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Présentée par :

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Spécialité : *Mathématiques*
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Intitulée

Etude des systèmes de files d'attente avec vacance du serveur et clients impatients

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Dedication

I dedicate my dissertation work to the soul of my dear father who was impatiently awaiting this achievement but unfortunately he is no longer there to see it. A special feeling of gratitude to my husband Ismail BENAMAR, for his support and encouragement to bring this work to an end. Without forgetting my beloved daughters Malek and Djamila.

Acknowledgments

At first, I thank the Almighty "ALLAH" who gave me the strength and the patience to establish this work.
I'd like to express my thanks to my patient and supportive supervisor, Prof. Amina Angelika BOUCHENTOUF. Her continued support, listening, guidance, and overall insights in this field have made this an inspiring experience for me.
Furthermore, I'd like to express my thanks to the members of my committee, Prof. Boubaker MECHAB, Prof. Toufik GUENDOUZI, and Dr. Hakim OUADJED for their time spent when evaluating this thesis.
I'd also like to thank my husband for all the unconditional support in the past four years of study and anyone who has helped me from near or far.

ملخص

نقوم في هذه الأطروحة بدراسة مختلف أنظمة طوابير الانتظار مع مختلف أنواع الإجازات ونفاذ صبر الزبائن. أولا، تحصلنا على احتمالات في حالة الاستقرار لنظام طابور انتظار أحادي الخادم مع تغذية راجعة تحت نوعين من الإجازات المتعددة المتباينة وعزوف الزبائن باستعمال الطريقة التراجعية. ثانيا، اعتبرنا نظام طابور ماركوف للانتظار أحادي الخادم مع تغذية بارنولي الراجعة، العزوف، تنصل الزبائن المتعلق بحالات الخوادم والاحتفاظ بالزبائن المتنصلين بموجب تعدد الإجازات المتتالية. ثالثا، تعاملنا مع نظام طابور انتظار متعدد الخوادم مع تعدد الإجازات المتتالية، تغذية بارنولي الراجعة، العزوف وتنصل الزبائن المتعلق بحالات الخوادم.

تم إنشاء حلول المراوحة بالنسبة للنظامين الثاني والثالث باستخدام وظائف توليد الاحتمال. كما تم اشتقاق مختلف الخصائص لأنظمة طوابير الانتظار المقترحة في هذه الأطروحة. تم أيضا تقديم تحليل الربح والتكلفة بالنسبة للنظام الأول. إضافة إلى ذلك، تم عرض بعض النتائج العددية من أجل توضيح تأثير بعض معاملات النظام على مقاييس أداء مختلف الأنظمة المدروسة في هذه الأطروحة.

كلمات مفتاحية

نماذج طوابير الانتظار، الإجازات، نفاذ الصبر، تغذية بارنولي الراجعة.

Abstract

In this thesis, we study various queueing systems with different types of vacation and customers' impatience. Firstly, we obtain the steady-state probabilities for a single server feedback queueing system under two differentiated multiple vacations and balked customers, using the recursive technique. Secondly, we consider a single server Markovian queueing system with Bernoulli feedback, balking, server's states-dependent reneging, and retention of reneged customers, under variant of multiple vacation policy. Thirdly, we deal with a multi-server queue with variant multiple vacations, Bernoulli feedback, balking and server's state-dependent reneging.
The stationary solutions for both second and third models are established via the probability generating functions (PGFs). Different characteristics of the queueing systems suggested in this thesis are derived. A cost-profit analysis for the first system is presented . In addition, some numerical results are presented in order to show the impact of some system parameters on the performance measures of the systems.

Keywords:
Queueing models, vacation, impatience, Bernoulli feedback.

Résumé

Dans cette thèse, nous étudions différents systèmes de files d'attente avec différents types de vacances et impatience des clients. Dans un Premier lieu, nous obtenons les probabilités d'état stable pour un système de files d'attente à serveur unique, feedback, deux types de vacances différentiées, et dérobade, en utilisant la récursivité. En second lieu, nous considérons un système de files d'attente Markovien à serveur unique avec K-vacances consécutives, Bernoulli feedback, dérobade, abandon dépendant de l'état du serveur, et rétention des clients abandonnés. En troisième lieu, nous traitons une file d'attente avec plusieurs serveurs, K-vacances consécutives, Bernoulli feedback, dérobade, et abandon des clients dépendant des états du serveur.
Les solutions stationnaires pour les deuxième et troisième modèles sont établies via les fonctions génératrices de probabilités (PGFs). Différentes caractéristiques des systèmes de files d'attente suggérées dans cette thèse sont dérivées. Une analyse coût-bénéfice du premier système est présentée. De plus, certains résultats numériques sont présentés afin de montrer l'impact de certains paramètres du système sur les mesures de performance des systèmes.

Mots clés:
Modèle de files d'attente, vacances du serveur, impatience, Bernoulli feedback.

List of works

During the preparation of this PhD thesis, a number of research works have been carried out.

List of research works

1 Bouchentouf, A. A. and Mejhahri, L. (2019). Performance and economic evaluation of differentiated multiple vacation queueing system with feedback and balked customers, *Appl. Appl. Math.*, **14**(1), 46–62.

2 Bouchentouf, A. A., Boualem, M., Cherfaoui, M. and Medjahri, L. (2021). Variant vacation queueing system with Bernoulli feedback, balking and server's states-dependent reneging, *Yugoslav Journal of Operations Research*, https://doi.org/10.2298/YJOR200418003B.

3 Bouchentouf, A. A., Medjahri, L., Boualem, M. and Amit K. (2021). Mathematical analysis of a Markovian multi-server feedback queue with a variant of multiple vacations, balking and reneging , *submitted*.

Contents

Chapter 1

Introduction and presentation

Queueing theory is a branch of mathematics, generally concerned with the mathematical modeling and the analysis of systems that offer a service to random requests. The utmost goal of queueing systems analysis is to comprehend the behavior of their underlying processes so that perspicacious decisions can be taken in their management. Then, the mathematical analysis of the models would give formulas that probably present the physical and stochastic parameters to some performance measures, such as server utilization, throughput, mean waiting time, the mean number of customers in the queue and/or in the system, busy and vacation periods of server, and so on. The art of applied queueing theory is to build a model, simple enough, yielding to mathematical analysis, yet contains enough detail so that its performance measures reflect the real system behavior. Queueing models are greatly employed in various fields such as IT, industrial engineering, emergency services, telecommunications, finance, military logistics, and several other areas that involve service systems whose demands are random.

A vacation in a queueing context represents a period when the server is absent or unavailable to offer a service. The situations that lead to a vacation are diverse, namely, system failures, system maintenance, or only for a break. Over the past decades, vacation queueing models have been extensively studied, either to solve particular problems in many practical situations, such as call centers, computers, growing industries, web services, etc.

Impatience (balking and/or reneging) is very important characteristic of queueing theory. Vacation queueing models with customers' impatience are considered to be very suitable tools for analyzing diverse complex service systems and important industries. In the traditional literature on vacation queues with impatient customers, studies of customer behavior have always been based on the assumption that customer impatience only occurs while the server is on vacation. This is the case where customers can see the status of the server. However, in many real-life situations, including call centers and production systems, it may not be possible to get information about the status of the server. In addition, a long wait in the queue is another factor that leads to customer impatience regardless of the state of the system (active or on vacation).

The objective of this thesis is to study different vacation queueing systems with impatient customers (balking and/or reneging) and feedback. In real situations, the absence of the server causing a long wait in the queue is a factor that leads to the loss of the customers resulting in significant loss for businesses. For this, companies give great importance and put a lot of effort into developing different strategies to keep the reneged customers.

The remainder of this chapter is organized as follows: In Section 1.1, we present a brief introduction of queueing theory, presenting some historical facts, applications, characteristics, and basic models with a single and multiple servers. In Sections 1.2 and 1.3, we provide literature reviews on queueing models with impatience and vacation queues, respectively. Then, in Section 1.4, we give a fairly large set of results on vacation queues and impatient customers in queue models. Section 1.5 is devoted to the presentation of a review of the literature on feedback queues. Finally, we present the contribution and the main lines of the thesis in sections 1.6 and 1.7, respectively.

1.1 Introduction to queueing theory

1.1.1 Some historical facts

Queuing theory is a mathematical theory in the domain of probabilities, which studies the optimal solutions for queue management, or tails. A queue is necessary and will be created of itself if it is not anticipated, in all the cases where the supply is lower than the demand, even temporarily. It can be applied to different situations: aircraft management on takeoff or landing, waiting for customers and users at the counters, or even storage of computer programs before processing. This field of research, born in 1917, from the work of the Danish engineer, statistician and mathematician: Agner Krarup Erlang on the management of telephone networks in Copenhagen between 1909 and 1920, studies in particular the arrival systems in a queue, the different priorities of each newcomer, and as statistical modeling of execution times. It is thanks to the contributions of mathematicians Khintchine, Palm, Kendall, Pollaczek and Kolmogorov that the theory has really developed, and is still the subject of many scientific publications. Kendall laid the foundations of the calculation of queueing systems in his work published in 1951 about embedded Markov chains. He also defined a naming convention for queueing systems which is still in use. Almost at the same time, Lindley developed an equation allowing for results of a queueing system under fairly general input and service conditions. A bit later (in 1957), Jackson started the investigation of networked queues thus leading to so called queueing network models. Then, with the advent of computers and computer networks, queueing systems and queue-

ing networks have been identified as a powerful analysis and design tool for various applications. Mathematical foundations for the analysis of queueing networks are due to Whittle (1967, 1968) and Kingman (1969), who treated them in the terminology of population processes. Complex queueing network problems have been investigated extensively since the beginning of the 1970's. Several survey papers and books summarize the major contribution made in this area. These include Guarguaglini et al. (1973), Kelly (1979), Whitt (1983a, b), Harrrison and Nguyen (1990, 1993) and Dai (1998). More investigations are by Bhat and Basawa (2002) who use queue length as well as waiting time data in estimating parameters in queueing systems. The year 2009 saw the publication of Optimal Design of Queueing Systems by Shaler Stidham where he surveyed subsequent research, including the use of tolls or prices in networks of queues, as well as applications to flow control in communication networks. Other researchers have used the matrix analytic method included; Alfa (2002), Lothar and Dieter (2005) and Madan (2011). An analysis of vehicular wireless channel communication via queueing theory model have investigated by Fowler et al. (2014). Invariably, queuing theory in our contemporary world has a very wide range of application as it is used in various operations in the computer systems to evaluate the level of performance so as to be able to stay competitive in the business of communication service as it grows at a very fast pace (Ammar (2016)). The impact of queuing theory as applied to solving a traffic junction congestion problem suggested the use of a mathematical models with respect to the queue is discussed by Kumaran et al. (2019). More realistic models have considered by He and Hu (2017), Ayyappan and Nirmala (2018) and Srinivas and Marathe (2020).
During the last fifty years, a lot of important books on queueing theory and its applications have been published including Kleinrock (1975, 1976, 1979), Daigle (2005), Gyorfi (2007), Anisimov (2008), Haghighi (2008) and, Mark (2010) to mention a few.

1.1.2 Applications

The queuing theory is used in diverse areas, certain significant concepts and applications are as follows:

- Bank ATMs: at an automated teller machine (ATM), the bank's customers arrive randomly and the time they take to complete a transaction is also random. Via a queueing models, different system characteristics are calculated including the utilization rate, the average waiting time in the queue, the average number of customers in the system, in the queue, and so on.

- Hospitals: one of the principal issues in hospitals is long queues because of low capacity and large patient rate. Queueing theory provide important performance metrics of servers and queues to reduce waiting times and queues. By measuring mean waiting times, waiting times and delays can be reduced in a number of ways, such as reducing service times or adding more capacity to servers in order to treat more patients. It is worth pointing out that controlling crowds is not easy. The complexity of service and the capacity increase stress levels for physicians, staff and patients. Patients have to wait in long queues. The structure as well as the behavior of the queues are different in each hospital department, and in different situations, customers can be sent from one department to another. Sometimes, the number of transfers between departments is very high and this can be modeled by the queueing theory.

- Traffic system: queueing theory is a powerful mathematical technique for solving diverse traffic problems of any system. For instance, vehicle traffic could be minimized by the use of queueing theory to reduce delays on the roads. The role of transport in human life cannot be overstated. This will determine the best times for red, amber and green lights to be on or off to reduce traffic congestion on the roads.

- Banking: nowadays, banks are one of the most important units of the public. Queueing theory of is often applied to determine an optimal number of servers. The impact of the queueing on the time spent by customers accessing banking services has increasingly been a matter of serious concern. It is very beneficial to avoid staying in a queue for a long time or on the wrong line. Queueing allows to generate a sequence of arrival time of customers and to choose between different services.

- Computer systems and communication networks: queueing theory has been greatly employed to analyze computer systems and communication networks. Simplified or analytical queueing models may offer the most cost-effective technique for computer system performance modeling and can be utilized to determine reasonable performance measures and reliability of the systems. As an example, we can cite the high speed operations that are frequently subject to bad performance because of a single bottleneck device such as CPU, communication ports, disk drive, graphics card, or bus system. So, using analytical models, the bottleneck device can be detected and upgraded.

1.1.3 Characteristics of queueing systems

A queuing system is characterised by the following five basic characteristics:

- The arrival process: it expresses the mode of arrival customers at the service facility governed by some probability law. The number of customers emanate from finite or infinite sources. Very often, the exponential distribution is assumed resulting in the arrival pattern to be measured as the average number of arrivals per unit of time.
  It is also necessary to know the reaction of a customer upon entering the system. A customer may decide to wait no matter how long the queue becomes, or if the queue is too long to suit him, may decide not to enter it. If a customer decides not

to enter the queue because of its huge length, he is said to have balked. On the other hand, a customer may enter the queue, but after some time loses patience and decides to leave. In this case he is said to have reneged. If a customer is not satisfied by the service, he can join the queue after he has left, in which case he is said to have made a feedback. In the case when there are two or more parallel queues, the customer may move from one queue to another for his personal economic gains, that is jockey for position.

- The service process: this means the arrangement of server-s facility to serve the customers. Again, exponentially is often assumed in practice due to intractabilities when releasing these assumptions. In opposite to the arrival process, the service process is highly dependent on the state of the system. In case, the queueing system is empty, the service facility is idle.

- The queueing discipline: it is a rule according to which customers are selected for service when a queue has been formed. The most common discipline is the "first come, first served" (FCFS), or "first in, first out" (FIFO) rule under which the customers are serviced in the strict order of their arrivals. Other queue disciplines include: "last in, first out" (LIFO) rule according to which the last arrival in the system is serviced first, "selection for service in random order" (SIRO) rule according to which the arrivals as serviced randomly irrespective of their arrivals in the system; and a variety of priority schemes-according to which a customer's service is done in preference over some other customer's service.

- The Capacity of the System: some of the queueing processes admit the physical limitation to the amount of waiting room, so that when the waiting line reaches a certain length, no further customers are allowed to enter until space becomes available by a service completion. Such types of situation are referred to as finite source queues, that is, there is a finite limit to the maximum queue size. The queue can also be viewed as one with forced balking.

- The number of servers: refers to the number of parallel channels, which can service customers simultaneously. A queueing system is called a one-server model when the system has one server only, and a multiple-server model when the system has a number of parallel channels each with one server.

1.1.4 Kendall notation

Different models in queueing theory are classified by using special (or standard) notations described initially by Kendall in 1953 in the form $(a/b/c)$. Later, Lee in 1966 added the symbols d and e to the Kendall notation. Now, in the literature of queuing theory, the standard format used to describe the main characteristics of parallel queues is as follows: $(a/b/c/d/e/f)$ where

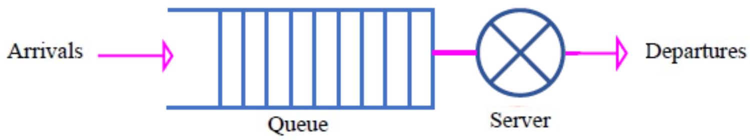

Figure 1.1: Basic queueing model

- a: Distribution of interarrival times of customers.
- b: Distribution of service times.
- c: Number of servers.
- d: Restriction on system capacity.
- e: Population size.
- f: Queue discipline.

1.1.5 Some examples of Markovian queues

1.1.5.1 $M/M/1$ model

The $M/M/1$ queue system is the most basic system of queueing theory. The flow of arrivals accord to a Poisson process (with parameter λ) and service time is exponential of parameter μ (with a single-server), the discipline queue is FIFO and the queue has infinite capacity. The density functions for the interarrival and service times are given respectively as

$$a(t) = \lambda e^{-\lambda t},$$
$$b(t) = \mu e^{-\mu t},$$

where $1/\lambda$ is the mean interarrival time and $1/\mu$ is the mean service time. Both the interarrival and service times are exponential, and the arrival and conditional service rates are Poisson:

$$P\{\text{an arrival occurs in an interval of length } \Delta t\} = \lambda \Delta t + o(\Delta t),$$

$$P\{\text{more than one arrival occurs in } \Delta t\} = o(\Delta t),$$

$$P\{\text{a service completion in } \Delta t \text{ given the system is not empty}\} = \mu \Delta t + o(\Delta t),$$

$$P\{\text{more than one service completion in } \Delta t \text{ given more than one in the system}\} = o(\Delta t).$$

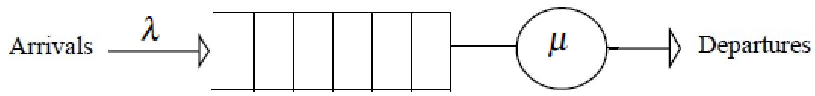

Figure 1.2: The $M/M/1$ queueing model

The $M/M/1$ queueing model is a simple birth-death process with $\lambda_n = \lambda$ and $\mu_n = \mu$, for all n. Arrivals are "births" to the system, if the system is in state n (the state refers to the number of customers in the system), an arrival increases it to state $n+1$. Similarly, departures are "deaths", moving from state n to state $n-1$.
Hence, the steady state equations are found to be

$$(\lambda+\mu)p_n = \mu p_{n+1} + \lambda p_{n-1},$$

$$\lambda p_0 = \mu p_1.$$

By recurrence, it follows that

$$p_n = p_0 \left(\frac{\lambda}{\mu}\right)^n, \ \forall n.$$

To find p_0, we have to use the normalisation condition: $\sum_{n=1}^{\infty} p_n = 1$. Then, we get

$$1 = \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n p_0.$$

We define $\rho = \frac{\lambda}{\mu}$ as the traffic intensity for single-server queues. Then, we find

$$p_0 = \frac{1}{\sum_{n=0}^{\infty} \rho^n}.$$

Now, $\sum_{n=0}^{\infty} \rho^n$ is an infinite geometric series which converges if and only if $\rho < 1$.
Thus, the steady-state solution exist if and only if $\rho < 1$.
Since we know the sum of the terms of a convergent geometric series,

$$\sum_{n=0}^{\infty} \rho^n = \frac{1}{1-\rho}, \quad (\rho < 1),$$

we find

$$p_0 = 1 - \rho.$$

Thus, the full steady-state solution for the $M/M/1$ queueing system is the geometric probability function

$$p_n = (1-\rho)\rho^n, \quad (\rho = \frac{\lambda}{\mu} < 1).$$

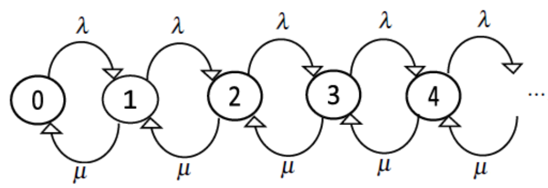

Figure 1.3: Rate diagram for the $M/M/1$ model

Some performance measures are given as:

- The average number of customers in the system:

$$L_s = \sum_{n=0}^{\infty} n p_n = \frac{\rho}{1-\rho}.$$

- The average number of customers in the queue:

$$L_q = \sum_{n=1}^{\infty} (n-1) p_n = \frac{\rho^2}{1-\rho}.$$

- The mean waiting time in the system:

$$W_s = \frac{L_s}{\lambda}.$$

- The mean waiting time in the queue:

$$W_q = \frac{L_q}{\lambda}.$$

1.1.5.2 $M/M/c$ model

The $M/M/c$ queue is a multi-server queueing model. It describes a system where arrivals form a single queue and are governed by a Poisson process (with rate λ), there are c servers operating independently of each other and job service times are exponentially distributed (with rate μ), the discipline queue is FIFO and the queue has infinite capacity. It is a generalisation of the $M/M/1$ queue which considers only a

single server.
$X(t)$, $t \geq 0$ is a birth-death process whose transition rates are

$$\lambda_n = \lambda, \forall n \geq 0,$$

$$\mu_n = \begin{cases} n\mu, & \text{if } 1 \leq n \leq c; \\ c\mu, & \text{if } n \geq c. \end{cases}$$

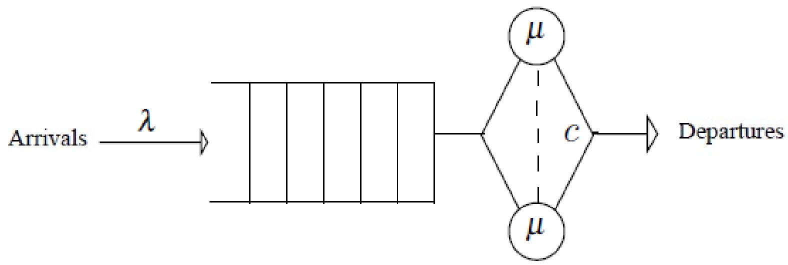

Figure 1.4: The $M/M/c$ model

Remark 1.1.1. *$c\mu$ is the overall service rate of the system.*
$\rho = \frac{\lambda}{c\mu}$ is the intensity of the global traffic.

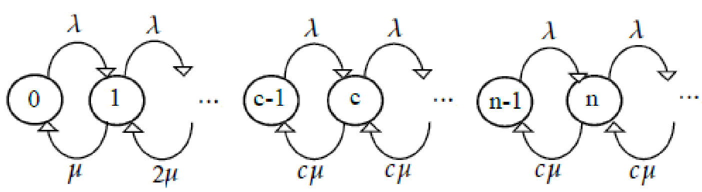

Figure 1.5: Rate diagram for the $M/M/c$ model

The steady-state solution are:

$$p_n = \begin{cases} \frac{(\frac{\lambda}{\mu})^n}{n!} p_0, & \text{if } n \leq c; \\ \frac{(\frac{\lambda}{\mu})^n}{c^{n-c} c!} p_0, & \text{if } n \geq c. \end{cases}$$

By considering $\sum_{n=0}^{\infty} p_n = 1$ and if $\rho = \frac{\lambda}{c\mu} < 1$, we have:

$$p_0 = \left\{ \sum_{n=0}^{c} \frac{(\frac{\lambda}{\mu})^n}{n!} + \frac{(\frac{\lambda}{\mu})^{c+1}}{(c - \frac{\lambda}{\mu})c!} \right\}^{-1}.$$

Further, let $P\{a\,customer\,who\,has\,to\,wait\} = P(X \geq c) = \sum_{n=c}^{\infty} \frac{p_c}{1-\rho}$, with $\rho = \frac{\lambda}{c\mu} < 1$ and $p_c = \frac{(\frac{\lambda}{\mu})^c}{c!} p_0$.

The main performance measures of this system are as follows:

- The average number of customers in the queue:

$$L_q = \sum_{n=c+1}^{\infty} (n-c)p_n = \frac{\rho}{(1-\rho)^2} p_c.$$

- The average number of customers in the system:

$$L_s = L_q + \frac{\lambda}{\mu}.$$

- The mean waiting time in the system:

$$W_s = \frac{p_c}{c\mu(1-\rho)^2} + \frac{1}{\mu}.$$

- The mean waiting time in the queue:

$$W_q = \frac{p_c}{c\mu(1-\rho)^2}.$$

1.2 Queueing models with impatience

To model an impatient queueing system, a constraint must be added to the system by specifying that customers are lost if the time they spend in the system is greater than a time allotted to them.

In many real-life situations, a customer can be impatient and leave the queue before being served, that is the objective of studying systems with impatience. Balking and reneging are most popular impatient phenomena. In addition, There exist other phenomena due to impatience, such as retial (a customer moves to some virtual place called orbit and try to get access to service after random intervals of time) and jockeying (a customer enters a line and then moves to another in order to reduce waiting time). In this chapter, we present some impatience behaviours which have been the basis of diverse research results in the literature while focusing on those presented in this thesis.

