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Study of stabilization and global existence to the linear and nonlinear evolutions equations

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Contents

Introduction

Stabilization of evolution problems

Problems of global existence and stability in time of Partial Differential Equations made object, recently, of many work. In this thesis we were interested in study of the global existence and the stabilization of some evolution equations.

The purpose of stabilization is to attenuate the vibrations by feedback, thus it consists in guaranteeing the decrease of energy of the solutions to 0 in a more or less fast way by a mechanism of dissipation.

More precisely, the problem of stabilization consists in determining the asymptotic behaviour of the energy denoted by $E(t)$, to study its limits in order to determine if this limit is null or not and if this limit is null, to give an estimate of the decay rate of the energy to zero.

This problem has been studied by many authors for various systems. In our study, we obtain several type of stabilization

1) Strong stabilization: $E(t) \to 0$, as $t \to \infty$.

2) Logarithmic stabilization: $E(t) \leq c(\log(t))^{-\delta}, \forall t > 0, (c, \delta > 0)$.

3) polynomial stabilization: $E(t) \leq ct^{-\delta}, \forall t > 0, (c, \delta > 0)$

4) uniform stabilization: $E(t) \leq ce^{-\delta t}, \forall t > 0, (c, \delta > 0)$.

For wave equation with dissipation of the form $u'' - \Delta_x u + g(u') = 0$, stabilization problems have been investigated by many authors:
When $g : \mathbb{R} \to \mathbb{R}$ is continuous and increasing function such that $g(0) = 0$, global existence of solutions is known for all initial conditions (u_0, u_1) given in $H_0^1(\Omega) \times L^2(\Omega)$. This result is, for instance, a consequence of the general theory of nonlinear semi-groups of contractions generated by a maximal monotone operator (see Brézis [**10**]).

Moreover, if we impose on the control the condition $\forall \lambda \neq 0, g(\lambda) \neq 0$, then strong asymptotic stability of solutions occurs in $H_0^1(\Omega) \times L^2(\Omega)$, i.e.,

$$(u, u') \to (0, 0) \text{ strongly in } H_0^1(\Omega) \times L^2(\Omega),$$

without speed of convergence. These results follows, for instance, from the invariance principle of Lasalle (see for example C. M. Dafermos [**14**], A. Haraux [**17**], , F. Conrad, M. Pierre [**13**]). If the solution goes to 0 as time goes to ∞, how to get energy decay rates?

Dafermos has written in 1978 "Another advantage of this approach is that it is so simplistic that it requires only quite weak assumptions on the dissipative mechanism. The corresponding drawback is that the deduced information is also weak, never yielding, for example, decay rates of solutions."

Many authors have worked since then on energy decay rates. First results were obtained for linear stabilization, then for polynomial stabilization (see M. Nakao A. Haraux [**17**], E. Zuazua and V. Komornik [**19**]) and then extended to arbitrary growing feedbacks (close to 0). In the same time, geometrical aspects were considered.

By combining the multiplier method with the techniques of micro-local analysis, Lasiecka et al [**12**], [**20**] have investigated different dissipative systems of partial differential equations (with Dirichlet and Neumann boundary conditions) under general geometrical conditions with nonlinear feedback without any growth restrictions near the origin or at infinity. The computation of decay rates is reduced to solving an appropriate explicitly given ordinary differential equation of monotone type. More precisely, the following explicit decay estimate of the energy is obtained:

$$E(t) \leq h(\frac{t}{t_0} - 1), \quad \forall t \geq t_0, \tag{1}$$

where $t_0 > 0$ and h is the solution of the following differential equation:

$$h'(t) + q(h(t)) = 0, \quad \forall t \geq 0 \quad \text{and} \quad h(0) = E(0) \tag{2}$$

and the function q is determined entirely from the behavior at the origin of the nonlinear feedback by proving that E satisfies

$$(Id - q)^{-1}\Big(E((m+1)t_0)\Big) \leq E(mt_0), \quad \forall m \in \mathbb{N}.$$

In this thesis, the main objective is to give a global existence and stabilization results.
This work consists in two chapter.

Chapter 1: Well-posedeness and asymptotic behavior of Timoshenko beam system with dynamic boundary dissipative feedback of fractional derivative type

In this chapter, we consider the Timoshenko beam system with dynamic controls of fractional derivative type, that is,

$$(P) \qquad \begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) = 0 & \text{in } (0,L)\times(0,+\infty), \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x+\psi)(x,t) = 0 & \text{in } (0,L)\times(0,+\infty), \end{cases}$$

where $(x,t) \in (0,L) \times (0,+\infty)$. This system is subject to the boundary conditions

$$\begin{array}{ll} \varphi(0,t)=0, \quad \psi(0,t)=0, & \text{in } (0,+\infty), \\ m_1\varphi_{tt}(L,t)+K(\varphi_x+\psi)(L,t)=-\gamma_1\partial_t^{\alpha,\eta}\varphi(L,t) & \text{in } (0,+\infty), \\ m_2\psi_{tt}(L,t)+b\psi_x(L,t)=-\gamma_2\partial_t^{\alpha,\eta}\psi(L,t) & \text{in } (0,+\infty), \end{array}$$

We prove a global existence result using the semi-group theory based on maximum monotone method. Furthermore, we show that our system is not uniformly stable in general, since it is the case of the interval, more precisely we show that an infinite number of eigenvalues approach the imaginary axis. Also, we look for a polynomial decay rate for smooth initial data for our system by applying a frequency domain approach combining with a multiplier method.

Chapter 2:Energy decay of solutions to a Timoshenko beam system with dynamic boundary feedbacks of fractional derivative type

In this chapter, we consider the same system as above

$$(P) \qquad \begin{cases} \rho_1\varphi_{tt}(x,t)-K(\varphi_x+\psi)_x(x,t)=0 & \text{in } (0,L)\times(0,+\infty), \\ \rho_2\psi_{tt}(x,t)-b\psi_{xx}(x,t)+K(\varphi_x+\psi)(x,t)=0 & \text{in } (0,L)\times(0,+\infty), \end{cases}$$

where $(x,t) \in (0,L) \times (0,+\infty)$. This system is subject to the boundary conditions

$$\begin{array}{ll} \varphi(0,t)=0, \quad \psi(0,t)=0, & \text{in } (0,+\infty), \\ m_1\varphi_{tt}(L,t)+K(\varphi_x+\psi)(L,t)=-\gamma_1\partial_t^{\alpha,\eta}\varphi(L,t) & \text{in } (0,+\infty), \\ m_2\psi_{tt}(L,t)+b\psi_x(L,t)=-\gamma_2\partial_t^{\alpha,\eta}\psi(L,t) & \text{in } (0,+\infty), \end{array}$$

By an explicit representation of the resolvant associated to the operator semi-group, we prove different optimal energy decay estimate following the speeds of propagation of coupled system.

Chapter 1

Preliminaries

1.1 Sobolev spaces

We denote by Ω an open domain in $\mathbb{R}^n, n \geq 1$, with a smooth boundary $\Gamma = \partial\Omega$. In general, some regularity of Ω will be assumed. We will suppose that either

$$\Omega \text{ is Lipschitz,}$$

i.e., the boundary Γ is locally the graph of a Lipschitz function, or

$$\Omega \text{ is of class } \mathcal{C}^r, r \geq 1,$$

i.e., the boundary Γ is a manifold of dimension $n \geq 1$ of class $\mathcal{C}^r$. In both cases we assume that Ω is totally on one side of Γ. These definitions mean that locally the domain Ω is below the graph of some function ψ, the boundary Γ is represented by the graph of ψ and its regularity is determined by that of the function ψ. Moreover, it is necessary to note that a domain with a continuous boundary is never on both sides of its boundary at any point of this boundary and that a Lipschitz boundary has almost everywhere a unit normal vector ν.

We will also use the following multi-index notation for partial differential derivatives of a function:

$$\partial_i^k u = \frac{\partial^k u}{\partial x_i^k} \text{ for all } k \in \mathbb{N} \text{ and } i = 1, ..., n,$$
$$D^\alpha u = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n} u = \frac{\partial^{\alpha_1 + \dots + \alpha_n} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}},$$
$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n, |\alpha| = \alpha_1 + \dots + \alpha_n.$$

We denote by $\mathcal{C}(D)$ (respectively $\mathcal{C}^k(D), k \in \mathbb{N}$ or $k = +\infty$) the space of real continuous functions on D (respectively the space of k times continuously differentiable functions on D), where D plays the role of Ω or its closure $\overline{\Omega}$. The space of real $\mathcal{C}^\infty$ functions on Ω with a compact support in Ω is denoted by $\mathcal{C}_0^\infty(\Omega)$ or $\mathcal{D}(\Omega)$ as in the distributions theory of Schwartz.The distributions space on Ω is denoted by $\mathcal{D}'(\Omega)$, i.e., the space of continuous linear form over $\mathcal{D}(\Omega)$.

For $1 \leq p \leq \infty$, we call $L^p(\Omega)$ the space of measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p dx\right)^{1/p} < +\infty \quad \text{for} \quad p < +\infty$$

$$\|f\|_{L^\infty(\Omega)} = \sup_\Omega |f(x)| < +\infty \quad \text{for} \quad p = +\infty$$

The space $L^p(\Omega)$ equipped with the norm $f \longrightarrow \|f\|_{L^p}$ is a Banach space: it is reflexive and separable for $1 < p < \infty$ (its dual is $L^{\frac{p}{p-1}}(\Omega)$), separable but not reflexive for $p = 1$ (its dual is $L^\infty(\Omega)$), and not separable, not reflexive for $p = \infty$ (its dual contains strictly $L^1(\Omega)$). In particular the space $L^2(\Omega)$ is a Hilbert space equipped with the scalar product defined by

$$(f, g)_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx.$$

We denote by $L^p_{loc}(\Omega)$ the space of functions which are L^p on any bounded sub-domain of Ω.

Similar space can be defined on any open set other than Ω, in particular, on the cylinder set $\Omega \times]a, b[$ or on the set $\Gamma \times]a, b[$, where $a, b \in \mathbb{R}$ and $a < b$.

Let U be a Banach space, $1 < p < +\infty$ and $-\infty \leq a < b \leq +\infty$, then $L^p(a, b; U)$ is the space of L^p functions f from (a, b) into U which is a Banach space for the norm

$$\|f\|_{L^p(a,b;U)} = \left(\int_a^b \|f(x)\|_U^p \, dt\right)^{1/p} < +\infty \quad \text{for} \quad p < +\infty$$

and for the norm

$$\|f\|_{L^\infty(a,b;U)} = \sup_{t\in(a,b)} \|f(x)\|_U < +\infty \quad \text{for} \quad p = +\infty$$

Similarly, for a Banach space $U, k \in \mathbb{N}$ and $-\infty < a < b < +\infty$, we denote by $C([a, b]; U)$ (respectively $C^k([a, b]; U)$) the space of continuous functions (respectively the space of k times continuously differentiable functions) f from $[a, b]$ into U, which are Banach spaces, respectively, for the norms

$$\|f\|_{\mathcal{C}(a,b;U)} = \sup_{t\in(a,b)} \|f(x)\|_U, \quad \|f\|_{\mathcal{C}^k(a,b;U)} = \sum_{i=0}^{k} \left\| \frac{\partial^i f}{\partial t^i} \right\|_{\mathcal{C}(a,b;U)}$$

1.1.1 Definition of Sobolev Spaces

Now, we will introduce the Sobolev spaces: The Sobolev space $W^{k,p}(\Omega)$ is defined to be the subset of L^p such that function f and its weak derivatives up to some order k have a finite L^p norm, for given $p \geq 1$.

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega); D^\alpha f \in L^p(\Omega). \ \ \forall\alpha; |\alpha| \leq k\} \ ,$$

With this definition, the Sobolev spaces admit a natural norm,

$$f \longrightarrow \|f\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \le m} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p}, \text{ for } p < +\infty$$

and

$$f \longrightarrow \|f\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha| \le m} \|D^\alpha f\|_{L^\infty(\Omega)}, \text{ for } p = +\infty$$

Space $W^{k,p}(\Omega)$ equipped with the norm $\| \, . \, \|_{W^{k,p}}$ is a Banach space. Moreover is a reflexive space for $1 < p < \infty$ and a separable space for $1 \le p < \infty$. Sobolev spaces with $p = 2$ are especially important because of their connection with Fourier series and because they form a Hilbert space. A special notation has arisen to cover this case:

$$W^{k,2}(\Omega) = H^k(\Omega)$$

the H^k inner product is defined in terms of the L^2 inner product:

$$(f, g)_{H^k(\Omega)} = \sum_{|\alpha| \le k} (D^\alpha f, D^\alpha g)_{L^2(\Omega)} .$$

The space $H^m(\Omega)$ and $W^{k,p}(\Omega)$ contain $\mathcal{C}^\infty(\overline{\Omega})$ and $\mathcal{C}^m(\overline{\Omega})$. The closure of $\mathcal{D}(\Omega)$ for the $H^m(\Omega)$ norm (respectively $W^{m,p}(\Omega)$ norm) is denoted by $H^m_0(\Omega)$ (respectively $W^{k,p}_0(\Omega)$).

Now, we introduce a space of functions with values in a space X (a separable Hilbert space).

The space $L^2(a, b; X)$ is a Hilbert space for the inner product

$$(f, g)_{L^2(a,b;X)} = \int_a^b (f(t), g(t))_X \, dt$$

We note that $L^\infty(a, b; X) = (L^1(a, b; X))'$.

Now, we define the Sobolev spaces with values in a Hilbert space X

For $k \in \mathbb{N}$, $p \in [1, \infty]$, we set:

$$W^{k,p}(a, b; X) = \left\{ v \in L^p(a, b; X); \frac{\partial v}{\partial x_i} \in L^p(a, b; X). \ \ \forall i \le k \right\} ,$$

The Sobolev space $W^{k,p}(a, b; X)$ is a Banach space with the norm

$$\begin{aligned} \|f\|_{W^{k,p}(a,b;X)} &= \left(\sum_{i=0}^k \left\| \frac{\partial f}{\partial x_i} \right\|^p_{L^p(a,b;X)} \right)^{1/p}, \text{ for } p < +\infty \\ \|f\|_{W^{k,\infty}(a,b;X)} &= \sum_{i=0}^k \left\| \frac{\partial v}{\partial x_i} \right\|_{L^\infty(a,b;X)}, \qquad \text{for } p = +\infty \end{aligned}$$

The spaces $W^{k,2}(a, b; X)$ form a Hilbert space and it is noted $H^k(0, T; X)$. The $H^k(0, T; X)$ inner product is defined by:

$$(u, v)_{H^k(a,b;X)} = \sum_{i=0}^k \int_a^b \left(\frac{\partial u}{\partial x^i}, \frac{\partial v}{\partial x^i} \right)_X dt .$$

Theorem 1.1.1 *Let $1 \leq p \leq n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$$

where p^* is given by $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}$ (where $p = n, p^* = \infty$). Moreover there exists a constant $C = C(p,n)$ such that

$$\|u\|_{L^{p^*}} \leq C\|\nabla u\|_{L^p(\mathbb{R}^n)} \forall u \in W^{1,p}(\mathbb{R}^n).$$

Corollary 1.1.1 *Let $1 \leq p < n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \quad \forall q \in [p, p^*]$$

with continuous imbedding.

For the case $p = n$, we have

$$W^{1,n}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \quad \forall q \in [n, +\infty[$$

Theorem 1.1.2 *Let $p > n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$$

with continuous imbedding.

Corollary 1.1.2 *Let Ω a bounded domain in $\mathbb{R}^n$ of C^1 class with $\Gamma = \partial\Omega$ and $1 \leq p \leq \infty$. We have*

$$\begin{array}{ll} if & 1 \leq p < \infty, \text{ then } W^{1,p}(\Omega) \subset L^{p^*}(\Omega) \text{ where } \dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}. \\ if & p = n, \text{ then } W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [p, +\infty[. \\ if & p > n, \text{ then } W^{1,p}(\Omega) \subset L^\infty(\Omega) \end{array}$$

with continuous imbedding.
Moreover, if $p > n$, we have: $\forall u \in W^{1,p}(\Omega)$,

$$|u(x) - u(y)| \leq C|x - y|^\alpha \|u\|_{W^{1,p}(\Omega)} \ a.e\ x, y \in \Omega$$

with $\alpha = 1 - \dfrac{n}{p} > 0$ and C is a constant which depend on p, n and Ω. In particular $W^{1,p}(\Omega) \subset C(\overline{\Omega})$.

Corollary 1.1.3 *Let Ω a bounded domain in $\mathbb{R}^n$ of C^1 class with $\Gamma = \partial\Omega$ and $1 \leq p \leq \infty$. We have*

$$\begin{array}{ll} if & p < n, \text{ then } W^{1,p}(\Omega) \subset L^q(\Omega) \forall q \in [1, p^*[\text{ where } \dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}. \\ if & p = n, \text{ then } W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [p, +\infty[. \\ if & p > n, \text{ then } W^{1,p}(\Omega) \subset C(\overline{\Omega}) \end{array}$$

with compact imbedding.

Remark 1.1.1 *We remark in particular that*

$$W^{1,p}(\Omega) \subset L^q(\Omega)$$

with compact imbedding for $1 \leq p \leq \infty$ *and for* $p \leq q < p^*$.

Corollary 1.1.4

$$\begin{array}{ll} if & \dfrac{1}{p} - \dfrac{m}{n} > 0, \ then\ W^{m,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \ where\ \dfrac{1}{q} = \dfrac{1}{p} - \dfrac{m}{n}. \\ if & \dfrac{1}{p} - \dfrac{m}{n} = 0, \ then\ W^{m,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \forall q \in [p, +\infty[. \\ if & \dfrac{1}{p} - \dfrac{m}{n} < 0, \ then\ W^{m,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n) \end{array}$$

with continuous imbedding.

1.2 Weak convergence

Let $(E; \|.\|_E)$ a Banach space and E' its dual space, i.e., the Banach space of all continuous linear forms on E endowed with the norm $\|.\|'_E$ defined by

$$\|f\|_{E'} =: \sup_{x\neq 0} \frac{|\langle f, x\rangle|}{\|x\|}$$

; where $\langle f, x\rangle$; denotes the action of f on$x$, i.e.$\langle f, x\rangle := f(x)$. In the same way, we can define the dual space of E' that we denote by E''. (The Banach space E'' is also called the bi-dual space of E.) An element x of E can be seen as a continuous linear form on E' by setting $x(f) := \langle x, f\rangle$, which means that $E \subset E''$:

Definition 1.2.1 *The Banach space E is said to be reflexive if $E = E''$.*

Definition 1.2.2 *The Banach space E is said to be separable if there exists a countable subset D of E which is dense in E, i.e. $\overline{D} = E$.*

Theorem 1.2.1 *(Riesz). If $(H; \langle .,.\rangle)$ is a Hilbert space, $\langle .,.\rangle$ being a scalar product on H, then $H' = H$ in the following sense: to each $f \in H'$ there corresponds a unique $x \in H$ such that $f = \langle x, .\rangle$ and $\|f\|'_H = \|x\|_H$*

Remark : From this theorem we deduce that $H'' = H$. This means that a Hilbert space is reflexive.

Proposition 1.2.1 *If E is reflexive and if F is a closed vector subspace of E, then F is reflexive.*

Corollary 1.2.1 *The following two assertions are equivalent: (i) E is reflexive; (ii) E' is reflexive.*

1.2.1 Weak, weak star and strong convergence

Definition 1.2.3 *(Weak convergence in E). Let $x \in E$ and let $\{x_n\} \subset E$. We say that $\{x_n\}$ weakly converges to x in E, and we write $x_n \rightharpoonup x$ in E, if*

$$\langle f, x_n\rangle \to \langle f, x\rangle$$

for all $f \in E'$.

Definition 1.2.4 *(weak convergence in E'). Let $f \in E'$ and let $\{f_n\} \subset E'$. We say that $\{f_n\}$ weakly converges to f in E', and we write $f_n \rightharpoonup f$ in E', if*

$$\langle f_n, x\rangle \to \langle f, x\rangle$$

for all $x \in E''$.

Definition 1.2.5 *(weak star convergence). Let $f \in E'$ and let $\{f_n\} \subset E'$. We say that $\{f_n\}$ weakly star converges to f in E', and we write $f_n \rightharpoonup *f$ in E' if;*

$$\langle f_n, x\rangle \to \langle f, x\rangle$$

for all $x \in E$.

Remark As $E \subset E''$ we have $f_n \rightharpoonup f$ in E' imply $f_n \rightharpoonup *f$ in E'. When E is reflexive, the last definitions are the same, i.e, weak convergence in E' and weak star convergence coincide.

Definition 1.2.6 *(strong convergence). Let $x \in E$(resp. $f \in E'$) and let $\{x_n\} \subset E$ (resp $\{f_n\} \subset E'$). We say that $\{x_n\}$ (resp. $\{f_n\}$) strongly converges to x (resp. f), and we write $x_n \to x$ in E (resp. $f_n \to f$ in E'), if*

$$\lim_n \|x_n - x\|_E = 0; (resp.\ \lim_n \|f_n - f\|'_E = 0)$$

Proposition 1.2.2 *Let $x \in E$, let $\{x_n\} \subset E$, let $f \in E'$ and let $\{f_n\} \subset E'$.*

i. If $x_n \to x$ in E then $x_n \rightharpoonup x$ in E.

ii. If $x_n \rightharpoonup x$ in E then $\{x_n\}$ is bounded.

iii. If $x_n \rightharpoonup x$ in E then $\liminf_{n\to\infty} \|x_n\|_E \geq \|x\|_E$

iv. If $f_n \to f$ in E' then $f_n \rightharpoonup f$ inE' (and so $f_n \overset{*}{\rightharpoonup} f$ in E').

v. If $f_n \rightharpoonup f$ in E' then $\{f_n\}$ is bounded.

vi. If $f_n \rightharpoonup f$ in E' then then $\liminf_{n\to\infty} \|f_n\|'_E \geq \|f\|'_E$

Proposition 1.2.3 *(finite dimension). If $\dim E < \infty$ then strong, weak and weak star convergence are equivalent.*

1.2.2 Weak and weak star compactness

In finite dimension, i.e, $\dim E < \infty$, we have Bolzano-Weierstrass's theorem (which is a strong compactness theorem).

Theorem 1.2.2 *(Bolzano-Weierstrass). If $\dim E < \infty$ and if $\{x_n\} \subset E$) is bounded, then there exist $\in E$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ strongly converges to x.*

The following two theorems are generalizations, in infinite dimension, of Bolzano- Weierstrass's theorem.

Theorem 1.2.3 *(weak star compactness, Banach-Alaoglu-Bourbaki). Assume that E is separable and consider $\{f_n\} \subset E'$) . If $\{x_n\}$ is bounded, then there exist $f \in E'$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}\}$ weakly star converges to f in E'.*

Theorem 1.2.4 *(weak compactness, Kakutani-Eberlein). Assume that E is reflexive and consider $\{x_n\} \subset E$). If $\{x_n\}$ is bounded, then there exist $x \in E$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ weakly converges to x in E.*

Weak, weak star convergence and compactness in $L^p(\Omega)$.

Definition 1.2.7 *(weak convergence in $L^p(\Omega)$ with $1 \leq p < \infty$). Let Ω an open subset of $\mathbb{R}^n$ .We say that the sequence $\{f_n\}$ of $L^p(\Omega)$ weakly converges to $f \in L^p(\Omega)$, if*

$$\lim_n \int_\Omega f_n(x)g(x)dx = \int_\Omega f(x)g(x)dx \text{ for all } g \in L^q;\ (\frac{1}{p} + \frac{1}{q} = 1)$$

Definition 1.2.8 *(weak star convergence in $L^\infty(\Omega)$). We say that the sequence $\{f_n\} \subset L^\infty(\Omega)$ weakly star converges to $f \in L^\infty(\Omega)$, if*

$$\lim_n \int_\Omega f_n(x)g(x)dx = \int_\Omega f(x)g(x)dx \text{ for all } g \in L^1(\Omega)$$

Theorem 1.2.5 *(weak compactness in $L^p(\Omega)$) with $1 < p < \infty$. Given $\{f_n\} \subset L^p(\Omega)$, if $\{f_n\}$ is bounded, then there exist $f \in L^p(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_n \rightharpoonup f$ in $L^p(\Omega)$.*

Theorem 1.2.6 *(weak star compactness in $L^\infty(\Omega)$.
Given $\{f_n\} \subset L^\infty(\Omega)$, if $\{f_n\}$ is bounded, then there exist $f \in L^\infty(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_n \overset{*}{\rightharpoonup} f$ in $L^\infty(\Omega)$.*

Generalities. In what follows, Ω is a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary and $1 \leq p \leq \infty$.

Weak and weak star convergence in Sobolev spaces

For $1 \leq p \leq \infty$, $W^{1;p}(\Omega)$ is a Banach space. Denote the space of all restrictions to Ω of C^1-differentiable functions from $\mathbb{R}^N$ to $\mathbb{R}$ with compact support in R^N by $C^1(\overline{\Omega})$.

Theorem 1.2.7 *for every $1 \leq p \leq \infty$ $C^1(\overline{\Omega}) \subset W^{1;p}(\Omega) \subset L^p(\Omega)$, and, for $1 < p < \infty$, $C^1(\overline{\Omega})$ is dense in $W^{1;p}(\Omega)$.*

Definition 1.2.9 *(weak convergence in $W^{1;p}(\Omega)$ with $1 \leq p < \infty$).)
We say the $\{f_n\} \subset W^{1;p}(\Omega)$ weakly converges to $f \in W^{1;p}(\Omega)$, and we write $f_n \rightharpoonup f$ in $W^{1;p}(\Omega)$, if $f_n \rightharpoonup f$ in $L^p(\Omega)$ and $\nabla f_n \rightharpoonup \nabla f$ in $L^p(\Omega;\ \mathbb{R}^N)$*

Definition 1.2.10 *(weak convergence in $W^{1;\infty}(\Omega)$
. We say the $\{f_n\} \subset W^{1;\infty}(\Omega)$ weakly star converges to $f \in W^{1;\infty}(\Omega)$, and we write $f_n \overset{*}{\rightharpoonup} f$ in $W^{1;\infty}(\Omega)$, if $f_n \overset{*}{\rightharpoonup} f$ in $L^p(\Omega)$ and $\nabla f_n \overset{*}{\rightharpoonup} \nabla f$ in $L^\infty(\Omega;\ \mathbb{R}^N)$*

Theorem 1.2.8 *(Rellich). Let $1 \leq p \leq \infty$, $\{f_n\} \subset W^{1;p}(\Omega)$ and $f \in W^{1;p}(\Omega)$; if $f_n \rightharpoonup f$ in $W^{1;p}(\Omega)$ when $1 \leq p < \infty$ (resp.$f_n \overset{*}{\rightharpoonup} f$ in $W^{1;\infty}(\Omega)$) when $p = \infty$) then $f_n \to f$ in $L^p(\Omega)$), which means that for every $1 \leq p \leq \infty$, the weak convergence in $W^{1;p}(\Omega)$ imply the strong convergence in $L^p(\Omega)$.*

Theorem 1.2.9 *Let $1 < p \leq \infty$ and let $\{f_n\} \subset W^{1;p}(\Omega)$. If $\{f_n\}$ is bounded, then there exist $f \in W^{1;p}(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \rightharpoonup f$ in $W^{1;p}(\Omega)$ when $1 < p < \infty$ (resp. $f_{n_k} \overset{*}{\rightharpoonup} f$ in $W^{1;\infty}(\Omega)$)*

As a consequence of this theorem we have

Corollary 1.2.2 *Let $1 < p \leq \infty$and let $\{f_n\} \subset W^{1;p}(\Omega)$. If $\{f_n\}$ is bounded, then there exist $f \in W^{1;p}(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \to f$ in $L^p(\Omega)$ and $\nabla f_{n_k} \rightharpoonup \nabla f$ in $L^p(\Omega)$ when $1 < p < \infty$ (resp.$\nabla f_{n_k} \overset{*}{\rightharpoonup} \nabla f$ in $L^\infty(\Omega)$)*

Theorem 1.2.10 . *If$N < p \leq \infty$ and if $\{f_n\} \subset W^{1;p}(\Omega)$ is bounded, then there exist $f \in W^{1;p}(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}\}$ converges uniformly to f, and $\nabla f_{n_k} \rightharpoonup \nabla f$ in $W^{1;p}(\Omega)$ when $N < p < \infty$ (resp. $\nabla f_{n_k} \overset{*}{\rightharpoonup} \nabla f$ in $W^{1;\infty}$)*

1.2.3 Gronwall lemma

Lemma 1.2.1 *Let $T > 0$, $g \in L^1(0,T)$, $g \geq 0$ a.e and c_1, c_2 are positives constants.Let $\varphi \in L^1(0,T)$ $\varphi \geq 0$ a.e such that $g\varphi \in L^1(0,T)$ and*

$$\varphi(t) \leq c_1 + c_2 \int_0^t g(s)\varphi(s)ds \quad a.e \ in \ (0,T).$$

then, we have

$$\varphi(t) \leq c_1 exp\ (c_2 \int_0^t g(s)ds) \quad a.e \ in \ (0,T).$$

1.2.4 Aubin -Lions lemma

The Aubin Lions lemma is a result in the theory of Sobolev spaces of Banach space-valued functions. More precisely, it is a compactness criterion that is very useful in the study of nonlinear evolutionary partial differential equations. The result is named after the French mathematicians Thierry Aubin and Jacques-Louis Lions. We complete the preliminaries by the useful inequalities of Gagliardo-Nirenberg and Sobolev-Poincaré.

Lemma 1.2.2 *LetX_0,X and X_1 be three Banach spaces with $X_0 \subseteq X \subseteq X_1$. Assume that$X_0$ is compactly embedded in X and that X is continuously embedded in X_1; assume also that X_0 and X_1 are reflexive spaces. For $1 < p, q < +\infty$, let*

$$W = \{u \in L^p([0,T]; X_0)/ \ \ \dot{u} \in L^q([0,T]; X_1)\}$$

Then the embedding of W into $L^p([0,T]; X)$ is also compact.

Lemma 1.2.3 (Gagliardo-Nirenberg) *Let $1 \leq r < q \leq +\infty$ and $p \leq q$. Then, the inequality*

$$\|u\|_{W^{m,q}} \leq C\|u\|^{\theta}_{W^{m,p}}\|u\|^{1-\theta}_r \quad for \quad u \in W^{m,p} \bigcap L^r$$

holds with some $C > 0$ *and*

$$\theta = \left(\frac{k}{n} + \frac{1}{r} - \frac{1}{q}\right)\left(\frac{m}{n} + \frac{1}{r} - \frac{1}{p}\right)^{-1}$$

provided that $0 < \theta \leq 1$ *(we assume* $0 < \theta < 1$ *if* $q = +\infty$*).*

Lemma 1.2.4 (Sobolev-Poincaré inequality) *Let* q *be a number with* $2 \leq q < +\infty$ $(n = 1, 2)$ *or* $2 \leq q \leq 2n/(n-2)$ $(n \geq 3)$, *then there is a constant* $c_* = c(\Omega, q)$ *such that*

$$\|u\|_q \leq c_* \|\nabla u\|_2 \quad \textit{for} \quad u \in H_0^1(\Omega).$$

1.3 Semigroup and spectral analysis theories

As the analysis done in this P.H.D thesis local on the semigroup and spectral analysis theories, we recall, in this chapter, some basic definitions and theorems which will be used in the following chapters.

1.3.1 Bounded and Unbounded linear operators

In this chapter we give some well known results abound bounded and undounded operators. We are not trying to give a complete development, but rather review the basic definitions and theorems, mostly without proof. Let $(E, \|.\|_E)$ and $(F, \|.\|_F)$ be two Banach spaces over $\mathbb{C}$, and H will always denote a Hilbert space equipped with the scalar product $< ., . >_H$ and the corresponding norm $\|.\|_H$. A linear operator $T : E \longrightarrow F$ is a transformation which maps linearly E in F, that is

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v), \quad \forall u, v \in E and \alpha, \beta \in \mathbb{C}.$$

Definition 1.3.1 *An operator* $T : E \longrightarrow F$ *is said to be bounded if there exists* $C \geq 0$ *such that*

$$\|Tu\|_F \leq C\|u\|_E \quad \forall u \in E.$$

The set of all bounded linear operators from E *into* F *is denoted by* $\mathcal{L}(E, F)$. *Moreover, the set of all bounded linear operators from* E *into* E *is denoted by* $\mathcal{L}(E)$.

Definition 1.3.2 *A bounded operator* $T \in \mathcal{L}(E, F)$ *is said to be compact if for each sequence* $(x_n)_{n\in\mathbb{N}} \in E$ *with* $\|x_n\|_E = 1$ *for each* $n \in \mathbb{N}$, *the sequence* $(Tx_n)_{n\in\mathbb{N}} \in E$ *has a subsequence which converges in* F. *The set of all compact operators from* E *into* F *is denoted by* $\mathcal{K}(E, F)$. *For simplicity one writes* $\mathcal{K}(E, E) = \mathcal{K}(E)$.

