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THESE

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Intitulée

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Dédicaces

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Contents

Introduction

The main objective of this thesis is the study of nonlinear boundary problems with delay which involve systems of ordinary differential equations and of fractional order . In this thesis existence and uniqueness of solutions to certain second order boundary value problems for delay differential equations is established by using some fixed point theorems, notably Leray-Schauder theorem, Krasnoselskii theorem, expansion and compression of a cone .

This thesis consists of five chapters,

In first chapter, we introduce notations, definitions , lemmas, and fixed point theorems that are used in the next chapters.

Chapter 2, we present some existence of positive solutions using the Krasnosel'skii fixed point theorem in cones for nonlocal boundary value problem for delay fractional boundary value problem following

$$\begin{cases} D_{0^+}^{\alpha} u(t) + r(t) f(u_t) = 0, \ 0 < t < 1, \quad \alpha \in (2,3] \\ u(t) = \phi(t), \ -\tau \leqslant t \leqslant 0, \\ u(0) = u'(0) = 0, \quad u'(1) = \beta u(\eta) \end{cases}$$

where $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractionnaire derivative, α, β, η and τ are positive constants such that $\eta \in (0,1), 0 < \tau \leq \frac{1}{2}$ and λ is a positive real parameter. By using Krasnosel'skii fixed point theorem, some sufficient conditions for the existence of positive solutions are obtained of nonlinear second order delay fractional boundary value problem

In chapter 3, we investigate the existence of positive solutions for boundary value problems composed by coupled systems of second order differential equations with nonlocal boundary conditions

$$
\begin{cases}
u''(t)+a_1(t)f(u_t,v_t)=0, & 0<t<1,\\
v''(t)+a_2(t)g(u_t,v_t)=0, & 0<t<1,\\
u(t)=\phi_1(t), & -\tau\leqslant t\leqslant 0,\\
u'(1)=\alpha u(\eta)+\beta u'(\eta) & \\
v(t)=\phi_2(t), & -\tau\leqslant t\leqslant 0,\\
v'(1)=\alpha v(\eta)+\beta v'(\eta) &
\end{cases}
$$

where $0<\eta<1$, $0<\alpha<1$ and $0<\beta<1-\alpha\eta$ are constants. By using fixed-point index theorem in cones, we establish the existence results of positive solutions for the boundary value problem

In chapter 4, a nonlinear second order multi-point boundary value problem is considered,

$$
\begin{cases}
u''(t)+f(t,u_t,u(t))=0, & 0<t<1,\\
u(t)=\phi(t), & -\tau\leqslant t\leqslant 0,\\
u(1)=\alpha u(\eta)+\beta u'(\eta) &
\end{cases}
$$

where $\eta\in(0,1)$, $0<\beta<1-\alpha\eta$, $0<\alpha<1$ are constants. Our results are based on the nonlinear alternative of Leray-Schauder fixed point theorem.

The purpose of Chapter 5, is to study the existence of positive solutions for three boundary value problems of fractional differential equations with delay. We consider the BVP of the form

$$
\begin{cases}
D^{\alpha}_{0^+}u(t)+f(t,u_t)=0, & 0<t<1,\quad \alpha\in(2,3]\\
u(t)=\phi(t), & -\tau\leqslant t\leqslant 0,\\
u(0)=u'(0)=0,\quad u(1)=\alpha u(\eta)+\beta u'(\eta) &
\end{cases}
$$

where $D^{\alpha}_{0^+}$ is the Riemann-Liouville fractionnaire derivative, α, η are positive constants. Our results are based on the nonlinear alternative of Leray-Schauder type and Krasnosel'skii fixed point theorem, the sufficient condition of existence of their solutions is derived.

Chapter 1

Preliminaries

In this chapter, we introduce notations, definitions and preliminary results that will be used in the sequel.

We shall consider the Banach space $E = C([a,b],\mathbb{R})$ endowed with the maximum norm

$$\|y\|_{[a,b]} = \max_{a \leq t \leq b} |y(t)| \text{ for } y \in E.$$

Definition 1.0.1. *An operator $T : E \to E$ is completely continuous if it is continuous and maps bounded sets into relatively compact sets.*

Theorem 1.0.1. *(Arzela-Ascoli Theorem). A subset of $C([a,b],\mathbb{R})$ is relatively compact if and only if it is bounded and equicontinuous.*

Definition 1.0.2. *Let X be a real Banach space. A nonempty, closed and convex set $P \subset X$ is a cone if it satisfies the following two conditions:*

1. *If $x \in P$ and $\lambda \geq 0$ then $\lambda x \in P$,*

2. *If $x \in P$ and $-x \in P$ then $x = 0$.*

The cone P induces an ordering $\leq$ on X by

$$x \leq y \text{ if and only if } y - x \in P.$$

Now we present the well-known Krasnosel'skii fixed point Theorem on cone.

Theorem 1.0.2. *(Krasnosel'skii)*

Let X be a Banach space, and let $K \subset X$ be a cone. Assume that Ω_1 and Ω_2 are open subsets of X with $0 \in \Omega_1$, $\Omega_1 \subset \Omega_2$ and let

$$A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$$

be a completely continuous operator such that

1. $\|Au\| \le \|u\|$ *for* $u \in K \cap \partial\Omega_1$ *and* $\|Au\| \ge \|u\|$ *for* $u \in K \cap \partial\Omega_2$, *or*

2. $\|Au\| \ge \|u\|$ *for* $u \in K \cap \partial\Omega_1$ *and* $\|Au\| \le \|u\|$ *for* $u \in K \cap \partial\Omega_2$.

Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Theorem 1.0.3. *(Nonlinear alternative of Leray-Schauder)*

Let E be a Banach space with $C \subset E$ closed and convex. Assume that U is a relatively open subset of C with $0 \in U$ and $T : \overline{U} \to C$ is completely continuous operator. Then either

1. *T has a fixed point in $\overline{U}$, or*

2. *there exists $u \in \partial U$ and $\lambda \in (0,1)$ such that $u = \lambda T u$.*

Lemma 1.0.1. *([19])*

Suppose that X is a Banach space and $K \subset X$ is a cone in X. Define $K_r = \{x \in K : ||x|| \le r\}$. Furthermore, assume $A : K \longrightarrow K$ be a completely continuous operator and $Ax \ne x$ for $x \in \partial K_r = \{x \in K : ||x|| = r\}$. Then, we have the following conclusions

1. *if $\| x \| \leqslant \| Ax \|$, for $x \in \partial K_r$, then $i(A, K_r, K) = 0$;*

2. *if $\| x \| \geqslant \| Ax \|$, for $x \in \partial K_r$, then $i(A, K_r, K) = 1$.*

Lemma 1.0.2. *Let Ω be a bounded open subset of X Banach space, with $0 \in K \cap \Omega$, and $K \cap \overline{\Omega} \ne K$. Assume that $T : K \cap \overline{\Omega} \longrightarrow K$ be a completely continuous mapping such that $u \ne Tu$ for $u \in K \cap \partial\Omega$. Then the fixed point index $i(T, K \cap \Omega, K)$ a has the following properties:*

(i) *If there exists* $v \in K \setminus \{\theta\}$*, such that* $u - Tu \neq \zeta v$ *for every* $u \in K \cap \partial\Omega$ *et tout* $\zeta > 0$*, then* $i(T, K \cap \Omega, K) = 0$.

If $\| Tu \| \geq \| u \|$ *for* $u \in K \cap \partial\Omega$*, then* $i(T, K \cap \Omega, K) = 0$.

(ii) *If* $\mu Tu \neq u$ *for every* $u \in K \cap \partial\Omega$ *and* $0 < \mu < 1$*, then* $i(T, K \cap \Omega, K) = 1$.

For example (ii) holds if $\| Tu \| \leq \| u \|$ *for* $u \in K \cap \partial\Omega$.

(iii) *Let* Ω' *be open in* X *such that* $K \cap \overline{\Omega'} \subset K \cap \Omega$*. if* $i(T, K \cap \Omega, K) = 1$ *and* $i(T, K \cap \overline{\Omega'}, K) = 0$*, then* T *has a fixed point in* $K \cap \Omega \backslash K \cap \overline{\Omega'}$*. The same holds* $i(T, K \cap \Omega, K) = 0$ *and* $i(T, K \cap \Omega', K) = 1$.

Lemma 1.0.3. *Let* Ω *be a bounded open subset of* X *Banach space, with* $\theta \in \Omega$*, and let* $T : K \cap \overline{\Omega} \longrightarrow K$ *be acompletely continuous mapping. If* $\mu Tu \neq u$ *for every* $u \in K \cap \partial\Omega$ *and* $0 < \mu \leq 1$*, then* $i(T, K \cap \Omega, K) = 1$.

Lemma 1.0.4. *Let* Ω *be a bounded open subset of* X *Banach space, and let* $T : K \cap \overline{\Omega} \longrightarrow K$ *be acompletely continuous mapping. If there existe an* $v \in K \setminus \{\theta\}$*, such that* $u - Tu \neq \zeta v$ *for every* $u \in K \cap \partial\Omega$ *and* $\zeta \geq 0$*, then* $i(T, K \cap \Omega, K) = 0$.

Lemma 1.0.5. *If* $i(T, \Omega, X) \neq 0$*, then* T *has at least one fixed point in* Ω.

Definition 1.0.3. [50] *The Riemann-Liouville fractional derivative of order* $\alpha > 0$ *of a continuous function* $y : (0, +\infty) \to \mathbb{R}$ *is given by*

$$D_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{d}{dt}\Big)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$ *and* $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt$.

Definition 1.0.4. *Let* $f \in L^1(\mathbb{R}^+)$ *The Riemann-Liouville fractional integral of order* α (> 0) *forf is defined by*

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(s)(t-s)^{\alpha-1} ds.$$

where $\Gamma(.)$ is the Euler gamma function.

Lemma 1.0.6. [50] *suppose that* $y \in C(0,1) \cap L(0,1)$ *with a fractional derivative of order*$\alpha (> 0)$*.Then*

$$I^{\alpha} D^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + \ldots + c_n t^{\alpha-n}.$$

for somme $c_i \in \mathbb{R}, i = 1, 2, \ldots, n$.

Chapter 2

Positive Solutions for Delay Fractional Differential Equations

In this chapter, we consider the existence of positive solutions for the following multi-point boundary value problem

$$\begin{cases} D_{0^+}^{\alpha}u(t)+r(t)f(u_t)=0 & 0\le t\le 1, \qquad \alpha\in(2,3], \\ u(t)=\phi(t) & -\tau\le t\ \le 0 \\ u(0)=u^{'}(0)=0 & u'(1)=\beta u(\eta), \end{cases} \tag{2.0.1}$$

where $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative, $0<\eta<1, \quad 0<\beta<\frac{\alpha-1}{\eta^{\alpha-1}}$, and $\phi(t)\in C([-\tau,0],[0,+\infty)), \phi(0)=0,\ f\in C(D_1;\mathbb{R}^+),\ D_1=C([-\tau,0],\mathbb{R})$ for each $t\in[0,1]$, $u_t(\theta)=u(t+\theta)], \theta\in[-\tau,0]$ and τ is the delay with $(0<\tau<\frac{1}{2})$.

2.1 Preliminaries

Lemma 2.1.1. *Let* $0 < \eta < 1$ *and* $\beta \neq \frac{\alpha-1}{\eta^{\alpha-1}}$ *If* $y(t) \in C[0,1]$ *,then the boundary value problem*

$$D_{0^+}^{\alpha} u(t) + y(t) = 0 \qquad 0 < t < 1,\ 2 < \alpha \leq 3 \tag{2.1.1}$$

$$u(0) = u^{'}(0) = 0 \qquad u'(1) = \beta u(\eta) \tag{2.1.2}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)y(s)ds + \frac{\beta t^{\alpha-1}}{[(\alpha-1) - \beta\eta^{\alpha-1}]}\int_0^1 G(\eta,s)y(s)ds.$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1} & for\ 0 \leq s \leq t \leq 1 \\ \\ t^{\alpha-1}(1-s)^{\alpha-2} & for\ 0 \leq t \leq s \leq 1 \end{cases} \tag{2.1.3}$$

Proof. By lemma 1.0.6 the solution of (2.1.1) can be written as

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds. \tag{2.1.4}$$

Using the bondary condition (2.1.2), we find that $c_2 = c_3 = 0$.

So, (2.1.4) becomes

$$u(t) = c_1 t^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y(s)ds.$$

We have

$$u'(1) = (\alpha-1)c_1 - \frac{\alpha-1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2} y(s)ds, \tag{2.1.5}$$

and

$$\beta u(\eta) = \beta c_1 \eta^{\alpha-1} - \frac{\beta}{\Gamma(\alpha)}\int_0^{\eta} (\eta-s)^{\alpha-1} y(s)ds. \tag{2.1.6}$$

According to $u'(1) = \beta u(\eta)$, we have

$$c_1 = \frac{(\alpha-1)}{\Gamma(\alpha)[(\alpha-1) - \beta\eta^{\alpha-1}]}\int_0^1 (1-s)^{\alpha-2} y(s)ds - \frac{\beta}{\Gamma(\alpha)[(\alpha-1) - \beta\eta^{\alpha-1}]}\int_0^{\eta} (\eta-s)^{\alpha-1} y(s)ds.$$

So,

$$u(t) = \frac{t^{\alpha-1}(\alpha-1)}{\Gamma(\alpha)[(\alpha-1)-\beta\eta^{\alpha-1}]}\int_0^1 (1-s)^{\alpha-2}y(s)ds - \frac{\beta t^{\alpha-1}}{\Gamma(\alpha)[(\alpha-1)-\beta\eta^{\alpha-1}]}\int_0^\eta (\eta-s)^{\alpha-1}y(s)ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds$$

$$= -\tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \left[\tfrac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{\beta\eta^{(\alpha-1)}t^{\alpha-1}}{\Gamma(\alpha)(\alpha-1)-\beta\eta^{\alpha-1}}\right]\int_0^1 (1-s)^{\alpha-2}y(s)ds$$
$$-\frac{\beta t^{\alpha-1}}{\Gamma(\alpha)[(\alpha-1)-\beta\eta^{\alpha-1}]}\int_0^\eta (\eta-s)^{\alpha-1}y(s)ds$$
$$= \tfrac{1}{\Gamma(\alpha)}\int_0^t [t^{\alpha-1}(1-s)^{\alpha-2}-(t-s)^{\alpha-1}]y(s)ds + \tfrac{1}{\Gamma(\alpha)}\int_t^1 t^{\alpha-1}(1-s)^{\alpha-2}y(s)ds$$
$$+\frac{\beta\eta^{(\alpha-1)}t^{\alpha-1}}{\Gamma(\alpha)[(\alpha-1)-\beta\eta^{\alpha-1}]}\int_0^1 (1-s)^{\alpha-2}y(s)ds - \frac{\beta t^{\alpha-1}}{\Gamma(\alpha)[(\alpha-1)-\beta\eta^{\alpha-1}]}\int_0^\eta (\eta-s)^{\alpha-1}y(s)ds$$
$$= \frac{1}{\Gamma(\alpha)}\Big(\int_0^t [t^{\alpha-1}(1-s)^{\alpha-2}-(t-s)^{\alpha-1}]y(s)ds + \int_t^1 t^{\alpha-1}(1-s)^{\alpha-2}y(s)ds\Big)$$
$$+\frac{\beta t^{\alpha-1}}{\Gamma(\alpha)[(\alpha-1)-\beta\eta^{\alpha-1}]}\left(\int_0^\eta [\eta^{\alpha-1}(1-s)^{\alpha-2}-(\eta-s)^{\alpha-1}]y(s)ds + \int_\eta^1 \eta^{\alpha-1}(1-s)^{\alpha-2}y(s)ds\right)$$
$$= \int_0^1 G(t,s)y(s)ds + \tfrac{\beta t^{\alpha-1}}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\int_0^1 G(\eta,s)y(s)ds.$$

. □

Lemma 2.1.2. *The function $G(t,s)$ defined by (2.1.3) satisfies*

i) $G(t,s) > 0, for\ t,s \in (0,1)$.

ii) $G(t,s) \le \max_{0\le t\le 1} G(t,s) = G(1,s),\ t,s \in [0,1]$.

iii) $\min_{\frac{1}{4}\le t\le \frac{3}{4}} G(t,s) \ge (\frac{1}{4})^{\alpha-1}G(1,s),\ s \in (0,1)$.