- Balking: a customer can make the decision to join or not the queue depending on the waiting time before it is going to be served. In addition, the customer can make his decision according to the length of the queue once being informed about the waiting time.

- Reneging: it occurs when a customer, who has entered in the queue, chooses to leave the system prior to receiving service. In fact, this choice is made if the customer considers that its maximum waiting time has been reached without being served.

- Retrial: the feature of retrial phenomena is that a customer who may balk upon arrival in the system or renege from it may join the virtual group of blocked customers, called orbit and retry for service after random amount of time. The queueing system with retrial phenomena is called retrial queue. The balked customer or reneged will join the orbit with a probability depending on the number of customer in service. Retrial queues are used in several areas such as online shopping, computer system, communication system,...etc.

- Jockeying: customers switch between queues in a tandem queue system, trying to orchestrate the shortest wait possible.

1.2.1 Literature review

The notion of customer impatience appeared in the queueing theory in the work of Haight in 1957. He considered a balking for an $M/M/1$ queue. Then, Haight studied a queue with reneging in which he presented the problem like how to make rational decision while waiting in the queue, the probable effect of this decision, etc. Ancker and Gafarian (1963a) examined a queuing system with balking and reneging and performed its steady state analysis. After that, Ancker and Gafarian (1963b) considered a queuing system with balking and reneging with a radical change in the balking behavior. In the same period, Finch (1959), Haight (1960), Singh (1970) and Yechiali (1971) studied queueing systems with the balking feature only, some problems of reneging feature were interesting for some researchers like Haight (1959), Finch (1960), Daley (1965), and Stanford (1979). Later, Shanthikumar (1988) proved that the queue size decomposition holds even for the $M/G/1$ models with bulk arrival, reneging, balking, etc. Abou-El-Ata (1991) considered an $M/M/1/N$ with reneging and general balk functions. Abou-El-Ata and Shawky (1992) dealt with a single-server Markovian overflow queue with balking, reneging and an additional server for longer queues. They derived an analytical explicit solution of the moments of a system of an overflow queue with balking, reneging and an additional server for longer queues. After that, Abou-El-Ata and Hariri (1992) studied the $M/M/c/N$ queueing system with balking and reneging concepts. In 2002, Wang and Chang focused on the cost analysis of a finite $M/M/R$ queue with balking, reneging and server breakdowns. Ward and Glynn (2005) presented a 'diffusion Approximation' for a queue with balking or reneging in a single-server queue with a renewal arrival process amount of time. Then, Lozano and Moreno (2008) dealt with an $Geo/Geo/1$ with finite and infinite capacity queueing model with two variants balking scenarios (constant and discouraged rate policies). Nasrallah (2009) analyzed an infinite space Markovian single server queue with preemptive priority and Poisson reneging. Further, customers' impatience in various Markovian queueing systems ($M/M/1$, $M/M/c$, and $M/M/\infty$ queues) in fast and slow phases Markovian random environment have been considered in Perel and Yechiali (2010). The transient solution of an infinite space single server Markovian queue with discouraged arrivals and reneging has been done in Ammar et al. (2012). Later, Goswami (2014) treated the multi-server Markovain queue with balking. The $M/M/1/K$ and the $M/M/1/K/L$ queueing models with fast and slow random environments of the service and customer's impatience have been studied in Singh et al. (2016). Recently, Danilyuk et al. (2019) considered the $M/GI/1$ retrial queueing model with collisions and impatient calls. Ammar et al. (2020) considered a single server queueing model with system disasters and impatient customers. You et al. (2020) proposed a pedestrian evacuation model for the crowd behavior of impatient customers. Sudhesh and Azhagappan (2020) analyzed the impatient behaviour in an infinite server queueing model with additional tasks assigned to the system. For an overview on queueing systems with balking and reneging, we can refer to Kapodistria (2011), Manoharan and Jose (2011), Choudhury and Medhi (2011), Ammar et al. (2013), Wang et al. (2018), and Diamant and Baron (2019).

Although there exists a considerable literature dealing queueing models with customers' impatience, but research papers on customers retention is limited. The pioneer papers on the retention have been done by Kumar and Sharma (2012a, 2012b, 2013). Then, Kumar (2013) extended the work given in Kumar and Sharma (2013) by incorporating the balking behaviour. After that, this idea has been used in different research papers dealing with customers' impatience in queueing models, see for instance, Bouchentouf et al. (2014), Madheswari et al. (2016), Kumar (2016), Yang and Wu (2017), Som and Seth (2017), Laxmi and Kassahun (2017), Kumar and Sharma (2018a), and Kumar and Sharma (2018b).

1.3 Vacation queues

In a vacation queueing system, a server may become unavailable for a random period of time due to many factors. In some cases the vacation can be because of a server breakdown, which means that the system must be repaired and brought back to service. It can also be a deliberate action taken to utilize the server in a secondary service center when there are no customers present at the primary service center. Thus, server vacations are useful for those systems in which the server wishes to utilize his idle time for different purposes, and this makes the queueing model very applicable to a variety of real world stochastic service systems.

1.3.1 Different types of vacation policies

Various kind of vacation policies has been seen in the literature:

- Single vacation queueing system: the server takes a vacation of a random duration when the queue is empty. At the end of the vacation the server returns to the system and stay there waiting for new arrivals.
- Multiple vacation queueing system: in this policy, the server goes on vacations successively until, at least one customer is present upon his return from the precedent vacation.
- Working vacation queues: the server works at a different rate instead of being completely idle during the vacation period.
- Gated vacation queues: the server will serve only those customers that he finds at the queue upon his return from vacation. At the end of their service, he will commence another vacation and any customers that arrive while the server was already serving at the station will be served when the server returns from the vacation.
- Limited service discipline: the server goes on vacation after serving a predefined maximum number of customers, or after a predefined time t or if he is idle. The single-service scheme in which exactly one customer is served is a special type of this policy.
- Exhaustive service discipline: the server will serve all waiting customers as well as those that arrive while he is still serving at the station. He takes another vacation when the queue becomes empty.
- Differentiated vacation queues: the server is subject to two types of vacation, vacation after the busy period and vacation taken immediately after the server has just returned from a previous vacation to find that there are no waiting customers.
- K-variant vacation queues: the server joins his post for a busy period immediately at the end of a vacation, if he finds customers in the system. Otherwise, it is allowed to take K-vacations consecutively. When the successive K-vacations end, the server must return to the busy period and wait for new customers.

1.3.2 Literature review

In the theory of queues, the situation where a server is unavailable for primary customers in occasional intervals of time is known as vacation. Vacation queueing theory was developed as an extension of the classical queueing theory. In a queueing system with vacations, other than serving randomly arriving customers, the server is allowed to take vacations. Servers' vacations may be due to lack of work, servers' failure, or some other tasks being assigned to the servers which occur in different applications like computer maintenance and testing, preventive maintenance job in a production system, priority queue, and so forth. Furthermore, allowing servers to take vacations makes queueing models more realistic and flexible in studying real world situations. Miller (1964) was the first to study the $M/G/1$ queueing system in which the server becomes idle and is unavailable during some random length of time. For about four decades, various researchers and practitioners have extensively studied such models and applied to many engineering systems such as computer network, digital communication, production or manufacturing, airline scheduling, inventory, and other stochastic systems. Intersting researches concerning this type of models, in different contexts, have been carried out by Doshi (1986), Takagi (1991, 1993), Tian and Zhang (2006), Ke et al. (2010), Upadhyaya (2016), Xu and Wang (2019) and Parimala (2020).

1.3.2.1 On single and multiple vacation queues

Queueing systems with single and multiple vacations have attracted the attention of many researchers since the idea was first discussed in the paper of Levy and Yechiali (1975), where they have found server's idle time utilization in the $M/G/1$ queue based on the assumption that as the queue becomes empty, the server takes vacation of exponential distribution during which he does some secondary work. They have given the expressions for the Laplace-Stieltjes transform of the system measures, like occupation period, vacation period, and waiting time. Then, Levy and Yechiali (1976) obtained expressions for the expected system length in an $M/M/S$ queueing model with vacation and have shown that the mean system size is a linear function of the mean vacation time. Van der Duyn Schouten (1978) considered an $M/G/1$ queueing model with vacation. Then, a study on the $M/G/1$ queueing model with server vacations has been presented by Fuhrmann (1984). Baba (1986) considered an $M^X/G/1$ queueing system with vacation, derived the queue length distribution at an arbitrary time. Servi (1986) studied a $D/G/1$ queue with vacations. He analyzed a model for investigating the waiting time at the primary queue, the probability distribution of the vacation duration and the processor's service schedule. Later, Kella and Yechiali (1988) studied an $M/G/1$ queue with single and multiple server vacations under preemptive and non-preemptive regimes. Tian et al. (1989) derived the probability distribution of the queue length at arrival epochs for the $GI/M/1$ queue with exhaustive service and multiple vacations using matrix-geometric method. An $GI/M/1$ queue with multiple vacation policy has been studied by Chatterjee and Mukherjee (1990). Kao and Narayanan (1991) studied an $M/M/N$ queueing model with asynchronous multiple vacation policies, via a matrix-geometric approach, the modelling of the system has been given, the stationary queue length distribution has been computed. An $M/M/s$ queueing model with repeated vacation has been studied by Afthab Begum and Nadarajan (1997). Zhang and Tian (2003) dealt with a $M/M/c$ queue with server vacations where the stationary distributions of queue length and waiting time have been obtained using matrix geometric method. An $M/M/2$ queueing system with heterogeneous servers and multiple vacations has been presented by Kumar and Madheswari (2005). Then, Ke et al. (2010) presented some developments in vacation gueueing models. Later, an $M/G/1$ queue with multiple vacation and balking for a class of disciplines was treated by Saffer and Yue (2015). An unreliable queueing system with single and multiple vacations was studied by Tadj (2017). Recently, Deepa and Azhagappan (2019) established the analysis of batch arrival single and bulk service queue with multiple vacation close down and repair. Padmavathi and Sivakumar (2019) compared three vacation policies in discrete time inventory system with postponed demands. After that, Joshi et al. (2020) studied the steady state analysis of $D/M/1$ and $M/G/1$ models with multiple vacation queueing system.

1.3.2.2 On differentiated and K-variant working vacation queues

Queueing systems with differentiated vacations can be used to model many physical systems. This type of models was introduced by Ibe and Isijola (2014) who studied a $M/M/1$ multiple vacation queueing systems with differentiated vacations. Then, Isijola and Ibe (2014) extended the model to include vacation interruption by forcing the server to return from a vacation. After, Ibe and Isijola (2015) considered an $M/M/1$ queue with differentiated multiple vacation where the vacations depend on service rates. Gupta et al. (2016) studied a multiple vacation queueing systems with deterministic service time. The transient analysis of an $M/M/1$ queueing model with two types of vacation is discussed by Vijayashree and Janani (2018). Recently, the study of multi-server queue differentiated vacations has been addressed in Unni and Rose Mary (2019). Then, Unni and Rose Mary (2020) investigated an $M/M/1$ multiple vacations queueing systems with differentiated vacations under mixed strategy of customers.

Under the variant vacation policy the server leaves for a working vacation as soon as the system becomes empty. The server takes at most K-working vacations until he finds at least one customer in the queue on return from a working vacation. If customers are not found in the queue by the end of Kth vacation, the server remains idle in the system. By doing a search on the literature corresponding to this type of vacation, we come across Zhang and Tian (2001) who considered a $Geo/G/1$ queue with multiple adaptive vacations. Ke et al. (2010) studied the operating characteristics of an $M/G/1$ queueing system with a randomized control policy and at most J-vacations. Wang et al. (2011) presented the analysis of a discrete-time $Geo/G/1$ queue, where the server may take at most J successive vacations. Later, Yue et al. (2014) examined the $M/M/1$ queue with impatient customers and a variant of multiple vacation policy. Then, Laxmi and Rajesh (2016) treated a batch arrival infinite-buffer single server queueing system with variant working vacations. Recently, Laxmi and Rajesh (2018) analyzed the performance measures of variant working vacations on batch arrival queue with server breakdowns. Then, Laxmi (2020) studied the performance characteristics of discrete-time queue with variant working vacations.

1.4 Vacation queues with impatient customers

In many real-life situations, the customer may be impatient because of the server vacation. Indeed, several researchers are interested in studying queueing models by considering these two phenomena. Yue et al. (2006) investigated an $M/M/c/N$ queue with balking, reneging and synchronous vacations of partial servers together, a matrix form is used to obtain the solution of the steady-state probabilities. A single server queue with impatient customers, server vacations, and a waiting server was studied by Padmavathy et al. (2011). After that, Yue et al. (2012) dealt with an $M/M/1$ queue with working vacations and impatient customers. An analysis of finite buffer Markovian queue with balking, reneging and working vacations was established by Laxmi and Jyothsna (2013). Later, Goswami (2015) dealt with a renewal input finite buffer queueing system with impatient customers and multiple working vacations. Goswami and Selvaraju (2016) studied a phase-type arrivals and impatient customers in multiserver queue with multiple working vacations. After that, Laxmi and Rajesh (2017) examined an $M/M/1$ queueing system with a variant of working vacations with balked customers. Recently, several researchers focused on such models. Among them, Bouchentouf and Guendouzi who have published many articles in this field. In 2018, they studied an infinite-buffer single-server queueing system with Bernoulli feedback, multiple vacations, differentiated vacations, vacation interruptions and impatient customers (balking and reneging). In the same year, Bouchentouf et al. presented a performance and economic analysis of Markovian Bernoulli feedback queueing system with vacations, waiting server and impatient customers. Then, Bouchentouf and Guendouzi (2019) considered an $M^X/M/c$ Bernoulli feedback queue with variant multiple working vacations, and impatient customers. Later, a cost optimization analysis for an $M^X/M/c$ vacation queueing system with waiting servers and impatient customers was examined by Bouchentouf and Guendouzi (2019). In the same year, Bouchentouf et al. presented a performance and economic study of heterogeneous $M/M/2/N$ feedback queue with working vacation and impatient customers. Very recently, in 2020, Laxmi and Kassahun examined an infinite capacity multi-server Markovian queue with synchronous multiple working vacations, and impatient customers.

1.5 Queueing models with feedback

The feedback in queueing systems represents customer dissatisfaction after the service. After the completion of service, each customer can return to the system as a feedback customer for receiving his service if the first one is incomplete or if his quality is unsatisfactory. When a customer decides to return to the system, he rejoins the end of the queue. This situation is seen in industrial operations, computer communications, banks, hospital management, and so on. If the customer is not satisfied, he can either join the system as a feedback customer with a certain probability, or leave the system permanently with an a complementary probability known as Bernoulli feedback.

1.5.1 Literature review

Feedback queues play an important role in real life service system, where tasks may require repeated services. The queueing systems which include the possibility of a customer to return to the service because of unsatisfied service or for requirement of additional service are called queues with feedback. The first who had the idea to study a queue with feedback was Takacs (1963) where he tackled a problem in the theory of telephone traffic. He considered a single-server queue with feedback. Then, a $M/G/1$ queue with state-dependent feedback is considered by D'Avignon and Disney (1976). After that, an $M/M/c$ retrial queue with geometric loss and feedback when $c = 1, 2$, has been considered in Choi et al. (1998). Later, an $M/M/1$ feedback queues with impatient customers and Bernoulli feedback were treated in Santhakumaran and Thangaraj (2000). Thangaraj and Vanitha (2009) analysed an $M/M/1$ feedback queue with catastrophes using continued fractions where they obtained the asymptotic behavior of the probability of the server being idle and mean system size and the steady state probability of the system size. Later, Bouchentouf and Belarbi (2013) investigated the performance evaluation of two Markovian retrial queueing model with balking and feedback, authors obtained the joint distribution of the server state and retrial queue lengths. After that, Kumar et al. (2014) presented the optimization of an $M/M/1/N$ feedback queue with retention of reneged customers, they developed a model for the costs incurred. Then, Ayyappan and Shyamala (2016) analysed the transient solution of an $M^{[X]}/G/1$ queueing model with feedback, random breakdowns, Bernoulli schedule server vacation and random setup time. Authors derived the probability generating function in terms of Laplace transforms and the corresponding steady state results explicitly. The single server vacation queue with geometric abandonments and Bernoulli's feedbacks was studied by Vijayalakshmi and Kalidass (2018). Recently, Shanmugasundaram and Vanitha (2020) dealt with a single server Markovian feedback queueing network with shared buffer and multi-queue nodes. For recent research works on the subject, we can refer to Kempa (2019), Sudhesh and Priya (2019), Shekhar et al. (2019), Rajadurai et al. (2020), and Rani and Radhika (2020).

1.6 Summary of results established in this thesis

▸ First Result: Performance and economic evaluation of differentiated multiple vacation queueing system with feedback and balked customers.

In this work, we deal with an $M/M/1$ Bernoulli feedback queueing system under differentiated multiple vacation and balked customers. The inter-arrival times are i.i.d and exponentially distributed with mean $1/\lambda$. The service time supposed to be exponentially distributed with mean $1/\mu$. The queue discipline is First-Come First-Served (FCFS). In this work, we consider two types of vacations: type 1 vacation taken after a busy period, in which a server has served at least one customer, and type 2 vacation taken when the server returns from a vacation and observe that the queue is empty. Suppose that the duration of type 1 vacation is independent of the busy period and is

exponentially distributed with mean $1/\gamma_1$. The duration of type 2 vacation is assumed to be exponentially distributed with mean $1/\gamma_2$.
On arrival, a customer who finds at least one customer in the system, either decides to join the queue with probability θ or balk with probability $\theta' = 1-\theta$. If a customer is unhappy with service, he may comes back to the system with probability β', or leave the system with probability β, where $\beta' + \beta = 1$. Note that $\frac{\lambda}{\beta\mu} < 1$ is the condition of the stability of the system. The inter-arrival times, vacation periods and service times are mutually independent.

Let (n,k) denote the status of the system, where n is the system size, and k represents the state of the sever:

$$k = \begin{cases} 0, & \text{the server is active,} \\ 1, & \text{the server is on type 1 vacation,} \\ 2, & \text{the server is on type 2 vacation.} \end{cases}$$

Let $P_{n,k}(t)$ be the probability that the system is in state (n,k) at time t. Then,

$$P_{n,k} = \lim_{t\to\infty} P_{n,k}(t)$$

is steady-state probability of the system.

– The steady-state-probabilities $P_{n,k}$ are given by:

$$P_{n,k} = \begin{cases} \phi\left\{\frac{\delta_1\chi_1(\chi_1^{n-1}-\phi^{n-1})}{\chi_1-\phi} + \frac{\delta_2\chi_2(\chi_2^{n-1}-\phi^{n-1})}{\chi_2-\phi} + \phi^{n-2}\right\}P_{1,0}, \text{ n=1,2,..., k=0,} \\ \theta\delta_1 P_{1,0}, \text{ n=0, k=1,} \\ \theta\delta_2 P_{1,0}, \text{ n=0, k=2,} \\ \delta_1\chi_1^n P_{1,0}, \text{ k=1,} \\ \delta_2\chi_2^n P_{1,0}, \text{ k=2,} \end{cases}$$

where

$$P_{1,0} = ((1-\chi_1)(1-\chi_2)(1-\phi))\Big\{\delta_1\chi_1(1-\chi_2) + \delta_2\chi_2(1-\chi_1)$$
$$+(1-\chi_1)(1-\chi_2) + \theta(\delta_1+\delta_2)(1-\chi_2)(1-\chi_1)(1-\phi)\Big\}^{-1},$$

with

$$\phi = \frac{\theta\lambda}{\beta\mu},$$

$$\delta_1 = \frac{\beta\mu}{\theta(\lambda+\gamma_1)}, \text{ and } \delta_2 = \frac{\gamma_1}{\theta\lambda}\frac{\beta\mu}{\lambda+\gamma_1} = \frac{\gamma_1}{\lambda}\cdot\delta_1,$$

and

$$\chi_1 = \left(\frac{\theta\lambda}{\theta\lambda+\gamma_1}\right), \text{ and } \chi_2 = \left(\frac{\theta\lambda}{\theta\lambda+\gamma_2}\right).$$

– Useful system characteristics include:
– The average number of customers in the system:

$$L_s = \left\{\delta_1\chi_1\phi\tfrac{2-\chi_1-\phi}{(1-\chi_1)^2(1-\phi)^2} + \delta_2\chi_2\phi\tfrac{2-\chi_2-\phi}{(1-\chi_2)^2(1-\phi)^2} + \tfrac{1}{(1-\phi)^2} + \tfrac{\delta_1\chi_1}{(1-\chi_1)^2} + \tfrac{\delta_2\chi_2}{(1-\chi_2)^2}\right\}P_{1,0}.$$

– The average number of customers in the queue:

$$L_q = \sum_{n=1}^{\infty}(n-1)P_{n,0} + \sum_{n=0}^{\infty}n(P_{n,1}+P_{n,2}).$$

– The average balking rate:

$$\lambda_{balk} = \lambda(1-\theta)\left\{\tfrac{\phi\delta_1\chi_1}{\chi_1-\phi}\left(\tfrac{1}{1-\chi_1}-\tfrac{1}{1-\phi}\right) + \tfrac{\phi\delta_2\chi_2}{\chi_2-\phi}\left(\tfrac{1}{1-\chi_2}-\tfrac{1}{1-\phi}\right) + \tfrac{1}{1-\phi} + \tfrac{\delta_1\chi_1}{1-\chi_1} + \tfrac{\delta_2\chi_2}{1-\chi_2}\right\}P_{1,0}.$$

– The probability that the server is in busy period:

$$P_B = \left\{\tfrac{\phi\delta_1\chi_1}{\chi_1-\phi}\left(\tfrac{1}{1-\chi_1}-\tfrac{1}{1-\phi}\right) + \tfrac{\phi\delta_2\chi_2}{\chi_2-\phi}\left(\tfrac{1}{1-\chi_2}-\tfrac{1}{1-\phi}\right) + \tfrac{1}{1-\phi}\right\}P_{1,0}.$$

Further,

$$P_{V1} = \tfrac{\delta_1(\theta(1-\chi_1)+\chi_1)}{1-\chi_1}P_{1,0},$$

and

$$P_{V2} = \tfrac{\delta_2(\theta(1-\chi_2)+\chi_2)}{1-\chi_2}P_{1,0}.$$

– The probability that the server is in vacation period:

$$P_V = \left\{\tfrac{\delta_1(\theta(1-\chi_1)+\chi_1)}{1-\chi_1} + \tfrac{\delta_2(\theta(1-\chi_2)+\chi_2)}{1-\chi_2}\right\}P_{1,0}.$$

After deriving explicit performance measures, a cost model is developed for considered queueing system. In addition, numerical examples are presented.

► **Second result: Variant vacation queueing system with bernoulli feedback, balking and server's states-dependent reneging.**

We analyze an infinite space single server Markovian queue with Bernoulli feedback under a variant of a multiple vacation policy with balking, reneging and retention. Customers arrive at the system according to a Poisson process of rate λ. During busy period, customers have i.i.d. service time, assumed to be exponential with parameter μ. Whenever the system becomes empty at a service completion instant, the sever goes on a vacation of random length which is assumed to be exponentially distributed with rate ϕ.

If at a vacation completion instant, there are some customers in the queue, they are immediately served. Otherwise, the server take K-vacations consecutively, then comes back to the busy period and remains there waiting for a new arrivals. It is supposed that, on arrival a customer either decides to join the queue with probability θ if the number of customers in the system is bigger or equal to one and may balk with probability $\overline{\theta} = 1-\theta$. Whenever a customer arrives at the system and finds the server on vacation (respectively, busy), he activates an impatience timer T_0 (respectively, T_1), which follow exponential distribution function with parameter ξ_0 (respectively, ξ_1). If the customer has not been serviced before the customer's timer expires, he may abandon the system. Further, each reneged customer may leave the system without getting service with probability α and may be retained in the queue with probability $\alpha' = (1-\alpha)$. Further, with probability $\beta^{'}$, a customer returns to the system as a Bernoulli feedback customer to get another regular service if he is not satisfied with the initial one. Otherwise, he leaves the system definitively with probability $\beta = 1-\beta^{'}$. The inter-arrival times, service times, impatience times, and vacation times are all mutually independent. The service order is supposed to be first come first served.

