with* $l^* = \frac{1}{e}$.
A simple computation shows that the condition (4.14) of Theorem 4.3 is satisfied, for

$$L = \frac{1}{e} + \frac{\sqrt{2\pi}(\sqrt{3}-1)^{\frac{1}{2}}}{(79e^5-1)} \approx 0.368062377 < 1.$$

Then the problem (4.24)–(4.26) has a unique solution in $PC_{\frac{1}{2},\frac{1}{2}}([1,3])$. *Also, hypothesis* **$(Ax_7)$** *is satisfied with* $\tau = 1$ *and*

$$\vartheta(t) = \begin{cases} 2(\sqrt{t}-\sqrt{s_k}), & if\ t \in I_0 \cup I_1, \\ e, & if\ t \in \tilde{I}_1, \end{cases}$$

and $\lambda_\vartheta = \frac{\sqrt{2}\Gamma(2)(\sqrt{2}-1)^{\frac{1}{2}}}{\Gamma(\frac{5}{2})}$. *Indeed, for each* $t \in I_0 \cup I_1$, *we get*

$$({}^{\frac{1}{2}}\mathcal{J}_{1^+}^{\frac{1}{2}}\vartheta)(t) \leq \frac{\sqrt{2}\Gamma(2)(\sqrt{2}-1)^{\frac{1}{2}}}{\Gamma(\frac{5}{2})}(2\sqrt{t}-2),$$

and

$$({}^{\frac{1}{2}}\mathcal{J}_{e^+}^{\frac{1}{2}}\vartheta)(t) \leq \frac{\sqrt{2}\Gamma(2)(\sqrt{3}-\sqrt{e})^{\frac{1}{2}}}{\Gamma(\frac{5}{2})}(2\sqrt{t}-2\sqrt{e}).$$

Consequently, Theorem 4.9 implies that the problem (4.24)–(4.26) is U-H-R stable.

Example 4.12. *Consider the following impulsive nonlocal initial value problem*

$$\left({}^1\mathcal{D}_{s_k^+}^{\frac{1}{2},0}u\right)(t) = \frac{1+|u(t)|+|{}^1\mathcal{D}_{s_k^+}^{\frac{1}{2},0}u(t)|}{107e^{-t+3}(1+|u(t)|+|{}^1\mathcal{D}_{s_k^+}^{\frac{1}{2},0}u(t)|)}, \quad t \in I_k, k = 0,\dots,4, \tag{4.27}$$

$$u(t) = \frac{|u(t)|}{10e^k+|u(t)|}, \ for\ each\ t \in \tilde{I}_k, k = 1,\dots,4, \tag{4.28}$$

$$\left({}^1\mathcal{J}_{1^+}^{\frac{1}{2}}u\right)(1^+) + \frac{1}{5}\frac{u(t)}{|u(t)|+1} = 1, \tag{4.29}$$

where

$$I_k = (s_k, t_{k+1}], \quad s_k = 1 + \frac{2k}{9} \ \ for \ \ k = 0,\dots,4$$

and

$$\tilde{I}_k = (t_k, s_k], \quad t_k = 1 + \frac{2k-1}{9}\ for\ k = 1,\dots,4,\ (m = 4),$$

and

$$a = s_0 = 1, \quad b = t_5 = 2.$$

Set

$$f(t,u,w) = \frac{1+|u|+|w|}{107e^{-t+3}(1+|u|+|w|)}, \ t \in I_k, k = 0,\dots,4,\ u,w \in \mathbb{R}.$$

We have

$$\begin{aligned} C^{\beta(1-\alpha)}_{\gamma,\rho}((s_k,t_{k+1}]) &= C^{0}_{\frac{1}{2},1}((s_k,t_{k+1}]) \\ &= \left\{ v : (s_k,t_{k+1}] \to \mathbb{R} : (\sqrt{t-s_k})v \in C([s_k,t_{k+1}],\mathbb{R}) \right\}, \end{aligned}$$

with $\gamma = \alpha = \frac{1}{2}$, $\rho = 1$, $\beta = 0$ *and* $k = 0,\ldots,4$. *Clearly, the continuous function* $f \in C^{0}_{\frac{1}{2},1}([s_k,t_{k+1}]); k = 0,\ldots,4$. *Hence the condition* ***(Ax₁)*** *is satisfied.*
For each $u,w \in \mathbb{R}$ *and* $t \in I_k; k = 0,\ldots,4$, *we have*

$$|f(t,u,w)| \leq \frac{1}{107e^{-t+3}}(1+|u|+|w|).$$

Hence condition ***(Ax₄)*** *is satisfied with*

$$p_1(t) = p_2(t) = p_3(t) = \frac{1}{107e^{-t+3}},$$

and

$$p_1^* = p_2^* = p_3^* = \frac{1}{107e}.$$

Let

$$g_k(u) = \frac{u}{10e^k + u}, k = 1,\ldots,4, u \in [0,\infty),$$

then we have

$$|g_k(u)| \leq \frac{1}{10e}|u| + 1, k = 1,\ldots,4,$$

and so the condition ***(Ax₅)*** *is satisfied with* $\Phi_1 = \dfrac{1}{10e}$ *and* $\Phi_2 = 1$.
And let

$$\xi(u) = \frac{1}{5}\frac{u}{|u|+1}$$

then we have

$$|g(u)| \leq 42 \sup\{u(t_k), k = 1,\ldots,4\},$$

and so the condition ***(Ax₆)*** *is satisfied with* $\tilde{M} = 42\sup\{u(t_k), k = 1,\ldots,4\}$.
The condition (4.21) of Theorem 4.6 is satisfied, for

$$\Phi_1 = \frac{1}{10e} < 1,$$

and

$$\left(\frac{p_2^*\Gamma(\gamma)}{(1-p_3^*)\Gamma(\gamma+\alpha)}\right)\left(\frac{b^\rho - a^\rho}{\rho}\right)^\alpha = \frac{\sqrt{\pi}}{(107e-1)} < 1.$$

Then the problem (4.27)–(4.29) has at least one solution in $PC_{\frac{1}{2},1}([1,2])$. *Also, hypothesis* ***(Ax₇)*** *is satisfied with* $\tau = 1$ *and*

$$\vartheta(t) = \begin{cases} (t-s_k)^2, & if\ t \in I_k, k = 0,\ldots,4, \\ 2, & if\ t \in \tilde{I}_k, k = 1,\ldots,4, \end{cases}$$

and $\lambda_\vartheta = \dfrac{\Gamma(3)}{\Gamma(\frac{7}{2})}$. *Indeed, for each* $t \in I_k, k = 0, \ldots, 4$, *we get*

$$\begin{aligned}({}^1\mathcal{J}_{s_k^+}^{\frac{1}{2}}\vartheta)(t) &\leq \frac{\Gamma(3)}{\Gamma(\frac{7}{2})}(t-s_k)^{\frac{5}{2}} \\ &\leq \frac{\Gamma(3)}{\Gamma(\frac{7}{2})}(t-s_k)^{2} \\ &= \lambda_\vartheta\vartheta(t).\end{aligned}$$

Consequently, Theorem 4.10 implies that the problem (4.27)–(4.29) is U-H-R stable.

4.3 Initial Value Problem for Nonlinear Implicit Generalized Hilfer-Type Fractional Differential Equations in Banach Spaces[2]

Motivated by the works mentioned in the introduction, in this section, we establish existence results to the initial value problem of nonlinear implicit generalized Hilfer-type fractional differential equation with non-instantaneous impulses:

$$\left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}u\right)(t)=f\left(t,u(t),\left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}u\right)(t)\right);\ t\in I_k,\ k=0,\dots,m, \tag{4.30}$$

$$u(t)=g_k(t,u(t));\ t\in\tilde{I}_k,\ k=1,\dots,m, \tag{4.31}$$

$$\left({}^{\rho}\mathcal{J}^{1-\gamma}_{a^+}u\right)(a^+)=\phi_0, \tag{4.32}$$

where ${}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}$, ${}^{\rho}\mathcal{J}^{1-\gamma}_{a^+}$ are the generalized Hilfer-type fractional derivative of order $\alpha\in(0,1)$ and type $\beta\in[0,1]$ and generalized fractional integral of order $1-\gamma,(\gamma=\alpha+\beta-\alpha\beta)$, respectively, $\rho>0$, $\phi_0\in E$, $I_k:=(s_k,t_{k+1}];k=0,\dots,m$, $\tilde{I}_k:=(t_k,s_k];k=1,\dots,m$, $a=s_0<t_1\leq s_1<t_2\leq s_2<\dots\leq s_{m-1}<t_m\leq s_m<t_{m+1}=b<\infty$, $u(t_k^+)=\lim\limits_{\epsilon\to0^+}u(t_k+\epsilon)$ and $u(t_k^-)=\lim\limits_{\epsilon\to0^-}u(t_k+\epsilon)$ represent the right and left hand limits of $u(t)$ at $t=t_k$, $f:I_k\times E\times E\to E$ is a given function and $g_k:\tilde{I}_k\times E\to E$; $k=1,\dots,m$ are given continuous functions such that $\left({}^{\rho}\mathcal{J}^{1-\gamma}_{s_k^+}g_k\right)(t,u(t))\,|_{t=s_k}=\phi_k\in E$, where $(E,\|\cdot\|)$ is a real Banach space.

4.3.1 Existence Results

Consider the weighted Banach space

$$C_{\gamma,\rho}(I_k)=\left\{u:I_k\to E:t\to\left(\frac{t^{\rho}-s_k^{\rho}}{\rho}\right)^{1-\gamma}u(t)\in C_E([s_k,t_{k+1}])\right\},$$

where $0\leq\gamma<1$, $k=0,\dots,m$, and

$$\begin{aligned}C^n_{\gamma,\rho}(I_k)&=\left\{u\in C^{n-1}(I_k):u^{(n)}\in C_{\gamma,\rho}(I_k)\right\},n\in\mathbb{N},\\ C^0_{\gamma,\rho}(I_k)&=C_{\gamma,\rho}(I_k).\end{aligned}$$

Also consider the Banach space

$$PC_{\gamma,\rho}(J)=\Big\{u:J\to E:u\in C_{\gamma,\rho}(I_k);k=0,\dots,m,\ \text{and }u\in C_E(\tilde{I}_k);k=1,\dots,m,$$
$$\text{and there exist }u(t_k^-),u(t_k^+),u(s_k^-),\ \text{and }u(s_k^+)\text{ with }u(t_k^-)=u(t_k)\Big\},$$

2. **A. Salim**, M. Benchohra, J. E. Lazreg and J. Henderson, Nonlinear implicit generalized Hilfer-type fractional differential equations with non-instantaneous impulses in Banach spaces, *Adv. Theory Nonlinear Anal. Appl.* **4** (2020), 332-348.

and

$$PC^n_{\gamma,\rho}(J) = \left\{u \in PC^{n-1}(J) : u^{(n)} \in PC_{\gamma,\rho}(J)\right\}, n \in \mathbb{N},$$
$$PC^0_{\gamma,\rho}(J) = PC_{\gamma,\rho}(J),$$

with the norm

$$\|u\|_{PC_{\gamma,\rho}} = \max\left\{\max_{k=0,\dots,m}\left\{\sup_{t\in[s_k,t_{k+1}]}\left\|\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}u(t)\right\|\right\}, \max_{k=1,\dots,m}\left\{\sup_{t\in[t_k,s_k]}\|u(t)\|\right\}\right\},$$

Also, we define the following Banach space

$$PC^\gamma_{\gamma,\rho}(J) = \Bigg\{u : J \to \mathbb{R} : u \in C^\gamma_{\gamma,\rho}(I_k); k = 0,\dots,m, \text{ and } u \in C_E(\tilde{I}_k); k = 1,\dots,m,$$
$$\text{and there exist } u(t_k^-), u(t_k^+), u(s_k^-), \text{ and } u(s_k^+) \text{ with } u(t_k^-) = u(t_k)\Bigg\}.$$

Lemma 4.13. *([61]) Let $D \subset PC_{\gamma,\rho}(J)$ be a bounded and equicontinuous set, then*
(i) the function $t \to \mu(D(t))$ is continuous on J, and

$$\mu_{PC_{\gamma,\rho}} = \max\left\{\max_{k=0,\dots,m}\left\{\sup_{t\in[s_k,t_{k+1}]}\mu\left(\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}u(t)\right)\right\}, \max_{k=1,\dots,m}\left\{\sup_{t\in[t_k,s_k]}\mu\left(u(t)\right)\right\}\right\},$$

(ii) $\mu\left(\int_a^b u(s)ds : u \in D\right) \le \int_a^b \mu(D(s))ds$, *where*

$$D(t) = \{u(t) : t \in D\},\ t \in J.$$

Same as the Last section, by following the same steps we can have the following result

Lemma 4.14. *Let $\gamma = \alpha + \beta - \alpha\beta$ where $0 < \alpha < 1$, $0 \le \beta \le 1$, and $k = 0,\dots,m$, let $f : I_k \times E \times E \to E$, be a function such that $f(\cdot, u(\cdot), w(\cdot)) \in C_{\gamma,\rho}(I_k)$, for any $u, w \in PC_{\gamma,\rho}(J)$. If $u \in PC^\gamma_{\gamma,\rho}(J)$, then u satisfies the problem (4.30) – (4.32) if and only if u is the fixed point of the operator $\Psi : PC_{\gamma,\rho}(J) \to PC_{\gamma,\rho}(J)$ defined by*

$$\Psi u(t) = \begin{cases} \dfrac{\phi_k}{\Gamma(\gamma)}\left(\dfrac{t^\rho - s_k^\rho}{\rho}\right)^{\gamma-1} + \left({}^\rho\mathcal{J}^\alpha_{s_k^+}h\right)(t), & t \in I_k,\ k = 0,\dots,m, \\ g_k(t, u(t)), & t \in \tilde{I}_k,\ k = 1,\dots,m. \end{cases} \tag{4.33}$$

where $\phi^* = max\{\|\phi_k\| : k = 0,\dots,m\}$ and $h \in C_{\gamma,\rho}(I_k)$, $k = 0,\dots,m$ be a function satisfying the functional equation

$$h(t) = f(t, u(t), h(t)).$$

Also, by Lemma 1.12, $\Psi u \in PC_{\gamma,\rho}(J)$.

The following hypotheses will be used in the sequel :

(Ax_1) The function $t \mapsto f(t,u,w)$ is measurable on $I_k; k = 0,\dots,m$, for each $u,w \in E$, and the functions $u \mapsto f(t,u,w)$ and $w \mapsto f(t,u,w)$ are continuous on E for a.e. $t \in I_k; k = 0,\dots,m$, and

$$f(\cdot,u(\cdot),w(\cdot)) \in C^{\beta(1-\alpha)}_{\gamma,\rho}(I_k) \ \text{ for any } u,w \in PC_{\gamma,\rho}(J).$$

(Ax_2) There exists a continuous function $p : [a,b] \longrightarrow [0,\infty)$ such that

$$\|f(t,u,w)\| \leq p(t), \text{ for a.e. } t \in I_k; k = 0,\dots,m, \text{ and for each } u,w \in E.$$

(Ax_3) For each bounded set $B \subset E$ and for each $t \in I_k; k = 0,\dots,m$, we have

$$\mu(f(t,B,({}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}B))) \leq p(t)\mu(B),$$

where ${}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}B = \{{}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}w : w \in B\}$.

(Ax_4) The functions $g_k \in C(\tilde{I}_k, E); k = 1,\dots,m$, and there exists $^* > 0$ such that

$$\|g_k(t,u)\| \leq l^*\|u\| \ \text{ for each } u \in E, k = 1,\dots,m.$$

(Ax_5) For each bounded set $B \subset E$ and for each $t \in \tilde{I}_k; k = 1,\dots,m$, we have

$$\mu(g_k(t,B)) \leq l^*\mu(B), k = 1,\dots,m.$$

We are now in a position to state and prove our existence result for the problem (4.30)$-$(4.32) based on Mönch's fixed point theorem. Set $p^* = \sup\limits_{t\in[a,b]} p(t)$.

Theorem 4.15. *Assume* **(Ax_1)** – **(Ax_5)** *hold. If*

$$L \ := \ max\left\{l^*, \frac{p^*\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^\alpha\right\} < 1, \tag{4.34}$$

then the problem (4.30)–(4.32) has at least one solution in $PC_{\gamma,\rho}(J)$.

Proof: Consider the operator $\Psi : PC_{\gamma,\rho}(J) \to PC_{\gamma,\rho}(J)$ defined in (4.33) and the ball $B_R := B(0,R) = \{w \in PC_{\gamma,\rho}(J) : \|w\|_{PC_{\gamma,\rho}} \leq R\}$, such that

$$R \geq \frac{\phi^*}{(1-l^*)\Gamma(\gamma)} + \frac{p^*}{(1-l^*)\Gamma(\alpha+1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha}.$$

For any $u \in B_R$, and each $t \in I_k$, $k = 0,\dots,m$, we have

$$\begin{aligned}\|\Psi u(t)\| &\leq \frac{\|\phi_k\|}{\Gamma(\gamma)}\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\gamma-1} + \left({}^{\rho}\mathcal{J}^{\alpha}_{s_k^+}\|h(s)\|\right)(t)\\ &\leq \frac{\phi^*}{\Gamma(\gamma)}\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\gamma-1} + p^*\left({}^{\rho}\mathcal{J}^{\alpha}_{s_k^+}(1)\right)(t).\end{aligned}$$

By Lemma 1.10, we have

$$\begin{aligned}\left\|\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\Psi u(t)\right\| &\leq \frac{\phi^*}{\Gamma(\gamma)} + \frac{p^*}{\Gamma(\alpha+1)}\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma+\alpha} \\ &\leq \frac{\phi^*}{\Gamma(\gamma)} + \frac{p^*}{\Gamma(\alpha+1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha}.\end{aligned}$$

And for $t \in \tilde{I}_k, k = 1, \ldots, m$, we have

$$\|(\Psi u)(t)\| \leq l^* \|u(t)\| \leq l^* R.$$

Hence,

$$\|\Psi u\|_{PC_{\gamma,\rho}} \leq l^* R + \frac{\phi^*}{\Gamma(\gamma)} + \frac{p^*}{\Gamma(\alpha+1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha} \leq R.$$

This proves that Ψ transforms the ball B_R into itself. We shall show that the operator $\Psi : B_R \to B_R$ satisfies all the assumptions of Theoreme 1.24. The proof will be given in several steps.

Step 1: $\Psi : B_R \to B_R$ is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $PC_{\gamma,\rho}(J)$. Then for each $t \in I_k, k = 0, \ldots, m$, we have,

$$\left\|((\Psi u_n)(t) - (\Psi u)(t))\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\right\| \leq \left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\left({}^\rho\mathcal{J}^\alpha_{s_k^+}\|h_n(s) - h(s)\|\right)(t),$$

where $h_n, h \in C_{\gamma,\rho}(I_k); k = 0, \ldots, m$, such that

$$\begin{aligned} h_n(t) &= f(t, u_n(t), h_n(t)), \\ h(t) &= f(t, u(t), h(t)). \end{aligned}$$

For each $t \in \tilde{I}_k, k = 1, \ldots, m$, we have,

$$\|((\Psi u_n)(t) - (\Psi u)(t))\| \leq \|(g_k(t, u_n(t)) - g_k(t, u(t)))\|.$$

Since $u_n \to u$, then we get $h_n(t) \to h(t)$ as $n \to \infty$ for each $t \in J$, and since f and g_k are continuous, then we have

$$\|\Psi u_n - \Psi u\|_{PC_{\gamma,\rho}} \to 0 \text{ as } n \to \infty.$$

Step 2: $\Psi(B_R)$ is bounded and equicontinuous.
Since $\Psi(B_R) \subset B_R$ and B_R is bounded, then $\Psi(B_R)$ is bounded.
Next, let $\epsilon_1, \epsilon_2 \in I_k, k = 0, \ldots, m$, $\epsilon_1 < \epsilon_2$, and let $u \in B_R$. Then

$$\begin{aligned} &\left\|\left(\frac{\epsilon_1^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}(\Psi u)(\epsilon_1) - \left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}(\Psi u)(\epsilon_2)\right\| \\ &\leq \left\|\left(\frac{\epsilon_1^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\left({}^\rho\mathcal{J}^\alpha_{s_k^+}h(\tau)\right)(\epsilon_1) - \left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\left({}^\rho\mathcal{J}^\alpha_{s_k^+}h(\tau)\right)(\epsilon_2)\right\| \\ &\leq \left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\left({}^\rho\mathcal{J}^\alpha_{\epsilon_1^+}\|h(\tau)\|\right)(\epsilon_2) + \frac{1}{\Gamma(\alpha)}\int_{s_k}^{\epsilon_1}\left\|\tau^{\rho-1}H(\tau)h(\tau)\right\| d\tau, \end{aligned}$$

where $H(\tau) = \left[\left(\frac{\epsilon_1^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\left(\frac{\epsilon_1^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} - \left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\left(\frac{\epsilon_2^\rho - \tau^\rho}{\rho}\right)^{\alpha-1}\right]$.

Then by Lemma 1.10, we have

$$\begin{aligned} &\left\| \left(\frac{\epsilon_1^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}(\Psi u)(\epsilon_1) - \left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}(\Psi u)(\epsilon_2)\right\| \\ &\le \frac{p^*}{\Gamma(1+\alpha)}\left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\left(\frac{\epsilon_2^\rho - \epsilon_1^\rho}{\rho}\right)^{\alpha} + p^*\int_{s_k}^{\epsilon_1}\left\|H(\tau)\frac{\tau^{\rho-1}}{\Gamma(\alpha)}\right\|\left(\frac{\tau^\rho - s_k^\rho}{\rho}\right)^{\gamma-1}d\tau, \end{aligned}$$

and for each $t \in \tilde{I}_k, k = 1, \dots, m$, we have

$$\|(\Psi u)(\epsilon_1) - (\Psi u)(\epsilon_2)\| \le \|(g_k(\epsilon_1, u(\epsilon_1))) - (g_k(\epsilon_2, u(\epsilon_2)))\|.$$

As $\epsilon_1 \to \epsilon_2$, the right-hand side of the above inequality tends to zero. Hence, $\Psi(B_R)$ is bounded and equicontinuous.

Step 3: The implication (1.1) of Theorem 1.24 holds.

Now let D be an equicontinuous subset of B_R such that $D \subset \overline{\Psi(D)} \cup \{0\}$, therefore the function $t \longrightarrow d(t) = \mu(D(t))$ are continuous on J. By ***(Ax₃)***, ***(Ax₅)*** and the properties of the measure μ, for each $t \in I_k, k = 0, \dots, m$, we have

$$\begin{aligned} \left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} d(t) &\le \mu\left(\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}(\Psi D)(t) \cup \{0\}\right) \\ &\le \mu\left(\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}(\Psi D)(t)\right), \end{aligned}$$

then,

$$\begin{aligned} \left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} d(t) &\le \left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\left({}^\rho\mathcal{J}_{s_k^+}^\alpha p(s)\mu(D(s))\right)(t) \\ &\le p^*\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma}\left({}^\rho\mathcal{J}_{s_k^+}^\alpha d(s)\right)(t) \\ &\le \left[\frac{p^*\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha}\right]\|d\|_{PC_{\gamma,\rho}}. \end{aligned}$$

And for each $t \in \tilde{I}_k, k = 1, \dots, m$, we have

$$d(t) \le \mu(g_k(t, D(t))) \le l^* d(t).$$

Thus for each $t \in J$, we have

$$\|d\|_{PC_{\gamma,\rho}} \le L\|d\|_{PC_{\gamma,\rho}}.$$

From (4.34), we get $\|d\|_{PC_{\gamma,\rho}} = 0$, that is $d(t) = \mu(D(t)) = 0$, for each $t \in J$, and then $D(t)$ is relatively compact in E. In view of the Ascoli-Arzela Theorem, D is relatively compact in B_R. Applying now Theorem 1.24, we conclude that Ψ has a fixed point $u^* \in PC_{\gamma,\rho}(J)$, which is solution of the problem (4.30)-(4.32).

Step 4: We show that such a fixed point $u^* \in PC_{\gamma,\rho}(J)$ is actually in $PC^\gamma_{\gamma,\rho}(J)$.

Since u^* is the unique fixed point of operator Ψ in $PC_{\gamma,\rho}(J)$, then for each $t \in J$, we have

$$\Psi u^*(t) = \begin{cases} \frac{\phi_k}{\Gamma(\gamma)}\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\gamma-1} + \left({}^\rho\mathcal{J}^\alpha_{s_k^+}h\right)(t), & t \in I_k,\ k = 0,\dots,m, \\ g_k(t, u^*(t)), & t \in \tilde{I}_k,\ k = 1,\dots,m. \end{cases}$$

where $h \in C_{\gamma,\rho}(I_k); k = 0,\dots,m$, such that

$$h(t) = f(t, u^*(t), h(t)).$$

For $t \in I_k; k = 0,\dots,m$, applying ${}^\rho\mathcal{D}^\gamma_{s_k^+}$ to both sides and by Lemma 1.10 and Lemma 1.16, we have

$$\begin{aligned} {}^\rho\mathcal{D}^\gamma_{s_k^+}u^*(t) &= \left({}^\rho\mathcal{D}^\gamma_{s_k^+}\ {}^\rho\mathcal{J}^\alpha_{s_k^+}f(s, u^*(s), h(s))\right)(t) \\ &= \left({}^\rho\mathcal{D}^{\beta(1-\alpha)}_{s_k^+}f(s, u^*(s), h(s))\right)(t). \end{aligned}$$

Since $\gamma \geq \alpha$, by **(*Ax₁*)**, the right hand side is in $C_{\gamma,\rho}(I_k)$ and thus ${}^\rho\mathcal{D}^\gamma_{s_k^+}u^* \in C_{\gamma,\rho}(I_k)$ which implies that $u^* \in C^\gamma_{\gamma,\rho}(I_k)$. And since $g_k \in C(\tilde{I}_k, E); k = 1,\dots,m$, then $u^* \in PC^\gamma_{\gamma,\rho}(J)$. As a consequence of Steps 1 to 4 together with Theorem 4.15, we can conclude that the problem (4.30)-(4.32) has at least one solution in $PC_{\gamma,\rho}(J)$. ■

Our second existence result for the problem (4.30)-(4.32) is based on Darbo's fixed point Theorem.

