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Présentée par : Horiya Kouadri-Habbaze

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Intitulée

Existence et multiplicité des solutions pour certains problèmes aux limites non linéaires

Soutenue le .....

Devant le jury composé de

Président : Mr. Mokeddem Soufiane, Prof. à l'université Djillali Liabes de S.B.A

Examinateurs :

Mr. Hammoudi Ahmed, Prof. au centre universitaire d'Ain-Temouchent

Mr. Helal Mohamed, M.C.A à l'université Djillali Liabes de S.B.A

Mr. Habib Habib, M.C.A à l'université Djillali Liabes de S.B.A

Rapporteur : Mr. Lakmeche Ahmed, Prof. à l'université Djillali Liabes de S.B.A

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Dédicace

Je dédie ce travail :

À mes chers parents, pour leur soutien, patience et leurs sacrifices durant mes études et tout long de ce projet.

À mes frères.

À mes sœurs.

À ma famille.

À mes amis...

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Publications

- **H. Kouadri-Habbaze**, Ahmed Lakmeche, and A. Lakmeche, *Positive solutions for a system of second order multi-point boundary value problem with delay,* Comm. Appl. Nonlinear Anal., Vol. 26, N 3, (2019), 1-20.

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Introduction

The objective of this thesis is the study of the existence of solutions of some problems to the boundary conditions associated with ordinary nonlinear differential equations with delay of order two and three. The study of delay differential equations has expanded rapidly in recent years due to numerous applications which have developed in engineering and the applied sciences. In this thesis existence and uniqueness of solutions to certain second and third order boundary value problems for delay differential equations is established by using some fixed point theorems, notably Leray-Schauder theorem, Krasnoselskii theorem, expansion and compression of a cone, and fixed point index.

This thesis consists of five chapters

In first chapter, we introduce notations, definitions , lemmas, and fixed point theorems that are used in the next chapters.

Chapter 2, we present some existence results of positive solutions for a class of nonlinear second order delay boundary value problem following

$$\begin{cases} u''(t) + \lambda a(t) f(t, u(t-\tau)) = 0, & 0 < t < 1, \\ u(t) = \alpha u(\eta), & -\tau \leqslant t \leqslant 0, \\ u(1) = \beta u(\eta) \end{cases}$$

where α, β, η and τ are positive constants such that $\eta \in (0,1), 0 < \tau \leq \frac{1}{2}$ and λ is a positive real parameter. By using the Leray-Schauder fixed point theorem, some sufficient conditions for the existence of positive solutions are obtained of nonlinear second order delay boundary value problem

In chapter 3, we investigate the existence of positive solutions for a system of second order two delay differential equations with twin-parameters.

$$\begin{cases} u''(t) + \lambda a(t) f(u(t-\tau_1), v(t-\tau_2)) = 0, & 0 < t < 1, \\ v''(t) + \mu a(t) h(u(t-\tau_1), v(t-\tau_2)) = 0, & 0 < t < 1, \\ u(t) = \alpha u(\eta), & -\tau_1 \leqslant t \leqslant 0, \\ u(1) = \beta u(\eta) & \\ v(t) = \alpha v(\eta), & -\tau_2 \leqslant t \leqslant 0, \\ v(1) = \beta v(\eta) & \end{cases}$$

where $0 < \eta < 1$, $0 < \alpha < \frac{1}{2}$ and $0 < \beta < \frac{1-\alpha\eta}{1-\eta}$ are constants, and λ, μ are positive real parameters. By using the fixed point theorem of Krasnoselskii's obtained sufficient conditions we obtain existence of positive solutions for a system of second order two delay differential equations with twin-parameters.

In chapter 4, the fixed-point index theorem in cones was used to discuss the third-order multi-point boundary value problem,

$$\begin{cases} u'''(t) + \lambda a(t) f(t, u(t-\tau)) = 0, & 0 < t < 1, \\ u(t) = \alpha u(\eta), & -\tau \leq t \leq 0; \\ u(1) = \beta u(\eta); & \\ u'(0) = 0; & \end{cases}$$

where $\eta \in (0,1)$, $0 < \beta < \dfrac{1}{\eta^2}$, $0 < \alpha < \dfrac{1-\beta\eta^2}{1-\eta^2}$ are constants, and λ is a positive real parameter. By using fixed-point index theorem in cones, we establish the existence results of positive solutions for the boundary value problem

The purpose of Chapter 5, is to study the boundary value problem for a class of quadratic mixed type of delay differential equations with eigenvalue

$$\begin{cases} u''(t) + \lambda p(t) f(t, u(t-\tau), \int_0^t k(t,s)u(s)ds) = 0, & 0 < t < 1, \\ u(t) = \alpha u(\eta), & -\tau \leqslant t \leqslant 0, \\ u(1) = \beta u(\eta) \end{cases}$$

where α, β, η and τ are positive constants and λ is a positive real parameter. By using a fixed-point theorem in cones to study the boundary value problem for a class of quadratic mixed type of delay differential equations with eigenvalue, the sufficient condition of existence of their solutions is derived.

Chapitre 1

Preliminary

In this chapter, we introduce notations, definitions, and preliminary facts that will be used in the sequel.