Definition 1.3.3 *Let* $T \in \mathcal{L}(E, F)$, *we define*

- *Range of T by*

$$\mathcal{R}(T) = \{Tu : \quad u \in E\} \subset F.$$

- *Kernel of T by*

$$ker(T) = \{u \in E : \quad Tu = 0\} \subset E.$$

Theorem 1.3.1 *(Fredholm alternative) if $T \in \mathcal{K}(E)$, then*

- *$ker(I - T)$ is finite dimension, (I is the identity operator on E).*
- *$\mathcal{R}(I - T)$ is closed.*
- *$ker(I - T) = 0 \Leftrightarrow \mathcal{R}(I - T) = E$.*

Definition 1.3.4 *Let $T : D(T) \subset E \longrightarrow F$ be an unbounded linear operator.*

- *The range of T is defined by*

$$\mathcal{R}(T) = \{Tu : \quad u \in D(T)\} \subset F.$$

- *The Kernel of T is defined by*

$$ker(T) = \{u \in D(T) : \quad Tu = 0\} \subset E.$$

- *The graph of T is defined by*

$$\mathcal{G}(T) = \{(u, Tu) : \quad u \in D(T)\} \subset E \times F.$$

Definition 1.3.5 *A map T is said to be closed if $\mathcal{G}(T)$ is closed in $E \times F$. The closedness of an unbounded linear operator T can be characterize as following if $u_n \in D(T)$ such that $u_n \longrightarrow u$ in E and $Tu_n \longrightarrow v$ in F, then $u \in D(T)$ and $Tu = v$.*

Definition 1.3.6 *Let $T : D(T) \subset E \longrightarrow F$ be a closed unbounded linear operator.*

- *The resolvent set of T is defined by*

$$\rho(T) = \{\lambda \in \mathbb{C} : \quad \lambda I - T \quad isbijectivefrom \quad D(T) \quad onto \quad F\}.$$

- *The resolvent of T is defined by*

$$R(\lambda, T) = \{(\lambda I - T)^{-1} : \quad \lambda \in \rho(T)\}.$$

- *The spectrum set of T is the complement of the resolvent set in $\mathbb{C}$, denoted by*

$$\sigma(T) = \mathbb{C}/\rho(T).$$

Definition 1.3.7 *Let $T : D(T) \subset E \longrightarrow F$ be a closed unbounded linear operator. we can split the spectrum $\sigma(T)$ of T into three disjoint sets, given by*

- *The ponctuel spectrum of T is define by*

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : \quad ker(\lambda I - T) \neq 0\}$$

in this case λ is called an eigenvalue of T.

- *The continuous spectrum of T is define by*

$$\sigma_c(T) = \{\lambda \in \mathbb{C} : \quad ker(\lambda I - T) = 0, \bar{\mathcal{R}(\lambda I - T)} = F \quad and (\lambda I - T)^{-1} is \quad not \quad bounded\}.$$

- *The residual spectrum of T is define by*

$$\sigma_r(T) = \{\lambda \in \mathbb{C} : \quad ker(\lambda I - T) = 0 and \mathcal{R}(\lambda I - T) \quad is \quad not \quad dense \quad in \quad F\}.$$

Definition 1.3.8 *Let $T : D(T) \subset E \longrightarrow F$ be a closed unbounded linear operator and let λ be an eigevalue of A. non-zero element $e \in E$ is called a generalized eigenvector of T associated with the eigenvalue value λ, if there exists $n \in \mathbb{N}^*$ such that*

$$(\lambda I - T)^n e = 0 \quad and \quad (\lambda I - T)^{n-1} e \neq 0.$$

if $n = 1$, then e is called an eigenvector.

Definition 1.3.9 *Let $T : D(T) \subset E \longrightarrow F$ be a closed unbounded linear operator. We say that T has compact resolvent, if there exist $\lambda_0 \in \rho(T)$ such that $(\lambda_0 I - T)^{-1}$ is compact.*

Theorem 1.3.2 *Let $(T, D(T))$ be a closed unbounded linear operator on H then the space $(D(T), \|.\|_{D(T)})$ where $\|u\|_{D(T)} = \|Tu\|_H + \|u\|_H \quad \forall u \in D(T)$ is banach space .*

Theorem 1.3.3 *Let $(T, D(T))$ be a closed unbounded linear operator on H then, $\rho(T)$ is an open set of $\mathbb{C}$.*

1.3.2 Semigroups, Existence and uniqueness of solution

In this section, we start by introducing some basic concepts concerning the semigroups. The vast majority of the evolution equations can be reduced to the form

$$(1.1) \qquad \begin{cases} U_t = AU, \quad t > 0, \\ U(0) = U_0 \end{cases}$$

where A is the infinitesimal generator of a C_0-semigroup $S(t)$ over a Hilbert space H. Lets start by basic definitions and theorems. Let $(X, \|.\|_X)$ be a Banach space, and H be a Hilbert space equipped with the inner product $< ., . >_H$ and the induced norm $\|.\|_H$.

Definition 1.3.10 *A family $S(t)_{t\geq 0}$ of bounded linear operators in X is called a strong continous semigroup (in short, a C_0-semigroup) if*

i) $S(0) = I_d$.

ii) $S(s+t) = S(s)S(t), \ \forall t \geq 0 \ \forall s \geq 0.$

iii) *For each $u \in H$, $S(t)u$ is continous in t on $[0, +\infty[$.*

Sometimes we also denote $S(t)$ by e^{At}.

Definition 1.3.11 *For a semigroup $S(t)_{t\geq 0}$, we define an linear operator A with domain $D(A)$ consisting of points u such that the limit*

$$Au = \lim_{t\to 0^+} \frac{S(t)u - u}{t} \quad \forall u \in D(A)$$

exists. Then A is called the infinitesimal generator of the semigroup $S(t)_{t\geq 0}$.

Proposition 1.3.1 *Let $S(t)_{t\geq 0}$ be a C_0-semigroup in X. Then there exist a constant $M \geq 1$ and $\omega \geq 0$ such that*

$$\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}. \ \ \forall t \geq 0$$

If $\omega = 0$ then the corresponding semigroup is uniformly bounded. Moreover, if $M = 1$ then $S(t)_{t\geq 0}$ is said to be a C_0-semigroup of contractions.

Definition 1.3.12 *An unbounded linear operator $(A, D(A))$ on H, is said to be dissipative if*

$$\Re < Au, u > \leq 0, \forall u \in D(A).$$

Definition 1.3.13 *An unbounded linear operator $(A, D(A))$ on X, is said to be m-dissipative if*

- *A is a dissipative operator.*
- *$\exists \lambda_O$ such that $\mathcal{R}(\lambda_0 I - A) = X$*

Theorem 1.3.4 *Let A be a m-dissipative operator, then*

- $\mathcal{R}(\lambda_0 I - A) = X, \quad \forall \lambda > 0$
- $]0, \infty[\subseteq \rho(A).$

Theorem 1.3.5 *(Hille-Yosida)An unbounded linear operator $(A, D(A))$ on X, is the infinitesimal generator of a C_0-semigroup of contractions $S(t)_{t\geq 0}$ if and only if*

- *A is closed and $D(\bar{A}) = X$.*

- *The resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$, and for all $\lambda > 0$,*

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \leq \lambda^{-1}$$

Theorem 1.3.6 *(Lumer-Phillips) Let $(A, D(A))$ be an unbounded linear operator on X, with dense domain $D(A)$ in X. A is the infinitesimal generator of a C_0-semigroup of contractions if and only if it is a m-dissipative operator.*

Theorem 1.3.7 *Let $(A, D(A))$ be an unbounded linear operator on X. If A is dissipative with $\mathcal{R}(I - A) = X$, and X is reflexive then $D(\bar{A}) = X$.*

Corollary 1.3.1 *Let $(A, D(A))$ be an unbounded linear operator on H. A is the infinitesimal generator of a C_0-semigroup of contractions if and only if A is a m-dissipative operator.*

Theorem 1.3.8 *Let A be a linear operator with dense domain $D(A)$ in a Hilbert space H. If A is dissipative and $0 \in \rho(A)$ then A is the infinitesimal generator of a C_0-semigroup of contractions on H.*

Theorem 1.3.9 *(Hille-Yosida) Let $(A, D(A))$ be an unbounded linear operator on H. Assume that A is the infinitesimal generator of a C_0-semigroup of contractions $S(t)_{t\geq 0}$.*

1. *For $U_0 \in D(A)$, the problem (1.1) admits a unique strong solution*

$$U(t) = S(t)U_0 \in C^1([0, \infty[; H) \cap C([0, \infty[; D(A))$$

2. *For $U_0 \in D(A)$, the problem (1.1) admits a unique weak solution*

$$U(t) \in C^0([0, \infty[; H).$$

1.3.3 Stability of semigroup

In this section we start by itroducing some definion about strong, exponential and polynomial stability of a C_0-semigroup. Then we collect some results about the stability of C_0-semigroup. Let $(X, \|.\|_X$ be a Banach space, and H be a Hilbert space equipped with the inner product $< ., . >_H$ and the induced norm $\|.\|_H$.

Definition 1.3.14 *Assume that A is the generator of a strongly continuous semigroup of contractions $S(t)_{t\geq 0}$ on X. We say that the C_0-semigroup $S(t)_{t\geq 0}$ is*

- *Strongly stable if*

$$\lim_{t\to+\infty} \|S(t)u\|_X = 0, \quad \forall u \in X.$$

- *Uniformly stable if*

$$\lim_{t\to+\infty} \|S(t)\|_{\mathcal{L}(X)} = 0$$

- *Exponentially stable if there exist two positive constants M and ϵ such that*

$$\|S(t)u\|_X \leq Me^{-\epsilon t}\|u\|_X, \quad \forall t > 0, \quad \forall u \in X.$$

- *Polynomially stable if there exist two positive constants C and α such that*

$$\|S(t)u\|_X \leq Ct^{-\alpha}\|u\|_X, \quad \forall t > 0, \quad \forall u \in X.$$

Proposition 1.3.2 *Assume that A is the generator of a strongly continuous semigroup of contractions $S(t)_{t\geq 0}$ on X. The following statements are equivalent*

- *$S(t)_{t\geq 0}$ is uniformly stable.*
- *$S(t)_{t\geq 0}$ is exponentially stable.*

First, we look for the necessary conditions of strong stability of a C_0-semigroup. The result was obtained by Arendt and Batty.

Theorem 1.3.10 *(Arendt and Batty) Assume that A is the generator of a strongly continuous semigroup of contractions $S(t)_{t\geq 0}$ on a reflexive Banach space X. If*

- *A has no pure imaginary eigenvalues.*
- *$\sigma(A) \cap i\ \mathbb{R}$ is countable.*

Then $S(t)$ is strongly stable.

Remark 1.3.1 *If the resolvent $(I - T)^{-1}$ of T is compact, then $\sigma(T) = \sigma_p(T)$. Thus, the state of Theorem 1.3.10 lessens to $\sigma_p(T) \cap i\ \mathbb{R} = \emptyset$ Next, when the C_0-semigroup is strongly stabe, we look for the necessary and suffient conditions of exponential stability of a C_0-semigroup. In fact, exponential stability results are obtained using different methods like : multipliers method, frequency domain approach, Riesz basis approach, Fourier analysis or a combination of them . In this thesis we will review only two methods. The first method is a frequency domain approach method was obtained by Huang- Pruss.*

Theorem 1.3.11 *(Huang-Pruss) Assume that A is the generator of a strongly continuous semigroup of contractions $S(t)_{t\geq 0}$ on H. $S(t)$ is uniformly stable if and only if*

- *$i\ \mathbb{R} \subset \rho(A)$.*
- *$\sup_{\beta\in\mathbb{R}} \|(i\beta I - A)^{-1}\|_{\mathcal{L}(H)} < +\infty$.*

The second one, is a classical method based on the spectrum analysis of the operator A.

Definition 1.3.15 *Let $(A, D(A))$ be an unbounded linear operator on H. Assume that A is the infinitesimal generator of a C_0-semigroup of contractions $S(t)_{t\geq 0}$.*

- *The growth bound of A is define by*

$$\omega_0(A) = inf\{\omega \in \mathbb{R} : \exists N_\omega \in \mathbb{R} such that \forall t \geq 0 we have \|S(t)\| \leq N_\omega e^{\omega t}\}.$$

- *The spectral bound of A is define by*

$$s(A) = sup\{\Re(\lambda) : \quad \lambda \in \sigma(A)\}.$$

Proposition 1.3.3 *Let $(A, D(A))$ be an unbounded linear operator on H. Assume that A is the infinitesimal generator of a C_0-semigroup of contractions $S(t)_{t\geq 0}$. Then $S(t)_{t\geq 0}$ is uniformly exponentially stable if and only if its growth bound $\omega_0(A) < 0$.*

Proposition 1.3.4 *Let $(A, D(A))$ be an unbounded linear operator on H. Assume that A is the infinitesimal generator of a C_0-semigroup of contractions $S(t)_{t\geq 0}$. Then, we have*

$$s(A) \leq \omega_0(A).$$

Corollary 1.3.2 *Let $(A, D(A))$ be an unbounded linear operator on H. Assume that $s(A) = 0$, then $S(t)_{t\geq 0}$ is not uniformly exponentially stable.*

In the case when the C_0-semigroup is not exponentialy stable we look for a polynomial one. In general, polynomial stability results also are obtained using different methods like : multipliers method, frequency domain approach, Riesz basis approach, Fourier analysis or a combination of them . In this thesis we will review only one method. The first method is a frequency domain approach method was obtained by Batty, A.Borichev and Y.Tomilov, Z. Liu and B. Rao.

Theorem 1.3.12 *(Batty , A.Borichev and Y.Tomilov, Z. Liu and B. Rao.) Assume that A is the generator of a strongly continuous semigroup of contractions $S(t)_{t\geq 0}$ on H. If $i\,\mathbb{R} \subset \rho(A)$, then for a fixed $l > 0$ the following conditions are equivalent*

1. $\lim_{|\lambda|\to+\infty} sup \frac{1}{\lambda^l} \|(\lambda I - A)^{-1}\|_{\mathcal{L}(H)} < +\infty$.
2. $\|S(t)U_0\|_H \leq \frac{C}{t^{l-1}} \|U_0\|_{D(A)}$ $\forall t > 0$, $U_0 \in D(A)$, *for some* $C > 0$.

1.3.4 Fractional Derivative Control

In this part, we introduce the necessary elements for the good understanding of this manuscript. It includes a brief reminder of the basic elements of the theory of fractional computation as well as some examples of applications of this theory in this scientific field. The concept of fractional computation is a generalization of ordinary derivation and integration to an arbitrary order. Derivatives of non-integer order are now widely applied in many domains, for example in economics, electronics, mechanics, biology, probability and viscoelasticity. A particular interest for fractional derivation is related to the mechanical modeling of gums and rubbers. In short, all kinds of materials that preserve the memory of previous deformations in particular viscoelastic. Indeed, the fractional derivation is introduced naturally. There exists a many mathematical definitions of fractional order integration and derivation. These definitions do not always lead to identical results but are equivalent for a wide large of functions. We introduce the fractional integration operator as well as the two most definitions of fractional derivatives, used, namely that Riemann-Liouville and Caputo, by giving the most important properties of the notions. Fractional systems appear in different fields of research. However, the progressive interest in their applications in the basic and applied sciences. It can be noted that for most of the domains presented (automatic, physics, mechanics of continuous media). The fractional operators are used to take into account memory effects. We can mention the works that reroute various applications of fractional computation. In physics, on of the most remarkable applications of fractional computation in physics was in the context of classical mechanics. Riewe, has shown that the Lagrangien of the motion of temporal derivatives of fractional orders leads to an equation of motion with friction forces and nonconservative are essential in macroscopic variational processing such as friction. This result are remarkable because friction forces and non conservative forces are essential in the usual macroscopic variational processing and therefore in the most advances methods classical mechanics. Riewe, has generalized the usual Lagrangian variation which depends on the fractional derivatives in order to deal with the usual non-conservative forces. On the another hand, several approaches have been developed to generalize the principle of least action and the Euler-Lagrange equation to the case of fractional derivative. The definition of the fractional order derivation is based on that of a fractional order integration, a fractional order derivation takes on a global character in contrast to an integral derivation. It turns out that the derivative of a fractional order of a function requires the knowledge of f(t) over the entire interval]a, b[, where in the whole case only the local knowledge of f around t is necessary. This property allows to interpret fractional order systems as long memory systems, the whole systems being then interpretable as systems with short memory. Now, we give the definition of the fractional derivatives in the sense of Riemann-Liouville as well as some essential properties.

Definition 1.3.16 *The fractional integral of order $\alpha > 0$, in sense Riemann-Liouville is given*

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad t > a.$$

Definition 1.3.17 *The fractional integral of order $\alpha > 0$, in sense Riemann-Liouville of a function f defined on the interval $[a,b]$ is given by*

$$D_{RL,a}^\alpha f(t) = D^n I_a^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{t^n} \int_a^t (t-s)^{n-\alpha-1} f(s) ds, \quad n = [\alpha]+1, \quad t > a.$$

In particular, if $\alpha = 0$, then

$$D_{RL,a}^0 f(t) = I_a^0 f(t) = f(t)$$

if $\alpha = n \in \mathrm{I\!N}$, *then*

$$D^{n}_{RL,a} f(t) = f^{(n)}(t)$$

Moreover, if $0 < \alpha < 1$, *then* $n = 1$, *then*

$$D^{\alpha}_{RL,a} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{t} \int_a^t (t-s)^{-\alpha} f(s) ds, \qquad t > a.$$

Example:

Let $\alpha > 0$,$\gamma > -1$ and $f(t) = (t-a)^{\gamma}$, then

$$I^{\alpha}_{a} f(t) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} (t-a)^{\gamma+\alpha},$$

$$D^{\alpha}_{RL,a} f(t) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha+1)} (t-a)^{\gamma-\alpha},$$

In particular, if $\gamma = 0$ and $\alpha > 0$, then $D^{\alpha}_{RL,a}(C) = C\frac{(t-a)^{-\alpha}}{\Gamma(1\alpha)}$. The derivatives of Riemann-Liouville have certain disadvantages when attempting to model real world phenomena. The problems studied require a definition of the fractional derivatives allowing the use of the physically interpretable initial conditions including $y(0)$, $y'(0)$, etc. There shortcomings led to an alternative definition of fractional derivatives that satisfies these demands in the last sixties. It was introduced by Caputo. In fact, Caputo and Minardi used this definition in their work on viscoelasticity. Now, we give the definition of the fractional derivatives in the sense of Caputo as well as some essential properties.

1.3.5 Geometric Condition

In this section, we present two different types on the geometric conditions.

Definition 1.3.18 *We say that the multiplier control condition* **MGC** *holds if there exist* $x_0 \in \mathrm{I\!R}^d$ *and a positive constant* $m_0 > 0$ *such that*

$$m.\nu \leq 0 \quad on \quad \Gamma_0 \quad and \quad m.\nu \geq m_0 \quad on \quad \Gamma_1,$$

whith $m(x) = x - x_0$, *for all* $x \in \mathrm{I\!R}^d$

We recall the Geometric Control condition **GCC** introduced by Bardos, Lebeau and Rauch [13] :

Definition 1.3.19 *We say that* Γ *satisfies the geometric condition named* **GCC**, *if every ray of geometrical optics, starting at any point* $x \in \Omega$ *at time* $t = 0$, *hits* Γ_1 *in finite time* T.

Remark 1.3.2 *In [13], Bardos et al. proved that* (H) *holds if* Γ *is smooth (of class* C^∞*),* $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ *and the* **GCC** *condition. For less regular domains, namely of class* C^2*,* (H) *holds if the vector field assumptions described in [33] (see* $(i),(ii),(iii)$ *of Theorem 1 in [33]) hold. Moreover, in Theorem 1.2 of [34] the authors prove that* (H) *holds for smooth domains under weaker geometric conditions than in [33] (without* (ii) *of Theorem 1). Finally, it is easy to see that the multiplier control condition* **MCC** *implies that the vector field assumptions described in [33] are satisfied and therefore the condition* (H) *holds if* **MCC** *holds.*

1.3.6 Appendix

Theorem 1.3.13 (see [26]) *Let* μ *be the function defined by*

$$\mu(\xi) = |\xi|^{\frac{2\alpha-d}{2}}, \quad \xi \in \mathbb{R}^d \quad \text{and} \quad 0 < \alpha < 1. \tag{1.2}$$

The relation between the "imput" U *and the "output"* O *of the following system*

$$\partial_t \omega(\xi,t) + (|\xi|^2 + \eta)\omega(\xi,t) - U(t)\mu(\xi) = 0, \quad \xi \in \mathbb{R}^d, t \in \mathbb{R}^+ \quad \text{and} \quad \eta \geq 0, \tag{1.3}$$

$$\omega(\xi,0) = 0, \tag{1.4}$$

$$O(t) = \frac{2\sin(\alpha\pi)\Gamma(\frac{d}{2}+1)}{d\pi^{\frac{d}{2}+1}} \int_{\mathbb{R}^d} \mu(\xi)\omega(\xi,t)d\xi, \tag{1.5}$$

is given by

$$O = I^{1-\alpha,\eta}U = D^{\alpha,\eta}U. \tag{1.6}$$

Lemma 1.3.1 *For all* $\lambda \in \mathbb{R}$ *and* $\eta > 0$*, we have*

$$A_1 = \int_{\mathbb{R}^d} \frac{|\xi|^{2\alpha-d}}{|\lambda| + \eta + |\xi|^2} d\xi = c(|\lambda| + \eta)^{\alpha-1} \quad \text{and} \quad A_2 = \Big(\int_{\mathbb{R}^d} \frac{|\xi|^{2\alpha-d}}{(|\lambda| + \eta + |\xi|^2)^2} d\xi \Big)^{\frac{1}{2}} = \tilde{c}(|\lambda| + \eta)^{\frac{\alpha}{2}-1}$$

where c*,* $\tilde{c}$ *are two positive constants given by*

$$c = \frac{d\pi^{\frac{d}{2}+1}}{2\Gamma(\frac{d}{2}+1)\sin(\alpha\pi)} \text{ and } \tilde{c} = \Big(\frac{d\pi^{\frac{d}{2}}}{2\Gamma(\frac{d}{2}+1)} \int_1^{+\infty} \frac{(y-1)^\alpha}{y^2} dy \Big)^{\frac{1}{2}}.$$

Lemma 1.3.2 *if* $\lambda \in D = \{\lambda \in \mathbb{C}, \lambda + \eta > 0\} \cup \{\lambda \in \mathbb{C}, \mathcal{F}(\lambda) \neq 0\}$*, then*

$$\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2} d\xi = \frac{\pi}{\sin(\alpha\pi)} (\lambda + \eta)^{\alpha-1}.$$

Chapter 2

WELL-POSEDENESS AND ASYMPTOTIC BEHAVIOR OF TIMOSHENKO BEAM SYSTEM WITH DYNAMIC BOUNDARY DISSIPATIVE FEEDBACK OF FRACTIONAL DERIVATIVE TYPE

2.1 Introduction

In this chapter we investigate the existence and decay properties of solutions for the initial boundary value problem of the linear Timoshenko beam system of the type

$$(P) \qquad \begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) = 0 & \text{in } (0,L)\times(0,+\infty), \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x+\psi)(x,t) = 0 & \text{in } (0,L)\times(0,+\infty), \end{cases}$$

where $(x,t) \in (0,L)\times(0,+\infty)$. This system is subject to the boundary conditions

$$\begin{array}{ll} \varphi(0,t) = 0, \quad \psi(0,t) = 0, & \text{in } (0,+\infty), \\ m_1\varphi_{tt}(L,t) + K(\varphi_x+\psi)(L,t) = -\gamma_1\partial_t^{\alpha,\eta}\varphi(L,t) & \text{in } (0,+\infty), \\ m_2\psi_{tt}(L,t) + b\psi_x(L,t) = -\gamma_2\partial_t^{\alpha,\eta}\psi(L,t) & \text{in } (0,+\infty), \end{array}$$

where $\gamma_i > 0, i = 1,2$. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order $\alpha, 0 < \alpha < 1$, with respect to the time variable. It is defined as follows

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\, ds, \ \eta \geq 0.$$

In other words, we investigate two dissipative effects at the boundary. The system is finally completed with initial conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad \psi(x,0) = \psi_0(x), \\ \psi_t(x,0) = \psi_1(x), \quad x \in (0,L), \end{cases}$$

where the initial data $(\varphi_0, \varphi_1, \psi_0, \psi_1)$ belong to a suitable function space.

A simple model describing the transverse vibration of a beam, which was developed in [**39**], is given by a system of coupled hyperbolic equations of the form

$$\begin{cases} \rho u_{tt}(x,t) = (K(u_x - \phi))_x & \text{in } (0,L) \times (0,+\infty), \\ \tilde{\rho}\phi_{tt}(x,t) = (EI\psi_x)_x + K(u_x - \phi) & \text{in } (0,L) \times (0,+\infty), \end{cases}$$

where t denotes the time variable, x is the space variable along the beam of length L, in its equilibrium configuration, u is the transverse displacement of the beam and ϕ is the rotation angle of the filament of the beam. The coefficients $\rho, \tilde{\rho}, E, I$ and K are respectively the density (the mass per unit length), the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus.

There are a number of publications concerning the stabilization of Timoshenko system with different kinds of damping (see [**1**], [**18**], [**30**], [**31**], [**33**] and [**36**]). Raposo et al. [**36**] proved the exponential decay of the solution for the following linear system of Timoshenko-type beam equations with linear frictional dissipative terms:

$$\begin{aligned} &\rho_1\varphi_{tt} - K(\varphi_x + \psi)_x + \mu_1\varphi_t = 0 \\ &\rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \widetilde{\mu_1}\psi_t = 0. \end{aligned}$$

Messaoudi and Mustafa [**30**] (see also [**33**]) considered the stabilization for the following Timoshenko system with nonlinear internal feedbacks:

$$\begin{aligned} &\rho_1\varphi_{tt} - K(\varphi_x + \psi)_x + g_1(\psi_t) = 0 \\ &\rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + g_2(\psi_t) = 0. \end{aligned}$$

Recently, Park and Kang [**33**] considered the stabilization of the Timoshenko system with weakly nonlinear internal feedbacks.

Kim and Renardy [**18**] considered (P) together with two boundary controls of the form

$$\begin{aligned} &K(\varphi_x + \psi)(L,t) = -\gamma_1\partial_t\varphi(L,t) && \text{in } (0,+\infty) \\ &b\psi_x(L,t) = -\gamma_2\partial_t\psi(L,t) && \text{in } (0,+\infty) \end{aligned}$$

and used the multiplier techniques to establish an exponential decay result for the natural energy of (P). In addition, a polynomial decay result was established by Yan [**43**] when considering two boundary frictional damping terms with polynomial growth near the origin. We also recall the result by G. Q. Xu, D. X. Feng [**42**], where the authors proved a result similar to the one in [**18**] by adopting the spectral analysis approach.

L. Zietsman, N.F.J. van Rensburg and A.J. van der Merwe [**44**] considered a one-dimensional hybrid structure consisting of a Timoshenko beam system (P) with a tip load attached to one free end. The beam is clamped at $x = 0$ while the tip load is fixed to the end $x = L$ in such a manner that the center of mass of the load is coincident with its point of attachment to the beam. We assume interaction between the beam and the load. Thus the forces and moments within the vibrating beam are transmitted to the tip load which moves in accordance with Newton's law. Dissipation is introduced into the coupled model by applying feedback boundary moment and force controls on the shear and displacement velocities ψ_t and φ_t at $x = L$. Hence the system (P) is subject to the following boundary conditions

$$\begin{array}{ll} m\varphi_{tt}(L,t) + K(\varphi_x + \psi)(L,t) = -\gamma_1 \partial_t \varphi(L,t) & \text{in } (0,+\infty), \\ I_m \psi_{tt}(L,t) + b\psi_x(L,t) = -\gamma_2 \partial_t \psi(L,t) & \text{in } (0,+\infty), \end{array}$$

where the coefficients m and I_m denote respectively the mass and the rotary inertia of the tip load. It is established an efficiency and accuracy of the finite element method for calculating the eigenvalues and eigenmodes.

In **[32]** J. E. Muñoz Rivera and Andrés I. Ávila, studied the same problem as in **[44]**. They proved that the decay of the energy is not exponential, but polynomial. They used the Weyls Theorem for lack of exponential stability and Borichev-Tomilov Theorem for establishing decay rate $E(t) \leq c/t, t \geq 0$.

Very recently in **[28]** D. Mercier and V. Régnier studied a more general problem than **[32]** (with constants k_1 and k_3 instead of K and b in boundary conditions). They proved that the decay of the energy is not exponential, but polynomial that is $E(t) \leq c/t, t \geq 0$. They used a semigroup theory with a frequency domain approach and Riesz basis property of the generalized eigenvector of the system.

The boundary feedback under the consideration here are of fractional type and are described by the fractional derivatives

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\, ds$$

The order of our derivatives is between 0 and 1. Very little attention has been paid to this type of feedback. In addition to being nonlocal, fractional derivatives involve singular and nonintegrable kernels ($t^{-\alpha}, 0 < \alpha < 1$). Therefore, the employment of mathematical analysis tools, such as stability analysis is very difficult.

It is well known (see **[27]**) that, as ∂_t, the fractional derivative ∂_t^α forces the system to become dissipative and the solution to converge the equilibrium state. Therefore, when applied on the boundary, we can consider them as controllers which help to suppress or attenuate the undesirable vibrations.

Nowadays, fractional calculus is a well-established theory with strong mathematical bases and its application has become a new interest in research areas such as electrical circuits, chemical processes, signal processing, bioengineering, viscoelasticity and obviously control systems (see **[34]**).

Control of fractional order type is not only important from the theoretical point of view but also for applications. It is the generalization of the classical integer order control theory, which could lead to a more adequate modeling and more robust control performance. Indeed, it has been observed by experiments that many concepts cannot be described in Newtonian terms. For example, in viscoelasticity, due to the nature of the material microstructure, both elastic solid and viscous fluid like response qualities are involved. More precisely, the stress at each point and at each instant does not depend only on the present value of the strain but also on the entire temporal prehistory of the motion from 0 up to time t. Viscoelastic response occurs in a variety of materials, such as soils, concrete, rubber, cartilage, biological tissue, glasses, and polymers (see **[4]**, **[5]**, **[6]** and **[24]**).

Our purpose in this chapter is to give a global solvability in Sobolev spaces and energy decay estimates of the strong solutions to the problem (P) for damping of fractional derivative type. To obtain global solutions to the problem (P), we use the argument combining the semigroup theory ([**10**]) with the energy estimate method. To prove decay estimates, we use a frequency domain approach and a Theorem of A. Borichev and Y. Tomilov.

2.2 Augmented model

This section is concerned with the reformulation of the model (P) into an augmented system. For that, we need the following claims.

Theorem 2.2.1 (see [26]) *Let μ be the function:*

$$\mu(\xi) = |\xi|^{(2\alpha-1)/2}, \quad -\infty < \xi < +\infty, \; 0 < \alpha < 1. \tag{2.1}$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t \phi(\xi,t) + \xi^2 \phi(\xi,t) + \eta\phi(\xi,t) - U(t)\mu(\xi) = 0, \quad -\infty < \xi < +\infty, \eta \geq 0, t > 0, \tag{2.2}$$

$$\phi(\xi, 0) = 0, \tag{2.3}$$

$$O(t) = (\pi)^{-1} \sin(\alpha\pi) \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,t)\, d\xi \tag{2.4}$$

is given by

$$O = I^{1-\alpha,\eta} U. \tag{2.5}$$

where

$$[I^{\alpha,\eta} f](t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{-\eta(t-\tau)} f(\tau)\, d\tau$$

Lemma 2.2.1 *If $\lambda \in D = \{\lambda \in \mathbb{C} : Re\lambda + \eta > 0\} \cup \{\lambda \in \mathbb{C} : Im\lambda \neq 0\}$ then*

$$F_\mu(\lambda) = \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2}\, d\xi = \frac{\pi}{\sin \alpha\pi} (\lambda + \eta)^{\alpha - 1}.$$

Proof Let us set

$$f_\lambda(\xi) = \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2}.$$

We have

$$\left| \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2} \right| \leq \begin{cases} \dfrac{\mu^2(\xi)}{Re\lambda + \eta + \xi^2} & \text{or} \\ \dfrac{\mu^2(\xi)}{|Im\lambda| + \eta + \xi^2} & \end{cases}$$

Then the function f_λ is integrable. Moreover

$$\left|\frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\right| \leq \begin{cases} \dfrac{\mu^2(\xi)}{\eta_0+\eta+\xi^2} \text{ for all } Re\lambda \geq \eta_0 > -\eta \\ \dfrac{\mu^2(\xi)}{\tilde{\eta}_0+\xi^2} \text{for all } |Im\lambda| \geq \tilde{\eta}_0 > 0 \end{cases}$$

From Theorem 1.16.1 in **[41]**, the function

$$F_\mu : D \to \mathbb{C} \text{ is holomorphic.}$$

For a real number $\lambda > -\eta$, we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\, d\xi = \int_{-\infty}^{+\infty} \frac{|\xi|^{2\alpha-1}}{\lambda+\eta+\xi^2}\, d\xi = \int_0^{+\infty} \frac{x^{\alpha-1}}{\lambda+\eta+x}\, dx \text{ (with } \xi^2 = x)\\
&= (\lambda+\eta)^{\alpha-1}\int_1^{+\infty} y^{-1}(y-1)^{\alpha-1}\, dy \text{ (with } y = x/(\lambda+\eta)+1)\\
&= (\lambda+\eta)^{\alpha-1}\int_0^1 z^{-\alpha}(1-z)^{\alpha-1}\, dz \text{ (with } z = 1/y)\\
&= (\lambda+\eta)^{\alpha-1}B(1-\alpha,\alpha) = (\lambda+\eta)^{\alpha-1}\Gamma(1-\alpha)\Gamma(\alpha) = (\lambda+\eta)^{\alpha-1}\frac{\pi}{\sin\pi\alpha}.
\end{aligned}$$

Both holomorphic functions F_μ and $\lambda \mapsto (\lambda+\eta)^{\alpha-1}\dfrac{\pi}{\sin\pi\alpha}$ coincide on the half line $]-\eta,\infty[$, hence on D following the principle of isolated zeroes.