The steady-state analysis of the system is carried out using a PGF method. Let $N(t)$ be the number of customers in the system at time t, and let $J(t)$ be the state of the server at time t :

$$J(t) = \begin{cases} j, & \text{the server is taking the } (j+1)th \text{ vacation at time } t \text{ for } j = 0,1,...,K-1; \\ K, & \text{the server is idle or busy at time } t. \end{cases}$$

The pair $\{(N(t); J(t)); t \geq 0\}$ defines a Markovian, continuous-time process where its state space is $\Omega = \{(n;j) : n \geq 0; j = \overline{0,K}\}$.

Let $P_{n,j} = \lim\limits_{t\to\infty} \mathbb{P}(N(t) = n; J(t) = j)$, $n \geq 0$; $j = \overline{0,K}$, denote the steady-state probabilities of the process $\{(N(t); J(t)); t \geq 0\}$.

– The steady-state-probabilities $P_{n,j}$ of system size are given by

$$\begin{aligned} P_{.,j} &= A^{j-1} P_{0,0}, \ j = \overline{0,K-1}, \\ P_{.,K} &= \Phi(1) P_{0,0}, \end{aligned}$$

where

$$P_{0,0} = \left(\frac{1-A^K}{A(1-A)} + \Phi(1) \right)^{-1},$$

with

$$A = \frac{\phi C_2(1)}{\alpha \xi_0 B},$$

and

$$\Phi(1)=e^{\frac{\lambda\theta}{\alpha\xi_1}}\left\{\left(\frac{B\xi_0}{C_2(1)\xi_1}+\frac{\phi}{\alpha\xi_1}\left(\frac{1-A^{K-1}}{1-A}\right)\right)H_1(1)+\frac{\phi}{\alpha\xi_1}\left(\frac{A^K}{A\lambda}H_2(1)-\left(\frac{1-A^K}{1-A}\right)H_3(1)\right)\right\},$$

with

$$B=\left(1+\frac{\lambda(1-\theta)C_1(1)}{\alpha\xi_0}\right),$$

$$C_1(1)=\int_0^1 e^{\frac{-\lambda\theta}{\alpha\xi_0}s}(1-s)^{\frac{\phi}{\alpha\xi_0}}ds,\quad C_2(1)=\int_0^1 e^{\frac{-\lambda\theta}{\alpha\xi_0}s}(1-s)^{\frac{\phi}{\alpha\xi_0}-1}ds,$$

$$H_1(1)=\int_0^1 s^{\frac{\beta\mu}{\alpha\xi_1}}e^{-\frac{\lambda\theta}{\alpha\xi_1}s}(1-s)^{-1}ds,\quad H_2(1)=\int_0^1(\lambda\overline{\theta}s+\beta\mu)s^{\frac{\beta\mu}{\alpha\xi_1}-1}e^{-\frac{\lambda\theta}{\alpha\xi_1}s}ds,$$

$$H_3(1)=\int_0^1 s^{\frac{\beta\mu}{\alpha\xi_1}}e^{-\frac{\lambda\theta}{\alpha\xi_1}s}(1-s)^{-1}\Psi(s)ds,$$

$$\Psi(s)=e^{\frac{\lambda\theta}{\alpha\xi_0}s}(1-s)^{-\frac{\phi}{\alpha\xi_0}}\left\{1+\frac{\lambda\overline{\theta}}{\alpha\xi_0}C_1(s)-\frac{B}{C_2(1)}C_2(s)\right\},$$

$$C_1(s)=\int_0^s e^{-\frac{\lambda\theta}{\alpha\xi_0}t}(1-t)^{\frac{\phi}{\alpha\xi_0}}dt,\ \text{and}\ C_2(s)=\int_0^s e^{-\frac{\lambda\theta}{\alpha\xi_0}t}(1-t)^{\frac{\phi}{\alpha\xi_0}-1}dt.$$

– Useful performance measures of the system that are of general interest.

- The average number of customers in the system when the server is in the state j, ($\mathbb{E}(L_j)$).

$$\mathbb{E}(L_j)=\sum_{n=1}^{\infty}nP_{n,j}\,,j=\overline{0,K}.$$

- The mean system size when the server is on vacation period ($\mathbb{E}(L_V)$).

$$\mathbb{E}(L_V)=\sum_{j=0}^{K-1}\mathbb{E}(L_j)=\sum_{j=0}^{K-1}\sum_{n=1}^{\infty}nP_{n,j}=\sum_{n=1}^{\infty}n\left(\sum_{j=0}^{K-1}P_{n,j}\right).$$

- The mean system size when the server is on busy period ($\mathbb{E}(L_K)$).

$$\mathbb{E}(L_K)=\sum_{n=1}^{\infty}nP_{n,K}.$$

- The mean size of the system ($\mathbb{E}(L)$).

$$\mathbb{E}(L)=\sum_{j=0}^{K}\sum_{n=0}^{\infty}nP_{n,j}=\mathbb{E}(L_V)+\mathbb{E}(L_K).$$

- The probability that the system is in a vacation period, (P_V).

$$P_V = \sum_{j=0}^{K-1} \sum_{n=0}^{\infty} P_{n,j} = \sum_{j=0}^{K-1} P_{.,j}.$$

- The probability that the server is idle and not in vacation period (P_I).

$$P_I = P_{0,K}.$$

- The probability that the system is busy (P_B).

$$P_B = 1 - P_V - P_{0,K}.$$

- The mean size of the queue $(\mathbb{E}(L_q))$.

$$\begin{aligned} \mathbb{E}(L_q) &= \sum_{j=0}^{K-1} \sum_{n=1}^{\infty} nP_{n,j} + \sum_{n=1}^{\infty} (n-1)P_{n,K} \\ &= \mathbb{E}(L) - (1 - P_V - P_{0,K}) \\ &= \mathbb{E}(L) - P_B. \end{aligned}$$

The expected number of customers served per unit of time $(\mathbb{E}_{cs})$.

$$\mathbb{E}_{cs} = \beta \mu P_B.$$

The average rate of balking (B_r).

$$B_r = \overline{\theta} \lambda \left(\sum_{j=0}^{K} \sum_{n=0}^{\infty} P_{n,j} \right).$$

The average rate of abandonment of a customer due to impatience (R_{ren}).

$$\begin{aligned} R_{ren} &= \alpha \xi_0 \sum_{j=0}^{K-1} \sum_{n=1}^{\infty} nP_{n,j} + \alpha \xi_1 \sum_{n=1}^{\infty} nP_{n,K} \\ &= \alpha \xi_0 \mathbb{E}(L_V) + \alpha \xi_1 \mathbb{E}(L_K). \end{aligned}$$

The average rate of retention of impatient customers (R_{ret}).

$$R_{ret} = \alpha' \xi_0 \mathbb{E}(L_V) + \alpha' \xi_1 \mathbb{E}(L_K).$$

After obtaining useful performance measures of the system. An extensive numerical analysis is done.

▶ **Third result: Mathematical analysis of a Markovian multi-server feedback queue with a variant of multiple vacations, balking and reneging.**

We examine an $M/M/c$ queue with K-variant vacation, bernoulli feedback, balking, servers's states-dependent reneging, and retention of reneged customers. Customers arrive one by one according to a Poisson process with rate λ. The service times is supposed to be exponentially distributed with rate μ. The service discipline is FCFS. Once the system is empty all the servers go synchronously for vacation once the system becomes empty, and they also return to the system as one at the same time. If the servers return from vacation period to find an empty queue, they immediately leave all together for another vacation; otherwise, they return to serve the queue. Vacation period is assumed to be exponentially distributed with rate ϕ.
If the servers find customer at a vacation completion instant, they all comeback to regular busy period; otherwise, they take all together K-vacations sequentially. When the K-consecutive vacations are complete, all servers switch to busy period and depending on the arrival customers, they stay idle or busy with the next arrivals. On arrival, if a customer finds the servers on vacation (resp. busy) period, he activates an impatience timer T_{Vac} (resp. T_{Busy}), which is exponentially distributed with parameter ξ_0 (resp. ξ_1). If the customer's service has not been completed before the customer's timer expires, this later may leave the system. Then, the system may use some strategy to keep the latter in the system, so with some probability α the customers can be kept in the queue; otherwise they leave with $\alpha' = 1-\alpha$. The customers timers are independent and identically distributed random variables and independent of the number of waiting customers. Further, if the customer is unhappy with the service, he can either leave the system with probability β or rejoin the end of the queue with probability β', where $\beta+\beta' = 1$.
A customer who on arrival finds at least one customer (resp. c customers) in the system, when the servers are on vacation period (resp. busy period) either decides to enter the queue with probability θ or balk with probability $\overline{\theta} = 1-\theta$.
It is supposed that all the random variables presented above are mutually independent.

Let $L(t)$ denote the number of customers in the system at time t, and let $J(t)$ be the status of the servers at time t, defined as follows:

$$J(t) = \begin{cases} j, & \text{the servers are taking the } (j+1)^{th} \text{ vacation at time } t, \\ & j = \overline{0, K-1}, \\ K, & \text{the servers are idle or busy at time } t. \end{cases}$$

Thus, the process $\{(L(t); J(t)); t \geq 0\}$ defines a continuous-time Markov process with state space $\Omega = \{(n; j) : n \geq 0; j = \overline{0, K}\}$.

Let $P_{n,j} = \lim\limits_{t\to\infty} \mathbb{P}(L(t) = n; J(t) = j)$, $n \geq 0$; $j = \overline{0, K}$, denote the steady-state probabilities of the process $\{(L(t); J(t)); t \geq 0\}$.

– Using the probability generating function (PGF), we get

$$G_K(1) = P_{\cdot,K} = \Phi(1)P_{0,0},$$

where

$$\Phi(1) = e^{\frac{\theta\lambda}{\alpha\xi_1}}\Big\{\Big((\beta\mu+\alpha\xi_1)\varpi_1 + \phi\left(\tfrac{1-A^{K-1}}{1-A}\right)\Big)H_1(1) + \tfrac{c\beta\mu\phi}{\lambda}A^{K-1}H_2(1)$$

$$-\phi\left(\tfrac{1-A^K}{1-A}\right)H_3(1) + \lambda\overline{\theta}H_4(1) - \beta\mu H_5(1)\Big\},$$

with

$$H_1(1) = \frac{1}{\alpha\xi_1}\int_0^1 s^{\frac{c\beta\mu}{\alpha\xi_1}}e^{-\frac{\theta\lambda}{\alpha\xi_1}s}(1-s)^{-1}ds, \quad H_2(1) = \frac{1}{\alpha\xi_1}\int_0^1 s^{\frac{c\beta\mu}{\alpha\xi_1}-1}e^{-\frac{\theta\lambda}{\alpha\xi_1}s}ds,$$

$$H_3(1) = \frac{1}{\alpha\xi_1}\int_0^1 s^{\frac{c\beta\mu}{\alpha\xi_1}}e^{-\frac{\theta\lambda}{\alpha\xi_1}s}\Psi(s)(1-s)^{-1}ds, \quad H_4(1) = \frac{1}{\alpha\xi_1}\int_0^1 s^{\frac{c\beta\mu}{\alpha\xi_1}}e^{-\frac{\theta\lambda}{\alpha\xi_1}s}\Theta_1(s)ds,$$

and

$$H_5(1) = \frac{1}{\alpha\xi_1}\int_0^1 s^{\frac{c\beta\mu}{\alpha\xi_1}-1}e^{-\frac{\theta\lambda}{\alpha\xi_1}s}\Theta_2(s)ds.$$

where

$$B = 1 + \frac{\lambda}{\alpha\xi_0}\overline{\theta}C_1(1) \quad \text{and} \quad \varpi_1 = \frac{\alpha\xi_0}{(\beta\mu+\alpha\xi_1)C_2(1)}B,$$

$$C_1(1) = \int_0^1 e^{-\frac{\theta\lambda}{\alpha\xi_0}s}(1-s)^{\frac{\phi}{\alpha\xi_0}}ds \quad \text{and} \quad C_2(1) = \int_0^1 e^{-\frac{\theta\lambda}{\alpha\xi_0}s}(1-s)^{\frac{\phi}{\alpha\xi_0}-1}ds,$$

$$\Psi(s) = e^{\frac{\theta\lambda}{\alpha\xi_0}s}(1-s)^{-\frac{\phi}{\alpha\xi_0}}\left\{1 + \frac{\lambda\overline{\theta}}{\alpha\xi_0}C_1(s) - \frac{B}{C_2(1)}C_2(s)\right\},$$

$$\Theta_1(s) = \sum_{n=0}^{c-1}s^n\varpi_n, \quad \text{and} \quad \Theta_2(s) = \sum_{n=1}^{c-1}(n-c)s^n\varpi_n,$$

$$C_1(s) = \int_0^s e^{-\frac{\theta\lambda}{\alpha\xi_0}t}(1-t)^{\frac{\phi}{\alpha\xi_0}}dt \quad \text{and} \quad C_2(s) = \int_0^s e^{-\frac{\theta\lambda}{\alpha\xi_0}t}(1-t)^{\frac{\phi}{\alpha\xi_0}-1}dt,$$

$$\varpi_0 = \frac{\phi}{\lambda}A^{K-1},$$

$$\varpi_n = \frac{1}{n(\beta\mu+\alpha\xi_1)}\left\{[\lambda+(n-1)(\beta\mu+\alpha\xi_1)]\varpi_{n-1} - \lambda\varpi_{n-2} - \phi\delta_{n-1}\right\},$$

$$\delta_n = \frac{1}{n\alpha\xi_0}\left\{[\theta\lambda+\phi+(n-1)\alpha\xi_0]\delta_{n-1} - \theta\lambda\delta_{n-2}\right\}$$

and

$$A = \frac{\phi C_2(1)}{\alpha\xi_0 B}.$$

– The steady-state probability $P_{.,j}$:

$$P_{.,j} = G_j(1) = A^{j-1} P_{0,0}, \quad j = \overline{0, K-1}.$$

where

$$P_{0,0} = \left(\frac{1 - A^K}{A(1-A)} + \Phi(1) \right)^{-1}.$$

– The performance measures that are of general interest include:
– The mean system size when the servers are on vacation:

$$\mathbb{E}(L_V) \quad = \quad \left(\frac{\lambda(\theta + \overline{\theta} A)}{\alpha \xi_0 + \phi} \right) \left\{ \frac{2 - (A + A^{K-1})}{A(1-A)} \right\} P_{0,0}.$$

– The mean system size when the servers are in busy period:

$$\mathbb{E}(L_K) = \tfrac{1}{\alpha \xi_1} \Big\{ (\theta\lambda - c\beta\mu)\Phi(1) + c\beta\mu \tfrac{\phi}{\lambda} A^{K-1} + \tfrac{\lambda\phi(\theta + \overline{\theta} A)}{\alpha \xi_0 + \phi} \Big(\tfrac{1 - A^K}{A(1-A)} \Big) + \overline{\theta} \lambda \Theta_1(1) \\ - \beta\mu \Theta_2(1) \Big\} P_{0,0},$$

where $\Theta_1(1) = \sum_{n=0}^{c-1} \varpi_n$ and $\Theta_2(1) = \sum_{n=1}^{c-1} (n-c)\varpi_n$.

– The mean size of the queue:

$$\mathbb{E}(L_q) \quad = \quad E(L) - c + \Big\{ c \Big[\tfrac{1 - A^K}{A(1-A)} + \tfrac{\phi}{\lambda} A^{K-1} \Big] - \Theta_2(1) \Big\} P_{0,0}.$$

– The expected number of customers served per unit of time:

$$E_{cs} \quad = \quad \beta\mu \left\{ c + \left[\Theta_2(1) - c \left(\frac{\phi}{\lambda} A^{K-1} + \frac{1 - A^K}{A(1-A)} \right) \right] P_{0,0} \right\}.$$

– The average rate of balking when the servers are in the state $j = \overline{0, K}$.

$$B_r \quad = \quad \overline{\theta} \lambda \Big\{ 1 - \Big[\tfrac{2 - A - A^{K-1} + (1-A)\Theta_1(1)}{(1-A)} \Big] P_{0,0} \Big\}.$$

– The average rate of abandonment of a customer due to reneging is as follows

$$R_{ren} \quad = \quad \alpha \xi_0 \mathbb{E}(L_V) + \alpha \xi_1 \mathbb{E}(L_K).$$

1.7 Outline of the thesis

This thesis consists of seven chapters including the introductory chapter.

Chapter 2 deals with a single server feedback queueing system under two differentiated multiple vacations and balked customers. The service times of the two vacation types are exponentially distributed with different means. The steady-state probabilities of the model are obtained. Some important performance measures of the system are derived. Then, a cost model is developed. Further, a numerical study is presented.

This chapter has been published in **Appl. Appl. Math., 14, 1, 46–62, 2019.**

Chapter 3 analyzes a single server Markovian queueing system with Bernoulli feedback, balking, server's states-dependent reneging, and retention of reneged customers under variant of multiple vacation policy. The steady-state solution of the considered queueing system are obtained based on the use of probability generating functions (PGFs). Then, the closed-form expressions of different system characteristics are derived. Finally, some numerical results are presented in order to show the impact of the parameters of impatience timer rates on performance measures of the system. This chapter has been published in **Yugoslav Journal of Operations Research, 2021, https://doi.org/10.2298/YJOR200418003B**

In chapter 4, we consider a multi-server queueing system with variant multiple vacations, Bernoulli feedback, balking and servers's states-dependent reneging, and retention of reneged customers. Using the probability generating functions (PGFs), the equations of the steady state probabilities of the model have been developed, their steady-state solutions are derived. Further, useful performance measures are obtained. This chapter is under consideration for possible consideration.

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Chapter 2

Performance and economic evaluation of differentiated multiple vacation queueing system with feedback and balked customers

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Performance and economic evaluation of differentiated multiple vacation queueing system with feedback and balked customers

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Abstract. The present chapter deals with a single server feedback queueing system under two differentiated multiple vacations and balked customers. It is assumed that the service times of the two vacation types are exponentially distributed with different means. The steady-state probabilities of the model are obtained. Some important performance measures of the system are derived. Then, a cost model is developed. Further, a numerical study is presented.

Keywords: Queueing models; Differentiated vacations; Balking; Bernoulli feedback.

MSC 2010 No.: 60K25, 68M20, 90B22

2.1 Introduction

Since the late 70's the queueing models with server vacations have been well studied and successfully applied in many areas such as manufacturing/service and computer/communication systems. Excellent surveys on the earlier works of vacation models have been given in Doshi (1986), Takagi (1991) and Tian and Zhang (2006). Zhang et al. (2001) presented the optimal service policies in an $M/G/1$ queueing model with multiple vacations. Choudhury (2002) analyzed the $M/G/1$ queue with multiple vacations of two types and obtained the stationary queue length waiting time distributions. Thangaraj and Vanitha (2009) studied a two-phase $M/G/1$ queue with Bernoulli feedback and multiple-vacation policy. Further, Li et al. (2009) used the matrix analytic method to analyze an $M/G/1$ queue with exponentially working vacations under a specific assumption. Yang et al. (2010) treated the F-policy $M/M/1/K$ queue with single working vacation and exponential startup times, authors derived the stationary distributions and related system performance measures. Jain and Jain (2010) investigated a single-server working-vacation model with server breakdowns of multiple types. An $M/M/1$ multiple vacation queueing systems with differentiated vacations was considered by Ibe and Isijola (2014). After that, Ibe (2015) studied the $M/G/1$ vacation queueing system with server timeout.

In recent years, extensive studies were conducted on the vacation models with impatient customers. Zhang et al. (2005) dealt with an $M/M/1/N$ queue with balking, reneging and server vacations. Both single server and multi-server vacation models with impatient (reneged) customers were discussed by Altman and Yechiali (2006). Yue et al. (2006) established optimal performance analysis of an $M/M/1/N$ queue system with balking, reneging and server vacation. Yue et al. (2006) studied a finite buffer multi-server queue with balking, reneging, and single synchronous vacation policy. Analysis of customers' impatience in an $M/M/1$ queue with working vacations was given in Yue et al. (2012). Zhang et al. (2013) presented the equilibrium balking strategies in Markovian queues with working vacations. Vijaya Laxmi et al. (2013) treated the $M/M/1/N$ queueing system with balking, reneging and working vacation. Selvaraju and Goswami (2013) gave an analysis of impatient customers in an $M/M/1$ queue with single and multiple working vacations. Sun and Li (2014) investigated the equilibrium and optimal behavior of customers in Markovian queues with multiple working vacations. Sun et al. (2014) presented the equilibrium balking strategies of customers in Markovian queues with two-stage working vacations. The study of a discrete-time working vacation queue with balking and reneging was given in Goswami (2014). Misra and Goswami (2015) considered a single server queue with multiple vacation and balking. Recently, Panda and Goswami (2016) studied the equilibrium balking strategies for a $GI/M/1$ queue with Bernoulli-schedule vacation and vacation interruption in the case, where a customer can only observe the state of the server and when there is no information available to a customer before taking decision to join the system or balk. Vijaya Laxmi and Jyothsna (2016) investigated a discrete-time impatient customer queue with Bernoulli-schedule vacation interruption.

In this work, we extend the work of Ibe (2014) by incorporating the concept of balking and feedback. We investigate performance and economic analysis of an $M/M/1$ Bernoulli feedback queueing system under differentiated multiple vacations, in which two types of vacations can be taken by the server (a type 1 vacation, taken immediately after the server has finished serving at least one customer and type 2 vacation, taken immediately after the server has just returned from a previous vacation to find that there are no customers waiting) and balked customers, in which on arrival, a customer who finds at least one customer in the system, either decides to join the queue with some probability or balk with a complimentary probability. Useful performance measures are given. Further, an economic analysis of the model is considered to study the effect of different parameters of model on total expected profit of the system.

The rest of the chapter is arranged as follows, in Section 2, the model is described. In Section 3, we obtain the steady state probabilities of the queueing system under consideration. In Section 4, important performance measures are derived. In Section 5, we develop a cost model. Section 6 is consecrated to the numerical analysis. Finally, we conclude the chapter in Section 7.

2.2 Description of the model

Consider a $M/M/1$ Bernoulli feedback queueing system under differentiated multiple vacation and balked customers.

– The inter-arrival times are independently, identically and exponentially distributed with mean $1/\lambda$.

– There is only one server, and service time is exponentially distributed with mean $1/\mu$.

– The queue discipline is First-Come First-Served (FCFS).

– Assume that there are two types of vacations: type 1 vacation taken after a busy period, in which a server has served at least one customer, and type 2 vacation taken when the server returns from a vacation and observe that the queue is empty. Suppose that the duration of type 1 vacation is independent of the busy period and is exponentially distributed with mean $1/\gamma_1$. The duration of type 2 vacation is assumed to be exponentially distributed with mean $1/\gamma_2$.

– On arrival, a customer who finds at least one customer in the system, either decides to join the queue with probability θ or balk with probability $\theta' = 1-\theta$.

– The inter-arrival times, vacation periods and service times are mutually independent.

– After getting incomplete (or unsatisfactory) service, with probability β', a customer may rejoin the system as a Bernoulli feedback customer to receive another regular service. Otherwise, he leaves the system definitively with probability β, where $\beta'+\beta=1$. Note that $\frac{\lambda}{\beta\mu}<1$ is the condition of the stability of the system.

2.3 Steady-state solution

In this section, we derive the steady state solution of our queueing model. Let (n,k) be the state of the system, where n is the number of customers in the system, k is the state of the sever, such that

$$k = \begin{cases} 0, & \text{the server is active,} \\ 1, & \text{the server is on type 1 vacation,} \\ 2, & \text{the server is on type 2 vacation.} \end{cases}$$

Thus, our system can be modeled by a continuous time Markov chain.

Let $P_{n,k}(t)$ be the probability that the system is in state (n,k) at time t. Then,

$$P_{n,k} = \lim_{t\to\infty} P_{n,k}(t), \tag{2.1}$$

is steady-state probability of the system.