1.1 Some notations and definitions

We shall consider the Banach space $X = C([a, b], \mathbb{R})$ be endowed with the maximum norm,

$$\|u\|_{[a,b]} = \max_{a \leq t \leq b} \{|u(t)| : a \leq t \leq b\}.$$

We need the following definitions :

Definition 1.1.1. *Let X be a real Banach space. A nonempty, closed, convex set $K \subset X$ is a cone if it satisfes the following two conditions :*

1. $u \in K, \lambda \geq 0$ *imply* $\lambda u \in K$,
2. $u \in K, -u \in K$ *imply* $u = 0$.

The cone K induces an ordering $\leq$ on X by

$$u \leq v \text{ if and only if } v - u \in K.$$

Definition 1.1.2. *An operator $T : X \longrightarrow X$ is completely continuous if it is continuous and $T(X)$ is relatively compact.*

Definition 1.1.3. *(Arzela-Ascoli Theorem). A subset A of $C([a, b], \mathbb{R})$ is relatively compact if and only if it is bounded and equicontinuous.*

1.2 Fixed point results

Fixed point theory plays a major role in many of our existence principles, therefore we shall state the fixed point theorems in Banach spaces.

Theorem 1.2.1. *(Leray-Schauder). Let Ω be a convex subset of a Banach space X, $0 \in \Omega$ and $T : \Omega \to \Omega$ be a completely continuous operator. Then either*

1. *T has at least one fixed point in Ω, or*
2. *the set $\{u \in \Omega / u = \mu T u,\ 0 < \mu < 1\}$ is unbounded.*

Theorem 1.2.2. *(Krasnosel'skii). Let X be a Banach space and $K \subset X$ be a cone. Assume Ω_1, Ω_2 are open subset of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K$ be a completely continuous operator such that either*

1. *$\| Tu \| \leqslant \| u \|, u \in K \cap \partial\Omega_1$, and $\| Tu \| \geqslant \| u \|, u \in K \cap \partial\Omega_2$; or*
2. *$\| Tu \| \geqslant \| u \|, u \in K \cap \partial\Omega_1$, and $\| Tu \| \leqslant \| u \|, u \in K \cap \partial\Omega_2$.*

Then T has a fixet point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 1.2.1. *Let Ω be a bounded open subset of X Banach space, with $\theta \in K \cap \Omega$, and $K \cap \overline{\Omega} \neq K$. Assume that $T : K \cap \overline{\Omega} \longrightarrow K$ be a completely continuous mapping such that $u \neq Tu$ for $u \in K \cap \partial\Omega$. Then the fixed point index $i(T, K \cap \Omega, K)$ has the following properties :*

(i) If there exists $v \in K \setminus \{\theta\}$, such that $u - Tu \neq \zeta v$ for every $u \in K \cap \partial\Omega$ and every $\zeta > 0$, then $i(T, K \cap \Omega, K) = 0$.
If $\| Tu \| \geq \| u \|$ for $u \in K \cap \partial\Omega$, then $i(T, K \cap \Omega, K) = 0$.

(ii) *If $\mu Tu \neq u$ for every $u \in K \cap \partial\Omega$ and $0 < \mu < 1$, then $i(T, K \cap \Omega, K) = 1$.*

For example (ii) holds if $\| Tu \| \leq \| u \|$ for $u \in K \cap \partial\Omega$.

(iii) *Let Ω' be open in X such that $K \cap \overline{\Omega'} \subset K \cap \Omega$. If $i(T, K \cap \Omega, K) = 1$ and $i(T, K \cap \overline{\Omega'}, K) = 0$, then T has a fixed point in $K \cap \Omega \backslash K \cap \overline{\Omega'}$. The same holds if $i(T, K \cap \Omega, K) = 0$ and $i(T, K \cap \Omega', K) = 1$.*

Lemma 1.2.2. *Let Ω be a bounded open subset of X Banach space, with $\theta \in \Omega$, and let $T : K \cap \overline{\Omega} \longrightarrow K$ be a completely continuous mapping. If $\mu Tu \neq u$ for every $u \in K \cap \partial\Omega$ and $0 < \mu \leq 1$, then $i(T, K \cap \Omega, K) = 1$.*

Lemma 1.2.3. *Let Ω be a bounded open subset of X Banach space, and let $T : K \cap \overline{\Omega} \longrightarrow K$ be a completely continuous mapping. If there exists an $v \in K \setminus \{\theta\}$, such that $u - Tu \neq \zeta v$ for every $u \in K \cap \partial\Omega$ and $\zeta \geq 0$, then $i(T, K \cap \Omega, K) = 0$.*

Lemma 1.2.4. *If $i(T, \Omega, X) \neq 0$, then T has at least one fixed point in Ω.*

The literature [8] studied the existence of positive solutions of boundary value problems (1.1)-(1.2) by using Krasnosels'kii fixed point theorem, they obtained the following results :