Proof.

i) If $0 \le s \le t \le 1$, we have

$$\begin{aligned} G(t,s) = \tfrac{1}{\Gamma(\alpha)}[t^{\alpha-1}(1-s)^{\alpha-2}-(t-s)^{\alpha-1}] &\ge \tfrac{1}{\Gamma(\alpha)}[t^{\alpha-1}(1-s)^{\alpha-2}-(t-ts)^{\alpha-1}] \\ &= \tfrac{1}{\Gamma(\alpha)}t^{\alpha-1}[(1-s)^{\alpha-2}-(1-s)^{\alpha-1}] \\ &= \tfrac{1}{\Gamma(\alpha)}t^{\alpha-1}[(1-s)^{\alpha-2}(1-(1-s))] \\ &= \tfrac{1}{\Gamma(\alpha)}t^{\alpha-1}[(1-s)^{\alpha-2}s] > 0, \end{aligned}$$

If $0 \le t \le s \le 1$ then we have $G(t,s) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1}(t-s)^{\alpha-1} > 0$.

Thus $G(t,s) > 0, \forall s,t \in (0,1)$.

ii)

$$\frac{\partial G(t,s)}{\partial t} = \frac{(\alpha-1)}{\Gamma(\alpha)} \begin{cases} t^{\alpha-2}(1-s)^{\alpha-2} - (t-s)^{\alpha-2} & if\ 0 \le s \le t \le 1 \\ \\ t^{\alpha-2}(1-s)^{\alpha-2} & if\ 0 \le t \le s \le 1 \end{cases}$$

If $0 \le s \le t \le 1$,we have

$$\begin{aligned} \tfrac{\partial G(t,s)}{\partial t} &= \tfrac{(\alpha-1)}{\Gamma(\alpha)}\left[t^{\alpha-2}(1-s)^{\alpha-2} - (t-s)^{\alpha-2}\right] \\ &\ge \tfrac{(\alpha-2)}{\Gamma(\alpha)}\left[t^{\alpha-2}(1-s)^{\alpha-2} - (t-ts)^{\alpha-2}\right] \\ &= \tfrac{(\alpha-2)}{\Gamma(\alpha)} t^{\alpha-2}\left[(1-s)^{\alpha-2} - (1-s)^{\alpha-2}\right] = 0. \end{aligned}$$

Then, for $0 \le t \le s \le 1$, we have $\frac{\partial G(t,s)}{\partial t} = \frac{\alpha-1}{\Gamma(\alpha)}[t^{\alpha-2}(1-s)^{\alpha-2}] > 0$.

Consequently $G(t,s)$ is increasing with respect to t. Then $\max\limits_{0\le t\le 1} G(t,s) = G(1,s)$, for $s \in [0,1]$.

iii) We have $\min\limits_{\frac{1}{4}\le t\le \frac{3}{4}} G(t,s) = G(\dfrac{1}{4}, s)$.

For $0 < s \le \frac{1}{4}$,

$$\begin{aligned} G(\tfrac{1}{4}, s) &= \tfrac{1}{\Gamma(\alpha)}(\tfrac{1}{4})^{\alpha-1}[(1-s)^{\alpha-2} - (1-4s)^{\alpha-1}] \\ &\ge \tfrac{1}{\Gamma(\alpha)}(\tfrac{1}{4})^{\alpha-1}[(1-s)^{\alpha-2} - (1-s)^{\alpha-1}] \\ &= (\tfrac{1}{4})^{\alpha-1} G(1,s). \end{aligned}$$

For $\frac{1}{4} \leq s < 1$,

$$\begin{aligned} G(\tfrac{1}{4},s) &= \tfrac{1}{\Gamma(\alpha)}(\tfrac{1}{4})^{\alpha-1}(1-s)^{\alpha-2} \\ &= (\tfrac{1}{4})^{\alpha-1}[\tfrac{1}{\Gamma(\alpha)}(1-s)^{\alpha-2}] \\ &= (\tfrac{1}{4})^{\alpha-1}G(1,s). \end{aligned}$$

Then $\min\limits_{\frac{1}{4}\leq t\leq \frac{3}{4}} G(t,s) \geq (\dfrac{1}{4})^{\alpha-1}G(1,s) \geq \dfrac{1}{16}G(1,s).$ □

Let $C := C([-\tau,0],\mathbb{R})$ a Banach space with the norm $\|\varphi\|_c = \sup\limits_{-\tau\leq\theta\leq 0}|\varphi(\theta)|$ and

$$C^+ := \{\varphi \in C : \varphi(\theta) \geq 0, \theta \in [-\tau,0]\},$$

Let

$$E := \{t \in [0,1] : \frac{1}{4} \leq t+\theta \leq \frac{3}{4}, -\tau \leq \theta \leq 0\} = \Big[\frac{1}{4}+\tau, \frac{3}{4}\Big]. \tag{2.1.7}$$

Assume the following hypotheses:

H_1) f is a positive continuous functional defined on C^+.

H_2) $r(t)$ is a measurable positive function defined on $[0,1]$ and satisfies

$$0 < \int_E G(1,s)r(s)ds < \int_0^1 G(1,s)r(s)ds < \infty.$$

Sppose that $u(t)$ is a solution of the bandary value problem (2.0.1), then

$$u(t) = \begin{cases} \int_0^1 G(t,s)r(s)f(u_s)ds + \frac{\beta t^{\alpha-1}}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\int_0^1 G(\eta,s)r(s)f(u_s)ds \quad for\ 0\leq t\leq 1 \\ \\ \phi(t) \quad pour\ -\tau \leq t \leq 0. \end{cases} \tag{2.1.8}$$

Suppose that $\bar{x}(t)$ is a solution of the bandary value problem (2.0.1) with $f \equiv 0$, then

$$\bar{x}(t) = \begin{cases} 0, & if\ 0 \le t \le 1, \\ \\ \phi(t), & if\ -\tau \le t \le 0. \end{cases} \tag{2.1.9}$$

Let $x(t) = u(t) - \bar{x}(t)$ then we have froms (2.1.8) and (2.1.9) that

$$x(t) = \begin{cases} \int_0^1 G(t,s)r(s)f(x_s + \bar{x}_s)ds + \frac{\beta t^{\alpha-1}}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\int_0^1 G(\eta,s)r(s)f(x_s+\bar{x}_s)ds, & 0 \le t \le 1, \\ 0, \qquad -\tau \le t \le 0. \end{cases} \tag{2.1.10}$$

Let P be a cone in Banach space $C[-\tau;1]$, such that

$$P := \{x \in C[-\tau,1] \mid x(t) = 0, t \in [-\tau,0]; \min_{\frac{1}{4}\le t\le\frac{3}{4}} x(t) \ge (\frac{1}{4})^{\alpha-1}\|x\|; x(t) > 0, t \in (0,1)\}.$$

where $\|x\| := \max\{|x(t)| : -\tau \le t \le 1\}$.

Define $T : P \longrightarrow C[-\tau,1]$ by

$$(Tx)(t) = \begin{cases} \int_0^1 G(t,s)r(s)f(x_s + \bar{x}_s)ds + \frac{\beta t^{\alpha-1}}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\int_0^1 G(\eta,s)r(s)f(x_s+\bar{x}_s)ds. & 0 \le t \le 1, \\ 0, \qquad -\tau \le t \le 0. \end{cases} \tag{2.1.11}$$

We define $\|x\|_{[0,1]} := \sup\{|x(t)| : 0 \le t \le 1\}$, then we have $\|x\| = \|x\|_{[0,1]}$ and $\|Tx\| = \|Tx\|_{[0,1]}$, $\forall x \in P$. Then, we have the following lemmas.

Lemma 2.1.3.

$$T(P) \subset P.$$

Proof. For $0 < t < 1, x \in P$, we get from (2.1.11) and lemma 2.1.2 that $(Tx)(t) > 0$

$$\begin{aligned} \|Tx\| &= \int_0^1 G(t,s)r(s)f(x_s+\bar{x}_s)ds + \frac{\beta t^{\alpha-1}}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\int_0^1 G(\eta,s)r(s)f(x_s+\bar{x}_s)ds \\ &\le \int_0^1 G(1,s)r(s)f(x_s+\bar{x}_s)ds + \frac{\beta t^{\alpha-1}}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\int_0^1 G(\eta,s)r(s)f(x_s+\bar{x}_s)ds. \end{aligned}$$

Further,we have

$$\begin{aligned}
\min_{\frac{1}{4}\leq t\leq\frac{3}{4}}(Tx)(t) &\geq (\tfrac{1}{4})^{\alpha-1}\int_0^1 G(1,s)r(s)f(x_s+\bar{x}_s)ds+\tfrac{\beta(\frac{1}{4})^{\alpha-1}}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\int_0^1 G(\eta,s)r(s)f(x_s+\bar{x}_s)ds\\
&\geq (\tfrac{1}{4})^{\alpha-1}[\int_0^1 G(1,s)r(s)f(x_s+\bar{x}_s)ds+\tfrac{\beta}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\int_0^1 G(\eta,s)r(s)f(x_s+\bar{x}_s)ds]\\
&\geq (\tfrac{1}{4})^{\alpha-1}\|Tx\|.
\end{aligned}$$

Then $T(P)\subset P$. □

Lemma 2.1.4. *$T:P\longrightarrow P$ is completly continuous.*

Proof. We can obtain the continuity of T from the continuity of f. In fact ,suppose $x^{(n)},x\in P$ and $\|x^{(n)}-x\|\longrightarrow 0$ as $n\longrightarrow\infty$, then we get

$$\|x_s^{(n)}-x_s\|_C=\sup_{-\tau\leq\theta\leq 0}|x^{(n)}(s+\theta)-x(s+\theta)|\longrightarrow 0, \text{ uniformaly for } s\in[0,1].$$

thus, for $t\in[-\tau,1]$ we have from (2.1.11) and lemma 2.1.2 that

$$|(Tx^n)(t)-(Tx)(t)|\leq\max_{0\leq s\leq 1}|f(x_s^{(n)}+\bar{x}_s)-f(x_s+\bar{x}_s)|\Big[1+\frac{\beta}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\Big]\int_0^1 G(1,s)r(s)ds.$$

This implies that $\|Tx^{(n)}-Tx\|\longrightarrow 0$ as $n\longrightarrow+\infty$ since f is continuous.

Next, we show that T is uniformly bounded and equi-continuous.

⋆ Let $D\subset P$ bounded there is a positive constant $M_1>0$ such taht $\|x\|\leq M_1$ for each $x\in D$ and $\|\bar{x}\|\leq M_2$ so $\|x+\bar{x}\|\leq M_1+M_2=M$. Defined set $S:=\{\varphi\in C^+;\|\varphi\|_C\leq M\}\subset C^+$,

Let $L:=\max_{\varphi\in S}|f(\varphi)|$.

First, for each $x\in D$ and $t\in[0,1]$,

$$\begin{aligned}
(Tx)(t) &\leq \int_0^1 G(1,s)r(s)f(x_s+\bar{x}_s)ds+\tfrac{\beta}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\int_0^1 G(1,s)r(s)f(x_s+\bar{x}_s)ds\\
&\leq L\Big[1+\tfrac{\beta}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\Big]\int_0^1 G(1,s)r(s)ds.
\end{aligned}$$

Therefore, $T(D)$is uniformly bounded.

⋆ Second, for each $\epsilon>0$, since $G(t,s)$ is uniformly continuous on $[0,1]\times[0,1]$and $t^{\alpha-1}$ is uniformly continuous on $[0,1]$, then there is $\eta > 0$ as for all $t_1, t_2 \in [0,1]$, when $|t_1 - t_2| < \eta$, we have

$$|G(t_1,s) - G(t_2,s)| < \frac{\epsilon}{2L\int_0^1 r(s)ds} \text{ and } |t_1^{\alpha-1} - t_2^{\alpha-1}| < \frac{\epsilon}{2L\frac{\beta}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\int_0^1 G(\eta,s)r(s)ds}$$

then for all $x \in D$

$$\begin{aligned} |Tx(t_1) - Tx(t_2)| &\leq \int_0^1 |(G(t_1,s) - G(t_2,s))r(s)| f(x_s + \bar{x}_s)ds \\ &\quad + \tfrac{\beta|t_1^{\alpha-1} - t_2^{\alpha-1}|}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\int_0^1 |G(\eta,s)r(s)f(x_s+\bar{x}_s)ds| \\ &\leq \epsilon. \end{aligned}$$

which means that $T(D)$ is equicontinuous.

By means of Arzela-Ascoli theorem, $T : P \longrightarrow P$ is completly continuous. □

2.2 Main Results

In this section, in order to establish some results of existence of positive solutions for BVP (2.0.1), we will impose growth conditions on f which allow us to apply theorem 1.0.2.

For convenience, we introduce the following notations.

$$f_0 := \limsup_{\|\varphi\|_C \to 0} \frac{f(\varphi)}{\|\varphi\|_C}, \quad f_0^* := \liminf_{\varphi\in C^* \|\varphi\|_C \to 0} \frac{f(\varphi)}{\|\varphi\|_C},$$
$$f_\infty := \limsup_{\|\varphi\|_C \to +\infty} \frac{f(\varphi)}{\|\varphi\|_C}, \quad f_\infty^* := \liminf_{\varphi\in C^* \|\varphi\|_C \to +\infty} \frac{f(\varphi)}{\|\varphi\|_C}.$$

Let

$m_1 := \frac{1}{3}\Big[\Big(1 + \dfrac{\beta}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\Big)\int_0^1 G(1,s)r(s)ds\Big]^{-1}$, and

$m_2 := \Big[(\frac{1}{4})^{\alpha-1}(\int_E G(1,s)r(s)ds + \dfrac{\beta}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\int_E G(\eta,s)r(s)ds\Big]^{-1}$

$$\phi_0 = \max_{-\tau\leq t\leq 0}|\phi(t)| \qquad C^* = \{\varphi \in C^+; 0 < \lambda\|\varphi\|_c \leq \varphi(\theta), \theta \in [-\tau,0]\}$$

where $0 < \lambda \leq (\frac{1}{4})^{\alpha-1}$ fixed.

Theorem 2.2.1. *Assume that the following conditions is satisfied:* $f_0 < m_1,\ f_\infty^* > m_2\lambda^{-1},\ \phi(t) \equiv 0$ *Then BVP (2.0.1) has at least one positive solution.*

Proof. By $\phi(t) := 0$, we have $\bar{x}_t = 0$, Since $f_0 = \limsup_{\|\varphi\|_C \to 0} \frac{f(\varphi)}{\|\varphi\|_C} < m_1$, There is a $r_1 > 0$ such that

$$f(\varphi) \leq m_1 \|\varphi\|_C, \text{ for } \varphi \in C^+, 0 \leq \|\varphi\|_C \leq r_1.$$

For any $x \in P$, satisfying $\|x\| = r_1$, we deduce that $\|x_s\|_C \leq \|x\| = r_1, s \in [0,1]$ and thus

$$\begin{aligned}
Tx(t) &\leq \textstyle\int_0^1 G(1,s)r(s)f(x_s)ds + \frac{\beta}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\int_0^1 G(1,s)r(s)f(x_s)ds \\
&\leq m_1 \|x_s\|_c \Big(1 + \frac{\beta}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\Big) \textstyle\int_0^1 G(1,s)r(s)ds \\
&\leq m_1 \|x\| \Big(1 + \frac{\beta}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\Big) \textstyle\int_0^1 G(1,s)r(s)ds < \|x\|
\end{aligned}$$

which leads to

$$\|Tx\| = \|Tx\|_{[0,1]} \leq \|x\|, \forall x \in P \cap \partial\Omega_{r_1},$$

where $\Omega_{r_1} = \{x \in C[-\tau,1] : \| x \| < r_1\}$.

