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BLOUHI TAYEB

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Une Contribution aux Equations et Inclusions Différentielles Stochastiques Avec Impulsion

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Devant le jury composé de :

Président :	*Yousfate Abderrahmane*	*Pr. Univ. Djilali Liabes SBA*
Examinateurs :		
	Tomás Caraballo	*Pr. Univ. Sevilla Espagne*
	Terbeche Mekki	*Pr. Univ. Oran 1 Oran*
	Benchohra Mouffak	*Pr. Univ. Djilali Liabes SBA*
	Bouguima Sidi Mohammed	*Pr.Uni.de Telemcen*
Directeur de thèse:	*Ouahab Abdelghani*	*Pr. Univ. Djilali Liabes SBA*

Une Contribution aux Equations et Inclusions Diffrentielles Stochastiques Avec Impulsion

Je dédie ce modeste travail à
mes chers parents
mes frères et sœurs
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Publications

- **T. Blouhi**,J. Nieto and A. Ouahab, , Existence and uniqueness results for systems of impulsive stochastic differential equations, *Ukrainian Math. J., to appear.*,
- **T. Blouhi**, T. Caraballo and A. Ouahab, Existence and Stability Results for Semilinear Systems of Impulsive Stochastic Differential Equations with Fractional Brownian Motion *Stoch.Ana.Appl* ,N.34,pp.792–834,(2016).
- **T. Blouhi**, T. Caraballo and A. Ouahab,Topological method for couple systems of impulsive stochastic semilinear differential inclusions with fractional Brownian motion, Fixed point theory, to appear.
- **T. Blouhi**, T. Caraballo and A. Ouahab, Existence and stability results for couple stochastic difference equations with delay , submitted September 2016.

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Introduction

Stochastic Integration and Stochastic Differential Equations with impulses have become an active area of investigation due to their applications in the fields such as mechanics, electrical engineering, medicine biology, ecology etc. It is highly considered that the evolution of many physical systems is described by an ordinary differential equation with impulses .In the case of differential equations, there is no ambiguity since the derivative $(x'(.), y'(.))$ of a solution $(x(.), y(.))$ to the differential equation

$$\left\{ \begin{array}{rcl} dx(t) & = & f^1(t, x(t), y(t)), \\ dy(t) & = & f^2(t, x(t), y(t)), \\ x(0) & = & x_0, \\ y(0) & = & y_0. \end{array} \right. \tag{0.0.1}$$

A great impetus to study differential inclusions came from the developent of control theory, i.e. of dynamical systems.

$$z'(t) = f(t, z(t), u(t)), \quad z(0) = z_0 \tag{0.0.2}$$

"controlled" by parameters $u(t)$ (the "controls") and $z = (x, y)$,$f = (f^1, f^2)$. Indeed, If we introduce the set valued map

$$F(t, z) = \{f(t, z, u)\}_{u \in U},$$

then solution to differential equations (0.0.2) are solutions to the differential inclusion

$$z'(t) \in F(t, z(t)), \quad z(0) = z_0, \tag{0.0.3}$$

in which the controls do not appear explicitly.

Differential inclusions provide a mathematical tool for studying differential equations

$$z'(t) = f(t, z(t)),$$

with discontinuous right-hand side, by embedding $f(t,z)$ into a set valued map $F(t,z)$ which, as a set valued map, enjoys enough regularities of the original differential equation.

However, in certain circumstances, physical systems are disturbed by random noise. One way to handle these disturbances is to alter equation (0.0.1) disturbing terms of the form $g^i(t)dB_Q^H(t)$, for each $i = 1,2$ where g^i characterizes the noise power. This leads to an evolution equation of the form:

$$\begin{cases} dx(t) &= f^1(t,x(t),y(t))dt + g^1(t,x(t),y(t))dB_Q^H(t) \\ dy(t) &= f^2(t,x(t),y(t))dt + g^2(t,x(t),y(t))dB_Q^H(t), \\ x(0) &= x_0, \\ y(0) &= y_0. \end{cases} \tag{0.0.4}$$

On the other hand, the Brownian motion is not differentiable, equation (0.0.4) should be discerned from the classical differential calculus. So, a meaning to $dB_Q^H(t)$ should be given in order to define $(x(t),y(t))$ as the solution of the equation:

$$\begin{cases} x(t) &= x_0 + \int_0^t f^1(s,x(s),y(s))ds \\ &+ \int_0^t g^1(t,x(s),y(s))dB^H(s) \quad \mathbb{P}.a.s, \quad t \in J \\ \\ y(t) &= y_0 + \int_0^t f^2(s,x(s),y(s))ds \\ &+ \int_0^t g^2(t,x(s),y(s))dB^H(s) \quad \mathbb{P}.a.s, \quad t \in J. \end{cases} \tag{0.0.5}$$

An equation of the form (0.0.5) is called stochastic differential equation.

Differential equations with impulses were considered for the first time by Milman and Myshkis [110] and then followed by a period of active research which culminated with the monograph by Halanay and Wexler [76]. Many phenomena and evolution processes in physics, chemical technology, population dynamics, and natural sciences may change state abruptly or be subject to short-term perturbations. These perturbations may be seen as impulses. Impulsive problems arise also in various applications in communications, mechanics (jump discontinuities in velocity), electrical engineering, medicine and biology. A comprehensive introduction to the basic theory is well developed in the monographs by Benchohra et al [24], Graef *et al* [71], Laskshmikantham *et al.* [17], Samoilenko and Perestyuk [139].

Random differential and integral equations play an important role in characterizing many social, physical, biological and engineering problems; see for instance the monographs by Da Prato and Zabczyk [52], Gard [65], Gikhman and Skorokhod [67], Sobzyk [143] and Tsokos and Padgett [152]. For example, a stochastic model for drug distribution in a biological system was described by Tsokos and Padgett [152] to a closed system with a simplified heat, one organ or capillary bed, and re-circulation of a blood with a constant rate of flow, where the heart is considered as a mixing chamber of constant volume. For the basic theory concerning stochastic differential equations see the monographs by Bharucha-Reid [26], Mao [112], Øksendal, [165], Tsokos and Padgett [152], Sobczyk [143] and Da Prato and Zabczyk [52].

The study of impulsive stochastic differential equations is a new research area. The existence and stability of stochastic of impulsive of differential equations were recently investigated, for example in [43, 71, 103–105, 123, 137, 157, 164].

Recently, stochastic differential and partial differential inclusions have been extensively studied. For instance, in [5, 21] the authors investigated the existence of solutions of nonlinear stochastic differential inclusions by means of a Banach fixed point theorem and a semigroup approach. Balasubramaniam [19] obtained existence of solutions of functional stochastic differential inclusions by Kakutani's fixed point theorem, Balasubramaniam et al. [21] initiated the study of existence of solutions of semilinear stochastic delay evolution inclusions in a Hilbert space by using the nonlinear alternative of Leray-Schauder type [59], some existence results for impulsive neutral stochastic evolution inclusions in Hilbert Space, where a class of first-order evolution inclusions with a convex and nonconvex cases are considered, is studied in [116] by using a fixed point theorem due to Dhage and Covitz, as well as Nadler's theorem for contraction multivalued maps.

In this thesis, we shall be concerned by semilinear stochastic differential equations and inclusions with impulsive , some existence results, among others things, are derived, Our results are based upon very recently fixed point theorems and using semigroups theory. We have arranged this thesis as follows:
In chapter 1 we give some basic concepts about stochastic processes, martin-

gale theory and Brownian motion, in the last section we show some recent applications of the Malliavin Calculus to develop a stochastic calculus with respect to the fractional Brownian motion
In chapter 2 we collect some preliminary materials on phase spaces used throughout this thesis, the next section is devoted to set-valued maps some notions,in section 3 we give some fixed point theorems , in the last section we present some propriety of semigroups.
In Chapter 3 we prove the existence of solutions for a first-order impulsive , an application of Schaefer and and Perov fixed point theorems in generalized Banach spaces ,driven by standard Brownian motion $H = \frac{1}{2}$

$$\begin{cases} dx(t) &= \sum_{l=1}^{\infty} f_l^1(t, x(t), y(t))dW^l(t) + g^1(t, x(t), y(t))dt, \ t \in J, t \neq t_k \\ dy(t) &= \sum_{l=1}^{\infty} f_l^2(t, x(t), y(t))dW^l(t) + g^2(t, x(t), y(t))dt, \ t \in J, t \neq t_k \\ x(t_k^+) &-x(t_k) = I_k(x(t_k)), \quad t = t_k \quad k = 1, 2, \ldots, m \\ y(t_k^+) &-y(t_k) = \overline{I}_k(y(t_k)), \\ x(0) &= x_0, \\ y(0) &= y_0, \end{cases} \tag{0.0.6}$$

where $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T, \ \ J := [0, T]$. $f_l^1, f_l^2 : J \times \mathbb{R}^2 \to \mathbb{R}$ are Carathéodory functions, $g^1, g^2 : J \times \mathbb{R}^2 \to \mathbb{R}$, and W^l is an infinite sequence of independent standard Brownian motions, $l = 1, 2, \ldots$ and $I_k, \overline{I}_k \in C(\mathbb{R}, \mathbb{R})\,(k = 1, \ldots, m)$, and $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$. The notations $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^+} y(t_k - h)$ stand for the right and the left limits of the function y at $t = t_k$, respectively. Set

$$\begin{cases} f_i(., x, y) = (f_1^i(., x, y), f_2^i(., x, y), \ldots), \\ \|f_i(., x, y)\| = \Big(\sum_{l=1}^{\infty} (f_l^i)^2(., x, y) \Big)^{\frac{1}{2}} \end{cases} \tag{0.0.7}$$

where $i = 1, 2, \ f_i(., x, y) \in l^2$ for all $x \in \mathbb{R}$.
In chapter 4 our main objective is to establish sufficient conditions for the local and global existence exponential stability of mild solutions of semi linear systems of stochastic differential equations with infinite fractional Brownian motions and impulses with the Hurst index $H > 1/2$.

$$\begin{cases} dx(t) & = (Ax(t) + f^1(t, x(t), y(t))dt \\ & \quad + \sum_{l=1}^{\infty} \sigma_l^1(t, x(t)), y(t))dB_l^H(t), \ t \in J, t \neq t_k, \\ dy(t) & = (Ay(t) + f^2(t, x(t), y(t)))dt \\ & \quad + \sum_{l=1}^{\infty} \sigma_l^2(t, x(t), y(t))dB_l^H(t), \ t \in J, t \neq t_k, \\ \Delta x(t) & = I_k(x(t_k)), \quad t = t_k \quad k = 1, 2, \ldots, m \\ \Delta y(t) & = \overline{I}_k(y(t_k)), \quad t = t_k \quad k = 1, 2, \ldots, m \\ x(0) & = x_0, \\ y(t) & = y_0, \end{cases} \tag{0.0.8}$$

where X is a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ induced by norm $\| \cdot \|$, $A : D(A) \subset X \longrightarrow X$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $(S(t))_{t \geq 0}$ in X and $f^1, f^2 : [0, T] \times X \times X \longrightarrow X$ are given functions, B_l^H is an infinite sequence of independent fractional Brownian motions, $l = 1, 2, \ldots$, with Hurst parameter H, $I_k, \overline{I}_k \in C(X, X)$ $(k = 1, 2, \ldots, m)$, $\sigma_l^1, \sigma_l^2 : J \times X \times X \to L_Q^0(Y, X)$. Here, $L_Q^0(Y, X)$ denotes the space of all Q-Hilbert-Schmidt operators from Y into X, which will be also defined in the next section. Moreover, the fixed times t_k satisfies $0 < t_1 < t_2 < \ldots < t_m < T$, $y(t_k^-)$ and $y(t_k^+)$ denotes the left and right limits of $y(t)$ at $t = t_k$.

$$\begin{cases} \sigma(\cdot, x, y) = (\sigma_1(\cdot, x, y), \sigma_2(\cdot, x, y), \ldots), \\ \|\sigma(\cdot, x, y)\|^2 = \sum_{l=1}^{\infty} \|\sigma_l(\cdot, x, y)\|_{L_Q^0}^2 < \infty \end{cases} \tag{0.0.9}$$

where $\sigma(\cdot, x) \in l^2$ for all $x \in X$, and l^2 is given as

$$l^2 = \{\phi = (\phi_l)_{l \geq 1} : \ [0, T] \times X \times X \to L_Q^0(Y, X) \quad : \|\phi(., x, y)\|^2 = \sum_{l=1}^{\infty} \|\phi_l(., x, y)\|_{L_Q^0}^2 < \infty\}.$$

It is obvious that system (5.0.1) can be seen as a fixed point problem:

$$\begin{cases} dz(t) & = A_* z(t) + f(t, z(t))dt + \sum_{l=1}^{\infty} \sigma_l(t, z))dB_l^H(t), \ t \in J, t \neq t_k, \\ \Delta z(t) & = I_k^*(z(t_k)), \quad t = t_k \quad k = 1, 2, \ldots, m \\ z(0) & = z_0, \end{cases} \tag{0.0.10}$$

where

$$z(t)=\left[\begin{array}{c} x(t)\\ y(t)\end{array}\right], A_*=\left[\begin{array}{cc} A & 0\\ 0 & A\end{array}\right],\ f(t,z)=\left[\begin{array}{c} f^1(t,x(t),y(t))\\ f^2(t,x(t),y(t))\end{array}\right],\ \sigma_l(t,z)=\left[\begin{array}{c} \sigma_l^1(t,x,y)\\ \sigma_l^2(t,x,y)\end{array}\right]$$

and $z_0=\left[\begin{array}{c} x_0\\ y_0\end{array}\right]$.

In Chapter 5 we prove some existence results based on a nonlinear alternative of Leray-Schauder type theorem in generalized Banach spaces in the convex case, we establish a multivalued version type of Perov's fixed point theorem [121] and prove another result on the existence of solution in a non-convex

$$\left\{\begin{array}{rcl} dx(t) & \in & (Ax(t)+F^1(t,x(t),y(t)))dt\\ & + & \displaystyle\sum_{l=1}^{\infty}\sigma_l^1(t,x(t)),y(t))dB_l^H(t),\ t\in[0,b],t\neq t_k,\\ dy(t) & \in & (Ay(t)+F^2(t,x(t),y(t)))dt\\ & + & \displaystyle\sum_{l=1}^{\infty}\sigma_l^2(t,x(t),y(t))dB_l^H(t),\ t\in[0,b],t\neq t_k,\\ \Delta x(t) & = & I_k(x(t_k)),\quad t=t_k\quad k=1,2,\ldots,m\\ \Delta y(t) & = & \overline{I}_k(y(t_k)),\\ x(0) & = & x(b),\\ y(0) & = & y(b),\end{array}\right. \tag{0.0.11}$$

where $J:=[0,b]$, X is a real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ induced by norm $\|\cdot\|$, $A:D(A)\subset X\longrightarrow X$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $(S(t))_{t\geq 0}$ in X and $F^1,F^2:[0,b]\times X\times X\longrightarrow\mathcal{P}(X)$ are given set-valued functions, where $\mathcal{P}(X)$ denotes the family of nonempty subsets of X, $I_k\in C(X,X)$ $(k=1,2,\ldots,m)$, $\sigma_l^1,\sigma_l^2:J\times X\times X\to L_Q^0(Y,X)$. Here, $L_Q^0(Y,X)$ denotes the space of all Q-Hilbert-Schmidt operators from Y into X, which will be also defined in the next section. Moreover, the fixed times t_k satisfies $0<t_1<t_2<\ldots<t_m<T$, $y(t_k^-)$ and $y(t_k^+)$ denotes the left and right limits of $y(t)$ at $t=t_k$.

$$\left\{\begin{array}{l} \sigma(.,x)=(\sigma_1(.,x),\sigma_2(.,x),\ldots),\\ \|\sigma(.,x)\|^2=\displaystyle\sum_{l=1}^{\infty}\|\sigma_l(.,x)\|^2_{L_Q^0}<\infty\end{array}\right. \tag{0.0.12}$$

with $\sigma(.,x) \in \ell^2$ for all $x \in X$, where

$$\ell^2 = \{\phi = (\phi_l)_{l\geq 1} : X \times X \to L^0_Q(Y,X) \quad : \|\phi(x)\|^2 = \sum_{l=1}^{\infty} \|\phi_l(x)\|^2_{L^0_Q} < \infty\}.$$

Finally, in Chapter 6,The aim of this chapter is to study the existence, uniqueness and exponential stability of solutions of stochastic difference equations with delays

$$\begin{cases} x(i+1) = F^1(i, x(i-h), \dots, x(i), y(i-h), \dots, y(i)) \\ \qquad +G^1(i, x(i-h), \dots, x(i-h), y(i-h), \dots, y(i))\xi_i, \ i \in \mathbb{N}(0, b+1), \\ y(i+1) = F^2(i, x(i-h), \dots, x(i), y(i-h), \dots, y(i)) \\ \qquad +G^2(i, x(i-h), \dots, x(i-h), y(i-h), \dots, y(i))\xi_i, \ i \in \mathbb{N}(0, b+1), \\ x(i) = \varphi_1(i), \ i \in Z_0, \\ y(i) = \varphi_2(i), \ i \in Z_0, \end{cases} \tag{0.0.13}$$

where $i \in Z_0 \cup \mathbb{N}(0, b+1)$, $Z_0 = \{-h, \dots, 0\}$, $\mathbb{N}(0, b+1) = \{0, \dots, b+1\}$, $\mathbb{N}(-h, b+1) = Z_0 \cup \mathbb{N}(0, b+1)$, h is a given nonnegative integer number, $F^l, G^l : \mathbb{N}(0, b+1) \times \mathbb{R}^{h+1} \times \mathbb{R}^{h+1} \to \mathbb{R}$ are continuous functions for each $l = 1, 2$ and $\varphi_i : Z_0 \to \mathbb{R}, \ i = 1, 2$. Let $\{\Omega, \mathcal{F}, P\}$ be a basic probability space, $\mathcal{F}_i \subset \mathcal{F}$, be a family of σ-algebras, $\mathbb{E}$ denote the mathematical expectation, $\xi_0, \xi_1, \dots$ be a sequence of real valued and mutually independent random variables, with ξ_i being $\mathcal{F}_{i+1}$-adapted and independent of $\mathcal{F}_i$, $\mathbb{E}(\xi_i) = 0$, $E(\xi_i)^2 = 1$, $i \in \mathbb{N}(0, b+1)$.

Motivation

The following examples give a more concrete notion of processes that can be described by impulsive differential equations stochastic.

Example 0.0.1. ***Black-Scholes model*** *[120]*
Consider a market consisting of one stock (risky asset) and one bond (risk

less asset). The price process of the risky asset is assumed to be of the form $S_t = S_0 e^{H_t}$ *,* $t \in [0,T]$*, with*

$$H_t = \int_0^t (\mu_s - \frac{\sigma_s^2}{2}) ds + \int_0^t \sigma_s dW_s, \qquad (0.0.14)$$

where $W = \{W_t, t \in [0,T]\}$ *is a Brownian motion defined in a complete probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*. We will denote by* $\mathcal{F}_t, t \in [0,T]$ *the filtration generated by the Brownian motion and completed by the* $\mathbb{P}$*-null sets. The mean rate of return* μ_t *and the volatility process* σ_t *are supposed to be measurable and adapted processes satisfying the following integrability conditions.*
By Itô is formula we obtain that S_t *satisfies a stochastic differential equations with impulses equation:*

$$\begin{cases} dS_t = S_t \mu_t dt + S_t \sigma_t dW_t, & t \neq t_k \\ S(t_k^+) - S(t_k^-) = I_k(x(t_k), & t = t_k \\ S(0) = S_0. \end{cases} \qquad (0.0.15)$$

where S_t *the price action at time* t *and* $I_k(x(t_k)$ *change the price action at time* t_k*. The price of the bond* $B_t, t \in [0,T]$*, evolves according to the differential equation*

$$dB_t = r_t B_t dt, \quad B_0 = 1,$$

where the interest rate process is a nonnegative measurable and adapted process satisfying the integrability condition

$$\int_0^T r_t dt < \infty$$

almost surely. That is,

$$B_t = \exp\Big(\int_0^T r_t dt\Big).$$

Imagine an investor who starts with some initial endowment $x \geq 0$ *and invests in the assets described above. Let* α_t *be the number of non-risky assets and* β_t *the number of stocks owned by the investor at time* t*. The couple* $\phi_t = (\alpha_t, \beta_t)$*,* $t \in [0,T]$*, is called a portfolio or trading strategy, and we assume that* α_t *and* β_t *are measurable and adapted processes such that*

$$\int_0^T |\beta_t \mu_t| dt < \infty, \quad \int_0^T \beta_t^2 \sigma_t dt, \quad \int_0^T |\alpha_t| r_t < \infty, \qquad (0.0.16)$$

almost surely. Then $x = \alpha_0 + \beta_0 S_0$*, and the investors wealth at time* t *(also called the value of the portfolio) is*

$$V_t(\phi) = \alpha_t B_t + \beta_t S_t.$$

The gain $G_t(\phi)$ *made by the investor via the portfolio* ϕ *up to time* t *is given by*

$$G_t(\phi) = \int_0^T \alpha_s dB_s + \int_0^T \beta_s dS_s.$$

Notice that both integrals are well defined thanks to condition (0.0.17)*.*

We say that the portfolio ϕ *is self-financing if there is no fresh investment and there is no consumption. This means that the value equals to the intial investment plus the gain:*

$$V_t(\phi) = x + \int_0^t \alpha_s dB_s + \int_0^t \beta_s dS_s. \qquad (0.0.17)$$

From now on we will consider only self-financing portfolios.

Completeness and hedging.

A derivative is a contract on the risky asset that produces a payoff H *at maturity time* T*. The payoff is, in general, an* FT *-measurable nonnegative random variable* H*.*

European Call-Option with maturity T *and exercise price* $K > 0$*: The buyer of this contract has the option to buy, at time T, one share of the stock at the specified price* K*. If* $S_T \leq K$ *the contract is worthless to him and he does not exercise his option. If* $S_T > K$*, the seller is forced to sell one share of the stock at the price* K*, and thus the buyer can make a profit* $S_T - K$ *by selling then the share at its market price. As a consequence, this contract effectively obligates the seller to a payment of* $H = (S_T - K)^+$ *at time* T*.*

Black-Scholes formula. *Suppose that the parameters* $\sigma_t = \sigma$*,* $\mu_t = \mu$ *and* $r_t = r$ *are constant. In that case we obtain that the dynamics of the stock price is described by a geometric Brownian motion:*

$$S_t = S_0 \exp\Big(\Big(\mu - \frac{\sigma^2}{2}\Big)t + \sigma W_t\Big).$$

Moreover,$\theta_t = \theta = \frac{\mu - r}{\sigma}$*, and*

$$Z_t = \exp\Big(-\theta W_t - \frac{\theta^2}{2} t\Big).$$

So, $\mathbb{E}(Z_T) = 1$, *and* $\widetilde{W}_t = W_t + \theta$ *is a Brownian motion under* Q, *with* $\frac{dQ}{dP} = Z_t$, *on the time interval* $[0,T]$.
This model is complete in the sense that any payoff $H \geq 0$ *satisfying* $\mathbb{E}_Q(H^2) < \infty$ *is replicable. In this case, we simply apply the integral representation theorem to the random variable* $e^{\cdot rT}H$, $L^2(\Omega, \mathcal{F}_T, Q)$ *with respect to the Wiener process* $\widetilde{W}$. *In this way we obtain*

$$e^{-rT}H = \mathbb{E}_Q(e^{-rT}H) + \int_0^T u_s d\widetilde{W}_s,$$

and the self-financing replicating portfolio is given by

$$\beta_t = \frac{\mu_t}{\sigma \widetilde{S}_t},$$

and

$$\alpha_t = M_t - \beta_t \widetilde{S}_t,$$

where

$$M_t = \mathbb{E}_Q(e^{-rT}H|\mathcal{F}_t) = \mathbb{E}_Q(e^{-rT}H) + \int_0^T u_s d\widetilde{W}_s.$$

Consider the particular case of an European option, that is, $H = \Phi(S_T)$, *where* Φ *is a measurable function with linear growth. The value of this derivative at time* t *will be*

$$\begin{aligned} V_t(\phi) &= \mathbb{E}_Q(e^{-r(T-t)}\Phi(S_T)|\mathcal{F}_t) \\ &= e^{-r(T-t)}\mathbb{E}_Q(\Phi(S_t e^{-r(T-t)} e^{\sigma(\widetilde{W}_T - \widetilde{W}_t) - \frac{\sigma^2}{2(T-t)}})|\mathcal{F}_t). \end{aligned}$$

Hence,

$$V_t = F(t, S_t), \tag{0.0.18}$$

where

$$F(t,x) = e^{-r(T-t)}\mathbb{E}_Q(\Phi(x_t e^{-r(T-t)} e^{\sigma(\widetilde{W}_T - \widetilde{W}_t) - \frac{\sigma^2}{2(T-t)}})). \tag{0.0.19}$$

Under general hypotheses on Φ *(for instance, if* Φ *has linear growth, is continuous and piece-wise differentiable) which include the cases*

$$\Phi(x) = (x - K)^+,$$

and

$$\Phi(x) = (K - x)^+.$$

Formula (0.0.19) can be written as

$$\begin{aligned} F(t,x) &= e^{-r(T-t)}\mathbb{E}_Q(\Phi(x_t e^{-r(T-t)} e^{\sigma(\widetilde{W}_T-\widetilde{W}_t)-\frac{\sigma^2}{2(T-t)}})) \\ &= e^{-r\tau}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-y^2/2}\Phi(xe^{r\tau-\frac{\sigma^2}{2}\tau+\sigma\sqrt{\tau}y})dy, \end{aligned}$$

where $\tau = T-t$ is the time to maturity.
In the particular case of an European call-option with exercise price K and maturity T, $\Phi(x) = (x-K)^+$, and we get

$$\begin{aligned} F(t,x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-y^2/2}\Phi(xe^{-\frac{\sigma^2}{2}\tau+\sigma\sqrt{\tau}y} - Ke^{-r\theta})^+ dy \\ &= x\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-), \end{aligned}$$

where

$$d_+ = \frac{\log\frac{x}{k} + \Big(r+\sigma^2/2\Big)\tau}{\sigma\sqrt{\tau}},$$

and

$$d_- = \frac{\log\frac{x}{k} + \Big(r-\sigma^2/2\Big)\tau}{\sigma\sqrt{\tau}}.$$

Example 0.0.2. ***Stochastic Navier-Stokes equations*** *[147]*
Fluids obey the general laws of continuum mechanics: conservation of mass, energy, and linear momentum. They can be written as mathematical equations once a representation for the state of a fluid is chosen. In the context of mathematics, there are two classical representations. One is the so-called Lagrangian representation, where the state of a fluid particle at a given time is described with reference to its initial position. The other representation is the so-called Eulerian representation, where at each time t and position x in space the state - in particular, the velocity $u(x,t)-$ of the fluid particle at that position and time is given. In the Eulerian representation of the flow, we also represent the density $\rho(x,t)$ as a function of the position x and time t. The conservation of mass is expressed by the continuity equation

$$\frac{\partial\rho}{\partial t} + div(\rho u) = 0. \tag{0.0.20}$$

The conservation of momentum is expressed in terms of the acceleration γ *and the Cauchy stress tensor* σ*:*

$$\rho\gamma_i = \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} + f_i, \qquad i = 1, 2, 3. \tag{0.0.21}$$

Here $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ *and* $\sigma = \sigma_{i,j=1,2,3}$ *, componentwise in the 3-dimensional case. Moreover,* $f = (f_i, f_2, f_3)$ *represents volume forces applied to the fluid. The acceleration vector* $\gamma = \gamma(x, t)$ *of the fluid at position* x *and time* t *can be expressed, using purely kinematic arguments, by the so-called material derivative*

$$\gamma = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + (u, \nabla)u, \tag{0.0.22}$$

or, componentwise,

$$\gamma_i = \frac{\partial u}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}, \qquad i = 1, 2, 3.$$

Inserting this expression into the left-hand side (LHS) of equation (0.0.21) yields the term $\rho(u.\nabla)u$*, which is the only nonlinear term in the Navier.Stokes equations; this term is also called the inertial term. The Navier.Stokes equations are among the very few equations of mathematical physics for which the nonlinearity arises not from the physical attributes of the system but rather from the mathematical (kinematical) aspects of the problem.*
Further transformations of the conservation of momentum equation necessitate additional physical arguments and assumptions. Rheology theory relates the stress tensor to the velocity field for different materials through the so-called stress.strain law and other constitutive equations. Assuming the fluid is Newtonian, which is the case of interest to us, amounts to assuming that the stress.strain law is linear. More precisely, for Newtonian fluids the stress tensor is expressed in terms of the velocity field by the formula

$$\sigma_{ij} = \mu\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) + \Big(\lambda\, divu - p\Big)\delta_{ij}, \tag{0.0.23}$$

where $p = p(x, t)$ *is the pressure. Here,* δ_{ij} *is the Kronecker symbol and* μ*,* λ *are constants. The constant* μ *is called the shear viscosity coefficient, and* $3\lambda + 2\mu$ *is the dilation viscosity coefficient. For thermodynamical reasons,*

$\mu > 0$ and $3\lambda + 2\mu \geq 0$. Inserting the stress.strain law (0.0.23) into the momentum equation (0.0.21), we obtain

$$\rho\Big(\frac{\partial u}{\partial t} + (u.\nabla)u\Big) = u\triangle u + (\lambda+\mu)\nabla divu - \nabla p + f. \qquad (0.0.24)$$

If we also assume that the fluid is incompressible and homogeneous, then the density is constant in space and time: $\rho(x,t) \equiv \rho_0$. In this case, the continuity equation is reduced to the divergence free condition:

$$divu = 0. \qquad (0.0.25)$$

Because the density is constant, we may divide the momentum equation (0.0.24) by ρ and consider the so-called kinematic viscosity $\nu = \mu/\rho_0$, we may then replace the pressure p and the volume force f by the kinetic pressure p/ρ_0 and the mass density of body forces f/ρ_0, respectively. In doing so, and taking into consideration the divergence-free condition (0.0.25), we obtain the Navier-Stokes equations for a viscous, incompressible, homogeneous flow:

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u.\nabla)u + \nabla p = f, \qquad (0.0.26)$$

$$\nabla.u = 0, \qquad (0.0.27)$$

where, for notational simplicity, we represent the divergence of u by $\nabla.u$. For all pratical purposes, the density has actually been normalized to unity; even so, we may sometimes replace (0.0.26) *by* (0.0.28), *remembering then that $\nabla.u = 0$ and ρ is constant.*
For a random variable ξ, we therefore consider the stochastic Navier-Stokes equations with stochasticity introduced in the body force by a Brownian motion $f = \sigma\frac{dW(t)}{dt}$ with $\sigma = \sigma(x)$, or

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u.\nabla)u + \nabla p = \xi. \qquad (0.0.28)$$

The noise $\xi = \xi(x,t)$ will be white (delta-correlated) in time, with some degree of correlation in space. In the usual language of applied sciences,

$$\langle \xi_\alpha(x,t), \xi_\alpha(y,t)\rangle = (t\wedge s)Q_{\alpha\beta}(x,y) \qquad \alpha,\beta = 1,\dots d,$$

where $Q(x,y)$ is the matrix-valued space-covariance of the noise. Assume $Q(x,y) = Q(x-y)$.

A rigorous mathematical model of ξ *is given by a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, *a sequence of independent Brownian motions* $W_k(t)$ *on* $(\Omega, \mathcal{F}, \mathbb{P})$, $k = 1, \ldots,$ *a sequence of divergence free vector fields such that*

$$Q_{\alpha\beta}(x, y) = \sum_{k=0}^{\infty} \sigma_\alpha^k \sigma_\beta^k.$$

Assumptions of summability and regularity of σ^k *are required, depending on the result. With these data, the time-distribution*

$$\xi(x, t) = \sum_{k=0}^{\infty} \sigma^k(x) \frac{dW(t)}{dt},$$

is a white noise in time, with covariance $Q(x, y)$ *in space.Transport type noise*

$$du + (u.\nabla u + \nabla p - \nu \Delta u)dt = \sum_{k=0}^{\infty} \sigma^k(x) dW(t),$$

(interpreted weakly: integrated in time and against smooth test functions).

Example 0.0.3. ***Analysis of a predatorprey model with stochastic perturbation*** *[72]. One of the first mathematical models which incorporate interaction between two species (Holling-Tanner predator-prey).The relationship between predator and prey has been and will long be the one of the most important hot topics in mathematical biology.First introduced a famous predatorprey system*

$$\begin{cases} \frac{dx}{dt} = x(t)(a_1 - b_1 x(t)) - p(x)y(t), \\ \\ \frac{dy}{dt} = y(t)\left(a_2 - b_2 \frac{y(t)}{x(t)}\right), \end{cases} \quad . \tag{0.0.29}$$

where $x(t), y(t)$ *are the density of prey and predator at time* t*, respectively.*$p(x)$ *is the functional response of predator to prey.Here the predator still grows logistically with intrinsic growth rate* a_2 *and carrying capacity* $x(t)/b_2$*. And the prey variable appears in the denominator,when* $p(x) = ax/(b + x)$ *is Holling-II functional response, the population dynamics of the type* (0.0.29) *is HollingTanner predator-prey system:*

$$\begin{cases} \frac{dx}{dt} = x(t)(a_1 - b_1 x(t)) - \frac{ax(t)y(t)}{b+x(t)}, \\ \\ \frac{dy}{dt} = y(t)\left(a_2 - b_2 \frac{y(t)}{x(t)}\right), \end{cases} \quad . \tag{0.0.30}$$

where $a, b > 0$ denote capturing rate and half capturing saturation constant, respectively.In addition, in the natural world, the birth rate, the death rate and carrying capacity of the species and other parameters will not remain constant, but exhibit more-or-less periodicity, as a result of seasonal change, life cycle and man-made factors.There have been many studies in literatures that investigate the population dynamics of the type (0.0.31) models. However, in the study of the dynamic relationship between spices, the effect of some impulsive factors, which exists widely in the real world, has been ignored. For example, the birth of many spices is an annual birth pulse or harvesting. Moreover, the human beings have been harvesting or stocking spices at some time, then the spices is affected by another impulsive type. Also impulsive reduction of the population density of a given spices is possible after its partial destruction by catching or poisoning with chemicals used at some transitory slots in fishing or agriculture. Such factors have a great impact on the population growth. If we incorporate this impulsive factors into the model of population interaction, the model must be governed by an impulsive differential system.

For example, if at the moment $t = t_k$ the population density of the predator is changed, then we can assume that

$$x(t_k^+) - x(t_k^-) = \alpha_k x(t_k), \qquad y(t_k^+) - y(t_k^-) = \beta_k y(t_k), \qquad k = 1, 2, \ldots$$

where $(x(t_k^-), y(t_k^-)) = (x(t_k), x(t_k))$ and $(x(t_k^+), y(t_k^+))$ are the population densities of prey and predator before and after perturbation. The corresponding periodic system with impulses is as follows:

$$\begin{cases} \frac{dx}{dt} = x(t)(a_1(t) - b_1(t)x(t)) - \frac{a(t)y(t)}{b(t)+x(t)}, \\ \frac{dy}{dt} = y(t)\Big(a_2(t) - b_2(t)\frac{y(t)}{x(t)}\Big), \\ x(t_k^+) - x(t_k^-) = \alpha_k x(t_k), \quad k = 1, 2, \ldots \\ y(t_k^+) - y(t_k^-) = \beta_k y(t_k), \quad k = 1, 2, \ldots \end{cases} \tag{0.0.31}$$

Simulated by the work above, we assume the intrinsic growth rates $a_1(t)$ and $a_2(t)$ of the prey and the predator are disturbed with

$$a_1(t) \to a_1(t) + \sigma_1(t)dB_1(t), \qquad a_2(t) \to a_2(t) + \sigma_2(t)dB_2(t)$$

where $B_i(t), i = 1, 2$ are independent Brownian motions. σ_1^2, σ_2^2 denote the intensity of the white noises. That is, we consider the following stochastic non-autonomous predator-prey system with impulsive effects:

$$\begin{cases} \frac{dx}{dt} = x(t)(a_1(t) - b_1(t)x(t)) - \frac{a(t)y(t)}{b(t)+x(t)} + x(t)\sigma_1(t)dB_1(t), \\ \frac{dy}{dt} = y(t)\Big(a_2(t) - b_2(t)\frac{y(t)}{x(t)}\Big) + y(t)\sigma_2(t)dB_2(t), \\ x(t_k^+) - x(t_k^-) = \alpha_k x(t_k), \quad k = 1, 2, \ldots \\ y(t_k^+) - y(t_k^-) = \beta_k y(t_k), \quad k = 1, 2, \ldots \end{cases} \tag{0.0.32}$$

where $a_i(t), b_i(t)$, $\sigma_1(t)(i = 1, 2)$, $a(t), b(t)$ are positive and continuous $T-$periodic functions. And $0 < t_1 < t_2 < \ \ldots < t_k < \ \ldots$ and $\lim_{k\to+\infty} t_k = +\infty$.

For biology purposes, we are only concerned with the positive solution of the equation. Therefore, it is a natural constraint that

$$1 + \alpha_k > 0, \qquad 1 + \beta_k > 0. \qquad k = 1, 2 \ldots$$

For example, if $\alpha_k, \beta_k > 0$, the impulsive effects denote the planting of the species, while $\alpha_k, \beta_k < 0$ denote harvesting.
In mathematical ecology the system (0.0.32) denotes a model of the dynamics of a stochastic predator-prey system, which is subject to impulsive effects at certain moments of time. By means of such models, it is possible to take into account the possible environmental changes or other exterior effects due to which population density of the predator is changed momentary.

Example 0.0.4. ***On pulse vaccine strategy in a stochastic SIR epidemic model:*** *([156])*
Vaccination strategies are designed and applied to control or eradicate infectious diseases, among which there are constant vaccination and pulse vaccination. Pulse vaccination strategy (PVS) consists of periodic repetitions of impulsive vaccinations in a population
Suppose an epidemic model with constant vaccination is given. We study a population that is composed of three classes of individuals: susceptibles (S), infectives (I), and recovereds (R), the SIRS model is described by the follow-

ing ordinary differential equation:

$$\begin{cases} \frac{dS(t)}{dt} = b - dS(t) - g(S(t), I(t)) + \gamma R(t), \\ \frac{dI(t)}{dt} = g(S(t), I(t)) - (d + \mu + \delta)I(t), \\ \frac{dR(t)}{dt} = \mu I(t) - (d + \gamma)R(t), \end{cases} \tag{0.0.33}$$

where b *is the recruitment rate of the population,* d *the natural death rate of the population,* μ *the natural recovery rate of the infective individuals,*γ *the rate at which recovered individuals lose immunity and return to the susceptible class,* δ *the disease inducing death rate, and* $g(S(t), I(t))$ *the transmission of the infection or called as incidence rate.The transmission function* $g(S(t), I(t))$ *plays a key role in determining disease dynamics.There are many different approaches to deriving a stochastic version from the deterministic SIRS model with the ratio-dependent transmission function* $g(S(t), I(t))$*.Thus, stochastic perturbation in our model is a white noise type that is directly proportional to* $S(t)$, $I(t)$,$R(t)$ *and is influenced on the* $\frac{dS(t)}{dt}$,$\frac{dI(t)}{dt}$,$\frac{dR(t)}{dt}$ *, respectively. Following this approach, we obtain the following SDE epidemic model* (0.0.35) *that is analog to its deterministic version* (0.0.33) *by introducing stochastic perturbation terms into the growth equations of susceptible, infective, recovered individuals to incorporate the effect of randomly fluctuating environments:*

$$\begin{cases} \frac{dS(t)}{dt} = b - dS(t) - g(S(t), I(t)) + \gamma R(t) + \sigma_1 S(t) dB_1(t), \\ \frac{dI(t)}{dt} = g(S(t), I(t)) - (d + \mu + \delta)I(t) + \sigma_2 I(t) dB_2(t), \\ \frac{dR(t)}{dt} = \mu I(t) - (d + \gamma)R(t) + \sigma_3 R(t) dB_3(t), \end{cases} \tag{0.0.34}$$

where $\sigma_1(i = 1, 2, 3)$ *are real constants and known as the intensity of environmental fluctuations,* $B_i(t)(i = 1, 2, 3)$ *independent standard Brownian motions,and the total population* $N = S + I + R$.
Assume the pulse scheme proposes to vaccinate a fraction ρ, $(0 < \rho < 1)$, *of the entire susceptible population in a single pulse, applied every year. When pulse vaccination is incorporated into the SIR model (0.0.35), the system can*

be rewritten as

$$\begin{cases} \frac{dS(t)}{dt} = b - dS(t) - g(S(t), I(t)) + \gamma R(t) + \sigma_1 S(t) dB_1(t), \\ \frac{dI(t)}{dt} = g(S(t), I(t)) - (d + \mu + \delta) I(t) + \sigma_2 I(t) dB_2(t), \\ \frac{dR(t)}{dt} = \mu I(t) - (d + \gamma) R(t) + \sigma_3 R(t) dB_3(t), \\ S(k^+) = (1 - \rho) S(k), \\ I(k^+) = I(k), \quad t = k \\ R(k^+) = R(k) + \rho S(k), \quad k = 0, 1, 2, \ldots \end{cases} \tag{0.0.35}$$

Here $f(k^+) = \lim_{t \to k^+} f(t)$, $f(k) = \lim_{t \to k^-} f(t)$.

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Chapter 1

Some Elements of Stochastic Analysis

1.1 Some selected topics from probability theory

The purpose of this section is to remember some familiar tunes and get warmed up. We just want to refresh our memory, recall some standard notions and facts, and introduce the notation to be used in the future. In Section 2 and 3, introduce all the background material used in herein such as stochastic calculus and some properties of generalized the Brownain motion and an application we are interested by definition of the stochastic integral corresponding to fBm with values in a Hilbert space.Finally , we show some recent applications of the Malliavin Calculus to develop a stochastic calculus with respect to the fractional Brownian motion.

Definition 1.1.1. Let Ω be a set and $\mathcal{F}$ a collection of its subsets. We say that $\mathcal{F}$ is a σ- field(orσ- algebra) if

(i) $\Omega \in \mathcal{F}$

(ii) for every $A_1, \ldots, A_n, \ldots$ such that $A_n \in \mathcal{F}$,we have $\cup_n A_n \in \mathcal{F}$

(iii) if $A \in \mathcal{F}$, then $A^c := \Omega \setminus A \in \mathcal{F}$.

In the case when $\mathcal{F}$ is a $\sigma-$ field the couple $(\Omega, \mathcal{F})$ is called a measurable space ,and elements of $\mathcal{F}$ are called events.

Example 1.1.1. *Let Ω be a set and $\mathcal{F} := \{\Omega, \emptyset\}$ is a $\sigma-$ field which is called the trivial $\sigma-$ field.*

Definition 1.1.2. *$(E, \mathcal{O})$ is a topological space, where $\mathcal{O}$ is the set of open sets in E. Then $\sigma(\mathcal{O})$ is called the Borel σ-algebra of the topological space. If $\mathcal{A} \subset \mathcal{B}$, then A is called a subalgebra of $\mathcal{B}$. A set B in $\mathcal{B}$ is also called a Borel set.*

The last property is called σ-additivity.

Definition 1.1.3. *A map X from a measure space $(\Omega, \mathcal{F})$ to an other measure space $(\Delta, \mathcal{B})$ is called measurable, if $X^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$. The set $X^{-1}(B)$ consists of all points $x \in \Omega$ for which $X(x) \in B$. This pull back set $X^{-1}(B)$ is defined even if X is non-invertible.*

Definition 1.1.4. We say that $\mathbb{P}(.)$ is probability measure on $(\Omega, \mathcal{F})$ or on $\mathcal{F}$ if

(i) $\mathbb{P}(A) \geq 0$ and $\mathbb{P}(\Omega) = 1$

(ii) for every sequence of pairwise disjoint $A_1, \ldots, A_n, \ldots \in \mathcal{F}$,we have

$$\mathbb{P}(\bigcup_n A_n) = \sum_n \mathbb{P}(A_n)$$

If on a measurable space $(\Omega, \mathcal{F})$ there is defined a probability measure $\mathbb{P}$,the triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space.

Example 1.1.2. *The triple,consisting of $[0, 1](= \Omega)$,the $\sigma-$ field $\mathcal{B}([0, 1])$ of Borel subsets of $[0, 1]$ (taken as $\mathcal{F}$) and Lebesgue measure ℓ as ($\mathbb{P}$) is a probability space.*

Definition 1.1.5. *The algebra of all random variables is denoted by $\mathcal{L}$. It is a vector space over the field $\mathbb{R}$ of the real numbers in which one can multiply. A elementary function or step function is an element of $\mathcal{L}$ which is of the form*

$$X = \sum_{i=1}^{n} \alpha_i \cdot 1_{A_i}$$

with $\alpha_i \in \mathbb{R}$ and where $A_i \in \mathcal{F}$ are disjoint sets. Denote by $\mathcal{S}$ the algebra of step functions. For $X \in \mathcal{S}$ we can define the integral

$$\mathbb{E}(X) = \int_{\Omega} X d\mathbb{P} = \sum_{i=1}^{n} \alpha_i \mathbb{P}(A_i).$$

Definition 1.1.6. *Define $\mathcal{L}^1 \subset \mathcal{L}$ as the set of random variables X, for which*

$$\sup_{Y \in \mathcal{S}, Y \leq X} \int Y d\mathbb{P}$$

is finite. For $X \in \mathcal{L}^1$, we can define the integral or expectation

$$\mathbb{E}(X) = \int X d\mathbb{P} = \sup_{Y \in \mathcal{S}, Y \leq X^+} \int Y d\mathbb{P} - \sup_{Y \in \mathcal{S}, Y \leq X^-} \int Y d\mathbb{P},$$

where $X^+ = X \vee 0 = \max(X, 0)$ and $X^- = -X \vee 0 = \max(-X, 0)$. The vector space $\mathcal{L}^1$ is called the space of integrable random variables. Similarly, for p ≥ 1 write $\mathcal{L}^p$ for the set of random variables X for which $\mathbb{E}(|X|^P) < \infty$.

Definition 1.1.7. *For $X, Y \in \mathcal{L}^2$ define the covariance*

$$Cov(X, Y) := \mathbb{E}\big[\big(X - \mathbb{E}(X)\big)\big(Y - \mathbb{E}(Y)\big)\big] = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y).$$

Two random variables in $\mathcal{L}^2$ are called uncorrected if $Cov(X, Y) = 0$.

Definition 1.1.8. *For $X \in \mathcal{L}^2$,we can define the variance*

$$Var(X) = Cov(x, x) = \mathbb{E}((X - E(X))^2).$$

Definition 1.1.9. *Write $J \subset I$ if J is a finite subset of I. A family $\{\mathcal{F}_i\}_{i \in I}$ of σ-sub-algebras of $\mathcal{F}$ is called independent, if for every $J \subset I$ and every choice $A_j \in \mathcal{F}_j$ $P[\bigcap_{j \in J} A_j] = \prod_{j \in J} \mathbb{P}(A_j)$. A family $\{X_j\}_{j \in J}$ of random variables is called independent, if $\{\sigma(X_j)\}_{j \in J}$ are independent σ-algebras. A family of sets $\{A_j\}_{j \in I}$ is called independent, if the σ-algebras $\mathcal{F}_j = \{\emptyset, A_j, A_j^c, \Omega\}$ are independent.*

Probability distribution function

The probability distribution function $F_x : \mathbb{R} \longrightarrow [0, 1]$ of a random variable X was defined as

$$F_X(x) = \mathbb{P}(X \leq x),$$

where $\mathbb{P}(X \leq x)$ is a short hand notation for $\mathbb{P}(\{\omega \in \Omega \mid X(\omega) < x\})$.

1.1.1 Some inequalities

[91] Let X and Y are random variables on $\mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})$

- `Chebychev inequality`: for all $\lambda > 0$

$$\mathbb{P}(|X| > \lambda) \leq \frac{1}{\lambda^p}\mathbb{E}(|X|^p).$$

- `Shwartz inequality`:

$$\mathbb{E}(XY) \leq \sqrt{\mathbb{E}(X^2)\mathbb{E}(Y^2)}.$$

- `Hölder inequality`:

$$\mathbb{E}(XY) \leq [\mathbb{E}|X|^p]^{\frac{1}{p}}[\mathbb{E}|Y|^q]^{\frac{1}{q}}$$

  where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

- `Jensen inequality`: For any convex function $h : \mathbb{R} \longrightarrow \mathbb{R}$, we have

$$h(\mathbb{E}(X)) \leq \mathbb{E}(hE(X)).$$

- `Kolmogorov inequality`:Let $X_1, \cdots, X_n$ are independent random variables with $E(X_i) = 0$ and $var(X_i) < \infty$. If $S_n = X_1 + \cdots + X_n$ then

$$\mathbb{P}(\max_{1\leq k\leq n} |S_k| \geq t) \leq \frac{1}{t^2}VarS_k, \forall t > 0.$$

1.1.2 Processes and filtrations

Definition 1.1.10. . *A collection of sub σ-algebras $\{\mathcal{F}_t; t \geq 0\}$ of the σ-algebra $\mathcal{F}$ is called a filtration if $s \leq t$ implies that $\mathcal{F}_s \subset \mathcal{F}_t$.*

For a given stochastic process X, write $\mathcal{F}_t^X$ for the filtration $\sigma\{X_s; 0 \leq s \leq t\}$. Call $\{\mathcal{F}_t^X; t \geq 0\}$ is called the natural filtration associated to the process X.

Definition 1.1.11. *A stochastic process is a parametrized collection of random variables $\{X_t\}_{t\in[0,T]}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and assuming values in $\mathbb{R}^n$.*

The parameter space T is usually the halfline $[0, \infty)$, but it may also be an interval $[a, b]$, the non-negative integers and even subsets of $\mathbb{R}^n$ for $n \geq 1$. Note that for each $t \in T$ fixed we have a random variable

$$\omega \longrightarrow X_t(\omega); \quad \omega \in \Omega.$$

On the other hand, fixing $\omega \in \Omega$ we can consider the function

$$t \longrightarrow X_t(\omega); \quad t \in T$$

which is called a path of X_t.

Definition 1.1.12. *The stochastic process X is adapted to the filtration $\{\mathcal{F}_t\}$ if, for each $t \geq 0$, X_t is an $\mathcal{F}_t$- measurable random variable.*

Definition 1.1.13. The parameter space T is usually the halfine $[0, \infty)$, but it may also be an interval $[a, b]$, the non-negative integers and even subsets of $\mathbb{R}^n$ for $n \geq 1$. Note that for each $t \in T$ fixed we have a random variable

$$w \rightarrow X_t(w); \quad w \in \Omega.$$

On the other hand, fixing $w \in \Omega$ we can consider the function

$$t \rightarrow X_t(w); \quad t \in T,$$

which is called a path of X_t.

Definition 1.1.14. The stochastic process X is adapted to the filtration $\{\mathcal{F}_t\}$ for each $t \geq 0$, X_t is an $\{\mathcal{F}_t\}$- measurable random variable.

Definition 1.1.15. A filtration $\mathcal{F}_t$ is said to satisfy the usual conditions if it is right-continuous and $\mathcal{F}_0$ contains all the $\mathbb{P}-$ negligible event in $\mathcal{F}$.

1.1.3 Stopping times

Definition 1.1.16. *A random variable $\tau : \Omega \longrightarrow [a, b]$ is called a stopping time with respect to a filtration $\{\mathcal{F}_t; a \leq t \leq b\}$ if $\{\omega; \tau(\omega) \leq t\} \in \mathcal{F}_t$ for all $t \in [a, b]$.*

Properties

- If the filtration is right continuous then τ is a stopping time if and only if, for all t, $\{\tau < t\} \in \mathcal{F}_t$.

- Let τ_1 and τ_1 be two stopping times. Then $\tau_1 + \tau_2$, $\tau_1 \wedge \tau_2$, $\tau_1 \vee \tau_2$ are stopping times.

Definition 1.1.17. *The σ-algebra $\mathcal{F}_\tau$ for a stopping time τ is given by*

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t \textit{ for all } \quad t\}.$$

The following holds:

1. *τ is measurable with respect to $\mathcal{F}_\tau$.*
2. *If $\tau(\omega) = t$ for almost all ω then $\mathcal{F}_\tau = \mathcal{F}_t$.*

1.1.4 Continuous time martingales

In this section we shall consider exclusively real -valued processes $X = \{X_t; 0 \leq t < \infty\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, adapted to a given filtration $\{\mathcal{F}_t\}$ and such that $\mathbb{E}|X_t| < \infty$ holds for every $t \geq 0$

Definition 1.1.18. (i) A stochastic process $X = (X_t)_{t\geq 0}$ is called a martingale (with respect to $\mathbb{P}$ and $\mathcal{F}$) if

1. X is adapted;
2. $\mathbb{E}(X_t) < \infty$ for all $t \geq 0$
3. For $0 \leq s \leq t < \infty$,

$$\mathbb{E}(X_t \mid \mathcal{F}_s) = X_s \qquad p.s$$

(ii) X is called a submartingale

$$\mathbb{E}(X_t \mid \mathcal{F}_s) \geq X_s \qquad p.s$$

(iii) X is called a supermartingale

$$\mathbb{E}(X_t \mid \mathcal{F}_s) \leq X_s \qquad p.s$$

Definition 1.1.19. The stochastic process X is adapted to $\mathcal{F}$ if, for each $t \geq 0$, X_t is $\mathcal{F}_t$-measurable.

Definition 1.1.20. A random variable $T : \Omega \longrightarrow \mathbb{R}^+ \cup \{\infty\}$ is called a stopping time with respect to $\mathcal{F}$ if

$$\{T \leq t\} \in \mathcal{F}_t \qquad \text{for all } t \in \mathbb{R}^+.$$

Definition 1.1.21. The process X is continuous if all its paths $t \rightsquigarrow X_t(w)$ are continuous. The process X is cadlag if all its paths are right-continuous with left-hand limits. If X is cadlag, we define the left-continuous process X_{t^-} as

$$X_{t^-} := \begin{cases} X_0, & t = 0 \\ \lim_{s\uparrow t} X_s, & t > 0 \end{cases}$$

and the jumps as

$$\Delta X_t := X_t - X_{t^-}, \quad t \geq 0.$$

For $T > 0$ we define the system of sets

$$\mathcal{G}_T = \{A \times \{0\} : \quad A \in \mathcal{F}\} \cup \{A \times (s,t] : \quad 0 \leq s \leq t \leq T \quad A \in \mathcal{F}_s\}$$

and the predictable σ-algebra $\mathcal{P}_T = \sigma(\mathcal{G}_T)$. An $\mathcal{F}$-valued process $X = (X_t)_{t\in[0,T]}$ is called predictable if it is $\mathcal{P}_T$ -measurable.

Definition 1.1.22. A modification Y of the stochastic process X is a stochastic process on the same probability space, with the same parameter set $\mathbb{R}^+$ such that

$$\mathbb{P}(X_t = Y_t) = 1 \qquad \text{for all} \qquad t \in \mathbb{R}_+.$$

Definition 1.1.23. For a stochastic process $X = (X_t)_{t\geq 0}$ and a stopping time T, the stopped process $X^T := (X_t^T)_{t\geq 0}$ is defined by

$$X_t^T := X_{T\wedge t} = \begin{cases} X_t, & \text{if} \quad t \leq T, \\ X_T, & \text{if} \quad t > T. \end{cases}$$

Definition 1.1.24. An adapted, right-continuous stochastic process $X = (X_t)_{t\geq 0}$ is called a local martingale, if there exists a sequence of stopping times (T_n) with $T_n \to \infty$ P.a.s, such that the stopped process

$$X^{T_n} I_{T_n > 0} = (X_{T_n\wedge t} I_{T_n>0})_{t\geq 0}$$

is a (uniformly integrable) martingale with respect to $(\mathcal{F}_t)$.

The following two estimates are known as Doobs L^p-inequalities.

Theorem 1.1.1. *Let $M = (M_t)_{t\geq 0}$ be an $\mathcal{F}$-valued martingale. Then, the following statements are valid:*

- *For all $p \geq 1$ and $\lambda > 0$ we have*

$$\mathbb{P}(\sup_{t\in[0,T]} \| M_t \|\geq \lambda) \leq \frac{1}{\lambda^p}\mathbb{E} \| M_T \|^p .$$

- *For every $p > 1$ we have*

$$\mathbb{E}(\sup_{t\in[0,T]} \| M_t \|) \leq \Big(\frac{p}{p-1}\Big)^p \mathbb{E} \| M_T \|^p .$$

Let us fix a number $T > 0$ and denote by $\mathcal{M}_T^2(E)$ the space of all E- valued continuous, square integrable martingales M. We will need the following.

Definition 1.1.25. (Square integrable martingales). A martingale (M_t) is called square integrable if it satisfies $\mathbb{E}(M_t^2) < \infty$, that is, $M_t \in L^2$, for all $t \geq 0$.

Notation 1.1.2. $\mathcal{M}^2$ = the collection of all cadlàg martingales (M_t) with

$$\mathbb{E}(\sup_{t\in[0,T]} \|M(t)\|^2) \leq \infty;$$

$\mathcal{M}^2 = \{M \in \mathcal{M}^2 : M \quad \text{is continuous in}\, t\}$

Properties 1.1.3. *The space $\mathcal{M}_T^2(E)$ equipped with the norm*

$$\|M\|_{\mathcal{M}_T^2} = \mathbb{E}(\sup_{t\in[0,T]} \|M(t)\|^2)^{1/2} \tag{1.1.1}$$

is a Banach space.

Proof. Since $\|M(t)\|$ is a submartingale, by Theorem 1.1.4 defines a norm. To prove completeness assume that M_n is a Cauchy sequence, i.e.

$$\mathbb{E}(\sup_{t\in[0,T]} \|M_n(t) - M_m(t)\|^2) \to 0 \quad \text{as} \quad n, m \to \infty.$$

It follows that one can find a subsequence M_{n_k} such that

$$\mathbb{E}(\sup_{t\in[0,T]} \|M_n(t) - M_m(t)\| \geq 2^{-k}) \leq 2^{-k}.$$

The Borel-Cantelli lemma implies that M_{n_k} converges $\mathbb{P}$.a.s. to a process $M(t), te[0,T]$, uniformly on $[0,T]$. So M is a continuous process. It is clear that, for arbitrary t e $[0,T]$, the sequence M_{n_k} converges to $M(t)$ in the mean square. if $0 \leq s \leq t \leq T$ and $k = 1, 2, \ldots,$ then

$$\mathbb{E}(M_{n_k}(t)/\mathcal{F}_s) = M_{n_k}(s) \qquad \mathbb{P}.a.s \tag{1.1.2}$$

and one can let k tend to infinity in 1.1.2 to get

$$\mathbb{E}(M(t)/\mathcal{F}_s) = M(s) \qquad \mathbb{P}.a.s.$$

So $M \in \mathcal{M}_T^2(E)$ and obviously $M_n \to M \in \mathcal{M}_T^2(E)$. □

Definition 1.1.26. Let $M, N \in \mathcal{M}^2$ be continuous local martingales. Then the product MN is not in general a local martingale. However, using the polarization identity

$$MN =:= \frac{1}{4}((M+N)^2 - (M-N)^2), \tag{1.1.3}$$

we see that the process

$$\langle M, N\rangle := \frac{1}{4}(\langle M+N\rangle - \langle M-N\rangle),$$

is called a cross variation (or quadratic covariation) of M and N.Note that $\langle M, M\rangle = \langle M\rangle$ is the quadratic variation process of M. Set

$$Q_\Delta(M,N) = \sum_{j=1}^{n}(M_{t_j} - M_{t_{j-1}})(N_{t_j} - N_{t_{j-1}}),$$

for each $t > 0$ and each partition $\Delta = \{0 = t_0 < ... < t_n = t\}$ of the interval $[0,t]$. The polarization identity 1.1.3 yields

$$Q_\Delta(M,N) = \frac{1}{4}(Q_\Delta(M+N) - Q_\Delta(M-N)).$$

It now follows that

$$Q_\Delta(M,N) \longrightarrow \langle M,N\rangle_t, \text{in probability}, as \quad \Delta \to 0.$$

Here the limit is taken over all partitions Δ of the interval $[0,t]$.

Lemma 1.1.4. *Let M_t be a square integrable martingale. Then*

$$\mathbb{E}((M_t - M_s)^2/\mathcal{F}_a) = \mathbb{E}((M_t^2 - M_s^2)/\mathcal{F}_a), \qquad 0 \leq a \leq s \leq t. \tag{1.1.4}$$

Proof. Let $0 \leq a \leq s \leq t$.Then

$$\mathbb{E}(M_s(M_t - M_s)/\mathcal{F}_s) = M_s\mathbb{E}((M_t - M_s)/\mathcal{F}_s) = 0,$$

and

$$\mathbb{E}(M_s(M_t - M_s)/\mathcal{F}_a) = 0.$$

Observing that

$$M_t^2 - M_s^2 - (M_t - M_s)^2 = M_s(M_t - M_s).$$

By (1.1.4), follows by taking the conditional expectation. □

1.2 Brownian motion and the Wiener process

In 1828 Robert Brown, observed that pollent grains suspended in water perform an unending chaotic motion .L.Bachelier (1900) derived the law governing the position W_t at time t of single grain performing a one -dimensional Brownain motion starting at $a \in \mathbb{R}^+$ at time $t = 0$,

$$\mathbb{P}_a\{W_t \in dx\} = p(t, a, x)dx, \tag{1.2.1}$$

where

$$p(t, a, x) = \frac{1}{\sqrt{2\pi t}} e^{-(x-a)^2/2t}$$

is the fundamental solution of the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial a^2}.$$

Bachelier (1900) also pointed out the Markovian nature of the Brownain path and used it to establish the of maximum displacement

$$\mathbb{P}_a\{\max_{s \leq t} W_t \leq b\} = \frac{2}{\sqrt{2\pi t}} \int_0^b e^{\frac{-x^2}{2t}} dx, \quad t > 0, b \geq 0.$$

Einstein (1905) also derived (1.2.1) from statistical machanis considerations and applied it to the determination of molecular diameters .Bachelier was unable to obtain a clear picture of the Brownain motion ,and has ideas were unappreciated at the time .This is not surprising ,because the precise mathematical definition of the Brownain motion involves a measure on the path space ,and even after the ideas of Borel, Lebesgue,and Daniell appeared ,N.Winer (1923) only constructed a Daniell integral on the path space which later was revealed to be the Lebesgue integral against a measure the so - called Wiener measure.

The simplest model describing movement of a particle subject to hists by much smaller particles is the following .Let η_k,$k = 1, 2, ...$ be independent identically distributed random variables which $\mathbb{E}\eta_k = 0$ and $E\eta_k^2 = 1$.Fix an integer n, and at times $1/n, 2/n, \ldots$ let our particle experience instant displacements by $\eta_1 n^{\frac{-1}{2}}, \eta_1 n^{\frac{-1}{2}}, \ldots$ at moment zero let our particle be at zero .If

$$S_k = \eta_1 + \eta_2 + \ldots + \eta_k,$$

then at moment k/n our particle will be at the point $S_k/\sqrt{n}$ and will stay there during the time interval $[k/n, (k + 1)/n)$ .Since real Brownain motion has continues paths ,we replace our piecewise constant trajectory by continuous piecewise linear one preserving its positions at times k/n .Thus we come to the process,

$$\xi_t^n = S_{[nt]}/\sqrt{n} + (nt - [nt])\eta_{[nt]+}/\sqrt{n}. \tag{1.2.2}$$

This process gives a very rough caricature of Brownain motion .Clearly ,to get a better model we have to let $n \to \infty$.By the way , precisely this necessity dictates the intervals of time between collisions to be $1/n$ and the displacements due to collisions to be $\eta_k/\sqrt{n}$,since then ξ_t^n is asymptotically normal with parameters $(0, 1)$

Definition 1.2.1. The standard Brownian motion is a stochastic process $(W_t)_{t\in\mathbb{R}^+}$ such that

(i) $W_0 = 0$ almost surely.

(ii) The sample trajectories $t \to W_t$ are continuous, with probability 1.

(iii) For any finite sequence of times $t_0 < t_1 < \ldots < t_n$, the increments

$$W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$$

are independent.

(iv) For any given times $0 \leq s \leq t$, $W_t - W_s$ has the Gaussian distribution $\mathcal{N}(0, t-s)$ with mean zero and variance $t - s$.

Theorem 1.2.1. ***(Bachelier)****. For every* $t \in (0,1]$ *we have* $\max_{s\leq t} W_s \sim |W_t|$*, which is to say that for every* $x \geq 0$

$$\mathbb{P}(\max_{s\leq t} W_s \leq x) = \frac{2}{\sqrt{2\pi t}} \int_0^x e^{-y^2/2t} dy.$$

Proof. Take independent identically distributed random variables η_k so that $\mathbb{P}(\eta_k = 1) = \mathbb{P}(\eta_k = -1) = 1/2$, and define ξ_t^n by 1.2.2. First we want to find the distribution of

$$\zeta^n = \max_{[0,1]} \xi_t^n = n^{\frac{-1}{2}} \max_{k\leq n} S_k.$$

Observe that, for each n, the sequence $(S_1, \ldots, S_n)$ takes its every particular value with the same probability 2^{-n}. In addition, for each integer $i > 0$, the number of sequences favorable for the events

$$\{\max_{k\leq n} S_k \geq i, S_n < i\} \quad \text{and} \quad \{\max_{k\leq n} S_k \geq i, S_n > i\}, \tag{1.2.3}$$

is the same. One proves this by using the reflection principle; that is, one takes each sequence favorable for the first event, keeps it until the moment when it reaches the level i and then reflects its remaining part about this level. This implies equality of the probabilities of the events in 1.2.3. Furthermore, due to the fact that i is an integer, we have

$$\{\zeta^n \geq in^{\frac{-1}{2}}, \xi_1^n < in^{\frac{-1}{2}}\} = \{\max_{k\leq n} S_k \geq i, S_n < i\}$$

and

$$\{\zeta^n \geq in^{\frac{-1}{2}}, \xi_1^n > in^{\frac{-1}{2}}\} = \{\max_{k\leq n} S_k \geq i, S_n > i\}.$$

Hence,

$$\mathbb{P}(\{\zeta^n \geq in^{\frac{-1}{2}}, \xi_1^n < in^{\frac{-1}{2}}\}) = \mathbb{P}(\{\max_{k\leq n} S_k \geq i, S_n < i\}).$$

Moreover, obviously,

$$\mathbb{P}(\{\zeta^n \geq in^{\frac{-1}{2}}, \xi_1^n > in^{\frac{-1}{2}}\}) = \mathbb{P}\{\xi_1^n > in^{-1/2}\},$$

$$\mathbb{P}(\{\zeta^n \geq in^{\frac{-1}{2}}\} = \mathbb{P}(\{\zeta^n \geq in^{\frac{-1}{2}}, \xi_1^n > in^{\frac{-1}{2}}\})$$

$$+\mathbb{P}\{\xi_1^n > in^{-1/2}, \xi_1^n < in^{\frac{-1}{2}}\} + \mathbb{P}\{\xi_1^n = in^{\frac{-1}{2}}\}.$$

It follows that

$$\mathbb{P}(\{\zeta^n \geq in^{\frac{-1}{2}}\} = \mathbb{P}(\{\xi_1^n > in^{\frac{-1}{2}}\}) + \mathbb{P}(\{\xi_1^n = in^{\frac{-1}{2}}\}), \tag{1.2.4}$$

for every integer $i > 0$. The last equality also obviously holds for $i = 0$. We see that for numbers a of type $in^{-1/2}$, where i is a nonnegative integer, we have

$$\mathbb{P}(\{\zeta^n \geq a\} = \mathbb{P}(\{\xi_1^n > a\}) + \mathbb{P}(\{\xi_1^n = a\}). \tag{1.2.5}$$

Certainly, the last probability goes to zero as $n \to \infty$ since ξ_1^n is asymptotically normal with parameters $(0, 1)$. Also, keeping in mind Donskerfor theorem, it is natural to think that

$$\mathbb{P}(\{\max_{s \leq 1} \xi_s^n \geq a\}) \longrightarrow \mathbb{P}(\{\max_{s \leq 1} W_s \geq a\}),$$

$$2\mathbb{P}(\{\xi_1^n \geq a\}) \longrightarrow \mathbb{P}(\{W_1 \geq a\}).$$

Therefore, 1.2.5 naturally leads to the conclusion that

$$\mathbb{P}(\{\max_{s \leq 1} W_s \geq a\}) = 2\mathbb{P}(\{W_1 \geq a\}) = \mathbb{P}(\{|W_1| \geq a\}) \quad \forall a \geq 0,$$

and this is our statement for $t = 1$.

$$\mathbb{P}(\{\zeta^n = in^{\frac{-1}{2}}\} = \mathbb{P}(\{\zeta^n \geq in^{\frac{-1}{2}}\} - \mathbb{P}(\{\zeta^n \geq (i+1)n^{\frac{-1}{2}}\} =$$

$$= \mathbb{P}(\{\xi_1^n = (i+1)n^{\frac{-1}{2}}\} + \mathbb{P}(\{\xi_1^n = (i)n^{\frac{-1}{2}}\} - \mathbb{P}(\{\xi_1^n = (i+1)n^{\frac{-1}{2}}\}$$

$$\mathbb{P}(\{\xi_1^n = (i+1)n^{\frac{-1}{2}}\} + \mathbb{P}(\{\xi_1^n = in^{\frac{-1}{2}}\}, \quad i \geq 0.$$

Now for every bounded continuous function $f(x)$ which vanishes for $x < 0$,we get

$$\mathbb{E}(f(\zeta^n)) = \sum_{i=0}^{\infty} f(in^{\frac{-1}{2}})\mathbb{P}(\{\zeta^n = in^{\frac{-1}{2}}\} = \mathbb{E}f(\xi_1^n - n^{-1/2}) + Ef(\xi_1^n).$$

By Donskerfs theorem and by the continuity of the function $x \longrightarrow \max_{[0,1]} x_t$ we have

$$\mathbb{E}f(\max_{[0,1]} W_t) = 2\mathbb{E}f(W_1) = \mathbb{E}f(|W_1|).$$

We have proved our statement for $t = 1$,saying that cW_{s/c^2} is a Wiener process for $s \in [0, 1]$ if $c \geq 1$. The theorem is proved. □

The continuity question of Brownian motion can be answered by using another famous theorem of Kolmogorov:

Theorem 1.2.2. *Suppose that the process ξ_t^n satisfies the following condition: For all $T > 0$ there exist positive constants α, β, N such that*

$$\mathbb{E}|\xi_t^n - \xi_s^n|^\alpha \leq N|t-s|^{1+\beta} \quad \forall s,t \in [0,1].$$

Then there exists a continuous version of ξ_t^n.

