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Etude de l'existence globale et de la stabilisation de certains problèmes d'évolution de type hyperbolique

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Contents

Notation-Guide to the reader

$|\cdot|$ Absolute value or the norm of a vector.

Ω Bounded domain in $\mathbb{R}^d, d \in \mathbb{N}^*$.

$\partial\Omega$ Topological boundary of Ω.

$x = (x_1, x_1, ..., x_N)$ Generic point of $\mathbb{R}^d$.

$dx = dx_1 dx_1 ... dx_d$ Lebesgue measuring on Ω.

∂_x^α Partial derivatives $\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} ... \partial_{x_d}^{\alpha_d}$ with a multi-index $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d)$, where α_i is nonnegative for all $i = 1, 2, ..., d$.

Δ Laplace operator in $\mathbb{R}^d$, i.e., $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + ... + \partial_{x_d}^2$.

$\nabla u(x) = \left(\frac{\partial u}{\partial x_i}(x) \right)_{1 \le i \le d}$.

$C_0^\infty(\Omega)$ (or $\mathcal{D}(\Omega)$) Space of infinitely differentiable functions having compact support.

$\mathcal{D}'(\Omega)$ Distribution space.

$C^k(\Omega)$ Space of functions k-times continuously differentiable in Ω.

$a.e.$ Almost everywhere.

$L^p(\Omega)$ Space of functions $p - th$ power integrated on Ω with measure of dx.

$\|u\|_p = \left(\int_\Omega |u(x)|^p \right)^{\frac{1}{p}}$.

$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega), \frac{\partial u}{\partial x_i} \in L^p(\Omega), i = 1, ..., d \right\}$.

$\|u\|_{1,p} = \left(\|u\|_p^p + \|\nabla u\|_p^p \right)^{\frac{1}{p}}$.

$W_0^{1,p}(\Omega)$ The closure of $D(\Omega)$ in $W^{1,p}(\Omega)$.

H Hilbert space.

$H^1(\Omega) = W^{1,2}(\Omega)$.

$H_0^1(\Omega)$ The closure of $D(\Omega)$ in $W^{1,2}(\Omega)$.

If X is a Banach space

$L^p(0, T; X) = \{f :]0, T[\to X$ is measurable; $\int_0^T \|f(t)\|_X^p dt < \infty\}$.

$L^\infty(0, T; X) = \{f : [0, T] \to X$ is measurable; $\sup_{t \in [0,T]} ess \|f(t)\|_X^p\}$.

$C^k([0, T]; X)$ Space of functions k-times continuously differentiable for $[0, T] \to X$.

$D([0;T];X)$ Space of functions continuously differentiable with compact support in $[0,T]$.

$B_X = \{x \in X; \|x\| \leq 1\}$ unit ball.

Introduction

Problems of global existence and stability in time of Partial Differential Equations made object, recently, of many work. In this thesis we were interested in study of the global existence and the stabilization of some evolution equations. The purpose of stabilization is to attenuate the vibrations by feedback, thus it consists in guaranteeing the decrease of energy of the solutions to 0 in a more or less fast way by a mechanism of dissipation.

More precisely, the problem of stabilization consists in determining the asymptotic behaviour of the energy by $E(t)$, to study its limits in order to determine if this limit is null or not and if this limit is null, to give an estimate of the decay rate of the energy to zero.

This problem has been studied by many authors for various systems. In our study, we obtain severals type of stabilization
1 Strong stabilization: $E(t) \to 0$, as $t \to \infty$.
2 Logarithmic stabilization: $E(t) \leq c(\log(t))^{-\delta}, \forall t > 0, (c, \delta > 0)$.
3 polynomial stabilization: $E(t) \leq ct^{-\delta}, \forall t > 0, (c, \delta > 0)$
4 uniform stabilization: $E(t) \leq ce^{-\delta t}, \forall t > 0, (c, \delta > 0)$.

In this thesis, the main objective is to give a global existence and stabilization results.
This work consists in three chapters,

• The **chapter 1**

This chapter is devoted to some notations and preliminaries, especially we recall some basic knowledge in functional analysis.

• The **chapter 2**

we consider the initial boundary value problem for the following integro-differential problem

$$\begin{cases} u_{tt} - \psi\left(\|\nabla u\|_2^2\right)\Delta u - \alpha\Delta u_t + g * \Delta u + M\left(\|\nabla u\|_2^2\right)u_t = f(u), \ x \in \Omega, \ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u(x,t) = 0, \quad x \in \partial\Omega, \ t > 0, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^d$ $(d \geq 1)$ with smooth boundary $\partial\Omega$. $\psi(r)$ is a positive locally Lipschitz function satisfying $\psi(r) \geq m_0 > 0$, for $r \geq 0$ like $\psi(r) = m_0 + br^\gamma$, $b \geq 0$ and $\gamma \geq 1$. $M(r)$ is a $C^1[0,\infty)$ -function satisfying $M(r) \geq m_1 > 0$ for $r \geq 0$, f is a non-linear function as similar to $|u|^{p-2}u$, $p > 2$ and $\alpha \geq 0$. The scalar function $g(s)$ (so-called relaxation kernel) is assumed to satisfy (2.1) and

$$g * \Delta u(t) = \int_0^t g(t-\tau)\Delta u(\tau)d\tau.$$

Firstly, we prove the local existence of solutions by using the Faedo-Galerkin approximation method and Contraction Mapping Theorem. By virtue of the potential well theory, we then prove that if the initial data enter into the stable set, then the solution globally exists. Furthermore, we study the asymptotic behavior of solution using a perturbed energy method.

• The **chapter 3**

This chapter is concerned with the study of the global existence and asymptotic behavior of solution for the Euler-Bernoulli viscoelastic equation:

$$\begin{cases} u_{tt} + \Delta^2 u - g_1 * \Delta^2 u + g_2 * \Delta u + u_t = 0, \quad x \in \Omega, \ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u = 0, \quad \frac{\partial u}{\partial \nu} = 0, x \in \partial\Omega, \ t > 0, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^d$ with a smooth boundary $\partial\Omega$, and ν is the unit outer normal on $\partial\Omega$. Here g_1 and g_2 are positive functions satisfying some conditions to be specified later, and

$$g_i * \chi(t) = \int_0^t g_i(t-\tau)\chi(\tau)d\tau, \quad i = 1, 2.$$

We study the existence of both strong and weak solutions of problem (1) for a bounded domain, then for $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ a increasing C^2 function such that

$$\xi(0) = 0, \quad \xi'(0) > 0, \quad \lim_{t\to+\infty} \xi(t) = +\infty, \quad \xi''(t) < 0 \quad \forall t \geq 0. \tag{1}$$

the solution features the asymptotic behavior

$$E(t) \leq E(0)e^{-\kappa\xi(t)}, \quad \forall t \geq 0.$$

Chapter 1

Preliminaries

In this chapter we introduce some definitions and notations which will often be used in the sequel. For a complete presentation we refer the interested reader to, e.g., Yosida (1974) or Brezis (1983) (see also Brezzi and Gilardi (1987) for a comprehensive and easy-to read proofless presentation).

1.1 Hilbert and Banach Spaces

Let V be a (real) linear space. A scalar product on V is a bilinear map
$(\cdot,\cdot) : V \times V \longrightarrow \mathbb{R}$ that $(w, v) = (v, w)$ for each $w, v \in V$ (symmetry),
$(v, v) \geq 0$ for each $v \in V$ (positivity),
$(v, v) = 0$ if and only if $v = 0$.
A semi-norm is a map $\|\cdot\| : V \longrightarrow \mathbb{R}$ such that $\|v\| \geq 0$ for each $v \in V$,
$\|cv\| = |c|\|v\|$ for each $c \in \mathbb{R}$ and $v \in V$,
$\|w + v\| \leq \|w\| + \|v\|$ for each $w, v \in V$ (triangular inequality).
A norm on V is a semi-norm satisfying the additional property that
$(\|v\| = 0) \Leftrightarrow (v = 0)$
Two norms $\|\cdot\|$ and $|||\cdot|||$ on V are equivalent if there exist two positive constants

M_1 and M_2 such that

$$M_1\|v\| \leq |||v||| \leq M_2\|v\| \quad \text{for each } v \in V.$$

It is readily verified that at any scalar product it is associated a norm through the following definition: $\|v\| = (v,v)^{\frac{1}{2}}$. Moreover, at any norm we can associate a distance: $d(w,v) = \|w - v\|$.
A linear space V endowed with a scalar (respectively, a norm) is called pre-hilbertian (respectively, normed) space. A sequence $(v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in a normed space V if it is a Cauchy sequence with respect to the distance $d(w,v) = \|w - v\|$. If any Cauchy sequence in a pre-hilbertian (normed) space V is convergent, the space V is called a Hilbert (respectively, Banach) space.
In a Hilbert space the Schwarz inequality holds:

$$|(w,v)| \leq \|w\|\|v\| \quad \text{for each } w, v \in V. \tag{1.1}$$

1.2 Dual spaces

If $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ are normed spaces, we denote by $\mathcal{L}(V,W)$ the set of linear continuous functionals from V into W, and for $L \in \mathcal{L}(V,W)$ we define the norm

$$\|L\||_{\mathcal{L}(V,W)} := \sup_{\substack{v\in V\\ v\neq 0}} \frac{\|Lv\|_W}{\|v\|_V}. \tag{1.2}$$

Thus $\mathcal{L}(V,W)$ is a normed space; if W is a Banach space, then $\mathcal{L}(V,W)$ is a Banach space, too. If $W = \mathbb{R}$, the space $\mathcal{L}(V,\mathbb{R})$ is called the dual space of V and is denoted by V'.
The bilinear form $\langle\cdot,\cdot\rangle$ from $V' \times V$ into $\mathbb{R}$ defined by $\langle L, v\rangle := L(v)$ is called the duality pairing between V' and V. As a consequence of the Riesz representation

theorem, if V is a Hilbert space, the dual V' is a Hilbert space which can be canonically identified to V.

1.2.1 weak and weak star convergence

In a normed space V it is possible to introduce another type of convergence, which is called weak convergence.

Definition 1.1. *A sequence v_n is called weakly convergent to $v \in V$ if $L(v_n)$ converges to $L(v)$ for each $L \in V'$.*

It can be proven that the weak limit v, if it exists, is unique. Clearly, converse is not true unless V is finite dimensional.

Definition 1.2. *A sequence of functionals $L_n \in V'$ is called weakly star convergent to $L \in V'$ if $L_n(v)$ converges to $L(v)$ for each $v \in V$.*

Also the weak star limit L, if it is exists, it is unique. Moreover, it can be shows that the weak convergence in V' implies the weak star convergence.

1.3 L^p spaces

We now introduce some spaces of functions which are the basis for the modern theory of partial differential equations. Let Ω be an open set contained in $\mathbb{R}^d$, $d \geq 1$, and consider in Ω the Lebesgue measure. A very important family of Banach spaces is the following one. Let $1 \leq p \leq \infty$, and consider the set of measurable functions u such that

$$\int_\Omega |u(x)|^p dx < \infty, \quad 1 \leq p < \infty, \tag{1.3}$$

or, when $p = \infty$,

$$\sup \{|u(x)|, x \in \Omega\}. \tag{1.4}$$

These spaces are usually denoted by $L^p(\Omega)$ and the associated norm is

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}}, \quad 1 \le p < \infty, \tag{1.5}$$

or, when $p = \infty$,

$$\|u\|_{L^\infty(\Omega)} := \sup \{|u(x)|, x \in \Omega\}. \tag{1.6}$$

More precisely, $L^p(\Omega)$ is indeed the space of classes of equivalence of measurable functions, satisfying (1.3) or (1.4) with respect to the equivalence relation: $u \equiv v$ if v and u are different on a subset having zero measure.
The space $L^2(\Omega)$ is indeed a Hilbert space, endowed with the scalar product

$$(w, v)_{L^2(\Omega)} = \int_\Omega w(x)v(x)dx.$$

For reasons which will be clear in the sequel, the norm in $L^2(\Omega)$ is denoted $\|\cdot\|_2$. Moreover, the scalar product $(\cdot,\cdot)_{L^2(\Omega)}$ is often indicated by $(\cdot,\cdot)$.
If $1 \le p < \infty$, the dual space $L^p(\Omega)$ is given by $L^q(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$ (and $q = \infty$ if $p = 1$). Moreover, the Hölder inequality holds:

$$\left| \int_\Omega w(x)v(x)dx \right| \le \|w\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}. \tag{1.7}$$

Notice that for $p = 2$ the Hölder inequality is the Schwarz inequality (1.1) for the Hilbert $L^2(\Omega)$.
Moreover, from (1.7) it easily follows that $L^q(\Omega) \subset L^p(\Omega)$ if $p \le q$ and Ω has finite measure.
Similarly, for a Banach space $V, k \in \mathbb{N}$ and $-\infty < a < b < +\infty$, we denote by $C([a, b]; V)$ (respectively $C^k([a, b]; V)$) the space of continuous functions (respectively the space of k times continuously differentiable functions) v from $[a, b]$ into V, which are Banach spaces, respectively, for the norms

$$\|v\|_{\mathcal{C}(a,b;V)} = \sup_{t \in (a,b)} \|v(t, .)\|_V, \quad \|v\|_{\mathcal{C}^k(a,b;V)} = \sum_{i=0}^{k} \left\| \frac{\partial^i v}{\partial t^i} \right\|_{\mathcal{C}(a,b;V)}$$

1.4 Distributions

Let us recall that $C_0^\infty(\Omega)$ (or $\mathcal{D}(\Omega)$) denotes the space of infinitely differentiable functions having compact support. The topology on $\mathcal{D}(\Omega)$ corresponds to the following notion of convergence of test functions: $v_n \in \mathcal{D}(\Omega)$ converges to $v \in \mathcal{D}(\Omega)$ if exists a compact set $K \subset \Omega$ such that supp $v_n \subset K$ for every $n \in \mathbb{N}$ and for every non-negative multi-index α the derivative $D^\beta v_n$ converges to $D^\beta v$ uniformly in Ω. We recall that if $\beta = (\beta_1, ..., \beta_d)$, β_i non-negative integers, then

$$D^\beta v := \frac{\partial^{|\beta|} v}{\partial x_1^{\beta_1} ....\partial x_d^{\beta_d}},$$

where $|\beta| := \beta_1 + ... + \beta_d$ is the length of β.

Definition 1.3. *The space of linear functionals on $\mathcal{D}(\Omega)$ which are continuous with respect to the convergence introduced above is denoted by $\mathcal{D}'(\Omega)$ and its elements are called distribution.*

If $L \in \mathcal{D}'(\Omega)$ and $v \in \mathcal{D}(\Omega)$, we usually denote $L(v)$ by the duality pairing $\langle L, v\rangle$.
If $L \in \mathcal{D}'(\Omega)$ and $f \in \mathcal{C}^\infty(\Omega)$, we define $fL \in \mathcal{D}'(\Omega)$ by

$$\langle fL, v\rangle = \langle L, fv\rangle.$$

If $\beta \in \mathbb{N}^d$ is any multi-index, we define the derivative $\partial^\beta L \in \mathcal{D}'(\Omega)$ by

$$\langle \partial^\beta L, v\rangle = (-1)^{|\beta|}\langle L, \partial^\beta v\rangle.$$

We say that a sequence of distributions (L_n) converges to a distribution L in $\mathcal{D}'(\Omega)$, written $L \rightharpoonup L$, if

$$\langle L_n, v\rangle = \langle L, v\rangle \quad \text{for every } v \in \mathcal{D}(\Omega).$$

Proposition 1.1. *The distribution* $L \in D'(\Omega)$ *is in* $L^p(\Omega)$ *if there exists a function* $\phi \in L^p(\Omega)$ *such that*

$$\langle L, \varphi \rangle = \int_\Omega \phi(x)\varphi(x)dx, \textit{ for all } \varphi \in D(\Omega),$$

where $1 \leq p \leq \infty$ *and it's well-known that* ϕ *is unique.*

1.5 Sobolev spaces

We finally introduce another class of functions, which will be most often used in the sequel, since they furnish the natural environment for the variational theory of partial differential equations. A comprehensive presentation of these spaces can be found in Adams (1975).

The Sobolev space $W^{m,p}(\Omega)$, m a non-negative integer and $1 \leq p \leq \infty$, is the space of functions $v \in WL^p(\Omega)$ such that all the distributional derivatives of v of order up to m are a function of $L^p(\Omega)$. In short

$$W^{m,p}(\Omega) := \{v \in L^p(\Omega),\ D^\beta v \in L^p(\Omega),\ \ \forall \beta \in \mathbb{N}^d,\ |\beta| \leq m\}.$$

Clearly, for each p, $1 \leq p \leq \infty$, $W^{0,p}(\Omega) = L^p(\Omega)$ and $W^{m_2,p}(\Omega) \subset W^{m_1,p}(\Omega)$ when $m_1 \leq m_2$. For $1 \leq P < \infty$, $W^{m,p}(\Omega)$ is a Banach space with respect to the norm

$$\|v\|_{W^{m,p}(\Omega)} := \Big(\sum_{|\beta| \leq m} \|D^\beta v\|^p_{L^p(\Omega)} \Big)^{1/p}.$$

On the other hand, $W^{m,\infty}(\Omega)$ is a Banach space with respect to the norm

$$\|v\|_{W^{m,\infty}(\Omega)} := \max_{|\beta|=m} \|D^\beta v\|_{L^\infty(\Omega)}.$$

Sobolev spaces with $p = 2$ are especially important because of their connection with Fourier series and because they form a Hilbert space. A special notation has arisen to cover this case:

$$W^{m,2}(\Omega) = H^m(\Omega)$$

the H^m inner product is defined in terms of the L^2 inner product:

$$(w,v)_{H^m(\Omega)} = \sum_{|\beta|\leq m} (D^\beta w, D^\beta v)_{L^2(\Omega)}.$$

The space $H^m(\Omega)$ and $W^{m,p}(\Omega)$ contain $\mathcal{C}^\infty(\overline{\Omega})$ and $\mathcal{C}^m(\overline{\Omega})$. The closure of $\mathcal{D}(\Omega)$ for the $H^m(\Omega)$ norm (respectively $W^{m,p}(\Omega)$ norm) is denoted by $H_0^m(\Omega)$ (respectively $W_0^{m,p}(\Omega)$).

Now, we introduce a space of functions with values in a space V (a separable Hilbert space).

The space $L^2(a,b;V)$ is a Hilbert space for the inner product

$$(w,v)_{L^2(a,b;V)} = \int_a^b (w(t),v(t))_V dt$$

We note that $L^\infty(a,b;V) = (L^1(a,b;V))'$.