Theorem 4.16. *Assume* **(*Ax₁*)** – **(*Ax₅*)** *hold. If*

$$L \ := \ max\left\{l^*, \frac{p^*\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^\alpha\right\} < 1,$$

then the problem (4.30)–(4.32) has at least one solution in $PC_{\gamma,\rho}(J)$.

Proof: Consider the operator Ψ defined in (4.33). We know that $\Psi : B_R \longrightarrow B_R$ is bounded and continuous and that $\Psi(B_R)$ is equicontinuous, we need to prove that the operator Ψ is a L-contraction.
Let $D \subset B_R$ and $t \in I_k, k = 0,\dots,m$. Then we have

$$\begin{aligned} &\mu\left(\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}(\Psi D)(t)\right) = \mu\left(\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}(\Psi u)(t) : u \in D\right) \\ \leq\ &\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma}\left\{\left({}^\rho\mathcal{J}^\alpha_{s_k^+}p^*\mu(u(s))\right)(t), u \in D\right\}. \end{aligned}$$

By Lemma 1.10, we have for $t \in I_k, k = 0,\dots,m$,

$$\mu\left(\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}(\Psi D)(t)\right) \leq \left[\frac{p^*\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^\alpha\right]\mu_{PC_{\gamma,\rho}}(D).$$

And for each $t \in \tilde{I}_k, k = 1,\dots,m$, we have

$$\mu\left((\Psi D)(t)\right) \leq \mu\left(g_k(t, D(t))\right) \leq l^*\mu\left(D(t)\right).$$

Hence, for each $t \in J$, we have

$$\mu_{PC_{\gamma,\rho}}(\Psi D) \leq L\mu_{PC_{\gamma,\rho}}(D).$$

So, By (4.34), the operator Ψ is a L-contraction. As consequence of Theorem 1.25 and using Step 4 of the last result, we deduce that Ψ has a fixed point which is a solution of the problem (4.30)-(4.32). ■

4.3.2 Ulam-Hyers-Rassias Stability

We are concerned with the Ulam-Hyers-Rassias stability of our problem (4.30)-(4.32). Let $u \in PC_{\gamma,\rho}(J)$, $\epsilon > 0$, $\tau > 0$ and $\vartheta : J \longrightarrow [0,\infty)$ be a continuous function. We consider the following inequality :

$$\begin{cases} \left\| \left({}^\rho\mathcal{D}^{\alpha,\beta}_{s_k^+}u\right)(t) - f\left(t, u(t), \left({}^\rho\mathcal{D}^{\alpha,\beta}_{s_k^+}u\right)(t)\right)\right\| \leq \epsilon\vartheta(t), \ t \in I_k, \ k = 0,\dots,m, \\ \\ \|u(t) - g_k(t, u(t))\| \leq \epsilon\tau, \ t \in \tilde{I}_k, \ k = 1,\dots,m. \end{cases} \tag{4.35}$$

Definition 4.17. *([108, 110]) Problem (4.30)–(4.32) is Ulam-Hyers-Rassias (U-H-R) stable with respect to (ϑ,τ) if there exists a real number $a_{f,\vartheta} > 0$ such that for each $\epsilon > 0$ and for each solution $u \in PC_{\gamma,\rho}(J)$ of inequality (4.35) there exists a solution $w \in PC_{\gamma,\rho}(J)$ of (4.30)–(4.32) with*

$$\|u(t) - w(t)\| \leq \epsilon a_{f,\vartheta}(\vartheta(t) + \tau), \quad t \in J.$$

Remark 4.18. *([108, 110]) A function $u \in PC_{\gamma,\rho}(J)$ is a solution of inequality (4.35) if and only if there exist $\sigma \in PC_{\gamma,\rho}(J)$ and a sequence σ_k, $k = 0,\dots,m$ such that*

1. $\|\sigma(t)\| \leq \epsilon\vartheta(t)$, $t \in I_k$, $k = 0,\dots,m$; *and* $\|\sigma_k\| \leq \epsilon\tau$, $t \in \tilde{I}_k$, $k = 1,\dots,m$,
2. $\left({}^\rho\mathcal{D}^{\alpha,\beta}_{s_k^+}u\right)(t) = f\left(t, u(t), \left({}^\rho\mathcal{D}^{\alpha,\beta}_{s_k^+}u\right)(t)\right) + \sigma(t)$, $t \in I_k$, $k = 0,\dots,m$,
3. $u(t) = g_k(t, u(t)) + \sigma_k$, $t \in \tilde{I}_k$, $k = 1,\dots,m$.

Theorem 4.19. *Assume that in addition to **(Ax1)** – **(Ax5)** and (4.34), the following hypothesis holds.*

(Ax6) *There exist a nondecreasing function $\vartheta : J \longrightarrow [0,\infty)$ and $\lambda_\vartheta > 0$ such that for each $t \in I_k; k = 0,\dots,m$, we have*

$$({}^\rho\mathcal{J}^{\alpha}_{s_k^+}\vartheta)(t) \leq \lambda_\vartheta\vartheta(t).$$

(Ax7) *There exists a continuous function $\chi : \bigcup_{k=1}^{m}[s_k, t_{k+1}] \longrightarrow [0,\infty)$ such that for each $t \in I_k; k = 0,\dots,m$, we have*

$$p(t) \leq \chi(t)\vartheta(t).$$

Then problem (4.30)–(4.32) is U-H-R stable with respect to (ϑ,τ).

Proof: Consider the operator Ψ defined in (4.33). Let $u \in PC_{\gamma,\rho}(J)$ be a solution if inequality (4.35), and let us assume that w is the unique solution of the problem

$$\begin{cases} \left({}^\rho\mathcal{D}^{\alpha,\beta}_{s_k^+} w\right)(t) = f\left(t, w(t), \left({}^\rho\mathcal{D}^{\alpha,\beta}_{s_k^+} w\right)(t)\right); \ t \in I_k, \ k = 0, \dots, m, \\ w(t) = g_k(t, w(t_k^-)); \ t \in \tilde{I}_k, \ k = 1, \dots, m, \\ \left({}^\rho\mathcal{J}^{1-\gamma}_{s_k^+} w\right)(s_k^+) = \left({}^\rho\mathcal{J}^{1-\gamma}_{s_k^+} u\right)(s_k^+) = \phi_k, \ k = 0, \dots, m. \end{cases}$$

By Lemma 4.14, we obtain for each $t \in (a, b]$

$$w(t) = \begin{cases} \dfrac{\phi_k}{\Gamma(\gamma)}\left(\dfrac{t^\rho - s_k^\rho}{\rho}\right)^{\gamma-1} + \left({}^\rho\mathcal{J}^{\alpha}_{s_k^+} h\right)(t) & if \ t \in I_k, k = 0, \dots, m, \\ g_k(t, w(t)) \qquad if \ t \in \tilde{I}_k, k = 1, \dots, m, \end{cases}$$

where $h \in C_{\gamma,\rho}(I_k); k = 0, \dots, m$, be a function satisfying the functional equation

$$h(t) = f(t, w(t), h(t)).$$

Since u is a solution of the inequality (4.35), by Remark 4.18, we have

$$\begin{cases} \left({}^\rho\mathcal{D}^{\alpha,\beta}_{s_k^+} u\right)(t) = f\left(t, u(t), \left({}^\rho\mathcal{D}^{\alpha,\beta}_{s_k^+} u\right)(t)\right) + \sigma(t), t \in I_k, k = 0, \dots, m; \\ u(t) = g_k(t, u(t)) + \sigma_k, t \in \tilde{I}_k, k = 1, \dots, m. \end{cases} \tag{4.36}$$

Clearly, the solution of (4.36) is given by

$$u(t) = \begin{cases} \dfrac{\phi_k}{\Gamma(\gamma)}\left(\dfrac{t^\rho - s_k^\rho}{\rho}\right)^{\gamma-1} + \left({}^\rho\mathcal{J}^{\alpha}_{s_k^+} g\right)(t) + \left({}^\rho\mathcal{J}^{\alpha}_{s_k^+} \sigma\right)(t), & t \in I_k, k = 1, \dots, m, \\ g_k(t, u(t)) + \sigma_k, \qquad t \in \tilde{I}_k, \ k = 1, \dots, m, \end{cases}$$

where $g : I_k \to E, k = 0, \dots, m$, be a function satisfying the functional equation

$$g(t) = f(t, u(t), g(t)).$$

Hence, for each $t \in I_k$,$k = 0, \dots, m$, we have

$$\begin{aligned} \|u(t) - w(t)\| &\leq \left({}^\rho\mathcal{J}^{\alpha}_{a^+} \|g(s) - h(s)\|\right)(t) + \left({}^\rho\mathcal{J}^{\alpha}_{a^+} \|\sigma(s)\|\right) \\ &\leq \epsilon\lambda_\vartheta\vartheta(t) + \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\frac{2\chi(t)\vartheta(t)}{\Gamma(\gamma)}ds \\ &\leq \epsilon\lambda_\vartheta\vartheta(t) + 2\chi^*\left({}^\rho\mathcal{J}^{\alpha}_{a^+}\vartheta\right)(t) \\ &\leq (\epsilon + 2\chi^*)\lambda_\vartheta\vartheta(t) \\ &\leq (1 + \frac{2\chi^*}{\epsilon})\lambda_\vartheta\epsilon(\tau + \vartheta(t)), \end{aligned}$$

where

$$\chi^* = \max_{k=0,\dots,m}\left\{\sup_{t\in[s_k,t_{k+1}]}\chi(t)\right\}.$$

For each $t \in \tilde{I}_k$,$k = 1, \dots, m$, we have

$$\begin{aligned} \|u(t) - w(t)\| &\leq \|g_k(t, u(t)) - g_k(t, w(t))\| + \|\sigma_k\| \\ &\leq l^*\|u(t) - w(t)\| + \epsilon\tau, \end{aligned}$$

then by (4.34),

$$\|u(t) - w(t)\| \leq \frac{\epsilon\tau}{1-l^*} \leq \frac{\epsilon}{1-l^*}(\tau + \vartheta(t)),$$

Then for each $t \in (a, b]$, we have

$$\|u(t) - w(t)\| \leq a_\vartheta \epsilon(\tau + \vartheta(t)),$$

where

$$a_\vartheta = \max\left\{(1 + \frac{2\chi^*}{\epsilon})\lambda_\vartheta, \frac{1}{1-l^*}\right\}.$$

Hence, problem (4.30)–(4.32) is U-H-R stable with respect to (ϑ, τ). ■

4.3.3 An Example

Let

$$E = l^1 = \left\{v = (v_1, v_2, \ldots, v_n, \ldots), \sum_{n=1}^{\infty} |v_n| < \infty\right\}$$

be the Banach space with the norm

$$\|v\| = \sum_{n=1}^{\infty} |v_n|.$$

Consider the following initial value problem with not instantaneous impulses

$$\left({}^1\mathcal{D}^{\frac{1}{2},0}_{s_k^+} u\right)(t) = f\left(t, u(t), \left({}^1\mathcal{D}^{\frac{1}{2},0}_{s_k^+} u\right)(t)\right), \ t \in (1,2] \cup (e,3], \ k \in \{0,1\} \tag{4.37}$$

$$u(t) = g(t, u(t)), \ t \in (2, e], \tag{4.38}$$

$$\left({}^1\mathcal{J}^{\frac{1}{2}}_{1^+} u\right)(1^+) = 0, \tag{4.39}$$

where

$$a = t_0 = s_0 = 1 < t_1 = 2 < s_1 = e < t_2 = 3 = b,$$
$$u = (u_1, u_2, \ldots, u_n, \ldots),$$
$$f = (f_1, f_2, \ldots, f_n, \ldots),$$
$${}^1\mathcal{D}^{\frac{1}{2},0}_{s_k^+} u = ({}^1\mathcal{D}^{\frac{1}{2},0}_{s_k^+} u_1, \ldots, {}^1\mathcal{D}^{\frac{1}{2},0}_{s_k^+} u_2, \ldots, {}^1\mathcal{D}^{\frac{1}{2},0}_{s_k^+} u_n, \ldots),$$
$$g = (g_1, g_2, \ldots, g_n, \ldots),$$
$$f_n(t, u_n(t), \left({}^1\mathcal{D}^{\frac{1}{2},0}_{s_k^+} u_n\right)(t)) = \frac{(2t^3 + 5e^{-2})|u_n(t)|}{183e^{-t+3}(1 + \|u(t)\| + \|\left({}^1\mathcal{D}^{\frac{1}{2},0}_{s_k^+} u\right)(t)\|)}, \ t \in (1,2] \cup (e,3],$$

with $k \in \{0, 1\}$, $n \in \mathbb{N}$, and

$$g_n(t, u_n(t)) = \frac{|u_n(t)|}{105e^{-t+5} + 1}, \ t \in (2, e], n \in \mathbb{N}.$$

We have

$$C_{\gamma,\rho}^{\beta(1-\alpha)}((1,2]) = C_{\frac{1}{2},1}^{0}((1,2]) = \left\{h : (1,2] \to E : (\sqrt{t-1})h \in C([1,2],E)\right\},$$

and

$$C_{\gamma,\rho}^{\beta(1-\alpha)}((e,3]) = C_{\frac{1}{2},1}^{0}((e,3]) = \left\{h : (e,3] \to E : (\sqrt{t-e})h \in C([e,3],E)\right\},$$

with $\gamma = \alpha = \frac{1}{2}$, $\rho = 1$, $\beta = 0$ and $k \in \{0,1\}$. Clearly, the continuous function $f \in C_{\frac{1}{2},1}^{0}((1,2]) \cap C_{\frac{1}{2},1}^{0}((e,3])$. Hence the condition ***(Ax1)*** is satisfied.
For each $u, w \in E$ and $t \in (1,2] \cup (e,3]$:

$$\|f(t,u,w)\| \leq \frac{2t^3 + 5e^{-2}}{183e^{-t+3}}.$$

Hence condition ***(Ax2)*** is satisfied with

$$p(t) = \frac{2t^3 + 5e^{-2}}{183e^{-t+3}},$$

and

$$p^* = \frac{54 + 5e^{-2}}{183}.$$

And for each $u \in E$ and $t \in (2,e]$ we have

$$\|g(t,u)\| \leq \frac{\|u\|}{105e^{5-e} + 1},$$

and so the condition ***(Ax4)*** is satisfied with $l^* = \dfrac{1}{105e^{5-e} + 1}$.
The condition (4.34) of Theorem 4.15 is satisfied, for

$$L := \max\left\{l^*, \frac{p^*\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^\alpha\right\} \approx 0.7489295248 < 1.$$

Let Ω be a bounded set in E where ${}^1\mathcal{D}_{s_k^+}^{\frac{1}{2},0}\Omega = \left\{{}^1\mathcal{D}_{s_k^+}^{\frac{1}{2},0}v : v \in \Omega\right\}; k \in \{0,1\}$, then by the properties of the Kuratowski measure of noncompactness, for each $u \in \Omega$ and $t \in (1,2] \cup (e,3]$, we have

$$\mu\left(f(t,\Omega,{}^1\mathcal{D}_{s_k^+}^{\frac{1}{2},0}\Omega)\right) \leq p(t)\mu(\Omega),$$

and for each $t \in (2,e]$,

$$\mu\left(g(t,\Omega)\right) \leq l^*\mu(\Omega).$$

Hence conditions ***(Ax3)*** and ***(Ax5)*** are satisfied. Then the problem (4.37)−(4.39) has at least one solution in $PC_{\frac{1}{2},1}([1,3])$.
Also, hypothesis ***(Ax6)*** is satisfied with $\tau = 1$ and

$$\vartheta(t) = \begin{cases} \dfrac{1}{\sqrt{t-s_k}}, & if\ t \in (1,2] \cup (e,3], \\ 1, & if\ t \in (2,e], \end{cases}$$

and $\lambda_\vartheta = \sqrt{\pi}$. Indeed, for each $t \in (1, 2]$, we get

$$({}^1\mathcal{J}_{1^+}^{\frac{1}{2}}\vartheta)(t) = \sqrt{\pi} \leq \frac{\sqrt{\pi}}{\sqrt{t-1}},$$

and for each $t \in (e, 3]$, we get

$$({}^1\mathcal{J}_{e^+}^{\frac{1}{2}}\vartheta)(t) = \sqrt{\pi} \leq \frac{\sqrt{\pi}}{\sqrt{t-e}}.$$

Let the function $\chi : [1, 2] \cup [e, 3] \longrightarrow [0, \infty)$ defined by :

$$\chi(t) = \frac{(2t^3 + 5e^{-2})\sqrt{t - s_k}}{183e^{-t+3}}; k \in \{0, 1\},$$

then, for each $t \in (1, 2] \cup (e, 3]$, we have

$$p(t) = \chi(t)\vartheta(t),$$

with $\chi^* = p^*$. Hence, the condition $(\boldsymbol{Ax_7})$ is satisfied. Consequently, Theorem 4.19 implies that the problem (4.37)–(4.39) is U-H-R stable.

Chapter 5

Boundary Value Problem for Fractional Order Generalized Hilfer-Type Fractional Derivative with Non-Instantaneous Impulses [1]

5.1 Introduction and Motivations

This chapter is devoted to proving some results concerning the existence of solutions for a class of boundary value problem for nonlinear implicit fractional differential equations with non-instantaneous impulses and Generalized Hilfer fractional derivative. The results are based on Banach contraction principle and Krasnoselskii's fixed point theorem. Further, for the justification of our results we provide two examples. Fractional calculus is a branch of classical mathematics, which generalizes the integer order differentiation and integration of a function to non-integer order [11, 12, 45]. There are several kinds of fractional derivatives, such as, Riemann-Liouville fractional derivative, Caputo fractional derivative, Hilfer fractional derivative, Hadamard fractional derivative and others. For some fundamental results in the theory of fractional calculus and fractional differential equations we refer the reader to [13, 14, 25, 27, 38]. Impulsive differential equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states. In the literature there are two popular types of impulses. In [45] the authors studied some new classes of abstract impulsive differential equations with instantaneous impulses, for more interesting results on the classes with not instantaneous impulses, one can see [4, 6, 24]. On the other hand, the stability investigation of differential and integral equations are important in applications. For basic results and recent development on Ulam stabilities of differential and integral equations, We refer the reader, for example, to references [10, 13, 46, 47, 65, 84, 89, 105].

Motivated by the works of the papers mentioned above, in this chapter, we establish existence and stability results to the boundary value problem with nonlinear implicit Generalized Hilfer-type fractional differential equation with non-instantaneous impulses:

$$\left({}^{\alpha}\mathcal{D}^{\vartheta,r}_{\tau_i^+}x\right)(t) = f\left(t, x(t), \left({}^{\alpha}\mathcal{D}^{\vartheta,r}_{\tau_i^+}x\right)(t)\right);\ t \in J_i,\ i = 0, \ldots, m, \tag{5.1}$$

$$x(t) = \psi_i(t, x(t));\ t \in \tilde{J}_i,\ i = 1, \ldots, m, \tag{5.2}$$

$$\phi_1\left({}^{\alpha}\mathcal{J}^{1-\xi}_{a^+}x\right)(a^+) + \phi_2\left({}^{\alpha}\mathcal{J}^{1-\xi}_{m^+}x\right)(b) = \phi_3, \tag{5.3}$$

1. **A. Salim**, M. Benchohra, J.R. Graef and J. E. Lazreg, Boundary Value Problem for Fractional Order Generalized Hilfer-Type Fractional Derivative with Non-Instantaneous Impulses. *Fractal Fract.* **5** (2021), 1-21.

where ${}^{\alpha}\mathcal{D}^{\vartheta,r}_{\tau_i^+}$, ${}^{\alpha}\mathcal{J}^{1-\xi}_{a^+}$ are the generalized Hilfer-type fractional derivative of order $\vartheta \in (0,1)$ and type $r \in [0,1]$ and generalized fractional integral of order $1-\xi, (\xi = \vartheta + r - \vartheta r)$ respectively, $\phi_1, \phi_2, \phi_3 \in \mathbb{R}$, $\phi_1 \neq 0$, $J_i := (\tau_i, t_{i+1}]; i = 0, \dots, m$, $\tilde{J}_i := (t_i, \tau_i]; i = 1, \dots, m$, $a = t_0 = \tau_0 < t_1 \leq \tau_1 < t_2 \leq \tau_2 < \dots \leq \tau_{m-1} < t_m \leq \tau_m < t_{m+1} = b < \infty$, $x(t_i^+) = \lim_{\epsilon \to 0^+} x(t_i + \epsilon)$ and $x(t_i^-) = \lim_{\epsilon \to 0^-} x(t_i + \epsilon)$ represent the right and left hand limits of $x(t)$ at $t = t_i$, $f : J_i \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function and $\psi_i : \tilde{J}_i \times \mathbb{R} \to \mathbb{R}; i = 1, \dots, m$ are given continuous functions such that $\left({}^{\alpha}\mathcal{J}^{1-\xi}_{\tau_i^+}\psi_i\right)(t, x(t))\,|_{t=\tau_i} = c_i \in \mathbb{R}$.

5.2 Existence of Solutions

We can use the preliminary details, essential notations, definitions and lemmata introduced in the first section of Chapter 4.

We consider the following linear fractional differential equation

$$\left({}^{\alpha}\mathcal{D}^{\vartheta,r}_{\tau_i^+}x\right)(t) = v(t), \quad t \in J_i, i = 0, \dots, m, \tag{5.4}$$

where $0 < \vartheta < 1, 0 \leq r \leq 1, \alpha > 0$, with the conditions

$$x(t) = \psi_i(t, x(\tau_i^-)); \; t \in \tilde{J}_i, \; i = 1, \dots, m, \tag{5.5}$$

$$\phi_1 \left({}^{\alpha}\mathcal{J}^{1-\xi}_{a^+}x\right)(a^+) + \phi_2 \left({}^{\alpha}\mathcal{J}^{1-\xi}_{\tau_m^+}x\right)(b) = \phi_3, \tag{5.6}$$

where $\xi = \vartheta + r - \vartheta r$, $\phi_1, \phi_2, \phi_3 \in \mathbb{R}$, $\phi_1 \neq 0$ and $c^* = max\{|c_i| : i = 1, \dots, m\}$.