Theorem 1.2.3. *[8]*
Let (H_1)-(H_3) hold and $f_0 > 0$, $f^\infty < \infty$, then there exists at least one positive solution to (2.1) *for*

$$\lambda \in \left(\frac{1}{f_0 \sup_{t\in J}(\beta \min\{\eta, 1-\eta\} \int_0^\tau G(t,s)a(s)ds + \theta \int_{b+\tau}^{c+\tau} G(t,s)a(s)ds)}, \frac{1-\alpha\eta-\beta(1-\eta)}{(1-\alpha(1-\eta)-\beta\eta) f_0 (\beta \int_0^\tau G(s,s)a(s)ds + \int_\tau^1 G(s,s)a(s)ds)} \right)$$

Theorem 1.2.4. *[8]*
Let (H_1)-(H_3) hold and $f_\infty > 0$, $f_0 < 0$, then there exists at least one positive solution to (2.1) *for*

$$\lambda \in \left(\frac{1}{f_\infty \sup_{t\in J}(\beta \min\{\eta, 1-\eta\} \int_0^\tau G(t,s)a(s)ds + \theta \int_{b+\tau}^{c+\tau} G(t,s)a(s)ds)}, \frac{1-\alpha\eta-\beta(1-\eta)}{(1-\alpha(1-\eta)-\beta\eta) f^\infty (\beta \int_0^\tau G(s,s)a(s)ds + \int_\tau^1 G(s,s)a(s)ds)} \right)$$

Chapitre 2

Positive solutions for a second order multi-point boundary value problem with delay

In this chapter, we study the existence of positive solutions, by using the Leray-Schauder fixed point, to the following nonlinear multi-point boundary value problem with delay

$$\begin{aligned} & u^{''}(t) + \lambda a(t) f(t, u(t-\tau)) = 0, \qquad t \in [0,1], \\ & u(t) = \beta u(\eta), \qquad -\tau \leqslant t \leqslant 0, \\ & u(1) = \alpha u(\eta) \end{aligned} \tag{2.0.1}$$

where α, β, η, and τ are positive constants such that $\eta \in (0,1)$, and λ is a positive real parameter.

We assume the following hypothesis :

(H_1) $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous

(H_2) $a : [0,1] \to [0,\infty)$ is continuous and does not vanish identically on any subinterval.

(H_3) $0 < \alpha < 1$ and $0 < \beta < \frac{1-\alpha\eta}{1-\eta}$, $0 < \tau < 1$

2.1 Preliminaries

In this section we give some preliminary results.

Definition 2.1.1.
$u(t)$ is called a positive solution of (2.0.1) *if* $u \in \mathrm{C}[-\tau, 1] \cap \mathrm{C}^2(0, 1)$, $u(t) \geq 0$ *for* $t \in (0, 1)$ *and satisfies* (2.0.1).

Lemma 2.1.1.
Let $\beta \neq \frac{1-\alpha\eta}{1-\eta}$. Then for $y \in C([0, 1],\ \mathbb{R})$, the boundary value problem

$$u''(t) + y(t) = 0, \quad t \in [0, 1], \tag{2.1.1}$$

$$u(0) = \beta u(\eta), \quad u(1) = \alpha u(\eta) \tag{2.1.2}$$

has a unique solution

$$u(t) = \int_0^1 G(t, s)y(s)ds \tag{2.1.3}$$

where

$$G(t, s) = g(t, s) + \frac{\beta + (\alpha - \beta)t}{(1 - \alpha\eta) - \beta(1 - \eta)} g(\eta, s) \tag{2.1.4}$$

and

$$g(t, s) = \begin{cases} s(1 - t), & 0 \leq s \leq t \leq 1, \\ t(1 - s), & 0 \leq t \leq s \leq 1. \end{cases}$$

Proof.
From (2.1.1), we have

$$u(t) = u(0) + u'(0)t - \int_0^t (t - s)y(s)ds \ := A + Bt - \int_0^t (t - s)y(s)ds$$

with

$$u(0) = A,$$
$$u(\eta) = A + B\eta - \int_0^\eta (\eta - s)y(s)ds.$$

and

$$u(1) = A + B - \int_0^1 (1-s)y(s)ds.$$

From $u(0) = \beta u(\eta)$, we have

$$(1-\beta)A - B\beta\eta = -\beta \int_0^\eta (\eta - s)y(s)ds.$$

From $u(1) = \alpha u(\eta)$, we have

$$(1-\alpha)A + B(1-\alpha\eta) = \int_0^1 (1-s)y(s)ds - \alpha \int_0^\eta (\eta - s)y(s)ds.$$

Therefore,

$$A = \frac{\beta\eta}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^1 (1-s)y(s)ds - \frac{\beta}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^\eta (\eta - s)y(s)ds$$

and

$$B = \frac{1-\beta}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^1 (1-s)y(s)ds - \frac{\alpha-\beta}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^\eta (\eta - s)y(s)ds.$$

From which it follows that

$$\begin{aligned} u(t) = {} & \frac{\beta\eta}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^1 (1-s)y(s)ds - \frac{\beta}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^\eta (\eta - s)y(s)ds \\ & + \frac{(1-\beta)t}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^1 (1-s)y(s)ds - \frac{(\alpha-\beta)t}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^\eta (\eta-s)y(s)ds - \int_0^t (t-s)y(s)ds \\ = {} & -\int_0^t (t-s)y(s)ds + \frac{(\beta-\alpha)t - \beta}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^\eta (\eta-s)y(s)ds + \frac{(1-\beta)t + \beta\eta}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^1 (1-s)y(s)ds \\ = {} & \int_0^1 g(t,s)y(s)ds + \frac{\beta + (\alpha-\beta)t}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^1 g(\eta,s)y(s)ds. \end{aligned} \tag{2.1.5}$$

Then, $u(t) = \int_0^1 G(t,s)y(s)ds$. The function u presented above is the unique solution to the problem (2.1.1), (2.1.2).