Now, we define the Sobolev spaces with values in a Hilbert space V

For $m \in \mathbb{N}$, $p \in [1,\infty]$, we set:

$$W^{m,p}(a,b;V) = \left\{v \in L^p(a,b;V); \frac{\partial^i v}{\partial t^i} \in L^p(a,b;V).\ \forall i \leq m\right\},$$

The Sobolev space $W^{m,p}(a,b;V)$ is a Banach space with the norm

$$\|v\|_{W^{m,p}(a,b;V)} = \left(\sum_{i=0}^m \left\|\frac{\partial^i v}{\partial t^i}\right\|^p_{L^p(a,b;V)}\right)^{1/p}, \text{ for } p < +\infty$$

$$\|v\|_{W^{m,\infty}(a,b;V)} = \sum_{i=0}^m \left\|\frac{\partial^i v}{\partial t^i}\right\|_{L^\infty(a,b;V)}, \qquad \text{for } p = +\infty.$$

The spaces $W^{m,2}(a,b;V)$ form a Hilbert space and it is noted $H^m(a,b;V)$. The $H^m(a,b;V)$ inner product is defined by:

$$(w,v)_{H^m(a,b;V)} = \sum_{i=0}^m \int_a^b \left(\frac{\partial^i w}{\partial t^i}, \frac{\partial^i v}{\partial t^i}\right)_V dt.$$

Theorem 1.1. *(Sobolev embedding theorem). Assume that Ω is a (bounded or unbounded) open set of $\mathbb{R}^d$ with a Lipschitz continuous boundary, and that $1 \leq p < \infty$.*

Then the following continuous embeddings hold:

a. If $0 \leq mp < d$, *then* $W^{m,p}(\Omega) \subset L^{p^*}(\Omega)$ *for* $p^* = \frac{dp}{d-mp}$*;*

b. If $mp = d$, *then* $W^{m,p}(\Omega) \subset L^q(\Omega)$ *for any* q *such that* $p \leq q < \infty$*;*

c. If $mp > d$, *then* $W^{m,p}(\Omega) \subset C^0(\overline{\Omega})$*.*

Theorem 1.2. *(Rellich-Kondrachov compactness theorem). Assume that* Ω *is a bounded open set of* $\mathbb{R}^d$ *with a Lipschitz continuous boundary, and that* $1 \leq p < \infty$*. Then the following embeddings are compact:*

a. If $0 < mp < d$, *then* $W^{m,p}(\Omega) \subset L^q(\Omega)$ *for any* q *such that* $1 \leq q < p^* = \frac{dp}{d-mp}$*;*

b. If $mp = d$, *then* $W^{m,p}(\Omega) \subset L^q(\Omega)$ *for any* q *such that* $1 \leq q < \infty$*;*

c. If $mp > d$, *then* $W^{m,p}(\Omega) \subset C^0(\overline{\Omega})$*;*

d. If $p > \frac{2d}{d+2}$, *then* $L^p(\Omega) \subset H^{-1}(\Omega)$*.*

In particular, $H^m(\Omega)$ *is compactly embedded into* $H^{m-1}(\Omega)$, m *a non-negative integer.*

We will introduce some basic results on the $L^p(0, T, V)$ space. These results, will be very useful in the other chapters of this thesis.

Lemma 1.1. . *Let* $u \in L^p(0, T, V)$ *and* $\frac{\partial u}{\partial t} \in L^p(0, T, V)$, $(1 \leq p \leq \infty)$, *then the function* u *is continuous from* $[0, T] \to V$. *i.e.* $u \in C^1(0, T, V)$.

Lemma 1.2. . *Let* $\mathcal{Q} =]0, T[\times\Omega$ *an open bounded domain in* $\mathbb{R} \times \mathbb{R}^d$, *and* v_μ, v *are two functions in* $L^q(]0, T[, L^q(\Omega))$, $1 < q < \infty$ *such that*

$$\|v_\mu\|_{L^q(]0,T[,L^q(\Omega))} \leq C, \forall \mu \in , \tag{1.8}$$

and $v_\mu \to v \quad in \quad \mathcal{Q}$, *then* $v_\mu \rightharpoonup v \quad in \quad L^q(\mathcal{Q})$.

Theorem 1.3. . $L^p(0, T, V)$ *equipped with the norm* $\| \cdot \|_{L^p(]0,T[,V)}$, $1 \leq p \leq \infty$ *is a Banach space.*

Proposition 1.2. . *Let* V *be a reflexive Banach space,* V' *it's dual, and* $1 \leq p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. *Then the dual of* $L^p(0, T, V)$ *is identify algebraically and topologically with* $L^q(0, T, V')$.

Proposition 1.3. *Let* V_1, V_2 *be Banach space,* $V_1 \subset V_2$ *with continuous embedding, then we have with continuous embedding*

$$L^p(0,T,V_1) \subset L^p(0,T,V_2),$$

The following compactness criterion will be useful for nonlinear evolution problem, especially in the limit of the nonlinear terms.

Proposition 1.4. *Let* B_0, B, B_1 *be Banach spaces with* $B_0 \subset B \subset B_1$*. Assume that the embedding* $B_0 \hookrightarrow B$ *is compact and* $B \hookrightarrow B_1$ *is continuous. Let* $1 < p, q < \infty$*. Assume further that* B_0 *and* B_1 *are reflexive. Define*

$$W \equiv \Big\{ u \in L^p(0,T,B_0) : u' \in L^q(0,T,B_1) \Big\}.$$

Then, the embedding $W \hookrightarrow L^p(0,T,B)$ *is compact.*

1.5.1 Gronwall lemma

Lemma 1.3. *Let* $f \in L^1(t_0,T)$ *be a non-negative function,* g *and* φ *be continuous functions on* $[t_0,T]$*. If* φ *satisfies*

$$\varphi(t) \leq g(t) + \int_{t_0}^{t} f(\tau) g(\tau) d\tau \quad \forall t \in [t_0,T],$$

then

$$\varphi(t) \leq g(t) + \int_{t_0}^{t} f(s) g(s) \exp\left(\int_s^t f(\tau) d\tau \right) \quad \forall t \in [t_0,T].$$

If moreover g *is non-increasing, then*

$$\varphi(t) \leq g(t) \exp\left(\int_{t_0}^{t} f(\tau) d\tau \right) \quad \forall t \in [t_0,T].$$

1.5.2 Nonlinear Generalisation of Gronwall's inequality

We can consider various nonlinear generalisations of Gronwall's inequality. The following theorem is proved in Perov [24]:

Theorem 1.4. *Let $u(t)$ be a nonnegative function that satisfies the integral inequality*

$$u(t) \leq c + \int_{t_0}^{t} (a(s)u(s) + b(s)u^{\sigma}(s))ds \quad c \geq 0, \sigma \geq 0,$$

where $a(t)$ and $b(t)$ are continuous nonnegative functions for $t \geq t_0$.
For $0 \leq \alpha < 1$ we have

$$u(t) \leq \left\{ c^{1-\sigma} \exp\left[(1-\sigma)\int_{t_0}^{t} a(s)ds\right] + (1-\sigma)\int_{t_0}^{t} b(s) \exp\left[(1-\sigma)\int_{s}^{t} a(r)dr\right] ds \right\}^{\frac{1}{1-\sigma}}.$$

1.6 Existence Methods

1.6.1 Faedo-Galerkin method

We consider the Cauchy problem abstract's for a second order evolution equation in the separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$.

$$\begin{cases} u_{tt}(t) + A(t)u(t) = f(t), \ \ t \in [0,T], \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x); \end{cases} \tag{1.9}$$

where u and f are unknown and given function, respectively, mapping the closed interval $[0,T] \subset \mathbb{R}$ into a real separable Hilbert space H ,$A(t)$ ($0 \leq t \leq T$) are linear bounded operators in H acting in the energy space $V \subset H$.
Assume that $\langle A(t)u(t), v(t) \rangle = a(t; u(t), v(t))$, for all $u, v \in V$; where $a(t; ., .)$ is a bilinear continuous in V.
The problem (1.9) can be formulated as: Found the solution $u(t)$ such that

$$\begin{cases} u \in C([0,T];V), u_t \in C([0,T];H), \\ \langle u_{tt}(t), v \rangle + a(t; u(t), v) = \langle f, v \rangle \text{ in } D'(]0,T[), \\ u_0 \in V \ , \ u_1 \in H. \end{cases} \tag{1.10}$$

This problem can be resolved with the approximation process of Fadeo-Galerkin.

1.6.2 General method

Let V^n a sub-space of V with the finite dimension d^n, and let $\{w_i^n\}$ one basis of V^n such that.
we define the solution u^n of the approximate problem

$$\begin{cases} u^n(t) = \sum_{i=1}^{d^n} g_i(t) w_i^n, \\ u^n \in C([0,T];V^n), u_t^n \in C([0,T];V^n) \ , u^n \in L^2(0,T;V^n), \\ \langle u_{tt}^n(t), w_i^n \rangle + a(t; u^n(t), w_i^n) = \langle f, w_i^n \rangle, \ \ 1 \le i \le d^n, \\ u^n(0) = \sum_{i=1}^{d^n} \xi_i(t) w_i^n \ , \ u_t^n(0) = \sum_{i=1}^{d^n} \eta_i(t) w_i^n. \end{cases} \tag{1.11}$$

where

$$\sum_{i=1}^{d^n} \xi_i(t) w_i^n \longrightarrow u_0 \text{ in V as } n \longrightarrow \infty \tag{1.12}$$

$$\sum_{i=1}^{d^n} \eta_i(t) w_i^n \longrightarrow u_1 \text{ in V as } n \longrightarrow \infty \tag{1.13}$$

1. $V^n \subset V(dim V^n < \infty), \forall n \in \mathbb{N}$
2. $V^n \to V$ such that, there exist a dense subspace ϑ in V and for all $v \in \vartheta$ we can get sequence $(u^n)_{n\in\mathbb{N}} \in V^n$ and $u^n \to u$ in V.
3. $V \subset V^{n+1}$ and $\overline{\bigcup_{n\in\mathbb{N}} V^n} = V$.

By virtue of the theory of ordinary differential equations, the system (1.11) has unique local solution which is extend to a maximal interval $[0, t_n[$ by Zorn lemma since the non-linear terms have the suitable regularity. In the next step, we obtain a priori estimates for the solution, so that can be extended outside $[0, t_n[$ to obtain one solution defined for all $t > 0$

1.6.3 A priori estimation and convergence

Using the following estimation

$$\|u_m\|^2 + \|u'_m\|^2 \le C \left(\|u_m(0)\|^2 + \|u'_m(0)\|^2 + \int_0^T |f(s)|^2 ds \right) \ ; \ \ 0 \le t \le T \tag{1.14}$$

and the Gronwall lemma we deduce that the solution u_m of the approximate problem (1.11) converges to the solution u of the initial problem (1.9).The uniqueness proves that u is the solution.

Chapter 2

Exponential Decay of the Viscoelastic Wave Equation of Kirchhoff Type with a Nonlocal Dissipation

2.1 Introduction

In this chapter, we shall consider the initial boundary value problem for the following integro-differential problem

$$\begin{cases} u_{tt} - \psi\left(\|\nabla u\|_2^2\right)\Delta u - \alpha\Delta u_t + g * \Delta u \\ +M\left(\|\nabla u\|_2^2\right)u_t = f(u),\ x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x),\ x \in \Omega, \\ u(x,t) = 0,\ x \in \partial\Omega,\ t > 0, \end{cases} \tag{2.1}$$

where Ω is a bounded domain in $\mathbb{R}^d$ $(d \geq 1)$ with smooth boundary $\partial\Omega$. Here, g is a positive function that represents the kernel of the memory term, $\psi(r)$

is a positive locally Lipschitz function satisfying $\psi(r) \geq m_0 > 0$, for $r \geq 0$ like $\psi(r) = m_0 + br^\gamma$, $b \geq 0$, $\gamma \geq 1$, $M(r)$ is a $C^1[0,\infty)$ -function satisfying $M(r) \geq m_1 > 0$ for $r \geq 0$, f is a non-linear function as similar to $|u|^{p-2}u$, $p > 2$. Here, $\alpha \geq 0$ and

$$g * \Delta u(t) = \int_0^t g(t-\tau)\Delta(\tau)d\tau.$$

The non-linear vibration of the elastic string are written in the forme of partial integro-differential equations:

$$\rho h \frac{\partial^2 u}{\partial t^2} + \delta \frac{\partial u}{\partial t} = \left\{ p_0 + \frac{Eh}{2L} \int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx \right\} \frac{\partial^2 u}{\partial x^2} + f, \tag{2.2}$$

for $0 \leq x \leq L$ and $t \geq 0$, where

$$\begin{cases} u = u(x,t) \text{ is the lateral deflection,} \\ x \text{ is the space coordinate variable while } t \text{ denotes the time variable,} \\ E \text{ represents the Young's modulus,} \\ \rho \text{ designates the mass density,} \\ \delta \text{ designates the resistance modulus,} \\ L \text{ indicates the string's length,} \\ h \text{ represents the cross section,} \\ p_0 \text{ denotes the axial tension,} \\ f \text{ the external force.} \end{cases}$$

When $\delta = f = 0$, Kirchhoff [1] [11] first introduced (2.2) in the study of oscillations of stretched string and plates.

In the absence of the term $M(\|\nabla u\|_2^2)u_t$. Wu and Tsai [32] studied (2.1) with $\alpha = 1$. The authors established the global existence and energy decay under the assumption $g'(t) \leq -rg(t), \quad \forall t \geq 0$ for some $r > 0$. Recently, this decay estimate of the energy function was improved by Wu in [33] under a weaker

[1] Gustav Kirchhoff Physicien Allemand 1824-1887

condition on g i.e. $g'(t) \leq 0, \quad \forall t \geq 0$.
If we consider (2.1) with $[\psi \equiv 1, f = \alpha = 0]$ and the the bi-harmonic instead of Laplace operator one we get the model

$$u_{tt} + \Delta^2 u - \int_0^t g(t-\tau)\Delta^2 u(\tau)d\tau + M(\|\nabla u\|_2^2)u_t = 0. \tag{2.3}$$

Cavalcanti et al. [6] investigated the global existence, uniqueness and stabilization of energy of (2.3). By taking a bounded or unbounded open set Ω, the authors showed in [6] that the energy goes to zero exponentially provided that g goes to zero at the same form.
The main interest of the present thesis is to examine whether there exists a global solution u of (2.1) under the presence of the nonlinear and nonlocal dissipation represented by $M\left(\int_\Omega |\nabla u(x,t|^2 dx\right) u_t$ and the real-value function $M : [0, +\infty) \to [m_1, +\infty)$, where $m_1 > 0$ will be considered of class C^1.
This kind of damping effect was firstly introduced by H. Lange and G. Perla Menzala [12] for the beam equation where the following model was considered

$$u_{tt} + \Delta^2 u + M\left(\int_\Omega |\nabla u(x,t|^2 dx\right) u_t = 0 \quad \text{in} \quad \mathbb{R}^d \times \mathbb{R}^+. \tag{2.4}$$

The nonlocal nonlinearity $M\left(\int_\Omega |\nabla u(x,t|^2 dx\right) u_t$ is indeed a damping term. It models a frictional mechanism acting on the body that depends on the average of u itself. Moreover, if such u does exist, we intend to investigate its asymptotic behavior as $t \to \infty$.
In this chapter we show that under some conditions the solution is global in time and the energy decays exponentially. We first use Faedo-Galerkin method to study the existence of the simpler problem (2.5). Then, we obtain the local existence Theorem 2.1 by using contraction mapping principle. We obtain global existence of the solutions of (2.1) given in Theorem 2.2. Our technique of proof is similar to the one in [30] with some necessary modifications due the nature of

the problem treated here. Moreover, the asymptotic behavior of global solutions is investigated under some assumptions on the initial data.

2.2 Preliminaries

In this section, we introduce some notations and some technical lemmas to be used throughout this chapter. Also, we give a local existence theorem. In order to state and prove our result, we formulate the following assumptions:
(H1) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded C^1 function satisfying

$$\begin{cases} 1 - \int_0^\infty g(\tau)d\tau = l_1 > 0, \\ g(0) - K_1 \int_0^\infty g(\tau)d\tau = l_2 > 0, \\ -K_1 g(t) \leq g'(t) \leq -K_2 g(t). \end{cases}$$

Here K_1 and K_2 are positive constants.
(H2) $f(0) = 0$ and there is a positive constant K_3 such that

$$\Big|f(u) - f(v)\Big| \leq K_3 \Big|u - v\Big| \left(\Big|u\Big|^{p-2} + \Big|v\Big|^{p-2}\right) \quad \text{for} \quad u, v \in \mathbb{R},$$

and

$$2 < p < \infty \quad \text{if} \quad d = 1, 2 \quad \text{and} \quad 2 < p \leq \frac{2(d-1)}{d-2} \quad \text{if} \quad d \geq 3.$$

(H3) The function $M(r)$ for $r \geq 0$ belongs to the class $C^1[0, \infty)$ and satisfies

$$M(r) \geq m_1 > 0 \quad \text{for} \quad r \geq 0.$$

Lemma 2.1. *(Sobolev-Poincaré inequality [17])*
If $2 \leq p \leq \frac{2d}{d-2}$, *then*

$$\|u\|_p \leq B_1 \|\nabla u\|_2,$$

for $u \in H_0^1(\Omega)$ *holds with some constant* B_1.

Lemma 2.2. *For any $h \in C^1$ and $k \in H^1(0,T)$, we have*

$$\begin{aligned} 2\int_0^t \int_\Omega h(t-\tau)kk_t dx d\tau &= -\frac{d}{dt}\Big\{(h\circ k)(t) - \int_0^t h(s)ds\|k\|_2^2\Big\} \\ &\quad +(h_t\circ k)(t) - h(t)\|k\|_2^2, \end{aligned}$$

where

$$(h\circ k)(t) = \int_0^t h(t-\tau)\|k(t)-k(\tau)\|_2^2 d\tau.$$

2.3 Local Existence of Solution

In this section, we shall discuss the local existence of solutions for (2.1) by using contraction mapping principle. An important step in the proof of local existence Theorem 3.1 below is the study of the following simpler problem:

$$\begin{cases} u_{tt} - \mu(t)\Delta u - \alpha\Delta u_t + g\star\Delta u \\ +\chi(t)u_t = f_1(x,t), \text{ in } \Omega\times(0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x\in\Omega, \\ u(x,t) = 0, \quad x\in\partial\Omega,\ t\geq 0. \end{cases} \tag{2.5}$$

Here, $T > 0$, $\alpha \geq 1$, f_1 is a fixed forcing term in $\Omega\times(0,T)$, $\mu(t)$ is a positive locally Lipschitz function on $[0,\infty)$ with $\mu(t) \geq m_0 > 0$ for $t \geq 0$ and $\chi(t)$ is C^1-function on $[0,\infty)$ such that $\chi(t) \geq 0$ for $t \geq 0$.

Lemma 2.3. *Suppose that $(H1)$ holds, and that $u_0 \in H^2(\Omega)\cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$ and $f_1 \in L^2\Big([0,T];L^2(\Omega)\Big)$ be given. Then the problem (2.5) admits a unique solution u such that*

$$u \in C([0,T];H^2(\Omega)\cap H_0^1(\Omega)),$$

$$u_t \in C([0,T];L^2(\Omega))\cap L^2([0,T];H_0^1(\Omega)),$$

$$u_{tt} \in L^2([0,T];L^2(\Omega)).$$

Proof. Let $(\omega_n)_{n\in\mathbb{N}}$ be a basis in $H^2(\Omega)\cap H_0^1(\Omega)$ and V^n be the space generated by $\omega_1, ..., \omega_n$, $n = 1, 2, ...$ Let us consider

$$u^n(t) = \sum_{k=1}^{n} d_k^n(t) w_k,$$

be the weak solution of the following approximate problem corresponding to (2.5)

$$\begin{aligned}
&\int_\Omega u_{tt}^n(t)\omega dx + \mu(t)\int_\Omega \nabla u^n(t)\cdot\nabla\omega dx - \int_0^t g(t-\tau)\int_\Omega \nabla u^n(\tau)\cdot\nabla\omega dx d\tau \\
&+\alpha\int_\Omega \nabla u_t^n(t)\cdot\nabla\omega dx + \chi(t)\int_\Omega u_t^n(t)\omega dx \\
&= \int_\Omega f_1(x,t) w dx \quad \text{for } \omega\in V^n, \qquad (2.6)
\end{aligned}$$

with initial conditions

$$u^n(0) = u_0^n = \sum_{k=1}^{n}\int_\Omega u_0 w_k dx w_k \longrightarrow u_0 \text{ in } H^2(\Omega)\cap H_0^1(\Omega), \qquad (2.7)$$

$$u_t^n(0) = u_1^n = \sum_{k=1}^{n}\int_\Omega u_1 w_k dx w_k \longrightarrow u_1 \text{ in } H_0^1(\Omega). \qquad (2.8)$$

By standard methods in differential equations, we prove the existence of solutions to $(2.6) - (2.8)$ on some
interval $[0, t_n)$, $0 < t_n < T$. In order to extend the solution of $(2.6) - (2.8)$ to the whole interval $[0, T]$, we need the following a priori estimate.