On the other hand since $f_\infty^* = \liminf_{\varphi \in C^* \|\varphi\|_C \to +\infty} \frac{f(\varphi)}{\|\varphi\|_C} > m_2\lambda^{-1}$, there existe a $r_2 > r_1$ such that

$$f(\varphi) \geq m_2\lambda^{-1}\|\varphi\|_C, \text{ for } \varphi \in C^*, \|\varphi\|_C \geq \lambda r_2.$$

Define $\Omega_{r_2} = \{x \in C[-\tau,1]; \| x \| < r_2\}$. For $x \in P$ and $\|x\| = r_2$,we deduce that

$$\lambda\|x_s\|_C \leq \lambda\|x\| \leq (\frac{1}{4})^{\alpha-1}\|x\| \leq \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} x(t) \leq x(t), for\ s \in E\ and\ t \in [\frac{1}{4}, \frac{3}{4}].$$

where E is given by(2.1.7). Then $x_s \in C^*$ and

$$\|x_s\|_C \geq \lambda\|x\| = \lambda r_2,\ for\ s \in E$$

. Thus, we have

$$
\begin{aligned}
(Tx)(\tfrac{1}{2}) &= \textstyle\int_0^1 G(\frac{1}{2},s)r(s)f(x_s)ds + \frac{\beta(\frac{1}{2})^{\alpha-1}}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\int_0^1 G(\eta,s)r(s)f(x_s)ds \\
&\geq (\tfrac{1}{4})^{\alpha-1}\Big(\textstyle\int_E G(1,s)r(s)f(x_s)ds + \frac{\beta}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\int_E G(\eta,s)r(s)f(x_s)ds\Big) \\
&\geq m_2\lambda^{-1}\|x_s\|_C(\tfrac{1}{4})^{\alpha-1}\Big(\textstyle\int_E G(1,s)r(s)ds + \frac{\beta}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\int_E G(\eta,s)r(s)ds\Big) \\
&\geq m_2\lambda^{-1}\lambda r_2(\tfrac{1}{4})^{\alpha-1}\Big(\textstyle\int_E G(1,s)r(s)ds + \frac{\beta}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\int_E G(\eta,s)r(s)ds\Big) \\
&= r_2 = \|x\|.
\end{aligned}
$$

Hence,we have

$$\|Tx\| \geq \|x\|, \forall x \in P \cap \partial\Omega_{r_2}.$$

According to the first part of Theorem 1.0.2 T has a fixed point $x \in P \cap (\overline{\Omega}_{r_2} \setminus \Omega_{r_1})$ that is $0 < r_1 \leq \|x\| \leq r_2$. □

Theorem 2.2.2. *Assume that the following conditions is satisfied:*$f_0^* > m_2\lambda^{-1}$ *and* $f_\infty < m_1$. *Then the boundary value problem (2.0.1) has at least one positive solution.*

Proof. Since $f_0^* := \liminf_{\varphi\in C^*\|\varphi\|_C\to 0}\frac{f(\varphi)}{\|\varphi\|_C} > m_2\lambda^{-1}$, there exist $r_1 > 0$ such that

$$f(\varphi) \geq m_2\lambda^{-1}|\varphi\|_C, \text{for } \varphi \in C^*, \|\varphi\|_C < r_1. \tag{2.2.1}$$

For $x \in P$ and $\|x\| = r_1$ where $r_1 > 0$ we have $\|x_s\|_C < \|x\| = r_1,\ s \in [0,1]$ and we have

$$\lambda\|x_s\|_C \leq \lambda\|x\| \leq \frac{1}{16}\|x\| \leq \min_{\frac{1}{4}\leq t\leq\frac{3}{4}} x(t) \leq x(t),\ \text{for } s \in E \textit{ and } t \in [\frac{1}{4}, \frac{3}{4}].$$

Then, $x_s \in C^*$ for $s \in E$ and $\|x_s\|_C \geq \lambda\|x\| = \lambda r_1,\ s \in E$.

Note that $\bar{x_s} \equiv 0$ on $[0,1]$,it follows from (2.2.1) that

$$\begin{aligned}
(Tx)(\tfrac{1}{2}) &= \textstyle\int_0^1 G(\frac{1}{2},s)r(s)f(x_s+\bar{x}_s)ds + \frac{\beta(\frac{1}{2})^{\alpha-1}}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\int_0^1 G(\eta,s)r(s)f(x_s+\bar{x}_s)ds \\
&\geq (\tfrac{1}{4})^{\alpha-1}\Big(\textstyle\int_E G(1,s)r(s)f(x_s)ds + \frac{\beta}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\int_E G(\eta,s)r(s)f(x_s)ds\Big) \\
&\geq m_2\lambda^{-1}\|x_s\|_C(\tfrac{1}{4})^{\alpha-1}\Big(\textstyle\int_E G(1,s)r(s)ds + \frac{\beta}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\int_E G(\eta,s)r(s)ds\Big) \\
&\geq m_2\lambda^{-1}\lambda r_2(\tfrac{1}{4})^{\alpha-1}\Big(\textstyle\int_E G(1,s)r(s)ds + \frac{\beta}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\int_E G(\eta,s)r(s)ds\Big) \\
&= r_1 = \|x\|
\end{aligned}$$

Hence,we obtain

$$\|Tx\| \geq \|x\|, \forall x \in P \cap \partial\Omega_{r_2}.$$

On the other hand,since $f_\infty = \limsup_{\|\varphi\|_C \to +\infty} \dfrac{f(\varphi)}{\|\varphi\|_C} < m_1$ there exist $N > r_1 + \|\bar{x}\|$ such that

$$f(\varphi) \leq m_1\|\varphi\|_C, \text{for } \varphi \in C^+, \|\varphi\|_C > N.$$

Let $r_2 > 0$ such that

$$r_2 > \max\{f(\varphi) : 0 \leq \|\varphi\|_C \leq N + \|\bar{x}\|\}\Big(1 + \frac{\beta}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\Big)\int_0^1 G(1,s)r(s)ds \tag{2.2.2}$$

For $x \in P$ and $\|x\| = r_2$,from the facts that $\bar{x}(t) \geq 0$ for $t \in [-\tau,1]$ for$s \in [0,1]$ we have $\|x_s + \bar{x}_s\|_C = \max_{\theta\in[-\tau,0]}(x(s+\theta)+\bar{x}(s+\theta)) > \max_{\theta\in[-\tau,0]} x(s+\theta) = \|x_s\|_C > N$, for $\|x_s\|_C > N$, and $\|x_s+\bar{x}_s\|_C \leq \|x_s\|_C + \|\bar{x}_s\|_C \leq N + \|\bar{x}_s\|_C$; for $\|x_s\|_C < N$.

It follows from (2.2.2) that

$$\begin{aligned}
Tx(t) &\leq \Big(1+\frac{\beta}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\Big)\int_0^1 G(1,s)r(s)f(x_s+\bar{x}_s)ds \\
&= \Big(1+\frac{\beta}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\Big)\big[\int_{\|x_s\|_C>N} G(1,s)r(s)f(x_s+\bar{x}_s)ds+\int_{0\leq\|x_s\|_C\leq N} G(1,s)r(s)f(x_s+\bar{x}_s)ds\big] \\
&\leq m_1(\|x\|+\|\bar{x}\|)\Big(1+\frac{\beta}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\Big)\int_0^1 G(1,s)r(s)ds \\
&\quad +\max\{f(\varphi):0\leq\|\varphi\|_C\leq N+\|\bar{x}\|\}\Big(1+\frac{\beta}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\Big)\int_0^1 G(1,s)r(s)ds \\
&< \tfrac{\|x\|}{3}+\tfrac{\|\bar{x}\|}{3}+\tfrac{r_2}{3}<r_2=\|x\|,\forall t\in[0,1].
\end{aligned}$$

Thus, $\|Tx\|<\|x\|\forall x\in P\cap\partial\Omega_{r_2}$, where $\Omega_{r_2}=\{x\in C[-\tau,1]:\|x\|<r_2\}$.
Therefore,by the second part of Theorem 1.0.2, T has a fixed point $x\in P\cap(\bar{\Omega}_{r_2}\backslash\Omega_{r_1})$ such that $0<r_1\leq\|x\|\leq r_2$. □

Suppose that x is the fixed point of T in $P\cap(\bar{\Omega}_{r_2}\backslash\Omega_{r_1})$, then

$$x(t)=\begin{cases}\int_0^1 G(t,s)r(s)f(x_s+\bar{x}_s)ds+\frac{\beta t^{\alpha-1}}{[(\alpha-1)-\beta\eta^{\alpha-1}]}\int_0^1 G(\eta,s)r(s)f(x_s+\bar{x}_s)ds. & 0\leq t\leq 1.\\ 0 & -\tau\leq t\leq 0.\end{cases} \tag{2.2.3}$$

Let $u(t)=x(t)+\bar{x}(t)$. By the facts $0<r_1<\|x\|\leq r_2$ and $\bar{x}\geq 0$, we conclude that $u(t)$ is a positive solution of (2.0.1).

2.3 An Example

$$\begin{cases} D_{0^+}^{\frac{11}{4}}u(t)+(1+t^2)u^{\frac{1}{3}}(t-\frac{1}{4})=0 & t\in[0,1]. \\ \\ u(t)=-sin(\pi t) & -\frac{1}{4}\leq t\ \leq 0 \\ \\ u(0)=u^{'}(0)=0, & u'(1)=2u(\frac{1}{5}), \end{cases}$$

Where $\tau=\frac{1}{4},\ f(\varphi)=\varphi^{\frac{1}{3}}(\frac{-1}{4}),E=[\frac{1}{2},\frac{3}{4}]$. Since

$$\frac{f(\varphi)}{\|\varphi\|_C}=\frac{\varphi^{\frac{1}{3}}(-\frac{1}{4})}{\|\varphi\|_C}\leq\frac{\|\varphi\|_C^{\frac{1}{3}}}{\|\varphi\|_C}\longrightarrow 0 \text{ when } \|\varphi\|_C\longrightarrow+\infty,$$

we have $f_\infty=0$. On the other hand, if $\varphi\in C^*$,ther is a constant $\lambda>0$ such that $\varphi(\theta)\geq\lambda\|\varphi\|_C$

$$\frac{f(\varphi)}{\|\varphi\|_C}\geq\frac{\lambda^{\frac{1}{3}}\|\varphi\|_C^{\frac{1}{3}}}{\|\varphi\|_C}\longrightarrow+\infty \ \text{ when } \|\varphi\|_C\longrightarrow 0.$$

Thus, $f_0^*=+\infty$ According to theorem 2.2.2, we conclude that BVP has at least one positive solution.

Chapter 3

Existence of positive solution for functional coupled system with nonlocal boundary conditions

In this chapter we investigate the existence of positive solutions for the following boundary value problem (BVP) of functional differential system

$$\begin{cases} x''(t) + a_1(t)f(x_t, y_t) = 0, & t \in [0,1], \\ y''(t) + a_2(t)g(x_t, y_t) = 0, & t \in [0,1], \\ x(t) = \phi_1(t) & -\tau \leq t \ \leq 0, \\ x'(1) = \alpha x(\eta) + \beta x'(\eta) & \\ y(t) = \phi_2(t) & -\tau \leq t \ \leq 0, \\ y'(1) = \alpha y(\eta) + \beta y'(\eta) & \end{cases} \tag{3.0.1}$$

where $0 < \eta < 1,\ 0 < \alpha < 1,\ \ 0 < \tau < 1$ and $0 < \beta < 1 - \alpha\eta$, are constants.

The system (3.0.1) is a generalization of the following boundary value problem

$$\begin{cases} z''(t)+a(t)f(z_t)=0, & 0<t<1, \\ z(t)=\phi(t), & -\tau \le t \ \le 0, \\ z'(1)=\alpha z(\eta)+\beta z'(\eta) \end{cases} \tag{3.0.2}$$

where $0<\eta<1,\ 0<\alpha<1$ and $0<\beta<1-\alpha\eta$. are constants.

3.1 Preliminaries

In this section we give some preliminary results.

Definition 3.1.1.

$(x(t),y(t))$ is called a positive solution of differential system (3.0.1) if it satisfies the following:

1. *$x(t),y(t)\in C[-\tau,1]\cap C^2(0,1)$;*

2. *$x(t)>0,\ y(t)>0$ for all $t\in(0,1)$ and satisfy (3.0.1).*

Lemma 3.1.1. *Let $\beta\neq 1-\alpha\eta$. If $h(t)\in C([0,1])$, then the boundary value problem*

$$z''(t)+h(t)=0, \quad 0<t<1, \tag{3.1.1}$$

$$z(0)=0,\ z'(1)=\alpha z(\eta)+\beta z'(\eta) \tag{3.1.2}$$

has a unique solution

$$z(t)=\int_0^1 G(t,s)h(s)ds+\frac{\alpha t}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)h(s)ds+\frac{\beta t}{1-\alpha\eta-\beta}\int_0^1 H(\eta,s)h(s)ds.$$

where

$$G(t,s)=\begin{cases} s, & s\le t, \\ t, & t\le s. \end{cases} \tag{3.1.3}$$

And

$$H(t,s):=\frac{\partial G(t,s)}{\partial t}=\begin{cases} 0, & s\le t, \\ 1, & t\le s. \end{cases} \tag{3.1.4}$$

Proof. From (3.1.1), we have

$$z(t) = z(0) + z'(0)t + \int_0^t (s-t)h(s)ds := B + At + \int_0^t (s-t)h(s)ds. \tag{3.1.5}$$

From $z(0) = 0$, we have $B = 0$. Since $z'(1) = \alpha z(\eta) + \beta z'(\eta)$, we have

$$-\int_0^1 h(s)ds + A = \alpha(\int_0^\eta (s-\eta)h(s)ds + A\eta) + \beta(-\int_0^\eta h(s)ds + A).$$

Then,

$$A = \frac{\alpha}{1-\alpha\eta-\beta}\int_0^\eta (s-\eta)h(s)ds - \frac{\beta}{1-\alpha\eta-\beta}\int_0^\eta h(s)ds + \frac{1}{1-\alpha\eta-\beta}\int_0^1 h(s)ds$$

Replacing these expression, in (3.1.5), we obtain

$$\begin{aligned} z(t) &= \int_0^t (s-t)h(s)ds + \frac{\alpha t}{1-\alpha\eta-\beta}\int_0^\eta (s-\eta)h(s)ds - \frac{\beta t}{1-\alpha\eta-\beta}\int_0^\eta h(s)ds + \frac{t}{1-\alpha\eta-\beta}\int_0^1 h(s)ds. \\ &= \int_0^1 G(t,s)h(s)ds + \frac{\alpha t}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)h(s)ds + \frac{\beta t}{1-\alpha\eta-\beta}\int_0^1 H(\eta,s)h(s)ds. \end{aligned}$$

□

Properties of $G(t,s)$ for which we will make use are

$$G(t,s) \leq G(s,s) = s, \quad 0 \leq t,s \leq 1, \tag{3.1.6}$$

$$G(t,s) \geq \frac{1}{2}G(s,s) = \frac{s}{2}, \quad \frac{1}{2} \leq t \leq 1, \quad 0 \leq s \leq 1. \tag{3.1.7}$$

In particular, from (3.1.7), we have

$$\min_{\frac{1}{2}\leq t\leq 1} G(t,s) \geq \frac{s}{2}, \quad 0 \leq s \leq 1. \tag{3.1.8}$$

Lemma 3.1.2. *If $h \in C([0,1])$ and $h \geqslant 0$, then the unique solution z of the boundary value problem (3.1.1), (3.1.2) satisfies*

$$\min_{\frac{1}{2}\leq t\leq 1} z(t) \geqslant \frac{1}{2}\|z\|.$$

where $\|z\| := \sup_{0\leq t\leq 1} |z(t)|$.

Proof. By (3.1.6), we have

$$z(t)=\int_0^1 G(t,s)h(s)ds+\frac{\alpha t}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)h(s)ds+\frac{\beta t}{1-\alpha\eta-\beta}\int_0^1 H(\eta,s)h(s)ds$$
$$\leq \int_0^1 G(s,s)h(s)ds+\frac{\alpha}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)h(s)ds+\frac{\beta}{1-\alpha\eta-\beta}\int_0^1 H(\eta,s)h(s)ds.$$

Thus

$$||z||\leq \int_0^1 G(s,s)h(s)ds+\frac{\alpha}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)h(s)ds+\frac{\beta}{1-\alpha\eta-\beta}\int_0^1 H(\eta,s)h(s)ds.$$

Moreover, from 3.1.7 for $t\in[\frac{1}{2},1]$, we have

$$z(t)=\int_0^1 G(t,s)h(s)ds+\frac{\alpha t}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)h(s)ds+\frac{\beta t}{1-\alpha\eta-\beta}\int_0^1 H(\eta,s)h(s)ds$$
$$\geq \frac{1}{2}\left(\int_0^1 G(s,s)h(s)ds+\frac{\alpha}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)h(s)ds+\frac{\beta}{1-\alpha\eta-\beta}\int_0^1 H(\eta,s)h(s)ds\right).$$

Therefore $\min\limits_{\frac{1}{2}\leq t\leq 1} z(t)\geqslant \frac{1}{2}\|z\|.$ ☐

Lemma 3.1.3.