Lemma 2.1.2.
Let $0 < \alpha < \frac{1}{\eta}$ *and* $0 \leq \beta < \frac{1-\alpha\eta}{1-\eta}$. *If* $y \in C([0,1],\ [0,\infty))$, *then the unique solution* u *of the problem* (2.1.1), (2.1.2) *satisfies*

$$u(t) \geq 0, \quad t \in [0,1].$$

Proof.
We know that if $u''(t) = -y(t) \leq 0$, for $t \in (0,1)$, $u(0) \geq 0$ and $u(1) \geq 0$, then $u(t) \geq 0$, for $t \in [0,1]$. We have

$$u(0) = \frac{-\beta}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^\eta(\eta-s)y(s)ds + \frac{\beta\eta}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^1(1-s)y(s)ds$$

$$= \frac{\beta}{(1-\alpha\eta)-\beta(1-\eta)}[-\int_0^\eta(\eta-s)y(s)ds + \eta\int_0^\eta(1-s)y(s)ds]$$

$$+\frac{\beta\eta}{(1-\alpha\eta)-\beta(1-\eta)}\int_\eta^1(1-s)y(s)ds$$

$$= \frac{\beta}{(1-\alpha\eta)-\beta(1-\eta)}[\int_0^\eta s(1-\eta)y(s)ds] + \frac{\beta\eta}{(1-\alpha\eta)-\beta(1-\eta)}\int_\eta^1(1-s)y(s)ds \geq 0$$

and

$$u(1) = -\int_0^1(1-s)y(s)ds + \frac{(\beta-\alpha)-\beta}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^\eta(\eta-s)y(s)ds$$

$$+\frac{(1-\beta)+\beta\eta}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^1(1-s)y(s)ds$$

$$= \frac{\alpha}{(1-\alpha\eta)-\beta(1-\eta)}[\eta\int_0^1(1-s)y(s)ds + \int_0^\eta(\eta-s)y(s)ds]$$

$$\geq \frac{\alpha}{(1-\alpha\eta)-\beta(1-\eta)}[\eta\int_0^\eta(1-s)y(s)ds + \int_0^\eta(\eta-s)y(s)ds]$$

$$= \frac{\alpha}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^\eta s(1-\eta)y(s)ds \geq 0.$$

Then, $u(t) \geq 0 \quad \forall t \in [0,1]$.

Lemma 2.1.3. *The function g has the following properties*

(i) $0 \leqslant g(t,s) \leqslant s(1-s) = g(s,s) \quad \forall t,s \in [0,1],$

(ii) Let $\theta \in [0, \frac{1}{2}]$. Then, for $t \in [\theta, 1-\theta]$ and $s \in [0,1]$, we have

$$g(t,s) \geq \min\{t, 1-t\} g(s,s) \geq \theta g(s,s).$$

Proof. For $0 \leq s \leq t \leq 1$, we have

$$0 \leq g(t,s) = s(1-t) \leq s(1-s) = g(s,s).$$

And for $0 \leq t \leq s \leq 1$, we have

$$g(t,s) = t(1-s) \leq s(1-s) = g(s,s).$$

Thus (i) holds.
If $s = 0$ or $s = 1$, we show that (ii) holds.
For $0 < s \leq t \leq 1$ and $s \neq 1$ we have

$$\frac{g(t,s)}{g(s,s)} = \frac{t(1-s)}{s(1-s)} = \frac{t}{s} \geq t \quad \forall t \in [0,1].$$

For $0 \leq t \leq s < 1$ and $s \neq 0$ we have

$$\frac{g(t,s)}{g(s,s)} = \frac{s(1-t)}{s(1-s)} = \frac{(1-t)}{(1-s)} \geq (1-t) \quad \forall t \in [0,1].$$

Then

$$g(t,s) \geq \min\{t, 1-t\} g(s,s).$$

Thus, there exist $\theta \in]0, \frac{1}{2}]$ such that

$$\frac{g(t,s)}{g(s,s)} \geq \theta, \quad \forall t \in [\theta, 1-\theta]$$

Thus (ii) holds.

Lemma 2.1.4. *The function G has the following properties*

(i) $G(t,s) \geqslant 0 \quad \forall t,s \in [0,1]$,

(ii) $G(t,s) \leqslant k_1 g(s,s) \quad \forall t,s \in [0,1]$ *and* $k_1 = 1 + \dfrac{\max\{\alpha,\beta\}}{(1-\alpha\eta)-\beta(1-\eta)}$,

(iii) $\min\limits_{\theta \leqslant t \leqslant 1-\theta} G(t,s) \geqslant k_2 g(s,s) \quad \forall t,s \in [0,1]$ *where* $\theta \in (0, \frac{1}{2})$ *and*

$$k_2 = \theta \left[1 + \frac{\beta + \min\{(\alpha-\beta)\theta, (\alpha-\beta)(1-\theta)\}}{(1-\alpha\eta)-\beta(1-\eta)}\right]$$

Proof.