Proof. For simplicity we assume that $m_4 = \mathbb{E}(\eta_k)^4 < \infty$,referring the reader to [Bi] for the proof in the general situation,it suffices to prove that

$$\mathbb{E}|\xi_t^n - \xi_s^n|^4 \leq N|t-s|^2 \quad \forall s,t \in [0,1], \tag{1.2.6}$$

where N is independent of n, t, s. Without loss of generality, assume that $s < t$. Denote $a_n = \mathbb{E}(S_n)^4$. By virtue of the independence of the η_k and the conditions$\mathbb{E}(\eta_k) = 0$ and $\mathbb{E}(\eta_k)^2 = 1$, we have

$$a_{n+1} = \mathbb{E}(\eta_{k+1}+S_n)^4 = a_n+4\mathbb{E}(S_n^3\eta_{n+1})+6\mathbb{E}(S_n^2\eta_{n+1}^2)+\mathbb{E}(S_n\eta_{n+1}^3)+m_4 = a_n+6n+m_4$$

Hence (for instance, by induction),

$$a_n = 3n(n-1) + nm_4 \leq 3n^2 + nm_4$$

Furthermore, if s and t belong to the same interval $[\frac{k}{n}, \frac{k+1}{n}]$,then

$$|\xi_t^n - \xi_s^n| = \sqrt{n}|\eta_{k+1}||t-s|,$$

$$\mathbb{E}|\xi_t^n - \xi_s^n|^4 = n^2 m_4|t-s|^4 \leq m_4|t-s|^2. \tag{1.2.7}$$

Now, consider the following picture, where s and t belong to different intervals of type $[k/n, (k+1)/n)$ and by crosses we denote points of type k/n and $s < s_1 < t_1 < t$.
Clearly

$$s_1 - s \leq t - s, \quad t - t_1 \leq t - s, \quad t_1 - s_1 \leq t - s \quad (t_1 - s_1)/n \leq (t_1 - s_1)^2.$$

Put, $s_1 = (([ns]+1)/n, \quad t_1 = ([nt])/n, \quad [nt] - ([ns]+1) = n(t_1 - s_1)$.
Hence and from (1.2.7) and the inequality $(a+b+c)^4 \leq 81(a^4+b^4+c^4)$ we

conclude that

$$\begin{aligned}
\mathbb{E}|\xi_t^n - \xi_s^n|^4 &\leq 81\Big(\mathbb{E}|\xi_t^n - \xi_{t_1}^n|^4 + \mathbb{E}|\xi_{t_1}^n - \xi_{s_1}^n|^4 + \mathbb{E}|\xi_{s_1}^n - \xi_s^n|^4\Big) \\
&\leq 162(t-s)^2 m_4 + 81\mathbb{E}|S_{[nt]}/\sqrt{n} - S_{[nt]+1/\sqrt{n}}| \\
&= 162(t-s)^2 m_4 + 81 n^{-2} a_{[nt]-([ns]+1)} \\
&\leq 162(t-s)^2 m_4 + 243(t-s)^2 + 81(t_1 - s_1)m_4/n \\
&\leq 243(m_4+1)|t-s|^2.
\end{aligned}$$

Thus for all positions of s and t we have (1.2.6) with $N = 243(m_4+1)$.

The lemma is proved. □

1.2.1 Functions of Bounded Variation

Definition 1.2.2. *The variation of a function* $f : [0,T] \longrightarrow \mathbb{R}$ *is defined to be*

$$\limsup_{\Delta t \to 0} \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|,$$

where $t = (t_0, t_1, \ldots, t_n)$ *is a partition of* $[0,T]$*, i.e.* $0 = t_0 < t_1 < \ldots < t_n = T$*,and where*

$$\Delta t = \max_{i=0,1,\ldots,n-1} |t_{i+1} - t_i|,$$

is bounded variation of f *on* $[0,T]$ *is defined to be* $V(f,[0,T])$*.*

Example 1.2.1. *If* f *is constant on* $[0,T]$ *then* f *is of bounded variation on* $[0,T]$*.*

Consider the constant function $f(x) = c$ *on* $[0,T]$*. Notice that*

$$\limsup_{\Delta t \to 0} \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|,$$

is zero for every partition of $[0,T]$*. Thus* $V(f;[0,T])$ *is zero.*

Another example of a function of bounded variation is a monotone function on $[0,T]$.

Theorem 1.2.3. *If f is increasing on $[0,T]$, then f is of bounded variation on $[0,T]$ and $V(f,[0,T]) = f(T) - f(0)$.*

Proof. Let $\{t_i : 0 \leq i \leq n-1\}$ be a partition of $[0,T]$. Consider

$$\sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)| = \sum_{i=0}^{n-1} (f(t_{i+1}) - f(t_i))$$

Because of the telescoping nature of this sum, it is the same for every partition of $[0,T]$. Thus we see that $V(f,[0,T]) = f(T) - f(0) < \infty$. Thus f is of bounded variation on $[0,T]$. □

Similarly, if f is decreasing on $[0,T]$ then $V(f,[0,T]) = f(0) - f(T)$. For the next example we first recall a theorem involving rational and irrational numbers.

Theorem 1.2.4. *Let $f : [a,b] \to \mathbb{R}$ be a function and let $c \in (a,b)$. If f is of bounded variation on $[a,c]$ and $[c,b]$, then f is of bounded variation on $[a,b]$ and $V(f,[a,b]) = V(f,[a,c]) + V(f,[c,b])$*

A proof of Theorem 1.2.4 is provided in Gordon's text [136].

Theorem 1.2.5. *The variation of the paths of $W(t)$ is infinite a.s.*

Proof. Consider the sequence of partitions $t^n = (t_0^n, t_1^n, ...t_n^n)$ of $[0,T]$ into a n equal parts .Then

$$\sum_{i=0}^{n-1} (W_{t_{i+1}^n} - W_{t_i^n})^2 \leq \max_{i=0,..,n-1} |W_{t_{i+1}^n} - W_{t_i^n}| \sum_{i=0}^{n-1} (W_{t_{i+1}^n} - W_{t_i^n}),$$

where

$$t_i^n = \frac{it}{n},$$

Since the paths of $W(t)$ are a.s. continuous on $[0,T]$,

$$\lim_{n\to\infty} \max_{i=0,..,n-1} |W_{t_{i+1}^n} - W_{t_i^n}| = 0\, a.s,$$

there is a subsequence
$t^{n_k} = (t_0^{n_k}, t_1^{n_k}, \ldots, t_{n_k}^{n_k})$ of partitions such that

$$\lim_{k\to\infty} \sum_{i=0}^{n_k-1} |W_{t_{i+1}^n} - W_{t_i^n}|^2 = T \quad a.s.$$

This is because every sequence of random variables convergent in L^2 has a subsequence convergent .a.s.follows that

$$\lim_{k\to\infty}\sum_{i=0}^{n_k-1}|W_{t^n_{i+1}} - W_{t^n_i}| = \infty \quad a.s,$$

while

$$\lim_{k\to\infty}\Delta t^{n_k} = \lim_{k\to\infty}\frac{T}{n_k} = 0,$$

which proves the theorem. □

Lemma 1.2.6. *Let $s \leq t$.*

$$Cov(W_s, W_t) = t \wedge s.$$

Proof.

$$\begin{aligned} Cov(W_s, W_t) &= \mathbb{E}[W_s W_t] - \mathbb{E}[W_s]E[W_t] \\ &= \mathbb{E}[W_s W_t] - 0 \\ &= \mathbb{E}[W_s(W_t - W_s)] + \mathbb{E}[W_s^2] \quad = \mathbb{E}[W_s]\mathbb{E}[(W_t - W_s)] + \mathbb{E}[W_s^2] \\ &= 0 + s \\ &= s. \end{aligned}$$

□

Lemma 1.2.7. *Let W Winer process is a d-dimensional and two constants $\lambda > 0$ and $r > 0$.*

1. *(self-similar) the process $X_t = \frac{1}{\sqrt{\lambda}}W_{\lambda t}$ is a Winer process.*
2. *the process $Y_t = W_{t+s} - W_s$ is a Winer process.*

Proof. the process X_t and Y_t p.s are continuous, zero at 0, Gaussian, centers. we have

$$\begin{aligned} \mathbb{E}(X_t^j X_s^k) &= \frac{1}{\lambda}\mathbb{E}(W_{\lambda t}^j W_{\lambda s}^k) \\ &= \frac{1}{\lambda}\delta_{jk}\min(\lambda t, \lambda s) \\ &= \delta_{jk}\min(t, s), \end{aligned}$$

$$\begin{aligned}\mathbb{E}(Y_t^j Y_s^k) &= \mathbb{E}(W_{r+t}^j W_{r+s}^k + W_r^j W_r^k - W_r^j W_{r+s}^k + W_{r+t}^j W_r^k)\\ &= \delta_{jk}(r + \min(t,s) + r - r - r)\\ &= \delta_{jk}\min(t,s).\end{aligned}$$

□

Lemma 1.2.8. *For every* $0 \le t_1 < t_2 < t_3 \ldots < t_k \le 1$ *the vectors* $(\xi_{t_1}^n, \xi_{t_2}^n \ldots \xi_{t_k}^n)$ *are asymptotically normal with parameters* $(0, t_i \wedge t_j)$.

Proof. We only consider the case $k = 2$ Other k are treated similarly. We have

$$\lambda_1 \xi_{t_1}^n + \lambda_2 \xi_{t_2}^n = (\lambda_1 + \lambda_2) S_{[nt_1]}/\sqrt{n} + \lambda_2 (S_{[nt_2]} - S_{[nt_1]+1})/\sqrt{n}$$
$$+\eta_{[nt_1]+1}\{(nt_1 - [nt_1])\}\lambda_1/\sqrt{n} + \lambda_2/\sqrt{n}\}$$
$$+\eta_{[nt_2+1]}(nt_2 - [nt_2])\lambda_2/\sqrt{n}.$$

On the right, we have a sum of independent terms. In addition, the coefficients of $\eta_{[nt1]+1}$ and $\eta_{[nt2]+1}$ go to zero and

$$\mathbb{E}\exp(ia_n \eta_{[nt+1]} = \mathbb{E}\exp(ia_n \eta_1) \longrightarrow 1 \quad \texttt{as} a_n \longrightarrow 0.$$

Finally, by the central limit theorem, for $\varphi(\lambda) = \mathbb{E}\exp(i\lambda\eta_1)$

$$\lim_{n\longrightarrow\infty} \varphi^n\Big(\frac{\lambda}{\sqrt{n}}\Big) = e^{-\lambda^2/2}.$$

Hence,

$$\begin{aligned}\lim_{n\longrightarrow\infty} E(e^{i(\lambda_1\xi_{t_1}^n + \lambda_2\xi_{t_2}^n}) &= \lim_{n\longrightarrow\infty} (\varphi(\lambda_1/\sqrt{n} + \lambda_2/\sqrt{n}))^{[nt_1]} (\varphi(\lambda_2/\sqrt{n}))^{[nt_2]-[nt_1]-1}\\ &= \exp\{-((\lambda_1+\lambda_2)^2 t_1 + \lambda_2^2(t_2 - t_1))/2\}\\ &= \exp\{-(\lambda_1^2(t_1\wedge t_2) + 2\lambda_1\lambda_2(t_1\wedge t_2) + \lambda_2^2(t_1\wedge t_2)\}.\end{aligned}$$

The lemma is proved. □

Theorem 1.2.9. *(Quadratic Variation) Let* $0 = t_{0n} \le t_{1n} \le \ldots \le t_{k_n n} = 1$ *be a sequence of partitions of* $[0,1]$ *such that* $\max_i(t_{i+1,n} - t_{i,n}) \to 0$ *a.s* $n \to \infty$ *Also let* $0 \le s \le t \le 1$. *Then, in probability as* $n \to \infty$

$$\sum_{s\le t_{i,n}\le t_{i+1,n}\le t} (W_{t_{i+1,n}} - W_{t_{i,n}})^2 \to t - s. \tag{1.2.8}$$

Proof. Let

$$\xi_n = \sum_{s \le t_{i,n} \le t_{i+1,n} \le t} (W_{t_{i+1,n}} - W_{t_{i,n}})^2,$$

and observe that ξ_n is a sum of independent random variables. Also use that if $\eta \sim \mathcal{N}(0, \sigma^2)$, then $\eta = \sigma\zeta$, where $\zeta \sim \mathcal{N}(0,1)$, and $Var(\eta^2) = \sigma^4 Var(\zeta)$.Then $N := Var(\zeta)$, we obtain

$$Var(\xi_n) = \sum_{s \le t_{i,n} \le t_{i+1,n} \le t} Var(W_{t_{i+1,n}} - W_{t_{i,n}})^2 = N \sum_{s \le t_{i,n} \le t_{i+1,n} \le t} (t_{i+1,n} - t_{i,n})^2$$

$$\le N \max_i (t_{i+1,n} - t_{i,n}) \sum_{s \le t_{i,n} \le t_{i+1,n} \le t} (t_{i+1,n} - t_{i,n}) = N \max_i (t_{i+1,n} - t_{i,n}) \to 0.$$

In particular,$\xi_n - \mathbb{E}\xi_n \to 0$ in probability. In addition,

$$\mathbb{E}\xi_n = \sum_{s \le t_{i,n} \le t_{i+1,n} \le t} (t_{i+1,n} - t_{i,n}) \to t - s.$$

Hence $\xi_n - (t-s) = \xi_n - \mathbb{E}\xi_n + \mathbb{E}\xi_n - (t-s) \to 0$ in probability, and the theorem is proved. □

Definition 1.2.3. *(Riemann-Stieltjes Integrals) Let* $V, f : \mathbb{R}^+ \to \mathbb{R}$ *and* $0 = t_0 < t_1 < t_2 < \ldots < t_n = T$ *is a partition of* $[0, T]$.

1. *A sum of the form*

$$\sum_{i=1}^{n} f(s_i^n)(V(t_i^n) - V(t_{i-1}^n)).$$

   is called a Riemann-Stieltjes sum of f *with respect to* V.

2. *A function* f *is Riemann-Stieltjes Integrable with respect to* V *on* $[a, b]$ *such that*

$$\int_0^t f(s)dV = \lim_{n\to\infty} \sum_{i=1}^{n} f(s_i^n)(V(t_i^n) - V(t_{i-1}^n)).$$

1.2.2 Itô Stochastic Integrals

Assume $W(t)$ is an infinite sequence of independent standard Brownian motions, defined on $(\Omega, \mathcal{F}, \mathbb{P})$ that is, $W(t) = (W^1(t), W^2(t), \ldots)^T$. An $\mathbb{R}$-valued random variable is an $\mathcal{F}$-measurable function $x(t) : \Omega \to \mathbb{R}$ and the collection of random variables

$$S = \{x(t, \omega) : \Omega \to \mathbb{R} |\ t \in J\}$$

is called a stochastic process. Generally, we just write $x(t)$ instead of $x(t, \omega)$.

Definition 1.2.4. *An $\mathcal{F}$-adapted process X on $[0, T] \times \Omega$ is elementary processes if for a partition $\phi = \{t = 0 < t_1 < \ldots < t_n = T\}$ and $(\mathcal{F}_{t_i})$-measurable random variables $(X_{t_i})_{i<n}$, X_t satisfies*

$$X_t(\omega) = \sum_{i=0}^{n-1} X_i(\omega)\chi_{[t_i,t_{i+1})}(t), \quad for \ \ 0 \le t \le T, \quad \omega \in \Omega.$$

The Itô integral of the simple process X is defined as

$$\int_0^T X(s)dW^l(s) = \sum_{i=0}^{n-1} X_l(t_i)(W^l(t_{i+1}) - W^l(t_i)), \tag{1.2.9}$$

whenever $X_{t_i} \in L^2(\mathcal{F}_{t_i})$ for all $i \le n$.

The following result is one of the elementary properties of square-integrable stochastic processes [112, 165].

Lemma 1.2.10. *(Itô Isometry for Elementary Processes) Let $(X_l)_{l\in\mathbb{N}}$ be a sequences of elementary processes. Assume that*

$$\int_0^T \mathbb{E}|X(s)|^2 ds < \infty,$$

where $|X| = (\sum_{l=1}^{\infty} X_l^2)$. Then

$$\mathbb{E}\Big(\sum_{l=1}^{\infty}\int_0^T X_l(s)dW^l(s)\Big)^2 = \mathbb{E}\Big(\sum_{l=1}^{\infty}\int_0^T X_l^2(s)ds\Big). \tag{1.2.10}$$

Proof. Set $\Delta W_i^l = W_{t_{i+1}}^l - W_{t_i}^l$. Let $M^k = \sum_{l=1}^k \int_0^T X_l(s)dW^l(s)$, observe that if $k = 1$ we have

$$\begin{aligned}\mathbb{E}\Big(\int_0^T X_1(s)dW(s)\Big)^2 &= \mathbb{E}\Big(\sum_{i=0}^{n-1} X_1(t_i)\Delta W_i^1\Big)^2 \\ &= \sum_{i=0}^{n-1} E\Big(X_1^2(t_i)(W_{t_{i+1}}^1 - W_{t_i}^1)^2\Big) \\ &\quad +2\sum_{i<j}\mathbb{E}\Big(X_1(t_i)X_1(t_j)(W_{t_{i+1}}^1 - W_{t_i}^1)(W_{t_{j+1}}^1 - W_{t_j}^1)\Big).\end{aligned}$$

For $i < j$, since $t_i < t_j$ and $(W_{t_{j+1}} - W_{t_j})$ is independent from

$$X_i(t_i)X_j(t_j)(W_{t_{j+1}} - W_{t_j}).$$

Hence

$$\begin{aligned}\mathbb{E}\Big(X_1(t_i)X_1(t_j)(W_{t_{i+1}}^1 - W_{t_i}^1)(W_{t_{j+1}}^1 - W_{t_j}^1)\Big) &= \mathbb{E}\Big(X_1(t_i)X_1(t_j)(W_{t_{i+1}}^1 - W_{t_i}^1\Big) \\ &\quad \times\mathbb{E}\Big(W_{t_{j+1}}^1 - W_{t_j}^1)\Big) \\ &= 0.\end{aligned}$$

This implies that

$$\begin{aligned}\mathbb{E}\Big(\int_0^T X_1(s)dW^1(s)\Big)^2 &= \sum_{i=0}^{n-1}\mathbb{E}\Big(X_1^2(t_i)(W_{t_{i+1}}^1 - W_{t_i}^1)^2\Big) \\ &= \sum_{i=0}^{n-1}\mathbb{E}\Big(X_1^2(t_i)\Big)\mathbb{E}\Big(W_{t_{i+1}}^1 - W_{t_i}^1)\Big)^2 \\ &= \sum_{i=0}^{n-1}\mathbb{E}\Big(X_1^2(t_i)\Big)(t_{i+1} - t_i) \\ &= \mathbb{E}\Big(\int_0^T X_1^2(s)ds\Big).\end{aligned}$$

Therefore

$$\mathbb{E}\Big(\int_0^T X_1(s)dW^1(s)\Big)^2 = \Big(\mathbb{E}\int_0^T X_1^2(s)ds\Big). \tag{1.2.11}$$

When $k=1$ the result is true, For $k=2$ we use the estimate (1.2.11), to get

$$\begin{aligned}
M^2 &= \mathbb{E}\Big(\sum_{l=1}^{2}\int_0^T X_l(s)dW^l(s)\Big)^2\\
&= \mathbb{E}\Big(\sum_{l=1}^{2}\sum_{i=0}^{n-1} X_l(t_i)\Delta W_i^l\Big)^2\\
&= \mathbb{E}\Big(\sum_{i=0}^{n-1} X_1(t_i)\Delta W_i^1\Big)^2 + \mathbb{E}\Big(\sum_{i=0}^{n-1} X_2(t_i)\Delta W_i^2\Big)^2\\
&+ 2\mathbb{E}\Big(\sum_{i=0}^{n-1} X_1(t_i)\Delta W_i^1\Big)\mathbb{E}\Big(\sum_{i=0}^{n-1} X_2(t_i)\Delta W_i^2\Big)\\
&= \sum_{i=0}^{n-1} E\Big(X_1^2(t_i)\Big)(t_{i+1}-t_i) + \sum_{i=0}^{n-1}\mathbb{E}\Big(X_2^2(t_i)\Big)(t_{i+1}-t_i)\\
&= \mathbb{E}\int_0^T X_1^2(s)ds + \mathbb{E}\int_0^T X_2^2(s)ds\\
&= \mathbb{E}\Big(\sum_{l=1}^{2}\int_0^T X_l^2(s)\Big)ds.
\end{aligned}$$

Suppose that

$$\mathbb{E}\Big(\sum_{l=1}^{k}\int_0^T X_l(s)dW^l(s)\Big)^2 = \sum_{l=1}^{k}\mathbb{E}\Big(\int_0^T X_l^2(s)ds\Big),$$

is true for fixed $k \in \mathbb{N}$. We show that

$$\mathbb{E}\Big(\sum_{l=1}^{k+1}\int_0^T X_l(s)dW^l(s)\Big)^2 = \sum_{l=1}^{k+1}\int_0^T \mathbb{E}(X_l^2(s))ds.$$

Since $(X_l)_{l=1}^{k+1}$ is a set of elementary stochastic process, then

$$\begin{aligned}
M^{k+1} = \mathbb{E}\Big(\sum_{l=1}^{k+1}\int_0^T X_l(s)dW^l(s)\Big)^2 &= \mathbb{E}\Big(\sum_{l=1}^{k+1}\sum_{i=0}^{n-1} X_l(t_i)\Delta W_i^l\Big)^2\\
&= \mathbb{E}\Big(\sum_{l=1}^{k}\sum_{i=0}^{n-1} X_l(t_i)\Delta W_i^l\Big)^2
\end{aligned}$$

$$+\,\mathbb{E}\Big(\sum_{i=0}^{n-1} X_{k+1}(t_i)\Delta W_i^{k+1}\Big)^2$$
$$+\,2\Big(\mathbb{E}\sum_{i=0}^{n-1} X_{k+1}(t_i)\Delta W_i^{k+1}\sum_{l=1}^{k+1}\sum_{i=0}^{n-1} X_l(t_i)\Delta W_i^l\Big).$$

Using the fact that $(W^l)_{l=1}^{k+1}$ is a set of independent standard Brownian motions, we have

$$\begin{aligned}\mathbb{E}\Big(\sum_{l=1}^{k+1}\int_0^T X_l(s)dW^l(s)\Big)^2 &= \mathbb{E}\sum_{l=1}^{k}\int_0^T X_l^2(s)ds + \mathbb{E}\int_0^T X_{k+1}^2(s)ds\\ &= \mathbb{E}\Big(\sum_{l=1}^{k+1}\int_0^T X_l^2(s)ds\Big).\end{aligned}$$

Hence the formula is true for $k+1$. This implies that for every $k \in \mathbb{N}$ we have

$$M^k = \mathbb{E}\Big(\sum_{l=1}^{k}\int_0^T X_l(s)dW^l(s)\Big)^2 = \mathbb{E}\Big(\sum_{l=1}^{k}\int_0^T X_l^2(s)ds\Big).$$

From Lemma 2.1, proved in [75], M^k is a convergent in $L^2(\mathcal{F}_t)$, then

$$\lim_{k\to\infty} M^k = \lim_{k\to\infty}\mathbb{E}\Big(\sum_{l=1}^{k}\int_0^T X_l(s)dW^l(s)\Big)^2 = \mathbb{E}\Big(\sum_{l=1}^{\infty}\int_0^T X_l^2(s)ds\Big).$$

Hence

$$\mathbb{E}\Big(\sum_{l=1}^{\infty}\int_0^T X_l(s)dW^l(s)\Big)^2 = \mathbb{E}\Big(\sum_{l=1}^{\infty}\int_0^T X_l^2(s)ds\Big).$$

□

Theorem 1.2.11. *[97] Let X, $Y \in S$, a, $b \in \mathbb{R}$. Then :*

- *(linearity) for all tat once with probability one*

$$\mathbb{E}\int_0^t \Big(aX(s)+bY(s)dW(s)\Big) = a\mathbb{E}\int_0^t X(s)dW(s)+b\mathbb{E}\int_0^t Y(s)dW(s).$$

- $\mathbb{E}(\int_0^t X(s)dW(s)) = 0$;
- *the process* $\int_0^t X(s)dW(s)$ *is a martingale relative to* $\mathcal{F}_t$;
- *Doobs inequality holds:*

$$\mathbb{E}\sup_t(\int_0^t X(s)dW(s))^2 \leq 4\mathbb{E}\sup_t(\int_0^t X^2(s)ds;$$

- *if* $A \in \mathcal{F}$, $T \in [0, \infty]$, *and* $X_t(w) = Y_t(w)$ *for all* $w \in A$ *and* $t \in [0, T)$, *then*

$$I_A \int_0^t X(s)dW(s) = I_A \int_0^t Y(s)dW(s)$$

for all $t \in [0, T]$ *at once with probability one.*

Theorem 1.2.12. *if* $f \in S$ *and* f *is continous ,then, for any sequence* Π_i *of* $0 = t_{n,0} < t_{n,1} < t_{n,2} < \ldots < t_{n,n} = T$ *is a partition of* $[0, T]$ *with* $|\Pi_i| \to 0$,

$$\mathbb{P}\Big(\int_0^T f(s)dW(s) = \lim_{n\to\infty} \sum_{k=0}^{m_n-1} X_l(t_i)(W(t_{n,k+1}) - W(t_{n,k}))\Big) = 1, \quad (1.2.12)$$

Proof. introduce the step function g_n

$$g_n(t) = f(t_{n,k}).$$

For $g_n(t) \to f(t)$ uniformaly in $t \in [0, T)$ as $n \to \infty$.
Hence

$$\int_0^T |g_n(t) - f(t)|^2 dt \to 0 \qquad \text{a.s.}$$

By (see Theoreme 2.7 [63]) we then have

$$\mathbb{P}\Big(\lim_{n\to\infty} \int_0^T g_n(t)dW(t) = \int_0^T f(t)dW(t)\Big) = 1.$$

Since

$$\int_0^T g_n(t)dW(t) = \sum_{k=0}^{m_n-1} X_l(t_i)(W(t_{n,k+1}) - W^l(n, t_n)),$$

the assertion (1.2.12) follows. □

Remark 1.2.1. *For an square integrable stochastic process X on $[0,T]$, its Itô integral is defined by*

$$\int_0^T X(s)dW(s) = \lim_{n\to\infty} \int_0^T X_n(s)dW(s),$$

taking the limit in L^2, with X_n is defined in definition 6.1.3. Then the Itô isometry holds for all Itô-integrable X.

The next result is known as the Burholder-Davis-Gundy inequalities. It was first proved for discrete martingales and $p > 0$ by Burkholder [40]. In 1968, Millar [109] extended the result to continuous martingales. In 1970, Davis [53] extended the result for discrete martingales to $p = 1$. The extension to $p > 0$ was obtained independently by Burkholder and Gundy [41] in 1970 and Novikov [118] in 1971.

Theorem 1.2.13. *[133] For each $p > 0$ there exist constants $c_p, C_p \in (0,\infty)$, such that for any progressive process x with the property that for some $t \in [0,\infty)$,$\int_0^t X_s^2 ds < \infty$,$\mathbb{P}.a.s$ we have*

$$c_p\mathbb{E}\big(\int_0^t X_s^2 ds\big)^{\frac{p}{2}} \leq \mathbb{E}\big(\sup_{s\in[0,t]} \int_0^t X_s^2 dW(s)\big)^p \leq C_p\mathbb{E}\big(\int_0^t X_s^2 ds\big)^{\frac{p}{2}}. \quad (1.2.13)$$

Example 1.2.2. *Mean and mean square of a stochastic integral Let $I(f) = \int_0^1 W(s)dW(s)$. Then, by the theorem 1.2.12 of It ô integrals,*

$$\mathbb{E}(I(f)) = 0 \quad \textit{and} \quad \mathbb{E}(|I(f)|^2) = \int_0^1 \mathbb{E}|W(s)|^2 ds = \frac{1}{2}.$$

Lemma 1.2.14. *(Stochastic Product Rule) Let $X, Y \in S$. Then*

$$X_tY_t = X_0Y_0 + \int_0^t Y_s dX_s + \int_0^t X(s)dY_s + < X, Y >_t, \quad (1.2.14)$$

for all $t \geq 0$.

Proof. Let $\Delta = 0 = t_0 < t_1 < \ldots < t_n = t$ be any partition of the interval $[0,t]$Sum the equalities

$$X_{t_j}Y_{t_j} - X_{t_{j-1}}Y_{t_{j-1}} = (X_{t_j} - X_{t_{j-1}})(X_{t_j} - X_{t_{j-1}}) + X_{t_{j-1}}(Y_{t_j} - Y_{t_{j-1}}).$$

$$Y_{t_{j-1}}(X_{t_j} - X_{t_{j-1}}).$$

By theorem 1.2.12 and $j = 1, 2, \dots, n$, we obtain

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + < X, Y >_t .$$

The lemma is proved. □

1.2.3 Gaussian processes in Hilbert spaces

Let X be a Gaussian process in a Hilbert space H. Let

$$m(t) = \mathbb{E}(X(t)), \quad Q(t) = \mathbb{E}\Big(X(t) - m(t)) \otimes (X(t) - m(t))\Big), t \geq 0.$$

and

$$\beta(t, s) = \mathbb{E}\Big(((X(t) - m(t)) \otimes (X(s) - m(s))\Big), \quad t, s \geq 0.$$

Properties 1.2.15. *([132],Prop. 2.1.4) Consider a $U-$valued Gaussian random variable X with mean $m \in U$ and covariance operator $Q \in L(U)$, where Q is self-adjoint, positive semidefinite and with finite trace, that is $P^{-1} \circ X = \mathcal{N}(m, Q)$.*
Then, for all $u \in U$, $(X, u)_U$ is a real-valued Gaussian random variable with

1. $\mathbb{E}((X, u)_U) = (m, u)_U$ *for all* $u \in U$.
2. $\mathbb{E}\Big((X - m, u)_U (X - m, v)_U)\Big) = (Qu, v)_U$ *for all* $u, v \in Q$.
3. $\mathbb{E}\|(X - m)\|_U^2 = Tr(Q)$.

Definition 1.2.5. *The process X is said to be stationary if*

$$\mathbb{E}(\exp(i \sum_{k=1}^{n} \langle X(t_k + r), h_k \rangle)) = \mathbb{E}(\exp(i \sum_{k=1}^{n} \langle X(t_k), h_k \rangle)),$$

for all $n \in \mathbb{N}$, $t_1, \dots, t_n \in [0, \infty[$, $h_1, \dots, h_n \in H$, and $r \in [0, \infty)$.

Properties 1.2.16. *[52] A Gaussian process X, is stationary if and only if:*

1. $m(t + r) = m(t), \quad t, r \geq 0.$
2. $\beta(t + r, s + r) = \beta(t, s), \quad t, s, r \geq 0.$

1.3 The stochastic integral in Hilbert spaces

Let H and U be two separable Hilbert spaces. This section is devoted to a construction of the stochastic Itô integral.

$$\int_0^t \Phi(s)dW(s), \quad t \in [0,T],$$

where $W(.)$ is a Wiener process on U and Φ is a process with values that are linear but not necessarily bounded operators from U into H. We will start by collecting basic facts on Hilbert space valued Wiener processes including cylindrical ones. Then we define the stochastic integral in steps starting from elementary processes and ending up with the most general. We also establish basic properties of the stochastic integral, including the Itô formula.

1.3.1 Hilbert space valued Wiener processe

We consider two Hilbert spaces H and U, and a symmetric non- negative operator $Q \in L(U)$. We will consider first the case when $TrQ < +\infty$. Then there exists a complete orthonormal system $\{e_k\}$ in U, and a bounded sequence of nonnegative real numbers λ_k such that

$$Qe_k = \lambda_k e_k, \quad k \in \mathbb{N}^*.$$

U-valued stochastic process $W(t), t > 0$, is called a Q-Wiener process if

1. $W(0) = 0$,
2. W has continuous trajectories,
3. W has independent increments,
4. $\mathcal{L}(W(t) - W(s)) = \mathcal{N}(0, t-s)Q)$,$t \geq s \geq 0$.

If a process $W(t)$,$t \in [0,T]$ satisfies (1)-(2) and (4) for $t, s \in [0,T]$, then we say that $W(t)$ is a Q-Wiener process on $[0,T]$.

Properties 1.3.1. *Assume that W is a Q-Wiener process, with $TrQ < +\infty$. Then the following statements hold.*

- *(i) W is a Gaussian process on U and*

$$\mathbb{E}(W(t)) = 0, \ Cov(W(t)) = tQ, \ t \geq 0, \tag{1.3.1}$$

- *(ii) For arbitrary t, W has the expansion*

$$W(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\beta_j e_j, \tag{1.3.2}$$

where

$$\beta_j(t) = \frac{1}{\sqrt{\lambda_j}}\langle W(t), e_j\rangle, ,\ \ j = 1, 2, \dots, \tag{1.3.3}$$

are real valued Brownian motions mutually independent on $(\Omega, \mathcal{F}, \mathbb{P})$ *and the series in (1.3.4)is convergent in* $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

Proof. Let $0 < t_1 <, \dots, < t_n$, let $u_1, , \dots, u_n \in U$. Let us consider the random variable Z.

$$\begin{aligned} Z = \sum_{j=1}^{n}\langle W(t_j), u_j\rangle &= \langle W(t_1), \sum_{k=1}^{n} u_k\rangle \\ &+ \langle W(t_2) - W(t_1), \sum_{k=2}^{n} u_k\rangle +, \dots, \\ &+ \langle W(t_n) - W(t_{n-1}), \sum_{k=2}^{n} u_k\rangle. \end{aligned}$$

Since W has independent increments, Z is Gaussian for any choice of $u_1, \dots, u_n$ and (i) follows.

We now prove (ii). Let $t > s > 0$; by (1.3.3) it follows that.

$$\begin{aligned} \mathbb{E}(\beta_i(t)\beta_j(s)) &= \frac{1}{\sqrt{\lambda_i\lambda_j}}\mathbb{E}\Big((\langle W(t), e_i\rangle)(\langle W(t), e_j\rangle)\Big) \\ &= \frac{1}{\sqrt{\lambda_i\lambda_j}}\mathbb{E}\Big((\langle W(t) - W(s), e_i\rangle)(\langle W(s), e_j\rangle)\Big) \\ &+ \frac{1}{\sqrt{\lambda_i\lambda_j}}\mathbb{E}\Big((\langle W(s), e_i\rangle)(\langle W(s), e_j\rangle)\Big) \\ &= \frac{1}{\sqrt{\lambda_i\lambda_j}}s\langle Qe_i, e_j\rangle = s\delta_{ij}. \end{aligned}$$

Therefore the independence of β_i, $i = 1, 2, \dots$, follows. To prove representation (1.3.4) it is enough to notice that, for $m \geq n \geq 1$,

$$\mathbb{E}\|\sum_{j=n}^{m} \sqrt{\lambda_j}\beta_j e_j\|^2 = t\sum_{j=n}^{m} \sqrt{\lambda_j}, \tag{1.3.4}$$

Recall that

$$\sum_{j=n}^{m} \sqrt{\lambda_j} < \infty.$$

□

1.3.2 Definition of the stochastic integral

We are given here a Q-Wiener process in $(\Omega, \mathcal{F}, \mathbb{P})$ having values in U. By Proposition 1.3.1, $W(t)$ is given by (1.3.4). For the sake of simplicity of notation, we require that $\lambda_k > 0$ for all $k = 1, 2, \ldots,$ We are also given a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ in $\mathcal{F}$ and we assume that.

- (i)$W(t)$ is $\mathcal{F}_t$ -measurable,
- (ii) $W(t+h) - W(t)$ is independent of $\mathcal{F}_t$, $\forall h, t \geq 0$.

If a Q-Wiener process W satisfies (i) and (ii) we say that W is a Q-Wiener process with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. However to shorten the formulation we usually avoid stressing the dependence on the filtration.

Definition 1.3.1. *Let us fix $T < \infty$. An $L = L(U, H)$-valued process $\Phi(t)$,$t \in [0, T]$ taking only a finite number of values is said to be elementary if there exists a sequence $0 = t_0 < t_1 <, \ldots, < t_k = T$ and a sequence $\Phi_0, \Phi_1, \ldots, , \Phi_{k-1}$ of L-valued random variables taking only a finite number of values such that Φ_m are $\mathcal{F}_{t_m}$ measurable and*

$$\Phi(t) = \Phi_m, \textit{ for } \quad t \in (t_m, t_{m+1}], m = 0, 1, \ldots, k-1.$$

For elementary processes Φ one defines the stochastic integral by the formula

$$\int_0^t \Phi(s) dW(s) = \sum_{m=0}^{k-1} \Phi_m W_{(t_{m+1}\wedge t)} - W_{(t_m \wedge t)}, \tag{1.3.5}$$

and denote it by $\Phi.W(t)$, $t \in [0, T]$.

It is useful, at this moment, to introduce the subspace $U_0 = Q^{1/2}(U)$ of U which, endowed with the inner product

$$\langle u, v\rangle_0 = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \langle u, e_k\rangle\langle v, e_k\rangle$$

$$= \langle Q^{1/2}u, Q^{1/2}v\rangle, \tag{1.3.6}$$

is a Hilbert space.

In the construction of the stochastic integral for more general processes an important role will be played by the space of all Hilbert

Definition 1.3.2. *(Hilbert-Schmidit operator) Let $e_j, f_j, j \in \mathbb{N}$ be an orthonormal bases of U_0.An operator $\Psi \in L_2^0$ is called Schmidt operators $L_2^0 = L_2(U_0, H)$ from U_0 into H. The space L_2^0 is also a separable Hilbert space, equipped with the norm*

$$\|\Psi\|^2_{L_2^0} = \sum_{h,k=1}^{\infty} |\langle \Psi g_h, f_k\rangle|^2 = \sum_{h,k=1}^{\infty} \lambda_h |\langle \Psi e_h, f_k\rangle|^2$$

$$= \|\Psi Q^{1/2}\|^2 = Tr(\Psi Q \Psi^*),$$

where $\{g_j\}$, with $g_j = \sqrt{\lambda_j} e_j$, $j = 1, 2, \dots$, e_j and f_j are complete orthonormal bases in U_0, U and H respectively. Clearly, $L \subset L_2^0$, but not all operators from L_2^0 can be regarded as restrictions of operators from L. The space L_2^0 contains genuinely unbounded operators on U.

Let $\Phi(t)$,$t \in [0, T]$, be a measurable L_2^0-valued process; we define the norms

$$\|\Phi\|_t = \Big\{\mathbb{E} \int_0^t \|\Phi\|^2_{L_2^0} ds\Big\}^{\frac{1}{2}}$$

$$= \Big\{\mathbb{E} \int_0^t Tr(\Phi Q^{1/2})(\Phi Q^{1/2})^* ds\Big\}^{\frac{1}{2}}, \quad t \in [0, T]. \tag{1.3.7}$$

Properties 1.3.2. *If a process Φ is elementary and $\mathbb{E} \int_0^t \|\Phi\|^2_{L_2^0} < \infty$ then the process $\int_0^t \Phi(s) dW(s)$ is a continuous, square integrable H-valued martingale on $[0, T]$ and*

$$\mathbb{E}\Big(\int_0^t \Phi(s) dW(s)\Big)^2 = \mathbb{E} \int_0^t \|\Phi\|^2_{L_2^0} ds, \quad t \in [0, T]. \tag{1.3.8}$$

Proof. The proof is straightforward. We will check for instance that (1.3.8) holds for $t = t_m < T$. Define $\Delta_j = W(t_{j+1}) - W_{t_j}$, $j = 1, \ldots, m-1$. Then

$$\begin{aligned}\mathbb{E}\Big(|\int_0^t \Phi(s)dW(s)|\Big)^2 &= \mathbb{E}|\sum_{j=}^{m-1} \Phi(t_j)\Delta_j|^2 \\ &= \sum_{j=}^{m-1} E|\Phi(t_j)\Delta_j|^2 \\ &+ 2\sum_{i<j=}^{n} \langle \Phi(t_j)\Delta_j, \Phi(t_i)\Delta_i\rangle.\end{aligned}$$

We will show first that:

$$\sum_{j=1}^{m-1} \mathbb{E}|\Phi(t_j)\Delta_j|^2 = \sum_{j=}^{m-1}(t_{j+1} - t_j)E\|\Phi(t_j)\|^2_{L_2^0}. \tag{1.3.9}$$

To this purpose note that the random variable $\Phi^*(t_j)$ is $\mathcal{F}_{t_j}$ measurable, and Δ_j is a random variable independent of $\mathcal{F}_{t_j}$.

$$\begin{aligned}\mathbb{E}|\Phi(t_j)\Delta_j|^2 &= \sum_{l=1}^{\infty} \mathbb{E}(|\langle \Phi(t_j)\Delta_j, f_l\rangle|^2) \\ &= \sum_{l=1}^{\infty} \mathbb{E}(\mathbb{E}(\langle \Delta_j, \Phi^\star(t_j)f_l\rangle)|\mathcal{F}_{t_j}) \\ &= (t_{j+1} - t_j)\sum_{l=1}^{\infty} \mathbb{E}(\langle Q\Phi^\star(t_j)f_l, \Phi^\star(t_j)f_l\rangle) \\ &= (t_{j+1} - t_j)\mathbb{E}\|\Phi(t_j)\|^2_{L_2^0}.\end{aligned}$$

This shows (1.3.9). Similarly one has,

$$\mathbb{E}\langle \Phi(t_i)\Delta_i, \Phi(t_j)\Delta_j\rangle = 0 \quad i \neq j,$$

and the conclusion follows. □

Remark 1.3.1. *Note that the stochastic integral is an isometric transformation from the space of all elementary processes equipped with the norm* $\|.\|_T$*into the space* $\mathcal{M}_T^2(H)$ *of* H*-valued martingales.*

Remark 1.3.2. *the mapping $h \longrightarrow W(h)$ is linear. Indeed, for any $\lambda, \mu\mathbb{R} \in$, and$h, g \in H$, we have*

$$\mathbb{E}(W(\lambda h + \mu g) - \lambda W(h) - \mu g(h))^2 = \| \lambda h + \mu g \|_t^2 + \lambda \| h \|_t^2$$
$$+ \lambda \| g \|_t^2 - 2\lambda \langle \lambda h + \mu g, h \rangle$$
$$- 2\mu \langle \lambda h + \mu g, g \rangle + 2\lambda\mu \langle h, g \rangle = 0.$$

We are able now to extend the definition of the stochastic integral to all L_2^0 predictable processes $\|\Phi\|_T$. Note that they form a Hilbert space denoted by $\mathcal{N}_W^2(0,T;L_2^0)$, more simply $\mathcal{N}_W^2(0,T)$ or $\mathcal{N}_W^2$, and, by the previous proposition, elementary processes form a dense set in $\mathcal{N}_W^2$. By Proposition 1.3.2 the stochastic integral $\int_0^t \Phi(s)dW(s)$ is an isometric transformation from that dense set into $\mathcal{M}_T^2(H)$, therefore the definition of the integral can be immediately extended to all elements of $\mathcal{N}_W^2$. Moreover (1.3.8) holds and $\int_0^t \Phi(s)dW(s)$ is a continuous square integrable martingale. As a final step we extend the definition of the stochastic integral to L_2^0-predictable processes satisfying the even weaker condition.

$$\mathbb{P}\Big(\int_0^t \|\Phi\|_{L_2^0}^2 ds, \quad t \in [0,T] \Big) = 1. \tag{1.3.10}$$

All such processes are called stochastically integrable on $[0,T]$. They form a linear space denoted by $\mathcal{N}_W^2(0,T;L_2^0)$, more simply $\mathcal{N}_W^2(0,T)$ or even $\mathcal{N}_W^2$. The extension can be accomplished by the so-called localization procedure. To do so we need the following.

Lemma 1.3.3. *Assume that $\Phi \in \mathcal{N}_W^2(0,T)$ and τ is an $\mathcal{F}_t$-stopping time such that $\mathbb{P}(\tau < T) = 1$. Then $\mathbb{P}$.a.s.*

Proof. Assume that Φ is elementary and that τ is a simple stopping time.if Φ is elementary and τ general, then there exists a sequence of simple stopping times $\{\tau_n\}$ such that $\tau_n \downarrow \tau$ and $\mathbb{P}$.a.s.$\int_0^t I_{[0,\tau_n]}\Phi(s)dW(s) \to \int_0^t I_{[0,\tau]}\Phi(s)dW(s)$. On the other hand

$$\|I_{[0,\tau]}\Phi - I_{[0,\tau_n]}\Phi\|_T^2 = \mathbb{E} \int_0^t I_{[0,\tau_n]} \|\Phi\|_{L_2^0}^2 ds \to 0$$

and therefore, for a subsequence, still denoted by $\{\tau_n\}$,

$$\int_0^t I_{[0,\tau_n]}\Phi(s)dW(s) \to \int_0^t I_{[0,\tau]}\Phi(s)dW(s), \quad \mathbb{P}.a.s. \quad \text{and uniformly in} \quad [0,T].$$

If Φ is general and $\|\Phi - \Phi_m\|_T$ for a sequence of elementary processes, we have $\int_0^t \Phi_m(s)dW(s) \to \int_0^t \Phi(s)dW(s)$ and, for an appropriate subsequence,

$$\int_0^t I_{[0,\tau_n]}\Phi_{m_k}(s)dW(s) \to \int_0^t I_{[0,\tau]}\Phi(s)dW(s).$$

□

Lemma 1.3.4. *Assume that condition (1.3.10) holds and define*

$$\tau_n = \inf\{t \in [0,T] \quad : \int_0^t \|\Phi\|^2_{L_2^0} ds \geq n\},$$

with the convention that the infimum of an empty set is T. Then $\{\tau_n\}$ is a sequence such that

$$\mathbb{E}\Big(\int_0^t \|I_{[0,\tau_n]}\Phi\|^2_{L_2^0} ds, < \infty. \tag{1.3.11}$$

Proof. Consequently stochastic integrals $\int_0^t I_{[0,\tau_n]}\Phi(s)dW(s)$, are well defined for all $n = 1, 2, \dots,$ Moreover if $n < m$ then $\mathbb{P}$.a.s.

$$\int_0^t I_{[0,\tau_n]}\Phi(s)dW(s) = \int_0^t (I_{[0,\tau_n]}I_{[0,\tau_m]})\Phi(s)dW(s)$$

$$= \int_0^{t\wedge\tau_n} (I_{[0,\tau_m]}I_{[0,\tau_m]})\Phi(s)dW(s)\, t \in [0,T]. \tag{1.3.12}$$

Therefore one can assume that (1.3.11) holds for all $w \in \Omega$, $n < m$,$t \in [0,T]$. For arbitrary $t \in [0,T]$ define

$$\int_0^t \Phi(s)dW(s) = \int_0^t I_{[0,\tau_n]}\Phi(s)dW(s), \tag{1.3.13}$$

where n is an arbitrary natural number such that $\tau_n > t$. Note that if also $\tau_m > t$ and $m > n$ then

$$\int_0^t I_{[0,\tau_m]}\Phi(s)dW(s) = \int_0^{t\wedge\tau_n} I_{[0,\tau_m]}\Phi(s)dW(s) = I_{[0,\tau_n]}\int_0^t \Phi(s)dW(s),$$

and therefore the definition 1.3.13 is consistent. By analogous reasoning if $\tau_n^{'} \to T$ is another sequence satisfying (1.3.11) then the definition 1.3.13

leads to a stochastic process identical $\mathbb{P}$.a.s. for all $t \in [0,T]$. Note that for arbitrary $n = 1, 2, \ldots, w \in \Omega,\ t \in [0,T]$.

$$\int_0^t I_{[0,\tau_n]}\Phi(s)dW(s) = \sum_{m=0}^{k-1} \Phi_m W_{\tau_n \wedge (t_{m+1}\wedge t)} - W_{\tau_n \wedge (t_m \wedge t)} = M_n(\tau_n \wedge t),$$

where M_n is a square integrable continuous H-valued martingale. This property will be referred to as the local martingale property of the stochastic integral. □

1.3.3 Stochastic integral for cylindrical Wiener processes

The construction of the stochastic integral required the assumption that Q was a nuclear operator; only then the Q-Wiener process had values in U. We can, however, extend the definition of the integral to the case of general bounded, self-adjoint, nonnegative operators Q on U. To avoid trivial complications we will assume that Q is strictly positive: $Qx \neq 0$ for $x \neq 0$. Let $U_0 = |Q^{1/2}U$ with the induced norm $\|u\|\|_0 = \|Q^{-1/2}u\|$, and let U_1 be an arbitrary Hilbert space such that U is embedded continuously into U_1 and the embedding of U_0 into U_1 is Hilbert - Schmidt. Let $\{g_j\}$ be an orthonormal and complete basis in U_0 and $\{\beta_j\}$ a family of independent real valued standard Wiener processes.

Proposition 1.3.5. *The formula*

$$W(t) = \sum_{j=1}^{\infty} g_j \beta_j(t), \quad t \geq 0, \tag{1.3.14}$$

defines aQ_1*-Wiener process on* U_1 *with* $TrQ_1 < \infty$*. For arbitrary* $a \in U$*, the process*

$$\langle a, W(t)\rangle = \sum_{j=1}^{\infty}\langle a, g_j\rangle \beta_j(t),$$

is a real valued Wiener process and

$$\mathbb{E}(\langle a, W(t)\rangle\langle b, W(s)\rangle) = (t \wedge s)\langle Qa, b\rangle.$$

Moreover we have $ImQ_1^{1/2} = U_0$ *and*

$$\|u\|_0 = \|Q^{-1/2}u\|_1. \tag{1.3.15}$$

Proof. The series defining $W(t)$ is convergent in $L^2(\Omega, \mathcal{F}, \mathbb{P}, U_1)$ because

$$\mathbb{E}\Big(\| \sum_{j=n}^{m} g_j \beta_j(t) \|_1^2\Big) = \sum_{j=n}^{m} \|g_j\|_1^2, \; m \geq n \geq 1,$$

and the embedding $J : U_0 \to U_1$ is Hilbert-Schmidt which means $\sum_{j=1}^{\infty} \|Jg_j\|_1^2 < \infty$. One can also construct U_1. The series defining $\langle a, W(t) \rangle$ does converge in $L^2(\Omega, \mathcal{F}, \mathbb{P}, U_1)$ and therefore $\mathbb{P}.a.s.$ since

$$\begin{aligned} \sum_{j=1}^{\infty} |\langle a, g_j \rangle|^2 &\leq |a|^2 \sum_{j=1}^{\infty} \|g_j\|^2 \\ &\leq C\|J\|^2 \|a\|^2 \sum_{j=1}^{\infty} \|g_j\|^2 < \infty. \end{aligned}$$

In addition if $t \geq s \geq 0$

$$\begin{aligned} \mathbb{E}(\langle a, W(t) \rangle \langle b, W(s) \rangle) &= \mathbb{E}(\langle a, W(s) \rangle \langle b, W(s) \rangle) \\ &= s \sum_{j=1}^{\infty} \langle a, g_j \rangle \langle b, g_j \rangle \\ &= s \sum_{j=1}^{\infty} \langle aQ^{1/2}, Q^{-1/2} g_j \rangle \langle Q^{1/2} b, Q^{-1/2} g_j \rangle \\ &= s \sum_{j=1}^{\infty} \langle aQ, g_j \rangle_0 \langle Qb, g_j \rangle_0 \\ &= s \langle aQ, Qb \rangle_0 \\ &= s \langle aQ, b \rangle. \end{aligned}$$

To prove the latter part of the proposition note that, by the very definition,

$$\begin{aligned} \langle aQ_1, b \rangle_1 &= \mathbb{E}(\langle a, W(1) \rangle \langle b, W(1) \rangle_1) \\ &= \sum_{j=1}^{\infty} \langle a, g_j \rangle_1 \langle b, g_j \rangle_1 \\ &= \sum_{j=1}^{\infty} \langle a, Jg_j \rangle_1 \langle b, Jg_j \rangle_1 \end{aligned}$$

$$= \sum_{j=1}^{\infty} \langle aJ^*, g_j \rangle_0 \langle J^* b, g_j \rangle_0$$
$$= \langle J^* a, J^* b \rangle_0 = \langle J^* J a, b \rangle_1.$$

Consequently $Q_1 = JJ^*$. In particular

$$\|Q_1^{1/2}\|_1^2 = \langle J^* J a, a \rangle_1 = \|J^* a\|_0^2 \quad a \in U_1. \tag{1.3.16}$$

Thus $ImQ_1^{1/2} = ImJ = U_o$ and the operator $G = Q^{-1/2}J$ is bounded from U_o onto U_1. It follows from (1.3.16) that $G^* = J^* Q^{-1/2}$ is an isometry, therefore G itself is an isometry. So

$$\|Q_1^{-1/2} u\|_1 = \|Q_1^{-1/2} J u\|_1 = \|u\|_0.$$

This provide completed . □

Proposition 1.3.6. *[52] For $\Phi \in \mathcal{N}_W^2(0, T, H)$ and $A \in L(H, H_1)$, where H_1 denotes a further Hilbert space $(H_1, \langle ., . \rangle_{H_1}, \| \cdot \|)$ it holds that $A \circ \Phi \in \mathcal{N}_W^2(0, T, H)$ and*

$$A \int_0^t \Phi(s) dW(s) = \int_0^t A\Phi(s) dW(s) \qquad \mathbb{P}.a.s.$$

1.3.4 Stochastic Integration With Respect to Continuous Semimartingales

Definition 1.3.3. *A stochastic process X is called a continuous semi martingale if there exist processes M_t is a local martingale and A_t denotes the family of all continuous and adapted processes with paths of finite variation which vanish at $t = 0$.*

$$X_t = X_0 + M_t + A_t \quad \text{for all} \quad t \geq 0.$$

Using the fact that there are no non-trivial continuous local martingales of finite variation one can show that that for a continuous semi martingale the decomposition $X = X_0 + M + A$ into the initial value, a continuous local martingale and a continuous adapted process of finite variation is unique.

This decomposition is called the semi martingale decomposition of X.

Definition 1.3.4. *(Quadratic variation and covariation.) If X, Y are continuous semi martingales, we define the covariation process $\langle X, Y \rangle$ as*

$$\langle X, Y \rangle = \langle M, N \rangle,$$

where M and N are the local martingale parts of X, Y respectively. In particular, if A is a continuous bounded variation process, then A is a continuous semi martingale with constant martingale part.It follows easily that $\langle X, A \rangle = 0$, for each continuous semi martingale X.The following justifies our definition of $\langle X, Y \rangle$:

Proposition 1.3.7. *Let X, Y be continuous semi martingales, A a continuous bounded variation process and $t \geq 0$. Then*

- *(a) $Q_\Delta(X, A) \to 0$, $\mathbb{P}.a.s$ $||\Delta|| \to 0$,*
- *(b) $Q_\Delta(X, Y) \to \langle X, Y \rangle_t$, $\mathbb{P}.a.s$ $||\Delta|| \to 0$.*

Here the limits are taken over all partitions Δ of the interval $[0, t]$.

Proof. (a) Let $w \in \Omega$ be such that the path $s \to A_s(w)$ is of bounded variation and the path $s \to X_s(w)$ is continuous and hence uniformly continuous on the interval $[0, t]$. This is the case for $\mathbb{P}$. a.s. $w \in \Omega$. Let $|A|_t(w) < \infty$ denote the total variation of the path s $\to A_s(w)$ on the interval $[0, t]$ and set, for any partition $\Delta = \{0 = t_0 < t_1 < \ldots < t_n = t\}$ of the interval $[0, t]$,

$$C_\Delta(w) = \sup_{1 \leq j \leq n} \mid X_{t_j}(w) - X_{t_{j-1}}(w) \mid .$$

The uniform continuity of the path $s \to X_s(w)$ on the interval $[0, t]$ implies that $\lim_{\Delta \to 0} C_\Delta(w) = 0$. Thus

$$|Q_\Delta(X, A)(w)| \leq \sum \mid X_{t_j}(w) - X_{t_{j-1}}(w) \mid\mid A_{t_j}(w) - A_{t_{j-1}}(w) \mid$$

$$\leq C_\Delta(w) \sum \mid A_{t_j}(w) - A_{t_{j-1}}(w) \mid \leq C_\Delta(w) |A|_t(w) \to 0,$$

as $\|\ \Delta\ \| \to 0$. This shows (a).
(b) Let $X = M + \widetilde{A}$. and $Y = N + B$ be the semi martingale decompositions of X, Y . Fix $t \geq 0$ and let $\Delta = \{0 = t_0 < t_1 < \ldots < t_n = t\}$ be a partition of the interval $[0, t]$. By elementary algebra

$$Q_\Delta(X, Y)(w) = \sum \Big((M_{t_j}(w) - M_{t_{j-1}}(w)) + (\widetilde{A}_{t_j}(w) - \widetilde{A}_{t_{j-1}}(w)) \Big)$$

$$\times\Big((N_{t_j}(w) - N_{t_{j-1}}(w)) + (B_{t_j}(w) - B_{t_{j-1}}(w))\Big)$$
$$= Q_\Delta(Y, \widetilde{A})(w) + Q_\Delta(M, N)(w) + Q_\Delta(M, B)(w).$$

Now let $\| \Delta \| \to 0$. Then we have $Q_\Delta(Y, \widetilde{A})(w) \to 0$, $Q_\Delta(M, B)(w) \to 0$ according to (a). $Q_\Delta(M, N)(w) \to \langle M, N\rangle_t$. It follows that $Q_\Delta(X, Y)(w) \to \langle M, N\rangle_t = \langle X, Y\rangle_t$, in probability. □

Lemma 1.3.8. *(Kunita-Watanabe Inequality) Let M, N be continuous local martingales, U, V be $\mathcal{B}\times\mathcal{F}$-measurable processes. Then*

$$\int_0^t \mid U_s V_s \mid d\langle M, N\rangle_s \leq \Big(\int_0^t \mid U_s \mid^2 \mid d\langle M\rangle_s \mid\Big)^{\frac{1}{2}}$$
$$\times\Big(\int_0^t \mid V_s \mid^2 d\langle M\rangle_s\Big)^{\frac{1}{2}}, \quad \mathbb{P}.a.s. \tag{1.3.17}$$

Proof. Let $w \in \Omega$ be such that lemma 1.3.8 holds for all nonnegative Borel measurable functions $f, g : [0, t] \to \overline{\mathbb{R}}$ and let $f(s) = U_s(w)$ and $g(s) = V_s(w)$. Recalling that $|\mu_w|(ds) = \mid d\langle M\rangle_s \mid$, $\sigma_w = d\langle M\rangle_s$ and $\nu_w = d\langle N\rangle_s$, lemma 1.3.8 shows that (1.3.17) holds at the point w. □

1.3.5 Itô Formula

Definition 1.3.5. *Let W_t be 1-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ (i.e.1-dimensional) Itô process (or stochastic integral) is a stochastic process X_t on $(\Omega, \mathcal{F}, \mathbb{P})$ of the form*

$$X_t = X_0 + \int_0^t u(s, w)ds + \int_0^t v(s, w)dW(s), \tag{1.3.18}$$

where $u(t, w), v(t, w)$ is $\mathcal{F}_t$-adapted.

$$\mathbb{P}\Big(\int_0^t v^2(s, w)ds < \infty \quad \textit{for all} \quad t \geq 0\Big) = 1,$$

and

$$\mathbb{P}\Big(\int_0^t u(s, w)ds < \infty \quad \textit{for all} \quad t \geq 0\Big) = 1.$$

If X_t is an Itô process of the form (1.3.18) is sometimes written in the shorter differential form

$$dX_t = udt + vdW(t).$$

1.3.6 The Multi-dimensional Itô Formula

Itô formula Let $X = (X^1, X^2, \dots, X^d)$ be an $\mathbb{R}^d$-valued process with continuously differentiable paths and consider the process $Y_t = f(X_t)$, where $f \in C^2(\mathbb{R}^d)$. Let us write

$$D_j f = \frac{\partial f}{\partial X_j} \quad \text{and} \quad D_{i,j} f = \frac{\partial^2 f}{\partial X_i \partial X_j}.$$

The process Y has continuously differentiable paths with

$$\frac{d}{dt} f(X_t) = \sum_{j=1}^{d} D_j f(X_t(w)) \frac{d}{dt} X_t^j(w).$$

Fixing $w \in \Omega$ and integrating yields

$$f(X_t(w)) - f(X_0(w)) = \sum_{j=1}^{d} \int_0^t D_j f(X_s(w)) \frac{d}{ds} X_s^j(w),$$

where this integral is to be interpreted pathwise. Written as

$$f(X_t(w)) - f(X_0(w)) = \sum_{j=1}^{d} \int_0^t D_j f(X_s(w)) dX_s^j(w). \tag{1.3.19}$$

This equation remains true if X is a continuous, bounded variation process. The situation becomes more complicated if the process X is a continuous semi martingale and hence no longer has paths which are of bounded variation on finite intervals in general. Then a new term appears on the right hand side of (1.3.19) (Itô's formula). We will give a very explicit derivation which shows clearly where the new term comes from.

Theorem 1.3.9. *Let $G \subset \mathbb{R}^d$ be an open set, $X = (X^1, X^2, \dots, X^d)$ a continuous semi martingale with values in G and $f \in C^2(G)$. Then*

$$f(X_t(w)) - f(X_0(w)) = \sum_{j=1}^{d} \int_0^t D_j f(X_s(w)) dX_s^j(w)$$

$$+\frac{1}{2} \sum_{i,j=1}^{d} \int_0^t D_{i,j} f(X_s(w)) d\langle X_s^i(w), X_s^j(w) \rangle \quad \mathbb{P}.a.s, \tag{1.3.20}$$

for each $t \geq 0$.

Proof. We may assume that the path $t \to X_t(w)$ is continuous, for every $w \in \Omega$. Assume first that F, K are compact sets such that $F \subseteq K^\circ \subseteq K \subseteq G$ and the range of X is contained in F. Fix $t \geq 0$, let (Δ_n) be a sequence of partitions of the interval $[0, t]$ such that $\|\Delta_n\| \to 0$ as $n \to \infty$. For $n \geq 1$ write $\Delta_n = \{0 = t_0^n < t_1^n \ldots t_{k_n}^n = t\}$.Set $\epsilon = dist(F, K^\circ) > 0$ and

$$\Omega_m = \{w \in \Omega \quad ||X_{t_k^n} - X_{t_{k-1}^n}|| < \epsilon, \quad \forall n \geq m, 1 \leq k \leq k_n\}.$$

If $w \in \Omega$, then the path $s \in [0, t] \to X_s(w)$ is uniformly continuous and so $w \in \Omega_m$, for some $m \geq 1$. Thus $\Omega_m \uparrow \Omega$, as $m \uparrow \infty$. It will thus suffice to show that (1.3.22) holds $\mathbb{P}$-as. on the set Ω_m, for each $m \geq 1$.
Fix $m \geq 1$. If $w \in \Omega$, $X_{t_k^n}(w) \in B_\epsilon(X_{t_k^{n-1}})$ and hence the line segment from $X_{t_{k-1}^n}(w)$ to $X_{t_k^n}(w)$ is contained in the ball $B_\epsilon(X_{t_k^{n-1}}) \subseteq K$ for all $n \geq m$and all $1 \leq k \leq k_n$. Let $n \geq m$ and write

$$f(X_t - f(X_0) = \sum_{k=1}^{k_n} (f(X_{t_k^n}) - f(X_{t_{k-1}^n})). \tag{1.3.21}$$

Consider $k = \{1, 2, ..., k_n\}$ and $w \in \Omega$. A second degree Taylor expansion for $f(x)$ centered at $x = X_{t_{k-1}^n}(w)$ yields

$$f(X_{t_k^n}) - f(X_{t_{k-1}^n}) = \sum_{j=1}^{d} D_j f(X_{t_{k-1}^n})(X^j_{t_k^n} - X^j_{t_{k1}^n})$$

$$+\frac{1}{2} \sum_{i,j=1}^{d} D_{i,j} f(\xi_{n,k})(X^j_{t_k^n} - X^i_{t_{k-1}^n})(X^j_{t_k^n} - X^i_{t_{k-1}^n}),$$

where the point $\xi_{n,k} = \xi_{n,k}(w)$ is on the line segment from $X_{t_{k-1}^n}(w)$ to $X_{t_k^n}(w)$. Note that this line segment is contained in K and that $D_{i,j}f$ is uniformly continuous on K. Entering the above expansion into (1.3.21) and commuting the order of summation, we can write

$$f(X_{t_k^n}) - f(X_{t_{k-1}^n}) = \sum_{j=1}^{d} A_j^n + \frac{1}{2} \sum_{i,j=1}^{d} B_{i,j}^n.$$

where

$$A_j^n = \sum_{k=1}^{k_n} D_j f(X_{t_{k-1}^n})(X^j_{t_k^n} - X^j_{t_{k-1}^n})$$

and

$$B_{i,j}^n = \sum_{k=1}^{k_n} D_{i,j}f(\xi_{n,k})(X_{t_k^n}^i - X_{t_{k-1}^n}^i)(X_{t_k^n}^i - X_{t_{k-1}^n}^i)$$

at all points $w \in \Omega$.by theorem 1.2.12.we have $A_j^n \to \int_0^t D_j f(X_s(w))dX_s^j(w)$ in probability, as $n \uparrow \infty$. Since limits in probability are uniquely determined $\mathbb{P}-$as,it will now suffice to show that $B_{i,j}^n \to \int_0^t D_{i,j}f(X_s(w))d\langle X^i(w), X^j(w)\rangle_s$ in probability on the set Ω_m as $n \uparrow \infty$. To see this we will compare $B_{i,j}^n$ to the similar term

$$\widetilde{B}_{i,j}^n = \sum_{k=1}^{k_n} D_{i,j}f(X_{t_{k-1}^n})(X_{t_k^n}^i - X_{t_{k-1}^n}^i)(X_{t_k^n}^j - X_{t_{k-1}^n}^j),$$

which is known to converge to $\int_0^t D_{i,j}f(X_s(w))d\langle X^i(w), X^j(w)\rangle_s$ in probability.

It will thus suffice to show that $|B_{i,j}^n - \widetilde{B}_{i,j}^n| \to 0$ in probability on the set Ω_m as $n \uparrow \infty$. Indeed, using the Cauchy Schwartz inequality,

$$|B_{i,j}^n - \widetilde{B}_{i,j}^n| = \left| \sum_{k=1}^{k_n}(D_{i,j}f(\xi_{n,k}) - D_{i,j}f(X_{t_{k-1}^n})))(X_{t_k^n}^i - X_{t_{k-1}^n}^i)(X_{t_k^n}^j - X_{t_{k1}^n}^j) \right|$$

$$\leq C_{i,j}^n \sum_{k=1}^{k_n} | X_{t_k^n}^i - X_{t_{k-1}^n}^i || X_{t_k^n}^j - X_{t_{k1}^n}^i | \leq C_{i,j}^n B_i^n B_j^n$$

where

$$(B_i^n)^2 = \sum_{k=1}^{k_n} | X_{t_k^n}^i - X_{t_{k-1}^n}^i |^2 \quad (B_j^n)^2 = \sum_{k=1}^{k_n} | X_{t_k^n}^j - X_{t_{k-1}^n}^j |^2$$

and

$$C_{i,j}^n = \sup_{1 \leq k \leq k_n} | D_{i,j}f(\xi_{n,k}) - D_{i,j}f(X_{t_{k-1}^n}) | .$$

From the uniform continuity of $D_{ij}f$ on K it follows that $C_{i,j}^n \to 0$, $\mathbb{P}$-a.s. and hence in probability on the set Ω_m,$n \uparrow \infty$ Moreover $B_i^n \to \sqrt{\langle X^i \rangle_t} < \infty$ and $B_j^n \to \sqrt{\langle X^j \rangle_t} < \infty$ in probability.It follows that $|B_{i,j}^n - \widetilde{B}_{i,j}^n| \to 0$ in probability on Ω_m as ,$n \uparrow \infty$ as desired. Let us now deal with the general case. Choose a sequence (K_m) of compact sets such that $K_m \subseteq K$ and $G = \cup_m K_m$ and set $T_m = \{t \geq 0 \quad : \quad X_t \in K_m\}$.By path continuity of X, (T_m) is a sequence of optional times such that $T_m \uparrow 0$ on all of Π as $m \uparrow \infty$. Since X_0 is constant we can choose m_0 such that $X_0 \in K^0_{m_0}$ and hence $X^{T_m}_t \in K_m$, for all $m \geq m_0$ and $t \geq 0$. Consider such m. The form of equation (1.3.22) to the process X^{T_m} and observing that $g(X^{T_m}) = g(X)^{T_m}$,$\langle Y^{T_m}, Z^{T_m}\rangle = \langle Y, Z\rangle^{T_m}$ we have

$$f(X^{T_m}_t(w)) - f(X_0(w)) = \sum_{j=1}^{d} \int_0^t D_j f(X_s(w)) dX^{j,T_m}_s(w) + \frac{1}{2}\sum_{i,j=1}^{d} \int_0^t D_{i,j} f(X_s(w)) d\langle X^i_s(w), X^j_s(w)\rangle^{T_m}.$$

Let $m \uparrow \infty$ to obtain

$$f(X_t(w)) - f(X_0(w)) = \sum_{j=1}^{d} \int_0^t D_j f(X_s(w)) dX^j_s(w) + \frac{1}{2}\sum_{i,j=1}^{d} \int_0^t D_{i,j} f(X_s(w)) d\langle X^i_s(w), X^j_s(w)\rangle, \ \mathbb{P}.a.s$$

for each $t \geq 0$. □

Theorem 1.3.10. *Let $G \subset \mathbb{R}^d$ be an open set, $X = (X^1, X^2, \ldots, X^d)$ a continuous semi martingale with values in G and $f \in C^{1,2}([0,T] \times G)$. Then*

$$f(t, X_t(w)) - f(0, X_0(w)) = \int_0^t \frac{\partial f(s, X_s(w))}{\partial s} ds + \sum_{j=1}^{d} \int_0^t D_j f(s, X_s(w)) dX^j_s(w)$$

$$+\frac{1}{2}\sum_{i,j=1}^{d} \int_0^t D_{i,j} f(s, X_s(w)) d\langle X^i_s(w), X^j_s(w)\rangle, \ \mathbb{P}.a.s \tag{1.3.22}$$

for each $t \geq 0$.