We are now in a position to reformulate system (P). Indeed, by using Theorem 2.2.1, system (P) may be recast into the augmented model:

$$(P')\quad \begin{cases} \rho_1\varphi_{tt} - K(\varphi_x+\psi)_x = 0\\ \partial_t\phi_1(\xi,t) + (\xi^2+\eta)\phi_1(\xi,t) - \varphi_t(L,t)\mu(\xi) = 0\\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x+\psi) = 0\\ \partial_t\phi_2(\xi,t) + (\xi^2+\eta)\phi_2(\xi,t) - \psi_t(L,t)\mu(\xi) = 0\\ \varphi(0,t) = 0, \quad \psi(0,t) = 0,\\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x),\\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x),\\ m_1\varphi_{tt}(L,t) + K(\varphi_x+\psi)(L,t) = -\zeta_1\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi_1(\xi,t)\, d\xi, \quad \zeta_1 = \gamma_1(\pi)^{-1}\sin(\alpha\pi)\\ m_2\psi_{tt}(L,t) + b\psi_x(L,t) = -\zeta_2\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi_2(\xi,t)\, d\xi, \qquad \zeta_2 = \gamma_2(\pi)^{-1}\sin(\alpha\pi). \end{cases}$$

We define the energy associated to the solution of the problem (P') by the following formula:

$$\begin{aligned}
E(t) &= \frac{\rho_1}{2}\|\varphi_t\|_2^2 + \frac{\rho_2}{2}\|\psi_t\|_2^2 + \frac{b}{2}\|\psi_x\|_2^2 + \frac{K}{2}\|\varphi_x+\psi\|_2^2\\
&+\frac{m_1}{2}|\varphi_t(L,t)|^2 + \frac{m_2}{2}|\psi_t(L,t)|^2 + (\pi)^{-1}\sin(\alpha\pi)\sum_{i=1}^{2}\frac{\gamma_i}{2}\int_{-\infty}^{+\infty}(\phi_i(\xi,t))^2\, d\xi.
\end{aligned} \tag{2.6}$$

Lemma 2.2.2 *Let $(\varphi, \phi_1, \psi, \phi_2)$ be a regular solution of the problem (P'). Then, the energy functional defined by (2.6) satisfies*

$$E'(t) = -(\pi)^{-1}\sin(\alpha\pi)\sum_{i=1}^{2}\gamma_i\int_{-\infty}^{+\infty}(\xi^2+\eta)(\phi_i(\xi,t))^2\,d\xi \leq 0. \tag{2.7}$$

Remark 2.2.1 *For an initial datum in $D(\mathcal{A})$ (see Theorem 2.3.1 below), we known that $(\varphi, \phi_1, \psi, \phi_2)$ is of class C^1 in time, thus we can derive the energy $E(t)$.*

Proof of Lemma 2.2.2. Multiplying the first equation in (P') by φ_t and the third equation by ψ_t, integrating over $(0, L)$ and using integration by parts, we get

$$\frac{1}{2}\rho_1\frac{d}{dt}\|\varphi_t\|_2^2 - K\int_0^L(\varphi_x+\psi)_x\varphi_t dx = 0,$$

$$\frac{1}{2}\rho_2\frac{d}{dt}\|\psi_t\|_2^2 - b\int_0^L \psi_{xx}\psi_t\,dx + K\int_0^L(\varphi_x+\psi)\psi_t dx = 0.$$

Then

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\rho_1}{2}\|\varphi_t\|_2^2+\frac{\rho_2}{2}\|\psi_t\|_2^2+\frac{b}{2}\|\psi_x\|_2^2+\frac{K}{2}\|\varphi_x+\psi\|_2^2+\frac{m_1}{2}|\varphi_t(L,t)|^2+\frac{m_2}{2}|\psi_t(L,t)|^2\right)\\&+\zeta_1\varphi_t(L,t)\textstyle\int_{-\infty}^{+\infty}\mu(\xi)\phi_1(\xi,t)\,d\xi+\zeta_2\psi_t(L,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi_2(\xi,t)\,d\xi = 0.\end{aligned} \tag{2.8}$$

Multiplying the second equation in (P') by $\gamma_1(\pi)^{-1}\sin(\alpha\pi)\phi_1$, the fourth equation in (P') by $\gamma_2(\pi)^{-1}\sin(\alpha\pi)\phi_2$ and integrating over $(-\infty, +\infty)$, to obtain:

$$\begin{cases}\frac{\zeta_1}{2}\frac{d}{dt}\|\phi_1\|_2^2+\zeta_1\int_{-\infty}^{+\infty}(\xi^2+\eta)(\phi_1(\xi,t))^2\,d\xi-\zeta_1\varphi_t(L,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi_1(\xi,t)\,d\xi = 0,\\ \frac{\zeta_2}{2}\frac{d}{dt}\|\phi_2\|_2^2+\zeta_2\int_{-\infty}^{+\infty}(\xi^2+\eta)(\phi_2(\xi,t))^2\,d\xi-\zeta_2\psi_t(L,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi_2(\xi,t)\,d\xi = 0.\end{cases} \tag{2.9}$$

From (2.6), (2.8) and (2.9) we get

$$E'(t) = -\sum_{i=1}^{2}\zeta_i\int_{-\infty}^{+\infty}(\xi^2+\eta)(\phi_i(\xi,t))^2\,d\xi.$$

This completes the proof of the lemma.

2.3 Global existence

In this section we will give well-posedness results for problem (P') using semigroup theory. Let us introduce the semigroup representation of the Timoshenko system (P'). We consider the following condition of the right end contour of wave

$$\varphi_t(L,t) = \theta(t),\;\; \psi_t(L,t) = \vartheta(t),\;\; \text{for } t > 0 \tag{2.10}$$

were θ and ϑ solve the system

$$\begin{aligned} &m_1\theta_t(t) + k(\varphi_x+\psi)(L,t) + \zeta_1 \int_{-\infty}^{+\infty} \mu(\xi)\phi_1(\xi,t)\,d\xi = 0, \\ &m_2\vartheta_t(t) + b\psi_x(L,t) + \zeta_2 \int_{-\infty}^{+\infty} \mu(\xi)\phi_2(\xi,t)\,d\xi = 0. \end{aligned} \tag{2.11}$$

Let $U = (\varphi, \varphi_t, \phi_1, \theta, \psi, \psi_t, \phi_2, \vartheta)^T$ and rewrite (P') as

$$\begin{cases} U' = \mathcal{A}U, \\ U(0) = (\varphi_0, \varphi_1, \phi_{01}, \theta_0, \psi_0, \psi_1, \phi_{02}, \vartheta_0), \end{cases} \tag{2.12}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} \varphi \\ u \\ \phi_1 \\ \theta \\ \psi \\ v \\ \phi_2 \\ \vartheta \end{pmatrix} = \begin{pmatrix} u \\ \dfrac{K}{\rho_1}(\varphi_x+\psi)_x \\ -(\xi^2+\eta)\phi_1 + u(L)\mu(\xi) \\ -\frac{K}{m_1}(\varphi_x+\psi)(L) - \frac{\zeta_1}{m_1}\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi_1(\xi)\,d\xi \\ v \\ \dfrac{b}{\rho_2}\psi_{xx} - \dfrac{K}{\rho_2}(\varphi_x+\psi) \\ -(\xi^2+\eta)\phi_2 + v(L)\mu(\xi) \\ -\frac{b}{m_2}\psi_x(L) - \frac{\zeta_2}{m_2}\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi_2(\xi)\,d\xi \end{pmatrix} \tag{2.13}$$

with domain

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (\varphi, u, \phi_1, \theta, \psi, v, \phi_2, \vartheta)^T \text{ in } \mathcal{H} : \varphi, \psi \in H^2(0,L) \cap H^1_*(0,L), u, v \in H^1_*(0,L), \\ \theta, \vartheta \in \mathbb{C}, -(\xi^2+\eta)\phi_1 + u(L)\mu(\xi), -(\xi^2+\eta)\phi_2 + v(L)\mu(\xi) \in L^2(-\infty,+\infty), \\ u(L) = \theta, v(L) = \vartheta, \\ |\xi|\phi_1, |\xi|\phi_2 \in L^2(-\infty,+\infty) \end{array} \right\}, \tag{2.14}$$

where the energy space $\mathcal{H}$ is defined as

$$\mathcal{H} = (H^1_*(0,L) \times L^2(0,L) \times L^2(-\infty,+\infty) \times \mathbb{C})^2$$

where

$$H^1_*(0,L) = \{\varphi \in H^1(0,L) : \varphi(0) = 0\}.$$

For $U = (\varphi, u, \phi_1, \theta, \psi, v, \phi_2, \vartheta)^T, \overline{U} = (\overline{\varphi}, \overline{u}, \overline{\phi}_1, \overline{\theta}, \overline{\psi}, \overline{v}, \overline{\phi}_2, \overline{\vartheta})^T$, we define the following inner product in $\mathcal{H}$

$$\begin{aligned} \langle U, \overline{U}\rangle_{\mathcal{H}} = &\int_0^L \Big(\rho_1 u\overline{u} + \rho_2 v\overline{v} + b\psi_x\overline{\psi}_x + K(\varphi_x+\psi)(\overline{\varphi}_x + \overline{\psi})\,dx \\ &+ \sum_{i=1}^{2}\zeta_i \int_{-\infty}^{+\infty} \phi_i\overline{\phi}_i\,d\xi + m_1\theta\overline{\theta} + m_2\vartheta\overline{\vartheta}. \end{aligned}$$

We show that the operator $\mathcal{A}$ generates a C_0- semigroup in $\mathcal{H}$. In this step, we prove that the operator $\mathcal{A}$ is dissipative. Let $U=(\varphi,u,\phi_1,\theta,\psi,v,\phi_2,\vartheta)^T$. Using (2.12), (2.7) and the fact that

$$E(t)=\frac{1}{2}\|U\|^2_{\mathcal{H}}, \tag{2.15}$$

we get

$$\langle \mathcal{A}U,U\rangle_{\mathcal{H}}=-\sum_{i=1}^{2}\zeta_i\int_{-\infty}^{+\infty}(\xi^2+\eta)(\phi_i(\xi))^2\,d\xi. \tag{2.16}$$

Consequently, the operator $\mathcal{A}$ is dissipative. Now, we will prove that the operator $\lambda I-\mathcal{A}$ is surjective for $\lambda>0$. For this purpose, let $(f_1,f_2,f_3,f_4,f_5,f_6,f_7,f_8)^T\in\mathcal{H}$, we seek $U=(\varphi,u,\phi_1,\theta,\psi,v,\phi_2,\vartheta)^T\in D(\mathcal{A})$ solution of the following system of equations

$$\begin{cases}\lambda\varphi-u=f_1,\\ \lambda u-\dfrac{K}{\rho_1}(\varphi_x+\psi)_x=f_2,\\ \lambda\phi_1+(\xi^2+\eta)\phi_1-u(L)\mu(\xi)=f_3,\\ \lambda\theta+\frac{K}{m_1}(\varphi_x+\psi)(L)+\frac{\zeta_1}{m_1}\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi_1(\xi)\,d\xi=f_4.\\ \lambda\psi-v=f_5,\\ \lambda v-\dfrac{b}{\rho_2}\psi_{xx}+\dfrac{K}{\rho_2}(\varphi_x+\psi)=f_6,\\ \lambda\phi_2+(\xi^2+\eta)\phi_2-v(L)\mu(\xi)=f_7,\\ \lambda\vartheta+\frac{b}{m_2}\psi_x(L)+\frac{\zeta_2}{m_2}\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi_2(\xi)\,d\xi=f_8.\end{cases} \tag{2.17}$$

Suppose that we have found φ and ψ. Therefore, the first and the fifth equations in (2.17) give

$$\begin{cases}u=\lambda\varphi-f_1,\\ v=\lambda\psi-f_5.\end{cases} \tag{2.18}$$

It is clear that $u\in H^1_*(0,L)$ and $v\in H^1_*(0,L)$. Furthermore, by (2.17) we can find ϕ_i $(i=1,2)$ as

$$\begin{cases}\phi_1 &= \dfrac{f_3(\xi)+\mu(\xi)u(L)}{\xi^2+\eta+\lambda},\\ \phi_2 &= \dfrac{f_7(\xi)+\mu(\xi)v(L)}{\xi^2+\eta+\lambda}.\end{cases} \tag{2.19}$$

By using (2.17) and (2.18) the functions φ and ψ satisfying the following system

$$\begin{cases}\lambda^2\varphi-\dfrac{K}{\rho_1}(\varphi_x+\psi)_x=f_2+\lambda f_1,\\ \lambda^2\psi-\dfrac{b}{\rho_2}\psi_{xx}+\dfrac{K}{\rho_2}(\varphi_x+\psi)=f_6+\lambda f_5,\end{cases} \tag{2.20}$$

Solving system (2.20) is equivalent to finding $(\varphi,\psi)\in(H^2\cap H^1_*(0,L))^2$ such that

$$\begin{cases}\displaystyle\int_0^L(\rho_1\lambda^2\varphi w-K(\varphi_x+\psi)_x w)\,dx &= \displaystyle\int_0^L\rho_1(f_2+\lambda f_1)w\,dx,\\ \displaystyle\int_0^L(\rho_2\lambda^2\psi\chi-b\psi_{xx}\chi+K(\varphi_x+\psi)\chi)\,dx &= \displaystyle\int_0^L\rho_2(f_6+\lambda f_5)\chi\,dx,\end{cases} \tag{2.21}$$

for all $(w,\chi) \in H^1_*(0,L) \times H^1_*(0,L)$. By using (2.21) and (2.19) the functions φ and ψ satisfying the following system

$$
\begin{cases}
\int_0^L (\rho_1\lambda^2\varphi w + K(\varphi_x+\psi)w_x)\,dx + (\lambda m_1 + \tilde{\zeta}_1)u(L)w(L) \\
= \int_0^L \rho_1(f_2+\lambda f_1)w\,dx - \zeta_1 \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_3(\xi)\,d\xi\ w(L) + m_1 f_4 w(L), \\
\int_0^L (\rho_2\lambda^2\psi\chi + b\psi_x\chi_x + K(\varphi_x+\psi)\chi)\,dx + (\lambda m_2 + \tilde{\zeta}_2)v(L)\chi(L) \\
\quad = \int_0^L \rho_2(f_6+\lambda f_5)\chi\,dx - \zeta_2 \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_7(\xi)\,d\xi\ \chi(L) + m_2 f_8 \chi(L)
\end{cases}
\tag{2.22}
$$

where $\tilde{\zeta}_i = \zeta_i \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2+\eta+\lambda}\,d\xi$. Using again (2.18), we deduce that

$$
\begin{cases}
u(L) = \lambda\varphi(L) - f_1(L), \\
v(L) = \lambda\psi(L) - f_5(L).
\end{cases}
\tag{2.23}
$$

Inserting (2.23) into (2.22), we get

$$
\begin{cases}
\int_0^L (\rho_1\lambda^2\varphi w + K(\varphi_x+\psi)w_x)\,dx + \lambda(\lambda m_1 + \tilde{\zeta}_1)\varphi(L)w(L) \\
= \int_0^L \rho_1(f_2+\lambda f_1)w\,dx - \zeta_1 \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_3(\xi)\,d\xi\ w(L) + (\lambda m_1 + \tilde{\zeta}_1)f_1(L)w(L) + m_1 f_4 w(L), \\
\int_0^L (\rho_2\lambda^2\psi\chi + b\psi_x\chi_x + K(\varphi_x+\psi)\chi)\,dx + \lambda(\lambda m_2 + \tilde{\zeta}_2)\psi(L)\chi(L) \\
\quad = \int_0^L \rho_2(f_6+\lambda f_5)\chi\,dx - \zeta_2 \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_7(\xi)\,d\xi\ \chi(L) + (\lambda m_2 + \tilde{\zeta}_2)f_5(L)\chi(L) + m_2 f_8 \chi(L).
\end{cases}
\tag{2.24}
$$

Consequently, problem (2.24) is equivalent to the problem

$$
a((\varphi,\psi),(w,\chi)) = L(w,\chi),
\tag{2.25}
$$

where the bilinear form $a : [H^1_*(0,L) \times H^1_*(0,L)]^2 \to \mathbb{R}$ and the linear form $L : H^1_*(0,L) \times H^1_*(0,L) \to \mathbb{R}$ are defined by

$$
\begin{aligned}
a((\varphi,\psi),(w,\chi)) &= \int_0^L (\rho_1\lambda^2\varphi w + K(\varphi_x+\psi)(w_x+\chi))\,dx \\
&+ \int_0^L (\rho_2\lambda^2\psi\chi + b\psi_x\chi_x)\,dx + \lambda(\lambda m_1 + \tilde{\zeta}_1)\varphi(L)w(L) + \lambda(\lambda m_2 + \tilde{\zeta}_2)\psi(L)\chi(L)
\end{aligned}
$$

and

$$
\begin{aligned}
L(w,\chi) &= \int_0^L \rho_1(f_2+\lambda f_1)w\,dx - \zeta_1 \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_3(\xi)\,d\xi\ w(L) + (\lambda m_1 + \tilde{\zeta}_1)f_1(L)w(L) \\
&+ m_1 f_4 w(L) + \int_0^L \rho_2(f_6+\lambda f_5)\chi\,dx - \zeta_2 \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_7(\xi)\,d\xi\ \chi(L) \\
&+ (\lambda m_2 + \tilde{\zeta}_2)f_5(L)\chi(L) + m_2 f_8 \chi(L).
\end{aligned}
$$

It is easy to verify that a is continuous and coercive, and L is continuous. So applying the Lax-Milgram theorem, we deduce that for all $(w,\chi) \in H^1_*(0,L) \times H^1_*(0,L)$ problem (2.25) admits a unique solution $(\varphi,\psi) \in H^1_*(0,L) \times H^1_*(0,L)$. Applying the classical elliptic regularity, it follows from (2.24) that $(\varphi,\psi) \in H^2(0,L) \times H^2(0,L)$. Therefore, the operator $\lambda I - \mathcal{A}$ is surjective for any $\lambda > 0$. Consequently, using Hille-Yosida theorem, we have the following results.

Theorem 2.3.1 (Existence and uniqueness)

(1) If $U_0 \in D(\mathcal{A})$, then system (2.12) has a unique strong solution

$$U \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

(2) If $U_0 \in \mathcal{H}$, then system (2.12) has a unique weak solution

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

2.4 Lack of exponential stability

We first state three well-known theorems.

Theorem 2.4.1 ([35]) *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on Hilbert space $\mathcal{H}$. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R}$$

and

$$\overline{\lim_{|\beta|\to\infty}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Theorem 2.4.2 ([9]) *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup on a Hilbert space $\mathcal{H}$. If*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \text{ and } \sup_{|\beta|\geq 1} \frac{1}{\beta^l} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < M$$

for some l, then there exist c such that

$$\|e^{\mathcal{A}t}U_0\|^2 \leq \frac{c}{t^{\frac{2}{l}}} \|U_0\|^2_{D(\mathcal{A})}.$$

Theorem 2.4.3 ([3]) *Let $\mathcal{A}$ be the generator of a uniformly bounded C_0 semigroup $\{S(t)\}_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. If:*

(i) $\mathcal{A}$ does not have eigenvalues on $i\,\mathbb{R}$.

(ii) The intersection of the spectrum $\sigma(\mathcal{A})$ with $i\,\mathrm{I\!R}$ is at most a countable set.

Then the semigroup $\{S(t)\}_{t\geq 0}$ is asymptotically stable, i.e, $\|S(t)z\|_{\mathcal{H}} \to 0$ as $t \to \infty$ for any $z \in \mathcal{H}$.

Our main result is the following

Theorem 2.4.4 *The semigroup generated by the operator $\mathcal{A}$ is not exponentially stable.*

Proof
We will examine two cases.
Case 1 $\eta = 0$: We shall show that $i\lambda = 0$ is not in the resolvent set of the operator $\mathcal{A}$. Indeed, noting that $(\sin x, 0, 0, 0, 0, 0, 0, 0,)^T \in \mathcal{H}$, and denoting by $(\varphi, u, \phi_1, \theta, \psi, v, \phi_2, \vartheta)^T$ the image of $(\sin x, 0, 0, 0, 0, 0, 0, 0)^T$ by $\mathcal{A}^{-1}$, we see that $\phi_1(\xi) = |\xi|^{\frac{2\alpha-5}{2}} \sin L$. But, then $\phi_1 \notin L^2(-\infty, +\infty)$, since $\alpha \in (0, 1)$ and so $(\varphi, u, \phi_1, \theta, \psi, v, \phi_2, \vartheta)^T \notin D(\mathcal{A})$.
Case 2 $\eta \neq 0$: We aim to show that an infinite number of eigenvalues of $\mathcal{A}$ approach the imaginary axis which prevents the Timoshenko system (P) from being exponentially stable. Indeed We first compute the characteristic equation that gives the eigenvalues of $\mathcal{A}$.

Let λ be an eigenvalue of $\mathcal{A}$ with associated eigenvector $U = (\varphi, u, \phi_1, \theta, \psi, v, \phi_2, \vartheta)^T$. Then $\mathcal{A}U = \lambda U$ is equivalent to

$$(2.26) \qquad \begin{cases} \lambda\varphi - u = 0, \\ \lambda u - \dfrac{K}{\rho_1}(\varphi_x + \psi)_x = 0, \\ \lambda\phi_1 + (\xi^2 + \eta)\phi_1 - u(L)\mu(\xi) = 0, \\ \lambda\theta + \frac{K}{m_1}(\varphi_x + \psi)(L) + \frac{\zeta_1}{m_1}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_1(\xi)\, d\xi, \\ \lambda\psi - v = 0, \\ \lambda v - \dfrac{b}{\rho_2}\psi_{xx} + \dfrac{K}{\rho_2}(\varphi_x + \psi) = 0, \\ \lambda\phi_2 + (\xi^2 + \eta)\phi_2 - v(L)\mu(\xi) = 0, \\ \lambda\vartheta + \frac{b}{m_2}\psi_x(L) + \frac{\zeta_2}{m_2}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_2(\xi)\, d\xi. \end{cases}$$

From $(2.26)_1 - (2.26)_2$ and $(2.26)_5 - (2.26)_6$ for such λ, we find

$$(2.27) \qquad \begin{cases} \lambda^2\varphi - \dfrac{K}{\rho_1}(\varphi_x + \psi)_x = 0, \\ \lambda^2\psi - \dfrac{b}{\rho_2}\psi_{xx} + \dfrac{K}{\rho_2}(\varphi_x + \psi) = 0. \end{cases}$$

Since $\theta = u(L)$ and $\vartheta = v(L)$, using $(2.26)_3 - (2.26)_4$ and $(2.26)_7 - (2.26)_8$, we get

$$(2.28) \qquad \begin{cases} \dfrac{K}{m_1}(\varphi_x + \psi)(L) + \left(\lambda + \dfrac{\gamma_1}{m_1}(\lambda + \eta)^{\alpha-1}\right)\lambda\varphi(L) = 0, \\ \dfrac{b}{m_2}\psi_x(L) + \left(\lambda + \dfrac{\gamma_2}{m_2}(\lambda + \eta)^{\alpha-1}\right)\lambda\psi(L) = 0, \\ \varphi(0) = \psi(0) = 0. \end{cases}$$

We set

$$\tilde{\varphi} = (\varphi_x + \psi), \quad \tilde{\psi} = \psi_x.$$

(2.27) is equivalent to

$$(2.29) \qquad \begin{cases} \lambda^2\varphi - \dfrac{K}{\rho_1}\tilde{\varphi}_x = 0, \\ \lambda^2\psi - \dfrac{b}{\rho_2}\tilde{\psi}_x + \dfrac{K}{\rho_2}\tilde{\varphi} = 0. \end{cases}$$

Then

$$(2.30) \qquad \begin{cases} (\lambda^2 + \dfrac{K}{\rho_2})\tilde{\varphi} - \dfrac{K}{\rho_1}\tilde{\varphi}_{xx} - \dfrac{b}{\rho_2}\tilde{\psi}_x = 0 \quad ((2.29)_1 x + (2.29)_2), \\ \lambda^2\tilde{\psi} - \dfrac{b}{\rho_2}\tilde{\psi}_{xx} + \dfrac{K}{\rho_2}\tilde{\varphi}_x = 0. \end{cases}$$

From $(2.29)_2$ we have

$$\begin{aligned} \tilde{\varphi} &= \frac{\rho_2}{K}(-\lambda^2\psi + \frac{b}{\rho_2}\tilde{\psi}_x) \\ \tilde{\varphi}_{xx} &= \frac{\rho_2}{K}(-\lambda^2\psi_{xx} + \frac{b}{\rho_2}\tilde{\psi}_{xxx}). \end{aligned}$$

Replacing this in $(2.30)_1$, we get

$$(2.31) \qquad \psi'''' - \lambda^2\left(\frac{\rho_1}{K} + \frac{\rho_2}{b}\right)\psi'' + \frac{\rho_1}{K}\frac{\rho_2}{b}\lambda^2\left(\lambda^2 + \frac{K}{\rho_2}\right)\psi = 0.$$

The characteristic polynomial of (2.31) is

$$s^4 - \left(\frac{\rho_1}{K} + \frac{\rho_2}{b}\right)\lambda^2 s^2 + \frac{\rho_1}{K}\frac{\rho_2}{b}\lambda^2\left(\lambda^2 + \frac{K}{\rho_2}\right) = 0.$$

The solution ψ is given by

$$(2.32) \qquad \psi(x) = \sum_{i=1}^{4} c_i e^{t_i x}$$

where $c_i \in \mathbb{C}$ for all $1 \le i \le 4$ and

$$\begin{cases} t_1(\lambda) = \lambda\sqrt{\dfrac{\left(\frac{\rho_1}{K}+\frac{\rho_2}{b}\right)+\sqrt{\left(\frac{\rho_1}{K}-\frac{\rho_2}{b}\right)^2-\frac{4\rho_1}{b\lambda^2}}}{2}}, \quad t_2(\lambda) = -t_1(\lambda), \\ t_3(\lambda) = \lambda\sqrt{\dfrac{\left(\frac{\rho_1}{K}+\frac{\rho_2}{b}\right)-\sqrt{\left(\frac{\rho_1}{K}-\frac{\rho_2}{b}\right)^2-\frac{4\rho_1}{b\lambda^2}}}{2}}, \quad t_4(\lambda) = -t_3(\lambda). \end{cases}$$

From $(2.29)_1$ and $(2.30)_2$, we have

$$\varphi = \frac{K}{\rho_1}\frac{1}{\lambda^2}\tilde{\varphi}_x = \frac{\rho_2}{\rho_1}\frac{1}{\lambda^2}\left(-\lambda^2\psi_x + \frac{b}{\rho_2}\tilde{\psi}_{xx}\right).$$

Thus the boundary conditions may be written as the following system:

$$\psi(0) = 0 \Longrightarrow \sum_{i=1}^{4} c_i = 0$$

$$\varphi(0) = 0 \Longrightarrow \sum_{i=1}^{4}\left(-\lambda^2 t_i + \frac{b}{\rho_2}\, t_i^3\right) c_i = 0$$

$$\frac{b}{m_2}\psi_x(L)+\left(\lambda+\frac{\gamma_2}{m_2}(\lambda+\eta)^{\alpha-1}\right)\lambda\psi(L)=0 \Longrightarrow \sum_{i=1}^{4}\left(\frac{b}{m_2}t_i+(\lambda+\frac{\gamma_2}{m_2}(\lambda+\eta)^{\alpha-1})\lambda\right)e^{t_iL}c_i=0$$

$$\frac{K}{m_1}(\varphi_x+\psi)(L)+\left(\lambda+\frac{\gamma_1}{m_1}(\lambda+\eta)^{\alpha-1}\right)\lambda\varphi(L)=0 \Longrightarrow$$

$$\sum_{i=1}^{4}\left(-\frac{1}{m_1}\lambda^2-\frac{1}{\rho_1}\lambda\left(\lambda+\frac{\gamma_1}{m_1}(\lambda+\eta)^{\alpha-1}\right)t_i+\frac{b}{m_1\rho_2}t_i^2+\frac{b}{\rho_1\rho_2}\frac{1}{\lambda}\left(\lambda+\frac{\gamma_1}{m_1}(\lambda+\eta)^{\alpha-1}\right)t_i^3\right)e^{t_iL}c_i=0$$

$$\mathcal{M}C(\lambda)=\begin{pmatrix} 1 & 1 & 1 & 1 \\ h_1(t_1) & h_1(t_2) & h_1(t_3) & h_1(t_4) \\ h_2(t_1)e^{t_1L} & h_2(t_2)e^{t_2L} & h_2(t_3)e^{t_3L} & h_2(t_4)e^{t_4L} \\ h_3(t_1)e^{t_1L} & h_3(t_2)e^{t_2L} & h_3(t_3)e^{t_3L} & h_3(t_4)e^{t_4L} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.33}$$

where

$$\begin{aligned} h_1(r) &= -\lambda^2 r+\tfrac{b}{\rho_2}\,r^3,\\ h_2(r) &= \frac{b}{m_2}r+(\lambda+\frac{\gamma_2}{m_2}(\lambda+\eta)^{\alpha-1})\lambda,\\ h_3(r) &= -\frac{1}{m_1}\lambda^2-\frac{1}{\rho_1}\lambda\left(\lambda+\frac{\gamma_1}{m_1}(\lambda+\eta)^{\alpha-1}\right)r+\frac{b}{m_1\rho_2}r^2+\frac{b}{\rho_1\rho_2}\frac{1}{\lambda}\left(\lambda+\frac{\gamma_1}{m_1}(\lambda+\eta)^{\alpha-1}\right)r^3. \end{aligned}$$

Set $r_1^2=\frac{\rho_2}{b}, r_2^2=\frac{\rho_1}{K}$ and $l=K/b$. We will examine two cases.