If $h\in C([0,1],\ [0,\infty))$, then the unique solution z of the problem (3.1.1), (3.1.2) satisfies

$$z(t)\geq 0,\quad t\in[0,1].$$

Proof. We know that if $z''(t)=-h(t)\leq 0$, for $t\in(0,1)$, $z(0)\geq 0$ and $z(1)\geq 0$, then $z(t)\geq 0$, for $t\in[0,1]$. We have $z(0)=0$, and

$$z(1)=\frac{1}{1-\alpha\eta-\beta}\left[\int_0^1 (s-1)(1-\alpha\eta-\beta)h(s)ds+\alpha\int_0^\eta (s-\eta)h(s)ds-\beta\int_0^\eta h(s)ds+\int_0^1 h(s)ds\right]$$
$$=\frac{1}{1-\alpha\eta-\beta}\int_0^1 s[\alpha(1-\eta)+(1-\beta)]h(s)ds\geq 0.$$

☐

3.2 Main results

Assume the following hypothesis

For $i=1,2,$

H_1) $\phi_i \in C_\tau$ are positive; $\phi_i(0) = 0$.

H_2) a_i is a measurable positive function defined on $[0,1]$ and satisfies

$$0 < \int_E G(s,s)a_i(s)ds < \int_0^1 G(s,s)a_i(s)ds < \infty,$$

where

$$E := \{t \in [0,1] : \frac{1}{2} \le t+\theta \le 1,\ -\tau \le \theta \le 0\} = [\frac{1}{2}+\tau, 1].$$

H_3) $0 < \eta < 1,\ 0 < \alpha < 1$ and $0 < \beta < 1-\alpha\eta$.

H_4) $f, g : C_\tau \times C_\tau \to (0, +\infty)$ are positive continuous functions.

Suppose $(x(t), y(t))$ is a solution of (3.0.1). Then

$$x(t) = \begin{cases} \int_0^1 G(t,s)a_1(s)f(x_s,y_s)ds + \frac{\alpha t}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a_1(s)f(x_s,y_s)ds \\ +\frac{\beta t}{1-\alpha\eta-\beta}\int_0^1 H(\eta,s)a_1(s)f(x_s,y_s)ds,\ 0 \le t \le 1, \\ \phi_1(t), \quad -\tau \le t \le 0, \end{cases} \tag{3.2.1}$$

$$y(t) = \begin{cases} \int_0^1 G(t,s)a_2(s)g(x_s,y_s)ds + \frac{\alpha t}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a_2(s)g(x_s,y_s)ds \\ +\frac{\beta t}{1-\alpha\eta-\beta}\int_0^1 H(\eta,s)a_2(s)g(x_s,y_s)ds,\ 0 \le t \le 1, \\ \phi_2(t), \quad -\tau \le t \le 0. \end{cases} \tag{3.2.2}$$

Suppose that $(x_0(t), y_0(t))$ is the solution of (3.0.1) with $f \equiv 0,\ g \equiv 0$. Then

$$x_0(t) = \begin{cases} 0, & 0 \le t \le 1, \\ \phi_1(t), & -\tau \le t \le 0, \end{cases}$$

$$y_0(t) = \begin{cases} 0, & 0 \le t \le 1, \\ \phi_2(t), & -\tau \le t \le 0. \end{cases}$$

If $(x(t), y(t))$ is the solution of (3.0.1) and

$$\begin{cases} u(t) = x(t) - x_0(t), \\ v(t) = y(t) - y_0(t). \end{cases}$$

Then for $0 \leq t \leq 1$, we have $(u(t), v(t)) = (x(t), y(t))$ and by (3.2.1) and (3.2.2) ,we have

$$u(t) = \begin{cases} \int_0^1 G(t,s)a_1(s)f(u_s + x_{0s}, v_s + y_{0s})ds + \dfrac{\alpha t}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a_1(s)f(u_s + x_{0s}, v_s + y_{0s})ds \\ + \dfrac{\beta t}{1-\alpha\eta-\beta}\int_0^1 H(\eta,s)a_1(s)f(u_s + x_{0s}, v_s + y_{0s})ds, \ 0 \leq t \leq 1, \\ 0, \ -\tau \leq t \leq 0. \end{cases}$$

$$v(t) = \begin{cases} \int_0^1 G(t,s)a_2(s)g(u_s + x_{0s}, v_s + y_{0s})ds + \dfrac{\alpha t}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a_2(s)g(u_s + x_{0s}, v_s + y_{0s})ds \\ + \dfrac{\beta t}{1-\alpha\eta-\beta}\int_0^1 H(\eta,s)a_2(s)g(u_s + x_{0s}, v_s + y_{0s})ds, \ 0 \leq t \leq 1, \\ 0, \ -\tau \leq t \leq 0. \end{cases}$$

Let $X = C([-\tau,1]) \times C([-\tau,1])$ with the norm $||(x,y)|| = \max\{||x||, ||y||\}$, where

$$||x|| = \sup\{|x(t)|/ \ t \in [-\tau,1]\}, \quad ||y|| = \sup\{|y(t)|/ \ t \in [-\tau,1]\}.$$

Then $(X, ||.||)$ is a Banach space. We will make use of the cone $K \subset X$ defined by

$$K = \{(x,y) \in C([-\tau,1]) \times C([-\tau,1])/ \min_{\frac{1}{2} \leq t \leq 1} x(t) \geqslant \frac{1}{2}\|x\|, \ \min_{\frac{1}{2} \leq t \leq 1} y(t) \geqslant \frac{1}{2}\|y\|\}.$$

Define an operate Φ on K by

$$\Phi(u,v) = (A(u,v), B(u,v)), \tag{3.2.3}$$

where

$$A(u,v)(t) = \begin{cases} \int_0^1 G(t,s)a_1(s)f(u_s + x_{0s}, v_s + y_{0s})ds + \dfrac{\alpha t}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a_1(s)f(u_s + x_{0s}, v_s + y_{0s})ds \\ + \dfrac{\beta t}{1-\alpha\eta-\beta}\int_0^1 H(\eta,s)a_1(s)f(u_s + x_{0s}, v_s + y_{0s})ds, \ 0 \leq t \leq 1, \\ 0, \ -\tau \leq t \leq 0. \end{cases} \tag{3.2.4}$$

$$B(u,v)(t) = \begin{cases} \int_0^1 G(t,s)a_2(s)g(u_s + x_{0s}, v_s + y_{0s})ds + \dfrac{\alpha t}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a_2(s)g(u_s + x_{0s}, v_s + y_{0s})ds \\ + \dfrac{\beta t}{1-\alpha\eta-\beta}\int_0^1 H(\eta,s)a_2(s)g(u_s + x_{0s}, v_s + y_{0s})ds, \ 0 \leq t \leq 1, \\ 0, \ -\tau \leq t \leq 0. \end{cases} \tag{3.2.5}$$

Lemma 3.2.1. *The map $\Phi : X \to X$ in (3.2.3) is completely continuous and $\Phi(K) \subset K$.*

Proof. We shall prove that $\Phi(K) \subset K$.

Choose some $(u,v) \in K$. Then by Lemma 3.1.2 we have

$$\min_{\frac{1}{2}\leq t\leq 1} A(u,v)(t) \geqslant \frac{1}{2}\|A(u,v)\|, \quad \min_{\frac{1}{2}\leq t\leq 1} B(u,v)(t) \geqslant \frac{1}{2}\|B(u,v)\|$$

Hence,

$$\Phi(u,v)(t) = (A(u,v)(t), B(u,v)(t)) \in K.$$

That is $\Phi(K) \subset K$.

We shall show that Φ is completely continuous.

1. Assume that $D \subset K$ is a bounded set, M is positive constant, such that for every $(u,v) \in D$, $||(u,v)|| \leq M$.

   Let $p_0 = \max\{||x_0||, ||y_0||\}$, then

   $$||(u_t + x_{0t}, v_t + y_{0t})||_{[-\tau,0]} = \sup_{t\in[0,1],\ \theta\in[-\tau,0]} \{|u(t+\theta)+x_0(t+\theta)|, |v(t+\theta)+y_0(t+\theta)|\} \leq M + p_0.$$

   By (3.2.4) and (H_4), if $0 \leq t \leq 1$, then
   $$\begin{aligned} ||A(u,v)|| \leq & \int_0^1 G(s,s)a_1(s)f(u_s+x_{0s}, v_s+y_{0s})ds + \frac{\alpha}{1-\alpha\eta-\beta}\int_0^1 G(s,s)a_1(s)f(u_s+x_{0s}, v_s+y_{0s})ds \\ & + \frac{\beta}{1-\alpha\eta-\beta}\int_0^1 H(\eta,s)a_1(s)f(u_s+x_{0s}, v_s+y_{0s})ds \\ & \leq L\left(\int_0^1 G(s,s)a_1(s)ds + \frac{\alpha}{1-\alpha\eta-\beta}\int_0^1 G(s,s)a_1(s)ds + \frac{\beta}{1-\alpha\eta-\beta}\int_0^1 H(\eta,s)a_1(s)ds\right) \end{aligned}$$
   where

   $$L = \sup\{|f(\psi,\varphi)|/\ ||(\psi,\varphi)||_{[-\tau,0]} \leq M + p_0\}.$$

   That is, $A(u,v)$ is uniformly bounded. In the similar way, we can show that $B(u,v)$ is uniformly bounded, so $\Phi(u,v) = (A(u,v), B(u,v))$ is uniformly bounded.

2. Now we prove that Φ is equi-continuous.

   For each $\epsilon > 0$, since $G(t,s)$ is uniformly continuous on $[0,1]\times[0,1]$ and the identity function t is uniformly continuous on $[0,1]$, then there is $\delta > 0$ as for all $t_1, t_2 \in [0,1]$, when $|t_1 - t_2| < \delta$, We have

   $$|G(t_1,s) - G(t_2,s)| < \frac{\epsilon}{3L\int_0^1 a_1(s)ds},$$

$$|t_1 - t_2| < \frac{\epsilon}{3L\frac{\alpha}{1-\eta\alpha-\beta}\int_0^1 G(\eta,s)a_1(s)ds} \quad \text{and} \quad |t_1 - t_2| < \frac{\epsilon}{3L\frac{\beta}{1-\eta\alpha-\beta}\int_0^1 H(\eta,s)a_1(s)ds}$$

then for all $(u,v) \in D$

$$\begin{aligned}
|A(u(t_1),v(t_1)) - A(u(t_2),v(t_2))| &\leq \int_0^1 |G(t_1,s) - G(t_2,s)|a_1(s)f(u_s + x_{0s}, v_s + y_{0s})ds \\
&+ \frac{\alpha|t_1 - t_2|}{1-\eta\alpha-\beta}\int_0^1 G(\eta,s)a_1(s)f(u_s + x_{0s}, v_s + y_{0s})ds \\
&+ \frac{\beta|t_1 - t_2|}{1-\eta\alpha-\beta}\int_0^1 H(\eta,s)a_1(s)f(u_s + x_{0s}, v_s + y_{0s})ds \\
&\leq \epsilon.
\end{aligned}$$

Which means that A is equicontinuous. Similarly, we can show that B is equicontinuous. Thus, Φ is equicontinous.

3. Assume that $(u_n, v_n) \to (u^*, v^*)$, where $(u_n, v_n), (u^*, v^*) \in C[-\tau, 1] \times C[-\tau, 1]$, then there exists $M_1 > 0$, such that $||(u_n, v_n)|| \leq M_1$, $||(u^*, v^*)|| \leq M_1$.

   For $t \in [-\tau, 1]$,

$$\begin{aligned}
|A(u_n,v_n)(t) - A(u^*,v^*)(t)| &\leq |\int_0^1 G(s,s)a_1(s)[f(u_{ns} + x_{0t}, v_{ns} + y_{0s}) - f(u_s^* + x_{0s}, v_s^* + y_{0s})]ds \\
&+ \frac{\alpha}{1-\eta\alpha-\beta}\int_0^1 G(s,s)a_1(s)[f(u_s + x_{0s}, v_s + y_{0s}) - f(u_s^* + x_{0s}, v_s^* + y_{0s})]ds \\
&+ \frac{\beta}{1-\eta\alpha-\beta}\int_0^1 H(\eta,s)a_1(s)[f(u_s + x_{0s}, v_s + y_{0s}) - f(u_s^* + x_{0s}, v_s^* + y_{0s})]ds| \\
&\leq \sup_{0\leq t\leq 1} |f(u_{ns} + x_{0s}, v_{ns} + y_{0s}) - f(u_s^* + x_{0s}, v_s^* + y_{0s})|(\int_0^1 G(s,s)a_1(s)ds \\
&+ \frac{\alpha}{1-\eta\alpha-\beta}\int_0^1 G(s,s)a_1(s)ds + \frac{\beta}{1-\eta\alpha-\beta}\int_0^1 H(\eta,s)a_1(s)ds).
\end{aligned}$$

   This implies that

$$||A(u_n,v_n)(t) - A(u^*,v^*)(t)|| \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, $A(u,v)$ is continuous on D. In a similar way, we can show that $B(u,v)$ is continuous. Hence, Φ is continuous.

The operator Φ is completely continuous by the mean of the Ascoli-Arzela theorem.

□

Theorem 3.2.1. *Assume that* (H_1)-(H_4) *holed. Then the differential system* (3.0.1) *has at last one positive solution, if the following conditions are satisfied*

i) there exist $p_1, p_0 > 0$, *such that if* $0 \le ||(\psi,\varphi)||_{[-\tau,0]} \le p_1 + p_0$, *then*

$$f(\psi,\varphi) < \eta_1 p_1, \quad g(\psi,\varphi) < \mu_1 p_1.$$

ii) there exist $q_1 > p_1$, *such that for* $||(\psi,\varphi)||_{[-\tau,0]} \ge q_1 + p_0$, *then*

$$f(\psi,\varphi) \ge \eta_2(q_1+p_0), \quad \text{or} \quad g(\psi,\varphi) \ge \mu_2(q_2+p_0).$$

where

$$\begin{aligned}
\eta_1^{-1} &\ge \int_0^1 G(s,s)a_1(s)ds + \frac{\alpha}{1-\eta\alpha-\beta}\int_0^1 G(s,s)a_1(s)ds + \frac{\beta}{1-\eta\alpha-\beta}\int_0^1 H(\eta,s)a_1(s)ds,\\
\mu_1^{-1} &\ge \int_0^1 G(s,s)a_2(s)ds + \frac{\alpha}{1-\eta\alpha-\beta}\int_0^1 G(s,s)a_2(s)ds + \frac{\beta}{1-\eta\alpha-\beta}\int_0^1 H(\eta,s)a_2(s)ds,\\
\eta_2^{-1} &\le \frac{1}{2}\left(\int_0^1 G(s,s)a_1(s)ds + \frac{\alpha}{1-\alpha\eta-\beta}\int_0^1 G(s,s)a_1(s)ds + \frac{2\beta}{1-\alpha\eta-\beta}\int_0^1 H(\eta,s)a_1(s)ds\right),\\
\mu_2^{-1} &\le \frac{1}{2}\left(\int_0^1 G(s,s)a_2(s)ds + \frac{\alpha}{1-\alpha\eta-\beta}\int_0^1 G(s,s)a_2(s)ds + \frac{2\beta}{1-\alpha\eta-\beta}\int_0^1 H(\eta,s)a_2(s)ds\right),
\end{aligned}$$

Proof.