Proof. The notation $f \in C^{1,2}([0,T] \times G)$ is to be interpreted as follows:the partial derivative $\partial f/\partial s$ exists on $(0,T) \times G$ and has a continuous extension to $[0,T] \times G$. Continuous partial derivatives $D_j f$, $D_{ij} f$ with respect to the remaining variables are assumed to exist on $[0,T] \times G$.This ensures that all partial derivatives are uniformly continuous on $[0,T] \times K$, for each compact subset $K \subseteq G$. Consider a partition $\Delta_n = \{0 = t_0^n < t_1^n \dots t_{k_n}^n = t\}$ of $[0,t]$ and write

$$\begin{aligned} f(t, X_t) - f(t, X_0) &= \sum_{k=1}^{k_n} \Big(f(t_k^n, X_{t_k^n}) - f(t_{k-1}^n, X_{t_{k-1}^n}) \Big). \\ &= \sum_{k=1}^{k_n} \Big(f(t_k^n, X_{t_k^n}) - f(t_{k-1}^n, X_{t_{k-1}^n}) + f(t_k^n, X_{t_k^n}) - f(t_{k-1}^n, X_{t_{k-1}^n}) \Big) \end{aligned}$$

and

$$f(t_k^n, X_{t_k^n}) - f(t_{k-1}^n, X_{t_{k-1}^n}) = \frac{\partial f}{\partial t}(\eta_{n,k}, X_{t_k^n})(t_k^n - t_{k-1}^n),$$

for some $\eta_{n,k} = \eta_{n,k}(w) \in (t_k^n, t_{k-1}^n)$). The summands $f(t_k^n, X_{t_k^n}) - f(t_{k-1}^n, X_{t_{k-1}^n})$ are dealt with exactly as in the proof of theorem1.3.13 (second degree Taylor expansion of $g(x) = f(t_{k-1}^n, x)$ around the point $x = X_{t_{k-1}^n}$). □

Let us write down the special case where $X \in H_{SP}$ is a scalar semi martingale ($d = 1$):

Theorem 1.3.11. *Let $G \subset \mathbb{R}$ be an open set, $X = (X^1)$ (i.e $d = 1$) a continuous semi martingale with values in G and $f \in C^2(G)$. Then*

$$\begin{aligned} f(X_t(w)) - f(X_0(w)) &= \int_0^t \frac{\partial f(X_s(w))}{\partial X_s} dX_s(w) \\ &\quad + \frac{1}{2} \int_0^t \frac{\partial^2 f(X_s(w))}{\partial X_s^2} d\langle X_s(w), X_s(w) \rangle, \mathbb{P}.a.s \end{aligned}$$

for each $t \geq 0$.

Proof. same evidence are dealt with exactly as in the proof of theorem 1.3.13 for $j = d = 1$ □

Theorem 1.3.12. *Let $G \subset \mathbb{R}$ be an open set, X a continuous semi martingale with values in G and $f \in C^{1,2}([0,T] \times G)$. Then*

$$\begin{aligned} f(t, X_t(w)) - f(0, X_0(w)) &= \int_0^t \frac{\partial f(s, X_s(w))}{\partial s} ds + \int_0^t \frac{\partial f(X_s(w))}{\partial X_s} dX_s(w) \\ &\quad + \frac{1}{2} \int_0^t \frac{\partial^2 f(X_s(w))}{\partial X_s^2} d\langle X_s(w), X_s(w) \rangle \; \mathbb{P}.a.s \end{aligned}$$

for each $t \geq 0$.

Proof. same evidence are dealt with exactly as in the proof of theorem 1.3.13 for $j = d = 1$. □

As a first consequence of Itô's formula we show that the family S of continuous semi martingales is not only a real algebra but is in fact closed under the application of twice continuously differentiable functions:

Theorem 1.3.13. *Let G be an open subset of $\mathbb{R}^d$, $X \in S^d$. Let S^d denote the family of all $\mathbb{R}^d$-valued with values in G, $f \in C^2(G)$. For each $i = 1, \dots, d$ let $X^i = M^i + A^i$ be the semi martingale decomposition of X^i, especially $A^i = u_{X^i}$ .Then $Z = f(X)$ is again a continuous semi martingale and its local martingale part M_t and compensator $u_Z(t)$ are given by,*

$$M = Z_0 + \sum_{j=1}^{d} \int_0^t D_j f(X_s(w)) dM_s^i(w).$$

and

$$u_Z = \sum_{j=1}^{d} \int_0^t D_j f(X_s(w)) dA_s^i(w) + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t D_{i,j} f(X_s(w)) d\langle X_s^i(w), X_s^j(w)\rangle.$$

Proof. Writing $f(X)$ to denote the process $f(X_t)$ and using the formula:

$$\begin{aligned} Z = f(X) = Z_0 &+ \int_0^t \frac{\partial f(X_s(w))}{\partial s} \langle X_s^i, X_s^j \rangle_s \\ &+ \sum_{i=1}^{d} \int_0^t D_j f(s, X_s(w)) dX_s^i(w) \\ &= Z_0 + \sum_{i=1}^{d} \int_0^t D_j f(s, X_s(w)) dM_s^i(w) \\ &+ \sum_{i=1}^{d} \int_0^t D_j f(s, X_s(w)) dA_s^i(w) \\ &+ \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t D_{i,j} f(X_s(w)) d\langle X_s^i(w), X_s^j(w)\rangle = M + A, \end{aligned}$$

where

$$M = Z_0 + \sum_{i=1}^{d} \int_0^t D_j f(s, X_s(w)) dM_s^i(w).$$

and

$$A = \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t D_{i,j}f(X_s(w))d\langle X_s^i(w), X_s^j(w)\rangle + \sum_{i=1}^{d}\int_0^t D_j f(s, X_s(w))dA_s^i(w).$$

Here M is a continuous local martingale, since so are the M^i, and A is a continuous bounded variation process vanishing at zero. This shows that Z is a continuous semi martingale with semi martingale decomposition $Z = M + A$, as desired. □

We can save this result with the table below.

$[d.,d.]$	ds	$dW^i(s)$	$dW^j(s)$
ds	0	0	0
$dW^i(s)$	0	ds	0
$dW^j(s)$	0	0	ds

1.4 Fractional Brownian motion

The fractional Brownian motion was first introduced within a Hilbert space framework by Kolmogorov in 1940 in [120], where it was called Wiener Helix. It was further studied by Yaglom in [160]. The name fractional Brownian motion is due to Mandelbrot and Van Ness, who in 1968 provided in [61] a stochastic integral representation of this process in terms of a standard Brownian motion.

Definition 1.4.1. *Given $H \in (0,1)$, a continuous centered Gaussian process B^H with the covariance function $R_H(t,s) = \mathbb{E}[B_l^H(t))B_k^H(s)]$*

$$R_{H_{lk}}(t,s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})\delta_{lk} \quad t,s \in [0,T],$$

where

$$\delta_{lk} = \begin{cases} 1 & l = k \\ 0, & l \neq k \end{cases}$$

is called a two-sided one-dimensional fractional Brownian motion (fBm), and H is the Hurst parameter.

For $H = 1/2$, the fBm is then a standard Brownian motion. By Definition 1.4.2 we obtain that a standard fBm $B^{(H)}$ has the following properties:

1. $B^{(H)}(0) = 0$ and $E(B^{(H)}(t)) = 0$ for all $t \geq 0$.

2. $B^{(H)}$ has homogeneous increments, i.e., $B^{(H)}(t+s) - B^{(H)}(s)$ has the same law of $B^{(H)}(t)$ for $s, t \geq 0$.

3. $B^{(H)}$ is a Gaussian process and $E(B^{(H)}(t)^2) = t^{2H}$, $t \geq 0$, for all $H \in (0,1)$.

4. $B^{(H)}$ has continuous trajectories.

1.4.1 Correlation between two increments

For $H = 1/2$, $B^{(H)}$ is a standard Brownian motion; hence, in this case the increments of the process are independent. On the contrary, for $H \neq 1/2$ the increments are not independent. More precisely, by Definition 1.4.2 we know that the covariance between $B^{(H)}(t+h) - B^{(H)}(t)$ and $B^{(H)}(s+h) - B^{(H)}(s)$ with $s + h \leq t$ and $t - s = nh$,

$$\rho_H(n) = \frac{1}{2} h^{2H}((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}).$$

In particular, we obtain that two increments of the form $B^{(H)}(t+h) - B^{(H)}(t)$ and $B^{(H)}(t+2h) - B^{(H)}(t+h)$ are positively correlated for $H > 1/2$, while they are negatively correlated for $H < 1/2$. In the first case the process presents an aggregation behavior and this property can be used in order to describe cluster phenomena (systems with memory and persistence). In the second case it can be used to model sequences with intermittency and antipersistence.

1.4.2 Long-range dependence

Definition 1.4.2. *A stationary sequence $(X_n)_{n \in N}$ exhibits long-range dependence if the autocovariance functions $\rho(n) := Cov(X_k, X_{k+n})$ satisfy:*

$$\lim_{n \to \infty} \frac{\rho(n)}{cn^{-\alpha}} = 1,$$

for some constant c and $\alpha \in (0,1)$. In this case, the dependence between X_k and X_{k+n} decays slowly as n tends to infinity and

$$\sum_{n=1}^{\infty} \rho(n) = \infty.$$

Hence, we obtain immediately that the increments $X_k := B^{(H)}(k) - B^{(H)}(k-1)$ *of* $B^{(H)}$ *and* $X_{k+n} := B^{(H)}(k+n) - B^{(H)}(k+n-1)$ *of* $B^{(H)}$ *have the long-range dependence property for* $H > 1/2$ *since*

$$\rho_H(n) = \frac{1}{2}h^{2H}((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}) \sim H(2H-1)n^{2H-2}$$

as n *goes to infinity. In particular,*

$$\lim_{n\to\infty} \frac{\rho(n)}{H(2H-1)n^{2H-2}} = 1.$$

Summarizing, we obtain

1. *For* $H > 1/2, \sum_{n=1}^{\infty} \rho_H(n) = \infty$.
2. *For* $H < 1/2, \sum_{n=1}^{\infty} \mid \rho_H(n) \mid < \infty$.

Definition 1.4.3. *We say that an* $\mathbb{R}^d$*-valued random process* $X = (X_t)_{t\geq 0}$ *is self-similar or satisfies the property of self-similarity if for every* $a > 0$ *there exists* $b > 0$ *such that*

$$Law(X_{at}, t \geq 0) = Law(bX_t, t \geq 0). \tag{1.4.1}$$

Note that (1.4.1) means that the two processes X_{at} *and* bX_t *have the same finite-dimensional distribution functions, i.e., for every choice* $t_0, \ldots, t_n$ *in* $\mathbb{R}$,

$$\mathbb{P}(X_{at_0} \leq x_0, \ldots, X_{at_n} \leq x_n) = \mathbb{P}(bX_{t_0} \leq x_0, \ldots, bX_{t_n} \leq x_n)$$

for every $x_0, \ldots, x_n$ *in* $\mathbb{R}$.

Definition 1.4.4. *If* $b = a^{-H}$ *in Definition 1.4.3, then we say that* $X = (X_t)_{t\geq 0}$ *is a self-similar process with Hurst index* H *or that it satisfies the property of (statistical) self-similarity with Hurst index* H*. The quantity* $D = 1/H$ *is called the statistical fractal dimension of* X*.*

1.4.3 Hölder continuite

We recall that according to the Kolmogorov criterion (see [228]), a process $X = (X_t)_{t\in\mathbb{R}}$ admits a continuous modification if there exist constants $0 < \alpha \leq 1$, and $k > 0$ such that

$$\mathbb{E}\Big(\mid X(t) - X(s) \mid \Big) \leq k \mid t - s \mid^{\alpha}, \quad \text{for all} \quad s, t \in \mathbb{R}.$$

Theorem 1.4.1. *Let $H \in (0,1)$. The fBm $B^{(H)}$ admits a version whose sample paths are almost surely Hölder continuous of order strictly less than H.*

Proof. We recall that a function $f : \mathbb{R} \to \mathbb{R}$ is Holder continuous of order $0 < \alpha \leq 1$, and write $f \in C^{\alpha}(\mathbb{R})$, if there exists $M > 0$ such that

$$|f(t) - f(s)| \leq M \mid t - s \mid^{\alpha},$$

for every $s, t \in \mathbb{R}$. For any $\alpha > 0$ we have

$$\mathbb{E}\Big(|B^{(H)}(t) - B^{(H)}(s))|^{\alpha}\Big) = \mathbb{E}(|B^{(H)}(1)|^{\alpha})|t - s|^{\alpha H}.$$

Hence, by the Kolmogorov criterion we get that the sample paths of $B^{(H)}$ are almost everywhere Hölder continuous of order strictly less than H. Moreover, by [14] we have ,

$$\limsup_{t \to 0^+} \frac{|B^{(H)}(t)|}{t^h \sqrt{\log \log t^{-1}}} = c_H,$$

with probability one, where c_H is a suitable constant . Hence $B^{(H)}$ cannot have sample paths with Hölder continuitys order greater than H.

□

1.4.4 Path differentiability

By [27] we also obtain that the process $B^{(H)}$ is not mean square differentiable and it does not have differentiable sample paths.

Properties 1.4.2. *Let $H \in (0,1)$. The fBm sample path $B^{(H)}(.)$ is not differentiable.*

In fact, for every $t_0 \in [0, \infty)$

$$\limsup_{t \to t_0} \mid \frac{B^{(H)}(t) - B^{(H)}(t_0)}{t - t_0} \mid = \infty,$$

with probability one.

Proof. Note that we assume $B^{(H)}(0) = 0$. The result is proved by exploiting the self-similarity of $B^{(H)}$. Consider the random variable,

$$\mathcal{R}_{t,t_0} := \frac{B^{(H)}(t) - B^{(H)}(t_0)}{t - t_0},$$

that represents the incremental ratio of$B^{(H)}$. Since $B^{(H)}$ is self-similar, we have that the law of $\mathcal{R}_{t,t_0}$ is the same of $(t - t_0)^{H-1}B^{(H)}(1)$. If one considers the event,

$$A(t, w) := \Big\{ \sup_{0 \le s \le t} \mid \frac{B^H(s)}{s} \mid > d \Big\}.$$

For any sequence $t \to 0$ we have,

$$A(t_n, w) \supseteq A(t_{n+1}, w).$$

Thus,

$$\mathbb{P}(\lim_{n\to\infty} A(t_n)) = \lim_{n\to\infty} \mathbb{P}(A(t_n)),$$

and

$$\mathbb{P}(A(t_n)) \ge \mathbb{P}\Big(\frac{B^H(t_n)}{(t_n)}\Big) = \mathbb{P}\Big(B^H(1) > t_n^{1-H} d\Big),$$

which tends to one as $n \to \infty$. Note that the proof goes through under the assumption of self-similarity. □

1.4.5 The fBm is not a semi martingale for $H \neq 1/2$

The definition of the Itô integral is a direct consequence of the martingale property of Brownian motion. But fBm does not exhibit this property, in fact, fBm is not even a semi-martingale. There are many different proofs revealing this fact (a rather nice one is given in [61]). We state the theorem and present a simple proof here. But first, we need to find the p-variation of $B^H(t)$. In order to verify that $B^{(H)}$ is not a semimartingale for $H \neq 1/2$, it is sufficient to compute the p-variation of $B^{(H)}$.

Definition 1.4.5. *Let $(X(t))_{t\in[0,T]}$ be a stochastic process and consider a partition $\pi = \{0 = t_0 < t_1 < \ldots < t_n = T\}$.Put*

$$S_p(X,\pi) := \sum_{k=1}^{n} |X(t_k) - X(t_{k-1})|^p.$$

The p-variation of X over the interval $[0,T]$ *is defined as,*

$$V_p(X,[0,T]) = \sup_p S_p(X,\pi),$$

where π *is a finite partition of* $[0,T]$. *The index of p-variation of a process is defined as,*

$$I(X,[0,T]) = \inf\{p>0; V_p(X,[0,T])<\infty\}.$$

Lemma 1.4.3. *[27] Let* $I\Big(B^H(t),[0,T]\Big) = \frac{1}{H}$.
Moreover,

$$V_p\Big(B^H(t),[0,T]\Big) = 0 \qquad when \qquad pH > 1$$

and

$$V_p\Big(B^H(t),[0,T]\Big) = \infty \quad when \qquad pH < 1.$$

Theorem 1.4.4. *Let* $\{B^H(t): t \geq 0\}$, *for* $H \neq 1/2$, *is not semi martingale.*

Proof. A process $\{X(t), \mathcal{F}_t, t \geq 0\}$ is called a semi martingale if it admits the Doob-Meyer decomposition $X(t) = X(0) + M(t) + A(t)$, where $M(t)$ is an $\mathcal{F}_t$ local martingale with $M(0) = 0$, $A(t)$ is a cadlag adapted process of locally bounded variation and $X(0)$ is $\mathcal{F}_0$-measurable. Moreover, any semi martingale has locally bounded quadratic variation [133].Now, let $X(t) = B^H(t)$
For $p > 0$ set.

$$Y_{n,p} = n^{pH-1} \sum_{j=1}^{n} |B_{j/n} - B_{(j-1)/n}|^p.$$

By the self-similar property of fBm, the sequence $\{Y_{n,p}, n \geq 1\}$ has the same distribution as $\{\widetilde{Y}_{n,p}, n \geq 1\}$, where

$$\widetilde{Y}_{n,p} = n^{-1} \sum_{j=1}^{n} |B_j - B_{j-1}|^p.$$

The stationary sequence $\{B_j - B_{j-1} j \geq 1\}$ is mixing. Hence, by the Ergodic Theorem $\widetilde{Y}_{n,p}$ converges almost surely and in $L^1(\Omega)$ to $\mathbb{E}(|B_1|^p)$ as n tends to infinity. As a consequence, $Y_{n,p}$ converges in probability as n tends to infinity to $\mathbb{E}(|B_1|^p)$. Therefore,

$$V_p(X, [0,T]) = \sup_p \sum_{j=1}^{n} |B_{j/n} - B_{(j-1)/n}|^p,$$

1. If $H < 1/2$, we can choose $p > 2$ such that $pH < 1$, and we obtain that the p-variation of fBm (defined as the limit in probability $\lim_{n\to\infty} V_{n,p}$) is infinite. Hence, the quadratic variation ($p = 2$) is also infinite.
2. If $H > 1/2$ 2 , we can choose p such that $\frac{1}{H} < p < 2$. Then the p variation is zero, and, as a consequence, the quadratic variation is also zero. On the other hand, if we choose p such that $1 < p < \frac{1}{H}$ we deduce that the total variation is infinite.

Therefore, we have proved that for $H \neq 1/2$ the fractional Brownian motion cannot be a semi martingale. □

In [92] Cheridito has introduced the notion of weak semimartingale as a stochastic process $\{X_t, t \geq 0\}$ such that for each $T > 0$, the set of random variables

$$\left\{ \sum_{j=1}^{n} f_j(X_{t_j} - X_{t_{j-1}}), n \geq 1, 0 = t_0 < t_1 < \ldots < t_n = T, |\ f_j \ |\leq 1, f_j \in \mathcal{F}^X_{t_{j-1}} \right\}$$

is bounded. Here $\mathcal{F}^X$ represents the natural filtration associated to the process X. Moreover, in [92] it is shown that if $B(t)$ is a standard Brownian motion independent of $B^{(H)}$, then the process:

$$M(t)^H = B^H(t) + B(t).$$

Then

$$\{M(t)^H, t \geq 0\}$$

is not a weak semi martingale if $H \in (0, 1/2) \cup (1/2, 3/4]$,and it is a semi martingale, equivalent in law to Brownian motion on any finite time interval $[0, T]$, if $H \in (3/4, 1)$.

1.4.6 Fractional integrals and derivatives

We recall the basic definitions and properties of the fractional calculus. For a detailed presentation of these notions we refer to [138].

Let $a, b \in \mathbb{R}$, $a < b$. Let $f \in L^1(a,b)$ and $\alpha > 0$. The left and right-sided fractional integrals of f of order α are defined for almost all $x \in (a,b)$ by

$$I_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y) dy \tag{1.4.2}$$

and

$$I_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y) dy \tag{1.4.3}$$

respectively. Let $I_{a^+}^{\alpha}(L^p)$ (resp. $I_{b^-}^{\alpha}(L^p)$) the image of $L^p(a,b)$ by the operator $I_{a^+}^{\alpha}$ (resp. $I_{b^-}^{\alpha}$).

If $f \in I_{a^+}^{\alpha}(L^p)$ (resp. $f \in I_{b^-}^{\alpha}(L^p)$) and $0 < \alpha < 1$ then the left and right-sided fractional derivatives are defined by

$$D_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha+1}} dy \right), \tag{1.4.4}$$

and

$$D_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_x^b \frac{f(x)-f(y)}{(y-x)^{\alpha+1}} dy \right), \tag{1.4.5}$$

for almost all $x \in (a,b)$ (the convergence of the integrals at the singularity $y = x$ holds point-wise for almost all $x \in (a,b)$ if $p = 1$ and moreover in L^p-sense if $1 < p < \infty$).

Recall the following properties of these operators:

- If $\alpha < \frac{1}{p}$ and $q = \frac{p}{1-\alpha p}$.then
$$I_{a^+}^{\alpha}(L^p) = I_{b^-}^{\alpha}(L^p) \subset L^q(a,b).$$
- If $\alpha > \frac{1}{p}$.Then
$$I_{a^+}^{\alpha}(L^p) \cup I_{b^-}^{\alpha}(L^p) \subset C^{\alpha - \frac{1}{p}}(a,b).$$

where $C^{\alpha-\frac{1}{p}}(a,b)$ denotes the space of $\left(\alpha-\frac{1}{p}\right)$-Hölder continuous functions of order $\alpha-\frac{1}{p}$ in the interval $[a,b]$.

The following inversion formulas hold:

$$I^{\alpha}_{a^+}(D^{\alpha}_{a^+}f)=f,$$

for all $f\in I^{\alpha}_{a^+}(L^p)$, and

$$D^{\alpha}_{a^+}(I^{\alpha}_{a^+}f)=f$$

for all $f\in L^1(a,b)$. Similar inversion formulas hold for the operators $I^{\alpha}_{b^-}$ and $D^{\alpha}_{b^-}$.

The following integration by parts formula holds:

$$\int_a^b (D^{\alpha}_{a^+}f)(s)g(s)ds=\int_a^b f(s)(D^{\alpha}_{b^-}g)(s)ds, \tag{1.4.6}$$

for any $f\in I^{\alpha}_{a^+}(L^p)$, $g\in I^{\alpha}_{b^-}(L^p)$ $\frac{1}{p}+\frac{1}{q}=1$. We start our tour through the different definitions of stochastic integration for fBm of Hurst index $H\in(0,1)$ with the Wiener integrals since they deal with the simplest case of deterministic integrands. We show how they can be expressed in terms of an integral with respect to the standard Brownian motion, extend their definition also to the case of stochastic integrands, and then proceed to define the stochastic integral by using the divergence operator. In both cases we need to distinguish between $H>1/2$ and $H<1/2$.

1.4.7 Integrals for fractional Brownian motion

Fix an interval $[0,T]$ and let $B^{(H)}(t)$, $t\in[0,T]$, be a fBm of Hurst index $H\in(0,1)$ on the probability space $(\Omega,\mathcal{F}^{(H)},\mathcal{F}^{(H)}_t,\mathbb{P}^H)$ endowed with the natural filtration $\mathcal{F}^{(H)}_t$ and the law $\mathbb{P}^H$ of $B^{(H)}$ (for a construction of the measure $\mathbb{P}^H$ we refer to section (1))

Definition 1.4.6. *An $\mathcal{F}$-adapted process ϕ on $[0,T]\times\Omega$ is elementary processes if for a partition $\psi=\{\bar{t}=0<\bar{t}_1<\ldots<\bar{t}_n=T\}$ and $(\mathcal{F}^H_{\bar{t}_i})$-measurable random variables $(\phi_{\bar{t}_i})_{i<n}$, ϕ_t satisfies*

$$\phi_t(\omega)=\sum_{i=1}^{n}\phi_i(\omega)\chi_{(\bar{t}_{i-1},\bar{t}_i]}(t),\quad for\ \ 0\le t\le T,\quad \omega\in\Omega.$$

The Itô integral of the simple process ϕ is defined as

$$I_H(\phi) = \int_0^T \phi_l(s) dB_l^H(s) = \sum_{i=1}^n \phi_l(\bar{t}_i)(B_l^H(\bar{t}_i) - B_l^H(\bar{t}_{i-1})), \qquad (1.4.7)$$

whenever $\phi_{\bar{t}_i} \in L^2(\mathcal{F}_{\bar{t}_i}^H)$ for all $i \leq n$.

If $H > \frac{1}{2}$

It is easy to see that the covariance of fBm can be written as

$$R_H(t,s) = \alpha_H \int_0^t \int_0^s |r-u|^{2H-2} du dr, \qquad (1.4.8)$$

where $\alpha_H = H(2H-1)$. Formula (1.4.8) implies that

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \alpha_H \int_0^T \int_0^T |r-u|^{2H-2} \varphi_r \psi_u du dr, \qquad (1.4.9)$$

for any pair of step functions φ and ψ in $\mathcal{E}$.

We can write

$$|r-u|^{2H-2} = \frac{(ru)^{H-\frac{1}{2}}}{\beta(2-2H, H-\frac{1}{2})} \times \int_0^{r \wedge u} v^{1-2H}(r-v)^{H-\frac{3}{2}}(u-v)^{H-\frac{3}{2}} dv, \qquad (1.4.10)$$

where β denotes the Beta function. Let us show Equation (1.4.10). Suppose $r > u$. By means of the change of variables $z = \frac{r-v}{u-v}$ and $x = \frac{r}{uz}$, we obtain

$$\begin{aligned} \int_0^u v^{1-2H}(r-v)^{H-\frac{3}{2}}(u-v)^{H-\frac{3}{2}} dv &= (r-u)^{2H-2} \int_{\frac{r}{u}}^{\infty} (zu-r)^{1-2H} z^{H-\frac{3}{2}} dz \\ &= (ru)^{\frac{1}{2}-H}(r-u)^{2H-2} \int_0^1 (1-x)^{1-2H} x^{H-\frac{3}{2}} dx \\ &= \beta(2-2H, H-\frac{1}{2})(ru)^{\frac{1}{2}-H}(r-u)^{2H-2}. \end{aligned}$$

Consider the square integrable kernel

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du, \qquad (1.4.11)$$

where $c_H = \left[\frac{H(2H-1)}{\beta(2-2H,H-\frac{1}{2})}\right]^{1/2}$ and $t > s$.

Taking into account formulas (1.4.8) and (1.4.10) we deduce that this kernel verifies

$$\begin{aligned}
\int_0^{t\wedge s} K_H(t,u)K_H(s,u)du &= c_H^2 \int_0^{t\wedge s} \Big(\int_u^t (y-u)^{H-\frac{3}{2}} y^{H-\frac{1}{2}} dy \Big) \\
&\quad \times \Big(\int_u^s (z-u)^{H-\frac{3}{2}} z^{H-\frac{1}{2}} dz \Big) u^{1-2H} du \\
&= c_H^2 \beta(2-2H, H-\frac{1}{2}) \int_0^t \int_0^s |y-z|^{2H-2} dz dy \\
&= R_H(t,s). \qquad (1.4.12)
\end{aligned}$$

and the latter sum coincides with the usual Rieman-Stieltjes sum. A question arises: in which sense can we consider formula 1.4.7 as the extension of the sum 1.4.7 Note, that for a step function. It is known that $B^H(t)$ with $H > \frac{1}{2}$ admits the following Volterra representation

$$B^H(t) = \int_0^t K_H(t,s) dB(s) \qquad (1.4.13)$$

where B is a standard Brownian motion and the Volterra kernel $K(t,s)$ is given by

$$K_H(t,s) = c_H \int_s^t (u-s)^{H-\frac{3}{2}} \left(\frac{u}{s}\right)^{H-\frac{1}{2}} du, \quad t \geq s,$$

where $c_H = \sqrt{\frac{H(2H-1)}{\beta(2H-2,H-\frac{1}{2})}}$ and $\beta(.,.)$ denotes the Beta function. We put $K(t,s) = 0$ if $t \leq s$.

$$\frac{\partial K_H}{\partial t}(t,s) = c_H \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{3}{2}}.$$

We denote by $\mathcal{E}$ the set of step functions on $[0,T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle \chi_{[0,t]}, \chi_{[0,s]} \rangle_{\mathcal{H}} = R_H(t,s).$$

Consider the linear operator K_H^* from $\mathcal{E}$ to $L^2([0,T])$ defined by,

$$(K_H^* \phi)(t) = \int_s^T \phi(t) \frac{\partial K_H}{\partial t}(t,s) dt.$$

Notice that,

$$(K_H^* \chi_{[0,t]})(s) = K_H(t,s) \chi_{[0,t]}(s).$$

The operator K_H^* is an isometry between $\mathcal{E}$ and $L^2([0,T])$ that can be extended to the Hilbert space $\mathcal{H}$. In fact, for any $s, t \in [0,T]$.
Then

$$\langle K_H^* \chi_{[0,t]}, K_H^* \chi_{[0,t]} \rangle_{L^2([0,T])} = \langle \chi_{[0,t]}, \chi_{[0,s]} \rangle_{\mathcal{H}} = R_H(t,s).$$

In addition, for any $\phi \in \mathcal{H}$,

$$\int_0^T \phi_l(s) dB_l^H(s) = \int_0^T (K_H^* \phi_l)(s) dB_l(s),$$

if and only if $K_H^* \phi_l \in L^2([0,T])$. Next we are interested in considering an fBm with values in a Hilbert space .
Consider the process $B_l = \{B_l(t), t \in [0,T]\}$ defined by

$$B_l(t) = B_l^H((K_H^*)^{-1} \chi_{[0,t]})$$

Indeed, for any $s, t \in [0,T]$ we have

$$\begin{aligned}
\mathbb{E}(B_l(t) B_l(s)) &= E\Big(B_l^H((K_H^*)^{-1} \chi_{[0,t]}) B_l^H((K_H^*)^{-1} \chi_{[0,s]})\Big) \\
&= \langle (K_H^*)^{-1} \chi_{[0,t]}), (K_H^*)^{-1} \chi_{[0,s]}) \rangle_{\mathcal{H}} \\
&= \langle \chi_{[0,t]}, \chi_{[0,s]} \rangle_{L^2[0,T]} \\
&= s \wedge t.
\end{aligned}$$

Let $(X, \langle \cdot, \cdot \rangle, |\cdot|_X)$, $(Y, \langle \cdot, \cdot \rangle, |\cdot|_Y)$ be separable Hilbert spaces. Let $\mathcal{L}(Y, X)$ denote the space of all linear bounded operators from Y into X. Let $e_n, n = 1, 2, \ldots$ be a complete orthonormal basis in Y and $Q \in \mathcal{L}(Y, X)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $trQ = \sum_{n=1}^{\infty} \lambda_n < \infty$ where λ_n, $n = 1, 2, \ldots$, are non-negative real numbers. Let $(\beta_n^H)_{n \in N}$ be a sequence of two-sided one-dimensional standard fractional Brownian motions mutually independent on $(\Omega, \mathcal{F}, \mathbb{P})$. If we define the infinite dimensional fBm on Y with covariance Q as

$$B^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta_n^H(t) e_n, \tag{1.4.14}$$

then it is well defined as an Y-valued Q-cylindrical fractional Brownian motion (see [165]). In order to define Wiener integrals with respect to a $Q-fBm$, we introduce the space $L_Q^0 := L_Q^0(Y, X)$ of all $Q-$Hilbert-Schmidt operators $\varphi : Y \longrightarrow X$. We recall that $\varphi \in L(Y, X)$ is called a $Q-$Hilbert-Schmidt operator, if

$$\|\varphi\|_{L_Q^0}^2 = \|\varphi Q^{1/2}\|_{HS}^2 = tr(\varphi Q \varphi^*) < \infty.$$

the following useful result holds

Lemma 1.4.5. *[165] There exists a positive constant $c_1(H)$ such that for any $\phi \in L^{1/H}([0, T])$ it holds*

$$H(2H-1) \int_0^T \int_0^T |\phi(y)||\phi(z)||y-z|^{2H-2} dy dz \leq c_1(H) \|\phi\|_{L^{1/H}([0,T])}^2. \tag{1.4.15}$$

Definition 1.4.7. *Let $\phi(s), s \in [0, T]$, be a function with values in $L_Q^0(Y, X)$. The Wiener integral of ϕ with respect to fBm given by* (1.4.14) *is defined by*

$$\begin{aligned} \int_0^T \phi(s) dB^H(s) &= \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} \phi(s) e_n d\beta_n^H \\ &= \sum_{n=1}^{\infty} \int_0^T \sqrt{\lambda_n} K_H^*(\phi e_n)(s) d\beta_n. \end{aligned} \tag{1.4.16}$$

Notice that if

$$\sum_{n=1}^{\infty} \|\phi Q^{1/2} e_n\|_{L^{1/H}([0,T];X)} < \infty, \tag{1.4.17}$$

the next result ensures the convergence of the series in the previous definition. It can be proved by similar arguments to those used to prove Lemma 2 in Caraballo *et al.* [44] and Lemma 2.1 in Blouhi *et al* [75].

Lemma 1.4.6. *For any $\phi : [0,T] \to L_Q^0(Y,X)$ such that (1.4.17) holds, and for any $\alpha, \beta \in [0,T]$ with $\alpha > \beta$,*

$$E\left|\int_\alpha^\beta \phi(s)dB^H(s)\right|_X^2 \le \overline{C}\sum_{n=1}^\infty \int_\alpha^\beta \left|\phi(s)Q^{1/2}e_n\right|_X^2 ds. \tag{1.4.18}$$

where $\overline{C} = c_2(H)H(2H-1)(\alpha-\beta)^{2H-1}$ and $c_2(H)$ is a constant depending on H. If, in addition,

$$\sum_{n=1}^\infty |\phi Q^{1/2}e_n|_X \text{ is uniformly convergent for } t \in [0,T],$$

then,

$$\mathbb{E}\left|\int_\alpha^\beta \phi(s)dB^H(s)\right|_X^2 \le \overline{C}\int_\alpha^\beta \|\phi(s)\|_{L_Q^0}^2 ds. \tag{1.4.19}$$

Corollary 1.4.7. *Let $(\mathcal{F}_t)_{t\ge 0}$ be a right-continuous complete σ-algebras filtration generated by a sequence of fBm B_l^H,l=1,2,..., mutually independent, and consider another sequence of $(\mathcal{F}_t)$-adapted processes ϕ_l. Denote*

$$M^k(t) = \sum_{l=1}^k \int_0^t \phi_l(y)dB_l^H(y),\ k \in \mathbb{N},$$

and assume that

$$\sum_{l=1}^\infty \int_0^T\int_0^T \|\phi_l(y)\|_{L_Q^0}\|\phi_l(z)\|_{L_Q^0}|y-z|^{2H-2}dydz < \infty,$$

for all $t \ge 0$. Then, M^k is a convergent in $L^2(\Omega, \mathcal{F}_t, \mathbb{P}; X)$

Proof. Let $M^k(t) = \Big(\sum_{l=1}^k \int_0^t \phi_l(y)dB_l^H(y)\Big)$ for any $t \in [0,T]$. By Lemma 1.4.6, we have that

$$\begin{aligned}
\mathbb{E}|M^n(t) - M^m(t)|_X^2 &= \mathbb{E}\left|\sum_{l=1}^n \int_0^t \phi_l(y)dB_l^H(y) - \sum_{l=1}^m \int_0^t \phi_l(y)dB_l^H(y)\right|_X^2 \\
&= \mathbb{E}\left|\sum_{l=n\wedge m+1}^{n\vee m} \int_0^t \phi_l(y)dB_l^H(y)\right|_X^2 \\
&\le c_3(H)H(2H-1)T^{2H-1}\sum_{l=n\wedge m+1}^{n\vee m}\int_0^T \|\phi_l(y)\|_{L_Q^0}^2 dy \to 0
\end{aligned}$$

as $n, m \to \infty$ where $c_3(H) > 0$. Consequently $M^n(\cdot)$ is a Cauchy sequence with respect to the norm $\sup_{0 \le t \le T} E|\cdot|_X^2$ and the limit is M. Then we can conclude

$$\lim_{n\to\infty} \sup_{0\le t\le T} E|M^n(t) - M(t)|_X^2 = 0, \tag{1.4.20}$$

and observe that

$$\sum_{l=1}^{\infty} \int_0^T \int_0^T \|\phi_l(y)\|_{L_Q^0} \|\phi_l(z)\|_{L_Q^0} |y-z|^{2H-2} dydz < \infty.$$

Then M in $L^2(\Omega, \mathcal{F}_t, \mathbb{P}; X)$. □

The following result is one of the elementary properties of square-integrable stochastic processes.

Lemma 1.4.8. *(Itô Isometry for Elementary Processes) Let $(\phi_l)_{l\in\mathbb{N}}$ be a sequence of elementary processes. If*

$$\sum_{l=1}^{\infty} \int_0^T \int_0^T \|\phi_l(y)\|_{L_Q^0} \|\phi_l(z)\|_{L_Q^0} |y-z|^{2H-2} dydz < \infty,$$

then

$$\mathbb{E}\left|\sum_{l=1}^{\infty} \int_0^T \phi_l(y) dB_l^H(y)\right|_X^2 \le H(2H-1) \sum_{l=1}^{\infty} \int_0^T \int_0^T \|\phi_l(y)\|_{L_Q^0} \|\phi_l(z)\|_{L_Q^0} |y-z|^{2H-2} dydz. \tag{1.4.21}$$

Proof. Let $M^k = \sum_{l=1}^{k} \int_0^T \phi_l(y) dB_l^H(y)$. We will first prove by induction that

$$\begin{aligned} &\mathbb{E}\left|\sum_{l=1}^{k} \int_0^T \phi_l(y) dB_l^H(y)\right|^2 \\ &\quad \le H(2H-1) \sum_{l=1}^{k} \int_0^T \int_0^T \|\phi_l(y)\|_{L_Q^0} \|\phi_l(z)\|_{L_Q^0} |y-z|^{2H-2} dydz, \end{aligned} \tag{1.4.22}$$

holds for all $k \in \mathbb{N}$.

Observe that if $k = 1$ we have

$$\begin{aligned}
\mathbb{E}\left|M^1\right|_X^2 &= E\left|\int_0^T \phi_1(y)dB_1^H(y)\right|_X^2 \\
&= \mathbb{E}\sum_{n=1}^{\infty}\left|\int_0^T \phi_1(y)Q^{\frac{1}{2}}e_n d\beta_{1,n}^H(y)\right|_X^2 \\
&= \mathbb{E}\sum_{n=1}^{\infty}\left\langle \int_0^T \phi_1(y)Q^{\frac{1}{2}}e_n d\beta_{1,n}^H(y), \int_0^T \phi_1(y)Q^{\frac{1}{2}}e_n d\beta_{1,n}^H(y)\right\rangle \\
&= \mathbb{E}\sum_{n=1}^{\infty}\left\langle \int_0^T (K_H^*(\phi Q^{\frac{1}{2}}e_n))(s)d\beta_{1,n}(s), \int_0^T (K_H^*(\phi Q^{\frac{1}{2}}e_n))(s)d\beta_{1,n}(s)\right\rangle.
\end{aligned}$$

Then

$$\mathbb{E}\left|\int_0^T \phi_1(y)dB_1^H(y)\right|_X^2 \leq H(2H-1)\int_0^T\int_0^T \|\phi_1(y)\|_{L_Q^0}\|\phi_1(z)\|_{L_Q^0}|y-z|^{2H-2}dydz. \tag{1.4.23}$$

Therefore the result holds for $k = 1$.

Suppose now that (**??**) holds for a fixed integer number $k > 1$, and let us prove it for $k+1$. Indeed, since $(\phi_l)_{l=1}^{k+1}$ is a set of elementary stochastic process, and on account of the induction hypothesis we have

$$\begin{aligned}
\mathbb{E}\left|M^{k+1}\right|_X^2 &= E\left|\sum_{l=1}^{k+1}\int_0^T \phi_l(y)dB_l^H(y)\right|_X^2 \\
&= \mathbb{E}\left|\sum_{l=1}^{k}\int_0^T \phi_l(y)dB_l^H(y)\right|_X^2 + \mathbb{E}\left|\int_0^T \phi_{k+1}(y)dB_{k+1}^H(y)\right|_X^2 \\
&+ 2E\Big\langle \sum_{l=1}^{k}\int_0^T \phi_l(y)dB_l^H(y), \int_0^T \phi_{k+1}(y)dB_{k+1}^H(y)\Big\rangle \\
&\leq \sum_{l=1}^{k} H(2H-1)\int_0^T\int_0^T \|\phi_l(y)\|_{L_Q^0}\|\phi_l(z)\|_{L_Q^0}|y-z|^{2H-2}dydz \\
&+ H(2H-1)\int_0^T\int_0^T \|\phi_{k+1}(y)\|_{L_Q^0}\|\phi_{k+1}(z)\|_{L_Q^0}|y-z|^{2H-2}dydz \\
&+ 2\mathbb{E}\Big\langle \sum_{l=1}^{k}\int_0^T \phi_l(y)dB_l^H(y), \int_0^T \phi_{k+1}(y)dB_{k+1}^H(y)\Big\rangle \\
&= \sum_{l=1}^{k} H(2H-1)\int_0^T\int_0^T \|\phi_l(y)\|_{L_Q^0}\|\phi_l(z)\|_{L_Q^0}|y-z|^{2H-2}dydz \\
&+ H(2H-1)\int_0^T\int_0^T \|\phi_{k+1}(y)\|_{L_Q^0}\|\phi_{k+1}(z)\|_{L_Q^0}|y-z|^{2H-2}dydz \\
&+ 2\mathbb{E}\Big\langle \sum_{l=1}^{k}\sum_{k=1}^{n}\phi_k(B_l^H(\bar{t}_k)^H - B_l^H(\bar{t}_{k-1})), \sum_{i=1}^{n}\phi_i(B_{k+1}^H(\bar{t}_i) - B_{k+1}^H(\bar{t}_{i-1}))\Big\rangle.
\end{aligned}$$

Thanks to (**??**) and the fact that $(B_l^H)_{l=1}^{k+1}$ is a set of independent standard fractional Brownian motions, it follows

$$\mathbb{E}\left|\sum_{l=1}^{k+1}\int_0^T \phi_l(y)dB_l^H(y)\right|_X^2 \leq \sum_{l=1}^{k+1} H(2H-1)\int_0^T\int_0^T \|\phi_l(y)\|_{L_Q^0}\|\phi_l(z)\|_{L_Q^0}|y-z|^{2H-2}dydz,$$

and hence the formula is true for $k+1$ as we wished. From Corollary 1.4.7, we know that M^k is convergent in $L^2(\Omega,\mathcal{F}_t,\mathbb{P};X)$, and thanks to the continuity of the norm,

$$\begin{aligned} E\left|\lim_{k\to\infty} M^k\right|_X^2 &= \lim_{k\to\infty} E\left|\sum_{l=1}^{k}\int_0^T \phi_l(y)dB_l^H(y)\right|_X^2 \\ &\leq \sum_{l=1}^{\infty} H(2H-1)\int_0^T\int_0^T \|\phi_l(y)\|_{L_Q^0}\|\phi_l(z)\|_{L_Q^0}|y-z|^{2H-2}dydz, \end{aligned}$$

and, consequently,

$$\mathbb{E}\left|\sum_{l=1}^{\infty}\int_0^T \phi_l(y)dB_l^H(y)\right|_X^2 \leq \sum_{l=1}^{\infty} H(2H-1)\int_0^T\int_0^T \|\phi_l(y)\|_{L_Q^0}\|\phi_l(z)\|_{L_Q^0}|y-z|^{2H-2}dydz.$$

□

For $H < 1/2$, the kernel

$$K_H(t,s) = b_H\Big(\Big(\frac{t}{s}\Big)^{H-1/2}(t-s)^{H-1/2}$$

$$-(H-1/2)s^{1/2-H}\int_s^t (u-s)^{H-1/2}u^{H-3/2}du\Big) \qquad (1.4.24)$$

with

$$b_H = \sqrt{\frac{2H}{(1-2H)\beta(1-2H, H+1/2)}}$$

and $t > s$, satisfies

$$R_H(t,s) = \int_0^{t\wedge s} K_H(t,u)K_H(s,u)du.$$

Consider the linear operator K_H^* from the space $\mathcal{E}$ of step functions on $[0,T]$ to $L^2([0,T])$ defined by

$$(K_H^*\phi)(s) := K(T,s)\phi(s)$$

$$+\int_s^T (\phi(t)-\phi(s))\frac{\partial K_H}{\partial t}(t,s)dt. \qquad (1.4.25)$$

Then (1.4.25) evaluated for $\phi = \chi_{[0,t]}$ gives,

$$(K_H^*\chi_{[0,t]})(s) = K_H(t,s)\chi_{[0,t]}(s),$$

and for $H < 1/2$, we have

$$\mathcal{H} = (K_H^*)^{-1}(L^2([0,T]) = I_-^{1/2-H}(L^2([0,T])).$$

This representation of $\mathcal{H}$ guarantees in addition that the inner product space $\mathcal{H}$ is complete when endowed with the inner product,

$$\langle f, g\rangle = \int_0^T K_H^* f(s) K_H^* g(s) ds.$$

For $H = 1/2$, we have $K_{1/2}(t,s) = \chi_{[0,t]}(s)$.

Lemma 1.4.9. *Let*

$$I_-^{1/2-H} f(u) = c_H \int_u^\infty (t-u)^{H-3/2} f(t) dt,$$

where $c_H = \sqrt{H(2H-1)\Gamma(3/2-H)/\Gamma(H-1/2)\Gamma(2-2H)}$,*and* Γ *denotes the gamma function. Then* $I_-^{1/2-H}$ *is an isometry from* $\overline{L([0,T],\langle,\rangle)} = \mathcal{H} = L^2_\phi(\mathbb{R}) \subseteq L^2_\phi(\mathbb{R})$ *to* $L^2(\mathbb{R})$.

Proof. By a limiting argument, we may assume that f and g are continuous with compact support. By definition,

$$\langle I_-^{1/2-H}(f), I_-^{1/2-H}(g)\rangle_{L^2(\mathbb{R})}.$$

This,we have

$$\begin{aligned}
&= \langle I_-^{1/2-H}(f), I_-^{1/2-H}(g)\rangle_{L^2(\mathbb{R})} \\
&= c_H^2 \int_{\mathbb{R}} \Big\{ \int_u^\infty (s-u)^{H-3/2} f(s) ds \int_u^\infty (s-u)^{H-3/2} g(t) dt \Big\} du \\
&= c_H^2 \int_{\mathbb{R}^2} f(s) g(t) \Big\{ \int_{-\infty}^{s\wedge t} (s-u)^{H-3/2} (t-u)^{H-3/2} du \Big\} ds dt \\
&= H(2H-1) \int_{\mathbb{R}^2} f(s) g(t) \mid s-t \mid^{2H-2},
\end{aligned}$$

where we have used the identity,

$$c_H^2 \int_{-\infty}^{s\wedge t} (s-u)^{H-3/2} (t-u)^{H-3/2} du = H(2H-1) \mid s-t \mid^{2H-2} .$$

□

Chapter 2

Some Elements of Functional Analysis

In this chapter, we introduce notations, definitions, lemmas and fixed point theorems which are used throughout this thesis

Let $J := [a, b]$ be an interval of $\mathbb{R}$. Let $(E, |\cdot|)$ be a real Banach space. $C(J, E)$ is the Banach space of all continuous functions from J into E with the norm

$$\|y\|_\infty = \sup_{t \in J} |y(t)|$$

$L^1([a, b], E)$ denotes the Banach space of measurable functions $y : [a, b] \longrightarrow E$ is Bochner integrable normed by

$$\|y\|_{L^1} = \int_a^b |y(t)| dt$$

if and only if $|y|$ is Lebesgue integrable. (For properties of the Bochner integral, see for instance, Yosida [62]).

We need the following definitions in the sequel.

Definition 2.0.8. *A map $f : J \times E \to E$ is said to be L^p-Carathéodory if*

(i) $t \mapsto f(t, v)$ *is measurable for each $v \in E$;*

(ii) $v \mapsto f(t, v)$ *is continuous for almost all $t \in J$;*

(iii) *for each $q > 0$, there exists $\alpha_q \in L^p(J, \mathbb{R}^+)$ such that*

$$\|f(t, v)\|^p \leq \alpha_q(t), \text{ for all } \|v\|_E^p \leq q \text{ and for a.e. } t \in J.$$

Definition 2.0.9. *A map f is said compact if the image is relatively compact. f is said completely continuous if is continuous and the image of every bounded is relatively compact.*

Lemma 2.0.10. *[30] Let $u,\, g\colon\, J \to \mathbb{R}$ be positive real continuous functions. Assume there exist $c > 0$ and a continuous nondecreasing function $h\colon\, \mathbb{R} \to (0, +\infty)$ such that*

$$u(t) \le c + \int_a^t g(s)h(u(s))\, ds, \quad \forall\, t \in J.$$

Then

$$u(t) \le H^{-1}\left(\int_a^t g(s)\, ds\right), \quad \forall\, t \in J$$

provided

$$\int_c^{+\infty} \frac{dy}{h(y)} > \int_a^b g(s)\, ds.$$

Here H^{-1} refers to the inverse of the function $H(u) = \int_c^u \frac{dy}{h(y)}$ for $u > c$.

2.1 Generalized metric and Banach spaces

In this section we define vector metric spaces and generalized Banach spaces and prove some properties. If, $x, y \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n)$, by $x \le y$ we mean $x_i \le y_i$ for all $i = 1, \ldots, n$. Also $|x| = (|x_1|, \ldots, |x_n|)$ and $\max(x, y) = \max(\max(x_1, y_1), \ldots, \max(x_n, y_n))$. If $c \in \mathbb{R}$, then $x \le c$ means $x_i \le c$ for each $i = 1, \ldots, n$. For $x \in \mathbb{R}^n,\ (x)_i = x_i,\ i = 1, \ldots, n$.

Definition 2.1.1. *Let X be a nonempty set. By a vector-valued metric on X we mean a map $d : X \times X \to \mathbb{R}^n$ with he following properties:*

(i) $d(u, v) \ge 0$ *for all* $u, v \in X$*; if* $d(u, v) = 0$ *then* $u = v$

(ii) $d(u, v) = d(v, u)$ *for all* $u, v \in X$

(iii) $d(u, v) \le d(u, w) + d(w, v)$ *for all* $u, v, w \in X$.

Note that for any $i \in \{1, \ldots, n\}$ $(d(u, v))_i = d_i(u, v)$ is a metric space in X.

We call the pair (X, d) generalized metric space .For $r = (r_1, r_2, \dots, r_n) \in \mathbb{R}^n_+$,we will denote by

$$B(x_0, r) = \{x \in X : \ d(x_0, x) < r\}$$

the open ball centrad in x_0 with radius r and

$$\overline{B(x_0, r)} = \{x \in X : d(x_0, x) \leq r\}$$

the closed ball centered in x_0 with radius r. We mention that for generalized metric space, the notation of open subset, closed set, convergence, Cauchy sequence and completeness are similar to those in usual metric spaces. If, $x, y \in \mathbb{R}^n$, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for all $i = 1, \dots, n$. Also $|x| = (|x_1|, \dots, |x_n|)$ and $\max(x, y) = \max(\max(x_1, y_1), \dots, \max(x_n, y_n))$. If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c$ for each $i = 1, \dots, n$.

Definition 2.1.2. *A generalized metric space (X, d), where*

$$d(x, y) := \begin{pmatrix} d_1(x, y) \\ \cdots \\ d_n(x, y) \end{pmatrix}.$$

is complete if for every $i = 1, \dots, n$, (X, d_i) is complete metric space.

Theorem 2.1.1. *Let (X, d) be a generalized metric space. For any compact set $A \subset X$ and for any closed set $B \subset X$ that is disjoint from A, there exists a continuous functions $f : X \to [0, 1]$, $g : X \to [0, 1] \times [0, 1] \times \dots [0, 1] := [0, 1]^n$ such that*

i) $f(x) = 0$ *for all* $x \in B$,

ii) $f(x) = 1$ *for all* $x \in A$.

iii) $g(x) = (1, \dots, 1)$ *for all* $x \in B$,

iv) $g(x) = (0, \dots, 0)$ *for all* $x \in A$.

Proof. Note that $d_i(x, B) = 0$ for any $x \in B$ and $d_i(x, A) = 0$ and $d_i(x, A) > 0$ for any $x \in A$. Thus we obtain i) and ii). Let $f : X \to [0, 1]$ be defined by

$$f(x) = \frac{\sum_{i=1}^{n} d_i(x, B)}{\sum_{i=1}^{n} d_i(x, A) + \sum_{i=1}^{n} d_i(x, B)}, \ x \in X.$$

To prove that f is continuous, let $(x_m)_{m\in\mathbb{N}}$ be a sequence convergent to $x \in X$. Then

$$\begin{aligned}
|f(x_m) - f(x)| &= \left| \frac{\sum_{i=1}^n d_i(x_m, B)}{\sum_{i=1}^n d_i(x_m, A) + \sum_{i=1}^n d_i(x_m, B)} - \frac{\sum_{i=1}^n d_i(x, B)}{\sum_{i=1}^n d_i(x, A) + \sum_{i=1}^n d_i(x, B)} \right| \\
&= \left| \frac{\sum_{i=1}^n d_i(x_m, B) \sum_{i=1}^n d_i(x, A) - \sum_{i=1}^n d_i(x_m, A) \sum_{i=1}^n d_i(x, B)}{(\sum_{i=1}^n d_i(x, A) + \sum_{i=1}^n d_i(x, B))(\sum_{i=1}^n d_i(x_m, A) + \sum_{i=1}^n d_i(x_m, B))} \right| \\
&\le \frac{\sum_{i=1}^n d_i(x, A) \sum_{i=1}^n |d_i(x_m, B) - d_i(x, B)|}{(\sum_{i=1}^n d_i(x, A) + \sum_{i=1}^n d_i(x, B))(\sum_{i=1}^n d_i(x_m, A) + \sum_{i=1}^n d_i(x_m, B))} \\
&\quad + \frac{\sum_{i=1}^n d_i(x, B) \sum_{i=1}^n |d_i(x_m, A) - d_i(x, A)|}{(\sum_{i=1}^n d_i(x, A) + \sum_{i=1}^n d_i(x, B))(\sum_{i=1}^n d_i(x_m, A) + \sum_{i=1}^n d_i(x_m, B))}.
\end{aligned}$$

Since for each $i = 1, \ldots, m$ we have

$$|d_i(x_m, B) - d_i(x, B)| \to 0, \ \ |d_i(x_m, A) - d_i(x, A)| \to 0 \text{ as } m \to \infty.$$

Therefore

$$\frac{\sum_{i=1}^n d_i(x, A) \sum_{i=1}^n |d_i(x_m, B) - d_i(x, B)|}{(\sum_{i=1}^n d_i(x, A) + \sum_{i=1}^n d_i(x, B))(\sum_{i=1}^n d_i(x_m, A) + \sum_{i=1}^n d_i(x_m, B))} \to 0 \text{ as } m \to \infty$$

and

$$\frac{\sum_{i=1}^{n} d_i(x,A) \sum_{i=1}^{n} |d_i(x_m,A) - d_i(x,A)|}{(\sum_{i=1}^{n} d_i(x,A) + \sum_{i=1}^{n} d_i(x,B))(\sum_{i=1}^{n} d_i(x_m,A) + \sum_{i=1}^{n} d_i(x_m,B))} \to 0 \text{ as } m \to \infty.$$

Thus, we get

$$|f(x_m) - f(x)| \to 0 \text{ as } m \to \infty.$$

We can easily prove that the following function $g : X \to [0,1]^n$ defined by

$$g(x) = \begin{pmatrix} \frac{d_1(x,A)}{d_1(x,B)+d_1(x,B)} \\ \dots \\ \frac{d_n(x,A)}{d_n(x,B)+d_n(x,B)} \end{pmatrix}, \quad x \in X$$

is continuous function and satisfied iii) and iv). □

Definition 2.1.3. *Let E be a vector space on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. By a vector-valued norm on E we mean a map $\|\cdot\| : E \to \mathbb{R}^n_+$ with the following properties:*

(i) $\|x\| \geq 0$ *for all* $x \in E$ *; if* $\|x\| = 0$ *then* $x = 0$

(ii) $\|\lambda x\| = |\lambda| \|x\|$ *for all* $x \in E$ *and* $\lambda \in \mathbb{K}$

(iii) $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in E$*.*

The pair $(E, \|\cdot\|)$ is called a generalized normed space . If the generalized metric generated by $\|\cdot\|$ (i.e $d(x,y) = \|x - y\|$) is complete then the space $(E, \|\cdot\|)$ is called a generalized Banach space, where

$$\|x - y\| = \begin{pmatrix} \|x-y\|_1 \\ \dots \\ \|x-y\|_n \end{pmatrix}.$$

Notice that $\|\cdot\|$ is a generalized Banach space on E if and only if $\|\cdot\|_i,\ i = 1,\dots,n$ are norms on E.

Definition 2.1.4. *Let $(E, \|\cdot\|)$ be a generalized Banach space and $U \subset E$ open subset such that $0 \in U$. The function $p_U : E \to \mathbb{R}_+$ defined by*

$$p_U(x) = \inf\{\alpha > 0 : x \in \alpha U\},$$

is called the Minkowski functional of U.

Lemma 2.1.2. *Let $(E, \|\cdot\|)$ be a generalized Banach space and $U \subset E$ open subset such that $0 \in U$. Then*

i) If $\lambda \geq 0$, then $p_U(\lambda x) = \lambda p_U(x)$.

ii) If U is convex we have

a)$p_U(x+y) \leq p_U(x) + p_U(y)$, for every $x, y \in U$.

b)$\{x \in E : p_U(x) < 1\} \subset U \subset \{x \in E : p_U(x) \leq 1\}$.

c) If U is symmetric; then $p_U(x) = p_U(-x)$.

iii) p_U is continuous.

Proof. $i)$ Let $x \in E$ be arbitrary and $\lambda \geq 0$. We have

$$\begin{aligned} p_U(\lambda x) &= \inf\{\alpha > 0 : \ \lambda x \in \alpha U\} \\ &= \inf\{\alpha > 0 : \ x \in \lambda^{-1}\alpha U\} \\ &= \inf\{\lambda\beta > 0 : \ x \in \beta U\} \\ &= \lambda \inf\{\beta > 0 : \ x \in \beta U\} \\ &= \lambda p_U(x). \end{aligned}$$

$ii) - a)$ Let $\alpha_1 > 0$ and $\alpha_2 > 0$ such that

$$x \in \alpha_1 U \text{ and } y \in \alpha_2 U,$$

then

$$x + y \in \alpha_1 U + \alpha_2 U \Rightarrow \frac{x+y}{\alpha_1 + \alpha_2} \in \frac{\alpha_1}{\alpha_1 + \alpha_2} U + \frac{\alpha_2}{\alpha_1 + \alpha_2} U$$

Hence

$$x + y \in (\alpha_1 + \alpha_2)U. \tag{2.1.1}$$

For every $\epsilon > 0$ there exist $\alpha_\epsilon > 0,\ \beta_\epsilon > 0$ such that

$$\alpha_\epsilon \leq p_U(x) + \epsilon \text{ and } \beta_\epsilon \leq p_U(y) + \epsilon$$

From (2.1.1) we have et

$$p_U(x+y) \leq p_U(x) + p_U(y) + 2\epsilon \Rightarrow p_U(x+y) \leq p_U(x) + p_U(y) + 2\epsilon.$$

Letting $\epsilon \to 0$ we obtain

$$p_U(x+y) \leq p_U(x) + p_U(y) \text{ for every } x, y \in U.$$

$b)$ Let $x \in E$ such that $p_U(x) < 1$, then there exists $\alpha \in (0,1)$ such that

$$p_U(x) \leq \alpha < 1 \text{ and } x \in \alpha U \Rightarrow x = \alpha a + (1-\alpha)0 \in U.$$

Therefore

$$\{x \in E : p_U(x) < 1\} \subset U.$$

For $x \in U$ we have

$$x = \alpha x \in U,\ \alpha = 1 \Rightarrow p_U(x) \leq 1.$$

Then

$$\{x \in E : p_U(x) < 1\} \subset U \subset \{x \in E : p_U(x) \leq 1\}.$$

iii) Since $0 \in U$ then there exist $r > 0$ such that

$$B(0,r) = \{x \in E : \|x\| < r_*\} \subset U,$$

where

$$\|x\| = \begin{pmatrix} \|x\|_1 \\ \dots \\ \|x\|_n \end{pmatrix} \text{ and } r_* = \begin{pmatrix} r \\ \dots \\ r \end{pmatrix}.$$

Given $\epsilon > 0$, then $x + \epsilon B(0, r_*)$ is a neighborhood of x. For every $y \in x + \epsilon B(0, r_*)$ we have

$$\frac{x-y}{\epsilon} \in B(0, r_*) \Rightarrow p_U\left(\frac{x-y}{\epsilon}\right) \leq 1.$$

It is clear that

$$|p_U(x) - p_U(y)| \leq p_U(x-y) = \epsilon p_U\left(\frac{x-y}{\epsilon}\right) \leq \epsilon.$$

Hence p_U is continuous. □

Remark 2.1.1. *In generalized metric space in the sense of Perov's, the notations of convergence sequence, Cauchy sequence, completeness, open subset and closed subset are similar to those for usual metric spaces.*

Definition 2.1.5. *Let E be a real generalized normed(or normed) space. A mapping $N : E \to E$ is called compact if T maps every bounded subset of E to a relatively compact subset in E. N is said to be completely continuous if N is continuous and compact.*

Definition 2.1.6. *A square matrix of real matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1 ,In other words ,this means that all the eigenvalues of M are in the open unit disc.*

2.2 Compactness criteria

Let $C([a, b], E)$,$C_b([a, \infty), E)$ and E be a real Banach space , $[a, b]$ be an interval.

Definition 2.2.1. *A family $\mathcal{A}$ in $C([a, b], E)$ is equicontinuous at t in $[a, b]$ if for each $\epsilon > 0$ there exists $\delta(\epsilon, t) > 0$ such that, for each $s \in [a, b]$ with $|t - s| < \delta(\epsilon, t)$, we have*

$$\|f(t) - f(s)\| < \epsilon,$$

uniformly with respect to $f \in \mathcal{A}$.

The family $\mathcal{A}$ is equicontinuous on $[a, b]$ if it is equicontinuous at each point $t \in [a, b]$ in the sense mentioned above.
The family $\mathcal{A}$ is uniformly equicontinuous on $[a, b]$ if it is equicontinuous on $[a, b]$, and $\delta(\epsilon, t)$ can be chosen independently of t in $[a, b]$.

Theorem 2.2.1. *(ArzelaAscoli [163]) A bounded subset $\mathcal{A}$ in $C([a, b], E)$ is relatively compact if and only if*

(i) $\mathcal{A}$ is equicontinuous on $[a, b]$;
there exists a dense subset D in $[a, b]$ such that, for each $t \in D$,

$$\mathcal{A}(t) = \{f(t) \quad | \quad f \in \mathcal{A}\}$$

is relatively compact in E.

The following compactness criterion for subsets of C_b is a consequence of the well-known ArzlaAscoli theorem (see Avramesu [15], Corduneanu [50], Przeradzki [131], Staikos [145])

Theorem 2.2.2. *Let $B \subset C([a,b],\mathbb{R}^n)$ a subset, assume conditions are satisfied:*

(i) for every $t \in \mathbb{R}^+$,the set $\{x(t) \mid x \in B\}$ is relatively compact,

(ii) for every $\alpha > 0$ the set B is equicontinuous on the interval $[0,\alpha]$,

(iii) for every $\epsilon > 0$ there exist $T = T(\epsilon)$ and $\delta = \delta(\epsilon)$ such that $x, y \in B$ with $\|x(T) - y(T)\| \leq \delta$, then $\|x(t) - y(t)\| \leq \epsilon$ for all $t \in [T,\infty)$. Then the set B is compact in $C_b([a,\infty),\mathbb{R}^n)$.

As a consequence, we have

Corollary 2.2.3. *Let $M \subset C_b$ be the space of functions which have limits at positive infinity. Then M is relatively compact in C_b if the following conditions hold:*

(a) M is uniformly bounded in C_b.

(b) The functions belonging to M are almost equicontinuous on $\mathbb{R}^+$, i.e., equicontinuous on every compact interval of $\mathbb{R}^+$.

(c) The functions from M are equiconvergent at ∞, that is, given $\epsilon > 0$, there corresponds $T = T(\epsilon) > 0$ such that $\|x(t) - x(\infty)\| < \epsilon$ for any $t \geq T(\epsilon)$ and $x \in M$.

Lemma 2.2.4. *[135] Let M be a square matrix of nonnegative numbers. The following assertions are equivalent:*

(i) M is convergent towards zero;

(ii) the matrix $I - M$ is non-singular and

$$(I - M)^{-1} = I + M + M^2 + \ldots + M^k + \ldots;$$

(iii) $\|\lambda\| < 1$ for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$

(iv) $(I - M)$ is non-singular and $(I - M)^{-1}$ has nonnegative elements;

Definition 2.2.2. *We say that a non-singular matrix* $A = (a_{ij})_{1\le i,j\le n} \in \mathcal{M}_{n\times n}(\mathbb{R})$ *has the absolute value property if*

$$A^{-1}|A| \le I,$$

where

$$|A| = (|a_{ij}|)_{1\le i,j\le n} \in \mathcal{M}_{n\times n}(\mathbb{R}_+).$$

Some examples of matrices convergent to zero are the following:

1) $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $\max(a, b) < 1$

2) $A = \begin{pmatrix} a & -c \\ 0 & b \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $a + b < 1,\ c < 1$

3) $A = \begin{pmatrix} a & -a \\ b & -b \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $|a - b| < 1,\ a > 1, b > 0$.

2.3 Some Properties of Set-Valued Maps

Let (X, d) be a metric space and Y be a subset of X. We denote:

$$\mathcal{P}_{cl}(X) = \{y \in \mathcal{P}(X) : y \text{ closed } \},$$

$$\mathcal{P}_{b}(X) = \{y \in \mathcal{P}(X) : y \text{ bounded } \},$$

$$\mathcal{P}_{c}(X) = \{y \in \mathcal{P}(X) : y \text{ convex } \},$$

$$\mathcal{P}_{cp}(X) = \{y \in \mathcal{P}(X) : y \text{ compact } \}.$$

Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}^n_+ \cup \{\infty\}$ defined by

$$H_d(A, B) \;:=\; \begin{pmatrix} H_{d_1}(A, B) \\ \cdots \\ H_{d_n}(A, B) \end{pmatrix}.$$

Let (X, d) be a generalized metric space with

$$d(x, y) \;:=\; \begin{pmatrix} d_1(x, y) \\ \cdots \\ d_n(x, y) \end{pmatrix}.$$

Notice that d is a generalized metric space on X if and only if $d_i,\ i = 1, \ldots, n$ are metrics on X, $H_d(A, B) = \ max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\}$, where $d(A, b) = \inf_{a \in A} d(a, b), d(a, B) = \inf_{b \in B} d(a, b)$. Then, $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space.

A multivalued map $F : X \longrightarrow \mathcal{P}(X)$ is convex (closed) valued if $F(y)$ is convex (closed) for all $y \in X$, F is bounded on bounded sets if $F(B) = \bigcup_{y \in B} F(y)$ is bounded in X for all $B \in \mathcal{P}_b(X)$. F is called upper semi-continuous (u.s.c. for short) on X if for each $y_0 \in X$ the set $F(y_0)$ is a nonempty, subset of X, and for each open set $\mathcal{U}$ of X containing $F(y_0)$, there exists an open neighborhood $\mathcal{V}$ of y_0 such that $F(\mathcal{V}) \in \mathcal{U}$. F is said to be completely continuous if $F(B)$ is relatively compact for every $B \in \mathcal{P}_b(X)$. F is quasicompact if, for each subset $A \subset X$,$F(A)$ is relatively compact .