Case 1 $r_1=r_2$:
We start by the expansion of t_1 and t_3:

$$t_1(\lambda)=r_1\lambda+\left(\frac{i}{2}\sqrt{l}\right)+\frac{1}{8}\frac{l}{r_1}\frac{1}{\lambda}-\left(\frac{i}{16}\frac{\sqrt{l}l}{r_1^2}\right)\frac{1}{\lambda^2}-\left(\frac{5}{128}\frac{l^2}{r_1^3}\right)\frac{1}{\lambda^3}+\left(\frac{7i}{256}\frac{l^2\sqrt{l}}{r_1^4}\right)\frac{1}{\lambda^4}+O\left(\frac{1}{\lambda^5}\right) \tag{2.34}$$

$$t_3(\lambda)=r_1\lambda-\left(\frac{i}{2}\sqrt{l}\right)+\frac{1}{8}\frac{l}{r_1}\frac{1}{\lambda}+\left(\frac{i}{16}\frac{\sqrt{l}l}{r_1^2}\right)\frac{1}{\lambda^2}-\left(\frac{5}{128}\frac{l^2}{r_1^3}\right)\frac{1}{\lambda^3}-\left(\frac{7i}{256}\frac{l^2\sqrt{l}}{r_1^4}\right)\frac{1}{\lambda^4}+O\left(\frac{1}{\lambda^5}\right) \tag{2.35}$$

Using (2.34) and (2.35), we find the asymptotic development of:

$$\begin{aligned} h_3(t_1) &= i\tfrac{\sqrt{l}}{\rho_1}\lambda^2+\left(-\tfrac{1}{2}\tfrac{l}{\rho_1r_1}+i\tfrac{\sqrt{l}}{m_1r_1}\right)\lambda+\tfrac{1}{8}i\tfrac{\left(\sqrt{l}\right)^3}{\rho_1r_1^2}+\gamma_1\tfrac{i\sqrt{l}}{m_1\rho_1}\lambda^{\alpha}+\gamma_1\tfrac{-l-2i(1-\alpha)\sqrt{l}\eta r_1}{2m_1\rho_1r_1}\lambda^{\alpha-1}+\tfrac{1}{16}\tfrac{l^2}{\rho_1r_1^3}\tfrac{1}{\lambda}\\ &\quad-\tfrac{5}{128}i\tfrac{l^{\frac{5}{2}}}{\rho_1r_1^4}\tfrac{1}{\lambda^2}+\tfrac{1}{8}il^{\frac{3}{2}}\lambda^{\alpha-2}\tfrac{\gamma_1}{m_1\rho_1r_1^2}+\tfrac{1}{2}i\lambda^{\alpha-2}\eta^2\sqrt{l}\gamma_1\left(\alpha-2\right)\tfrac{\alpha-1}{m_1\rho_1}-\tfrac{1}{2}\lambda^{\alpha-2}l\eta\gamma_1\tfrac{\alpha-1}{m_1\rho_1r_1}+o\left(\tfrac{1}{\lambda^2}\right). \end{aligned} \tag{2.36}$$

$$\begin{aligned} h_3(t_2) &= -i\tfrac{\sqrt{l}}{\rho_1}\lambda^2+\left(\tfrac{1}{2}\tfrac{l}{\rho_1r_1}+i\tfrac{\sqrt{l}}{m_1r_1}\right)\lambda-\tfrac{1}{8}i\tfrac{l^{\frac{3}{2}}}{\rho_1r_1^2}+\gamma_1\tfrac{-i\sqrt{l}}{m_1\rho_1}\lambda^{\alpha}-\gamma_1\tfrac{-l-2i(1-\alpha)\sqrt{l}\eta r_1}{2m_1\rho_1r_1}\lambda^{\alpha-1}-\tfrac{1}{16}\tfrac{l^2}{\rho_1r_1^3}\tfrac{1}{\lambda}\\ &\quad+\tfrac{5}{128}i\tfrac{l^{\frac{5}{2}}}{\rho_1r_1^4}\tfrac{1}{\lambda^2}-\tfrac{1}{8}il^{\frac{3}{2}}\lambda^{\alpha-2}\tfrac{\gamma_1}{m_1\rho_1r_1^2}-\tfrac{1}{2}i\lambda^{\alpha-2}\eta^2\sqrt{l}\gamma_1\left(\alpha-1\right)\tfrac{\alpha-2}{m_1\rho_1}+\tfrac{1}{2}\lambda^{\alpha-2}l\eta\gamma_1\tfrac{\alpha-1}{m_1\rho_1r_1}+o\left(\tfrac{1}{\lambda^2}\right). \end{aligned} \tag{2.37}$$

$$\begin{aligned} h_3(t_3) &= -i\tfrac{\sqrt{l}}{\rho_1}\lambda^2+\left(-\tfrac{1}{2}\tfrac{l}{\rho_1r_1}-i\tfrac{\sqrt{l}}{m_1r_1}\right)\lambda-\tfrac{1}{8}i\tfrac{l^{\frac{3}{2}}}{\rho_1r_1^2}-\gamma_1\tfrac{i\sqrt{l}}{m_1\rho_1}\lambda^{\alpha}+\gamma_1\tfrac{-l+2i(1-\alpha)\sqrt{l}\eta r_1}{2m_1\rho_1r_1}\lambda^{\alpha-1}+\tfrac{1}{16\rho_1r_1^3}l^2\tfrac{1}{\lambda}\\ &\quad+\tfrac{5}{128}i\tfrac{l^{\frac{5}{2}}}{\rho_1r_1^4}\tfrac{1}{\lambda^2}-\tfrac{1}{8}il^{\frac{3}{2}}\lambda^{\alpha-2}\tfrac{\gamma_1}{m_1\rho_1r_1^2}-\tfrac{1}{2}i\lambda^{\alpha-2}\eta^2\sqrt{l}\gamma_1\left(\alpha-2\right)\tfrac{\alpha-1}{m_1\rho_1}-\tfrac{1}{2}\lambda^{\alpha-2}l\eta\gamma_1\tfrac{\alpha-1}{m_1\rho_1r_1}+o\left(\tfrac{1}{\lambda^2}\right). \end{aligned} \tag{2.38}$$

$$\begin{aligned} h_3(t_4) &= i\frac{\sqrt{l}}{\rho_1}\lambda^2 + \left(\frac{1}{2}\frac{l}{\rho_1 r_1} - i\frac{\sqrt{l}}{m_1 r_1}\right)\lambda + \frac{1}{8}i\frac{l^{\frac{3}{2}}}{\rho_1 r_1^2} + \gamma_1\frac{i\sqrt{l}}{m_1\rho_1}\lambda^\alpha - \gamma_1\frac{-l+2i(1-\alpha)\sqrt{l}\eta r_1}{2m_1\rho_1 r_1}\lambda^{\alpha-1} - \frac{1}{16\rho_1 r_1^3}l^2\frac{1}{\lambda} \\ &\quad -\frac{5}{128}i\frac{l^{\frac{5}{2}}}{\rho_1 r_1^4}\frac{1}{\lambda^2} + \frac{1}{8}il^{\frac{3}{2}}\lambda^{\alpha-2}\frac{\gamma_1}{m_1\rho_1 r_1^2} + \frac{1}{2}i\lambda^{\alpha-2}\eta^2\sqrt{l}\gamma_1(\alpha-1)\frac{\alpha-2}{m_1\rho_1} + \frac{1}{2}\lambda^{\alpha-2}l\eta\gamma_1\frac{\alpha-1}{m_1\rho_1 r_1} + o\left(\frac{1}{\lambda^2}\right). \end{aligned} \tag{2.39}$$

$$\begin{aligned} h_2(t_1) &= \lambda^2 + b\frac{\lambda}{m_2}r_1 + \frac{1}{2}ib\frac{\sqrt{l}}{m_2} + \frac{\gamma_2}{m_2}\lambda^\alpha + \eta\gamma_2\frac{\alpha-1}{m_2}\lambda^{\alpha-1} + \frac{1}{8}b\frac{l}{m_2 r_1}\frac{1}{\lambda} + \frac{1}{2}\lambda^{\alpha-2}\eta^2\gamma_2(\alpha-1)\frac{\alpha-2}{m_2} \\ &\quad -\frac{1}{16}ib\frac{l^{\frac{3}{2}}}{m_2 r_1^2}\frac{1}{\lambda^2} + o\left(\frac{1}{\lambda^2}\right). \end{aligned} \tag{2.40}$$

$$\begin{aligned} h_2(t_2) &= \lambda^2 - b\frac{\lambda}{m_2}r_1 - \frac{1}{2}ib\frac{\sqrt{l}}{m_2} + \frac{\gamma_2}{m_2}\lambda^\alpha + \eta\gamma_2\frac{\alpha-1}{m_2}\lambda^{\alpha-1} - \frac{1}{8}b\frac{l}{m_2 r_1}\frac{1}{\lambda} + \frac{1}{2}\lambda^{\alpha-2}\eta^2\gamma_2(\alpha-1)\frac{\alpha-2}{m_2} \\ &\quad +\frac{1}{16}ib\frac{l^{\frac{3}{2}}}{m_2 r_1^2}\frac{1}{\lambda^2} + o\left(\frac{1}{\lambda^2}\right). \end{aligned} \tag{2.41}$$

$$\begin{aligned} h_2(t_3) &= \lambda^2 + b\frac{\lambda}{m_2}r_1 - \frac{1}{2}ib\frac{\sqrt{l}}{m_2} + \frac{\gamma_2}{m_2}\lambda^\alpha + \eta\gamma_2\frac{\alpha-1}{m_2}\lambda^{\alpha-1} + \frac{1}{8}b\frac{l}{m_2 r_1}\frac{1}{\lambda} + \frac{1}{2}\lambda^{\alpha-2}\eta^2\gamma_2(\alpha-1)\frac{\alpha-2}{m_2} \\ &\quad +\frac{1}{16}ib\frac{l^{\frac{3}{2}}}{m_2 r_1^2}\frac{1}{\lambda^2} + o\left(\frac{1}{\lambda^2}\right). \end{aligned} \tag{2.42}$$

$$\begin{aligned} h_2(t_4) &= \lambda^2 - b\frac{\lambda}{m_2}r_1 + \frac{1}{2}ib\frac{\sqrt{l}}{m_2} + \frac{\gamma_2}{m_2}\lambda^\alpha + \eta\gamma_2\frac{\alpha-1}{m_2}\lambda^{\alpha-1} - \frac{1}{8}b\frac{l}{m_2 r_1}\frac{1}{\lambda} + \frac{1}{2}\lambda^{\alpha-2}\eta^2\gamma_2(\alpha-1)\frac{\alpha-2}{m_2} \\ &\quad -\frac{1}{16}ib\frac{l^{\frac{3}{2}}}{m_2 r_1^2}\frac{1}{\lambda^2} + o\left(\frac{1}{\lambda^2}\right). \end{aligned} \tag{2.43}$$

$$h_1(t_1) = i\sqrt{l}\lambda^2 - \frac{1}{2}l\frac{\lambda}{r_1} + \frac{1}{8}i\frac{l^{\frac{3}{2}}}{r_1^2} + \frac{1}{16}\frac{l^2}{\lambda r_1^3} - \frac{5}{128}i\frac{l^{\frac{5}{2}}}{\lambda^2 r_1^4} + O\left(\frac{1}{\lambda^3}\right). \tag{2.44}$$

$$h_1(t_2) = -i\sqrt{l}\lambda^2 + \frac{1}{2}l\frac{\lambda}{r_1} - \frac{1}{8}i\frac{l^{\frac{3}{2}}}{r_1^2} - \frac{1}{16}\frac{l^2}{\lambda r_1^3} + \frac{5}{128}i\frac{l^{\frac{5}{2}}}{\lambda^2 r_1^4} + O\left(\frac{1}{\lambda^3}\right). \tag{2.45}$$

$$h_1(t_3) = -i\sqrt{l}\lambda^2 - \frac{1}{2}l\frac{\lambda}{r_1} - \frac{1}{8}i\frac{l^{\frac{3}{2}}}{r_1^2} + \frac{1}{16}\frac{l^2}{\lambda r_1^3} + \frac{5}{128}i\frac{l^{\frac{5}{2}}}{\lambda^2 r_1^4} + O\left(\frac{1}{\lambda^3}\right). \tag{2.46}$$

$$h_1(t_4) = i\sqrt{l}\lambda^2 + \frac{1}{2}l\frac{\lambda}{r_1} + \frac{1}{8}i\frac{l^{\frac{3}{2}}}{r_1^2} - \frac{1}{16}\frac{l^2}{\lambda r_1^3} - \frac{5}{128}i\frac{l^{\frac{5}{2}}}{\lambda^2 r_1^4} + O\left(\frac{1}{\lambda^3}\right). \tag{2.47}$$

Using the asymptotic development (2.36)-(2.47)

$$
\begin{aligned}
f(\lambda) &= e^{t_3+t_4}\Big(h_1(t_2)-(h_1(t_1)\Big)\Big(h_2(t_3)h_3(t_4)-h_3(t_3)h_2(t_4)\Big)+\\
&\quad e^{t_1+t_3}\Big(h_1(t_2)-(h_1(t_4)\Big)\Big(h_2(t_1)h_3(t_3)-h_3(t_1)h_2(t_3)\Big)+\\
&\quad e^{t_1+t_4}\Big(h_1(t_2)-(h_1(t_3)\Big)\Big(h_3(t_1)h_2(t_4)-h_2(t_1)h_3(t_4)\Big)+\\
&\quad e^{t_2+t_3}\Big(h_1(t_4)-(h_1(t_1)\Big)\Big(h_2(t_2)h_3(t_3)-h_2(t_3)h_3(t_2)\Big)+\\
&\quad e^{t_2+t_4}\Big(h_1(t_1)-(h_1(t_3)\Big)\Big(h_2(t_2)h_3(t_4)-h_3(t_2)h_2(t_4)\Big)+\\
&\quad e^{t_1+t_2}\Big(h_1(t_4)-(h_1(t_3)\Big)\Big(h_2(t_1)h_3(t_2)-h_2(t_2)h_3(t_1)\Big)\\
&= -4l\tfrac{e^{L(t_1+t_3)}+e^{L(t_2+t_4)}-2}{\rho_1}\lambda^6-4l\left(m_2\rho_1+m_1\rho_2\right)\tfrac{e^{L(t_1+t_3)}-e^{L(t_2+t_4)}}{m_1m_2\rho_1r_1}\lambda^5\\
&\quad -4l\left(\gamma_1m_2+\gamma_2m_1\right)\tfrac{e^{L(t_1+t_3)}+e^{L(t_2+t_4)}-2}{m_1m_2\rho_1}\lambda^{4+\alpha}\\
&\quad +\Big(-l^2\tfrac{e^{L(t_1+t_3)}+e^{L(t_1+t_4)}+e^{L(t_2+t_3)}+e^{L(t_2+t_4)}-4}{\rho_1r_1^2}-4bl\tfrac{e^{L(t_1+t_3)}+e^{L(t_2+t_4)}+2}{m_1m_2}\\
&\quad +2ibl^{\frac{3}{2}}(\tfrac{1}{m_1\rho_2}-\tfrac{1}{m_2\rho_1})\left(e^{L(t_1+t_4)}-e^{L(t_2+t_3)}\right)\Big)\lambda^4\\
&\quad -4l\left(\eta\left(\gamma_1m_2+\gamma_2m_1\right)(\alpha-1)\tfrac{e^{L(t_1+t_3)}+e^{L(t_2+t_4)}-2}{m_1m_2\rho_1}+\left(\gamma_2\rho_1+\gamma_1\rho_2\right)\tfrac{e^{L(t_1+t_3)}-e^{L(t_2+t_4)}}{m_1m_2\rho_1r_1}\right)\lambda^{3+\alpha}\\
&\quad -\frac{1}{2}\frac{bl^2}{r_1}\left(\frac{1}{m_1\rho_2}+5\frac{1}{m_2\rho_1}\right)\left(e^{L(t_1+t_3)}-e^{Lt_2+Lt_4}\right)\lambda^3\\
&\quad -4l\frac{\gamma_1\gamma_2}{m_1m_2\rho_1}(e^{L(t_1+t_3)}+e^{L(t_2+t_4)}-2)\lambda^{2+2\alpha}+\frac{8l\eta(1-\alpha)\gamma_1\gamma_2}{m_1m_2\rho_1}(e^{L(t_1+t_3)}+e^{L(t_2+t_4)}-2)\lambda^{1+2\alpha}\\
&\quad +\Bigg[\frac{1}{\rho_1}(\frac{\gamma_1}{m_1}+\frac{\gamma_2}{m_2})(-4l\eta^2-\frac{l^2}{r_1^2}+6ld\eta^2-2ld^2\eta^2)(e^{L(t_1+t_3)}+e^{L(t_2+t_4)}-2)\\
&\quad +\frac{4lb(1-\alpha)\eta r_1}{m_1m_2}(\frac{\gamma_1}{\rho_1}+\frac{\gamma_2}{\rho_2})(e^{L(t_1+t_3)}-e^{L(t_2+t_4)})-\frac{l^2}{r_1^2\rho_1}(\frac{\gamma_1}{m_1}+\frac{\gamma_2}{m_2})(e^{L(t_1+t_4)}+e^{L(t_2+t_3)}-2)\\
&\quad -2ib\frac{l^{3/2}}{m_1m_2}(\frac{\gamma_1}{\rho_1}+\frac{\gamma_2}{\rho_2})(e^{L(t_1+t_4)}-e^{L(t_2+t_3)})\Bigg]\lambda^{2+\alpha}\\
&\quad +\Bigg[\frac{l^3}{\rho_1r_1^4}(e^{L(t_1+t_3)}+e^{L(t_2+t_4)}+e^{L(t_1+t_4)}+e^{L(t_2+t_3)}-4)\\
&\quad -\frac{bl^2}{\rho_1r_1^4}(e^{L(t_1+t_3)}+e^{L(t_2+t_4)}+e^{L(t_1+t_4)}+e^{L(t_2+t_3)}+4)\\
&\quad -\frac{1}{4}i\frac{l^{5/2}b}{r_1^2}(\frac{1}{m_1\rho_2}+\frac{3}{m_2\rho_1})(e^{L(t_1+t_4)}-e^{L(t_2+t_3)})\Bigg]\lambda^2+o(\lambda^{1+\alpha})\\
&= -\frac{4l}{\rho_1}\lambda^6\Bigg[(e^{L(t_1+t_3)}+e^{L(t_2+t_4)}-2)+\frac{(m_2\rho_1+m_1\rho_2)}{m_1m_2r_1}\frac{e^{L(t_1+t_3)}-e^{L(t_2+t_4)}}{\lambda}\\
&\quad +\tfrac{(\gamma_1m_2+\gamma_2m_1)}{m_1m_2}\tfrac{e^{L(t_1+t_3)}+e^{L(t_2+t_4)}-2}{\lambda^{2-\alpha}}\\
&\quad +\Bigg(l\tfrac{e^{L(t_1+t_3)}+e^{L(t_1+t_4)}+e^{L(t_2+t_3)}+e^{L(t_2+t_4)}-4}{4r_1^2}+b\rho_1\tfrac{e^{L(t_1+t_3)}+e^{L(t_2+t_4)}+2}{m_1m_2}\\
&\quad -ib\tfrac{\rho_1}{2}l^{\frac{1}{2}}(\tfrac{1}{m_1\rho_2}-\tfrac{1}{m_2\rho_1})\left(e^{L(t_1+t_4)}-e^{L(t_2+t_3)}\right)\Bigg)\tfrac{1}{\lambda^2}
\end{aligned}
$$

(2.48)

$$+\left(\eta\left(\gamma_1 m_2+\gamma_2 m_1\right)(\alpha-1)\,\tfrac{e^{L(t_1+t_3)}+e^{L(t_2+t_4)}-2}{m_1m_2}-\left(\gamma_2\rho_1+\gamma_1\rho_2\right)\tfrac{e^{L(t_1+t_3)}-e^{L(t_2+t_4)}}{m_1m_2r_1}\right)\tfrac{1}{\lambda^{3-\alpha}}$$
$$+\frac{1}{8}\frac{bl\rho_1}{r_1}\left(\frac{1}{m_1\rho_2}+5\frac{1}{m_2\rho_1}\right)\left(e^{L(t_1+t_3)}-e^{Lt_2+Lt_4}\right)\frac{1}{\lambda^3}$$
$$+\frac{\gamma_1\gamma_2}{m_1m_2}(e^{L(t_1+t_3)}+e^{L(t_2+t_4)}-2)\frac{1}{\lambda^{4-2\alpha}}-2\eta(1-\alpha)\frac{\gamma_1\gamma_2}{m_1m_2}(e^{L(t_1+t_3)}+e^{L(t_2+t_4)}-2)\frac{1}{\lambda^{5-2\alpha}}$$
$$-\Bigg[\frac{1}{4}(\frac{\gamma_1}{m_1}+\frac{\gamma_2}{m_2})(-4\eta^2-\frac{l}{r_1^2}+6d\eta^2-2d^2\eta^2)(e^{L(t_1+t_3)}+e^{L(t_2+t_4)}-2)$$
$$+b(1-\alpha)\eta\rho_1\frac{r_1}{m_1m_2}(\frac{\gamma_1}{\rho_1}+\frac{\gamma_2}{\rho_2})(e^{L(t_1+t_3)}-e^{L(t_2+t_4)})-\frac{l}{4r_1^2}(\frac{\gamma_1}{m_1}+\frac{\gamma_2}{m_2})(e^{L(t_1+t_4)}+e^{L(t_2+t_3)}-2)$$
$$-ib\rho_1\frac{l^{1/2}}{2m_1m_2}(\frac{\gamma_1}{\rho_1}+\frac{\gamma_2}{\rho_2})(e^{L(t_1+t_4)}-e^{L(t_2+t_3)})\Bigg]\frac{1}{\lambda^{4-\alpha}}$$
$$-\Bigg[\frac{l^2}{4r_1^4}(e^{L(t_1+t_3)}+e^{L(t_2+t_4)}+e^{L(t_1+t_4)}+e^{L(t_2+t_3)}-4)$$
$$-\frac{bl}{4r_1^4}(e^{L(t_1+t_3)}+e^{L(t_2+t_4)}+e^{L(t_1+t_4)}+e^{L(t_2+t_3)}+4)$$
$$-\frac{1}{16}i\frac{l^{3/2}b\rho_1}{r_1^2}(\frac{1}{m_1\rho_2}+\frac{3}{m_2\rho_1})(e^{L(t_1+t_4)}-e^{L(t_2+t_3)})\Bigg]\tfrac{1}{\lambda^4}\Bigg]+o\left(\tfrac{1}{\lambda^{5-\alpha}}\right).$$

We set

$$\tilde f(\lambda) = f_0(\lambda)+\tfrac{f_1(\lambda)}{\lambda}+\tfrac{f_2(\lambda)}{\lambda^{2-\alpha}}+\tfrac{f_3(\lambda)}{\lambda^2}+\tfrac{f_4(\lambda)}{\lambda^{3-\alpha}}+\tfrac{f_5(\lambda)}{\lambda^3}+\tfrac{f_6(\lambda)}{\lambda^{4-\alpha}} +\tfrac{f_7(\lambda)}{\lambda^{4-2\alpha}}+\tfrac{f_8(\lambda)}{\lambda^{5-2\alpha}}+\tfrac{f_9(\lambda)}{\lambda^4}+o\left(\tfrac{1}{\lambda^{5-\alpha}}\right) \tag{2.49}$$

$$f_0(\lambda)=e^{L(t_1+t_3)}+e^{L(t_2+t_4)}-2=e^{-L(t_1+t_3)}(e^{L(t_1+t_3)}-1)^2 \tag{2.50}$$

$$\begin{aligned} f_1(\lambda) &= \tfrac{(m_2\rho_1+m_1\rho_2)}{m_1m_2r_1}(e^{(t_1+t_3)L}-e^{-(t_1+t_3)L})\\ &= \tfrac{(m_2\rho_1+m_1\rho_2)}{m_1m_2r_1}e^{-(t_1+t_3)L}(e^{(t_1+t_3)L}-1)(e^{(t_1+t_3)L}+1)\end{aligned} \tag{2.51}$$

$$\begin{aligned} f_2(\lambda) &= \tfrac{(\gamma_1m_2+\gamma_2m_1)}{m_1m_2}(e^{L(t_1+t_3)}+e^{L(t_2+t_4)}-2)\\ &= \tfrac{(\gamma_1m_2+\gamma_2m_1)}{m_1m_2}e^{-L(t_1+t_3)}(e^{L(t_1+t_3)}-1)^2\end{aligned} \tag{2.52}$$

$$\begin{aligned} f_3(\lambda) &= \Bigg(l\tfrac{e^{L(t_1+t_3)}+e^{L(t_1+t_4)}+e^{L(t_2+t_3)}+e^{L(t_2+t_4)}-4}{4r_1^2}+b\rho_1\tfrac{e^{L(t_1+t_3)}+e^{L(t_2+t_4)}+2}{m_1m_2}\\ &\qquad -ib\tfrac{\rho_1}{2}l^{\frac12}(\tfrac{1}{m_1\rho_2}-\tfrac{1}{m_2\rho_1})\left(e^{L(t_1+t_4)}-e^{L(t_2+t_3)}\right)\Bigg)\\ &= e^{-L(t_1+t_3)}\Bigg(\tfrac{l}{4r_1^2}((e^{L(t_1+t_3)}-1)^2+(e^{Lt_1}-e^{Lt_3})^2)+\tfrac{b\rho_1}{m_1m_2}(e^{L(t_1+t_3)}+1)^2\\ &\qquad -ib\tfrac{\rho_1}{2}l^{\frac12}(\tfrac{1}{m_1\rho_2}-\tfrac{1}{m_2\rho_1})\left(e^{2Lt_1}-e^{2Lt_3}\right)\Bigg)\end{aligned} \tag{2.53}$$

$$f_4(\lambda)=e^{-L(t_1+t_3)}\left(\frac{\eta(\alpha-1)\left(\gamma_1m_2+\gamma_2m_1\right)}{m_1m_2}(e^{L(t_1+t_3)}-1)^2+\frac{(\gamma_2\rho_1+\gamma_1\rho_2)}{m_1m_2r_1}(e^{2L(t_1+t_3)}-1)\right). \tag{2.54}$$

$$f_5(\lambda)=-\frac{1}{2}\frac{bl^2}{r_1}\left(\frac{1}{m_1\rho_2}+5\frac{1}{m_2\rho_1}\right)e^{-L(t_1+t_3)}\left(e^{2L(t_1+t_3)}-1\right) \tag{2.55}$$

$$
\begin{aligned}
f_6(\lambda) &= -\Big[\frac{1}{4}(\frac{\gamma_1}{m_1}+\frac{\gamma_2}{m_2})(-4\eta^2-\frac{l}{r_1^2}+6d\eta^2-2d^2\eta^2)(e^{L(t_1+t_3)}-1)^2 \\
&\quad +b(1-\alpha)\eta\rho_1\frac{r_1}{m_1m_2}(\frac{\gamma_1}{\rho_1}+\frac{\gamma_2}{\rho_2})(e^{2L(t_1+t_3)}-1)-\frac{l}{4r_1^2}(\frac{\gamma_1}{m_1}+\frac{\gamma_2}{m_2})(e^{Lt_1}-e^{Lt_3})^2 \\
&\quad -ib\rho_1\frac{l^{1/2}}{2m_1m_2}(\frac{\gamma_1}{\rho_1}+\frac{\gamma_2}{\rho_2})(e^{2Lt_1}-e^{2Lt_3})\Big]
\end{aligned} \tag{2.56}
$$

$$
f_7(\lambda)=\frac{\gamma_1\gamma_2}{m_1m_2}e^{-L(t_1+t_3)}(e^{L(t_1+t_3)}-1)^2 \tag{2.57}
$$

$$
f_8(\lambda)=-2\eta(1-\alpha)\frac{\gamma_1\gamma_2}{m_1m_2}e^{-L(t_1+t_3)}(e^{L(t_1+t_3)}-1)^2 \tag{2.58}
$$

$$
\begin{aligned}
f_9(\lambda) &= -e^{-L(t_1+t_3)}\Big[\frac{l^2}{4r_1^4}((e^{L(t_1+t_3)}-1)^2+(e^{Lt_1}-e^{Lt_3})^2) \\
&\quad -\frac{bl}{4r_1^4}((e^{L(t_1+t_3)}+1)^2+(e^{Lt_1}+e^{Lt_3})^2)-\frac{1}{16}i\frac{l^{3/2}b\rho_1}{r_1^2}(\frac{1}{m_1\rho_2}+\frac{3}{m_2\rho_1})(e^{2Lt_1}-e^{2Lt_3})\Big].
\end{aligned} \tag{2.59}
$$

Lemma 2.4.1 *(Asymptotic behavior of the large eigenvalues of $\mathcal{A}$) The large eigenvalues of $\mathcal{A}$ can be split into two families $(\lambda_k^j)_{k\in\mathbb{Z},|k|\geq k_0}$, $j=1,2$, ($k_0\in\mathbb{N}$ chosen large enough). The following asymptotic expansions hold:*

$$
\lambda_k^1=\frac{i}{Lr_1}k\pi+o(1),\quad \lambda_k^2=\frac{i}{Lr_1}k\pi+o(1). \tag{2.60}
$$

Either $\lambda_k^1=\lambda_k^2$ and this root is of order 2, or $\lambda_k^1\neq\lambda_k^2$ and these two roots are simple.

Proof. The multiplicity of the roots of f_0 given by (2.60) is two and λ is a root of f_0 if and only if

$$
(t_1+t_3)L=2ik\pi.
$$

Since $t_1+t_3=2r_1\lambda+\frac{1}{4}\frac{1}{r_1}l\frac{1}{\lambda}+o(\frac{1}{\lambda})$. we deduce that, for each $k\in\mathbb{Z}$, with $|k|$ large enough, corresponds a double root of f_0; denoted by λ_k^0 which satisfies

$$
\lambda_k^0=\frac{i}{Lr_1}k\pi+O\left(\frac{1}{k}\right).
$$

We will now use Rouché's Theorem. Let $B_k=B(\frac{i}{Lr_1}k\pi,r_k)$ be the ball of centrum $ik\pi$ and radius $r_k=\frac{1}{k^{\frac{1}{4}}}$ and $\lambda\in\partial B_k$ (i.e $\lambda=\frac{i}{Lr_1}k\pi+r_ke^{i\theta},\theta\in[0,2\pi]$). Then we successively have:

$$
L(t_1+t_3)(\lambda)=2ik\pi+2Lr_1r_ke^{i\theta}+O\left(\frac{1}{k}\right)
$$

$$
\begin{aligned}
e^{L(t_1+t_3)(\lambda)} &= e^{2Lr_1r_ke^{i\theta}+O\left(\frac{1}{k}\right)} \\
&= 1+2Lr_1r_ke^{i\theta}+O(r_k^2).
\end{aligned}
$$

and

$$\begin{array}{rcl} f_0(\lambda) & = & (1-2Lr_1r_ke^{i\theta}+O(r_k^2))(2Lr_1r_ke^{i\theta}+O(r_k^2))^2 \\ & = & (1-2Lr_1r_ke^{i\theta}+O(r_k^2))(4L^2r_1^2r_k^2e^{2i\theta}+O(r_k^3)) \\ & = & 4r_1^2L^2r_k^2e^{2i\theta}+O(r_k^3). \end{array}$$

It follows that there exists a positive constant c such that

$$\forall \lambda \in \partial B_k,\ |f_0(\lambda)| \geq cr_k^2 = \frac{c}{\sqrt{k}}.$$

Then we deduce from (2.49) that $|f(\lambda)-f_0(\lambda)| = O\left(\frac{1}{\lambda}\right) = O\left(\frac{1}{k}\right)$. It follows that, for $|k|$ large enough

$$\forall \lambda \in \partial B_k,\ |f(\lambda)-f_0(\lambda)| < |f_0(\lambda)|.$$

Since the imaginary axis is an asymptote for the spectrum of $\mathcal{A}$ then system (2.33) is not uniformly stable.
More information concerning the asymptotic behavior of the spectrum of $\mathcal{A}$ is given by:

Proposition 2.4.1 *(Asymptotic expansions for the eigenvalues of $\mathcal{A}$) Assume Condition*

$$(H) \qquad \frac{\rho_1}{m_1} \neq \frac{\rho_2}{m_2} \quad or \quad L\sqrt{l} \neq 2k\pi, \quad k \in \mathbb{N}^*.$$

Then the large eigenvalues of the dissipative operator $\mathcal{A}$ are simple and can be split into two families $(\lambda_k^j)_{k\in\mathbb{Z},|k|\geq k_0}$, $j=1,2,$ $(k_0 \in \mathbb{N}$, chosen large enough). Moreover, we have the following asymptotic expansions for the eigenvalues of $\mathcal{A}$:

$$\lambda_k^1 = \frac{i}{Lr_1}k\pi + \frac{iq_1}{k} + \frac{\tilde{\alpha}_1}{k^{3-\alpha}} + \frac{\tilde{q}_1}{|k|^{3-\alpha}} + o(\frac{1}{k^{3-\alpha}}),\ q_1 \in \mathbb{R}, \tilde{\alpha}_1 \in i\mathbb{R}, \tilde{q}_1 \in \mathbb{R}, \tilde{q}_1 < 0, k \geq k_0$$
$$\lambda_k^1 = \overline{\lambda_{-k}^1}, \quad k \leq -k_0.$$

$$\lambda_k^2 = \frac{i}{Lr_1}k\pi + \frac{iq_2}{k} + \frac{\tilde{\alpha}_2}{k^{3-\alpha}} + \frac{\tilde{q}_2}{|k|^{3-\alpha}} + o(\frac{1}{k^{3-\alpha}}),\ q_2 \in \mathbb{R}, \tilde{\alpha}_2 \in i\mathbb{R}, \tilde{q}_2 \in \mathbb{R}, \tilde{q}_2 < 0, k \geq k_0$$
$$\lambda_k^2 = \overline{\lambda_{-k}^2}, \quad k \leq -k_0.$$

Proof. Let $\lambda_k = \lambda_k^j$ with $j=1$ or $j=2$. It follows

$$\lambda_k = \frac{i}{Lr_1}k\pi + \varepsilon_k. \tag{2.61}$$

Using (2.34)-(2.35), we get

(2.62)
$$
\begin{aligned}
(t_1+t_3)L &= 2lr_1\lambda_k + \frac{lL}{4r_1}\frac{1}{\lambda_k} - \frac{5}{64}\frac{L^2l^2}{r_1^3}\frac{1}{\lambda^3} + O\left(\frac{1}{\lambda^3}\right)\\
&= 2ik\pi + 2Lr_1\varepsilon_k - i\frac{lL^2}{4k\pi} + o\left(\frac{1}{k^2}\right) + o(\varepsilon_k)\\
2t_1L &= 2r_1\lambda_k + iL\sqrt{l} + \frac{lL}{4r_1}\frac{1}{\lambda_k} - \frac{i}{8}\frac{Ll\sqrt{l}}{r_1^2}\frac{1}{\lambda^2} + O\left(\frac{1}{\lambda^3}\right)\\
&= 2ik\pi + 2Lr_1\varepsilon_k + iL\sqrt{l} - i\frac{lL^2}{4k\pi} + \frac{i}{8}\frac{L^3l\sqrt{l}}{\pi^2k^2} + o\left(\frac{1}{k^2}\right) + o(\varepsilon_k)\\
2t_3L &= 2r_1\lambda_k - iL\sqrt{l} + \frac{lL}{4r_1}\frac{1}{\lambda_k} + \frac{i}{8}\frac{Ll\sqrt{l}}{r_1^2}\frac{1}{\lambda^2} + O\left(\frac{1}{\lambda^3}\right)\\
&= 2ik\pi + 2Lr_1\varepsilon_k - iL\sqrt{l} - i\frac{lL^2}{4k\pi} - \frac{i}{8}\frac{L^3l\sqrt{l}}{\pi^2k^2} + o\left(\frac{1}{k^2}\right) + o(\varepsilon_k).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
e^{L(t_1+t_3)} &= 1 + 2Lr_1\varepsilon_k - i\frac{lL^2}{4k\pi} - \frac{l^2L^4}{32\pi^2k^2} - i\frac{lL^3r_1}{2\pi}\frac{\varepsilon_k}{k} + o(\varepsilon_k) + o(\frac{\varepsilon_k}{k}) + o(\frac{1}{k^2})\\
e^{2Lt_1} &= e^{iL\sqrt{l}}\left(1 + 2Lr_1\varepsilon_k - i\frac{lL^2}{4k\pi} - \frac{l^2L^4}{32\pi^2k^2} - i\frac{lL^3r_1}{2\pi}\frac{\varepsilon_k}{k} + \frac{i}{8}\frac{L^3l\sqrt{l}}{\pi^2k^2} + o(\frac{\varepsilon_k}{k}) + o(\frac{1}{k^2})\right)\\
e^{2Lt_3} &= e^{-iL\sqrt{l}}\left(1 + 2Lr_1\varepsilon_k - i\frac{lL^2}{4k\pi} - \frac{l^2L^4}{32\pi^2k^2} - i\frac{lL^3r_1}{2\pi}\frac{\varepsilon_k}{k} - \frac{i}{8}\frac{L^3l\sqrt{l}}{\pi^2k^2} + o(\frac{\varepsilon_k}{k}) + o(\frac{1}{k^2})\right).
\end{aligned}
$$