(i) From (2.1.4) and (i) of Lemma 2.1.3, we get

$$G(t,s) \geq 0, \ \forall t,s \in [0,1].$$

(ii) By (2.1.4) and (i) of Lemma 2.1.3, we have

$$G(t,s) = g(t,s) + \frac{\beta + (\alpha-\beta)t}{(1-\alpha\eta)-\beta(1-\eta)} g(\eta,s)$$

$$\leq g(s,s) + \frac{\max(\alpha,\beta)}{(1-\alpha\eta)-\beta(1-\eta)} g(s,s) = k_1 g(s,s).$$

(ii) From (ii) of Lemma 2.1.3, for $t \in [\theta, 1-\theta]$ we have

$$\begin{aligned} G(t,s) &= g(t,s) + \frac{\beta + (\alpha-\beta)t}{(1-\alpha\eta)-\beta(1-\eta)} g(\eta,s) \\ &\geqslant \theta g(s,s) + \frac{\beta + \min\{(\alpha-\beta)\theta, (\alpha-\beta)(1-\theta)\}}{(1-\alpha\eta)-\beta(1-\eta)} \theta g(s,s) \\ &\geqslant \theta \left[1 + \frac{\beta + \min\{(\alpha-\beta)\theta, (\alpha-\beta)(1-\theta)\}}{(1-\alpha\eta)-\beta(1-\eta)}\right] g(s,s) = k_2 g(s,s). \end{aligned}$$

Lemma 2.1.5. *If* $y \in C([0,1])$ *and* $y \geqslant 0$*, then the unique solution* u *of the boundary value problem (2.1.1),(2.1.2) satisfies* $\min\limits_{\theta \leqslant t \leqslant 1-\theta} u(t) \geqslant \gamma \|u\|_1$ *where* $\|u\|_1 := \sup\{|u(t)|;\ 0 \leq t \leq 1\}$ *and* $\gamma := \dfrac{k_2}{k_1}$.

Proof. For any $t \in [0,1]$, by Lemma 2.1.4 we have

$$u(t) = \int_0^1 G(t,s)y(s)ds \leqslant k_1 \int_0^1 g(s,s)y(s)ds,$$

thus $||u||_1 \leqslant k_1 \int_0^1 g(s,s)y(s)ds$. Moreover, from (iii) of Lemma 2.1.4 for $t \in [\theta, 1-\theta]$, we have

$$u(t) = \int_0^1 G(t,s)y(s)ds \ \geqslant k_2 \int_0^1 g(s,s)y(s)ds \ \geqslant \frac{k_2}{k_1}||u||_1.$$

Therefore $\min\limits_{\theta \leqslant t \leqslant 1-\theta} u(t) \geqslant \gamma \|u\|_1$.

By Lemma 2.1.1, we can show that the BVP (2.1.1), (2.1.2) has a solution $u(t)$ if and only if $u(t)$ is a solution of the operator equation $u(t) = Tu(t)$, where

$$Tu(t) = \begin{cases} \beta u(\eta), & -\tau \leq t \leq 0, \\ \lambda \int_0^1 G(t,s)a(s)f(s,u(s-\tau))ds, & 0 \leq t \leq 1. \end{cases}$$

Let define,

$$f^0 = \limsup_{u \to 0} \max_{t \in [0,1]} \frac{f(t,u)}{u}, \qquad f^\infty = \limsup_{u \to \infty} \max_{t \in [0,1]} \frac{f(t,u)}{u}.$$

And

$$M_1 := \beta \int_0^\tau g(s,s)a(s)ds + \int_\tau^1 g(s,s)a(s)ds, \text{ and } M_2 := \int_0^1 g(s,s)a(s)ds.$$

The proof of our main results is based upon an application of the following Leray-Schauder fixed point theorem.

Theorem 2.1.1. *([10])*
Let Ω be a convex subset of a Banach space X, $0 \in \Omega$ and $T : \Omega \to \Omega$ be a completely continuous operator. Then either

1. *T has at least one fixed point in Ω, or*
2. *the set $\{u \in \Omega / u = \mu T u, \ 0 < \mu < 1\}$ is unbounded.*

2.2 Main results

Let $X = \{u \in C[-\tau, 1] : u(t) = \beta u(\eta) \text{ when } t \in [-\tau, 0], u(1) = \alpha u(\eta)\}$. With the norm $\| \,.\, \|$ given by $||u|| = \sup_{-\tau \le t \le 1} |u(t)|$. Then $(X, \| \,.\, \|)$ is a Banach space.

Theorem 2.2.1. *[16]*
Assume (H_1) and (H_2) hold. If $f^0 < \infty$, then the boundary value problem (2.0.1) has at least one positive solution.