Let $\Omega_{p_1} = \{(u,v) \in K/\ ||(u,v)|| \le p_1\}$. For every $(u,v) \in \partial\Omega_{p_1}$ and $t \in [-\tau,1]$, by (3.2.1) and the condition (i),

$$\begin{aligned}
||A(u,v)|| \le &\int_0^1 G(s,s)a_1(s)f(u_s+x_{0s},v_s+y_{0s})ds + \frac{\alpha}{1-\alpha\eta-\beta}\int_0^1 G(s,s)a_1(s)f(u_s+x_{0s},v_s+y_{0s})ds\\
&+\frac{\beta}{1-\alpha\eta-\beta}\int_0^1 H(\eta,s)a_1(s)f(u_s+x_{0s},v_s+y_{0s})ds\\
&< \eta_1 p_1\left(\int_0^1 G(s,s)a_1(s)ds + \frac{\alpha}{1-\alpha\eta-\beta}\int_0^1 G(s,s)a_1(s)ds + \frac{\beta}{1-\alpha\eta-\beta}\int_0^1 H(\eta,s)a_1(s)ds\right)\\
&< p_1 = ||(u,v)||.
\end{aligned}$$

In the similar way, we may show that $||B(u,v)|| < ||(u,v)||$.

Hence $||\Phi(u,v)|| = \max(||A(u,v)||, ||B(u,v)||) < ||(u,v)||$. by Lemma 1.0.1, we have

$$i(\Phi, \Omega_{p_1}, K) = 1. \tag{3.2.6}$$

Let $\Omega_{q_1+p_0} = \{(u,v) \in K/\ ||(u,v)|| \leq q_1 + M_0\}$. For every $(u,v) \in \partial\Omega_{q_1+p_0}$ and by (3.1.7) and the condition (ii),

$$\begin{aligned}||A(u,v)|| &\geq \frac{1}{2}\Big(\int_0^1 G(s,s)a_1(s)f(u_s + x_{0s}, v_s + y_{0s})ds + \frac{\alpha}{1-\alpha\eta-\beta}\int_0^1 G(s,s)a_1(s)f(u_s + x_{0s}, v_s + y_{0s})ds \\ &+ \frac{2\beta}{1-\alpha\eta-\beta}\int_0^1 H(\eta,s)a_1(s)f(u_s + x_{0s}, v_s + y_{0s})ds\Big) \\ &> \frac{\eta_2(q_1+p_0)}{2}\left(\int_E G(s,s)a_1(s)ds + \frac{\alpha}{1-\alpha\eta-\beta}\int_E G(s,s)a_1(s)ds + \frac{2\beta}{1-\alpha\eta-\beta}\int_E H(\eta,s)a_1(s)ds\right) \\ &\geq q_1 + p_0 = ||(u,v)||.\end{aligned}$$

In the similar way, we may show that $||B(u,v)|| > ||(u,v)||$. Hence $||\Phi(u,v)|| > ||(u,v)||$. by Lemma 1.0.1, we have

$$i(\Phi, \Omega_{q_1+p_0}, K) = 0. \tag{3.2.7}$$

by (3.2.6), (3.2.7) and the properties of fixed point index we have

$$i(\Phi, \Omega_{q_1+p_0} \setminus \overline{\Omega}_{p_1}, K) = i(\Phi, \Omega_{q_1+p_0}, K) - i(\Phi, \Omega_{p_1}, K) = -1. \tag{3.2.8}$$

So there exist a fixed point (u,v) in $\Omega_{q_1+p_0} \setminus \overline{\Omega}_{p_1}$. Notice that $(x,y) = (u,v)$ for $t \in [0,1]$ and so the differential system (3.0.1) has a positive solution (x,y) and satisfies $p_1 < ||(x,y)||_{[0,1]} < q_1 + p_0$. ☐

Theorem 3.2.2. *Assume that* (H_1)-(H_4) *holed. Then the differential system* (3.0.1) *has at last one positive solution, if the following conditions are satisfied*

$A_1)$ $f_0 = 0, \quad g_0 = 0$

$A_2)$ $f_\infty = \infty,$ *or* $g_\infty = \infty$

where

$$f_0 = \lim_{||(\psi,\varphi)||_{[-\tau,0]} \to 0} \frac{f(\psi,\varphi)}{||(\psi,\varphi)||_{[-\tau,0]}}, \qquad f_\infty = \lim_{||(\psi,\varphi)||_{[-\tau,0]} \to \infty} \frac{f(\psi,\varphi)}{||(\psi,\varphi)||_{[-\tau,0]}}$$

$$g_0 = \lim_{||(\psi,\varphi)||_{[-\tau,0]} \to 0} \frac{g(\psi,\varphi)}{||(\psi,\varphi)||_{[-\tau,0]}}, \qquad g_\infty = \lim_{||(\psi,\varphi)||_{[-\tau,0]} \to \infty} \frac{g(\psi,\varphi)}{||(\psi,\varphi)||_{[-\tau,0]}}.$$

Proof. By (A_1), assume that $f_0 = 0$, that is

$$\lim_{||(\psi,\varphi)||_{[-\tau,0]}\to 0} \frac{f(\psi,\varphi)}{||(\psi,\varphi)||_{[-\tau,0]}} = 0.$$

Then for $M = \eta_1$, there exists $\rho_* \in (0, p_1)$, such that

$$f(\psi,\varphi) < M\rho_*, \quad 0 < ||(\psi,\varphi)||_{[-\tau,0]} \leq \rho_*.$$

Let $\Omega_{\rho_*} = \{(u,v) \in K/\ ||(u,v)|| \leq \rho_*\}$. For every $(u,v) \in \partial\Omega_{\rho_*}$, by (3.1.6), we have

$$\begin{aligned}
||A(u,v)|| \leq & \int_0^1 G(s,s)a_1(s)f(u_s + x_{0s}, v_s + y_{0s})ds + \frac{\alpha}{1-\alpha\eta-\beta}\int_0^1 G(s,s)a_1(s)f(u_s + x_{0s}, v_s + y_{0s})ds \\
& + \frac{\beta}{1-\alpha\eta-\beta}\int_0^1 H(\eta,s)a_1(s)f(u_s + x_{0s}, v_s + y_{0s})ds \\
& \leq M\rho_*\Big(\int_0^1 G(s,s)a_1(s)ds + \frac{\alpha}{1-\alpha\eta-\beta}\int_0^1 G(s,s)a_1(s)ds + \frac{\beta}{1-\alpha\eta-\beta}\int_0^1 H(\eta,s)a_1(s)ds\Big) \\
& < \rho_* = ||(u,v)||.
\end{aligned}$$

In the similar way, we can show that $||B(u,v)|| \leq ||(u,v)||$. Hence, $||\Phi(u,v)|| < ||(u,v)||$. By Lemma 1.0.1, $i(\Phi, \Omega_{\rho_*}, K) = 1$.

By (A_2), assume that $f_\infty = \infty$, Then for $N = \eta_2$, there exists $\rho^* > p_1$, such that for $||(\psi,\varphi)||_{[-\tau,0]} \geq \rho^* + p_0$, $f(\psi,\varphi) > N(\rho^* + p_0)$. Let $\Omega_{\rho^*+p_0} = \{(u,v) \in K/\ ||(u,v)|| \leq \rho^* + p_0\}$. Then for every $(u,v) \in \partial\Omega_{\rho^*+p_0}$, by (3.1.6), we have

$$\begin{aligned}
||A(u,v)|| \geq & \frac{1}{2}\Big(\int_0^1 G(s,s)a_1(s)f(u_s + x_{0s}, v_s + y_{0s})ds + \frac{\alpha}{1-\alpha\eta-\beta}\int_0^1 G(s,s)a_1(s)f(u_s + x_{0s}, v_s + y_{0s})ds \\
& + \frac{2\beta}{1-\alpha\eta-\beta}\int_0^1 H(\eta,s)a_1(s)f(u_s + x_{0s}, v_s + y_{0s})ds\Big) \\
& > \frac{N(\rho^* + p_0)}{2}\Big(\int_0^1 G(s,s)a_1(s)ds + \frac{\alpha}{1-\alpha\eta-\beta}\int_0^1 G(s,s)a_1(s)ds + \frac{2\beta}{1-\alpha\eta-\beta}\int_0^1 H(\eta,s)a_1(s)ds\Big) \\
& \geq \rho^* + p_0 = ||(u,v)||.
\end{aligned}$$

Hence, $||\Phi(u,v)|| > ||(u,v)||$. By Lemma 1.0.1, $i(\Phi, \Omega_{\rho^*+p_0}, K) = 0$.

So,

$$i(\Phi, \Omega_{\rho^*+p_0} \setminus \Omega_{\rho_*}, K) = -1.$$

Hence, Φ has fixed point $(u,v) \in \Omega_{\rho^*+p_0} \setminus \Omega_{\rho_*}$, which satisfy $0 < ||(u,v)|| < \infty$. □

Chapter 4

Positive solutions for second order boundary value problems with dependence on the first order derivative

In this chapter, We study the existence of positive solutions for nonlocal boundary value problems for functional differential equations

$$\begin{aligned} & u''(t) + f(t, u_t, u'(t)) = 0, \quad 0 \le t \le 1, \\ & u(t) = \phi(t), \quad -\tau \le t \le 0, \\ & u(1) = \alpha u(\eta) + \beta u'(\eta) \end{aligned} \tag{4.0.1}$$

where $\phi \in C$, $f : [0,1] \times C \times \mathbb{R} \to \mathbb{R}$ is continuos functions and $\eta \in (0,1)$,

Our results are based on the nonlinear alternative of Leray-Schauder fixed point theorem.

Let $C = C([-\tau, 0]; \mathbb{R})$, with $\tau > 0$ is a fixed constant, be the Banach space of all continuous functions $\phi : [-\tau, 0] \to \mathbb{R}$, with the norm

$$\|\phi\| = \sup\{|\phi(\theta)| : -\tau \le \theta \le 0\}.$$

For any continuous function $u : [-r, 1] \to \mathbb{R}$ and for any $t \in [0,1]$, we denote by u_t the element of C

defined by $u_t(\theta) = u(t+\theta), \theta \in [-\tau, 0]$.

4.1 Preliminaries

We denote by $C[0,1]$ and $C^1[0,1]$, respectively, the Banach spaces of continuous real functions and continuously differentiable real functions on $[0,1]$, with the norms:

$$\|u\|_0 = \sup\{|u(t)| : 0 \le t \le 1\},$$
$$\|u\|_1 = \max\{\|u\|_0, \|u'\|_0\},$$

where $\|u'\|_0 = \sup\{|u'(t)| : 0 \le t \le 1\}$, and by $L^1[0,1]$ the space of all real functions $x(t)$ such that $|x(t)|$ is Lebesgue integrable on $[0,1]$.

The proofs of our theorems are based on the following theorems result.

Lemma 4.1.1.

Let $\beta \neq 1 - \alpha\eta$. If $y(t) \in C([0,1])$, then the boundary value problem

$$u''(t) + y(t) = 0, \quad 0 < t < 1, \tag{4.1.1}$$

$$u(0) = 0, \;\; u(1) = \alpha u(\eta) + \beta u'(\eta) \tag{4.1.2}$$

has a unique solution

$$u(t) = \int_0^t (s-t)y(s)ds + \frac{1}{1-\alpha\eta-\beta}\int_0^1 t(1-s)y(s)ds - \frac{\alpha t}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)y(s)ds - \frac{\beta t}{1-\alpha\eta-\beta}\int_0^\eta y(s)ds$$

Proof.

From (4.1.1), we have

$$u(t) = u(0) + u'(0)t + \int_0^t (s-t)y(s)ds := B + At + \int_0^t (s-t)y(s)ds. \tag{4.1.3}$$

From $u(0) = 0$, we have $B = 0$. Since $u(1) = \alpha u(\eta) + \beta u'(\eta)$, we have

$$\int_0^1 (s-1)y(s)ds + A = \alpha(\int_0^\eta (s-\eta)g(s)ds + A\eta) + \beta(-\int_0^\eta y(s)ds + A).$$

Then,

$$A = \frac{1}{1-\alpha\eta-\beta}\int_0^1 (1-s)y(s)ds - \frac{\alpha}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)y(s)ds - \frac{\beta}{1-\alpha\eta-\beta}\int_0^\eta y(s)ds.$$

Replacing these expression, in (4.1.3), we obtain

$$u(t) = \int_0^t (s-t)y(s)ds + \frac{1}{1-\alpha\eta-\beta}\int_0^1 t(1-s)y(s)ds - \frac{\alpha t}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)y(s)ds - \frac{\beta t}{1-\alpha\eta-\beta}\int_0^\eta y(s)ds.$$

□

4.2 Main Results

In this section, we present our existence results for the boundary-value problem (4.0.1).

Theorem 4.2.1. *Let $f:[0,1]\times C\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Assume that there exist nonnegative functions p, q, $r\in L^1[0,1]$ such that*

$(H1)$ $|f(t,u,v)| \leq p(t)\|u\| + q(t)|v| + r(t)$, *for all* $(t,u,v)\in[0,1]\times C\times\mathbb{R}$

$(H2)$ $\frac{2-(\alpha\eta+\beta)}{1-\alpha\eta-\beta}\int_0^1 (1-s)p(s)ds + \frac{\alpha}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)p(s)ds + \frac{\beta}{1-\alpha\eta-\beta}\int_0^\eta p(s)ds < 1,$

$(H3)$ $\int_0^1 [p(s)+q(s)]ds + \frac{1}{1-\alpha\eta-\beta}\int_0^1 (1-s)[p(s)+q(s)]ds + \frac{\alpha}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)[p(s)+q(s)]ds + \frac{\beta}{1-\alpha\eta-\beta}\int_0^\eta [p(s)+q(s)]ds < 1.$

Then the boundary-value problem (4.0.1) has at least one solution.

Proof.

Consider first the case $\phi(0)=0$. Put $C_0=\{u\in C^1[0,1]:u(0)=0\}$. Then C_0 is the subspace of $C^1[0,1]$. We note that for all $u\in C_0$, $u(t)=\int_0^t u'(s)ds$, so

$$\|u\|_0 \leq \|u'\|_0. \tag{4.2.1}$$

For a function $u\in C_0$, we define the function $\widehat{u}:[-r,1]\to\mathbb{R}$ by

$$\widehat{u}(t) = \begin{cases} \phi(t), & t\in[-r,0], \\ u(t), & t\in[0,1]. \end{cases}$$

We also note that

$$\|\widehat{u}_t\|^k \leq \max\{\|u\|_0^k, \|\phi\|^k\} \leq \|u\|_0^k + \|\phi\|^k, \forall t \in [0,1], \forall k \geq 0. \tag{4.2.2}$$

Define the integral operator $T : C_0 \to C^1[0,1]$ by

$$Tu(t) = \int_0^t (s-t)f(s,u_s,u'(s))ds + \frac{1}{1-\alpha\eta-\beta}\Big(\int_0^1 t(1-s)f(s,u_s,u'(s))ds - \alpha\int_0^\eta t(\eta-s)f(s,u_s,u'(s))ds$$

$$-\beta\int_0^\eta tf(s,u_s,u'(s))ds.\Big) \tag{4.2.3}$$

It is easy to know that fixed points of T are solutions of the boundary-value problem (4.0.1).