The following theorem shows that the problem (5.4)–(5.6) has a unique solution given by

$$x(t) = \begin{cases} \frac{1}{\Gamma(\xi)}\left(\frac{t^\alpha - a^\alpha}{\alpha}\right)^{\xi-1}\left[\frac{\phi_3}{\phi_1} - \frac{c_m\phi_2}{\phi_1} - \frac{\phi_2}{\phi_1}\left({}^{\alpha}\mathcal{J}^{1-\xi+\vartheta}_{\tau_m^+}v\right)(b)\right] + \left({}^{\alpha}\mathcal{J}^{\vartheta}_{a^+}v\right)(t) & if \; t \in J_0, \\ \frac{c_i}{\Gamma(\xi)}\left(\frac{t^\alpha - \tau_i^\alpha}{\alpha}\right)^{\xi-1} + \left({}^{\alpha}\mathcal{J}^{\vartheta}_{\tau_i^+}v\right)(t) \qquad if \; t \in J_i, i = 1, \dots, m, \\ \psi_i(t, x(t)) \qquad if \; t \in \tilde{J}_i, i = 1, \dots, m. \end{cases} \tag{5.7}$$

Theorem 5.1. *Let $\xi = \vartheta + r - \vartheta r$, where $0 < \vartheta < 1$ and $0 \leq r \leq 1$. If $v : J_i \to \mathbb{R}; i = 0, \dots, m$, is a function such that $v(\cdot) \in C_{\xi,\alpha}(J_i)$, then $x \in PC^{\xi}_{\xi,\alpha}(J)$ satisfies the problem (5.4)–(5.6) if and only if it satisfies (5.7).*

Proof: Assume x satisfies (5.4)–(5.6). If $t \in J_0$, then

$$\left({}^{\alpha}\mathcal{D}^{\vartheta,r}_{a^+}x\right)(t) = v(t),$$

Lemma 1.21 implies we have the solution can be written as

$$x(t) = \frac{\left({}^{\alpha}\mathcal{J}^{1-\xi}_{a^+}x\right)(a)}{\Gamma(\xi)}\left(\frac{t^\alpha - a^\alpha}{\alpha}\right)^{\xi-1} + \frac{1}{\Gamma(\vartheta)}\int_a^t \left(\frac{t^\alpha - \tau^\alpha}{\alpha}\right)^{\vartheta-1}\tau^{\alpha-1}v(\tau)d\tau.$$

If $t \in \tilde{J}_1$, then we have $x(t) = \psi_1(t, x(t))$.
If $t \in J_1$, then Lemma 1.21 implies

$$\begin{aligned} x(t) &= \frac{\left({}^{\alpha}\mathcal{J}_{\tau_1^+}^{1-\xi} x\right)(\tau_1)}{\Gamma(\xi)} \left(\frac{t^{\alpha} - \tau_1^{\alpha}}{\alpha}\right)^{\xi-1} + \frac{1}{\Gamma(\vartheta)} \int_{\tau_1}^{t} \left(\frac{t^{\alpha} - \tau^{\alpha}}{\alpha}\right)^{\vartheta-1} \tau^{\alpha-1} v(\tau) d\tau \\ &= \frac{c_1}{\Gamma(\xi)} \left(\frac{t^{\alpha} - \tau_1^{\alpha}}{\alpha}\right)^{\xi-1} + \left({}^{\alpha}\mathcal{J}_{\tau_1^+}^{\vartheta} v\right)(t). \end{aligned}$$

If $t \in \tilde{J}_2$, then we have $x(t) = \psi_2(t, x(t))$.
If $t \in J_2$, then Lemma 1.21 implies

$$\begin{aligned} x(t) &= \frac{\left({}^{\alpha}\mathcal{J}_{\tau_2^+}^{1-\xi} x\right)(\tau_2)}{\Gamma(\xi)} \left(\frac{t^{\alpha} - \tau_2^{\alpha}}{\alpha}\right)^{\xi-1} + \frac{1}{\Gamma(\vartheta)} \int_{\tau_2}^{t} \left(\frac{t^{\alpha} - \tau^{\alpha}}{\alpha}\right)^{\vartheta-1} \tau^{\alpha-1} v(\tau) d\tau \\ &= \frac{c_2}{\Gamma(\xi)} \left(\frac{t^{\alpha} - \tau_2^{\alpha}}{\alpha}\right)^{\xi-1} + \left({}^{\alpha}\mathcal{J}_{\tau_2^+}^{\vartheta} v\right)(t). \end{aligned}$$

Repeating the process in this way, the solution $x(t)$ for $t \in J$, can be written as

$$x(t) = \begin{cases} \dfrac{\left({}^{\alpha}\mathcal{J}_{a^+}^{1-\xi} x\right)(a)}{\Gamma(\xi)} \left(\dfrac{t^{\alpha} - a^{\alpha}}{\alpha}\right)^{\xi-1} + \left({}^{\alpha}\mathcal{J}_{a^+}^{\vartheta} v\right)(t) & if\ t \in J_0, \\ \dfrac{c_i}{\Gamma(\xi)} \left(\dfrac{t^{\alpha} - \tau_i^{\alpha}}{\alpha}\right)^{\xi-1} + \left({}^{\alpha}\mathcal{J}_{\tau_i^+}^{\vartheta} v\right)(t) & if\ t \in J_i, i = 1, \ldots, m, \\ \psi_i(t, x(t)) \quad if\ t \in \tilde{J}_i, i = 1, \ldots, m. \end{cases} \tag{5.8}$$

Applying ${}^{\alpha}\mathcal{J}_{\tau_m^+}^{1-\xi}$ on both sides of (5.8), using Lemma 1.10 and taking $t = b$, we obtain

$$\left({}^{\alpha}\mathcal{J}_{\tau_m^+}^{1-\xi} x\right)(b) = c_m + \left({}^{\alpha}\mathcal{J}_{\tau_m^+}^{1-\xi+\vartheta} v\right)(b),$$

using the condition (5.6), we get

$$\left({}^{\alpha}\mathcal{J}_{a^+}^{1-\xi} x\right)(a) = \frac{\phi_3}{\phi_1} - \frac{c_m \phi_2}{\phi_1} - \frac{\phi_2}{\phi_1} \left({}^{\alpha}\mathcal{J}_{\tau_m^+}^{1-\xi+\vartheta} v\right)(b). \tag{5.9}$$

Substituting (5.9) in (5.8) we get (5.7).

Reciprocally, for $t \in J_i; i = 0, \ldots, m$, applying ${}^{\alpha}\mathcal{J}_{\tau_i^+}^{1-\xi}$ on both sides of (5.7) and using Lemma 1.10 and Theorem 1.9, we get

$$\left({}^{\alpha}\mathcal{J}_{\tau_i^+}^{1-\xi} x\right)(t) = \begin{cases} \dfrac{\phi_3}{\phi_1} - \dfrac{c_m \phi_2}{\phi_1} - \dfrac{\phi_2}{\phi_1} \left({}^{\alpha}\mathcal{J}_{\tau_m^+}^{1-\xi+\vartheta} v\right)(b) + \left({}^{\alpha}\mathcal{J}_{a^+}^{1-\xi+\vartheta} v\right)(t) & if\ t \in J_0, \\ c_i + \left({}^{\alpha}\mathcal{J}_{\tau_i^+}^{1-\xi+\vartheta} v\right)(t) & if\ t \in J_i, i = 1, \ldots, m. \end{cases} \tag{5.10}$$

Next, taking the limit $t \to a^+$ of (5.10) and using Lemma 1.13, with $1-\xi < 1-\xi+\vartheta$, we obtain

$$\left({}^{\alpha}\mathcal{J}_{a^+}^{1-\xi}u\right)(a^+) = \frac{\phi_3}{\phi_1} - \frac{c_m\phi_2}{\phi_1} - \frac{\phi_2}{\phi_1}\left({}^{\alpha}\mathcal{J}_{\tau_m^+}^{1-\xi+\vartheta}v\right)(b). \tag{5.11}$$

Now taking $t = b$ in (5.10), we get

$$\left({}^{\alpha}\mathcal{J}_{\tau_m^+}^{1-\xi}u\right)(b) = c_m + \left({}^{\alpha}\mathcal{J}_{\tau_m^+}^{1-\xi+\vartheta}v\right)(b). \tag{5.12}$$

From (5.11) and (5.12) we obtain

$$\phi_1\left({}^{\alpha}\mathcal{J}_{a^+}^{1-\xi}x\right)(a^+) + \phi_2\left({}^{\alpha}\mathcal{J}_{\tau_m^+}^{1-\xi}x\right)(b) = \phi_3,$$

which shows that the boundary condition (5.6) is satisfied.
Next, for $t \in J_i; i = 0,\dots,m$, apply operator ${}^{\alpha}\mathcal{D}_{\tau_i^+}^{\xi}$ on both sides of (5.7). Then, from Lemma 1.10 and Lemma 1.16 we obtain

$$({}^{\alpha}\mathcal{D}_{\tau_i^+}^{\xi}x)(t) = \left({}^{\alpha}\mathcal{D}_{\tau_i^+}^{r(1-\vartheta)}v\right)(t). \tag{5.13}$$

Since $x \in C_{\xi,\alpha}^{\xi}(J_i)$ and by definition of $C_{\xi,\alpha}^{\xi}(J_i)$, we have ${}^{\alpha}\mathcal{D}_{\tau_i^+}^{\xi}x \in C_{\xi,\alpha}(J_i)$, then (5.13) implies that

$$({}^{\alpha}\mathcal{D}_{\tau_i^+}^{\xi}x)(t) = \left(\delta_\alpha\ {}^{\alpha}\mathcal{J}_{\tau_i^+}^{1-r(1-\vartheta)}v\right)(t) = \left({}^{\alpha}\mathcal{D}_{\tau_i^+}^{r(1-\vartheta)}v\right)(t) \in C_{\xi,\alpha}(J_i). \tag{5.14}$$

As $v(\cdot) \in C_{\xi,\alpha}(J_i)$ and from Lemma 1.12, follows

$$\left({}^{\alpha}\mathcal{J}_{\tau_i^+}^{1-r(1-\vartheta)}v\right) \in C_{\xi,\alpha}(J_i), i = 0,\dots,m. \tag{5.15}$$

From (5.14), (5.15) and by the definition of the space $C_{\xi,\alpha}^{n}(J_i)$, we obtain

$$\left({}^{\alpha}\mathcal{J}_{\tau_i^+}^{1-r(1-\vartheta)}v\right) \in C_{\xi,\alpha}^{1}(J_i), i = 0,\dots,m.$$

Applying operator ${}^{\alpha}\mathcal{J}_{\tau_i^+}^{r(1-\vartheta)}$ on both sides of (5.13) and using Lemma 1.15, Lemma 1.13 and Property 1.11, we have

$$\begin{aligned}\left({}^{\alpha}\mathcal{D}_{\tau_i^+}^{\vartheta,r}x\right)(t) = {}^{\alpha}\mathcal{J}_{\tau_i^+}^{r(1-\vartheta)}\left({}^{\alpha}\mathcal{D}_{\tau_i^+}^{\xi}x\right)(t) &= v(t) - \frac{\left({}^{\alpha}\mathcal{J}_{\tau_i^+}^{1-r(1-\vartheta)}v\right)(\tau_i)}{\Gamma(r(1-\vartheta))}\left(\frac{t^\alpha - \tau_i^\alpha}{\alpha}\right)^{r(1-\vartheta)-1} \\ &= v(t),\end{aligned}$$

that is, (5.4) holds.
Also, we can easily have

$$x(t) = \psi_i(t, x(t));\ t \in \tilde{J}_i,\ i = 1,\dots,m.$$

This completes the proof. ■

As a consequence of Theorem 5.1, we have the following result

Lemma 5.2. *Let $\xi = \vartheta + r - \vartheta r$ where $0 < \vartheta < 1$ and $0 \leq r \leq 1$, and $i = 0, \dots, m$, let $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, be a function such that $f(\cdot, x(\cdot), y(\cdot)) \in C_{\xi,\alpha}(J_i)$, for any $x, y \in PC_{\xi,\alpha}(J)$. If $x \in PC^{\xi}_{\xi,\alpha}(J)$, then x satisfies the problem (5.1)-(5.3) if and only if x is the fixed point of the operator $\Im : PC_{\xi,\alpha}(J) \to PC_{\xi,\alpha}(J)$ defined by*

$$\Im x(t) = \begin{cases} \dfrac{\bar{c}}{\Gamma(\xi)}\left(\dfrac{t^\alpha - a^\alpha}{\alpha}\right)^{\xi-1} + \left({}^\alpha\mathcal{J}^{\vartheta}_{a^+}\varphi\right)(t) & \text{if } t \in J_0, \\ \dfrac{c_i}{\Gamma(\xi)}\left(\dfrac{t^\alpha - \tau_i^\alpha}{\alpha}\right)^{\xi-1} + \left({}^\alpha\mathcal{J}^{\vartheta}_{\tau_i^+}\varphi\right)(t) & \text{if } t \in J_i, i = 1, \dots, m, \\ \psi_i(t, x(t)) & \text{if } t \in \tilde{J}_i, i = 1, \dots, m. \end{cases} \tag{5.16}$$

where φ be a function satisfying the functional equation

$$\varphi(t) = f(t, x(t), \varphi(t)),$$

and $\bar{c} = \dfrac{\phi_3}{\phi_1} - \dfrac{c_m \phi_2}{\phi_1} - \dfrac{\phi_2}{\phi_1}\left({}^\alpha\mathcal{J}^{1-\xi+\vartheta}_{\tau_m^+}\varphi\right)(b)$. Also, by Lemma 1.12, $\Im u \in PC_{\xi,\alpha}(J)$.

The following hypotheses will be used in the sequel :

(Ax_1) The function $f : J_i \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous on $J_i; i = 0, \dots, m$, and

$$f(\cdot, x(\cdot), y(\cdot)) \in C^{r(1-\vartheta)}_{\xi,\alpha}(J_i), i = 0, \dots, m, \quad \text{for any } x, y \in PC_{\xi,\alpha}(J).$$

(Ax_2) There exist constants $\eta_1 > 0$ and $0 < \eta_2 < 1$ such that

$$|f(t, x, y) - f(t, \bar{x}, \bar{y})| \leq \eta_1 |x - \bar{x}| + \eta_2 |y - \bar{y}|$$

for any $x, y, \bar{x}, \bar{y} \in \mathbb{R}$ and $t \in J_i$, $i = 0, \dots, m$.

(Ax_3) The functions ψ_i are continuous and there exists a constant $K^* > 0$ such that

$$|\psi_i(x) - \psi_i(\bar{x})| \leq K^* |x - \bar{x}|, \ x, \bar{x} \in \mathbb{R}, \ i = 1, \dots, m.$$

Remark 5.3. *By the hypothesis **(Ax_2)** we have*

$$\begin{aligned} |f(t, x, y)| &\leq |f(t, x, y) - f(t, 0, 0)| + |f(t, 0, 0)| \\ &\leq \eta_1 |x| + \eta_2 |y| + f_0. \end{aligned}$$

where $f_0 = \sup_{t \in [a,b]} |f(t, 0, 0)|$.

We are now in a position to state and prove our existence result for the problem (5.1)-(5.3) based on based on Banach's fixed point theorem.

Theorem 5.4. *Assume* (Ax_1) – (Ax_3) *hold. If*

$$\tilde{\ell} = \max\left\{K^*, \frac{\eta_1}{1-\eta_2}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^\vartheta \left[\frac{|\phi_2|}{|\phi_1|\Gamma(\vartheta+1)} + \frac{\Gamma(\xi)}{\Gamma(\xi+\vartheta)}\right]\right\} < 1, \tag{5.17}$$

then the problem (5.1)–(5.3) has a unique solution in $PC_{\xi,\alpha}(J)$.

Proof: The proof will be given in two steps.
Step 1: We show that the operator $\Im$ defined in (5.16) has a unique fixed point x^* in $PC_{\xi,\alpha}(J)$. Let $x, y \in PC_{\xi,\alpha}(J)$ and $t \in J$.
For $t \in J_0$ we have

$$\begin{aligned} |\Im x(t) - \Im y(t)| &\leq \frac{|\phi_2|}{|\phi_1|\Gamma(\xi)}\left(\frac{t^\alpha - a^\alpha}{\alpha}\right)^{\xi-1}\left({}^\alpha\mathcal{J}_{\tau_m^+}^{1-\xi+\vartheta}|\varphi(\tau)-\tilde{\varphi}(\tau)|\right)(b) \\ &+ \left({}^\alpha\mathcal{J}_{a^+}^{\vartheta}|\varphi(\tau)-\tilde{\varphi}(\tau)|\right)(t), \end{aligned}$$

and for $t \in J_i, i = 1, \ldots, m$, we have

$$|\Im x(t) - \Im y(t)| \leq \left({}^\alpha\mathcal{J}_{\tau_i^+}^{\vartheta}|\varphi(\tau)-\tilde{\varphi}(\tau)|\right)(t),$$

where $\varphi, \tilde{\varphi} \in C_{\xi,\alpha}(J_i); i = 0, \ldots, m$, such that

$$\begin{aligned} \varphi(t) &= f(t, x(t), \varphi(t)), \\ \tilde{\varphi}(t) &= f(t, y(t), \tilde{\varphi}(t)). \end{aligned}$$

By (Ax_2), we have

$$\begin{aligned} |\varphi(t) - \tilde{\varphi}(t)| &= |f(t, x(t), \varphi(t)) - f(t, y(t), \tilde{\varphi}(t))| \\ &\leq \eta_1|x(t) - y(t)| + \eta_2|\varphi(t) - \tilde{\varphi}(t)|. \end{aligned}$$

Then,

$$|\varphi(t) - \varphi(t)| \leq \frac{\eta_1}{1-\eta_2}|x(t) - y(t)|.$$

Therefore, for each $t \in J_i, i = 1, \ldots, m$,

$$|\Im x(t) - \Im y(t)| \leq \frac{\eta_1}{1-\eta_2}\left({}^\alpha\mathcal{J}_{\tau_i^+}^{\vartheta}|x(\tau)-y(\tau)|\right)(t).$$

Thus

$$|\Im x(t) - \Im y(t)| \leq \left[\frac{\eta_1}{1-\eta_2}\left({}^\alpha\mathcal{J}_{\tau_i^+}^{\vartheta}\left(\frac{\tau^\alpha - \tau_i^\alpha}{\alpha}\right)^{\xi-1}\right)(t)\right]\|x-y\|_{PC_{\xi,\alpha}}.$$

By Lemma 1.10, we have

$$|\Im x(t) - \Im y(t)| \leq \left[\frac{\eta_1\Gamma(\xi)}{(1-\eta_2)\Gamma(\xi+\vartheta)}\left(\frac{t^\alpha - \tau_i^\alpha}{\alpha}\right)^{\vartheta+\xi-1}\right]\|x-y\|_{PC_{\xi,\alpha}},$$

hence

$$\begin{aligned} \left|(\Im x(t) - \Im y(t))\left(\frac{t^\alpha - \tau_i^\alpha}{\alpha}\right)^{1-\xi}\right| &\leq \left[\frac{\eta_1\Gamma(\xi)}{(1-\eta_2)\Gamma(\xi+\vartheta)}\left(\frac{t^\alpha - \tau_i^\alpha}{\alpha}\right)^{\vartheta}\right]\|x-y\|_{PC_{\xi,\alpha}} \\ &\leq \left[\frac{\eta_1\Gamma(\xi)}{(1-\eta_2)\Gamma(\xi+\vartheta)}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^{\vartheta}\right]\|x-y\|_{PC_{\xi,\alpha}} \\ &\leq \tilde{\ell}\|x-y\|_{PC_{\xi,\alpha}}. \end{aligned}$$

And for $t \in J_0$ we have

$$\begin{aligned}
&|\Im x(t) - \Im y(t)| \\
&\leq \frac{|\phi_2|}{|\phi_1|\Gamma(\xi)} \left(\frac{t^\alpha - a^\alpha}{\alpha}\right)^{\xi-1} \left({}^\alpha\mathcal{J}^{1-\xi+\vartheta}_{\tau_m^+}|\varphi(\tau) - \tilde{\varphi}(\tau)|\right)(b) + \left({}^\alpha\mathcal{J}^{\vartheta}_{a^+}|\varphi(\tau) - \tilde{\varphi}(\tau)|\right)(t) \\
&\leq \frac{\eta_1}{1-\eta_2}\left[\frac{|\phi_2|}{|\phi_1|\Gamma(\vartheta+1)}\left(\frac{b^\alpha - \tau_m^\alpha}{\alpha}\right)^{\vartheta+\xi-1} + \frac{\Gamma(\xi)}{\Gamma(\xi+\vartheta)}\left(\frac{t^\alpha - a^\alpha}{\alpha}\right)^{\vartheta+\xi-1}\right]\|x-y\|_{PC_{\xi,\alpha}},
\end{aligned}$$

hence

$$\begin{aligned}
\left|(\Im x(t) - \Im y(t))\left(\frac{t^\alpha - a^\alpha}{\alpha}\right)^{1-\xi}\right| &\leq \frac{\eta_1 (b^\alpha - a^\alpha)^\vartheta}{(1-\eta_2)\alpha^\vartheta}\left[\frac{|\phi_2|}{|\phi_1|\Gamma(\vartheta+1)} + \frac{\Gamma(\xi)}{\Gamma(\xi+\vartheta)}\right]\|x-y\|_{PC_{\xi,\alpha}} \\
&\leq \tilde{\ell}\|x-y\|_{PC_{\xi,\alpha}}.
\end{aligned}$$

For $t \in \tilde{J}_i, i = 1, \ldots, m$, we have

$$\begin{aligned}
|\Im x(t) - \Im y(t)| &\leq |(\psi_i(t, x(t)) - \psi_i(t, y(t)))| \\
&\leq K^*\|x-y\|_{PC_{\xi,\alpha}} \\
&\leq \tilde{\ell}\|x-y\|_{PC_{\xi,\alpha}}.
\end{aligned}$$

Then, for each $t \in J$, we have

$$\|\Im x - \Im y\|_{PC_{\xi,\alpha}} \leq \tilde{\ell}\|u-w\|_{PC_{\xi,\alpha}}.$$

By (5.17), the operator $\Im$ is a contraction. Hence, by Theorem 1.26, $\Im$ has a unique fixed point $x^* \in PC_{\xi,\alpha}(J)$.

Step 2: We prove that the fixed point $x^* \in PC_{\xi,\alpha}(J)$ is actually in $PC^\xi_{\xi,\alpha}(J)$.
Since x^* is the unique fixed point of operator $\Im$ in $PC_{\xi,\alpha}(J)$, then for each $t \in J$, we have

$$\Im x^*(t) = \begin{cases}
\frac{\bar{c}}{\Gamma(\xi)}\left(\frac{t^\alpha - a^\alpha}{\alpha}\right)^{\xi-1} + \left({}^\alpha\mathcal{J}^\vartheta_{a^+}\varphi\right)(t) & if\ t \in J_0, \\
\frac{c_i}{\Gamma(\xi)}\left(\frac{t^\alpha - \tau_i^\alpha}{\alpha}\right)^{\xi-1} + \left({}^\alpha\mathcal{J}^\vartheta_{\tau_i^+}\varphi\right)(t) & if\ t \in J_i, i = 1, \ldots, m, \\
\psi_i(t, x^*(t)) \qquad if\ t \in \tilde{J}_i, i = 1, \ldots, m,
\end{cases}$$

where $\varphi \in C_{\xi,\alpha}(J_i); i = 0, \ldots, m$, such that

$$\varphi(t) = f(t, x^*(t), \varphi(t)),$$

For $t \in J_i; i = 0, \ldots, m$, applying ${}^\alpha\mathcal{D}^\xi_{\tau_i^+}$ to both sides and by Lemma 1.10 and Lemma 1.16, we have

$$\begin{aligned}
{}^\alpha\mathcal{D}^\xi_{\tau_i^+}x^*(t) &= \left({}^\alpha\mathcal{D}^\xi_{\tau_i^+}\,{}^\alpha\mathcal{J}^\vartheta_{\tau_i^+}f(\tau, x^*(\tau), \varphi(\tau))\right)(t) \\
&= \left({}^\alpha\mathcal{D}^{r(1-\vartheta)}_{\tau_i^+}f(\tau, x^*(\tau), \varphi(\tau))\right)(t).
\end{aligned}$$

Since $\xi \geq \vartheta$, by ***(Ax₁)***, the right hand side is in $C_{\xi,\alpha}(J_i)$ and thus ${}^\alpha\mathcal{D}^\xi_{\tau_i^+}x^* \in C_{\xi,\alpha}(J_i)$. And since $\psi_i \in C(\tilde{J}_i, \mathbb{R}); i = 1, \ldots, m$, then $x^* \in PC^\xi_{\xi,\alpha}(J)$. As a consequence of Steps 1 and 2 together with Theorem 5.4, we can conclude that the problem (5.1)-(5.3) has a unique solution in $PC_{\xi,\alpha}(J)$. ■

Our second result is based on Krasnoselskii's fixed point theorem.

Theorem 5.5. *Assume that* $(\boldsymbol{Ax_1})$, $(\boldsymbol{Ax_2})$ *and the following condition hold :*

$(\boldsymbol{Ax_4})$ *The functions* ψ_i *are continuous and there exist constants* $1 > \Phi_1 > 0, \Phi_2 > 0$ *such that*

$$|\psi_i(x)| \leq \phi_1|x| + \phi_2 \quad \textit{for each } x \in \mathbb{R}, i = 1, \dots, m.$$

If

$$\frac{|\phi_2|\eta_1}{|\phi_1|\Gamma(\vartheta+1)(1-\eta_2)} \left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^\vartheta < 1, \tag{5.18}$$

then the problem (5.1)-(5.3) *has at least one solution in* $PC_{\xi,\alpha}(J)$.

Proof: Consider the set

$$B_\omega = \{x \in PC_{\xi,\alpha}(J) : ||x||_{PC_{\xi,\alpha}} \leq \omega\},$$

where $\omega \geq r_1 + r_2$, with

$$r_1 := \max\left\{\frac{c^*}{\Gamma(\xi)}, \frac{|\phi_3 - c_m\phi_2|}{\Gamma(\xi)|\phi_1|} + \frac{A|\phi_2|}{\Gamma(\vartheta+1)|\phi_1|}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^\vartheta\right\},$$

$$r_2 := \max\left\{\Phi_1 r + \Phi_2, A\left(\frac{\Gamma(\xi)}{\Gamma(\xi+\vartheta)}\right)\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^\vartheta\right\}.$$

We define the operators N_1 and N_2 on B_ω by

$$N_1x(t) = \begin{cases} \frac{1}{\Gamma(\xi)}\left(\frac{t^\alpha - a^\alpha}{\alpha}\right)^{\xi-1}\left[\frac{\phi_3}{\phi_1} - \frac{c_m\phi_2}{\phi_1} - \frac{\phi_2}{\phi_1}\left({}^\alpha\mathcal{J}_{\tau_m^+}^{1-\xi+\vartheta}\varphi\right)(b)\right] & \textit{if } t \in J_0, \\ \frac{c_i}{\Gamma(\xi)}\left(\frac{t^\alpha - \tau_i^\alpha}{\alpha}\right)^{\xi-1} \qquad \textit{if } t \in J_i, i = 1, \dots, m, \\ 0 \qquad \textit{if } t \in \tilde{J}_i, i = 1, \dots, m. \end{cases} \tag{5.19}$$

and

$$N_2x(t) = \begin{cases} \left({}^\alpha\mathcal{J}_{\tau_i^+}^{\vartheta}\varphi\right)(t) & \textit{if } t \in J_i, 0 = 1, \dots, m, \\ \psi_i(t, x(t)) & \textit{if } t \in \tilde{J}_i, i = 1, \dots, m. \end{cases} \tag{5.20}$$

where $i = 0, \dots, m$ and $\varphi : J_i \to \mathbb{R}$ be a function satisfying the functional equation

$$\varphi(t) = f(t, x(t), \varphi(t)).$$

Then the fractional integral equation (5.16) can be written as operator equation

$$\Im x(t) = N_1x(t) + N_2x(t), \quad x \in PC_{\xi,\alpha}(J).$$

We shall use Krasnoselskii's fixed point theorem to prove in several steps that the operator $\Im$ defined in (5.16) has a fixed point.