If the multivalued map F is completely continuous with nonempty compact valued, then F is $u.s.c.$ if and only if F has a closed graph, i.e., $x_n \longrightarrow x_*$, $y_n \longrightarrow y_*$, $y_n \in F(x_n)$ imply $y_* \in F(x_*)$.

A multi-valued map $F : J \longrightarrow \mathcal{P}_{cp,c}(X)$ is said to be measurable if for each $y \in X$, the mean-square distance between y and $F(t)$ is measurable.

Definition 2.3.1. *The set-valued map $F_1, F_2 : J \times X \times X \to \mathcal{P}(X)$ is said to be L^2-Caratheodory if*

(i) *$t \mapsto F(t, v)$ is measurable for each $v \in X \times X$;*

(ii) *$v \mapsto F(t, v)$ is u.s.c. for almost all $t \in J$;*

(iii) *for each $q > 0$, there exists $h_q \in L^1(J, \mathbb{R}^+)$ such that*

$$\|F(t, v)\|^2 := \sup_{f \in F(t,v)} \|f\|^2 \leq h_q(t), \quad \text{for all } \|v\|^2 \leq q \text{ and for a.e. } t \in J.$$

Remark 2.3.1. **(a)** *For each $x \in C(J, X)$, the set $S_{F,x}$ is closed whenever F has closed values. It is convex if and only if $F(t, x(t))$ is convex for a.e. $t \in J$.*

(b) *From [155], Theorem 5.10 (see also [102] when X is finite dimensional), we know that $S_{F,x}$ is nonempty if and only if the mapping $t \longmapsto \inf\{\|v\| : v \in F(t, x(t))\}$ belongs to $L^2(J)$.*

Lemma 2.3.1. *[102] Let J be a compact interval and X be a Hilbert space. Let F be an L^2-Carathéodory multi-valued map with $S_{F,y} \neq \emptyset$. and let Γ be a linear continuous mapping from $L^2(J,X)$ to $C(J,X)$. Then, the operator*

$$\Gamma \circ S_F : C(J,X) \longrightarrow \mathcal{P}_{cp,c}(L^2(J,X)), \quad y \longmapsto (\Gamma \circ S_F)(y) = \Gamma(S_F, y),$$

is a closed graph operator in $C(J,X) \times C(J,X)$, where $S_{F,y}$ is known as the selectors set from F and given by

$$f \in S_{F,y} = \{f \in L^2([0,T],X) : f(t) \in F(t,y) \quad for\ a.e.t \in J\}.$$

We denote the graph of G to be the set

$$gr(G) = \{(x,y) \in X \times Y, \quad y \in G(x)\}$$

.

Lemma 2.3.2. *[55] If $G : X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $gr(G)$ is a closed subset of $X \times Y$. Conversely, if G is locally compact and has nonempty compact values and a closed graph, then it is u.s.c.*

Lemma 2.3.3. *[57] If $G : X \to \mathcal{P}_{cp}(Y)$ is quasicompact and has a closed graph, then G is u.s.c.*

The following two results are easily deduced from the limit properties

Lemma 2.3.4. *(See e.g. [12], Theorem 1.4.13) If $G : X \to \mathcal{P}_{cp}(X)$ is u.s.c., then for any $x_0 \in X$,*

$$\limsup_{x \to x_0} G(x) = G(x_0)$$

Lemma 2.3.5. *(See e.g. [12], Lemma 1.1.9) If Let $(K_n)_{n \in N} \subset K \subset X$ be a sequence of subsets where K is compact in the separable Banach space X. Then*

$$\overline{co}(\limsup_{n \to \infty} K_n) = \cap_{N>0} \overline{co}(\cup_{n \geq N} K_n)$$

where $\overline{co}A$ refers to the closure of the convex hull of A.

The second one is due to Mazur, 1933:

Lemma 2.3.6. *(Mazur's Lemma, ([69] [Theorem 21.4])) Let X be a normed space and $\{x_k\}_{k\in N} \subset X$ be a sequence weakly converging to a limit $x \in X$. Then there exists a sequence of convex combinations $y_m = \sum_{k=1}^{m} \alpha_{mk} x_k$ with $\alpha_{mk} > 0$ for $k = 1, 2, \ldots, m$ and $\sum_{k=1}^{m} \alpha_{mk} = 1$, which converges strongly to x.*

Definition 2.3.2. *A sequence $(\upsilon_n)_{n\in\mathbb{N}}$is said to be semi-compact if*

(1) it is integrable bounded, i.e. there exists $q \in L^1(J, \mathbb{R})$ such that

$$|\upsilon_n|_X \leq q(t)$$

for a.e. $t \in J$ and every $n \in \mathbb{N}$,

(2) the image sequence $(\upsilon_n)_{n\in\mathbb{N}}$ is relatively compact in X for a.e. $t \in J$.

Our next result describes a basic theorem of reflexive spaces:

Theorem 2.3.7. *[39] E is reflexive if and only if $B_E = \{x \in E; ||x|| \leq 1\}$ is compact in the weak topology.*

This result is of particular importance if X is reflexive in which case (1) implies (2) in Definition 2.3.2.

Lemma 2.3.8. *Every semi-compact sequence $L^1(J, X)$ is weakly compact in $L^1(J, X)$.*

Definition 2.3.3. *A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called*

a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leq \gamma d(x, y), \quad \text{for each } x,\ y \in X,$$

b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Let $A : E \to E$ be a linear operator.

Definition 2.3.4. *The resolvent set $\Lambda(A)$ of A consists of all complex numbers λ for which the linear operator $\lambda I - A$ is invertible, i.e. $(\lambda I - A)^{-1}$ is a bounded linear operator in E: The family $R(\lambda, A) = (\lambda I - A)^{-1}$, $\lambda \in \Lambda(A)$ is called the resolvent of A. All complex numbers λ not in $\Lambda(A)$ form a set called the spectrum of A.*

2.4 Fixed point results

The classical Banach contraction principle was extended for contractive maps on spaces endowed with vector-valued metric space by Perov in 1964 [125], Perov and Kibenko [129] and Precup [129]. For a version of Schauder fixed point, see Cristescu [51]. The purpose of this section is to present the version of Schaefer's fixed point theorem and nonlinear alternative of Leary-Schauder type in generalized Banach spaces.

Theorem 2.4.1. *[125]Let (X, d) be a complete generalized metric space with $d : X \times X \longrightarrow \mathbb{R}^n$ and let $N : X \longrightarrow X$ be such that*

$$d(N(x), N(y)) \leq Md(x, y)$$

for all $x, y \in X$ and some square matrix M of nonnegative numbers. If the matrix M is convergent to zero, that is $M^k \longrightarrow 0$ as $k \longrightarrow \infty$, then N has a unique fixed point $x_* \in X$

$$d(N^k(x_0), x_*) \leq M^k(I - M)^{-1}d(N(x_0), x_0)$$

for every $x_0 \in X$ and $k \geq 1$.

Theorem 2.4.2. *[51] Let E be a generalized Banach space, $C \subset E$ be a nonempty closed convex subset of E and $N : C \to C$ be a continuous operator with relatively compact range. Then N has at least fixed point in C.*

As a consequence of Schauder fixed point theorem we present the version of Schaefer's fixed point theorem and nonlinear alternative Leary-Schauder type theorem in generalized Banach space.

Theorem 2.4.3. *Let $(E, \|\cdot\|)$ be a generalized Banach space and $N : E \to E$ is a continuous compact mapping. Moreover assume that the set*

$$\mathcal{A} = \{x \in E : x = \lambda N(x) \quad \textit{for some } \lambda \in (0, 1)\}$$

is bounded. Then N has a fixed point.

Proof. Let $K > 0$ such that

$$\sum_{i=1}^{n} \|x\|_i < nK \text{ for each } x \in \mathcal{A}.$$

Set $M_* = (nK, \dots, nK)$ and we, define $N_* : \overline{B(0, M_*)} \to \overline{B(0, M_*)}$ by

$$N_*(x) = \begin{cases} N(x) & \text{if } \sum_{i=1}^{n} \|N(x)\|_i \leq nK \\ \\ \frac{KnN(x)}{\sum_{i=1}^{n} \|N(x)\|_i} & \text{if } \sum_{i=1}^{n} \|N(x)\|_i > nK. \end{cases}$$

We show that that N_* is continuous.

Let $x \in \overline{B(0, M_*)}$ such that $\sum_{i=1}^{n} \|N(x)\|_i < nK$ then $N_*(x) = N(x)$.

If $(x_m)_{m\in\mathbb{N}} \in \overline{B(0, M_*)}$ and $\sum_{i=1}^{n} \|N(x_m)\|_i \leq M_*$, then the continuity of N implies that $\|N(x_m) - N(x)\| \to 0$ as $m \to \infty \Rightarrow \|N_*(x_n) - N_*(x)\| \to 0$ as $m \to \infty$.

Now let $x \in \overline{B(0, M_*)}$ such that $\sum_{i=1}^{n} \|N(x)\|_i > nK$. Then $N_*(x) = \frac{KnN(x)}{\sum_{i=1}^{n} \|N(x)\|_i}$.

If $(x_m)_{m\in\mathbb{N}} \in \overline{B(0, M_*)}$ and $\sum_{i=1}^{n} \|N(x_m)\|_i > nK$, then $N_*(x_m) = \frac{KnN(x_m)}{\sum_{i=1}^{n} \|N(x_m)\|_i}$.

By the continuity of N we have, for every $j = 1, \dots, n$

$$\begin{aligned} \|N_*(x_m) - N_*(x)\|_j &= \left\| \frac{KnN(x_m)}{\sum_{i=1}^{n} \|N(x_m)\|_i} - \frac{KnN(x)}{\sum_{i=1}^{n} \|N(x)\|_i} \right\|_j \\ &= \left\| \frac{KnN(x_m) \sum_{i=1}^{n} \|N(x)\|_i - KnN(x) \sum_{i=1}^{n} \|N(x_m)\|_i}{\sum_{i=1}^{n} \|N(x_m)\|_i \sum_{i=1}^{n} \|N(x)\|_i} \right\|_j \\ &\leq \frac{Kn\|N(x_m) - N(x)\|_j}{\sum_{i=1}^{n} \|N(x_m)\|_i} + \frac{Kn\|N(x)\|_j \sum_{i=1}^{n} \|N(x_m) - N(x)\|_i}{\sum_{i=1}^{n} \|N(x_m)\|_i \sum_{i=1}^{n} \|N(x)\|_i}. \end{aligned}$$

Since $\sum_{i=1}^{n} \|N(x_m)\|_i > nM$, thus $\lim_{m\to\infty} \sum_{i=1}^{n} \|N(x_m)\|_i \geq nK$, hence

$$\|N_*(x_m) - N_*(x)\| \to 0 \text{ as } m \to \infty.$$

Let $x \in \overline{B(0, M_*)}$ such that $\sum_{i=1}^{n} \|N(x)\|_i = nK$ then $N_*(x) = N(x)$.

If $(x_m)_{m\in\mathbb{N}} \in \overline{B(0, M_*)}$ and $\sum_{i=1}^{n} \|N(x_m)\|_i \leq nK$ then the continuity of N implies that

$$\|N(x_m) - N(x)\| \to 0 \text{ as } m \to \infty \Rightarrow \|N_*(x_m) - N_*(x)\| \to 0 \text{ as } m \to \infty.$$

If $(x_m)_{m\in\mathbb{N}} \in \overline{B(0, M_*)}$ and $\sum_{i=1}^{n} \|N(x_m)\|_i > M_*$ then $N_*(x_m) = \frac{KnN(x_m)}{\sum_{i=1}^{n} \|N(x_m)\|_i}$,

hence

$$\begin{aligned}
\|N_*(x_m) - N_*(x)\|_j &= \left\| \frac{KnN(x_m)}{\sum_{i=1}^{n} \|N(x_m)\|_i} - N(x) \right\|_j \\
&= \left\| \frac{KnN(x_m) - N(x) \sum_{i=1}^{n} \|N(x_m)\|_i}{\sum_{i=1}^{n} \|N(x_m)\|_i} \right\|_j \\
&\leq \frac{Kn\|N(x_m) - N(x)\|_j}{\sum_{i=1}^{n} \|N(x_m)\|_i} + \frac{\|N(x)\|_j \left| Kn - \sum_{i=1}^{n} \|N(x_m)\|_i \right|}{\sum_{i=1}^{n} \|N(x_m)\|_i}.
\end{aligned}$$

It is clear that

$$\lim_{m\to\infty} \sum_{i=1}^{n} \|N(x_m)\|_i = \sum_{i=1}^{n} \|N(x)\|_i = Kn.$$

Therefore
$$\|N_*(x_m) - N_*(x)\|_j \to 0 \text{ as } m \to \infty.$$

Thus, we conclude that N_* is continuous. Consider the following map
$$\rho : \overline{B(0, M_*)} \to \overline{B(0, M_*)} \text{ defined by}$$

$$\rho(x) = \begin{cases} x & \text{if } \sum_{i=1}^{n} \|x\|_i \le nK \\ \\ \frac{Knx}{\sum_{i=1}^{n} \|x\|_i} & \text{if } \sum_{i=1}^{n} \|x\|_i > nK. \end{cases}$$

It is evident that ρ is continuous and $N_* = \rho \circ N$. The compactness of N implies that N_* is compact. By Theorem 2.4.2 there exists $x \in \overline{B(0, M_*)}$ such that $x = N_*(x)$. Notice $\sum_{i=1}^{n} \|N(x)\|_i \le Kn$ for otherwise,

$$x = \lambda N(x),\ \lambda = \frac{Kn}{\sum_{i=1}^{n} \|N(x)\|_i} \text{ with } 0 < \lambda < 1 \Rightarrow x \in \mathcal{A}.$$

This implies that
$$\sum_{i=1}^{n} \|x\|_i < Kn,$$

but
$$x = \frac{KnN(x)}{\sum_{i}^{n} \|N(x)\|_i} \Rightarrow \sum_{i=1}^{n} \|x\|_i = Kn.$$

This yields a contradiction with $x \in \mathcal{A}$. Hence we get that
$$x = N_*(x) = N(x).$$

□

Next we state the nonlinear alternative of Leray-Schauder type.

Lemma 2.4.4. *Let X be a generalized Banach space, $U \subset E$ be a bounded, convex open neighborhood of zero and let $G : \overline{U} \to E$ be a continuous compact map. If G satisfies the boundary condition*

$$x \neq \lambda G(x)$$

for all $x \in \partial U$ and $0 \leq \lambda \leq 1$, then the set $Fix(G) = \{x \in U : x = G(x)\}$ is nonempty.

Proof. Let p is the Minkowski function of U and since $\overline{U}$ is bounded, then there exists $M > 0$ such that

$$G(\overline{U}) \subseteq \frac{1}{2} B(0, M_*), \quad M_* = (K, \dots, K).$$

Consider $G_* : \overline{B(0, M_*)} \to \overline{B(0, M_*)}$ defined by

$$G_*(x) = \begin{cases} G(x) & \text{if } x \in \overline{U} \\ \\ \frac{1}{p(x)} G(\frac{x}{p(x)}) & \text{if } x \in E \backslash \overline{U}. \end{cases}$$

Clear that $\overline{B(0, M_*)}$ is closed, convex, bounded subset of E and G_* is continuous compact operator. Then from Theorem 2.4.2 there exists $x \in \overline{B(0, M_*)}$ such that $\overline{G}(x) = x$. If $x \in E \backslash \overline{U}$ then

$$x = \frac{G\left(\frac{x}{p(x)}\right)}{p(x)} \Rightarrow \frac{x}{p(x)} = \frac{1}{p^2(x)} G\left(\frac{x}{p(x)}\right).$$

Since $x \in E \backslash \overline{U}$, then

$$p(x) = 1 \text{ or } p(x) > 1 \Rightarrow x \in \partial U, \ \frac{x}{p(x)} \in \partial U.$$

This is a contradiction with

$$z \neq \lambda G(z), \quad \text{for each, } \lambda \in [0, 1], \ z \in \partial U.$$

Consequently, there exist $x_* \in U$ such that $G(x_*) = x_*$. □

Theorem 2.4.5. *Let $(E, \|\cdot\|)$ be a Banach space, $C \subset E$ a closed convex subset, $U \subset C$ a bounded set, open (with respect to the topology C) and such that $0 \in U$. Let $G : \overline{U} \to C$ be a compact continuous mapping. If the following assumption is satisfied:*

$$x \neq \lambda G(x), \textit{ for all } x \in \partial_C U \textit{ and all } \lambda \in (0,1),$$

then f has a fixed point in U.

Proof. Let $C_* = \{x \in \overline{U} : x = \lambda G(x) \text{ for some; } \lambda \in [0,1]\}$. Since $0 \in U$ then C_* is nonempty set and by the continuity of G we concluded that C_* is closed. Clear that $\partial_C U \cap C_* = \emptyset$. From Theorem 2.1.1 there exists $f : \overline{U} \to [0,1]$ such that

$$f(x) = \begin{cases} 0 & \text{if } x \in \partial_C U \\ \\ 1 & \text{if } x \in C_*. \end{cases}$$

Consider $G_* : C \to C$ defined by

$$G_*(x) = \begin{cases} f(x)G(x) & \text{if } x \in U \\ \\ 0 & \text{if } x \in C \backslash U. \end{cases}$$

Since $G_*(x) = 0,$ for each $x \in \partial_C U$, and G_* is continuous on U, $E \backslash U$, then G_* is continuous. Set $\Omega = \overline{co}(\{0\} \cup G(\overline{U}))$ is convex and compact. We can easily prove that

$$G_*(\Omega) \subset \Omega.$$

Then from Theorem 2.4.2 there exists $x \in \Omega$ such that $G_*(x) = x$. From the definition of G_* we have $G(x) = x$. □

From above theorem we obtain the following:

Theorem 2.4.6. *Let $C \subset E$ be a closed convex subset and $U \subset C$ a bounded open neighborhood of zero(with respect to topology of C). If $G : \overline{U} \to E$ is compact continuous then*

i) either G has a fixed point in $\overline{U}$, or

ii) there exists $x \in \partial U$ such that $x = \lambda G(x)$ or some $\lambda \in (0,1)$.

By above lemma we can easily prove the following so-called nonlinear alternatives of Leray and Schauder will be needed in the proofs of our results [**?**]).

Lemma 2.4.7. *Let $(X, \|\cdot\|)$ be a generalized Banach space with $C \subset X$ a closed and convex subset of X. Assume U is an open subset of C, with $0 \in U$, and let $G : \overline{U} \longrightarrow C$ is a compact map. Then either,*

(a) *G has a fixed point in $\overline{U}$, or*

(b) *there is a point $u \in \partial U$ and $\lambda \in (0,1)$, with $u \in \lambda G(u)$.*

The single-valued version may be stated as follows:

Lemma 2.4.8. *[59] Let $(X, \|\cdot\|)$ be a generalized Banach and $G : X \longrightarrow \mathcal{P}_{cl,cv}(X)$ be an upper semi continuous and compact map. Then either,*

(a) *F has at least one fixed point, or*

(b) *the set $\mathcal{M} = \{x \in X \text{ and } \lambda \in (0,1), \text{with} \quad x \in \lambda G(u)\}$ is unbounded.*

Theorem 2.4.9. *[121] Let (X, d) be a generalized complete metric space, and let $F : X \to \mathcal{P}_{cl}(X)$ be a multivalued map. Assume that there exist $A, B, C \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ such that*

$$H_d(F(x), F(y)) \leq Ad(x,y) + Bd(y, F(x)) + Cd(x, F(x)) \tag{2.4.1}$$

where $A + C$ converge to zero. Then there exist $x \in X$ such that $x \in F(x)$.

2.5 Semi-groups

Let X be a Banach space and let $\mathcal{L}(X)$ be the set of all linear bounded operators from X to X. Endowed with the operator norm $\|.\|_{\mathcal{L}(X)}$, defined by

$$\|U\|_{\mathcal{L}(X)} = \sup_{\|x\|\leq 1} \|Ux\|$$

for each $U \in \mathcal{L}(X), \mathcal{L}(X)$ is a Banach space.

Definition 2.5.1. *A semigroup is a one-parameter family $\{S(t) : t \geq 0\} \subset \mathcal{L}(X)$ satisfying the conditions:*

(a) $S(t) \circ S(s) = S(t+s)$, *for* $t, s \geq 0$,

(b) $S(0) = I$ *where* I *denotes the identity operator in* X.

Definition 2.5.2. *A semigroup* $S(t)$ *is uniformly continuous if*

$$\lim_{t \to 0^+} \|S(t) - I\|_{\mathcal{L}(X)}$$

that is if

$$\lim_{|t-s| \to 0^+} \|S(t) - S(s)\|_{\mathcal{L}(X)} = 0.$$

Definition 2.5.3. *We say that the semigroup* $\{S(t)\}_{t \geq 0}$ *is strongly continuous (or a* C_0*-semigroup) if the map* $t \to S(t)(x)$ *is strongly continuous, for each* $x \in E$*, i.e.*

$$\lim_{t \to 0^+} S(t)x = S(0)x \quad \textit{for} \quad x \in E.$$

Definition 2.5.4. *Let* $S(t)$ *be a* C_0*-semigroup defined on* X*. The infinitesimal generator* A *in the linear operator of* $S(t)$ *is the linear operator defined by*

$$Ax = \lim_{t \to 0^+} \frac{S(t)x - S(0)x}{t}, \textit{for} \quad x \in D(A),$$

where $D(A) = \{x \in X \quad \lim_{t \to 0^+} \frac{S(t)x - x}{t} \quad \textit{exists in} \quad X\}$*. and*

$$Ax = \lim_{t \downarrow 0} \frac{1}{t}(S(t)x - x).$$

Equivalently, we say that A *generates* $\{S(t); t > 0\}$*.*

Remark 2.5.1. *If* $A : D(A) \subseteq X \to X$ *is the infinitesimal generator of a semigroup of linear operators then* $D(A)$ *is a vector subspace of* X *and* A *is a possibly unbounded linear operator.*

Example 2.5.1. *A first significant example of uniformly continuous semigroup is given by* $t \longmapsto e^{tA}$ *where* e^{tA} *is the exponential of the matrix* tA*. Namely, let* $A \in \mathcal{M}_n(\mathbb{R})$ *and let* $S(t) = e^{tA}$ *for each* $t > 0$*, where*

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n.$$

We can easily see that $\{S(t); t > 0\}$ *is a uniformly continuous semigroup of linear operators.More that this, straightforward computations show that that* $t \longmapsto S(t)$ *is of class* C^1 *from* $[0, +\infty)$ *to* X*, and satisfies the first-order differential equation*

$$\frac{d}{dt}S(t) = AS(t) = S(t)A, \tag{2.5.1}$$

for each $t > 0$*.*

The next example shows that there exist semigroups which are not uniformly continuous.

Example 2.5.2. *Let* $X = C_{ub}(\mathbb{R}_+)$ *be the space of all bounded and uniformly continuous functions from* $\mathbb{R}_+$ *to* $\mathbb{R}$*, endowed with the sup-norm* $\|.\|_\infty$*, and let* $\{S(t); t \geq 0\} \subset \mathcal{L}(X)$ *be defined by*

$$[S(t)f](s) = f(t + s)$$

for each $f \in X$*and each* $t, s \in \mathbb{R}_+$*. One may easily verify that* $\{S(t); t \geq 0\}$ *satisfies (i) and (ii) in Definition 2.5.1, and therefore it is a semigroup of linear operators. As in this specific case, the uniform continuity of the semigroup is equivalent to the equicontinuity of the unit ball in* X*, property which obviously is not satisfied, the semigroup is not uniformly continuous. The generator of the semigroup is given by*

$$D(A) = \{f \in X; \quad \exists \lim_{t \downarrow 0^+} \frac{f(t + .) - f(.)}{t} = f^{'} \exists \quad \textit{strongly in} \quad X\},$$

and

$$Af = f^{'}.$$

Proposition 2.5.1. *If* $\{S(t); t > 0\}$ *is a uniformly continuous semigroup of linear operators then, for each* $t > 0$*,* $S(t)$ *is invertible.*

Proof. Inasmuch as

$$\lim_{t \downarrow 0} S(t) - I = 0,$$

in the norm topology of $\mathcal{L}(X)$, there exists ($\delta > 0$ such that

$$\|S(t) - I\|_{\mathcal{L}(X)} \leq 1$$

for each $t \in (0, \delta]$. Thus, for each $t \in (0, \delta]$, $S(t)$ is invertible. Let $t > \delta$. Then there exist $n \in \mathbb{N}^*$ and $\eta \in [0, \delta)$ such that $t = n\delta + \eta$. Therefore $S(t) = S(\delta)^n S(\eta)$, and so $S(t)$ is invertible. The proof is complete, □

2.5.1 Generators of uniformly continuous semigroups

Theorem 2.5.2. *linear operator* $A : D(A) \subseteq X \to X$ *is the generator of a uniformly continuous semigroup if and only if* $D(A) = X$ *and* $A \in \mathcal{L}(X)$.

Proof. The "only if" part. Let $\{S(t); t > 0\}$ be uniformly continuous. Since

$$\lim_{t \downarrow 0} S(t) = I$$

in the norm topology of $\mathcal{L}(X)$, there exists $\rho > 0$ such that

$$\Big\| \frac{1}{\rho} \int_0^\rho S(t)dt - I \Big\|_{\mathcal{L}(X)} < 1.$$

We notice that the integral here is the Riemann integral of the continuous function $S : [0, \rho] \to \mathcal{L}(X)$, which is defined by a simple analogy with 1 its scalar counterpart. Consequently, the operator $\frac{1}{\rho} \int_0^\rho S(t)dt$ is invertible and accordingly $\int_0^\rho S(t)dt$ has the same property. Let $h > 0$. Let us remark that

$$\frac{1}{h}(S(h) - I) \int_0^\rho S(t)dt = \frac{1}{h} \int_0^\rho S(t + h)dt - \frac{1}{h} \int_0^\rho S(t)dt$$

The change of variable $t + h = s$ in the first integral on the right-hand side yields

$$\begin{aligned} \frac{1}{h}(S(h) - I) \int_0^\rho S(t)dt &= \frac{1}{h} \int_h^{\rho+h} S(s)ds - \frac{1}{h} \int_0^\rho S(s)ds \\ &= \frac{1}{h} \int_h^{\rho+h} S(s)ds - \frac{1}{h} \int_0^h S(s)ds. \end{aligned}$$

Then

$$\frac{1}{h}(S(h) - I) = \Big(\frac{1}{h} \int_\rho^{\rho+h} S(s)ds - \frac{1}{h} \int_0^h S(s)ds \Big) \Big(\frac{1}{h} \int_0^\rho S(t)dt \Big)^{-1}.$$

But, the right-hand side of the equality above converges for h tending to 0 by positive values, and thus, the left-hand side enjoys the same property. As the convergence in the uniform operator topology of $\mathcal{L}(X)$ implies the pointwise convergence, letting h to tend to 0 by positive values, we deduce

$$A = (S(\rho) - I) \Big(\int_0^\rho S(t)dt \Big)^{-1} \Big)$$

Hence $A \in \mathcal{L}(X)$, which proves the necessity.
The "if" part. Let $A \in \mathcal{L}(X)$, $t > 0$ and let

$$S(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n,$$

where $A^n = A.A \ldots A$ times and $A^0 = I$.
We can easily see that $\{S(t); t > 0\}$ is a semigroup of linear operators. In order to prove that this semigroup is uniformly continuous let us remark that

$$\Big\| S(t) - I \Big\|_{\mathcal{L}(X)} = \Big\| \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n - I \Big\|_{\mathcal{L}(X)}$$

$$= \Big\| \sum_{n=1}^{\infty} \frac{t^n}{n!} A^n \Big\|_{\mathcal{L}(X)} \leq t \sum_{n=1}^{\infty} \frac{t^{n-1}}{n!} \|A^n\|_{\mathcal{L}(X)}.$$

Since

$$\sum_{n=1}^{\infty} \frac{t^{n-1}}{n!} \|A^n\|_{\mathcal{L}(X)} \leq \|A\| e^{t\|A\|},$$

we conclude that

$$\lim_{t \downarrow 0} S(t) = I$$

in the norm topology of $\mathcal{L}(X)$, and thus $\{S(t); t > 0\}$ is a uniformly continuous semigroup. To achieve the proof we have to show that A is the infinitesimal generator of this semigroup. To this aim it suffices to verify that

$$\lim_{t \downarrow 0} \Big\| \frac{1}{t}(S(t) - I) - A \Big\|_{\mathcal{L}(X)} = 0.$$

But this follows from the obvious inequality

$$\Big\| \frac{1}{t}(S(t) - I) - A \Big\|_{\mathcal{L}(X)} = t \Big\| \sum_{n=1}^{\infty} \frac{t^{n-2}}{n!} \|A^n \Big\|_{\mathcal{L}(X)} \leq t \|A^2\| e^{t\|A\|},$$

there by completing the proof. □

2.5.2 C_0-Semigroups. General Properties

In this section we introduce a class of semigroups of linear operators, strictly larger than that of uniformly continuous semigroups, class which proves very useful in the study of many partial differential equations of parabolic or hyperbolic type. The following properties are classical (see [85, 124]).

Definition 2.5.5. *A semigroup of linear operators $\{S(t); t > 0\}$ is called a semigroup of class C_0, or C_0-semigroup if for each $x \in X$ we have*

$$\lim_{t \downarrow 0} S(t)x = x.$$

Remark 2.5.2. *Each uniformly continuous semigroup is of class C_0 but not conversely as we can state from the example below.*

Example 2.5.3. *Let $X = C_{ub}(\mathbb{R}^+)$ be the space of all functions which are uniformly continuous and bounded from $\mathbb{R}^+$ to $\mathbb{R}$ endowed with the sup-norm $\|.\|_\infty$, and let $\{S(t); t > 0\}$ be defined by*

$$[S(t)f](s) = f(t+s)$$

for each $f \in X$ and each $t, s \in$. We know from Example 2.5.1 that $\{S(t); t > 0\}$ is a semigroup. In addition, this is of class C_0 . On the other hand, as we mentioned in Example 2.5.1, it is not uniformly continuous because the unit ball in X is not equicontinuous.

Theorem 2.5.3. *If $\{S(t); t > 0\}$ is a C_0-semigroup, then there exist $M > 1$, and $w \in \mathbb{R}$ such that*

$$\|S(t)\|_{\mathcal{L}(X)} \leq Me^{wt}, \quad for t \geq 0 \tag{2.5.2}$$

for each $t > 0$.

Proof. First, we will show that there exist $\eta > 0$ and $M > 1$ such that

$$\|S(t)\|_{\mathcal{L}(X)} \leq M, \quad for t \geq 0 \tag{2.5.3}$$

for each $t \in [0, \eta]$. To this aim, let us assume by contradiction that this is not the case. Then there exists at least one C_0-semigroup $\{S(t); t > 0\}$ with the property that, for each $\eta > 0$ and each $M > 1$, there exists $t_{\eta,M} \in [0, \eta]$, such that

$$\|S(t)\|_{\mathcal{L}(X)} > M. \tag{2.5.4}$$

Taking $\eta = 1/n, M = n$ and denoting $t_{\eta,M} = t_n$ for $n \in \mathbb{N}^*$, we deduce

$$\|S(t_n)\|_{\mathcal{L}(X)} > n, \tag{2.5.5}$$

where $t_n \in [0, 1/n]$ for each $n \in \mathbb{N}^+$. Recalling that, for each $x \in X$, $\lim_{n\to\infty} S(t_n)x = x$, it follows that the family $\{S(t_n); n \in \mathbb{N}^+\}$ of linear bounded operators is pointwise bounded, i.e., for each $x \in X$, the set $\{S(t_n)x; n \in \mathbb{N}^+\}$ is bounded. By the uniform boundedness principle (see Dunford and Schwartz [60], Corollary 21, p. 66), it follows that this family is bounded in the uniform operator norm $\|.\|_{\mathcal{L}(X)}$ which contradicts (2.5.5). This contradiction can be eliminated only if 2.5.4 holds. Next, let $t > 0$. Then there exist $n \in \mathbb{N}^+$ and $\delta \in [0, \eta)$, such that $t = n\eta + \delta$. We have

$$\|S(t)\|_{\mathcal{L}(X)} = \|S^n(\eta)S(\delta)\|_{\mathcal{L}(X)} \leq \|S(\eta)\|^n_{\mathcal{L}(X)}\|S(\delta)\|_{\mathcal{L}(X)} \leq MM^n.$$

But $n = \frac{t-\delta}{\eta} < \frac{t}{\eta}$ and thus$\|S(t)\|_{\mathcal{L}(X)} \leq MM^{\frac{t}{\eta}} = Me^{tw}$ where $w = \frac{1}{\eta} lnM$. The proof is complete. □

Remark 2.5.3. *If $\{S(t); t > 0\}$ is a uniformly continuous semigroup whose generator is A, then (2.5.4) holds with $M = 1$ and $w = \|A\|_{\mathcal{L}(X)}$.*

Definition 2.5.6. *A C_0-semigroup, $\{S(t); t > 0\}$ is called of type (M, w) with $M > 1$ and $w \in \mathbb{R}$, if for each $t > 0$, we have*

$$\|S(t)\|_{\mathcal{L}(X)} \leq Me^{wt}, \quad \textit{for} \qquad t \geq 0.$$

A C_0-semigroup $\{S(t); t > 0\}$ is called a C_0-semigroup of contractions, or of nonexpansive operatorsi.e., if for each $t > 0$, we have

$$\|S(t)\|_{\mathcal{L}(X)} \leq 1, \quad \textit{for} \qquad t \geq 0.$$

We shall use also the term of contraction semigroup.

Corollary 2.5.4. *If $\{S(t); t > 0\}$ is a C_0-semigroup, then the mapping $(t, x) = S(t)x$ is jointly continuous from $[0, +\infty) \times X$ to X.*

Proof. Let $x, y \in X$, $t > 0$ and $h \in \mathbb{R}^*$ with $t + h > 0$. We distinguish between two cases$h > 0$, or $h < 0$. If $h > 0$, we have

$$\begin{aligned}\|S(t+h)y - S(t)x\| &\leq \|S(t+h)y - S(t+h)x\| + \|S(t+h)x - S(t)x\| \\ &\leq \|S(t+h)\|_{\mathcal{L}(X)}\|y - x\| + \|S(t+h)x - S(t)x\| \\ &\leq Me^{(t+h)w}\|y - x\| + \|S(t)\|_{\mathcal{L}(X)}\|S(h)x - x\|,\end{aligned}$$

which shows that

$$\lim_{(t,y)\to(t+0,x)} S(\tau)y = S(t)x.$$

If $h < 0$, by Theorem 2.5.3, we deduce between two cases$h > 0$, or $h < 0$. If $h > 0$, we have

$$\begin{aligned}\|S(t+h)y - S(t)x\| &= \|S(t+h)y - S(t+h)S(-h)x\| \\ &\leq \|S(t+h)\|_{\mathcal{L}(X)}\|y - S(-h)x\| \\ &\leq Me^{(t+h)w}(\|y - x\| + \|S(-h)x - x\|),\end{aligned}$$

which implies that

$$\lim_{(t,y)\to(t-0,x)} S(\tau)y = S(t)x.$$

The proof is complete. □

Some basic properties of C_0-semigroups are listed below.

Theorem 2.5.5. *Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a C_0 semigroup $\{S(t); t > 0\}$. Then*

- *(i) for each $x \in X$ and each $t > 0$, we have*

$$\lim_{h\downarrow 0} \frac{1}{h}\int_{\tau}^{\tau+h} S(\tau)xd\tau = S(\tau)x;$$

- *(ii) for each $\in X$ and each $t > 0$, we have*

$$\int_0^{\tau} S(\tau)d\tau x \in D(A)$$

and

$$A\Big(\int_0^{\tau} S(\tau)xd\tau\Big) = S(\tau)x - x;$$

- *(iii) for each $x \in D(A)$ and each $t > 0$, we have $S(t)x \in D(A)$. In addition, the mapping $t \longmapsto S(t)x$ is of class C^1 on $[0, +\infty)$, and satisfies*

$$\frac{d}{dt}(S(t)x) = AS(t)x = S(t)Ax$$

- *(iv) for each $x \in D(A)$ and each $0 \le s \le t < +\infty$, we have*

$$\int_s^t AS(\tau)x d\tau = \int_s^t S(\tau)Ax d\tau = S(t)x - S(s)x.$$

Proof. In order to prove (i), let us observe that

$$\left\| \frac{1}{h} \int_\tau^{\tau+h} S(\tau)x - S(t)x d\tau \right\| \le \frac{1}{h} \int_\tau^{\tau+h} \left\| S(\tau)x - S(t)x \right\| d\tau.$$

The conclusion follows from Corollary 2.5.4.
Let $x \in X$, $t > 0$ and $h > 0$. We remark that

$$\frac{1}{h}(S(h) - I) \int_0^h S(\tau) d\tau x = \frac{1}{h} \int_0^t S(\tau + h)x d\tau - \frac{1}{h} \int_0^t S(\tau)x d\tau.$$

The change of variable $\tau + h = s$ in the first integral on the right-hand side yields

$$\frac{1}{h}(S(h) - I) \int_0^h S(\tau) d\tau x = \frac{1}{h} \int_h^{t+h} S(s)x ds - \frac{1}{h} \int_0^t S(s)x ds$$

$$- \frac{1}{h} \int_0^h S(s)x ds + \frac{1}{h} \int_t^{t+h} S(s)x ds.$$

From this equality and from (i), we deduce that there exists

$$\lim_{h \downarrow 0} \frac{1}{h} \int_\tau^{\tau+h} S(\tau)x d\tau = S(\tau)x,$$

which proves (ii).
Next, let $x \in D(A)$, $t > 0$ and $h > 0$. We have

$$\left\| \frac{1}{h}(S(t+h)x - S(t)x) - S(t)Ax \right\| \le \|S(t)\| \left\| \frac{1}{h}(S(h)x - x) - Ax \right\|,$$

inequality which proves that $S(t)x \in D(A)$, $t \longmapsto S(t)x$ is differentiable from the right, and that

$$\frac{d^+}{dt}(S(t)x) = AS(t)x = S(t)Ax. \tag{2.5.6}$$

On the other hand, for each $t > 0$ and $h < 0$ with $t + h > 0$, we have

$$\begin{aligned}\left\|\frac{1}{h}(S(t+h)x - S(t)x) - S(t)Ax\right\| &\leq \|S(t+h)\|\left\|\frac{1}{h}(x - S(-h)x) - S(-h)Ax\right\| \\ &\leq \|S(t+h)\|\Big(\left\|\frac{-1}{h}(S(-h)x - x) - Ax\right\| \\ &+ \|S(-h)Ax - Ax\|\Big).\end{aligned}$$

This inequality shows that $t \longmapsto S(t)x$is differentiable from the left as well. From (2.5.6) and the continuity of the function $t \longmapsto S(t)Ax$ on $[0, +\infty)$, we deduce that $t \longmapsto S(t)x$ is of class C^1 on $[0, +\infty)$, which completes the proof of (iii).
Since (iv) follows from (iii) by integrating from s to t both sides in 2.5.6, the proof is complete. □

2.5.3 The infinitesimal generator

In this section we shall prove two basic properties of the generator of a C_0-semigroup: the density of the domain and the closedness of the graph. First, let us recall the following:

Definition 2.5.7. *An operator $A : D(A) \subseteq X \longrightarrow X$ is called closed, if its graph is closed in $X \times X$.*

Theorem 2.5.6. *Let$A : D(A) \subseteq X \longrightarrow X$ be the infinitesimal generator of a C_0-semigroup $\{S(t); t > 0\}$. Then $D(A)$ is dense in X, and A is a closed operator.*

Proof. Let $x \in X$ and $\epsilon > 0$. Then, we have and by virtue of (i) in Theorem 2.5.5,we have

$$\frac{1}{\epsilon}\int_0^{\epsilon} S(\tau)x d\tau \in D(A)$$

and

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon}\int_0^{\epsilon} S(\tau)x d\tau = x.$$

Consequently $D(A)$ is dense in X.
Next, let $(x_n)_{n\in\mathbb{N}^*}$ be a sequence in $D(A)$ such that

$$\lim_{n\to} x_n = x \, and \, \lim_{n\to} Ax_n = y$$

From (iv) in Theorem 2.5.5, it follows that

$$S(h)x - x = \int_0^h S(\tau) y d\tau.$$

By virtue of (i) in Theorem 2.5.5, it follows that there exists

$$\lim_{h \downarrow 0} \frac{1}{h} \int_0^\epsilon S(\tau) y d\tau = y.$$

From this relation, and the preceding one, we deduce that $x \in D(A)$ and $Ax = y$. The proof is complete. □

2.5.4 Hille-Yosida Theorem.

This section begins with the presentation of the most fundamental result within the theory of C_0-semigoups as: the Hille-Yosida generation theorem. This gives a very precise delimitation of the class of linear operators A, acting in a Banach space X, that generate C_0-semigroups containing only operators whose norms do not exceed .

2.5.5 The Hille-Yosida Theorem.Necessity

The goal of the next two sections is to prove the most important result in the theory of C_0-semigroups: the Hille-Yosida theorem. More precisely, we shall present a necessary and sufficient condition in order that a linear operator A generate a C_0-semigroup of contractions. We recall that, if $A : D(A) \subseteq X \longrightarrow X$ is a linear operator, the resolvent set $\rho(A)$ is the set of all those complex numbers A, called regular values, for which $R(\lambda I - A)$ is dense in X and $R(\lambda; A) = (\lambda I - A)^{-1}$ is continuous from $R(\lambda I - A)$ to X.

Theorem 2.5.7. *(Hille-Yosida) A linear operator $A : D(A) \subseteq X \longrightarrow X$ is the infinitesimal generator of a C_0-semigroup of contractions if and only if:*

- *(i) A is densely defined and closed and*
- *(ii)* $(0, +\infty) \subseteq \rho(A)$ *and for each* $\lambda > 0$

$$\|R(\lambda; A)\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda}.$$

Proof. We begin with the necessity. Let $A : D(A) \subseteq X \longrightarrow X$ be the infinitesimal generator of a C_0-semigroup of contractions $\{S(t); t > 0\}$. In view of Theorem 2.5.6, A is densely defined and closed. Thus (i) holds. In order to prove (ii), let $\lambda > 0$, $x \in X$, and let us define

$$R(\lambda)x = \int_0^\infty e^{-\lambda x} S(t)x dt.$$

We notice that the integral on the right-hand side of the equality above is convergent. Indeed, for each $a, b > 0$, $a < b$, we have

$$\left\| \int_a^b e^{-\lambda x} S(t)x dt \right\| \leq \int_a^b e^{-\lambda t} \|S(t)\|_{\mathcal{L}(X)} \|x\| dt,$$

and

$$\int_a^b e^{-\lambda t} \|x\| dt = \frac{e^{-\lambda a} - e^{-\lambda b}}{\lambda} \|x\|.$$

Accordingly, we are in the hypotheses of the Catchy test, and thus the integral is convergent. Clearly $R(\lambda) \in \mathcal{L}(X)$ and

$$\begin{aligned} \|R(\lambda)x\| &= \left\| \int_0^\infty e^{-\lambda t} S(t)x dt \right\| \\ &\leq \int_0^\infty e^{-\lambda t} \|S(t)\|_{\mathcal{L}(X)} \|x\| dt \\ &\leq \frac{1}{\lambda} \|x\|. \end{aligned}$$

Hence

$$\|R(\lambda)\| \leq \frac{1}{\lambda}.$$

We prove next that $R(\lambda)$ coincides with $R(\lambda; A)$. To this aim we show that $R(\lambda)$ is both the right and the left inverse of the operator $\lambda I - A$. Let $x \in X$, $\lambda > 0$ and $h > 0$. We have

$$\begin{aligned} \frac{1}{h}(S(h) - I)R(\lambda)x &= \frac{1}{h} \int_0^\infty e^{-\lambda t} S(t+h)x dt - \frac{1}{h} \int_0^\infty e^{-\lambda t} S(t)x dt \\ &= \frac{e^{\lambda h} - 1}{h} \int_0^\infty e^{-\lambda t} S(t)x dt - \frac{e^{\lambda h}}{h} \int_0^\infty e^{-\lambda t} S(t)x dt \end{aligned}$$

As the right-hand side of the above equality converges to $\lambda R(\lambda)x - x$, it follows that $R(\lambda)x \in D(A)$, and

$$AR(\lambda) = \lambda R(\lambda) - I,$$

which proves that

$$(\lambda I - A)R(\lambda) = I.$$

So, $R(\lambda)$ is the right inverse of $I\lambda - A$. Next, let $x \in D(A)$. Let us remark that

$$\begin{aligned} R(\lambda)Ax &= \int_0^\infty e^{-\lambda t} S(t)Axdt \\ &= \int_0^\infty e^{-\lambda t} \frac{d}{dt}(S(t)x)dt \\ &= \lim_{t\to\infty} e^{-\lambda t} S(t)x - x + \lambda \int_0^\infty e^{-\lambda t} S(t)xdt \\ &= \lambda R(\lambda)x - x. \end{aligned}$$

This equality may be equivalently rewritten as

$$R(\lambda)(\lambda I - A) = I,$$

which shows that $R(\lambda)$ is the left inverse of $\lambda I - A$, and this completes the proof of the necessity. □

Remark 2.5.4. *Using similar arguments, one may prove that, whenever A generates a C_0-semigroup of contractions, then $\{\lambda \in \mathbb{C}; Re(\lambda) > 0\} \subseteq \rho(A)$ and for each $\lambda \in \mathbb{C}$ with $Re(\lambda) > 0$, we have*

$$\|R(\lambda; A)\|_{\mathcal{L}(X)} \leq \frac{1}{Re(\lambda)}.$$

2.5.6 The Hille-Yosida Theorem. Sufficiency

In order to prove the sufficiency, some preliminary lemmas are needed. First, let us observe that, by (i) , it follows that, for each $\lambda > 0$, $R(\lambda; A) \in \mathcal{L}(X)$.

Definition 2.5.8. *Let $A : D(A) \subseteq X \longrightarrow X$ be a linear operator satisfying (i) and (ii) in Theorem 2.5.7, and let $\lambda > 0$. The operator $A_\lambda : X \to X$, defined by $A_\lambda = \lambda AR(\lambda, A)$, is called the Yosida approximation of A.*

Lemma 2.5.8. *Let $A : D(A) \subseteq X \longrightarrow X$ be a linear operator which satisfies (i) and (ii) in Theorem 2.5.7. Then:*

$$\lim_{\lambda\to\infty} \lambda R(\lambda; A)x = x \tag{2.5.7}$$

for each $x \in X$,

$$A_\lambda x = \lambda^2 R(\lambda; A)x - x \tag{2.5.8}$$

for each $x \in X$, and

$$\lim_{\lambda\to\infty} A_\lambda x = Ax \tag{2.5.9}$$

for each $x \in D(A)$.

Proof. Let $x \in D(A)$ and $\lambda > 0$. We have

$$\|\lambda R(\lambda; A)x - x\| = \|AR(\lambda; A)x\| = \|R(\lambda; A)Ax\| \leq \frac{1}{\lambda}\|Ax\|,$$

and consequently

$$\lim_{\lambda\to\infty} \lambda R(\lambda; A)x = x$$

for each $x \in D(A)$. Since $D(A)$ is dense in X and $\|R(\lambda; A)x\| \leq 1$, from the last relation, we deduce 2.5.7. To check (2.5.8), let us remark that we have successively

$$\lambda^2 R(\lambda; A) - \lambda I = \lambda^2 R(\lambda; A) - \lambda(\lambda I - A)R(\lambda; A) = \lambda AR(\lambda; A) = A_\lambda.$$

Finally, if $x \in D(A)$, by (2.5.7), we have

$$\lim_{\lambda\to\infty} A_\lambda x = \lim_{\lambda\to\infty} \lambda AR(\lambda; Ax) = \lambda R(\lambda; A)Ax = Ax,$$

which concludes the proof of Lemma 2.5.8. □

Lemma 2.5.9. *Let $A : D(A) \subseteq X \longrightarrow X$ be a linear operator which satisfies (i) and (ii) in Theorem 2.5.7. Then, for each $\lambda > 0$, A_λ is the infinitesimal generator of a uniformly continuous semigroup $\{e^{tA_\lambda}; t \geq 0\}$ satisfying*

$$\|e^{tA_\lambda}\|_{\mathcal{L}(X)} \leq 1 \tag{2.5.10}$$

for each $t \geq O$. In addition, for each $x \in X$ and each $\lambda, \mu > O$, we have

$$\|e^{tA_\lambda}x - e^{tA_\mu}x\|_{\mathcal{L}(X)} \leq t\|A_\lambda x - A_\mu x\|. \tag{2.5.11}$$

Proof. As $A_\lambda \in \mathcal{L}(X)$, by Theorem 2.5.2, it follows that it generates a uniformly continuous semigroup $\{e^{tA_\lambda}; t \geq 0\}$. In order to check 2.5.10, let us remark that, by virtue of 2.5.8 and (ii), we have

$$\|e^{tA_\lambda}\|_{\mathcal{L}(X)} = \|e^{t\lambda^2 R(\lambda;A) - t\lambda I}\|_{\mathcal{L}(X)} \leq \|e^{t\lambda^2 R(\lambda;A)}\|_{\mathcal{L}(X)}\|e^{-t\lambda I}\|_{\mathcal{L}(X)}$$

$$e^{t\lambda^2\|R(\lambda;A)\|_{\mathcal{L}(X)}}e^{-t\lambda} \leq e^{-t\lambda}e^{t\lambda} = 1.$$

Since A_λ, A_μ, e^{tA_λ} and e^{tA_μ} commute each to another, we have

$$\begin{aligned} \|e^{tA_\lambda}x - e^{tA_\mu}x\|_{\mathcal{L}(X)} &= \Big\| \int_0^1 \frac{d}{ds}(e^{stA_\lambda})e^{(1-s)tA_\mu}xds\Big\| \\ &\leq \int_0^1 t\Big\|(e^{stA_\lambda})e^{(1-s)tA_\mu}(A_\lambda x - A_\mu x)\Big\|ds \\ &\leq t\|A_\lambda x - A_\mu x\|, \end{aligned}$$

which completes the proof. □

We proceed to the proof of the sufficiency of Theorem 2.5.7.

Proof. of Theorem 2.5.7 (continued). From (2.5.9) and (2.5.11), it follows that, for each $t \geq 0$, there exists a linear operator $S(t) : D(A) \subseteq X \longrightarrow X$ such that, for each $x \in D(A)$,

$$\lim_{\lambda\to\infty} e^{tA_\lambda}x = S(t)x,$$

uniformly on compact subsets in $\mathbb{R}^+$. By (2.5.10) we deduce that

$$\|S(t)x\| \leq \|x\|,$$

for each $t \geq 0$ and $x \in D(A)$. Since $D(A)$ is dense in X, it follows that $S(t)$ can be extended by continuity to the whole space X. It is easy to see that the family of linear bounded operators thus obtained is a semigroup, denoted for simplicity again by $\{S(t); t > 0\}$. Clearly it satisfies

$$\|S(t)\| \leq 1.$$

In addition, for each $t > 0$ and $x, y \in X$, we have

$$\begin{aligned} \|S(t)x - x\| &\leq \|S(t)x - S(t)y\| + \|S(t)y - e^{tA_\lambda}y\| + \|e^{tA_\lambda}y - y\| + \|y - x\| \\ &\leq \|S(t)y - e^{tA_\lambda}y\| + \|e^{tA_\lambda}y - y\| + 2\|y - x\| \end{aligned}$$

Let $T > 0$ and $\epsilon > 0$. Fix $y = x_\epsilon \in D(A)$, with $\|x - x_\epsilon\| < \epsilon$, and a sufficiently large λ, such that

$$\|S(t)x_\epsilon - e^{tA_\lambda}x_\epsilon\| \leq \epsilon$$

for each $t \in [0, T]$. By this inequality, we deduce

$$\|S(t)x - x\| \leq 3\epsilon + \|e^{tA_\lambda} x_\epsilon - x_\epsilon\|. \tag{2.5.12}$$

Inasmuch as $\{e^{tA_\lambda} x_\epsilon; t > 0\}$ is a uniformly continuous semigroup, for the very same $\epsilon > 0$, there exists $\delta(\epsilon) > 0$, such that $\|e^{tA_\lambda} - I\|_{\mathcal{L}(x)} \leq \epsilon$ for each $t \in (0, \delta(\epsilon))$. Consequently

$$\begin{aligned} \|e^{tA_\lambda} x_\epsilon - x_\epsilon\| &\leq \|e^{tA_\lambda} - I\|_{\mathcal{L}(x)} \|x_\epsilon\| \\ &\leq \epsilon \|x_\epsilon\| \end{aligned}$$

for each $t \in (0, \delta(\epsilon))$. Since$\{x_\epsilon; \epsilon > 0\}$ is bounded, this inequality, along with (2.5.12), shows that $\{S(t); t > 0\}$is a semigroup of class C_0. To conclude the proof, we have merely to show that the infinitesimal generator, $B : D(B) \subseteq X \to X$, of this semigroup coincides with $A : D(A) \subseteq X \to X$. To this aim, let $x \in D(A)$ and $h > 0$. We have

$$\lim_{\lambda \to \infty} e^{tA_\lambda} A_\lambda x = S(t)Ax$$

uniformly on compact subsets in $\mathbb{R}_+$. Indeed,

$$\|e^{tA_\lambda} A_\lambda x - S(t)Ax\| \leq \|e^{tA_\lambda} A_\lambda x - e^{tA_\lambda} Ax\| + \|e^{tA_\lambda} Ax - S(t)Ax\|$$

$$\leq \|e^{tA_\lambda}\|_{\mathcal{L}(X)} \|A_\lambda x - Ax\| + \|e^{tA_\lambda} Ax - S(t)Ax\|.$$

But this relation, along with (2.5.9) and with the partial conclusions above, proves that

$$S(h)x - x = \lim_{\lambda \to \infty} (e^{hA_\lambda} x - x) = \lim_{\lambda \to \infty} \int_0^h e^{tA_\lambda} A_\lambda x dt = \int_0^h S(t)Ax dt.$$

Dividing both sides this equality by h and letting h tend to 0 by positive values, we deduce that $x \in D(B)$ and $Bx = Ax$. Finally, we show that $D(A) = D(B)$. Since B is the infinitesimal generator of a C_0-semigroup of contractions, from the necessity it follows that $1 \in \rho(B)$. Accordingly $I - B$is invertible and $(I - B)^{-1}X = D(B)$. As $(I - B)D(A) = (I - A)D(A)$ and, by (ii), $(I - A)D(A) = X$, it follows that $(I - B)D(A) = X$,or equivalently $(I - B)^{-1}X = D(A)$. Hence $D(A) = D(B)$, which completes the proof of Theorem 2.5.7. □

Chapter 3

Stochastic Differential Equations with Impulses

In this chapter, our main objective is to establish sufficient conditions for the existence of solutions for system of stochastic impulsive functional equation with infinite Brownian motions. Our approach based on Perov fixed point theorem and a new version of Schaefer's fixed point in generalized Banach spaces.Consider the problem following

$$\begin{cases} dx(t) & = \sum_{l=1}^{\infty} f_l^1(t, x(t), y(t))dW^l(t) + g^1(t, x(t), y(t))dt, \ t \in J, t \neq t_k \\ dy(t) & = \sum_{l=1}^{\infty} f_l^2(t, x(t), y(t))dW^l(t) + g^2(t, x(t), y(t))dt, \ t \in J, t \neq t_k \\ x(t_k^+) & -x(t_k) = I_k(x(t_k)), \quad t = t_k \quad k = 1, 2, \dots, m \\ y(t_k^+) & -y(t_k) = \overline{I}_k(y(t_k)), \\ x(0) & = x_0, \\ y(0) & = y_0 \end{cases} \tag{3.0.1}$$

where $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$, $J := [0, T]$. $f_l^1, f_l^2 : J \times \mathbb{R}^2 \to \mathbb{R}$ are Carathéodory functions, $g^1, g^2 : J \times \mathbb{R}^2 \to \mathbb{R}$, and W^l is an infinite sequence of independent standard Brownian motions, $l = 1, 2, \ldots$ and $I_k, \overline{I}_k \in C(\mathbb{R}, \mathbb{R})$ $(k = 1, \ldots, m)$, and $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$. The notations $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^+} y(t_k - h)$ stand for the right and the left limits of the function y at $t = t_k$, respectively. Set

$$\begin{cases} f_i(.,x,y) = (f_1^i(.,x,y), f_2^i(.,x,y), \ldots), \\ \|f_i(.,x,y)\| = \Big(\sum_{l=1}^{\infty} (f_l^i)^2(.,x,y) \Big)^{\frac{1}{2}} \end{cases} \tag{3.0.2}$$

where $i = 1, 2,\ f_i(.,x,y) \in l^2$ for all $x \in \mathbb{R}$.

This chapter is motivated by [75, 157] and we generalize the existence and uniqueness of the solution to impulsive stochastic differential equations under non-Lipschitz condition and Lipschitz condition.

3.1 Existence and Uniqueness Result

Let $J_k = (t_k, t_{k+1}],\ k = 1, 2, \ldots, m$. In order to define a solutions for Problem (5.0.1), consider the space of pice-wise continuous functions

$$\begin{aligned} PC &= \{x :\ \Omega \times J \longrightarrow \mathbb{R},\ x(w,.) \in C(J_k, \mathbb{R}),\ k = 1, \ldots, m \text{ such that} \\ &\quad x(t_k^+,.) \text{ and } x(t_k^-,.) \text{ exist with } x(t_k^-,.) = x(t_k,.)\}, \end{aligned}$$

Endowed with the norm

$$\|x\|_{PC}^2 = \sup_{t \in J} \mathbb{E}|x(t,.)|^2$$

PC is a Banach space with norm $\|\cdot\|_{PC}$.

Definition 3.1.1. $\mathbb{R}-$ *valued stochastic process* $u = (x, y) \in PC \times PC$ *is said to be a solution of* (5.0.1) *with respect to the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$,*if:*

1) $u(t)$ *is* $\mathcal{F}_t$*-adapted for all* $t \in J_k = (t_k, t_{k+1}] \quad k = 1, 2, \ldots, m$

2) $u(t)$ *is right continuous and has limit on the left;*

3) $u(t)$ satisfies that

$$\begin{cases} x(t) &= x_0 + \sum_{l=1}^{\infty} \int_0^t f_l^1(s, x(s), y(s)) dW^l(s) \\ & \quad + \int_0^t g^1(s, x(s), y(s)) ds + \sum_{0 \le t_k \le t} I_k(x(t_k)), \quad t \in J \\ \\ y(t) &= y_0 + \sum_{l=1}^{\infty} \int_0^t f_l^2(s, x(s), y(s)) dW^l(s) \\ & \quad + \int_0^t g^2(s, x(s), y(s)) ds + \sum_{0 \le t_k \le t} \overline{I}_k(y(t_k)), \ t \in J. \end{cases}$$

We are now in a position to state and prove our existence result for the problem (5.0.1). First we will list the following hypotheses which will be imposed in our main theorem.:

(H_1) There exist nonnegative numbers a_i and b_i for each $i \in \{1, 2\}$

$$\begin{cases} \mathbb{E}(\|f^1(t, x, y) - f^1(t, \overline{x}, \overline{y})\|)^2 \le a_1 \mathbb{E}(|x - \overline{x}|)^2 + b_1 \mathbb{E}(|y - \overline{y}|)^2 \\ \mathbb{E}(\|f^2(t, x, y) - f^2(t, \overline{x}, \overline{y})\|)^2 \le a_2 \mathbb{E}(|x - \overline{x}|)^2 + b_2 \mathbb{E}(|y - \overline{y}|)^2 \end{cases}$$

for all x, y, $\overline{x}$, $\overline{y} \in \mathbb{R}$.

(H_2) There exist positive constants α_i and β_i for each $i = 1, 2$

$$\begin{cases} \mathbb{E}(|g^1(t, x, y) - g^1(t, \overline{x}, \overline{y})|)^2 \le \alpha_1 \mathbb{E}(|x - \overline{x}|)^2 + \beta_1 \mathbb{E}(|y - \overline{y}|)^2 \\ \mathbb{E}(|g^2(t, x, y) - g^2(t, \overline{x}, \overline{y})|)^2 \le \alpha_2 \mathbb{E}(|x - \overline{x}|)^2 + \beta_2 \mathbb{E}(|y - \overline{y}|)^2 \end{cases}$$

for all x, y, $\overline{x}$, $\overline{y} \in \mathbb{R}$ and $t \in J$.

(H_3) there exist constants $d_k \ge 0$ and $\overline{d}_k \ge 0$, $k = 1, \ldots, m$ such that

$$\mathbb{E}(|I_k(x) - I_k(\overline{x})|)^2 \le d_k \mathbb{E}(|x - \overline{x}|)^2$$

$$\mathbb{E}(|\overline{I}_k(y) - \overline{I}_k(\overline{y})|)^2 \le \overline{d}_k \mathbb{E}(|y - \overline{y}|)^2$$

for all x, y, $\overline{x}$, $\overline{y} \in \mathbb{R}$.

For our main consideration of Problem (5.0.1), a Preov fixed point is used to investigate the existence and uniqueness of solutions for system of impulsive stochastic differential equations.

Theorem 3.1.1. *Assume that* $(H_1)-(H_3)$ *are satisfied and the matrix*

$$M = \sqrt{3}\begin{pmatrix} \sqrt{C_2a_1+\alpha_1 T+l_1} & \sqrt{C_2b_1+\beta_1 T} \\ \sqrt{C_2a_2+\alpha_2 T} & \sqrt{C_2b_2+\beta_2 T+l_2} \end{pmatrix},\ l_1=\sum_{k=1}^{m} d_k,\ l_2=\sum_{k=1}^{m}\overline{d}_k,$$

where $C_2 \geq 0$ *is defined in Lemma 1.2.13. If* M *converges to zero. Then the problem* (5.0.1) *has unique solution.*

Proof. Consider the operator $N : PC\times PC \to PC\times PC$ defined by

$$N(x,y)=(N_1(x,y),N_2(x,y)),\ (x,y)\in PC\times PC$$

where

$$N_1(x,y)=x_0+\sum_{l=1}^{\infty}\int_0^t f_l^1(s,x(s),y(s))dW^l(s)+\int_0^t g^1(s,x(s),y(s))ds+\sum_{0<t_k\leq t} I_k(x(t_k))$$

and

$$N_2(x,y)=y_0+\sum_{l=1}^{\infty}\int_0^t f_l^2(s,x(s),y(s))dW^l(s)+\int_0^t g^2(s,x(s),y(s))ds+\sum_{0<t_k\leq t} \overline{I}_k(y(t_k)).$$

We shall use Theorem 2.4.1to prove that N has a fixed point. Indeed, let $(x,y),(\overline{x},\overline{y})\in PC\times PC$. Then we ave for each $t\in[0,T]$

$$\begin{aligned}|N_1(x(t),y(t))-N_1(\overline{x}(t),\overline{y}(t))|^2 = & \left|\sum_{l=1}^{\infty}\int_0^t f_l^1(s,x(s),y(s))dW^l(s)\right.\\ & +\int_0^t g^1(s,x(s),y(s))ds+\sum_{k=1}^{m} I_k(x(t_k))\\ & -\sum_{l=1}^{\infty}\int_0^t f_l^1(s,\overline{x}(s),\overline{y}(s))dW^l(s)\\ & \left.-\int_0^t g^1(s,\overline{x}(s),\overline{y}(s))ds-\sum_{k=1}^{m} I_k(\overline{x}(t_k))\right|^2.\end{aligned}$$

Then

$$\begin{array}{rcl} |N_1(x(t),y(t)) - N_1(\overline{x}(t),\overline{y}(t))|^2 & \leq & 3|\sum_{l=1}^{\infty}\int_0^t (f_l^1(s,x(s),y(s)) \\ & & -f_l^1(s,\overline{x}(s),\overline{y}(s)))dW^l(s)|^2 \\ & & +3\left|\int_0^t (g^1(s,x(s),y(s) - g^1(s,\overline{x}(s),\overline{y}(s))ds\right|^2 \\ & & +3\sum_{k=1}^{m}|I_k(x(t_k)) - I_k(\overline{x}(t_k))|^2. \end{array}$$

By the g B-D-G inequality(lemma (1.2.13)),we get

$$\begin{array}{rcl} \mathbb{E}|N_1(x(t),y(t)) - N_1(\overline{x}(t),\overline{y}(t))|^2 & \leq & 3C_2\int_0^t \mathbb{E}|f^1(s,x(s),y(s)) - f^1(s,\overline{x}(s),\overline{y}(s))|^2 ds \\ & & +3t\int_0^t \mathbb{E}|g^1(s,x(s),y(s)) - g^1(s,\overline{x}(s),\overline{y}(s))|^2 ds \\ & & +\sum_{k=1}^{m}\mathbb{E}|I_k(x(t_k)) - I_k(\overline{x}(t_k))|^2. \end{array}$$

Thus

$$\begin{array}{rl} \leq & C_2\int_0^t a_1\mathbb{E}(|x(s) - \overline{x}(s)|)^2 + b_1\mathbb{E}(|y(s) - \overline{y}(s)|)^2 ds \\ & +3t\int_0^t \alpha_1 E(|x(s) - \overline{x}(s)|)^2 + \beta_1\mathbb{E}(|y(s) - \overline{y}(s)|)^2 ds \\ & +3\sum_{k=1}^{m} d_k\mathbb{E}(|x(t_k) - \overline{x}(t_k)|)^2. \end{array}$$

Therefore,

$$\begin{array}{rcl} \sup_{t\in J}\mathbb{E}|N_1(x(t),y(t)) - N_1(\overline{x}(t),\overline{y}(t))|^2 & \leq & 3(C_2a_1 + \alpha_1 T + l_1)\|x - \overline{x}\|_{PC}^2 \\ & & +3(C_2b_1 + \beta_1 T)\|y - \overline{y}\|_{PC}^2. \end{array}$$

Similarly we have

$$\begin{array}{rcl} \|N_2(x,y) - N_2(\overline{x},\overline{y})\|_{PC} & \leq & 3(C_2a_2 + \alpha_2 T)\|x - \overline{x}\|_{PC} \\ & & +(C_2b_2 + \beta_2 T + l_2)\|y - \overline{y}\|_{PC}. \end{array}$$

Hence

$$\begin{aligned}\|N(x,y)-N(\overline{x},\overline{y})\|_{PC} &= \begin{pmatrix}\|N_1((x,y)-N_1(\overline{x},\overline{y})\|_{PC}\\ \|N_2(x,y)-N_2(\overline{x},\overline{y})\|_{PC}\end{pmatrix}\\ &\leq \sqrt{3}\begin{pmatrix}\sqrt{C_2a_1+\alpha_1T+l_1} & \sqrt{C_2b_1+\beta_1T}\\ \sqrt{C_2a_2+\alpha_2T} & \sqrt{C_2b_2+\beta_2T+l_2}\end{pmatrix}\begin{pmatrix}\|x-\overline{x}\|_{PC}\\ \|y-\overline{y}\|_{PC}\end{pmatrix}.\end{aligned}$$

Therefore

$$\|N(x,y)-N(\overline{x},\overline{y})\|_{PC}\leq M\begin{pmatrix}\|x-\overline{x}\|_{PC}\\ \|y-\overline{y}\|_{PC}\end{pmatrix},\ \text{for all},(x,y),(\overline{x},\overline{y})\in PC\times PC.$$

From Preov fixed point theorem, the mapping N has a unique fixed $(x,y)\in PC\times PC$ which is unique solution of problem (5.0.1). □

3.1.1 Existence Results

In this section we present the existence result under a nonlinearity f^i and $g^i,\ i=1,2$ satisfying a Nagumo type growth conditions:

(H_4) There exist a function $p_i\in L^1(J,\mathbb{R}^+)$ and a continuous nondecreasing function $\psi_i:[0,\infty)\to(0,\infty)$for each $i=1,2$ such that

$$\begin{cases}\mathbb{E}(\|f^1(t,x,y)\|)^2\leq p_1(t)\psi_1(\mathbb{E}(|x|^2+|y|^2))\\ \mathbb{E}(\|f^2(t,x,y)\|)^2\leq p_2(t)\psi_2(\mathbb{E}(|x|^2+|y|^2))\end{cases}$$

with

$$\int_0^T m_1(s)ds<\int_{v_1}^{\infty}\frac{ds}{\psi_1(s)+\psi_2(s)}$$

where $m_1(t)=\max\{4C_2p_1(t),\ 4Tp_2(t)\}$,$v_1=4E|x_0|^2+4\sum_{k=1}^m c_k$

(H_5) There exist a function $p_i\in L^1(J,\mathbb{R}^+)$ and a continuous nondecreasing function $\psi_i:[0,\infty)\to(0,\infty)$ for each $i=3,4$ such that

$$\begin{cases}\mathbb{E}(|g^1(t,x,y)|)^2\leq p_3(t)\psi_3(\mathbb{E}(|x|^2+|y|^2))\\ \mathbb{E}(|g^2(t,x,y)|)^2\leq p_4(t)\psi_4(\mathbb{E}(|x|^2+|y|^2))\end{cases}$$

with

$$\int_0^T m_2(s)ds<\int_{v_2}^{\infty}\frac{ds}{\psi_3(s)+\psi_4(s)}$$

where $m_2(t)=\max\{4C_2p_3(t),4Tp_4(t)\}$,$v_2=4\mathbb{E}|y_0|^2+\sum_{k=1}^m\widetilde{c}_k$.

(H_6) There exist positive constants $c_k, \widetilde{c}_k, k = 1, l, m$ such that

$$\mathbb{E}(|I_k(x_k)|)^2 \leq c_k \quad \text{for all} \quad x \in \mathbb{R}.$$

$$\mathbb{E}(|\overline{I}_k(y_k)|)^2 \leq \widetilde{c}_k \quad \text{for all} \quad y \in \mathbb{R}.$$

Theorem 3.1.2. *Assume that* $(H_4) - (H_6)$ *hold. Then* (5.0.1)*has at least one solution on* J.