(2.63)

Using (2.49), inserting (2.63) into $f(\lambda_k)$ and keeping only the terms greater than or equal to $O(\frac{1}{k^2})$, we obtain after calculations

(2.64)
$$
\begin{aligned}
f(\lambda_k) &= 4L^2r_1^2\varepsilon_k^2 - \left(4\tfrac{i}{\pi}\tfrac{L^2}{k}mr_1^2 + \tfrac{i}{\pi}\tfrac{L^3}{k}lr_1\right)\varepsilon_k - \left(\tfrac{1}{16\pi^2}\tfrac{L^4}{k^2}l^2 + \tfrac{1}{2\pi^2}\tfrac{L^3}{k^2}lmr_1\right.\\
&\quad\left. + \frac{L^2r_1^2}{\pi^2k^2}\Big(4\widetilde{\widetilde{m}} - 4A\sin^2(\frac{L\sqrt{l}}{2}) - B\sin(L\sqrt{l})\Big)\right) + o(\varepsilon_k^2) + o(\frac{\varepsilon_k}{k}) + o(\frac{1}{k^2}) = 0,
\end{aligned}
$$

where

$$
m = \frac{m_2\rho_1 + m_1\rho_2}{m_1m_2r_1},\ \widetilde{\widetilde{m}} = \frac{b\rho_1}{m_1m_2},\ A = \frac{l}{4r_1^2},\ B = b\rho_1\sqrt{l}\left(\frac{1}{m_1\rho_2} - \frac{1}{m_2\rho_1}\right).
$$

Multiplying (2.64) by k^2 leads to:

$$
\begin{aligned}
&4L^2r_1^2(k\varepsilon_k)^2 - i\left(4\tfrac{1}{\pi}L^2mr_1^2 + \tfrac{1}{\pi}L^3lr_1\right)(k\varepsilon_k)\\
&-\left(\tfrac{L^4l^2}{16\pi^2} + \tfrac{L^3lmr_1}{2\pi^2} + \frac{L^2r_1^2}{\pi^2}\Big(4\widetilde{\widetilde{m}} - 4A\sin^2\left(\frac{L\sqrt{l}}{2}\right) - B\sin(L\sqrt{l})\Big)\right) + o(1) + o(k\varepsilon_k) + o(k^2\varepsilon_k^2) = 0.
\end{aligned}
$$

Thus $k\varepsilon_k$ is bounded and

$$
\begin{aligned}
&4L^2r_1^2(k\varepsilon_k)^2 - i\left(4\tfrac{L^2mr_1^2}{\pi} + \tfrac{L^3lr_1}{\pi}\right)(k\varepsilon_k)\\
&-\left(\tfrac{L^4l^2}{16\pi^2} + \tfrac{L^3lmr_1}{2\pi^2} + \frac{L^2r_1^2}{\pi^2}\Big(4\widetilde{\widetilde{m}} - 4A\sin^2\left(\frac{L\sqrt{l}}{2}\right) - B\sin(L\sqrt{l})\Big)\right) + o(1) = 0.
\end{aligned}
$$

The previous equation has two solutions

$$k\varepsilon_k = \tfrac{1}{8\pi r_1}\left(4imr_1 - 4r_1\sqrt{4\widetilde{\widetilde{m}} - 2A + 2A\cos L\sqrt{l} - B\sin L\sqrt{l} - m^2} + iLl\right) \text{ or}$$
$$k\varepsilon_k = \tfrac{1}{8\pi r_1}\left(4imr_1 + 4r_1\sqrt{4\widetilde{\widetilde{m}} - 2A + 2A\cos L\sqrt{l} - B\sin L\sqrt{l} - m^2} + iLl\right).$$

It holds:

$$\varepsilon_k = \tfrac{1}{8\pi r_1 k}\left(4imr_1 - 4r_1\sqrt{4\widetilde{\widetilde{m}} - 2A + 2A\cos L\sqrt{l} - B\sin L\sqrt{l} - m^2} + iLl\right) \text{ or}$$
$$\varepsilon_k = \tfrac{1}{8\pi r_1 k}\left(4imr_1 + 4r_1\sqrt{4\widetilde{\widetilde{m}} - 2A + 2A\cos L\sqrt{l} - B\sin L\sqrt{l} - m^2} + iLl\right).$$

Set

$$\begin{aligned} P &= 4\widetilde{\widetilde{m}} - 2A + 2A\cos(L\sqrt{l}) - B\sin(L\sqrt{l}) - m^2 \\ &= 4\widetilde{\widetilde{m}} - m^2 - 2A + 2A\cos(L\sqrt{l}) - B\sin(L\sqrt{l}). \end{aligned}$$

As $r_1^2 = r_2^2 = \dfrac{\rho_2}{b}$, we deduce that

$$4\widetilde{\widetilde{m}} - m^2 = -\frac{1}{r_1^2}\left(\frac{\rho_1}{m_1} - \frac{\rho_2}{m_2}\right)^2.$$

Then

$$\begin{aligned} P &= -\frac{1}{r_1^2}\left(\frac{\rho_1}{m_1} - \frac{\rho_2}{m_2}\right)^2 - \frac{\sqrt{l}}{r_1^2}\left(\frac{\rho_1}{m_1} - \frac{\rho_2}{m_2}\right)\sin(L\sqrt{l}) - \frac{1}{2}\frac{l}{r_1^2} + \frac{1}{2}\frac{l}{r_1^2}\cos(L\sqrt{l}) \\ &= -\frac{1}{r_1^2}\left(\frac{\sqrt{l}}{2}\sin(L\sqrt{l}) + \left(\frac{\rho_1}{m_1} - \frac{\rho_2}{m_2}\right)\right)^2 - \frac{l}{4r_1^2}(\cos(L\sqrt{l}) - 1)^2. \end{aligned}$$

Hence

$$\varepsilon_k = \tfrac{i}{8\pi r_1 k}\left(4mr_1 - 2\sqrt{\left(\sqrt{l}\sin(L\sqrt{l}) + 2\left(\frac{\rho_1}{m_1} - \frac{\rho_2}{m_2}\right)\right)^2 + (\cos(L\sqrt{l}) - 1)^2} + Ll\right) + o\left(\tfrac{1}{k}\right) \text{ or}$$
$$\varepsilon_k = \tfrac{i}{8\pi r_1 k}\left(4mr_1 + 2\sqrt{\left(\sqrt{l}\sin(L\sqrt{l}) + 2\left(\frac{\rho_1}{m_1} - \frac{\rho_2}{m_2}\right)\right)^2 + (\cos(L\sqrt{l}) - 1)^2} + Ll\right) + o\left(\tfrac{1}{k}\right).$$

Step 2. From Step 1, we can write

$$\begin{aligned} \lambda_k^1 &= \frac{i}{Lr_1}k\pi + i\frac{q_1}{k} + \frac{\varepsilon_k^1}{k}, \\ \lambda_k^2 &= \frac{i}{Lr_1}k\pi + i\frac{q_2}{k} + \frac{\varepsilon_k^2}{k}, \end{aligned}$$

where $\varepsilon_k^j = o(1)$.

$$(t_1 + t_3)L = i\frac{2r_1 L q_j}{k} + \frac{2r_1 L\varepsilon_k^j}{k} - i\frac{lL^2}{4k\pi} + i\frac{lL^3 r_1}{4k^3\pi^2}q_j + \frac{lL^3 r_1}{4k^3\pi^2}\varepsilon_k^j - i\frac{5}{64}\frac{l^2L^4}{k^3\pi^3}$$

$$\begin{aligned}
Lt_1 &= k\pi i + i\frac{L\sqrt{l}}{2} + i\frac{L}{k}r_1q_j - \frac{1}{8}\frac{i}{\pi}\frac{L^2}{k}l - \frac{1}{2}\frac{L^2}{k^2}r_1^2q_j^2 + \frac{L}{k}\varepsilon_k^j r_1 + \frac{i}{8}\frac{L^3l\sqrt{l}}{\pi^2k^2}\\
Lt_3 &= k\pi i - i\frac{L\sqrt{l}}{2} + i\frac{L}{k}r_1q_j - \frac{1}{8}\frac{i}{\pi}\frac{L^2}{k}l - \frac{1}{2}\frac{L^2}{k^2}r_1^2q_j^2 + \frac{L}{k}\varepsilon_k^j r_1 - \frac{i}{8}\frac{L^3l\sqrt{l}}{\pi^2k^2}\\
e^{(t_1+t_3)L} - 1 &= 2i\tfrac{L}{k}r_1q_j + 2\tfrac{L^2}{k^2}(\varepsilon_k^j)^2r_1^2 - \tfrac{1}{4}\tfrac{i}{\pi}\tfrac{L^2}{k}l - 2\tfrac{L^2}{k^2}r_1^2q_j^2 + 2\tfrac{L}{k}\varepsilon_k^j r_1 - \tfrac{1}{32\pi^2}\tfrac{L^4}{k^2}l^2 - \tfrac{5}{64}\tfrac{i}{\pi^3}\tfrac{L^4}{k^3}l^2\\
&\quad -\tfrac{5}{256\pi^4}\tfrac{L^6}{k^4}l^3 + 4i\tfrac{L^2}{k^2}\varepsilon_k^j r_1^2q_j + \tfrac{7}{32\pi^3}\tfrac{L^5}{k^4}l^2r_1q_j - \tfrac{1}{2}\tfrac{i}{\pi}\tfrac{L^3}{k^2}l\varepsilon_k^j r_1 + \tfrac{1}{4\pi^2}\tfrac{L^3}{k^3}l\varepsilon_k^j r_1 - \tfrac{1}{2\pi^2}\tfrac{L^4}{k^4}lr_1^2q_j^2\\
&\quad +\tfrac{1}{2\pi}\tfrac{L^3}{k^2}lr_1q_j + \tfrac{1}{4}\tfrac{i}{\pi^2}\tfrac{L^3}{k^3}lr_1q_j.
\end{aligned}$$

$$\begin{aligned}
e^{t_1L} &= (-1)^k e^{\frac{iL\sqrt{l}}{2}}\Big(1 + i\tfrac{L}{k}r_1q_j - \tfrac{1}{8}\tfrac{i}{\pi}\tfrac{L^2}{k}l - \tfrac{1}{2}\tfrac{L^2}{k^2}r_1^2q_j^2 + \tfrac{L}{k}\varepsilon_k^j r_1 - \tfrac{1}{128\pi^2}\tfrac{L^4}{k^2}l^2 + \tfrac{1}{16}\tfrac{i}{\pi^2}\tfrac{L^3}{k^2}l^{\frac{3}{2}} + i\tfrac{L^2}{k^2}\varepsilon_k^j r_1^2q_j\\
&\quad -\tfrac{1}{8}\tfrac{i}{\pi}\tfrac{L^3}{k^2}l\varepsilon_k^j r_1 + \tfrac{1}{8\pi}\tfrac{L^3}{k^2}lr_1q_j\Big)\\
e^{t_3L} &= (-1)^k e^{-\frac{iL\sqrt{l}}{2}}\Big(1 + i\tfrac{L}{k}r_1q_j - \tfrac{1}{8}\tfrac{i}{\pi}\tfrac{L^2}{k}l - \tfrac{1}{2}\tfrac{L^2}{k^2}r_1^2q_j^2 + \tfrac{L}{k}\varepsilon_k^j r_1 - \tfrac{1}{128\pi^2}\tfrac{L^4}{k^2}l^2 - \tfrac{1}{16}\tfrac{i}{\pi^2}\tfrac{L^3}{k^2}l^{\frac{3}{2}} + i\tfrac{L^2}{k^2}\varepsilon_k^j r_1^2q_j\\
&\quad -\tfrac{1}{8}\tfrac{i}{\pi}\tfrac{L^3}{k^2}l\varepsilon_k^j r_1 + \tfrac{1}{8\pi}\tfrac{L^3}{k^2}lr_1q_j\Big).
\end{aligned}$$

Using (2.49), Taylor series and simplification in the term of order $1/k^2$ coming from Step 1, we get

$$\begin{aligned}
f(\lambda_k) &= \Big(4\tfrac{L^2}{k^2}r_1^2\Big)(\varepsilon_k^j)^2 + \Big(8i\tfrac{L^2}{k^2}r_1^2q_j - 24\tfrac{L^3}{k^3}r_1^3q_j^2 - 4\tfrac{i}{\pi}\tfrac{L^2}{k^2}mr_1^2 - \tfrac{3}{8\pi^2}\tfrac{L^5}{k^3}l^2r_1 - \tfrac{i}{\pi}\tfrac{L^3}{k^2}lr_1 - \tfrac{2}{\pi^2}\tfrac{L^4}{k^3}lmr_1^2\\
&+\tfrac{6}{\pi}\tfrac{L^4}{k^3}lr_1^2q_j + \tfrac{16}{\pi}\tfrac{L^3}{k^3}mr_1^3q_j\Big)\varepsilon_k^j + (4\tfrac{L^4}{k^4}r_1^4q_j^4 - 8i\tfrac{L^3}{k^3}r_1^3q_j^3 + \tfrac{1}{64}\tfrac{i}{\pi^3}\tfrac{L^6}{k^3}l^3 - \tfrac{5}{128\pi^4}\tfrac{L^6}{k^4}l^3 + \tfrac{1}{1024\pi^4}\tfrac{L^8}{k^4}l^4\\
&+\tfrac{1}{8}\tfrac{i}{\pi^3}\tfrac{L^5}{k^3}l^2mr_1 - \tfrac{5}{32\pi^4}\tfrac{L^5}{k^4}l^2mr_1 + \tfrac{1}{64\pi^4}\tfrac{L^7}{k^4}l^3mr_1 + \tfrac{3}{8\pi^2}\tfrac{L^6}{k^4}l^2r_1^2q_j^2 - \tfrac{3}{8}\tfrac{i}{\pi^2}\tfrac{L^5}{k^3}l^2r_1q_j + \tfrac{7}{16\pi^3}\tfrac{L^5}{k^4}l^2r_1q_j\\
&-\tfrac{1}{32\pi^3}\tfrac{L^7}{k^4}l^3r_1q_j + 3\tfrac{i}{\pi}\tfrac{L^4}{k^3}lr_1^2q_j^2 + 8\tfrac{i}{\pi}\tfrac{L^3}{k^3}mr_1^3q_j^2 - \tfrac{1}{\pi^2}\tfrac{L^4}{k^4}lr_1^2q_j^2 - \tfrac{2}{\pi}\tfrac{L^5}{k^4}lr_1^3q_j^3 - \tfrac{4}{\pi^2}\tfrac{L^3}{k^4}mr_1^3q_j^2\\
&-\tfrac{8}{\pi}\tfrac{L^4}{k^4}mr_1^4q_j^3 - 2\tfrac{i}{\pi^2}\tfrac{L^4}{k^3}lmr_1^2q_j + \tfrac{1}{\pi^3}\tfrac{L^4}{k^4}lmr_1^2q_j + \tfrac{3}{\pi^2}\tfrac{L^5}{k^4}lmr_1^3q_j^2 - \tfrac{3}{8\pi^3}\tfrac{L^6}{k^4}l^2mr_1^2q_j\\
&+s_{2-\alpha}(\tfrac{Lr_1}{i\pi})^{2-\alpha}(-4\tfrac{L^2}{k^{4-\alpha}}r_1^2q_j^2 - \tfrac{1}{16\pi^2}\tfrac{L^4}{k^{4-\alpha}}l^2 + \tfrac{1}{\pi}\tfrac{L^3}{k^{4-\alpha}}lr_1q_j) + s_{3-\alpha}(\tfrac{Lr_1}{i\pi})^{3-\alpha}(4i\tfrac{L}{k^{4-\alpha}}r_1q_j - \tfrac{1}{2}\tfrac{i}{\pi}\tfrac{L^2}{k^{4-\alpha}}l)\\
&-s_{4-\alpha}(\tfrac{Lr_1}{ik\pi})^{4-\alpha} + s_{41}(\tfrac{Lr_1}{ik\pi})^4 + s_{42}(\tfrac{Lr_1}{ik\pi})^4 + s_{43}(\tfrac{Lr_1}{ik\pi})^4 + +s_3(4i\tfrac{L}{k^4}r_1q_j - \tfrac{1}{2}\tfrac{i}{\pi}\tfrac{L^2}{k^4}l)(\tfrac{Lr_1}{i\pi})^3\\
&+\frac{l}{4r_1^2}(\frac{Lr_1}{i\pi})^2(2i\sin(\frac{L\sqrt{l}}{2}))^2\left(2i\frac{L}{k^3}r_1q_j - \frac{1}{4}\frac{i}{\pi}\frac{L^2}{k^3}l\right)\\
&+\tfrac{b\rho_1}{m_1m_2}(8i\tfrac{L}{k^3}r_1q_j - \tfrac{i}{\pi}\tfrac{L^2}{k^3}l)(\tfrac{Lr_1}{i\pi})^2 + \tfrac{l}{4r_1^2}(\tfrac{Lr_1}{i\pi})^2(-4\tfrac{L^2}{k^4}r_1^2q_j^2 - \tfrac{1}{16\pi^2}\tfrac{L^4}{k^4}l^2 + \tfrac{1}{\pi}\tfrac{L^3}{k^4}lr_1q_j - \sin(L\sqrt{l})\tfrac{1}{2}\tfrac{1}{\pi^2}\tfrac{L^3}{k^4}l^{\frac{3}{2}}\\
&-4\sin^2(\tfrac{L\sqrt{l}}{2})(-3\tfrac{L^2}{k^4}r_1^2q_j^2 - \tfrac{1}{32\pi^2}\tfrac{L^4}{k^4}l^2 + \tfrac{1}{2\pi}\tfrac{L^3}{k^4}lr_1q_j)) + \tfrac{b\rho_1}{m_1m_2}(-12\tfrac{L^2}{k^4}r_1^2q_j^2 - \tfrac{3}{16\pi^2}\tfrac{L^4}{k^4}l^2 + \tfrac{3}{\pi}\tfrac{L^3}{k^4}lr_1q_j)(\tfrac{Lr_1}{i\pi})^2\\
&+2is_2(\tfrac{Lr_1}{\pi})^2\sin(L\sqrt{l})(2i\tfrac{L}{k^3}r_1q_j - \tfrac{1}{4}\tfrac{i}{\pi}\tfrac{L^2}{k^3}l) + 2is_2(\tfrac{Lr_1}{\pi})^2\sin(L\sqrt{l})(-2\tfrac{L^2}{k^4}r_1^2q_j^2 - \tfrac{1}{32\pi^2}\tfrac{L^4}{k^4}l^2 + \tfrac{1}{2\pi}\tfrac{L^3}{k^4}lr_1q_j)\\
&+s_2(\tfrac{Lr_1}{i\pi})^2\tfrac{i}{4}L^3\sqrt{l}l\tfrac{\cos L\sqrt{l}}{\pi^2k^4} + s_{2-\alpha}(\tfrac{Lr_1}{i\pi})^{2-\alpha}(-\tfrac{i}{\pi}\tfrac{L^3}{k^{4-\alpha}}l\varepsilon_k^j r_1 + 8i\tfrac{L^2}{k^{4-\alpha}}\varepsilon_k^j r_1^2q_j) + s_{3-\alpha}(\tfrac{Lr_1}{i\pi})^{3-\alpha}\tfrac{4L}{k^{4-\alpha}}\varepsilon_k^j r_1\\
&+2\tfrac{L^3l}{\pi^2k^3}r_1\sin^2(\tfrac{L\sqrt{l}}{2})\varepsilon_k^j + \tfrac{b\rho_1}{m_1m_2}(8\tfrac{L}{k^3}\varepsilon_k^j r_1)(\tfrac{Lr_1}{i\pi})^2 + s_{2-\alpha}(\tfrac{Lr_1}{i\pi})^{2-\alpha}(4\tfrac{L^2}{k^{4-\alpha}}(\varepsilon_k^j)^2r_1^2) + 4is_2\tfrac{L^3r_1^3}{\pi^2k^3}\sin(L\sqrt{l})\varepsilon_k^j,
\end{aligned}$$

where

$$\begin{aligned}
s_2 &= -ib\frac{\rho_1}{2}\sqrt{l}(\frac{1}{m_1\rho_2} - \frac{1}{m_2\rho_1}),\ s_{2-\alpha} = \frac{\gamma_1m_2 + \gamma_2m_1}{m_1m_2},\ m = \frac{\rho_1m_2 + \rho_2m_1}{m_1m_2r_1},\ \ s_{3-\alpha} = \frac{\gamma_1\rho_2 + \gamma_2\rho_1}{m_1m_2r_1},\\
&\quad s_3 = \frac{1}{8}\frac{lb\rho_1}{r_1}(\frac{1}{m_1\rho_2} + \frac{5}{m_2\rho_1}),\ s_{4-\alpha} = \frac{b\sqrt{l}\rho_1}{m_1m_2}(\frac{\gamma_1}{\rho_1} + \frac{\gamma_2}{\rho_2})\sin(L\sqrt{l})\\
&\quad s_{41} = \frac{l^2}{r_1^4}\sin^2(\frac{L\sqrt{l}}{2}),\ s_{42} = \frac{bl^2}{r_1^4}(\cos^2(\frac{L\sqrt{l}}{2}) + 1),\ s_{43} = -\frac{bl^2\rho_1}{8r_1^2}\sin(L\sqrt{l}).
\end{aligned}$$

Considering only the dominant terms of $\frac{1}{k}$, the following is obtaining:

$$f(\lambda_k) = \left(4\tfrac{L^2}{k^2}r_1^2\right)(\varepsilon_k^j)^2 + \left(8i\tfrac{L^2}{k^2}r_1^2q_j - 4\tfrac{i}{\pi}\tfrac{L^2}{k^2}mr_1^2 - \tfrac{i}{\pi}\tfrac{L^3}{k^2}lr_1\right)\varepsilon_k^j - 8i\tfrac{L^3}{k^3}r_1^3q_j^3 + \tfrac{1}{64}\tfrac{i}{\pi^3}\tfrac{L^6}{k^3}l^3$$
$$+\tfrac{1}{8}\tfrac{i}{\pi^3}\tfrac{L^5}{k^3}l^2mr_1 - \tfrac{3}{8}\tfrac{i}{\pi^2}\tfrac{L^5}{k^3}l^2r_1q_j + 3\tfrac{i}{\pi}\tfrac{L^4}{k^3}lr_1^2q_j^2 + 8\tfrac{i}{\pi}\tfrac{L^3}{k^3}mr_1^3q_j^2 - 2\tfrac{i}{\pi^2}\tfrac{L^4}{k^3}lmr_1^2q_j$$
$$+s_{2-\alpha}(\tfrac{Lr_1}{i\pi})^{2-\alpha}(-4\tfrac{L^2}{k^{4-\alpha}}r_1^2q_j^2 - \tfrac{1}{16\pi^2}\tfrac{L^4}{k^{4-\alpha}}l^2 + \tfrac{1}{\pi}\tfrac{L^3}{k^{4-\alpha}}lr_1q_j) + s_{3-\alpha}(\tfrac{Lr_1}{i\pi})^{3-\alpha}(4i\tfrac{L}{k^{4-\alpha}}r_1q_j - \tfrac{1}{2}\tfrac{i}{\pi}\tfrac{L^2}{k^{4-\alpha}}l)$$
$$-s_{4-\alpha}(\tfrac{Lr_1}{ik\pi})^{4-\alpha} + \frac{l}{4r_1^2}(\frac{Lr_1}{i\pi})^2(2i\sin(\frac{L\sqrt{l}}{2}))^2\left(2i\frac{L}{k^3}r_1q_j - \frac{1}{4}\frac{i}{\pi}\frac{L^2}{k^3}l\right)$$
$$+\tfrac{b\rho_1}{m_1m_2}(8i\tfrac{L}{k^3}r_1q_j - \tfrac{i}{\pi}\tfrac{L^2}{k^3}l)(\tfrac{Lr_1}{i\pi})^2 + 2is_2(\tfrac{Lr_1}{\pi})^2\sin(L\sqrt{l})(2i\tfrac{L}{k^3}r_1q_j - \tfrac{1}{4}\tfrac{i}{\pi}\tfrac{L^2}{k^3}l).$$

We remark that

$$-8i\tfrac{L^3}{k^3}r_1^3q_j^3 + \tfrac{1}{64}\tfrac{i}{\pi^3}\tfrac{L^6}{k^3}l^3 + \tfrac{1}{8}\tfrac{i}{\pi^3}\tfrac{L^5}{k^3}l^2mr_1 - \tfrac{3}{8}\tfrac{i}{\pi^2}\tfrac{L^5}{k^3}l^2r_1q_j + 3\tfrac{i}{\pi}\tfrac{L^4}{k^3}lr_1^2q_j^2 + 8\tfrac{i}{\pi}\tfrac{L^3}{k^3}mr_1^3q_j^2$$
$$-2\tfrac{i}{\pi^2}\tfrac{L^4}{k^3}lmr_1^2q_j + \frac{l}{4r_1^2}(\frac{Lr_1}{i\pi})^2(2i\sin(\frac{L\sqrt{l}}{2}))^2\left(2i\frac{L}{k^3}r_1q_j - \frac{1}{4}\frac{i}{\pi}\frac{L^2}{k^3}l\right)$$
$$+\tfrac{b\rho_1}{m_1m_2}(8i\tfrac{L}{k^3}r_1q_j - \tfrac{i}{\pi}\tfrac{L^2}{k^3}l)(\tfrac{Lr_1}{i\pi})^2 + 2is_2(\tfrac{Lr_1}{\pi})^2\sin(L\sqrt{l})(2i\tfrac{L}{k^3}r_1q_j - \tfrac{1}{4}\tfrac{i}{\pi}\tfrac{L^2}{k^3}l) = 0.$$

Then ε_k^j satisfy

$$\begin{aligned} f(\lambda_k) \;=\; & \left(4\tfrac{L^2}{k^2}r_1^2\right)(\varepsilon_k^j)^2 + \left(8i\tfrac{L^2}{k^2}r_1^2q_j - 4\tfrac{i}{\pi}\tfrac{L^2}{k^2}mr_1^2 - \tfrac{i}{\pi}\tfrac{L^3}{k^2}lr_1\right)\varepsilon_k^j \\ & +s_{2-\alpha}(\tfrac{Lr_1}{i\pi})^{2-\alpha}(-4\tfrac{L^2}{k^{4-\alpha}}r_1^2q_j^2 - \tfrac{1}{16\pi^2}\tfrac{L^4}{k^{4-\alpha}}l^2 + \tfrac{1}{\pi}\tfrac{L^3}{k^{4-\alpha}}lr_1q_j) \\ & +s_{3-\alpha}(\tfrac{Lr_1}{i\pi})^{3-\alpha}(4i\tfrac{L}{k^{4-\alpha}}r_1q_j - \tfrac{1}{2}\tfrac{i}{\pi}\tfrac{L^2}{k^{4-\alpha}}l) - s_{4-\alpha}(\tfrac{Lr_1}{ik\pi})^{4-\alpha} \end{aligned} \tag{2.65}$$

Multiplying (2.65) by k^4 leads to:

$$f(\lambda_k) = (4L^2r_1^2)(k\varepsilon_k^j)^2 + k\left(8iL^2r_1^2q_j - 4\tfrac{i}{\pi}L^2mr_1^2 - \tfrac{i}{\pi}L^3lr_1\right)(k\varepsilon_k^j) + s_{2-\alpha}(\tfrac{Lr_1}{i\pi})^{2-\alpha}k^\alpha(-4L^2r_1^2q_j^2$$
$$-\tfrac{1}{16\pi^2}L^4l^2 + \tfrac{1}{\pi}L^3lr_1q_j) + s_{3-\alpha}(\tfrac{Lr_1}{i\pi})^{3-\alpha}k^\alpha(4iLr_1q_j - \tfrac{1}{2}\tfrac{i}{\pi}L^2l) - s_{4-\alpha}k^\alpha(\tfrac{Lr_1}{i\pi})^{4-\alpha} + o(1) = 0.$$

Hence ε_k^1 and ε_k^2 satisfy

$$\left(4L^2r_1^2\right)(k\varepsilon_k^1)^2 - 4k\frac{i}{\pi}L^2\sqrt{\theta}r_1^2(k\varepsilon_k^1) + I_1k^\alpha(\frac{Lr_1}{i\pi})^{2-\alpha}\frac{L^2r_1^2}{\pi^2} + o(1) = 0$$

$$\left(4L^2r_1^2\right)(k\varepsilon_k^2)^2 + 4k\frac{i}{\pi}L^2\sqrt{\theta}r_1^2(k\varepsilon_k^2) + I_2k^\alpha(\frac{Lr_1}{i\pi})^{2-\alpha}\frac{L^2r_1^2}{\pi^2} + o(1) = 0,$$

where

$$\begin{aligned} I_1 &= 2s_{3-\alpha}(m-\sqrt{\theta}) - s_{2-\alpha}(m-\sqrt{\theta})^2 + s_{4-\alpha} \\ I_2 &= 2s_{3-\alpha}(m+\sqrt{\theta}) - s_{2-\alpha}(m+\sqrt{\theta})^2 + s_{4-\alpha} \\ \theta &= \frac{1}{r_1^2}\left(\frac{\sqrt{l}}{2}\sin(L\sqrt{l}) + \left(\frac{\rho_1}{m_1} - \frac{\rho_2}{m_2}\right)\right)^2 + \frac{l}{4r_1^2}(\cos(L\sqrt{l})-1)^2 \\ k\varepsilon_k^1 &= -\frac{I_1}{4\pi\sqrt{\theta}}\left(\frac{Lr_1}{\pi}\right)^{2-\alpha}(\cos(1-\alpha)\frac{\pi}{2} - i\cos(1-\alpha)\frac{\pi}{2})\frac{1}{k^{1-\alpha}} + o(1) \\ k\varepsilon_k^2 &= \frac{I_2}{4\pi\sqrt{\theta}}\left(\frac{Lr_1}{\pi}\right)^{2-\alpha}(\cos(1-\alpha)\frac{\pi}{2} - i\cos(1-\alpha)\frac{\pi}{2})\frac{1}{k^{1-\alpha}} + o(1). \end{aligned}$$

Then

$$\begin{aligned}\varepsilon_k^1 &= -\frac{I_1}{4\pi\sqrt{\theta}}\left(\frac{Lr_1}{\pi}\right)^{2-\alpha}(\cos(1-\alpha)\frac{\pi}{2} - i\cos(1-\alpha)\frac{\pi}{2})\frac{1}{k^{2-\alpha}} + o\left(\frac{1}{k^{2-\alpha}}\right)\\ \varepsilon_k^2 &= \frac{I_2}{4\pi\sqrt{\theta}}\left(\frac{Lr_1}{\pi}\right)^{2-\alpha}(\cos(1-\alpha)\frac{\pi}{2} - i\cos(1-\alpha)\frac{\pi}{2})\frac{1}{k^{2-\alpha}} + o\left(\frac{1}{k^{2-\alpha}}\right).\end{aligned}$$

Since all the eigenvalues of $\mathcal{A}$ are on the left of the imaginary axis, necessarily $I_1 > 0$ and $I_2 < 0$. Note that, if $\gamma_1 = \gamma_2 = 0$ then $I_1 = I_2 = 0$.