Proof.
Choose $\epsilon > 0$ such that $(f^0 + \epsilon)\lambda k_1 M_1 \le 1$. By the definition of $f^0 < \infty$, we know that there exists constant $B > 0$, such that $f(s, u) < (f^0 + \epsilon)u$ for $0 < u \le B$.
Let

$$\Omega = \{u \,/ u \in X, u \ge 0, \|u\| \le B, \min_{\theta \le t \le 1-\theta} u(t) \ge \gamma \|u\|\}.$$

Then Ω is a convex subset of X.
For $u \in \Omega$, by Lemmas 2.1.2 and 2.1.5, we know that $Tu(t) \ge 0$ and $\min_{\theta \le t \le 1-\theta} (Tu)(t) \ge \gamma \|Tu\|$.
Moreover,

$$\begin{aligned}
Tu &\le \lambda k_1 \int_0^1 g(s, s)a(s)f(s, u(s-\tau))ds \\
&\le \lambda(f^0 + \epsilon)k_1 \int_0^1 g(s, s)a(s)u(s-\tau)ds \\
&= \lambda(f^0 + \epsilon)k_1 \left(\int_0^\tau g(s, s)a(s)\beta u(\eta)ds + \int_\tau^1 g(s, s)a(s)u(s-\tau)ds \right) \\
&\le \lambda(f^0 + \epsilon)k_1 \left(\beta \int_0^\tau g(s, s)a(s)ds + \int_\tau^1 g(s, s)a(s)ds \right) \|u\| \\
&\le \|u\| \le B.
\end{aligned}$$

Thus, $\|Tu\| \le B$. Hence, $T\Omega \subset \Omega$.

We shall show that T is completely continuous.

Suppose $u_n \to u$ $(n \to \infty)$ and $u_n \in \Omega$ $\forall n \in \mathbb{N}$, then there exists $M > 0$ such that $\|u_n\| \leq M$.

Since f is continuous on $[0,1] \times [0,M]$, it is uniformly continuous.

Therefore, $\forall \varepsilon > 0$ there exists $\delta > 0$ such that $|x-y| < \delta$ implies $|f(s,x)-f(s,y)| < \epsilon$ $\forall s \in [0,1]$, $x, y \in [0,M]$ and there exists N such that $\|u_n - u\| < \delta$ for $n > N$, so $|f(s,u_n(s-\tau)) - f(s,u(s-\tau))| < \varepsilon$, for $n > N$ and $s \in [0,1]$.

This implies

$$|Tu_n(t) - Tu(t)| \leq \lambda k_1 \int_0^1 g(s,s)a(s)|f(s,u_n(s-\tau)) - f(s,u(s-\tau)|ds$$

$$\leq \lambda\varepsilon k_1 \int_0^1 g(s,s)a(s)ds.$$

Therefore T is continuous.

Let D be any bounded subset of Ω, then there exists $\gamma > 0$ such that $||u|| \leq \gamma$ for all $u \in D$.

Since f is continuous on $[0,1] \times [0,\gamma]$ there exists $L > 0$ such that $|f(t,v)| < L$ $\forall (t,v) \in [0,1] \times [0,\gamma]$.

Consequently, for all $u \in D$ and $t \in [0.1]$ we have

$$|Tu(t)| \leq \left|\lambda k_1 \int_0^1 g(s,s)a(s)f(s,u(s-\tau))ds\right|$$
$$\leq \lambda k_1 L \int_0^1 g(s,s)a(s)ds.$$

Which implies the boundedness of TD.

Since G is continuous on $[0,1] \times [0,1]$, it is uniformly continuous.

Then $\forall \varepsilon > 0$ there exists $\delta > 0$ such that $|t_1 - t_2| < \delta$ implies that $|G(t_1,s) - G(t_2,s)| < \varepsilon$ $\forall s \in [0,1]$. So, if $u \in D$, $|Tu(t_1) - Tu(t_2)| \leq \lambda \int_0^1 |G(t_1,s) - G(t_2,s)|a(s)f(s,u_n(s-\tau))ds \leq \lambda L\varepsilon \int_0^1 a(s)ds$.

From the arbitrariness of ε, we get the equicontinuity of TD.
The operator T is completely continuous by the mean of the Ascoli-Arzela theorem.
For $u \in \Omega$ and $u = \mu Tu$, $0 < \mu < 1$, we have $u(t) = \mu Tu(t) < Tu(t) < B$, which implies $\|u\| \leq B$. So, $\{u \in \Omega/u = \mu Tu,\ 0 < \mu < 1\}$ is bounded. By theorem 2.1.1, we deduce that operator T has at least one fixed point in Ω. Thus the boundary value problem (2.0.1) has at least one positive solution.

Remarque 2.2.1. *The conditions of Theorem* 2.2.1 *are weaker than those of Theorem 1.2.3.*

Theorem 2.2.2. *[16]*
Assume $(H_1) - (H_2)$ hold. If $f^\infty < \infty$ is satisfied, then the boundary value problem (2.0.1) has at least one positive solution.