Step1 (The first priori estimate) Replacing w by $2u_t^n(t)$ in (2.6), we have

$$\begin{aligned}
&\frac{d}{dt}\left[\|u_t^n(t)\|_2^2 + \mu(t)\|\nabla u^n(t)\|_2^2\right] + 2\alpha\|\nabla u_t^n(t)\|_2^2 + 2\chi(t)\|u_t^n(t)\|_2^2 \\
&= \mu'(t)\|\nabla u^n(t)\|_2^2 + 2\int_\Omega f_1(x,t)u_t^n(t)dx + 2\int_0^t g(t-\tau)\int_\Omega \nabla u^n(\tau)\cdot\nabla u_t^n(t)dx d\tau \\
&\le \mu'(t)\|\nabla u^n(t)\|_2^2 + \|\nabla u_t^n(t)\|_2^2 + \|g\|_{L^1}\int_0^t g(t-\tau)\|\nabla u^n(\tau)\|_2^2 d\tau \\
&+\|f_1\|_2^2 + \|u_t^n(t)\|_2^2. \qquad (2.9)
\end{aligned}$$

Then, integrating (2.9) from 0 to t, we get

$$\begin{aligned}&\|u_t^n(t)\|_2^2+\mu(t)\|\nabla u^n(t)\|_2^2+(2\alpha-1)\int_0^t\|\nabla u_\tau^n(\tau)\|_2^2d\tau\\&\le c_1+\int_0^t\left[1+\frac{1}{\mu(\tau)}\left(|\mu'(\tau)|+\|g\|_{L^1}^2\right)\right]\left[\|u_\tau^n(\tau)\|_2^2+\mu(\tau)\|\nabla u^n(\tau)\|_2^2\right]d\tau,\end{aligned}$$

where

$$c_1=\|u_1^n\|_2^2+\mu(0)\|\nabla u_0^n\|_2^2+\int_0^t\|f_1\|_2^2dt.$$

Taking into account (2.7) and (2.8), we obtain from Gronwall's Lemma the first priori estimate

$$\|u_t^n(t)\|_2^2+\mu(t)\|\nabla u^n(t)\|_2^2+\int_0^t\|\nabla u_t^n(\tau)\|_2^2d\tau\le L_1, \tag{2.10}$$

for all $t\in[0,T]$. Here L_1 is a positive constant independent of $n\in\mathbb{N}$ and $t\in[0,T]$.

Step2: (The second priori estimate) Replacing ω by $u_{tt}^n(t)$ in (2.6), we have

$$\begin{aligned}&\|u_{tt}^n(t)\|_2^2+\frac{d}{dt}\left[\mu(t)\int_\Omega\nabla u^n(t)\cdot\nabla u_t^n(t)dx+\frac{\alpha}{2}\|\nabla u_t^n(t)\|_2^2+\frac{\chi(t)}{2}\|u_t^n(t)\|_2^2\right]\\&=\mu'(t)\int_\Omega\nabla u^n(t)\cdot\nabla u_t^n(t)dx+\mu(t)\|\nabla u_t^n(t)\|_2^2+\frac{\chi'(t)}{2}\|u_t^n(t)\|_2^2\\&+\frac{d}{dt}\left(\int_0^t g(t-\tau)\int_\Omega\nabla u^n(\tau)\cdot\nabla u_t^n(t)dxd\tau\right)-\int_0^t g'(t-\tau)\int_\Omega\nabla u^n(\tau)\cdot\nabla u_t^n(t)dxd\tau\\&-g(0)\int_\Omega\nabla u^n(t)\cdot\nabla u_t^n(t)dx+\int_\Omega f_1(x,t)u_{tt}^n(t))dx.\end{aligned} \tag{2.11}$$

By $(H1)$, Hölder inequality and Young's inequality, we have

$$\begin{aligned}&-\int_0^t g'(t-\tau)\int_\Omega\nabla u^n(\tau)\cdot\nabla u_t^n(t)dxd\tau\\&\le\frac{1}{2}\|\nabla u_t^n(t)\|_2^2+\frac{\xi_1^2\|g\|_{L^1}}{2}\int_0^t g(t-\tau)\|\nabla u^n(\tau)\|_2^2d\tau.\end{aligned} \tag{2.12}$$

Since $\mu(t)\ge m_0$ and from (2.10) we obtain

$$\begin{aligned}-g(0)\int_\Omega\nabla u^n(t)\cdot\nabla u_t^n(t)dx&\le\frac{1}{2}\|\nabla u_t^n(t)\|_2^2+\frac{g(0)^2}{2}\|\nabla u^n(t)\|_2^2\\&\le\frac{1}{2}\|\nabla u_t^n(t)\|_2^2+\frac{g(0)^2L_1}{2m_0}.\end{aligned} \tag{2.13}$$

Since $\chi(t)$ is C^1-function on $[0,\infty)$ and using (2.10) we infer that

$$\begin{aligned}\frac{\chi'(t)}{2}\|u_t^n(t)\|_2^2 &\leq \frac{A_1}{2}\|u_t^n(t)\|_2^2\\ &\leq \frac{A_1}{2}L_1.\end{aligned} \tag{2.14}$$

Moreover,

$$\begin{aligned}\left|\mu'(t)\int_\Omega \nabla u^n(t)\cdot\nabla u_t^n(t)dx\right| &\leq \frac{1}{2}\|\nabla u_t^n(t)\|_2^2+\frac{M_1^2}{2}\|\nabla u^n(t)\|_2^2\\ &\leq \frac{1}{2}\|\nabla u_t^n(t)\|_2^2+\frac{M_1^2L_1}{2m_0},\end{aligned} \tag{2.15}$$

where $M_1 = \sup\limits_{0\leq t\leq T}\{|\mu'(t)|\}$ and $A_1 = \max\limits_{0\leq t\leq T}\{|\chi'(t)|\}$. Then, by using $(2.12)-(2.15)$, we obtain from (2.11)

$$\begin{aligned}&\frac{1}{2}\|u_{tt}^n(t)\|_2^2+\frac{d}{dt}\left[\mu(t)\int_\Omega \nabla u^n(t)\cdot\nabla u_t^n(t)dx+\frac{\alpha}{2}\|\nabla u_t^n(t)\|_2^2+\frac{\chi(t)}{2}\|u_t^n(t)\|_2^2\right]\\ &\leq c_2+\left(\frac{3}{2}+M_2\right)\|\nabla u_t^n(t)\|_2^2+\frac{\xi_1^2\|g\|_{L^1}}{2}\int_0^t g(t-\tau)\|\nabla u^n(\tau)\|_2^2d\tau\\ &+\frac{d}{dt}\left(\int_0^t g(t-\tau)\int_\Omega \nabla u^n(\tau)\cdot\nabla u_t^n(t)dxd\tau\right),\end{aligned} \tag{2.16}$$

where $c_2 = \left(\frac{g(0)^2+M_1^2+A_1m_0}{2m_0}\right)L_1+\frac{1}{2}\|f_1\|_2^2$ and $M_2 = \sup\limits_{0\leq t\leq T}\{|\mu(t)|\}$.
Thus, integrating (2.16) over $(0,t)$, we obtain

$$\begin{aligned}&\frac{\alpha}{2}\|\nabla u_t^n(t)\|_2^2+\frac{1}{2}\int_0^t\|u_{\tau\tau}^n(\tau)\|_2^2dt+\frac{\chi(t)}{2}\|u_t^n(t)\|_2^2\\ &\leq c_3+\mu(t)\left|\int_\Omega \nabla u^n(t)\cdot\nabla u_t^n(t)dx\right|+\mu(0)\left|\int_\Omega \nabla u_0^n\cdot\nabla u_1^n dx\right|\\ &+\int_0^t g(t-\tau)\int_\Omega \nabla u^n(\tau)\cdot\nabla u_t^n(t)dxd\tau\\ &+\left(M_2+\frac{3}{2}\right)\int_0^t\|\nabla u_\tau^n(\tau)\|_2^2d\tau,\end{aligned} \tag{2.17}$$

where $(c_3 = c_2+\xi_1^2\|g\|_{L^1}^2L_1)\,T+\frac{\alpha}{2}\|\nabla u_1^n\|_2^2+\frac{\chi(0)}{2}\|u_1^n\|_2^2$.
We note that using the inequality $ab\leq \frac{1}{4\eta}a^2+\eta b^2$, where $\eta>0$ is arbitrary, it

follows that

$$\begin{aligned}
&\int_0^t g(t-\tau)\int_\Omega \nabla u^n(\tau)\cdot\nabla u_t^n(t)dxd\tau \\
&\leq \frac{1}{4\eta}\|g\|_{L^1(0,\infty)}\|g\|_{L^\infty(0,\infty)}\int_0^t \|\nabla u^n(\tau)\|_2^2 d\tau \\
&+\eta\|\nabla u_t^n(t)\|_2^2 \\
&\leq \eta\|\nabla u_t^n(t)\|_2^2 + \frac{\|g\|_{L^1}\|g\|_{L^\infty}}{4\eta m_0}L_1 T, \qquad (2.18)
\end{aligned}$$

and

$$\begin{aligned}
\mu(t)\left|\int_\Omega \nabla u^n(t)\cdot\nabla u_t^n(t)dx\right| &\leq \eta\|\nabla u_t^n(t)\|_2^2 + \frac{M_2^2}{4\eta}\|\nabla u^n(t)\|_2^2 \\
&\leq \eta\|\nabla u_t^n(t)\|_2^2 + \frac{M_2^2}{4\eta m_0}L_1. \qquad (2.19)
\end{aligned}$$

Putting (2.18) and (2.19) in (2.17) with $0<\eta\leq\frac{\alpha}{4}$, we obtain from $\chi(t)\geq 0$ that

$$\begin{aligned}
&\left(\frac{\alpha}{2}-2\eta\right)\|\nabla u_t^n(t)\|_2^2 + \frac{1}{2}\int_0^t \|u_{\tau\tau}^n(\tau)\|_2^2 dt \\
&\leq c_4 + \left(M_2+\frac{3}{2}\right)\int_0^t \|\nabla u_\tau^n(\tau)\|_2^2 d\tau, \qquad (2.20)
\end{aligned}$$

where

$$c_4 = c_3 + \mu(0)\|\nabla u_0^n\|_2\|\nabla u_1^n\|_2 + \frac{M_2^2}{4\eta m_0}L_1 + \frac{\|g\|_{L^1}\|g\|_{L^\infty}}{4\eta m_0}L_1 T. \qquad (2.21)$$

Taking into account $(2.7)-(2.8)$, we obtain from Gronwall's Lemma the second priori estimate

$$\|\nabla u_t^n(t)\|_2^2 + \int_0^t \|u_{\tau\tau}^n(\tau)\|_2^2 d\tau \leq L_2, \qquad (2.22)$$

for all $t\in[0,T]$. Here L_2 is a positive constant independent of $n\in\mathbb{N}$ and $t\in[0,T]$.

Step3. (The third priori estimate) Replacing ω by $-\Delta u^n(t)$ in (2.6), we have

$$\begin{aligned}
&\frac{d}{dt}\left[-\int_\Omega u_t^n(t)\Delta u^n(t)dx + \frac{\alpha}{2}\|\Delta u^n(t)\|_2^2 + \frac{\chi(t)}{2}\|\nabla u^n(t)\|_2^2\right] \\
&-\|\nabla u_t^n(t)\|_2^2 + \mu(t)\|\Delta u^n(t)\|_2^2 \\
&= \frac{\chi'(t)}{2}\|\nabla u^n(t)\|_2^2 + \int_0^t g(t-\tau)\int_\Omega \Delta u^n(\tau)\cdot\Delta u^n(t)dxd\tau \\
&+ \int_\Omega f_1(x,t)(-\Delta u^n(t))dx \\
&\leq \frac{A_1}{2}\|\nabla u^n(t)\|_2^2 + 2\eta\|\Delta u^n(t)\|_2^2 + \frac{1}{4\eta}\|f_1\|_2^2 \\
&+\frac{\|g\|_{L^1}}{4\eta}\int_0^t g(t-\tau)\|\Delta u^n(\tau)\|_2^2 d\tau, \qquad (2.23)
\end{aligned}$$

where $0 < \eta \leq \frac{m_0}{2}$ is some positive constant. From $\mu(t) \geq m_0 > 0$, we deduce by integration

$$\begin{aligned}
&\frac{\alpha}{2}\|\Delta u^n(t)\|_2^2 + (m_0 - 2\eta)\int_0^t \|\Delta u^n(\tau)\|_2^2 d\tau \\
&+\frac{\chi(t)}{2}\|\nabla u^n(t)\|_2^2 \\
&\leq \int_0^t \|\nabla u_\tau^n(\tau)\|_2^2 dt + \frac{A_1}{2}\int_0^t \|\nabla u^n(\tau)\|_2^2 d\tau \\
&+\left|\int_\Omega u_t^n(t)\Delta u^n(t)dx\right| + \left|\int_\Omega u_t^n(0)\Delta u^n(0)dx\right| \\
&+\frac{1}{4\eta}\int_0^t \|f_1\|_2^2 dt + \frac{\alpha}{2}\|\Delta u_0^n\|_2^2 + \frac{\chi(0)}{2}\|\nabla u_0^n\|_2^2 \\
&+\frac{\|g\|_{L^1}^2}{4\eta}\int_0^t \|\Delta u^n(\tau)\|_2^2 d\tau \\
&\leq c_5 + \left|\int_\Omega u_t^n(t)\Delta u^n(t)dx\right| + \frac{\|g\|_{L^1}^2}{4\eta}\int_0^t \|\Delta u^n(\tau)\|_2^2 d\tau, \qquad (2.24)
\end{aligned}$$

where

$$\begin{aligned}
c_5 &= \|u_1^n\|_2\|\Delta u_0^n\|_2 + \frac{\alpha}{2}\|\Delta u_0^n\|_2^2 + \frac{1}{4\eta}\int_0^t \|f_1\|_2^2 d\tau \\
&\quad +\frac{\chi(0)}{2}\|\nabla u_0^n\|_2^2 + \left(\frac{A_1}{m_0}L_1 + L_2\right)T.
\end{aligned}$$

We note that using the inequality $ab \leq \frac{1}{4}a^2 + b^2$, it follows that

$$\int_\Omega u_t^n(t)\Delta u^n(t)dx \leq \frac{1}{4}\|\Delta u^n(t)\|_2^2 + \|u_t^n(t)\|_2^2. \qquad (2.25)$$

Putting (2.25) in (2.24), we obtain from $\chi(t) \geq m_1 > 0$ that

$$\left(\frac{\alpha}{2} - \frac{1}{4}\right) \|\Delta u^n(t)\|_2^2 + (m_0 - 2\eta) \int_0^t \|\Delta u^n(\tau)\|_2^2 d\tau + \frac{\chi(t)}{2} \|\nabla u^n(t)\|_2^2$$
$$\leq c_6 + \frac{\|g\|_{L^1}^2}{4\eta} \int_0^t \|\Delta u_m(\tau)\|_2^2 d\tau, \tag{2.26}$$

where

$$c_6 = c_5 + L_1.$$

Taking into account $(2.7) - (2.8)$, we obtain from Gronwall's Lemma the third priori estimate,

$$\|\Delta u^n(t)\|_2^2 + \int_0^t \|\Delta u^n(\tau)\|_2^2 d\tau \leq L_3, \tag{2.27}$$

for all $t \in [0, T]$ and L_3 is a positive constant independent of $n \in \mathbb{N}$ and $t \in [0, T]$.
Step4. Let $j \geq n$ be two natural numbers, and consider $z^n = u^j - u^n$. Then, applying the same way as in the estimate step 1 and step 3 and observing that $\{u_0^n\}$ and $\{u_1^n\}$ are Cauchy sequence in $H_0^1(\Omega) \cap H^2(\Omega)$ and $H_0^1(\Omega)$, respectively, we deduce for all $t \in [0, T]$

$$\|z_t^n(t)\|_2^2 + \mu(t)\|\nabla z^n(t)\|_2^2 + \int_0^t \|\nabla z_\tau^n(\tau)\|_2^2 d\tau \to 0, \tag{2.28}$$

and

$$\|\Delta z^n(t)\|_2^2 + \int_0^t \|\Delta z^n(\tau)\|_2^2 d\tau \to 0, \quad \text{as} \quad n \to \infty. \tag{2.29}$$

Therefore, (2.10), (2.22), (2.27), (2.28) and (2.29), we see that

$$u^n \to u \;\text{ strongly in }\; C(0, T; H_0^1(\Omega)), \tag{2.30}$$

$$u_t^n \to u_t \;\text{ strongly in }\; C(0, T; L^2(\Omega)). \tag{2.31}$$

$$u_t^n \to u_t \;\text{ strongly in }\; L^2(0, T; H_0^1(\Omega)), \tag{2.32}$$

$$u_{tt}^n \to u_{tt} \;\text{ weakly in }\; L^2(0, T; L^2(\Omega)). \tag{2.33}$$

Then $(2.30) - (2.33)$ are sufficient to pass the limit in (2.6) to obtain

$$\begin{aligned} & u_{tt} - \mu(t)\Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau - \alpha\Delta u_t + \chi(t)u_t \\ & = f_1(x,t) \text{ in } L^2(0,T;H^{-1}(\Omega)). \end{aligned} \tag{2.34}$$

Next, we want to show the uniqueness of (2.5). Let $u^{(1)}$ and $u^{(2)}$ be two solutions of (2.5). Then $y = u^{(1)} - u^{(2)}$ satisfies

$$\begin{aligned} & \int_\Omega y_{tt}(t)\omega dx + \mu(t)\int_\Omega \nabla y(t)\cdot\nabla\omega dx - \int_0^t g(t-\tau)\int_\Omega \nabla y(\tau)\cdot\nabla\omega dx d\tau \\ & +\alpha\int_\Omega \nabla y_t(t)\cdot\nabla\omega dx + \chi(t)\int_\Omega y_t(t)\omega dx = 0 \quad \text{for } \omega \in H_0^1(\Omega), \end{aligned} \tag{2.35}$$

$$y(x,0) = 0, \quad y_t(x,0) = 0, \quad x \in \Omega,$$

$$y(x,t) = 0, \quad x \in \partial\Omega, \ t \geq 0.$$

Setting $w = 2y_t(t)$ in (2.35), then as in deriving (2.10), we see that

$$\begin{aligned} & \|y_t(t)\|_2^2 + \mu(t)\|\nabla y(t)\|_2^2 + (2\alpha - 1)\int_0^t \|\nabla y_\tau(\tau)\|_2^2 d\tau \\ & \leq \int_0^t \left[1 + \frac{1}{\mu(\tau)}\left(|\mu'(\tau)| + \|g\|_{L^1}^2\right)\right]\left[\|y_\tau(\tau)\|_2^2 + \mu(\tau)\|\nabla y(\tau)\|_2^2\right] d\tau. \end{aligned} \tag{2.36}$$

Thus employing Gronwall's Lemma, we conclude that

$$\|y_t(t)\|_2 = \|\nabla y(t)\|_2 = 0 \quad \text{for all} \quad t \in [0,T]. \tag{2.37}$$

Therefore, we have the uniqueness. This finishes the proof of Lemma 2.3. □

Now, let us prove the local existence of the problem (2.1).