Remark 2.4.1 *If condition (H) does not hold, we can study the asymptotic behavior of the spectrum of $\mathcal{A}$ but the calculation is long.*

Case 2 $r_1 \neq r_2$:
We start by the expansion of t_1 and t_3:

$$t_1 = r_1\lambda - \frac{l}{2}\frac{r_2^2}{r_1(r_1^2-r_2^2)}\frac{1}{\lambda} - \frac{l^2}{8}\frac{r_2^4(5r_1^2-r_2^2)}{r_1^3(r_1^2-r_2^2)^3}\frac{1}{\lambda^3} + O\left(\frac{1}{\lambda^5}\right). \tag{2.66}$$

$$t_3 = r_2\lambda + \frac{l}{2}\frac{r_2}{(r_1^2-r_2^2)}\frac{1}{\lambda} + \frac{l^2}{8}\frac{r_2(5r_2^2-r_1^2)}{(r_1^2-r_2^2)^3}\frac{1}{\lambda^3} + O\left(\frac{1}{\lambda^5}\right). \tag{2.67}$$

$$h_1(t_1) = -l\lambda\frac{r_2^2}{r_1(r_1^2-r_2^2)} - \frac{1}{2}l^2r_2^4\frac{r_1^2+r_2^2}{r_1^3\left(r_1^2-r_2^2\right)^3}\frac{1}{\lambda} + O\left(\frac{1}{\lambda^3}\right). \tag{2.68}$$

$$h_1(t_2) = l\lambda\frac{r_2^2}{r_1(r_1^2-r_2^2)} + \frac{1}{2}l^2r_2^4\frac{r_1^2+r_2^2}{r_1^3(r_1^2-r_2^2)^3}\frac{1}{\lambda} + O\left(\frac{1}{\lambda^3}\right). \tag{2.69}$$

$$h_1(t_3) = -\lambda^3(r_1^2-r_2^2)\frac{r_2}{r_1^2} - \frac{1}{2}lr_2\frac{r_1^2-3r_2^2}{r_1^2\left(r_1^2-r_2^2\right)}\lambda + \frac{1}{8}l^2r_2\frac{r_1^4+9r_2^4-2r_1^2r_2^2}{r_1^2\left(r_1^2-r_2^2\right)^3}\frac{1}{\lambda} + O\left(\frac{1}{\lambda^3}\right). \tag{2.70}$$

$$h_1(t_4) = \lambda^3\left(r_1^2-r_2^2\right)\frac{r_2}{r_1^2} + \frac{1}{2}lr_2\frac{r_1^2-3r_2^2}{r_1^2\left(r_1^2-r_2^2\right)}\lambda - \frac{1}{8}l^2r_2\frac{r_1^4+9r_2^4-2r_1^2r_2^2}{r_1^2\left(r_1^2-r_2^2\right)^3}\frac{1}{\lambda} + O\left(\frac{1}{\lambda^3}\right). \tag{2.71}$$

$$h_2(t_1) = \lambda^2 + \frac{br_1}{m_2}\lambda + \frac{\gamma_2}{m_2}\lambda^\alpha + \eta\lambda^{\alpha-1}\gamma_2\frac{\alpha-1}{m_2} - \frac{1}{2}\frac{blr_2^2}{m_2r_1\left(r_1^2-r_2^2\right)}\frac{1}{\lambda} + O\left(\frac{1}{\lambda^{2-\alpha}}\right). \tag{2.72}$$

$$h_2(t_2) = \lambda^2 - \frac{br_1}{m_2}\lambda + \frac{\gamma_2}{m_2}\lambda^\alpha + \eta\lambda^{\alpha-1}\gamma_2\frac{\alpha-1}{m_2} + \frac{1}{2}\frac{blr_2^2}{m_2r_1\left(r_1^2-r_2^2\right)}\frac{1}{\lambda} + O\left(\frac{1}{\lambda^{2-\alpha}}\right). \tag{2.73}$$

$$h_2(t_3) = \lambda^2 + \frac{br_2}{m_2}\lambda + \frac{\gamma_2}{m_2}\lambda^\alpha + \eta\lambda^{\alpha-1}\gamma_2\frac{\alpha-1}{m_2} + \frac{1}{2}bl\frac{r_2}{m_2\left(r_1^2-r_2^2\right)}\frac{1}{\lambda} + O\left(\frac{1}{\lambda^{2-\alpha}}\right). \tag{2.74}$$

$$h_2(t_4) = \lambda^2 - \frac{br_2}{m_2}\lambda + \frac{\gamma_2}{m_2}\lambda^\alpha + \eta\lambda^{\alpha-1}\gamma_2\frac{\alpha-1}{m_2} - \frac{1}{2}bl\frac{r_2}{m_2\left(r_1^2-r_2^2\right)}\frac{1}{\lambda} + O\left(\frac{1}{\lambda^{2-\alpha}}\right). \tag{2.75}$$

$$\begin{aligned}&h_3(t_1) = -\frac{lr_2^2}{\rho_1r_1\left(r_1^2-r_2^2\right)}\lambda - \frac{lr_2^2}{m_1r_1^2\left(r_1^2-r_2^2\right)} - l\lambda^{\alpha-1}\gamma_1\frac{r_2^2}{m_1\rho_1r_1\left(r_1^2-r_2^2\right)}\\ &-\frac{1}{2}l^2r_2^4\frac{r_1^2+r_2^2}{\rho_1r_1^3\left(r_1^2-r_2^2\right)^3}\frac{1}{\lambda} + o\left(\frac{1}{\lambda}\right).\end{aligned} \tag{2.76}$$

$$h_3(t_2) = \frac{lr_2^2}{\rho_1 r_1\left(r_1^2-r_2^2\right)}\lambda - l\frac{r_2^2}{m_1 r_1^2\left(r_1^2-r_2^2\right)} + l\lambda^{\alpha-1}\gamma_1\frac{r_2^2}{m_1\rho_1 r_1\left(r_1^2-r_2^2\right)} + \frac{1}{2}l^2 r_2^4\frac{r_1^2+r_2^2}{\rho_1 r_1^3\left(r_1^2-r_2^2\right)^3}\frac{1}{\lambda} + o\left(\frac{1}{\lambda}\right). \tag{2.77}$$

$$\begin{aligned} h_3(t_3) \;=\; & -\lambda^3\left(r_1^2-r_2^2\right)\frac{r_2}{\rho_1 r_1^2} - \lambda^2\left(r_1^2-r_2^2\right)\frac{1}{m_1 r_1^2} - \lambda^{1+\alpha}\left(r_1^2-r_2^2\right)\frac{\gamma_1 r_2}{m_1\rho_1 r_1^2} - \frac{1}{2}l\lambda r_2\frac{r_1^2-3r_2^2}{\rho_1 r_1^2\left(r_1^2-r_2^2\right)} \\ & -\eta\lambda^\alpha\gamma_1 r_2\left(r_1^2-r_2^2\right)\frac{\alpha-1}{m_1\rho_1 r_1^2} + l\frac{r_2^2}{r_1^2 m_1\left(r_1^2-r_2^2\right)} - \frac{1}{2}l\lambda^{\alpha-1}\gamma_1 r_2\frac{r_1^2-3r_2^2}{m_1\rho_1 r_1^2\left(r_1^2-r_2^2\right)} \\ & -\frac{1}{2}\eta^2\lambda^{\alpha-1}\gamma_1 r_2\left(r_1^2-r_2^2\right)(\alpha-1)\frac{\alpha-2}{m_1\rho_1 r_1^2} + o\left(\frac{1}{\lambda^{1-\alpha}}\right). \end{aligned} \tag{2.78}$$

$$\begin{aligned} h_3(t_4) \;=\; & \lambda^3\left(r_1^2-r_2^2\right)\frac{r_2}{\rho_1 r_1^2} - \lambda^2\left(r_1^2-r_2^2\right)\frac{1}{m_1 r_1^2} + \lambda^{1+\alpha}\gamma_1 r_2\left(r_1-r_2\right)\frac{r_1+r_2}{m_1\rho_1 r_1^2} + \frac{1}{2}l\lambda r_2\frac{r_1^2-3r_2^2}{\rho_1 r_1^2\left(r_1^2-r_2^2\right)} \\ & +\eta\lambda^\alpha\gamma_1 r_2\left(r_1^2-r_2^2\right)\frac{\alpha-1}{m_1\rho_1 r_1^2} + l\frac{r_2^2}{r_1^2 m_1\left(r_1^2-r_2^2\right)} + \frac{1}{2}l\lambda^{\alpha-1}\gamma_1 r_2\frac{r_1^2-3r_2^2}{m_1\rho_1 r_1^2\left(r_1^2-r_2^2\right)} \\ & +\frac{1}{2}\eta^2\lambda^{\alpha-1}\gamma_1 r_2\left(r_1^2-r_2^2\right)(\alpha-1)\frac{\alpha-2}{m_1\rho_1 r_1^2} + o\left(\frac{1}{\lambda^{1-\alpha}}\right). \end{aligned} \tag{2.79}$$

Using the asymptotic development (2.68)-(2.79)

$$\begin{aligned} f(\lambda) \;=\; & \lambda^8\frac{r_2^2}{\rho_1 r_1^4}(r_1-r_2)^2(r_1+r_2)^2[e^{L(t_1+t_3)} - e^{L(t_1+t_4)} - e^{L(t_2+t_3)} + e^{L(t_2+t_4)}] \\ & +\lambda^7\frac{r_2}{m_1 m_2\rho_1 r_1^4}(r_1-r_2)^2(r_1+r_2)^2[(m_2\rho_1 + bm_1 r_1 r_2)(e^{L(t_1+t_3)} - e^{L(t_2+t_4)}) \\ & +(m_2\rho_1 - bm_1 r_1 r_2)(e^{L(t_1+t_4)} - e^{L(t_2+t_3)})] \\ & +\lambda^{6+\alpha}\frac{r_2^2(\gamma_1 m_2 + \gamma_2 m_1)}{m_1 m_2\rho_1 r_1^4}(r_1^2-r_2^2)^2[e^{L(t_1+t_3)} - e^{L(t_1+t_4)} - e^{L(t_2+t_3)} + e^{L(t_2+t_4)}] + O(\lambda^2) \\ & = \lambda^8\frac{r_2^2}{\rho_1 r_1^4}(r_1-r_2)^2(r_1+r_2)^2\Big[(e^{L(t_1+t_3)} - e^{L(t_1+t_4)} - e^{L(t_2+t_3)} + e^{L(t_2+t_4)}) \\ & +((m_2\rho_1 + bm_1 r_1 r_2)(e^{L(t_1+t_3)} - e^{L(t_2+t_4)}) + (m_2\rho_1 - bm_1 r_1 r_2)(e^{L(t_1+t_4)} - e^{L(t_2+t_3)}))\frac{1}{m_1 m_2 r_2\lambda} \\ & +\frac{(\gamma_1 m_2 + \gamma_2 m_1)}{m_1 m_2}(e^{L(t_1+t_3)} - e^{L(t_1+t_4)} - e^{L(t_2+t_3)} + e^{L(t_2+t_4)})\frac{1}{\lambda^{2-\alpha}} + O\left(\frac{1}{\lambda^2}\right)\Big]. \end{aligned} \tag{2.80}$$

We set

$$\tilde{\tilde{f}}(\lambda) = f_0(\lambda) + \frac{f_1(\lambda)}{\lambda} + \frac{f_2(\lambda)}{\lambda^{2-\alpha}} + +O\left(\frac{1}{\lambda^3}\right), \tag{2.81}$$

where

$$f_0(\lambda) = e^{L(t_1+t_3)} - e^{L(t_1+t_4)} - e^{L(t_2+t_3)} + e^{L(t_2+t_4)} = e^{-L(t_1+t_3)}(e^{2Lt_1}-1)(e^{2Lt_3}-1) \tag{2.82}$$

$$\begin{aligned} f_1(\lambda) \;=\; & \frac{(m_2\rho_1 + bm_1 r_1 r_2)}{m_1 m_2 r_2}(e^{L(t_1+t_3)} - e^{L(t_2+t_4)}) + \frac{(m_2\rho_1 - bm_1 r_1 r_2)}{m_1 m_2 r_2}(e^{L(t_1+t_4)} - e^{L(t_2+t_3)}) \\ =\; & e^{-L(t_1+t_3)}[\frac{(m_2\rho_1 + bm_1 r_1 r_2)}{m_1 m_2 r_2}(e^{2L(t_1+t_3)}-1) + \frac{(m_2\rho_1 - bm_1 r_1 r_2)}{m_1 m_2 r_2}(e^{2Lt_1} - e^{2Lt_3})] \end{aligned} \tag{2.83}$$

$$f_2(\lambda) = \frac{(\gamma_1 m_2 + \gamma_2 m_1)}{m_1 m_2}e^{-L(t_1+t_3)}(e^{2Lt_1}-1)(e^{2Lt_3}-1) \tag{2.84}$$

Lemma 2.4.2 *(Asymptotic behavior of the large eigenvalues of $\mathcal{A}$) The large eigenvalues of $\mathcal{A}$ can be split into two families $(\lambda_k^j)_{k\in\mathbb{Z},|k|\geq k_0}$, $j=1,2$, ($k_0 \in \mathbb{N}$ chosen large enough). The following asymptotic expansions hold:*

$$\lambda_k^1 = \frac{i}{Lr_1}k\pi + o(1), \quad \lambda_k^2 = \frac{i}{Lr_2}k\pi + o(1) \tag{2.85}$$

and these two roots are simple.

Proof. From (2.85), f_0 has two families of roots that we denote λ_k^0 and μ_k^0. Now, we prove that

$$f_0(\lambda) = 0 \text{ if and only if } 2t_1L = 2ik\pi \text{ and } 2t_3L = 2ik'\pi,\ k,k' \in \mathbb{Z}.$$

Indeed, Suppose that

$$t_1L = ik\pi \text{ and } t_3L \neq ik'\pi,\ k,k' \in \mathbb{Z}.$$

Then

$$\mathcal{M} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ h_1(t_1) & -h_1(t_1) & h_1(t_3) & -h_1(t_3) \\ h_2(t_1)(-1)^k & h_2(t_2)(-1)^k & h_2(t_3)e^{t_3L} & h_2(t_4)e^{-t_3L} \\ h_3(t_1)(-1)^k & h_3(t_2)(-1)^k & h_3(t_3)e^{t_3L} & h_3(t_4)e^{-t_3L} \end{pmatrix}$$

We can check that $h_1(t_1) \neq 0$ and $h_1(t_3) \neq 0$ for λ large enough. Since $t_3L \neq ik'\pi$ for all $k' \in \mathbb{Z}$, then using Gaussian elimination for $\mathcal{M}$, we get

$$c_1 = c_2 = c_3 = c_4 = 0.$$

which is a contradiction with $\|U\|_{\mathcal{H}} = 1$. Similarly if

$$t_1L \neq ik\pi \text{ and } t_3L = ik'\pi,\ k,k' \in \mathbb{Z}.$$

we get $U \equiv 0$. We conclude that

$$f_0(\lambda) = 0 \text{ if and only if } t_1L = ik\pi \text{ and } t_3L = ik'\pi,\ k,k' \in \mathbb{Z}.$$

Then from (2.66) and (2.67), the large roots of f_0 satisfy the following asymptotic equations

$$\begin{aligned} \lambda_k^0 &= \frac{i}{Lr_1}k\pi + O\left(\frac{1}{k}\right) \forall |k| \geq k_0 \\ \lambda_k^1 &= \frac{i}{Lr_2}k'\pi + O\left(\frac{1}{k'}\right) |k'| \geq k_0'. \end{aligned}$$

We will now use Rouché's Theorem. Let $B_k = B(\frac{i}{Lr_1}k\pi, r_k)$ be the ball of centrum $\frac{ik\pi}{Lr_1}$ and radius $r_k = \frac{1}{k^{\frac{1}{4}}}$ and $\lambda \in \partial B_k$ (i.e $\lambda = \frac{i}{Lr_1}k\pi + r_k e^{i\theta}, \theta \in [0, 2\pi]$). Then we successively have:

$$2Lt_1(\lambda) = 2ik\pi + 2Lr_1r_ke^{i\theta} + O\left(\frac{1}{k}\right)$$

$$\begin{aligned} e^{2Lt_1(\lambda)} &= e^{2Lr_1r_ke^{i\theta}+O\left(\frac{1}{k}\right)} \\ &= 1+2Lr_1r_ke^{i\theta}+O(r_k^2). \end{aligned}$$

and

$$\begin{aligned} f_0(\lambda) &= (2Lr_1r_ke^{i\theta}+O(r_k^2))\left((2Lr_2r_ke^{i\theta}+O\left(\tfrac{1}{k}\right))\right) \\ &= 4L^2r_1r_2r_k^2e^{2i\theta}+O(r_k^3). \end{aligned}$$

It follows that there exists a positive constant c such that

$$\forall\lambda\in\partial B_k,\ |f_0(\lambda)|\geq cr_k^2=\frac{c}{\sqrt{k}}.$$

Then we deduce from (2.49) that $|f(\lambda)-f_0(\lambda)|=O\left(\frac{1}{\lambda}\right)=O\left(\frac{1}{k}\right)$. It follows that, for $|k|$ large enough

$$\forall\lambda\in\partial B_k,\ |f(\lambda)-f_0(\lambda)|<|f_0(\lambda)|,$$

Since the imaginary axis is an asymptote for the spectrum of $\mathcal{A}$ then system (2.33) is not uniformly stable.

2.5 Asymptotic stability

2.5.1 Strong stability of the system

In this part, we use a general criteria of Theorem 2.4.3 to show the strong stability of the C_0-semigroup $e^{t\mathcal{A}}$ associated to the wave system (P') in the absence of the compactness of the resolvent of $\mathcal{A}$. Our main result is the following theorem:

Theorem 2.5.1 *The C_0-semigroup $e^{t\mathcal{A}}$ is strongly stable in $\mathcal{H}$; i.e, for all $U_0\in\mathcal{H}$, the solution of (2.12) satisfies*

$$\lim_{t\to\infty}\|e^{t\mathcal{A}}U_0\|_{\mathcal{H}}=0.$$

Lemma 2.5.1 *$\mathcal{A}$ does not have eigenvalues on $i\,\mathbb{R}$.*

Proof
We will argue by contraction. Let us suppose that there $\lambda\in\ \mathbb{R},\lambda\neq0$ and $U\neq0$, such that $\mathcal{A}U=i\lambda U$. Then, we get

$$\left\{\begin{array}{l} i\lambda\varphi-u=0,\\ i\lambda u-\dfrac{K}{\rho_1}(\varphi_x+\psi)_x=0,\\ i\lambda\phi_1+(\xi^2+\eta)\phi_1-u(L)\mu(\xi)=0,\\ i\lambda\theta+\frac{K}{m_1}(\varphi_x+\psi)(L)+\frac{\zeta_1}{m_1}\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi_1(\xi)\,d\xi=0.\\ i\lambda\psi-v=0,\\ i\lambda v-\dfrac{b}{\rho_2}\psi_{xx}+\dfrac{K}{\rho_2}(\varphi_x+\psi)=0,\\ i\lambda\phi_2+(\xi^2+\eta)\phi_2-v(L)\mu(\xi)=0,\\ i\lambda\vartheta+\frac{b}{m_2}\psi_x(L)+\frac{\zeta_2}{m_2}\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi_2(\xi)\,d\xi=0. \end{array}\right. \tag{2.86}$$

Then, from (2.16) we have

$$\phi_i \equiv 0, \ i = 1, 2. \tag{2.87}$$

From $(2.86)_3$ and $(2.86)_7$, we have

$$u(L) = v(L) = 0. \tag{2.88}$$

Hence, from $(2.86)_1, (2.86)_5, (2.86)_4$ and $(2.86)_8$ we obtain

$$\varphi(L) = \psi(L) = 0 \text{ and } \varphi_x(L) = \psi_x(L) = 0. \tag{2.89}$$

From (2.86), we have

$$\begin{cases} -\lambda^2\rho_1\varphi - K(\varphi_x + \psi)_x = 0, \\ -\lambda^2\rho_2\psi - b\psi_{xx} + K(\varphi_x + \psi) = 0, \end{cases} \tag{2.90}$$

Consider $X = (\varphi, \psi, \varphi_x, \psi_x)$. Then we can rewrite (2.89) and (2.90) as the initial value problem

$$\begin{array}{l} \dfrac{d}{dx}X = \mathcal{A}X \\ X(L) = 0 \end{array} \tag{2.91}$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-\lambda^2\rho_1}{K} & 0 & 0 & -1 \\ 0 & \frac{-\rho_2\lambda^2 + K}{b} & \frac{K}{b} & 0 \end{pmatrix}$$

By the Picard Theorem for ordinary differential equations the system (2.91) has a unique solution $X = 0$. Therefore $\varphi = 0, \psi = 0$. It follows from (2.86), that $u = 0, v = 0, \theta = 0, \vartheta = 0$, i.e., $U = 0$.

The condition (ii) of Theorem 2.4.3 will be satisfied if we show that $\sigma(\mathcal{A}) \cap \{i\,\mathbb{R}\}$ is at most a countable set. We will prove that the operator $i\lambda I - \mathcal{A}$ is surjective for $\lambda \neq 0$. For this purpose, let $(f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)^T \in \mathcal{H}$, we seek $U = (\varphi, u, \phi_1, \theta, \psi, v, \phi_2, \vartheta)^T \in D(\mathcal{A})$ solution of the following equation

$$(i\lambda - \mathcal{A})U = F.$$

Equivalently, we have the following system

$$\begin{cases} i\lambda\varphi - u = f_1, \\ i\lambda u - \dfrac{K}{\rho_1}(\varphi_x + \psi)_x = f_2, \\ i\lambda\phi_1 + (\xi^2 + \eta)\phi_1 - u(L)\mu(\xi) = f_3, \\ i\lambda\theta + \frac{K}{m_1}(\varphi_x + \psi)(L) + \frac{\zeta_1}{m_1}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_1(\xi)\, d\xi = f_4. \\ i\lambda\psi - v = f_5, \\ i\lambda v - \dfrac{b}{\rho_2}\psi_{xx} + \dfrac{K}{\rho_2}(\varphi_x + \psi) = f_6, \\ i\lambda\phi_2 + (\xi^2 + \eta)\phi_2 - v(L)\mu(\xi) = f_7, \\ i\lambda\vartheta + \frac{b}{m_2}\psi_x(L) + \frac{\zeta_2}{m_2}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_2(\xi)\, d\xi = f_8. \end{cases} \tag{2.92}$$

We get

$$\tag{2.93} \begin{cases} -\lambda^2\varphi - \dfrac{K}{\rho_1}(\varphi_x+\psi)_x = f_2 + i\lambda f_1, \\ -\lambda^2\psi - \dfrac{b}{\rho_2}\psi_{xx} + \dfrac{K}{\rho_2}(\varphi_x+\psi) = f_6 + i\lambda f_5. \end{cases}$$

Solving system (2.93) is equivalent to finding $(\varphi,\psi) \in (H^2 \cap H^1_*(0,L))^2$ such that

$$\tag{2.94} \begin{cases} \displaystyle\int_0^L (-\rho_1\lambda^2\varphi w - K(\varphi_x+\psi)_x w)\,dx = \int_0^L \rho_1(f_2+i\lambda f_1)w\,dx, \\ \displaystyle\int_0^L (-\rho_2\lambda^2\psi\chi - b\psi_{xx}\chi + K(\varphi_x+\psi)\chi)\,dx = \int_0^L \rho_2(f_6+i\lambda f_5)\chi\,dx, \end{cases}$$

for all $(w,\chi) \in H^1_*(0,L) \times H^1_*(0,L)$. By using (2.21) and (2.19) the functions φ and ψ satisfying the following system

$$\tag{2.95} \begin{cases} \displaystyle\int_0^L (-\rho_1\lambda^2\varphi w + K(\varphi_x+\psi)w_x)\,dx + (i\lambda m_1 + \tilde{\zeta}_1)u(L)w(L) \\ \displaystyle= \int_0^L \rho_1(f_2+i\lambda f_1)w\,dx - \zeta_1\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+i\lambda}f_3(\xi)\,d\xi\ w(L) + m_1 f_4 w(L), \\ \displaystyle\int_0^L (-\rho_2\lambda^2\psi\chi + b\psi_x\chi_x + K(\varphi_x+\psi)\chi)\,dx + (i\lambda m_2 + \tilde{\zeta}_2)v(L)\chi(L) \\ \displaystyle= \int_0^L \rho_2(f_6+i\lambda f_5)\chi\,dx - \zeta_2\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+i\lambda}f_7(\xi)\,d\xi\ \chi(L) + m_2 f_8\chi(L), \end{cases}$$

where $\tilde{\zeta}_i = \zeta_i \displaystyle\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta+i\lambda}\,d\xi$. Using again (2.18), we deduce that

$$\tag{2.96} \begin{cases} u(L) = i\lambda\varphi(L) - f_1(L), \\ v(L) = i\lambda\psi(L) - f_5(L). \end{cases}$$

Inserting (2.96) into (2.95), we get

$$\tag{2.97} \begin{cases} \displaystyle\int_0^L (-\rho_1\lambda^2\varphi w + K(\varphi_x+\psi)w_x)\,dx + i\lambda(i\lambda m_1 + \tilde{\zeta}_1)\varphi(L)w(L) \\ \displaystyle= \int_0^L \rho_1(f_2+i\lambda f_1)w\,dx - \zeta_1\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+i\lambda}f_3(\xi)\,d\xi\ w(L) + (i\lambda m_1+\tilde{\zeta}_1)f_1(L)w(L) + m_1 f_4 w(L), \\ \displaystyle\int_0^L (-\rho_2\lambda^2\psi\chi + b\psi_x\chi_x + K(\varphi_x+\psi)\chi)\,dx + i\lambda(i\lambda m_2 + \tilde{\zeta}_2)\psi(L)\chi(L) \\ \displaystyle= \int_0^L \rho_2(f_6+i\lambda f_5)\chi\,dx - \zeta_2\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+i\lambda}f_7(\xi)\,d\xi\ \chi(L) + (i\lambda m_2+\tilde{\zeta}_2)f_5(L)\chi(L) + m_2 f_8\chi(L). \end{cases}$$

We can rewrite (2.97) as

$$\tag{2.98} -(L_\lambda U, V)_{H^1_R} + (U,V)_{H^1_R} = l(V)$$

where

$$H^1_R(0,L) = H^1_*(0,L) \times H^1_*(0,L),$$

with the inner product defined by

$$(U,V)_{H^1_R} = \int_0^L K(\varphi_x+\psi)(w_x+\chi) + b\psi_x\chi_x\,dx$$

$$(L_\lambda U, V)_{H^1_R} = \lambda^2 \int_0^L (\rho_1 \varphi w + \rho_2 \psi \chi + dx - i\lambda((i\lambda m_1 + \tilde{\zeta}_1)\varphi(L)w(L) + (i\lambda m_2 + \tilde{\zeta}_2)\psi(L)\chi(L)).$$

Using the compactness embedding from $L^2(0,L)$ into $H^{-1}(0,L)$ and from $H^1_*(0,L)$ into $L^2(0,L)$ we deduce that the operator L_λ is compact from $L^2(0,L)$ into $L^2(0,L)$. Consequently, by Fredholm alternative, proving the existence of U solution of (2.98) reduces to proving that 1 is not an eigenvalue of L_λ. Indeed if 1 is an eigenvalue, then there exists $U \neq 0$, such that

$$(L_\lambda U, V)_{H^1_R} = (U, V)_{H^1_R} \quad \forall V \in H^1_R. \tag{2.99}$$

In particular for $V = U$, it follows that

$$\lambda^2 \left[\rho_1 \|\varphi\|^2_{L^2(0,L)} + \rho_2 \|\psi\|^2_{L^2(0,L)}\right] - i\lambda((i\lambda m_1 + \tilde{\zeta}_1)|\varphi(L)|^2 + (i\lambda m_2 + \tilde{\zeta}_2)|\psi(L)|^2)$$
$$= K\|\varphi_x + \psi\|^2_{L^2(0,L)} + b\|\psi_x\|^2_{L^2(0,L)}.$$

Hence, we have

$$\varphi(L) = \psi(L) = 0. \tag{2.100}$$

From (2.99), we obtain

$$v_x(L) = 0 \tag{2.101}$$

and

$$\begin{cases} -\lambda^2 \varphi - \dfrac{K}{\rho_1}(\varphi_x + \psi)_x = 0, \\ -\lambda^2 \psi - \dfrac{b}{\rho_2}\psi_{xx} + \dfrac{K}{\rho_2}(\varphi_x + \psi) = 0. \end{cases} \tag{2.102}$$

Consider $X = (\varphi, \psi, \varphi_x, \psi_x)$. Then we can rewrite (2.102), (2.100) and (2.101) as the initial value problem

$$\begin{aligned} \frac{d}{dx}X &= \mathcal{B}X \\ X(L) &= 0 \end{aligned} \tag{2.103}$$

where

$$\mathcal{B} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-\lambda^2\rho_1}{K} & 0 & 0 & -1 \\ 0 & \frac{-\rho_2\lambda^2 + K}{b} & \frac{K}{b} & 0 \end{pmatrix}$$

By the Picard Theorem for ordinary differential equations the system (2.103) has a unique solution $X = 0$. Therefore $\varphi = 0, \psi = 0$. It follows from (2.86), that $u = 0, v = 0, \theta = 0, \vartheta = 0$, i.e., $U = 0$.

Lemma 2.5.2 *If $\eta \neq 0$, we have*

$$0 \in \rho(\mathcal{A}).$$

Proof
From (2.92)

$$\begin{cases} -u = f_1, \\ -\dfrac{K}{\rho_1}(\varphi_x+\psi)_x = f_2, \\ (\xi^2+\eta)\phi_1 - u(L)\mu(\xi) = f_3, \\ \frac{K}{m_1}(\varphi_x+\psi)(L) + \frac{\zeta_1}{m_1}\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi_1(\xi)\,d\xi = f_4. \\ -v = f_5, \\ -\dfrac{b}{\rho_2}\psi_{xx} + \dfrac{K}{\rho_2}(\varphi_x+\psi) = f_6, \\ (\xi^2+\eta)\phi_2 - v(L)\mu(\xi) = f_7, \\ \frac{b}{m_2}\psi_x(L) + \frac{\zeta_2}{m_2}\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi_2(\xi)\,d\xi = f_8. \end{cases} \tag{2.104}$$

$$\begin{cases} \displaystyle\int_0^L K(\varphi_x+\psi)w_x\,dx \\ = \displaystyle\int_0^L \rho_1 f_2 w\,dx - \zeta_1\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta}f_3(\xi)\,d\xi\ w(L) + \tilde{\zeta}_1 f_1(L)w(L) + m_1 f_4 w(L), \\ \displaystyle\int_0^L (b\psi_x\chi_x + K(\varphi_x+\psi)\chi)\,dx \\ = \displaystyle\int_0^L \rho_2 f_6\chi\,dx - \zeta_2\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta}f_7(\xi)\,d\xi\ \chi(L) + \tilde{\zeta}_2 f_5(L)\chi(L) + m_2 f_8\chi(L). \end{cases} \tag{2.105}$$

Consequently, problem (2.105) is equivalent to the problem

$$a_\eta((\varphi,\psi),(w,\chi)) = L_\eta(w,\chi) \tag{2.106}$$

where the bilinear form $a_\eta : [H^1_*(0,L)\times H^1_*(0,L)]^2 \to$ IR and the linear form $L_\eta : H^1_*(0,L)\times H^1_*(0,L) \to$ IR are defined by

$$a_\eta((\varphi,\psi),(w,\chi)) = \int_0^L K(\varphi_x+\psi)(w_x+\chi)\,dx + \int_0^L b\psi_x\chi_x\,dx.$$

and

$$\begin{aligned} L_\eta(w,\chi) &= \int_0^L \rho_1 f_2 w\,dx - \zeta_1\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta}f_3(\xi)\,d\xi\ w(L) + \tilde{\zeta}_1 f_1(L)w(L) \\ &+ \int_0^L \rho_2 f_6\chi\,dx - \zeta_2\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta}f_7(\xi)\,d\xi\ \chi(L) + \tilde{\zeta}_2 f_5(L)\chi(L) \\ &+ m_1 f_4 w(L) + m_2 f_8\chi(L). \end{aligned}$$

It is easy to verify that a_η is continuous and coercive, and L_η is continuous. So applying the Lax-Milgram theorem, we deduce that for all $(w,\chi,\zeta) \in H^1_*(0,L)\times H^1_*(0,L)$ problem (2.25) admits a unique solution $(\varphi,\psi) \in H^1_*(0,L)\times H^1_*(0,L)$. Applying the classical elliptic regularity, it follows from (2.24) that $(\varphi,\psi) \in H^2(0,L)\times H^2(0,L)$. Therefore, the operator $\mathcal{A}$ is surjective.