Proof. Clearly, the fixe point of N are solutions to (5.0.1), where N is defined in Theorem 6.1.1. In order to apply Lemma 2.4.3, we first show that N is completely continuous. The proof will be given in several steps.

- **Step 1.** $N = (N_1, N_2)$ is continuous.
  Let (x_n, y_n) be a sequence such that $(x_n, y_n) \to (x, y) \in PC \times PC$ as $n \to \infty$. Then

$$\begin{aligned} |N_1(x_n(t), y_n(t)) - N_1(x(t), y(t))|^2 &= \Bigg| \sum_{l=1}^{\infty} \int_0^t f_l^1(s, x_n(s), y_n(s)) dW^l(s) \\ &\quad + \int_0^t g^1(s, x_n(s), y_n(s)) ds \\ &\quad + \sum_{0<t_k\leq t} I_k(x_n(t_k)) \\ &\quad - \sum_{l=1}^{\infty} \int_0^t f_l^1(s, x(s), y(s)) dW^l(s) \\ &\quad - \int_0^t g^1(s, x(s), y(s)) ds - \sum_{0<t_k\leq t} I_k(x(t_k)) \Bigg|^2. \end{aligned}$$

  Since f^1, g^1 is an Carathéodory function and $I_k, \overline{I}_k$ are continuous functions. By Lebesgue dominated convergence theorem, we get

$$\begin{aligned} \sup_{t\in J} \mathbb{E}|N_1(x_n(t), y_n(t)) - N_1(x(t), y(t))|^2 &\leq 3C_2\mathbb{E}\|f^1(., x_n, y_n) - f^1(., x, y)\|^2_{L^2} \\ &\quad +3TE\|g^1(., x_n, y_n) - g^1(., x, y)\|^2_{L^2} \\ &\quad +3\sum_{k=1}^{m} \mathbb{E}|I_k(x_n(t_k)) \\ &\quad -I_k(x(t_k))|^2 \to 0 \text{ as } n \to \infty. \end{aligned}$$

Similarly

$$\begin{aligned}\sup_{t\in J}\mathbb{E}|N_2(x_n(t),y_n(t))-N_1(x(t),y(t))|^2 &\leq 3C_2E\|f^2(.,x_n,y_n)-f^2(.,x,y)\|^2_{L^2}\\ &+3TE\|g^2(.,x_n,y_n)-g^2(.,x,y)\|_{L^2}\\ &+3\sum_{k=1}^{m}\mathbb{E}|\overline{I}_k(y_n(t_k))\\ &-\overline{I}_k(y(t_k))|^2\to 0 \text{ as } n\to\infty.\end{aligned}$$

Thus N is continuous.

- **Step 2.** N maps bounded sets into bounded sets in $PC\times PC$. Indeed, it is enough to show that for any $q>0$ there exists a positive constant l such that for each $(x,y)\in B_q=\{(x,y)\in PC\times PC : \|x\|_{PC}\leq q,\ \|y\|\leq q\}$, we have

$$\|N(x,y)\|_{PC}\leq l=(l_1,l_2).$$

Then for each $t\in J$, we get

$$\begin{aligned}|N_1(x(t),y(t))|^2 &= |x_0+\sum_{l=1}^{\infty}\int_0^t f_l^1(s,x(s),y(s))dW^l(s)\\ &\quad+\int_0^t g^1(s,x(s),y(s))ds+\sum_{k=1}^{m}I_k(x(t_k))|^2\\ &\leq 4|x_0|^2+4|\sum_{l=1}^{\infty}\int_0^t f_l^1(s,x(s),y(s))dW^l(s)|^2\\ &\quad+4|\int_0^t g^1(s,x(s),y(s))ds|^2+4|\sum_{k=1}^{m}I_k(x(t_k))|^2.\end{aligned}$$

Using B-D-G inequality(1.2.13), so we get

$$\begin{aligned}\mathbb{E}|N_1(x(t),y(t))|^2 &\leq 4\mathbb{E}|x_0|^2 + 4C_2\int_0^t E\|f^1(s,x(s),y(s))\|^2 ds \\ &\quad +4T\int_0^t E|g^1(s,x(s),y(s))|^2 d(s) \\ &\quad +4E\sum_{k=1}^{m}|I_k(x(t_k))|^2 \\ &\leq 4\mathbb{E}|x_0|^2 + 4C_2\|p_1\|_{L^1}\psi_1(2q) \\ &\quad +4T\|p_3\|_{L^1}\psi_2(2q)ds + 4\sum_{k=1}^{m} c_k.\end{aligned}$$

Therefore

$$\|N_1(x,y)\|_{PC} \leq 4\mathbb{E}|x_0|^2 + 4C_2\|p_1\|_{L^1}\psi_1(2q) + 4\|p_2\|_{L^1}\psi_2(2q)ds + 4\sum_{k=1}^{m} c_k := l_1.$$

Similarly, we have

$$\|N_2(x,y)\|_{PC} \leq 4\mathbb{E}|x_0|^2 + 4C_2\|p_3\|_{L^1}\psi_2(q) + 4\|p_4\|_{L^1}\psi_4(q)ds + 4\sum_{k=1}^{m} \widetilde{c}_k := l_2.$$

- **Step 3.** N maps bounded sets into equicontinuous sets of $PC \times PC$. Let B_q be a bounded set in $PC\times PC$ as in Step 2. Let $r_1, r_2 \in J, r_1 < r_2$ and $u \in B_q$. Thus we have

$$\begin{aligned}|N_1(x(r_2),y(r_2)) - N_1(x(r_1),y(r_1))|^2 &\leq 3|\sum_{l=1}^{\infty}\int_{r_1}^{r_2}(f_l^1(s,x(s),y(s))dW^l(s)|^2 \\ &\quad +3|\int_{r_1}^{r_2} g^1(s,x(s),y(s))ds|^2 \\ &\quad +3\sum_{r_1\leq t_k\leq r_2}|I_k(x(t_k))|^2.\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{E}|N_1(x(r_2),y(r_2))-N_1(x(r_1),y(r_1))|^2 &\leq 3C_2\int_{r_1}^{r_2}\mathbb{E}\|f^1(s,x(s),y(s))\|^2ds\\ &\quad +3T\int_{r_1}^{r_2}\mathbb{E}|g^1(s,x(s),y(s))|^2ds\\ &\quad +3\sum_{r_1\leq t_k\leq r_2}E|I_k(x(t_k))|^2\\ &\leq 3C_2\psi_1(q)\int_{r_1}^{r_2}p_1(s)ds\\ &\quad +T\psi_2(q)\int_{r_1}^{r_2}p_2(s)ds+3\sum_{r_1\leq t_k\leq t_2}c_k.\end{aligned}$$

The right-hand term tends to zero as $|r_2-r_1|\to 0$. As a consequence of Steps 1 to 3 together with the Arzela-Ascoli, we conclude that N maps B_q into a precompact set in $PC\times PC$.

- **Step 4.** It remains to show that

$$\mathcal{A}=\{(x,y)\in PC\times PC:\ (x,y)=\lambda N(x,y),\lambda\in(0,1)\}$$

is bounded.
Let $(x,y)\in\mathcal{A}$. Then $x=\lambda N_1(x,y)$ and $y=\lambda N_2(x,y)$ for some $0<\lambda<1$. Thus, for $t\in J$, we have

$$\begin{aligned}\mathbb{E}|x(t)|^2 &\leq 4\mathbb{E}|x_0|^2+4\mathbb{E}|\sum_{l=1}^{\infty}\int_0^t f_l^1(s,u(s))dW^l(s)|^2\\ &\quad +4\mathbb{E}|\int_0^t g^1(s,x(s),y(s))ds|^2+4\mathbb{E}|\sum_{k=1}^{m}I_k(x(t_k))|^2\\ &\leq 4\mathbb{E}|x_0|^2+4C_2\int_0^t E\|f^1(s,x(s),y(s))\|^2ds\\ &\quad +4T\int_0^t\mathbb{E}|g^1(s,x(s),y(s))|^2ds+E\sum_{k=1}^{m}|I_1(x(t_k)|^2\\ &\leq 4\mathbb{E}|x_0|^2+4C_2\int_0^t p_1(s)\psi_1(\mathbb{E}|x(s)|^2+\mathbb{E}|y(s)|^2)ds\\ &\quad +4T\int_0^t p_2(s)\psi_2(\mathbb{E}|x(s)|^2+\mathbb{E}|y(s)|^2)ds+4\sum_{k=1}^{m}c_k\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{E}|x(t)|^2 &\leq 4\mathbb{E}|x_0|^2 + 4C_2\int_0^t p_1(s)\psi_1(\mathbb{E}|x(s)|^2 + \mathbb{E}|y(s)|^2)ds \\ &\quad +4T\int_0^t p_2(s)\psi_2(\mathbb{E}|x(s)|^2 + \mathbb{E}|y(s)|^2)ds + 4\sum_{k=1}^m c_k\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}|y(t)|^2 &\leq 4\mathbb{E}|x_0|^2 + 4C_2\int_0^t p_3(s)\psi_2(\mathbb{E}|x(s)|^2 + \mathbb{E}|y(s)|^2)ds \\ &\quad +4T\int_0^t p_4(s)\psi_3(\mathbb{E}|x(s)|^2 + \mathbb{E}|y(s)|^2)ds + 4\sum_{k=1}^m \widetilde{c}_k.\end{aligned}$$

Therefore

$$\mathbb{E}|x(t)|^2 + \mathbb{E}|y(t)|^2 \leq \gamma + \int_0^t p(s)\phi(\mathbb{E}|x(s)|^2 + \mathbb{E}|y(s)|^2)ds,$$

where

$$\gamma = 8\mathbb{E}|x_0|^2 + 4\sum_{k=1}^m (c_k + \widetilde{c}_k),\ p(t) = m_1(t) + m(t),\ \text{and}\ \phi(t) = \sum_{i=1}^m \psi_i(t).$$

By Lemma 6.0.8, we have

$$\mathbb{E}|x(t)|^2 + \mathbb{E}|y(t)|^2 \leq \Gamma^{-1}\left(\int_\gamma^T p(s)ds\right) := K,\ \text{for each}\ t \in J,$$

where

$$\Gamma(z) = \int_\gamma^z \frac{du}{\phi(u)}.$$

Consequently

$$\|x\|_{PC} \leq K \quad \text{and}\ \|y\|_{PC} \leq K.$$

This shows that $\mathcal{E}$ is bounded. As a consequence of Theorem 2.4.3 we deduce that N has a fixed point (x, y) which is a solution to the problem (5.0.1).

□

The goal of the second result of this section is to apply Schauder fixed point. For the study of this problem we first introduce the following hypotheses:

(H_7) There exist nonnegative numbers $\overline{a}_i$ and $\overline{b}_i, c_i$ for each $i = 1, 2$

$$\begin{cases} \mathbb{E}(|f_1(t,x,y)|^2) \leq \overline{a}_1 \mathbb{E}(|x|)^2 + \overline{b}_1 E(|y|)^2 + c_1 \\ \mathbb{E}(|f_2(t,x,y)|^2) \leq \overline{a}_2 \mathbb{E}(|x|)^2 + \overline{b}_2 \mathbb{E}(|y|)^2 + c_2. \end{cases}$$

(H_8) There exist positive constants $\overline{\alpha}_i$ and $\overline{\beta}_i, \lambda_i$ for each $i = 1, 2$

$$\begin{cases} \mathbb{E}(|g_1(t,x,y)|^2) \leq \overline{\alpha}_1 \mathbb{E}(|x|)^2 + \overline{\beta}_1 \mathbb{E}(|y|)^2 + \lambda_1 \\ E(|g_2(t,x,y)|^2) \leq \overline{\alpha}_2 \mathbb{E}(|x|)^2 + \overline{\beta}_2 \mathbb{E}(|y|)^2 + \lambda_2. \end{cases}$$

(H_9) There exist constants $d \geq 0, \overline{d} \geq 0$ and $e_i \geq 0$ for each $i \in \{1, 2\}$ and $k = 1, \dots, m$ such that

$$\begin{cases} \sum_{k=1}^{m} \mathbb{E}|I_k(x)|^2 \leq d\mathbb{E}|x|^2 + e_1 \\ \sum_{k=1}^{m} \mathbb{E}|\overline{I}_k(2)|^2 \leq \overline{d}\mathbb{E}|x|^2 + e_2. \end{cases}$$

for all $x, y, \in \mathbb{R}$.

Theorem 3.1.3. *Assume* $(H_7) - (H_9)$ *hold and*

$$M_{a,b} = \begin{pmatrix} C_2\overline{a}_1 + \overline{\alpha}_1 T + d & C_2\overline{b}_1 + \overline{\beta}_1 T \\ C_2\overline{a}_2 + \overline{\alpha}_2 T & C_2\overline{b}_2 + \overline{\beta}_2 T + \overline{d} \end{pmatrix}$$

converges to zero, then problem (5.0.1) *has at least one solution.*

Proof. Consider the operator $N = (N_1, N_2) : PC \times \times PC \longrightarrow PC \times \times PC$ defined for $x, y \in PC$ by

$$N_1(x,y) = x_0 + \sum_{l=1}^{\infty} \int_0^t f_l^1(s, x(s), y(s))dW^l(s) + \int_0^t g^1(s, x(s), y(s))ds + \sum_{0<t_k<t} I_k(x(t_k)$$

and

$$N_2(x,y) = y_0 + \sum_{l=1}^{\infty} \int_0^t f_l^2(s, x(s), y(s))dW^l(s) + \int_0^t g^2(s, x(s), y(s))ds + \sum_{0<t_k<t} \overline{I}_k(y(t_k)).$$

$$D=\{(x,y)\in PC\times PC:\|x\|_{PC}\le R_1,\|y\|_{PC}\le R_2\},$$

Obviously, the set D is a bounded closed convex set in space $PC\times PC$.
Using B-D-G inequality(1.2.13), we have

$$\begin{aligned}|N_1(x,y)|^2 \quad &= |x_0+\textstyle\sum_{l=1}^{\infty}\int_0^t f_l^1(s,x(s),y(s))dW^l(s)\\ &+\textstyle\int_0^t g^1(s,x(s),y(s))ds+\sum_{0<t_k<t}I_k(x(t_k))|^2\\ &\le 4|x_0|^2+4|\textstyle\sum_{l=1}^{\infty}\int_0^t f_l^1(s,x(s),y(s))dW^l(s)|^2\\ &+4|\textstyle\int_0^t g^1(s,x(s),y(s))ds|^2+|4\displaystyle\sum_{k=1}^{m}I_k(x(t_k))|^2\end{aligned}$$

Using B-D-G inequality(1.2.13), so we get

$$\begin{aligned}\mathbb{E}|N_1(x,y)|^2 &\le 4\mathbb{E}|x_0|^2+4E|\sum_{l=1}^{\infty}\int_0^t f_l^1(s,x(s),y(s))dW^l(s)|^2\\ &\quad +4E|\int_0^t g^1(s,x(s),y(s))ds|^2+4\mathbb{E}|\sum_{k=1}^{m}I_k(x(t_k))|^2\\ &\le 4\mathbb{E}|x_0|^2+4C_2\int_0^t \mathbb{E}|f^1(s,x(s),y(s))|^2ds\\ &\quad +4T\int_0^t E|g^1(s,x(s),y(s))|^2ds+4E\sum_{k=1}^{m}|I_k(x(t_k)|^2\\ &\le 4E|x_0|^2+4C_2\int_0^t(\overline{a}_1\mathbb{E}|x(s)|^2+4\overline{b}_1\mathbb{E}|y(s)|^2+c_1)ds\\ &\quad +4T\int_0^t(\overline{\alpha}_1\mathbb{E}|x(s)|^2+\overline{\beta}_1E|y(s)|^2+\lambda_1)ds\\ &\quad +4dE|x|^2+4e_1\\ &\le 4E|x_0|^2+4C_2\overline{a}_1\int_0^t\mathbb{E}|x(s)|^2ds+4\overline{b}_1C_2\int_0^t\mathbb{E}|y(s)|^2ds+4c_1T\\ &\quad +4T\overline{\alpha}_1\int_0^t\mathbb{E}|x(s)|^2d(s)+4\overline{\beta}_1T\int_0^t\mathbb{E}|y(s)|^2ds+4\lambda_1T\\ &\quad +4d\mathbb{E}|x|^2+4e_1,\end{aligned}$$

thus

$$\sup_{t\in J}E|N_1(x,y)|^2 \quad \le 4(C_2\overline{a}_1+\overline{\alpha}_1T+d)\|x\|_{PC}^2+4(C_2\overline{b}_1+\overline{\beta}_1T)\|y\|_{PC}^2$$

$$+4E|x_0|^2 + 4e_1 + 4Tc_1 + 4T\lambda_1 \tag{3.1.1}$$

From (3.1.1) we obtain that

$$\|N_1(x,y))\|_{PC}^2 \leq \widetilde{a}_1\|x\|_{PC}^2 + \widetilde{b}_1\|y\|_{PC}^2 + \widetilde{c}_1 \tag{3.1.2}$$

where

$\widetilde{a}_1 = 4C_2\overline{a}_1 + 4\overline{\alpha}_1 T + 4d$, $\widetilde{b}_1 = 4C_2\overline{b}_1 + 4\overline{\beta}_1 T$, $\widetilde{c}_1 = 4E|x_0$ and $|^2 + 4e_1 + 4Tc_1 + 4T\lambda_1$.

Similarly we have

$$\|N_2(x,y)\|_{PC}^2 \leq \widetilde{a}_2\|x\|_{PC}^2 + \widetilde{b}_2\|y\|_{PC}^2 + \widetilde{c}_2 \tag{3.1.3}$$

where

$\widetilde{a}_2 = 4C_2\overline{a}_2 + 4\overline{\alpha}_2 T$, $\widetilde{b}_2 = 4C_2\overline{b}_2 + 4\overline{\beta}_2 T + 4\overline{d}$, and $\widetilde{c}_2 = 4E|y_0|^2 + 4e_2 + 4Tc_2 + 4T\lambda_2$.

Now (3.1.2), (3.1.3) can be put together as

$$\begin{aligned}\|N(x,y)\|_{PC} &= \begin{pmatrix}\|N_1(x,y)\|_{PC}\\ \|N_2(x,y)\|_{PC}\end{pmatrix}\\ &\leq 2\begin{pmatrix}\sqrt{C_2\overline{a}_1 + \overline{\alpha}_1 T + d} & \sqrt{C_2\overline{b}_1 + \overline{\beta}_1 T}\\ \sqrt{C_2\overline{a}_2 + \overline{\alpha}_2 T} & \sqrt{C_2\overline{b}_2 + \overline{\beta}_2 T + \overline{d}}\end{pmatrix}\begin{pmatrix}\|x\|_{PC}\\ \|y\|_{PC}\end{pmatrix} + \begin{pmatrix}\sqrt{\widetilde{c}_1}\\ \sqrt{\widetilde{c}_2}\end{pmatrix}.\end{aligned}$$

Therefore

$$\|N(x,y)\|_{PC} \leq M_{a,b}\begin{pmatrix}\|x\|_{PC}\\ \|y\|_{PC}\end{pmatrix} + \begin{pmatrix}\sqrt{\widetilde{c}_1}\\ \sqrt{\widetilde{c}_2}\end{pmatrix}.$$

Since $M_{a,b} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ converges to zero. Next, we look for two positive numbers R_1, R_2 such that if $\|x\|_{PC}^2 \leq R_1$, $\|y\|_{PC}^2 \leq R_2$, then $\|N_1(x,y)\|_{PC}^2 \leq R_1$, $\|N_2(x,y)\|_{PC}^2 \leq R_1$. To this end it is sufficient that

$$\begin{pmatrix}\sqrt{R_1}\\ \sqrt{R_2}\end{pmatrix} \leq M_{a,b}\begin{pmatrix}\sqrt{R_1}\\ \sqrt{R_2}\end{pmatrix} + \begin{pmatrix}\sqrt{\widetilde{c}_1}\\ \sqrt{\widetilde{c}_2}\end{pmatrix}$$

whence

$$(I - M_{a,b})\begin{pmatrix}\sqrt{R_1}\\ \sqrt{R_2}\end{pmatrix} \leq \begin{pmatrix}\sqrt{\widetilde{c}_1}\\ \sqrt{\widetilde{c}_2}\end{pmatrix}$$

that is

$$\begin{pmatrix} \sqrt{R_1} \\ \sqrt{R_2} \end{pmatrix} \leq (I - M_{a,b})^{-1} \begin{pmatrix} \sqrt{\widetilde{c}_1} \\ \sqrt{\widetilde{c}_2}. \end{pmatrix}$$

Set

$$D = \{(x,y) \in PC \times PC : \|x\|_{PC}^2 \leq R_1, \|y\|_{PC}^2 \leq R_2\},$$

It clear that $N(D) \subset D$. Hence by Theorem (2.2.4),the operator N has at least one fixe point which is solution of (5.0.1) □

3.1.2 Example

Now we an example of application for our main result. We consider the following problem

$$\begin{cases} dx(t) & = \sum_{l=1}^{\infty}(a_{2l+1}\sin k^2 x + a_{2l}\cos l^2 y)dW^l(t) \\ & \quad +d_1(t + x(t) + y(t))dt,\ t \in [0,1], t \neq \frac{1}{2} \\ dy(t) & = \sum_{l=1}^{\infty}(b_{2l+1}\sin k^2 x + b_{2l}\cos l^2 y)dW^l(t) \\ & \quad +d_2(t + x(t) + y(t))dt,\ t \in [0,1], t \neq \frac{1}{2} \\ \Delta x(t) & = c_1 \dfrac{x(t)}{1+|x(t)|}, \quad t = \dfrac{1}{2} \\ \Delta y(t) & = c_1 \dfrac{y(t)}{1+|y(t)|}, \quad t = \dfrac{1}{2} \\ x(0) & = x_0, \\ y(0) & = y_0 \end{cases} \tag{3.1.4}$$

where $c_1, c_2 \in \mathbb{R},\ (a_l)_{l\in\mathbb{N}}, (b_l)_{l\in\mathbb{N}} \in l^2,\ f_1, f_2 : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$f_1(t,x,y) = \sum_{k=1}^{\infty}(a_{2k+1}\sin k^2 x + a_{2k}\cos k^2 y),\ f_2(t,x,y) = \sum_{k=1}^{\infty}(b_{2k+1}\sin k^2 x + b_{2k}\cos k^2 y).$$

We deduce

$$\begin{aligned} \|f_1(t,x,y)\|^2 &= \sum_{k=1}^{\infty}(a_{2k+1}\sin k^2 x + a_{2k}\cos k^2 y)^2 \\ &\leq 2\sum_{k=1}^{\infty}(a_{2k+1}^2 + a_{2k}^2) \\ &\leq 4\sum_{k=1}^{\infty} a_k^2 < \infty. \end{aligned}$$

and

$$\begin{aligned} \|f_1(t,x,y)\|^2 &= \sum_{k=1}^{\infty}(b_{2k+1}\sin k^2x + b_{2k}\cos k^2y)^2 \\ &\leq 2\sum_{k=1}^{\infty}(b_{2k+1}^2 + b_{2k}^2) \\ &\leq 4\sum_{k=1}^{\infty} b_k^2 < \infty. \end{aligned}$$

Hence

$$\mathbb{E}|f_1(t,x,y)|^2 \leq 4\sum_{k=1}^{\infty} a_k^2 + \mathbb{E}(|x|^2 + |y|^2) \text{ for all } x,y \in \mathbb{R},$$

and

$$\mathbb{E}|f_2(t,x,y)|^2 \leq 4\sum_{k=1}^{\infty} b_k^2 + \mathbb{E}(|x|^2 + |y|^2) \text{ for all } x,y \in \mathbb{R}.$$

Also we have

$$I_1(x) = c_1\frac{x(t)}{1+|x(t)|},\ I_2(y) = c_2\frac{y}{1+|y|} \Rightarrow E|I_1(x)|^2 \leq c_1,\ \mathbb{E}|I_2(x)|^2 \leq c_2,$$

and

$$g^1(t,x,y) = d_1(t+x+y),\ g^2(t,x,y) = d_2(t+x+y),\ x,y \in \mathbb{R},\ t \in [0,1].$$

Hence

$$\mathbb{E}|g^1(t,x,y)|^2 \leq 3d_1^2(1+\mathbb{E}|x|^2+\mathbb{E}|y|^2),\ \mathbb{E}|g^2(t,x,y)|^2 \leq 3d_2^2(1+E|x|^2+E|y|^2).$$

Thus all the conditions of Theorem 5.1.3 hold, then Problem (6.0.1) has at least one solution.

Remark 3.1.1. *In the case where $f_2 = 0$, $g^1 = g^2 = 0$, and $I_1 = I_2 = 0$ in our example we includes the equation for Brownian motion on the group of diffeomorphism of the circle(see [8, 106]).*

Chapter 4

Semilinear Systems of Impulsive Stochastic Differential Equations

In this chapter, we investigate the following first order stochastic impulsive equation , driven by fractional Brownian motion with the Hurst index $H > 1/2$:

$$\begin{cases} dx(t) &= (Ax(t) + f^1(t, x(t), y(t))dt \\ &\quad + \sum_{l=1}^{\infty} \sigma_l^1(t, x(t)), y(t))dB_l^H(t), \;\; t \in [0, T], t \neq t_k, \\ dy(t) &= (Ay(t) + f^2(t, x(t), y(t)))dt \\ &\quad + \sum_{l=1}^{\infty} \sigma_l^2(t, x(t), y(t))dB_l^H(t), \;\; t \in J, t \neq t_k, \\ \Delta x(t) &= I_k(x(t_k)), \quad t = t_k \quad k = 1, 2, \ldots, m \\ \Delta y(t) &= \overline{I}_k(y(t_k)), \quad t = t_k \quad k = 1, 2, \ldots, m \\ x(0) &= x_0, \\ y(t) &= y_0, \end{cases} \tag{4.0.1}$$

where X is a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$, $A : D(A) \subset X \longrightarrow X$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $(S(t))_{t \geq 0}$ in X and $f^1, f^2 : [0, T] \times X \times X \longrightarrow X$ are given functions, B_l^H is an infinite sequence of mutually independent fractional Brownian motions, $l = 1, 2, \ldots$, with Hurst parameter H, $I_k, \overline{I}_k \in C(X, X)$ $(k = 1, 2, \ldots, m)$, $\sigma_l^1, \sigma_l^2 : J \times X \times X \to L_Q^0(Y, X)$. Here, $L_Q^0(Y, X)$ denotes the space of all Q-Hilbert-Schmidt operators from Y into X, which will be also defined in the next section. Moreover, the fixed times t_k satisfy $0 < t_1 < t_2 < \ldots < t_m < T$, and $y(t_k^-)$ and $y(t_k^+)$ denote the left and right limits of $y(t)$ at $t = t_k$.

$$\begin{cases} \sigma(t,x,y) = (\sigma_1(t,x,y), \sigma_2(t,x,y), \ldots), \\ \|\sigma(t,x,y)\|^2 = \sum_{j=1}^{\infty} \|\sigma_j(t,x,y)\|^2_{L_Q^0} < \infty \end{cases} \tag{4.0.2}$$

where $\sigma(\cdot,\cdot,\cdot) \in \ell^2$, and ℓ^2 is given by

$$\ell^2 = \{\phi = (\phi_j)_{j\geq 1} : \ [0,T]\times X\times X \to L_Q^0(Y,X) \quad : \|\phi(t,x,y)\|^2 = \sum_{j=1}^{\infty} \|\phi_j(t,x,y)\|^2_{L_Q^0} < \infty\}.$$

We denote $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$. It is obvious that system (5.0.1) can be seen as a fixed point problem for the model

$$\begin{cases} dz(t) &= A_* z(t) + f(t,z(t))dt + \sum_{l=1}^{\infty} \sigma_l(t,z))dB_l^H(t), \ t \in [0,T], t \neq t_k, \\ \Delta z(t) &= I_k^*(z(t_k)), \quad t = t_k \quad k = 1, 2, \ldots, m \\ z(0) &= z_0, \end{cases} \tag{4.0.3}$$

where

$$z(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, A_* = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \ f(t,z) = \begin{bmatrix} f^1(t,x(t),y(t)) \\ f^2(t,x(t),y(t)) \end{bmatrix}, \ \sigma_l(t,z) = \begin{bmatrix} \sigma_l^1(t,x,y) \\ \sigma_l^2(t,x,y) \end{bmatrix}$$

and $z_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.

4.1 Existence and uniqueness of mild solution

Let $J_k = (t_k, t_{k+1}],\ k = 1, 2, \dots, m$. In order to define a solution for Problem (5.0.1), consider the following space of pice-wise continuous functions

$$\begin{aligned} PC &= \{x : \Omega \times [0,T] \longrightarrow X,\ x \in C(J_k, L^2(\Omega, X)),\ k = 1, \dots, m \text{ such that} \\ &\quad x(t_k^+, .) \text{ and } x(t_k^-, .) \text{ exist with } x(t_k^-, .) = x(t_k, .) \quad \text{and} \\ &\quad \sup_{t\in[0,T]} E\|x(t, .)\|^2 < \infty \text{ almost surely}\}, \end{aligned}$$

endowed with the norm

$$\|x\|_{PC} = \sup_{s\in[0,T]} (E\|x(s, .)\|^2)^{\frac{1}{2}}.$$

It is not difficult to check that PC is a Banach space with norm $\|\cdot\|_{PC}$. First, we will list the following hypotheses which will be imposed in our main theorem. In this section, we assume that there exists $M > 0$ such that

$$\|S(t)\| \leq M, \quad \text{for every} \quad t \in [0, T].$$

(H_1) There exist functions $a_i,\ b_i \in L^1([0, T], \mathbb{R}^+)$ such that

$$|f^i(t, x, y) - f^i(t, \overline{x}, \overline{y})|_X^2 \leq a_i(t)|x - \overline{x}|_X^2 + b_i(t)|y - \overline{y}|_X^2, \quad i = 1, 2$$

for all $x,\ y,\ \overline{x},\ \overline{y} \in X$.

(H_2) There exist functions $\alpha_i, \beta_i \in L^1([0, T], \mathbb{R}^+)$ such that

$$\|\sigma^i(t, x, y) - \sigma^i(t, \overline{x}, \overline{y})\|^2 \leq \alpha_i(t)|x - \overline{x}|_X^2 + \beta_i(t)|y - \overline{y}|_X^2$$

and

The function $\sigma : J \times X \times X \longrightarrow L_Q^0(Y, X)$ satisfies

$$\sum_{l=1}^{\infty} \int_0^T \|\sigma_l^i(s, x(s), y(s))\|_{L_Q^0}^2 ds < \infty. \tag{4.1.1}$$

for all $x,\ y,\ \overline{x},\ \overline{y} \in X$ and a.e $t \in J$.

Now, we first define the concept of mild solution to our problem.

Definition 4.1.1. *An $X-$ valued stochastic process $u=(x,y)\in PC\times PC$ is said to be a solution of* (5.0.1) *with respect to the probability space $(\Omega,\mathcal{F},\mathbb{P})$, if:*

1) $u(t)$ is $\mathcal{F}_t$-adapted for all $t\in J_k=(t_k,t_{k+1}]\quad k=1,2,\ldots,m$

2) $u(t)$ is right continuous and has limit on the left almost surely;

3) $u(0)=(x_0,y_0)$

4) $u(t)$ satisfies for all $t\in[0,T]$ and almost surely that

$$\begin{cases} x(t) &= S(t)x_0+\displaystyle\int_0^t S(t-s)f^1(s,x(s),y(s))ds \\ &+ \displaystyle\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(t,x(s),y(s))dB_l^H(s) \\ &\quad+\displaystyle\sum_{0<t_k<t} S(t-t_k)I_k(x(t_k)),\quad \mathbb{P}-a.s,\quad t\in J \\ \\ y(t) &= S(t)y_0+\displaystyle\int_0^t S(t-s)f^2(s,x(s),y(s))ds \\ &+ \displaystyle\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^2(t,x(s),y(s))dB_l^H(s) \\ &\quad+\displaystyle\sum_{0<t_k<t} S(t-t_k)I_k(y(t_k)),\quad \mathbb{P}-a.s,\quad t\in J. \end{cases} \tag{4.1.2}$$

Notice that, thanks to (4.1.1) and the fact that $H\in(1/2,1)$, (1.4.17) holds, which implies that the stochastic integrals in (4.1.2) are well-defined since $S(\cdot)$ is a strongly continuous semigroup, for every $t\in[0,T]$, and that this concept of solution can be considered as more general than the classical concept of solution to equation (5.0.1). A continuous solution of (4.1.2) is called a mild solution of (5.0.1).

Definition 4.1.2. *The map $f:J\times X\to X$ is said to be L^2-Caratheodory if*

i) $t\mapsto f(t,u)$ is measurable for each $u\in X$;

ii) $u\mapsto f(t,u)$ is continuous for almost all $t\in J$;

iii) for each $q>0$, there exists $\alpha_q\in L^1(J,\mathbb{R}^+)$ such that

$E|f(t,u)|_X^2 \leq \alpha_q$, *for all* $u \in X$ *such that* $E|u|_X^2 \leq q$ *and for a.e.* $t \in J$.

Let us now prove the existence and uniqueness of mild solution for (5.0.1) by using Perov's fixed point theorem.

Theorem 4.1.1. *Assume that* (H_1) *and* (H_2) *hold. Then, problem* (5.0.1) *possesses a unique mild solution on* $[0,T]$.

Proof. The proof will be split into several steps.

Step 1. Consider the problem

$$\begin{cases} dx(t) &= (Ax(t) + f^1(t,x(t),y(t))dt \\ &\quad + \sum_{l=1}^{\infty} \sigma_l^1(t,x(t)),y(t))dB_l^H(t),\ t \in [0,t_1], \\ dy(t) &= (Ay(t) + f^2(t,x(t),y(t)))dt \\ &\quad + \sum_{l=1}^{\infty} \sigma_l^2(t,x(t),y(t))dB_l^H(t),\ t \in [0,t_1], \\ x(0) &= x_0, \\ y(0) &= y_0. \end{cases} \tag{4.1.3}$$

Let

$$D_{t_0} = \{x:\ \Omega\times[0,t_1] \longrightarrow X,\ x \in C(J_1,L^2(\Omega,X)) :\quad \sup_{t\in[0,t_1]} E|x(t,\cdot)|_X^2 < \infty\},$$

Consider the operator

$$P^0 : D_{t_0} \times D_{t_0} \to D_{t_0} \times D_{t_0}$$

defined by

$$P^0(x,y) = (P_1^0(x,y), P_2^0(x,y)),\ (x,y) \in D_{t_0} \times D_{t_0}$$

where

$$\begin{cases} P_1^0(x,y) &= S(t)x_0 + \int_0^t S(t-s)f^1(s,x(s),y(s))ds \\ & \quad + \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(t,x(s),y(s))dB_l^H(s), \quad t\in[0,t_1]\,. \\ \\ P_2^0(x,y) &= S(t)y_0 + \int_0^t S(t-s)f^2(s,x(s),y(s))ds \\ & \quad + \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^2(t,x(s),y(s))dB_l^H(s), \quad t\in[0,t_1]. \end{cases} \tag{4.1.4}$$

The operators in (4.1.4) are well-defined. In other words, given $(x,y)\in D_{t_0}\times D_{t_0}$, we see that $P^0(x,y)\in D_{t_0}\times D_{t_0}$ as well. We will use Theorem 2.4.1 to prove that P^0 possesses a fixed point. Let $(x,y),(\overline{x},\overline{y})\in D_{t_0}\times D_{t_0}$, then for each $t\in[0,t_1]$,by Lemma 1.4.8 and assumptions $(H_1),(H_2)$ imply

$$\begin{aligned} & E|P_1^0(x(t),y(t)) - P_1^0(\overline{x}(t),\overline{y}(t))|_X^2 \\ & \quad \le 2M^2t_1\int_0^t a_1(s)E|x(s)-\overline{x}(s)|_X^2 + b_1(s)E|y(s)-\overline{y}(s)|_X^2 ds \\ & \quad\quad 2c_H H(2H-1)t_1^{2H-1}\int_0^t M^2ds\int_0^t \alpha_1(s)E|x(s)-\overline{x}(s)|_X^2ds \\ & \quad\quad +2c_H H(2H-1)t_1^{2H-1}\int_0^t M^2ds\int_0^t \beta_1(s)E|y(s)-\overline{y}(s)|_X^2ds \\ & \le \int_0^t \alpha(s)e^{\tau\widehat{\alpha}(s)}e^{-\tau\widehat{\alpha}(s)}\sup_{s\in[0,t_1]}E|x(s)-\overline{x}(s)|_X^2ds \\ & \quad + \int_0^t \alpha(s)e^{\tau\widehat{\alpha}(s)}e^{-\tau\widehat{\alpha}(s)}\sup_{s\in[0,t_1]}E|y(s)-\overline{y}(s)|_X^2ds \\ & \le \int_0^t \alpha(s)e^{\tau\widehat{\alpha}(s)}ds\|x-\overline{x}\|_*^2 + \int_0^t \alpha(s)e^{\tau\widehat{\alpha}(s)}ds\|y-\overline{y}\|_*^2 \\ & \le \tfrac{1}{\tau}\int_0^t (e^{\tau\widehat{\alpha}(s)})'ds\|x-\overline{x}\|_*^2 + \frac{1}{\tau}\int_0^t (e^{\tau\widehat{\alpha}(s)})'ds\|y-\overline{y}\|_*^2 \\ & \le \tfrac{1}{\tau}e^{\tau\widehat{\alpha}(t)}\|x-\overline{x}\|_*^2 + \tfrac{1}{\tau}e^{\tau\widehat{\alpha}(t)}\|y-\overline{y}\|_*^2. \end{aligned}$$

Therefore

$$e^{-\tau\widehat{\alpha}(t)}E|L_1(x(t),y(t)) - L_1(\overline{x}(t),\overline{y}(t))|_X^2 \quad \le \quad \tfrac{1}{\tau}\|x-\overline{x}\|_*^2 + \tfrac{1}{\tau}\|y-\overline{y}\|_*^2,$$

where $\|\cdot\|_*$ is the Bielecki-type norm on D_{t_0} defined by

$$\|x\|_*^2 = \sup_{t\in[0,t_1]} E|x(t,.)|_X^2 e^{-\tau\widehat{\alpha}(t)}$$

where

$$\widehat{\alpha}(t) = \int_0^t \alpha(s)ds, \quad t \in [0,t_1],$$

and

$$\alpha(s) = \max\{2M^2 t_1 a_1(s) + 2c_H H(2H-1)t_1^{2H} M^2 \alpha_1(s), 2M^2 t_1 b_1(s) \\ +2c_H H(2H-1)t_1^{2H} M^2 \beta_1(s)\}.$$

Hence

$$\|P_1^0(x,y) - P_1^0(\overline{x},\overline{y})\|_*^2 \;\leq\; \tfrac{1}{\tau}\|x-\overline{x}\|_*^2 + \tfrac{1}{\tau}\|y-\overline{y}\|_*^2.$$

Using the fact that for all $a,b \geq 0$ we have $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, we conclude that

$$\|P_1^0(x,y) - P_1^0(\overline{x},\overline{y})\|_* \;\leq\; \tfrac{1}{\sqrt{\tau}}\|x-\overline{x}\|_* + \tfrac{1}{\sqrt{\tau}}\|y-\overline{y}\|_*.$$

Similar computations for N_1 yield

$$\|P_2^0(x,y) - P_2^0(\overline{x},\overline{y})\|_* \;\leq\; \tfrac{1}{\sqrt{\tau}}\|x-\overline{x}\|_* + \tfrac{1}{\sqrt{\tau}}\|y-\overline{y}\|_*.$$

Thus

$$\begin{aligned}\|P^0(x,y) - P^0(\overline{x},\overline{y})\|_* &= \begin{pmatrix}\|P_1^0((x,y) - P_1^0(\overline{x},\overline{y})\|_* \\ \|P_2^0(x,y) - P_2^0(\overline{x},\overline{y})\|_*\end{pmatrix} \\ &\leq \frac{1}{\sqrt{\tau}}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\|x-\overline{x}\|_* \\ \|y-\overline{y}\|_*\end{pmatrix}.\end{aligned}$$

Hence

$$\|P^0(x,y) - P^0(\overline{x},\overline{y})\|_* \leq \frac{1}{\sqrt{\tau}} M_{\alpha,\beta}\begin{pmatrix}\|x-\overline{x}\|_* \\ \|y-\overline{y}\|_*\end{pmatrix},$$

for all $(x,y), (\overline{x},\overline{y}) \in D_{t_0} \times D_{t_0}$, where

$$M_{\alpha,\beta} = \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}.$$

If we choose a suitable $\sqrt{\tau} > 2$ such that the matrix

$$\frac{\|M_{\alpha,\beta}\|}{\sqrt{\tau}} < 1,$$

then $\frac{M_{\alpha,\beta}}{\tau}$ is nonnegative, $I - \frac{\|M_{\alpha,\beta}\|}{\tau}$ is non singular and

$$\left(I - \frac{M_{\alpha,\beta}}{\sqrt{\tau}}\right)^{-1} = I + \frac{M_{\alpha,\beta}}{\sqrt{\tau}} + \frac{M^2_{\alpha,\beta}}{\tau} + \dots$$

From Lemma 2.2.4, we obtain that $\frac{M_{\alpha,\beta}}{\sqrt{\tau}}$ converges to zero. As a consequence of Perov's fixed point theorem, P^0 has a unique fixed $(x, y) \in D_{t_0} \times D_{t_0}$ which is the unique solution of problem (4.1.3). Let us denote this solution by (x_1, y_1).

Step 2. Now consider the problem

$$\begin{cases} dx(t) &= (Ax(t) + f^1(t, x(t), y(t))dt \\ &\quad + \displaystyle\sum_{l=1}^{\infty} \sigma_l^1(t, x(t)), y(t))dB_l^H(t), \ t \in (t_1, t_2], \\ dy(t) &= (Ay(t) + f^2(t, x(t), y(t)))dt \\ &\quad + \displaystyle\sum_{l=1}^{\infty} \sigma_l^2(t, x(t), y(t))dB_l^H(t), \ t \in (t_1, t_2], \\ x(t_1^+) &= x_1(t_1^-) + I_1(x_1(t_1)), \\ y(t_1^+) &= y_1(t_1^-) + \overline{I}_1(y_1(t_1)). \end{cases} \tag{4.1.5}$$

Let

$$D_{t_1} = \{\, x \in C((t_1, t_2], L^2(\Omega, X)) \text{ such that } \ x(t_1^+, .) \text{ and } x(t_1^-, .)$$

$$\text{exist with } x(t_1^-, .) = x(t_1, .) \text{and } \sup_{t \in (t_1, t_2]} E|x(t, .)|_X^2 < \infty\},$$

Set

$$C_1 = D_{t_0} \cap D_{t_1}$$

Consider the operator $P : C_1 \times C_1 \to C_1 \times C_1$ defined by

$$P^1(x, y) = (P_1^1(x, y), P_2^1(x, y)), \ (x, y) \in C_1 \times C_1$$

where

$$\begin{cases} P_1^1(x,y) &= S(t-t_1)[x_1(t_1)+I_1(x_1(t_1^-))]+\int_{t_1}^t S(t-s)f^1(s,x(s),y(s))ds \\ & \quad +\sum_{l=1}^{\infty}\int_{t_1}^t S(t-s)\sigma_l^1(t,x(s),y(s))dB_l^H(s), \quad t\in[t_1,t_2]. \\ \\ P_2^1(x,y) &= S(t-t_1)[y_1(t_1)+I_1(y_1(t_1^-))]+\int_{t_1}^t S(t-s)f^2(s,x(s),y(s))ds \\ & \quad +\sum_{l=1}^{\infty}\int_{t_1}^t S(t-s)\sigma_l^2(t,x(s),y(s))dB_l^H(s), \quad t\in[t_1,t_2]. \end{cases}$$

As in Step 1, we can show that P^1 is well defined and the integral equation possesses a unique fixed point (x,y) which is a solution to problem (4.1.5). Denote this solution by (x_2,y_2).

Step 3 We continue this process taking into account that $(x_m,y_m)=(x\mid_{[t_m,T]}, y\mid_{[t_m,T]})$ is a solution of the problem

$$\begin{cases} dx(t) &= (Ax(t)+f^1(t,x(t),y(t))dt \\ & \quad +\sum_{l=1}^{\infty}\sigma_l^1(t,x(t)),y(t))dB_l^H(t), \ t\in(t_m,T], \\ dy(t) &= (Ay(t)+f^2(t,x(t),y(t)))dt \\ & \quad +\sum_{l=1}^{\infty}\sigma_l^2(t,x(t),y(t))dB_l^H(t), \ t\in(t_m,T], \\ x(t_m^+) &= x_m(t_m^-)+I_m(x_m(t_m)), \\ y(t_m^+) &= y_m(t_m^-)+\overline{I}_m(y_m(t_1)). \end{cases} \tag{4.1.6}$$

Let

$$\begin{aligned} D_{t_m} = \{ & x\in C((t_m,T],L^2(\Omega,X))\text{such that } x(t_m^+,.)\text{ and } x(t_m^-,.)\text{ exist with} \\ & x(t_m^-,.)=x(t_m,.)\text{and} \sup_{t\in(t_m,T]} E|x(t,.)|_X^2<\infty \text{ almost surely}\}, \end{aligned}$$

Set

$$C_m=\cap_{k=0}^m D_{t_k}$$

Then, there exists a fixed point $(x_m(t),y_m(t))$ of $P^m=(P_1^m,P_2^m):C_m\times C_m\to C_m\times C_m$. The unique solution (x,y) of problem (5.0.1) is then defined by

$$\begin{cases} P_1^m(x,y) &= S(t-t_m)[x_m(t_m)+I_m(x_1(t_m^-))] + \displaystyle\int_{t_m}^t S(t-s)f^1(s,x(s),y(s))ds \\ & \quad + \displaystyle\sum_{l=1}^{\infty}\int_{t_m}^t S(t-s)\sigma_l^1(t,x(s),y(s))dB_l^H(s), \quad t\in[t_m,T]. \\ \\ P_2^m(x,y) &= S(t-t_m)[y_m(t_m)+I_m(y_1(t_m^-))] + \displaystyle\int_{t_m}^t S(t-s)f^2(s,x(s),y(s))ds \\ & \quad + \displaystyle\sum_{l=1}^{\infty}\int_{t_m}^t S(t-s)\sigma_l^2(t,x(s),y(s))dB_l^H(s), \quad t\in[t_m,T]. \end{cases}$$

$$(x(t),y(t)) = \begin{cases} (x_1(t),y_1(t)), & \text{if, } t\in[0,t_1], \\ (x_2(t),y_2(t)), & \text{if, } t\in(t_1,t_2], \\ \dots & \\ (x_{m+1}(t),y_{m+1}(t)), & \text{if, } t\in(t_m,T] \end{cases} \tag{4.1.7}$$

and the proof is finished. □

The second result in this section dealing with the existence of solutions to our problem will be obtained by applying the Leary-Schauder fixed point theorem. To this end we first need to introduce the following hypotheses:

(H_3) There exist functions $\overline{a}_i, \overline{b}_i, c_i \in L^1([0,T],\mathbb{R}^+)$ such that each

$$|f^i(t,x,y)|_X^2 \le \overline{a}_i(t)|x|_X^2 + \overline{b}_i(t)|y|_X^2 + c_i(t), \ i=1,2.$$

for all $x,y\in X$, and a.e. $t\in J$.

(H_4) There exist functions $\overline{\alpha}_i \in L^1([0,T],\mathbb{R}^+)$ and $\overline{\beta}_i, \overline{c}_i \in L^1([0,T],\mathbb{R}^+)$ such that

$$\|\sigma^i(t,x,y)\|^2 \le \overline{\alpha}_i(t)|x|_X^2 + \overline{\beta}_i(t)|y|_X^2 + \overline{c}_i(t), \quad i=1,2$$

for all $x,y\in X$, and a.e. $t\in J$.

(H_5) The semigroup $\{S(t)\}_{t>0}$ is compact in X

(H_6) there exist constants $d_k, \lambda_k \ge 0$ and $\overline{d}_k, \overline{\lambda}_k \ge 0$, $k=1,\dots,m$ such that

$$|I_k(x)|_X^2 \le d_k|x|_X^2 + \lambda_k$$
$$|\overline{I}_k(y)|^2 \le \overline{d}_k|y|_X^2 + \overline{\lambda}_k$$

for all x, $y\in X$.

(H_7) f^i, g^i are L^2-Carathéodory maps.

We now prove our second goal of this section.

Theorem 4.1.2. *Assume conditions* $(H_3)-(H_7)$ *hold. Then, problem* (5.0.1) *has at least one solution.*

Proof. We transform problem (4.1.3) into a fixed point problem (4.1.3). Consider the operator $N : PC \times PC \to PC \times PC$ defined by

$$N(x,y) = (N_1(x,y), N_2(x,y)),\ (x,y) \in PC \times PC$$

where

$$N_1(x,y) = S(t)x_0 + \int_0^t S(t-s)f^1(s,x(s),y(s))ds + \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(t,x(s),y(s))dB_l^H(s)$$

$$+ \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k)), \quad t \in [0,T].$$

$$N_2(x,y) = S(t)y_0 + \int_0^t S(t-s)f^2(s,x(s),y(s))ds + \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^2(t,x(s),y(s))dB_l^H(s)$$

$$+ \sum_{0<t_k<t} S(t-t_k)\overline{I}_k(y(t_k)), \quad t \in [0,T].$$

Clearly, the fixed points of N are solutions to (5.0.1). In order to apply Theorem 2.4.3, we first show that N is completely continuous. The proof will be carried out in several steps.

Step 1. N is continuous.
Let (x_n, y_n) be a sequence such that $(x_n, y_n) \to (x,y) \in PC \times PC$ as $n \to \infty$, and observe that thanks to $(H_3)-(H_6)$ and (H_7), I_k, $\overline{I}_k$, $k = 1, 2, .., m$, are continuous.

Then

$$\begin{aligned}
&|N_1(x_n(t),y_n(t))-N_1(x(t),y(t))|_X^2\\
&\quad\le 3\left|\sum_{l=1}^{\infty}\int_0^t S(t-s)(\sigma_l^1(s,x_n(s),y_n(s))\sigma_l^1(s,x(s),y(s)))dB_l^H(s)\right|_X^2\\
&\qquad+3\left|\int_0^t S(t-s)(f^1(s,x_n(s),y_n(s))-f^1(s,x(s),y(s)))ds\right|_X^2\\
&\qquad+3\left|\sum_{0<t_k<t}S(t-t_k)I_k(x_n(t_k))-\sum_{0<t_k<t}S(t-t_k)I_k(x(t_k))\right|_X^2.
\end{aligned}$$

Thus, we deduce

$$\begin{aligned}
&E|N_1(x_n(t),y_n(t))-N_1(x(t),y(t))|_X^2\\
&\quad\le 3M^2c_2(H)H(2H-1)T^{2H-1}\int_0^t E\|\sigma^1(s,x_n(s),y_n(s))-\sigma^1(s,x(s),y(s))\|^2ds\\
&\qquad+3M^2\int_0^t E|f^1(s,x_n(s),y_n(s))-f^1(s,x(s),y(s))|_X^2ds\\
&\qquad+3M^2\sum_{0<t_k<t}E|I_k(x_n(t_k))-I_k(x(t_k))|_X^2.
\end{aligned}$$

By the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
&\sup_{t\in J}E|N_1(x_n(t),y_n(t))-N_1(x(t),y(t))|_X^2\\
&\quad\le \bar{c}_H\int_0^T E\|\sigma^1(s,x_n(s),y_n(s))-\sigma^1(s,x(s),y(s))\|^2ds\\
&\qquad+3M^2\int_0^T E|f^1(s,x_n(s),y_n(s))-f^1(s,x(s),y(s))|_X^2ds\\
&\qquad+3M^2\sum_{k=1}^{m}E|I_k(x_n(t_k))-I_k(x(t_k))|_X^2\to 0 \text{ as } n\to\infty,
\end{aligned}$$

where

$$\bar{c}_H=3M^2c_2(H)H(2H-1)T^{2H-1}.$$

Similarly

$$\begin{aligned}
&\sup_{t\in J} E|N_2(x_n(t),y_n(t)) - N_1(x(t),y(t))|_X^2 \\
&\leq \bar{c}_H \int_0^T E\|\sigma^2(s,x_n(s),y_n(s)) - \sigma^2(s,x(s),y(s))\|^2 ds \\
&\quad +2M^2\int_0^T E|f^2(s,x_n(s),y_n(s)) - f^2(s,x(s),y(s))|_X^2 ds \\
&\quad +3M^2\sum_{k=1}^{m} E|\overline{I}_k(y_n(t_k)) - \overline{I}_k(y(t_k))|_X^2 \to 0 \text{ as } n\to\infty.
\end{aligned}$$

Therefore, N is continuous.

Step 2. N maps bounded sets into bounded sets in $PC\times PC$. Indeed, it is enough to show that for any $q>0$, there exists a positive constant κ such that for each $(x,y)\in B_q=\{(x,y)\in PC\times PC : \|x\|_{PC}\leq q,\ \|y\|\leq q\}$, we have

$$\|N(x,y)\|_{PC}\leq \kappa = (\kappa_1,\kappa_2).$$

Then, for each $t\in J$ and thanks to Lemma 1.4.6,

$$\begin{aligned}
E|N_1(x(t),y(t))|_X^2 &\leq 4M^2E|x_0|_X^2 + 4M^2\|c_1\|_{L^1} + 4M^2c_2(H)H(2H-1)T^{2H-1}\|\overline{c}_1\|_{L^1} \\
&\quad +4M^2\Big(\sum_{0<t_k<t}(d_kE|x|_X^2+\lambda_k)\Big) \\
&\quad +4M^2\int_0^t (c_2(H)H(2H-1)T^{2H-1}\overline{\alpha}_1(s)+\overline{a}_1(s))E|x(s)|_X^2 ds \\
&\quad +4M^2\int_0^t (c_2(H)H(2H-1)T^{2H-1}\overline{\beta}_1(s)+\overline{b}_1(s))E|y(s)|_X^2 ds \\
&\leq 4M^2E|x_0|_X^2 + 4M^2\|c_1\|_{L^1} + 4M^2c_2(H)H(2H-1)T^{2H-1}\|\overline{c}_1\|_{L^1} \\
&\quad +4M^2\sum_{0<t_k<t}\lambda_k \\
&\quad +4M^2\int_0^t (c_2(H)H(2H-1)T^{2H-1}\overline{\alpha}_1(s)+\overline{a}_1(s))q\,ds \\
&\quad +4M^2\int_0^t (c_2(H)H(2H-1)T^{2H-1}\overline{\beta}_1(s)+\overline{b}_1(s))q\,ds \\
&\quad +4M^2\sum_{0<t_k<t} d_kq = \kappa_1.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}E|N_2(x,y)|_X^2 \leq\ & 4M^2E|y_0|_X^2 + 4M^2\|c_2\|_{L^1} + 4M^2c_2(H)H(2H-1)T^{2H-1}\|\overline{c}_2\|_{L^1}\\ &+4M^2\sum_{0<t_k<t}\overline{\lambda}_k\\ &+4M^2\int_0^t (c_2(H)H(2H-1)T^{2H-1}\overline{\alpha}_2(s)+\overline{a}_2(s))q\,ds\\ &+4M^2\int_0^t (c_2(H)H(2H-1)T^{2H-1}\overline{\beta}_2(s)+\overline{b}_2(s))q\,ds\\ &+4M^2\sum_{0<t_k<t}\overline{d}_kq=\kappa_2.\end{aligned}$$

Step 3 N maps bounded sets into equicontinuous sets of $PC\times PC$. Let B_q be a bounded set in $PC\times PC$ as in Step 2. Let $\tau_1,\tau_2\in J,\tau_1<\tau_2$ and $u\in B_q$. Then, for i=1,2, we obtain

$$\begin{aligned}&E|N_i(\tau_2,x(\tau_2),y(\tau_2))-N_i(\tau_1,x(\tau_1),y(\tau_1))|_X^2\\ &\leq 7\|(S(\tau_2)-S(\tau_1))\|^2E|x_0|_X^2\\ &\quad+7E\left|\int_0^{\tau_1}(S(\tau_2-s)-S(\tau_1-s))f^i(s,x(s),y(s))ds\right|_X^2\\ &\quad+7E\left|\int_{\tau_1}^{\tau_2}S(\tau_2-s)f^i(s,x(s),y(s))ds\right|_X^2\\ &\quad+7E\left|\sum_{l=1}^{\infty}\int_0^{\tau_1}(S(\tau_2-s)-S(\tau_1-s))\sigma_l^i(s,x(s),y(s))dB_l^H(s)\right|_X^2\\ &\quad+7E\left|\sum_{l=1}^{\infty}\int_{\tau_1}^{\tau_2}S(\tau_2-s)\sigma_l^i(s,x(s),y(s))dB_l^H(s)\right|_X^2\\ &\quad+7E\left|\sum_{0<t_k<\tau_2}(S(\tau_1-t_k)-S(\tau_2-t_k))I_k(x(t_k))\right|_X^2\\ &\quad+7E\left|\sum_{\tau_1<t_k<\tau_2}(S(\tau_2-t_k))I_k(x(t_k))\right|_X^2.\end{aligned}$$

From Lemma 1.4.6 we deduce

$$
\begin{aligned}
&E|N_i(\tau_2, x(\tau_2), y(\tau_2)) - N_i(\tau_1, x(\tau_1), y(\tau_1))|_X^2 \\
&\quad \leq 7\|(S(\tau_2-\tau_1) - Id))\|^2 E|x_0|_X^2 \\
&\qquad +7\tau_2 \int_0^{\tau_1} \|(S(\tau_2-s) - S(\tau_1-s)\|^2 E|f^i(s, x(s), y(s))|_X^2 ds \\
&\qquad +7(\tau_2-\tau_1) \int_{\tau_1}^{\tau_2} \|S(\tau_2-s)\|^2 E|f^i(s, x(s), y(s))|_X^2 ds \\
&\qquad +7c_2(H)H(2H-1)\tau_2^{2H-1} \int_0^{\tau_1} \|S(\tau_2-s) - S(\tau_1-s)\|^2 E\|\sigma^i(s, x(s), y(s))\|^2 ds \\
&\qquad +7c_2(H)H(2H-1)|\tau_2-\tau_1|^{2H-1} \int_{\tau_1}^{\tau_2} |(S(\tau_1-s)\|^2 E\|\sigma^i(s, x(s), y(s))\|^2 ds \\
&\qquad +7 \sum_{0<t_k<\tau_2} \|S(\tau_2-\tau_1) - I_d)\|^2 E|I_k(x(t_k))|_X^2 \\
&\qquad +7 \sum_{\tau_1<t_k<\tau_2} \|S(\tau_2-t_k))\|^2 E|I_k(x(t_k))|_X^2.
\end{aligned}
$$

From $(H_3) - (H_4)$ and (H_6) we have

$$
\begin{aligned}
&E|N_i(\tau_2, x(\tau_2), y(\tau_2)) - N_i(\tau_1, x(\tau_1), y(\tau_1))|_X^2 \\
&\quad \leq 7\|S(\tau_2-\tau_1) - Id\|^2 E|x_0|_X^2 \\
&\qquad +7\tau_2 \int_0^{\tau_2} \|S(\tau_2-s) - S(\tau_1-s)\|^2 \Big(\overline{a}_i(s)E|x(s)|_X^2 + \overline{b}_i(s)E|y(s)|_X^2 + c_i(s)\Big) ds \\
&\qquad +7(\tau_2-\tau_1) \int_{\tau_1}^{\tau_2} \|S(\tau_1-s)\|^2 \Big(\overline{a}_i(s)E|x(s)|_X^2 + \overline{b}_i(s)E|y(s)|_X^2 + c_i(s)\Big) ds \\
&\qquad +7c_2(H)H(2H-1)\tau_2^{2H-1} \times \\
&\qquad \times \int_0^{\tau_2} \|S(\tau_2-s) - S(\tau_1-s)\|^2 \Big(\overline{\alpha}_i(s)E|x(s)|_X^2 + \overline{\beta}_i(s)E|y(s)|_X^2 + \overline{c}_i(s)\Big) ds \\
&\qquad +7c_2(H)H(2H-1)(\tau_2-\tau_1)^{2H-1} \times \\
&\qquad \times \int_{\tau_1}^{\tau_2} \|(S(\tau_1-s)\|^2 \Big(\overline{\alpha}_i(s)E|x(s)|_X^2 + \overline{\beta}_i(s)E|y(s)|_X^2 + \overline{c}_i(s)\Big) ds \\
&\qquad +7 \sum_{0<t_k<\tau_2} \|S(\tau_2-\tau_1) - I_d)\|^2 \Big(d_k E|x(t_k)|_X^2 + \lambda_k\Big) \\
&\qquad +7 \sum_{\tau_1<t_k<\tau_2} \|S(\tau_2-t_k)\|^2 \Big(d_k E|x(t_k)|_X^2 + \lambda_k\Big).
\end{aligned}
$$

Now, it is straightforward to see that the right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$ since the compactness of $S(t)$ for $t > 0$ implies the continuity in the uniform operator topology (see [124]). This proves the equicontinuity.

Step 4 ($N(B_q)(t)$ is precompact in $X \times X$.
As a consequence of Steps 2 to 3, and at light of the Arzelá-Ascoli theorem, it is sufficient to show that N maps B_q into a precompact set in $X \times X$. Let $0 < t < b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $(x, y) \in B_q$ we define

$$\begin{aligned}
N_i^\epsilon(x,y) &= S(t)z_0^i + \int_0^{t-\epsilon} S(t-s)f^i(s,x(s),y(s))ds \\
&\quad + \sum_{l=1}^{\infty}\int_0^{t-\epsilon} S(t-s)\sigma_l^i(t,x(s),y(s))dB_l^H(s) \\
&\quad + \sum_{0<t_k<t-\epsilon} S(t-t_k)I_k^i(z(t_k)) \\
&= S(\epsilon)S(t-\epsilon)z_0^i + S(\epsilon)\int_0^{t-\epsilon} S(t-s-\epsilon)f^i(s,x(s),y(s))ds \\
&\quad + S(\epsilon)\sum_{l=1}^{\infty}\int_0^{t-\epsilon} S(t-s-\epsilon)\sigma_l^i(t,x(s),y(s))dB_l^H(s) \\
&\quad + \sum_{0<t_k<t-\epsilon} S(t-t_k)I_k^i(z^i(t_k)).
\end{aligned}$$

Since $S(t)$ is a compact operator, the set

$$H_\epsilon = \{N^\epsilon(x,y)(t) = (N_1^\epsilon(x,y)(t), N_2^\epsilon(x,y)(t)) \ \ (x,y) \in B_q\}$$

is precompact in $X \times X$ for every ϵ such that $0 < \epsilon < t$ Moreover, for every $(x, y) \in B_q$, and $i = 1, 2$ we have

$$\begin{aligned}
E\|N_i(x,y) - N_i^\epsilon(x,y)\|^2 &\leq 3\int_{t-\epsilon}^{t} (\overline{a}_i(s)q + \overline{b}_i(s)q + c_i(s))ds \\
&\quad + 3M^2 c_2(H)H(2H-1)T^{2H-1}\int_{t-\epsilon}^{t} (\overline{\alpha}_i(s)q + \overline{\beta}_i(s)q + \overline{c}_i(s))ds \\
&\quad + 3M^2 \sum_{t-\epsilon<t_k<t} (d_k q + \lambda_k).
\end{aligned}$$

Therefore, there are precompact sets arbitrarily close to the set $H_\epsilon = \{N^\epsilon(x,y)(t) = (N_1^\epsilon(x,y)(t), N_2^\epsilon(x,y)(t)),\ \ (x,y) \in B_q\}$.
Hence, the set $H = \{N(x,y)(t) = (N_1(x,y)(t), N_2(x,y)(t)),\ \ (x,y) \in B_q\}$ is precompact in $X \times X$ and the right-hand side tends to 0 uniformly in t as $\epsilon \to 0^+$. Hence we can conclude the relative compactness of $N(B_q)(t)$ for $t \geq 0$. By the Arzelá-Ascoli theorem, we conclude that $N : PC \times PC \to PC \times PC$ is a completely continuous operator.

Step 5 A priori bounds on solutions.
The set

$$U = \{(x,y) \in PC\times PC : \quad x = \lambda N_1(x,y) \text{ and } y = \lambda N_2(x,y) \text{ for some } 0 < \lambda < 1\}$$

is bounded.
Let $(x,y) \in PC \times PC$ be a solution of the abstract nonlinear equation $x = \lambda N_1(x,y)$ and $y = \lambda N_2(x,y)$ for some $0 < \lambda < 1$. Thus, for $t \in [0,t_1]$,

$$\begin{aligned} E|x(t)|_X^2 &\leq 3M^2E|x_0|_X^2 + 3M^2\int_0^t \Big(\overline{a}_1(s)E|x(s)|_X^2 + \overline{b}_1(s)E|y(s)|_X^2 + c_1(s)\Big)ds \\ &+ 3M^2c_2(H)H(2H-1)t_1^{2H-1}\int_0^t \Big(\overline{\alpha}_1(s)E|x(s)|_X^2 + \overline{\beta}_1(s)E|y(s)|_X^2 + \overline{c}_1(s)\Big)ds \\ &\leq 3M^2E|x_0|^2 + 3M^2\|c_1\|_{L^1} + 3M^2c_2(H)H(2H-1)t_1^{2H-1}\|\overline{c}_1\|_{L^1} \\ &\quad +3M^2\int_0^t \Big(c_2(H)H(2H-1)^2t_1^{2H-1}\overline{\alpha}_1(s) + \overline{a}_1(s)\Big)E|x(s)|_X^2ds \\ &\quad +3M^2\int_0^t \Big(c_2(H)H(2H-1)^2t_1^{2H-1}\overline{\beta}_1(s) + \overline{b}_1(s)\Big)E|y(s)|_X^2ds \end{aligned}$$

$$E|x(t)|_X^2 \leq A_1 + \int_0^t B_1(s)E|x(s)|_X^2ds + \int_0^t C_1(s)E|y(s)|_X^2\,ds$$

and similarly

$$E|y(t)|_X^2 \leq A_2 + \int_0^t B_2(s)E|x(s)|_X^2ds + \int_0^t C_2(s)E|y(s)|_X^2,$$

where for each $j = 1,2$

$$A_j = 3M^2E|z_0^j|^2 + 3M^2\|c_j\|_{L^1} + 3M^2c_2(H)H(2H-1)t_1^{2H-1}\|\overline{c}_j\|_{L^1}$$

$$B_j(s) = 3M^2\Big(c_2(H)H(2H-1)^2 t_1^{2H-1}\overline{\alpha}_i(s) + \overline{a}_i(s))\Big) \qquad s \in [0, t_1]$$

and

$$C_j(s) = 3M^2\Big(c_2(H)H(2H-1)^2 t_1^{2H-1}\overline{\beta}_j(s) + \overline{b}_j(s)\Big),$$

and $z_0^1 = x_0$ and $z_0^2 = y_0$.