Using $(H1)$ and (4.2.2), for all $u \in C_0$, for all $t \in [0,1]$, we obtain

$$\begin{aligned}|Tu(t)| \leq &\int_0^1 (1-s)[p(s)\|\widehat{u}_s\| + q(s)|u'(s)| + r(s)]ds + \frac{1}{1-\alpha\eta-\beta}\Big(\int_0^1 (1-s)[p(s)\|\widehat{u}_s\| + q(s)|u'(s)| + r(s)]ds \\ &+ \alpha\int_0^\eta (\eta-s)[p(s)\|\widehat{u}_s\| + q(s)|u'(s)| + r(s)]ds + \beta\int_0^\eta [p(s)\|\widehat{u}_s\| + q(s)|u'(s)| + r(s)]ds\Big) \\ &\leq A_1\|u\|_0 + B_1\|u'\|_0 + C_1,\end{aligned}$$

where

$$\begin{aligned}A_1 &= \frac{2-(\alpha\eta+\beta)}{1-\alpha\eta-\beta}\int_0^1 (1-s)p(s)ds + \frac{\alpha}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)p(s)ds + \frac{\beta}{1-\alpha\eta-\beta}\int_0^\eta p(s)ds, \\ B_1 &= \frac{2-(\alpha\eta+\beta)}{1-\alpha\eta-\beta}\int_0^1 (1-s)q(s)ds + \frac{\alpha}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)q(s)ds + \frac{\beta}{1-\alpha\eta-\beta}\int_0^\eta q(s)ds, \\ C_1 &= \Big(\frac{2-(\alpha\eta+\beta)}{1-\alpha\eta-\beta}\int_0^1 (1-s)p(s)ds + \frac{\alpha}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)p(s)ds + \frac{\beta}{1-\alpha\eta-\beta}\int_0^\eta p(s)ds\Big)\|\phi\| \\ &+ \frac{2-(\alpha\eta+\beta)}{1-\alpha\eta-\beta}\int_0^1 (1-s)r(s)ds + \frac{\alpha}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)r(s)ds + \frac{\beta}{1-\alpha\eta-\beta}\int_0^\eta r(s)ds.\end{aligned}$$

Hence

$$\|Tu\|_0 \leq A_1\|u\|_0 + B_1\|u'\|_0 + C_1, \quad \forall u \in C_0. \tag{4.2.4}$$

On the other hand,

$$\begin{aligned}(Tu)'(t) = &-\int_0^t f(s,\widehat{u}_s,u'(s))ds + \frac{1}{1-\alpha\eta-\beta}\Big(\int_0^1 (1-s)f(s,\widehat{u}_s,u'(s))ds - \alpha\int_0^\eta (\eta-s)f(s,\widehat{u}_s,u'(s))ds \\ &-\beta\int_0^\eta f(s,\widehat{u}_s,u'(s))ds\Big), \quad t \in [0,1].\end{aligned} \tag{4.2.5}$$

Similarly, it follows from $(H1)$ and (4.2.2) that

$$\|(Tu)'\|_0 \le A_2\|u\|_0 + B_2\|u'\|_0 + C_2, \quad \forall u \in C_0, \tag{4.2.6}$$

where

$$\begin{aligned}
A_2 &= \int_0^1 p(s)ds + \frac{1}{1-\alpha\eta-\beta}\int_0^1 (1-s)p(s)ds + \frac{\alpha}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)p(s)ds + \frac{\beta}{1-\alpha\eta-\beta}\int_0^\eta p(s)ds,\\
B_2 &= \int_0^1 q(s)ds + \frac{1}{1-\alpha\eta-\beta}\int_0^1 (1-s)q(s) + \frac{\alpha}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)q(s)ds + \frac{\beta}{1-\alpha\eta-\beta}\int_0^\eta q(s)ds,\\
C_2 &= \Big(\int_0^1 p(s)ds + \frac{1}{1-\alpha\eta-\beta}\int_0^1 (1-s)p(s)ds + \frac{\alpha}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)p(s)ds + \frac{\beta}{1-\alpha\eta-\beta}\int_0^\eta p(s)ds\Big)\|\phi\|\\
&\quad + \int_0^1 r(s)ds + \frac{1}{1-\alpha\eta-\beta}\int_0^1 (1-s)r(s)ds + \frac{\alpha}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)r(s)ds + \frac{\beta}{1-\alpha\eta-\beta}\int_0^\eta r(s)ds.
\end{aligned}$$

Put

$$A = \max\{A_1, A_2 + B_2\}. \tag{4.2.7}$$

From $(H2)$-$(H3)$, it follows that $A_1 < 1$, $A_2 + B_2 < 1$, so $A < 1$. We now choose a constant $B > 0$ such that

$$B \ge \max\{\frac{B_1 C_2}{1 - A_2 - B_2} + C_1, C_2\}, \tag{4.2.8}$$

and put

$$m = \frac{B}{1-A}, \quad \Omega = \{u \in C_0 : \|u\|_1 < m\}. \tag{4.2.9}$$

Then Ω be a bounded open subset of C_0, $0 \in \Omega$, and $\partial\Omega = \{u \in C_0 : \|u\|_1 = m\}$.

We shall show that $T : \overline{\Omega} = \Omega \cup \partial\Omega \to C^1[0,1]$ has a fixed point $u \in \Omega$ by applying Theorem 1.0.3.

(a) First, T is continuous. Indeed, for each $u_0 \in \overline{\Omega}$, let $\{u_n\}$ be a sequence in $\overline{\Omega}$ such that $\lim_{n\to\infty} u_n = u_0$. For all $t \in [0,1]$, from (4.1.2), we get

$$\begin{aligned}
Tu_n(t) - Tu_0(t) = &-\int_0^t (t-s)\Big[f(s,(\widehat{u}_n)_s, u_n'(s)) - f(s,(\widehat{u}_0)_s, u_0'(s))\Big]ds\\
&+ \frac{t}{1-\alpha\eta-\beta}\int_0^1 (1-s)\Big[f(s,(\widehat{u}_n)_s, u_n'(s)) - f(s,(\widehat{u}_0)_s, u_0'(s))\Big]ds\\
&- \frac{\alpha t}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)\Big[f(s,(\widehat{u}_n)_s, u_n'(s)) - f(s,(\widehat{u}_0)_s, u_0'(s))\Big]ds\\
&- \frac{\beta t}{1-\alpha\eta-\beta}\int_0^\eta \Big[f(s,(\widehat{u}_n)_s, u_n'(s)) - f(s,(\widehat{u}_0)_s, u_0'(s))\Big]ds.
\end{aligned}$$

Put $D = \{(\widehat{u}_n)_s : s \in [0,1], n = 0,1,2,\dots\}$, then D is compact in C. Since $f : [0,1] \times C \times \mathbb{R} \to \mathbb{R}$ is continuous, f is uniformly continuous on the compact subset $[0,1] \times D \times [-m,m]$. This implies that, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for each (s_1, ϕ_1, ν_1), $(s_2, \phi_2, \nu_2) \in [0,1] \times D \times [-m, m]$,

$$\begin{aligned} &|s_1 - s_2| < \delta, \quad \|\phi_1 - \phi_2\| < \delta, \quad |\nu_1 - \nu_2| < \delta \\ &\Rightarrow \quad |f(s_1, \phi_1, \nu_1) - f(s_2, \phi_2, \nu_2)| < \frac{\varepsilon}{2\gamma}, \end{aligned}$$

with $\gamma = 1 + \dfrac{1+\alpha+\beta}{1-\alpha\eta-\beta} > 0$. Since $\lim\limits_{n\to\infty} u_n = u_0$ in $\overline{\Omega}$, with respect to $\|\cdot\|_1$, there exists n_0 such that for all $n \geq n_0$,

$$\|(\widehat{u}_n)_s - (\widehat{u}_0)_s\| < \delta, |u_n'(s) - u_0'(s)| < \delta, \quad \forall s \in [0,1].$$

On the other hand, for all $s \in [0,1]$, $(s, (\widehat{u}_n)_s, u_n'(s)), (s, (\widehat{u}_0)_s, u_0'(s)) \in [0,1] \times D \times [-m, m]$, therefore, for all $n \geq n_0$,

$$\begin{aligned} |Tu_n(t) - Tu_0(t)| &\leq (1 + \frac{1+\alpha+\beta}{1-\alpha\eta-\beta}) \int_0^1 \left| f(s, (\widehat{u}_n)_s, u_n'(s)) - f(s, (\widehat{u}_0)_s, u_0'(s)) \right| ds \\ &< (1 + \frac{1+\alpha+\beta}{1-\alpha\eta-\beta}) \frac{\varepsilon}{2\gamma} = \frac{\varepsilon}{2}, \quad \forall t \in [0,1]. \end{aligned}$$

Similarly

$$|(Tu_n)'(t) - (Tu_0)'(t)| < \frac{\varepsilon}{2}, \quad \forall t \in [0,1].$$

This implies that for all $n \geq n_0$,

$$\|Tu_n - Tu_0\|_1 = \max\left\{ \|Tu_n - Tu_0\|_0, \|(Tu_n)' - (Tu_0)'\|_0 \right\} \leq \frac{\varepsilon}{2} < \varepsilon.$$

(b) Next, we show that $T(\overline{\Omega})$ is relatively compact. Let $\{Tu_n\}$ be a bounded sequence of $T(\overline{\Omega})$, corresponding $\{u_n\} \subset \overline{\Omega}$, we shall show that $\{Tu_n\}$ contains a convergence subsequence in $C^1[0,1]$, with respect to $\|.\|_1$. The proof of this fact is obtained as follows. For all n, it follows from (4.2.4), (4.2.6), (4.2.9) that

$$\begin{aligned} \|Tu_n\|_0 &\leq A_1 \|u_n\|_0 + B_1 \|u_n'\|_0 + C_1 \leq A_1 m + B_1 m + C_1, \\ \|(Tu_n)'\|_0 &\leq A_2 \|u_n\|_0 + B_2 \|u_n'\|_0 + C_2 \leq A_2 m + B_2 m + C_2. \end{aligned}$$

Hence, the sequences $\{Tu_n\}, \{(Tu_n)'\}$ are uniformly bounded. On the other hand, combining (4.2), (4.2.5), (4.2.9) and $(H1)$, for all n, for all $t_1, t_2 \in [0,1]$, we have

$$\begin{aligned}|Tu_n(t_1) - Tu_n(t_2)| &\le \Big| \int_{t_1}^{t_2} (1-s)[(m+\|\phi\|)p(s) + mq(s) + r(s)]ds \Big| \\ &+ \frac{1}{1-\alpha\eta-\beta}\Big(\int_0^1 [(m+\|\phi\|)p(s) + mq(s) + r(s)]ds\Big)|t_1 - t_2| \\ &+ \frac{\alpha}{1-\alpha\eta-\beta}\Big(\int_0^\eta (\eta - s)[(m+\|\phi\|)p(s) + mq(s) + r(s)]ds\Big)|t_1 - t_2| \\ &+ \frac{\beta}{1-\alpha\eta-\beta}\Big(\int_0^\eta [(m+\|\phi\|)p(s) + mq(s) + r(s)]ds\Big)|t_1 - t_2| \\ &\le K_1 |t_1 - t_2|,\end{aligned}$$

$$\begin{aligned}|(Tu_n)'(t_1) - (Tu_n)'(t_2)| &\le \Big| \int_{t_1}^{t_2} [(m+\|\phi\|)p(s) + mq(s) + r(s)]ds \Big| \\ &\le K_2 |t_1 - t_2|,\end{aligned}$$

where K_1, K_2 are independent of t_1, t_2 and n. This implies that the sequences $\{Tu_n\}, \{(Tu_n)'\}$ are equi-continuous. By using the Ascoli-Arzela theorem, we have $\{Tu_n\}$, $\{(Tu_n)'\}$ are relatively compact in $C[0,1]$. Therefore, there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$, such that

$$Tu_{n_k} \to u \quad \text{and} \quad (Tu_{n_k})' \to v, \quad \text{as} \quad k \to \infty,$$

with respect to $\|.\|_0$. Then u is differentiable and $u' = v$, so $Tu_{n_k} \to u$, as $k \to \infty$, in $C^1[0,1]$, with respect to $\|.\|_1$. Thus T is completely continuous.

(c) Finally, suppose that there exists $u^* \in \partial\Omega$, such that $T(u^*) = \lambda u^*$, for some $\lambda > 1$. Then, we have the following set is bounded

$$\{u^* \in \partial\Omega : T(u^*) = \lambda u^*, \lambda > 1\}.$$

Indeed, it follows from (4.2.6) that

$$\|(u^*)'\|_0 = \frac{1}{\lambda}\|(Tu^*)'\|_0 \le \|(Tu^*)'\|_0 \le A_2\|u^*\|_0 + B_2\|(u^*)'\|_0 + C_2. \tag{4.2.10}$$

Combining (4.2.1), (4.2.10), we get

$$(1 - A_2 - B_2)\|(u^*)'\|_0 \le C_2.$$

Since $A_2 + B_2 < 1$, this implies that

$$\|(u^*)'\|_0 \le M, \tag{4.2.11}$$

where $M = C_2/(1-A_2-B_2)$ is a constant. Thus, combining (4.2.1), (4.2.4), (4.2.6)-(4.2.8), (4.2.10) and (4.2.11), we obtain

$$\begin{aligned} \|Tu^*\|_0 &\le A_1\|u^*\|_0 + B_1\|(u^*)'\|_0 + C_1 \\ &\le A_1\|u^*\|_0 + B_1 M + C_1 \\ &\le A\|u^*\|_0 + B, \qquad (4.2.12) \\ \|(Tu^*)'\|_0 &\le A_2|u^*\|_1 + B_2\|u^*\|_1 + C_2 \\ &\le A\|u^*\|_1 + B. \end{aligned}$$

Consequently

$$\lambda\|u^*\|_1 = \|Tu^*\|_1 \le A\|u^*\|_1 + B,$$

which implies

$$\lambda m \le Am + B \quad \text{or} \quad \lambda \le A + \frac{B}{m}, \quad \text{i.e. } \lambda \le 1,$$

this contradicts $\lambda > 1$. The proof of step 1 is complete.

Step 2. The case $\phi(0) \neq 0$. By the transformation $v = u - \phi(0)$, the boundary-value problem (4.0.1) reduces to the boundary-value problem

$$\begin{aligned} v'' + f(t, v_t + \phi(0), v'(t)) = 0, &\quad 0 \le t \le 1, \\ v_0 = \phi - \phi(0) \equiv \widetilde{\phi}, &\quad v(1) = v(\eta), \end{aligned}$$

with $\widetilde{\phi} \in C$ and $\widetilde{\phi}(0) = 0$. By step 1, this boundary-value problem has at least one solution. Step 2 follows and Theorem 4.2.1 is proved. □

Theorem 4.2.2. *Let $f : [0,1] \times C \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that there exist nonnegative functions p, q, $r \in L^1[0,1]$ and reals constants $k, l \in [0,1]$ such that (H2) holds and*

(H_1) $|f(t,u,v)| \le p(t)\|u\|^k + q(t)|v|^l + r(t)$, *for all* $(t,u,v) \in [0,1] \times C \times \mathbb{R}$,

(H_2) $h(k)A_2 + h(l)B_2 < 1$,

where

$$A_2 = \int_0^1 p(s)ds + \frac{1}{1-\alpha\eta-\beta}\int_0^1 (1-s)p(s)ds + \frac{\alpha}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)p(s)ds + \frac{\beta}{1-\alpha\eta-\beta}\int_0^\eta p(s)ds,$$
$$B_2 = \int_0^1 q(s)ds + \frac{1}{1-\alpha\eta-\beta}\int_0^1 (1-s)q(s) + \frac{\alpha}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)q(s)ds + \frac{\beta}{1-\alpha\eta-\beta}\int_0^\eta q(s)ds,$$

and

$$h(\mu) = \begin{cases} 0, & 0 \le \mu < 1, \\ 1, & \mu = 1. \end{cases}$$

Then the boundary-value problem (4.0.1) has at least one solution.