Step 1: We prove that $N_1x + N_2y \in B_\omega$ for any $x, y \in B_\omega$.
By Remarque (5.3) and from (5.16), We have for each $t \in J_i, i = 0, \dots, m$,

$$\begin{aligned}\left|\left(\frac{t^\alpha-\tau_i^\alpha}{\alpha}\right)^{1-\xi}\varphi(t)\right| &= \left|\left(\frac{t^\alpha-\tau_i^\alpha}{\alpha}\right)^{1-\xi}f(t,x(t),\varphi(t))\right|\\ &\leq \left(\frac{t^\alpha-\tau_i^\alpha}{\alpha}\right)^{1-\xi}\left(\eta_1|x(t)|+\eta_2|\varphi(t)|+f_0\right),\end{aligned}$$

which implies that

$$\left|\left(\frac{t^\alpha-\tau_i^\alpha}{\alpha}\right)^{1-\xi}\varphi(t)\right| \leq \eta_1\left(\frac{b^\alpha-a^\alpha}{\alpha}\right)^{1-\xi}\omega+\eta_2\left|\left(\frac{t^\alpha-\tau_i^\alpha}{\alpha}\right)^{1-\xi}\varphi(t)\right|+f_0\left(\frac{b^\alpha-a^\alpha}{\alpha}\right)^{1-\xi}.$$

Then

$$\max_{i=0,\dots,m}\left\{\sup_{t\in J_i}\left|\left(\frac{t^\alpha-\tau_i^\alpha}{\alpha}\right)^{1-\xi}\varphi(t)\right|\right\} \leq \frac{(\eta_1\omega+f_0)\left(\frac{b^\alpha-a^\alpha}{\alpha}\right)^{1-\xi}}{1-\eta_2} := A.$$

Thus, for $t \in J_0$, by (5.19) and Lemma 1.10,

$$\begin{aligned}\left|\left(\frac{t^\alpha-a^\alpha}{\alpha}\right)^{1-\xi}(N_1x)(t)\right| &\leq \frac{|\phi_3-c_m\phi_2|}{\Gamma(\xi)|\phi_1|}+\frac{|\phi_2|}{\Gamma(\xi)|\phi_1|}\left({}^\alpha\mathcal{J}_{\tau_m^+}^{1-\xi+\vartheta}|\varphi(\tau)|\right)(b)\\ &\leq \frac{|\phi_3-c_m\phi_2|}{\Gamma(\xi)|\phi_1|}+\frac{A|\phi_2|}{\Gamma(\vartheta+1)|\phi_1|}\left(\frac{b^\alpha-a^\alpha}{\alpha}\right)^{\vartheta},\end{aligned}$$

and for $t \in J_i, i = 1, \dots, m$, we have

$$\left|\left(\frac{t^\alpha-\tau_i^\alpha}{\alpha}\right)^{1-\xi}(N_1x)(t)\right| \leq \frac{|c_i|}{\Gamma(\xi)} \leq \frac{c^*}{\Gamma(\xi)},$$

then for each $t \in J$ we get

$$\|N_1x\|_{PC_{\xi,\alpha}} \leq \max\left\{\frac{c^*}{\Gamma(\xi)}, \frac{|\phi_3-c_m\phi_2|}{\Gamma(\xi)|\phi_1|}+\frac{A|\phi_2|}{\Gamma(\vartheta+1)|\phi_1|}\left(\frac{b^\alpha-a^\alpha}{\alpha}\right)^{\vartheta}\right\}. \tag{5.21}$$

For $t \in J_i, i = 0, \dots, m$, by (5.20) and Lemma 1.10, we have

$$\begin{aligned}\left|\left(\frac{t^\alpha-\tau_i^\alpha}{\alpha}\right)^{1-\xi}(N_2y)(t)\right| &\leq \left(\frac{t^\alpha-\tau_i^\alpha}{\alpha}\right)^{1-\xi}\left({}^\alpha\mathcal{J}_{\tau_i^+}^{\vartheta}|\varphi(\tau)|\right)(t)\\ &\leq A\left(\frac{\Gamma(\xi)}{\Gamma(\xi+\vartheta)}\right)\left(\frac{b^\alpha-a^\alpha}{\alpha}\right)^{\vartheta},\end{aligned}$$

and for each $t \in \tilde{J}_i, i = 1, \dots, m$, we have,

$$\begin{aligned}|(N_2y)(t)| &\leq |\psi_i(t,y(t))|\\ &\leq \Phi_1 r+\Phi_2,\end{aligned}$$

then for each $t \in J$ we get

$$\|N_2 y\|_{PC_{\xi,\alpha}} \leq \max\left\{\Phi_1 r + \Phi_2, A\left(\frac{\Gamma(\xi)}{\Gamma(\xi+\vartheta)}\right)\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^\vartheta\right\}. \tag{5.22}$$

From (5.21) and (5.22), for each $t \in J$ we have,

$$\begin{aligned}\|N_1 x + N_2 y\|_{PC_{\xi,\alpha}} &\leq \|N_1 x\|_{PC_{\xi,\alpha}} + \|N_2 y\|_{PC_{\xi,\alpha}}\\ &\leq r_1 + r_2\\ &\leq \omega,\end{aligned}$$

which infers that $N_1 x + N_2 y \in B_\omega$.
Step 2: N_1 is a contraction.
Let $x, y \in PC_{\xi,\alpha}(J)$ and $t \in J$.
By **(Ax_2)**, we have

$$\begin{aligned}|\varphi(t) - \tilde{\varphi}(t)| &= |f(t, x(t), \varphi(t)) - f(t, y(t), \tilde{\varphi}(t))|\\ &\leq \eta_1 |x(t) - y(t)| + \eta_2 |\varphi(t) - \tilde{\varphi}(t)|.\end{aligned}$$

where $\varphi, \tilde{\varphi} \in C_{\xi,\alpha}(J_i); i = 0, \ldots, m$, such that

$$\begin{aligned}\varphi(t) &= f(t, x(t), \varphi(t)),\\ \tilde{\varphi}(t) &= f(t, y(t), \tilde{\varphi}(t)).\end{aligned}$$

Then,

$$|\varphi(t) - \varphi(t)| \leq \frac{\eta_1}{1-\eta_2}|x(t) - y(t)|.$$

Therefore, for $t \in J_0$ we have

$$\begin{aligned}|N_1 x(t) - N_1 y(t)| &\leq \frac{|\phi_2|}{|\phi_1|\Gamma(\xi)}\left(\frac{t^\alpha - a^\alpha}{\alpha}\right)^{\xi-1}\left({}^\alpha\mathcal{J}_{\tau_m^+}^{1-\xi+\vartheta}|\varphi(\tau) - \tilde{\varphi}(\tau)|\right)(b)\\ &\leq \frac{\eta_1}{1-\eta_2}\left[\frac{|\phi_2|}{|\phi_1|\Gamma(\vartheta+1)}\left(\frac{b^\alpha - \tau_m^\alpha}{\alpha}\right)^{\vartheta+\xi-1}\right]\|x-y\|_{PC_{\xi,\alpha}}\end{aligned}$$

hence

$$\left|(N_1 x(t) - N_1 y(t))\left(\frac{t^\alpha - a^\alpha}{\alpha}\right)^{1-\xi}\right| \leq \frac{|\phi_2|\eta_1}{|\phi_1|\Gamma(\vartheta+1)(1-\eta_2)}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^\vartheta \|x-y\|_{PC_{\xi,\alpha}}.$$

Then, for each $t \in J$, we have

$$\|N_1 x - N_1 y\|_{PC_{\xi,\alpha}} \leq \frac{|\phi_2|\eta_1}{|\phi_1|\Gamma(\vartheta+1)(1-\eta_2)}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^\vartheta \|x-y\|_{PC_{\xi,\alpha}}.$$

Then by (5.18), the operator N_1 is a contraction.
Step 3: N_2 is continuous and compact. Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $PC_{\xi,\alpha}(J)$.
Then for each $t \in J_i, i = 0, \ldots, m$, we have,

$$\left|(N_2 x_n)(t) - (N_2 x)(t))\left(\frac{t^\alpha - \tau_i^\alpha}{\alpha}\right)^{1-\xi}\right| \leq \left(\frac{t^\alpha - \tau_i^\alpha}{\alpha}\right)^{1-\xi}\left({}^\alpha\mathcal{J}_{\tau_i^+}^{\vartheta}|\varphi_n(\tau) - \varphi(\tau)|\right)(t),$$

where $h_n, h \in C(J_i, \mathbb{R})$, such that

$$\varphi_n(t) = f(t, x_n(t), \varphi_n(t)),$$
$$\varphi(t) = f(t, x(t), \varphi(t)).$$

For each $t \in \tilde{J}_i, i = 1, \ldots, m$, we have,

$$|(N_2 x_n)(t) - (N_2 x)(t)| \leq |(\psi_i(t, x_n(t)) - \psi_i(t, x(t)))|.$$

Since $x_n \to x$, then we get $\varphi_n(t) \to \varphi(t)$ as $n \to \infty$ for each $t \in J_i; i = 0, \ldots, m$. By Lebesgue's dominated convergence Theorem and since ψ_i are continuous, we have

$$\|N_2 x_n - N_2 x\|_{PC_{\xi,\alpha}} \to 0 \text{ as } n \to \infty.$$

Then N_2 is continuous. Next we prove that N_2 is uniformly bounded on B_ω. Let any $y \in B_\omega$. We have form step 1 that for each $t \in J$

$$\|N_2 y\|_{PC_{\xi,\alpha}} \leq \max\left\{\Phi_1 r + \Phi_2, A\left(\frac{\Gamma(\xi)}{\Gamma(\xi+\vartheta)}\right)\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^\vartheta\right\}.$$

This prove that the operator N_2 is uniformly bounded on B_ω. To prove the compactness of N_2, we take $y \in B_\omega$ and $a < \varepsilon_1 < \varepsilon_2 \leq b$. Then for $\varepsilon_1, \varepsilon_2 \in J_i; i = 0, \ldots, m$,

$$\begin{aligned}
&\left|\left(\frac{\varepsilon_1^\alpha - \tau_i^\alpha}{\alpha}\right)^{1-\xi}(N_2 y)(\varepsilon_1) - \left(\frac{\varepsilon_2^\alpha - \tau_i^\alpha}{\alpha}\right)^{1-\xi}(N_2 y)(\varepsilon_2)\right| \\
&\leq \left|\left(\frac{\varepsilon_1^\alpha - \tau_i^\alpha}{\alpha}\right)^{1-\xi}\left({}^\alpha\mathcal{J}_{\tau_i^+}^\vartheta \varphi(\tau)\right)(\varepsilon_1) - \left(\frac{\varepsilon_2^\alpha - \tau_i^\alpha}{\alpha}\right)^{1-\xi}\left({}^\alpha\mathcal{J}_{\tau_i^+}^\vartheta \varphi(\tau)\right)(\varepsilon_2)\right| \\
&\leq \left(\frac{\varepsilon_2^\alpha - \tau_i^\alpha}{\alpha}\right)^{1-\xi}\left({}^\alpha\mathcal{J}_{\varepsilon_1^+}^\vartheta |\varphi(\tau)|\right)(\varepsilon_2) + \frac{1}{\Gamma(\vartheta)}\int_{\tau_i}^{\varepsilon_1}\left|\tau^{\alpha-1}H(\tau)\varphi(\tau)\right| d\tau,
\end{aligned}$$

where $H(\tau) = \left[\left(\frac{\varepsilon_1^\alpha - \tau_i^\alpha}{\alpha}\right)^{1-\xi}\left(\frac{\varepsilon_1^\alpha - \tau^\alpha}{\alpha}\right)^{\vartheta-1} - \left(\frac{\varepsilon_2^\alpha - \tau_i^\alpha}{\alpha}\right)^{1-\xi}\left(\frac{\varepsilon_2^\alpha - \tau^\alpha}{\alpha}\right)^{\vartheta-1}\right]$.
Then by Lemma 1.10, we have

$$\begin{aligned}
&\left|\left(\frac{\varepsilon_1^\alpha - \tau_i^\alpha}{\alpha}\right)^{1-\xi}(N_2 y)(\varepsilon_1) - \left(\frac{\varepsilon_2^\alpha - \tau_i^\alpha}{\alpha}\right)^{1-\xi}(N_2 y)(\varepsilon_2)\right| \\
&\leq \frac{A\Gamma(\xi)}{\Gamma(\vartheta+\xi)}\left(\frac{\varepsilon_2^\alpha - \tau_i^\alpha}{\alpha}\right)^{1-\xi}\left(\frac{\varepsilon_2^\alpha - \varepsilon_1^\alpha}{\alpha}\right)^{\vartheta+\xi-1} + A\int_{\tau_i}^{\varepsilon_1}\left|H(\tau)\frac{\tau^{\alpha-1}}{\Gamma(\vartheta)}\right|\left(\frac{\tau^\alpha - \tau_i^\alpha}{\alpha}\right)^{\xi-1} d\tau,
\end{aligned}$$

note that

$$\left|\left(\frac{\varepsilon_1^\alpha - \tau_i^\alpha}{\alpha}\right)^{1-\xi}(N_2 y)(\varepsilon_1) - \left(\frac{\varepsilon_2^\alpha - \tau_i^\alpha}{\alpha}\right)^{1-\xi}(N_2 y)(\varepsilon_2)\right| \to 0 \quad \text{as} \quad \varepsilon_1 \to \varepsilon_2.$$

And for $\varepsilon_1, \varepsilon_2 \in \tilde{J}_i; i = 1, \ldots, m$,

$$|(N_2 y)(\varepsilon_1) - (N_2 y)(\varepsilon_2)| \leq |\psi_i(\varepsilon_1, y(\varepsilon_1)) - \psi_i(\varepsilon_2, y(\varepsilon_2))|,$$

note since ψ_i are continuous that

$$|(N_2y)(\varepsilon_1) - (N_2y)(\varepsilon_2)| \to 0 \quad \text{as} \quad \varepsilon_1 \to \varepsilon_2.$$

This proves that N_2B_ω is equicontinuous on J. Therefore N_2B_ω is relatively compact. By $PC_{\xi,\alpha}$ type Arzela-Ascoli Theorem N_2 is compact. As a consequence of Theorem 1.28, we deduce that $\Im$ has at least a fixed point $x^* \in PC_{\xi,\alpha}(J)$ and by the same way of the proof of Theorem 5.4, we can easily show that $x^* \in PC^{\xi}_{\xi,\alpha}(J)$. Using Lemma 5.2, we conclude that the problem (5.1)-(5.3) has at least one solution in the space $PC_{\xi,\alpha}(J)$. ■

5.3 Ulam-Hyers-Rassias Stability

Now, we consider the Ulam stability for problem (5.1)−(5.3). Let $x \in PC_{\xi,\alpha}(J)$, $\theta > 0$, $\mu > 0$ and $\chi : J \longrightarrow [0, \infty)$ be a continuous function. We consider the following inequality :

$$\begin{cases} \left|\left({}^{\alpha}\mathcal{D}^{\vartheta,r}_{\tau_i^+}x\right)(t) - f\left(t, x(t), \left({}^{\alpha}\mathcal{D}^{\vartheta,r}_{\tau_i^+}x\right)(t)\right)\right| \le \theta, t \in J_i, i = 0, \dots, m, \\ |x(t) - \psi_i(t, x(t))| \le \theta, t \in \tilde{J}_i, i = 1, \dots, m. \end{cases} \tag{5.23}$$

$$\begin{cases} \left|\left({}^{\alpha}\mathcal{D}^{\vartheta,r}_{\tau_i^+}x\right)(t) - f\left(t, x(t), \left({}^{\alpha}\mathcal{D}^{\vartheta,r}_{\tau_i^+}x\right)(t)\right)\right| \le \chi(t), t \in J_i, i = 0, \dots, m, \\ |x(t) - \psi_i(t, x(t))| \le \mu, t \in \tilde{J}_i, i = 1, \dots, m. \end{cases} \tag{5.24}$$

and

$$\begin{cases} \left|\left({}^{\alpha}\mathcal{D}^{\vartheta,r}_{\tau_i^+}x\right)(t) - f\left(t, x(t), \left({}^{\alpha}\mathcal{D}^{\vartheta,r}_{\tau_i^+}x\right)(t)\right)\right| \le \theta\chi(t), t \in J_i, i = 0, \dots, m, \\ |x(t) - \psi_i(t, x(t))| \le \theta\mu, t \in \tilde{J}_i, i = 1, \dots, m. \end{cases} \tag{5.25}$$

Definition 5.6. *([108, 110]) Problem (5.1)-(5.3) is Ulam-Hyers (U-H) stable if there exists a real number $a_f > 0$ such that for each $\theta > 0$ and for each solution $x \in PC_{\xi,\alpha}(J)$ of inequality (5.23) there exists a solution $y \in PC_{\xi,\alpha}(J)$ of (5.1)-(5.3) with*

$$|x(t) - y(t)| \le \theta a_f, \qquad t \in J.$$

Definition 5.7. *([108, 110]) Problem (5.1)-(5.3) is generalized Ulam-Hyers (G.U-H) stable if there exists $K_f : C([0, \infty), [0, \infty))$ with $K_f(0) = 0$ such that for each $\theta > 0$ and for each solution $x \in PC_{\xi,\alpha}(J)$ of inequality (5.23) there exists a solution $y \in PC_{\xi,\alpha}(J)$ of (5.1)-(5.3) with*

$$|x(t) - y(t)| \le K_f(\theta), \qquad t \in J.$$

Definition 5.8. *([108, 110]) Problem (5.1)-(5.3) is Ulam-Hyers-Rassias (U-H-R) stable with respect to (χ, μ) if there exists a real number $a_{f,\chi} > 0$ such that for each $\theta > 0$ and for each solution $x \in PC_{\xi,\alpha}(J)$ of inequality (5.25) there exists a solution $y \in PC_{\xi,\alpha}(J)$ of (5.1)-(5.3) with*

$$|x(t) - y(t)| \le \theta a_{f,\chi}(\chi(t) + \mu), \qquad t \in J.$$

Definition 5.9. *([108,110]) Problem (5.1)-(5.3) is generalized Ulam-Hyers-Rassias (G.U-H-R) stable with respect to (χ,μ) if there exists a real number $a_{f,\chi}>0$ such that for each solution $x\in PC_{\xi,\alpha}(J)$ of inequality (5.25) there exists a solution $y\in PC_{\xi,\alpha}(J)$ of (5.1)-(5.3) with*

$$|x(t)-y(t)|\leq a_{f,\chi}(\chi(t)+\mu),\qquad t\in J.$$

Remark 5.10. *It is clear that :*

1. *Definition 5.6 $\Longrightarrow$ Definition 5.7*
2. *Definition 5.8 $\Longrightarrow$ Definition 5.9*
3. *Definition 5.8 for $\chi(.)=\mu=1\Longrightarrow$ Definition 5.6*

Remark 5.11. *([108,110]) A function $x\in PC_{\xi,\alpha}(J)$ is a solution of inequality (5.25) if and only if there exist $\upsilon\in PC_{\xi,\alpha}(J)$ and a sequence $\upsilon_i, i=0,\ldots,m$ such that*

1. $|\upsilon(t)|\leq\theta\chi(t)$, $t\in J_i, i=0,\ldots,m$; *and* $|\upsilon_i|\leq\theta\mu$, $t\in\tilde{J}_i, i=1,\ldots,m$,
2. $\left({}^{\alpha}\mathcal{D}^{\vartheta,r}_{\tau_i^+}x\right)(t)=f\left(t,x(t),\left({}^{\alpha}\mathcal{D}^{\vartheta,r}_{\tau_i^+}x\right)(t)\right)+\upsilon(t)$, $t\in J_i, i=0,\ldots,m$,
3. $x(t)=\psi_i(t,x(t))+\upsilon_i$, $t\in\tilde{J}_i, i=1,\ldots,m$.

Theorem 5.12. *Assume that in addition to **(Ax1)** – **(Ax3)** and (5.17), the following hypothesis holds:*

(Ax5) *There exist a nondecreasing function $\chi:J\longrightarrow[0,\infty)$ and $\kappa_\chi>0$ such that for each $t\in J_i; i=0,\ldots,m$, we have*

$$({}^{\alpha}\mathcal{J}^{\vartheta}_{\tau_i^+}\chi)(t)\leq\kappa_\chi\chi(t).$$

Then the problem (5.1)-(5.3) is U-H-R stable with respect to (χ,μ).

Proof: Let $x\in PC_{\xi,\alpha}(J)$ be a solution if inequality (5.25), and let us assume that y is the unique solution of the problem

$$\begin{cases}\left({}^{\alpha}\mathcal{D}^{\vartheta,r}_{\tau_i^+}y\right)(t)=f\left(t,y(t),\left({}^{\alpha}\mathcal{D}^{\vartheta,r}_{\tau_i^+}y\right)(t)\right);\ t\in J_i,\ \ i=0,\ldots,m,\\ y(t)=\psi_i(t,y(t));\ t\in\tilde{J}_i,\ \ i=1,\ldots,m,\\ \phi_1\left({}^{\alpha}\mathcal{J}^{1-\xi}_{a^+}y\right)(a^+)+\phi_2\left({}^{\alpha}\mathcal{J}^{1-\xi}_{m^+}y\right)(b)=\phi_3,\\ \left({}^{\alpha}\mathcal{J}^{1-\xi}_{\tau_i^+}y\right)(\tau_i)=\left({}^{\alpha}\mathcal{J}^{1-\xi}_{\tau_i^+}x\right)(\tau_i),\ \ i=0,\ldots,m.\end{cases}$$

By Lemma 5.2, we obtain for each $t\in J$

$$y(t)=\begin{cases}\dfrac{\bar{c}}{\Gamma(\xi)}\left(\dfrac{t^\alpha-a^\alpha}{\alpha}\right)^{\xi-1}+\left({}^{\alpha}\mathcal{J}^{\vartheta}_{a^+}\varphi\right)(t) & if\ t\in J_0,\\[2ex] \dfrac{\left({}^{\alpha}\mathcal{J}^{1-\xi}_{\tau_i^+}y\right)(\tau_i)}{\Gamma(\xi)}\left(\dfrac{t^\alpha-\tau_i^\alpha}{\alpha}\right)^{\xi-1}+\left({}^{\alpha}\mathcal{J}^{\vartheta}_{\tau_i^+}\varphi\right)(t) & if\ t\in J_i, i=1,\ldots,m,\\[2ex] \psi_i(t,y(t)) & if\ t\in\tilde{J}_i, i=1,\ldots,m.\end{cases}$$

where $\varphi \in C_{\xi,\alpha}(J_i); i = 0, \dots, m$, be a function satisfying the functional equation

$$\varphi(t) = f(t, y(t), \varphi(t)).$$

and $\bar{c} = \frac{\phi_3}{\phi_1} - \frac{\phi_2}{\phi_1}\left({}^\alpha\mathcal{J}^{1-\xi}_{\tau_m^+} y\right)(\tau_m) - \frac{\phi_2}{\phi_1}\left({}^\alpha\mathcal{J}^{1-\xi+\vartheta}_{\tau_m^+}\varphi\right)(b).$

Since x is a solution of the inequality (5.25), by Remark 5.11, we have

$$\begin{cases} \left({}^\alpha\mathcal{D}^{\vartheta,r}_{\tau_i^+}x\right)(t) = f\left(t, x(t), \left({}^\alpha\mathcal{D}^{\vartheta,r}_{\tau_i^+}x\right)(t)\right) + \upsilon(t), t \in J_i, i = 0, \dots, m; \\ x(t) = \psi_i(t, x(t)) + \upsilon_i, t \in \tilde{J}_i, i = 1, \dots, m. \end{cases} \tag{5.26}$$

Clearly, the solution of (5.26) is given by

$$x(t) = \begin{cases} \frac{\left({}^\alpha\mathcal{J}^{1-\xi}_{\tau_i^+}x\right)(\tau_i)}{\Gamma(\xi)}\left(\frac{t^\alpha - \tau_i^\alpha}{\alpha}\right)^{\xi-1} + \left({}^\alpha\mathcal{J}^{\vartheta}_{\tau_i^+}(\tilde{\varphi} + \upsilon)\right)(t) \quad if\ t \in J_i, i = 1, \dots, m, \\ \\ \psi_i(t, x(t)) + \upsilon_i \qquad if\ t \in \tilde{J}_i, i = 1, \dots, m, \end{cases}$$

where $\tilde{\varphi} : J_i \to \mathbb{R}, i = 0, \dots, m$, be a function satisfying the functional equation

$$\tilde{\varphi}(t) = f(t, x(t), \tilde{\varphi}(t)).$$

Hence, for each $t \in J_i$,$i = 0, \dots, m$, we have

$$\begin{aligned} |x(t) - y(t)| &\leq \left({}^\alpha\mathcal{J}^{\vartheta}_{\tau_i^+}|\tilde{\varphi}(\tau) - \varphi(\tau)|\right)(t) + \left({}^\alpha\mathcal{J}^{\vartheta}_{\tau_i^+}|\upsilon(\tau)|\right) \\ &\leq \theta\kappa_\chi\chi(t) + \frac{\eta_1}{(1-\eta_2)}\int_{\tau_i}^t \tau^{\alpha-1}\left(\frac{t^\alpha - \tau^\alpha}{\alpha}\right)^{\vartheta-1}\frac{|x(\tau) - y(\tau)|}{\Gamma(\vartheta)}d\tau. \end{aligned}$$

We apply Lemma 1.20 to obtain

$$\begin{aligned} |x(t) - y(t)| &\leq \theta\kappa_\chi\chi(t) + \int_{\tau_i}^t\sum_{k=1}^{\infty}\frac{\left(\frac{\eta_1}{1-\eta_2}\right)^k}{\Gamma(k\vartheta)}\tau^{\alpha-1}\left(\frac{t^\alpha - \tau^\alpha}{\alpha}\right)^{k\vartheta-1}(\theta\kappa_\chi\chi(\tau))d\tau \\ &\leq \theta\kappa_\chi\chi(t)E_\vartheta\left[\frac{\eta_1}{1-\eta_2}\left(\frac{t^\alpha - \tau_i^\alpha}{\alpha}\right)^\vartheta\right] \\ &\leq \theta\kappa_\chi\chi(t)E_\vartheta\left[\frac{\eta_1}{1-\eta_2}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^\vartheta\right]. \end{aligned}$$

And for each $t \in \tilde{J}_i$,$i = 1, \dots, m$, we have

$$\begin{aligned} |x(t) - y(t)| &\leq |\psi_i(t, x(t)) - \psi_i(t, y(t))| + |\upsilon_i| \\ &\leq K^*|x(t) - y(t)| + \theta\mu, \end{aligned}$$

then by 5.17, we have

$$|x(t) - y(t)| \leq \frac{\theta\mu}{1 - K^*}.$$

Then for each $t \in J$, we have

$$|x(t) - y(t)| \leq a_\chi\theta(\mu + \chi(t)),$$

where

$$a_\chi = \frac{1}{1 - K^*} + \kappa_\chi E_\vartheta\left[\frac{\eta_1}{1-\eta_2}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^\vartheta\right].$$

Hence, the problem (5.1)–(5.3) is U-H-R stable with respect to (χ, τ). ■

Remark 5.13. *If the conditions* $(\boldsymbol{Ax_1})$ – $(\boldsymbol{Ax_3})$, $(\boldsymbol{Ax_5})$ *and (5.17) are satisfied, then by Theorem 5.12 and Remark 5.10, it is clear that problem* (5.1)-(5.3) *is U-H-R stable and G.U-H-R stable. And if* $\chi(.) = \mu = 1$, *then problem* (5.1)-(5.3) *is also G.U-H stable and U-H stable.*

Remark 5.14. *Our results for the boundary value problem (5.1)–(5.3) apply in the following cases :*

- *Initial value problems :* $\phi_1 = 1,\ \phi_2 = 0$.
- *Anti-periodic problems :* $\phi_1 = 1,\ \phi_2 = 1,\ \phi_3 = 0$.
- *Periodic problems :* $\phi_1 = 1,\ \phi_2 = -1,\ \phi_3 = 0$.