Proof. Choose $\epsilon > 0$ such that $(f^\infty + \epsilon)\lambda k_1 M_1 \leq \dfrac{1}{2}$. By the definition of $f^\infty < \infty$, we know that there exists constant $N > 0$, such that $f(s, u) < (f^\infty + \epsilon)u$ for $u > N$.
Let $B > 0$ such that

$$B \geq N + 1 + 2\lambda k_1 M_2 \max_{\substack{0\leq s\leq 1\\ 0\leq u\leq N}} f(s, u).$$

Let

$$\Omega = \{u/u \in X,\ u \geq 0\ \|u\| \leq B, \min_{\theta\leq t\leq 1-\theta} u(t) \geq \gamma\|u\|\}.$$

Then Ω is a convex subset of X.
For $u \in \Omega$, by Lemmas 2.1.2 and 2.1.5, we know that $Tu(t) \geq 0$ and $\min_{\theta\leq t\leq 1-\theta}(Tu)(t) \geq \gamma\|Tu\|$.
Moreover, for $u \in \Omega$, we have

$$Tu(t) = \lambda \int_0^1 G(t, s)a(s)f(s, u(s-\tau))ds \leq \lambda k_1 \int_0^1 g(s, s)a(s)\ f(s, u(s-\tau))ds$$

$$= \lambda k_1 \left(\int_{J_1=\{s\in[0,1]/u>N\}} g(s, s)a(s)f(s, u(s-\tau))ds + \int_{J_2=\{s\in[0,1]/u\leq N\}} g(s, s)a(s)f(s, u(s-\tau))ds \right)$$

$$\leq \lambda k_1 \left(\int_0^1 g(s,s)a(s)(f^\infty + \epsilon)u(s-\tau)ds + \int_0^1 g(s,s)a(s) \max_{\substack{0\leq s\leq 1\\ 0\leq u\leq N}} f(s,u(s-\tau))ds \right)$$

$$\leq \lambda k_1 \left((f^\infty + \epsilon) \left[\beta \int_0^\tau g(s,s)a(s)ds + \int_\tau^1 g(s,s)a(s)ds \right] \|u\| + \int_0^1 g(s,s)a(s) \max_{\substack{0\leq s\leq 1\\ 0\leq u\leq N}} f(s,u(s-\tau))ds \right)$$

$$\leq \lambda k_1(f^\infty+\epsilon)M_1 B + \lambda k_1 M_2 \max_{\substack{0\leq s\leq 1\\ 0\leq u\leq N}} f(s,u(s-\tau)) \leq \frac{B}{2} + \frac{B}{2} = B.$$

Thus, $\|Tu\| \leq B$. Hence, $T\Omega \subset \Omega$.

We can show that $T : \Omega \to \Omega$ is completely continuous.
For $u \in \Omega$ and $u = \mu Tu$, $0 < \mu < 1$, we have $u(t) = \mu Tu(t) < Tu(t) < B$, which implies $\|u\| \leq B$. So, $\{u \in \Omega/u = \mu Tu,\ 0 < \mu < 1\}$ is bounded. By theorem 2.1.1, we show that the operator T has at least one fixed point in Ω. Thus the boundary value problem (2.0.1) has at least one positive solution.

Remarque 2.2.2. *The conditions of Theorem* 2.2.2 *are weaker than those of Theorem 1.2.4.*

Chapitre 3

Positive solutions for a system of second order multi-point boundary value problem with delay

In this chapter, we study the existence of positive solutions, by using the Krasnosel'skii fixed-point, to the following second-order delay differential system,

$$\begin{cases} x''(t) + \lambda a(t) f(x(t-\tau_1), y(t-\tau_2)) = 0, & 0 < t < 1, \\ y''(t) + \mu b(t) h(x(t-\tau_1), y(t-\tau_2)) = 0, & 0 < t < 1; \\ x(t) = \alpha x(\eta), & -\tau_1 \leqslant t \leqslant 0, \\ y(t) = \alpha y(\eta), & -\tau_2 \leqslant t \leqslant 0, \\ x(1) = \beta x(\eta), & \\ y(1) = \beta y(\eta). & \end{cases} \tag{3.0.1}$$

where $0 < \eta < 1$, $0 < \alpha < \frac{1}{2}$ and $0 < \beta < \frac{1-\alpha\eta}{1-\eta}$ are constants, and λ, μ are positive real parameters.
This system is a generalisation of the following boundary value problem

$$\begin{cases} u''(t) + \lambda a(t) f(u(t-\tau)) = 0 & 0 < t < 1, \qquad \tau > 0, \\ u(t) = \alpha u(\eta) & -\tau \leqslant t \leqslant 0, \\ u(1) = \beta u(\eta) \end{cases} \tag{3.0.2}$$

where $0 < \eta < 1$, $0 < \alpha < \frac{1}{2}$ and $0 < \beta < \frac{1-\alpha\eta}{1-\eta}$ are constants, and λ positive real parameter.

3.1 Preliminaries

In this section we give the definition of positive solution of (3.0.1), and we will state some important preliminary lemmas.

Definition 3.1.1.

$(x(t), y(t))$ is called a positive solution of (3.0.1) if it satisfies the following :

1. *$(x, y) \in (C[-\tau_1, 1] \cap C^2(0,1)) \times (C[-\tau_2, 1] \cap C^2(0,1))$;*
2. *$x(t) > 0$, $y(t) > 0$ for all $t \in (0,1)$ and satisfy (3.0.1).*

Lemma 3.1.1.