2.5.2 Residual spectrum of $\mathcal{A}$

Lemma 2.5.3 *Let $\mathcal{A}$ be defined by (2.13). Then*

$$\mathcal{A}^*\begin{pmatrix}\varphi\\ u\\ \phi_1\\ \theta\\ \psi\\ v\\ \phi_2\\ \vartheta\end{pmatrix}=\begin{pmatrix}-u\\ -\frac{K}{\rho_1}(\varphi_x+\psi)_x\\ -(\xi^2+\eta)\phi_1-u(L)\mu(\xi)\\ \frac{K}{m_1}(\varphi_x+\psi)(L)+\frac{\zeta_1}{m_1}\int_{-\infty}^{+\infty}\mu(\xi)\phi_1(\xi)\,d\xi\\ -v\\ -\frac{b}{\rho_2}\psi_{xx}+\frac{K}{\rho_2}(\varphi_x+\psi)\\ -(\xi^2+\eta)\phi_2-v(L)\mu(\xi)\\ \frac{b}{m_2}\psi_x(L)+\frac{\zeta_2}{m_2}\int_{-\infty}^{+\infty}\mu(\xi)\phi_2(\xi)\,d\xi\end{pmatrix}\tag{2.107}$$

with domain

$$\mathcal{D}(\mathcal{A}^*)=\left\{\begin{array}{l}(\varphi,u,\phi_1,\theta,\psi,v,\phi_2)\ \text{in}\ \mathcal{H}:\varphi,\psi\in H^2(0,L)\cap H^1_*(0,L),u,v\in H^1_*(0,L),\\ \theta,\vartheta\in\mathbb{C},u(L)=\theta,v(L)=\vartheta,\\ -(\xi^2+\eta)\phi_1-u(L)\mu(\xi),-(\xi^2+\eta)\phi_2-v(L)\mu(\xi)\in L^2(-\infty,+\infty),\\ |\xi|\phi_1,|\xi|\phi_2\in L^2(-\infty,+\infty)\end{array}\right\}\tag{2.108}$$

Proof

Let $U=(\varphi,u,\phi_1,\theta,\psi,v,\phi_2,\vartheta)^T$ and $V=(\tilde{\varphi},\tilde{u},\tilde{\phi}_1,\tilde{\theta},\tilde{\psi},\tilde{v},\tilde{\phi}_2,\tilde{\vartheta})^T$. We have
$<\mathcal{A}U,V>_{\mathcal{H}}=<U,\mathcal{A}^*V>_{\mathcal{H}}$.

$$\begin{aligned}<\mathcal{A}U,V>_{\mathcal{H}}\ &=\ K\int_0^L\tilde{u}(\varphi_x+\psi)_x\,dx+b\int_0^L\tilde{v}\psi_{xx}\,dx-K\int_0^L\tilde{v}(\varphi_x+\psi)\,dx\\ &\quad+K\int_0^L(\tilde{\varphi}_x+\tilde{\psi})(u_x+v)\,dx\\ &\quad+b\int_0^L\tilde{\psi}_xv_x\,dx+\zeta_1\int_{-\infty}^{+\infty}[-(\xi^2+\eta)\phi_1+u(L)\mu(\xi)]\tilde{\phi}_1\,d\xi\\ &\quad+\zeta_2\int_{-\infty}^{+\infty}[-(\xi^2+\eta)\phi_2+v(L)\mu(\xi)]\tilde{\phi}_2\,d\xi\\ &\quad+m_1\left(-\frac{K}{m_1}(\varphi_x+\psi)(L)-\frac{\zeta_1}{m_1}\int_{-\infty}^{+\infty}\mu(\xi)\phi_1(\xi)\,d\xi\right)\tilde{\theta}\\ &\quad+m_2\left(-\frac{b}{m_2}\psi_x(L)-\frac{\zeta_2}{m_2}\int_{-\infty}^{+\infty}\mu(\xi)\phi_2(\xi)\,d\xi\right)\tilde{\vartheta}\\ &=\ -K\int_0^L(\tilde{u}_x+\tilde{v})(\varphi_x+\psi)\,dx+K(\varphi_x+\psi)(L)\tilde{u}(L)-b\int_0^L\tilde{v}_x\psi_x\,dx\\ &\quad+b\psi_x(L)\tilde{v}(L)+K(\tilde{\varphi}_x+\tilde{\psi})(L)u(L)-K\int_0^L(\tilde{\varphi}_x+\tilde{\psi})_xu\,dx+K\int_0^L(\tilde{\varphi}_x+\tilde{\psi})_xv\,dx\\ &\quad-b\int_0^L\tilde{\psi}_{xx}v\,dx+b\tilde{\psi}_x(L)v(L)+\zeta_1u(L)\int_{-\infty}^{+\infty}\mu(\xi)\tilde{\phi}_1\,d\xi+\zeta_2u(L)\int_{-\infty}^{+\infty}\mu(\xi)\tilde{\phi}_2\,d\xi\\ &\quad-\zeta_1\int_{-\infty}^{+\infty}\phi_1[(\xi^2+\eta)\tilde{\phi}_1+\tilde{\theta}\mu(\xi)]\,d\xi-\zeta_2\int_{-\infty}^{+\infty}\phi_2[(\xi^2+\eta)\tilde{\phi}_2+\tilde{\vartheta}\mu(\xi)]\,d\xi\\ &\quad-K(\varphi_x+\psi)(L)\tilde{\theta}-b\psi_x(L)\tilde{\vartheta}\end{aligned}$$

As $\theta = u(L), \vartheta = v(L)$ and if we set $\tilde{\theta} = \tilde{u}(L), \tilde{\vartheta} = \tilde{v}(L)$, we find

$$\begin{aligned} < \mathcal{A}U, V >_{\mathcal{H}} \quad = \quad & -K\int_0^L (\tilde{u}_x + \tilde{v})(\varphi_x + \psi)\, dx - b\int_0^L \tilde{v}_x \psi_x \, dx - K\int_0^L (\tilde{\varphi}_x + \tilde{\psi})_x u\, dx \\ & + \int_0^L (-b\tilde{\psi}_{xx} + K(\tilde{\varphi}_x + \tilde{\psi})_x) v\, dx \\ & + u(L)\left(K(\tilde{\varphi}_x + \tilde{\psi})(L) + \zeta_1 \int_{-\infty}^{+\infty} \mu(\xi)\tilde{\phi}_1\, d\xi\right) + v(L)\left(b\tilde{\psi}_x(L) + \zeta_2 \int_{-\infty}^{+\infty} \mu(\xi)\tilde{\phi}_2\, d\xi\right) \\ & -\zeta_1 \int_{-\infty}^{+\infty} \phi_1[(\xi^2+\eta)\tilde{\phi}_1 + \tilde{u}(L)\mu(\xi)]\, d\xi - \zeta_2 \int_{-\infty}^{+\infty} \phi_2[(\xi^2+\eta)\tilde{\phi}_2 + \tilde{v}(L)\mu(\xi)]\, d\xi. \end{aligned}$$

Theorem 2.5.2 *$\sigma_r(\mathcal{A}) = \emptyset$, where $\sigma_r(\mathcal{A})$ denotes the set of residual spectrum of $\mathcal{A}$.*

Since $\lambda \in \sigma_r(\mathcal{A}), \overline{\lambda} \in \sigma_p(\mathcal{A}^*)$ the proof will be accomplished if we can show that $\sigma_p(\mathcal{A}) = \sigma_p(\mathcal{A}^*)$. This is because obviously the eigenvalues of A are symmetric on the real axis. From (2.107), the eigenvalue problem $\mathcal{A}^* Z = \lambda Z$ for $\lambda \in \mathbb{C}$ and $0 \neq Z = (\varphi, u, \phi_1, \theta, \psi, v, \phi_2, \vartheta) \in D(\mathcal{A}^*)$ we have

$$(2.109) \qquad \begin{cases} \lambda\varphi + u = 0, \\ \lambda u + \dfrac{K}{\rho_1}(\varphi_x + \psi)_x = 0, \\ \lambda\phi_1 + (\xi^2+\eta)\phi_1 + u(L)\mu(\xi) = 0, \\ \lambda\theta - \frac{K}{m_1}(\varphi_x+\psi)(L) - \frac{\zeta_1}{m_1}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_1(\xi)\, d\xi = 0. \\ \lambda\psi + v = 0, \\ \lambda v + \dfrac{b}{\rho_2}\psi_{xx} - \dfrac{K}{\rho_2}(\varphi_x + \psi) = 0, \\ \lambda\phi_2 + (\xi^2+\eta)\phi_2 + v(L)\mu(\xi) = 0, \\ \lambda\vartheta - \frac{b}{m_2}\psi_x(L) - \frac{\zeta_2}{m_2}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_2(\xi)\, d\xi = 0. \end{cases}$$

From $(2.109)_1$ and $(2.109)_2$, $(2.109)_5$ and $(2.109)_6$, we get

$$(2.110) \qquad \begin{cases} -\lambda^2 u + \dfrac{K}{\rho_1}(\varphi_x + \psi)_x = 0, \\ -\lambda^2 v + \dfrac{b}{\rho_2}\psi_{xx} - \dfrac{K}{\rho_2}(\varphi_x+\psi) = 0, \end{cases}$$

As $\theta = u(L) = -\lambda\varphi(L)$ and $\vartheta = v(L) = -\lambda\psi(L)$, we deduce from $(2.109)_3$ and $(2.109)_4$, $(2.109)_7$ and $(2.109)_8$ that

$$(2.111) \qquad \begin{aligned} &\left(\lambda + \frac{\gamma_1}{m_1}(\lambda+\eta)^{\alpha-1}\right)\lambda\varphi(L) + \frac{K}{m_1}(\varphi_x+\psi)(L) = 0 \\ &\left(\lambda + \frac{\gamma_2}{m_2}(\lambda+\eta)^{\alpha-1}\right)\lambda\psi(L) + \frac{b}{m_2}\psi_x(L) = 0 \end{aligned}$$

System (2.110)-(2.111) is exactly the eigenvalue problem of $\mathcal{A}$. Hence $\mathcal{A}^*$ has the same eigenvalues as $\mathcal{A}$. The proof is complete.

2.5.3 Polynomial Stability (for $\eta \neq 0$)

Theorem 2.5.3 *The semigroup $S_{\mathcal{A}}(t)_{t\geq 0}$ is polynomially stable and*

$$\|S_{\mathcal{A}}(t)U_0\|_{\mathcal{H}} \leq \frac{1}{t^{1/(4-2\alpha)}}\|U_0\|_{D(\mathcal{A})}$$

Proof
We will need to study the resolvent equation $(i\lambda - \mathcal{A})U = F$, for $\lambda \in \mathbb{R}$, namely

$$\begin{cases} i\lambda\varphi - u = f_1, \\ i\lambda u - \dfrac{K}{\rho_1}(\varphi_x + \psi)_x = f_2, \\ i\lambda\phi_1 + (\xi^2+\eta)\phi_1 - u(L)\mu(\xi) = f_3, \\ i\lambda\theta + \frac{k}{m_1}(\varphi_x+\psi)(L) + \frac{\zeta_1}{m_1}\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi_1(\xi)\,d\xi = f_4. \\ i\lambda\psi - v = f_5, \\ i\lambda v - \dfrac{b}{\rho_2}\psi_{xx} + \dfrac{K}{\rho_2}(\varphi_x+\psi) = f_6, \\ i\lambda\phi_2 + (\xi^2+\eta)\phi_2 - v(L)\mu(\xi) = f_7, \\ i\lambda\vartheta + \frac{b}{m_2}\psi_x(L) + \frac{\zeta_2}{m_2}\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi_2(\xi)\,d\xi = f_8. \end{cases} \tag{2.112}$$

where $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)^T$. Taking inner product in $\mathcal{H}$ with U and using (2.16) we get

$$|Re\langle \mathcal{A}U, U\rangle| \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{2.113}$$

This implies that

$$\sum_{i=1}^{2}\zeta_i\int_{-\infty}^{+\infty}(\xi^2+\eta)(\varphi_i(\xi,t))^2\,d\xi \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{2.114}$$

and, applying $(3.1)_{1,4,7}$, we obtain

$$||\lambda||\varphi(L)| - |f_1(L)||^2 \leq |u(L)|^2.$$

We deduce that

$$|\lambda|^2|\varphi(L)|^2 \leq c|f_1(L)|^2 + c|u(L)|^2.$$

Moreover, from $(3.1)_4$, we have

$$K(\varphi_x+\psi)(L) = -im_1\lambda u(L) - \zeta_1\int_{-\infty}^{+\infty}\mu(\xi)\phi_1(\xi)\,d\xi + m_1 f_4.$$

Then

$$\begin{aligned} K^2|(\varphi_x+\psi)(L)|^2 &\leq 2m_1^2|\lambda|^2|u(L)|^2 + 2m_1^2 f_4^2 + 2\zeta_1^2\left|\int_{-\infty}^{+\infty}\mu(\xi)\phi_1(\xi)\,d\xi\right|^2 \\ &\leq 2m_1^2|\lambda|^2|u(L)|^2 + 2m_1^2 f_4^2 + 2\zeta_1^2\left(\int_{-\infty}^{+\infty}(\xi^2+\eta)^{-1}|\mu(\xi)|^2\,d\xi\right)\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi_1(\xi)|^2\,d\xi \\ &\leq 2m_1^2|\lambda|^2|u(L)|^2 + c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c'\|F\|_{\mathcal{H}}^2. \end{aligned} \tag{2.115}$$

From $(3.1)_3$, we obtain

(2.116) $$u(L)\mu(\xi) = (i\lambda + \xi^2 + \eta)\phi_1 - f_3(\xi).$$

By multiplying $(2.116)_1$ by $(i\lambda + \xi^2 + \eta)^{-1}\mu(\xi)$, we get

(2.117) $$(i\lambda + \xi^2 + \eta)^{-1}u(L)\mu^2(\xi) = \mu(\xi)\phi_1 - (i\lambda + \xi^2 + \eta)^{-1}\mu(\xi)f_3(\xi).$$

Hence, by taking absolute values of both sides of (2.117), integrating over the interval $]-\infty, +\infty[$ with respect to the variable ξ and applying Cauchy-Schwartz inequality, we obtain

(2.118) $$S|u(L)| \leq U\left(\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi_1|^2\,d\xi\right)^{\frac{1}{2}} + V\left(\int_{-\infty}^{+\infty}|f_3(\xi)|^2\,d\xi\right)^{\frac{1}{2}}$$

where

$$S = \int_{-\infty}^{+\infty}(|\lambda| + \xi^2 + \eta)^{-1}|\mu(\xi)|^2\,d\xi$$

$$U = \left(\int_{-\infty}^{+\infty}(\xi^2 + \eta)^{-1}|\mu(\xi)|^2\,d\xi\right)^{\frac{1}{2}}$$

$$V = \left(\int_{-\infty}^{+\infty}(|\lambda| + \xi^2 + \eta)^{-2}|\mu(\xi)|^2\,d\xi\right)^{\frac{1}{2}}.$$

Thus, by using again the inequality $2PQ \leq P^2 + Q^2, P \geq 0, Q \geq 0$, we get

(2.119) $$S^2|u(L)|^2 \leq 2U^2\left(\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi_1|^2\,d\xi\right) + 2V^2\left(\int_{-\infty}^{+\infty}|f_3(\xi)|^2\,d\xi\right).$$

We deduce that

(2.120) $$|u(L)|^2 \leq c|\lambda|^{2-2\alpha}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c\|F\|^2_{\mathcal{H}}.$$

Similarly, we have

(2.121) $$b^2|\psi_x(L)|^2 \leq 2m_2^2|\lambda|^2|v(L)|^2 + c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c'\|F\|^2_{\mathcal{H}}.$$

(2.122) $$|v(L)|^2 \leq c|\lambda|^{2-2\alpha}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c\|F\|^2_{\mathcal{H}}.$$

Let us introduce the following notation

$$\begin{aligned}\mathcal{I}_\varphi(\alpha) &= \rho_1|u(\alpha)|^2 + K|\varphi_x(\alpha)|^2\\ \mathcal{I}_\psi(\alpha) &= \rho_2|v(\alpha)|^2 + b|\psi_x(\alpha)|^2\end{aligned}$$

$$\mathcal{E}_\varphi(L) = \int_0^L q(x)\mathcal{I}_\varphi(s)\,ds, \quad \mathcal{E}_\psi(L) = \int_0^L \mathcal{I}_\psi(s)\,ds.$$

Lemma 2.5.4 *Let $q \in H^1(0, L)$. We have that*

(2.123) $$\mathcal{E}_\varphi(L) = [q\mathcal{I}_\varphi]_0^L + 2K\ Re\int_0^L q\psi_x\overline{\varphi}_x\,dx + R_1.$$

and

(2.124) $$\mathcal{E}_\psi(L) = [q\mathcal{I}_\psi]_0^L - K[q|\psi|^2]_0^L - 2K\ Re\int_0^L q\varphi_x\overline{\psi}_x\,dx + K\int_0^L q'|\psi|^2\,dx + R_2.$$

where R_i satisfies

$$\begin{aligned} |R_1| &\leq C\mathcal{E}_\varphi(L) + \|q^{1/2}F\|^2_{\mathcal{H}} \\ |R_2| &\leq C\mathcal{E}_\psi(L) + \|q^{1/2}F\|^2_{\mathcal{H}}. \end{aligned}$$

for a positive constant C.

Proof

To get (2.123), let us multiply the equation $(3.1)_2$ by $q\overline{\varphi}_x$ Integrating on $(0, L)$ we obtain

$$i\lambda\rho_1 \int_0^L uq\overline{\varphi}_x\,dx - K\int_0^L (\varphi_x+\psi)_x q\overline{\varphi}_x\,dx = \rho_1\int_0^L f_2 q\overline{\varphi}_x\,dx$$

or

$$-\rho_1\int_0^L uq(\overline{i\lambda\varphi_x})\,dx - K\int_0^L q\varphi_{xx}\overline{\varphi}_x\,dx - K\int_0^L q\psi_x\overline{\varphi}_x\,dx = \rho_1\int_0^L f_2 q\overline{\varphi}_x\,dx.$$

Since $i\lambda\varphi_x = u_x + f_{1x}$ taking the real part in the above equality results in

$$\begin{aligned} &-\frac{\rho_1}{2}\int_0^L q\frac{d}{dx}|u|^2\,dx - \frac{K}{2}\int_0^L q\frac{d}{dx}|\varphi_x|^2\,dx = \rho_1 Re\int_0^L f_2 q\overline{\varphi}_x\,dx + \rho_1 Re\int_0^L uq\overline{f}_{1x}\,dx \\ &+K\ Re\int_0^L q\psi_x\overline{\varphi}_x\,dx. \end{aligned}$$

Performing an integration by parts we get

$$\begin{aligned} &\int_0^L q'(s)[\rho_1|u(s)|^2 + K|\varphi_x(s)|^2]\,ds \\ &= [q\mathcal{I}_\varphi]_0^L + 2K\ Re\int_0^L q\psi_x\overline{\varphi}_x\,dx + R_1 \end{aligned}$$

where

$$R_1 = 2\rho_1 Re\int_0^L f_2 q\overline{\varphi}_x\,dx + 2\rho_1 Re\int_0^L uq\overline{f}_{1x}\,dx.$$

Similarly, multiplying equation $(3.1)_5$ by $q\overline{\varphi}_x$, integrating on $(0, L)$ and taking the real part we obtain

$$i\lambda\rho_2\int_0^L vq\overline{\psi}_x\,dx - b\int_0^L \psi_{xx}q\overline{\psi}_x\,dx + K\int_0^L (\varphi_x+\psi)q\overline{\psi}_x\,dx = \rho_2\int_0^L f_6 q\overline{\psi}_x\,dx$$

or

$$\begin{aligned} &-\rho_2\int_0^L vq(\overline{i\lambda\psi_x})\,dx - b\int_0^L q\psi_{xx}\overline{\psi}_x\,dx + K\int_0^L q\varphi_x\overline{\psi}_x\,dx \\ &+K\int_0^L q\psi\overline{\psi}_x\,dx = \rho_2\int_0^L f_6 q\overline{\psi}_x\,dx. \end{aligned}$$

Since $i\lambda\psi_x = v_x + f_{5x}$ taking the real part in the above equality results in

$$\begin{aligned} &-\frac{\rho_2}{2}\int_0^L q\frac{d}{dx}|v|^2\,dx - \frac{b}{2}\int_0^L q\frac{d}{dx}|\psi_x|^2\,dx = \rho_2 Re\int_0^L f_6 q\overline{\psi}_x\,dx \\ &+\rho_2\ Re\int_0^L qv\overline{f}_{5x}\,dx - K\ Re\int_0^L q\varphi_x\overline{\psi}_x\,dx - \frac{K}{2}\int_0^L q\frac{d}{dx}|\psi|^2\,dx. \end{aligned}$$

Performing an integration by parts we get

$$\begin{aligned}&\int_0^L q'(s)[\rho_2|v(s)|^2+b|\psi_x(s)|^2]\,ds\\&=[q\mathcal{I}_\psi]_0^L-K[q|\psi|^2]_0^L-2K\ Re\int_0^L q\varphi_x\overline{\psi}_x\,dx\\&+K\int_0^L q'|\psi|^2\,dx+R_2\end{aligned}$$

where

$$R_2=2\rho_2 Re\int_0^L f_6 q\overline{\psi}_x\,dx+2\rho_2\ Re\int_0^L qv\overline{f}_{5x}\,dx.$$

If we take $q(x)=\int_0^x e^{ns}\,ds=\frac{e^{nx}-1}{n}$ (Here n will be chosen large enough) in Lemma 2.5.4 we arrive at

$$\begin{aligned}&\mathcal{E}_\varphi(L)+\mathcal{E}_\psi(L)\\&=q(L)\mathcal{I}_\varphi(L)+2K\ Re\int_0^L q\psi_x\overline{\varphi}_x\,dx\\&q(L)\mathcal{I}_\psi(L)-Kq(L)|\psi(L)|^2+K\int_0^L q'(x)|\psi|^2\,dx-2K\ Re\int_0^L q\varphi_x\overline{\psi}_x\,dx\\&+R_1+R_2\\&=q(L)\mathcal{I}_\varphi(L)+q(L)\mathcal{I}_\psi(L)-Kq(L)|\psi(L)|^2+K\int_0^L q'(x)|\psi|^2\,dx\\&+R_1+R_2\end{aligned}$$

Also, we have

$$\begin{aligned}|R_1|&\leq 2\rho_1\int_0^L q(x)(|u(s)|^2+|\varphi_x(s)|^2)\,ds+2\rho_1\int_0^L q(x)(|f_2(s)|^2+|f_{1x}(s)|^2)\,ds\\&\leq C\frac{e^{Ln}}{n}\|F\|_{\mathcal{H}}^2+\frac{c'}{n}\mathcal{E}_\varphi(L)\end{aligned}\tag{2.125}$$

and

$$\begin{aligned}|R_2|&\leq 2\rho_2\int_0^L q(x)(|v(s)|^2+|\psi_x(s)|^2)\,ds+2\rho_1\int_0^L q(x)(|f_6(s)|^2+|f_{5x}(s)|^2)\,ds\\&\leq C\frac{e^{Ln}}{n}\|F\|_{\mathcal{H}}^2+\frac{c'}{n}\mathcal{E}_\psi(L)\end{aligned}\tag{2.126}$$

Using Lemma 2.5.4 and the Young inequality we get

$$\begin{aligned}&\mathcal{E}_\varphi(L)+\mathcal{E}_\psi(L)\\&\leq q(L)\mathcal{I}_\varphi(L)+q(L)\mathcal{I}_\psi(L)+K\int_0^L q'(x)|\psi|^2\,dx\\&+c\|F\|_{\mathcal{H}}^2\end{aligned}$$

for a positive constant C. It results by (2.115), (2.120), (2.121) and (2.122) that we can find a positive constant C such that

$$\begin{aligned}&\mathcal{E}_\varphi(L)+\mathcal{E}_\psi(L)\\&\leq K\int_0^L|\psi|^2\,dx+c(|\lambda|^{4-2\alpha}+|\lambda|^{2-2\alpha}+1)\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+c(|\lambda|^2+1)\|F\|_{\mathcal{H}}^2\end{aligned}$$

for $\lambda \neq 0$. Since that $\varphi = \frac{u+f_1}{i\lambda}$ and $\psi = \frac{v+f_4}{i\lambda}$ we obtain

$$\begin{aligned}&\mathcal{E}_\varphi(L) + \mathcal{E}_\psi(L)\\ &\leq c(|\lambda|^{4-2\alpha} + |\lambda|^{2-2\alpha} + 1)\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c(|\lambda|^2 + 1)\|F\|^2_{\mathcal{H}} + \frac{c}{|\lambda|^2}\|U\|^2_{\mathcal{H}} + \frac{c}{|\lambda|^2}\|F\|^2_{\mathcal{H}}.\end{aligned}$$

Since that

$$\int_{-\infty}^{+\infty} (\phi_i(\xi))^2\, d\xi \leq C \int_{-\infty}^{+\infty} (\xi^2 + \eta)(\phi_i(\xi))^2\, d\xi$$

for $\lambda \neq 0$. If $|\lambda| > 1$ we get

$$\|U\|^2_{\mathcal{H}} \leq |\lambda|^{8-4\alpha}\|F\|^2_{\mathcal{H}}.$$

It follows that

$$\frac{1}{|\lambda|^{(4-2\alpha)}}\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C, \quad \forall \lambda \in \mathbb{R},$$

for a positive constant C. The conclusion then follows by applying Theorem 2.4.2.

Remark 2.5.1 **1)** *By Proposition 2.4.1, the spectrum of $\mathcal{A}$ is at the left of the imaginary axis, but approaches this axis. Hence, the decay of the energy depends on the asymptotic behavior of the real part of these eigenvalues, since Proposition 2.4.1 shows a behavior like $k^{-(3-\alpha)}$, we can expect a decay rate (optimal) of the energy of order $t^{-2/(3-\alpha)}$. We unfortunately were not able to prove this optimal decay rate by Borichev-Tomilov Theorem. In theorem 2.5.3, we obtain decay rate of order $t^{-1/(2-\alpha)}$ which is less better. But, it is interesting to remark that both energy decay in Theorem 2.5.3 and Proposition 2.4.1 approach t^{-1} (as $\alpha \to 1$) which is the energy decay given in* **[32]** *and* **[28]**.

2) *Estimation of decay rate in the case $\eta = 0$ is open. As $\lambda = 0$ is a spectral value, both technic used in* **[32]** *and* **[28]** *do not work. In the futur, we try other methods, in particular some tools from observability theory. Another technic is the use of Laplace transform and representation of solutions by Mittag-Leffler Functions.*

3) *It seems to be interesting to study a global decaying solutions of hyperbolic systems (strong and weakly) under control of fractional derivative type. We think that the interaction of the hyperbolicity (order of multiplicity) and the number of dissipative terms have an effect on the result.*

Chapter 3

OPTIMAL ENERGY DECAY OF SOLUTIONS TO A TIMOSHENKO BEAM SYSTEM WITH DYNAMIC BOUNDARY FEEDBACKS OF FRACTIONAL DERIVATIVE TYPE

3.1 Introduction

In this chapter we investigate the decay properties of solutions for the initial boundary value problem of the linear Timoshenko beam system of the type

$$(P) \qquad \begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) = 0 & \text{in } (0,L)\times(0,+\infty), \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x+\psi)(x,t) = 0 & \text{in } (0,L)\times(0,+\infty), \end{cases}$$

where $(x,t) \in (0,L)\times(0,+\infty)$. This system is subject to the boundary conditions

$$\begin{array}{ll} \varphi(0,t) = 0, \quad \psi(0,t) = 0, & \text{in } (0,+\infty), \\ m_1\varphi_{tt}(L,t) + K(\varphi_x+\psi)(L,t) = -\gamma_1\partial_t^{\alpha,\eta}\varphi(L,t) & \text{in } (0,+\infty), \\ m_2\psi_{tt}(L,t) + b\psi_x(L,t) = -\gamma_2\partial_t^{\alpha,\eta}\psi(L,t) & \text{in } (0,+\infty), \end{array}$$

where $\gamma_i > 0, i = 1,2$. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order $\alpha, 0 < \alpha < 1$, with respect to the time variable.

The problem of stabilization for the initial boundary value problem

$$(P') \qquad \begin{cases} u_{tt} - \Delta u = 0 \text{ on } \Omega\times(0,+\infty), \\ u = 0 \text{ on } \Gamma_D\times(0,+\infty), \\ \dfrac{\partial u}{\partial \nu} + a(x)u_t = 0 \text{ on } \Gamma_N\times(0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \text{ on } \Omega, \end{cases}$$

was investigated by several authors. In Haraux [**16**], Bardos, G. Lebeau and J. Rauch [**7**], Lebeau and Robbiano [**7**], Burq [**11**] and Xiaoyu Fu [**15**].

First, A. Haraux has shown that if $a \in L^{\infty}(\Gamma_N), a \not\equiv 0$, then any solution of (P') tends to 0 in $H^1_*(\Omega)$ strong as $t \to +\infty$.

C. Bardos, G. Lebeau and J. Rauch [**7**] introduced a geometric control condition witch is a necessary and sufficient condition for the uniform exponential decay rate of the energy.

Moreover, Lebeau and Robbiano (see [**22**]) have shown that, in the case where the Neumann boundary condition is applied on the entire boundary, a weak condition on the feedback (which does not satisfy Geometric Control Condition) provides logarithmic decay of regular solutions. The optimal result without geometrical hypothesis is given in [**11**]. We also recall the result by Fu [**15**], where the author proved a result similar to the one in [**22**] for less regular conditions ($\partial\Omega \in C^2$) by adopting the global Carleman estimate.

In [**26**] Mbodje studies the energy decay of the wave equation with a boundary control of fractional derivative type (CF). The major inconvenient associated to the fractional operators is the hereditary behavior. Therefore, the employment of mathematical analysis tools, such as stability analysis and numerical approximation is very difficult. He used a new approach called "diffusive representation" to reduce these difficulties. The original model is transformed into an augmented system which can be more easily tackled by the energy method. The author showed strong asymptotic stability of solutions when $\eta = 0$ and polynomial type decay rate $\mathcal{E}(t) \leq C/t$ for $t \geq 0$ when $\eta \neq 0$.

Recently in [**8**], benaissa and Benkhedda considered the stabilization for the following wave equation with dynamic boundary control of fractional derivative type (CF):

$$(P) \qquad \begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0 & \text{in }]0,L[\times]0,+\infty[\\ u(0,t) = 0 & \text{in } (0,+\infty) \\ mu_{tt}(L,t) + u_x(L,t) = -\gamma \partial_t^{\alpha,\eta} u(L,t) & \text{in } (0,+\infty). \end{cases}$$

They proved that the decay of the energy is not exponential, but polynomial. They used the spectrum method for lack of exponential stability and Borichev-Tomilov Theorem for establishing polynomial decay rate $\mathcal{E}(t) \leq c/t^{1/(2-\alpha)}$.

Very recently in [**8**], benaissa and Benazzouz considered the problem (P). The author showed strong asymptotic stability of solutions when $\eta = 0$ and polynomial type decay rate $\mathcal{E}(t) \leq C/t^{1/(2-\alpha)}$ for $t \geq 0$ when $\eta \neq 0$.

Our purpose in this chapter is to prove an optimal decay estimate following the wave propagation speeds. We use an explicit representation of the resolvent associate to the semi-group operator and an application of the recent theorem of Borichev-Tomilov.