From these previous expressions we easily deduce

$$\sup_{r\in[0,t]}\Big(E|y(r)|_X^2 + E|x(r)|_X^2\Big) \;\le\; K_1 + \int_0^t K_2(s)\sup_{\eta\in[0,s]}\Big(E|x(\eta)|_X^2 + E|y(\eta)|_X^2\Big)ds$$

where

$$K_1 = A_1 + A_2, \quad K_2(s) = \max\{B_1(s) + B_2(s), C_1(s) + C_2(s)\}.$$

Using now the Gronwall inequality

$$\sup_{r\in[0,t]}\Big(E|y(r)|_X^2 + E|x(r)|_X^2\Big) \;\le\; K_1 \exp\left(\int_0^{t_1} K_2(s)ds\right) = \overline{M}_0,$$

whence

$$\sup_{t\in[0,t_1]} E|x(t)|_X^2 \;\le\; \overline{M}_0, \quad \text{and} \quad \sup_{t\in[0,t_1]} E|y(t)|_X^2 \le \overline{M}_0,$$

where $\overline{M}_0$ depends only on t_1. For $t \in (t_1, t_2]$, we have

$$\begin{aligned} x(t) \;&=\; S(t-t_1)[x(t_1) + I_1(x_1(t_1^-))] + \int_{t_1}^t S(t-s)f^1(s, x(s), y(s))ds \\ &\quad + \sum_{l=1}^{\infty}\int_{t_1}^t S(t-s)\sigma_l^1(t, x(s), y(s))dB_l^H(s). \end{aligned}$$

Then, by a similar argument we obtain

$$\begin{aligned} E|x(t)|_X^2 \;&=\; E|S(t-t_1)[x_1(t_1) + I_1(x_1(t_1^-))] + \int_{t_1}^t S(t-s)f^1(s, x(s), y(s))ds \\ &\quad + \sum_{l=1}^{\infty}\int_{t_1}^t S(t-s)\sigma_l^1(t, x(s), y(s))dB_l^H(s)|_X^2 \\ &\le\; \|S(t-t_1)\|^2\Big(E|x_1(t_1)|_X^2 + E|I_1(x_1(t_1^-))|_X^2\Big) \\ &\quad + E|\int_{t_1}^t S(t-s)f^1(s, x(s), y(s))ds|_X^2 \\ &\quad + E|\sum_{l=1}^{\infty}\int_{t_1}^t S(t-s)\sigma_l^1(t, x(s), y(s))dB_l^H(s)|_X^2 \end{aligned}$$

Consequently,

$$\begin{aligned} E|x(t)|_X^2 &\leq 4M^2(\overline{M}^2 + d_1 E|x(t_1|_X^2 + \lambda) \\ &+ 4M^2 \int_{t_1}^t \Big(\overline{a}_1(s)E|x(s)|_X^2 + \overline{b}_1(s)E|y(s)|_X^2 + c_1(s)\Big)ds \\ & \quad +4M^2 c_2(H)H(2H-1)t_2^{2H-1} \int_{t_1}^t \Big(\overline{\alpha}_1(s)E|x(s)|_X^2 \\ & \qquad +\overline{\beta}_1(s)E|y(s)|_X^2 + \overline{c}_1(s)\Big)ds \\ &\leq A_3 + \int_{t_1}^t B_3(s)E|x(s)|_X^2 ds + \int_{t_1}^t C_3(s)E|y(s)|_X^2 ds. \end{aligned}$$

Then we deduce

$$E|y(t)|_X^2 \leq A_4 + \int_{t_1}^t B_4(s)E|x(s)|_X^2 ds + \int_{t_1}^t C_4(s)E|y(s)|_X^2,$$

where

$$A_3 = 4M^2(\overline{M}^2 + \lambda) + 4M^2\|c_1\|_{L^1} + 4M^2 c_2(H)H(2H-1)t_2^{2H-1}\|\overline{c}_1\|_{L^1},$$

and

$$A_4 = 4M^2(\overline{M}^2 + \overline{\lambda}) + 4M^2\|c_2\|_{L^1} + 4M^2 c_2(H)H(2H-1)t_2^{2H-1}\|\overline{c}_2\|_{L^1},$$

and

$$B_3(s) = 4M^2 d_1 + 4M^2\Big(c_2(H)H(2H-1)^2 t_2^{2H-1}\overline{\alpha}_1(s) + \overline{a}_1(s)\Big)$$

$$B_4(s) = 4M^2 + \Big(c_2(H)H(2H-1)^2 t_2^{2H-1}\overline{\alpha}_2(s) + \overline{a}_2(s)\Big), \qquad s \in (t_1, t_2],$$

and

$$C_3(s) = 4M^2\Big(c_2(H)H(2H-1)^2 t_2^{2H-1}\overline{\beta}_1(s) + \overline{b}_1(s)\Big), \qquad s \in (t_1, t_2],$$

$$C_4(s) = 4M^2\overline{d}_1 + 4M^2\Big(c_2(H)H(2H-1)^2 t_2^{2H-1}\overline{\beta}_1(s) + \overline{b}_2(s)\Big), \qquad s \in (t_1, t_2].$$

Combining $E|x(t)|_X^2$ and $E|y(t)|_X^2$,

$$\sup_{r\in[t_1,t]}\Big(E|y(r)|_X^2 + E|x(r)|_X^2\Big) \leq K_3 + \int_{t_1}^t K_4(s) \sup_{\eta\in[t_1,s]}\Big(E|x(\eta)|_X^2 + E|y(\eta)|_X^2\Big)ds$$

where

$$K_3 = A_3 + A_4, \quad K_4(s) = \max\{B_3(s) + B_4(s), C_3(s) + C_4(s)\}.$$

Using once more the Gronwall inequality,

$$\sup_{t\in(t_1,t_2]} \left(E|y(t)|_X^2 + E|x(t)|_X^2 \right) \leq K_3 \exp\left(\int_{t_1}^{t_2} K_4(s)ds \right) = \overline{M}_1.$$

Consequently, there exists a constant $\overline{M}_1$ which only depends on t_1, t_2 such that

$$\sup_{t\in(t_1,t_2]} E|x(t)|_X^2 \leq \overline{M}_1, \text{ and } \sup_{t\in(t_1,t_2]} E|y(t)|_X^2 \leq \overline{M}_1.$$

If we iterate this procedure in every interval $(t_i, t_{i+1}]$ we can prove that there exists a positive constant $\overline{M}_i$, which only depends on t_i, t_{i+1}, such that

$$\sup_{t\in(t_i,t_{i+1}]} E|x(t)|_X^2 \leq \overline{M}_i, \text{ and } \sup_{t\in(t_i,t_{i+1}]} E|y(t)|_X^2 \leq \overline{M}_i, \quad i = 0, 1, 2, \ldots, m-1.$$

Set

$$U = \{(x, y) \in PC \times PC : \quad \|x\|_{PC} < M^*, \quad \|y\|_{PC} < M^*\}$$

where

$$M^* = \sup\{\overline{M}_i : \quad i = 0, 1, 2, \ldots, m-1\}.$$

Then, U is an open subset of $PC \times PC$ and it is straightforward to see that there is no $(x, y) \in \partial U$ such that $x = \lambda N_1(x, y)$ and $y = \lambda N_2(x, y)$ for some $0 < \lambda < 1$. By Theorem 2.4.3, N possesses at least one fixed point (x, y). $\square$

Remark 4.1.1. *We can replace conditions* (H_3) *and* (H_5) *by the following hypotheses*

$(\widetilde{H})$ *there exist integrable functions* $\eta : [0, T] \longrightarrow \mathbb{R}^+$ *and* $\psi : [0, \infty) \longrightarrow (0, \infty)$ *which are continuous and nondecreasing such that, for all* $t \in [0, T]$ *and* $x, y \in X$,

$$E\|f^i(t, x, y)\|^2 \leq \eta(t)\psi(E\|x\|^2+E\|y\|^2), \;\; E\|\sigma^i(t, x, y)\|^2 \leq \eta(t)\psi(E|x|^2+E|y|^2),$$

where

$$\sigma^i(t, x, y) = \sum_{l=1}^{\infty} \sigma_l^i(t, x, y), \quad i = 1, 2.$$

4.1.1 Weak solutions

In this section we prove that mild solutions to system (5.0.1) are also weak solutions. First, we recall the definition of weak solution according to Da Prato and Zabczyk [52]. To shorten the notation, we will use $\langle .,. \rangle$ instead of $(.,.)$ below since no confusion is possible.

Definition 4.1.3. *An $X-$ valued stochastic process $u(t) = (x(t), y(t)), t \in [0, T]$ is called a weak solution of* (5.0.1) *if for each $\varphi \in D(A^*)$*

$$\langle x(t), \varphi \rangle = \langle x_0, \varphi \rangle + \int_0^t \langle x(s), A^*\varphi \rangle d\tau + \int_0^t \langle f^1(\tau, x(\tau), y(\tau)), \varphi \rangle d\tau$$

$$+ \sum_{l=1}^{\infty} \int_0^t \langle \sigma_l^1(\tau, x(\tau), y(\tau)), \varphi \rangle dB_l^H(\tau) + \sum_{0<t_k<t} \langle I_k(x(t_k)), \varphi \rangle, \quad t \in J$$

and

$$\langle y(t), \varphi \rangle = \langle y_0, \varphi \rangle + \int_0^t \langle y(s), A^*\varphi \rangle d\tau + \int_0^t \langle f^2(\tau, x(\tau), y(\tau)), \varphi \rangle d\tau$$

$$+ \sum_{l=1}^{\infty} \int_0^t \langle \sigma_l^2(\tau, x(\tau), y(\tau)), \varphi \rangle dB_l^H(\tau) + \sum_{0<t_k<t} \langle \overline{I}_k(y(t_k)), \varphi \rangle, \quad t \in J.$$

Theorem 4.1.3. *Assume conditions $(H_3) - (H_6)$ hold. The mild solution of* (5.0.1) *is also a weak solution.*

Proof. Let $(x(t), y(t))$ be a mild solution to (5.0.1). Then, for each $\varphi \in D(A^*)$

and $t \in [0, t_1]$, we obtain

$$
\begin{aligned}
& E\bigg|\int_0^t \langle x(s), A^*\varphi\rangle ds - \int_0^t \langle S(s)x_0, A^*\varphi\rangle ds \\
& - \int_0^t \int_0^s \langle S(s-\tau) f^1(\tau, x(\tau), y(\tau)), A^*\varphi\rangle d\tau ds \\
& \qquad - \sum_{l=1}^{\infty} \int_0^t \int_0^s \langle S(s-\tau)\sigma_l^1(\tau, x(\tau), y(\tau)), A^*\varphi\rangle dB_l^H(\tau) ds\bigg| \\
& \quad \le \int_0^t E\bigg|\langle x(s), A^*\varphi\rangle - \langle S(t)x_0, A^*\varphi\rangle - \int_0^s \langle S(s-\tau) f^1(\tau, x(\tau), y(\tau)), A^*\varphi\rangle d\tau \\
& \qquad - \sum_{l=1}^{\infty} \int_0^s \langle S(s-\tau)\sigma_l^1(\tau, x(\tau), y(\tau)), A^*\varphi\rangle dB_l^H(\tau) ds\bigg| ds \\
& \quad = \int_0^t E\bigg|\langle x(s) - S(s)x_0 - \int_0^s S(s-\tau) f^1(\tau, x(\tau), y(\tau)) d\tau \\
& \qquad - \sum_{l=1}^{\infty} \int_0^s S(s-\tau)\sigma_l^1(\tau, x(\tau), y(\tau)) dB_l^H(\tau) ds) ds, A^*\varphi\rangle\bigg| ds \\
& \quad = 0.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\int_0^t \langle x(s), A^*\varphi\rangle ds &= \int_0^t \langle S(t)x_0, A^*\varphi\rangle ds + \int_0^t \int_0^s \langle S(s-\tau) f^1(\tau, x(\tau), y(\tau)), A^*\varphi\rangle d\tau ds \\
&\quad + \sum_{l=1}^{\infty} \int_0^t \int_0^s \langle S(s-\tau)\sigma_l^1(\tau, x(\tau), y(\tau)), A^*\varphi\rangle dB_l^H(\tau) ds \\
&= \sum_{j=1}^{3} \widetilde{P}_j.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\int_0^t \langle y(s), A^*\varphi\rangle ds &= \int_0^t \langle S(t)y_0, A^*\varphi\rangle ds + \int_0^t \int_0^s \langle S(s-\tau) f^2(\tau, x(\tau), y(\tau)), A^*\varphi\rangle d\tau ds \\
&\quad + \sum_{l=1}^{\infty} \int_0^t \int_0^s \langle S(s-\tau)\sigma_l^2(\tau, x(\tau), y(\tau)), A^*\varphi\rangle dB_l^H(\tau) ds \\
&= \sum_{j=1}^{3} \widehat{P}_j.
\end{aligned}
$$

Now for $\varphi \in D(A^*)$ and $t \in [0, t_1]$ we use the fact that $\frac{d}{dt}S^*(t)\varphi = S^*(t)A^*\varphi$, which yields

$$\begin{aligned}\widetilde{P}_1 = \int_0^t \langle S(s)x_0, A^*\varphi\rangle ds &= \int_0^t \langle x_0, S^*(s)A^*\varphi\rangle ds \\ &= \int_0^t \langle x_0, \frac{d}{ds}S^*(s)\varphi\rangle ds \\ &= \langle S(t)x_0 - x_0, \varphi\rangle,\end{aligned}$$

and analogously,

$$\widehat{P}_1 = \int_0^t \langle S(s)y_0, A^*\varphi\rangle = \langle S(t)y_0 - y_0, \varphi\rangle.$$

In addition, using Fubini's Theorem for $t \in [0, t_1]$,

$$\begin{aligned}\widetilde{P}_2 &= \int_0^t \int_0^s \langle S(s-\tau)f^1(\tau, x(\tau), y(\tau)), A^*\varphi\rangle d\tau ds \\ &= \int_0^t \int_\tau^t \langle 1_{(0,s]}f^1(\tau, x(\tau), y(\tau)), S^*(s-\tau)A^*\varphi\rangle d\tau ds \\ &= \int_0^t \langle S(t-\tau)f^1(\tau, x(\tau), y(\tau)) - f^1(\tau, x(\tau), y(\tau)), \varphi\rangle d\tau \\ &= \int_0^t \langle S(t-\tau)f^1(\tau, x(\tau), y(\tau)), y(\tau)), \varphi\rangle d\tau - \int_0^t \langle f^1(\tau, x(\tau), y(\tau)), \varphi\rangle d\tau\end{aligned}$$

and

$$\begin{aligned}\widehat{P}_2 &= \int_0^t \int_0^s \langle S(s-\tau)f^2(\tau, x(\tau), y(\tau)), A^*\varphi\rangle d\tau ds \\ &= \int_0^t \langle S(t-\tau)f^2(\tau, x(\tau), y(\tau)), y(\tau)), \varphi\rangle d\tau - \int_0^t \langle f^2(\tau, x(\tau), y(\tau)), \varphi\rangle d\tau.\end{aligned}$$

As for the estimates of the third terms we have for $t \in [0, t_1]$

$$\begin{aligned}\widetilde{P}_3 &= \sum_{l=1}^{\infty} \int_0^t \int_0^s \langle S(s-\tau)\sigma_l^1(\tau, x(\tau), y(\tau)), A^*\varphi\rangle dB_l^H(\tau) ds \\ &= \sum_{l=1}^{\infty} \int_0^t \int_\tau^t \langle 1_{(0,s]}\sigma_l^1(\tau, x(\tau), y(\tau)), S^*(s-\tau)A^*\varphi\rangle dB_l^H(\tau) ds \\ &= \sum_{l=1}^{\infty} \int_0^t \langle S(t-\tau)\sigma_l^1(\tau, x(\tau), y(\tau)), \varphi\rangle dB_l^H(\tau) - \sum_{l=1}^{\infty} \int_0^t \langle \sigma_l^1(\tau, x(\tau), y(\tau)), \varphi\rangle dB_l^H(\tau),\end{aligned}$$

and

$$\begin{aligned}
\widehat{P}_3 &= \sum_{l=1}^{\infty}\int_0^t\int_0^s \langle S(s-\tau)\sigma_l^2(\tau,x(\tau),y(\tau)), A^*\varphi\rangle dB_l^H(\tau)ds \\
&= \sum_{l=1}^{\infty}\int_0^t \langle S(t-\tau)\sigma_l^2(\tau,x(\tau),y(\tau)),\varphi\rangle dB_l^H(\tau) - \sum_{l=1}^{\infty}\int_0^t \langle \sigma_l^2(\tau,x(\tau),y(\tau)),\varphi\rangle dB_l^H(\tau).
\end{aligned}$$

Now, taking into account all the previous estimates we can write

$$\begin{aligned}
\int_0^t \langle x(s), A^*\varphi\rangle ds &= \int_0^t \langle Ax(s),\varphi\rangle ds = \langle S(t)x_0 - x_0,\varphi\rangle \\
&\quad + \int_0^t \langle S(t-\tau)f^1(\tau,x(\tau),y(\tau)),y(\tau)),\varphi\rangle d\tau \\
&\quad - \int_0^t \langle f^1(\tau,x(\tau),y(\tau)),\varphi\rangle d\tau \\
&\quad + \sum_{l=1}^{\infty}\int_0^t \langle S(t-\tau)\sigma_l^1(\tau,x(\tau),y(\tau)),\varphi\rangle dB_l^H(\tau) \\
&\quad - \sum_{l=1}^{\infty}\int_0^t \langle \sigma_l^1(\tau,x(\tau),y(\tau)),\varphi\rangle dB_l^H(\tau) \\
&= \langle x(t)-x_0,\varphi\rangle - \int_0^t \langle f^1(\tau,x(\tau),y(\tau)),\varphi\rangle d\tau \\
&\quad - \sum_{l=1}^{\infty}\int_0^t \langle \sigma_l^1(\tau,x(\tau),y(\tau)),\varphi\rangle dB_l^H(\tau).
\end{aligned}$$

Finally,

$$\begin{aligned}
\langle x(t),\varphi\rangle &= \langle x_0,\varphi\rangle + \int_0^t \langle x(s),A^*\varphi\rangle d\tau + \int_0^t \langle f^1(\tau,x(\tau),y(\tau)),\varphi\rangle d\tau \\
&+ \sum_{l=1}^{\infty}\int_0^t \langle \sigma_l^1(\tau,x(\tau),y(\tau)),\varphi\rangle dB_l^H(\tau),
\end{aligned}$$

and similar computations for y for $t\in[0,t_1]$ imply

$$\begin{aligned}
\langle y(t),\varphi\rangle &= \langle y_0,\varphi\rangle + \int_0^t \langle y(s),A^*\varphi\rangle d\tau + \int_0^t \langle f^2(\tau,x(\tau),y(\tau)),\varphi\rangle d\tau \\
&+ \sum_{l=1}^{\infty}\int_0^t \langle \sigma_l^2(\tau,x(\tau),y(\tau)),\varphi\rangle dB_l^H(\tau).
\end{aligned}$$

Therefore the mild solution is also a weak solution in the interval $[0, t_1]$.

We repeat this scheme in every subinterval $(t_{m-1}, t_m])$ until we reach the points $t \in (t_m, T]$. As the arguments are the same as in the previous case, we prefer to omit the details. □

4.1.2 Exponential stability

As in this section we are interested in the exponential decay to zero in mean square of the mild solutions to (5.0.1), we will assume that solutions are defined globally in time.

$$
\begin{aligned}
PC &= \{x: \ \Omega \times [0,T] \longrightarrow X,\ x \in C(J_k, L^2(\Omega, X)),\ k = 1, \ldots, m \text{ such that} \\
&\quad x(t_k^+, .) \text{ and } x(t_k^-, .) \text{ exist with } x(t_k^-, .) = x(t_k, .) \quad \text{and} \\
&\quad \sup_{t \in [0,T]} E\|x(t, .)\|^2 < \infty \text{ almost surely}\},
\end{aligned}
$$

endowed with the norm

$$\|x\|_{PC} = \sup_{s \in [0,T]} (E\|x(s, .)\|^2)^{\frac{1}{2}}.$$

It is not difficult to check that PC is a Banach space with norm $\|\cdot\|_{PC}$.

Consider the Banach space

$$PC_b = \{y \in PC_*(\mathbb{R}^+ \times \Omega, X) : y \text{ is bounded almost surely}\},$$

where

$$
\begin{aligned}
&PC_*(\mathbb{R}^+ \times \Omega, X) = \{y\colon\ [0, \infty) \times \Omega \to X,\ y_k \in C((t_k, t_{k+1}), L^2(\Omega, X)),\ k = 0, 1, 2 \ldots, \\
&y(t_k^-) \ \text{ and } \ y(t_k^+) \text{ exist and satisfy } \ y(t_k) = y(t_k^-) \ \text{ for } \ k = 1, \ldots\}
\end{aligned}
$$

and $y_k := y_{|(t_k, t_{k+1}]}$. Endowed with the norm

$$\|y\|_b^2 = \sup\{E|y(t)|_X^2 : t \in [0, \infty)\},$$

PC_b is a Banach space.

For the study of this problem we first introduce the following hypotheses:

(H_7) There exist $M > 0$ and $\gamma > 0$ such that

$$\|S(t)\| \leq Me^{-\gamma t}$$

for all $t > 0$.

(H_8) There exist nonnegative numbers a_i and b_i for each $i \in \{1, 2\}$, and for all $0 \leq \delta \leq \gamma$ it holds

$$\int_0^t e^{\delta s} E|f^i(s, x(s), y(s)) - f^i(s, \overline{x}(s), \overline{y}(s))|_X^2 ds \leq a_i \int_0^t e^{\delta s} E|x(s) - \overline{x}(s)|_X^2 ds$$
$$+ b_i \int_0^t e^{\delta s} E|y(s) - \overline{y}(s)|_X^2$$

and

$$\int_0^\infty e^{\gamma s} |f^i(s, 0, 0)|_X^2 ds < \infty.$$

for all x, y, $\overline{x}$, $\overline{y} \in PC_b$.

(H_9) There exist functions $\alpha_i, \beta_i \in C(\mathbb{R}^+, \mathbb{R}^+)$, $i = 1, 2$ such that, for all $0 \leq \delta \leq \gamma$, $x, y, \overline{x}, \overline{y} \in PC_b$ and $t \in \mathbb{R}^+$, we have

$$\begin{aligned}\int_0^t e^{\delta s} E\|\sigma^i(s, x(s), y(s)) - \sigma^i(s, \overline{x}(s), \overline{y}(s))\|^2 ds \;&\leq\; \alpha_i(t) \int_0^t e^{\delta s} E|x(s) - \overline{x}(s)|_X^2 ds \\ &\quad + \beta_i(t) \int_0^t e^{\delta s} E|y(s) - \overline{y}(s)|_X^2 ds\end{aligned}$$

and

$$\int_0^\infty e^{\gamma s} E\|\sigma^i(s, 0, 0)\|^2 ds < \infty, \ \sup_{t \in \mathbb{R}^+} t^{2H-1} \alpha_i(t) < \infty \quad \sup_{t \in \mathbb{R}^+} t^{2H-1} \beta_i(t) < \infty.$$

(H_{10}) there exist constants $d_k \geq 0$ and $\overline{d}_k \geq 0$, $k = 1, \ldots,$ such that

$$E|I_k(x) - I_k(\overline{x})|_X^2 \leq d_k E|x - \overline{x}|_X^2$$
$$E|\overline{I}_k(y) - \overline{I}_k(\overline{y})|_X^2 \leq \overline{d}_k E|y - \overline{y}|_X^2,$$

and

$$\sum_{k=1}^\infty e^{\gamma t_k} E|I_k(0)|_X^2 < \infty, \ 4M^2 \sum_{k=1}^\infty d_k < 1,$$

and

$$\sum_{k=1}^{\infty} e^{\gamma t_k} E|\overline{I}_k(0)|_X^2 < \infty, \; 4M^2 \sum_{k=1}^{\infty} \overline{d}_k < 1$$

for all $x,\, y,\, \overline{x},\, \overline{y} \in PC_b$.

Theorem 4.1.4. *Assume* $(H_7) - (H_{10})$ *hold and that* $\gamma > A_2 > 0$ *where*

$$A_2 = \max\Big\{\frac{4M^2 c_2(H)H(2H-1)(\bar{c}_2+\bar{c}_1)}{\widetilde{k}}, \frac{4\gamma^{-1}M^2(c_2+c_1)}{\widetilde{k}}\Big\},$$

$$c_i = \max\{a_i, b_i\}\,, \bar{c}_i = \max\{\sup_{t\in\mathbb{R}_+} t^{2H-1}\alpha_i(t), \sup_{t\in\mathbb{R}_+} t^{2H-1}\beta_i(t)\},$$

and

$$\bar{k}_* = 4M^2 \sum_{k=1}^{\infty} \overline{d}_k, \; k_* = 4M^2 \sum_{k=1}^{\infty} d_k, \; \frac{1}{\widetilde{\widetilde{k}}} = \max\Big(\frac{1}{1-k_*}, \frac{1}{1-\bar{k}_*}\Big).$$

Then, there exists a unique mild solution (x,y) *to problem* (5.0.1) *which converges to zero in mean square, i.e.*

$$\lim_{t\to+\infty} \begin{pmatrix} E\|x(t)\|^2 \\ E\|y(t)\|^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Proof. It is clear that each mild solution to Problem (5.0.1) is a fixed point of the operator N defined in Theorem 6.1.1. By using $(H_7) - (H_{10})$, we can easily prove that $N(PC_b) \subset PC_b$, and from Perov's fixed there exists a unique $(x,y) \in PC_b \times PC_b$ which is a fixed point of N. Now, we show that

$$\begin{aligned}
\lim_{t\to+\infty} E|N(x,y)|_X^2 &= \lim_{t\to+\infty} (E|N_1(x,y)|_X^2, E|N_2(x,y)|_X^2) \\
&= \lim_{t\to+\infty} (E|x(t)|_X^2, E|y(t)|_X^2) \\
&= (0,0).
\end{aligned}$$

First, observe that we have

$$\begin{aligned}
E|x(t)|_X^2 \;\leq\; & 4E|S(t)x_0|_X^2 + 4E\Big|\int_0^t S(t-s)f^1(s,x(s),y(s))ds\Big|_X^2 \\
& +4E\left|\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(s,x(s),y(s))dB_l^H(s)\right|_X^2 \\
& +4E\Big|\sum_{0<t_k<t} S(t-t_k)I_k(x(t_k))\Big|_X^2
\end{aligned}$$

and

$$\begin{aligned} E|y(t)|_X^2 &\le 4E|S(t)y_0|_X^2 + 4E\left|\int_0^t S(t-s)f^1(s,x(s),y(s))ds\right|_X^2 \\ &+4E\left|\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(s,x(s),y(s))dB_l^H(s)\right|_X^2 \\ &+4E\left|\sum_{0<t_k<t} S(t-t_k)\overline{I}_k(y(t_k))\right|_X^2. \end{aligned}$$

Therefore, by Lemma 1.4.19 and (H_7),

$$\begin{aligned} E|x(t)|_X^2 &\le 4M^2e^{-2\gamma t}E|x_0|_X^2 + 4\gamma^{-1}M^2\int_0^t e^{-\gamma(t-s)}E|f^1(s,x(s),y(s))|_X^2 ds \\ &+4M^2c_2(H)H(2H-1)t^{2H-1}\sum_{l=1}^{\infty}\int_0^t e^{-2\gamma(t-s)}\|\sigma_l^1(s,x(s),y(s))\|^2 ds \\ &+4M^2\sum_{0<t_k<t} e^{-2\gamma(t-t_k)}E|I_k(x(t_k))|_X^2 \end{aligned}$$

and, consequently,

$$\begin{aligned} e^{\gamma t}E|x(t)|_X^2 &\le 4M^2E|x_0|_X^2 + 4\gamma^{-1}M^2\int_0^t e^{\gamma s}E|f^1(s,x(s),y(s))|_X^2 ds \\ &+4M^2c_2(H)H(2H-1)t^{2H-1}\sum_{l=1}^{\infty}\int_0^t e^{\gamma s}\|\sigma_l^1(s,x(s),y(s))\|^2 ds \\ &+4M^2\sum_{0<t_k<t} e^{\gamma t_k}E|I_k(x(t_k))|_X^2. \end{aligned}$$

Thanks to the fact that $A_2 - \gamma < 0$ we can choose $\theta > 0$ such that $\eta =$

$A_2 - \gamma + \theta < 0$, and for these constants we have

$$
\begin{aligned}
e^{(\gamma-\theta)t}E|x(t)|_X^2 \leq\; & 4M^2e^{-\theta t}E|x_0|_X^2 + 4e^{-\theta t}\gamma^{-1}M^2\int_0^t e^{\gamma s}E|f^1(s,x(s),y(s))|_X^2 ds \\
& +4M^2c_2(H)H(2H-1)t^{2H-1}e^{-\theta t}\sum_{l=1}^{\infty}\int_0^t e^{\gamma s}\|\sigma_l^1(s,x(s),y(s))\|^2 ds \\
& +4M^2e^{-\theta t}\sum_{0<t_k<t} e^{\gamma t_k}E|I_k(x(t_k))|_X^2 \\
\leq\; & 4M^2e^{-\theta t}E|x_0|_X^2 + 4\gamma^{-1}M^2\int_0^t e^{(\gamma-\theta)s}E|f^1(s,x(s),y(s))|_X^2 ds \\
& +4M^2c_H H(2H-1)t^{2H-1}\int_0^t e^{(\gamma-\theta)s}E\|\sigma^1(s,x(s),y(s))\|^2 ds \\
& +4M^2e^{-\theta t}\sum_{0<t_k<t} e^{\gamma t_k}E|I_k(x(t_k))|_X^2 \\
=\; & 4\sum_{i=1}^{4}P_i.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
e^{(\gamma-\theta)t}E|y(t)|_X^2 \leq\; & 4M^2e^{-\theta t}E|y_0|_X^2 + 4\gamma^{-1}M^2\int_0^t e^{(\gamma-\theta)s}E|f^2(s,x(s),y(s))|_X^2 ds \\
& +4M^2c_2(H)H(2H-1)t^{2H-1}\int_0^t e^{(\gamma-\theta)s}E\|\sigma^2(s,x(s),y(s))\|^2 ds \\
& +4M^2e^{-\theta t}\sum_{0<t_k<t} e^{\gamma t_k}E|\overline{I}_k(y(t_k))|_X^2 \\
=\; & 4\sum_{i=1}^{4}\overline{P}_i.
\end{aligned}
$$

Now it is easy to see, for any $t \geq 0$, that

$$\begin{aligned} P_2 &= \int_0^t e^{(\gamma-\theta)s} E|f^1(s,x(s),y(s))|_X^2 ds \\ &\leq \int_0^t e^{(\gamma-\theta)s} E|f^1(s,x(s),y(s)) - f^1(s,0,0) + f^1(s,0,0)|_X^2 ds \\ &\leq \int_0^t e^{(\gamma-\theta)s} E|f^1(s,x(s),y(s)) - f^1(s,0,0)|_X^2 ds + \int_0^t e^{(\gamma-\theta)s} E|f^1(s,0,0)|_X^2 ds \\ &\leq a_1 \int_0^t e^{(\gamma-\theta)s} E|x(s)|_X^2 ds + b_1 \int_0^t e^{(\gamma-\theta)s} E|y(s)|_X^2 ds + \int_0^t e^{(\gamma-\theta)s} E|f^1(s,0,0)|_X^2 ds, \end{aligned}$$

and similarly,

$$\begin{aligned} \overline{P}_2 &= \int_0^t e^{(\gamma-\theta)s} E|f^2(s,x(s),y(s))|_X^2 ds \\ &\leq a_2 \int_0^t e^{(\gamma-\theta)s} E|x(s)|_X^2 ds + b_2 \int_0^t e^{(\gamma-\theta)s} E|y(s)|_X^2 ds + \int_0^t e^{(\gamma-\theta)s} E|f^2(s,0,0)|_X^2 ds. \end{aligned}$$

By condition (H_8) we deduce the existence of positive constants $K_1, \overline{K}_1$ such that

$$4\gamma^{-1} M^2 \int_0^t e^{(\gamma-\theta)s} E|f^1(s,0,0)|_X^2 ds \leq K_1$$

and

$$4\gamma^{-1} M^2 \int_0^t e^{(\gamma-\theta)s} E|f^2(s,0,0)|_X^2 ds \leq \overline{K}_1$$

As for the term P_3 we have

$$\begin{aligned} P_3 &= \int_0^t e^{(\gamma-\theta)s} E\|\sigma^1(s,x(s),y(s))\|^2 ds \\ &\leq \int_0^t e^{(\gamma-\theta)s} E\|\sigma^1(s,x(s),y(s)) - \sigma^1(s,0,0) + \sigma^1(s,0,0)\|^2 ds \\ &\leq \int_0^t e^{(\gamma-\theta)s} E\|\sigma^1(s,x(s),y(s)) - \sigma^1(s,0,0)\|^2 ds + \int_0^t e^{(\gamma-\theta)s} E\|\sigma^1(s,0,0)\|^2 ds \\ &\leq \alpha_1(t) \int_0^t e^{(\gamma-\theta)s} E|x(s)|_X^2 ds + \beta_1(t) \int_0^t e^{(\gamma-\theta)s} E|y(s)|_X^2 ds \\ &+ \int_0^t e^{(\gamma-\theta)s} E\|\sigma^1(s,0,0)\|^2 ds, \end{aligned}$$

and similarly,

$$\begin{aligned}
\overline{P}_3 &= \int_0^t e^{(\gamma-\theta)s} E\|\sigma^2(s,x(s),y(s))\|^2 ds \\
&\le \alpha_2(t)\int_0^t e^{(\gamma-\theta)s} E|x(s)|_X^2 ds + \beta_2(t)\int_0^t e^{(\gamma-\theta)s} E|y(s)|_X^2 ds \\
&+ \int_0^t e^{(\gamma-\theta)s} E\|\sigma^1(s,0,0)\|^2 ds.
\end{aligned}$$

From (H_9) we deduce the existence of positive constants $K_2, \overline{K}_2$, such that

$$4M^2 c_2(H)H(2H-1)t^{2H-1}\int_0^t e^{(\gamma-\theta)s} E\|\sigma^1(s,0,0)\|^2 ds \le K_2$$

and

$$4M^2 c_2(H)H(2H-1)t^{2H-1}\int_0^t e^{(\gamma-\theta)s} E\|\sigma^2(s,0,0)\|^2 ds \le \overline{K}_2.$$

And for the last terms

$$\begin{aligned}
P_4 &= e^{-\theta t}\sum_{0<t_k<t} e^{\gamma t_k} E|I_k(x(t_k))|_X^2 \\
&\le e^{-\theta t}\sum_{0<t_k<t} e^{\gamma t_k} E|I_k(x(t_k)) - I_k(0) + I_k(0)|_X^2 \\
&\le e^{-\theta t}\sum_{0<t_k<t} e^{\gamma t_k} E|I_k(x(t_k)) - I_k(0)|_X^2 + e^{-\theta t}\sum_{0<t_k<t} e^{\gamma t_k} E|I_k(0)|_X^2 \\
&\le e^{-\theta t}\sum_{0<t_k<t} e^{\gamma t_k} d_k E|x(t_k)|_X^2 + e^{-\theta t}\sum_{0<t_k<t} e^{\gamma t_k} E|I_k(0)|_X^2 \\
&\le e^{(\gamma-\theta)t}\sum_{0<t_k<t} e^{\gamma(t_k-t)} d_k E|x(t_k)|_X^2 + e^{-\theta t}\sum_{0<t_k<t} e^{\gamma t_k} E|I_k(0)|_X^2,
\end{aligned}$$

and analogously

$$\begin{aligned}
\overline{P}_4 &= e^{-\theta t}\sum_{0<t_k<t} e^{\gamma t_k} E|\overline{I}_k(y(t_k))|_X^2 \\
&\le e^{(\gamma-\theta)t}\sum_{0<t_k<t} e^{\gamma(t_k-t)} d_k E|x(t_k)|_X^2 + e^{-\theta t}\sum_{0<t_k<t} e^{\gamma t_k} E|\overline{I}_k(0)|_X^2.
\end{aligned}$$

Now, thanks to (H_{10}) we have the existence of positive constants $K_3, \overline{K}_3$ such that, for all $t \geq 0$,

$$e^{-\theta t} \sum_{0<t_k<t} e^{\gamma t_k} E|I_k(0)|_X^2 \leq K_3 \quad \text{and} \quad e^{-\theta t} \sum_{0<t_k<t} e^{\gamma t_k} E|\overline{I}_k(0)|_X^2 \leq \overline{K}_3,$$

which further imply

$$\begin{aligned} e^{(\gamma-\theta)t} E|x(t)|_X^2 \leq\ & 4M^2 e^{-\theta t} E|x_0|_X^2 + \sum_{i=1}^{3} K_i + 4\gamma^{-1} M^2 a_1 \int_0^t e^{(\gamma-\theta)s} E|x(s)|_X^2 ds \\ & + 4\gamma^{-1} M^2 b_1 \int_0^t e^{(\gamma-\theta)s} E|y(s)|_X^2 ds \\ & + 4M^2 c_2(H) H(2H-1) t^{2H-1} \alpha_1 \int_0^t e^{(\gamma-\theta)s} E|x(s)|_X^2 ds \\ & + 4M^2 c_2(H) H(2H-1) t^{2H-1} \beta_1 \int_0^t e^{(\gamma-\theta)s} E|y(s)|_X^2 ds \\ & + 4M^2 e^{(\gamma-\theta)t} \sum_{0<t_k<t} e^{\gamma(t_k-t)} d_k E|x(t_k)|_X^2, \end{aligned}$$

and similarly

$$\begin{aligned} e^{(\gamma-\theta)t} E|y(t)|_X^2 \leq\ & 4M^2 e^{-\theta t} E|y_0|_X^2 + \sum_{i=1}^{3} \overline{K}_i + 4\gamma^{-1} M^2 a_2 \int_0^t e^{(\gamma-\theta)s} E|x(s)|_X^2 ds \\ & + 4\gamma^{-1} M^2 b_2 \int_0^t e^{(\gamma-\theta)s} E|y(s)|_X^2 ds \\ & + 4M^2 c_2(H) H(2H-1) t^{2H-1} \alpha_2 \int_0^t e^{(\gamma-\theta)s} E|x(s)|_X^2 ds \\ & + 4M^2 c_2(H) H(2H-1) t^{2H-1} \beta_2 \int_0^t e^{(\gamma-\theta)s} E|y(s)|_X^2 ds \\ & + 4M^2 e^{(\gamma-\theta)t} \sum_{0<t_k<t} e^{\gamma(t_k-t)} \overline{d}_k E|y(t_k)|_X^2. \end{aligned}$$

Consider functions $\mu, \overline{\mu}$ defined on $[0, +\infty)$ by

$$\mu(t) = \sup\{E|x(s)|_X^2 : 0 \leq s \leq t\}, \qquad \overline{\mu}(t) = \sup\{E|y(s)|_X^2 : 0 \leq s \leq t\}.$$

Together with (H_9), it follows that, for each $t > 0$, the constant $k_*(t) = 4M^2 \sum_{0<t_k<t} e^{\gamma(t_k-t)} d_k \leq k_* < 1$. Then

$$\begin{aligned}
e^{(\gamma-\theta)t}\mu(t) &\leq \frac{4M^2}{1-k_*}e^{-\theta t}E|x_0|_X^2 + \sum_{i=1}^{3}\frac{K_i}{1-k_*} + \frac{4\gamma^{-1}M^2a_1}{1-k_*}\int_0^t e^{(\gamma-\theta)s}\mu(s)ds \\
&+ 4\frac{\gamma^{-1}M^2b_1}{1-k_*}\int_0^t e^{(\gamma-\theta)s}\overline{\mu}(s)ds \\
&+ \frac{4M^2c_2(H)H(2H-1)t^{2H-1}\alpha_1(t)}{1-k_*}\int_0^t e^{(\gamma-\theta)s}\mu(s)ds \\
&+ \frac{4M^2c_2(H)H(2H-1)t^{2H-1}\beta_1(t)}{1-k_*}\int_0^t e^{(\gamma-\theta)s}\overline{\mu}(s)ds.
\end{aligned}$$

Analogously, the constant $\bar{k}_*(t) = 4M^2 \sum_{0<t_k<t} e^{\gamma(t_k-t)}\overline{d}_k \leq \bar{k}_* < 1$ and we have

$$\begin{aligned}
e^{(\gamma-\theta)t}\overline{\mu}(t) &\leq \frac{4M^2}{1-\bar{k}_*}e^{-\theta t}E|y_0|_X^2 + \sum_{i=1}^{3}\frac{\overline{K}_i}{1-\bar{k}_*} + \frac{4\gamma^{-1}M^2a_2}{1-\bar{k}_*}\int_0^t e^{(\gamma-\theta)s}\mu(s)ds \\
&+ 4\frac{\gamma^{-1}M^2b_2}{1-\bar{k}_*}\int_0^t e^{(\gamma-\theta)s}\overline{\mu}(s)ds \\
&+ \frac{4M^2c_2(H)H(2H-1)t^{2H-1}\alpha_2}{1-\bar{k}_*}\int_0^t e^{(\gamma-\theta)s}\mu(s)ds \\
&+ \frac{4M^2c_2(H)H(2H-1)t^{2H-1}\beta_2}{1-\bar{k}_*}\int_0^t e^{(\gamma-\theta)s}\overline{\mu}(s)ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
e^{(\gamma-\theta)t}(\mu(t)+\overline{\mu}(t)) \leq\ & \frac{4M^2}{1-k_*}e^{-\theta t}E|x_0|_X^2 + \frac{4M^2}{1-\bar{k}_*}e^{-\theta t}E|y_0|_X^2 + \sum_{i=1}^{3}\frac{K_i}{1-k_*} + \sum_{i=1}^{3}\frac{\overline{K}_i}{1-\bar{k}_*} \\
+\ & \frac{4\gamma^{-1}M^2}{1-k_*}\Big(a_1\int_0^t e^{(\gamma-\theta)s}\mu(s)ds + b_1\int_0^t e^{(\gamma-\theta)s}\overline{\mu}(s)ds\Big) \\
+\ & \frac{4M^2c_2(H)H(2H-1)}{1-k_*}\Big(t^{2H-1}\alpha_1(t)\int_0^t e^{(\gamma-\theta)s}\mu(s)ds \\
+\ & t^{2H-1}\beta_1(t)\int_0^t e^{(\gamma-\theta)s}\overline{\mu}(s)ds\Big) \\
+\ & \frac{4\gamma^{-1}M^2}{1-\bar{k}_*}\Big(a_2\int_0^t e^{(\gamma-\theta)s}\mu(s)ds + b_2\int_0^t e^{(\gamma-\theta)s}\overline{\mu}(s)ds\Big) \\
+\ & \frac{4M^2c_2(H)H(2H-1)}{1-\bar{k}_*}\Big(t^{2H-1}\alpha_2(t)\int_0^t e^{(\gamma-\theta)s}\mu(s)ds \\
+\ & t^{2H-1}\beta_2(t)\int_0^t e^{(\gamma-\theta)s}\overline{\mu}(s)ds\Big),
\end{aligned}$$

the maximum being taken componentwise for each $i=1,2$, and

$$c_i = \max\{a_i, b_i\}\,, \bar{c}_i = \max\{t^{2H-1}\alpha_i(t), t^{2H-1}\beta_i(t)\}, \quad \frac{1}{\widetilde{k}} = \max\{\frac{1}{1-\bar{k}_*}, \frac{1}{1-k_*}\}.$$

$$\begin{aligned}
e^{(\gamma-\theta)t}(\mu(t)+\overline{\mu}(t)) \leq\ & \frac{4M^2}{\widetilde{k}}e^{-\theta t}\Big(E|x_0|_X^2 + E|y_0|_X^2\Big) + \sum_{i=1}^{3}\frac{K_i+\overline{K}_i}{\widetilde{k}} \\
+\ & \frac{4\gamma^{-1}M^2c_1}{\widetilde{k}}\int_0^t e^{(\gamma-\theta)s}\Big(\mu(s)+\overline{\mu}(s)\Big)ds \\
+\ & \frac{4M^2c_2(H)H(2H-1)\bar{c}_1}{\widetilde{k}}\int_0^t e^{(\gamma-\theta)s}\Big(\mu(s)+\overline{\mu}(s)\Big)ds \\
+\ & \frac{4\gamma^{-1}M^2c_2}{\widetilde{k}}\int_0^t e^{(\gamma-\theta)s}\Big(\mu(s)+\overline{\mu}(s)\Big)ds \\
+\ & \frac{4M^2c_2(H)H(2H-1)\bar{c}_2}{\widetilde{k}}\int_0^t e^{(\gamma-\theta)s}\Big(\mu(s)+\overline{\mu}(s)\Big)ds,
\end{aligned}$$

where

$$A_2 = \max\Big\{\frac{4M^2c_2(H)H(2H-1)(\bar{c}_2+\bar{c}_1)}{\widetilde{k}}, \frac{4\gamma^{-1}M^2(c_2+c_1)}{\widetilde{k}}\Big\}$$

and

$$A_1 = \frac{4M^2}{\widetilde{k}} e^{-\theta t}\Big(E|x_0|_X^2 + E|y_0|_X^2\Big) + \sum_{i=1}^{3} \frac{K_i + \overline{K}_i}{\widetilde{k}}$$

$$e^{(\gamma-\theta)t}\Big(\mu(t) + \overline{\mu}(t)\Big) \;\; \leq \;\; A_1 + A_2 \int_0^t e^{(\gamma-\theta)s}\Big(\mu(s) + \overline{\mu}(s)\Big) ds,$$

where A_1 is a suitable positive constant. Gronwall's Lemma conduces us to

$$e^{(\gamma-\theta)t}(\mu(t) + \overline{\mu}(t)) \leq A_1 e^{A_2 t},$$

and, consequently,

$$\mu(t) + \overline{\mu}(t) \leq A_1 e^{(A_2-\gamma+\theta)t} = A_1 e^{\eta t}.$$

Then

$$\lim_{t\to\infty} [\mu(t) + \bar{\mu}(t)] = 0 \Rightarrow \lim_{t\to\infty} E|x(t)|_X^2 = 0, \;\; \lim_{t\to\infty} E|y(t)|_X^2 = 0.$$

The proof is therefore complete. □

4.1.3 An example

In this section we present an example to illustrate the usefulness and applicability of our results. We consider with finite or infinite fractional Browian motion.

Example 4.1.1. *Consider the following couple stochastic partial differential equation with impulsive effects*

$$\begin{cases} du(t,\xi) &= \frac{\partial^2}{\partial \xi^2} u(t,\xi) + F(t, u(t,\xi), v(t,\xi)) \\ & \quad +\sigma(t)\frac{dB_Q^H}{dt}, \quad t \geq 0, \quad t \neq t_k, \quad 0 \leq \xi \leq \pi, \\ dv(t,\xi) &= \frac{\partial^2}{\partial \xi^2} v(t,\xi) + G(t, u(t,\xi), v(t,\xi)) \\ & \quad +\sigma(t)\frac{dB_Q^H}{dt}, \quad t \geq 0, \quad t \neq t_k, \quad 0 \leq \xi \leq \pi, \\ u(t_k^+,\xi) &- \;\; u(t_k^-,\xi) = \alpha_k u(t_k^-,\xi), \quad k = 1, \cdots, m, \\ v(t_k^+,\xi) &- \;\; v(t_k^-,\xi) = \bar{\alpha}_k v(t_k^-,\xi), \quad k = 1, \cdots, m, \\ u(t,0) &= u(t,\pi) = 0, t \geq 0, \\ v(t,0) &= v(t,\pi) = 0, t \geq 0, \\ u(0,\xi) &= u_0(\xi), \;\; 0 \leq \xi \leq \pi, \\ v(0,\xi) &= v_0(\xi), \;\; 0 \leq \xi \leq \pi, \end{cases} \tag{4.1.8}$$

where $\alpha_k > 0$, B_Q^H *denotes a fractional Brownian motion,* $G, F : [0, \pi] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *are continuous functions.*

Let

$$x(t)(\xi) = u(t,\xi), y(t)(\xi) = v(t,\xi) \quad t \in J, \quad \xi \in [0,\pi],$$

$$I_k(x(t_k))(\xi) = \alpha_k u(t_k^-,\xi), \bar{I}_k(y(t_k))(\xi) = \alpha_k v(t_k^-,\xi) \;\; \xi \in [0,\pi], \;\; k = 1, \cdots, m,$$

$$f(t, x(t), y(t))(\xi) = F(t, u(t,\xi), v(t,\xi)), \quad , \quad \xi \in [0,\pi],$$

$$g(t, x(t), y(t))(\xi) = G(t, u(t,\xi), v(t,\xi)), \quad , \quad \xi \in [0,\pi].$$

$$u_0(\xi) = u(0,\xi), \; v_0(\xi) = v(0,\xi) \quad , \quad \xi \in [0,\pi],$$

Take $\mathcal{K} = \mathcal{H} = L^2([0,\pi])$. *We define the operator* A *by* $Au = u''$, *with domain* $D(A) = \{u \in \mathcal{H}, u', u'' \in \mathcal{H} \quad \text{and} \quad u(0) = u(\pi) = 0\}$.

Then, it is well known that

$$Az = -\sum_{n=1}^{\infty} e^{-n^2 t} \langle z, e_n \rangle e_n, \quad z \in \mathcal{H},$$

and A *is the infinitesimal generator of an analytic semigroup* $\{S(t)\}_{t \geq 0}$ *on* $\mathcal{H}$, *which is given by*

$S(t)u = \sum_{n=1}^{\infty} e^{-n^2 t} \langle u, e_n \rangle e_n$, $u \in \mathcal{H}$, *and* $e_n(u) = (2/\pi)^{1/2} \sin(nu), n = 1, 2, \cdots$, *is the orthogonal set of eigenvectors of* A. *The analytic semigroup* $\{S(t)\}_{t>0}$, $t \in J$, *is compact, and there exists a constant* $M \geq 1$ *such that* $\|S(t)\|^2 \leq M$.

In order to define the operator $Q : \mathcal{K} \longrightarrow \mathcal{K}$, *we choose a sequence* $\{\sigma_n\}_{n \geq 1} \subset \mathbb{R}^+$, *set* $Qe_n = \sigma_n e_n$, *and assume that*

$$tr(Q) = \sum_{n=1}^{\infty} \sqrt{\sigma_n} < \infty.$$

Define the process $B_Q^H(s)$ *by*

$$B_Q^H = \sum_{n=1}^{\infty} \sqrt{\sigma_n} \gamma_n^H(t) e_n,$$

where $H \in (1/2, 1)$, *and* $\{\gamma_n^H\}_{n \in \mathbb{N}}$ *is a sequence of two-sided one-dimensional mutually independent fractional Brownian motions. Assume now that*

(i) *There exist some positive number* $d_k, \bar{d}_k \quad k \in \{1, \cdots, m\}$ *such that*

$$|I_k(\xi)| \leq d_k, \quad |\bar{I}_k(\xi)| \leq \bar{d}_k$$

for any $\xi \in \mathbb{R}$.

(ii) *The functions* $f, g : [0,T] \times \mathcal{H} \longrightarrow \mathcal{H}$ *defined by* $f(t,u)(.) = F(t, u(.)), g(t,u)(.) = G(t, u(.))$ *are continuous and we impose suitable conditions on* F *and* G *to verify assumption* $(H2)$.

(iii) *Assume that there exists an integrable function* $\eta : [0,T] \longrightarrow \mathbb{R}^+$ *such that*

$$|F(t,x,y)|^2 \leq \eta(t)\psi(|x|^2 + |y|^2), \quad |G(t,x,y)|^2 \leq \eta(t)\psi(|x|^2 + |y|^2)$$

for any $t \in [0,T]$ *and* $x, y \in \mathbb{R}$, *where* $\psi : [0,\infty) \longrightarrow (0,\infty)$ *is continuous, nondecreasing and concave with*

$$\int_1^\infty \frac{ds}{\psi(s)} = +\infty.$$

(iv) *The function* $\sigma : [0,T] \longrightarrow L_Q^2(\mathcal{K}, \mathcal{H})$ *is bounded, that is, there exists a positive constant* L *such that*

$$\int_0^T \|\sigma(s)\|_{L_Q^2}^2 ds < L, \quad \forall T > 0.$$

Thus, problem (6.1.4) can be written in the abstract form

$$\begin{cases} dx(t) = [Ax(t) + f(t,x,y)]dt + \sigma(t)dB_Q^H(t), \quad t \in J := [0,T], \\ dy(t) = [Ay(t) + g(t,x,y)]dt + \sigma(t)dB_Q^H(t), \quad t \in J := [0,T], \\ x(t_k^+) - x(t_k) = I_k(x(t_k)), \; k = 1, \ldots, m; \\ y(t_k^+) - y(t_k) = I_k(y(t_k)), \; k = 1, \ldots, m; \\ x(0) = x_0, \\ y(0) = y_0. \end{cases} \tag{4.1.9}$$

Thanks to these assumptions, it is straightforward to check that $(H_1) - (H_6)$ *hold true and, then, assumptions in Theorem 5.1.3 are fulfilled, and we can conclude that system (6.1.4) possesses a mild solution on* $[0,T]$.

In the case that we consider the problem for $t \in [0,\infty)$, *we observe that*

(i') $\|S(t)\| \le e^{-\pi^2 t}$,

(ii') *The functions* $f, g : [0,\infty)\times\mathcal{H} \longrightarrow \mathcal{H}$ *are continuous and under suitable conditions on* F*, assumption (H2) holds in* $[0,+\infty)$.

(iii') *The function* $\sigma : [0,+\infty) \longrightarrow L^2_Q(\mathcal{K},\mathcal{H})$ *is bounded, that is, there exists a positive constant* L *such that*

$$\int_0^\infty e^{\gamma s}\|\sigma(s)\|^2_{L^0_Q}\, ds < L.$$

(iv') *There exist some positive number* d_k, $\quad k \in \{1,\cdots,m,\cdots\}$ *such that*

$$|I_k(\xi)| \le d_k, \quad |\bar{I}_k(\xi)| \le \bar{d}_k \text{ and } \sum_{k=1}^{\infty} d_k < \infty,\ \sum_{k=1}^{\infty} \bar{d}_k < \infty$$

for any $\xi \in \mathbb{R}$.

Thus the problem (6.1.4) can be written in the abstract form

$$\begin{cases} dx(t) = [Ax(t) + f(t,x,y)]dt + \sigma(t)dB^H_Q(t), \quad t \in J := [0,\infty) \quad ; \\ dy(t) = [Ay(t) + g(t,x,y)]dt + \sigma(t)dB^H_Q(t), \quad t \in J := [0,\infty) \quad ; \\ x(t_k^+) - x(t_k) = I_k(x(t_k)),\ k = 1,2,3\ldots \\ y(t_k^+) - y(t_k) = I_k(y(t_k)),\ k = 1,2,3\ldots \\ x(0) = x_0, \\ y(0) = y_0. \end{cases} \tag{4.1.10}$$

Corollary 4.1.5. *Let* $\sigma_1, \sigma_2 : [0,\pi] \times L^2([0,\pi]) \times L^2([0,\pi]) \to L^2([0,\pi])$ *be defined by*

$$\sigma^1(t,x,y) = \sum_{k=1}^{\infty} \sigma^1_k(t,x,y),\ \sigma^2(t,x,y) = \sum_{k=1}^{\infty} \sigma^2_k(t,x,y),$$

where

$$\sigma^1_k(t,x,y)v(s) = a_{2k+1}\int_0^t [\sin k^2 x(t)] f(s)ds + a_{2k}\int_0^t [\cos k^2 y(t)] f(s)ds,\ v \in L^2([0,\pi])$$

and

$$\sigma_k^2(t,x,y)v(s) = b_{2k+1}\int_0^t [\sin k^2 x(s)]f(s)ds + b_{2k}\int_0^t [\cos k^2 y(s)]f(s)ds,\ v \in L^2([0,\pi]).$$

If $\sum a_k^2 < \infty,\ \sum b_k^2 < \infty$, *then for each* $k \in \mathbb{N}$ *we have*

$$\sigma_k^1,\ \sigma_k^2 \in L_Q^0(L^2([0,\pi], L^2([0,\pi]))$$

and

$$\sum_{l=1}^{\infty} \|\sigma_l^i(.,x,y)\|_{L_Q^0}^2 < \infty,\ i = 1,2.$$

Proof. Clearly, for every $k \in \mathbb{N}$, $\sigma_k^1,\ \sigma_k^2$ are linear continuous operators. Let $(e_i)_{i\in\mathbb{N}}$ be an orthonormal basis of $L^2([0,\pi])$

$$\begin{aligned} &\|\sigma_k^1(t,x,y)\|_{L_Q}^2 \\ &= \sum_{k=1}^{\infty}\int_0^{\pi}\left(\left|\sqrt{\sigma_n}a_{2k+1}\int_0^t [\sin k^2 x(s)]e_n(s)ds + \sqrt{\sigma_n}a_{2k}\int_0^t [\cos k^2 y(s)]e_n(s)ds\right|^2\right)dt \\ &\le 2\pi^2(a_{2k+1}^2 + a_{2k}^2)\sum_{n=1}^{\infty}\sigma_n. \end{aligned}$$

Therefore

$$\|\sigma_k^1(t,x,y)\|_{L_Q^0} \ \le\ \pi\sqrt{2(a_{2k+1}^2 + a_{2k}^2)}\sum_{n=1}^{\infty}\sqrt{\sigma_n} < \infty.$$

$$\|\sigma_k^1(t,x,y)\|_{L_Q^0}^2 \ \le\ L_*(a_{2k+1}^2 + a_{2k}^2), \quad L_* = 2\pi^2\sum_{n=1}^{\infty}\sqrt{\sigma_n}.$$

We deduce

$$\begin{aligned} \|\sigma^1(t,x,y)\|^2 &= \sum_{k=1}^{\infty}\|\sigma_l^1\|_{L_Q^0} \\ &\le L_*\sum_{k=1}^{\infty}(a_{2k+1}^2 + a_{2k}^2) \\ &\le L_*\sum_{k=1}^{\infty}a_k^2 < \infty. \end{aligned}$$

and

$$\begin{aligned}\|\sigma^2(t,x,y)\|^2 &= \sum_{k=1}^{\infty}\|\sigma_l^2\|_{L_Q^0}^2 \\ &\le L_*\sum_{k=1}^{\infty} b_k^2 < \infty.\end{aligned}$$

Hence

$$E\|\sigma_1(t,x,y)\|^2 \le L_*\sum_{k=1}^{\infty} a_k^2 + E(\|x\|_{L^2}^2 + \|y\|_{L^2}^2) \text{ for all } x,y \in L^2([0,\pi]),$$

and

$$E\|\sigma_2(t,x,y)\|^2 \le L_*\sum_{k=1}^{\infty} b_k^2 + E(\|x\|_{L^2}^2 + \|y\|_{L^2}^2) \text{ for all } x,y \in L^2([0,\pi]).$$

□

Corollary 4.1.6. *Let $K : [0,\pi]\times[0,\pi]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a continuous function such that there exist $a, b \ge 0$ such that*

$$|K(t,s,x,y) - K(t,s,\bar{x},\bar{y})| \le a|x-\bar{x}| + b|y-\bar{y}|, \quad \textit{for all } x,\bar{x},y,\bar{y} \in \mathbb{R}.$$

Then $K_* : [0,\pi]\times L^2([0,\pi])\times L^2([0,\pi]) \to L_Q^0(L^2([0,\pi]), L^2([0,\pi]))$ defined by

$$K_*(t,x,y)f(t) = \int_0^t K(t,s,x(s),y(s))f(s)ds, \quad f \in L^2([0,\pi]),$$

is a Hilbert-Schmidt operator and there exist $\alpha, \beta \ge 0$ such that

$$\|K_*(t,,x,y) - K(t,s,\bar{x},\bar{y})\|_{L_Q^0} \le \alpha\|x-\bar{x}\|_{L^2} + \beta\|y-\bar{y}\|_{L^2} \quad \textit{for each } x,\bar{x},y,\bar{y} \in L^2([0,\pi]).$$

Proof. It is clear that $K_*(t,x,y)$ is a bounded linear operator. Let $(e_i)_{i\in\mathbb{N}}$ be an orthonormal basis of $L^2([0,\pi])$

$$\begin{aligned}
&\|K_*(t,x,y)\|^2_{L^0_Q} \\
&= \sum_{k=1}^{\infty}\int_0^{\pi}\left(\left|\sqrt{\sigma_n}\int_0^t [K(t,s,x(s),y(s))]e_n(s)ds\right|^2\right)dt \\
&\leq \sum_{n=1}^{\infty}\int_0^{\pi}\left(\sigma_n\left(\int_0^{\pi}[a|x(s)|+b|y(s)|]|e_n(s)|ds+\int_0^{\pi}|K(t,s,0,0)||e_n(s)|ds\right)^2\right)dt \\
&\leq 4\pi a^2\sum_{n=1}^{\infty}\sigma_n\|x\|^2_{L^2}+4\pi b^2\sum_{n=1}^{\infty}\sigma_n\|y\|^2_{L^2}+2\pi^2\sup_{(t,s)\in[0,\pi]\times[0;\pi]}|K(t,s,0,0)|\sum_{n=1}^{\infty}\sigma_n.
\end{aligned}$$

Thus

$$\|K_*(t,x,y)\|_{L^0_Q} \leq \alpha_1\|x\|_{L^2}+\beta_1\|y\|_{L^2}+\gamma_1<\infty,$$

where

$$\alpha_1^2 = 4\pi a^2\sum_{n=1}^{\infty}\sigma_n,\ \beta_1^2 = 4\pi b^2\sum_{n=1}^{\infty}\sigma_n,\ \gamma_1^2 = 2\pi^2\sup_{(t,s)\in[0,\pi]\times[0,\pi]}|K(t,s,0,0)|\sum_{n=1}^{\infty}\sigma_n.$$

We can easily prove

$$\|K_*(t,,x,y)-K(t,s,\bar{x},\bar{y})\|^2_{L_Q} \leq \alpha\|x-\bar{x}\|^2_{L^2}+\beta\|y-\bar{y}\|^2_{L^2} \quad \text{for each } x,\bar{x},y,\bar{y}\in L^2([0,\pi]),$$

where $\alpha = \alpha_1^2,\ \beta = \beta_1^2$. □

Corollary 4.1.7. *Let* $\{K_n\}_{n\in\mathbb{R}} : [0,\pi]\times[0,\pi]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *be a summability kernel on* $[0,\pi]$ *i.e it holds*

$$\sum_{n=1}^{\infty}\|K_n(t,.,x,y)\|^2_{L^2}<\infty,$$

and K *is a continuous function such that there exists* $a_n, b_n \geq 0$ *such that*

$$|K(t,s,x,y)-K(t,s,\bar{x},\bar{y})| \leq a_n|x-\bar{x}|+b_n|y-\bar{y}|, \quad \textit{for all } x,\bar{x},y,\bar{y}\in\mathbb{R}.$$

If

$$\sum_{n=1}^{\infty}(a_n^2+b_n^2)<\infty.$$

Then $\bar{K}_n : [0,\pi]\times L^2([0,\pi])\times L^2([0,\pi]) \to L^0_Q(L^2([0,\pi]), L^2([0,\pi]))$ *be defined by*

$$\bar{K}_n(t,x,y)f(t) = \int_0^t K_n(t,s,x(s),y(s))f(s)ds, \quad f \in L^2([0,\pi]),$$

is a Hilbert-Schmidt operator and there exists $\alpha, \beta \geq 0$ *such that*

$$\sum_{n=1}^{\infty} \|\bar{K}_n(t,x,y) - \bar{K}_n(t,s,\bar{x},\bar{y})\|^2_{L^0_Q} \leq \alpha\|x-\bar{x}\|^2_{L^2} + \beta\|y-\bar{y}\|^2_{L^2} \quad \text{for each } x,\bar{x},y,\bar{y} \in L^2([0,\pi]).$$

Proof. To prove the result we use the same method in the proof of Corollary 4.1.6. □

Lemma 4.1.8. *Assume that* σ *satisfies conditions of Corollary 6.1.5 or (Corollary 6.1.5 or 4.1.6), and* (iii) *or* F, G *are Lipschitz functions and* I_k, I_k *satisfy* (i) *or* (iv'). *Then, problem 4.1.9 has at least one solution and problem (4.1.10) has exponentially stable mild solutions.*

Proof. If we use that F, G are Lipchitz functions and assumptions in Corollary 4.1.6 or 4.1.7 hold, then by Theorem 6.1.1, problem 4.1.9 possesses a unique mild solution. By the same method we can prove that all the conditions of Theorem 5.1.3 or 4.1.4 are satisfied. □

Chapter 5

Systems of impulsive stochastic semilinear differential inclusions

In this chapter we prove the existence of mild solutions for a first-order impulsive semilinear stochastic functional differential inclusions driven by a fractional Brownian motion. We consider the cases in which the right hand side is convex or nonconvex-valued. The results are obtained by using two different fixed point theorems for multivalued mappings. We are interested in the existence problem of the following stochastic differential inclusions:

$$\begin{cases} dx(t) & \in & (Ax(t) + F^1(t, x(t), y(t)))dt \\ & + & \displaystyle\sum_{l=1}^{\infty} \sigma_l^1(t, x(t)), y(t))dB_l^H(t), \ t \in J := [0, b], t \neq t_k, \\ dy(t) & \in & (Ay(t) + F^2(t, x(t), y(t)))dt \\ & + & \displaystyle\sum_{l=1}^{\infty} \sigma_l^2(t, x(t), y(t))dB_l^H(t), \ t \in [0, b], t \neq t_k, \\ \Delta x(t) & = & I_k(x(t_k)), \quad t = t_k \quad k = 1, 2, \dots, m \\ \Delta y(t) & = & \overline{I}_k(y(t_k)), \\ x(0) & = & x(b), \\ y(0) & = & y(b), \end{cases} \tag{5.0.1}$$

where $J := [0, b]$, X is a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ induced by norm $\|\cdot\|$, $A : D(A) \subset X \longrightarrow X$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $(S(t))_{t\geq 0}$ in

X and $F^1, F^2 : [0,b] \times X \times X \longrightarrow \mathcal{P}(X)$ are given set-valued functions, where $\mathcal{P}(X)$ denotes the family of nonempty subsets of X, $I_k \in C(X,X)$ $(k = 1,2,\ldots,m)$, $\sigma_l^1, \sigma_l^2 : J \times X \times X \to L_Q^0(Y,X)$. Here, $L_Q^0(Y,X)$ denotes the space of all Q-Hilbert-Schmidt operators from Y into X, which will be also defined in the next section. Moreover, the fixed times t_k satisfies $0 < t_1 < t_2 < \ldots < t_m < T$, $y(t_k^-)$ and $y(t_k^+)$ denotes the left and right limits of $y(t)$ at $t = t_k$.

$$\begin{cases} \sigma(.,x) = (\sigma_1(.,x), \sigma_2(.,x), \ldots), \\ \|\sigma(.,x)\|^2 = \sum_{l=1}^{\infty} \|\sigma_l(.,x)\|_{L_Q^0}^2 < \infty \end{cases} \tag{5.0.2}$$

with $\sigma(.,x) \in \ell^2$ for all $x \in X$, where

$$\ell^2 = \{\phi = (\phi_l)_{l\geq 1} : X \times X \to L_Q^0(Y,X) \quad : \|\phi(x)\|^2 = \sum_{l=1}^{\infty} \|\phi_l(x)\|_{L_Q^0}^2 < \infty\}.$$

5.1 Existence results

In this section we prove the existence of mild solution of the problem (5.0.1). Our approach is based on multivalued versions of Schaefer's fixed point theorem.

5.1.1 The convex case

In this section, we will show same results concerning the existence results of mild solutions for convex case of system (5.0.1)in the convex case. Let $J_k = (t_k, t_{k+1}]$, $k = 1,2,\ldots,m$. In order to define a solution for Problem (5.0.1), consider the following space of pice-wise continuous functions

$$\begin{aligned} PC = \{ & x : \Omega \times [0,b] \longrightarrow X,\ x \in C(J_k, X)),\ k = 1,\ldots,m \text{ such that} \\ & x(t_k^+,.) \text{ and } x(t_k^-,.) \text{ exist with } x(t_k^-,.) = x(t_k,.) \quad \text{and} \\ & \sup_{t\in[0,b]} E|x(t,.)|_X^2 < \infty \text{ almost surely}\}. \end{aligned}$$

Endowed with the norm

$$\|x\|_{PC} = \left(\sup_{s\in[0,b]} E|x(s,.)|_X^2 \right)^{\frac{1}{2}},$$

it is not difficult to check that PC is a Banach space with norm $\|\cdot\|_{PC}$. $AC^i(J,X)$ is the space of functions $y: J \to X$ $i-$ differentiable in whose ith derivative, $y^{(i)}$, is absolutely continuous.

Let A be the infinitesimal generator of a C_0-semigroup $\{S(t)\}_{t\geq 0}$ such that $1 \in \Lambda(S(b))$ and let $f: J \to X$ be a continuous function.

Lemma 5.1.1. *If $x, y \in PC$ is a mild solution of the problem*

$$\begin{cases} dx(t) &= (f^1(t) + Ax(t))dt + \sum_{l=1}^{\infty} \sigma_l^1(t, x(t), y(t))dB_l^H(t), \ t \in J, t \neq t_k \\ x(t_k^+) - x(t_k) &= I_k(x(t_k)), \quad t = t_k \quad k = 1, 2, \ldots, m \\ y(t_k^+) - y(t_k) &= \overline{I}_k(y(t_k)), \\ x(0) &= x(b), \end{cases} \tag{5.1.1}$$

then it is given by

$$x(t) = S(t)(I - S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b - t_k))I_k(x(t_k)) + \int_0^b S(b-s)f^1(s)ds$$

$$+\sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^1(t, x(s), y(s))dB_l^H(s)\Big) + \int_0^t S(t-s)f^1(s)ds$$

$$+\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(t, x(s), y(s))dB_l^H(s) + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k)), \ \textit{for} \ \ t \in J$$

and

$$y(t) = S(t)(I - S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b - t_k))\overline{I}_k(y(t_k)) + \int_0^b S(b-s)f^2(s)ds$$

$$+\sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^2(t, x(s), y(s))dB_l^H(s)\Big) + \int_0^t S(t-s)f^2(s)ds$$

$$+\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^2(t, x(s), y(s))dB_l^H(s) + \sum_{0<t_k<t} S(t-t_k)\overline{I}_k(y(t_k)), \ \textit{for} \ \ t \in J$$

Proof. Let (x, y) be a solution of Problem (6.0.1) and $L_1(s) = S(t-s)x(s)$ and $L_2(s) = S(t-s)y(s)$ for fixed $t \in J$. We have

$$\begin{aligned} L_1^{'}(s) &= -S^{'}(t-s)x(s) + S(t-s)x^{'}(s) \\ &= -AS(t-s)x(s) + S(t-s)x^{'}(s) \\ &= S(t-s)(x^{'}(s) - Ax(s)) \\ &= S(t-s)(f^1(s)ds + \sum_{l=1}^{\infty} \sigma_l^1(t, x(s), y(s))dB_l^H(s) \end{aligned}$$

Let $0 < t < t_1$. Integrating the previous equation, we deduce for $k = 1$

$$L_1(t) - L_1(0) = \int_0^t S(t-s)f^1(s)ds + \sum_{l=1}^{\infty} \int_0^t S(t-s)\sigma_l^1(t, x(s), y(s))dB_l^H(s)$$

Hence

$$x(t) = S(t)x(0) + \int_0^t S(t-s)f^1(s)ds + \sum_{l=1}^{\infty} \int_0^t S(t-s)\sigma_l^1(t, x(s), y(s))dB_l^H(s).$$

More generally, for $t_k < t < t_{k+1}$

$$\begin{aligned} \int_0^{t_1} L_1^{'}(s) + \int_{t_2}^{t_1} L_1^{'}(s) + ... + \int_{t_k}^{t} L_1^{'}(s) &= \int_0^t S(t-s)f^1(s)ds \\ &+ \sum_{l=1}^{\infty} \int_0^t S(t-s)\sigma_l^1(t, x(s), y(s))dB_l^H(s) \\ &= L_1(t_1^-) - L_1(0) + L_1(t_2^-) - L_1(t_1^+) + ... + L_1(t) - L_1(t_k^+) \\ &= \int_0^t S(t-s)f^1(s)ds + \sum_{l=1}^{\infty} \int_0^t S(t-s)\sigma_l^1(t, x(s), y(s))dB_l^H(s). \end{aligned}$$

Therefore

$$\begin{aligned} x(t) = S(t)x(0) + \sum_{0<t_k<t} (L_1(t_k^+ - L_1(t_k^-)) + \int_0^t S(t-s)f^1(s)ds \\ + \sum_{l=1}^{\infty} \int_0^t S(t-s)\sigma_l^1(s, x(s), y(s))dB_l^H(s). \end{aligned}$$

Since $x(0)=x(b)$ and $1\in\rho(S(T))$, then $(I-S(b))$ is invertible. Hence we obtain after substitution

$$\begin{aligned} x(t) &= S(t)(I-S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b-t_k))I_k(x(t_k))+\int_0^b S(b-s)f^1(s)ds \\ &\quad +\sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^1(s,x(s),y(s))dB_l^H(s)\Big)+\int_0^t S(t-s)f^1(s)ds \\ &\quad +\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(s,x(s),y(s))dB_l^H(s) \\ &\quad +\sum_{0<t_k<t} S(t-t_k)I_k(x(t_k)),\ \text{for}\ \ t\in J. \end{aligned}$$

and,we can proceed similarly with the y-component.

$$\begin{aligned} y(t) &= S(t)(I-S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b-t_k))\overline{I}_k(y(t_k))+\int_0^b S(b-s)f^2(s)ds \\ &\quad +\sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^2(s,x(s),y(s))dB_l^H(s)\Big)+\int_0^t S(t-s)f^2(s)ds \\ &\quad +\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^2(s,x(s),y(s))dB_l^H(s)+\sum_{0<t_k<t} S(t-t_k)\overline{I}_k(y(t_k)),\ \text{for}\ \ t\in J. \end{aligned}$$

□

This lemma leads to the definition of a mild solution.