5.4 An Example

Consider the following impulsive periodic problem of generalized Hilfer fractional differential equation

$$\left({}^{\frac{1}{2}}\mathcal{D}^{\frac{1}{2},0}_{\tau_i^+}x\right)(t) = \frac{|cos(t)|e^{-2t} + |sin(t)|}{122e^{t+2}(1 + |x(t)| + |{}^{\frac{1}{2}}\mathcal{D}^{\frac{1}{2},0}_{\tau_i^+}x(t)|)}, \text{ for each } t \in J_0 \cup J_1, \tag{5.27}$$

$$x(t) = \frac{|x(t)|}{5e^t + 3|x(t)|}, \text{ for each } t \in \tilde{J}_1, \tag{5.28}$$

$$\left({}^{\frac{1}{2}}\mathcal{J}^{\frac{1}{2}}_{1^+}x\right)(1^+) = \left({}^{\frac{1}{2}}\mathcal{J}^{\frac{1}{2}}_{3^+}x\right)(\pi), \tag{5.29}$$

where $J_0 = (1, e]$, $J_1 = (3, \pi]$, $\tilde{J}_1 = (e, 3]$, $s_0 = 1$, $t_1 = e$ and $s_1 = 3$.
Set

$$f(t, u, w) = \frac{|cos(t)|e^{-2t} + |sin(t)|}{122e^{t+2}(1 + |x| + |y|)}, \ t \in J_0 \cup J_1, \ x, y \in \mathbb{R}.$$

We have

$$\begin{aligned} C^{r(1-\vartheta)}_{\xi,\alpha}((1, e]) &= C^{0}_{\frac{1}{2},\frac{1}{2}}((1, e]) \\ &= \left\{u : (1, e] \to \mathbb{R} : \sqrt{2}\left(\sqrt{t} - 1\right)^{\frac{1}{2}} u \in C([1, e], \mathbb{R})\right\}, \end{aligned}$$

and

$$\begin{aligned} C^{r(1-\vartheta)}_{\xi,\alpha}((3, \pi]) &= C^{0}_{\frac{1}{2},\frac{1}{2}}((3, \pi]) \\ &= \left\{u : (3, \pi] \to \mathbb{R} : \sqrt{2}\left(\sqrt{t} - \sqrt{3}\right)^{\frac{1}{2}} u \in C([3, \pi], \mathbb{R})\right\}, \end{aligned}$$

with

$$\xi = \vartheta = \tfrac{1}{2},\ \alpha = \tfrac{1}{2},\ r = 0, \text{ and } i \in \{0, 1\}.$$

Clearly, the continuous function $f \in C^0_{\frac{1}{2},\frac{1}{2}}((1,e]) \cap C^0_{\frac{1}{2},\frac{1}{2}}((3,\pi])$. Hence the condition ***(Ax₁)*** is satisfied.
For each $x, \bar{x}, y, \bar{y} \in \mathbb{R}$ and $t \in J_0 \cup J_1$, we have

$$\begin{aligned}|f(t,x,y) - f(t,\bar{x},\bar{y})| &\leq \frac{|cos(t)|e^{-2t} + |sin(t)|}{122e^{t+2}}(|x-\bar{x}| + |y-\bar{y}|)\\ &\leq \frac{1+e^2}{122e^5}\left(|x-\bar{x}| + |y-\bar{y}|\right).\end{aligned}$$

Hence condition ***(Ax₂)*** is satisfied with $\eta_1 = \eta_2 = \dfrac{1+e^2}{122e^5}$.
And let

$$\psi(x) = \frac{x}{5e^t + 3x}, u \in [0,\infty).$$

Let $x, y \in [0,\infty)$. Then we have

$$|\psi(x) - \psi(y)| = |\frac{x}{5e^t+3x} - \frac{y}{5e^t+3y}| = \frac{5e^t|x-y|}{(5e^t+3x)(5e^t+3y)} \leq \frac{1}{5e}|x-y|,$$

and so the condition ***(Ax₃)*** is satisfied with $K^* = \dfrac{1}{5e}$.
Also, the condition (5.17) of Theorem 5.4 is satisfied, for

$$\begin{aligned}\tilde{\ell} &= \max\left\{K^*, \frac{\eta_1}{1-\eta_2}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^\vartheta \left[\frac{|\phi_2|}{|\phi_1|\Gamma(\vartheta+1)} + \frac{\Gamma(\xi)}{\Gamma(\xi+\vartheta)}\right]\right\}\\ &= \max\left\{\frac{1}{5e}, \frac{\sqrt{2}(1+e^2)}{122e^5 - e^2 - 1}\left(\sqrt{\pi}-1\right)^{\frac{1}{2}}\left[\frac{1}{\Gamma(\frac{3}{2})} + \sqrt{\pi}\right]\right\}\\ &\approx \max\{0.0735758882, 0.00167130655\}\\ &= 0.00167130655 < 1.\end{aligned}$$

Then the problem (5.27)–(5.29) has a unique solution in $PC_{\frac{1}{2},\frac{1}{2}}([1,\pi])$.
Hypothesis ***(Ax₅)*** is satisfied with $\mu = 1$ and

$$\chi(t) = \begin{cases} \dfrac{1}{\sqrt{2(\sqrt{t} - \sqrt{\tau_i})}}, & if\ t \in J_0 \cup J_1,\\[2ex] \pi, \qquad if\ t \in \tilde{J}_1, \end{cases}$$

and $\kappa_\chi = \sqrt{2\pi}(\sqrt{e}-1)^{\frac{1}{2}}$. Indeed, for each $t \in J_0 \cup J_1$, we get

$$({}^{\frac{1}{2}}\mathcal{J}^{\frac{1}{2}}_{1^+}\chi)(t) \leq \frac{\sqrt{2\pi}(\sqrt{\pi}-\sqrt{3})^{\frac{1}{2}}}{\sqrt{2(\sqrt{t}-1)}},$$

and

$$({}^{\frac{1}{2}}\mathcal{J}^{\frac{1}{2}}_{3^+}\chi)(t) \leq \frac{\sqrt{2\pi}(\sqrt{e}-1)^{\frac{1}{2}}}{\sqrt{2(\sqrt{t}-\sqrt{3})}}.$$

Consequently, Theorem 5.12 implies that the problem (5.27)–(5.29) is U-H-R stable.

Chapter 6

Initial Value Problem for Hybrid Fractional Implicit Differential Equations

6.1 Introduction and Motivations

In this chapter, we prove some existence results of solutions for a class of initial value problem for nonlinear fractional hybrid implicit differential equations . First, the problem studied is with Generalized Hilfer fractional derivative. Next, we deals with a problem of ψ-Hilfer fractional derivative. The results are based on fixed point theorems due to Dhage. Further, examples are provided to illustrate our results. We took as motivation the following papers [1, 10–14, 25, 27, 38, 44, 46, 91, 92, 95] and the references therein. One should hope to find in these listed papers some fundamental results in the theory of fractional calculus and fractional differential equations. Another interesting class of problems involves hybrid fractional differential equations appeared recently and has achieved a great deal of interest and attention of several researchers. For some recent results on this type of problems, we refer the reader, for example, to references [28, 36, 55, 64, 112].

In [43], the authors discussed the following terminal value problem for fractional differential equations with generalized Hilfer fractional derivative :

$$\begin{cases} \left({}^{\alpha}\mathcal{D}_{a^+}^{\vartheta,r}x\right)(t) = f\left(t, x(t), \left({}^{\alpha}\mathcal{D}_{a^+}^{\vartheta,r}x\right)(t)\right), & t \in I := [a,T],\ a > 0, \\ x(T) = c \in \mathbb{R}, \end{cases}$$

where ${}^{\alpha}\mathcal{D}_{a^+}^{\vartheta,r}$ is the generalized Hilfer type fractional derivative of order $\vartheta \in (0,1)$ and type $r \in [0,1]$ and $f : (a,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function.

Wang and Zhang [111] proved some existence results for the following nonlocal initial value problem for differential equations involving Hilfer's fractional derivative :

$$\begin{cases} D_{a^+}^{\vartheta,r}u(t) = f(t,u(t)), & t \in (a,b], \\ \left(I_{a^+}^{1-\xi}u\right)(a^+) = \displaystyle\sum_{i=1}^{m} \lambda_i u(\tau_i), \end{cases}$$

where $D_{a^+}^{\vartheta,r}, I_{a^+}^{1-\xi}$ are the left-sided Hilfer fractional derivative of order $\vartheta \in (0,1)$ and type $r \in [0,1]$ and the left-sided Riemann-Liouville fractional integral of order $1-\xi, (\xi = \vartheta + r - \vartheta r)$ respectively, $f : (a,b] \times \mathbb{R} \to \mathbb{R}$ is a given function, $\lambda_i, i = 1,\dots,m$, are real numbers and $\tau_i, i = 1,\dots,m$, are pre-fixed points satisfying $a < \tau_1 \leq \ldots \leq \tau_m < b$.

Zhao *et al.* [112] discussed the the following hybrid differential equations involving Riemann-Liouville fractional derivative :

$$\begin{cases} {}^{RL}\mathbb{D}^r\left(\dfrac{u(t)}{g(t,u(t))}\right) = h(t,u(t)),\ t \in I := [0,T], \\ u(0) = 0, \end{cases}$$

where $0 < r < 1$, $g \in C(I \times \mathbb{R}, \mathbb{R}\backslash\{0\})$ and $h \in C(I \times \mathbb{R}, \mathbb{R})$.

Derbazi *et al.* [55] studied the existence and uniqueness of solutions of the following three-point boundary value problem for fractional hybrid differential equations with Caputo's fractional derivative :

$$\begin{cases} {}^c\mathcal{D}^{\vartheta}_{0^+}\left(\frac{u(t)-f(t,u(t))}{g(t,u(t))}\right) = h(t,u(t)),\ t \in J := [0,T], \\ a_1\left(\frac{u(0)-f(0,u(0))}{g(0,u(0))}\right) + b_1\left(\frac{u(T)-f(T,u(T))}{g(T,u(T))}\right) = c_1, \\ a_2{}^c\mathcal{D}^{\beta}_{0^+}\left(\frac{u(t)-f(t,u(t))}{g(t,u(t))}\right)_{t=\eta} + b_2{}^c\mathcal{D}^{\beta}_{0^+}\left(\frac{u(t)-f(t,u(t))}{g(t,u(t))}\right)_{t=T} = c_2, \end{cases}$$

where $1 < \vartheta \le 2$, $0 < \beta \le 1$, $0 < \eta < T$, $g \in C(I \times \mathbb{R}, \mathbb{R}\backslash\{0\})$, $f, h \in C(I \times \mathbb{R}, \mathbb{R})$ and $a_i, b_i, c_i \in \mathbb{R}$, with $i = 1, 2$ such that $a_1 + b_1 \neq 0$, $a_2\eta^{1-\beta} + b_2T^{1-\beta} \neq 0$.

In [101], Sousa and Oliveira proved some existence, uniqueness and stability results for following initial value problem for fractional differential equations involving ψ-Hilfer derivative :

$$\begin{cases} {}^H\mathbb{D}^{\vartheta,r;\psi}_{a^+}y(t) = f\left(t, y(t), {}^H\mathbb{D}^{\vartheta,r;\psi}_{a^+}y(t)\right) \\ \mathbb{J}^{1-\xi;\psi}_{a^+}y(a) = y_a, \end{cases}$$

where ${}^H\mathbb{D}^{\vartheta,r;\psi}_{a^+}, \mathbb{J}^{1-\xi;\psi}_{a^+}$ are the ψ-Hilfer fractional derivative of order $\vartheta \in (0,1)$ and type $r \in [0,1]$ and ψ-Riemann-Liouville fractional integral of order $1-\xi, (\xi = \vartheta + r - \vartheta r)$ respectively, $y_a \in \mathbb{R}$ and $f \in C([a,T] \times \mathbb{R}^2, \mathbb{R})$. the existence result is based on Banach's contraction principle.

6.2 Nonlocal Initial Value Problem for Hybrid Generalized Hilfer-type Fractional Implicit Differential Equations [1]

Motivated by the works of the papers mentioned in the introduction of the chapter, in this section, we establish existence results to the nonlocal initial value problem with nonlinear implicit hybrid Generalized Hilfer-type fractional differential equation :

$$ {}^{\alpha}\mathcal{D}_{a^+}^{\vartheta,r}\left(\frac{x(t)-\chi(t,x(t))}{f(t,x(t))}\right)=\varphi\left(t,x(t),{}^{\alpha}\mathcal{D}_{a^+}^{\vartheta,r}\left(\frac{x(t)-\chi(t,x(t))}{f(t,x(t))}\right)\right),\quad t\in(a,b], \tag{6.1}$$

$$\left({}^{\alpha}\mathcal{J}_{a^+}^{1-\xi}\left(\frac{x(\tau)-\chi(t,x(t))}{f(\tau,x(\tau))}\right)\right)(a^+)=\sum_{i=1}^{m}c_i\left(\frac{x(\epsilon_i)-\chi(\epsilon_i,x(\epsilon_i))}{f(\epsilon_i,x(\epsilon_i))}\right), \tag{6.2}$$

where ${}^{\alpha}\mathcal{D}_{a^+}^{\vartheta,r}, {}^{\alpha}\mathcal{J}_{a^+}^{1-\xi}$ are the generalized Hilfer fractional derivative of order $\vartheta\in(0,1)$ and type $r\in[0,1]$ and generalized fractional integral of order $1-\xi$, $(\xi=\vartheta+r-\vartheta r)$ respectively, $c_i, i=1,\dots,m$, are real numbers, $\epsilon_i, i=1,\dots,m$, are pre-fixed points satisfying $a<\epsilon_1\le\dots\le\epsilon_m<b$, $f\in C([a,b]\times\mathbb{R},\mathbb{R}\backslash\{0\})$, $\chi\in C([a,b]\times\mathbb{R},\mathbb{R})$, $\varphi\in C([a,b]\times\mathbb{R}^2,\mathbb{R})$ and $\sum_{i=1}^{m}c_i\bar{\Psi}_{\xi}(\epsilon_i,a)\neq 1$, for further details see the definitions in the following subsection.

6.2.1 Existence Results

Consider the following parameters ϑ, r, ξ satisfying

$$\xi=\vartheta+r-\vartheta r,\quad 0<\vartheta,r,\xi<1.$$

Consider the weighted Banach space

$$C_{\xi,\alpha}(J)=\left\{x:J\to\mathbb{R}:\ \text{the function } t\to\Psi_{\xi}(t,a)x(t)\in C([a,b],\mathbb{R})\right\},$$

where $\bar{\Psi}_{\vartheta}(t,a)=\dfrac{\alpha^{1-\vartheta}}{\Gamma(\vartheta)}\left(t^{\alpha}-a^{\alpha}\right)^{\vartheta-1}$, $\Psi_{\xi}(t,a)=\alpha^{\xi-1}\left(t^{\alpha}-a^{\alpha}\right)^{1-\xi}$, and

$$\begin{aligned}&C_{\xi,\alpha}^{n}(J)=\left\{x\in C^{n-1}(J):x^{(n)}\in C_{\xi,\alpha}(J)\right\},n\in\mathbb{N},\\&C_{\xi,\alpha}^{0}(J)=C_{\xi,\alpha}(J).\end{aligned}$$

with the norm

$$\|x\|_{C_{\xi,\alpha}}=\sup_{t\in[a,b]}\left|\Psi_{\xi}(t,a)x(t)\right|.$$

We consider the following fractional differential equation

$${}^{\alpha}\mathcal{D}_{a^+}^{\vartheta,r}\left(\frac{x(t)-\chi(t,x(t))}{f(t,x(t))}\right)=v(t),\quad t\in J, \tag{6.3}$$

where $0<\vartheta<1, 0\le r\le 1, \alpha>0$, with the nonlocal condition

$$\left({}^{\alpha}\mathcal{J}_{a^+}^{1-\xi}\left(\frac{x(\tau)-\chi(t,x(t))}{f(\tau,x(\tau))}\right)\right)(a^+)=\sum_{i=1}^{m}c_i\left(\frac{x(\epsilon_i)-\chi(\epsilon_i,x(\epsilon_i))}{f(\epsilon_i,x(\epsilon_i))}\right), \tag{6.4}$$

1. **A. Salim**, M. Benchohra, J. E. Lazreg, J. J. Nieto and Y. Zhou, Nonlocal Initial Value Problem for Hybrid Generalized Hilfer-type Fractional Implicit Differential Equations. *Nonauton. Dyn. Syst.* **8** (2021), 87-100.

where $\xi = \vartheta + r - \vartheta r$, $c_i, i = 1, \ldots, m$, are real numbers, $f \in C([a,b] \times \mathbb{R}, \mathbb{R}\backslash\{0\})$, $\chi \in C([a,b] \times \mathbb{R}, \mathbb{R})$, $\epsilon_i, i = 1, \ldots, m$, are pre-fixed points satisfying $a < \epsilon_1 \leq \ldots \leq \epsilon_m < b$, $v \in C_{\xi,\alpha}(J)$ and $\sum_{i=1}^{m} c_i \bar{\Psi}_\xi(\epsilon_i, a) \neq 1$. The following theorem shows that the problem (6.3)-(6.4) have a solution given by

$$x(t) = f(t,x(t)) \left[\bar{\Psi}_\xi(t,a) \frac{\sum_{i=1}^{m} c_i \left({}^{\alpha}\mathcal{J}_{a^+}^{\vartheta} v(\tau) \right)(\epsilon_i)}{1 - \sum_{i=1}^{m} c_i \bar{\Psi}_\xi(\epsilon_i, a)} + \left({}^{\alpha}\mathcal{J}_{a^+}^{\vartheta} v(\tau) \right)(t) \right] + \chi(t, x(t)). \quad (6.5)$$

Theorem 6.1. *Let $\xi = \vartheta + r - \vartheta r$, where $0 < \vartheta < 1$ and $0 \leq r \leq 1$. If $v : J \to \mathbb{R}$ is a function such that $v(\cdot) \in C_{\xi,\alpha}(J)$, $f \in C([a,b] \times \mathbb{R}, \mathbb{R}\backslash\{0\})$, and the function $\chi \in C([a,b] \times \mathbb{R}, \mathbb{R})$, then x satisfies equations (6.3) and (6.4) if and only if it satisfies (6.5).*

Proof: Assume x satisfies the equations (6.3) and (6.4) and such that the function $\sigma : t \longrightarrow \left(\frac{x(t) - \chi(t,x(t))}{f(t,x(t))} \right) \in C_{\xi,\alpha}^{\xi}(J)$. We prove that x is a solution to the equation (6.5). From the definition of the space $C_{\xi,\alpha}^{\xi}(J)$ and by using Lemma 1.12 and Definition 1.5, we have

$$\left({}^{\alpha}\mathcal{J}_{a^+}^{1-\xi} \sigma(\tau) \right)(t) \in C_{\xi,\alpha}(J),$$

and

$${}^{\alpha}\mathcal{D}_{a^+}^{\xi} \sigma(t) = \left(\delta_\alpha \, {}^{\alpha}\mathcal{J}_{a^+}^{1-\xi} \sigma(\tau) \right)(t) \in C_{\xi,\alpha}(J).$$

By the definition of the space $C_{\xi,\alpha}^{n}(J)$, we have

$$\left({}^{\alpha}\mathcal{J}_{a^+}^{1-\xi} \sigma(\tau) \right)(t) \in C_{\xi,\alpha}^{1}(J).$$

Hence, Lemma 1.15 implies that

$$\left({}^{\alpha}\mathcal{J}_{a^+}^{\xi} \, {}^{\alpha}\mathcal{D}_{a^+}^{\xi} \sigma(\tau) \right)(t) = \sigma(t) - \bar{\Psi}_\xi(t,a) \left({}^{\alpha}\mathcal{J}_{a^+}^{1-\xi} \sigma(\tau) \right)(a), \quad \text{for all} \quad t \in (a,b].$$

Using Lemma 1.16 we have

$$\begin{aligned} \left({}^{\alpha}\mathcal{J}_{a^+}^{\xi} \, {}^{\alpha}\mathcal{D}_{a^+}^{\xi} \sigma(\tau) \right)(t) &= \left({}^{\alpha}\mathcal{J}_{a^+}^{\vartheta} \, {}^{\alpha}\mathcal{D}_{a^+}^{\vartheta,r} \sigma(\tau) \right)(t) \\ &= \left({}^{\alpha}\mathcal{J}_{a^+}^{\vartheta} v(\tau) \right)(t). \end{aligned}$$

Then,

$$\frac{x(t) - \chi(t,x(t))}{f(t,x(t))} = \bar{\Psi}_\xi(t,a) \left({}^{\alpha}\mathcal{J}_{a^+}^{1-\xi} \sigma(\tau) \right)(a) + \left({}^{\alpha}\mathcal{J}_{a^+}^{\vartheta} v(\tau) \right)(t),$$

wich implies that

$$x(t) = f(t,x(t)) \left[\bar{\Psi}_\xi(t,a) \left({}^{\alpha}\mathcal{J}_{a^+}^{1-\xi} \left(\frac{x(\tau) - \chi(\tau, x(\tau))}{f(\tau, x(\tau))} \right) \right)(a) + \left({}^{\alpha}\mathcal{J}_{a^+}^{\vartheta} v(\tau) \right)(t) \right] + \chi(t, x(t)), \quad (6.6)$$

where $t \in J$. Next, we substitute $t = \epsilon_i$ into (6.6), then we multiply c_i to both sides, we obtain

$$c_i \left(\frac{x(\epsilon_i) - \chi(\epsilon_i, x(\epsilon_i))}{f(\epsilon_i, x(\epsilon_i))} \right) = c_i \bar{\Psi}_\xi(\epsilon_i, a) \left({}^\alpha\mathcal{J}_{a^+}^{1-\xi} \sigma(\tau) \right)(a) + c_i \left({}^\alpha\mathcal{J}_{a^+}^{\vartheta} v(\tau) \right)(\epsilon_i).$$

Then by using condition (6.4), we have

$$\begin{aligned} \left({}^\alpha\mathcal{J}_{a^+}^{1-\xi} \sigma(\tau) \right)(a^+) &= \sum_{i=1}^{m} c_i \left(\frac{x(\epsilon_i) - \chi(\epsilon_i, x(\epsilon_i))}{f(\epsilon_i, x(\epsilon_i))} \right) \\ &= \left({}^\alpha\mathcal{J}_{a^+}^{1-\xi} \sigma(\tau) \right)(a) \sum_{i=1}^{m} c_i \bar{\Psi}_\xi(\epsilon_i, a) + \sum_{i=1}^{m} c_i \left({}^\alpha\mathcal{J}_{a^+}^{\vartheta} v(\tau) \right)(\epsilon_i), \end{aligned}$$

which implies

$$\left({}^\alpha\mathcal{J}_{a^+}^{1-\xi} \sigma(\tau) \right)(a^+) = \frac{\sum_{i=1}^{m} c_i \left({}^\alpha\mathcal{J}_{a^+}^{\vartheta} v(\tau) \right)(\epsilon_i)}{1 - \sum_{i=1}^{m} c_i \bar{\Psi}_\xi(\epsilon_i, a)}. \tag{6.7}$$

Substituting (6.7) into (6.6), we obtain (6.5).
Reciprocally, assume x satisfies the equation (6.5) such that the function $\sigma : t \longrightarrow \left(\frac{x(t) - \chi(t, x(t))}{f(t, x(t))} \right) \in C_{\xi,\alpha}^{\xi}(J)$. We prove that x is a solution to the problem (6.3)-(6.4). Apply operator ${}^\alpha\mathcal{D}_{a^+}^{\xi}$ on both sides of (6.5). And since $f(t, x(t)) \neq 0$ for all $t \in J$, then, from Lemma 1.10 and Lemma 1.16 we obtain

$$({}^\alpha\mathcal{D}_{a^+}^{\xi} \sigma(\tau))(t) = \left({}^\alpha\mathcal{D}_{a^+}^{r(1-\vartheta)} v(\tau) \right)(t). \tag{6.8}$$