The differential-difference equations of the model are as follows:

$$\frac{dP_{0,1}(t)}{dt} = -(\lambda+\gamma_1)P_{0,1}(t) + \beta\mu P_{1,0}(t),\ \ n = 0, \tag{2.2}$$

$$\frac{dP_{0,2}(t)}{dt} = -\lambda P_{0,2}(t) + \gamma_1 P_{0,1}(t),\ \ n = 0, \tag{2.3}$$

$$\frac{dP_{0,1}(t)}{dt} = -\lambda P_{0,1}(t) + (\theta\lambda+\gamma_1)P_{1,1}(t),\ \ n = 0, \tag{2.4}$$

$$\frac{dP_{0,2}(t)}{dt} = -\lambda P_{0,2}(t) + (\theta\lambda+\gamma_2)P_{1,2}(t),\ \ n = 0, \tag{2.5}$$

$$\frac{dP_{n,1}(t)}{dt} = -\theta\lambda P_{n,1}(t) + (\theta\lambda+\gamma_1)P_{n+1,1}(t),\ \ n = 1,2,..., \tag{2.6}$$

$$\frac{dP_{n,2}(t)}{dt} = -\theta\lambda P_{n,2}(t) + (\theta\lambda+\gamma_2)P_{n+1,2}(t),\ \ n = 1,2,..., \tag{2.7}$$

$$\frac{dP_{n+1,0}(t)}{dt} = -\beta\mu P_{n+1,0}(t) + \theta\lambda P_{n,0}(t) + \theta\lambda P_{n,1}(t) + \theta\lambda P_{n,2}(t),\ \ n = 1,2,.... \tag{2.8}$$

From Equations (2.2)-(2.8), as $t \to \infty$ taking into consideration Equation (2.1) and assuming that

$$\lim_{t\to\infty} \frac{P_{n,k}(t)}{dt} = 0,$$

which is always satisfied for a continuous time Markov chain, we respectively get the relations

$$(\lambda+\gamma_1)P_{0,1} = \beta\mu P_{1,0},\ \ n = 0, \tag{2.9}$$

$$\lambda P_{0,2} = \gamma_1 P_{0,1},\ \ n = 0, \tag{2.10}$$

$$\lambda P_{0,1} = (\theta\lambda+\gamma_1)P_{1,1},\ \ n = 0, \tag{2.11}$$

$$\lambda P_{0,2} = (\theta\lambda+\gamma_2)P_{1,2},\ \ n = 0, \tag{2.12}$$

$$\theta\lambda P_{n,1} = (\theta\lambda+\gamma_1)P_{n+1,1},\ \ n = 1,2,..., \tag{2.13}$$

$$\theta\lambda P_{n,2} = (\theta\lambda+\gamma_2)P_{n+1,2},\ \ n = 1,2,..., \tag{2.14}$$

$$\theta\lambda P_{n,0} + \theta\lambda P_{n,1} + \theta\lambda P_{n,2} = \beta\mu P_{n+1,0},\ \ n = 1,2,.... \tag{2.15}$$

Theorem 2.3.1. *The steady-state-probabilities $P_{n,k}$ are given by:*

$$P_{n,k} = \begin{cases} \phi\left\{\frac{\delta_1\chi_1(\chi_1^{n-1}-\phi^{n-1})}{\chi_1-\phi} + \frac{\delta_2\chi_2(\chi_2^{n-1}-\phi^{n-1})}{\chi_2-\phi} + \phi^{n-2}\right\}P_{1,0},\ \ n{=}1,2,...,\ \ k{=}0, \\ \theta\delta_1 P_{1,0},\ \ n{=}0,\ \ k{=}1, \\ \theta\delta_2 P_{1,0},\ \ n{=}0,\ \ k{=}2, \\ \delta_1\chi_1^n P_{1,0},\ \ k{=}1, \\ \delta_2\chi_2^n P_{1,0},\ \ k{=}2, \end{cases} \tag{2.16}$$

where

$$P_{1,0} = ((1-\chi_1)(1-\chi_2)(1-\phi))\Big\{\delta_1\chi_1(1-\chi_2)+\delta_2\chi_2(1-\chi_1) \\ +(1-\chi_1)(1-\chi_2)+\theta(\delta_1+\delta_2)(1-\chi_2)(1-\chi_1)(1-\phi)\Big\}^{-1}, \tag{2.17}$$

with

$$\phi = \frac{\theta\lambda}{\beta\mu}, \tag{2.18}$$

$$\delta_1 = \frac{\beta\mu}{\theta(\lambda+\gamma_1)},\ \textit{and}\ \delta_2 = \frac{\gamma_1}{\theta\lambda}\frac{\beta\mu}{\lambda+\gamma_1} = \frac{\gamma_1}{\lambda}\cdot\delta_1, \tag{2.19}$$

and

$$\chi_1 = \left(\frac{\theta\lambda}{\theta\lambda+\gamma_1}\right),\ \textit{and}\ \chi_2 = \left(\frac{\theta\lambda}{\theta\lambda+\gamma_2}\right). \tag{2.20}$$

Proof. From Equations (4.1) and (4.2), we get easily

$$P_{0,1} = \frac{\beta\mu}{\lambda+\gamma_1}P_{1,0} = \theta\delta_1 P_{1,0},$$

and

$$P_{0,2} = \frac{\gamma_1}{\lambda}\frac{\beta\mu}{\lambda+\gamma_1}P_{1,0} = \theta\delta_2 P_{1,0},$$

respectively.

Then, resolving recursively Equations (4.3)-(4.6), we get for $n = 1, 2, 3, ...$

$$\begin{cases} P_{n,1} = \frac{\beta\mu}{\theta(\lambda+\gamma_1)}\left(\frac{\theta\lambda}{\theta\lambda+\gamma_1}\right)^n P_{1,0} = \delta_1\chi_1^n P_{1,0}, \\ P_{n,2} = \frac{\gamma_1}{\theta\lambda}\frac{\beta\mu}{\lambda+\gamma_1}\left(\frac{\theta\lambda}{\theta\lambda+\gamma_2}\right)^n P_{1,0} = \delta_2\chi_2^n P_{1,0}. \end{cases}$$

From Equation (4.7), it yields

$$P_{n+1,0} = \phi P_{n,0} + \phi P_{n,1} + \phi P_{n,2},\ \ n = 1, 2, ..., \tag{2.21}$$

with

$$\phi = \frac{\theta\lambda}{\beta\mu}.$$

Then, solving recursively Equation (2.21), we get

$$P_{n,0} = \phi\left\{\frac{\delta_1\chi_1(\chi_1^{n-1}-\phi^{n-1})}{\chi_1-\phi} + \frac{\delta_2\chi_2(\chi_2^{n-1}-\phi^{n-1})}{\chi_2-\phi} + \phi^{n-2}\right\}P_{1,0},\ \ n = 1, 2, ...$$

Finally, using the normalization condition

$$\sum_{n=1}^{\infty} P_{n,0} + \sum_{n=0}^{\infty} P_{n,1} + \sum_{n=0}^{\infty} P_{n,2} = 1,$$

we get easily Equation (4.9). $\square$

2.4 Performance Measures

In this part of chapter, some important performance indices of the proposed system will be discussed.

– The average number of customers in the system:

$$\begin{aligned} L_s &= \sum_{n=1}^{\infty} n(P_{n,0} + P_{n,1} + P_{n,2}) \\ &= \Big\{ \delta_1 \chi_1 \phi \tfrac{2-\chi_1-\phi}{(1-\chi_1)^2(1-\phi)^2} + \delta_2 \chi_2 \phi \tfrac{2-\chi_2-\phi}{(1-\chi_2)^2(1-\phi)^2} + \tfrac{1}{(1-\phi)^2} + \tfrac{\delta_1\chi_1}{(1-\chi_1)^2} + \tfrac{\delta_2\chi_2}{(1-\chi_2)^2} \Big\} P_{1,0}. \end{aligned}$$

– The average number of customers in the queue:

$$L_q = \sum_{n=1}^{\infty} (n-1) P_{n,0} + \sum_{n=0}^{\infty} n(P_{n,1} + P_{n,2}).$$

– The average balking rate:

$$\begin{aligned} \lambda_{balk} &= \lambda \cdot P_{balk} \\ &= \sum_{n=1}^{\infty} \lambda(1-\theta)(P_{n,0} + P_{n,1} + P_{n,2}) \\ &= \lambda(1-\theta) \Big\{ \tfrac{\phi\delta_1\chi_1}{\chi_1-\phi}\big(\tfrac{1}{1-\chi_1} - \tfrac{1}{1-\phi}\big) + \tfrac{\phi\delta_2\chi_2}{\chi_2-\phi}\big(\tfrac{1}{1-\chi_2} - \tfrac{1}{1-\phi}\big) + \tfrac{1}{1-\phi} + \tfrac{\delta_1\chi_1}{1-\chi_1} + \tfrac{\delta_2\chi_2}{1-\chi_2} \Big\} P_{1,0}. \end{aligned}$$

– The probability that the server is in busy period:

$$\begin{aligned} P_B &= \mathbb{P}(\text{normal busy period}) \\ &= \sum_{n=1}^{\infty} P_{n,0} \\ &= \Big\{ \tfrac{\phi\delta_1\chi_1}{\chi_1-\phi}\big(\tfrac{1}{1-\chi_1} - \tfrac{1}{1-\phi}\big) + \tfrac{\phi\delta_2\chi_2}{\chi_2-\phi}\big(\tfrac{1}{1-\chi_2} - \tfrac{1}{1-\phi}\big) + \tfrac{1}{1-\phi} \Big\} P_{1,0}. \end{aligned}$$

Further,

$$\begin{aligned} P_{V1} &= \mathbb{P}(\text{vacation period of type 1}) \\ &= \sum_{n=0}^{\infty} P_{n,1} \\ &= \tfrac{\delta_1(\theta(1-\chi_1)+\chi_1)}{1-\chi_1} P_{1,0}, \end{aligned}$$

and

$$\begin{aligned} P_{V2} &= \mathbb{P}(\text{vacation period of type 2}) \\ &= \sum_{n=0}^{\infty} P_{n,2} \\ &= \tfrac{\delta_2(\theta(1-\chi_2)+\chi_2)}{1-\chi_2} P_{1,0}. \end{aligned}$$

Thus, the probability that the server is in vacation period

$$\begin{aligned} P_V &= \mathbb{P}(\text{vacation period of type 1 and 2}) \\ &= \sum_{n=0}^{\infty} (P_{n,1} + P_{n,2}) = P_{V1} + P_{V2} \\ &= \left\{ \tfrac{\delta_1(\theta(1-\chi_1)+\chi_1)}{1-\chi_1} + \tfrac{\delta_2(\theta(1-\chi_2)+\chi_2)}{1-\chi_2} \right\} P_{1,0}. \end{aligned}$$

2.5 Cost model

In this part of chapter, we develop a model for the costs incurred in the queueing system using the following elements:

- C_b : Cost per unit time when the server is busy.
- C_{v1} : Cost per unit time when the server is on vacation of type 1.
- C_{v2} : Cost per unit time when the server is on vacation of type 2.
- C_q : Cost per unit time when a customer joins the queue and waits for service.
- C_{balk} : Cost per unit time when a customer balks.
- C_s : Cost per service per unit time.
- C_{s-f} : Cost per unit time when a customer returns to the system as a feedback customer.

Next, let

- R be the revenue earned by providing service to a customer.
- Γ be the total expected cost per unit time of the system.

$$\Gamma = C_{balk}\lambda_{balk} + C_q L_q + C_b \mathbb{P}(\text{normal busy period}) + \mu(C_s + \beta' C_{s-f})$$

$$+ C_{v1}\mathbb{P}(\text{vacation period of type 1}) + C_{v2}\mathbb{P}(\text{vacation period of type 2}).$$

- Δ be the total expected revenue per unit time of the system.

$$\Delta = R\mu(1 - \mathbb{P}(\text{vacation period of type 1}) - \mathbb{P}(\text{vacation period of type 2})).$$

  And

- Θ be the total expected profit per unit time of the system.

$$\Theta = \Delta - \Gamma.$$

2.6 Numerical analysis

2.6.1 Performance analysis

To bring out the qualitative aspects of the queueing model under consideration, some numerical results are presented in the form of Tables and Graphs. To this end, we consider the following items:

- Table 1: $\lambda = 0.20 : 0.15 : 1.40, \mu = 3.00, \gamma_1 = 0.50, \gamma_2 = 3.00, \theta' = 0.40, \beta' = 0.40.$
- Table 2: $\lambda = 3.00, \mu = 4.00 : 0.50 : 8.00, \gamma_1 = 0.50, \gamma_2 = 3.00, \theta' = 0.30, \beta' = 0.30.$
- Table 3: $\lambda = 1.50, \mu = 3.00, \gamma_1 = 0.50 : 0.25 : 2.50, \gamma_2 = 3.00, \theta' = 0.40, \beta' = 0.40.$
- Table 4: $\lambda = 1.50, \mu = 3.00, \gamma_1 = 1.00, \gamma_2 = 2.00 : 0.50 : 6.00, \theta' = 0.40, \beta' = 0.40,$
- Table 5: $\lambda = 1.50, \mu = 3.00, \gamma_1 = 0.50, \gamma_2 = 3.00, \theta' = 0.00 : 0.10 : 0.90, \beta' = 0.30,$
- Table 6: $\lambda = 1.00, \mu = 7.00, \gamma_1 = 0.50, \gamma_2 = 3.00, \theta' = 0.40, \beta' = 0.00 : 0.10 : 0.90.$

Table 2.1: Performance measures vs. λ.

λ	P_V	P_B	L_s	L_q	λ_{balk}
0.20	0.8989	0.1011	0.2668	0.1657	0.0181
0.35	0.8354	0.1646	0.5228	0.3582	0.0538
0.50	0.7787	0.2213	0.7918	0.5705	0.1016
0.65	0.7260	0.2740	1.0665	0.7925	0.1568
0.80	0.6758	0.3242	1.3467	1.0225	0.2164
0.95	0.6271	0.3729	1.6349	1.2620	0.2788
1.10	0.5794	0.4206	1.9348	1.5142	0.3430
1.25	0.5324	0.4676	2.2511	1.7835	0.4084
1.40	0.4859	0.5141	2.5900	2.0759	0.4746

Table 2.2: Performance measures vs. μ.

μ	P_V	P_B	L_s	L_q	λ_{balk}
4.00	0.2378	0.7622	7.1330	6.3708	0.8659
4.50	0.3188	0.6812	6.1500	5.4688	0.8543
5.00	0.3843	0.6157	5.6638	5.0480	0.8450
5.50	0.4382	0.5618	5.3751	4.8133	0.8373
6.00	0.4835	0.5165	5.1846	4.6681	0.8308
6.50	0.5220	0.4780	5.0499	4.5719	0.8253
7.00	0.5552	0.4448	4.9497	4.5049	0.8205
7.50	0.5841	0.4159	4.8724	4.4565	0.8164
8.00	0.6094	0.3906	4.8111	4.4205	0.8127

Table 2.3: Performance measures vs. γ_1.

γ_1	P_{V1}	P_{V2}	P_B	L_s	L_q	λ_{balk}
0.50	0.4045	0.0506	0.5449	2.8326	2.2876	0.5191
0.75	0.3529	0.0882	0.5588	2.2294	1.6706	0.4941
1.00	0.3082	0.1233	0.5685	1.9466	1.3781	0.4767
1.25	0.2709	0.1539	0.5752	1.7921	1.2169	0.4646
1.50	0.2400	0.1800	0.5800	1.7000	1.1200	0.4560
1.75	0.2145	0.2021	0.5834	1.6418	1.0584	0.4499
2.00	0.1933	0.2209	0.5859	1.6034	1.0175	0.4454
2.25	0.1754	0.2368	0.5877	1.5772	0.9895	0.4421
2.50	0.1604	0.2506	0.5891	1.5589	0.9698	0.4396

Table 2.4: Performance measures vs. γ_2.

γ_2	P_{V1}	P_{V2}	P_B	L_s	L_q	λ_{balk}
2.00	0.2961	0.1382	0.5658	1.9783	1.4125	0.4816
2.50	0.3032	0.1294	0.5674	1.9581	1.3907	0.4787
3.00	0.3082	0.1233	0.5685	1.9466	1.3781	0.4767
3.50	0.3119	0.1188	0.5693	1.9393	1.3700	0.4752
4.00	0.3147	0.1154	0.5699	1.9344	1.3645	0.4741
4.50	0.3169	0.1127	0.5704	1.9310	1.3606	0.4732
5.00	0.3187	0.1105	0.5708	1.9284	1.3576	0.4725
5.50	0.3202	0.1087	0.5712	1.9265	1.3554	0.4719
6.00	0.3214	0.1071	0.5714	1.9250	1.3536	0.4714

General comments.

From Tables 2.1-2.6 and Figures 2.1-2.3, we observe that

1. With the increase in the arrival rate λ, the probability of normal busy period P_B, the mean size of the system L_s, the mean queue length L_q and the average rate of

Table 2.5: Performance measures vs. θ'.

θ'	P_V	P_B	L_s	L_q	λ_{balk}
0.00	0.2857	0.7143	5.2222	4.5079	0.0000
0.10	0.3497	0.6503	4.2889	3.6386	0.1345
0.20	0.4112	0.5888	3.5983	3.0095	0.2635
0.30	0.4701	0.5299	3.0500	2.5201	0.3873
0.40	0.5268	0.4732	2.5934	2.1202	0.5063
0.50	0.5813	0.4187	2.2003	1.7816	0.6208
0.60	0.6338	0.3662	1.8535	1.4873	0.7310
0.70	0.6843	0.3157	1.5421	1.2264	0.8371
0.80	0.7330	0.2670	1.2585	0.9916	0.9394
0.90	0.7800	0.2200	0.9976	0.7776	1.0380

Table 2.6: Performance measures vs. β'.

β'	P_V	P_B	L_s	L_q	λ_{balk}
0.00	0.8934	0.1066	1.3425	1.2359	0.2538
0.10	0.8819	0.1181	1.3534	1.2352	0.2557
0.20	0.8675	0.1325	1.3673	1.2348	0.2580
0.30	0.8492	0.1508	1.3859	1.2350	0.2610
0.40	0.8250	0.1750	1.4117	1.2367	0.2650
0.50	0.7916	0.2084	1.4501	1.2416	0.2705
0.60	0.7423	0.2577	1.5132	1.2555	0.2785
0.70	0.6627	0.3373	1.6361	1.2988	0.2916
0.80	0.5116	0.4884	1.9779	1.4895	0.3163
0.90	0.1158	0.8842	7.2063	6.3221	0.3811

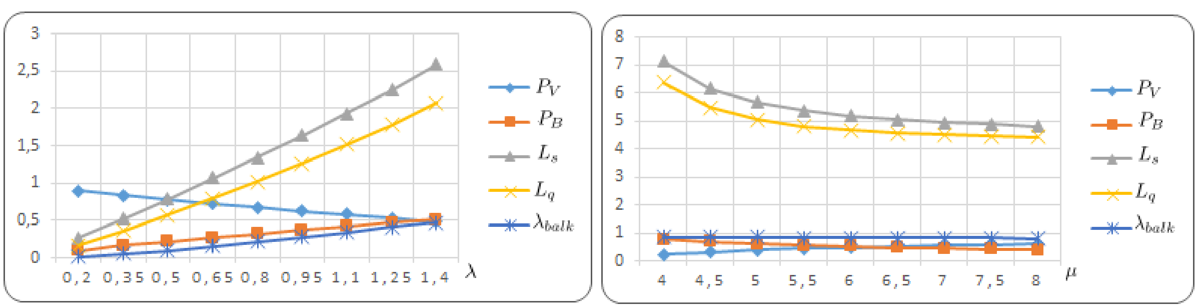

Figure 2.1: Performance measures curves vs. λ and μ.

balking λ_{balk} all increase. While the probability of vacation period P_V decreases. This can be explained by the fact that

– When the arrival rates increases, the queue size becomes large. Thus, the aver-

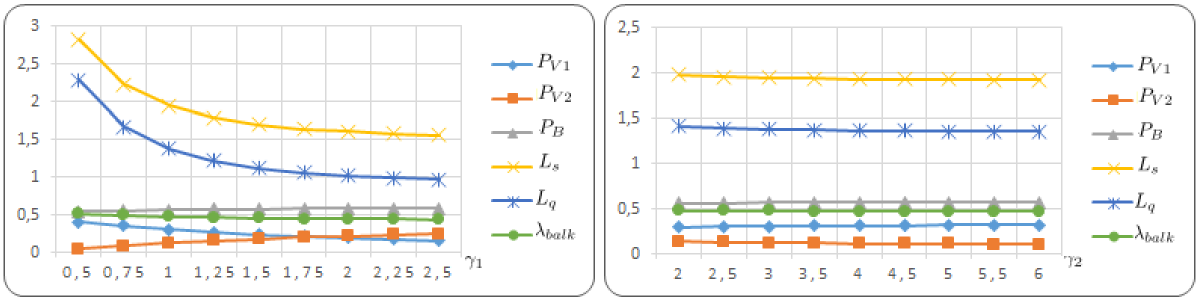

Figure 2.2: Performance measures curves vs. γ_1 and γ_2.

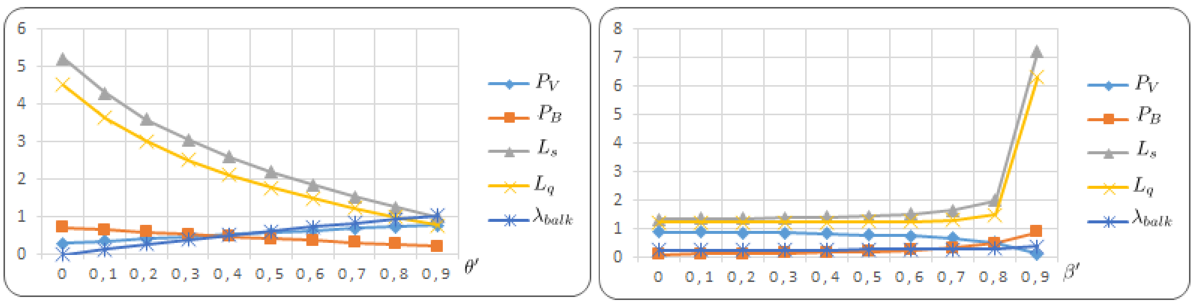

Figure 2.3: Performance measures curves vs. θ' and β'.

age rate of balking increases accordingly.

– High number of customers in the system generates a big probability of normal busy period and small probability of vacation period (vacation periods of types 1 and 2).

2. Along the increasing of the service rate μ, the customers are served faster, this engenders a decrease in the probability of normal busy period P_B. Consequently, the mean number of customers in the system L_s and the mean number of customers waiting for service L_q decrease significantly. Therefore, the average balking rate λ_{balk} is reduced. However, the probability of vacation period P_V increases, as intuitively expected.

3. With the increase in the vacation rate of type 1, γ_1, the probability of vacation of type 1, P_{V1}, the mean system size L_s, the mean queue length L_q, and the average balking rate λ_{balk} all decrease, as it should be. While the vacation probability of type 2, P_{V2} and the probability of normal busy period P_B increase. This can be explained by the fact that the increase of the vacation rate of type 1 leads to the increase in the probability of busy period. Therefore, significant number of

customers will be served. Then, the mean size of the system becomes small. Consequently, the average rate of balking is reduced.

4. The increases of the vacation rate of type 2, γ_2 has the same effect as γ_1 on the mean size of the system, the mean queue length, the average balking rate, and the probability of normal busy period. Otherwise, the increasing of the vacation rate of type 2 implies a decrease in the vacation probability of type 2 and an increase in the probability of vacation type 1, as it should be.

5. Along the increasing of the balking probability θ', the average balking rate λ_{balk} and the probability that the system is in vacation period P_V increase monotonically. While the probability that the system is on normal busy period P_B, the mean number of customers in the system L_s and the mean number of customers in the queue L_q all decrease. This is due to the fact that when the balking probability increases, the probability that the customers do not enter the system grows. Consequently, the mean number of customers in the system is reduced. Thus, the probability that the system is on busy period decreases, while the probability that the server goes on vacation becomes high.

6. When the probability of feedback β' increases, the probability of vacation period P_V decreases, whereas the probability of normal busy period P_B, the mean size of the system L_s and the mean queue length L_q increase significantly which implies an increase in the average balking rate λ_{balk}.