Theorem 2.1. *Assume that $(H1)$, $(H2)$ and $(H3)$ are fulfilled. Suppose that $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$ be given. Then there exists a unique solution u of (2.1) satisfying*

$$u \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega))$$

and

$$u_t \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)),$$

and at least one of the following statements is valid:

$$\begin{aligned} &(i)\ T=\infty,\\ &(ii)\ e\Big(u(t)\Big)\equiv\|u_t(t)\|_2^2+\|\Delta u(t)\|_2^2\to\infty\ \ as\quad t\to T^-. \end{aligned} \tag{2.38}$$

Proof. Define the following two-parameter space:

$$X_{T,R_0}=\left\{\begin{array}{l} v\in C([0,T];H_0^1(\Omega)\cap H^2(\Omega)),\\ v_t\in C([0,T];L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega)):\\ e\Big(v(t)\Big)\le R_0^2,\ t\in[0,T],\\ \text{with}\quad v(0)=u_0,\quad v_t(0)=u_1. \end{array}\right\},$$

for $T>0$, $R_0>0$. Then X_{T,R_0} is a complete metric space with the distance

$$d(y,z)=\sup_{0\le t\le T} e\left(y(t)-z(t)\right)^{\frac{1}{2}}, \tag{2.39}$$

where $y,z\in X_{T,R_0}$. Given $v\in X_{T,R_0}$, we consider the following problem

$$\begin{cases} u_{tt}-\psi(\|\nabla v\|_2^2)\Delta u-\alpha\Delta u_t+g*\Delta u\\ +M(\|\nabla v\|_2^2)u_t=f(v),\text{in }\Omega\times(0,T),\\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x),\ x\in\Omega,\\ u(x,t)=0,\ x\in\partial\Omega,\ t\ge 0. \end{cases} \tag{2.40}$$

By $(H2)$, we see that $f(v)\in L^2\Big(0,T;L^2(\Omega)\Big)$. Thus, by Lemma 2.3, we derive that problem (2.40) admits a unique solution $u\in C([0,T];H^2(\Omega)\cap H_0^1(\Omega))$ and $u_t\in C([0,T];L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$. Then, we define the nonlinear mapping $Sv=u$, and we would like to show that there exist $T>0$ and $R_0>0$ such that S is a contraction mapping from X_{T,R_0} into itself. For this, we multiply the first equation of (2.40) by $2u_t$, integrate it over Ω and using Lemma 2.2, we get

$$\begin{aligned} &\frac{d}{dt}\left[\|u_t(t)\|_2^2+\left(\psi(\|\nabla v\|_2^2)-\int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2+(g\circ\nabla u)(t)\right]\\ &+2\alpha\|\nabla u_t(t)\|_2^2+2M(\|\nabla v\|_2^2)\|u_t(t)\|_2^2-(g'\circ\nabla u)(t)+g(t)\|\nabla u(t)\|_2^2\\ &=\left(\frac{d}{dt}\psi(\|\nabla v\|_2^2)\right)\|\nabla u(t)\|_2^2+2\int_\Omega f(v)u_t dx. \end{aligned} \tag{2.41}$$

Next, multiplying the first equation of (2.40) by $-2\Delta u$, and integrating it over Ω, we have

$$\begin{aligned}
&\frac{d}{dt}\left[\alpha\|\Delta u(t)\|_2^2 - 2\int_\Omega u_t \Delta u dx + M(\|\nabla v\|_2^2)\|\nabla u(t)\|_2^2\right] \\
&+2\psi(\|\nabla v\|_2^2)\|\Delta u(t)\|_2^2 - 2\|\nabla u_t(t)\|_2^2 \\
&= \left(\frac{d}{dt}M(\|\nabla v\|_2^2)\right)\|\nabla u(t)\|_2^2 - 2\int_\Omega f(v)\Delta u dx \\
&+2\int_0^t g(t-\tau)\Delta u(\tau)\cdot \Delta u(t) dx d\tau. \qquad (2.42)
\end{aligned}$$

Multiplying (2.42) by ϵ, $0 \le \epsilon \le 1$, adding (2.41) together and taking into account $(H1)$, $(H3)$, we obtain

$$\begin{aligned}
\frac{d}{dt}e^*(u(t)) \quad + \quad & 2(\alpha-\epsilon)\|\nabla u_t(t)\|_2^2 + 2\epsilon\psi(\|\nabla v\|_2^2)\|\Delta u(t)\|_2^2 \\
& \le I_1 + I_2 + I_3, \qquad (2.43)
\end{aligned}$$

where

$$\begin{aligned}
e^*(u(t)) \quad = \quad & \|u_t(t)\|_2^2 + \left(\psi(\|\nabla v\|_2^2) - \int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2 \\
& +(g\circ\nabla u)(t) + \epsilon\alpha\|\Delta u(t)\|_2^2 - 2\epsilon\int_\Omega u_t\Delta u dx \\
& +\epsilon M(\|\nabla v\|_2^2)\|\nabla u(t)\|_2^2. \qquad (2.44)
\end{aligned}$$

$$I_1 = 2\int_\Omega f(v)\Big(u_t - \epsilon\Delta u\Big)dx,$$

$$I_2 = \left(\frac{d}{dt}\psi(\|\nabla v\|_2^2) + \epsilon\frac{d}{dt}M(\|\nabla v\|_2^2)\right)\|\nabla u(t)\|_2^2,$$

and

$$I_3 = 2\epsilon\int_0^t g(t-\tau)\Delta u(\tau)\cdot\Delta u(t)dxd\tau.$$

Estimate for $I_1 = 2\int_\Omega f(v)(u_t - \epsilon\Delta u)dx$.

From $(H2)$ and making use of Hölder's inequality and Lemma 2.1, we have

$$
\begin{aligned}
I_1 &= 2 \int_\Omega f(v)(u_t - \epsilon \Delta u)dx \\
&\leq 2\int_\Omega |f(v)u_t|dx + 2\epsilon \int_\Omega |f(v)\Delta u|dx \\
&\leq 2K_3 \int_\Omega |v|^{p-1}|u_t|dx + 2\epsilon K_3 \int_\Omega |v|^{p-1}|\Delta u|dx \\
&\leq 2K_3 B_1^{2(p-1)}\|\Delta v\|_2^{p-1}\|u_t\|_2 + 2\epsilon K_3 B_1^{2(p-1)}\|\Delta v\|_2^{p-1}\|\Delta u\|_2 \\
&\leq 2K_3 B_1^{2(p-1)} R_0^{p-1} e(u(t))^{\frac{1}{2}} + 2\epsilon K_3 B_1^{2(p-1)} R_0^{p-1} e(u(t))^{\frac{1}{2}} \\
&= 2K_3(1+\epsilon) B_1^{2(p-1)} R_0^{p-1} e(u(t))^{\frac{1}{2}}. \qquad (2.45)
\end{aligned}
$$

Estimate for $I_2 = \left(\frac{d}{dt}\psi(\|\nabla v\|_2^2) + \epsilon\frac{d}{dt}M(\|\nabla v\|_2^2)\right)\|\nabla u(t)\|_2^2$.
First of all, we observe that

$$
\begin{aligned}
\frac{d}{dt}\psi(\|\nabla v\|_2^2) &= 2\psi'(\|\nabla v\|_2^2)\int_\Omega \nabla v \cdot \nabla v_t dx \\
&\leq 2M_3\|\Delta v\|_2\|v_t\|_2 \\
&\leq 2M_3 R_0^2, \qquad (2.46)
\end{aligned}
$$

where $M_3 = \sup\{|\psi'(s)|;\ 0 \leq s \leq B_1^2 R_0^2\}$, and

$$
\begin{aligned}
\epsilon\frac{d}{dt}M(\|\nabla v\|_2^2) &= 2\epsilon M'(\|\nabla v\|_2^2)\int_\Omega \nabla v \cdot \nabla v_t dx \\
&\leq 2\epsilon A_2\|\Delta v\|_2\|v_t\|_2 \\
&\leq 2\epsilon A_2 R_0^2, \qquad (2.47)
\end{aligned}
$$

where $A_2 = \max\{|M'(s)|;\ 0 \leq s \leq B_1^2 R_0^2\}$. Then, from (2.46) ,(2.47) and using (2.38) we arrive at

$$I_2 \leq 2B_1^2 R_0^2 (M_3 + \epsilon A_2) e(u(t)). \qquad (2.48)$$

Estimate for $I_3 = 2\epsilon\int_0^t g(t-\tau)\Delta u(\tau)\cdot\Delta u(t)dxd\tau$.
Using the inequality $ab \leq \frac{1}{4\eta}a^2 + \eta b^2$, where $\eta > 0$ is arbitrary,

we get

$$\begin{aligned} I_3 &= 2\epsilon \int_0^t g(t-\tau) \int_\Omega \Delta u(\tau) \cdot \Delta u(t) dx d\tau \\ &\leq 2\epsilon\eta \|\Delta u(t)\|_2^2 + \epsilon \frac{\|g\|_{L^1}}{2\eta} \int_0^t g(t-\tau) \|\Delta u(\tau)\|_2^2 d\tau. \end{aligned} \tag{2.49}$$

Combining these inequalities with $0 < \eta < \frac{\|g\|_{L^1}}{2}$, we get

$$\begin{aligned} &\frac{d}{dt} e^*(u(t)) + 2(\alpha - \epsilon)\|\nabla u_t(t)\|_2^2 + 2\epsilon \left(\psi(\|\nabla v\|_2^2) - \eta\right) \|\Delta u(t)\|_2^2 \\ &\leq 2B_1^2 R_0^2 (M_3 + \epsilon A_2) e(u(t)) + 2K_3(1+\epsilon) B_1^{2(p-1)} R_0^{p-1} e(u(t))^{\frac{1}{2}} \\ &+\epsilon \frac{\|g\|_{L^1}}{2\eta} \int_0^t g(t-\tau)\|\Delta u(\tau)\|_2^2 d\tau. \end{aligned} \tag{2.50}$$

When we take $\epsilon = 0$ in (2.50), we see that

$$\begin{aligned} &\frac{d}{dt}\left[\|u_t(t)\|_2^2 + \left(\psi(\|\nabla v\|_2^2) - \int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t)\right] \\ &+2\alpha \|\nabla u_t(t)\|_2^2 \\ &\leq 2B_1^2 R_0^2 M_3 e(u(t)) + 2K_3 B_1^{2(p-1)} R_0^{p-1} e(u(t))^{\frac{1}{2}}. \end{aligned} \tag{2.51}$$

By Young's inequality, we get

$$2\epsilon \int_\Omega u_t \Delta u dx \leq 2\epsilon \|u_t\|_2^2 + \frac{\epsilon}{2}\|\Delta u(t)\|_2^2.$$

Hence

$$\begin{aligned} e^*(u(t)) &\geq (1-2\epsilon)\|u_t\|_2^2 + \epsilon\left(\alpha - \frac{1}{2}\right)\|\Delta u(t)\|_2^2 \\ &\quad + \left(\psi(\|\nabla v\|_2^2) - \int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2 \\ &\quad +(g \circ \nabla u)(t) + \epsilon M(\|\nabla v\|_2^2)\|\nabla u(t)\|_2^2. \end{aligned}$$

Choosing $\epsilon = \frac{2}{5}$ and taking into account $(H1)$ and $(H3)$, we have

$$e^*(u(t)) \geq \frac{1}{5} e(u(t)). \tag{2.52}$$

and

$$\begin{aligned} e^*(u_0) &\leq (1+2\epsilon)\|u_1\|_2^2 + \epsilon\left(\alpha+\frac{1}{2}\right)\|\Delta u_0\|_2^2 + \psi(\|\nabla u_0\|_2^2)\|\nabla u_0\|_2^2 \\ &\quad +\epsilon M(\|\nabla u_0\|_2^2)\|\nabla u_0\|_2^2 \\ &\leq 2\|u_1\|_2^2 + (\alpha+\frac{1}{2})\|\Delta u_0\|_2^2 + \psi(\|\nabla u_0\|_2^2)\|\nabla u_0\|_2^2 \\ &\quad +M(\|\nabla u_0\|_2^2)\|\nabla u_0\|_2^2 \\ &= c^*. \end{aligned} \tag{2.53}$$

Integrating (2.50) over $(0,t)$, we get

$$\begin{aligned} & e^*(u(t)) + \frac{4}{5}\left(m_0 - \eta - \frac{\|g\|_{L^1}^2}{4\eta}\right)\int_0^t \|\Delta u(\tau)\|_2^2 d\tau \\ & \leq e^*(u_0) + \int_0^t \left[C_1 e^*(u(\tau)) + C_2 e^*(u(\tau))^{\frac{1}{2}}\right] d\tau, \end{aligned} \tag{2.54}$$

where $C_1 = 10B_1^2R_0^2(M_3 + \frac{2}{5}A_2)$ and $C_2 = \frac{14\sqrt{5}}{5}K_1B_1^{2(p-1)}R_0^{p-1}$.
Taking $\eta = \frac{\|g\|_{L^1}}{2\eta}$ in (2.54), then from $(H1)$, we deduce

$$\begin{aligned} e^*(u(t)) &\leq e^*(u_0) + \int_0^t \left[C_1 e^*(u(\tau)) + C_2 e^*(u(\tau))^{\frac{1}{2}}\right] d\tau \\ &\leq c^* + \int_0^t \left[C_1 e^*(u(\tau)) + C_2 e^*(u(\tau))^{\frac{1}{2}}\right] d\tau. \end{aligned} \tag{2.55}$$

Hence, by Gronwall's inequality, we have

$$e^*(u(t)) \leq \left(\sqrt{c^*} + \frac{C_2}{2}T\right)^2 e^{C_1T}. \tag{2.56}$$

Then, by (2.52), we obtain

$$e(u(t)) \leq 5\left(\sqrt{c^*} + \frac{C_2}{2}T\right)^2 e^{C_1T}. \tag{2.57}$$

for any $t \in (0,T]$. Therefore, we see that for parameters T and R_0 satisfy

$$5\left(\sqrt{c^*} + \frac{C_2}{2}T\right)^2 e^{C_1T} \leq R_0^2. \tag{2.58}$$

That means S maps X_{T,R_0} into itself. Moreover, by Lemma 2.3,

$$u \in C^0([0,T]; H^2(\Omega)\cap H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)).$$

On the other hand, it follows from (2.51) and (2.57) that

$$u_t \in L^2(0,T;H_0^1(\Omega)).$$

Next, we shall verify that S is a contraction mapping with respect to the metric $d(\cdot,\cdot)$. We take $v_1, v_2 \in X_{T,R_0}$, and denote $u^{(1)} = Sv_1$ and $u^{(2)} = Sv_2$. Hereafter we suppose that (2.58) is valid, thus $u^{(1)}, u^{(2)} \in X_{T,R_0}$. Putting $w(t) = (u^{(1)} - u^{(2)})(t)$, then w satisfies

$$\begin{cases} w_{tt} - \psi\left(\|\nabla v_1\|_2^2\right)\Delta w + g\star\Delta w - \alpha\Delta w_t + M\left(\|\nabla v_1\|_2^2\right) w_t \\ = f(v_1) - f(v_2) + \left[\psi\left(\|\nabla v_1\|_2^2\right) - \psi\left(\|\nabla v_2\|_2^2\right)\right]\Delta u^{(2)} \\ + \left[M\left(\|\nabla v_2\|_2^2\right) - M\left(\|\nabla v_1\|_2^2\right)\right] u_t^{(2)}, \\ w(0) = 0, \quad w_t(0) = 0, \\ w(x,t) = 0,\ x\in\partial\Omega,\ t\ge 0. \end{cases} \tag{2.59}$$

We multiply the first equation of (2.59) by $2w_t$ and integrate it over Ω to get

$$\begin{aligned} &\frac{d}{dt}\left[\|w_t(t)\|_2^2 + \left(\psi\Big(\|\nabla v_1\|_2^2\Big) - \int_0^t g(\tau)d\tau\right)\|\nabla w(t)\|_2^2 + (g\circ\nabla w)(t)\right] \\ &+2\alpha\|\nabla w_t(t)\|_2^2 \\ &\le I_4 + I_5 + I_6 + I_7. \end{aligned} \tag{2.60}$$

We now estimate I_4-I_7 (defined as below), respectively.

$$\begin{aligned} I_4 &= \left(\frac{d}{dt}\psi(\|\nabla v_1\|_2^2)\right)\|\nabla w(t)\|_2^2 \\ &\le 2M_3B_1^2R_0^2 e\Big(w(t)\Big), \end{aligned} \tag{2.61}$$

$$\begin{aligned} I_5 &= 2\int_\Omega [f(v_1) - f(v_2)]w_t dx \\ &\le 2K_3\int_\Omega (|v_1|^{p-2} + |v_2|^{p-2})|v_1 - v_2|w_t dx \\ &\le 2K_3[\|v_1\|_{d(p-2)}^{p-2} + \|v_2\|_{d(p-2)}^{p-2}]\|v_1 - v_2\|_{\frac{2d}{d-2}}\|w_t\|_2 \\ &\le 4K_3B_1^{2(p-1)}R_0^{p-2}e(v_1 - v_2)^{\frac{1}{2}}e(w(t))^{\frac{1}{2}}, \end{aligned} \tag{2.62}$$

$$\begin{aligned} I_6 &= 2\left[\psi(\|\nabla v_1\|_2^2) - \psi(\|\nabla v_2\|_2^2)\right]\int_\Omega \Delta u^{(2)} w_t dx \\ &\leq 2L(\|\nabla v_1\|_2 + \|\nabla v_2\|_2)\|\nabla v_1 - \nabla v_2\|_2 \|\Delta u^{(2)}\|_2 \|w_t\|_2 \\ &\leq 4LB_1^2R_0^2 e(v_1 - v_2)^{\frac{1}{2}} e(w(t))^{\frac{1}{2}}, \end{aligned} \tag{2.63}$$

where $L = L(R)$ is the Lipschitz constant of $\psi(s)$ in $[0, R_0]$.

Estimate for $I_7 = 2\left[M(\|\nabla v_2\|_2^2) - M(\|\nabla v_1\|_2^2)\right]\int_\Omega u_t^{(2)} w_t dx$.