Since $\sigma \in C_{\xi,\alpha}^{\xi}(J)$ and by definition of $C_{\xi,\alpha}^{\xi}(J)$, we have ${}^\alpha\mathcal{D}_{a^+}^{\xi} \sigma \in C_{\xi,\alpha}(J)$, then (6.8) implies that

$$({}^\alpha\mathcal{D}_{a^+}^{\xi} \sigma(\tau))(t) = \left(\delta_\alpha \, {}^\alpha\mathcal{J}_{a^+}^{1-r(1-\vartheta)} v(\tau) \right)(t) = \left({}^\alpha\mathcal{D}_{a^+}^{r(1-\vartheta)} v(\tau) \right)(t) \in C_{\xi,\alpha}(J). \tag{6.9}$$

As $v(\cdot) \in C_{\xi,\alpha}(J)$ and from Lemma 1.12, follows

$$\left({}^\alpha\mathcal{J}_{a^+}^{1-r(1-\vartheta)} v \right) \in C_{\xi,\alpha}(J). \tag{6.10}$$

From (6.9), (6.10) and by the definition of the space $C_{\xi,\alpha}^{n}(J)$, we obtain

$$\left({}^\alpha\mathcal{J}_{a^+}^{1-r(1-\vartheta)} v \right) \in C_{\xi,\alpha}^{1}(J).$$

Applying operator ${}^\alpha\mathcal{J}_{a^+}^{r(1-\vartheta)}$ on both sides of (6.9) and using Lemma 1.15, Lemma 1.13 and Property 1.11, we have

$$\begin{aligned} \left({}^\alpha\mathcal{D}_{a^+}^{\vartheta,r} \sigma(\tau) \right)(t) &= {}^\alpha\mathcal{J}_{a^+}^{r(1-\vartheta)} \left({}^\alpha\mathcal{D}_{a^+}^{\xi} \sigma(\tau) \right)(t) \\ &= v(t) - \bar{\Psi}_{r(1-\vartheta)}(t, a) \left({}^\alpha\mathcal{J}_{a^+}^{1-r(1-\vartheta)} v(\tau) \right)(a) \\ &= v(t), \end{aligned}$$

that is, (6.3) holds. Now, applying ${}^{\alpha}\mathcal{J}_{a^+}^{1-\xi}$ on both sides of (6.5) and using Lemma 1.10 and Theorem 1.9, we get

$$\left({}^{\alpha}\mathcal{J}_{a^+}^{1-\xi}\sigma(\tau)\right)(t)=\frac{\sum_{i=1}^{m}c_i\left({}^{\alpha}\mathcal{J}_{a^+}^{\vartheta}v(\tau)\right)(\epsilon_i)}{1-\sum_{i=1}^{m}c_i\bar{\Psi}_{\xi}(\epsilon_i,a)}+\left({}^{\alpha}\mathcal{J}_{a^+}^{1-\xi+\vartheta}v(\tau)\right)(t). \tag{6.11}$$

Taking the limit $t\to a^+$ of (6.11) and using Lemma 1.13, with $1-\xi<1-\xi+\vartheta$, we obtain

$$\left({}^{\alpha}\mathcal{J}_{a^+}^{1-\xi}\left(\frac{x(\tau)-\chi(\tau,x(\tau))}{f(\tau,x(\tau))}\right)\right)(a^+)=\frac{\sum_{i=1}^{m}c_i\left({}^{\alpha}\mathcal{J}_{a^+}^{\vartheta}v(\tau)\right)(\epsilon_i)}{1-\sum_{i=1}^{m}c_i\bar{\Psi}_{\xi}(\epsilon_i,a)}. \tag{6.12}$$

Substituting $t=\epsilon_i$ into (6.5), we have

$$\frac{x(\epsilon_i)-\chi(\epsilon_i,x(\epsilon_i))}{f(\epsilon_i,x(\epsilon_i))}=\bar{\Psi}_{\xi}(\epsilon_i,a)\frac{\sum_{i=1}^{m}c_i\left({}^{\alpha}\mathcal{J}_{a^+}^{\vartheta}v(\tau)\right)(\epsilon_i)}{1-\sum_{i=1}^{m}c_i\bar{\Psi}_{\xi}(\epsilon_i,a)}+\left({}^{\alpha}\mathcal{J}_{a^+}^{\vartheta}v(\tau)\right)(\epsilon_i).$$

Then, we have

$$\sum_{i=1}^{m}c_i\left(\frac{x(\epsilon_i)-\chi(\epsilon_i,x(\epsilon_i))}{f(\epsilon_i,x(\epsilon_i))}\right)=\sum_{i=1}^{m}c_i\bar{\Psi}_{\xi}(\epsilon_i,a)\frac{\sum_{i=1}^{m}c_i\left({}^{\alpha}\mathcal{J}_{a^+}^{\vartheta}v(\tau)\right)(\epsilon_i)}{1-\sum_{i=1}^{m}c_i\bar{\Psi}_{\xi}(\epsilon_i,a)}+\sum_{i=1}^{m}c_i\left({}^{\alpha}\mathcal{J}_{a^+}^{\vartheta}v(\tau)\right)(\epsilon_i),$$

thus,

$$\sum_{i=1}^{m}c_i\left(\frac{x(\epsilon_i)-\chi(\epsilon_i,x(\epsilon_i))}{f(\epsilon_i,x(\epsilon_i))}\right)=\frac{\sum_{i=1}^{m}c_i\left({}^{\alpha}\mathcal{J}_{a^+}^{\vartheta}v(\tau)\right)(\epsilon_i)}{1-\sum_{i=1}^{m}c_i\bar{\Psi}_{\xi}(\epsilon_i,a)}. \tag{6.13}$$

From (6.12) and (6.13), we find that

$$\left({}^{\alpha}\mathcal{J}_{a^+}^{1-\xi}\left(\frac{x(\tau)-\chi(\tau,x(\tau))}{f(\tau,x(\tau))}\right)\right)(a^+)=\sum_{i=1}^{m}c_i\left(\frac{x(\epsilon_i)-\chi(\epsilon_i,x(\epsilon_i))}{f(\epsilon_i,x(\epsilon_i))}\right),$$

which shows that the initial condition (6.4) is satisfied. This completes the proof. ■

As a consequence of Theorem 6.1, we have the following result

Lemma 6.2. *Let* $\xi=\vartheta+r-\vartheta r$ *where* $0<\vartheta<1$ *and* $0\leq r\leq 1$, *let* $f\in C([a,b]\times$

$\mathbb{R}, \mathbb{R}\backslash\{0\})$, $\chi \in C([a,b]\times\mathbb{R},\mathbb{R})$ *and let* $\varphi : J\times\mathbb{R}^2 \to \mathbb{R}$, *be a function such that* $\varphi(\cdot,x(\cdot),y(\cdot)) \in C_{\xi,\alpha}(J)$, *for any* $x,y \in C_{\xi,\alpha}(J)$. *If the function* $t \longrightarrow \left(\frac{x(t)-\chi(t,x(t))}{f(t,x(t))}\right) \in C^{\xi}_{\xi,\alpha}(J)$, *then* x *satisfies the problem* (6.1) – (6.2) *if and only if* x *is the fixed point of the operator* $\Im : C_{\xi,\alpha}(J) \to C_{\xi,\alpha}(J)$ *defined by*

$$\Im x(t) = f(t,x(t))\left[K\bar{\Psi}_{\xi}(t,a)\sum_{i=1}^{m} c_i\left({}^{\alpha}\mathcal{J}^{\vartheta}_{a^+}v(\tau)\right)(\epsilon_i) + \left({}^{\alpha}\mathcal{J}^{\vartheta}_{a^+}v(\tau)\right)(t)\right] + \chi(t,x(t)), \quad (6.14)$$

where $K = \left[1 - \sum_{i=1}^{m} c_i\bar{\Psi}_{\xi}(\epsilon_i,a)\right]^{-1}$ *and* $v : J \to \mathbb{R}$ *be a function satisfying the functional equation*

$$v(t) = \varphi(t,x(t),v(t)).$$

Since the functions f and χ are continuous and $\varphi(\cdot,x(\cdot),y(\cdot)) \in C_{\xi,\alpha}(J)$, then, by Lemma 1.12, we have $\Im x \in C_{\xi,\alpha}(J)$.

The following hypotheses will be used in the sequel :

(*Ax₁*) The function $\varphi : J\times\mathbb{R}^2 \to \mathbb{R}$ be continuous on J and

$$\varphi(\cdot,x(\cdot),y(\cdot)) \in C^{r(1-\vartheta)}_{\xi,\alpha}(J), \quad \text{for any } x,y \in C_{\xi,\alpha}(J).$$

(*Ax₂*) The functions $f : [a,b]\times\mathbb{R} \to \mathbb{R}\backslash\{0\}$ and $\chi : [a,b]\times\mathbb{R} \to \mathbb{R}$ are continuous and there exist two functions $p,q \in C([a,b],[0,\infty))$ such that

$$|f(t,x) - f(t,\bar{x})| \le p(t)\Psi_{\xi}(t,a)|x-\bar{x}|$$

and

$$|\chi(t,x) - \chi(t,\bar{x})| \le q(t)|x-\bar{x}|$$

for any $x,\bar{x} \in \mathbb{R}$ and $t \in J$.

(*Ax₃*) There exists functions $\lambda_1,\lambda_2,\lambda_3 \in C([a,b],[0,\infty))$ such that

$$|\varphi(t,x,y)| \le \lambda_1(t) + \lambda_2(t)|x| + \lambda_3(t)|y| \quad \text{for } t\in J, \text{ and } x,y \in \mathbb{R}.$$

(*Ax₄*) There exists a number $\ell > 0$ such that

$$\ell \ge \frac{f^*M + \chi^*}{1 - p^*M - q^*},$$

where

$$p^* = \sup_{t\in[a,b]} p(t), \quad q^* = \sup_{t\in[a,b]} q(t),$$

$$\lambda_i^* = \sup_{t\in[a,b]} \lambda_i(t), i = 1,2, \quad \lambda_3^* = \sup_{t\in[a,b]} \lambda_3(t) < 1,$$

$$f^* = \sup_{t\in[a,b]} |f(t,0)|, \quad \chi^* = \sup_{t\in[a,b]} \Psi_{\xi}(t,a)|\chi(t,0)|, \quad \Lambda := \frac{\Psi_{\xi}(b,a)\lambda_1^* + \lambda_2^*\ell}{1-\lambda_3^*}$$

and

$$M = \frac{\Lambda\alpha^{-\vartheta}}{\Gamma(\vartheta+\xi)}\left[|K|\alpha^{1-\xi}\sum_{i=1}^{m}|c_i|\left(\epsilon_i^{\alpha} - a^{\alpha}\right)^{\vartheta+\xi-1} + \Gamma(\xi)\left(b^{\alpha} - a^{\alpha}\right)^{\vartheta}\right].$$

We are now in a position to state and prove our existence result for the problem (6.1)–(6.2) based on based on Lemma 1.31.

Theorem 6.3. *Assume* $(\boldsymbol{Ax_1})$–$(\boldsymbol{Ax_4})$ *hold. If*

$$\max\{p^*M, p^*\Psi_\xi(b,a)M\} + q^* < 1, \tag{6.15}$$

then the problem (6.1)–(6.2) has at least one solution in $C_{\xi,\alpha}(J)$.

Proof: We define a subset Ω of $C_{\xi,\alpha}(J)$ by

$$\Omega = \{x \in C_{\xi,\alpha}(J) : \|x\|_{\xi,\alpha} \leq \ell\}.$$

We consider the operator $\Im$ defined in (6.14), and define three operators $\mathcal{S}, \mathcal{N} : C_{\xi,\alpha}(J) \to C_{\xi,\alpha}(J)$ by

$$(\mathcal{S}x)(t) = f(t, x(t)), \quad t \in J, \tag{6.16}$$

$$(\mathcal{N}x)(t) = \chi(t, x(t)), \quad t \in J, \tag{6.17}$$

and $\mathcal{T} : \Omega \to C_{\xi,\alpha}(J)$ by

$$(\mathcal{T}x)(t) = K\bar{\Psi}_\xi(t,a)\sum_{i=1}^{m} c_i \left({}^{\alpha}\mathcal{J}_{a^+}^{\vartheta} v(\tau)\right)(\epsilon_i) + \left({}^{\alpha}\mathcal{J}_{a^+}^{\vartheta} v(\tau)\right)(t), \quad t \in J. \tag{6.18}$$

Then we get $\Im x = \mathcal{S}x\mathcal{T}x + \mathcal{N}x$. We shall show that the operators $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{N}$ satisfie all the conditions of Lemma 1.31. The proof will be given in several steps.

Step 1: The operators $\mathcal{S}$ and $\mathcal{N}$ are Lipschitzian on $C_{\xi,\alpha}(J)$.
Let $x, y \in C_{\xi,\alpha}(J)$ and $t \in J$. Then by $(\boldsymbol{Ax_2})$ we have

$$\begin{aligned}
|((\mathcal{S}x)(t) - (\mathcal{S}y)(t))\,\Psi_\xi(t,a)| &\leq \Psi_\xi(t,a)|f(t,x(t)) - f(t,y(t))|,\\
&\leq p(t)\Psi_\xi(t,a)\|x(t) - y(t)\|_{\xi,\alpha},\\
&\leq p^*\Psi_\xi(b,a)\|x(t) - y(t)\|_{\xi,\alpha},
\end{aligned}$$

then for each $t \in J$ we obtain

$$\|\mathcal{S}x - \mathcal{S}y\|_{\xi,\alpha} \leq p^*\Psi_\xi(b,a)\|x(t) - y(t)\|_{\xi,\alpha}.$$

Also, for each $t \in J$ we have

$$\begin{aligned}
|((\mathcal{N}x)(t) - (\mathcal{N}y)(t))\,\Psi_\xi(t,a)| &\leq \Psi_\xi(t,a)|\chi(t,x(t)) - \chi(t,y(t))|,\\
&\leq q(t)\|x(t) - y(t)\|_{\xi,\alpha},\\
&\leq q^*\|x(t) - y(t)\|_{\xi,\alpha},
\end{aligned}$$

then,

$$\|\mathcal{N}x - \mathcal{N}y\|_{\xi,\alpha} \leq q^*\|x(t) - y(t)\|_{\xi,\alpha}.$$

Step 2: The operator $\mathcal{T}$ is completely continuous on Ω.

We firstly show that the operator $\mathcal{T}$ is continuous on Ω. Let $\{x_n\}$ be sequence in Ω such that $x_n \to x$ in Ω. Let $x, y \in C_{\xi,\alpha}(J)$.
Then for each $t \in J$, we have

$$\begin{aligned}
|(\mathcal{T}x_n)(t) - (\mathcal{T}x)(t))\,\Psi_\xi(t,a)| &\leq \frac{|K|}{\Gamma(\xi)} \sum_{i=1}^{m} |c_i| \left({}^{\alpha}\mathcal{J}_{a^+}^{\vartheta} |v_n(\tau) - v(\tau)|\right)(\epsilon_i) \\
&+ \Psi_\xi(t,a) \left({}^{\alpha}\mathcal{J}_{a^+}^{\vartheta} |v_n(\tau) - v(\tau)|\right)(t),
\end{aligned}$$

where $v_n, v \in C_{\xi,\alpha}(J)$ such that

$$\begin{aligned}
v_n(t) &= \varphi(t, x_n(t), v_n(t)), \\
v(t) &= \varphi(t, x(t), v(t)).
\end{aligned}$$

Since $x_n \to x$ and φ is continuous function on J then we get $v_n(t) \to v(t)$ as $n \to \infty$ for each $t \in J$, so by Lebesgue's dominated convergence theorem, we have

$$\|\mathcal{T}x_n - \mathcal{T}x\|_{C_{\xi,\alpha}} \to 0 \text{ as } n \to \infty.$$

Then $\mathcal{T}$ is continuous.
Next we prove that $\mathcal{T}(\Omega)$ is uniformly bounded on $C_{\xi,\alpha}(J)$. Let any $x \in \Omega$. By $(\boldsymbol{Ax_3})$, we have for each $t \in J$

$$\begin{aligned}
|\Psi_\xi(t,a)v(t)| &= |\Psi_\xi(t,a)\varphi(t, x(t), v(t))| \\
&\leq \Psi_\xi(t,a)(\lambda_1(t) + \lambda_2(t)|x(t)| + \lambda_3(t)|v(t)|) \\
&\leq \Psi_\xi(b,a)\lambda_1^* + \lambda_2^*\ell + \lambda_3^*|\Psi_\xi(t,a)v(t)|.
\end{aligned}$$

Wich implies that

$$|\Psi_\xi(t,a)v(t)| \leq \frac{\Psi_\xi(b,a)\lambda_1^* + \lambda_2^*\ell}{1 - \lambda_3^*}.$$

Then, we have

$$\sup_{t \in (a,b]} |\Psi_\xi(t,a)v(t)| \leq \frac{\Psi_\xi(b,a)\lambda_1^* + \lambda_2^*\ell}{1 - \lambda_3^*} := \Lambda.$$

For $t \in J$, by (6.18) and Lemma 1.10, we have

$$\begin{aligned}
&|\Psi_\xi(t,a)(\mathcal{T}x)(t)| \\
&\leq \frac{|K|}{\Gamma(\xi)} \sum_{i=1}^{m} |c_i| \left({}^{\alpha}\mathcal{J}_{a^+}^{\vartheta} |v(\tau)|\right)(\epsilon_i) + \Psi_\xi(t,a) \left({}^{\alpha}\mathcal{J}_{a^+}^{\vartheta} |v(\tau)|\right)(t) \\
&\leq \Lambda|K| \sum_{i=1}^{m} |c_i| \left({}^{\alpha}\mathcal{J}_{a^+}^{\vartheta} \bar{\Psi}_\xi(\tau,a)\right)(\epsilon_i) + \Lambda\Psi_\xi(t,a)\Gamma(\xi) \left({}^{\alpha}\mathcal{J}_{a^+}^{\vartheta} \bar{\Psi}_\xi(\tau,a)\right)(t) \\
&\leq \Lambda|K| \sum_{i=1}^{m} |c_i| \bar{\Psi}_{\vartheta+\xi}(\epsilon_i, a) + \Lambda\Psi_\xi(t,a)\Gamma(\xi)\bar{\Psi}_{\vartheta+\xi}(t,a) \\
&\leq \frac{\Lambda|K|}{\Gamma(\vartheta+\xi)} \sum_{i=1}^{m} |c_i| \left(\frac{\epsilon_i^\alpha - a^\alpha}{\alpha}\right)^{\vartheta+\xi-1} + \frac{\Lambda\Gamma(\xi)}{\Gamma(\vartheta+\xi)} \left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^{\vartheta}.
\end{aligned}$$

Then for $t \in J$, we obtain
$$\|\mathcal{T}x\|_{C_{\xi,\alpha}} \leq M.$$
This prove that the operator $\mathcal{T}$ is uniformly bounded on Ω. Next we prove that the operator $\mathcal{T}\Omega$ equicontinuous. We take $x \in \Omega$ and $a < \varepsilon_1 < \varepsilon_2 \leq b$. Then,

$$\begin{aligned}
& |\Psi_\xi(\varepsilon_1,a)(\mathcal{T}x)(\varepsilon_1) - \Psi_\xi(\varepsilon_2,a)(\mathcal{T}x)(\varepsilon_2)| \\
\leq & \left|\Psi_\xi(\varepsilon_1,a)\left({}^\alpha\mathcal{J}^\vartheta_{a^+}v(\tau)\right)(\varepsilon_1) - \Psi_\xi(\varepsilon_2,a)\left({}^\alpha\mathcal{J}^\vartheta_{a^+}v(\tau)\right)(\varepsilon_2)\right| \\
\leq & \int_a^{\varepsilon_1} |\Psi_\xi(\varepsilon_1,a)\bar{\Psi}_\vartheta(\varepsilon_1,\tau) - \Psi_\xi(\varepsilon_2,a)\bar{\Psi}_\vartheta(\varepsilon_2,\tau)| \left|\tau^{\alpha-1}v(\tau)\right| d\tau, \\
+ & \Psi_\xi(\varepsilon_2,a)\left({}^\alpha\mathcal{J}^\vartheta_{\varepsilon_1^+}|v(\tau)|\right)(\varepsilon_2).
\end{aligned}$$

Then by Lemma 1.10, we have for each $t \in (a,b]$

$$\begin{aligned}
& |\Psi_\xi(\varepsilon_1,a)(\mathcal{T}x)(\varepsilon_1) - \Psi_\xi(\varepsilon_2,a)(\mathcal{T}x)(\varepsilon_2)| \\
\leq & \Lambda\Gamma(\xi)\int_a^{\varepsilon_1} \tau^{\alpha-1}|\Psi_\xi(\varepsilon_1,a)\bar{\Psi}_\vartheta(\varepsilon_1,\tau) - \Psi_\xi(\varepsilon_2,a)\bar{\Psi}_\vartheta(\varepsilon_2,\tau)|\bar{\Psi}_\xi(\tau,a)d\tau, \\
& +\Lambda\Gamma(\xi)\Psi_\xi(\varepsilon_2,a)\bar{\Psi}_{\vartheta+\xi}(\varepsilon_2,\varepsilon_1).
\end{aligned}$$

Note that

$$|\Psi_\xi(\varepsilon_1,a)(\mathcal{T}x)(\varepsilon_1) - \Psi_\xi(\varepsilon_2,a)(\mathcal{T}x)(\varepsilon_2)| \to 0 \quad \text{as} \quad \varepsilon_1 \to \varepsilon_2.$$

This proves that $\mathcal{T}\Omega$ is equicontinuous on J. Therefore by the Arzela-Ascoli Theorem, $\mathcal{T}$ is completely continuous on Ω.

Step 3: Now we show that the third hypothesis of Lemma 1.31 is satisfied. Let $x \in C_{\xi,\alpha}(J)$ and $y \in \Omega$ be arbitrary such that $x = \mathcal{S}x\mathcal{T}y + \mathcal{N}x$ and $\tilde{v} \in C_{\xi,\alpha}(J)$ with

$$\tilde{v}(t) = \varphi(t, y(t), \tilde{v}(t)).$$

Then, for $t \in J$ we have

$$\begin{aligned}
& |\Psi_\xi(t,a)x(t)| \\
= & |\Psi_\xi(t,a)(\mathcal{S}x\mathcal{T}y)(t) + \Psi_\xi(t,a)(\mathcal{N}x)(t)| \\
\leq & \Psi_\xi(t,a)\,|(\mathcal{S}x)(t)|\,|(\mathcal{T}y)(t)| + |\Psi_\xi(t,a)(\mathcal{N}x)(t)| \\
\leq & |f(t,x(t))| \left[\frac{|K|}{\Gamma(\xi)}\sum_{i=1}^m |c_i|\left({}^\alpha\mathcal{J}^\vartheta_{a^+}|\tilde{v}(\tau)|\right)(\epsilon_i) + \Psi_\xi(t,a)\left({}^\alpha\mathcal{J}^\vartheta_{a^+}|\tilde{v}(\tau)|\right)(t)\right] \\
& +\Psi_\xi(t,a)|\chi(t,x(t))| \\
\leq & M\left(|f(t,x(t)) - f(t,0)| + |f(t,0)|\right) + \Psi_\xi(t,a)\left(|\chi(t,x(t)) - \chi(t,0)| + |\chi(t,0)|\right) \\
\leq & M\left(p^*\|x\|_{C_{\xi,\alpha}} + f^*\right) + q^*\|x\|_{C_{\xi,\alpha}} + \chi^*,
\end{aligned}$$

then,

$$\|x\|_{C_{\xi,\alpha}} = \frac{f^*M + \chi^*}{1 - p^*M - q^*} \leq \ell.$$

Then $x \in \Omega$, thus the third hypothesis of Lemma 1.31 is satisfied.

Step 4: Now, we show that $p^*\Psi_\xi(b,a)L + q^* < 1$, where

$$L = \|\mathcal{T}(\Omega)\|_{C_{\xi,\alpha}} = \sup\{\|\mathcal{T}y\|_{C_{\xi,\alpha}} : y \in \Omega\}.$$

Since $L \leq M$, we have

$$p^*\Psi_\xi(b,a)L + q^* \leq p^*\Psi_\xi(b,a)M + q^* < 1.$$

That is, the last hypothesis of Lemma 1.31 is satisfied. Thus, the operator equation $\Im x = \mathcal{S}x\mathcal{T}x + \mathcal{N}x = x$ has at least one solution $x^* \in C_{\xi,\alpha}$, wich is a point fixe for the operator $\Im$.

Step 5: We prove that for such fixed point $x^* \in C_{\xi,\alpha}(J)$, the function $\sigma : t \to \dfrac{x^*(t) - \chi(t, x^*(t))}{f(t, x^*(t))}$ is in $C^{\xi}_{\xi,\alpha}(J)$.