2.6.2 Economic analysis

This subsection is devoted to study numerically the cost profit aspects associated with the model. More precisely, we present the variation in total expected cost, total expected revenue and total expected profit with the change in balking probability θ', feedback probability β', and vacation rates of type 1 and 2 γ_1 and γ_2, respectively. Indeed, using a program implemented under R, we present some numerical examples to illustrate the effect of these parameters on Γ, Δ and Θ. To this end, we fix the different costs as follows: $C_s = 2$, $C_{s-f} = 2$, $C_{balk} = 2$, $C_q = 3$, $C_b = 3$, $C_{v1} = 2$, $C_{v1} = 2$, and $R = 250$.

2.6.2.1 Case 1: Impact of balking probability θ'

We check the behavior of total expected cost, total expected revenue and total expected profit for various values of θ' by keeping all other variables fixed. Let $\lambda = 2.00$, $\mu = 3.00$, $\gamma_1 = 0.30$, $\gamma_2 = 1.10$ and $\beta' = 0.20$.

From Table 2.7 and Figure 2.4, it can be observed that with the increase in the balking probability θ', total expected cost Γ, total expected revenue Δ and total expected profit Θ of the system decrease significatively. This is due to the fact that the larger the balking probability, the smaller the mean size of the system and the lower the number of customers served. Clearly, one can deduce that balking probability has a negative impact of the rentability of the system.

2.6.2.2 Case 2: Impact of feedback probability β'

We examine the behavior of Γ, Δ and Θ for various values of β'. To this end, we fixe the other parameters as $\lambda = 0.55$, $\mu = 6.00$, $\gamma_1 = 2.00$, $\gamma_2 = 1.00$ and $\theta' = 0.30$.

From Table 2.8 and Figure 2.5, it can be seen that total expected cost Γ, total expected revenue Δ, and total expected profit Θ increase significantly along the increasing of the feedback probability β'. Obviously, when the feedback probability increases, the mean number of customers in the system L_s becomes large. Thus, important number of customers will be served. Therefore, the positive impact of this probability is quite clear on the economy of the system.

2.6.2.3 Case 3: Impact of vacation rates γ_1 and γ_2

– Firstly, we analyze the impact of γ_1 on Γ, Δ and Θ. To this end, we put $\lambda = 1.20$, $\mu = 6.00$, $\gamma_2 = 3.00$, $\theta' = 0.30$ and $\beta' = 0.40$.

– Secondly, we examine the impact of γ_2 on Γ, Δ and Θ by keeping all other variables fixed. Put $\lambda = 1.20$, $\mu = 6.00$, $\gamma_1 = 2.00$, $\beta' = 0.40$ and $\theta' = 0.30$.

From Tables 2.9-2.10 and Figure 2.6, it is clearly seen that the decrease in the mean vacation times $1/\gamma_1$ and $1/\gamma_2$ leads to the increase in total expected revenue Δ and in total expected profit Θ. While the total expected cost Γ decreases. This can be explained by the fact that when vacation rates γ_1 and γ_2 increase, the probability that the system is on busy period becomes large. Consequently, the mean number of customers served increases.

Table 2.7: Γ, Δ and Θ vs. θ'.

θ'	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
Γ	23.784	21.234	19.403	17.989	16.841	15.875	15.039	14.303	13.643	13.043
Δ	535.714	487.694	441.624	397.388	354.878	313.995	274.648	236.751	200.226	165.000
Θ	511.930	466.460	422.221	379.399	338.037	298.121	259.608	222.448	186.584	151.957

Table 2.8: Γ, Δ and Θ vs. β'.

β'	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
Γ	11.591	12.214	12.843	13.483	14.140	14.827	15.574	16.475	17.971	33.918
Δ	79.949	88.608	99.369	113.106	131.250	156.328	193.252	253.012	366.279	663.158
Θ	68.358	76.394	86.526	99.623	117.110	141.501	177.678	236.537	348.308	629.240

Table 2.9: Γ, Δ and Θ vs. γ_1.

γ_1	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50
Γ	18.096	16.276	15.430	14.974	14.708	14.544	14.439	14.370	14.324
Δ	408.708	419.118	426.370	431.430	435.000	437.557	439.417	440.789	441.815
Θ	390.612	402.841	410.940	416.455	420.292	423.013	424.978	426.419	427.491

Table 2.10: Γ, Δ and Θ vs. γ_2.

γ_2	2.00	2.50	3.00	3.50	4.00	4.50	5.00	5.50	6.00
Γ	15.549	15.475	15.430	15.401	15.381	15.366	15.355	15.346	15.339
Δ	424.342	425.539	426.370	426.980	427.448	427.817	428.116	428.364	428.571
Θ	408.793	410.064	410.940	411.579	412.067	412.451	412.761	413.017	413.232

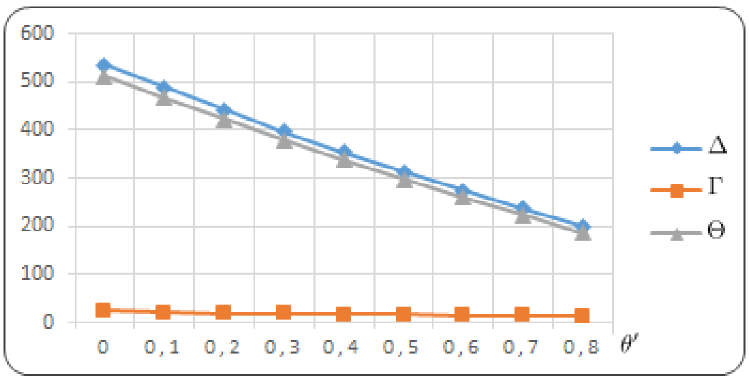

Figure 2.4: Γ, Δ and Θ for different values of θ'.

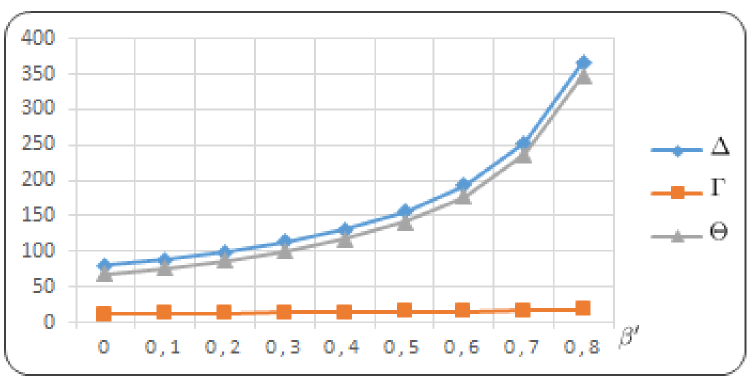

Figure 2.5: Γ, Δ and Θ for different values of β'.

2.7 Conclusion

In this work, we studied a single server Markovian Bernoulli feedback queueing system under two differentiated multiple vacations and balked customers. The steady-

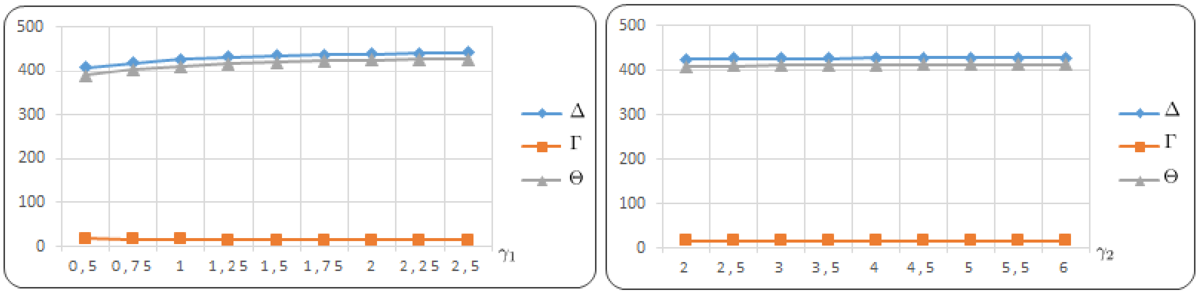

Figure 2.6: Γ, Δ and Θ for different values of γ_1 and γ_2.

state solution was obtained. Important performance measures were derived and the economic model analysis has been carried out. For further work, it will be interesting to study the effect of the reneging in such system. Moreover extension of our results for a non-Markovian models is a pointer to future research.

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Chapter 3

Variant vacation queueing system with Bernoulli feedback, balking and server's states-dependent reneging

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Variant vacation queueing system with Bernoulli feedback, balking and server's states-dependent reneging

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Abstract. We consider a single server Markovian feedback queue with variant of multiple vacation policy, balking, server's states-dependent reneging, and retention of reneged customers. We obtain the steady-state solution of the considered queue based on the use of probability generating functions. Then, the closed-form expressions of different system characteristics are derived. Finally, we present some numerical results in order to show the impact of the parameters of impatience timers on the performance measures of the system.

Keywords: Queueing models, Vacation, Impatience, Bernoulli feedback, Simulation.

MSC: 60K25, 68M20, 90B22.

3.1 Introduction

Vacation queue modeling is being employed in a large variety of day-to-day congestion issues as well as industrial scenarios including communication and manufacturing systems, transportation systems, call centers, web services and so on. A vast literature has been devoted to vacation queueing models in different frame-works. Research works on single vacation (once the vacation is ended, the server switches to the busy period and stays there waiting for a new arrival) and multiple vacation (the server continues his vacation as long as there is no customers in the system) have a significant developments (cf. Doshi (1989), Tian (1989), Takagi (1991), Tian and Zhang (2006), Zhang and Tian (2001), Ke *et al.* (2010), and Upadhyaya (2016)). In recent years, queueing models with variant of multiple vacations have drawn a significant attention, different from the above policies, this new concept considers the case wherein at the vacation completion instant, if the system is still empty, the server is permitted to take a certain number of successive vacations, and once the vacations are ended, the server has to come back to the busy period and remains there, busy or idle, depending on the presence of customers in the system (cf. Banik (2009), Yue *et al.* (2010), Wang *et al.* (2011), and Laxmi and Rajesh (2016)).

Impatience (balking and/or reneging) is the most prominent feature in queueing theory. Vacation queueing models with impatient customers are considered to be very appropriate tools in analysing various complex service systems and important industries. Therefore, it has generated a fundamental results with extensive bibliographical references on this area (cf. Zhang *et al.* (2005), Altman and Yechaili (2006, 20008), Ke (2007), Ke *et al.* (2010), Adan *et al.* (2009), Yue *et al.* (2014), Ammar (2015), Laxmi and Rajesh (2017), Bouchentouf *et al.* (2019, 2020), and Kumar (2020)).

In traditional vacation queueing literature with impatient customers, the studies of customer's behavior was always based on the hypothesis that customers' impatience happens only during the absence of the server. This is the case where the customers can see the state of the server. However, in many real-life situations including call center and production systems, it may not possible to have information on the server's state. Further, a long wait in the queue is another factor which leads to customer's impatience whatever the state of the system (active or on vacation). Despite the rapid growth of the literature about customers's impatience in vacation queueing models, there is very limited literature to deal with the customers' impatience during both vacation and busy periods. The authors can refer to Yue *et al.* (2016) and Bouchentouf and Guendouzi (2018, 2020), and Cherfaoui *et al.* (2020).

Customer feedback has an utmost importance in queueing systems at which if the customer in not satisfied with the service, he can return to the system asking for another one. He can retry several times until he gets a satisfactory service. Such queues usually occur in our everyday life. As a concrete example, we cite multiple access telecommunication systems, where the data packet with errors at the destination will be sent over and over again until the data packet is transmitted with success. Queueing models with feedback have been widely studied. The pioneer research work on the subject have been done by Takacs (1963) who dealt with an $M/M/1$ queue with feedback, where he determined the stationary process of the queue length as well as a customer waiting distribution in the system. The literature on the related topic is abundant, for a comprehensive overview, the readers may refer to Davignon and Disney (1976), Bengtsson (1982), Santhakumaran and Thangaraj (2000), Atencia *et al.*

(2009), Upadhyaya (2016), Liu and Whitt (2017), and Shekhar *et al.* (2019).

In regard to the mathematical solution techniques, it is worth pointing out that a distinction can be made between algorithms that allow efficient calculation of the system characteristics numerically; i.e. matrix-analytic approaches; a very powerful numerical technique employed when it is not easy to obtain clear and closed-form analytical solutions for queueing problems (see Neuts (1981), Blondia and Casals (1992)) and analytical approaches that give rise to (exact or approximate) closed-form expressions; i.e. probability generating functions based techniques "special case of the z-transform" (see Takagi (1993), Wittevrongel and Bruneel (2000)), or approaches based on the maximum entropy approach (see Kouvatsos *et al.* (1994)). Via this method, the unbiased distribution of the concerned queueing system is obtained by maximizing the defined entropy function in terms of known performance measures.

In line with the above, we consider in this paper a queueing model under K-variant of multiple vacations, Bernoulli feedback, server's states-dependent reneging, and retention. The analysis of such a model is complicated. Generally, the problems encountered by processes describing this kind of systems are facing a big challenge. The behavior of the suggested queueing system is studied analytically by means of a generating-functions approach. The probability generating functions (PGFs) is a powerful tools for presenting the solution of a difference equations set and solving probability problems. Through the use of PGFs, we can without difficulty convert the discrete sequence of numbers i.e., probabilities into a function of dummy variable. This results in closed-form expressions for the steady-state distributions of the system. Further, the PGFs are utilized to deduce diverse system characteristics. The obtained expressions are easy to be numerically evaluated. To the best of authors' knowledge, the existing literatures mainly focuses on impatient customers during vacation period, whereas, there have been no results presenting the analytic and computational aspects of the $M/M/1$ queueing model with Bernoulli feedback, balking, server's states-dependent reneging, and retention of reneged customers under variant of multiple vacation policy. This motivate us to investigate such a queueing model, where the stationary analysis of the queueing model is established via probability generating functions (PGFs) method. Then, various measures of effectiveness including the mean system size, the mean queue length, the mean number of customers served, and average rates of balking and reneging, are derived in terms of steady-state probabilities. Further, we carried out numerical experiments that can be very beneficial to examine the effects of the parameters of impatience timers on the performance measures in different contexts.

The paper is arranged as follows. Section 2 describes the queueing model. In Section 3, we present the theoretical analysis of the suggested queueing system. In Section 4, we deduce useful system characteristics. Section 5 is devoted to some particular cases. Section 6 presents numerical computation of the analytical results in order to show the effect of customers's impatience on the performance measures of the queueing system under consideration. Section 7 concludes the paper.

3.2 Model formulation

We analyze a $M/M/1$ Bernoulli feedback queueing system under a variant of a multiple vacation policy with balking, reneging and retention of reneging in which customers arrive at the system according to a Poisson process of rate λ.

During busy period, all customers have i.i.d. service time, assumed to be exponential with parameter μ.

Whenever the system becomes empty at a service completion instant, the sever goes on a vacation of random length, which is assumed to be exponentially distributed with rate ϕ.

If at a vacation completion instant, some customers are present in the queue, the server immediately begins the busy period. Otherwise, it will take vacations consecutively until the server has taken a maximum number of vacations (denoted by K-vacations), then the server switches to the busy period and remains idle waiting for a new arrival if the system is still empty.

In addition, we suppose that on arrival a customer either decides to join the queue with probability θ if the number of customers in the system is bigger or equal to one and may balk with probability $\overline{\theta} = 1 - \theta$.

Whenever a customer arrives at the system and finds the server on vacation (respectively, busy), he activates an impatience timer T_0 (respectively, T_1), which follows exponential distribution function with parameter ξ_0 (respectively, ξ_1). If the customer's service has not been finished before the customer's timer expires, the customer may abandon the system. That is, the customer's deadline is effective until the end of his service. This sort of customer behavior occurs often in practice including a situation where a customer's deadline corresponds to a fundamentally irreversible property such as failure or death, at which, the absence of a deadline by the customer may be seen as a permanent phenomenon, and may occur at any time including the time a customer is served. Further, each reneged customer may leave the system without getting service with probability α and may be retained in the queue with probability $\alpha' = 1 - \alpha$.

With probability $\beta^{'}$, a customer rejoin the system as a Bernoulli feedback customer to receive another regular service if the initial one is unsatisfactory or incomplete. Otherwise, he leaves the system definitively with probability $\beta = 1 - \beta^{'}$.

We also suppose that inter-arrival times, service times, impatience times, and vacation times are all mutually independent. The service order is supposed to be First-Come-First-Served (FCFS).

3.3 Steady-state solution of the queueing system

Let $N(t)$ denote the number of customers in the system at time t, and let $J(t)$ denote the state of the server at time t, which is defined as follows:

$$J(t) = \begin{cases} j, & \text{the server is taking the } (j+1)^{th} \text{ vacation at time } t \text{ for } j = 0, 1, ..., K-1, \\ K, & \text{the server is idle or busy at time } t. \end{cases}$$

Figure 3.1 depicts the state transition diagram of the queueing model under consideration.

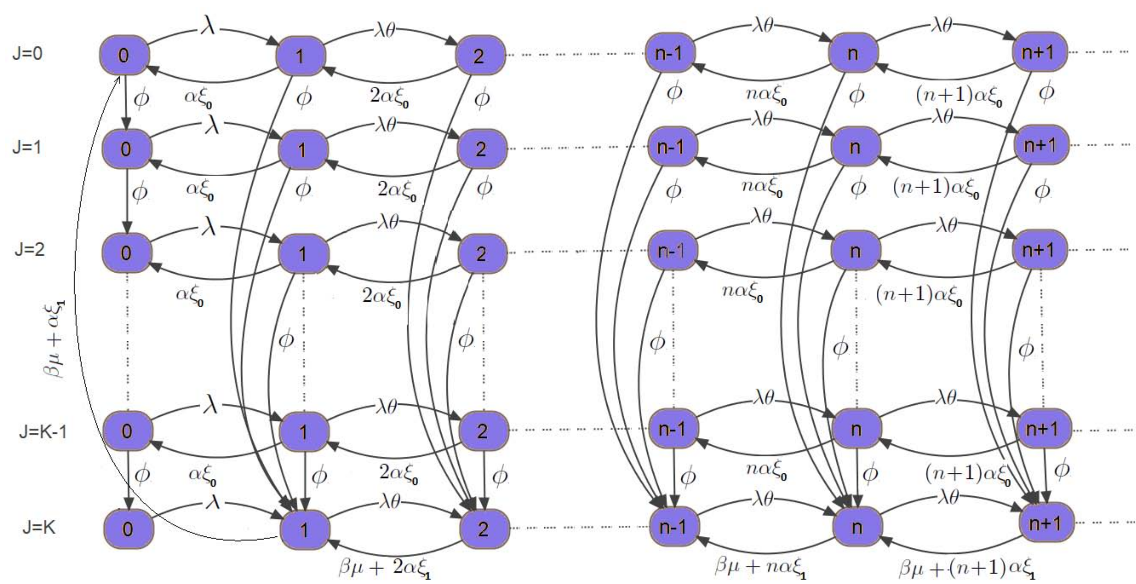

Figure 3.1: The state-transition diagram.

The pair $\{(N(t);\ J(t));\ t \geq 0\}$ defines a Markovian, continuous-time process where its state space is $\Omega = \{(n, j):\ n \geq 0;\ j = \overline{0, K}\}$.

Let $P_{n,j} = \lim\limits_{t\to\infty} \mathbb{P}(N(t) = n;\ J(t) = j,\ n \geq 0,\ j = \overline{0, K})$, denote the steady-state probabilities of the process $\{(N(t),\ J(t));\ t \geq 0\}$.

Then, based on the theory of Markov process, it is easy to show that the steady-state equations of the model are:

$$
\begin{aligned}
(\lambda+\phi)P_{0,0} &= \alpha\xi_0 P_{1,0}+(\beta\mu+\alpha\xi_1)P_{1,K}, && (3.1)\\
(\lambda+\phi+\alpha\xi_0)P_{1,0} &= \lambda P_{0,0}+2\alpha\xi_0 P_{2,0},\ n=1, && (3.2)\\
(\theta\lambda+\phi+n\alpha\xi_0)P_{n,0} &= \theta\lambda P_{n-1,0}+(n+1)\alpha\xi_0 P_{n+1,0},\ n\geq 2, && (3.3)\\
(\lambda+\phi)P_{0,j} &= \alpha\xi_0 P_{1,j}+\phi P_{0,j-1},\ j=\overline{1,K-1}, && (3.4)\\
(\theta\lambda+\phi+\alpha\xi_0)P_{1,j} &= \lambda P_{0,j}+2\alpha\xi_0 P_{2,j}, j=\overline{1,K-1},\ n=1, && (3.5)\\
(\theta\lambda+\phi+n\alpha\xi_0)P_{n,j} &= \theta\lambda P_{n-1,j}+(n+1)\alpha\xi_0 P_{n+1,j}, j=\overline{1,K-1}, n\geq 2, && (3.6)\\
\lambda P_{0,K} &= \phi P_{0,K-1}, && (3.7)\\
(\theta\lambda+\beta\mu+\alpha\xi_1)P_{1,K} &= \lambda P_{0,K}+(\beta\mu+2\alpha\xi_1)P_{2,K}+\phi\sum_{j=0}^{K-1}P_{1,j},\ n=1, && (3.8)\\
(\theta\lambda+\beta\mu+n\alpha\xi_1)P_{n,K} &= \theta\lambda P_{n-1,K}+(\beta\mu+(n+1)\alpha\xi_1)P_{n+1,K}+ \\
&\quad +\phi\sum_{j=0}^{K-1}P_{n,j}, n\geq 2. && (3.9)
\end{aligned}
$$

Next, the steady-state-probabilities are given in the following theorem.