Assumption $(H3)$ gives

$$\begin{aligned} \left|M(\|\nabla v_2\|_2^2) - M(\|\nabla v_1\|_2^2)\right| &= \left|\int_{\|\nabla v_1\|_2^2}^{\|\nabla v_2\|_2^2} M'(r)dr\right| \\ &\leq \int_{\|\nabla v_1\|_2^2}^{\|\nabla v_2\|_2^2} |M'(r)|\, dr \\ &\leq C_* \left|\|\nabla v_2\|_2^2 - \|\nabla v_1\|_2^2\right| \\ &\leq C_*(\|\nabla v_1\|_2 + \|\nabla v_1\|_2)\|\nabla v_2 - \nabla v_1\|_2, \end{aligned} \tag{2.64}$$

where C_* is a positive constant. From (2.64) and (2.38), we have

$$\begin{aligned} I_7 &= 2\left[M(\|\nabla v_2\|_2^2) - M(\|\nabla v_1\|_2^2)\right]\int_\Omega u_t^{(2)} w_t dx \\ &\leq 2C_*(\|\nabla v_1\|_2 + \|\nabla v_1\|_2)\|\nabla(v_2 - v_1)\|_2 \|u_t^{(2)}\|_2 \|w_t\|_2 \\ &\leq 2C_* B_1^2 R_0^2 e(v_1 - v_2)^{\frac{1}{2}} e(w(t))^{\frac{1}{2}}. \end{aligned} \tag{2.65}$$

Inserting $(2.61) - (2.65)$ in (2.60), we get

$$\begin{aligned} &\frac{d}{dt}\left[\|w_t(t)\|_2^2 + \left(\psi(\|\nabla v_1\|_2^2) - \int_0^t g(\tau)d\tau\right)\|\nabla w(t)\|_2^2 + (g \circ \nabla w)(t)\right] \\ &+2\alpha\|\nabla w_t(t)\|_2^2 \\ &\leq C_3 e(w(t)) + C_4 e(v_1 - v_2)^{\frac{1}{2}} e(w(t))^{\frac{1}{2}}, \end{aligned} \tag{2.66}$$

where $C_3 = 2M_3B_1^2R_0^2$ and $C_4 = 4K_3B_1^{2(p-1)}R_0^{p-2} + 4LB_1^2R_0^2 + 2C_*B_1^2R_0^2$.

On the other hand, multiplying the first equation in (2.59) by $-2\Delta w$, and integrating it over Ω, we get

$$\frac{d}{dt}\left\{\alpha\|\Delta w(t)\|_2^2 - 2\int_\Omega w_t \Delta w dx + M(\|\nabla v_1\|_2^2)\|\nabla w(t)\|_2^2\right\}$$
$$+2\psi(\|\nabla v_1\|_2^2)\|\Delta w(t)\|_2^2 - 2\|\nabla w_t\|_2^2$$
$$= I_8 + I_9 + I_{10} + I_{11} + I_{12}. \tag{2.67}$$

We now estimate I_8-I_{12} (defined as below), respectively.

Applying the similar arguments as in estimating I_i, $i = 2, 3, 5, 6, 7$, we observe that

$$\begin{aligned} I_8 &= \left(\frac{d}{dt}M(\|\nabla v_1\|_2^2)\right)\|\nabla w(t)\|_2^2 \\ &\leq 2A_2 R_0^2 B_1^2 e(w(t)), \end{aligned} \tag{2.68}$$

$$\begin{aligned} I_9 &= -2\int_\Omega [f(v_1) - f(v_2)]\Delta w dx \\ &\leq 4K_3 B_1^{2(p-1)} R_0^{p-2} e(v_1 - v_2)^{\frac{1}{2}} e(w(t))^{\frac{1}{2}}, \end{aligned} \tag{2.69}$$

$$\begin{aligned} I_{10} &= 2\left[\psi(\|\nabla v_1\|_2^2) - \psi(\|\nabla v_2\|_2^2)\right]\int_\Omega \Delta u^{(2)} \Delta w dx \\ &\leq 4LB_1^2 R_0^2 e(v_1 - v_2)^{\frac{1}{2}} e(w(t))^{\frac{1}{2}}, \end{aligned} \tag{2.70}$$

$$\begin{aligned} I_{11} &= 2\left[M(\|\nabla v_2\|_2^2) - M(\|\nabla v_1\|_2^2)\right]\int_\Omega \Delta u^{(2)} \Delta w dx \\ &\leq 2C_* B_1^2 R_0^2 e(v_1 - v_2)^{\frac{1}{2}} e(w(t))^{\frac{1}{2}}, \end{aligned} \tag{2.71}$$

and

$$\begin{aligned} I_{12} &= 2\int_0^t g(t-\tau)\int_\Omega \Delta w(\tau)\cdot\Delta w(t) dx d\tau \\ &\leq 2\eta\|\Delta w(t)\|_2^2 + \frac{\|g\|_{L^1}}{2\eta}\int_0^t g(t-\tau)\|\Delta w(\tau)\|_2^2 d\tau, \end{aligned} \tag{2.72}$$

where $\eta > 0$ is arbitrary. Combining these inequalities with $0 < \eta \leq \frac{\|g\|_{L^1}}{2}$, we get

$$\frac{d}{dt}\left\{\alpha\|\Delta w(t)\|_2^2 - 2\int_\Omega w_t\Delta w dx + M(\|\nabla v_1\|_2^2)\|\nabla w(t)\|_2^2\right\}$$
$$+2\left(\psi(\|\nabla v_1\|_2^2) - 2\eta\right)\|\Delta w(t)\|_2^2$$
$$\leq C_5 e(w(t)) + C_4 e(v_1 - v_2)^{\frac{1}{2}} e(w(t))^{\frac{1}{2}}$$
$$+\frac{\|g\|_{L^1}}{2\eta}\int_0^t g(t-\tau)\|\Delta w(\tau)\|_2^2 d\tau + 2\|\nabla w_t\|_2^2, \tag{2.73}$$

where $C_5 = 2A_2B_1^2R^2$. Multiplying (2.73) by ϵ, $0 < \epsilon \leq 1$, and adding (2.66) together, we obtain

$$\frac{d}{dt}e^{**}(w(t)) + 2(\alpha - \epsilon)\|\nabla w_t\|_2^2 + 2\epsilon\left(\psi(\|\nabla v_1\|_2^2) - 2\eta\right)\|\Delta w(t)\|_2^2$$
$$\leq (C_3 + \epsilon C_5)e(w(t)) + (1+\epsilon)C_4 e(v_1 - v_2)^{\frac{1}{2}} e(w(t))^{\frac{1}{2}}$$
$$+\epsilon\frac{\|g\|_{L^1}}{2\eta}\int_0^t g(t-\tau)\|\Delta w(\tau)\|_2^2 d\tau, \tag{2.74}$$

where

$$\begin{aligned} e^{**}(w(t)) &= \|w_t(t)\|_2^2 + \left(\psi(\|\nabla v_1\|_2^2) - \int_0^t g(\tau)d\tau\right)\|\nabla w(t)\|_2^2 \\ &\quad +(g\circ\nabla w)(t) + \epsilon\alpha\|\Delta w(t)\|_2^2 - 2\epsilon\int_\Omega w_t\Delta w dx \\ &\quad +\epsilon M(\|\nabla v_1\|_2^2)\|\nabla w(t)\|_2^2. \end{aligned} \tag{2.75}$$

By using Young's inequality on the fifth term of right hand side of (2.75), we get

$$\begin{aligned} e^{**}(w(t)) &\geq (1-2\epsilon)\|w_t(t)\|_2^2 + \epsilon(\alpha - \frac{1}{2})\|\Delta w(t)\|_2^2 \\ &\quad + \left(\psi(\|\nabla v_1\|_2^2) - \int_0^t g(\tau)d\tau\right)\|\nabla w(t)\|_2^2 \\ &\quad +(g\circ\nabla w)(t) + \epsilon M(\|\nabla v_1\|_2^2)\|\nabla w(t)\|_2^2. \end{aligned} \tag{2.76}$$

Choosing $\epsilon = \frac{2}{5}$ and by $(H1)$, $(H3)$, we have

$$e^{**}(w(t)) \geq \frac{1}{5}e(w(t)). \tag{2.77}$$

Then, applying the some way as in obtained (2.54) and taking $\eta = \frac{\|g\|_{L^1}}{2\eta}$,
we deduce

$$e^{**}(w(t)) \leq e^{**}(w(0)) + \int_0^t \left[5\left(C_3 + \frac{2}{5}C_5 \right) e^{**}(w(t)) + \frac{7\sqrt{5}}{5} C_4 e(v_1 - v_2)^{\frac{1}{2}} e^{**}(w(t))^{\frac{1}{2}} \right] d\tau. \tag{2.78}$$

Thus, applying Gronwall's Lemma and noting that $e^{**}(w(0)) = 0$, we have

$$e^{**}(w(t)) \leq \frac{49}{20} C_4^2 T^2 e^{5\left(C_3 + \frac{2}{5}C_5\right)T} \sup_{0 \leq t \leq T} e(v_1 - v_2). \tag{2.79}$$

By (2.39) and (2.77), we have

$$d(u^{(1)}, u^{(2)}) \leq C(T, R_0)^{\frac{1}{2}} d(v_1, v_2), \tag{2.80}$$

where

$$C(T, R_0)^{\frac{1}{2}} = \frac{49}{4} C_4^2 T^2 e^{5\left(C_3 + \frac{2}{5}C_5\right)T}. \tag{2.81}$$

Hence, under inequality (2.58), S is a contraction mapping if $C(T, R_0) < 1$. Indeed, we choose R_0 sufficient large and T sufficient small so that (2.58) and (2.80) are satisfied at the same time. By applying Banach fixed point theorem, we obtain the local existence result.

The second statement of the theorem is proved by a standard continuation argument. Indeed, let $[0, T)$ be a maximal existence interval on which the solution of (2.1) exists. Suppose that $T < \infty$ and $\lim_{t \to T^-} \left(\|u_t(t)\|_2^2 + \|\Delta u(t)\|_2^2 \right) < \infty$. Then, there are a sequence $\{t_n\}$ and a constant $K > 0$ such that $t_n \to T^-$ as $n \to \infty$ and $\|u_t(t_n)\|_2^2 + \|\Delta u(t_n)\|_2^2 \leq K$, $n = 1, 2, ...$ Since for all $n \in \mathbb{N}$, there exists a unique solution of (2.1) with initial data $(u(t_n), u_t(t_n))$ on $[t_n, t_{n+\rho}]$, $\rho > 0$ depending on K and independent of $n \in \mathbb{N}$. Thus, we can get $T < t_n + \rho$ for $n \in \mathbb{N}$ large enough. It contradicts to the maximality of T. The proof of theorem 2.1 is now completed. □

2.4 Global Existence and Energy Decay

In this section, we consider the global existence and energy decay of solutions for a kind of the problem (2.1):

$$\begin{cases} u_{tt} - \psi(\|\nabla u\|_2^2)\Delta u - \alpha\Delta u_t + g\star\Delta u + M(\|\nabla u\|_2^2)u_t \\ = |u|^{p-2}u,\ x\in\Omega,\ t>0, \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x),\ x\in\Omega, \\ u(x,t)=0,\ x\in\partial\Omega,\ t>0, \end{cases} \tag{2.82}$$

where $2 < p \le \frac{2(d-1)}{d-2}$, $\alpha \ge 1$ and $\psi(r) = 1 + br^\gamma$, $b \ge 0$, $\gamma \ge 1$ and $r \ge 0$.

Lemma 2.4. *The energy for* (2.82) *is defined by*

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\left(1-\int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2 \\ &\quad + \frac{1}{2}(g\circ\nabla u)(t) + \frac{b}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} \\ &\quad - \frac{1}{p}\|u(t)\|_p^p. \end{aligned} \tag{2.83}$$

and its derivative satisfies the following

$$\begin{aligned} \frac{d}{dt}\{E(t)\} &= \frac{1}{2}(g'\circ\nabla u)(t) - M(\|\nabla u\|_2^2)\|u_t(t)\|_2^2 - \alpha\|\nabla u_t(t)\|_2^2 \\ &\quad - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 \\ &\le 0, \quad \forall t\ge 0. \end{aligned} \tag{2.84}$$

Proof. Multiplying the first equation in (2.82) by u_t and integrating the result over Ω and adding Green's formula, we get

$$\begin{aligned} &(u_{tt},u_t)_{L^2(\Omega)} - \Big(\psi(\|\nabla u\|_2^2)\Delta u, u_t\Big)_{L^2(\Omega)} - \alpha(\Delta u_t, u_t)_{L^2(\Omega)} \\ &+ \Big(\int_0^t g(t-\tau)\Delta u(\tau)d\tau, u_t(t)\Big)_{L^2(\Omega)} + \Big(M(\|\nabla u\|_2^2)u_t, u_t\Big)_{L^2(\Omega)} \\ &= (|u|^{p-2}u, u_t)_{L^2(\Omega)}. \end{aligned} \tag{2.85}$$

By a direct calculation, it follows that

$$(u_{tt}, u_t)_{L^2(\Omega)} = \frac{1}{2}\frac{d}{dt}\|u_t\|_2^2, \tag{2.86}$$

$$\begin{aligned} -\alpha(\Delta u_t, u_t)_{L^2(\Omega)} &= \alpha(\nabla u_t, \nabla u_t)_{L^2(\Omega)} \\ &= \alpha\|\nabla u_t\|_2^2, \end{aligned} \tag{2.87}$$

$$\Big(M(\|\nabla u\|_2^2)u_t, u_t\Big)_{L^2(\Omega)} = M(\|\nabla u\|_2^2)\|u_t\|_2^2, \tag{2.88}$$

$$(|u(t)|^{p-2}u, u_t)_{L^2(\Omega)} = \frac{1}{p}\frac{d}{dt}\|u\|_p^p, \tag{2.89}$$

and

$$\begin{aligned} -\Big(\psi(\|\nabla u\|_2^2)\Delta u, u_t\Big)_{L^2(\Omega)} &= -\Big(1 + b\|\nabla u\|_2^{2\gamma}\Big)\int_\Omega \Delta u u_t dx \\ &= \Big(1 + b\|\nabla u\|_2^{2\gamma}\Big)\int_\Omega \nabla u \nabla u_t dx \\ &= \Big(1 + b\|\nabla u\|_2^{2\gamma}\Big)\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega |\nabla u|^2 dx\right\} \\ &= \frac{1}{2}\frac{d}{dt}\left\{\|\nabla u(t)\|_2^2\right\} + \frac{b}{2}\|\nabla u(t)\|_2^{2\gamma}\frac{d}{dt}\left\{\|\nabla u\|_2^2\right\} \\ &= \frac{1}{2}\frac{d}{dt}\left\{\|\nabla u\|_2^2\right\} + \frac{b}{2(\gamma+1)}\frac{d}{dt}\left\{\|\nabla u\|_2^{2(\gamma+1)}\right\} \\ &= \frac{d}{dt}\left\{\frac{1}{2}\|\nabla u\|_2^2 + \frac{b}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)}\right\}, \end{aligned} \tag{2.90}$$

Also using Lemma 2.2, we get

$$\begin{aligned} \left(\int_0^t g(t-\tau)\Delta u(\tau)d\tau, u_t(t)\right)_{L^2(\Omega)} &= -\int_0^t g(t-\tau)\int_\Omega \nabla u(\tau)\nabla u_t(t) dx d\tau \\ &= \frac{1}{2}\frac{d}{dt}\left\{(g \circ \nabla u)(t) - \int_0^t g(\tau)d\tau\|\nabla u(t)\|_2^2\right\} \\ &\quad -\frac{1}{2}(g' \circ \nabla u)(t) + \frac{1}{2}g(t)\|\nabla u(t)\|_2^2. \end{aligned} \tag{2.91}$$

By replacing $(2.86) - (2.91)$ in (2.85), we get

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\|u_t(t)\|_2^2\right\} + \frac{d}{dt}\left\{\frac{1}{2}\|\nabla u\|_2^2 + \frac{b}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)}\right) \\
&-\frac{d}{dt}\left\{\frac{1}{2}(g\circ\nabla u)(t) - \int_0^t g(\tau)d\tau\|\nabla u(t)\|_2^2\right\} - \frac{d}{dt}\left\{\frac{1}{p}\|u(t)\|_p^p\right\} \\
&+\alpha\|\nabla u_t(t)\|_2^2 + M(\|\nabla u\|_2^2)\|u_t(t)\|_2^2 - \frac{1}{2}(g'\circ\nabla u)(t) \\
&+\frac{1}{2}g(t)\|\nabla u(t)\|_2^2). \qquad (2.92)
\end{aligned}$$

Then (2.92) inspires us to define the energy functional $E(t)$ as (2.83).

To obtain the results of this section, we now define some functionals as follows:

$$\begin{aligned}
I_1(t) &= \left(1 - \int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2 + (g\circ\nabla u)(t) \\
&\quad -\|u(t)\|_p^p, \qquad (2.93)
\end{aligned}$$

$$I_2(t) = I_1(t) + b\|\nabla u(t)\|_2^{2(\gamma+1)}, \qquad (2.94)$$

$$\begin{aligned}
J(t) &= \frac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}(g\circ\nabla u)(t) \\
&\quad +\frac{b}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} - \frac{1}{p}\|u(t)\|_p^p. \qquad (2.95)
\end{aligned}$$

We define the energy of the solution u of (2.82) by

$$\begin{aligned}
E(t) &= \frac{1}{2}\|u_t(t)\|_2^2 + J(t) \\
&= \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2 \\
&\quad +\frac{1}{2}(g\circ\nabla u)(t) + \frac{b}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} \\
&\quad -\frac{1}{p}\|u(t)\|_p^p. \qquad (2.96)
\end{aligned}$$

Lemma 2.5. *$E(t)$ is a non-increasing function for $t \geq 0$, that is*

$$\begin{aligned}
E'(t) &\leq -\left[m_1\|u_t(t)\|_2^2 + \alpha\|\nabla u_t(t)\|_2^2 + \frac{K_2}{2}(g\circ\nabla u)(t) + \frac{1}{2}g(t)\|\nabla u(t)\|_2^2\right] \\
&\leq 0, \quad \forall t \geq 0. \qquad (2.97)
\end{aligned}$$

Multiplying the differential equation in (2.82) by u_t, integrating by parts over Ω and using $(H3)$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\|u_t(t)\|_2^2+\frac{1}{2}\|\nabla u(t)\|_2^2+\frac{b}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)}-\frac{1}{p}\|u(t)\|_p^p\right]\\
&=-\alpha\|\nabla u_t(t)\|_2^2-M(\|\nabla u\|_2^2)\|u_t(t)\|_2^2+\int_0^t\int_\Omega g(t-\tau)\nabla u(\tau)\cdot\nabla u_t(t)dxd\tau\\
&\leq-\alpha\|\nabla u_t(t)\|_2^2-m_1\|u_t(t)\|_2^2+\int_0^t\int_\Omega g(t-\tau)\nabla u(\tau)\cdot\nabla u_t(t)dxd\tau.
\end{aligned}$$

Exploiting Lemma 2.2 on the third term on the right hand side of the above inequality and using $(H1)$, we have the result. □

Lemma 2.6. *Let u be the solution of (2.82). Assume the conditions of Theorem 2.1 hold. If $I_1(0)>0$ and*

$$\sigma=\frac{B_1^p}{l_1}\left(\frac{2p}{l_1(p-2)}E(0)\right)^{\frac{p-2}{2}}<1, \tag{2.98}$$

then $I_2(t)>0$, for all $t\geq 0$.