Since x^* is a fixed point of operator $\Im$ in $C_{\xi,\alpha}(J)$, then for each $t \in J$, we have

$$\Im x^*(t) = f(t, x^*(t)) \left[K\bar{\Psi}_\xi(t,a) \sum_{i=1}^{m} c_i \left({}^{\alpha}\mathcal{J}^{\vartheta}_{a^+} v(\tau)\right)(\epsilon_i) + \left({}^{\alpha}\mathcal{J}^{\vartheta}_{a^+} v(\tau)\right)(t) \right] + \chi(t, x^*(t)). \tag{6.19}$$

where $v \in C_{\xi,\alpha}(J)$ such that

$$v(t) = \varphi(t, x^*(t), v(t)).$$

Applying ${}^{\alpha}\mathcal{D}^{\xi}_{a^+}$ to both sides of 6.19, and by Lemma 1.10 and Lemma 1.16, we have

$$\begin{aligned} {}^{\alpha}\mathcal{D}^{\xi}_{a^+} \left(\frac{x^*(t) - \chi(t, x^*(t))}{f(t, x^*(t))} \right) &= \left({}^{\alpha}\mathcal{D}^{\xi}_{a^+} \, {}^{\alpha}\mathcal{J}^{\vartheta}_{a^+} v(\tau) \right)(t) \\ &= \left({}^{\alpha}\mathcal{D}^{r(1-\vartheta)}_{a^+} v(\tau) \right)(t). \end{aligned}$$

Since $\xi \geq \vartheta$, by ***(Ax₁)***, the right hand side is in $C_{\xi,\alpha}(J)$ and thus ${}^{\alpha}\mathcal{D}^{\xi}_{a^+}\sigma \in C_{\xi,\alpha}(J)$. Its clear that $\sigma \in C_{\xi,\alpha}(J)$, since $f \in C([a,b] \times \mathbb{R} \to \mathbb{R}\backslash\{0\})$ and $\chi \in C([a,b] \times \mathbb{R} \to \mathbb{R})$, then $\sigma \in C^{\xi}_{\xi,\alpha}(J)$. As a consequence of Steps 1 and 5 with Theorem 6.3, we can conclude that the problem (6.1) − (6.2) has at least a solution in $C_{\xi,\alpha}(J)$. ■

6.2.2 Example

Example 6.4. *Consider the nonlocal initial value problem of hybrid generalized type Hilfer Fractional differential equation*

$$ {}^{1}\mathcal{D}^{\frac{1}{2},0}_{1^+} \left(\tfrac{x(t)-\chi(t,x(t))}{f(t,x(t))} \right) = \frac{\sqrt{t-1}\left(x(t) + {}^{1}\mathcal{D}^{\frac{1}{2},0}_{1^+} \left(\tfrac{x(t)-\chi(t,x(t))}{f(t,x(t))} \right) + 1 \right)}{111e^{-t+2}(1+\sqrt{t-1}|x(t)|)}, \ \textit{for each } t \in (1,2], \tag{6.20}$$

$$\left({}^{1}\mathcal{J}^{\frac{1}{2}}_{1^+} \left(\frac{x(\tau) - \chi(\tau, x(\tau))}{f(\tau, x(\tau))} \right) \right)(1^+) = 2 \left(\frac{x(\frac{3}{2}) - \chi(\frac{3}{2}, x(\frac{3}{2}))}{f(\frac{3}{2}, x(\frac{3}{2}))} \right), \tag{6.21}$$

where $I = (1,2]$, $a = 1$, $b = 2$ *and*

$$f(t, x(t)) = \frac{|sin(\pi t)|(t-1)|x(t)| + 1}{41e^{-t+4}}, \ t \in [1,2], \ x \in C_{\frac{1}{2},1}([1,2]),$$

and

$$\chi(t, x(t)) = \frac{\sqrt{t-1}\ln(|cos(t)|+1)x(t)}{33e^3\sqrt{6-t}} + \frac{1}{55e^{-t+2}}, \ t \in [1,2], \ x \in C_{\frac{1}{2},1}([1,2]).$$

Set

$$\varphi(t,x,y)=\frac{\sqrt{t-1}\,(x+y+1)}{111e^{-t+2}(1+|x|\sqrt{t-1})},\ t\in I,\ x,y\in \mathbb{R}.$$

We have

$$C^{r(1-\vartheta)}_{\xi,\alpha}(I)=C^{0}_{\frac{1}{2},1}(I)=\left\{v:I\to\mathbb{R}:t\to(\sqrt{t-1})v(t)\in C([1,2],\mathbb{R})\right\},$$

with $\xi=\vartheta=\frac{1}{2}$, $\alpha=1$, $r=0$. *Clearly, the continuous function* $\varphi\in C^{0}_{\frac{1}{2},1}(I)$. *Hence the condition* **(*Ax₁*)** *is satisfied.*
For each $x,\bar{x}\in\mathbb{R}$ *and* $t\in I$, *we have*

$$|f(t,x)-f(t,\bar{x})| \ \leq\ \frac{|sin(\pi t)|(t-1)}{41e^{-t+4}}|x-\bar{x}|,$$

and

$$|\chi(t,x)-\chi(t,\bar{x})| \ \leq\ \frac{\sqrt{t-1}\ln(|cos(t)|+1)}{33e^{3}\sqrt{6-t}}|x-\bar{x}|.$$

Hence condition **(*Ax₂*)** *is satisfied with*

$$p(t)=\frac{|sin(\pi t)|\sqrt{t-1}}{41e^{-t+4}},\quad and\quad q(t)=\frac{\sqrt{t-1}\ln(|cos(t)|+1)}{33e^{3}\sqrt{6-t}},$$

so we have

$$p^{*}\leq\frac{1}{41e^{2}},\quad and\quad q^{*}\leq\frac{ln(2)}{66e^{3}}.$$

Let $x,y\in\mathbb{R}$. *Then we have*

$$|\varphi(t,x,y)|\leq\frac{\sqrt{t-1}}{111e^{-t+2}}(|x|+|y|+1),\ t\in I,$$

and so the condition **(*Ax₃*)** *is satisfied with*

$$\lambda_1(t)=\lambda_2(t)=\lambda_3(t)=\frac{\sqrt{t-1}}{111e^{-t+2}},$$

and

$$\lambda_1^{*}=\lambda_2^{*}=\lambda_3^{*}=\frac{1}{111}.$$

Also, the condition **(*Ax₄*)** *and the condition (6.15) of Theorem 6.3 is satisfied if we take*

$$2956\approx 400e^{2}\leq\ell<\frac{4510e^{2}(2\sqrt{2}-\sqrt{\pi})}{(2\sqrt{2\pi}+2\sqrt{\pi}-\pi)}-1\approx 6496,$$

where

$$f^{*}=\tfrac{1}{41e^{2}},\quad \chi^{*}=\tfrac{1}{55},\quad k=\tfrac{\sqrt{\pi}}{\sqrt{\pi}-2\sqrt{2}}$$

and

$$\Lambda=\frac{1+\ell}{110}.$$

Then the problem (6.22)-(6.23) has at least one solution in $C_{\frac{1}{2},1}(I)$.

Example 6.5. *In this example, we change the boundary condition (6.23) which give us the following nonlocal initial value problem of hybrid generalized Hilfer Fractional differential equation*

$$ {}^{1}\mathcal{D}_{1^+}^{\frac{1}{2},0}\left(\frac{x(t)-\chi(t,x(t))}{f(t,x(t))}\right)=\frac{\sqrt{t-1}\left(x(t)+{}^{1}\mathcal{D}_{1^+}^{\frac{1}{2},0}\left(\frac{x(t)-\chi(t,x(t))}{f(t,x(t))}\right)+1\right)}{111e^{-t+2}(1+\sqrt{t-1}|x(t)|)},\ \textit{for each } t\in(1,2], \tag{6.22}$$

$$\left({}^{1}\mathcal{J}_{1^+}^{\frac{1}{2}}\left(\frac{x(\tau)-\chi(\tau,x(\tau))}{f(\tau,x(\tau))}\right)\right)(1^+)=3\left(\frac{x(\frac{5}{4})-\chi(\frac{5}{4},x(\frac{5}{4}))}{f(\frac{5}{4},x(\frac{5}{4}))}\right)+2\left(\frac{x(\frac{4}{3})-\chi(\frac{4}{3},x(\frac{4}{3}))}{f(\frac{4}{3},x(\frac{4}{3}))}\right). \tag{6.23}$$

All the hypothesis of Theorem 6.3 are satisfied, indeed, we have

$$2956\approx 400e^2\le \ell<\frac{4510e^2(66e^3-\ln(2))(2\sqrt{3}-\sqrt{\pi}+6)}{66e^3(11\sqrt{\pi}+2\sqrt{3\pi}-\pi)}-1\approx 11387,$$

where

$$m=2,\quad f^*=\frac{1}{41e^2},\quad \chi^*=\frac{1}{55},\quad K=\frac{\sqrt{\pi}}{\sqrt{\pi}-2\sqrt{3}-6},$$

and

$$\Lambda=\frac{1+\ell}{110}.$$

Then the problem (6.22)-(6.23) has at least one solution in $C_{\frac{1}{2},1}(I)$.

6.3 Initial Value Problem for Hybrid ψ-Hilfer Fractional Implicit Differential Equations[2]

In this section, we consider the initial value problem with nonlinear implicit hybrid ψ-Hilfer type fractional differential equation :

$$ {}^{H}\mathbb{D}_{a^+}^{\vartheta,r;\psi}\left(\frac{x(t)}{g(t,x(t))}\right)=f\left(t,x(t),{}^{H}\mathbb{D}_{a^+}^{\vartheta,r;\psi}\left(\frac{x(t)}{g(t,x(t))}\right)\right),\quad t\in(a,b], \tag{6.24}$$

$$\left(\mathbb{J}_{a^+}^{1-\xi;\psi}\left(\frac{x(\tau)}{g(\tau,x(\tau))}\right)\right)(a^+)=x_0, \tag{6.25}$$

where ${}^{H}\mathbb{D}_{a^+}^{\vartheta,r;\psi},\mathbb{J}_{a^+}^{1-\xi;\psi}$ are the ψ-Hilfer fractional derivative of order $\vartheta\in(0,1)$ and type $r\in[0,1]$ and ψ-Riemann-Liouville fractional integral of order $1-\xi,(\xi=\vartheta+r-\vartheta r)$ respectively, $x_0\in\mathbb{R}$, $g\in C([a,b]\times\mathbb{R},\mathbb{R}\backslash\{0\})$ and $f\in C([a,b]\times\mathbb{R}^2,\mathbb{R})$.

6.3.1 Existence Results

Consider the weighted Banach space

$$C_{\xi;\psi}(J)=\left\{x:J\to\mathbb{R}:t\to(\psi(t)-\psi(a))^{1-\xi}x(t)\in C([a,b],\mathbb{R})\right\},0\le\xi<1,$$

2. **A. Salim**, M. Benchohra and J. E. Lazreg, Initial Value Problem for Hybrid ψ-Hilfer Fractional Implicit Differential Equations. (**Submitted**)

with the norm

$$\|x\|_{C_{\xi;\psi}} = \sup_{t\in[a,b]} \left|(\psi(t)-\psi(a))^{1-\xi}\, x(t)\right|,$$

and

$$C^n_{\xi;\psi}(J) = \left\{x \in C^{n-1}(J) : x^{(n)} \in C_{\xi;\psi}(J)\right\}, n \in \mathbb{N},$$
$$C^0_{\xi;\psi}(J) = C_{\xi;\psi}(J),$$

with the norm

$$\|x\|_{C^n_{\xi;\psi}} = \sum_{i=0}^{n-1} \|x^{(i)}\|_\infty + \|x^{(n)}\|_{C_{\xi;\psi}}.$$

The weighted space $C^{\vartheta,r}_{\xi;\psi}(J)$ is defined by

$$C^{\vartheta,r}_{\xi;\psi}(J) = \left\{x \in C_{\xi;\psi}(J),\ {}^H\mathbb{D}^{\vartheta,r;\psi}_{a^+}x \in C_{\xi;\psi}(J)\right\}.$$

Lemma 6.6. *([99]) Let $\vartheta > 0$, $0 \le \xi < 1$. Then, $\mathbb{J}^{\vartheta;\psi}_{a^+}$ is bounded from $C_{\xi;\psi}(J)$ into $C_{\xi;\psi}(J)$. In addition, if $\xi \le \vartheta$, then $\mathbb{J}^{\vartheta;\psi}_{a^+}$ is bounded from $C_{\xi;\psi}(J)$ into $C([a,b],\mathbb{R})$.*

Lemma 6.7. *([100]) Let $0 < a < b < \infty, \vartheta > 0, 0 \le \xi < 1$, $x \in C_{\xi;\psi}(J)$. If $\vartheta > 1-\xi$, then $\mathbb{J}^{\vartheta;\psi}_{a^+}x \in C([a,b],\mathbb{R})$ and*

$$\left(\mathbb{J}^{\vartheta;\psi}_{a^+}x\right)(a) = \lim_{t\to a^+}\left(\mathbb{J}^{\vartheta;\psi}_{a^+}x\right)(t) = 0.$$

Lemma 6.8. *([70, 100]) Let $t > a$. Then, for $\vartheta \ge 0$ and $r > 0$, we have*

$$\left[\mathbb{J}^{\vartheta;\psi}_{a^+}(\psi(\tau)-\psi(a))^{r-1}\right](t) = \frac{\Gamma(r)}{\Gamma(\vartheta+r)}(\psi(t)-\psi(a))^{\vartheta+r-1},$$
$$\left[\mathbb{D}^{\vartheta;\psi}_{a^+}(\psi(\tau)-\psi(a))^{\vartheta-1}\right](t) = \frac{\Gamma(r)}{\Gamma(\vartheta-r)}(\psi(t)-\psi(a))^{\vartheta+r-1}.$$

Lemma 6.9. *([100, 101]) Let $t > a$, $\vartheta > 0, 0 \le r \le 1$. Then for $0 < \xi < 1; \xi = \vartheta + r - \vartheta r$, we have*

$$\left[\mathbb{D}^{\xi;\psi}_{a^+}(\psi(\tau)-\psi(a))^{\xi-1}\right](t) = 0,$$

and

$$\left[{}^H\mathbb{D}^{\vartheta,r;\psi}_{a^+}(\psi(\tau)-\psi(a))^{\xi-1}\right](t) = 0.$$

Lemma 6.10. *([100, 100]) Let $\vartheta > 0, 0 \le r \le 1$, and $h \in C^1_{\xi;\psi}(J)$. Then,*

$$\left({}^H\mathbb{D}^{\vartheta,r;\psi}_{a^+}\ \mathbb{J}^{\vartheta;\psi}_{a^+}h\right)(t) = h(t), \quad \text{for all} \quad t \in (a,b].$$

Lemma 6.11. *([100, 101]) Let $\vartheta > 0, 0 \le r \le 1$, and $h \in C^1_{\xi;\psi}(J)$. Then,*

$$\left(\mathbb{J}^{\vartheta;\psi}_{a^+}\ {}^H\mathbb{D}^{\vartheta,r;\psi}_{a^+}h\right)(t) = h(t) - \frac{\left(\mathbb{J}^{1-\xi;\psi}_{a^+}h\right)(a)}{\Gamma(\xi)}(\psi(t)-\psi(a))^{\xi-1}, \quad \text{for all} \quad t \in (a,b].$$

We consider the following fractional differential equation

$$ {}^{H}\mathbb{D}_{a^+}^{\vartheta,r;\psi}\left(\frac{x(t)}{g(t,x(t))}\right) = v(t), \quad t \in (a,b], \tag{6.26}$$

where $0 < \vartheta < 1, 0 \leq r \leq 1$, with the condition

$$\left(\mathbb{J}_{a^+}^{1-\xi;\psi}\left(\frac{x(\tau)}{g(\tau,x(\tau))}\right)\right)(a^+) = \phi_0, \tag{6.27}$$

where $\xi = \vartheta + r - \vartheta r$, $\phi_0 \in \mathbb{R}$, $g \in C([a,b]\times\mathbb{R},\mathbb{R}\backslash\{0\})$. The following theorem shows that the equations (6.26) and (6.27) have a unique solution given by

$$x(t) = g(t,x(t))\left[\frac{\phi_0\left(\psi(t)-\psi(a)\right)^{\xi-1}}{\Gamma(\xi)} + \left(\mathbb{J}_{a^+}^{\vartheta;\psi}v(\tau)\right)(t)\right]. \tag{6.28}$$

Theorem 6.12. *Let $\xi = \vartheta + r - \vartheta r$, where $0 < \vartheta < 1$ and $0 \leq r \leq 1$. If $v : J \to \mathbb{R}$ is a given function such that $v \in C^1_{\xi,\psi}(J)$ and the function $g \in C([a,b]\times\mathbb{R},\mathbb{R}\backslash\{0\})$ then x satisfies problem (6.26)-(6.27) if and only if it satisfies (6.28).*

Proof: Assume x satisfies the equations (6.26) and (6.27) such that the function $h : t \longrightarrow \left(\frac{x(t)}{g(t,x(t))}\right) \in C^1_{\xi,\psi}(J)$. We prove that x is a solution to the equation (6.28). Applying the fractional integral $\mathbb{J}_{a^+}^{\vartheta;\psi}$ to both sides of equation (6.26) and using Lemma 6.11, we get

$$h(t) - \frac{(\psi(t)-\psi(a))^{\xi-1}}{\Gamma(\xi)}\left(\mathbb{J}_{a^+}^{1-\xi;\psi}h(\tau)\right)(a) = \left(\mathbb{J}_{a^+}^{\vartheta;\psi}v(\tau)\right)(t).$$

Thus,

$$x(t) = g(t,x(t))\left[\frac{(\psi(t)-\psi(a))^{\xi-1}}{\Gamma(\xi)}\left(\mathbb{J}_{a^+}^{1-\xi;\psi}\left(\tfrac{x(\tau)}{g(\tau,x(\tau))}\right)\right)(a) + \left(\mathbb{J}_{a^+}^{\vartheta;\psi}v(\tau)\right)(t)\right].$$

By using the condition (6.27), we obtain equation (6.28).
Reciprocally, assume x satisfies the equation (6.28). We prove that x is a solution to the equations (6.26) and (6.27). Applying operator ${}^{H}\mathbb{D}_{a^+}^{\vartheta,r;\psi}$ to both sides of (6.28) and using Lemma 6.9 and Lemma 6.10, we have

$$\begin{aligned} {}^{H}\mathbb{D}_{a^+}^{\vartheta,r;\psi}\left(\frac{x(t)}{g(t,x(t))}\right) &= {}^{H}\mathbb{D}_{a^+}^{\vartheta,r;\psi}\frac{\phi_0\left(\psi(t)-\psi(a)\right)^{\xi-1}}{\Gamma(\xi)} + {}^{H}\mathbb{D}_{a^+}^{\vartheta,r;\psi}\left(\mathbb{J}_{a^+}^{\vartheta;\psi}v(\tau)\right)(t) \\ &= v(t), \end{aligned}$$

that is, (6.26) holds.
Now, applying $\mathbb{J}_{a^+}^{1-\xi}$ to both sides of (6.28) and using Lemma 1.19 and Lemma 6.8, we get

$$\left(\mathbb{J}_{a^+}^{1-\xi;\psi}h(\tau)\right)(t) = \phi_0 + \left(\mathbb{J}_{a^+}^{1-\xi+\vartheta;\psi}v(\tau)\right)(t). \tag{6.29}$$

Next, taking the limit $t \to a^+$ of (6.29) and using Lemma 6.7, with $1-\xi < 1-\xi+\vartheta$, we obtain

$$\left(\mathbb{J}_{a^+}^{1-\xi;\psi}\left(\frac{x(\tau)}{g(\tau,x(\tau))}\right)\right)(a^+) = \phi_0. \tag{6.30}$$

which shows that the initial condition (6.27) is satisfied. This completes the proof. ■

As a consequence of Theorem 6.12, we have the following result

Lemma 6.13. *Let $\xi = \vartheta + r - \vartheta r$ where $0 < \vartheta < 1$ and $0 \leq r \leq 1$, let $g \in C([a,b] \times \mathbb{R}, \mathbb{R}\backslash\{0\})$ and let $f : J \times \mathbb{R}^2 \to \mathbb{R}$, be a function such that $f(\cdot, x(\cdot), y(\cdot)) \in C^1_{\xi,\psi}(J)$, for any $x, y \in C_{\xi,\psi}(J)$. If the function $t \longrightarrow \left(\frac{x(t)}{g(t,x(t))}\right) \in C^1_{\xi,\psi}(J)$, then x satisfies the problem (6.24)-(6.25) if and only if x is the fixed point of the operator $\Im : C_{\xi,\psi}(J) \to C_{\xi,\psi}(J)$ defined by*

$$\Im x(t) = g(t, x(t)) \left[\frac{x_0 \left(\psi(t) - \psi(a)\right)^{\xi-1}}{\Gamma(\xi)} + \left(\mathbb{J}^{\vartheta;\psi}_{a^+} v(\tau) \right)(t) \right], \tag{6.31}$$

where $v : J \longrightarrow \mathbb{R}$ be function satisfying the functional equation

$$v(t) = f(t, x(t), v(t)).$$

Since the function g is continuous and $f(\cdot, x(\cdot), y(\cdot)) \in C_{\xi,\psi}(J)$, then, by Lemma 6.6, we have $\Im x \in C_{\xi,\psi}(J)$.

The following hypotheses will be used in the sequel :

(*Ax₁*) The function $f : J \times \mathbb{R} \to \mathbb{R}$ be continuous on J and

$$f(\cdot, x(\cdot), y(\cdot)) \in C^1_{\xi;\psi}(J), \quad \text{for any } x, y \in C_{\xi;\psi}(J).$$

(*Ax₂*) The function $g : [a,b] \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$ is continuous and there exists function $p \in C([a,b], [0,\infty))$ that

$$|g(t,x) - g(t,\bar{x})| \leq p(t) \left(\psi(t) - \psi(a)\right)^{1-\xi} |x - \bar{x}|$$

for any $x, \bar{x} \in \mathbb{R}$ and $t \in (a,b]$.

(*Ax₃*) There exist functions $\eta_1, \eta_2, \eta_3 \in C([a,b], [0,\infty))$ such that

$$|f(t,x,y)| \leq \eta_1(t) + \eta_2(t)|x| + \eta_3(t) \quad \text{for } t \in (a,b], \text{ and } x, y \in \mathbb{R}.$$

(*Ax₄*) There exists a number $R > 0$ such that

$$R \geq \frac{g^*}{1-\ell} \left[\frac{|x_0|}{\Gamma(\xi)} + \frac{\eta \Gamma(\xi)}{\Gamma(\vartheta + \xi)} \left(\psi(b) - \psi(a)\right)^{\vartheta} \right],$$

where

$$p^* = \sup_{t \in [a,b]} p(t), \quad \eta_i^* = \sup_{t \in [a,b]} \eta_i(t), \quad i = 1, 2, \quad \eta_3^* = \sup_{t \in [a,b]} \eta_3(t) < 1,$$

$$g^* = \sup_{t \in [a,b]} |g(t,0)|, \quad \eta = \frac{\left(\psi(b) - \psi(a)\right)^{1-\xi} \eta_1^* + \eta_2^* R}{1 - \eta_3^*},$$

and

$$\ell = p^* \left[\frac{|x_0|}{\Gamma(\xi)} + \frac{\eta \Gamma(\xi)}{\Gamma(\vartheta + \xi)} \left(\psi(b) - \psi(a)\right)^{\vartheta} \right] < 1.$$

We are now in a position to state and prove our existence result for the problem (6.24)−(6.25) based on based on Lemma 1.30.

Theorem 6.14. *Assume* **(Ax_1)**–**(Ax_4)** *hold. If*

$$(\psi(b)-\psi(a))^{1-\xi}\,\ell<1, \tag{6.32}$$

then the problem (6.24)–(6.25) has at least one solution in $C_{\xi;\psi}(J)$.

Proof: We define a subset D of $C_{\xi;\psi}(J)$ by

$$D=\{x\in C_{\xi;\psi}(J):\|x\|_{\xi;\psi}\leq R\}.$$

We consider the operator $\Im$ defined in (6.31), and define two operators $\mathcal{N}_1: C_{\xi;\psi}(J)\to C_{\xi;\psi}(J)$ by

$$(\mathcal{N}_1x)(t)=g(t,x(t)),\quad t\in(a,b], \tag{6.33}$$

and $\mathcal{N}_2: D\to C_{\xi;\psi}(J)$ by

$$(\mathcal{N}_2x)(t)=\frac{x_0\,(\psi(t)-\psi(a))^{\xi-1}}{\Gamma(\xi)}+\left(\mathbb{J}_{a^+}^{\vartheta;\psi}v(\tau)\right)(t),\quad t\in(a,b]. \tag{6.34}$$

Then we get $\Im x=\mathcal{N}_1x\mathcal{N}_2x$. We shall show that the operators $\mathcal{N}_1$ and $\mathcal{N}_2$ satisfies all the conditions of Lemma 1.30. The proof will be given in several steps.