Theorem 3.3.1. *The steady-state-probabilities $P_{n,j}$ of system size are given by*

$$
\begin{aligned}
P_{\bullet,j} &= A^{j-1}P_{0,0},\ j=\overline{0,K-1}, && (3.10)\\
P_{\bullet,K} &= \Phi(1)P_{0,0}, && (3.11)
\end{aligned}
$$

where

$$P_{0,0}=\left(\frac{1-A^K}{A(1-A)}+\Phi(1)\right)^{-1}, \tag{3.12}$$

with

$$A=\frac{\phi C_2(1)}{\alpha\xi_0 B},$$

and

$$
\begin{aligned}
\Phi(1) &= e^{\frac{\lambda\theta}{\alpha\xi_1}}\left\{\left(\frac{B\xi_0}{C_2(1)\xi_1}+\frac{\phi}{\alpha\xi_1}\left(\frac{1-A^{K-1}}{1-A}\right)\right)H_1(1)\right.\\
&\quad \left.+\frac{\phi}{\alpha\xi_1}\left(\frac{A^K}{A\lambda}H_2(1)-\left(\frac{1-A^K}{1-A}\right)H_3(1)\right)\right\},
\end{aligned}
$$

with

$$
\begin{aligned}
B &= \left(1+\frac{\lambda(1-\theta)C_1(1)}{\alpha\xi_0}\right),\\
C_1(1) &= \int_0^1 e^{\frac{-\lambda\theta}{\alpha\xi_0}s}(1-s)^{\frac{\phi}{\alpha\xi_0}}ds,\\
C_2(1) &= \int_0^1 e^{\frac{-\lambda\theta}{\alpha\xi_0}s}(1-s)^{\frac{\phi}{\alpha\xi_0}-1}ds,\\
H_1(1) &= \int_0^1 s^{\frac{\beta\mu}{\alpha\xi_1}}e^{-\frac{\lambda\theta}{\alpha\xi_1}s}(1-s)^{-1}ds,\\
H_2(1) &= \int_0^1 (\lambda\overline{\theta}s+\beta\mu)s^{\frac{\beta\mu}{\alpha\xi_1}-1}e^{-\frac{\lambda\theta}{\alpha\xi_1}s}ds,\\
H_3(1) &= \int_0^1 s^{\frac{\beta\mu}{\alpha\xi_1}}e^{-\frac{\lambda\theta}{\alpha\xi_1}s}(1-s)^{-1}\Psi(s)ds,\\
\Psi(s) &= e^{\frac{\lambda\theta}{\alpha\xi_0}s}(1-s)^{-\frac{\phi}{\alpha\xi_0}}\left\{1+\frac{\lambda\overline{\theta}}{\alpha\xi_0}C_1(s)-\frac{B}{C_2(1)}C_2(s)\right\},\\
C_1(s) &= \int_0^s e^{-\frac{\lambda\theta}{\alpha\xi_0}t}(1-t)^{\frac{\phi}{\alpha\xi_0}}dt,\\
C_2(s) &= \int_0^s e^{-\frac{\lambda\theta}{\alpha\xi_0}t}(1-t)^{\frac{\phi}{\alpha\xi_0}-1}dt.
\end{aligned}
$$

Proof. We investigate the steady-state probabilities of the system through the use of PGFs. Define the probability generating functions (PGFs) as

$$
\begin{aligned}
G_j(z) &= \sum_{n=0}^{\infty} z^n P_{n,j},\\
G_j^{'}(z) &= \frac{d}{dz}G_j(z),\ \ j=\overline{0,K}.
\end{aligned}
$$

The normalizing condition is given as

$$
\sum_{n=0}^{\infty}\sum_{j=0}^{K} P_{n,j} = 1.
$$

Multiplying equation (4.3) by z^n, using equations (4.1)–(4.2) and summing all possible values of n, we get

$$
\alpha\xi_0(1-z)G_0'(z)-[\lambda\theta(1-z)+\phi]G_0(z) = -(\beta\mu+\alpha\xi_1)P_{1,K}+\lambda\overline{\theta}(1-z)P_{0,0}. \tag{3.13}
$$

In the same manner, we obtain from equations (4.6) and (4.9), respectively.

$$
\alpha\xi_0(1-z)G_j'(z)-[\lambda\theta(1-z)+\phi]G_j(z) = \lambda\overline{\theta}(1-z)P_{0,j}-\phi P_{0,j-1},\ \ j=\overline{1,K-1}, \tag{3.14}
$$

and

$$\begin{aligned}\alpha\xi_1 z(1-z)G_K'(z) &- (1-z)(\theta\lambda z-\beta\mu)G_K(z) = (1-z)[\lambda\overline{\theta}z+\beta\mu]P_{0,K} \\ &+ z(\beta\mu+\alpha\xi_1)P_{1,K} - \phi z\sum_{j=0}^{K-1} G_j(z) + \phi z\sum_{j=0}^{K-2} P_{0,j}. \qquad (3.15)\end{aligned}$$

For $z \neq 1$, equation (4.12) can be written as follows :

$$G_0'(z) - \left[\frac{\lambda\theta}{\alpha\xi_0} + \frac{\phi}{\alpha\xi_0(1-z)}\right]G_0(z) = -\frac{\beta\mu+\alpha\xi_1}{\alpha\xi_0(1-z)}P_{1,K} + \frac{\lambda\overline{\theta}}{\alpha\xi_0}P_{0,0}. \qquad (3.16)$$

Multiply both sides of equation (4.15) by $e^{-\frac{\lambda\theta}{\alpha\xi_0}z}(1-z)^{\frac{\phi}{\alpha\xi_0}}$, we get

$$\frac{d}{dz}\left[e^{-\frac{\lambda\theta}{\alpha\xi_0}z}(1-z)^{\frac{\phi}{\alpha\xi_0}}G_0(z)\right] = e^{-\frac{\lambda\theta}{\alpha\xi_0}z}(1-z)^{\frac{\phi}{\alpha\xi_0}}\left[\frac{\lambda\overline{\theta}}{\alpha\xi_0}P_{0,0} - \frac{(\beta\mu+\alpha\xi_1)}{\alpha\xi_0(1-z)}P_{1,K}\right].$$

Then, integrating from 0 to z, we obtain

$$G_0(z) = e^{\frac{\lambda\theta}{\alpha\xi_0}z}(1-z)^{-\frac{\phi}{\alpha\xi_0}}\left\{G_0(0) + \frac{\lambda\overline{\theta}}{\alpha\xi_0}P_{0,0}C_1(z) - \frac{\beta\mu+\alpha\xi_1}{\alpha\xi_0}P_{1,K}C_2(z)\right\}, \qquad (3.17)$$

where

$$\begin{aligned}C_1(z) &= \int_0^z e^{-\frac{\lambda\theta}{\alpha\xi_0}s}(1-s)^{\frac{\phi}{\alpha\xi_0}}ds, \\ C_2(z) &= \int_0^z e^{-\frac{\lambda\theta}{\alpha\xi_0}s}(1-s)^{\frac{\phi}{\alpha\xi_0}-1}ds.\end{aligned}$$

Since $G_0(1) = \sum_{n=0}^{\infty} P_{n,0} > 0$ and $z = 1$ is the root of denominator of the right hand side of equation (4.16), we have that $z = 1$ must be the root of the nominator of the right hand side of equation (4.16). So, we obtain

$$G_0(0) = P_{0,0} = \frac{(\beta\mu+\alpha\xi_1)P_{1,K}}{\alpha\xi_0}C_2(1) - \frac{\lambda\overline{\theta}P_{0,0}}{\alpha\xi_0}C_1(1). \qquad (3.18)$$

Next, equation (4.17) implies

$$P_{1,K} = \frac{\alpha\xi_0}{(\beta\mu+\alpha\xi_1)C_2(1)}BP_{0,0}, \qquad (3.19)$$

with

$$B = \left[1 + \frac{\lambda}{\alpha\xi_0}\overline{\theta}C_1(1)\right].$$

Substituting equation (4.18) into equation (4.16), we obtain

$$G_0(z) = e^{\frac{\lambda\theta}{\alpha\xi_0}z}(1-z)^{-\frac{\phi}{\alpha\xi_0}}\left\{1+\frac{\lambda\overline{\theta}}{\alpha\xi_0}C_1(z)-\frac{B}{C_2(1)}C_2(z)\right\}P_{0,0}. \tag{3.20}$$

Equation (4.13) can be written as

$$G_j'(z)-\left[\frac{\lambda\theta}{\alpha\xi_0}+\frac{\phi}{\alpha\xi_0(1-z)}\right]G_j(z) = \frac{\lambda\overline{\theta}}{\alpha\xi_0}P_{0,j}-\frac{\phi}{\alpha\xi_0(1-z)}P_{0,j-1}. \tag{3.21}$$

In a similar manner used for solving equation (4.13), we get

$$G_j(z) = e^{\frac{\lambda\theta}{\alpha\xi_0}z}(1-z)^{-\frac{\phi}{\alpha\xi_0}}\left\{G_j(0)+\frac{\lambda\overline{\theta}}{\alpha\xi_0}C_1(z)P_{0,j}-\frac{\phi}{\alpha\xi_0}C_2(z)P_{0,j-1}\right\},\ j=\overline{1,K-1}. \tag{3.22}$$

Since $G_j(1)=\sum_{n=0}^{\infty}P_{n,j}>0$ ($G_j(1)=P_{\bullet,j}$ represents the probability that the server is taking the $(j+1)^{th}$ vacation), and $z=1$ is the root of denominator of the right hand side of equation (4.21), we have that $z=1$ must be the root of the nominator of the right hand side of equation (4.21). So, we get

$$G_j(0) = P_{0,j} = \frac{\phi C_2(1)}{\alpha\xi_0 B}P_{0,j-1} = AP_{0,j-1},\ j=\overline{1,K-1}. \tag{3.23}$$

Remark 3.3.1. *It is easy to check that* $0<\phi C_2(1)<\alpha\xi_0$ *and* $\lambda\overline{\theta}C_1(1)>0$. *Thus,* $0<\phi C_2(1)<\alpha\xi_0+\lambda\overline{\theta}C_1(1)$. *Consequently, we have* $0<A<1$.

Using equation (4.22) repeatedly, we find

$$P_{0,j} = A^jP_{0,0},\ j=\overline{1,K-1}. \tag{3.24}$$

Substituting equation (4.23) into equation (4.21), we obtain

$$G_j(z) = e^{\frac{\lambda\theta}{\alpha\xi_0}z}(1-z)^{-\frac{\phi}{\alpha\xi_0}}A^j\left\{1+\frac{\lambda\overline{\theta}}{\alpha\xi_0}C_1(z)-\frac{B}{C_2(1)}C_2(z)\right\}P_{0,0},\ j=\overline{1,K-1}. \tag{3.25}$$

Using equations (4.7) and (4.23), we obtain

$$P_{0,K} = \frac{\phi}{\lambda}A^{K-1}P_{0,0}. \tag{3.26}$$

Next, equation (4.14) can be written as

$$\begin{aligned} G_K'(z)-\left[\frac{\theta\lambda}{\alpha\xi_1}-\frac{\beta\mu}{\alpha\xi_1 z}\right]G_K(z) &= \left[\frac{\beta\mu+\alpha\xi_1}{\alpha\xi_1(1-z)}\right]P_{1,K}+\frac{\lambda\overline{\theta}z+\beta\mu}{\alpha\xi_1 z}P_{0,K} \\ &\quad +\frac{\phi}{\alpha\xi_1(1-z)}\left[\sum_{j=0}^{K-2}P_{0,j}-\sum_{j=0}^{K-1}G_j(z)\right]. \end{aligned} \tag{3.27}$$

Then, multiplying by $e^{\frac{-\theta\lambda}{\alpha\xi_1}z}z^{\frac{\beta\mu}{\alpha\xi_1}}$, we find

$$\begin{aligned}\frac{d}{dz}\left(e^{\frac{-\theta\lambda}{\alpha\xi_1}z}z^{\frac{\beta\mu}{\alpha\xi_1}}G_K(z)\right) &= e^{\frac{-\theta\lambda}{\alpha\xi_1}z}z^{\frac{\beta\mu}{\alpha\xi_1}}\left\{\left[\frac{\beta\mu+\alpha\xi_1}{\alpha\xi_1(1-z)}\right]P_{1,K}+\frac{\lambda\overline{\theta}z+\beta\mu}{\alpha\xi_1 z}P_{0,K}\right.\\ &\quad\left.+\frac{\phi}{\alpha\xi_1(1-z)}\left[\sum_{j=0}^{K-2}P_{0,j}-\sum_{j=0}^{K-1}G_j(z)\right]\right\}. \qquad (3.28)\end{aligned}$$

Then, integrating from 0 to z and using equations (4.18) and (4.23)-(4.25), we obtain

$$\begin{aligned}G_K(z) &= e^{\frac{\lambda\theta}{\alpha\xi_1}z}z^{-\frac{\beta\mu}{\alpha\xi_1}}\left\{\left(\frac{B\xi_0}{C_2(1)\xi_1}+\frac{\phi}{\alpha\xi_1}\left(\frac{1-A^{K-1}}{1-A}\right)\right)H_1(z)\right.\\ &\quad\left.+\frac{\phi}{\alpha\xi_1}\left(\frac{A^{K-1}}{\lambda}H_2(z)-\left(\frac{1-A^K}{1-A}\right)H_3(z)\right)\right\}P_{0,0}, \qquad (3.29)\end{aligned}$$

where

$$\begin{aligned}H_1(z) &= \int_0^z s^{\frac{\beta\mu}{\alpha\xi_1}}e^{-\frac{\lambda\theta}{\alpha\xi_1}s}(1-s)^{-1}ds,\\ H_2(z) &= \int_0^z(\lambda\overline{\theta}s+\beta\mu)s^{\frac{\beta\mu}{\alpha\xi_1}-1}e^{-\frac{\lambda\theta}{\alpha\xi_1}s}ds,\\ H_3(z) &= \int_0^z s^{\frac{\beta\mu}{\alpha\xi_1}}e^{-\frac{\lambda\theta}{\alpha\xi_1}s}\Psi(s)(1-s)^{-1}ds,\\ \Psi(s) &= e^{\frac{\lambda\theta}{\alpha\xi_0}s}(1-s)^{-\frac{\phi}{\alpha\xi_0}}\left\{1+\frac{\lambda\overline{\theta}}{\alpha\xi_0}C_1(s)-\frac{B}{C_2(1)}C_2(s)\right\}.\end{aligned}$$

Thus, we get $G_K(1)$; the probability that the server is busy or idle:

$$G_K(1)=P_{\bullet,K}=\Phi(1)P_{0,0}, \qquad (3.30)$$

where,

$$\begin{aligned}\Phi(1) &= e^{\frac{\lambda\theta}{\alpha\xi_1}}\left\{\left(\frac{B\xi_0}{C_2(1)\xi_1}+\frac{\phi}{\alpha\xi_1}\left(\frac{1-A^{K-1}}{1-A}\right)\right)H_1(1)\right.\\ &\quad\left.+\frac{\phi}{\alpha\xi_1}\left(\frac{A^{K-1}}{\lambda}H_2(1)-\left(\frac{1-A^K}{1-A}\right)H_3(1)\right)\right\},\\ H_1(1) &= \int_0^1 s^{\frac{\beta\mu}{\alpha\xi_1}}e^{-\frac{\lambda\theta}{\alpha\xi_1}s}(1-s)^{-1}ds,\\ H_2(1) &= \int_0^1(\lambda\overline{\theta}s+\beta\mu)s^{\frac{\beta\mu}{\alpha\xi_1}-1}e^{-\frac{\lambda\theta}{\alpha\xi_1}s}ds,\\ H_3(1) &= \int_0^1 s^{\frac{\beta\mu}{\alpha\xi_1}}e^{-\frac{\lambda\theta}{\alpha\xi_1}s}(1-s)^{-1}\Psi(s)ds.\end{aligned}$$

From equations (4.12)–(4.13), for $z = 1$, we have

$$P_{\bullet,j} = G_j(1) = A^{j-1} P_{0,0}, \; j = \overline{0, K-1}. \tag{3.31}$$

By the definition of $P_{\bullet,j}$ and using the normalizing condition, we get

$$\sum_{j=0}^{K} P_{\bullet,j} = 1.$$

From equations (4.28)–(4.31), we get

$$P_{0,0} = \left(\frac{1 - A^K}{A(1-A)} + \Phi(1) \right)^{-1}.$$

This completes the proof. □

3.4 Performance measures

Now, we present some important performance measures of the queueing model.

The mean number of customers in the system when the server is in the state j :

$$E(L_j) = \sum_{n=1}^{\infty} n P_{n,j}, j = \overline{0, K}.$$

The mean system size when the server is on vacation period :

$$E(L_V) = \sum_{j=0}^{K-1} E(L_j) = \sum_{j=0}^{K-1} \sum_{n=1}^{\infty} n P_{n,j} = \sum_{n=1}^{\infty} n \left(\sum_{j=0}^{K-1} P_{n,j} \right).$$

The mean system size when the server is on busy period :

$$E(L_K) = \sum_{n=1}^{\infty} n P_{n,K}.$$

The mean size of the system :

$$E(L) = \sum_{j=0}^{K} \sum_{n=0}^{\infty} n P_{n,j} = E(L_V) + E(L_K).$$

The probability that the system is in a vacation period :

$$P_V = \sum_{j=0}^{K-1}\sum_{n=0}^{\infty} P_{n,j} = \sum_{j=0}^{K-1} P_{\bullet,j}.$$

The probability that the server is idle and not in vacation period :

$$P_I = P_{0,K}.$$

The probability that the system is busy :

$$P_B = 1 - P_V - P_{0,K}.$$

The mean size of the queue :

$$\begin{aligned} E(L_q) &= \sum_{j=0}^{K-1}\sum_{n=1}^{\infty} nP_{n,j} + \sum_{n=1}^{\infty}(n-1)P_{n,K} \\ &= E(L) - (1 - P_V - P_{0,K}) \\ &= E(L) - P_B. \end{aligned}$$

The expected number of customers served per unit of time :

$$E_{cs} = \beta\mu P_B.$$

The average rate of balking :

$$B_r = \overline{\theta}\lambda\left(\sum_{j=0}^{K}\sum_{n=0}^{\infty} P_{n,j}\right).$$

The average rate of abandonment of a customer due to impatience :

$$\begin{aligned} R_{ren} &= \alpha\xi_0 \sum_{j=0}^{K-1}\sum_{n=1}^{\infty} nP_{n,j} + \alpha\xi_1 \sum_{n=1}^{\infty} nP_{n,K} \\ &= \alpha\xi_0 E(L_V) + \alpha\xi_1 E(L_K). \end{aligned}$$

The average rate of retention of impatient customers :

$$R_{ret} = \alpha'\xi_0 E(L_V) + \alpha'\xi_1 E(L_K).$$

3.5 Special cases

Case 1: If $\xi_1 = 0$, $K = 1$, $\theta = 1$, $\alpha = 1$ and $\beta = 1$, then, the steady-state probabilities $P_{\bullet,0}$ and $P_{\bullet,1}$ are as:

$$\begin{aligned} P_{\bullet,0} &= \frac{\xi_0}{\phi C(1)} P_{0,0}, \\ P_{\bullet,1} &= \frac{1}{\mu-\lambda}\left(\frac{\lambda\xi_0}{\phi+\xi_0} + \frac{\phi\mu C(1)}{\lambda}\right) P_{0,0}, \end{aligned}$$

where

$$\begin{aligned} P_{0,0} &= (\mu-\lambda)\left(\frac{\lambda\xi_0}{(\xi_0+\phi)C(1)} + \frac{(\mu-\lambda)}{\phi C(1)} + \frac{\phi\mu}{\lambda}\right)^{-1}, \\ C(1) &= \int_0^1 (1-s)^{\frac{\phi}{\xi_0}-1} e^{\frac{-\lambda}{\xi_0}s} ds, \end{aligned}$$

which coincide with Equations (5.8) and (5.12) of Altman and Yechiali (2006).

Case 2: When $\xi_1 = 0$, $\theta = 1$, $\alpha = 1$ and $\beta = 1$, the steady-state-probabilities of the number of customers in the system have the following form:

$$\begin{aligned} P_{\bullet,j} &= A^{j-1} P_{0,0}, \quad j = \overline{0, K-1}, \\ P_{\bullet,K} &= \frac{\phi}{\mu-\lambda}\left(\frac{\lambda(1-A^K)}{(\phi+\xi_0)A(1-A)} + \frac{\mu}{\lambda}A^{K-1}\right) P_{0,0}, \end{aligned}$$

where

$$P_{0,0} = \left\{\frac{(\mu\phi + (\mu-\lambda)\xi_0)(1-A^K)}{(\mu-\lambda)(\phi+\xi_0)A(1-A)} + \frac{\mu\phi A^{K-1}}{\lambda(\mu-\lambda)}\right\}^{-1},$$

with

$$A = \frac{\phi C(1)}{\xi_0},$$

such that

$$C(1) = \int_0^1 e^{\frac{-\lambda}{\xi_0}s} (1-s)^{\frac{\phi}{\xi_0}-1} ds.$$

The obtained results match with Equations (26), (33), and (35) in Yue *et al.* (2012).

Case 3: If $K = 1$, $\theta = 1$, $\alpha = 1$ and $\beta = 1$, the steady-state probabilities $P_{\bullet,0}$ and $P_{\bullet,1}$ are as:

$$\begin{aligned} P_{\bullet,0} &= \frac{\xi_0}{\phi C_0(1)} P_{00}, \\ P_{\bullet,1} &= \frac{e^{\frac{\lambda}{\xi_1}}}{\xi_1}\left(\xi_0 \frac{C_1(1)}{C_0(1)} - \phi C_2(1) + \frac{\phi\mu}{\lambda} C_3(1)\right) P_{0,0}, \end{aligned}$$

where

$$P_{0,0} = \left(\frac{\xi_0}{\phi C_0(1)} + \frac{e^{\frac{\lambda}{\xi_1}}}{\xi_1} \left(\xi_0 \frac{C_1(1)}{C_0(1)} - \phi C_2(1) + \frac{\phi\mu}{\lambda} C_3(1) \right) \right)^{-1},$$

with

$$\begin{aligned} C_0(1) &= \int_0^1 (1-s)^{\frac{\phi}{\xi_0}-1} e^{\frac{-\lambda}{\xi_0}s} ds, \\ C_1(1) &= \int_0^1 (1-s)^{-1} s^{\frac{\mu}{\xi_1}} e^{\frac{-\lambda}{\xi_1}s} ds, \\ C_2(1) &= \int_0^1 \left(1 - \frac{C_0(t)}{C_0(1)}\right) s^{\frac{\mu}{\xi_1}} (1-s)^{-(\frac{\phi}{\xi_0}+1)} e^{\frac{\lambda}{\xi_0} - \frac{\lambda}{\xi_1}s} ds, \\ C_3(1) &= \int_0^1 s^{\frac{\mu}{\xi_1}-1} e^{\frac{\lambda}{\xi_1}s} ds, \end{aligned}$$

which coincide with Equations (44), (43), and (45) in Yue *et al.* (2016).

3.6 Numerical analysis

To show the applicability of the theoretical results obtained previously, we present some numerical results pointing out the impact of the impatience rates on different performance measures of the considered queueing system. Numerical works have been carried out using MATLAB program. To this end, we put $\lambda = 3$, $\mu = 4$, $\phi = 0.5$, $K = 3$, $\beta = 0.4$ and $\alpha = 0.6$. The obtained results are presented in Table 3.1 and Figures 3.2–3.6.

From the numerical results given in Table 3.1 and Figures 3.2–3.7, we have

* The monotonicity of P_B, P_V, P_I, $E(L_K)$, $E(L)$, $E(L_q)$, B_r, R_{ren}, and R_{ret} with regard to ξ_0 is similar to the monotonicity of P_B, P_V, P_I, $E(L_K)$, $E(L)$, $E(L_q)$, B_r, R_{ren}, and R_{ret} with regard to ξ_1. However, $E(L_V)$ increases with the increasing of ξ_1 and decreases with ξ_0 (see Figure 3.5).

* As intuitively expected, the increasing of the impatience rates during both vacation and busy periods generate a decrease in the mean number of customers in the queue $E(L_q)$ as well as in the system $E(L)$ (see Figures 3.2–3.3). Consequently, the probability that the server is idle during busy period monotonically increase. This leads to a decrease in the average rate of balking B_r.

* As it should be, the increasing of the impatience rates ξ_0 and ξ_1 implies a diminution in the mean number of customers in the system during vacation $E(L_V)$ and busy $E(L_K)$ period, respectively (see Figures 3.5–3.6). This implies a decreasing in the probability of busy period P_B and an increasing in the vacation period P_V (see Figure 3.7).

* Obviously, the increase of ξ_0 and ξ_1 implies an increase in the average rate of reneging R_{ren} (see Figure 3.4). In this situation, the system uses certain persuasive mechanism in order to convince customers not to leave the system; R_{ret} monotonically increases with ξ_0 and ξ_1.

ξ_0	ξ_1	P_B	P_V	P_I	$E(L_V)$	$E(L_K)$	$E(L)$	$E(L_q)$	B_r	R_{ren}	R_{ret}
	1.00	0.5228	0.4765	0.0006	0.8160	1.3195	2.1355	1.6127	0.5546	0.9676	0.6451
1.00	2.00	0.4218	0.5774	0.0009	0.9827	0.8843	1.8670	1.4452	0.4913	1.1446	0.7631
	3.00	0.3749	0.6242	0.0008	1.0654	0.7178	1.7832	1.4083	0.4708	1.2564	0.8376
	4.00	0.3494	0.6497	0.0010	1.1033	0.6326	1.7358	1.3865	0.4589	1.3417	0.8944
	1.00	0.4499	0.5475	0.0026	0.5922	1.0804	1.6725	1.2226	0.4413	1.0888	0.7259
2.00	2.00	0.3530	0.6440	0.0031	0.6982	0.6856	1.3838	1.0308	0.3724	1.2370	0.8246
	3.00	0.3122	0.6845	0.0032	0.7403	0.5456	1.2859	0.9736	0.3482	1.3084	0.8723
	4.00	0.2880	0.7087	0.0034	0.7668	0.4703	1.2371	0.9492	0.3359	1.3578	0.9052
	1.00	0.4041	0.5902	0.0056	0.4672	0.9512	1.4184	1.0142	0.3762	1.1691	0.7794
3.00	2.00	0.3129	0.6806	0.0065	0.5408	0.5909	1.1317	0.8187	0.3070	1.3069	0.8713
	3.00	0.2725	0.7207	0.0067	0.5724	0.4598	1.0322	0.7596	0.2822	1.3673	0.9116
	4.00	0.2507	0.7425	0.0068	0.5904	0.3940	0.9844	0.7336	0.2701	1.4065	0.9377
	1.00	0.3748	0.6169	0.0084	0.3873	0.8739	1.2612	0.8864	0.3351	1.2290	0.8193
4.00	2.00	0.2872	0.7031	0.0096	0.4415	0.5357	0.9771	0.6899	0.2661	1.3576	0.9051
	3.00	0.2487	0.7412	0.0101	0.4657	0.4128	0.8784	0.6298	0.2413	1.4130	0.9420
	4.00	0.2274	0.7620	0.0106	0.4787	0.3506	0.8293	0.6019	0.2288	1.4446	0.9630

Table 3.1: Model characteristics vs. ξ_0 and ξ_1

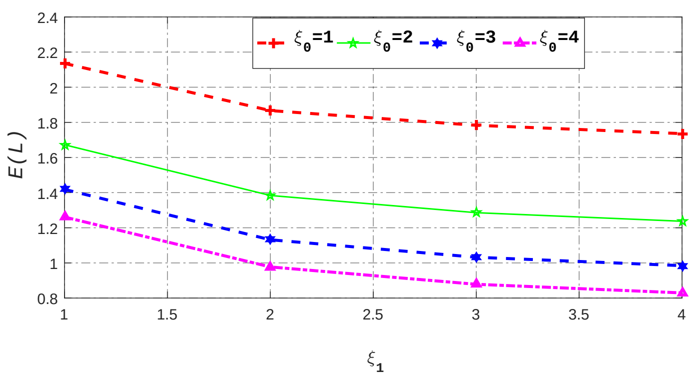

Figure 3.2: $E(L)$ vs. ξ_0 and ξ_1.