3.2 Optimality of energy decay when $\eta > 0$

We will need to study the resolvent equation $(i\lambda - \mathcal{A})U = F$, for $\lambda \in \mathbb{R}$, namely

$$(3.1)\quad \begin{cases} i\lambda\varphi - u = f_1, \\ i\lambda u - \dfrac{K}{\rho_1}(\varphi_x + \psi)_x = f_2, \\ i\lambda\phi_1 + (\xi^2+\eta)\phi_1 - u(L)\mu(\xi) = f_3, \\ i\lambda\theta + \frac{k}{m_1}(\varphi_x+\psi)(L) + \frac{\zeta_1}{m_1}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_1(\xi)\,d\xi = f_4. \\ i\lambda\psi - v = f_5, \\ i\lambda v - \dfrac{b}{\rho_2}\psi_{xx} + \dfrac{K}{\rho_2}(\varphi_x+\psi) = f_6, \\ i\lambda\phi_2 + (\xi^2+\eta)\phi_2 - v(L)\mu(\xi) = f_7, \\ i\lambda\vartheta + \frac{b}{m_2}\psi_x(L) + \frac{\zeta_2}{m_2}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_2(\xi)\,d\xi = f_8. \end{cases}$$

where $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)^T \in \mathcal{H}$. The first and fifth equations of (3.1) being equivalent to

$$(3.2)\quad u = i\lambda\varphi - f_1, \quad v = i\lambda\psi - f_5$$

and by substitution in the second and sixth equations, we obtain the following system

$$(3.3)\quad \begin{cases} \lambda^2\rho_1\varphi + K\varphi_{xx} + K\psi_x = -\rho_1(f_2 + i\lambda f_1) \\ \lambda^2\rho_2\psi + b\psi_{xx} - K\varphi_x - K\psi = -\rho_2(f_6 + i\lambda f_5). \\ i\lambda\phi_1 + (\xi^2+\eta)\phi_1 - u(1)\mu(\xi) = f_3, \\ i\lambda\phi_2 + (\xi^2+\eta)\phi_2 - v(1)\mu(\xi) = f_7, \\ i\lambda\theta + \frac{k}{m_1}(\varphi_x+\psi)(L) + \frac{\zeta_1}{m_1}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_1(\xi)\,d\xi = f_4. \\ i\lambda\vartheta + \frac{b}{m_2}\psi_x(L) + \frac{\zeta_2}{m_2}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_2(\xi)\,d\xi = f_8. \end{cases}$$

Now the system $(3.3)_1 - (3.3)_2$ takes the form

$$U' = BU + F \text{ where } U = \begin{pmatrix} \varphi \\ \varphi_x \\ \psi \\ \psi_x \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{\rho_1}{K}\lambda^2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{K}{b} & \frac{K}{b} - \frac{\rho_2}{b}\lambda^2 & 0 \end{pmatrix} \text{ and } F = \begin{pmatrix} 0 \\ -\frac{\rho_1}{K}(f_2 + i\lambda f_1) \\ 0 \\ -\frac{\rho_2}{b}(f_6 + i\lambda f_5) \end{pmatrix}$$

The case $r_1^2 = r_2^2$

It is not restrictive to suppose $\rho_1 = \rho_2 = K = b = 1$. A simple computation shows that the eigenvalues μ_i of the matrix B are the roots of the following equation

$$(3.4)\quad x^4 + 2\lambda^2x^2 + \lambda^2(\lambda^2 - 1) = 0$$

Thus (3.4) has only pure imaginary solutions when λ is large enough. Applying the classical method of variation of constants formula, we obtain

$$(3.5)\quad U = e^{Bx}U(0) + \int_0^x e^{B(x-s)}F(s)\,ds \qquad U(0) = (0, \varphi_x(0), 0, \psi_x(0))^T,$$

where e^{Bx} is the solution of the homogeneous equation

$$\frac{dY}{dx} = BY, \quad Y(0) = I. \tag{3.6}$$

To obtain an explicit expression of (3.5), we consider the initial value problem

$$\begin{cases} \lambda^2\varphi + \varphi_{xx} + \psi_x = 0 \\ \lambda^2\psi + \psi_{xx} - \varphi_x - \psi = 0 \\ \varphi(0) = c_1, \varphi_x(0) = c_2, \psi(0) = c_3, \psi_x(0) = c_4 \end{cases} \tag{3.7}$$

Then a straightforward computation gives that:

$$\begin{cases} \varphi = Ae^{\mu_1 x} + Be^{-\mu_1 x} + Ce^{\mu_2 x} + De^{-\mu_2 x} \\ \psi = -A\left(\frac{\lambda^2}{\mu_1} + \mu_1\right)e^{\mu_1 x} + B\left(\frac{\lambda^2}{\mu_1} + \mu_1\right)e^{-\mu_1 x} - C\left(\frac{\lambda^2}{\mu_2} + \mu_2\right)e^{\mu_1 x} + D\left(\frac{\lambda^2}{\mu_2} + \mu_2\right)e^{-\mu_2 x} \end{cases} \tag{3.8}$$

where

$$\mu_1 = i\sqrt{\lambda^2 - \lambda}, \quad -\mu_1, \quad \mu_2 = i\sqrt{\lambda^2 + \lambda}, \quad -\mu_2$$

are the roots of (3.4). Then

$$\begin{cases} A + B + C + D = c_1 \\ \mu_1 A - \mu_1 B + \mu_2 C - \mu_2 D = c_2 \\ -\left(\frac{\lambda^2}{\mu_1} + \mu_1\right)A + \left(\frac{\lambda^2}{\mu_1} + \mu_1\right)B - \left(\frac{\lambda^2}{\mu_2} + \mu_2\right)C + \left(\frac{\lambda^2}{\mu_2} + \mu_2\right)D = c_3 \\ -(\mu_1^2 + \lambda^2)A - (\mu_1^2 + \lambda^2)B - (\mu_2^2 + \lambda^2)C - (\mu_2^2 + \lambda^2)D = c_4. \end{cases} \tag{3.9}$$

Solving system (3.9), we find that

$$\begin{cases} A &=& -\frac{1}{2}\frac{\mu_2^2 + \lambda^2}{\mu_1^2 - \mu_2^2}c_1 + \frac{1}{2}\frac{\mu_2^2 + \lambda^2}{(\mu_1^2 - \mu_2^2)\lambda^2}\mu_1 c_2 + \frac{1}{2}\frac{\mu_2^2}{(\mu_1^2 - \mu_2^2)\lambda^2}\mu_1 c_3 - \frac{1}{2}\frac{c_4}{\mu_1^2 - \mu_2^2} \\ B &=& -\frac{1}{2}\frac{\mu_2^2 + \lambda^2}{\mu_1^2 - \mu_2^2}c_1 - \frac{1}{2}\frac{\mu_2^2 + \lambda^2}{(\mu_1^2 - \mu_2^2)\lambda^2}\mu_1 c_2 - \frac{1}{2}\frac{\mu_2^2}{(\mu_1^2 - \mu_2^2)\lambda^2}\mu_1 c_3 - \frac{1}{2}\frac{c_4}{\mu_1^2 - \mu_2^2} \\ C &=& \frac{1}{2}\frac{\mu_2^2 + \lambda^2}{\mu_1^2 - \mu_2^2}c_1 - \frac{1}{2}\frac{\mu_2^2 + \lambda^2}{(\mu_1^2 - \mu_2^2)\lambda^2}\mu_1 c_2 - \frac{1}{2}\frac{\mu_2^2}{(\mu_1^2 - \mu_2^2)\lambda^2}\mu_1 c_3 + \frac{1}{2}\frac{c_4}{\mu_1^2 - \mu_2^2} \\ D &=& \frac{1}{2}\frac{\mu_2^2 + \lambda^2}{\mu_1^2 - \mu_2^2}c_1 + \frac{1}{2}\frac{\mu_2^2 + \lambda^2}{(\mu_1^2 - \mu_2^2)\lambda^2}\mu_1 c_2 + \frac{1}{2}\frac{\mu_2^2}{(\mu_1^2 - \mu_2^2)\lambda^2}\mu_1 c_3 + \frac{1}{2}\frac{c_4}{\mu_1^2 - \mu_2^2} \end{cases} \tag{3.10}$$

Setting (c_1, c_2, c_3, c_4) to be the unit vectors e_i for $i = 1, ..., 4$, we obtain

$$\begin{cases} \varphi_1(x) &=& -\frac{\mu_2^2 + \lambda^2}{\mu_1^2 - \mu_2^2}\cosh(\mu_1 x) + \frac{\mu_1^2 + \lambda^2}{\mu_1^2 - \mu_2^2}\cosh(\mu_2 x) \\ \psi_1(x) &=& \frac{(\mu_1^2 + \lambda^2)(\mu_2^2 + \lambda^2)}{\mu_1(\mu_1^2 - \mu_2^2)}\sinh(\mu_1 x) - \frac{(\mu_1^2 + \lambda^2)(\mu_2^2 + \lambda^2)}{\mu_2(\mu_1^2 - \mu_2^2)}\sinh(\mu_2 x) \end{cases} \tag{3.11}$$

$$\begin{cases} \varphi_2(x) &=& \frac{(\mu_2^2 + \lambda^2)\mu_1}{\lambda^2(\mu_1^2 - \mu_2^2)}\sinh(\mu_1 x) - \frac{(\mu_1^2 + \lambda^2)\mu_2}{\lambda^2(\mu_1^2 - \mu_2^2)}\sinh(\mu_2 x) \\ \psi_2(x) &=& -\frac{(\mu_1^2 + \lambda^2)(\mu_2^2 + \lambda^2)}{\lambda^2(\mu_1^2 - \mu_2^2)}\cosh(\mu_1 x) + \frac{(\mu_1^2 + \lambda^2)(\mu_2^2 + \lambda^2)}{\lambda^2(\mu_1^2 - \mu_2^2)}\cosh(\mu_2 x) \end{cases} \tag{3.12}$$

$$(3.13)\quad \begin{cases} \varphi_3(x) &= \dfrac{\mu_2^2\mu_1}{\lambda^2(\mu_1^2-\mu_2^2)}\sinh(\mu_1 x) - \dfrac{\mu_1^2\mu_2}{\lambda^2(\mu_1^2-\mu_2^2)}\sinh(\mu_2 x) \\ \psi_3(x) &= -\dfrac{(\mu_1^2+\lambda^2)\mu_2^2}{\lambda^2(\mu_1^2-\mu_2^2)}\cosh(\mu_1 x) + \dfrac{(\mu_2^2+\lambda^2)\mu_1^2}{\lambda^2(\mu_1^2-\mu_2^2)}\cosh(\mu_2 x) \end{cases}$$

$$(3.14)\quad \begin{cases} \varphi_4(x) &= -\dfrac{1}{\mu_1^2-\mu_2^2}\cosh(\mu_1 x) + \dfrac{1}{\mu_1^2-\mu_2^2}\cosh(\mu_2 x) \\ \psi_4(x) &= \dfrac{\mu_1^2+\lambda^2}{\mu_1(\mu_1^2-\mu_2^2)}\sinh(\mu_1 x) - \dfrac{\mu_2^2+\lambda^2}{\mu_2(\mu_1^2-\mu_2^2)}\sinh(\mu_2 x). \end{cases}$$

From (3.11), (3.12), (3.13) and (3.14), we have

$$(3.15)\quad e^{Bx} = \begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 \\ \varphi_{1x} & \varphi_{2x} & \varphi_{3x} & \varphi_{4x} \\ \psi_1 & \psi_2 & \psi_3 & \psi_4 \\ \psi_{1x} & \psi_{2x} & \psi_{3x} & \psi_{4x} \end{pmatrix}.$$

We deduce from (3.5) and (3.15) that

$$(3.16)\quad \begin{cases} \varphi(x) = \varphi_x(0)\varphi_2 + \psi_x(0)\varphi_4 - \displaystyle\int_0^x ((f_2+i\lambda f_1)\varphi_2(x-s) + (f_6+i\lambda f_5)\varphi_4(x-s))\,ds, \\ \psi(x) = \varphi_x(0)\psi_2 + \psi_x(0)\psi_4 - \displaystyle\int_0^x ((f_2+i\lambda f_1)\psi_2(x-s) + (f_6+i\lambda f_5)\psi_4(x-s))\,ds. \end{cases}$$

Then

$$(3.17)\quad \begin{cases} \varphi_x(x) = \varphi_x(0)\varphi_{2x} + \psi_x(0)\varphi_{4x} - \displaystyle\int_0^x ((f_2+i\lambda f_1)\varphi_{2x}(x-s) + (f_6+i\lambda f_5)\varphi_{4x}(x-s))\,ds, \\ \psi_x(x) = \varphi_x(0)\psi_{2x} + \psi_x(0)\psi_{4x} - \displaystyle\int_0^x ((f_2+i\lambda f_1)\psi_{2x}(x-s) + (f_6+i\lambda f_5)\psi_{4x}(x-s))\,ds. \end{cases}$$

With third and forth equations of (3.3), we get

$$(3.18)\quad \phi_1(\xi) = \frac{u(L)\mu(\xi)+f_3(\xi)}{i\lambda+\xi^2+\eta}, \quad \phi_2(\xi) = \frac{v(L)\mu(\xi)+f_7(\xi)}{i\lambda+\xi^2+\eta}.$$

Inserting (3.18) in last equations of (3.3), we get

$$(3.19)\quad \begin{aligned} \frac{b}{m_2}\psi_x(L) + (i\lambda + \frac{\gamma_2}{m_2}(i\lambda+\eta)^{\alpha-1})i\lambda\psi(L) = f_8 + (i\lambda + \frac{\gamma_2}{m_2}(i\lambda+\eta)^{\alpha-1})f_5(L) \\ -\zeta_2\int_{-\infty}^{\infty}\frac{\mu(\xi)f_7(\xi)}{i\lambda+\xi^2+\eta}\,d\xi, \end{aligned}$$

$$(3.20)\quad \begin{aligned} \frac{K}{m_1}(\varphi_x+\psi)(L) + (i\lambda + \frac{\gamma_1}{m_1}(i\lambda+\eta)^{\alpha-1})i\lambda\varphi(L) = f_4 + (i\lambda + \frac{\gamma_1}{m_1}(i\lambda+\eta)^{\alpha-1})f_1(L) \\ -\zeta_1\int_{-\infty}^{\infty}\frac{\mu(\xi)f_3(\xi)}{i\lambda+\xi^2+\eta}\,d\xi. \end{aligned}$$

Using (3.16), we can rewrite (3.19) and (3.20) as equations in the unknowns $\varphi_x(0)$and $\psi_x(0)$

$$\varphi_x(0)\left[\psi_{2x}(L)+(im_2\lambda+\gamma_2(i\lambda+\eta)^{\alpha-1})i\lambda\psi_2(L)\right]+\psi_x(0)\left[\psi_{4x}(L)+(im_2\lambda+\gamma_2(i\lambda+\eta)^{\alpha-1})i\lambda\psi_4(L)\right]$$
$$=m_2f_8+(im_2\lambda+\gamma_2(i\lambda+\eta)^{\alpha-1})f_5(L)-\zeta_2\int_{-\infty}^{\infty}\frac{\mu(\xi)f_7(\xi)}{i\lambda+\xi^2+\eta}\,d\xi$$
$$+(im_2\lambda+\gamma_2(i\lambda+\eta)^{\alpha-1})i\lambda\int_0^L((f_2+i\lambda f_1)\psi_2(L-s)+(f_6+i\lambda f_5)\psi_4(L-s))\,ds$$
$$+\int_0^L((f_2+i\lambda f_1)\psi_{2x}(L-s)+(f_6+i\lambda f_5)\psi_{4x}(L-s))\,ds, \tag{3.21}$$

$$\varphi_x(0)\Big[\varphi_{2x}(L)+\psi_2(L)+(im_1\lambda+\gamma_1(i\lambda+\eta)^{\alpha-1})i\lambda\varphi_2(L)\Big]$$
$$+\psi_x(0)\Big[\varphi_{4x}(L)+\psi_4(L)+(im_1\lambda+\gamma_1(i\lambda+\eta)^{\alpha-1})i\lambda\varphi_4(L)\Big]$$
$$=m_1f_4+(im_1\lambda+\gamma_1(i\lambda+\eta)^{\alpha-1})f_1(L)-\zeta_1\int_{-\infty}^{\infty}\frac{\mu(\xi)f_3(\xi)}{i\lambda+\xi^2+\eta}\,d\xi$$
$$+(im_1\lambda+\gamma_1(i\lambda+\eta)^{\alpha-1})i\lambda\int_0^L((f_2+i\lambda f_1)\varphi_2(L-s)+(f_6+i\lambda f_5)\varphi_4(L-s))\,ds \tag{3.22}$$
$$+\int_0^L((f_2+i\lambda f_1)\psi_2(L-s)+(f_6+i\lambda f_5)\psi_4(L-s))\,ds$$
$$+\int_0^L((f_2+i\lambda f_1)\varphi_{2x}(L-s)+(f_6+i\lambda f_5)\varphi_{4x}(L-s))\,ds.$$

Using (3.21) and (3.22), a linear system in $\varphi_x(0)$ and $\psi_x(0)$ is obtained

$$\begin{pmatrix} m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix}\begin{pmatrix}\varphi_x(0)\\ \psi_x(0)\end{pmatrix}=\begin{pmatrix}I_1\\ I_2\end{pmatrix} \tag{3.23}$$

where

$$\begin{aligned}
m_{11} &= \psi_{2x}(L)+(im_2\lambda+\gamma_2(i\lambda+\eta)^{\alpha-1})i\lambda\psi_2(L)\\
&= -\tfrac{1}{2\lambda}(z_1\sin z_1L-z_2\sin z_2L)+\tfrac12(-m_2\lambda+\gamma_2q)(\cos z_1L-\cos z_2L)\\
m_{12} &= \psi_{4x}(L)+(im_2\lambda+\gamma_2(i\lambda+\eta)^{\alpha-1})i\lambda\psi_4(L)\\
&= \tfrac12(\cos z_1L+\cos z_2L)+(-m_2\lambda+\gamma_2q)\lambda\tfrac12(\tfrac{1}{z_1}\sin z_1L+\tfrac{1}{z_2}\sin z_2L)\\
m_{21} &= \varphi_{2x}(L)+\psi_2(L)+(im_1\lambda+\gamma_1(i\lambda+\eta)^{\alpha-1})i\lambda\varphi_2(L)\\
&= \tfrac{1}{2\lambda^2}(z_1^2\cos z_1L+z_2^2\cos z_2L)+\tfrac{1}{2\lambda}(\cos z_1L-\cos z_2L)+(-m_1\lambda+\gamma_1q)\tfrac{1}{2\lambda}(z_1\sin z_1L+z_2\sin z_2L)\\
m_{22} &= \varphi_{4x}(L)+\psi_4(L)+(im_1\lambda+\gamma_1(i\lambda+\eta)^{\alpha-1})i\lambda\varphi_4(L)\\
&= \tfrac{1}{2\lambda}(z_1\sin z_1L-z_2\sin z_2L)+\tfrac12(\tfrac{1}{z_1}\sin z_1L+\tfrac{1}{z_2}\sin z_2L)-\tfrac12(-m_1\lambda+\gamma_1q)(\cos z_1L-\cos z_2L),
\end{aligned}$$

where $z_1=\Im(\mu_1)=\sqrt{\lambda^2-\lambda}$ and $z_2=\Im(\mu_2)=\sqrt{\lambda^2+\lambda}$.

Let the determinant of the linear system given in (3.23) be denoted by D. Then the following is obtained:

$$\begin{aligned}
D = {} & -\tfrac12x^2m_2m_1-\tfrac14\tfrac{x^2}{z_1}z_2m_2\left(\sin z_1\right)m_1\sin z_2-\tfrac14x^2\tfrac{z_1}{z_2}m_2\left(\sin z_2\right)m_1\sin z_1+\tfrac12x^2m_2\left(\cos z_1\right)m_1\cos z_2\\
&+\tfrac14\tfrac{z_1^2}{z_2}m_2\sin z_2\cos z_1+\tfrac{1}{4z_1}z_2^2m_2\sin z_1\cos z_2+\tfrac14z_1\left(\cos z_2\sin z_1\right)(2m_1+m_2)\\
&+\tfrac14z_2\left(\cos z_1\sin z_2\right)(2m_1+m_2)+\tfrac12xq\left(a_2m_1+m_2a_1\right)+\tfrac14x\left(\sin z_1\right)q\left(\sin z_2\right)\left(z_2^2+z_1^2\right)\tfrac{a_2m_1+m_2a_1}{z_1z_2}\\
&-\tfrac12xq\left(\cos z_1\cos z_2\right)\left(a_2m_1+m_2a_1\right)-\tfrac12-\tfrac12\left(\cos z_1\cos z_2\right)+\tfrac{1}{2x^2}z_1z_2\sin z_1\sin z_2\\
&-\tfrac12q\left(\sin z_2\cos z_1\right)\tfrac{a_2x^2+z_2^2a_1}{xz_2}-\tfrac12q\left(\sin z_1\cos z_2\right)\tfrac{a_2x^2+z_1^2a_1}{xz_1}\\
&-\tfrac12a_2q^2a_1-\tfrac14a_2q^2\left(\sin z_1\right)a_1\left(\sin z_2\right)\tfrac{z_2^2+z_1^2}{z_1z_2}+\tfrac12a_2q^2\left(\cos z_1\right)a_1\cos z_2+\tfrac12\left(\sin z_2\sin z_1\right)\tfrac{1}{z_1z_2},
\end{aligned}$$

where $q(\lambda) = i(i\lambda + \eta)^{\alpha-1}$. We can easily prove that

$$|D| \geq C|\lambda|^{2(\alpha-1)} \text{ for } |\lambda| \text{ large }. \tag{3.24}$$

This estimation is optimal. Indeed suppose that (3.24) is not optimal. This means that there exists $\varepsilon > 0$ such that

$$|D| \geq C|\lambda|^{2(\alpha-1)+\varepsilon} \text{ for } |\lambda| \text{ large }. \tag{3.25}$$

This is a contradiction because, we can construct sequences λ_ς large such that $\cos(z_1+z_2)L = -1$ and then $\sin(z_1+z_2)L = 0$ which implies that

$$|D| \leq C|\lambda_\varsigma|^{2(\alpha-1)},$$

contradicting (3.25). As

$$\begin{aligned} \varphi_x(0) &= \frac{1}{D}\left(m_{22}I_1 - m_{21}I_2\right) \\ \psi_x(0) &= \frac{1}{D}\left(m_{11}I_2 - m_{12}I_1\right) \end{aligned}$$

Then, we conclude that

$$|\varphi_x(0)|, |\psi_x(0)| \leq c|\lambda|^{2(1-\alpha)}.$$

From (3.16) and (3.17), we deduce that

$$\|\varphi_x+\psi\|_{L^2(0,L)}, \|\psi_x\|_{L^2(0,L)} \leq c|\lambda|^{2(1-\alpha)}\left(\|f_1\|_{H^1(0,L)} + \|f_2\|_{L^2(0,L)} + \|f_4\|_{H^1(0,L)} + \|f_5\|_{L^2(0,L)}\right).$$

From (3.2), we get

$$\|u\|_{L^2(0,L)}, \|v\|_{L^2(0,L)} \leq c|\lambda|^{2(1-\alpha)}\left(\|f_1\|_{H^1(0,L)} + \|f_2\|_{L^2(0,L)} + \|f_4\|_{H^1(0,L)} + \|f_5\|_{L^2(0,L)}\right).$$

From (3.18), we get

$$\begin{aligned} \|\phi_1\|_{L^2(-\infty,\infty)} &\leq |u(L)|\left\|\frac{\mu(\xi)}{i\lambda+\xi^2+\eta}\right\|_{L^2(-\infty,\infty)} + \left\|\frac{f_3(\xi)}{i\lambda+\xi^2+\eta}\right\|_{L^2(-\infty,\infty)} \\ &\leq c|\lambda|^{1-\frac{3\alpha}{2}}\left(\|f_1\|_{H^1(0,L)} + \|f_2\|_{L^2(0,L)} + \|f_4\|_{H^1(0,L)} + \|f_5\|_{L^2(0,L)}\right) + c\frac{1}{|\lambda|}\|f_3\|_{L^2(-\infty,\infty)} \end{aligned}$$

Thus, we conclude that

$$\|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{H}} \leq c|\lambda|^{2(1-\alpha)} \text{ as } |\lambda| \to \infty.$$

This estimation is optimal following (3.24). Then, we deduce that

$$E(t) \sim \frac{1}{t^{\frac{1}{1-\alpha}}}.$$

The case $r_1^2 \neq r_2^2$

We consider only the case $r_1^2 > r_2^2$, the case $r_1^2 < r_2^2$ is similar. It is not restrictive to suppose $\rho_1 = \rho_2 = K = 1$ and then $b < 1$.

The eigenvalues μ_i of the matrix B are the roots of the following equation

$$x^4 + (a+1)\lambda^2 x^2 + a\lambda^2(\lambda^2 - 1) = 0, \tag{3.26}$$

where $a = 1/b$.
Expansion of $\mu_i, i = 1, 2$ for $a > 1$. We have

$$\mu_{1,2}^2 = -\frac{1}{2}(a+1)\lambda^2 \pm \frac{1}{2}\sqrt{(a-1)^2\lambda^4 + 4\lambda^2}.$$

It follows that

$$\mu_1 = i\lambda - \frac{1}{2(a-1)\lambda} + i\frac{5-a}{8(a-1)^3\lambda^3} + i\frac{6a - a^2 - 21}{16(a-1)^5\lambda^5} + O\left(\frac{1}{\lambda^7}\right). \tag{3.27}$$

$$\mu_2 = i\sqrt{a}\lambda + \frac{1}{2\sqrt{a}(a-1)\lambda} - i\frac{5a-1}{8\sqrt{a}a(a-1)^3\lambda^3} + i\frac{21a^2 - 6a + 1}{16\sqrt{a}a^2(a-1)^5\lambda^5} + O\left(\frac{1}{\lambda^7}\right). \tag{3.28}$$

Therefore

$$\begin{cases} \varphi_2(x) &= \dfrac{1}{i\lambda}\sinh(\mu_1 x) + O\left(\dfrac{1}{\lambda^3}\right) \\ \psi_2(x) &= \left(\dfrac{1}{(a-1)\lambda^2} - \dfrac{2}{(a-1)^3\lambda^4}\right)(\cosh(\mu_1 x) - \cosh(\mu_2 x) + O\left(\dfrac{1}{\lambda^6}\right) \\ \varphi_4(x) &= \dfrac{-1}{(a-1)\lambda^2}\cosh(\mu_1 x) + \dfrac{1}{(a-1)\lambda^2}\cosh(\mu_2 x) + O\left(\dfrac{1}{\lambda^2}\right) \\ \psi_4(x) &= \dfrac{1}{i(a-1)^2\lambda^3}\sinh(\mu_1 x) + \dfrac{1}{i\sqrt{a}\lambda}\sinh(\mu_2 x) + O\left(\dfrac{1}{\lambda^5}\right) \end{cases} \tag{3.29}$$

We have

$$\begin{aligned} m_{11} &= \tfrac{1}{a}\psi_{2x}(L) + (im_2\lambda + \gamma_2(i\lambda+\eta)^{\alpha-1})i\lambda\psi_2(L) \\ &\sim \tfrac{1}{a(a-1)\lambda}(-\sin z_1 L + \sqrt{a}\sin z_2 L) + (-m_2\lambda + \gamma_2 q)\tfrac{1}{(a-1)\lambda}(\cos z_1 L - \cos z_2 L) \\ m_{12} &= \tfrac{1}{a}\psi_{4x}(L) + (im_2\lambda + \gamma_2(i\lambda+\eta)^{\alpha-1})i\lambda\psi_4(L) \\ &\sim \tfrac{1}{a(a-1)^2\lambda^2}\cos z_1 L + \tfrac{1}{a}\cos z_2 L + (-m_2\lambda + \gamma_2 q)(\tfrac{1}{(a-1)^2\lambda^2}\sin z_1 L + \tfrac{1}{\sqrt{a}}\sin z_2 L) \\ m_{21} &= \varphi_{2x}(L) + \psi_2(L) + (im_1\lambda + \gamma_1(i\lambda+\eta)^{\alpha-1})i\lambda\varphi_2(L) \\ &\sim \cos z_1 + \tfrac{1}{(a-1)\lambda^2}(\cos z_1 L - \cos z_2 L) + (-m_1\lambda + \gamma_1 q)\sin z_1 L \\ m_{22} &= \varphi_{4x}(L) + \psi_4(L) + (im_1\lambda + \gamma_1(i\lambda+\eta)^{\alpha-1})i\lambda\varphi_4(L) \\ &\sim \tfrac{1}{(a-1)\lambda}(\sin z_1 L - \sqrt{a}\sin z_2 L) + (\tfrac{1}{(a-1)^2\lambda^3}\sin z_1 L + \tfrac{1}{\sqrt{a}\lambda}\sin z_2 L) - \tfrac{(-m_1\lambda + \gamma_1 q)}{(a-1)\lambda}(\cos z_1 L - \cos z_2 L) \end{aligned}$$

Then

$$D = -a\cos z_1 L\cos z_2 L - \frac{q}{\sqrt{a}}(a\sqrt{a}\gamma_1\cos z_2 L\sin z_1 L + \gamma_2\cos z_1 L\sin z_2 L) - \frac{\gamma_1\gamma_2}{\sqrt{a}}q^2\sin z_1 L\sin z_2 L + o(q^2).$$

We remark that

$$|D| \geq c\, q^2 \Leftrightarrow \cos z_1 L = 0 \text{ and } \cos z_2 L = 0.$$

Now, we solve system

$$\begin{cases} \cos z_1 L &= 0 \\ \cos z_2 L &= 0 \end{cases} \tag{3.30}$$

Then, using (3.2)-(3.28), it follows from (3.30) that there exist

$$\begin{cases} \lambda - \dfrac{1}{2(a-1)\lambda} &= \dfrac{1}{L}(m+\dfrac{1}{2})\pi + O\left(\dfrac{1}{\lambda^2}\right) \\ \sqrt{a}\lambda + \dfrac{1}{2\sqrt{a}(a-1)\lambda} &= \dfrac{1}{L}(k+\dfrac{1}{2})\pi + O\left(\dfrac{1}{\lambda^2}\right) \end{cases} \tag{3.31}$$

Since $m \sim k \sim \lambda$, (3.31) can be written as

$$\begin{cases} \lambda^2 &= \dfrac{1}{L}(m+\dfrac{1}{2})^2\pi^2 + \dfrac{1}{(a-1)} + O\left(\dfrac{1}{\lambda^2}\right), \\ a\lambda^2 &= \dfrac{1}{L}(k+\dfrac{1}{2})^2\pi^2 - \dfrac{1}{\sqrt{a}(a-1)} + O\left(\dfrac{1}{\lambda^2}\right). \end{cases} \tag{3.32}$$

Finally we obtain

$$a\left(m+\frac{1}{2}\right)^2 - \left(k+\frac{1}{2}\right)^2 = -\frac{L^2(a\sqrt{a}+1)}{\sqrt{a}(a-1)\pi^2} + O\left(\frac{1}{\lambda^2}\right). \tag{3.33}$$

Let us set

$$c = -\frac{L^2(a\sqrt{a}+1)}{\sqrt{a}(a-1)\pi^2}.$$

Now, we assume that $\sqrt{a} \in \mathbb{Q}$. Then, $a = \frac{p^2}{q^2}$ for some $p, q \in \mathbb{N}$, we deduce

$$\frac{p\left(m+\frac{1}{2}\right) - q\left(k+\frac{1}{2}\right)}{q^2} = \frac{c}{p\left(m+\frac{1}{2}\right) + q\left(k+\frac{1}{2}\right)} + \frac{o(1)}{p\left(m+\frac{1}{2}\right) + q\left(k+\frac{1}{2}\right)}. \tag{3.34}$$

If $p\left(m+\frac{1}{2}\right) - q\left(k+\frac{1}{2}\right) = 0$ for an infinity number of pairs (m, k), then $c = o(1)$ and this is a contradiction.
Else $p\left(m+\frac{1}{2}\right) - q\left(k+\frac{1}{2}\right) \neq 0$ for λ large enough and then

$$\frac{1}{q^2} \leq \left|\frac{c}{p\left(m+\frac{1}{2}\right) + q\left(k+\frac{1}{2}\right)}\right| + \left|\frac{o(1)}{p\left(m+\frac{1}{2}\right) + q\left(k+\frac{1}{2}\right)}\right|.$$

which cannot be true. Then, if $\sqrt{a} \in \mathbb{Q}$, we have only the following possibilities:

$$\cos z_1 L = 0 \text{ and } \cos z_2 L \neq 0 \text{ or } \cos z_2 L = 0 \text{ and } \cos z_1 L \neq 0.$$

In this case, we get

$$|D| \geq c|q|.$$

Using (3.16) and (3.29), we conclude that

$$\|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{H}} \leq c|\lambda|^{(1-\alpha)} \text{ as } |\lambda| \to \infty.$$

We deduce an optimal decay rate of the energy of order $t^{-2/(1-\alpha)}$.

3.3 Conclusions

1) In this paper we have studied the Timoshenko beam with two feedbacks of fractional derivative type. We have considered two cases: $\eta = 0$ and $\eta > 0$.

For the case $\eta = 0$, we have prove only strong asymptotic stability. The decay rate is polynomial but we did not obtain any exponent depending on parameter α. As $\lambda = 0$ is a spectral value, the method based on multiplier technic and Borichev-Tomilov method do not work. In the future, we try other technic as a representation of solution by Mittag-Leffler Functions.

For the case $\eta > 0$, we have succeed to prove decay rate depending on parameter α using multiplier technic and Borichev-Tomilov method. This tool is flexible and can be adapted to the multi-dimensional case and other complex systems. But, in general do not give optimal decay rate. In our case, we have prove the optimality of the energy decay when the wave propagation speeds are equal (weakly hyperbolic case). However, in the natural physical case when the speeds of propagation are different (strongly hyperbolic case), we obtain optimal and better decay rate (if $\sqrt{1/b}$ is a rational number) witch is consistent with the asymptotic expansion of eigenvalues. This is a surprisely fact because Timoshenko beam system can be stabilized uniformly by only one internal frictional feedback in the weakly hyperbolic case.

2) In the future, we will consider Timoshenko beam system with only one boundary feedback of of fractional derivative type, that is

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) = 0 & \text{in } (0,L) \times (0,+\infty), \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x + \psi)(x,t) = 0 & \text{in } (0,L) \times (0,+\infty), \\ \varphi(0,t) = 0, \quad \psi(0,t) = 0 & \text{in } (0,+\infty), \\ K(\varphi_x + \psi)(L,t) = 0 & \text{in } (0,+\infty), \\ b\psi_x(L,t) = -\gamma_2\partial_t^{\alpha,\eta}\psi(L,t) & \text{in } (0,+\infty). \end{cases}$$

Publications

The following results were published:

1. A. Benaissa and S. Benazzouz *Energy decay of solutions to the Cauchy problem for a nondissipative wave equation,* Journal of Mathematical Physics. Vol. 51, 123504, 2010.

2. A. Benaissa and S. Benazzouz *well-posedeness and asymptotic behavior of Timoshenko beam system with dynamic boundary dissipative feedback of fractional derivative type* Z. Angew. Math.Phys. 68-94, 2017.

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Résumé

Ma thèse de doctorat intitulé " Etude de la stabilisation et d'existence globale des équations d'évolutions linaires et non linéaires.

Dans les préliminaires, on rappelle des définitions et des résultats utiles pour notre travail.
Ces résultats concernent essentiellement la théorie de semi-groupe ,
. On rappelle aussi les types de stabilité et des résultats généraux connus dans la littérature et appliquées pour certaines équations dissipatives. Dans le chapitre deux, on considère un système Timoshenko de type dynamique avec un contrôle au frontière de type dérivée fractionnaire. On montre l'existence globale de la solution dans des espaces de Sobolev et on détermine la vitesse de décroissance de l'énergie associée aux solutions. Dans le chapitre trois, on considère le même système que dans le chapitre un

mots clés: système Timoshenko, Existence globale, stabilisation, Méthode de semi groupe, Méthode des multiplicateurs.

Abstract

My thesis is devoted to the study of stabilization and global existence, to linear and nonlinear evolutions equations.
This work consists of three chapters:
In chapter 1 we give some preliminaries about some functional spaces in particular semi groupe theory and the different result for the stabilization.
In chapter 2, we consider the Timoshenko beam system with dynamic controls of fractional derivative type We prove a global existence result using the semi-group theory, we show that our system is not uniformly stable in general,. Also, we look for a polynomial decay rate for smooth initial data for our system by applying a frequency domain approach combining with a multiplier method.
In chapter 3, we consider the same system as above ,By an explicit representation of the resolvant associated to the operator semi-group, we prove different optimal energy decay
Key words: Timoshenko beam system, global existence , stabilization , semi-group method, multiplier Method .

الملخص

الرسالة التي بين أيديكم تحمل عنوان : **"دراسة استقرار ووجود الحل لمعادلات التطور الخطية وغير الخطية"** تنقسم إلى ثلاثة فصول :

في البداية : نذكر بالتعاريف و النتائج المستعملة في هذا العمل، هذه النتائج تخص بالأساس : نظريات الزمر الجزئية ،والنتائج المختلفة لدراسة الاستقرار

في الفصل الثاني : نعتبر جملة تيموشنكو مع ضوابط ديناميكية ذات مشتقات كسرية ، أثبتنا وجود حل كلي باستخدام نظريات الزمر الجزئية ،أما الاستقرار الجملة غير مستقرة أسيا عموما ، كما توصلنا إلى أن الطاقة متناقصة على شكل كثير حدود .

في الفصل الثالث : فإننا نعتبر نفس الجملة المذكورة أعلاه ، من خلال تمثيل صريح للحال المربوط إلى شبه المشغل ، فإننا نثبت تباين الطاقة الأمثل

الكلمات المفتاحية : جملة تيموشنكو ، الوجود الكلي ، الإستقرار ، طريقة الزمر الجزئية ، طريقة المضاعفات.