Proof. Since $I_1(0)>0$, it follows from the continuity of $u(t)$ that

$$I_1(t)>0, \tag{2.99}$$

for some interval near $t=0$. Let $t_{max}>0$ be a maximal time (possibly $t_{max}=T$), when (2.99) holds on $[0,t_{max})$.
From (2.93) and (2.95), we have

$$\begin{aligned}
J(t) &\geq \frac{1}{2}\left(1-\int_0^t g(\tau)d\tau\right)\|\nabla u\|_2^2+\frac{1}{2}(g\circ\nabla u)(t)-\frac{1}{p}\|u\|_p^p\\
&\geq \frac{p-2}{2p}\left[\left(1-\int_0^t g(\tau)d\tau\right)\|\nabla u\|_2^2+(g\circ\nabla u)(t)\right]+\frac{1}{p}I_1(t)\\
&\geq \frac{p-2}{p}\left(1-\int_0^t g(\tau)d\tau\right)\|\nabla u\|_2^2\\
&\geq \left(\frac{p-2}{2p}\right)l_1\|\nabla u\|_2^2.
\end{aligned} \tag{2.100}$$

Using (2.96), (2.100) and $E(t)$ is non-increasing by (2.97), we get

$$\begin{aligned} l_1\|\nabla u\|_2^2 &\leq \frac{2p}{p-2}J(t) \\ &\leq \frac{2p}{p-2}E(t) \\ &\leq \frac{2p}{p-2}E(0). \end{aligned} \tag{2.101}$$

Exploiting Lemma 2.1 and (2.98), we obtain from (2.101)

$$\begin{aligned} \|u\|_p^p &\leq B_1^p\|\nabla u\|_2^p = B_1^p\|\nabla u\|_2^{p-2}\|\nabla u\|_2^2 \\ &\leq \frac{B_1^p}{l_1}\left(\frac{2p}{l_1(p-2)}E(0)\right)^{\frac{p-2}{2}} l_1\|\nabla u\|_2^2 = \sigma l_1\|\nabla u\|_2^2 \\ &< \left(1-\int_0^t g(\tau)d\tau\right)\|\nabla u\|_2^2 \quad \text{on} \quad [0, t_{max}). \end{aligned} \tag{2.102}$$

Thus

$$I_1(t) = \left(1-\int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2 + (g\circ\nabla u)(t) - \|u(t)\|_p^p > 0 \quad \text{on} \quad [0, t_{max}). \tag{2.103}$$

This implies that we can take $t_{max} = T$. But, from (2.93) and (2.94), we see that

$$I_2(t) \geq I_1(t) > 0, \quad t \in [0, T]. \tag{2.104}$$

Therefore, we have $I_2(t) > 0$, $t \in [0, T]$.

Next, we want to show that $T = \infty$. Multiplying the first equation in (2.82) by $-2\Delta u$, and integrating it over Ω, we get

$$\begin{aligned} &\frac{d}{dt}\left\{\alpha\|\Delta u\|_2^2 - 2\int_\Omega u_t\Delta u dx + M(\|\nabla u\|_2^2)\|\nabla u\|_2^2\right\} \\ &+ \left(2\psi(\|\nabla u\|_2^2) - 2\eta\right)\|\Delta u\|_2^2 \\ \leq\ & 2\|\nabla u_t\|_2^2 - 2\int_\Omega |u|^{p-2}u\Delta u dx + \frac{\|g\|_{L^1}}{2\eta}\int_0^t g(t-\tau)\|\Delta u(\tau)\|_2^2 d\tau \\ &+ \left(\frac{d}{dt}M(\|\nabla u\|_2^2)\right)\|\nabla u\|_2^2, \end{aligned} \tag{2.105}$$

where $0 < \eta \leq \frac{\|g\|_{L^1}}{2}$. On the other hand, Multiplying the first equation in (2.82) by $2u_t$, and integrating it over Ω,
we get

$$\frac{d}{dt}(2E(t)) + 2\alpha\|\nabla u_t\|_2^2 = (g' \circ \nabla u)(t) - g(t)\|\nabla u(t)\|_2^2 - 2M(\|\nabla u\|_2^2)\|u_t\|_2^2. \quad (2.106)$$

Multiplying (2.105) by ϵ, $0 < \epsilon \leq 1$, and adding (2.106) together, we obtain

$$\begin{aligned} &\frac{d}{dt}E^*(t) + 2(\alpha - \epsilon)\|\nabla u_t\|_2^2 + 2\epsilon\left(\psi(\|\nabla u\|_2^2) - 2\eta\right)\|\Delta u\|_2^2 \\ &\leq -2\epsilon\int_\Omega |u|^{\alpha-2}u\Delta u dx + \epsilon\left(\frac{d}{dt}M(\|\nabla u\|_2^2)\right)\|\nabla u\|_2^2 \\ &+2\epsilon\frac{\|g\|_{L^1}}{2\eta}\int_0^t g(t-\tau)\|\Delta u(\tau)\|_2^2 d\tau, \end{aligned} \quad (2.107)$$

where

$$E^*(t) = 2E(t) - 2\epsilon\int_\Omega u_t\Delta u dx + \epsilon\alpha\|\Delta u\|_2^2 + \epsilon M(\|\nabla u\|_2^2)\|\nabla u\|_2^2. \quad (2.108)$$

By young's inequality, we get

$$\left|2\epsilon\int_\Omega u_t\Delta u dx\right| \leq 2\epsilon\|u_t\|_2^2 + \frac{\epsilon}{2}\|\Delta u\|_2^2. \quad (2.109)$$

Hence, choosing $\epsilon = \frac{2}{5}$ and by (2.103), we see that

$$E^*(t) \geq \frac{1}{5}\left(\|u_t\|_2^2 + \|\Delta u\|_2^2\right). \quad (2.110)$$

Let us estimate $I_{13} = \left(\frac{d}{dt}M(\|\nabla u\|_2^2)\right)\|\nabla u\|_2^2$.
Since $M \in C^1([0,\infty)$, using (2.101) and (2.110) we infer that

$$\begin{aligned} I_{13} &= \left(\frac{d}{dt}M(\|\nabla u\|_2^2)\right)\|\nabla u\|_2^2 \\ &= 2M'(\|\nabla u\|_2^2)\left(\int_\Omega \nabla u \cdot \nabla u_t dx\right)\|\nabla u\|_2^2 \\ &\leq 2A_3\|\Delta u\|_2\|u_t\|_2\|\nabla u\|_2^2 \\ &\leq 10A_3\left(\frac{2p}{l_1(p-2)}\right)E(0)E^*(t) = c_7E^*(t), \end{aligned} \quad (2.111)$$

where $c_7 = 10A_3\left(\frac{2p}{l_1(p-2)}\right)E(0)$ and $A_3 = \max\left\{M'(r),\ 0 \le r \le \left(\frac{2p}{l_1(p-2)}\right)E(0)\right\}$. Moreover, we note that

$$\begin{aligned} 2\left|\int_\Omega |u|^{p-2}u\Delta u dx\right| &\le 2(p-1)\int_\Omega |u|^{p-2}|\nabla u|^2 dx \\ &\le 2(p-1)\|u\|^{p-2}_{(p-2)\theta_1}\|\nabla u\|^2_{2\theta_2}, \end{aligned} \tag{2.112}$$

where $\frac{1}{\theta_1}+\frac{1}{\theta_2}=1$, so that, we put $\theta_1 = 1$ and $\theta_2 = \infty$, if $d = 1$;
$\theta_1 = 1+\epsilon_1$ (for arbitrary small $\epsilon_1 > 0$), if $d = 2$; and $\theta_2 = \frac{d}{d-2}$, if $d \ge 3$.
Then, by Lemma 2.1, (2.101) and (2.110), we have

$$\begin{aligned} 2\left|\int_\Omega |u|^{p-2}u\Delta u dx\right| &\le 2B_1^p(p-1)\|\nabla u\|_2^{p-2}\|\Delta u\|_2^2 \\ &\le c_8 E^*(t), \end{aligned} \tag{2.113}$$

where $c_8 = 10B_1^p(p-1)\left(\frac{2p}{l_1(p-2)}E(0)\right)^{\frac{p-2}{2}}$. Inserting (2.111) and (2.113) into (2.107), and then integrating it over $(0,t)$, we obtain

$$\begin{aligned} &E^*(t) + \frac{4}{5}\left(m_0 - \eta - \frac{\|g\|^2_{L^1}}{4\eta}\right)\int_0^t \|\Delta u(\tau)\|_2^2 d\tau \\ &\le E^*(0) + \int_0^t c_9 E^*(\tau)d\tau, \end{aligned} \tag{2.114}$$

where $c_9 = c_7 + c_8$. Taking $\eta = \frac{\|g\|_{L^1}}{2}$ in (2.114), and by Gronwall's Lemma, we deduce

$$E^*(t) \le E^*(0)e^{c_9 t}, \tag{2.115}$$

for any $t \ge 0$. Therefore by Theorem 2.1, we have $T = \infty$. □

Lemma 2.7. *If u satisfies the assumptions of Lemma 2.5, then there exists $B > 0$ such that*

$$\|u\|_p^p \le BE(t). \tag{2.116}$$

Proof. Using Lemma 2.1 and (2.101), we have

$$\begin{aligned} \|u\|_p^p &\le B_1^p\|\nabla u\|_2^p = B_1^p\|\nabla u\|_2^{p-2}\|\nabla u\|_2^2 \\ &\le \frac{B_1^p}{l_1}\left(\frac{2p}{l_1(p-2)}E(0)\right)^{\frac{p-2}{2}} l_1\|\nabla u\|_2^2 = \sigma l_1\|\nabla u\|_2^2 \\ &\le \sigma\left(\frac{2p}{p-2}\right)E(t). \end{aligned}$$

Let $B = \sigma\left(\frac{2p}{p-2}\right)$, then we have (2.116). □

Theorem 2.2. *(Global existence and Energy decay) Suppose that* $(H1)$ *and* $(H3)$ *hold. Assume* $I_1(u_0) > 0$ *and* (2.98) *holds, then the problem* (2.82) *admits a global solution* u *if* $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ *and* $u_1 \in H_0^1(\Omega)$*. Moreover, we have the following decay estimates*

$$E(t) \le ce^{-\kappa\epsilon t}, \quad \forall t \ge 0 \ \textit{and} \quad \epsilon \in (0, \epsilon_1],$$

where c*,* κ *and* ϵ_1 *are positive constants.*

Proof. Defining the perturbed energy by

$$E_\epsilon(t) = E(t) + \epsilon\varphi(t), \tag{2.117}$$

where

$$\varphi(t) = \int_\Omega u(t)u_t(t)dx, \tag{2.118}$$

we can show that for ϵ small enough, there exist two positive constants β_1 and β_2 such that

$$\beta_1 E(t) \le E_\epsilon(t) \le \beta_2 E(t). \tag{2.119}$$

In fact

$$\begin{aligned} E_\epsilon(t) &\le E(t) + \frac{\epsilon}{2}\|u_t\|_2^2 + \frac{\epsilon}{2}\|u\|_2^2 \\ &\le (1+\epsilon)E(t) + \frac{\epsilon}{2}B_1^2\|\nabla u\|_2^2 \\ &\le (1+\epsilon)E(t) + \frac{\epsilon}{2}B_1^2\left(\frac{2p}{l_1(p-2)}\right)E(t) \\ &\le \beta_2 E(t), \end{aligned} \tag{2.120}$$

and

$$\begin{aligned} E_\epsilon(t) &\geq E(t) - \frac{\epsilon}{4\delta}\|u_t\|_2^2 - \epsilon\delta\|u\|_2^2 \\ &\geq E(t) - \frac{\epsilon}{4\delta}\|u_t\|_2^2 - \epsilon\delta B_1^2\|\nabla u\|_2^2. \end{aligned} \tag{2.121}$$

By choosing δ small enough, we have

$$\begin{aligned} E_\epsilon(t) &\geq E(t) - \frac{\epsilon}{4\delta}\|u_t\|_2^2 \\ &\geq J\Big(u(t)\Big) + \left(\frac{1}{2} - \frac{\epsilon}{4\delta}\right)\|u_t\|_2^2. \end{aligned} \tag{2.122}$$

Once δ is chosen, we take ϵ so small that

$$\begin{aligned} E_\epsilon(t) &\geq J\Big(u(t)\Big) + \frac{\beta_1}{2}\|u_t\|_2^2 \\ &\geq \beta_1 E(t), \end{aligned} \tag{2.123}$$

where $\frac{\beta_1}{2} \leq \frac{1}{2} - \frac{\epsilon}{4\delta}$. Now taking the derivative of $\varphi(t)$ defined in (2.118) and substituting

$$\begin{aligned} u_{tt} &= \psi(\|\nabla u\|_2^2)\nabla u + \alpha\Delta u_t - \int_0^t g(t-\tau)\Delta u(\tau)d\tau \\ &\quad -M(\|\nabla u\|_2^2)u_t + |u|^{p-2}u, \end{aligned} \tag{2.124}$$

in the obtained expression, it results that

$$\begin{aligned} \varphi'(t) &= \|u_t\|_2^2 - \|\nabla u\|_2^2 - b\|\nabla u\|_2^{2(\gamma+1)} - \alpha(\nabla u_t, \nabla u) \\ &\quad + \int_0^t g(t-\tau)\int_\Omega \nabla u(\tau)\cdot\nabla u(t)dxd\tau \\ &\quad -M(\|\nabla u\|_2^2)(u_t, u) + \|u\|_p^p. \end{aligned} \tag{2.125}$$

Adding and subtracting $2E(t)$, and taking (2.96) into account, from (2.125) we infer

$$\begin{aligned}\varphi'(t) &= -2E(t)+2\|u_t\|_2^2-\left(\int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2\\ &\quad +(g\circ\nabla u)(t)-b(1-\frac{1}{\gamma+1})\|\nabla u\|_2^{2(\gamma+1)}\\ &\quad +(1-\frac{2}{p})\|u\|_p^p-\alpha(\nabla u_t,\nabla u)-M(\|\nabla u\|_2^2)(u_t,u)\\ &\quad +\int_0^t g(t-\tau)\int_\Omega \nabla u(\tau)\cdot\nabla u(t)dxd\tau. \qquad (2.126)\end{aligned}$$

Estimate for $J_1=\alpha(\nabla u_t,\nabla u)$. Considering Cauchy-Schwartz inequality, we have

$$|J_1|\leq \frac{\alpha^2}{2}\|\nabla u_t(t)\|_2^2+\frac{1}{2}\|\nabla u(t)\|_2^2. \qquad (2.127)$$

Let us estimate $J_2=M(\|\nabla u\|_2^2)(u_t,u)$.
Noting that $\|\nabla u(t)\|_2^2\leq \frac{2p}{l_1(p-2)}E(0)=\beta_3$ for all $t\geq 0$, we have that

$$M(\|\nabla u\|_2^2)\leq \xi, \quad \forall t\geq 0, \qquad (2.128)$$

where $\xi=\max\{M(r);\ r\in[0,\beta_3]\}$. From (2.228) we conclude that

$$\begin{aligned}|J_2| &\leq \frac{\xi^2}{2}\|u_t(t)\|_2^2+\frac{1}{2}\|u(t)\|_2^2\\ &\leq \frac{\xi^2}{2}\|u_t(t)\|_2^2+\frac{1}{2}B_1^2\|\nabla u(t)\|_2^2. \qquad (2.129)\end{aligned}$$

Estimate $J_3=\int_0^t g(t-\tau)\int_\Omega \nabla u(\tau)\cdot\nabla u(t)dxd\tau$. From assumption $(H1)$ and making use of the Cauchy-Schwarz inequality, we have

$$\begin{aligned}J_3 &= \int_0^t g(t-\tau)\int_\Omega \nabla u(\tau)\cdot\nabla u(t)dxd\tau\\ &= \int_0^t g(t-\tau)\int_\Omega [\nabla u(\tau)-\nabla u(t)+\nabla u(t)]\cdot\nabla u(t)dxd\tau\\ &\leq \int_0^t g(t-\tau)|\nabla u(t)-\nabla u(\tau)||\nabla u(t)|dxd\tau+\left(\int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2\\ &\leq \|\nabla u(t)\|_2^2\int_0^t g(t-\tau)\|\nabla u(t)-\nabla u(\tau)\|_2^2d\tau+\left(\int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2\\ &\leq \frac{1}{2}\|\nabla u(t)\|_2^2+\frac{1}{2}\|g\|_{L^1(0,\infty)}(g\circ\nabla u)(t)+\left(\int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2\\ &\leq \frac{1}{2}\|\nabla u(t)\|_2^2+\frac{1}{2}(g\circ\nabla u)(t)+\left(\int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2. \qquad (2.130)\end{aligned}$$

Utilizing Lemma 2.7 and inserting (2.227), (2.229) and (2.130) in (2.226), we have

$$\begin{aligned}\varphi'(t) &\le \left(\frac{\xi^2}{2}+2\right)\|u_t\|_2^2 + \left(1+\frac{B_1^2}{2}\right)\|\nabla u\|_2^2 + \frac{3}{2}(g\circ\nabla u)(t)\\ &\quad + \left[\left(1-\frac{2}{p}\right)B-2\right]E(t) - b\left(1-\frac{1}{\gamma+1}\right)\|\nabla u\|_2^{2(\gamma+1)}\\ &\quad + \frac{\alpha^2}{2}\|\nabla u_t(t)\|_2^2. \qquad (2.131)\end{aligned}$$

Then, from (2.97), (2.117), (2.118) and (2.131) we arrive at

$$\begin{aligned}E'_\epsilon(t) &= E'(t)+\epsilon\varphi'(t)\\ &\le -(m_1-\lambda_1\epsilon)\|u_t\|_2^2 + \lambda_2\epsilon\|\nabla u\|_2^2\\ &\quad -\left(\frac{K_2}{2}-\frac{3}{2}\epsilon\right)(g\circ\nabla u)(t) - \left(\alpha-\frac{\alpha^2}{2}\epsilon\right)\|\nabla u_t(t)\|_2^2\\ &\quad -\epsilon(-\lambda_3)E(t) - b\epsilon\left(1-\frac{1}{\gamma+1}\right)\|\nabla u\|_2^{2(\gamma+1)}\\ &\quad -\frac{1}{2}g(t)\|\nabla u(t)\|_2^2, \qquad (2.132)\end{aligned}$$

where $\lambda_1 = \frac{\xi^2}{2}+2>0$, $\lambda_2 = \frac{B_1^2}{2}+1>0$
and $\lambda_3 = \left(1-\frac{2}{p}\right)B-2 = \left(1-\frac{2}{p}\right)\left(\frac{2p}{p-2}\right)\sigma-2 = 2\sigma-2<0$.
On the other hand, since

$$\int_0^t g'(\tau)d\tau = g(t)-g(0),$$

then

$$-g(t)\|\nabla u(t)\|_2^2 = -g(0)\|\nabla u(t)\|_2^2 - \left(\int_0^t g'(\tau)d\tau\right)\|\nabla u(t)\|_2^2.$$

From $(H1)$ the last inequality yields

$$-\frac{1}{2}g(t)\|\nabla u(t)\|_2^2 \le -\frac{1}{2}g(0)\|\nabla u(t)\|_2^2 + \frac{K_1}{2}\|g\|_{L^1(0,\infty)}\|\nabla u(t)\|_2^2. \qquad (2.133)$$

Combining (2.133) and (2.132) we conclude that

$$\begin{aligned} E'_\epsilon(t) &\le -\epsilon(-\lambda_3)E(t) - (m_1 - \lambda_1\epsilon)\|u_t\|_2^2 - \left(\frac{K_2}{2} - \frac{3}{2}\epsilon\right)(g\circ\nabla u)(t) \\ &\quad - \left(\alpha - \frac{\alpha^2}{2}\epsilon\right)\|\nabla u_t(t)\|_2^2 - b\epsilon\left(1 - \frac{1}{\gamma+1}\right)\|\nabla u\|_2^{2(\gamma+1)} \\ &\quad -\frac{1}{2}\left[g(0) - K_1\|g\|_{L^1(0,\infty)} - 2\lambda_2\epsilon\right]\|\nabla u(t)\|_2^2. \end{aligned} \quad (2.134)$$