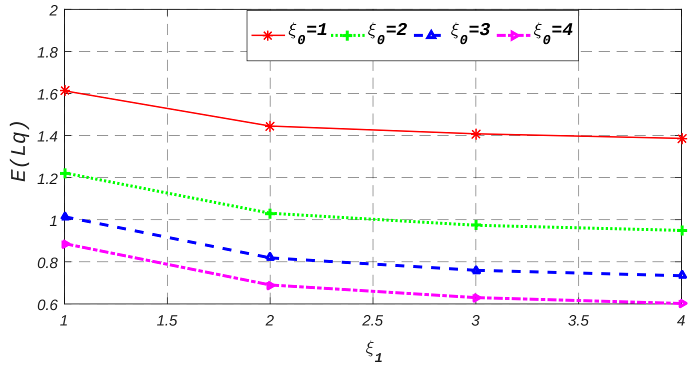

Figure 3.3: $E(L_q)$ vs. ξ_0 and ξ_1.

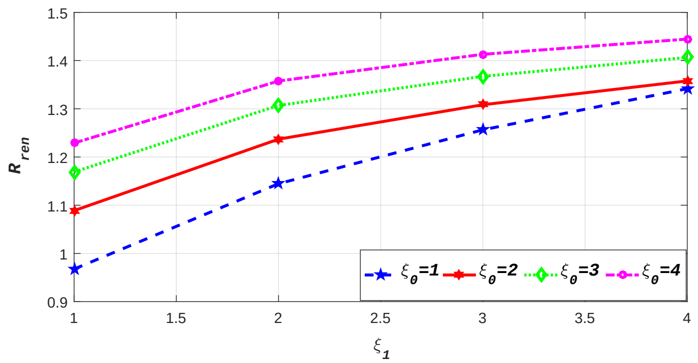

Figure 3.4: R_{ren} vs. ξ_0 and ξ_1.

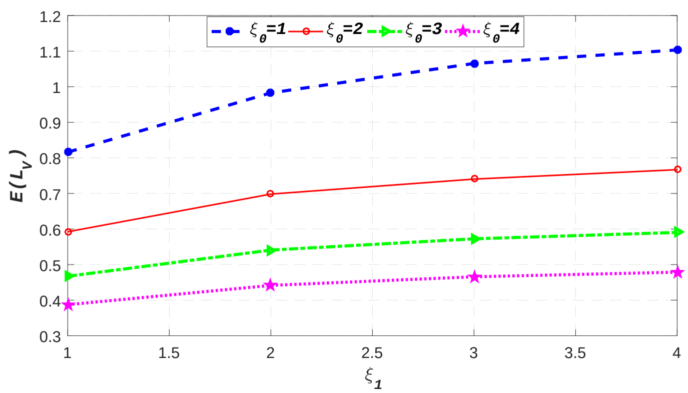

Figure 3.5: $E(L_V)$ vs. ξ_0 and ξ_1.

3.7 Conclusion

In this paper, we analyzed an $M/M/1$ Bernoulli feedback queue with balking, reneging which depends on the state of the server, and retention of reneged customers under K-variant vacation policy. The steady-state probabilities of the queueing system have

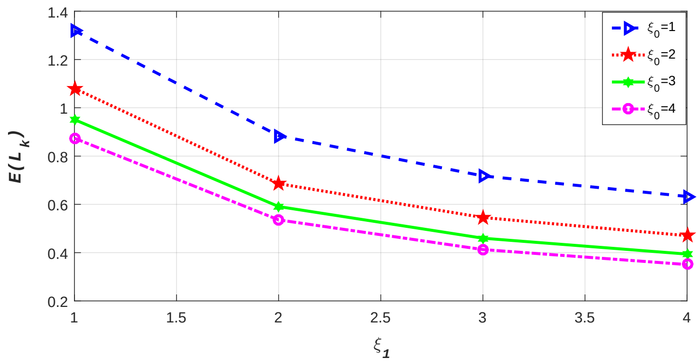

Figure 3.6: $E(L_K)$ vs. ξ_0 and ξ_1.

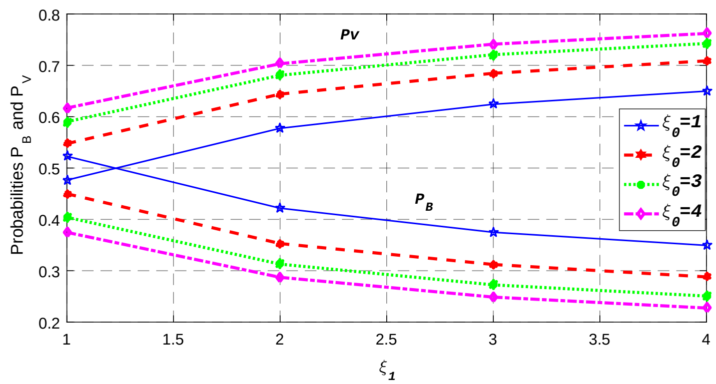

Figure 3.7: P_B and P_V vs. ξ_0 and ξ_1.

been obtained, using probability generating functions (PGFs). Then, important system characteristics have been derived. An illustrative numerical example is presented to confirm the theoretical results. Our queueing system can be considered as a generalized version of different existing queueing models presented by Altman and Yechiali (2006), Yue *et al.* (2014), and Yue *et al.* (2016). Other variations can be done on the considered queueing system, e.g., the queueing model can be extended to a state dependent arrival, state dependent service, and state dependent vacation.

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Chapter 4

Mathematical analysis of a Markovian multi-server feedback queue with a variant of multiple vacations, balking and reneging

This chapter is under consideration for possible publication.

Mathematical analysis of a Markovian multi-server feedback queue with a variant of multiple vacations, balking and reneging

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Abstract. In this paper, we analyze a multi-server queue with customers' impatience and Bernoulli feedback under a variant of multiple vacations. On arrival, a customer decides whether to join or balk the system, based on the observation of the system size as well as the status of the servers. It is supposed that customer impatience can arise both during busy and vacation period because of the long wait already experienced in the system. The latter can be retained via certain mechanism used by the system. The feedback occurs as returning a part of serviced customers to get a new service. The queue under consideration can be used to model the processes of information transmission in telecommunication networks. We develop the Chapman-Kolmogorov equations for the steady-state probabilities and solve the differential equations by using the probability generating function method. In addition, we obtain explicit expressions of some important system characteristics. Different queueing indices are derived such as the probabilities when the servers are in different states, the mean number of customers served per unit of time, and the average rates of balking and reneging.

Keywords: Markovian multi-server queue; probability generating function; impatient phenomena; sever vacations; Bernoulli feedback.

AMS Subject Classification: 60K25, 68M20, 90B22.

4.1 Introduction

Queueing models with server vacation have been efficiently studied by many researchers in the last decades and successfully applied in various practical problems such as telecommunication system design and control, manufacturing industries, and other related systems. There are two basic vacation queueing models namely, multiple vacation, and single vacation. In multiple vacation queueing models, the server continues to take successive vacations until it finds at least one customer waiting in a queue at a vacation completion epoch Arumuganathan and Ramaswami (2005), Boualem *et al.* (2009). Nevertheless, in single vacation queueing models, the server precisely takes one vacation between two consecutive busy periods. These two types of vacation models were first introduced by Levy and Yechiali (1976). Eminent literature on the subject is found in Doshi (1986), Gupta (1997), Tian and Zhang (2006), Zhang (2003a), Zhang (2003b) and others.

Over the past few years, queueing models with Bernoulli feedback have increasingly attracted the attention of many researchers Bochentouf *et al.* (2019, 2020), Bochentouf and Guendouzi (2018, 2020), Boualem *et al.* (2013), Melikov *et al.* (2020). Taking into account the feedback effect makes it possible to bring the considered models closer to a real situation, where the claims once serviced may require repeat service for different reasons. For example, in communication networks erroneously transmitted, a data is retransmitted.

In recent years, a growing body of literature has emerged on the analysis of queueing systems with impatient customers Bochentouf *et al.* (2020). This is due to their potential applications in many related areas, see for instance Benjaafar *et al.* (2010), Gans *et al.* (2003). Balking is one form of impatience, which is the reluctance of a customer to join a queue upon arrival Afroun *et al.* (2018), Boualem (2020). The other forms are reneging, the reluctance to remain in line after joining and waiting, and jockeying between lines when each of a number of parallel lines has its own queue Bochentouf *et al.* (2021), Manoharan (2011). When the impatience becomes sufficiently strong, the manager of the firm concerned has to take some measures to diminish the congestion to levels that customers can tolerate.

In most queueing situations, customers seem to get discouraged from receiving service when the server is absent and tend to leave the system without receiving service. This phenomenon is very precisely observed when the server is on vacation. This results in a potential loss of customers and customer goodwill for a service provider. For a comprehensive overview of the subject, authors may refer to Altman and Yechiali (2006, 2008), Antonis *et al.* (2011), Bouchentouf and Guendouzi (2019), Sun *et al.* (2016), Yue *et al.* (2014) , Yue *et al.* (2014), Yue *et al.* (2006). Most of the literature mentioned here studies reneging during the vacation state of the server. However, in many real-life situations, the abandonment may occur even when the system is in the busy state. For instance, incoming customers can not have any information about the state of the server, or when they are not satisfied with the service time (in particular, when they find that the server takes too much time to serve the customers). This paper contributes in this sense. In fact, only a few research papers have been done treating this case Bochentouf *et al.*(2019, 2020), Bouchentouf and Guendouzi (2018, 2020), Yue *et al.* (2016).

In this paper, we provide the analysis of a multi-server feedback queue with a variant multiple vacation policy, balking and server's states-dependent reneging. When all the customers present in the system have been served, the servers immediately leave for a vacation. If they return from a vacation to find an empty queue, they leave for another vacation; otherwise, the servers, synchronously, return to serve the queue. These latter are permitted to take a finite number, say K, of sequential vacations. It is assumed that an arriving customer who finds the system (all the servers) on vacation period (respectively, on busy period) activates an impatience timer T_{Vac} (respectively, T_{Busy}). If the customer's service has not been completed before the customer's impatience timer expires, the customer abandons the queue. The latter can be convinced to stay in the system (retained) using certain strategy. In addition, if the customer is unhappy with the service, he can rejoin the end of the queue for another one with some probability. That's what we call a feedback customer. To the best of the researchers' knowledge, the model under consideration has so far not treated in the literature of queues. Moreover, our model can be considered as a generalized version of existing queueing model given by Yue *et al.* (2014) and Bochentouf *et al.* (2021) equipped with many features and associated with many practical situations.

The rest of the paper is arranged as follows. In Section 2, we introduce the mathematical description of the model and we give a practical application. In Section 3, we develop the differential equations for the probability generating functions of the steady-state probabilities. In Section 4, we give the solution of the differential equations. In Section 5, we give the probabilities when the servers are in different states. Some essential system performance measures of this model are obtained in Section 6. Finally, we conclude the paper in Section 7.

4.2 Mathematical model description

We consider a multi-server feedback queueing system with K-variant vacation, balking and server's states-dependent reneging. The following assumptions and notations are taken into account to structure the proposed queueing system:

1. The suggested queueing system consists of c servers. Customers arrive into the system according to a Poisson process with rate $\lambda > 0$, they are served according to First-Come-First-Served (FCFS) discipline. The service times are assumed to be exponentially distributed with rate μ.

2. A multiple synchronous vacation policy is considered; once all the customers present in the system are served, the servers, all together, leave for a vacation. At the end of the vacation period, if the queue is still empty, they immediately leave for another vacation; otherwise, they return to serve the queue. The servers

are allowed to take all together K vacations sequentially. When the K consecutive vacations are complete, the servers switch to a busy period and, depending on the arrival of new customers, they stay idle or busy. The vacation period is assumed to be exponentially distributed with rate ϕ.

3. Whenever a customer arrives at the system and finds the servers on vacation period (resp. busy period), it activates an impatience timer T_{Vac} (resp. T_{Busy}), which is exponentially distributed with parameter ξ_0 (resp. ξ_1). If the customer's service has not been completed before the customer's timer expires, this later may leave the system. We suppose that the customers timers are independent and identically distributed random variables and independent of the number of waiting customers.

4. It is supposed that a system employs a certain mechanism in order to keep impatient customers in the system, that is, with some probability α', a customer may be retained in the system, and with a complementary probability α it may decide to leave to never return.

5. If, after completion of service, a customer is not happy with the quality of the service, he can return to the system with some probability β' for another service, or decide to leave the system with probability $\beta = 1 - \beta'$.

6. A customer who on arrival finds at least one customer (resp. c customers) in the system, when the servers are on vacation period (resp. busy period) either decides to enter the queue with probability θ or balk with probability $\overline{\theta} = 1 - \theta$.

All random variables presented above are mutually independent of each other.

4.2.1 Practical application of the model

The operation mode of a call center with vacation and impatience provides an initial motivation for our study; a central offices is used for receiving or transmitting a large volume of enquiries. A private branch exchange (PBX) is a private telephone network used within a company or organizations that offers various features such as transfer calls, voicemail, call recording, interactive voice menus (IVR), and call queues. It helps in making an organization's communication simpler and more robust. The incoming calls are routed to an available customer support manager drawn from the group of agents. Assume that the service facility consists in a group of c channels (servers) available to meet the demands of the requests. If an arriving call finds some servers free it immediately occupies the channel and leaves the system after service. However, the behavior of a call may vary depending on the waiting expectations provided by the call center and the personal preferences of each specific customer. Therefore, each call may decide either to balk or to wait for a while. The servers commute between busy and vacation periods in groups. When there is no demands to be handled, the latter, all together, go synchronously on vacation and come back as one station to the busy period, once the idle period ends. If there are some waiting calls at the end of the vacation period, they will be immediately served. Alternatively, they quit for another vacation period. The calls have no information on the queue length nor the state of the servers, then, an increase in the mean waiting time of a customer in the system can anticipate an increase in the average rate of reneging. Thus, to avoid losing potential customers, the system should employ some strategies by choosing the system parameter to further encourage customers to stay in the system. In the case that the service is not successful, the customer can repeat its request again and again until the service succeeds.

4.3 Governing equations

At an arbitrary time, the system state is defined by a continuous time Markov chain $\{(L(t);\ J(t));\ t \geq 0\}$ on the state space $\Omega = \{(n;j):\ n \geq 0;\ j = \overline{0,K}\}$, where $L(t)$ is the number of customers in the system and $J(t)$ is the state of the servers, i.e.,

$$J(t) = \begin{cases} j, & \text{if the servers are taking the } (j+1)^{th} \text{ vacation at time } t, \\ & j = \overline{0, K-1}, \\ K, & \text{if the servers are idle or busy at time } t. \end{cases}$$

Let $P_{n,j} = \lim\limits_{t\to\infty} \mathbb{P}(L(t) = n;\ J(t) = j)$, $n \geq 0$; $j = \overline{0,K}$, denote the steady-state probabilities of the process $\{(L(t);\ J(t));\ t \geq 0\}$. The state-transition diagram is illustrated in Figure 4.1.

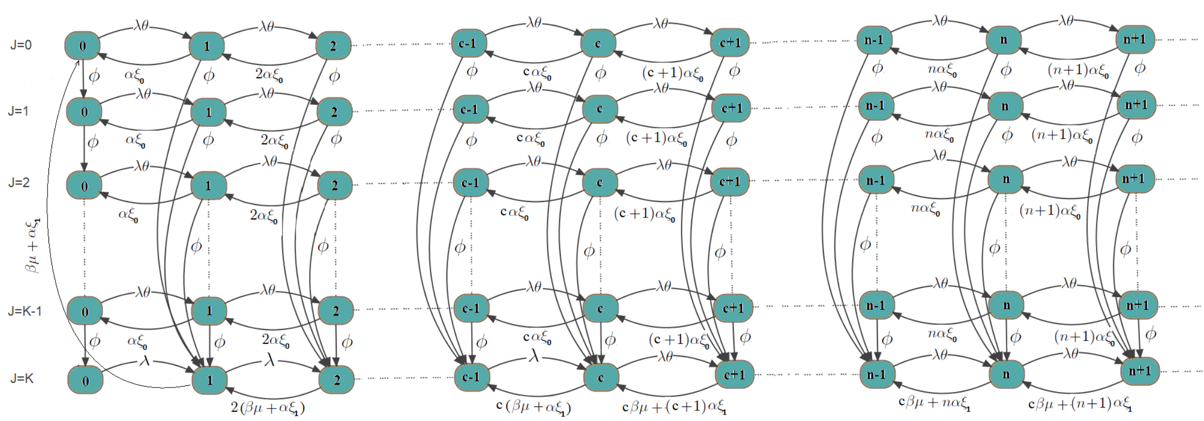

Figure 4.1: Transition plot

Using Chapman-Kolmogorov equations, we can formulate the balance equations for the suggested queueing model as:

$$(\lambda + \phi)P_{0,0} = \alpha\xi_0 P_{1,0} + (\beta\mu + \alpha\xi_1)P_{1,K},\ \ n = 0, \tag{4.1}$$

$$(\theta\lambda+\phi+\alpha\xi_0)P_{1,0}=\lambda P_{0,0}+2\alpha\xi_0P_{2,0},\ \ n=1, \tag{4.2}$$

$$(\theta\lambda+\phi+n\alpha\xi_0)P_{n,0}=\theta\lambda P_{n-1,0}+(n+1)\alpha\xi_0P_{n+1,0},\ \ n\geq 2, \tag{4.3}$$

$$(\lambda+\phi)P_{0,j}=\alpha\xi_0P_{1,j}+\phi P_{0,j-1},\ \ j=\overline{1,K-1},\ \ n=0, \tag{4.4}$$

$$(\theta\lambda+\phi+\alpha\xi_0)P_{1,j}=\lambda P_{0,j}+2\alpha\xi_0P_{2,j},\ \ j=\overline{1,K-1},\ \ n=1, \tag{4.5}$$

$$(\theta\lambda+\phi+n\alpha\xi_0)P_{n,j}=\theta\lambda P_{n-1,j}+(n+1)\alpha\xi_0P_{n+1,j},\ \ j=\overline{1,K-1},\ \ n\geq 2, \tag{4.6}$$

$$\lambda P_{0,K}=\phi P_{0,K-1},\ \ n=0, \tag{4.7}$$

$$(\lambda+\beta\mu+\alpha\xi_1)P_{1,K}=\lambda P_{0,K}+2(\beta\mu+\alpha\xi_1)P_{2,K}+\phi\sum_{j=0}^{K-1}P_{1,j},\ \ n=1, \tag{4.8}$$

$$\begin{aligned}(\lambda+n(\beta\mu+\alpha\xi_1))P_{n,K} &= \lambda P_{n-1,K}+(n+1)(\beta\mu+\alpha\xi_1)P_{n+1,K}\\ &\quad+\phi\sum_{j=0}^{K-1}P_{n,j},\ \ 2\leq n\leq c-1,\end{aligned} \tag{4.9}$$

$$\begin{aligned}(\theta\lambda+c\beta\mu+n\alpha\xi_1)P_{n,K} &= \lambda P_{n-1,K}+(c\beta\mu+(n+1)\alpha\xi_1)P_{n+1,K}\\ &\quad+\phi\sum_{j=0}^{K-1}P_{n,j},\ \ n=c,\end{aligned} \tag{4.10}$$

$$\begin{aligned}(\theta\lambda+c\beta\mu+n\alpha\xi_1)P_{n,K} &= \theta\lambda P_{n-1,K}+(c\beta\mu+(n+1)\alpha\xi_1)P_{n+1,K}\\ &\quad+\phi\sum_{j=0}^{K-1}P_{n,j},\ \ n>c.\end{aligned} \tag{4.11}$$

Consider the probability generating functions (PGFs) as:

$$G_j(z)=\sum_{n=0}^{\infty}z^nP_{n,j},$$

and define

$$G_j'(z)=\frac{d}{dz}G_j(z),\ \ j=\overline{0,K}.$$

The normalizing condition is defined as

$$\sum_{n=0}^{\infty}\sum_{j=0}^{K}P_{n,j}=1.$$

Multiplying Equation (4.3) by z^n, summing all possible values of n, and using Equations (4.1) and (4.2), we get

$$\alpha\xi_0(1-z)G_0^{'}(z) - (\theta\lambda(1-z)+\phi)G_0(z) = -(\beta\mu+\alpha\xi_1)P_{1,K} + \overline{\theta}\lambda(1-z)P_{0,0}. \tag{4.12}$$

In the same manner, from Equations (4.4)–(4.6) and (4.7)–(4.11) respectively, we obtain

$$\alpha\xi_0(1-z)G_j'(z) - [\theta\lambda(1-z)+\phi]G_j(z) = \overline{\theta}\lambda(1-z)P_{0,j} - \phi P_{0,j-1}, \quad j = \overline{1, K-1}, \tag{4.13}$$

and

$$\begin{aligned}
&\alpha\xi_1 z(1-z)G_K^{'}(z) - (1-z)(\theta\lambda z - c\beta\mu)G_K(z) = c\beta\mu(1-z)P_{0,K} \\
&+z(\beta\mu+\alpha\xi_1)P_{1,K} - \phi z\sum_{j=0}^{K-1} G_j(z) + \phi z\sum_{j=0}^{K-2} P_{0,j} + \lambda\overline{\theta}z(1-z)\Gamma_1(z) \\
&-\beta\mu(1-z)\Gamma_2(z),
\end{aligned} \tag{4.14}$$

where

$$\Gamma_1(z) = \sum_{n=0}^{c-1} z^n P_{n,K} \text{ and } \Gamma_2(z) = \sum_{n=1}^{c-1}(n-c)z^n P_{n,K}.$$

4.4 Solution of the differential equations

For $z \neq 1$, Equation (4.12) can be written as follows:

$$G_0'(z) - \left[\frac{\theta\lambda}{\alpha\xi_0} + \frac{\phi}{\alpha\xi_0(1-z)}\right]G_0(z) = -\frac{\beta\mu+\alpha\xi_1}{\alpha\xi_0(1-z)}P_{1,K} + \frac{\overline{\theta}\lambda}{\alpha\xi_0}P_{0,0}. \tag{4.15}$$

Multiply both sides of Equation (4.15) by $e^{-\frac{\theta\lambda}{\alpha\xi_0}z}(1-z)^{\frac{\phi}{\alpha\xi_0}}$, we get

$$\frac{d}{dz}\left(e^{-\frac{\theta\lambda}{\alpha\xi_0}z}(1-z)^{\frac{\phi}{\alpha\xi_0}}G_0(z)\right) = e^{-\frac{\theta\lambda}{\alpha\xi_0}z}(1-z)^{\frac{\phi}{\alpha\xi_0}}\left(\frac{\overline{\theta}\lambda}{\alpha\xi_0}P_{0,0} - \frac{(\beta\mu+\alpha\xi_1)}{\alpha\xi_0(1-z)}P_{1,K}\right).$$