Definition 5.1.1. *A $X-$valued stochastic process $u=(x,y)\in PC\times PC$ is said to be a mild solution of* (5.0.1) *with respect to the probability space $(\Omega,\mathcal{F},\mathbb{P})$, if:*

1) $u(t)$ is $\mathcal{F}_t$-adapted for all $t\in J_k=(t_k,t_{k+1}]\quad k=1,2,\dots,m$

2) $u(t)$ is right continuous and has limit on the left, and there exists selections f^i, $i=1,2$, such that $f^i(t)\in F^i(t,u(t))$ a.e. $t\in J$.

3) $u(t)$ satisfies

$$\begin{cases} x(t) &= S(t)(I-S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b-t_k))I_k(x(t_k)) + \int_0^b S(b-s)f^1(s)ds \\ &+ \sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^1(t,x(s),y(s))dB_l^H(s)\Big) + \int_0^t S(t-s)f^1(s)ds \\ &+ \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(t,x(s),y(s))dB_l^H(s) \\ &+ \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k)), \quad t\in J, \\ y(t) &= S(t)(I-S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b-t_k))\overline{I}_k(y(t_k)) + \int_0^b S(b-s)f^2(s)ds \\ &+ \sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^2(t,x(s),y(s))dB_l^H(s)\Big) + \int_0^t S(t-s)f^2(s)ds \\ &+ \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^2(t,x(s),y(s))dB_l^H(s) \\ &+ \sum_{0<t_k<t} S(t-t_k)\overline{I}_k(y(t_k)), \quad t\in J. \end{cases}$$

In this section, we assume again that $1 \in \Lambda(S(b)))$.We are now in a position to state and prove our existence result for the problem (5.0.1). First we will list the following hypotheses which will be imposed in our main theorem.
Consider the following assumptions: In all this part, we assume that $S(t)$ is compact for $t>0$ and that there exists $M>0$ such that

$$\|S(t)\| \le M, \quad \text{for every} \quad t\in[0,b].$$

(H_1) The function $\sigma_l^i : J\times X\times X \to L_Q^0(Y,X)$. There exist positive constants α_i and $\beta_i$and c_i for each $i=1,2$ such that

$$\|\sigma^1(t,x,y)\|^2 \le \alpha_1|x|_X^2+\beta_1|y|_X^2+c_1, \quad \|\sigma^2(t,x,y)\|^2 \le \alpha_2|x|_X^2+\beta_2|y|_X^2+c_2$$

and

$$\sum_{l=1}^{\infty}\int_0^b \|\sigma_l^1(t,x,y)\|_{L_Q^0}^2 dt < \infty$$

for all x, $y\in X$ and $t\in J$.

(H_2) $F^i : [0, b] \times X \times X \longrightarrow \mathcal{P}_{cv,cp}(X)$ is an integrably bounded multi-valued map,i.e.,there exists $p_i \in L^2(J, X), i = 1, 2$ such that

$$|F^i(t, x, y)|_X^2 = \sup_{f^i \in F^i(t,x,y)} |f^i(t)|_X^2 \le p_i(t), \quad \forall t \in J, \quad \forall (x, y) \in X \times X.$$

(H_3) There exist constants $d_k, \overline{d}_k \ge 0$ and $e_k, \overline{e}_k \ge 0$ for each $k = 1, \dots, m$ such that

$$|I_k(x)|_X^2 \le d_k |x|_X^2 + e_k, \quad |\overline{I}_k(y)|_X^2 \le \overline{d}_k |y|_X^2 + \overline{e}_k, \text{ for all } x, y \in X.$$

Consider the following operator $N(x, y) = (N_1(x, y), N_2(x, y)),\ (x, y) \in PC \times PC$ defined by

$$N(x, y) = \big\{(h, \overline{h}) \in PC \times PC\big\}$$

given by

$$\begin{aligned}
h(t) &= S(t)(I - S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b - t_k))I_k(x(t_k)) \\
&+ \int_0^b S(t - s)f^1(s)ds + \sum_{l=1}^{\infty}\int_0^b S(t - s)\sigma_l^1(t, x(s), y(s))dB_l^H(s)\Big) \\
&+ \int_0^t S(t - s)f^1(s)ds + \sum_{l=1}^{\infty}\int_0^t S(t - s)\sigma_l^1(t, x(s), y(s))dB_l^H(s) \\
&+ \sum_{0<t_k<t} S(t - t_k)I_k(x(t_k)), \quad \text{if} \quad t \in [0, b] \\
\overline{h}(t) &= S(t)(I - S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b - t_k))\overline{I}_k(y(t_k)) \\
&+ \int_0^b S(t - s)f^2(s)ds + \sum_{l=1}^{\infty}\int_0^b S(t - s)\sigma_l^2(t, x(s), y(s))dB_l^H(s)\Big) \\
&+ \int_0^t S(t - s)f^2(s)ds + \sum_{l=1}^{\infty}\int_0^t S(t - s)\sigma_l^2(t, x(s), y(s))dB_l^H(s) \\
&+ \sum_{0<t_k<t} S(t - t_k)\overline{I}_k(y(t_k)), \quad \text{if} \quad t \in J,
\end{aligned}$$

where

$$f^i \in S_{F^i,u} = \{f^i \in L^2(J, X) : f^i(t) \in F^i(t, x, y) \quad \text{for a.e} \quad t \in J\}.$$

Lemma 5.1.2. *Assume that $F^i : J \times X \times X \longrightarrow \mathcal{P}_{cv,cp}(X)$ is a Caratheòdory map satisfying $(H_1)-(H_3)$ hold. Then the operator is completely continuous and u.s.c.*

Proof. Firstly we show that $N = (N_1, N_2)$ completely continuous

Step 1. N maps bounded sets into bounded sets in $PC \times PC$.
Indeed, it is enough to show that for any $q > 0$ there exists a positive constant $l = (l_1, l_2)$ such that for each $(x, y) \in B_q = \{(x, y) \in PC \times PC : E|x|_X^2 \leq q,\ E|y|_X^2 \leq q\}$ one has

$$\|h\|_X^2 \leq l_1, \quad \|\overline{h}\|_X^2 \leq l_2.$$

Let $(h, \overline{h}) \in (N_1, N_2)$ there exists $f^i(t) \in F^i(t, x, y)$ for each $t \in J$, we get

$$\begin{aligned} E|h(t)|_X^2 &= E\Big|S(t)(I - S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b-t_k))I_k(x(t_k)) \\ &+ \int_0^b S(t-s)f^1(s)ds \\ &+ \sum_{l=1}^{\infty}\int_0^b S(t-s)\sigma_l^1(t, x(s), y(s))dB_l^H(s)\Big) \\ &+ \int_0^t S(t-s)f^1(s)ds \\ &+ \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(t, x(s), y(s))dB_l^H(s) \\ &+ \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k))\Big|_X^2. \end{aligned}$$

Also

$$\begin{aligned}
E|h(t)|_X^2 &\leq 4E\Big|S(t)(I-S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b-t_k))I_k(x(t_k)) \\
&+ \int_0^b S(t-s)f^1(s)ds \\
&+ \sum_{l=1}^{\infty}\int_0^b S(t-s)\sigma_l^1(t,x(s),y(s))dB_l^H(s)\Big)\Big|_X^2 \\
&+ 4E\Big|\int_0^t S(t-s)f^1(s)ds\Big|_X^2 \\
&+ 4E\Big|\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(t,x(s),y(s))dB_l^H(s)\Big|_X^2 \\
&+ 4E\Big|\sum_{0<t_k<t} S(t-t_k)I_k(x(t_k))\Big|_X^2
\end{aligned}$$

Using (H_1)-(H_3) and (1.4.19) we have

$$\begin{aligned}
E|h(t)|_X^2 &\leq 12M^4\Big\|(I-S(b))^{-1}\Big\|^2\Big(m\sum_{k=1}^{m}\sup_{z\in B(0,q)} E|I_k(z)|_X^2 + \|p_1\|_{L^1}^2 \\
&+ c_H H(2H-1)b^{2H}(\alpha_1 E|x(t)|_X^2 + \beta_1 E|y(t)|_X^2 + c_1)\Big) + 4M^2\|p_1\|_{L^1} \\
&+ 4M^2(c_H H(2H-1)b^{2H}(\alpha_1 E|x(t)|_X^2 + \beta_1 E|y(t)|_X^2 + c_1)) \\
&+ 4M^2 m\sum_{k=1}^{m}(d_k q + e_k) \\
&\leq 4M^4\Big\|(I-S(b))^{-1}\Big\|^2\Big(\sum_{k=1}^{m}(d_k q + e_k) + \|p_1\|_{L^1} \\
&+ c_H H(2H-1)b^{2H}(\alpha_1 q + \beta_1 q + c_1)\Big) + 4M^2\|p_1\|_{L^1} \\
&+ 4M^2(c_H H(2H-1)b^{2H}(\alpha_1 q + \beta_1 q + c_1)) \\
&+ 4M^2 m\sum_{k=1}^{m}(d_k q + e_k) := l_1.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
E|\overline{h}(t)|_X^2 &\leq 4M^4\Big\|(I-S(b))^{-1}\Big\|^2\Big(\sum_{k=1}^{m}\overline{d}_k q+\overline{e}_k)+\|p_2\|_{L^1}\\
&+ c_H H(2H-1)b^{2H}(\alpha_2 q+\beta_2 q+c_2)\Big)+4M^2\|p_2\|_{L^1}\\
&+ 4M^2(c_H H(2H-1)b^{2H}(\alpha_2 q+\beta_2 q+c_2))\\
&+ 4M^2 m\sum_{k=1}^{m}(\overline{d}_k q+\overline{e}_k)) := l_2.
\end{aligned}$$

Therefore

$$\begin{pmatrix} \mathbb{E}|h(t)|_X^2 \\ \mathbb{E}|\overline{h}(t)|_X^2 \end{pmatrix} \leq \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}$$

Step 2 N maps bounded sets into equicontinuous sets of $PC\times PC$.
Let B_q be a bounded set in $PC\times PC$ as in $Step1$. Let $\tau_1,\tau_2\in J,\tau_1<\tau_2$ and $(x,y)\in B_q$,there exists $f^i(t)\in F^i(t,x,y),\ i=1,2$, such that

$$\begin{aligned}
&E\Big|h(\tau_2)-h(\tau_1)\Big|_X^2\\
&\leq 12M^2\Big\|(S(\tau_2)-S(\tau_1)\Big\|^2\Big\|(I-S(b))^{-1}\Big\|^2\Big(m\sup_{z\in B(0,q)}\sum_{k=1}^{m}E|I_k(z)|_X^2+\|p_1\|_{L^1}\\
&\quad +c_H H(2H-1)b^{2H-1}(\alpha_1 E|x(t)|_X^2+\beta_1 E|y(t)|_X^2+c_1)\Big)\\
&\quad +12\int_0^{\tau_1}\Big\|S(\tau_2-s)-S(\tau_1-s)\Big\|^2 p_1(s)ds\\
&\quad +12\int_{\tau_1}^{\tau_2}\Big\|S(\tau_2-s)\Big\|^2 p_1(s)ds\\
&\quad +12c_H H(2H-1)(\tau_2)^{2H-1}\int_0^{\tau_2}\Big\|S(\tau_2-s)-S(\tau_1-s)\Big\|^2\Big(\alpha_1 E|x(t)|_X^2\\
&\quad +\beta_1 E|y(t)|_X^2+c_1)ds\Big)\\
&\quad +12c_H H(2H-1)(t_2-t_1)^{2H-1}\int_{\tau_1}^{\tau_2}\Big\|S(\tau_1-s)\Big\|^2\Big(\alpha_1 E|x(t)|_X^2
\end{aligned}$$

$$+\beta_1 E|y(t)|_X^2 + c_1)ds\Big)$$
$$+12m \sum_{0<t_k<\tau_1} \Big\|S(\tau_2 - t_k) - S(\tau_1 - t_k)\Big\|^2 \sup_{z\in B(0,q)} E|I_k(z)|_X^2$$
$$+12m \sum_{\tau_1<t_k<\tau_2} \Big\|S(\tau_2 - t_k)\Big\|^2 \sup_{z\in B(0,q)} E|I_k(z)|_X^2.$$

The right-hand term tends to zero as $|\tau_2 - \tau_1| \to 0$ since $S(t)$ is strongly continuous operator and the compactness of $S(t)$ for $t > 0$ implies the continuity in the uniform operator topology [74]. This proves the equicontinuity.

Step 3 $(N(B_q)(t)$ is precompact in $X \times X$. As a consequence of Steps a to b, together with the Arzelá-Ascoli theorem, it suffices to show that N maps B_q into a precompact set in $X \times X$. Let $0 < t < b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $(x, y) \in B_q$ we define

$$\begin{aligned} h_\epsilon(t) &= S(\epsilon)S(t-\epsilon)(I - S(b))^{-1}\Big(\sum_{k=0}^{m} S(b-t_k)I_k(x(t_k)) + \int_0^b S(b-s)f^1(s)ds \\ &+ \sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^1(s,(s),y(s))dB_l^H(s)\Big) + S(\epsilon)\int_0^{t-\epsilon} S(t-s)f^1(s)ds \\ &+ \sum_{l=1}^{\infty} S(\epsilon)\int_0^{t-\epsilon} S(t-s)\sigma_l^1(s,x(s),y(s))dB_l^H(s) \\ &+ S(\epsilon)\sum_{0<t_k<t-\epsilon} S(t-\epsilon-t_k)I_k(x(t_k)). \end{aligned}$$

Since $S(t)$ is a compact operator, the set

$$H_\epsilon = \{\widetilde{h}_\epsilon(t) = (h_\epsilon(t), \overline{h}_\epsilon(t)) : \widetilde{h}_\epsilon \in N_\epsilon(x,y) \;\; (x,y) \in B_q\}.$$

$$\begin{aligned}
E\Big|h(t)-h_\epsilon(t)\Big|_X^2 &\leq 3E\Big|\int_{t-\epsilon}^{t} S(t-s)f^1(s)ds\Big|_X^2\\
&+ 3E\Big|\sum_{l=1}^{\infty}\int_{t-\epsilon}^{t} S(t-s)\sigma_l^1(s,x(s),y(s))dB_l^H(s)\Big|_X^2\\
&+ 3E\Big|\sum_{t-\epsilon<t_k<t} S(t-t_k)I_k(x(t_k))\Big|_X^2\\
&\leq 3M^2\int_{t-\epsilon}^{t} p_1(s)ds\\
&+ 3M^2(c_H H(2H-1)\epsilon^{2H-1}\int_{t-\epsilon}^{t}(\alpha_1 q+\beta_1 q+c_1))ds\\
&+ 3M^2(m\sum_{t-\epsilon<t_k<t}\sup_{z\in B(0,q)} E\Big|I_k(z)\Big|_X^2).
\end{aligned}$$

Similarly,

$$\begin{aligned}
E\Big|\overline{h}(t)-\overline{h}_\epsilon(t)\Big|_X^2 &\leq 3M^2\int_{t-\epsilon}^{t} p_2(s)ds\\
&+ 3M^2(c_H H(2H-1)\epsilon^{2H-1}\int_{t-\epsilon}^{t}(\alpha_2 q+\beta_2 q+c_2))ds\\
&+ 3M^2(m\sum_{t-\epsilon<t_k<t}\sup_{\overline{z}\in B(0,q)} E\Big|\overline{I}_k(\overline{z})\Big|_X^2).
\end{aligned}$$

The right-hand side tends to 0, as $\epsilon\to 0$. Therefore, there are precompact sets arbitrarily close to the set $H=\{\widetilde{h}(t)=(h(t),\overline{h}(t)):\widetilde{h}\in N(x,y)\ \ (x,y)\in B_q\}$. This set is then precompact in $X\times X$.

Step 2. $N=(N_1,N_2)$ has a closed graph. Let $u_n=(x_n,y_n)\longrightarrow z_*=(x_*,y_*)$,$(h_n,\overline{h}_n)\in N(u_n)$ and $(h_n,\overline{h}_n)\longrightarrow(h_*,\overline{h}_*)$ as $n\longrightarrow\infty$, we shall prove that $h_*\in N_1(u_*)$. The fact that $h_n\in N_1(u_n)$ and $\overline{h}_n\in N_2(u_n)$ means that there exists $f_n^i \in S_{F^i,u_n}$ for each $i = 1, 2$ such that

$$\begin{aligned} h_n(t) &= S(t)(I - S(b))^{-1}\Big(\sum_{0<t_k<t} S(b - t_k)I_k(x_n(t_k)) + \int_0^b S(b - s)f_n^1(s)ds \\ &+ \sum_{l=1}^{\infty}\int_0^b S(b - s)\sigma_l^1(s, x_n(s), y_n(s))dB_l^H(s)\Big) + \int_0^t S(t - s)f_n^1(s)ds \\ &+ \sum_{l=1}^{\infty}\int_0^t S(t - s)\sigma_l^1(s, x_n(s), y_n(s))dB_l^H(s) + \sum_{0<t_k<t} S(t - t_k)I_k(x_n(t_k)). \end{aligned}$$

First, notice that, as $n \to \infty$

$$\begin{aligned} \Big\|h_n - S(t)(I &- S(b))^{-1}\Big(\sum_{k=1}^{m} S(b - t_k)I_k(x_n(t_k)) \\ &+ \sum_{l=1}^{\infty}\int_0^b S(b - s)\sigma_l^1(s, x_n(s), y_n(s))dB_l^H(s)\Big) \\ &\quad -\sum_{l=1}^{\infty}\int_0^t S(t - s)\sigma_l^1(s, x_n(s), y_n(s))dB_l^H(s) - \sum_{0<t_k<k} S(t - t_k)I_k(x_n(t_k)) \\ &- h_* + S(t)(I - S(b))^{-1}\Big(\sum_{k=1}^{m} S(b - t_k)I_k(x_*(t_k)) \\ &\quad -\sum_{l=1}^{\infty}\int_0^t S(t - s)\sigma_l^1(s, x(s), y(s))dB_l^H(s) \\ &+ \sum_{l=1}^{\infty}\int_0^b S(b - s)\sigma_l^1(s, x_*(s), y_*(s))dB_l^H(s)\Big) \\ &\quad + \sum_{0<t_k<k} S(t - t_k)I_k(x(t_k))\Big\|_{PC} \longrightarrow 0, \quad n \longrightarrow 0. \end{aligned}$$

Now, consider the continuous linear operator $\Gamma : L^2(J, X) \longrightarrow PC$ defined for each $i = 1, 2$, by

$$\Gamma(v^i)(t) = S(t)(I - S(b))^{-1}\int_0^b S(b - s)v^i(s)ds + \int_0^t S(t - s)v^i(s)ds.$$

From the definition of Γ we know that

$$\Big(h_n(t) - S(t)(I-S(b))^{-1}\Big(\sum_{k=1}^{m} S(b-t_k)I_k(x_n(t_k)) + \sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^1(s,x_n(s),y_n(s))dB_l^H(s)\Big) - \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(s,x_n(s),y_n(s))dB_l^H(s) - \sum_{0<t_k<t} S(t-t_k)I_k(x_n(t_k))\Big) \in \Gamma(S_{F^1,u_n}),$$

and

$$(\overline{h}_n(t) - S(t)(I-S(b))^{-1}\Big(\sum_{k=1}^{m} S(b-t_k)\overline{I}_k(y_n(t_k)) + \sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^2(s,x_n(s),y_n(s))dB_l^H(s)\Big) - \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^2(s,x_n(s),y_n(s))dB_l^H(s) - \sum_{0<t_k<t} S(t-t_k)\overline{I}_k(y_n(t_k))\Big) \in \Gamma(S_{F^2,u_n}).$$

Since $u_n = (x_n, y_n) \longrightarrow z_* = (x_*, y_*)$ and $\Gamma \circ S_{F^i}$ is a closed graph operator by Lemma (2.3.1), then there exists $f_*^i \in S_{F^i,u_*}$ for each $i = 1, 2$ such that

$$\begin{aligned} h_*(t) &= S(t)(I-S(b))^{-1}\Big(\sum_{k=1}^{m} S(b-t_k)I_k(x_*(t_k)) + \int_0^b S(b-s)f_*^1(s)ds \\ &+ \sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^1(s,x_*(s),y_*(s))dB_l^H(s)\Big) + \int_0^t S(t-s)f_*^1(s)ds \\ &+ \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(s,x_*(s),y_*(s))dB_l^H(s) + \sum_{0<t_k<t} S(t-t_k)I_k(x_*(t_k)). \end{aligned}$$

Similarly,

$$\begin{aligned}
\overline{h}_*(t) &= S(t)(I-S(b))^{-1}\Big(\sum_{k=1}^{m} S(b-t_k)\overline{I}_k(\overline{y}_*(t_k)) + \int_0^b S(b-s)f_*^2(s)ds \\
&+ \sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^2(s,x_*(s),y_*(s))dB_l^H(s)\Big) + \int_0^t S(t-s)f_*^2(s)ds \\
&+ \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^2(s,x_*(s),y_*(s))dB_l^H(s) + \sum_{0<t_k<t} S(t-t_k)\overline{I}_k(y_*(t_k)).
\end{aligned}$$

Hence $(h_*, h_*) \in (N_1(u_*), N_2(u_*))$, proving our claim. Lemma 2.3.3 yields that N is upper semicontinuous. □

Now, we present the fist our existence and compactness of solution set of the Problem (5.0.1).

Theorem 5.1.3. *Assume that $F^i : [0,b] \times X \times X \longrightarrow \mathcal{P}_{cv,cp}(X)$ is a Carathèodory map satisfying (H_1)-(H_3) hold. Then the* (5.0.1) *has at least one mild solution on J. If further X is a reflexive space, then the solution set is compact in $PC \times PC$.*

Proof. **Part 1**. Existence of solutions.
We transform the problem (5.0.1) into a fixed point problem. Consider the multi- valued operator $N : PC \times PC \to \mathcal{P}(PC \times PC)$ defined in lemma 5.1.2. It is clear that all solutions of Problem (5.0.1) are fixed points of the multi-valued operator defined by We shall show that N satisfies assumptions of Lemma 2.4.8. Since for each $(x, y) \in PC \times PC$, the nonlinearity F^i takes convex values, the selection set $S_{F^i,u}$ is convex, and therefore N has convex values. From lemma 5.1.2, N is completely continuous and u.s.c.

Step 3 A priori bounds on solutions. Let $(x, y) \in PC \times PC$ be a solution of the abstract nonlinear equation $x \in N_1(x, y)$ and $y \in N_2(x, y)$. Then there exists $f^i \in S_{F^i}$ for $t \in [0,b]$ for each $i = \{1,2\}$, namely

$$\begin{aligned} x(t) &= S(t)(I-S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b-t_k))I_k(x(t_k)) \\ &\quad + \int_0^b S(b-s)f^1(s)ds + \sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^1(s,x(s),y(s))dB_l^H(s)\Big) \\ &\quad + \int_0^t S(t-s)f^1(s)ds + \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(s,x(s),y(s))dB_l^H(s) \\ &\quad + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k)), \end{aligned}$$

and

$$\begin{aligned} y(t) &= S(t)(I-S(b))^{-1}\Big(\sum_{k=1}(S(b-t_k))\overline{I}_k(y(t_k)) \\ &\quad + \int_0^b S(b-s)f^2(s)ds + \sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^2(s,x(s),y(s))dB_l^H(s)\Big) \\ &\quad + \int_0^t S(t-s)f^2(s)ds + \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^2(s,x(s),y(s))dB_l^H(s) \\ &\quad + \sum_{0<t_k<t} S(t-t_k)\overline{I}_k(y(t_k)). \end{aligned}$$

We first give an estimation for the third part,

$$\begin{aligned} E|x(t)|_X^2 &\leq 12M^4\Big\|(I-S(b))^{-1}\Big\|^2\Big(m\sum_{k=1}^{m}(d_kE|x(t_k)|_X^2+e_k)+\|p_1\|_{L^1} \\ &\quad +c_HH(2H-1)b^{2H}(\alpha_1E|x(t)|_X^2+\beta_1E|y(t)|_X^2+c_1)\Big)+4M^2\|p_1\|_{L^1} \\ &\quad +4M^2c_HH(2H-1)b^{2H}(\alpha_1E|x(t)|_X^2+\beta_1E|y(t)|_X^2+c_1) \\ &\quad +4M^2m\sum_{k=1}^{m}(d_kE|x(t_k)|_X^2+e_k). \end{aligned}$$

Similarly,

$$
\begin{aligned}
E|y(t)|_X^2 \;\leq\; & 12M^4\Big\|(I-S(b))^{-1}\Big\|^2\Big(m\sum_{k=1}^{m}(\overline{d}_kE|y(t_k)|_X^2+\overline{e}_k)+\|p_2\|_{L^1}\\
&+c_HH(2H-1)b^{2H}(\alpha_2E|x(t)|_X^2+\beta_2E|y(t)|_X^2+c_2)\Big)+4M^2\|p_1\|_{L^1}\\
&+4M^2c_HH(2H-1)b^{2H}(\alpha_2E|x(t)|_X^2+\beta_2E|y(t)|_X^2+c_2)\\
&+4M^2m\sum_{k=1}^{m}(\overline{d}_kE|y(t_k)|_X^2+\overline{e}_k).
\end{aligned}
$$

Consider the function $\mu,\overline{\mu}$ defined on J by

$$\mu(t)=\sup\{E|x(s)|_X^2\ :0\leq s\leq t\},$$

and

$$\overline{\mu}(t)=\sup\{E|y(s)|_X^2\ :0\leq s\leq t\}.$$

This implies, for each $t\in J$,

$$
\begin{aligned}
\mu(t) \;\leq\; & 12M^4\Big\|(I-S(b))^{-1}\Big\|^2\Big(m\sum_{k=1}^{m}(d_k\mu(t)+e_k)+\|p_1\|_{L^1}\\
&+c_HH(2H-1)b^{2H}(\alpha_1\mu(t)+\beta_1\overline{\mu}(t)+c_1)\Big)+4M^2\|p_1\|_{L^1}\\
&+4M^2(c_HH(2H-1)b^{2H}(\alpha_1\mu(t)+\beta_1\overline{\mu}(t)+c_1)\\
&+4M^2m\sum_{k=1}^{m}(d_k\mu(t)+e_k)\\
=\;& 12M^4\Big\|(I-S(b))^{-1}\Big\|^2\Big(m\sum_{k=1}^{m}e_k+\|p_1\|_{L^1}+c_HH(2H-1)b^{2H}c_1\Big)\\
&+\mu(t)\Big(12M^4m\Big\|(I-S(b))^{-1}\Big\|^2(\sum_{k=1}^{m}d_k+c_HH(2H-1)b^{2H}\alpha_1)\\
&+4M^2(c_HH(2H-1)b^{2H}\alpha_1+4M^2m\sum_{k=1}^{m}d_k\Big)\\
&+\overline{\mu}(t)\Big(12M^4\Big\|(I-S(b))^{-1}\Big\|^2c_HH(2H-1)b^{2H}\beta_1\\
&+4M^2c_HH(2H-1)b^{2H}\beta_1\Big)\\
=\;& K_1+K_2\mu(t)+K_3\overline{\mu}(t).
\end{aligned}
$$

There exists a constant $K_j, \overline{K}_j$ for each j=1,2,3 defined as follows

$$K_1 = 12M^4\Big\|(I - S(b))^{-1}\Big\|^2\Big(m\sum_{k=1}^{m} e_k + \|p_1\|_{L^1} + c_H H(2H-1)b^{2H}c_1\Big),$$

and

$$K_2 = 4M^4\Big\|(I - S(b))^{-1}\Big\|^2(m\sum_{k=1}^{m} d_k + c_H H(2H-1)b^{2H}\alpha_1)$$

$$+4M^2(c_H H(2H-1)b^{2H}\alpha_1 + 4M^2 m\sum_{k=1}^{m} d_k,$$

and

$$K_3 = 12mM^4\Big\|(I-S(b))^{-1}\Big\|^2 c_H H(2H-1)b^{2H}\beta_1 + 4M^2(c_H H(2H-1)b^{2H}\beta_1.$$

Similarly,

$$\begin{aligned}
\overline{\mu}(t) &\leq 12M^4\Big\|(I - S(b))^{-1}\Big\|^2\Big(m\sum_{k=1}^{m}\overline{e}_k + \|p_2\|_{L^1} + c_H H(2H-1)b^{2H}c_2\Big) \\
&+ \overline{\mu}(t)\Big(12M^4\Big\|(I - S(b))^{-1}\Big\|^2(m\sum_{k=1}^{m}\overline{d}_k + c_H H(2H-1)b^{2H}\beta_2) \\
&+ 4M^2(c_H H(2H-1)b^{2H}\beta_2 + 4M^2\sum_{0<t_k<t}\overline{d}_k\Big) \\
&+ \mu(t)\Big(4M^4\Big\|(I - S(b))^{-1}\Big\|^2 c_H H(2H-1)b^{2H}\alpha_2 + 4M^4 c_H H(2H-1)b^{2H}\alpha_2\Big) \\
&= \overline{K}_1 + \overline{K}_2\mu(t) + \overline{K}_3\overline{\mu}(t),
\end{aligned}$$

where

$$\overline{K}_1 = 12mM^4\Big\|(I-S(b))^{-1}\Big\|^2\Big(\sum_{0<t_k<t}\overline{e}_k + \|p_2\|_{L^1} + c_H H(2H-1)b^{2H}c_2\Big)$$

and

$$\overline{K}_2 = 12M^4\Big\|(I-S(b))^{-1}\Big\|^2 c_H H(2H-1)b^{2H}\alpha_2 + 4M^4 c_H H(2H-1)b^{2H}\alpha_2$$

and

$$\overline{K}_3 = \Big(12M^4 \Big\|(I - S(b))^{-1}\Big\|^2 (m\sum_{k=1}^{m} \overline{d}_k + c_H H(2H-1)b^{2H}\beta_2)\Big)$$

$$+4M^2\Big(c_H H(2H-1)b^{2H}\beta_2 + 4M^2 m\sum_{k=1}^{m}\overline{d}_k\Big).$$

On the other hand,

$$\mu(t) + \overline{\mu}(t) \quad \leq \quad \widetilde{K}_1 + \widetilde{K}_2(\mu(t) + \overline{\mu}(t))$$

Thus, we have

$$\mu(t) + \overline{\mu}(t) \quad \leq \quad \frac{\widetilde{K}_1}{1 - \widetilde{K}_2} = M,$$

the maximum being taken componentwise, and $\widetilde{K}_2$ is a suitable value lower than 1

$$\widetilde{K}_1 = \overline{K}_1 + K_1 \quad \widetilde{K}_2 = \max\{K_2 + \overline{K}_2, K_3 + \overline{K}_3\} < 1$$

$$E|x(t)|_X^2 + E|y(t)|_X^2 \quad \leq \quad M.$$

Consequently

$$\|x\|_{PC}^2 \leq M \quad \text{and} \quad \|y\|_{PC}^2 \leq M.$$

Let

$$U = \{(x, y) \in PC \times PC \quad : \|x\|_{PC}^2 < M + 1 \quad \text{and} \quad \|y\|_{PC}^2 < M + 1\},$$

and consider the operator $N : \overline{U} \to \mathcal{P}_{cv,cp}(PC \times PC)$ From the choice of U, there is no $(x, y) \in \partial U$ such that $x \in \lambda N_1(x, y)$ and $y \in \lambda N_2(x, y)$ for some $\lambda \in (0, 1)$ As a consequence of the Leray and Schauder nonlinear alternative (Lemma 2.4.8), we deduce that N has a fixed point (x, y) in U, solution of Problem (5.0.1).

Part 2 Compactness of the solution set. Let

$$S_F = \{(x, y) \in PC \times PC : \quad (x, y) \quad \text{is a solution of Problem(5.0.1)}\}.$$

From Part 1, $S_F \neq \emptyset$ and there exists M such that for every $(x, y) \in S_F, \|x\|^2_{PC} \leq M$ and $\|y\|^2_{PC} \leq M$. Since N is completely continuous, then $N(S_F) = (N_1(S_{F^1}), N_2(S_{F^2}))$ is relatively compact in $PC \times PC$. Let $(x, y) \in S_F$ then $(x, y) \in N(x, y)$ and $S_F \subset \overline{N(S_F)}$. It remains to prove that S_F is a closed set in $PC \times PC$. Let $(x_n, y_n) \in S_F$ such that (x_n, y_n) converge to (x, y). For every $n \in N$, there exists $\upsilon^i_n(t) \in F^i(t, x_n, y_n)$ a.e. $t \in J$ for each $i \in \{1, 2\}$ such that

$$\begin{aligned}
x_n(t) &= S(t)(I - S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b - t_k))I_k(x_n(t_k)) \\
&\quad + \int_0^b S(b - s)\upsilon^1_n(s)ds + \sum_{l=1}^{\infty}\int_0^b S(b - s)\sigma^1_l(t, x_n(s), y_n(s))dB^H_l(s)\Big) \\
&\quad + \int_0^t S(t - s)\upsilon^1_n(s)ds + \sum_{l=1}^{\infty}\int_0^t S(t - s)\sigma^1_l(s, x_n(s), y_n(s))dB^H_l(s) \\
&\quad + \sum_{0<t_k<t} S(t - t_k)I_k(x_n(t_k)),
\end{aligned}$$

and

$$\begin{aligned}
y_n(t) &= S(t)(I - S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b - t_k))\overline{I}_k(y_n(t_k)) \\
&\quad + \int_0^b S(-s)\upsilon^2_n(s)ds + \sum_{l=1}^{\infty}\int_0^b S(b - s)\sigma^2_l(t, x_n(s), y_n(s))dB^H_l(s)\Big) \\
&\quad + \int_0^t S(t - s)\upsilon^2_n(s)ds + \sum_{l=1}^{\infty}\int_0^t S(t - s)\sigma^2_l(s, x_n(s), y_n(s))dB^H_l(s) \\
&\quad + \sum_{0<t_k<t} S(t - t_k)\overline{I}_k(y_n(t_k)).
\end{aligned}$$

(H_3) implies that for a.e. $t \in J$,$\upsilon^i_n \in p_i(t)B(0, 1)$, $i = 1, 2$ hence $(\upsilon^i_n)_{n\in N}$ is integrably bounded. Note that this still remains true holds for S_F is a bounded set. Since X is reflexive, by Theorem 2.3.7, there exists a subsequence, still denoted by $(\upsilon^i_n)_{n\in N}$, which converges weakly to some limit $\upsilon^i \in L^2(J, X)$. Moreover, the mapping $\Gamma : L^2(J, X) \longrightarrow PC$ defined by

$$\Gamma(g^i)(t) = \int_0^t S(t - s)g^i(s)ds$$

is a continuous linear operator. Then it remains continuous if these spaces are endowed with their weak topologies [39]. Therefore for a.e. $t \in J$, the sequence $(x_n(t), y_n(t))$ converges to $(x(t), y(t))$ and by the continuity of $(I_k, \overline{I}_k)$ it follows that

$$\begin{aligned} x(t) &= S(t)(I-S(b))^{-1}\Big(\sum_{0<t_k<t}(S(b-t_k))I_k(x(t_k)) \\ &\quad + \int_0^b S(t-s)v^1(s)ds + \sum_{l=1}^{\infty}\int_0^b S(t-s)\sigma_l^1(t,x(s),y(s))dB_l^H(s)\Big) \\ &\quad + \int_0^t S(t-s)v^1(s)ds + \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(s,x(s),y(s))dB_l^H(s) \\ &\quad + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k)), \end{aligned}$$

and

$$\begin{aligned} y(t) &= S(t)(I-S(b))^{-1}\Big(\sum_{0<t_k<t}(S(b-t_k))\overline{I}_k(y(t_k)) \\ &\quad + \int_0^b S(t-s)v^2(s)ds + \sum_{l=1}^{\infty}\int_0^b S(t-s)\sigma_l^2(t,x(s),y(s))dB_l^H(s)\Big) \\ &\quad + \int_0^t S(t-s)v^2(s)ds + \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^2(s,x(s),y(s))dB_l^H(s) \\ &\quad + \sum_{0<t_k<t} S(t-t_k)\overline{I}_k(y(t_k)). \end{aligned}$$

Now we need to prove that $v^i(t) \in F^i(t, x(t), y(t))$, for a.e. $t \in J$. Lemma 2.3.6 yields the existence of constants $\alpha_i^n \geq 0$, $j = 1, 2 \ldots, k(n)$ and $i = 1, 2$ such that $\sum_{j=1}^{k(n)} \alpha_i^n = 1$ and the sequence of convex combinations $g_n^i(.) = \sum_{j=1}^{k(n)} \alpha_j^n v_j^i(.)$ converges strongly to some limit $v^i \in L^2(J, X)$. Since F^i takes convex values, using Lemma 2.3.5, we obtain that

$$\begin{aligned} v^i(t) &\in \bigcap_{n\geq 1} \overline{\{g_k^i(t) : k \geq n\}}, \quad a.e \quad t \in J \\ &\subseteq \bigcap_{n\geq 1} \overline{co}\{v_k^i(t), \quad k \geq n\} \\ &\subseteq \bigcap_{n\geq 1} \overline{co}\{\bigcup_{k\geq n} F^i(t, x_k(t), y_k(t))\} \end{aligned}$$

$$\subseteq \overline{co}\{\limsup_{k\to\infty} F^i(t, x_k(t), y_k(t))\} \tag{5.1.2}$$

Since F^i is u.s.c. and has compact values, then by Lemma 2.3.4, we have

$$\limsup_{n\to\infty} F^i(t, x_n(t), y_n(t)) \subseteq F^i(t, x(t), y(t)) \quad \text{for a.e} \quad t \in J.$$

This and (5.1.2) imply that $v^i(t) \in \overline{co}(F^i(t, x(t), y(t)))$. Since, for each $i = 1, 2$, $F^i(.,.)$ has closed, convex values, we deduce that $v^i(t) \in F^i(t, x(t), y(t))$ for a.e. $t \in J$, for each $i = 1, 2$ as claimed. Hence $(x, y) \in S_{F^i}$ which proves that S_{F^i}, for each $i = 1, 2$, is closed, hence compact in $PC \times PC$.

□

5.1.2 The nonconvex case

Now we present a second result for the problem (5.0.1) with a non convex valued right-hand side. Our considerations are based on a multivalued version of Perov's fixed point theorem given by by Petre and Petruşel [126](see also Ouahab [121]).

Let us introduce the following hypotheses:

(H4) $F^i : J \times X \times X \longrightarrow \mathcal{P}_{cp}(X); (t, y) \longrightarrow F^i(t, x, y)$ is measurable for each $(x, y) \in X \times X$.

(H5) There exist functions $a_i,\ b_i \in L^1([0, T], \mathbb{R}^+)$ such that

$$\begin{cases} H_{d_1}^2(F^1(t, x, y), F^1(t, \overline{x}, \overline{y})) \leq a_1(t)|x - \overline{x}|_X^2 + b_1(t)|y - \overline{y}|_X^2 \\ H_{d_2}^2(F^2(t, x, y), F^2(t, \overline{x}, \overline{y})) \leq a_2(t)|x - \overline{x}|_X^2 + b_2(t)|y - \overline{y}|_X^2 \end{cases}$$

with

$$d_i(0, F^i(t, 0, 0) \leq a_i(t)$$

for all x, y, $\overline{x}$, $\overline{y} \in X$ for each $i = 1, 2$

(H6) There exist functions $\alpha_i,\ \beta_i \in L^1([0,T], \mathbb{R}^+)$ for each $i = 1, 2$ such that

$$\begin{cases} \|\sigma^1(t,x,y) - \sigma^1(t,\overline{x},\overline{y})\|^2 \leq \alpha_1(t)\|x-\overline{x}\|^2 + \beta_1(t)\|y-\overline{y}\|^2 \\ \|\sigma^2(t,x,y) - \sigma^2(t,\overline{x},\overline{y})\|^2 \leq \alpha_2(t)\|x-\overline{x}\|^2 + \beta_2(t)\|y-\overline{y}\|^2 \end{cases}$$

for all x, y, $\overline{x}$, $\overline{y} \in X$ and $t \in J$.

(H7) there exist constants $d_k \geq 0$ and $\overline{d}_k \geq 0,\ k = 1, \ldots, m$ such that

$$|I_k(x) - I_k(\overline{x})|_X^2 \leq d_k|x-\overline{x}|_X^2,$$

and

$$|\overline{I}_k(y) - \overline{I}_k(\overline{y})|^2 \leq \overline{d}_k|y-\overline{y}|_X^2,$$

for all x, y, $\overline{x}$, $\overline{y} \in X$.

Theorem 5.1.4. *Assume that hypotheses (H4)-(H7) are fulfilled. If the matrix*

$$M_{\alpha,\beta} = \begin{pmatrix} A_1 & A_2 \\ B_1 & B_1 \end{pmatrix}$$

converges to zero, then problem (5.0.1) *has at least one mild solution.*

Proof. In order to transform the problem (5.0.1) into a fixed point problem, let the multi-valued operator $N : PC \times PC \to \mathcal{P}(PC \times PC)$ be as defined in lemma 5.1.2. We shall show that N satisfies the assumptions of theorem **??**. Note that $(H4)$ implies that F^i for each $i = 1, 2$ has at most linear growth, i.e.

$$E|F^i(t,x,y)|_X^2 \leq a_i(t)E|x|_X^2 + b_i(t)E|y|_X^2,$$

for a.e. $t \in J$ and all $x, y \in X$

(a) $N(x,y) \in \mathcal{P}_{cl}(PC \times PC)$ for each $(x,y) \in PC \times PC$. The proof is similar to that in Theorem 5.1.3, Part 1, Step 2 and is omitted.

(b) There exists $M_{\alpha,\beta} \in \mathcal{M}_{2\times 2}(\mathbb{R}_+)$ convergent matric to zero, such that

$$H_d(N(x,y),N(\overline{x},\overline{y})) \leq M_{\alpha,\beta}\begin{pmatrix}\|x-\overline{x}\|_{PC}\\ \|y-\overline{y}\|_{PC}\end{pmatrix}, \quad \text{for all} \quad x,y,\overline{x},\overline{y}\in PC.$$

Let $x,y,\overline{x},\overline{y} \in PC$ and $h_i \in N_i(x,y)$, $i = 1,2$. Then there exists $f^i(\cdot) \in S_{F^i,x,y}$ such that for each $t \in J$, we have

$$\begin{aligned} h_i(t) &= S(t)(I-S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b-t_k))\widetilde{I}_k(\overline{z^i}(t_k)) \\ &\quad + \int_0^b S(b-s)f^i(s)ds + \sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^i(t,x(s),y(s))dB_l^H(s)\Big) \\ &\quad + \int_0^t S(t-s)f^i(s)ds + \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^i(t,x(s),y(s))dB_l^H(s) \\ &\quad + \sum_{0<t_k<t} S(t-t_k)\widetilde{I}_k(\overline{z^i}(t_k)), \end{aligned}$$

where

$$\widetilde{I}_k(\overline{z^1}(t_k)) = I_k(x(t_k)), \quad \text{and } \widetilde{I}_k(\overline{z^2}(t_k)) = \overline{I}_k(y(t_k)),\ k = 1,\ldots,m.$$

From $(H5)$, tells us that

$$\begin{cases} EH_{d_1}^2(F(t,x,y),F(t,\overline{x},\overline{y})) \leq a_1(t)E|x-\overline{x}|_X^2 + b_1(t)E|y-\overline{y}|_X^2, & \text{a.e.}t\in J,\\ EH_{d_2}^2(F(t,x,y),F(t,\overline{x},\overline{y})) \leq a_2(t)E|x-\overline{x}|_X^2 + b_2(t)E|y-\overline{y}|_X^2, & \text{a.e.}t\in J. \end{cases}$$

Hence there is $(w,\overline{w}) \in F^1(t,\overline{x}(t),\overline{y}(t)) \times F^2(t,\overline{x}(t),\overline{y}(t))$ such that

$$E|f^1(t)-w|_X^2 \leq a_1(t)E|x-\overline{x}|_X^2 + b_1(t)E|y-\overline{y}|_X^2, \quad t\in J,$$

and

$$E|f^2(t)-\overline{w}|^2 \leq a_2(t)E|x-\overline{x}|_X^2 + b_2(t)E|y-\overline{y}|_X^2, \quad t\in J.$$

Consider the multi-valued maps $U_i : J \to \mathcal{P}(X)$, $i = 1,2$ defined by

$$U_1(t) = \{w \in F^1(t,\overline{x}(t),\overline{y}(t)) : E|f^1(t)-w|^2 \leq a_1(t)E|x-\overline{x}|_X^2$$
$$+b_1(t)E|y-\overline{y}|_X^2\, a.e\, t\in J\},$$

and

$$U_2(t) = \{\overline{w} \in F^2(t,\overline{x}(t),\overline{y}(t)) : E|f^2(t)-\overline{w}|^2 \leq a_2(t)E|x-\overline{x}|_X^2$$

$$+b_2(t)E|y-\overline{y}|_X^2,\ a.e\ t\in[0,b]\},$$

that is $U_1=\overline{B}(f^1(t),a_1(t)E|x-\overline{x}|_X^2+b_1(t)E|y-\overline{y}|_X^2)$ and $U_2=\overline{B}(f^2(t),a_2(t)E|x-\overline{x}|_X^2+b_2(t)E|y-\overline{y}|_X^2)$. Since $f^i,a_i,b_i,x,y,\overline{x},\overline{y}$ are measurable for each $i=1,2$, Theorem $III.4.1$ in [47], tells us that the closed ball U_i is measurable. In addition $(H4)$ and $(H5)$ imply that for each $(x,y)\in PC\times PC$and $F^i(t,x(t),y(t))$ is measurable .Finally the set $V_i(.)=U_i(.)\cap F^i(.,\overline{x}(.),\overline{y}(.))$ is nonempty. Therefore the intersection multi-valued operator V_i is measurable with nonempty, closed values (see [69]), there exists a function $\overline{f}^i(t)$ which is a measurable selection for $V_i(.)$. Thus

$$\overline{f}^i(t)\in F^i(t,\overline{x}(t),\overline{y}(t))\quad\text{for a.e.}t\in J.$$

Hence

$$E|f^1(t)-\overline{f}^1(t)|_X^2\le a_1(t)E|x-\overline{x}|_X^2+b_1(t)E|y-\overline{y}|_X^2,\quad\text{for a.e.}t\in J.$$

and

$$E|f^2(t)-\overline{f}^2(t)|_X^2\le a_2(t)E|x-\overline{x}|_X^2+b_2(t)E|y-\overline{y}|_X^2,\quad\text{for a.e.}t\in J.$$

So

$$\begin{aligned}\overline{h}_i(t)\ &=\ S(t)(I-S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b-t_k))\widetilde{I}_k(\overline{z^i}(t_k))+\int_0^b S(b-s)\overline{f}^i(s)ds\\&\quad+\sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^i(t,\overline{x}(s),\overline{y}(s))dB_l^H(s)\Big)+\int_0^t S(t-s)\overline{f}^i(s)ds\\&\quad+\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^i(t,\overline{x}(s),\overline{y}(s))dB_l^H(s)+\sum_{0<t_k<t}S(t-t_k)\widetilde{I}_k(\overline{z^i}(t_k)).\end{aligned}$$

From (1.4.19) and $(H5)-(H8)$,

$$\begin{aligned}
E|h_1(t)-\overline{h}_1(t)|_X^2 \ \leq\ & 12M^4m\|(I-S(b))^{-1}\|^2\sum_{k=1}^m d_kE|x(t_k)-\overline{x}(t_k)|_X^2\\
&+12M^4\|(I-S(b))^{-1}\|^2\Big(\int_0^b a_1(s)E|x(s)-\overline{x}(s)|_X^2\\
&+b_1(s)E|y(s)-\overline{y}(s)|_X^2ds\Big)\\
&+12M^4\|(I-S(b))^{-1}\|^2c_HH(2H-1)b^{2H-1}\\
&\times\Big(\int_0^b \alpha_1(s)E|x(s)-\overline{x}(s)|_X^2\\
&+\beta_1(s)E|y(s)-\overline{y}(s)|_X^2ds\Big)\\
&+4M^2\Big(\int_0^t a_1(s)E|x(s)-\overline{x}(s)|_X^2+b_1(s)E|y(s)-\overline{y}(s)|_X^2ds\Big)\\
&+4M^2c_HH(2H-1)b^{2H-1}\int_0^t \alpha_1(s)E|x(s)-\overline{x}(s)|_X^2\\
&+\beta_1(s)E|y(s)-\overline{y}(s)|_X^2ds+4mM^2\sum_{k=1}^m d_kE|x(t_k)-\overline{x}(t_k)|_X^2.
\end{aligned}$$

Taking the supremum, we have

$$\begin{aligned}
\sup_{t\in J}E|h_1(t)-\overline{h}_1(t)|_X^2 \ \leq\ & 4M^2\Big(3m\|(I-S(b))^{-1}\|^2\sum_{k=1}^m d_k+3M^2\|(I-S(b))^{-1}\|^2\|a_1\|_{L^1}\\
&+\ 3M^2\|(I-S(b))^{-1}\|^2c_HH(2H-1)b^{2H-1}\|\alpha_1\|_{L^1}\\
&+\ \|a_1\|_{L^1}+c_HH(2H-1)b^{2H-1}\|\alpha_1\|_{L^1}\\
&+\ m\sum_{k=1}^m d_k\Big)\sup_{t\in J}E|x(t)-\overline{x}(t)|_X^2\\
&+4M^2\Big(\|b_1\|_{L^1}+3M^2\|(I-S(b))^{-1}\|^2c_HH(2H-1)b^{2H-1}\|\beta_1\|_{L^1}\\
&+\|b_1\|_{L^1}+c_HH(2H-1)b^{2H-1}\|\beta_1\|_{L^1}\Big)\sup_{t\in J}E|y(t)-\overline{y}(t)|_X^2.
\end{aligned}$$

Hence

$$\|h_1-\overline{h}_1\|_{PC} \ \leq\ A_1\|x-\overline{x}\|_{PC}+A_2\|y-\overline{y}\|_{PC}.$$

where

$$A_1 = 2M\sqrt{\begin{array}{l} 3m\|(I-S(b))^{-1}\|^2\sum_{k=1}^{m} d_k + 3M^2\|(I-S(b))^{-1}\|^2\|a_1\|_{L^1} \\ +3M^2\|(I-S(b))^{-1}\|^2 c_H H(2H-1)b^{2H-1}\|\alpha_1\|_{L^1} \\ +\|a_1\|_{L^1} + c_H H(2H-1)b^{2H-1}\|\alpha_1\|_{L^1} + m\sum_{k=1}^{m} d_k \end{array}}$$

and

$$A_2 = 2M\sqrt{\begin{array}{l} \|b_1\|_{L^1} + 3M^2\|(I-S(b))^{-1}\|^2 c_H H(2H-1)b^{2H-1}\|\beta_1\|_{L^1} \\ +\|b_1\|_{L^1} + c_H H(2H-1)b^{2H-1}\|\beta_1\|_{L^1} \end{array}}$$

and similarly

$$\begin{aligned} \sup_{t\in J} E|h_2(t)-\overline{h}_2(t)|_X^2 \ \leq\ & 4M^2\Big(3m\|(I-S(b))^{-1}\|^2\sum_{k=1}^{m}\bar{d}_k + 3M^2\|(I-S(b))^{-1}\|^2\|a_2\|_{L^1} \\ & + \ 3M^2\|(I-S(b))^{-1}\|^2 c_H H(2H-1)b^{2H-1}\|\alpha_2\|_{L^1} \\ & + \ \|a_2\|_{L^1} + c_H H(2H-1)b^{2H-1}\|\alpha_2\|_{L^1} \\ & + \ m\sum_{k=1}^{m}\bar{d}_k\Big)\sup_{t\in J} E|y(t)-\overline{y}(t)|_X^2 \\ & +4M^2\Big(\|b_2\|_{L^1} + 3M^2\|(I-S(b))^{-1}\|^2 c_H H(2H-1)b^{2H-1}\|\beta_2\|_{L^1} \\ & +\|b_2\|_{L^1} + c_H H(2H-1)b^{2H-1}\|\beta_1\|_{L^1}\Big)\sup_{t\in J} E|x(t)-\overline{x}(t)|_X^2. \end{aligned}$$

Therefore

$$\|h_2-\overline{h}_2\|_{PC} \ \leq\ B_1\|x-\overline{x}\|_{PC} + B_2\|y-\overline{y}\|_{PC}.$$

where

$$B_1 = 2M\sqrt{\begin{array}{l} 3m\|(I-S(b))^{-1}\|^2\sum_{k=1}^{m} \bar{d}_k + 3M^2\|(I-S(b))^{-1}\|^2\|a_2\|_{L^1} \\ +3M^2\|(I-S(b))^{-1}\|^2 c_H H(2H-1)b^{2H-1}\|\alpha_2\|_{L^1} \\ +\|a_2\|_{L^1} + c_H H(2H-1)b^{2H-1}\|\alpha_2\|_{L^1} + m\sum_{k=1}^{m} \bar{d}_k \end{array}}$$

and

$$B_2 = 2M\sqrt{\begin{matrix} \|b_2\|_{L^1} + 3M^2\|(I-S(b))^{-1}\|^2 c_H H(2H-1)b^{2H-1}\|\beta_2\|_{L^1} \\ +\|b_2\|_{L^1} + c_H H(2H-1)b^{2H-1}\|\beta_1\|_{L^1} \end{matrix}}.$$

By an analogous relation, obtained by interchanging the roles of x, y and $\overline{x}, \overline{y}$, we finally arrive at

$$H_d(N(x,y), N(\overline{x},\overline{y})) \leq M_{\alpha,\beta} \begin{pmatrix} \|x-\overline{x}\|_{PC} \\ \|y-\overline{y}\|_{PC} \end{pmatrix},$$

where

$$M_{\alpha,\beta} = \begin{pmatrix} A_1 & A_2 \\ B_1 & B_1 \end{pmatrix}.$$

Since $M_{\alpha,\beta}$ converges to zero. Thanks to theorem 2.4.9, we can ensure that N has a fixed point (x, y), which is a mild solution to (5.0.1).

□

5.1.3 An example

In this section we us the abstract result proved in the above section to study the existence of mild solution for impulsive Stokes differential inclusions.

Let $D \subset \mathbb{R}^3$ be a bounded open domain with the smooth boundary ∂D and and let $n(x)$ be the outward normal to D at the point $x \in \partial D$. Let

$$X = \{u \in (C_c^\infty(D))^3 : \nabla u = 0 \text{ in } \Omega \text{ and } n.u = 0 \text{ on } \partial D\}$$

and $E = \overline{Y}^{(L^2(D))^3}$ be the closure of Y in $(L^2(D))^3$. Clear that, endowed with the standard inner product of the space $(L^2(D))^3$, defined by

$$\langle u, v\rangle = \sum_{i=1}^{3} \langle u_i, v_i\rangle_{L^2(D)},$$

E is a Hilbert space. Let $P : (L^2(D))^3 \to X$ denote the orthogonal projection of $(L^2(D))^3$ in X.

Consider the following system of impulsive stochastic Stokes type partial differential inclusions:

$$\begin{cases} u_t - P(\Delta u) & \in & F(t,u(t,x),v(t,x)) + \sigma_1(t)\frac{dB_Q^H}{dt}, \quad \text{a.e. } \in [0,b], \;\; x \in D, \\ v_t - P(\Delta v) & \in & G(t,u(t,x),v(t,x)) + \sigma_2(t)\frac{dB_Q^H}{dt}, \quad \text{a.e.} t \in [0,b], \;\; x \in D, \\ u(t_k^+,x) - u(t_k^-,x) & = & I_k(u(t_k,x)), \\ v(t_k^+,x) - v(t_k,x) & = & \overline{I}_k(v(t_k,x)), \qquad k = 1,\dots,m \\ \nabla u = \nabla v = 0, & & (t,x) \in [0,b]\times \partial D \\ u = v = 0, & & (t,x) \in [0,b]\times \partial D \\ u(0,x) = u(b,x), & & v(0,x) = v(b,x) \qquad x \in D, \end{cases} \tag{5.1.3}$$

where $P(\Delta)$ is the Stokes operator. Let $A : D(A) \subset X \to X$ defined by

$$\begin{cases} D(A) & = & (H^2(D) \cap H_0^1(D))^3 \cap X \\ Au & = & -P(\Delta u), \;\; u \in D(A). \end{cases}$$

Lemma 5.1.5. *(Fujita-Kato)(Theorem 7.3.4, [154]) The operator A, defined as above, is the generator of a compact and analytic C_0-semigroup of contractions in X.*

Let us assume that

($\mathcal{K}_1$) Let $f_i, g_i : [0,b] \times D \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i = 1,2$ are functions such that

$$f_1(t,x,u,v) \le f_2(t,x,u,v), \;\; g_1(t,x,u,v) \le g_2(t,x,u,v)$$

for all $(t,x,u,v) \in [0,b] \times D \times \mathbb{R} \times \mathbb{R}$.

($\mathcal{K}_2$) there exist $\phi_i, \psi_i \in L^1([0,b],\mathbb{R}_+) \cap L^\infty([0,b],\mathbb{R}_+)$ such that

$$|f_i(t,x,u,v)| \le \phi_i(t) \text{ and } |g_i(t,x,u,v)| \le \psi_i(t), \;\; i = 1,2$$

for each $(t,x,u,v) \in [0,b] \times D \times \mathbb{R} \times \mathbb{R}$.

($\mathcal{K}_3$) f_1, g_1 are l.s.c and f_2, g_2 are u.s.c.

($\mathcal{K}_4$) The function $\sigma_, \sigma_1 : [0,b] \longrightarrow L_Q^2(\mathcal{K},\mathcal{H})$ is bounded, that is, there exists a positive constant L such that

$$\int_0^b \|\sigma_i(s)\|_{L_Q^2}^2 ds < L_i, \quad i = 1,2.$$

Lemma 5.1.6. *[48] Let $f_i, g_i : [0,b] \times D \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ i = 1,2$ are functions satisfying $(\mathcal{K}_2) - (\mathcal{K}_3)$. Let $F : G : [0,b] \times L^2(D) \times L^2(D) \to \mathcal{P}(L^2(D))$ be multivalued map defined by*

$$F(t,u,v) = \{f \in L^2(D) : f(x) \in [f_1(t,x,u,v), f_2(t,x,u,v]\}$$

and

$$G(t,u,v) = \{g \in L^2(D) : g(x) \in [g_1(t,x,u,v), g_2(t,x,u,v)])\}.$$

Then F and G are nonempty, u.s.c. with weakly compact and convex values. Moreover $F(.,.,.), G(.,.,.) \in \mathcal{P}_{cl,b,cv}(L^2(D))$.

Let

$$x(t)(\xi) = u(t,\xi) \quad t \in J, \quad \xi \in D,$$

$$I_k(x(t_k)) = K_k \frac{u(t_k^-,\xi)}{1 + |u(t_k^-,.)|_X}, \quad \xi \in D, \quad k = 1, \cdots, m,$$

$$I_k(y(t_k)) = \bar{K}_k \frac{v(t_k^-,\xi)}{1 + |v(t_k^-,)|_X}, \quad \xi \in \Omega, \quad k = 1, \cdots, m,$$

$$x(0)(\xi) = u(0,\xi) = u(b,\xi) = x(b)(\xi), \;\; y(0)(\xi) = v(0,\xi) = v(b,\xi) = y(b)(\xi) \;\; \xi \in D,$$

where $K_k, \bar{K}_k \in \mathbb{R},\ k = 1, \ldots, m$. Assume that $(\mathcal{K}_1) - (\mathcal{K}_4)$ are satisfied. Thus the problem (5.1.3) can be written in the abstract form

$$\left\{\begin{array}{rcl} x'(t) - A_1 x(t) & \in & F_1(t, x(t), y(t)) + \sigma_1(t)\frac{dB_Q^H}{dt}, \quad t \in [0,b] \\ y'(t) - A_2 y(t) & \in & F_2(t, x(t), y(t)) + \sigma_2(t)\frac{dB_Q^H}{dt}, \; t \in [0,b], \\ x(t_k^+) - x(t_k^-) & \in & I_k(x(t_k)), \\ y(t_k^+) - y(t_k^-) & \in & \overline{I}_k(y(t_k)), \quad k = 1, \ldots, m \\ x(0) = x_0, & & y(0) = y_0. \end{array}\right. \tag{5.1.4}$$

where $A_1 = A_2 = A$. Since for each $k = 1, \ldots, m$ we have

$$|I_k(x)| = \left| K_k \frac{x}{1 + |x|_X} \right|_X \leq |K_k|, \quad |\bar{I}_k(x)| = \left| \bar{K}_k \frac{x}{1 + |x|_X} \right|_X \leq |\bar{K}_k|, \quad \text{for all } x \in X.$$

Then from Theorem 5.1.3 the problem (5.1.3) has at least on solution.

Chapter 6

Coupled stochastic difference equations with delay

Hereditary systems (or systems with delays, or aftereffect) are those whose future development depends not only on their present state but also on their previous history. Such systems are widely used to model processes in physics, mechanics, automatic regulation, economy, biology, ecology, etc.
It is worth mentioning that difference equations usually appear in the investigation of systems with discrete time or in the numerical approximation of solutions for systems with continuous time [94].
Some good literature for difference delay and or neutral equations are the books [**?**, 4, 11, 28] and the references therein. In particular, some stability results for systems of difference equations with delay were proved by Krasovskii [95, 96] and this topic is being widely studied currently (see, e.g. [2, 98]).
This chapter is concerned with the existence of solutions, as well as their asymptotic behavior, for a perturbed stochastic difference equation of the

following type:

$$\begin{cases} x(i+1) = F^1(i, x(i-h), \dots, x(i), y(i-h), \dots, y(i)) \\ \qquad +G^1(i, x(i-h), \dots, x(i-h), y(i-h), \dots, y(i))\xi_i, \ i \in \mathbb{N}(0, b+1), \\ y(i+1) = F^2(i, x(i-h), \dots, x(i), y(i-h), \dots, y(i)) \\ \qquad +G^2(i, x(i-h), \dots, x(i-h), y(i-h), \dots, y(i))\xi_i, \ i \in \mathbb{N}(0, b+1), \\ x(i) = \varphi_1(i), \, i \in Z_0, \\ y(i) = \varphi_2(i), \, i \in Z_0, \end{cases} \tag{6.0.1}$$

where $i \in Z_0 \cup \mathbb{N}(0, b+1)$, $Z_0 = \{-h, \dots, 0\}$, $\mathbb{N}(0, b+1) = \{0, \dots, b+1\}$, $\mathbb{N}(-h, b+1) = Z_0 \cup \mathbb{N}(0, b+1)$, h is a given nonnegative integer number, $F^l, G^l : \mathbb{N}(0, b+1) \times \mathbb{R}^{h+1} \times \mathbb{R}^{h+1} \to \mathbb{R}$ are continuous functions for each $l = 1, 2$ and $\varphi_i : Z_0 \to \mathbb{R}$, for $i \in \mathbb{N}(0, b+1)$ and $b \in \mathbb{N}$. Let $\{\Omega, \mathcal{F}, \mathbb{P}\}$ be a basic probability space, $\mathcal{F}_i \in \mathcal{F}$, be a family of σ-algebras, $\mathbb{E}$ denote the mathematical expectation, $\xi_0, \xi_1, \dots$ be a sequence of real valued and mutually independent random variables, with ξ_i being $\mathcal{F}_{i+1}$-adapted and independent of $\mathcal{F}_i$, $\mathbb{E}(\xi_i) = 0$, $\mathbb{E}(\xi_i)^2 = 1$, $i \in \mathbb{N}(0, b+1)$. An $\mathbb{R}$-valued random variable is an $\mathcal{F}$-measurable function $x_i : \Omega \to \mathbb{R}$ and the collection of random variables

$$S = \{x(i, \omega) : \Omega \to \mathbb{R} | \ i \in \mathbb{N}(-h, b+1)\}$$

is called a stochastic process. Generally, we just write x_i instead of $x_i(\omega)$.

6.0.4 Discrete calculus

This section is essentially introductory in nature .its main aim is to introduce certain well known basic concepts in difference calculus and to present some important results.In this we develop the theory of difference inequalities and prove a variety of theorem Arzela-Ascoli that play a prominent role in the development of this chapiter.

Definition 6.0.2. *Let* $y : \mathbb{N}(0, b+1) \to \mathbb{R}$*. Then* Δ *is the difference operator, defined by*

$$\Delta y(k) = y(k+1) - y(k)$$

$\mathbb{N}(0, b+1) = \{0, 1, \ldots, b+1\}$

The following result is known as Gronwall-Bihari's Theorem. We have the following well-known properties that have been proved in [101].

Theorem 6.0.7. *(Discrete Bihari inequality [122](pp 142) Theorem 2.5.7) Suppose that $u(n)$ and $b(n)$ be nonnegative functions defined for $n \in \mathbb{N}_0$ where $\mathbb{N}_0 = \{0, 1, ..\}$ and c be a nonnegative constant .Let $g(u)$ be a nondecreasing continuous functions defined on $\mathbb{R}^+$ with $g(u) > 0$ for $u > 0$. If*

$$u(n) \leq c + \sum_{s=0}^{n-1} b(s) g(u(s)),$$

for $n \in \mathbb{N}_0$,then for $0 \leq n \leq n_1$, $n_1 \in \mathbb{N}_0$

$$u(n) \leq G^{-1}(G(c) + \sum_{s=0}^{n-1} b(s)),$$

where

$$G(r) = \int_{r_0}^{r} \frac{ds}{g(s)}, \qquad r > 0,$$

where $r_0 > 0$ is arbitrary , G^{-1} is the inverse of G and $n_1 \in \mathbb{N}_0$ be chosen so that

$$G(c) + \sum_{s=0}^{n-1} b(s) \in Dom(G^{-1})$$

for all $n \in \mathbb{N}_0$ such that $0 \leq n \leq n_1$.

Lemma 6.0.8. *[101] Let $p, q, f, u : \mathbb{N}(a) \to \mathbb{R}^+$ are nonnegative functions such that*

$$u(k) \leq p(k) + q(k) \sum_{l=a}^{l=k-1} f(l) u(l), \text{ for all } k \in N(a).$$

Then

$$u(k) \leq p(k) + q(k) \sum_{l=a}^{l=k-1} f(l) p(l) \prod_{\tau=l+1}^{\tau=k-1} (1 + q(\tau)) f(\tau)).$$

We now state a theorem needed later on in our analysis.

Theorem 6.0.9. *[2](Arzela-Ascoli) Let* Γ *be a closed subset of* $C(\mathbb{N}(0,b+1),\mathbb{R})$*. If* Γ *is uniformly bounded and the set*

$$\Gamma(k)=\{y(k): y\in\Gamma\}$$

is relatively compact for each $k\in\mathbb{N}(0,b+1)$*, then* Γ *is compact.*

6.1 Existence and uniqueness of solutions

Recall that $\mathbb{N}(0,b+1)=\{0,1,\ldots,b+1\}$. In order to define a solutions for Problem (6.0.1), let us first consider the space

$$C=\{x:\ \mathbb{N}(-h,b+1)\times\Omega\longrightarrow\mathbb{R}:\ x(\cdot,\omega)\in C(\mathbb{N}(-h,b+1),\mathbb{R}),\ \text{for almost all }\omega\in\Omega$$

$$\text{and}\sup_{i\in\mathbb{N}(-h,b+1)}E|x(i,\cdot)|^2<\infty\}.$$

Endowed with the norm

$$\|x\|^2=\max_{i\in\mathbb{N}(0,b+1)}E|x(i,.)|^2,$$

C is a Banach space with norm $\|\cdot\|$. Now, we define the concept of solution to our problem.