Proof. It is obvious that the Theorem 4.2.1 is a special case of this theorem with $k = l = 1$. Here, we consider only the case $\phi(0) = 0$ and let the subspace C_0, the function $\widehat{u}$ and the operator T be defined as in Theorem 4.2.1. Using (H_1) and (4.2.2), for all $u \in C_0$ and all $t \in [0,1]$, we have

$$\begin{aligned} |Tu(t)| \le & \int_0^1 (1-s)[p(s)\|\widehat{u}_s\|^k + q(s)|u'(s)|^l + r(s)]ds \\ & + \frac{1}{1-\alpha\eta-\beta}\int_0^1 (1-s)[p(s)\|\widehat{u}_s\|^k + q(s)|u'(s)|^l + r(s)]ds \\ & + \frac{\alpha}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)[p(s)\|\widehat{u}_s\|^k + q(s)|u'(s)|^l + r(s)]ds \\ & + \frac{\beta}{1-\alpha\eta-\beta}\int_0^\eta [p(s)\|\widehat{u}_s\|^k + q(s)|u'(s)|^l + r(s)]ds \\ \le & A_1\|u\|_0^k + B_1\|u'\|_0^l + C_3, \end{aligned}$$

where A_1 and B_1 as in Theorem 4.2.1, and

$$\begin{aligned} C_3 = & \Big(\frac{2-(\alpha\eta+\beta)}{1-\alpha\eta-\beta}\int_0^1 (1-s)p(s)ds + \frac{\alpha}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)p(s)ds + \frac{\beta}{1-\alpha\eta-\beta}\int_0^\eta p(s)ds\Big)\|\phi\|^k \\ & + \frac{2-(\alpha\eta+\beta)}{1-\alpha\eta-\beta}\int_0^1 (1-s)r(s)ds + \frac{\alpha}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)r(s)ds + \frac{\beta}{1-\alpha\eta-\beta}\int_0^\eta r(s)ds. \end{aligned}$$

It follows that for all $u \in C_0$,

$$\|Tu\|_0 \le A_1\|u\|_0^k + B_1\|u'\|_0^l + C_3. \tag{4.2.13}$$

Similarly, for all $u \in C_0$, we obtain

$$\begin{aligned} \|(Tu)'\|_0 & \le A_2\|u\|_0^k + B_2\|u'\|_0^l + C_4 \\ & \le A_2\|u'\|_0^k + B_2\|u'\|_0^l + C_4, \end{aligned} \tag{4.2.14}$$

where A_2 and B_2 are as above and

$$\begin{aligned} C_4 = & \Big(\int_0^1 p(s)ds + \frac{1}{1-\alpha\eta-\beta}\int_0^1 (1-s)p(s)ds + \frac{\alpha}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)p(s)ds + \frac{\beta}{1-\alpha\eta-\beta}\int_0^\eta p(s)ds\Big)\|\phi\|^k \\ & + \int_0^1 r(s)ds + \frac{1}{1-\alpha\eta-\beta}\int_0^1 (1-s)r(s)ds + \frac{\alpha}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)r(s)ds + \frac{\beta}{1-\alpha\eta-\beta}\int_0^\eta r(s)ds. \end{aligned}$$

Clearly, as the proof of the Theorem 4.2.1, if we show the boundedness of the following set

$$\{u^* \in \partial\Omega : T(u^*) = \lambda u^*, \lambda > 1\}, \tag{4.2.15}$$

then, combining the assume (H_2), the proof of Theorem 4.2.2 will be completely. That is proved as follows.

Suppose that there exists $u^* \in \partial\Omega$ such that $T(u^*) = \lambda u^*$ for some $\lambda > 1$. We consider three cases.

Case 1:

$0 \le k < 1$, $0 \le l < 1$. If $\|(u^*)'\|_0 > 1$, then from (3.14), we have

$$\|(Tu^*)'\|_0 \le (A_2 + B_2)\|(u^*)'\|_0^p + C_4, \tag{4.2.16}$$

where $p = \max\{k, l\}$. It follows that

$$\|(u^*)'\|_0 = \frac{1}{\lambda}\|(Tu^*)'\|_0 \le \|(Tu^*)'\|_0 \le (A_2 + B_2)\|(u^*)'\|_0^p + C_4. \tag{4.2.17}$$

Here, let us note that if $K \ge 0$, $H > 0$, $0 \le \gamma < 2$ are given constants, then there exists a constant $C > 0$ such that

$$Kx^\gamma \le \frac{Hx^2}{2} + C, \quad \forall x \ge 0. \tag{4.2.18}$$

Hence, with $x = \sqrt{\|(u^*)'\|_0}$, $K = A_2 + B_2$, $\gamma = 2p, H = 1$, the inequality (4.2.18) implies that

$$(A_2 + B_2)\|(u^*)'\|_0^p + C_4 \le \frac{1}{2}\|(u^*)'\|_0 + C_4 + C.$$

Combining the above inequalities,

$$\|(u^*)'\|_0 \le \frac{1}{2}\|(u^*)'\|_0 + C_4 + C \quad \text{or} \quad \|(u^*)'\|_0 \le 2C_4 + 2C.$$

We can choose C such that $2C_4 + 2C > 1$; therefore,

$$\|(u^*)'\|_0 \le 2C_4 + 2C,$$

although $\|(u^*)'\|_0 \le 1$ or $\|(u^*)'\|_0 > 1$. Thus, in case 1, there exists a positive constant $\widetilde{M} = 2C_4 + 2C$, such that

$$\|(u^*)'\|_0 \le \widetilde{M}. \tag{4.2.19}$$

Case 2:

$k=1$, $0 \le l < 1$. From (4.2.14), we have

$$\|(Tu^*)'\|_0 \le A_2\|(u^*)'\|_0 + B_2\|(u^*)'\|_0^l + C_2,$$

where $C_4 = C_2$, since $k=1$. So we have

$$(1-A_2)\|(u^*)'\|_0 \le B_2\|(u^*)'\|_0^l + C_2.$$

Clearly, from (H_2), $A_2 < 1$. Using (4.2.18) again, with $x = \sqrt{\|(u^*)'\|_0}$, $K = B_2$, $\gamma = 2l$, $H = 1-A_2$, we get

$$B_2\|(u^*)'\|_0^l + C_2 \le \frac{1}{2}(1-A_2)\|(u^*)'\|_0 + C_2 + \widetilde{C},$$

and so

$$(1-A_2)\|(u^*)'\|_0 \le \frac{1}{2}(1-A_2)\|(u^*)'\|_0 + C_2 + \widetilde{C} \Leftrightarrow \|(u^*)'\|_0 \le \frac{2C_2 + 2\widetilde{C}}{1-A_2},$$

where $\widetilde{C}$ is a positive constant. We deduce that (4.2.19) also holds in the second case, in which $\widetilde{M} = \frac{2C_2+2\widetilde{C}}{1-A_2}$. **Case 3:**

$0 \le k < 1$, $l = 1$. We conclude from the hypothesis (H_2) that $B_2 < 1$, hence that it is similar to the above cases, (4.2.19) also holds. Therefore, Theorem 4.2.2 is proved. □

Chapter 5

Existence of Solutions for Boundary Value Problem of Fractional Functional Differential Equations with Delay

5.1 Introduction

In this chapter, we study the existence of positive solutions for three boundary value problems of fractional differential equations with delay. We consider the BVP of the form

$$\begin{cases} D_{0^+}^{\alpha} u(t) + f(t,u_t) = 0, & t \in [0,1], \qquad \alpha \in (2,3], \\ u(t) = \phi(t), & -\tau \leq t \ \leq 0, \\ u(0) = u^{'}(0) = 0, & u(1) = \beta u(\eta) + \gamma u'(\eta), \end{cases} \tag{5.1.1}$$

where $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractionnaire derivative, $0 < \beta\eta^{\alpha-1} - \gamma(\alpha-1)\eta^{\alpha-2} < 1$ and $0 < \eta < 1$. Our results are based on the nonlinear alternative of Leray-Schauder type and Krasnosel'skii fixed point theorem.

5.2 Preliminaries

Let $\tau > 0$, and denote by C_τ the Banach space of all continuous functions $\psi : [-\tau, 0] \to \mathbb{R}$ endowed with the sup-norm

$$\|\psi\|_{[-\tau,0]} := \sup\{|\psi(s)| : s \in [-\tau, 0]\}.$$

For any continuous function u defined on the interval $[-\tau, 1]$ and any $t \in [0,1]$, the symbol u_t is used to denote the element of $C([-\tau, 0])$ defined by

$$u_t(s) := u(t+s), \quad s \in [-\tau, 0].$$

And ϕ is an element of the space

$$C_\tau^+(0) := \{\psi \in C_\tau : \ \psi(s) \geq 0, \ s \in [-\tau, 0], \psi(0) = 0\}.$$

where $\Gamma(.)$ is the Euler gamma function.

Lemma 5.2.1. *Let $D = 1 - \beta\eta^{\alpha-1} - \gamma(\alpha-1)\eta^{\alpha-2}$, $0 < \eta < 1$ and $0 < \beta\eta^{\alpha-1} - \gamma(\alpha-1)\eta^{\alpha-2} < 1$ If $y(t) \in C[0,1]$, then the boundary value problem*

$$D_{0+}^{\alpha} u(t) + y(t) = 0 \qquad 0 < t < 1, \ 2 < \alpha \leq 3 \tag{5.2.1}$$

$$u(0) = u'(0) = 0 \qquad u(1) = \beta u(\eta) + \gamma u'(\eta) \tag{5.2.2}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)y(s)ds + \frac{\beta t^{\alpha-1}}{D}\int_0^1 G(\eta,s)y(s)ds + \frac{\gamma t^{\alpha-1}}{D}\int_0^1 H(\eta,s)y(s)ds \tag{5.2.3}$$

where

$$G(t,s) := \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}, & s \leq t, \\ \\ t^{\alpha-1}(1-s)^{\alpha-2}, & t \leq s. \end{cases} \tag{5.2.4}$$

and

$$H(t,s) := \frac{\partial G(t,s)}{\partial t} = \frac{(\alpha-1)}{\Gamma(\alpha)}\begin{cases} t^{\alpha-2}(1-s)^{\alpha-2} - (t-s)^{\alpha-2}, & s \leq t, \\ \\ t^{\alpha-2}(1-s)^{\alpha-2}, & t \leq s. \end{cases} \tag{5.2.5}$$

Proof. By Lemma 1.0.6 the solution of (5.2.1) can be written as

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s).ds \tag{5.2.6}$$

Using the boundary conditions $u(0) = u'(0) = 0$, we must have $c_2 = c_3 = 0$.

So, (5.2.6) becomes

$$u(t) = c_1 t^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds.$$

We have

$$u'(1) = (\alpha-1)c_1 - \frac{\alpha-1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} y(s) ds, \tag{5.2.7}$$

and

$$\beta u(\eta) = \beta c_1 \eta^{\alpha-1} - \frac{\beta}{\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} y(s) ds. \tag{5.2.8}$$

According to $u(1) = \beta u(\eta) + \gamma u'(\eta)$, we have

$$c_1 = \frac{1}{D\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds - \frac{\beta}{D\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} y(s) ds - \frac{\gamma(\alpha-1)}{D\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-2} y(s) ds.$$

So,

$$u(t) = \frac{t^{\alpha-1}}{D\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds - \frac{\beta t^{\alpha-1}}{D\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} y(s) ds - \frac{\gamma(\alpha-1)t^{\alpha-1}}{D\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-2} y(s) ds$$

$$- \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds$$

$$= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds - \frac{\beta t^{\alpha-1}}{D\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} y(s) ds - \frac{\gamma(\alpha-1)t^{\alpha-1}}{D\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-2} y(s) ds$$

$$+ \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{\beta \eta^{(\alpha-1)} t^{\alpha-1} + \gamma(\alpha-1)\eta^{\alpha-2} t^{\alpha-1}}{D\Gamma(\alpha)} \right] \int_0^1 (1-s)^{\alpha-1} y(s) ds$$

$$= \frac{1}{\Gamma(\alpha)} \left(\int_0^t [t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}] y(s) ds + \int_t^1 t^{\alpha-1}(1-s)^{\alpha-1} y(s) ds \right)$$

$$+ \frac{\beta t^{\alpha-1}}{D\Gamma(\alpha)} \left(\int_0^\eta [\eta^{\alpha-1}(1-s)^{\alpha-1} - (\eta-s)^{\alpha-1}] y(s) ds + \int_\eta^1 \eta^{\alpha-1}(1-s)^{\alpha-1} y(s) ds \right)$$

$$+\frac{\gamma(\alpha-1)t^{\alpha-1}}{D\Gamma(\alpha)}\left(\int_0^{\eta}[\eta^{\alpha-2}(1-s)^{\alpha-1}-(\eta-s)^{\alpha-2}]y(s)ds+\int_{\eta}^{1}\eta^{\alpha-2}(1-s)^{\alpha-1}y(s)ds\right)$$

$$=\int_0^1 G(t,s)y(s)ds+\frac{\beta t^{\alpha-1}}{D}\int_0^1 G(\eta,s)y(s)ds+\frac{\gamma t^{\alpha-1}}{D}\int_0^1 H(\eta,s)y(s)ds.$$

□

The proof our main results is based upon an application of the following known results.

Lemma 5.2.2. *The function $G(t,s)$ defined by (5.2.4) satisfies*

1. $0\le G(t,s)\le h(s),$ *for* $t,s\in[0,1],$ *where*

$$h(s)=\frac{(1-s)^{\alpha-2}-(1-s)^{\alpha-1}}{\Gamma(\alpha)};$$

2. $G(t,s)\ge t^{\alpha-1}h(s),$ *for* $t,s\in[0,1].$

Proof.

It is easy to check that (i) holds. Next, we prove (ii) holds. If $t\ge s$, then

$$\begin{aligned}\frac{G(t,s)}{h(s)} &= \frac{t^{\alpha-1}(1-s)^{\alpha-2}-(t-s)^{\alpha-1}}{(1-s)^{\alpha-2}-(1-s)^{\alpha-1}}\\ &= \frac{t(t-ts)^{\alpha-2}-(t-s)(t-s)^{\alpha-2}}{(1-s)^{\alpha-2}-(1-s)^{\alpha-1}}\\ &\ge \frac{t(t-ts)^{\alpha-2}-(t-s)(t-ts)^{\alpha-2}}{(1-s)^{\alpha-2}-(1-s)^{\alpha-1}}\\ &= \frac{(t-ts)^{\alpha-2}}{(1-s)^{\alpha-2}}=t^{\alpha-2}\ge t^{\alpha-1}.\end{aligned}$$

□

Definition 5.2.1.

$u(t)$ is called a positive solution of (5.1.1) *if* $u\in \mathrm{C}[-\tau,1]\cap\mathrm{C}^2(0,1)$, $u(t)\ge 0$ *for* $t\in[-\tau,1]$ *and satisfies* (5.1.1).

Lemma 5.2.3. *Let* $y\in C^+[0,1]:=\{u\in C[0,1],u(t)\ge 0,t\in[0,1]\}$. *If* $0<\beta\eta^{\alpha-1}-\gamma(\alpha-1)\eta^{\alpha-2}<1$, *then the unique solution u of the boundary value problem (5.2.1), (5.2.2) satisfies*

$$\min_{\sigma\leqslant t\leqslant 1}u(t)\geqslant\sigma^{\alpha-1}\|u\|$$

where $\|u\|:=\sup\limits_{0\le t\le 1}|u(t)|$.

Proof. Let $y \in C^+[0,1]$, we have from (5.2.3)-(5.2.5) that $u(t) \geq 0$. By Lemma 5.2.2, we get

$$||u(t)|| = \int_0^1 G(t,s)y(s)ds + \frac{\beta t^{\alpha-1}}{D}\int_0^1 G(\eta,s)y(s)ds + \frac{\gamma t^{\alpha-1}}{D}\int_0^1 H(\eta,s)y(s)ds$$

$$\leq \int_0^1 h(s)y(s)ds + \frac{\beta}{D}\int_0^1 G(\eta,s)y(s)ds + \frac{\gamma}{D}\int_0^1 H(\eta,s)y(s)ds. \tag{5.2.9}$$

On the other hand, from Lemma 5.2.2 and (5.2.9), we obtain for each $t \in [0,1]$ that

$$u(t) = \int_0^1 G(t,s)y(s)ds + \frac{\beta t^{\alpha-1}}{D}\int_0^1 G(\eta,s)y(s)ds + \frac{\gamma t^{\alpha-1}}{D}\int_0^1 H(\eta,s)y(s)ds$$

$$\geq t^{\alpha-1}\int_0^1 h(s)y(s)ds + \frac{\beta t^{\alpha-1}}{D}\int_0^1 G(\eta,s)y(s)ds + \frac{\gamma t^{\alpha-1}}{D}\int_0^1 H(\eta,s)y(s)ds \tag{5.2.10}$$

$$\geq t^{\alpha-1}\left[\int_0^1 h(s)y(s)ds + \frac{\beta}{D}\int_0^1 G(\eta,s)y(s)ds + \frac{\gamma}{D}\int_0^1 H(\eta,s)y(s)ds\right] \tag{5.2.11}$$

$$\geq t^{\alpha-1}||u||. \tag{5.2.12}$$

Therefore $\min_{\sigma \leq t \leq 1} u(t) \geqslant \sigma^{\alpha-1}\|u\|$, $0 < \sigma < 1$. □

5.3 Main results

In the sequel we shall denote by $C_0([0,1])$ the space all continuous functions $u:[0,1] \to \mathbb{R}$ with $u(0) = 0$. The set $C_0([0,1])$ is a Banach space with sup-norm

$$\|u\| := \sup_{0 \leq t \leq 1} |u(t)|.$$

For each $\phi \in C_0^+(0)$ and $u \in C_0([0,1])$ we define

$$u_t(s) = u_t(s;\phi) := \begin{cases} u(t+s), & t+s > 0, \\ \phi(t+s), & t+s \leq 0, \ s \in [-\tau, 0]. \end{cases}$$

and observe that $u_t(s;\phi) \in C_\tau$.