Next, integrating the above equation from 0 to z, we obtain

$$G_0(z) = e^{\frac{\theta\lambda}{\alpha\xi_0}z}(1-z)^{-\frac{\phi}{\alpha\xi_0}}\left\{G_0(0) + \frac{\overline{\theta}\lambda}{\alpha\xi_0}P_{0,0}C_1(z) - \frac{\beta\mu+\alpha\xi_1}{\alpha\xi_0}P_{1,K}C_2(z)\right\}, \tag{4.16}$$

with

$$C_1(z) = \int_0^z e^{-\frac{\theta\lambda}{\alpha\xi_0}s}(1-s)^{\frac{\phi}{\alpha\xi_0}}ds \text{ and } C_2(z) = \int_0^z e^{-\frac{\theta\lambda}{\alpha\xi_0}s}(1-s)^{\frac{\phi}{\alpha\xi_0}-1}ds.$$

Since $G_0(1) = \sum_{n=0}^{\infty} P_{n,0} > 0$ and $z = 1$ is the root of denominator of the right hand side of Equation (4.16), we have that $z = 1$ must be the root of the nominator of the right hand side of Equation (4.16). So, we obtain

$$G_0(0) = \frac{(\beta\mu + \alpha\xi_1)P_{1,K}}{\alpha\xi_0} C_2(1) - \frac{\overline{\theta}\lambda P_{0,0}}{\alpha\xi_0} C_1(1), \tag{4.17}$$

where

$$C_1(1) = \int_0^1 e^{-\frac{\theta\lambda}{\alpha\xi_0}s}(1-s)^{\frac{\phi}{\alpha\xi_0}} ds \quad \text{and} \quad C_2(1) = \int_0^1 e^{-\frac{\theta\lambda}{\alpha\xi_0}s}(1-s)^{\frac{\phi}{\alpha\xi_0}-1} ds.$$

Noting $G_0(0) = P_{0,0}$. Then, Equation (4.17) implies

$$P_{1,K} = \frac{\alpha\xi_0}{(\beta\mu + \alpha\xi_1)C_2(1)} B P_{0,0} = \varpi_1 P_{0,0}, \tag{4.18}$$

with

$$B = 1 + \frac{\lambda}{\alpha\xi_0}\overline{\theta}C_1(1) \text{ and } \varpi_1 = \frac{\alpha\xi_0}{(\beta\mu + \alpha\xi_1)C_2(1)} B.$$

Substituting Equation (4.18) into Equation (4.16), we obtain

$$G_0(z) = e^{\frac{\theta\lambda}{\alpha\xi_0}z}(1-z)^{-\frac{\phi}{\alpha\xi_0}} \left\{ 1 + \frac{\overline{\theta}\lambda}{\alpha\xi_0} C_1(z) - \frac{B}{C_2(1)} C_2(z) \right\} P_{0,0}. \tag{4.19}$$

Next, Equation (4.13) can be written as

$$G_j'(z) - \left[\frac{\theta\lambda}{\alpha\xi_0} + \frac{\phi}{\alpha\xi_0(1-z)} \right] G_j(z) = \frac{\overline{\theta}\lambda}{\alpha\xi_0} P_{0,j} - \frac{\phi}{\alpha\xi_0(1-z)} P_{0,j-1}. \tag{4.20}$$

Similarly, as for Equation (4.15), we multiply both sides of Equation (4.20) by $e^{-\frac{\theta\lambda}{\alpha\xi_0}z}(1-z)^{\frac{\phi}{\alpha\xi_0}}$. Then, we find

$$G_j(z) = e^{\frac{\theta\lambda}{\alpha\xi_0}z}(1-z)^{-\frac{\phi}{\alpha\xi_0}} \left\{ G_j(0) + \frac{\lambda\overline{\theta}}{\alpha\xi_0} C_1(z) P_{0,j} - \frac{\phi}{\alpha\xi_0} C_2(z) P_{0,j-1} \right\}, \quad j = \overline{1, K-1}. \tag{4.21}$$

Since $G_j(1) = \sum_{n=0}^{\infty} P_{n,j} > 0$ ($G_j(1) = P_{\bullet,j}$ represents the probability that the servers are taking the $(j+1)^{th}$ vacation) and $z = 1$ is the root of denominator of the right hand side of Equation (4.21), we have that $z = 1$ must be the root of the nominator of the right hand side of Equation (4.21). So, we obtain

$$G_j(0) = P_{0,j} = A P_{0,j-1}, \quad j = \overline{1, K-1}, \tag{4.22}$$

where $A = \frac{\phi C_2(1)}{\alpha\xi_0 B}$. Using Equation (4.22) repeatedly, we get

$$P_{0,j} = A^j P_{0,0}, \quad j = \overline{1, K-1}. \tag{4.23}$$

Now, by substituting Equation (4.23) into Equation (4.21), we find

$$G_j(z) = e^{\frac{\theta\lambda}{\alpha\xi_0}z}(1-z)^{-\frac{\phi}{\alpha\xi_0}}A^j\left\{1+\frac{\lambda\overline{\theta}}{\alpha\xi_0}C_1(z)-\frac{B}{C_2(1)}C_2(z)\right\}P_{0,0}, \quad j=\overline{1,K-1}. \tag{4.24}$$

To find $P_{0,K}$; the probability that the servers are idle during the busy period, we use Equations (4.7) and (4.23). Thus

$$P_{0,K} = \varpi_0 P_{0,0}, \tag{4.25}$$

where $\varpi_0 = \frac{\phi}{\lambda}A^{K-1}$.

Remark 4.4.1. *It is easy to see that* $0 < \phi C_2(1) < \alpha\xi_0$, *and* $\overline{\theta}\lambda C_1(1) > 0$. *Thus,* $0 < \phi C_2(1) < \alpha\xi_0 + \overline{\theta}\lambda C_1(1)$. *Consequently, we have* $0 < A < 1$.

Next, Equation (4.14) can be written as:

$$\begin{aligned} G'_K(z)-\left(\frac{\theta\lambda}{\alpha\xi_1}-\frac{c\beta\mu}{\alpha\xi_1 z}\right)G_K(z) &= \frac{\beta\mu+\alpha\xi_1}{\alpha\xi_1(1-z)}P_{1,K}+\frac{c\beta\mu}{\alpha\xi_1 z}P_{0,K}+\frac{\lambda\overline{\theta}}{\alpha\xi_1}\Gamma_1(z) \\ &\quad -\frac{\beta\mu}{\alpha\xi_1 z}\Gamma_2(z)+\frac{\phi}{\alpha\xi_1(1-z)}\left(\sum_{j=0}^{K-2}P_{0,j}-\sum_{j=0}^{K-1}G_j(z)\right). \end{aligned} \tag{4.26}$$

In the same way, by multiplying Equation (4.13) by $\Upsilon(z) = e^{-\frac{\theta\lambda}{\alpha\xi_1}z}z^{\frac{c\beta\mu}{\alpha\xi_1}}$, we get

$$\begin{aligned} \frac{d}{dz}\left(\Upsilon(z)G_K(z)\right) &= \Upsilon(z)\left\{\frac{\beta\mu+\alpha\xi_1}{\alpha\xi_1(1-z)}P_{1,K}+\frac{c\beta\mu}{\alpha\xi_1 z}P_{0,K}+\frac{\lambda\overline{\theta}}{\alpha\xi_1}\Gamma_1(z)\right. \\ &\quad \left.-\frac{\beta\mu}{\alpha\xi_1 z}\Gamma_2(z)+\frac{\phi}{\alpha\xi_1(1-z)}\left(\sum_{j=0}^{K-2}P_{0,j}-\sum_{j=0}^{K-1}G_j(z)\right)\right\}. \end{aligned} \tag{4.27}$$

Then, integrating from 0 to z and using Equations (4.18) and (4.23)–(4.25), we obtain

$$\begin{aligned} G_K(z) &= e^{\frac{\theta\lambda}{\alpha\xi_1}z}z^{-\frac{c\beta\mu}{\alpha\xi_1}}\left\{\left((\beta\mu+\alpha\xi_1)\varpi_1+\phi\left(\frac{1-A^{K-1}}{1-A}\right)\right)H_1(z)+\frac{c\beta\mu\phi}{\lambda}A^{K-1}H_2(z)\right. \\ &\quad -\phi\left(\frac{1-A^K}{1-A}\right)H_3(z)+\frac{1}{\alpha\xi_1}\left(\lambda\overline{\theta}\int_0^z s^{\frac{c\beta\mu}{\alpha\xi_1}}e^{-\frac{\theta\lambda}{\alpha\xi_1}s}\Gamma_1(s)ds\right. \\ &\quad \left.\left.-\beta\mu\int_0^z s^{\frac{c\beta\mu}{\alpha\xi_1}-1}e^{-\frac{\theta\lambda}{\alpha\xi_1}s}\Gamma_2(s)ds\right)\right\}P_{0,0}, \end{aligned} \tag{4.28}$$

where

$$\begin{aligned}
H_1(z) &= \frac{1}{\alpha\xi_1}\int_0^z s^{\frac{c\beta\mu}{\alpha\xi_1}} e^{-\frac{\theta\lambda}{\alpha\xi_1}s}(1-s)^{-1}ds,\\
H_2(z) &= \frac{1}{\alpha\xi_1}\int_0^z s^{\frac{c\beta\mu}{\alpha\xi_1}-1} e^{-\frac{\theta\lambda}{\alpha\xi_1}s}ds,\\
H_3(z) &= \frac{1}{\alpha\xi_1}\int_0^z s^{\frac{c\beta\mu}{\alpha\xi_1}} e^{-\frac{\theta\lambda}{\alpha\xi_1}s}\Psi(s)(1-s)^{-1}ds,\\
\Psi(s) &= e^{\frac{\theta\lambda}{\alpha\xi_0}s}(1-s)^{-\frac{\phi}{\alpha\xi_0}}\left\{1+\frac{\lambda\overline{\theta}}{\alpha\xi_0}C_1(s)-\frac{B}{C_2(1)}C_2(s)\right\}.
\end{aligned}$$

4.5 Evaluation of probabilities $P_{\bullet,K}$, $P_{\bullet,j}$ and $P_{0,0}$

From Equations (4.18) and (4.25), we have

$$P_{1,K} = \varpi_1 P_{0,0} \ \text{ and } \ P_{0,K} = \varpi_0 P_{0,0}.$$

Making use of Equations (4.4)–(4.6), we recursively get

$$\sum_{j=0}^{K-1} P_{n,j} = \delta_n P_{0,0},$$

where

$$\delta_n = \frac{1}{n\alpha\xi_0}\{[\theta\lambda+\phi+(n-1)\alpha\xi_0]\delta_{n-1}-\theta\lambda\delta_{n-2}\}.$$

Similarly, from Equations (4.8)–(4.9), we recursively obtain

$$P_{n,K} = \varpi_n P_{0,0},$$

where

$$\varpi_n = \frac{1}{n(\beta\mu+\alpha\xi_1)}\{[\lambda+(n-1)(\beta\mu+\alpha\xi_1)]\varpi_{n-1}-\lambda\varpi_{n-2}-\phi\delta_{n-1}\}.$$

Thus, Equation (4.28) can be written as

$$\begin{aligned}
G_K(z) &= e^{\frac{\theta\lambda}{\alpha\xi_1}z} z^{-\frac{c\beta\mu}{\alpha\xi_1}}\left\{\left[\frac{\alpha\xi_0 B}{C_2(1)}+\phi\left(\frac{1-A^{K-1}}{1-A}\right)\right]H_1(z)+\frac{c\beta\mu\phi}{\lambda}A^{K-1}H_2(z)\right.\\
&\quad \left.-\phi\left(\frac{1-A^K}{1-A}\right)H_3(z)+\lambda\overline{\theta}H_4(z)-\beta\mu H_5(z)\right\}P_{0,0}, \qquad (4.29)
\end{aligned}$$

with

$$H_4(z) = \frac{1}{\alpha\xi_1}\int_0^z s^{\frac{c\beta\mu}{\alpha\xi_1}} e^{-\frac{\theta\lambda}{\alpha\xi_1}s}\Theta_1(s)ds,\ \ H_5(z) = \frac{1}{\alpha\xi_1}\int_0^z s^{\frac{c\beta\mu}{\alpha\xi_1}-1} e^{-\frac{\theta\lambda}{\alpha\xi_1}s}\Theta_2(s)ds,$$

$$\Theta_1(z) = \sum_{n=0}^{c-1} z^n \varpi_n, \quad \text{and} \quad \Theta_2(z) = \sum_{n=1}^{c-1} (n-c) z^n \varpi_n.$$

Thus, for $z = 1$ (noting that $G_K(1) = P_{\bullet,K}$ represents the probability that the servers are busy or idle), we get

$$G_K(1) = P_{\bullet,K} = \Phi(1) P_{0,0}, \tag{4.30}$$

where

$$\begin{aligned}\Phi(1) \quad = \quad & e^{\frac{\theta\lambda}{\alpha\xi_1}} \left\{ \left((\beta\mu + \alpha\xi_1)\varpi_1 + \phi\left(\frac{1-A^{K-1}}{1-A}\right) \right) H_1(1) + \frac{c\beta\mu\phi}{\lambda} A^{K-1} H_2(1) \right. \\ & \left. - \phi\left(\frac{1-A^K}{1-A}\right) H_3(1) + \lambda\overline{\theta} H_4(1) - \beta\mu H_5(1) \right\},\end{aligned}$$

with

$$\begin{aligned}
H_1(1) &= \frac{1}{\alpha\xi_1}\int_0^1 s^{\frac{c\beta\mu}{\alpha\xi_1}} e^{-\frac{\theta\lambda}{\alpha\xi_1}s} (1-s)^{-1} ds, \\
H_2(1) &= \frac{1}{\alpha\xi_1}\int_0^1 s^{\frac{c\beta\mu}{\alpha\xi_1}-1} e^{-\frac{\theta\lambda}{\alpha\xi_1}s} ds, \\
H_3(1) &= \frac{1}{\alpha\xi_1}\int_0^1 s^{\frac{c\beta\mu}{\alpha\xi_1}} e^{-\frac{\theta\lambda}{\alpha\xi_1}s} \Psi(s)(1-s)^{-1} ds, \\
H_4(1) &= \frac{1}{\alpha\xi_1}\int_0^1 s^{\frac{c\beta\mu}{\alpha\xi_1}} e^{-\frac{\theta\lambda}{\alpha\xi_1}s} \Theta_1(s) ds, \\
H_5(1) &= \frac{1}{\alpha\xi_1}\int_0^1 s^{\frac{c\beta\mu}{\alpha\xi_1}-1} e^{-\frac{\theta\lambda}{\alpha\xi_1}s} \Theta_2(s) ds.
\end{aligned}$$

Now, from Equations (4.12) and (4.13), for $z = 1$, we have

$$P_{\bullet,j} = G_j(1) = A^{j-1} P_{0,0}, \quad j = \overline{0, K-1}. \tag{4.31}$$

By the definition of $P_{\bullet,j}$, using the normalizing condition, we get

$$\sum_{j=0}^{K} P_{\bullet,j} = 1.$$

Finally, from Equations (4.30) and (4.31), we get

$$P_{0,0} = \left(\frac{1-A^K}{A(1-A)} + \Phi(1) \right)^{-1}.$$

4.6 Performance measures

The prime aim of determining probabilities in previous section is to formulate different metrics in order to examine the performance of the concerned system.

4.6.1 Mean system sizes

Systematic observations of the system state is very important to enhance the performance and to improve the decision-making.

Let L_j be the system size when the servers are in the state j ($j = \overline{0,K}$). Thus, $\mathbb{E}(L_j)$ is the mean system size when the servers are in the state j, defined by

$$\mathbb{E}(L_j) = G_j'(1) = \sum_{n=1}^{\infty} nP_{n,j}, \quad j = \overline{0,K},$$

that is, for $j = \overline{0,K-1}$, $\mathbb{E}(L_j)$ represents the mean system size when the servers are taking the $(j+1)th$ vacation, and $\mathbb{E}(L_K)$ represents the mean system size when the servers are busy. We first derive $\mathbb{E}(L_j)$ for $j = \overline{0,K-1}$.

From Equation (4.15), using the Hospital rule, we get

$$\begin{aligned}\mathbb{E}(L_0) = G_0^{'}(1) &= \lim_{z\to 1} \frac{-\theta\lambda G_0(z) + [\theta\lambda(1-z)+\phi]G_0^{'}(z) - \lambda\overline{\theta}P_{0,0}}{-\alpha\xi_0} \\ &= \frac{\theta\lambda G_0(1) - \phi G_0^{'}(1) + \overline{\theta}\lambda P_{0,0}}{\alpha\xi_0}.\end{aligned}$$

Thus, we get

$$G_0'(1) = \frac{\theta\lambda G_0(1) + \lambda\overline{\theta}P_{0,0}}{\alpha\xi_0 + \phi}. \tag{4.32}$$

Similarly, from Equation (4.13), we find

$$(\alpha\xi_0 + \phi)G_j'(1) = \theta\lambda G_j(1) + \lambda\overline{\theta}P_{0,j}, \quad j = \overline{1,K-1}. \tag{4.33}$$

Then, from Equations (4.32) and (4.33), we have

$$\mathbb{E}(L_j) = G_j'(1) = \frac{\lambda[\theta G_j(1) + \overline{\theta}P_{0,j}]}{\alpha\xi_0 + \phi}, \quad j = \overline{0,K-1}. \tag{4.34}$$

By substituting Equation (4.31) and (4.34), we get

$$\mathbb{E}(L_j) = \frac{\lambda}{\alpha\xi_0 + \phi}\left[\frac{\theta + \overline{\theta}A}{A}\right]A^j P_{0,0}, \quad j = \overline{0,K-1}.$$

Thus, the mean system size when the servers are on vacation is obtained as

$$\begin{aligned}
\mathbb{E}(L_V) &= \sum_{j=0}^{K-1}\mathbb{E}(L_j) = \mathbb{E}(L_0) + \sum_{j=1}^{K-1}\mathbb{E}(L_j) \\
&= \frac{\lambda(\theta A^{-1}+\overline{\theta})}{(\alpha\xi_0+\phi)}P_{0,0} + \frac{\lambda}{(\alpha\xi_0+\phi)}\left[\frac{\theta+\overline{\theta}A}{A}\right]\sum_{j=1}^{K-1}A^j P_{0,0} \\
&= \left(\frac{\lambda(\theta+\overline{\theta}A)}{\alpha\xi_0+\phi}\right)\left\{\frac{2-(A+A^{K-1})}{A(1-A)}\right\}P_{0,0}.
\end{aligned}$$

Next, from Equation (4.26) and by using the Hospital rule, we get

$$\begin{aligned}
\mathbb{E}(L_K) &= \lim_{z\to 1} G_K^{'}(z) \\
&= \frac{1}{\alpha\xi_1}\left\{(\theta\lambda - c\beta\mu)\Phi(1) + c\beta\mu\frac{\phi}{\lambda}A^{K-1} + \frac{\lambda\phi(\theta+\overline{\theta}A)}{\alpha\xi_0+\phi}\left(\frac{1-A^K}{A(1-A)}\right)\right\}P_{0,0} \\
&\quad + \frac{1}{\alpha\xi_1}\left\{\overline{\theta}\lambda\Theta_1(1) - \beta\mu\Theta_2(1)\right\}P_{0,0},
\end{aligned}$$

where $\Theta_1(1) = \sum_{n=0}^{c-1}\varpi_n$ and $\Theta_2(1) = \sum_{n=1}^{c-1}(n-c)\varpi_n$.

4.6.2 Queueing model indices

The expressions for the mean queue length, the mean number of customers served and the average rates of impatient customers are established as follows:

- The mean size of the queue is calculated as

$$\begin{aligned}
\mathbb{E}(L_q) &= \sum_{j=0}^{K-1}\sum_{n=1}^{\infty} nP_{n,j} + \sum_{n=c}^{\infty}(n-c)P_{n,K} \\
&= \mathbb{E}(L) - c + \left\{c\left[\frac{1-A^K}{A(1-A)} + \frac{\phi}{\lambda}A^{K-1}\right] - \Theta_2(1)\right\}P_{0,0}.
\end{aligned}$$

- The mean number of customers served per unit of time is given as

$$\begin{aligned}
E_{cs} &= \beta\mu\sum_{n=1}^{c-1} nP_{n,K} + c\beta\mu\sum_{n=c}^{\infty}P_{n,K} \\
&= \beta\mu\left\{c + \left[\Theta_2(1) - c\left(\frac{\phi}{\lambda}A^{K-1} + \frac{1-A^K}{A(1-A)}\right)\right]P_{0,0}\right\}.
\end{aligned}$$

• The average rate of balking when the servers are in the state $j = \overline{0,K}$ is calculated as

$$\begin{aligned} B_r &= \overline{\theta}\lambda\left(\sum_{j=0}^{K-1}\sum_{n=1}^{\infty}P_{n,j} + \sum_{n=c}^{\infty}P_{n,K}\right) \\ &= \overline{\theta}\lambda\left\{1 - \left[\frac{2 - A - A^{K-1} + (1-A)\Theta_1(1)}{(1-A)}\right]P_{0,0}\right\}. \end{aligned}$$

• The average rate of abandonment of a customer due to reneging is as follows

$$\begin{aligned} R_{ren} &= \sum_{j=0}^{K-1}\sum_{n=1}^{\infty} n\alpha\xi_0 P_{n,j} + \sum_{n=1}^{\infty} n\alpha\xi_1 P_{n,K} \\ &= \alpha\xi_0\mathbb{E}(L_V) + \alpha\xi_1\mathbb{E}(L_K). \end{aligned}$$

4.7 Conclusion

In this paper, we studied an $M/M/c$ feedback queue under synchronous K-variant vacations, balking, server's states-dependent reneging and retention of reneged customers. We developed the Chapman-Kolmogorov equations for the steady-state probabilities and solved the differential equations by using the probability generating function method. Based on these results, we obtained the probability generating function of the number of customers in the system when the system is on vacation period (resp. on busy period). In addition, we derived explicit expressions of some useful performance measures for the system. Furthermore, we presented closed-form expressions of some important other queueing indices such as the probabilities when the servers are in different states, the proportion of customers served per unit of time, and the average rates of balking and reneging.

It would be interesting to investigate a similar model with two-phase services and multiple vacation policy, server breakdown and repair, and customers' impatience. Further, one can evaluate the optimality of service and repair rates to minimize the waiting time of the customers in the system.

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Conclusion and future work

In this thesis, we dealt with different queueing models with server vacation and impatient customers, we combine various features, including reneging during busy and vacation periods, feedback, and retention of reneged customers. Then, we analysed the impact of these features on the system performance measures such as mean system sizes during busy and vacation periods, mean queue length, mean number of customers served and average rate of balking.
In the following, we first recapitulate the pricipal conclusions of this thesis, then we suggest few possible extensions that may contribute to the literature on the queueing systems with vacation and impatient customers.

In Chapter 2, we studied a single server Markovian Bernoulli feedback queueing system under two differentiated multiple vacations and balked customers. The steady-state solution was obtained. Important performance measures were derived and the economic model analysis has been carried out.
In Chapter 3, we analyzed a single server Markovian Bernoulli feedback queue with balking, reneging which depends on the state of the server, and retention of reneged customers, under K-variant vacation policy. The steady-state probabilities of the queueing system have been obtained, using probability generating functions. Then, important system characteristics have been derived. Our queueing system can be considered as a generalized version of different existing queueing models presented by Altman and Yechiali (2006), Yue et al. (2014) and Yue et al. (2016).
In Chapter 4, we considered an infinite-buffer multi-server queueing system subjected to synchronous K-variant vacations, Bernoulli feedback, balking and server's states-dependent reneging. We derived the PGF of the number of customers in the system when the system is in busy/vacation period and calculated values of key performance measures such as mean system size during busy and vacation periods, mean queue length, mean number of customers served, average rate of balking and average rate of reneging.

The studies presented in this thesis can be extended to more complex queueing models including queueing systems with K-variant of multiple working vacations, vacation interruption, and breackdowns, $G^X/G/c$ vacation queueing systems with K-variant vacation, and failures, etc.

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