Definition 6.1.1. *An* $\mathbb{R}-$ *valued stochastic process* $u=(x,y)\in C\times C$ *is said to be a solution of* (6.0.1) *with respect to the probability space* $(\Omega,\mathcal{F},\mathbb{P})$ *if:*

1) *$u(i)$ is $\mathcal{F}_i$-adapted for all $i\in\mathbb{N}(-h,b+1)$, where $\mathcal{F}_i=\mathcal{F}_0$ for $i=-h,-(h-1),\ldots,-1$.*

2) *$u(i)$ satisfies that*

$$\begin{cases} x(i+1) &= F^1(i,x(i-h),\ldots,x(i),y(i-h),\ldots,y(i))\\ &+\ G^1(k,x(i-h),\ldots,x(i-h),y(i-h),\ldots,y(i))\xi_i,\quad i\in\mathbb{N}(0,b+1)\\ x(i) &= \varphi_i,\ i\in Z_0,\\ \\ y(i+1) &= F^2(i,x(i-h),\ldots,x(i),y(i-h),\ldots,y(i))\\ &+\ G^2(k,x(i-h),\ldots,x(i-h),y(i-h),\ldots,y(i))\xi_i,\quad i\in\mathbb{N}(0,b+1),\\ y(i) &= \varphi_i^{'},\ i\in Z_0.\end{cases}$$

We are now in a position to state and prove our existence result for the problem (6.0.1). First, we will list the following hypotheses which will be imposed in our main theorem.

(H_1) There exist nonnegative numbers $a_j(i), \overline{a}_j(i)$ and $b_j(i), \overline{b}_j(i)$ for each $i \in \mathbb{N}(0, b+1)$

$$\Big|F^1\Big(i, x_0, \ldots, x_h, y_0, \ldots, y_h\Big) - F^1\Big(i, \overline{x}_0, \ldots, \overline{x}_h, \overline{y}_0, \ldots, \overline{y}_h\Big)\Big| \leq \sum_{j=0}^{h} a_j(i)|x_j - \overline{x}_j| + \sum_{j=0}^{h} b_j(i)|y_j - \overline{y}_j|$$

and

$$\Big|F^2\Big(i, x_0, \ldots, x_h, y_0, \ldots, y_h\Big) - F^2\Big(i, \overline{x}_0, \ldots, \overline{x}_h, \overline{y}_0, \ldots, \overline{y}_h\Big)\Big| \leq \sum_{j=0}^{h} \overline{a}_j(i)|x_j - \overline{x}_j| + \sum_{j=0}^{h} \overline{b}_j(i)|y_j - \overline{y}_j|$$

for each $x_j, y_j, \overline{x}_j, \overline{y_j} \in \mathbb{R},\ j = 0, \ldots, h.$ where

$$a(i) = \sum_{j=0}^{h} a_j(i), \overline{a}(i) = \sum_{j=0}^{h} a_j(i), \quad i \in \mathbb{N}(0, b+1)$$

and

$$b(i) = \sum_{j=0}^{h} b_j(i), \overline{b}(i) = \sum_{j=0}^{h} b_j(i), \quad i \in \mathbb{N}(0, b+1).$$

(H_2) There exist positive constants $\alpha_j(i), \overline{\alpha}_j(i)$ and $\beta_j(i), \overline{\beta}_j(i)$ for each $i \in \mathbb{N}(0, b+1)$

$$\Big|G^1\Big(i, x_0, \ldots, x_h, y_0, \ldots, y_h\Big) - G^1\Big(i, \overline{x}_0, \ldots, \overline{x}_h, \overline{y}_0, \ldots, \overline{y}_h)\Big| \leq \sum_{j=0}^{h} \alpha_j(i)|x_j - \overline{x}_j| + \sum_{j=0}^{h} \beta_j(i)|y_j - \overline{y}_j|$$

and

$$\Big|G^2\Big(i, x_0, \ldots, x_h, y_0, \ldots, y_h\Big) - G^2\Big(i, \overline{x}_0, \ldots, \overline{x}_h, \overline{y}_0, \ldots, \overline{y}_h\Big)\Big| \leq \sum_{j=0}^{h} \overline{\alpha}_j(i)|x_j - \overline{x}_j| + \sum_{j=0}^{h} \overline{\beta}_j(i)|y_j - \overline{y}_j|$$

for each $x_j, y_j, \overline{x}_j, \overline{y_j} \in \mathbb{R},\ j = 0, \dots, h.$ where

$$\alpha(i) = \sum_{j=0}^{h} \alpha_j(i), \overline{\alpha}(i) = \sum_{j=0}^{h} \overline{\alpha}_j(i), \quad i \in \mathbb{N}(0, b+1)$$

and

$$\beta(i) = \sum_{j=0}^{h} \beta_j(i), \overline{\beta}(i) = \sum_{j=0}^{h} \overline{\beta}_j(i), \quad i \in \mathbb{N}(0, b+1).$$

For our main consideration of Problem (6.0.1), a Perov fixed point theorem is used to investigate the existence and uniqueness of solutions for our system of stochastic difference equations.

Theorem 6.1.1. *Assume that hypotheses* $(H_1) - (H_2)$ *are satisfied and the matrix*

$$M \ = \begin{pmatrix} \sqrt{\mu_1} & \sqrt{\mu_2} \\ \sqrt{\mu_3} & \sqrt{\mu_4} \end{pmatrix}$$

converges to zero, where

$$\mu_1 = \left(2\sum_{j=0}^{h} a_j(i)a(i) + 2\sum_{j=0}^{h} \alpha_j(i)\alpha(i) + 4\sum_{j=0}^{h} a_j^2(i) + 4\sum_{j=0}^{h} \alpha_j^2(i) \right)$$

and

$$\mu_2 = \left(2\sum_{j=0}^{h} b_j(i)b(i) + 2\sum_{j=0}^{h} \beta_j(i)\beta(i) + 4\sum_{j=0}^{h} b_j^2(i) + 4\sum_{j=0}^{h} \beta_j^2(i) \right)$$

and

$$\mu_3 = \left(2\sum_{j=0}^{h} \overline{a}_j(i)\overline{a}(i) + 2\sum_{j=0}^{h} \overline{\alpha}_j(i)\overline{\alpha}(i) + 4\sum_{j=0}^{h} \overline{a}_j^2(i) + 4\sum_{j=0}^{h} \overline{\alpha}_j^2(i) \right)$$

and

$$\mu_4 = \left(2\sum_{j=0}^{h} \overline{b}_j(i)\overline{b}(i) + 2\sum_{j=0}^{h} \overline{\beta}_j(i)\overline{\beta}(i) + 4\sum_{j=0}^{h} \overline{b}_j^2(i) + 4\sum_{j=0}^{h} \overline{\beta}_j^2(i) \right).$$

Then, the problem (6.0.1) *possesses a unique solution.*

Proof. Consider the operator $N : C \times C \to C \times C$ defined by

$$N(x,y) = (N_1(x,y), N_2(x,y)),\ (x,y) \in C \times C$$

where

$$N_1(x,y) = \begin{cases} F^1(i, x(i-h), \dots, x(i), y(i-h), \dots, y(i)) \\ +G^1(k, x(i-h), \dots, x(i-h), y(i-h), \dots, y(i))\xi_i, & i \in \mathbb{N}(0, b+1) \\ \varphi_1(i), & i \in Z_0, \end{cases}$$

and

$$N_2(x,y) = \begin{cases} F^2(i, x(i-h), \dots, x(i), y(i-h), \dots, y(i)) \\ +G^2(k, x(i-h), \dots, x(i-h), y(i-h), \dots, y(i))\xi_i, & i \in \mathbb{N}(0, b+1), \\ \varphi_2(i), & i \in Z_0. \end{cases}$$

We shall use Theorem 2.4.1 to prove that N has a fixed point. Indeed, let $(x,y), (\overline{x}, \overline{y}) \in C \times C$. Then, from (6.0.1), conditions (H_1) and (H_2) and the Hölder inequality, we have for each $i \in Z_0 \cup \mathbb{N}(0, b+1)$,

$$\begin{aligned} E|N_1(x(i), y(i)) - N_1(\overline{x}(i), \overline{y}(i))|^2 \ \le\ & 2\sum_{j=0}^{h} a_j(i) \sum_{j=0}^{h} a_j(i) E|x(i-j) - \overline{x}(i-j)|^2 \\ & +2\sum_{j=0}^{h} b_j(i) \sum_{j=0}^{h} b_j(i) E|y(i-j) - \overline{y}(i-j)|^2 \\ & +2\sum_{j=0}^{h} \alpha_j(i) \sum_{j=0}^{h} \alpha_j(i) E|x(i-j) - \overline{x}(i-j)|^2 \\ & +2\sum_{j=0}^{h} \beta_j(i) \sum_{j=0}^{h} \beta_j(i) E|y(i-j) - \overline{y}(i-j)|^2 \end{aligned}$$

$$
\begin{aligned}
& +4\sum_{j=0}^{h} a_j^2(i)E|x(i-j)-\overline{x}(i-j)|^2 \\
& +4\sum_{j=0}^{h} b_j^2(i)E|y(i-j)-\overline{y}(i-j)|^2 \\
& +4\sum_{j=0}^{h} \alpha_j^2(i)E|x(i-j)-\overline{x}(i-j)|^2 \\
& +4\sum_{j=0}^{h} \beta_j^2(i)E|y(i-j)-\overline{y}(i-j)|^2 \\
= \; & \mu_1 E|x(i-j)-\overline{x}(i-j)|^2 + \mu_2 E|y(i-j)-\overline{y}(i-j)|^2
\end{aligned}
$$

where

$$\mu_1 = \left(2\sum_{j=0}^{h} a_j(i)a(i) + 2\sum_{j=0}^{h}\alpha_j(i)\alpha(i) + 4\sum_{j=0}^{h} a_j^2(i) + 4\sum_{j=0}^{h}\alpha_j^2(i)\right)$$

and

$$\mu_2 = \left(2\sum_{j=0}^{h} b_j(i)b(i) + 2\sum_{j=0}^{h}\beta_j(i)\beta(i) + 4\sum_{j=0}^{h} b_j^2(i) + 4\sum_{j=0}^{h}\beta_j^2(i)\right).$$

Therefore

$$
\begin{aligned}
\sup_{i\in\mathbb{N}(-h,b+1)} \mathbb{E}|N_1(x(i),y(i)) - N_1(\overline{x}(i),\overline{y}(i))|^2 \;\le\; & \mu_1 \sup_{i\in\mathbb{N}(-h,b+1)} E|x(i-j)-\overline{x}(i-j)|^2 \\
+ \; & \mu_2 \sup_{i\in\mathbb{N}(-h,b+1)} E|y(i-j)-\overline{y}(i-j)|^2.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\|N_2(x,y) - N_2(\overline{x},\overline{y})\|^2 \;\le\; & \mu_3 \sup_{i\in\mathbb{N}(-h,b+1)} E|x(i-j)-\overline{x}(i-j)|^2 \\
+ \; & \mu_4 \sup_{i\in\mathbb{N}(-h,b+1)} E|y(i-j)-\overline{y}(i-j)|^2.
\end{aligned}
$$

where

$$\mu_3 = \left(2\sum_{j=0}^{h} \overline{a}_j(i)\overline{a}(i) + 2\sum_{j=0}^{h}\overline{\alpha}_j(i)\overline{\alpha}(i) + 4\sum_{j=0}^{h} \overline{a}_j^2(i) + 4\sum_{j=0}^{h}\overline{\alpha}_j^2(i)\right)$$

and

$$\mu_4 = \left(2\sum_{j=0}^{h} \overline{b}_j(i)\overline{b}(i) + 2\sum_{j=0}^{h} \overline{\beta}_j(i)\overline{\beta}(i) + 4\sum_{j=0}^{h} \overline{b}_j^2(i) + 4\sum_{j=0}^{h} \overline{\beta}_j^2(i) \right).$$

It is clear that

$$\begin{aligned} \|N(x,y) - N(\overline{x},\overline{y})\| &= \begin{pmatrix} \|N_1(x,y) - N_1(\overline{x},\overline{y})\| \\ \|N_2(x,y) - N_2(\overline{x},\overline{y})\| \end{pmatrix} \\ &\leq \begin{pmatrix} \sqrt{\mu_1} & \sqrt{\mu_2} \\ \\ \sqrt{\mu_3} & \sqrt{\mu_4} \end{pmatrix} \begin{pmatrix} \|x-\overline{x}\| \\ \|y-\overline{y}\| \end{pmatrix}. \end{aligned}$$

Therefore

$$\|N(x,y) - N(\overline{x},\overline{y})\| \leq M \begin{pmatrix} \|x-\overline{x}\| \\ \|y-\overline{y}\| \end{pmatrix}, \text{ for all}, (x,y), (\overline{x},\overline{y}) \in C \times C.$$

From Perov's fixed point theorem, the mapping N has a unique fixed point $(x,y) \in C \times C$ which is the unique solution of problem (6.0.1). □

Now we present an existence result under nonlinearities F^i and G^i, $i = 1,2$ satisfying a Nagumo type growth condition:

(H_4) There exist a function $p_k, \overline{p}_k : Z_0 \to \mathbb{R}_+$ and nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ for each $k = 1,2$ such that

$$E(|F^k(i, x_0, \ldots, x_h, y_0, \ldots, y_h)|)^2 \leq \sum_{j=0}^{h} p_k(j)\psi(E|x_j|^2 + E|y_j|^2)$$

and

$$E(|G^k(i, x_0, \ldots, x_h, y_0, \ldots, y_h)|)^2 \leq \sum_{j=0}^{h} \overline{p}_k(j)\psi(E|x_j|^2 + E|y_j|^2)$$

for any random variables $x_j, y_j \in S,\ j = 0, \ldots h$.

Now, we present our result on the existence of solutions.

Theorem 6.1.2. *Assume that condition* (H_4) *holds. Then problem* (6.0.1) *has at least one solution.*

Proof. Clearly, the fixed points of N are solutions to (6.0.1), where N is defined in Theorem 6.1.1. In order to apply Theorem 2.4.2, we first show that N is completely continuous. The proof will be given in several steps.

- **Step 1.** $N = (N_1, N_2)$ is continuous.
  Let (x_n, y_n) be a sequence in $C \times C$ such that $(x_n, y_n) \to (x, y) \in C \times C$ as $n \to \infty$. Then

$$\begin{aligned}
&\mathbb{E}|N_1(x_n(i), y_n(i)) - N_1(x(i), y(i))|^2 \\
&\quad \leq 2\mathbb{E}\Big|F^1(i, x_n(i-h), \ldots, x_n(i), y_n(i-h), \ldots, y_n(i)) \\
&\qquad -F^1(i, x(i-h), \ldots, x(i), y(i-h), \ldots, y(i))\Big|^2 \\
&\qquad +2\mathbb{E}\Big|\Big(G^1(i, x_n(i-h), \ldots, x_n(i), y_n(i-h), \ldots, y_n(i)) \\
&\qquad -G^1(i, x(i-h), \ldots, x(i), y(i-h), \ldots, y(i))\Big)\xi_i\Big|^2
\end{aligned}$$

Since F^k, G^k are continuous functions for each $k = 1, 2$, we obtain

$$\begin{aligned}
&\sup_{i \in \mathbb{N}(-h, b+1)} \mathbb{E}|N_1(x_n(i), y_n(i)) - N_1(x(i), y(i))|^2 \\
&\quad \leq 2\mathbb{E}\Big|F^1(i, x_n(i-h), \ldots, x_n(i), y_n(i-h), \ldots, y_n(i)) \\
&\quad -F^1(i, x(i-h), \ldots, x(i), y(i-h), \ldots, y(i))\Big|^2 \\
&\quad +2\mathbb{E}\Big|G^1(i, x_n(i-h), \ldots, x_n(i), y_n(i-h), \ldots, y_n(i)) \\
&\quad -G^1(i, x(i-h), \ldots, x(i), y(i-h), \ldots, y(i))\Big|^2 \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Similarly

$$\begin{aligned}
&\sup_{i \in \mathbb{N}(-h, b+1)} \mathbb{E}|N_2(x_n(i), y_n(i)) - N_2(x(i), y(i))|^2 \\
&\quad \leq 2\mathbb{E}\Big|F^2(i, x_n(i-h), \ldots, x_n(i), y_n(i-h), \ldots, y_n(i)) \\
&\quad -F^2(i, x(i-h), \ldots, x(i), y(i-h), \ldots, y(i))\Big|^2 \\
&\quad +2\mathbb{E}\Big|G^2(i, x_n(i-h), \ldots, x_n(i), y_n(i-h), \ldots, y_n(i)) \\
&\quad -G^2(i, x(i-h), \ldots, x(i), y(i-h), \ldots, y(i))\Big|^2 \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus N is continuous.

- **Step 2.** N maps bounded sets into bounded sets in $C \times C$. Indeed, it is enough to show that for any $q > 0$ there exists a positive constant l such that for each $(x, y) \in B_q = \{(x, y) \in C \times C : \|x\| \leq q,\ \|y\| \leq q\}$, we have
$$\|N(x, y)\| \leq l = (l_1, l_2).$$
Then for each $i \in \mathbb{N}(0, b+1)$, we deduce

$$\begin{aligned}\mathbb{E}|N_1(x(i), y(i))|^2 &\leq 2\mathbb{E}\Big|F^1(i, x(i-h), \ldots, x(i), y(i-h), \ldots, y(i))\Big|^2 \\ &\quad +2\mathbb{E}\Big|G^1(i, x(i-h), \ldots, x(i), y(i-h), \ldots, y(i))\Big|^2\end{aligned}$$

Thanks to (H_4),

$$\begin{aligned}\mathbb{E}|N_1(x(i), y(i))|^2 &\leq 2\mathbb{E}\Big|F^1(i, x(i-h), \ldots, x(i), y(i-h), \ldots, y(i))\Big|^2 \\ &\quad +2\mathbb{E}\Big|G^1(i, x(i-h), \ldots, x(i), y(i-h), \ldots, y(i))\Big|^2 \mathbb{E}(\xi_i)^2 \\ &\leq 2\sum_{j=0}^{h} p_1(j)\Big(\psi(\mathbb{E}|x(i-j))|^2 + \mathbb{E}|y(i-j)|^2)\Big) \\ &+ 2\sum_{j=0}^{h} p_1(j)\Big(\psi(\mathbb{E}|x(i-j)|^2 + \mathbb{E}|y(i-j)|^2)\Big) \\ &\leq 2\sum_{j=0}^{h} p_1(j)\psi(2q) + 2\sum_{j=0}^{h} \overline{p}_1(j)\psi(2q)\end{aligned}$$

Therefore

$$\|N_1(x, y)\|^2 \leq 2\sum_{j=0}^{h} p_1(j)\psi(2q) + 2\sum_{j=0}^{h} \overline{p}_1(j)\psi(2q) := l_1,$$

and, similarly,

$$\|N_2(x, y)\|^2 \leq 2\sum_{j=0}^{h} p_2(j)\psi(2q) + 2\sum_{j=0}^{h} \overline{p}_2(j)\psi(2q) := l_2.$$

For each $i \in \mathbb{N}(-h, b+1)$ we can easily prove that the set $N(B_q)(i)$ defined by

$$N(B_q)(i) = \{N(x(i), y(i)) : x, y \in B_q\}$$

is relatively compact in $\mathbb{R} \times \mathbb{R}$. By the Arzela-Ascoli Theorem 6.0.9, we conclude that $N(B_q)$ is relatively compact in $C \times C$. Hence $N : C \times C \to C \times C$ is a completely continuous operator.

- **Step 4.** It remains to show that

$$\mathcal{A} = \{(x, y) \in C \times C : \ (x, y) = \lambda N(x, y), \lambda \in (0, 1)\}$$

is bounded.
Let $(x, y) \in \mathcal{A}$. Then $x = \lambda N_1(x, y)$ and $y = \lambda N_2(x, y)$ for some $0 < \lambda < 1$. Thus, for $i \in \mathbb{N}(0, b+1)$, we have

$$\begin{aligned}
\mathbb{E}|x(i+1)|^2 &\le 2\mathbb{E}\Big|F^1(i, x(i-h), \ldots, x(i), y(i-h), \ldots, y(i))\Big|^2 \\
&\quad +2\mathbb{E}\Big|G^1(i, x(i-h), \ldots, x(i), y(i-h), \ldots, y(i))\Big|^2 \mathbb{E}(\xi_i)^2 \\
&\le 2\sum_{j=0}^{h} p_1(j)\Big(\psi(\mathbb{E}|x(i-j)|^2 + \mathbb{E}|y(i-j)|^2)\Big) \\
&+ 2\sum_{j=0}^{h} \overline{p}_1(j)\Big(\psi(\mathbb{E}|x(i-j)|^2 + \mathbb{E}|y(i-j)|^2)\Big)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}|y(i+1)|^2 &\le 2\mathbb{E}\Big|F^2(i, x(i-h), \ldots, x(i), y(i-h), \ldots, y(i))\Big|^2 \\
&\quad +2\mathbb{E}\Big|G^2(i, x(i-h), \ldots, x(i), y(i-h), \ldots, y(i))\Big|^2 \mathbb{E}(\xi_i)^2 \\
&\le 2\sum_{j=0}^{h} p_2(j)\Big(\psi(\mathbb{E}|x(i-j)|^2 + \mathbb{E}|y(i-j)|^2)\Big) \\
&\quad +2\sum_{j=0}^{h} \overline{p}_2(j)\Big(\psi(\mathbb{E}|x(i-j)|^2 + \mathbb{E}|y(i-j)|^2)\Big)
\end{aligned}$$

Therefore

$$\mathbb{E}|x(i+1)|^2 + \mathbb{E}|y(i+1)|^2 \le p(s)\phi(\mathbb{E}|x(i-j)|^2 + \mathbb{E}|y(i-j)|^2),$$

where

$$p(t) = 2\Big(\sum_{i=1}^{2}\sum_{j=0}^{h} p_i(j) + \sum_{i=1}^{2}\sum_{j=0}^{h} \overline{p}_i(j)\Big), \text{ and } \phi(t) = \sum_{i=1}^{m} \psi(i).$$

We consider the function μ_1 and μ_2 defined by

$$\mu_1(i) = \sup\{\mathbb{E}|x(k)|^2 : \quad k \in \mathbb{N}(-h, i+1)\}, \quad i+1 \in \mathbb{N}(0, b+1)$$

and

$$\mu_2(i) = \sup\{\mathbb{E}|y(k)|^2 : \quad k \in \mathbb{N}(-h, i+1)\}, \quad i+1 \in \mathbb{N}(0, b+1)$$

Let $i^* \in \mathbb{N}(-h, i+1)$ be such that $\mu_1(i) = \mathbb{E}|x(i^*)|^2$. If $i^* \in \mathbb{N}(0, b+1)$, by the previous inequality we have

$$\mu_1(i) \le \mathbb{E}|\varphi_1(i)|^2 + 2\sum_{j=0}^{h} p_1(j)\Big(\psi(\mu_1(k) + \mu_2(k))\Big) + 2\sum_{j=0}^{h} \overline{p}_1(j)\Big(\psi(\mu_1(k) + \mu_2(k))\Big)$$

and

$$\mu_2(i) \le \mathbb{E}|\varphi_2(i)|^2 + 2\sum_{j=0}^{h} p_2(j)\Big(\psi(\mu_1(k) + \mu_2(k))\Big) + 2\sum_{j=0}^{h} \overline{p}_2(j)\Big(\psi(\mu_1(k) + \mu_2(k))\Big).$$

Therefore

$$\mu_1(i) + \mu_2(i) \quad \le \quad \mathbb{E}|\varphi_1(i)|^2 + \mathbb{E}|\varphi_2(i)|^2 + p(k)\psi(\mu_1(k) + \mu_2(k)),$$

If $i^* \in \mathbb{N}(-h, 0)$, then $\mu_1(i) = \mathbb{E}|\varphi_1(i)|^2$ and $\mu_2(i) = \mathbb{E}|\varphi_2(i)|^2$ and the previous inequality holds. Let us take the right-hand side of the above inequality as $\upsilon(i)$. Then we have

$$\upsilon(0) = \mathbb{E}|\varphi_1(0)|^2 + \mathbb{E}|\varphi_2(0)|^2$$

and

$$\mu = \mu_1(i) + \mu_2(i) \le \upsilon(i)$$

where

$$\upsilon(i) = \mathbb{E}|\varphi_1(i)|^2 + \mathbb{E}|\varphi_2(i)|^2 + p(k)\psi(\mu_1(k) + \mu_2(k))$$

Using the nondecreasing character of ψ we get

$$\mu(i) \leq \gamma + p(k)\psi(\mu(i)), \quad i \in \mathbb{N}(0, b+1)$$

where

$$\gamma = \mathbb{E}|\varphi_1(0)|^2 + \mathbb{E}|\varphi_2(0)|^2.$$

By Theorem6.0.7, we have

$$\mathbb{E}|x(i)| + \mathbb{E}|y(i)| \leq G^{-1}(p(k)) := K, \text{ for each } i \in \mathbb{N}(-h, b+1),$$

Consequently

$$\|x\| \leq K \text{ and } \|y\| \leq K.$$

This shows that $\mathcal{A}$ is bounded. As a consequence of Theorem 2.4.2 we deduce that N has a fixed point (x, y) which is a solution to problem (6.0.1)

□

6.1.1 Exponentially stability

As in this section we are interested in the exponential decay to zero in mean square of the solutions to (6.0.1), we will assume that solutions are defined globally in time.

Consider the Banach space

$$C_* = \{x : \Omega \times Z_0 \cup Z \longrightarrow \mathbb{R}, \; x(\cdot, \omega) \in C(Z_0 \cup Z, \mathbb{R}) \quad \text{a.s. and } \sup_{i \in Z_0 \cup Z} \mathbb{E}|x(i, .)|^2 < \infty \},$$

where $Z_0 = \{-h, \ldots, 0\}$ and $Z = \{0, 1, 2, \ldots\}$, h is a given nonnegative integer number.

Next, we will introduce some basic definitions.

Definition 6.1.2. *The zero solution of probleme* (6.0.1) *is said to be mean square stable if for any* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that* $\mathbb{E}|x(i)|^2 < \epsilon$, $i \in Z$, *when the initial condition* $\varphi = (\varphi_{-h}, .., \varphi_0)^t$ *satisfies* $\|\varphi\| = \sup_{i \in Z_0} \mathbb{E}|\varphi_i| < \delta$. *If, besides,* $\lim_{i \to \infty} \mathbb{E}|x(i)|^2 = 0$, $i \in Z$, *for all initial condition* φ *then the zero solution of Eq.* (6.0.1) *is called mean square asymptotically stable.*

Definition 6.1.3. *The zero solution of Eq.* (6.0.1) *is said to be mean square exponential stable if there are positive constants* λ *and* M *such that for any initial condition* φ, φ',

$$\mathbb{E}|x(i)|^2 \leq M\|\varphi_1\|^2 e^{-\lambda i}, \qquad i \in Z \tag{6.1.1}$$

and

$$\mathbb{E}|y(i)|^2 \leq M\|\varphi_2\|^2 e^{-\lambda i}, \qquad i \in Z \tag{6.1.2}$$

Here λ *is called the exponential convergence rate.*

Theorem 6.1.3. *[146] Suppose* $c_j(i) \in \mathbb{R}^+$, $i \in Z$, $j \in \mathbb{N}(0,h)$ *and* $\sup_{i\in Z}\{\sum_{j=0}^{h} c_j(i)\} = \mu_1 < 1$. *Let* $\{u(i)\}$ *be a sequence of real numbers satisfying the following difference inequality:*

$$u(i+1) \leq \sum_{j=0}^{h} c_j(i)u(i-j).$$

Then

$$u(i) \leq de^{-\lambda i} \; i \geq i', i \in Z,$$

where $d \in \mathbb{R}^+$ *and* λ *satisfies*

$$0 < \lambda < \frac{1}{h+1}\ln\frac{1}{\mu}.$$

To establish the main results on stability for system (6.0.1), we will impoaw the following assumptions.

(H_5) For any $i \in Z, j \in Z_0$, there exist positive constants $a_j(i), \overline{a}_j(i)$ and $b_j(i), \overline{b}_j(i)$ such that

$$\left|F^1(i, x_0, \ldots, x_h, y_0, \ldots, y_h)\right|^2 \leq \sum_{j=0}^{h} a_j(i)|x_j|^2 + \sum_{j=0}^{h} b_j(i)|y_j|^2$$

and

$$\left|F^2(i, x_0, \ldots, x_h, y_0, \ldots, y_h))\right|^2 \leq \sum_{j=0}^{h} \overline{a}_j(i)|x_j|^2 + \sum_{j=0}^{h} \overline{b}_j(i)|y_j|^2$$

for each $x_j, y_j \in \mathbb{R},\ j = 0, \ldots, h$.

(H_6) For any $i \in Z\,, j \in Z_0$, there exist positive constants $\alpha_j(i), \overline{\alpha}_j(i)$ and $\beta_j(i), \overline{\beta}_j(i)$

$$\Big|G^1((i, x_0, \ldots, x_h, y_0, \ldots, y_h)\Big|^2 \leq \sum_{j=0}^{h} \alpha_j(i)|x_j|^2 + \sum_{j=0}^{h} \beta_j(i)|y_j|^2$$

and

$$\Big|G^2(i, x_0, \ldots, x_h, y_0, \ldots, y_h)\Big|^2 \leq \sum_{j=0}^{h} \overline{\alpha}_j(i)|x_j|^2 + \sum_{j=0}^{h} \overline{\beta}_j(i)|y_j|^2$$

for each $x_j, y_j \in \mathbb{R},\ j = 0, \ldots, h$.

(H_7) There exists constant $\lambda > 0$ such that

$$0 < \lambda \leq \frac{1}{h+1} \ln \frac{1}{\mu}$$

where

$$\sup_{i \in Z}\{k_1^*(i), k_2^*(i)\} = \mu < 1$$

with

$$k_1^*(i) = \sum_{j=0}^{h}(a_j(i) + \alpha_j(i) + \overline{a}_j(i) + \overline{\alpha}_j(i)) < \infty$$

and

$$k_2^*(i) = \sum_{j=0}^{h}(b_j(i) + \beta_j(i) + \overline{b}_j(i) + \overline{\beta}_j(i)) < \infty$$

for each $i \in Z$.

Theorem 6.1.4. *Assume that conditions* $(H_5) - (H_7)$ *hold, then the zero solution of probleme* (6.0.1) *is mean square exponentially stable and the exponential convergence rate is equal to* λ*.*

Proof. By virtue of the Hölder inequality and the mean value inequality, we obtain

$$\begin{aligned}
\mathbb{E}|x(i+1)|^2 &= \mathbb{E}\Big|F^1(i, x(i-h), \ldots, x(i), y(i-h), \ldots, y(i)) \\
&\quad + G^1(k, x(i-h), \ldots, x(i-h), y(i-h), \ldots, y(i))\Big|^2 \\
&\leq 2\sum_{j=0}^{h} a_j(i)\mathbb{E}|x(i-j)|^2 + 2\sum_{j=0}^{h} b_j(i)\mathbb{E}|y(i-j)|^2 \\
&\quad + 2\sum_{j=0}^{h} \alpha_j(i)\mathbb{E}|x(i-j)|^2 + 2\sum_{j=0}^{h} \beta_j(i)\mathbb{E}|y(i-j)|^2.
\end{aligned}$$

which implies

$$\begin{aligned}
\mathbb{E}|x(i+1)|^2 &= 2(\sum_{j=0}^{h} a_j(i) + \sum_{j=0}^{h} \alpha_j(i))\mathbb{E}|x(i-j)|^2 \\
&\quad + 2(\sum_{j=0}^{h} b_j(i) + \sum_{j=0}^{h} \beta_j(i))\mathbb{E}|y(i-j)|^2, \qquad (6.1.3)
\end{aligned}$$

and, in a similar way,

$$\begin{aligned}
\mathbb{E}|y(i+1)|^2 &\leq 2(\sum_{j=0}^{h} \overline{a}_j(i) + \sum_{j=0}^{h} \overline{\alpha}_j(i))\mathbb{E}|x(i-j)|^2 \\
&\quad + 2(\sum_{j=0}^{h} \overline{b}_j(i) + \sum_{j=0}^{h} \overline{\beta}_j(i))\mathbb{E}|y(i-j)|^2. \qquad (6.1.4)
\end{aligned}$$

Now, combining (6.1.3) and (6.1.4) we can rewrite them as

$$\begin{aligned}
\mathbb{E}|x(i+1)|^2 + \mathbb{E}|y(i+1)|^2 &\leq 2\sum_{j=0}^{h}(a_j(i) + \alpha_j(i) + \overline{a}_j(i) + \overline{\alpha}_j(i))\mathbb{E}|x(i-j)|^2 \\
&\quad + 2\sum_{j=0}^{h}(b_j(i) + \beta_j(i) + \overline{b}_j(i) + \overline{\beta}_j(i))\mathbb{E}|y(i-j)|^2. \qquad (6.1.5)
\end{aligned}$$

where

$$k_1^*(i) = \sum_{j=0}^{h}(a_j(i) + \alpha_j(i) + \overline{a}_j(i) + \overline{\alpha}_j(i)),$$

and

$$k_2^*(i) = \sum_{j=0}^{h}(b_j(i) + \beta_j(i) + \overline{b}_j(i) + \overline{\beta}_j(i)).$$

From condition (H_7), we obtain

$$\sup_{i \in Z}\{k_1^*(i), k_2^*(i)\} = \mu < 1 \tag{6.1.6}$$

For the initial condition $x(i) = \varphi_1(i)$ and $y(i) = \varphi_2(i)$, $i \in Z_0$, we have

$$\mathbb{E}|x(i)|^2 + \mathbb{E}|y(i)|^2 \leq (\|\varphi_1\|^2 + \|\varphi_2\|^2)e^{-\lambda i}, \tag{6.1.7}$$

where $\|\varphi_1\|^2 = \sup_{i \in Z_0} \mathbb{E}|\varphi_1(i)|^2 < \delta$ and $\|\varphi_2\|^2 = \sup_{i \in Z_0} \mathbb{E}|\varphi_2(i)|^2 < \delta$. Then, all the conditions of Theorem 6.1.3 are fulfilled by (6.1.5) and (6.1.7). Consider the functions $\mu, \overline{\mu}$ defined on $i \in Z$ by

$$\mu(i) = \sup\{\mathbb{E}|x(i)|^2 \ : i \in Z\}$$

and

$$\overline{\mu}(i) = \sup\{\mathbb{E}|y(i)|^2 \ : i \in Z\},$$

consequently

$$\mu(i) + \overline{\mu}(i) \leq (\|\varphi_1\|^2 + \|\varphi_2\|^2)e^{-\lambda i}. \tag{6.1.8}$$

Then

$$\mu(t) \leq (\|\varphi_1\|^2 + ||\varphi_2\|^2)e^{-\lambda i},$$

and

$$\overline{\mu}(t) \leq (\|\varphi_1\|^2 + \|\varphi_2\|^2)e^{-\lambda i}.$$

This implies that the conclusion of Theorem 6.1.3 holds. □

As a straightforward consequence we can establish the following result.

Corollary 6.1.5. *Assume that conditions* $(H_5) - (H_7)$ *hold, then the zero solution of Eq.* (6.0.1) *is mean square asymptotically stable.*

6.1.2 An example

Consider the following system of stochastic difference equation with delays:

$$\begin{cases} x(i+1) & = \frac{1}{5}(x(i)+y(i))\sin(x(i)+y(i)) - \frac{1}{6}(x(i-1)+y(i-1)) \\ & +\frac{1}{5}(x(i)+y(i))\xi_i, \\ y(i+1) & = \frac{1}{5}(x(i)+y(i))(\cos(x(i)+y(i))-1) - \frac{1}{6}(x(i-1)+y(i-1)) \\ & +\frac{1}{5}(x(i)+y(i))\xi_i, \end{cases} \tag{6.1.9}$$

where $h=1$,

$$F^1(i,x_0,\dots,x_h,y_0,\dots,y_h) = \frac{1}{5}(x(i)+y(i))\sin(x_i+y_i) - \frac{1}{6}(x_{i-1}+y_{i-1})$$

and

$$F^2(i,x_0,\dots,x_h,y_0,\dots,y_h) = \frac{1}{5}(x(i)+y(i))(\cos(x_i+y_i)-1) - \frac{1}{6}(x_{i-1}+y_{i-1})$$

and

$$G^l(i,x_0,\dots,x_h,y_0,\dots,y_h) = \frac{1}{5}(x_i+y_i), \ \ l=1,2.$$

Then

$$|F^l(i,x_0,\dots,x_h,y_0,\dots,y_h)|^2 \le \frac{4}{25}(|x_i|^2+|y_i|^2) + \frac{1}{9}(|x_{i-1}|^2+|y_{i-1}|^2)$$

and

$$|G^l(i,x_0,\dots,x_h,y_0,\dots,y_h)|^2 \le \frac{4}{25}(|x_i|^2+|y_i|^2)$$

So, the parameters of conditions $(H_5)-(H_7)$ for each $l=1,2$ are as follows:

$$a_0(i) = \frac{4}{25}, a_1(i) = \frac{1}{9} \quad b_0(i) = \frac{4}{25}, \quad b_1(i) = \frac{1}{9}$$
$$\alpha_0(i) = \frac{4}{25}, \quad \beta_0(i) = \frac{4}{25} \quad \beta_1(i) = \alpha_1(i) = 0.$$

For their values $l=2$

$$\overline{a}_0(i) = \frac{4}{25}, \overline{a}_1(i) = \frac{1}{9} \quad \overline{b}_0(i) = \frac{4}{25}, \quad \overline{b}_1(i) = \frac{1}{9}$$

$$\overline{\alpha}_0(i) = \frac{4}{25}, \quad \overline{\beta}_0(i) = \frac{4}{25} \qquad \overline{\alpha}_1(i) = \overline{\beta}_1(i) = 0.$$

It is easy to compute that

$$k_1^*(i) = \sum_{j=0}^{1}(a_j(i) + \alpha_j(i) + \overline{a}_j(i) + \overline{\alpha}_j(i)) = 0.862$$

and

$$k_2^*(i) = \sum_{j=0}^{1}(b_j(i) + \beta_j(i) + \overline{b}_j(i) + \overline{\beta}_j(i)) = 0.862$$

From condition (H_7), we obtain

$$\sup_{i \in Z}\{k_1^*(i), k_2^*(i)\} = \mu = 0.862 < 1 \tag{6.1.10}$$

we can choose $\lambda = 0, 16$ such that

$$0 < \lambda \leq \frac{1}{h+1} \ln \frac{1}{\mu} = \frac{1}{2} \ln 1.16.$$

Then condition (H_7) is satisfied and thanks to Theorem 6.1.3 we deduce that the zero solution of problem (6.1.9) is mean square exponentially stable.

Conclusion and Perspectives

In this thesis, we have presented some results to the theory of existence and exponential stability of mild solutions of semilinear systems of stochastic differential equations with infinite fractional Brownian motions and impulses and inclusions driven by infinite fractional Brownian motions with the Hurst index $H > \frac{1}{2}$. In most of these works suffcient conditions were considered to get the existence of solution by reducing the research to the search of the existence of fixed points by applying different fixed points . Existence results give for some classes by Perov's and Schaefer's fixed point theorems in generalized Banach spaces.

We plan to extend for the existence of solutions for singularϕ -Laplacian impulsive differential equations stochastic and ϕ-Laplacian impulsive differential equations stochastic with parameter.We will study global existence and energy decay for a weak viscoelastic wave equations stochastic with a dynamic boundary and nonlinear delay term.Using several methods (the Faedo–Galerkin approximations combined with a contraction mapping theorem,Lax-Milgram).We will study the system stochastic semilinear functional differential equations and inclusions .Some sufficient conditions are obtained by using the notion of measure of noncompactness and the LeraySchauder type fixed theorem.

Bibliography

[1] R. A.Adams *Sobolev spaces* , Academic Press, Boston San Diego New York London Sidney Tokyo Toronto, 1978.

[2] R.P. Agarwal and D. O'Regan, Infinite interval problems for differential, difference and integral equations. Kluwer Academic, Dordrecht, 2001.

[3] R.P. Agarwal, D. O'Regan and P.J.Y. Wong, *Positive solutions of differential, difference and integral equations.* Kluwer Academic, Dordrecht, 1999.

[4] R.P. Agarwal, S.R .Grace and D. O'Regan, Oscillation theory for difference and functional differential equations, Kluwer Academic, Dordrecht; 2000.

[5] N. U. Ahmed, Nonlinear stochastic differential inclusions on Bananch space, *Stochastic Anal. Appl.* **12** (1994), 1-10.

[6] N. U. Ahmed, Existence of solutions of nonlinear stochastic differential inclusions on Banach space , in: V. Lakshmikantham (Ed.), Proceedings of the First World Congress of Nonlinear Analysis, Tampa, FL, August (1992) 19-26, de Gruyter, Berlin, (1996), pp. 1699-1712.

[7] G. Ahmadi, On the mean square stability of linear difference equations, *Appl. Math. Comput.* **5** (1979) 233-241.

[8] H. Airault and J. Ren, Modulus of continuity of the canonic Brownian motion "on the group of diffeomorphisms of the circle *J. Funct. Anal.* **196** (2002), 95-426.

[9] E. Alòs, O. Mazet and D. Nualart, *Stochastic calculus with respect to Gaussian processes.*

[10] E. Alòs and D. Nualart, An extension of Itô's formula for anticipating processes. *J. Theoret. Probab.* 11 (1998) 493-514. *Ann Probab*, **29** (1999), 766-801.

[11] E. Andreyeva and V.B. Kolmanovskii, *Shaikhet LE Control of hereditary systems*, Nauka, Moscow, (1992),(in Russian).

[12] J.P. Aubin, H. Frankowska, *Set-Valued Analysis*, Birkhauser, Boston, 1990.

[13] A. Arara, M. Benchohra, L. Górniewicz and A. Ouahab, Controllability results for semilinear functional differential inclusions with unbounded delay, *Math. Bulletin*, **3** (2006), 157–183.

[14] M.A.Arcones, On the law of the iterated logarithm for gaussian processes. *Journal of Theoretical Probability*, **8** (4), 877–904, 1995

[15] C. Avramescu, Sur lexistence des solutions convergentes des systmes dquations diffrentielles non linaire, *Ann. Mat. Pura. Appl.,* 81 (1969), 147 168.

[16] C. Avramescu, Some remarks on a fixed point theorem of Krasnoselskii, *Electron. J. Qual. Theory Differ. Equ.* **5** (2003), 1-15.

[17] D.D. Bainov, V. Lakshmikantham, P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[18] D.D. Bainov and P.S. Simeonov, *Systems with Impulsive Effect*, Horwood, Chichester, 1989.

[19] P. Balasubramaniam, Existence of solutions of functional stochastic differential inclusions, *Tamkang J. Math.* 33 (2002), no. 1, 35–43.

[20] P. Balasubramaniam, Existence of solutions of functional stochastic differential inclusions, *Tamkang J. Math.* **33** (2002), 35-43.

[21] P. Balasubramaniam, S. K. Ntouyas and D. Vinayagam, Existence of solutions of semilinear stochastic delay evolution inclusions in a Hilbert space. *J. Math. Anal. Appl.* **305** (2005), 438–451.

[22] P. Balasubramaniam and D. Vinayagam, Existence of solutions of nonlinear neutral stochastic differential inclusions in a Hilbert Space. *Stochastic Analysis and Applications*, **23** (2005), 137-151.

[23] M.Benchohra, S. K.Ntouyas , A.Ouahab . On nondensely defined semilinear stochastic functional differential equations with nonlocal conditions. *J Appl Math Stoch Anal*, 2006, Art ID 69584

[24] M. Benchohra, J. Henderson and S. K. Ntouyas, *Impulsive differential equations and inclusions*, **2**, Hindawi Publishing Corporation, New York, 2006.

[25] M. Benchohra and S. K. Ntouyas, Existence of mild solutions for certain delay semilinear evolution inclusion with nonlocal conditions, *Dynam. systems Appl.* **9** (3) (2000), 405–412.

[26] A. T. Bharucha-Reid, *Random Integral Equations*, Academic Press, New York, 1972.

[27] F.Biagini,et al. Stochastic Calculus for Fractional Brownian Motion and Applications, Springer, (2008). Print.

[28] Y. Biran and B. Innis, Optimal control of bilinear systems: time-varying effects of cancer drugs, *Automatica* **15**, (1979).

[29] A. Bielecki, Une remarque sur la mtode de Banach-Cacciopoli-Tikhonov dans la thorie des quations diffrentielles ordinaires, *Bull. Acad. Polon. Sci.* **4** (1956), 261-264.

[30] I. Bihari, A generalisation of a lemma of Bellman and its application to uniqueness problems of differential equations, *Acta Math. Acad. Sci.* Hungar.,**7** (1956), 81-94.

[31] T. Björk and H. Hult, A note on Wick products and the fractional Black-Scholes model. *Finance and Stochastics*, **9** (2005) 197-209.

[32] T. Blouhi, T. Caraballo, A. Ouahab, Existence and stability results for semilinear systems of impulsive stochastic differential equations with fractional Brownian motion, *Stoch. Anal. Appl.* (2016), to appear.

[33] O. Bolojan-Nica, G. Infante and R. Precup, Existence results for systems with coupled nonlocal initial conditions, *Nonlinear Anal.* **94** (2014), 231–242.

[34] A. Boudaoui, T. Caraballo and A. Ouahab, Impulsive neutral functional differential equations driven by a fractional Brownian motion with unbounded delay. *Applicable Analysis* (2015).

[35] A. Boudaoui, T. Caraballo and A. Ouahab, Impulsive stochastic functional differential inclusions driven by a fractional Brownian motion with infinite delay. *Mathematical methods in the Applied Sciences* (2015).

[36] A. Boudaoui, T. Caraballo and A. Ouahab, Existence of mild solutions to stochastic delay evolution equations with a fractional Brownian motion and impulses. *Stoch. Anal. Appl.* (2015), 244-258.

[37] B. Boufoussi and S. Hajji, Neutral stochastic functional differential equations driven by a fractional Brownian motion in a Hilbert space. *Statist Probab Lett*, **82** (2012), 1549-1558.

[38] A. Bressan and G. Colombo, Extensions and selections of maps with decomposable values, *Studia Math.* **90** (1988), 69-86.

[39] H. Brèzis, *Analyse Fonctionnelle*, Thèorie et Applications, Masson, Paris, 1983.

[40] D. L. Burkholder, Martingale transforms, *Ann. Math. Statist.* **37** (1966) 1494-1504.

[41] D. L. Burkholder and R. F. Gundy, Extrapolation and interpolation of quasi-linear operators on martingales, *Acta Math.*, 124 (1970), 249-304.

[42] T. A. Burton and C. Kirk, A fixed point theorem of Krasnoselskii type, *Math. Nachrichten*, **189** (1998), 23–31.

[43] J. Cao, Q. Yang, Z. Huang and Q. Liu, Asymptotically almost periodic solutions of stochastic functional differential equations, *Appl. Math. Comput.* **218** (2011), 1499-1511.

[44] T. Caraballo, M. Garrido-Atienza, T. Taniguchi, The existence and exponential behavior of solutions to stochastic delay evolution equations

with a fractional Brownian motion, *Nonlinear Anal.* 74 (2011), 3671–3684.

[45] T.Caraballo , Mamadou A. Diop, Neutral stochastic delay partial functional integro-differential equations driven by a fractional Brownian motion Front. Math. China. 2013;8:745760.

[46] P. Cheridito: Mixed fractional Brownian motion. *Bernoulli* 7 (2001) 913 934.

[47] C. Castaing, M. Valadier, *Convex Analysis and Measurable Multifunctions*, in: Lecture Notes in Mathematics, vol. 580.

[48] O. Cârjă, M. Necula and I. I.Vrabie, *Viability, Invariance and Applications.* North-Holland Mathematics Studies, 207. Elsevier Science B.V., Amsterdam, 2007.

[49] P.Cheridito, Mixed fractional Brownian motion. *Bernoulli* 7, 913-934, 2001.

[50] C.Corduneanu, *Integral Equations and Stability of Feedback Systems,* Academic Press, New York, 1973.

[51] R. Cristescu, *Order Structures in Normed Vector Spaces*, Editura Ştiin ţifică şi Enciclopedică, Bucureşti, 1983 (in Romanian).

[52] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, 1992.

[53] B. Davis, On the integrability of the martingale square function, *Israel J. Math.* **8** (1970) 187-190.

[54] L. Decreusefond and A. S. Üstünel, Stochastic analysis of the fractional Brownian motion. *Potential Analysis* **10** (1998) 177214.

[55] K. Deimling, *Multi-valued Differential Equations*, De Gruyter, Berlin, New York, 1992.

[56] B.C. Dhage and V. Lakshmikantham, On global existence and attractivity results for nonlinear functional integral equations, *Nonlinear Anal.*, **72** (2010) 2219–2227

[57] S. Djebali, L. Górniewicz, and A. Ouahab, First order periodic impulsive semilinear differential inclusions: existence and structure of solution sets, *Math. and Comput. Mod.*, 52 (2010), 683–714.

[58] S. Djebali, L. Gorniewicz and A. Ouahab, *Solutions Sets for Differential Equations and Inclusions*, De Gruyter Series in Nonlinear Analysis and Applications **18**. Berlin: de Gruyter, 2013.

[59] J. Dugundji and A. Granas, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[60] N.Dunford, J. T.Schwartz, *Linear operators part I: general theory*, Interscience Publishers, Inc. New York, Interscience Publishers Ltd. London, 1958.

[61] P.Embrechts , Maejima, Makoto, Selfsimilar Processes. Princeton, NJ: Princeton University Press, 2002.

[62] L. Erbe, H. I. Freedman, X. Z. Liu and J. H. Wu, Comparison principales for impulsive parabolic equations with applications to models of singles species growth, *J. Austral. Math. Soc.* Ser. B **32** (1991), 382-400.

[63] A.ferldman, Stochastic Differential Equations and applications , *Probablity and mathmatical statistics series.,* **90** 11698–11702 (1993).

[64] M. Frigon and A. Granas, Théorèmes d'existence pour des inclusions différentielles sans convexit, *C. R. Acad. Sci. Paris, Ser. 1* **310** (1990), 819-822.

[65] T.C. Gard, *Introduction to Stochastic Differential Equations*, Marcel Dekker, New York, 1988.

[66] R. K. George, D. N. Chalishajar and A. K. Nandakumaran, Controllability of second order semi-linear neutral functional differential inclusions in Banach spaces, *Mediterr. J. Math.*, **1** (2004), 463-477.

[67] I.I. Gikhman and A. Skorokhod, *Stochastic Differential Equations*, Springer-Velag, 1972.

[68] J. A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford Univ. Press, New York, 1985.

[69] L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings*, Mathematics and its Applications, 495, Kluwer Academic Publishers, Dordrecht, 1999.

[70] J.R. Graef, A. Ouahab, Structure of solutions sets and a continuous version of Filippov's Theorem for first order impulsive differential inclusions with periodic conditions,*Electronic J. Qual* . **24** (2009), 1-23.

[71] J. R. Graef, J. Henderson and A. Ouahab, *Impulsive differential inclusions. A fixed point approach.* De Gruyter Series in Nonlinear Analysis and Applications 20. Berlin: de Gruyter, 2013.

[72] H. Guo, X. Song, An impulsive predatorprey system with modified LeslieGower and Holling type II schemes, *Chaos Solitons Fractals.* **36** (2008) 13201331.

[73] P. Guasoni, No arbitrage under transaction costs, with fractional Brownian motion and beyond. *Math. Finance* **16** (2006) no. 3, 569-582.

[74] P. Guasoni, M. Rasonyi, W. Schachermayer, The fundamental theorem of asset pricing for continuous processes under small transaction costs. *Ann. Finance.* **6** (2010), no. 2, 157-191.

[75] C. Guilan and H. Kai, On a type of stochastic differential equations driven by countably many Brownian motions, *J. Funct. Anal.* **203**, (2003), 262-285.

[76] A. Halanay , D. Wexler, *Teoria calitativa a systeme cu impulduri*, Editura Republicii Socialiste Romania, Bucharest, 1968.

[77] J. K. Hale, *Theory of Functional Differential Equations,* Springer-Verlag, New York, 1977.

[78] J.K. Hale and J. Kato, Phase space for retarded equations with infinite delay, *Funkcial. Ekvac.*, **21** (1978), 11-41.

[79] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equation,* Applied Mathematical Sciences 99, Springer-Verlag, New York, 1993.

[80] J. K. Hale and K. R. Meyer, A class of functional equations of neutral type, *Mem. Amer. Math. Soc.* **76** (1967), 1-65.

[81] S. Heikkila, V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Marcel Dekker, Inc., New York, 1994.

[82] E. Hernández, Existence results for partial neutral functional integrodifferential equations with unbounded delay, *J. Math. Anal. Appl.* **292** (2004), 194-210.

[83] E. Hernández, D. O'Regan, On a new class of abstract impulsive differential equations, *Proc. Amer. Math. Soc.* S 00029939 (2012) 11613-11612.

[84] E. Hernández, A. Prokopczyk and L. Ladeira, A note on partial functional differential equations with state-dependent delay, *Nonlin. Anal.* **7** (2006), 510–519.

[85] R. Hille, R.S. Philips, Functional Analysis and Semi-Groups, Amer. Math. Soc. Provid, RI,1957.

[86] Y. Hino, S. Murakami and T. Naito, *Functional Differential Equations with Infinite Delay*, in: Lecture Notes in Mathematics, vol. 1473, Springer-Verlag, Berlin, 1991.

[87] Sh. Hu and N. Papageorgiou, *Handbook of Multivalued Analysis*, Volume I: Theory, Kluwer Academic Publishers, Dordrecht, 1997.

[88] B.S. Jensen, *The Dynamic Systems of Basic Economic Growth Models*, Kluwer, Dordrecht, 1994.

[89] F. Jiang and Y. Shen, A note on the existence and uniqueness of mild solutions to neutral stochastic partial functional differential equations with non-Lipschitz coefficients. *Comput Math Appl*, **61** (2011), 1590-1594.

[90] M. Kisielewicz, *Differential Inclusions and Optimal Control*, Kluwer, Dordrecht, The Netherlands, 1991.

[91] O. Knill, *Probability and Stochastic Processes with Applications*, Overseas Press, 2009.

[92] A.N.Kolmogorov, Wienersche Spiralen und einige andere interessante Kurven im Hilbertschen Raum. C.R.(Doklady) Acad. URSS (N.S) 26, 115118, 1940.

[93] V. Kolmanovskii, and A. Myshkis, *Introduction to the Theory and Application of Functional-Differential Equations,* Kluwer Academic Publishers, Dordrecht, 1999.

[94] V.B. Kolmanovskii and L.E. Shaikhet, *Control of systems with aftereffect*, in: Translations of Mathematical Monographs, vol. 157, American Mathematical Society, Providence, RI, 1996.

[95] N.N. Krasovskii, On the asymptotic stability of systems with aftereffect. *Prikl MatMeh* **20**(3):513-518 (in Russian). Translated in *J. Appl. Math. Mekh.* **20** (1956).

[96] N.N. Krasovskii, *Some problems of dynamic stability theory.* Fizmatgiz, Moscow ,(1959) (in Russian)

[97] N.V. Krylov, Introduction to the Theory of Random Processes, *the American Mathematical Society. USA.,* **90** (1993) 11698–11702.

[98] N. Kuchkina and L. Shaikhet, Optimal control of Volterra type stochastic difference equations, *Comput. and Math. Appl.* **36** (1998), 251-259.

[99] V.Lakshmikantham, D.D. Bainov and P.S. Simeonov, *Theory of Impulsive Differential Equations. World Scientific*, Singapore 1989.

[100] V. Lakshmikantham, L. Wen and B. Zhang, *Theory of Differential Equations with Unbounded Delay*, Kluwer Acad. Publ., Dordrecht, 1994.

[101] V. Lakshmikantham and D. Trigiante, *Theory of Difference Equations,Numerical Methods and Applications,Mathematics in Science and Engineering*,181 ,Academic Press,Inc.,Boston,MA, 1988.

[102] A. Lasota , Z. Opial, An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations, *Bull. Acad. Pol. Sci. Ser. Sci. Math.Astronom. Phys.* **13** (1965), 781-786.

[103] C. Li, J. Shi and J. Sun, Stability of impulsive stochastic differential delay systems and its application to impulsive stochastic neural networks, *Nonlinear Anal.* **74**, (2011), 3099-3111.

[104] J. Liu, X. Liu and W.-C. Xie, Existence and uniqueness results for impulsive hybrid stochastic delay systems, *Commun. Appl. Nonlinear Anal.* **17** (2010) 37-54.

[105] M. Liu and K. Wang, On a stochastic logistic equation with impulsive perturbations, *Comput. Math. Appl.* **63** (2012), 871-886.

[106] P. Malliavin, The canonic diffusion above the diffeomorphism group of the circle. *C. R. Acad. Sci. Paris Sér. I Math.* **329** (1999), 325-329.

[107] B. B. Mandelbrot and J.W. Van Ness, Fractional Brownian motions, fractional noises and applications. *SIAM Rev.* **10**, 422-437, 1968

[108] J. Memin, Y. Mishura, and E. Valkeila, Inequalities for the moments of Wiener integrals with respecto to fractional Brownian motions. *Statist. Prob. Letters* **55** (2001) 421-430.

[109] P. Millar, Warwick Martingale integrals, *Trans. Amer. Math. Soc.* **133** (1968) 145-166.

[110] V.D. Milman and A.A. Myshkis, On the stability of motion in the presence of impulses, *Sib. Math. J.* (in Russian) **1** (1960) 233–237.

[111] V. D. Milman and A. A. Myshkis, Randorn impulses in linear dynamical systems, in "Approximante Methods for Solving Differential Equations," *Publishing house of the Academy of Sciences of Ukainian SSR, Kiev*, in Russian, (1963) 64-81.

[112] X. Mao, *Stochastic Differential Equations and Applications*, Horwood, Chichester, (1997).

[113] Y. Mishura, *Stochastic Calculus for Fractional Brownian Motion and Related Topics*. Lecture Notes in Mathematics, Vol 1929. Berlin: Springer-Verlag, 2008.

[114] J.D. Murray, *Mathematical Biology*, Springer, Berlin, 1989.

[115] J. Musielak, *Introduction to Functional Analysis*, PWN, Warszawa, 1976 (in Polish).

[116] H. W. Ning and B. Liu, Existence results for impulsive neutral stochastic evolution inclusions in Hilbert Space, *Acta Mathematica Sinica,* **27** (2011), 1405-1418.

[117] I. Norros, E. Valkeila, and J. Virtamo, An elementary approach to a Girsanov formula and other analytical resuls on fractional Brownian motion. *Bernoulli* **5** (1999) 571587.

[118] A. A. Novikov, The moment inequalities for stochastic integrals. (Russian) *Teor. Verojatnost. i Primenen.*, **16** (1971) 548-551.

[119] D. Nualart, Stochastic integration with respect to fractional Brownian motion and applications. Stochastic Models (Mexico City, 2002), Contemp. Math. 336 *Amer. Math. Soc. Providence, RI*, 3-39, 2003.

[120] D. Nualart, *The Malliavin Calculus and Related Topics.* 2nd ed. Berlin: Springer- Verlag, 2006 .

[121] A. Ouahab, Some Pervo's and Krasnosel'skii type fixed point results and application, *Comm. Appl. Nonlinear Anal.*, **19** (2015), 623-642.

[122] B.G.Pachpatte, *Inequalities for finite difference equations.* CRC Press (2001).

[123] L. Pan and J. Cao, Exponential stability of impulsive stochastic functional differential equations, *J. Math. Anal. Appl.* **382** (2011), 672-685.

[124] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations,* Springer-Verlag, New York, 1983.

[125] A.I. Perov, On the Cauchy problem for a system of ordinary differential equations, *Pviblizhen. Met. Reshen. Differ. Uvavn.,* **2**, (1964), 115-134. (in Russian).

[126] I.R. Petre and A. Petruşel, Krasnoselskii's theorem in generalized Banach spaces and applications, *Electron. J. Qual. Theory Differ. Equ.* (2012), 85, 20 pp.

[127] V. Pipiras and M. S. Taqqu, Integration questions related to fractional Brownian motion. *Probab. Theory Rel. Fields* 118 (2000) 121-291.

[128] V. Pipiras and M. S. Taqqu, Are classes of deterministic integrands for fractional Brownian motion on a interval complete *Bernoulli* 7 (2001) 873897

[129] R. Precup, *Methods in Nonlinear Integral Equations*, Kluwer, Dordrecht, 2000.

[130] P. Protter, *Stochastic Integration and Differential Equations*, Springer Verlag, 1990.

[131] B. Przeradzki, On a two-point boundary value problem for differential equations on the half-line, *Ann. Polon. Math.,* **50** (1989), 5361.

[132] C. Prvt, M. Rckner, *A Concise Course on Stochastic Partial Differential Equations.* Lecture Notes in Mathematics, vol. 1905 (Springer, Berlin, 2007)

[133] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, Springer. Third edition 1999.

[134] P. Revathi, R. Sakthivel, D. Y. Song, Y. Ren and P. Zhang, Existence and stability results for second-order stochastic equations driven by fractional Brownian motion, *Transport Theory Statist. Phys.* **42** (2013), 299–317.

[135] I.A. Rus, The theory of a metrical fixed point theorem: theoretical and applicative relevances, *Fixed Point Theory* **9** (2008), 541–559.

[136] A.Russell ,Gordon, *Real Analysis: A First Course* 2nd edition. Boston, Pearson Education Inc. 2002.

[137] R. Sakthivel and J. Luo, Asymptotic stability of nonlinear impulsive stochastic differential equations, *Statist. Probab. Lett.* **79** (2009) 1219-1223.

[138] S. G. Samko, A. A. Kilbas O. L. and Marichev: *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, 1993.

[139] A.M. Samoilenko and N.A. Perestyuk, *Impulsive Differential Equations,* World Scientific, Singapore, 1995.

[140] G. Samorodnitsky, and M. S. Taqqu, *Stable non-Gaussian random processes. Stochastic models with infinite variance.* Stochastic Modeling. Chapman and Hall, New York, 1994.

[141] M.L. Sinacer,J.J. Nieto and A. Ouahab, Random fixed point theorem in generalized Banach space and applications, *Ran. Ope. Sto. Eq.*, to appear.

[142] G. V. Smirnov, *Introduction to the theory of differential inclusions*, Graduate studies in mathematics, vol. 41, American Mathematical Society, Providence, 2002.

[143] H. Sobczyk, *Stochastic Differential Equations with Applications to Physics and Engineering*, Kluwer Academic Publishers, London, 1991.

[144] T. Sottinen, E. Valkeila, On arbitrage and replication in the fractional Black-Scholes pricing model. *Statistics & Decisions*, **21** (2003), 93-107.

[145] V. A. Staikos, *Differential equations with deviating arguments-Oscillation theory.*

[146] T. Taniguchi, Stability theorems of stochastic difference equations, *J. Math. Anal. Appl.* **147** (1) (1990); 8196.

[147] R.Temam, *Navier-Stokes Equations, Theory and Numerical Analysis*, North-Holland, Amsterdam (1984).

[148] S. Tindel, C. Tudor and F. Viens, Stochastic evolution equations with fractional Brownian motion. *Probab Theory Related Fields*,**127** (2003), 186-204.

[149] A. Tolstonogov, *Differential Inclusions in a Banach Space*, Kluwer Academic Publishers, Dordrecht, (2000).

[150] C. C. Travis and G. F. Webb, Existence and stability for partial functional differential equations, *Trans. Amer. Math. Soc.*, **200** (1974), 395–418.

[151] C. C. Travis and G.F. Webb, Second order differential equations in Banach spaces, in: Nonlinear Equations in Abstract Spaces, *Proc. Internat. Sympos. (Univ. Texas, Arlington, TX, 1977), Academic Press, New-York*, 1978, 331–361.

[152] C. P. Tsokos and W.J. Padgett, *Random Integral Equations with Applications to Life Sciences and Engineering*, Academic Press, New York, 1974.

[153] A. Viorel, *Contributions to the study of nonlinear evolution equations*, Ph.D. thesis, Babeş-Bolyai University Cluj-Napoca Department of Mathematics, 2011.

[154] I. I. Vrabie, C_0-*Semigroups and Applications*, Elsevier, 2003.

[155] D. Wagner, Survey of measurable selection theorems, *SIAM J. Control Optim.* 15 (1977), 859–903.

[156] F.Wang, X.Wang,S. Zhang , C Ding. On pulse vaccine strategy in a periodic stochastic SIR epidemic model. *Chaos, Solitons Fractals*. **66**, 127-135,(2014).

[157] S.J. Wu, X.L. Guo and S. Q. Lin, Existence and uniqueness of solutions to random impulsive differential systems, *Acta Mathematicae Applicatae Sinica*, **22** (2006), 595-600.

[158] J. Wu, *Theory and Application of Partial Functional Differential Equations,* Springer-Verlag, New York, 1996.

[159] D.Y. Xu, Z. Ma and Q. Guo, Difference inequality and attraction domain of discrete-time systems, *Ann. of Math.* **20** (A(1)) (1999), 21-26 (in Chinese).

[160] A.M. Yaglom, Correlation theory of processes with random stationary nth increments. *AMS Transl.* **2**, 8, (1958), 87-141.

[161] Z. Yan and F. Lu, Existence results for a new class of fractional impulsive partial interro-differential equation with infinity delay. *J. Applied Analysis and Computation.* **5**, 3, (2015), 329-346.

[162] Z. Yang and D. Xu, Mean square exponential stability of impulsive difference equations, *Appl. Math. Lett.*, **20** (2007), 938-945.

[163] T. Yoshizawa, *Stability Theorey by Liapunovs Second Method*, The Mathematical Society of Japan, Tokyo, 1966.

[164] Q. Zhu and B. Song, Exponential stability for impulsive nonlinear stochastic differential equations with mixed delays, *Nonlinear Anal. RWA* **12** (2011), 2851-2860.

[165] B. Øksendal, *Stochastic Differential Equations:An Introduction with Applications* (Fourth Edition) Springer-Verlag, Berlin, 1995.

Index of Symbols

The most frequently used notations, symbols, and abbreviations are listed below.

$\|A\|$ norm of the mapping to A,
$A^{'}$ derived set of A,
A^{-1} inverse function of the mapping A,
$\sqrt{A}$ square root of the mapping A,
$\langle x, y \rangle$ inner product of x and y,
$\mathbb{R}$ reel number
$C([a, b])$ continuous functions on $[a, b]$ $\mathbb{R}^n$ space of n-tuples of reel numbers,
$f(A)$ image of set A under function J,
$d(x, y)$ distance (metric) from x to y,
$d(x, A)$ distance from the point x to the set A,
$d(A, C)$ distance between set B and set C,
$\overline{B(x_0, r)}$ the closed ball centered in x_0 with radius r,
$B(x_0, r)$ be the open ball centrad in x_0 with radius r ,
∂B_n boundary of B_n unit sphere in $\mathbb{R}^n$
$\alpha, \beta, \gamma, \delta$ members of the underlying scalar field,
$L^0_Q(Y, X)$ be the space of all Q-Hilbert-Schmidt operators from Y into X,
$y(t_k^-)$ and $y(t_k^+)$ be the left and right limits of $y(t)$ at $t = t_k$,
l^2 square-summable sequences,
$AC^i(J; X)$ be the space of functions $y : J \to X$,$i-$ differentiable in whose $i-$th derivative, $y^{(i)}$, is absolutely continuous
$F_X(x)$ be the probability distribution function $\mathbb{R}$ to $[0, 1]$ of a random variable X
A^+,A^- functions associated with A
$A \geq 0$ mapping A greater than or equal to 0,
$\mathcal{O}$ open sets of a topological space,
$\mathcal{F}$ be the collection of its subsets

$\mathbb{P}$ is probability measure on $(\Omega, \mathcal{F})$

$\mathbb{E}$ be the expectation mathematica on $(\Omega, \mathcal{F})$

$(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a complete probability space

$W(t)$ be a scalar Brownian motion defined on the probability space

B_l^H be an infinite sequence of mutually independent fractional Brownian motions, $l = 1, 2, \ldots$ with Hurst parameter H,

$Q_\Delta(M, M)$ be the quadratic variation process of M

$\mathcal{N}(0, t)$ be the Gaussian distribution with mean zero and variance t

$p_U(x)$ be the Minkowski functional of U,

$\mathcal{P}_{cl}(X)$ be the a multivalued map is convex

$\mathbb{Z}$ integers,

$\mathbb{Z}^+$ positive integers,

$Cov(X, Y)$ covariance between X and Y

$Var(X)$ variance of random variable X

$s \wedge t$ the minimum of s and t $(= \min(s, t))$

$s \vee t$ the maximum of s and t $(= \max(s; t))$

$\mathcal{F}_\infty$ the -algebra generated by $\cup_{t>0} \mathcal{F}_t$

I_n the $n \times n$ identity matrix

$B^{(H)}$ Fractional Brownian motion

$p_U(x)$ the Minkowski functional of U.

$\mathcal{P}_{cl}(X)$ multivalued map a closed

$\mathcal{P}_b(X)$ multivalued map a bounded

$\mathcal{P}_c(X)$ multivalued map a convex

$\mathcal{P}_{cp}(X)$ multivalued map a compact

$:=$ equal to by definition

$I_{a^+}^\alpha f(x)$ The left fractional integrals of f of order α are defined for almost all $x \in (a, b)$

$I_{b^-}^\alpha f(x)$ The right-sided fractional integrals of f of order α are defined for almost all $x \in (a, b)$

$C_0^k(U)$ the functions in $C^k(U)$ with compact support in U

$\mathcal{L}(X)$ be the set of all linear bounded operator from X

$C(J, E)$ be the Banach space of all continuous functions from J into E

$L^1(J, E)$ be the Banach space of all continuous functions from J into E

$\mathcal{L} \otimes \mathcal{B}$ product-measurable

$S_F(y)$ selection set of the multi-valued map F

$\overline{co}(A) = \overline{conv}(A)$ closure of the convex hull of the set A

$dimX$ dimension of the space X

$clA = \overline{A}$ closure of the set A.

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