From $(H1)$ we have $l_2 = g(0) - K_1\|g\|_{L^1(0,\infty)} > 0$. Defining

$$\epsilon_1 = \min\left\{\frac{m_1}{\lambda_1}, \frac{K_2}{3}, \frac{2}{\alpha}, \frac{l_2}{2\lambda_2}\right\},$$

we conclude by taking $\epsilon \in (0, \epsilon_1]$ in (2.134) that

$$E'_\epsilon(t) \le -\epsilon(-\lambda_3)E(t).$$

Thus, we see that

$$\begin{aligned} E'_\epsilon(t) &\le -\epsilon(-\lambda_3)E(t) \\ &\le -\frac{-\lambda_3}{\beta_2}\epsilon E_\epsilon(t), \quad \forall t \ge 0 \text{ and } \epsilon \in (0, \epsilon_1]. \end{aligned}$$

By the Gronwall inequality, we see that

$$E_\epsilon(t) \le E_\epsilon(0)e^{-\kappa\epsilon t}, \quad \forall t \ge 0 \text{ and } \epsilon \in (0, \epsilon_1],$$

where $\kappa = \frac{-\lambda_3}{\beta_2}$. Combining with (2.119), we obtain

$$\beta_1 E(t) \le E_\epsilon(t) \le E_\epsilon(0)e^{-\kappa\epsilon t}, \quad \forall t \ge 0 \text{ and } \epsilon \in (0, \epsilon_1],$$

and

$$E(t) \le ce^{-\kappa\epsilon t}, \quad \forall t \ge 0 \text{ and } \epsilon \in (0, \epsilon_1],$$

where $c = \frac{E_\epsilon(0)}{\beta_1}$. Thus, the proof of the theorem is completed. □

Chapter 3

Global Existence, Uniqueness, and Asymptotic Behavior of Solution for the Euler-Bernoulli Viscoelastic Equation

3.1 Introduction

This chapter is concerned with the global existence, uniqueness, and asymptotic behavior of solution for the Euler-Bernoulli viscoelastic equation

$$\begin{cases} u_{tt} + \Delta^2 u - g_1 * \Delta^2 u + g_2 * \Delta u + u_t = 0,\ x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x),\ x \in \Omega, \\ u = 0, \quad \frac{\partial u}{\partial \nu} = 0, x \in \partial\Omega,\ t > 0, \end{cases} \tag{3.1}$$

where Ω is a bounded domain of $\mathbb{R}^d$ with a smooth boundary $\partial\Omega$, and ν is the unit outer normal on $\partial\Omega$. Here g_1 and g_2 are positive functions satisfying some conditions to be specified later, and

$$g_i * \chi(t) = \int_0^t g_i(t-\tau)\chi(\tau)d\tau, \quad i = 1, 2.$$

The Euler[1]-Bernoulli[2] equation

$$u_{tt}(x,t) + \Delta^2 u(x,t) + h(u_t) = f(u), \quad (x,t) \in \mathbb{R}^d \times (0,\infty), \tag{3.2}$$

describes the deflection $u(x,t)$ of a beam (when $d = 1$) or a plate (when $d = 2$). Where

$$\Delta^2 u := \Delta(\Delta u) = \sum_{j=1}^{d} \left(\sum_{i=1}^{d} u_{x_i} u_{x_i} \right)_{x_j x_i},$$

h and f represent the friction damping and the source respectively.
H. Lange and G. Perla Menzala [12] considered

$$u_{tt}(x,t) + \Delta^2 u(x,t) + a(t)u_t(x,t) = 0 \tag{3.3}$$

where $x \in \mathbb{R}^d$, $t \geq 0$, $a(t) = m(\|\nabla v(\cdot,t)\|^2_{L^2(\mathbb{R}^d)})$ and the real-valued function $m : [0,+\infty) \to [1,+\infty)$ will be assumed to be of class C^1 satisfying the condition $m(s) \geq 1+s$ for all $s \geq 0$. They remarked that the imaginary part of the solutions of Schrödinger's equation

$$iw_t = \Delta w + im\left(\|\nabla(Imw)\|^2_{L^2(\mathbb{R}^d)}\right) Rew = 0,$$

are precisely the solutions for (3.3). Then, using Fourier transform, the existence of global classical solutions and algebraic decay rate were proved for initial data whose regularity depends on the spacial dimension d. Messaoudi [20] studied the equation

$$u_{tt}(x,t) + \Delta^2 u(x,t) + a|u_t|^{m-2}u_t = b|u|^{p-2}u, \tag{3.4}$$

[1] Leonhard Euler 1707-1783 mathématicien et physicien suisse

[2] Daniel Bernoulli 1700-1782 physicien et mathématicien suisse

where $a, b > 0, p, m > 2$ and established an existence result for (3.4) and showed that the solution continued to exist globally if $m \geq p$. When we take the viscoelastic materials into consideration, the model (3.2) becomes

$$u_{tt}(x,t) + \Delta^2 u(x,t) - \int_0^t g(t-s)\Delta^2 u(x,s)ds + h(u_t) = f(u), \qquad (3.5)$$

where g is so-called viscoelastic kernel. The term $\int_0^t g(t-s)\Delta^2 u(x,s)ds$ describes the hereditary properties of the viscoelastic materials [8]. It expresses the fact that the stress at any instant t depends on the past history of strains which the material has undergone from time 0 up to t.

When $h = f = 0$, Tatar [28] obtained the property of the energy decay of the model (3.5). And from this, we know that the term $\int_0^t g(t-s)\Delta^2 u(x,s)ds$, similar to the friction damping, can cause the inhibition of the energy.

Messaoudi and Mukiawa [22] studied the fourth-order viscoelastic plate equation

$$u_{tt}(x,t) + \Delta^2 u(x,t) - \int_0^t g(t-s)\Delta^2 u(x,s)ds = 0,$$

in the bounded domain $\Omega = (0,\pi) \times (-l,l) \subset \mathbb{R}^2$ with nontraditional boundary conditions. The authors established the well-posedness of the solution and a decay result.

Rivera et al. [26] investigated the plate model:

$$u_{tt} + \Delta^2 u - \sigma\Delta u_{tt} + \int_0^t g(t-s)\Delta^2 u(s)ds = 0,$$

in a bounded domain $\Omega \subset \mathbb{R}^2$ with mixed boundary condition, suitable geometrical hypotheses on $\partial\Omega$. They established that the energy decays to zero with the same rate of the kernel g such as exponential and polynomial decay. To do so in the second case they made assumptions on g, g' and g'' which means that $g \simeq (1+t)^{-p}$ for $p > 2$. Then they obtained the same decay rate for the energy. However, their approach can not be applied to prove similar results for

$1 < p \leq 2$.

Cavalcanti et al. [6] investigated the global existence, uniqueness and stabilization of energy of

$$u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)ds + a(t)u_t = 0$$

where

$$a(t) = M\left(\int_\Omega |\nabla u(x,t)|^2 dx\right) \quad \text{with} \quad M \in C^1([0,+\infty)).$$

By taking a bounded or unbounded open set Ω where $M(s) > m_0 > 0$ for all $s \geq 0$, the authors showed in [6] that the energy goes to zero exponentially provided that g goes to zero at the same form.

The aim of this work is to study the global existence of regular and weak solutions of problem (1) for a bounded domain, then for $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ a increasing C^2 function such that

$$\xi(0) = 0, \quad \xi'(0) > 0, \quad \lim_{t\to+\infty} \xi(t) = +\infty, \quad \xi''(t) < 0 \quad \forall t \geq 0. \tag{3.6}$$

the solution features the asymptotic behavior

$$E(t) \leq E(0)e^{-\kappa\xi(t)}, \quad \forall t \geq 0,$$

where $E(t)$ is defined in (3.38) and κ is a positive constant independent of the initial energy $E(0)$.

3.2 Preliminaries and main results

We begin by introducing some notation that will be used throughout this work. We define the Hilbert space

$$X = \left\{u \in H_0^2(\Omega); \Delta^2 u \in L^2(\Omega)\right\}$$

Then, X is a Hilbert space endowed with the natural inner product

$$(u, v)_X = (u, v)_{H_0^2} + (\Delta^2 u, \Delta^2 v).$$

Now let us precise the hypotheses on g_1 and g_2.

(H1) $g_1 : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded function satisfying

$$g_1(t) \in C^2(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \quad g_1(0) > 0.$$

(H2) There exist positive constants α_1, α_2 and α_3 such that

$$-\alpha_1 g_1(t) \leq g_1'(t) \leq -\alpha_2 g_1(t), \quad \forall t \geq 0,$$

(H3)

$$0 \leq g_1''(t) \leq \alpha_3 g_1(t), \quad \forall t \geq 0,$$

(H4) $g_2 : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded function satisfying

$$g_2(t) \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \quad g_2(0) > 0.$$

(H5) There exist positive constants η_1 and η_2 such that

$$-\eta_1 g_2(t) \leq g_2'(t) \leq -\eta_2 g_2(t), \quad \forall t \geq 0,$$

(H6)

$$1 - \int_0^t \Big(g_1(s) + \lambda_1^{-1} g_2(s)\Big)\, ds = l > 0,$$

where $\lambda_1 > 0$ is the first eigenvalue of the spectral Dirichlet problem

$$\Delta^2 u = \lambda_1 u \quad \text{in } \Omega, \quad u = \frac{\partial u}{\partial \nu} = 0 \text{ in } \partial\Omega,$$

$$\|\nabla u\|_2 \leq \frac{1}{\sqrt{\lambda_1}} \|\Delta u\|_2.$$

Theorem 3.1. *Assume that* $(H1) - (H6)$ *hold, and that* $\{u_0, u_1\}$ *belong to* $H_0^2(\Omega) \times L^2(\Omega)$. *Then, problem* (3.1) *admits a unique weak solution* u *in the class*

$$u \in C^0([0,\infty); H_0^2(\Omega)) \cap C^1([0,\infty); L^2(\Omega)).$$

Moreover, for $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ *a increasing* C^2 *function satisfying* (3.6) *and, if* $\|g_1\|_{L^1(0,\infty)}$ *is sufficiently small we have for* $\kappa > 0$

$$E(t) \leq E(0)e^{-\kappa\xi(t)}, \quad \forall t \geq 0.$$

3.3 Existence of Solutions

In this section we first prove the existence and uniqueness of regular solutions to problem (3.1). Then, we extend the same result to weak solutions using density arguments.

Regular solutions. Let (w_j) be a Galerkin basis in X, and let V^n be the subspace generated by the first n vectors $w_1, ..., w_n$. We search for a function

$$u^n(t) = \sum_{i=1}^{n} k_i^n(t) w_i(x), \; n = 1, 2, ...$$

satisfying the approximate Cauchy problem

$$\begin{aligned} &(u_{tt}^n(t), v) + (\Delta u^n(t), \Delta v) - \int_0^t g_1(t-s)(\Delta u^n(s), \Delta v) ds \\ &- \int_0^t g_2(t-s)(\nabla u^n(s), \nabla v) ds + (u_t^n(t), v) = 0, \quad \forall v \in V^n, \end{aligned} \tag{3.7}$$

$$u^n(0) = u_0^n \longrightarrow u_0 \text{ in } X \text{ and } u_t^n(0) = u_1^n \longrightarrow u_1 \text{ in } H_0^2(\Omega). \tag{3.8}$$

By standard methods in differential equations, we can prove the existence of solutions to the problem $(3.7) - (3.8)$ on $[0, t_n)$ with $0 < t_n < T$. In order to extend the solution of $(3.7) - (3.8)$ to the whole $[0, T]$, we need the following a priori estimate.

Estimate 1. Taking $v = 2u_t^n(t)$ in (3.7), we have

$$\begin{aligned}
&\frac{d}{dt}\left[\|u_t^n(t)\|_2^2 + \|\Delta u^n(t)\|_2^2\right] + 2\|u_t^n(t)\|_2^2 \\
&-2\int_0^t g_1(t-s)\Delta u^n(s)\cdot \Delta u_t^n(t)dxds \\
&-2\int_0^t g_2(t-s)\nabla u^n(s)\cdot \nabla u_t^n(t)dxds = 0. \qquad (3.9)
\end{aligned}$$

Exploiting Lemma 2.2, we obtain

$$\begin{aligned}
&-2\int_0^t g_1(t-s)\int_\Omega \Delta u^n(s)\cdot \Delta u_t^n(t)dxds \\
&= \frac{d}{dt}\left\{(g_1 \circ \Delta u^n)(t) - \left(\int_0^t g_1(s)ds\right)\|\Delta u^n(t)\|_2^2\right\} \\
&-(g_1' \circ \Delta u^n)(t) + g_1(t)\|\Delta u^n(t)\|_2^2, \qquad (3.10)
\end{aligned}$$

and

$$\begin{aligned}
&-2\int_0^t g_2(t-s)\int_\Omega \nabla u^n(s)\cdot \nabla u_t^n(t)dxds \\
&= \frac{d}{dt}\left\{(g_2 \circ \nabla u^n)(t) - \left(\int_0^t g_2(s)ds\right)\|\nabla u^n(t)\|_2^2\right\} \\
&-(g_2' \circ \nabla u^n)(t) + g_2(t)\|\nabla u^n(t)\|_2^2, \qquad (3.11)
\end{aligned}$$

Inserting (3.10) and (3.11) into (3.9) and integrating over $[0,t] \subset [0,T]$, we obtain

$$\begin{aligned}
&\|u_t^n(t)\|_2^2 + \left(1 - \int_0^t g_1(s)ds\right)\|\Delta u^n(t)\|_2^2 + (g_1 \circ \Delta u^n)(t) \\
&-\left(\int_0^t g_2(s)ds\right)\|\nabla u^n(t)\|_2^2 + (g_2 \circ \nabla u^n)(t) + 2\int_0^t \|u_t^n(s)\|_2^2 ds \\
&-\int_0^t (g_1' \circ \Delta u^n)(s)ds + \int_0^t\int_\Omega g_1(s)|\Delta u^n(s)|^2 dxds \\
&-\int_0^t (g_2' \circ \nabla u^n)(s)ds + \int_0^t\int_\Omega g_2(s)|\nabla u^n(s)|^2 dxds \\
&= \|u_1^n(t)\|_2^2 + \|\Delta u_0^n\|_2^2. \qquad (3.12)
\end{aligned}$$

By using the fact that

$$\begin{aligned}
&(g_1 \circ \Delta u^n)(t) + (g_2 \circ \nabla u^n)(t) - \int_0^t (g_1' \circ \Delta u^n)(s)ds - \int_0^t (g_2' \circ \nabla u^n)(s)ds \\
&+\int_0^t\int_\Omega g_1(s)|\Delta u^n(s)|^2 dxds + \int_0^t\int_\Omega g_2(s)|\nabla u^n(s)|^2 dxds \geq 0,
\end{aligned}$$

and

$$\begin{aligned}&\left(1-\int_0^t g_1(s)ds\right)\|\Delta u^n(t)\|_2^2-\left(\int_0^t g_2(s)ds\right)\|\nabla u^n(t)\|_2^2\\&\geq\left(1-\int_0^t\left[g_1(s)+\lambda_1^{-1}g_2(s)\right]ds\right)\|\Delta u^n(t)\|_2^2\\&\geq l\|\Delta u^n(t)\|_2^2,\end{aligned}$$

estimate (3.12) yields

$$\|u_t^n(t)\|_2^2+l\|\Delta u^n(t)\|_2^2+2\int_0^t\|u_t^n(s)\|_2^2ds\leq\|u_1^n(t)\|_2^2+\|\Delta u_0^n\|_2^2. \quad (3.13)$$

Taking the convergence in (3.8) into consideration, we arrive at

$$\|u_t^n(t)\|_2^2+l\|\Delta u^n(t)\|_2^2+2\int_0^t\|u_t^n(s)\|_2^2ds\leq L_1. \quad (3.14)$$

where $L_1=\|u_1\|_2^2+\|\Delta u_0\|_2^2$.

Estimate 2. Firstly, we obtain an estimate for $u_{tt}^n(0)$ in the L^2 norm. indeed, setting $v=u_{tt}^n(0)$ and $t=0$ in (3.7), we obtain

$$\|u_{tt}^n(0)\|_2^2\leq\left[\|\Delta^2u_0^n\|_2+\|u_1^n\|_2\right]\|u_{tt}^n(0)\|_2. \quad (3.15)$$

From (3.8), (3.14) and (3.15), it follows that

$$\|u_{tt}^n(0)\|_2\leq L_2,\quad\forall n\in\mathbb{N}, \quad (3.16)$$

where L_2 is a positive constant independent of $n \in \mathbb{N}$. Differentiating equation (3.7) with respect to t, and setting $v = u^n_{tt}(t)$, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\|u^n_{tt}(t)\|_2^2 + \frac{1}{2}\|\Delta u^n_t(t)\|_2^2\right] + \|u^n_{tt}(t)\|_2^2 \\
&= -g_1(0)\int_\Omega \Delta^2 u^n(t) u^n_{tt}(t)dx - \int_\Omega\int_0^t g_1'(t-s)\Delta^2 u^n(s) u^n_{tt}(t)dsdx \\
&-g_2(0)\int_\Omega \Delta u^n(t) u^n_{tt}(t)dx - \int_\Omega\int_0^t g_2'(t-s)\Delta u^n(s) u^n_{tt}(t)dsdx \\
&= -g_2(0)\int_\Omega \Delta u^n(t) u^n_{tt}(t)dx - \int_\Omega\int_0^t g_2'(t-s)\Delta u^n(s) u^n_{tt}(t)dsdx \\
&-g_1(0)\|\Delta u^n_t(t)\|_2^2 + g_1(0)\frac{d}{dt}\int_\Omega \Delta u^n(t)\cdot\Delta u^n_t(t)dx \\
&+\frac{d}{dt}\left\{\int_0^t g_1'(t-s)\int_\Omega \Delta u^n(t)\cdot\Delta u^n_t(t)dxds\right\} - g_1'(0)\int_\Omega \Delta u^n(t)\cdot\Delta u^n_t(t)dx \\
&-\int_0^t g_1''(t-s)\int_\Omega \Delta u^n(s)\cdot\Delta u^n_t(t)dxds.
\end{aligned}
\tag{3.17}
$$

By $(H5)$, Hölder's inequality and Young's inequality give

$$
-\int_\Omega\int_0^t g_2'(t-s)\Delta u^n(s) u^n_{tt}(t)dsdx \le \frac{1}{2}\|u^n_{tt}(t)\|_2^2 + \frac{\eta_1^2\|g_2\|_{L^1}}{2}\int_0^t g_2(t-s)\|\Delta u^n(s)\|_2^2 ds.
\tag{3.18}
$$

From (3.14) we obtain

$$
-g_2(0)\int_\Omega \Delta u^n(t) u^n_{tt}(t)dx \le \frac{1}{2}\|u^n_{tt}(t)\|_2^2 + \frac{[g_2(0)]^2 L_1}{2l}
\tag{3.19}
$$

and

$$
\begin{aligned}
-g_1'(0)\int_\Omega \Delta u^n(t)\cdot\Delta u^n_t(t)dx &\le \frac{|g_1'(0)|}{2}\left(\|\Delta u^n(t)\|_2^2 + \|\Delta u^n_t(t)\|_2^2\right) \\
&\le \frac{|g_1'(0)|L_1}{2l} + \frac{|g_1'(0)|}{2}\|\Delta u^n_t(t)\|_2^2.
\end{aligned}
\tag{3.20}
$$