Step 1: The operator $\mathcal{N}_1$ is a Lipschitz on $C_{\xi;\psi}(J)$.
Let $x,y\in C_{\xi;\psi}(J)$ and $t\in(a,b]$. Then by **(Ax_2)** we have

$$\begin{aligned}\left|(\psi(t)-\psi(a))^{1-\xi}\left((\mathcal{N}_1x)(t)-(\mathcal{N}_1y)(t)\right)\right| &\leq(\psi(t)-\psi(a))^{1-\xi}\,|g(t,x(t))-g(t,y(t))|\\ &\leq p(t)\,(\psi(t)-\psi(a))^{1-\xi}\,\|x-y\|_{\xi,\psi}\\ &\leq p^*\,(\psi(b)-\psi(a))^{1-\xi}\,\|x-y\|_{\xi,\psi},\end{aligned}$$

then for each $t\in(a,b]$ we obtain

$$\|\mathcal{N}_1x-\mathcal{N}_1y\|_{\xi,\psi}\leq p^*\,(\psi(b)-\psi(a))^{1-\xi}\,\|x-y\|_{\xi,\psi}.$$

Step 2: The operator $\mathcal{N}_2$ is completely continuous on D.
We firstly show that the operator $\mathcal{N}_2$ is continuous on D. Let $\{x_n\}$ be sequence in D such that $x_n\to x$ in D. Let $x,y\in C_{\xi;\psi}(J)$.
Then for each $t\in(a,b]$, we have

$$\left|(\psi(t)-\psi(a))^{1-\xi}\,(\mathcal{N}_2x_n)(t)-(\mathcal{N}_2x)(t))\right|\leq(\psi(t)-\psi(a))^{1-\xi}\left(\mathbb{J}_{a^+}^{\vartheta;\psi}|v_n(\tau)-v(\tau)|\right)(t),$$

where $v_n,v\in C_{\xi,\psi}(J)$ such that

$$\begin{aligned}v_n(t)&=f(t,x_n(t),v_n(t)),\\ v(t)&=f(t,x(t),v(t)).\end{aligned}$$

Since $x_n\to x$ and f is continuous function on J then we get $v_n(t)\to v(t)$ as $n\to\infty$ for each $t\in(a,b]$, so by Lebesgue's dominated convergence theorem, we have

$$\|\mathcal{N}_2x_n-\mathcal{N}_2x\|_{C_{\xi;\psi}}\to 0 \text{ as } n\to\infty.$$

Then $\mathcal{N}_2$ is continuous. Next we prove that $\mathcal{N}_2(D)$ is uniformly bounded on $C_{\xi;\psi}(J)$. Let any $x \in D$.
By $(\boldsymbol{Ax_3})$, we have for each $t \in (a,b]$

$$\begin{aligned}\left|(\psi(t)-\psi(a))^{1-\xi} v(t)\right| &= \left|(\psi(t)-\psi(a))^{1-\xi} f(t,x(t),v(t))\right| \\ &\leq (\psi(t)-\psi(a))^{1-\xi}\left(\eta_1(t)+\eta_2(t)|x(t)|+\eta_3(t)|v(t)|\right) \\ &\leq (\psi(b)-\psi(a))^{1-\xi}\eta_1^* + \eta_2^* R + \eta_3^* |(\psi(t)-\psi(a))^{1-\xi} v(t)|.\end{aligned}$$

Witch implies that

$$\left|(\psi(t)-\psi(a))^{1-\xi} v(t)\right| \leq \frac{(\psi(b)-\psi(a))^{1-\xi}\eta_1^* + \eta_2^* R}{1-\eta_3^*}.$$

Then, we have

$$\sup_{t\in(a,b]}\left|(\psi(t)-\psi(a))^{1-\xi} v(t)\right| \leq \frac{(\psi(b)-\psi(a))^{1-\xi}\eta_1^* + \eta_2^* R}{1-\eta_3^*} := \eta.$$

For $t \in (a,b]$, by (6.34), $(\boldsymbol{Ax_3})$ and Lemma 6.8, we have

$$\begin{aligned}&\left|(\psi(t)-\psi(a))^{1-\xi}(\mathcal{N}_2x)(t)\right| \\ &\leq \frac{|x_0|}{\Gamma(\xi)} + (\psi(t)-\psi(a))^{1-\xi}\left(\mathbb{J}_{a^+}^{\vartheta;\psi}|v(\tau)|\right)(t) \\ &\leq \frac{|x_0|}{\Gamma(\xi)} + \eta\,(\psi(t)-\psi(a))^{1-\xi}\left(\mathbb{J}_{a^+}^{\vartheta;\psi}(\psi(\tau)-\psi(a))^{\xi-1}\right)(t) \\ &\leq \frac{|x_0|}{\Gamma(\xi)} + \frac{\eta\Gamma(\xi)}{\Gamma(\vartheta+\xi)}(\psi(t)-\psi(a))^{\vartheta} \\ &\leq \frac{|x_0|}{\Gamma(\xi)} + \frac{\eta\Gamma(\xi)}{\Gamma(\vartheta+\xi)}(\psi(b)-\psi(a))^{\vartheta}.\end{aligned}$$

Then for $t \in (a,b]$, we obtain

$$\|\mathcal{N}_2x\|_{C_{\xi;\psi}} \leq \frac{|x_0|}{\Gamma(\xi)} + \frac{\eta\Gamma(\xi)}{\Gamma(\vartheta+\xi)}(\psi(b)-\psi(a))^{\vartheta}.$$

This prove that the operator $\mathcal{N}_2$ is uniformly bounded on D. Next we prove that the operator $\mathcal{N}_2D$ equicontinuous. We take $x \in D$ and $a<\varepsilon_1<\varepsilon_2\leq b$. Then,

$$\begin{aligned}&\left|(\psi(\varepsilon_1)-\psi(a))^{1-\xi}(\mathcal{N}_2x)(\varepsilon_1)-(\psi(\varepsilon_2)-\psi(a))^{1-\xi}(\mathcal{N}_2x)(\varepsilon_2)\right| \\ &\leq \left|(\psi(\varepsilon_1)-\psi(a))^{1-\xi}\left(\mathbb{J}_{a^+}^{\vartheta;\psi}v(\tau)\right)(\varepsilon_1)-(\psi(\varepsilon_2)-\psi(a))^{1-\xi}\left(\mathbb{J}_{a^+}^{\vartheta;\psi}v(\tau)\right)(\varepsilon_2)\right| \\ &\leq (\psi(\varepsilon_2)-\psi(a))^{1-\xi}\left(\mathbb{J}_{\varepsilon_1^+}^{\vartheta;\psi}|v(\tau)|\right)(\varepsilon_2) + \frac{1}{\Gamma(\vartheta)}\int_a^{\varepsilon_1}|\psi'(\tau)\Psi(\tau)v(\tau)|\,d\tau,\end{aligned}$$

where

$$\Psi(\tau) = \left[(\psi(\varepsilon_1)-\psi(a))^{1-\xi}(\psi(\varepsilon_1)-\psi(\tau))^{\vartheta-1}-(\psi(\varepsilon_2)-\psi(a))^{1-\xi}(\psi(\varepsilon_2)-\psi(\tau))^{\vartheta-1}\right].$$

Then by Lemma 6.8, we have for each $t \in (a, b]$

$$\begin{aligned}
&\left|(\psi(\varepsilon_1)-\psi(a))^{1-\xi}(\mathcal{N}_2x)(\varepsilon_1)-(\psi(\varepsilon_2)-\psi(a))^{1-\xi}(\mathcal{N}_2x)(\varepsilon_2)\right|\\
&\leq \frac{\eta\Gamma(\xi)}{\Gamma(\vartheta+\xi)}(\psi(\varepsilon_2)-\psi(a))^{1-\xi}(\psi(\varepsilon_2)-\psi(\varepsilon_1))^{\vartheta+\xi-1}\\
&+\eta\int_a^{\varepsilon_1}\left|\Psi(\tau)\frac{\psi'(\tau)}{\Gamma(\vartheta)}\right|(\psi(\tau)-\psi(a))^{\xi-1}d\tau,
\end{aligned}$$

note that

$$\left|(\psi(\varepsilon_1)-\psi(a))^{1-\xi}(\mathcal{N}_2x)(\varepsilon_1)-(\psi(\varepsilon_2)-\psi(a))^{1-\xi}(\mathcal{N}_2x)(\varepsilon_2)\right| \to 0 \quad \text{as} \quad \varepsilon_1 \to \varepsilon_2.$$

This proves that $\mathcal{N}_2D$ is equicontinuous on J. Therefore by the Arzela-Ascoli Theorem, $\mathcal{N}_2$ is completely continuous.

Step 3: Now we show that the third hypothesis of Lemma 1.30 is satisfied. Let $x \in C_{\xi;\psi}(J)$ and $y \in D$ be arbitrary such that $x = \mathcal{N}_1x\mathcal{N}_2y$. Then, for $t \in (a, b]$ we have

$$\begin{aligned}
&\left|(\psi(t)-\psi(a))^{1-\xi}x(t)\right|\\
&=\left|(\psi(t)-\psi(a))^{1-\xi}(\mathcal{N}_1x\mathcal{N}_2y)(t)\right|\\
&=(\psi(t)-\psi(a))^{1-\xi}\left|(\mathcal{N}_1x)(t)\right|\left|(\mathcal{N}_2y)(t)\right|\\
&=|g(t,x(t))|\left|\frac{x_0}{\Gamma(\xi)}+(\psi(t)-\psi(a))^{1-\xi}\left(\mathbb{J}_{a^+}^{\vartheta;\psi}\left|f\left(\tau,y(\tau),{}^H\mathbb{D}_{a^+}^{\vartheta,r;\psi}\left(\frac{y(\tau)}{g(\tau,y(\tau))}\right)\right)\right|\right)(t)\right|\\
&\leq(|g(t,x(t))-g(t,0)|+|g(t,0)|)\left[\frac{|x_0|}{\Gamma(\xi)}+\frac{\eta\Gamma(\xi)}{\Gamma(\vartheta+\xi)}(\psi(b)-\psi(a))^{\vartheta}\right]\\
&\leq\left(p^*\|x\|_{C_{\xi;\psi}}+g^*\right)\left[\frac{|x_0|}{\Gamma(\xi)}+\frac{\eta\Gamma(\xi)}{\Gamma(\vartheta+\xi)}(\psi(b)-\psi(a))^{\vartheta}\right],
\end{aligned}$$

then,

$$\begin{aligned}
\|x\|_{C_{\xi;\psi}} &= \frac{g^*\left[\frac{|x_0|}{\Gamma(\xi)}+\frac{\eta\Gamma(\xi)}{\Gamma(\vartheta+\xi)}(\psi(b)-\psi(a))^{\vartheta}\right]}{1-p^*\left[\frac{|x_0|}{\Gamma(\xi)}+\frac{\eta\Gamma(\xi)}{\Gamma(\vartheta+\xi)}(\psi(b)-\psi(a))^{\vartheta}\right]}\\
&\leq R.
\end{aligned}$$

Then $x \in D$, thus the third hypothesis of Lemma 1.30 is satisfied.

Step 4: Now, we show that $p^*(\psi(b)-\psi(a))^{1-\xi}L < 1$, where $L = \|\mathcal{N}_2(D)\|_{C_{\xi;\psi}} = \sup\{\|\mathcal{N}_2y\|_{C_{\xi;\psi}} : y \in D\}$.
Since

$$L \leq \frac{|x_0|}{\Gamma(\xi)}+\frac{\eta\Gamma(\xi)}{\Gamma(\vartheta+\xi)}(\psi(b)-\psi(a))^{\vartheta},$$

then $p^*(\psi(b)-\psi(a))^{1-\xi}L \leq (\psi(b)-\psi(a))^{1-\xi}\ell < 1$. That is, the last hypothesis of Lemma 1.30 is satisfied. Thus, the operator equation $\Im x = \mathcal{N}_1x\mathcal{N}_2x = x$ has at least

one solution $x \in C_{\xi;\psi}$, witch is a point fixe for the operator $\Im$.
It is clear that by $(\boldsymbol{Ax_1})$ we have $t \longrightarrow \left(\frac{x(t)}{g(t,x(t))}\right) \in C^1_{\xi,\psi}(J)$. Then, as a consequence of Steps 1 to 4 with Theorem 6.14, we can conclude that the problem (6.24)-(6.25) has at least a solution in $C_{\xi;\psi}(J)$. ■

6.3.2 Examples

Example 6.15. *Taking $r \to 0$, $\vartheta = \frac{1}{2}$, $\psi(t) = t$, $a = 1$, $b = 2$ and $x_0 = 0$, we obtain a particular case of problem (6.24)-(6.25) with Riemann-Liouville fractional derivative, given by*

$$ {}^{RL}\mathbb{D}_{1^+}^{\frac{1}{2},0;t}\left(\frac{x(t)}{g(t,x(t))}\right) = f\left(t, x(t), {}^{RL}\mathbb{D}_{1^+}^{\frac{1}{2},0;t}\left(\frac{x(t)}{g(t,x(t))}\right)\right), \quad t \in (1,2], \tag{6.35}$$

$$\left(\mathbb{J}_{1^+}^{\frac{1}{2};t}\left(\frac{x(\tau)}{g(\tau,x(\tau))}\right)\right)(1^+) = 0, \tag{6.36}$$

where $J = (1,2]$. Set

$$g(t,x(t)) = \frac{\sqrt{t-1}}{33e^{-t+2}}\left(|x(t)sin(t)| + 3\right), \ t \in [1,2], \ x \in C_{\frac{1}{2},t}(J).$$

and

$$f(t,x,y) = \frac{\sqrt{t-1}|cos(t)|(1+x+y)}{55e^{-t+4}(2+|x|)}, \ t \in J, \ x,y \in \mathbb{R}.$$

We have

$$C_{\xi,\psi}(J) = C_{\frac{1}{2},t}(J) = \left\{u : J \to \mathbb{R} : (\sqrt{t-1})u \in C([a,b],\mathbb{R})\right\},$$

and

$$C^1_{\xi,\psi}(J) = C^1_{\frac{1}{2},t}(J) = \left\{u \in C_{\frac{1}{2},t}(J) : u' \in C_{\frac{1}{2},t}(J)\right\},$$

with $\xi = \vartheta = \frac{1}{2}$, $\psi(t) = t$, $r = 0$. Clearly, the function $f \in C^1_{\frac{1}{2},t}(J)$. Hence the condition $(\boldsymbol{Ax_1})$ is satisfied.
For each $x, \bar{x} \in \mathbb{R}$ and $t \in J$, we have

$$|g(t,x) - g(t,\bar{x})| \leq \frac{\sqrt{t-1}|sin(t)|}{33e^{-t+2}}|x - \bar{x}|.$$

Hence condition $(\boldsymbol{Ax_2})$ is satisfied with

$$p(t) = \frac{|sin(t)|}{33e^{-t+2}} \quad and \quad p^* = \frac{1}{33}.$$

Let $x, y \in \mathbb{R}$. Then we have

$$|f(t,x,y)| \leq \frac{\sqrt{t-1}|cos(t)|(1+|x|+|y|)}{55e^{-t+4}}, \ t \in J,$$

and so the condition $(\boldsymbol{Ax_3})$ is satisfied with

$$\eta_1(t) = \eta_2(t) = \eta_3(t) = \frac{\sqrt{t-1}|cos(t)|}{55e^{-t+4}},$$

and

$$\eta_1^* = \eta_2^* = \eta_3^* = \frac{1}{55e^2}.$$

Also, the condition $(\boldsymbol{Ax_4})$ *and the condition (6.32) of Theorem 6.14 is satisfied if we take*

$$2515 \approx \frac{605e^2 - \sqrt{\pi} - 11}{\sqrt{\pi}} \leq R < \frac{1815e^2 - \sqrt{\pi} - 33}{\sqrt{\pi}} \approx 7547.$$

Then the problem (6.35)-(6.36) *has at least one solution in* $C_{\frac{1}{2},t}(J)$.

Example 6.16. *Taking* $r \to 1$, $\vartheta = \frac{1}{2}$, $\psi(t) = t$, $a = 1$, $b = \pi$ *and* $x_0 = 0$, *we obtain a particular case of problem* (6.24)-(6.25) *involving Caputo fractional derivative, given by*

$$^C\mathbb{D}_{1^+}^{\frac{1}{2},1;t}\left(\frac{x(t)}{g(t,x(t))}\right) = f\left(t, x(t), {}^C\mathbb{D}_{1^+}^{\frac{1}{2},1;t}\left(\frac{x(t)}{g(t,x(t))}\right)\right), \quad t \in (1,\pi], \tag{6.37}$$

$$\left(\mathbb{J}_{1^+}^{1;t}\left(\frac{x(\tau)}{g(\tau,x(\tau))}\right)\right)(1^+) = 0, \tag{6.38}$$

where $J = (1,\pi]$. *Set*

$$g(t,x(t)) = \frac{1}{115}\left(|x(t)| + |tan^{-1}(t)|\right), \ t \in [1,\pi], \ x \in C_{1,t}(J).$$

and

$$f(t,x,y) = \frac{|sin^2(t)|}{77^t}\left(\frac{x}{1+|x|} + \frac{y}{2+|y|}\right) + |tan^{-1}(t)| + 5\pi, \ t \in J, \ x, y \in \mathbb{R}.$$

We have

$$C_{\xi,\psi}(J) = C_{1,t}(J) = \{u : J \to \mathbb{R} : u \in C([1,\pi],\mathbb{R})\},$$

and

$$C^1_{\xi,\psi}(J) = C^1_{1,t}(J) = \{u \in C_{1,t}(J) : u' \in C_{1,t}(J)\},$$

with $\vartheta = \frac{1}{2}$, $\xi = 1$, $\psi(t) = t$, $r = 1$. *Clearly, the function* $f \in C^1_{1,t}(J)$. *Hence the condition* $(\boldsymbol{Ax_1})$ *is satisfied.*
For each $x, \bar{x} \in \mathbb{R}$ *and* $t \in J$, *we have*

$$|g(t,x) - g(t,\bar{x})| \leq \frac{1}{115}|x - \bar{x}|.$$

Hence condition $(\boldsymbol{Ax_2})$ *is satisfied with* $p(t) = \frac{1}{115}$.
Let $x, y \in \mathbb{R}$. *Then we have*

$$|f(t,x,y)| \leq \frac{|sin^2(t)|(|x| + |y|)}{77^t} + |tan^{-1}(t)| + 5\pi, \ t \in J,$$

and so the condition $(\boldsymbol{Ax_3})$ *is satisfied with*

$$\eta_1(t) = |tan^{-1}(t)| + 5\pi, \quad \eta_2(t) = \eta_3(t) = \frac{|sin^2(t)|}{77^t},$$

and

$$\eta_1^* = \frac{11\pi}{2}, \quad \eta_2^* = \eta_3^* = \frac{1}{77}.$$

Same as the last example, the conditions $(\boldsymbol{Ax_4})$ *and (6.32) of Theorem 6.14 is satisfied if we choose a convenient constant* R. *Then the problem* (6.37)-(6.38) *has at least one solution in* $C_{1,t}([1,\pi])$.

Example 6.17. *Taking $r \to 0$, $\vartheta = \frac{1}{2}$, $\psi(t) = \ln t$, $a = 1$, $b = e$ and $x_0 = e$, we get a particular case of problem (6.24)-(6.25) using the Hadamard fractional derivative, given by*

$$^{HD}\mathbb{D}_{1^+}^{\frac{1}{2},0;\ln t}\left(\frac{x(t)}{g(t,x(t))}\right) = f\left(t, x(t), ^{HD}\mathbb{D}_{1^+}^{\frac{1}{2},0;\ln t}\left(\frac{x(t)}{g(t,x(t))}\right)\right), \quad t \in (1,e], \tag{6.39}$$

$$\left(\mathbb{J}_{1^+}^{\frac{1}{2};\ln t}\left(\frac{x(\tau)}{g(\tau,x(\tau))}\right)\right)(1^+) = e, \tag{6.40}$$

where $J = (1,e]$. Set

$$g(t,x(t)) = \frac{e^{-t+1}|cos(\pi t)|}{12\pi + 111e^{2t}}\left((\sqrt{\ln t})|x(t)| + e^{-t+e}\right) + e\sqrt{\pi},\ t \in [1,e],\ x \in C_{\frac{1}{2},\ln t}(J).$$

and

$$f(t,x,y) = \frac{e + x + y}{22e^t},\ t \in J,\ x, y \in \mathbb{R}.$$

We have

$$C_{\xi,\psi}(J) = C_{\frac{1}{2},\ln t}(J) = \left\{u : J \to \mathbb{R} : (\sqrt{\ln t})u \in C([1,e],\mathbb{R})\right\},$$

and

$$C^1_{\xi,\psi}(J) = C^1_{\frac{1}{2},\ln t}(J) = \left\{u \in C_{\frac{1}{2},\ln t}(J) : u' \in C_{\frac{1}{2},\ln t}(J)\right\},$$

with $\vartheta = \xi = 1$, $\psi(t) = \ln t$, $r = 0$. Clearly, the continuous function $f \in C^1_{\frac{1}{2},\ln t}(J)$. Hence the condition ***(Ax₁)*** *is satisfied.*
For each $x, \bar{x} \in \mathbb{R}$ and $t \in J$, we have

$$|g(t,x) - g(t,\bar{x})| \leq \frac{e^{-t+1}|cos(\pi t)|\sqrt{\ln t}}{12\pi + 111e^{2t}}|x - \bar{x}|.$$

Hence condition ***(Ax₂)*** *is satisfied with*

$$p(t) = \frac{e^{-t+1}|cos(\pi t)|}{12\pi + 111e^{2t}}, \quad \text{and} \quad p^* = \frac{1}{12\pi + 111e^2}.$$

Let $x, y \in \mathbb{R}$. Then we have

$$|f(t,x,y)| \leq \frac{e + |x| + |y|}{22e^t},\ t \in J,$$

and so the condition ***(Ax₃)*** *is satisfied with*

$$\eta_1(t) = \frac{e}{22e^t}, \quad \eta_2(t) = \eta_3(t) = \frac{1}{22e^t},$$

and

$$\eta_1^* = \frac{e}{22e}, \quad \eta_2^* = \eta_3^* = \frac{1}{22e}.$$

Same as before, we choose a suitable constant R so the conditions ***(Ax₄)*** *and (6.32) of Theorem 6.14 be satisfied. Then the problem (6.39)-(6.40) has at least one solution in $C_{\frac{1}{2},\ln t}(J)$.*

CONCLUSION AND PERSPECTIVE

In this thesis, we have presented some results to the theory of existence, uniqueness and Ulam-Hyers-Rassias stability results for a class of initial value problem and boundary value problem for differential equations with generalized Hilfer type fractional derivative with and without impulses (both instantaneous and non-instantaneous), we also delved in a class of initial value problem for nonlinear fractional Hybrid implicit differential equations with generalized Hilfer and ψ-Hilfer fractional derivative. The tools used include the fixed point theorems of Krasnoselskii, Dhage and Schaefer and Banach contraction principle. Also we have considered in this thesis the same problems but in Banach spaces, with results based on the fixed point theorems of Darbo and Mönch associated with the technique of measure of noncompactness.

Since the concept of proportional fractional and Integral derivatives is very recent, in the future research, we plan to study some problems for nonlinear implicit fractional equations with generalized proportional fractional derivative. Indeed, in recent years, various researchers have studied integrals and so-called conformable derivatives to recover certain properties that are not satisfied by the fractional case. Based on this notion, some authors have used modified conformable derivatives (proportional derivatives or proportional fractional and integral derivatives) to generate integrals and nonlocal fractional derivatives, called integrals and proportional derivatives, which contain exponential functions in their kernels. For more information on this subject, see [17, 66, 83, 103] and the references therein.

In terms of context and potential generalization, it will be useful to expand the findings of the present study by considering differential inclusions, the case of hybrid equations and the nonlinear coupled systems. In addition, the problems studied in Banach spaces can be extended to Fréchet spaces with other methods, other fixed point theorems and the conditions that are ideally suited to achieve the best results. We may add delay and advance arguments, study the random problems or even go further and generalize our problems with the recently studied generalized Mittag-Leffler functions.

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Abstract

In this thesis, we present some results on existence, uniqueness, and stability of Ulam-Hyers-Rassias for a class of initial value problem and boundary value problem for differential equations with generalized Hilfer-type fractional derivative with and without impulses (both instantaneous and non-instantaneous), We have also discussed the class of initial value problem for nonlinear fractional Hybrid implicit differential equations with generalized Hilfer and ψ-Hilfer fractional derivative. The methods used are the fixed point theorems of Krasnoselskii, Dhage and Schaefer and Banach contraction principle. We also took into account the same problems, albeit in Banach Spaces, with results based on the fixed point theorems of Darbo and Mönch associated with the technique of measure of noncompactness.

Résumé

Dans cette thèse, nous présentons quelques résultats d'existence, d'unicité et de stabilité au sens d'Ulam pour une classe de problèmes à valeurs initiales et de problème aux limites pour des équations différentielles avec des dérivées fractionnaires généralisées de type Hilfer, avec et sans impulsions (instantanées et non-instantané), Nous avons également discuté la classe de problèmes à valeur initiale pour les équations différentielles, implicites hybrides, non linéaires avec la dérivée fractionnaire généralisée de Hilfer et la dérivée fractionnaire de ψ-Hilfer. Les méthodes utilisées sont les théorèmes de points fixes de Krasnoselskii, Dhage, Schaefer et le principe de contraction de Banach. Nous avons également pris en compte les mêmes problèmes mais dans les espaces de Banach, avec des résultats basés sur en utilisent les théorèmes de point fixe de Darbo et de Mönch associés à la technique de la mesure de non-compacité.

ملخص

في هذه الأطروحة ، نقدم بعض النتائج حول وجود الحلول لبعض الفئات من المعادلات التفاضلية ذات الرتب الكسرية و وحدانيتها. تم ايضا شمل دراسة مسألة الإستقرار (Stability) بمفهوم Ulam. تم التعرف على المعادلات ذات القيمة الأولية و المعادلات ذات القيم الحدية ذات المشتق الكسري المعمم من نوع Hilfer مع و بدون نبضات (فورية و غير فورية) . ناقشنا أيضا فئة مشكلة القيمة الأولية للمعادلات التفاضلية الهجينة (hybrid) غير الخطية مع مشتق Hilfer و ψ-Hilfer. إستخدمنا في هذه الدراسة مبرهنات النقطة الصامدة ل: Krasnoselskii و Dhage و Schaefer و Banach. لقد أخذنا في الاعتبار أيضا المشكلات نفسها ، ولكن في فضاءات باناخ ، مع نتائج تستند إلى مبرهنات النقطة الصامدة ل: Darbo و Mönch المرتبطة بتقنية قياس عدم التراص.