Suppose that $u(t)$ is a solution of the boundary value problem (5.1.1), then it can be written as follow

$$u(t)=\int_0^1 G(t,s)f(s,u_t(.;\phi))ds+\frac{\beta t^{\alpha-1}}{D}\int_0^1 G(\eta,s)f(s,u_t(.;\phi))ds+\frac{\gamma t^{\alpha-1}}{D}\int_0^1 H(\eta,s)f(s,u_t(.;\phi))ds,\ \ t\in[0,1].$$

Define the cone $P\subset C_0([0,1])$ by

$$P:=\{u\in C_0^+([0,1]):\min_{\sigma\leqslant t\leqslant 1}u(t)\geqslant\sigma^{\alpha-1}\parallel u\parallel\}$$

For $u\in P$, we define the operator T as follows:

$$Tu(t)=\int_0^1 G(t,s)f(s,u_t(.;\phi))ds+\frac{\beta t^{\alpha-1}}{D}\int_0^1 G(\eta,s)f(s,u_t(.;\phi))ds+\frac{\gamma t^{\alpha-1}}{D}\int_0^1 H(\eta,s)f(s,u_t(.;\phi))ds. \tag{5.3.1}$$

It is clear that fixed points of T are solutions of the BVP (5.1.1).

We assume that $0<\tau<1$, $\phi\in C_0^+([0,1])$, and we make use of the following assumption:

H_1) $f:[0,1]\times C_0^+(0)\to[0,+\infty)$ is a continuous function.

Lemma 5.3.1. *Let* (H_1) *holds. Then* $T:P\to P$ *is completely continuous.*

Proof.

By (H_1), we have $Tu(t)\geq 0$, for $u\in P$ and $t\in[0,1]$. It follows form (5.3.1) and Lemma 5.2.2 that

$$||Tu(t)||=\max_{0\leq t\leq 1}|Tu(t)|\leq\int_0^1 h(s)f(s,u_t(.;\phi))ds+\frac{\beta}{D}\int_0^1 G(\eta,s)f(s,u_t(.;\phi))ds+\frac{\gamma}{D}\int_0^1 H(\eta,s)f(s,u_t(.;\phi))ds.$$

In view of Lemma 5.2.2, we have

$$\begin{aligned}Tu(t)&\geq t^{\alpha-1}\int_0^1 h(s)f(s,u_t(.;\phi))ds+\frac{\beta t^{\alpha-1}}{D}\int_0^1 G(\eta,s)f(s,u_t(.;\phi))ds+\frac{\gamma t^{\alpha-1}}{D}\int_0^1 H(\eta,s)f(s,u_t(.;\phi))ds.\\&\geq t^{\alpha-1}\left[\int_0^1 h(s)f(s,u_t(.;\phi))ds+\frac{\beta t^{\alpha-1}}{D}\int_0^1 G(\eta,s)f(s,u_t(.;\phi))ds+\frac{\gamma t^{\alpha-1}}{D}\int_0^1 H(\eta,s)f(s,u_t(.;\phi))ds\right]\end{aligned}$$

So,

$$\min_{\sigma\leq t\leq 1}Tu(t)\geq\sigma^{\alpha-1}||Tu||,$$

which shows that $T(P)\subset P$.

Using similar arguments of the proof of Theorem 2.1 in 6, we get T is completely continuous. □

Lemma 5.3.2. *If $0 < \sigma < 1$ and $u \in P$, then we have*

$$||u_t(.;\phi)||_{[-\tau,0]} \geq \sigma^{\alpha-1}||u||, \quad t \in [\sigma, 1].$$

Proof. From the definition of $u_t(s;\phi)$, for $t \geq \sigma$ we have

$$u_t(s;\phi) = u(t+s), \quad s \in [-\tau, 0].$$

Thus, we get for $u \in P$ that

$$||u_t(.;\phi)||_{[-\tau,0]} = \max_{s\in[-\tau,0]} u(t+s) \geq u(t) \geq t^{\alpha-1}||u|| \geq \sigma^{\alpha-1}||u||, \quad t \in [\sigma, 1].$$

□

Theorem 5.3.1. *Let $(H1)$ holds. Suppose that the following conditions are satisfied*

1. *there exist a continuous function $a : [0,1] \to [0,+\infty)$ and a continuous nondecreasing function $F : [0,+\infty) \to (0,+\infty)$ such that*

$$f(t,\psi) \leq a(t)F(||\psi||_{[-\tau,0]}), \quad (t,\psi) \in [0,1] \times C_\tau^+(0),$$

2. *there exists $r > ||\phi||_{[-\tau,0]}$, with*

$$\frac{r}{F(r)} > \int_0^1 h(s)a(s)ds + \frac{\beta}{D}\int_0^1 G(\eta,s)a(s)ds + \frac{\gamma}{D}\int_0^1 H(\eta,s)a(s)ds. \tag{5.3.2}$$

Then BVP (5.1.1) has at least one positive solution.

Proof. We shall apply Theorem 1.0.3 to prove that T has at least one positive solution.

Let $U := \{u \in P : ||u|| < r\}$, where r is as in 2. Assume that there exist $u \in P$ and $\lambda \in (0,1)$ such that $u = \lambda T u$, we claim that $||u|| \neq r$. In fact, if $||u|| = r$, we have

$$u(t) = \lambda \int_0^1 G(t,s)f(s,u_t(.;\phi))ds + \frac{\lambda\beta t^{\alpha-1}}{D}\int_0^1 G(\eta,s)f(s,u_t(.;\phi))ds + \frac{\lambda\gamma t^{\alpha-1}}{D}\int_0^1 H(\eta,s)f(s,u_t(.;\phi))ds$$

$$\leq \int_0^1 h(s)a(s)F(||u_s(.;\phi)||_{[-\tau,0]})ds + \frac{\beta}{D}\int_0^1 G(\eta,s)a(s)F(||u_s(.;\phi)||_{[-\tau,0]})ds + \frac{\gamma}{D}\int_0^1 H(\eta,s)a(s)F(||u_s(.;\phi)||_{[-\tau,0]})ds \tag{5.3.3}$$

By the definition of $u_s(.;\phi))$, we have

$$||u_s(.;\phi)||_{[-\tau,0]} \leq \max\{||u||, ||\phi||_{[-\tau,0]}\} = \max\{r, ||\phi||_{[-\tau,0]}\} = r. \tag{5.3.4}$$

Thus by (5.3.3), (5.3.4) and the nondecreasing of F, we get that

$$r = ||u|| \leq \int_0^1 h(s)a(s)F(r)ds + \frac{\beta}{D}\int_0^1 G(\eta,s)a(s)F(r)ds + \frac{\gamma}{D}\int_0^1 H(\eta,s)a(s)F(r)ds$$

Consequently,

$$\frac{r}{F(r)} \leq \int_0^1 h(s)a(s)ds + \frac{\beta}{D}\int_0^1 G(\eta,s)a(s)ds + \frac{\gamma}{D}\int_0^1 H(\eta,s)a(s)ds$$

which contradict (5.3.2).

Hence $u \notin \partial U$. By Theorem 1.0.3, T has a fixed point $u \in \overline{U}$. Therefore, BVP (5.1.1) has at least one positive solution. □

Theorem 5.3.2. *Let $(H1)$ holds. Suppose that the following conditions are satisfied*

1. *there exist constant $r_1 > ||\phi||_{[-\tau,0]}$, as well as a continuous function $p \in C([0,1])$ and nondecreasing continuous function $L : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$f(t,\psi) \leq p(t)L(||\psi||_{[-\tau,0]}), \quad (t,\psi) \in [0,1]\times C_0^+([-\tau,0]), \quad ||\psi||_{[-\tau,0]} \leq r_1. \tag{5.3.5}$$

2. *there exists functions $\omega : [0,1] \to [0,\tau]$, $c : [0,1] \to \mathbb{R}^+$ continuous, and $k : \mathbb{R}^+ \to \mathbb{R}^+$ nondecreasing such that*

$$F(t,\psi) \geq c(t)k(\psi(-\omega(t))), \quad (t,\psi) \in [0,1]\times C_0^+([-\tau,0]). \tag{5.3.6}$$

If r_1 satisfies

$$\frac{L(r_1)}{r_1}\left(\int_0^1 h(s)p(s)ds + \frac{\beta}{D}\int_0^1 G(\eta,s)p(s)ds + \frac{\gamma}{D}\int_0^1 H(\eta,s)p(s)ds\right) \leq 1, \tag{5.3.7}$$

and there exists a constant $r_2 > 0$ $(r_2 < r_1)$ satisfying

$$\frac{k(\sigma^{\alpha-1}r_2)}{r_2}\int_\sigma^1 \sigma^{\alpha-1}h(s)c(s)ds \geq 1. \tag{5.3.8}$$

Then BVP (5.1.1) has a positive solution.

Proof. If $u \in P$ with $||u|| = r_1$, then from (5.3.4), (5.3.5) and Lemma 5.2.2, we get for any $t \in [0,1]$ that

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)f(s,u_t(.;\phi))ds + \frac{\beta t^{\alpha-1}}{D}\int_0^1 G(\eta,s)f(s,u_t(.;\phi))ds + \frac{\gamma t^{\alpha-1}}{D}\int_0^1 H(\eta,s)f(s,u_t(.;\phi))ds \\ &\leq \int_0^1 h(s)f(s,u_t(.;\phi))ds + \frac{\beta}{D}\int_0^1 G(\eta,s)f(s,u_t(.;\phi))ds + \frac{\gamma}{D}\int_0^1 H(\eta,s)f(s,u_t(.;\phi))ds \\ &\leq \int_0^1 h(s)p(s)L(||u_s(.;\phi)||_{[-\tau,0]})ds + \frac{\beta}{D}\int_0^1 G(\eta,s)p(s)L(||u_s(.;\phi)||_{[-\tau,0]})ds \\ &\quad + \frac{\gamma}{D}\int_0^1 H(\eta,s)p(s)L(||u_s(.;\phi)||_{[-\tau,0]})ds \\ &\leq L(r_1)\left(\int_0^1 h(s)p(s)ds + \frac{\beta}{D}\int_0^1 G(\eta,s)p(s)ds + \frac{\gamma}{D}\int_0^1 H(\eta,s)p(s)ds\right) \\ &\leq r_1. \end{aligned}$$

Now, if we set $\Omega_1 := \{u \in C([0,1]) : \ ||u|| < r_1\}$, then, we have $||Tu|| \leq ||u||$ for $u \in P \cap \partial\Omega_1$.

For $u \in P$ with $||u|| = r_2$, we have from (5.3.4), (5.3.8) and Lemmas 5.2.2 and 5.3.2 that

$$\begin{aligned} Tu(t) &\geq \int_0^1 G(t,s)f(s,u_s(.;\phi))ds \\ &\geq \int_\sigma^1 G(t,s)f(s,u_s(.;\phi))ds \\ &\geq \int_\sigma^1 \sigma^{\alpha-1}h(s)f(s,u_s(.;\phi))ds \\ &\geq \int_\sigma^1 \sigma^{\alpha-1}h(s)c(s)k(u_s(-\omega(s);\phi))ds \\ &\geq k(\sigma^{\alpha-1}||u||)\int_a^b h(s)c(s)ds \\ &\geq k(\sigma^{\alpha-1}r_2)\int_a^b h(s)c(s)ds \\ &\geq r_2. \end{aligned}$$

Now if we set $\Omega_2 := \{u \in C([0,1]) : \ ||u|| < r_2\}$, then, we have $||Tu|| \geq ||u||$ for $u \in P \cap \partial\Omega_2$. □

Hence, by the second part of Theorem (1.0.2), T has a fixed point $u \in P \cap (\overline{\Omega_1} \setminus \Omega_2)$, and accordingly, u is a solution of BVP (5.1.1).

5.4 Example

Consider the boundary value problem of second order functional differential equations

$$\begin{aligned} &u^{2.2}(t) + \sqrt{t}\sin^2 t.u^4(t - \tfrac{1}{4}) = 0, \quad 0 < t < 1, \\ &u(t) = \phi(t) \ \ -\tfrac{1}{4} \leq t \ \leq 0, \\ &u(1) = \alpha u(\eta) + \beta u'(\eta) \end{aligned} \tag{5.4.1}$$

where $\eta = \frac{1}{2}$, $\beta = \frac{1}{2}$, $\gamma = \frac{1}{4}$ and $\phi \in C_0^+$ with $||\phi||_{[-\tau,0]} < \frac{3}{4}$.

Let $f(t,\psi) = \sqrt{t}\sin^2 t.\psi^4(-\frac{1}{4})$, $(t,\psi) \in [0,1] \times C_0^+$.

Obviously, $0 \le \beta \le \alpha(1-\eta) = \frac{1}{4}$, and

$$f(t,\psi) = \sqrt{t}\sin^2 t.\psi^4(-\frac{1}{4}) \le a(t)F(||\psi||_{[-\tau,0]}),$$

where $a(t) = \sqrt{t}$ and $F(x) = x^4$, which implies that conditions 1. of Theorem 5.3.1 holds.

By simple calculation, we obtain

$$\int_0^1 h(s)a(s)ds = \int_0^1 \max\{1, \alpha\eta + \beta\} \frac{s(1-s)}{1-\alpha\eta-\beta}\sqrt{s}ds = 0,228.$$

Choosing $r = \frac{3}{4}$, then $r > ||\phi||_{[-\tau,0]}$ and

$$\frac{r}{F(r)} = \frac{1}{r^3} = 2,370 > 0,228 = \int_0^1 h(s)a(s)ds,$$

that is conditions 2. of Theorem 5.3.1 holds. By Theorem 5.3.1, BVP 5.1.1 has at least one positive solution.

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Résumé

Cette thèse aborde les questions d'existence et de multiplicité de solutions positives pour des problèmes aux limites non locaux avec retard et d'ordre fractionnaire, considérés sur des intervalles bornés avec des non linéarités changeant de signe. La méthode utilisée est topologique : théorème de Krasnoselski (théorème de point fixe dans le cônes) et la théorie de l'indice de point fixe sur les cônes.

Mots clés

Théorème de point fixe de Krasnoselski, cône, solution positive, indice de point fixe, fonction de Green, problème aux limites non local, ordre fractionnaire, condition à multipoints.

Abstract

In this thesis, we study the questions of existence and multiplicity of positive solutions for a nonlocal boundary value problems with delay and fractional order, considered on bounded intervals, when the nonlinearities are allowed to change sign. We use topological methods: Krasnoselskii theorem (fixed point theorem in cones) and fixed point index theory in cones.

Key words

Krasnoselskii fixed point theorem, cone, positive solution, fixed point index, Green function, non-local boundary value problem, fractional order, multi-point condition.

ملخص

ناقشنا في هذه الأطروحة إشكالية وجود وتعدد الحلول الموجبة لصنف من المسائل الحدية ذات رتبة كسرية ٬ معرفة على مجلا ت محدودة٬ حين تكون الدالة الغير خطية اللاموقعية والمرتبطة بمعادلات تفاضلية متغيرة إلا شارة وذالك باستعمال طرق طوبولوجية مبرهنة النقطة الثابتة لكراسنوساسكي (مبرهنة النقطة الثابتة داخل مخروط) ونظرية دليل النقطة الثابتة.

الكلمات المفتاحية

مبرهنة النقطة الثابتة لكراسنوساسكي٬ مخروط٬ حل موجب٬ دليل النقطة الثابتة٬ دالة غرين٬ المسائل الحدية اللاموقعية٬ شرط تكاملي٬ شرط متعدد النقاط٬ ر تبة زوجية٬ ذات رتبة كسرية.