From $(H3)$ we deduce

$$
\int_\Omega\int_0^t g_1''(t-s)\Delta u^n(s)\Delta u^n_t(t)dsdx \le \frac{1}{2}\|\Delta u^n_t(t)\|_2^2 + \frac{\alpha_3^2\|g_1\|_{L^1}}{2}\int_0^t g_1(t-s)\|\Delta u^n(s)\|_2^2 ds
\tag{3.21}
$$

Then inserting $(3.18) - (21)$ in (3.17) we infer that

$$\frac{1}{2}\|u_{tt}^n(t)\|_2^2 + \frac{1}{2}\|\Delta u_t^n(t)\|_2^2 \leq \|u_{tt}^n(0)\|_2^2 + \|\Delta u_1^n\|_2^2 + C_3 + g_1(0)\int_\Omega \Delta u^n(t)\cdot\Delta u_t^n(t)dx$$
$$+\int_0^t g_1'(t-s)\int_\Omega \Delta u^n(t)\cdot\Delta u_t^n(t)dxds$$
$$+C_4\int_0^t \|\Delta u_t^n(s)\|_2^2 ds, \quad (3.22)$$

where

$$C_3 = \left[\frac{\eta_1^2\|g_2\|_{L^1}}{2} + \frac{[g_2(0)]^2 L_1}{2l} + \frac{|g_1'(0)|L_1}{2l}\right]T$$
$$+\left[\frac{\eta_1^2\|g_2\|_{L^1(0,\infty)}\|g_2\|_{L^\infty(0,\infty)}}{2} + \frac{\alpha_1^2\|g_1\|_{L^1(0,\infty)}\|g_1\|_{L^\infty(0,\infty)}}{2}\right]\frac{L_1 T}{l}$$

and $C_4 = \frac{|g_1'(0)|}{2} + \frac{1}{2}$.
Using Hölder's inequality, we know that, for any $\delta > 0$,

$$g_1(0)\int_\Omega \Delta u^n(t)\cdot\Delta u_t^n(t)dx + \int_0^t g_1'(t-s)\int_\Omega \Delta u^n(t)\cdot\Delta u_t^n(t)dxds$$
$$\leq 2\delta\|\Delta u_t^n(t)\|_2^2 + \frac{[g_1(0)]^2}{4\delta}\|\Delta u^n(t)\|_2^2 + \frac{\alpha_1^2}{4\delta}\|g_1\|_{L^1(0,\infty)}\|g_1\|_{L^\infty(0,\infty)}\int_0^t \|\Delta u^n(s)\|_2^2 ds$$
$$\leq 2\delta\|\Delta u_t^n(t)\|_2^2 + C_5, \quad (3.23)$$

where

$$C_5 = \left[\frac{[g_1(0)]^2}{4\delta} + \frac{\alpha_1^2}{4\delta}\|g_1\|_{L^1(0,\infty)}\|g_1\|_{L^\infty(0,\infty)}T\right]\frac{L_1}{l}.$$

Combining (3.22) and (3.23) we conclude that

$$\frac{1}{2}\|u_{tt}^n(t)\|_2^2 + \left(\frac{1}{2} - 2\delta\right)\|\Delta u_t^n(t)\|_2^2 \leq \|u_{tt}^n(0)\|_2^2 + \|\Delta u_{1n}^n\|_2^2 + C_3 + C_5$$
$$+C_4\int_0^t \|\Delta u_t^n(s)\|_2^2 ds, \quad (3.24)$$

Fixing $\delta > 0$ sufficiently small so that $\frac{1}{2} - 2\delta > 0$ in (3.24), and taking into account (3.8) and (3.16), we get from Gronwall's Lemma the second estimate,

$$\|u_{tt}^n(t)\|_2^2 + \|\Delta u_t^n(t)\|_2^2 \leq L_3, \quad (3.25)$$

where L_3 is a positive constant independent of $n \in \mathbb{N}$ and $t \in [0, T]$.

Estimate 3. Let $n_1 \geq n_2$ be two natural numbers, and consider $z^n = u^{n_1} - u^{n_2}$. Then, applying the same way as in the estimate 1 and observing that $\{u_0^n\}$ and $\{u_1^n\}$ are Cauchy sequence in X and $H_0^2(\Omega)$, respectively, we deduce

$$\|z_t^n(t)\|_2^2 + \|\Delta z^n(t)\|_2^2 + 2\int_0^t \|z_t^n(s)\|_2^2 ds \to 0, \ \text{ as } \ n \to +\infty, \tag{3.26}$$

for all $t \in [0, T]$.

Therefore, from (3.14), (3.25) and (3.26), we deduce that there exist a subsequence $\{u^\mu\}$ of $\{u^n\}$ and u such that

$$u_t^\mu \to u_t \ \text{ strongly in } \ C^0([0, T]; L^2(\Omega)), \tag{3.27}$$

$$u^\mu \to u \ \text{ strongly in } \ C^0([0, T]; H_0^2(\Omega)), \tag{3.28}$$

$$u_{tt}^\mu \to u_{tt} \ \text{ weakly star in } \ L^\infty(0, T; L^2(\Omega)). \tag{3.29}$$

The above convergences $(3.27) - (3.29)$ are enough to pass to the limit in (3.7), to obtain

$$\begin{array}{l} u_{tt} + \Delta^2 u - \int_0^t g_1(t-s)\Delta^2 u(s) ds + \int_0^t g_2(t-s)\Delta u(s) ds + u_t = 0 \ \text{ in } \ L^\infty(0, \infty; L^2(\Omega)), \\ u(0) = u_0, \quad u_t(0) = u_1. \end{array}.$$

Next, we want to show the uniqueness of $(3.7) - (3.8)$. Let $u^{(1)}$, $u^{(2)}$ be two solutions of $(3.7) - (3.8)$. Then $z = u^{(1)} - u^{(2)}$ satisfies

$$\begin{aligned} &(z_{tt}(t), v) + (\Delta z(t), \Delta v) - \int_0^t g_1(t-s)(\Delta z(s), \Delta v) ds \\ &- \int_0^t g_2(t-s)(\nabla z(s), \nabla v) ds + (z_t(t), v) = 0, \quad \forall v \in H_0^2(\Omega), \end{aligned} \tag{3.30}$$

$$z(x, 0) = z_t(x, 0) = 0, \quad x \in \Omega,$$

$$z = 0, \quad \frac{\partial z}{\partial \nu} = 0, x \in \partial\Omega, \ t > 0.$$

Setting $v = 2z'(t)$ in (30), then as in deriving (3.14), we see that

$$\|z_t(t)\|_2 = \|\Delta z(t)\|_2 = 0 \ \text{ for all } \ t \in [0, T]. \tag{3.31}$$

Therefore, we have the uniqueness.

Weak solutions. Let $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$. Then, since $X \times H_0^2(\Omega)$ is dense in $H_0^2(\Omega) \times L^2(\Omega)$ there exists $(u_{0\mu}, u_{1\mu}) \subset X \times H_0^2(\Omega)$ such that

$$u_{0\mu} \to u_0 \ \text{in} \ \ H_0^2(\Omega) \ \text{and} \ \ u_{1\mu} \to u_1 \ \text{in} \ \ L^2(\Omega). \tag{3.32}$$

Then, for each $\mu \in \mathbb{N}$, there exists a unique regular solution u_μ of problem (3.1) in the class

$$u_\mu \in L^\infty(0, \infty; H_0^2(\Omega)), \quad u'_\mu \in L^\infty(0, \infty; H_0^2(\Omega)), \quad u''_\mu \in L^\infty(0, \infty; L^2(\Omega)). \tag{3.33}$$

In view of (3.33) and using an analogous argument to that in Estimate 1 and Estimate 3, we find a sequence $\{u^\mu\}$ of solutions to problem (3.1) such that

$$u_t^\mu \to u' \ \text{ weak star in } \ L^\infty(0, T; L^2(\Omega)), \tag{3.34}$$

$$u^\mu \to u \ \text{ weak star in } \ L^\infty(0, T; H_0^2(\Omega)), \tag{3.35}$$

$$u^\mu \to u \ \text{ strongly in } \ C^0([0, T]; H_0^2(\Omega)), \tag{3.36}$$

$$u_t^\mu \to u_t \ \text{ strongly in } \ C^0([0, T]; L^2(\Omega)), \tag{3.37}$$

The convergences $(3.33) - (3.36)$ are sufficient to pass to the limit in order to obtain a weak solution of (1) which satisfies

$$\begin{aligned} &u_{tt} + \Delta^2 u - \textstyle\int_0^t g_1(t-s)\Delta^2 u(s)ds + \int_0^t g_2(t-s)\Delta u(s)ds + u_t = 0 \ \text{ in } \ L^2(0, \infty; H^{-2}(\Omega)), \\ &u(0) = u_0, \quad u_t(0) = u_1. \end{aligned}$$

The uniqueness of weak solutions requires a regularization procedure and can be obtained using the standard method of Visik-Ladyzhenskaya, c.f. Lions and Magenes [13, Chap. 3, Sec. 8.2.2].

3.4 Asymptotic Behaviour

In this section, we discuss the asymptotic behavior of the above-mentioned weak solutions. Let us define the energy associated to (3.1) as

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g_1(s)ds\right)\|\Delta u(t)\|_2^2 + \frac{1}{2}(g_1 \circ \Delta u)(t) \\ &\quad - \frac{1}{2}\left(\int_0^t g_2(s)ds\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}(g_2 \circ \nabla u)(t). \end{aligned} \tag{3.38}$$

To demonstrate our decay result, the lemmas below are essential.

Lemma 3.1. *For any* $t > 0$

$$0 \leq E(t) \leq \frac{1}{2}\Big[\|u_t(t)\|_2^2 + \|\Delta u(t)\|_2^2 + (g_1 \circ \Delta u)(t) + (g_2 \circ \nabla u)(t)\Big].$$

proof. Using the fact that $\|\nabla u(t)\|_2^2 \leq \lambda_1^{-1}\|\Delta u(t)\|_2^2$, we have

$$\begin{aligned} &\left(1 - \int_0^t g_1(\tau)d\tau\right)\|\Delta u(t)\|_2^2 - \left(\int_0^t g_2(\tau)d\tau\right)\|\nabla u(t)\|_2^2 \\ &\geq \left(1 - \int_0^t \left[g_1(s) + \lambda_1^{-1}g_2(s)\right]ds\right)\|\Delta u(t)\|_2^2 \end{aligned}$$

and according to $(H6)$ we have $E(t) \geq 0$,

and

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\Delta u(t)\|_2^2 + \frac{1}{2}(g_1 \circ \Delta u)(t) + \frac{1}{2}(g_2 \circ \nabla u)(t) \\ &- \frac{1}{2}\left\{\left(\int_0^t g_1(s)ds\right)\|\Delta u(t)\|_2^2 + \left(\int_0^t g_2(s)ds\right)\|\nabla u(t)\|_2^2\right\} \\ &\leq \frac{1}{2}\Big[\|u_t(t)\|_2^2 + \|\Delta u(t)\|_2^2 + (g_1 \circ \Delta u)(t) + (g_2 \circ \nabla u)(t)\Big]. \end{aligned}$$

Lemma 3.2. *The energy* $E(t)$ *satisfies*

$$\begin{aligned} E'(t) &\leq -\|u_t(t)\|_2^2 - \frac{1}{2}\alpha_2(g_1 \circ \Delta u)(t) - \frac{1}{2}\eta_2(g_2 \circ \nabla u)(t) \\ &\quad - \frac{1}{2}\left[g_1(0) - \alpha_1\|g_1\|_{L^1(0,\infty)}\right]\|\Delta u(t)\|_2^2 \leq 0. \end{aligned} \tag{3.39}$$

proof. Multiplying the first equation in (3.1) by u_t, integrating over Ω, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\Delta u(t)\|_2^2\right] + \|u_t(t)\|_2^2 \\
&= \int_0^t g_1(t-\tau)\Delta u(\tau)\cdot \Delta u_t(t)dxd\tau \\
&+ \int_0^t g_2(t-\tau)\nabla u(\tau)\cdot \nabla u_t(t)dxd\tau.
\end{aligned}$$

Exploiting $(3.10)-(3.11)$ and by $(H1)-(H5)$, we deduce

$$\begin{aligned}
E'(t) &= -\|u_t(t)\|_2^2 + \frac{1}{2}(g_1' \circ \Delta u)(t) - \frac{1}{2}g_1(t)\|\Delta u(t)\|_2^2 \\
&\quad +\frac{1}{2}(g_2' \circ \nabla u)(t) - \frac{1}{2}g_2(t)\|\nabla u(t)\|_2^2 \\
&\leq -\|u_t(t)\|_2^2 - \frac{1}{2}\alpha_2(g_1 \circ \Delta u)(t) - \frac{1}{2}\eta_2(g_2 \circ \nabla u)(t) \\
&\quad -\frac{1}{2}g_1(t)\|\Delta u(t)\|_2^2. \qquad (3.40)
\end{aligned}$$

From assumptions $(H2)$ and since $\int_0^t g_1'(\tau)d\tau = g_1(t) - g_1(0)$, we obtain

$$\begin{aligned}
-\frac{1}{2}g_1(t)\|\Delta u(t)\|_2^2 &= -\frac{1}{2}g_1(0)\|\Delta u(t)\|_2^2 - \frac{1}{2}\left(\int_0^t g_1'(s)ds\right)\|\Delta u(t)\|_2^2 \\
&\leq -\frac{1}{2}g_1(0)\|\Delta u(t)\|_2^2 + \frac{\alpha_1}{2}\|g_1\|_{L^1(0,\infty)}\|\Delta u(t)\|_2^2 \\
&= -\frac{1}{2}\left[g_1(0) - \alpha_1\|g_1\|_{L^1(0,\infty)}\right]\|\Delta u(t)\|_2^2. \qquad (3.41)
\end{aligned}$$

Combining (3.40) and (3.41) we conclude that

$$\begin{aligned}
E'(t) &\leq -\|u_t(t)\|_2^2 - \frac{1}{2}\alpha_2(g_1 \circ \Delta u)(t) - \frac{1}{2}\eta_2(g_2 \circ \nabla u)(t) \\
&\quad -\frac{1}{2}\left[g_1(0) - \alpha_1\|g_1\|_{L^1(0,\infty)}\right]\|\Delta u(t)\|_2^2 \leq 0.
\end{aligned}$$

Multiplying (3.39) by $e^{\kappa\xi(t)}$ $(\kappa > 0)$ and utilizing Lemma 3.1, we have

$$\begin{aligned}
\frac{d}{dt}\left(e^{\kappa\xi(t)}E(t)\right) &\leq -e^{\kappa\xi(t)}E(t)\|u_t(t)\|_2^2 - \frac{1}{2}\left[g_1(0) - \alpha_1\|g_1\|_{L^1(0,\infty)}\right]e^{\kappa\xi(t)}E(t)\|\Delta u(t)\|_2^2 \\
&\quad -\frac{1}{2}\alpha_2(g_1\circ\Delta u)(t)e^{\kappa\xi(t)}E(t) - \frac{1}{2}\eta_2(g_2\circ\nabla u)(t)e^{\kappa\xi(t)}E(t) \\
&\quad +\kappa\xi'(t)e^{\kappa\xi(t)}E(t) \\
&\leq -\frac{1}{2}\left[2-\kappa\xi'(t)\right]e^{\kappa\xi(t)}E(t)\|u_t(t)\|_2^2 \\
&\quad -\frac{1}{2}\left[\alpha_2-\kappa\xi'(t)\right]e^{\kappa\xi(t)}E(t)(g_1\circ\Delta u)(t) \\
&\quad -\frac{1}{2}\left[\eta_2-\kappa\xi'(t)\right]e^{\kappa\xi(t)}E(t)(g_2\circ\nabla u)(t) \\
&\quad -\frac{1}{2}\left[g_1(0)-\alpha_1\|g_1\|_{L^1(0,\infty)}-\kappa\xi'(t)\right]e^{\kappa\xi(t)}E(t)\|\Delta u(t)\|_2^2. \qquad (3.42)
\end{aligned}$$

Using the fact that ξ' is decreasing we arrive at

$$\begin{aligned}
\frac{d}{dt}\left(e^{\kappa\xi(t)}E(t)\right) &\leq -\frac{1}{2}\left[2-\kappa\xi'(0)\right]e^{\kappa\xi(t)}E(t)\|u_t(t)\|_2^2 \\
&\quad -\frac{1}{2}\left[\alpha_2-\kappa\xi'(0)\right]e^{\kappa\xi(t)}E(t)(g_1\circ\Delta u)(t) \\
&\quad -\frac{1}{2}\left[\eta_2-\kappa\xi'(0)\right]e^{\kappa\xi(t)}E(t)(g_2\circ\nabla u)(t) \\
&\quad -\frac{1}{2}\left[g_1(0)-\alpha_1\|g_1\|_{L^1(0,\infty)}-\kappa\xi'(0)\right] \\
&\quad \times e^{\kappa\xi(t)}E(t)\|\Delta u(t)\|_2^2. \qquad (3.43)
\end{aligned}$$

Choosing $\|g_1\|_{L^1(0,\infty)}$ sufficiently small so that

$$g_1(0) - \alpha_1\|g_1\|_{L^1(0,\infty)} = L > 0,$$

and choosing κ sufficiently small in order to have

$$2-\kappa\xi'(0) > 0, \quad \alpha_2-\kappa\xi'(0) > 0, \quad \eta_2-\kappa\xi'(0) >, \quad L-\kappa\xi'(0) > 0.$$

from (3.43) we arrive at

$$\frac{d}{dt}\left(e^{\kappa\xi(t)}E(t)\right) \leq 0, \quad t > 0.$$

Integrating the above inequality over $(0,t)$, it follows that

$$E(t) \leq E(0)e^{-\kappa\xi(t)}, \quad t > 0.$$

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الملخص

في هذه الرسالة قمنا بدراسة بعض المعادلات الرياضية من نوع قطوع زائدة (معادلة كيرشوف و معادلة أولر- برنولي) مع وجود آليات للتبديد، في ظل بعض الفرضيات على المعطيات الأولية و على شروط التبديد. قد أثبتنا وجود حلول عامة لهذه المعادلات و حددنا معدل انخفاض الطاقة المرتبطة بهذه الحلول.

الكلمات المفتاحية : معادلة الأمواج، آليات التبديد، حلول عامة، معدل انخفاض الطاقة

Résumé

Dans cette thèse, nous avons considéré le problème aux dérivées partielles de type hyperbolique (l'équation d'onde viscoélastique de Kirchhoff type et l'équation d'Euler-Bernoulli viscoélastique) avec la présence de différents mécanismes de dissipation. Sous quelques hypothèses sur les données initiales, conditions sur les termes de dissipation et les termes viscoélastiques, nous avons montré l'existence globale et le comportement asymptotique des solutions.

Les mots clés : équation des ondes viscoélastique, termes d'amortissement, existence globale, comportement asymptotique.

Abstract

In this thesis, we considered some hyperbolic type problems (the viscoelastic wave equation of Kirchhoff type and the Euler-Bernoulli viscoelastic equation) with the presence of different mechanisms of dissipation. Under some assumptions on initial data, conditions on damping and viscoelastic terms, we proved the global existence and asymptotic behavior of the solutions.

Keywords: viscoelastic wave equation, damping terms, global existence, asymptotic behavior.