Let $\beta \neq \frac{1-\alpha\eta}{1-\eta}$. Then for $y \in C([0,1],\ \mathbb{R})$, the boundary value problem

$$u''(t) + y(t) = 0, \quad t \in [0,1], \tag{3.1.1}$$

$$u(0) = \beta u(\eta), \quad u(1) = \alpha u(\eta) \tag{3.1.2}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s) y(s) ds \tag{3.1.3}$$

where

$$G(t,s) = g(t,s) + \frac{\beta + (\alpha - \beta)t}{(1-\alpha\eta) - \beta(1-\eta)} g(\eta, s) \tag{3.1.4}$$

and

$$g(t,s) = \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1, \\ t(1-s), & 0 \leq t \leq s \leq 1. \end{cases}$$

Proof.
From (3.1.1), we have

$$u(t) = u(0) + u'(0)t - \int_0^t (t-s)y(s)ds$$
$$:= A + Bt - \int_0^t (t-s)y(s)ds$$

with

$$u(0) = A,$$
$$u(\eta) = A + B\eta - \int_0^\eta (\eta - s)y(s)ds.$$

and

$$u(1) = A + B - \int_0^1 (1-s)y(s)ds.$$

From $u(0) = \beta u(\eta)$, we have

$$(1-\beta)A - B\beta\eta = -\beta \int_0^\eta (\eta - s)y(s)ds.$$

From $u(1) = \alpha u(\eta)$, we have

$$(1-\alpha)A + B(1-\alpha\eta) = \int_0^1 (1-s)y(s)ds - \alpha \int_0^\eta (\eta - s)y(s)ds.$$

Therefore,

$$A = \frac{\beta\eta}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^1 (1-s)y(s)ds - \frac{\beta}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^\eta (\eta - s)y(s)ds$$

and

$$B = \frac{1-\beta}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^1 (1-s)y(s)ds - \frac{\alpha-\beta}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^\eta (\eta - s)y(s)ds.$$

From which it follows that

$$u(t) = \frac{\beta\eta}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^1 (1-s)y(s)ds - \frac{\beta}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^\eta (\eta - s)y(s)ds$$

$$+\frac{(1-\beta)t}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^1(1-s)y(s)ds-\frac{(\alpha-\beta)t}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^\eta(\eta-s)y(s)ds-\int_0^t(t-s)y(s)ds$$
$$=-\int_0^t(t-s)y(s)ds+\frac{(\beta-\alpha)t-\beta}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^\eta(\eta-s)y(s)ds+\frac{(1-\beta)t+\beta\eta}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^1(1-s)y(s)ds$$
$$=\int_0^1 g(t,s)y(s)ds+\frac{\beta+(\alpha-\beta)t}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^1 g(\eta,s)y(s)ds$$
$$=\int_0^1 G(t,s)y(s)ds. \qquad (3.1.5)$$

The function u presented above is the unique solution to the problem (3.1.1), (3.1.2).

Lemma 3.1.2. *Let $0\le\alpha<\frac{1}{\eta}$ and $0\le\beta<\frac{1-\alpha\eta}{1-\eta}$. If $y\in C([0,1],\ [0,\infty))$, then the unique solution u of the problem (3.1.1), (3.1.2) satisfies*

$$u(t)\ge 0,\quad t\in[0,1].$$

Proof. We know that if $u''(t)=-y(t)\le 0$, for $t\in(0,1)$, $u(0)\ge 0$ and $u(1)\ge 0$, then $u(t)\ge 0$, for $t\in[0,1]$. We have

$$u(0)=\frac{-\beta}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^\eta(\eta-s)y(s)ds+\frac{\beta\eta}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^1(1-s)y(s)ds$$
$$=\frac{\beta}{(1-\alpha\eta)-\beta(1-\eta)}[-\int_0^\eta(\eta-s)y(s)ds+\eta\int_0^\eta(1-s)y(s)ds]+\frac{\beta\eta}{(1-\alpha\eta)-\beta(1-\eta)}\int_\eta^1(1-s)y(s)ds$$
$$=\frac{\beta}{(1-\alpha\eta)-\beta(1-\eta)}[\int_0^\eta s(1-\eta)y(s)ds]+\frac{\beta\eta}{(1-\alpha\eta)-\beta(1-\eta)}\int_\eta^1(1-s)y(s)ds\ge 0$$

and

$$u(1)=-\int_0^1(1-s)y(s)ds+\frac{(\beta-\alpha)-\beta}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^\eta(\eta-s)y(s)ds+\frac{(1-\beta)+\beta\eta}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^1(1-s)y(s)ds$$
$$=\frac{\alpha}{(1-\alpha\eta)-\beta(1-\eta)}[\eta\int_0^1(1-s)y(s)ds+\int_0^\eta(\eta-s)y(s)ds]$$
$$\ge\frac{\alpha}{(1-\alpha\eta)-\beta(1-\eta)}[\eta\int_0^\eta(1-s)y(s)ds+\int_0^\eta(\eta-s)y(s)ds]$$
$$=\frac{\alpha}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^\eta s(1-\eta)y(s)ds\ge 0.$$

Then, $u(t)\ge 0\quad\forall t\in[0,1]$.

Lemma 3.1.3. *The function g has the following properties*

(i) $0 \leqslant g(t,s) \leqslant s(1-s) = g(s,s) \quad \forall t, s \in [0,1]$,

(ii) *Let $\theta \in [0, \frac{1}{2}]$. Then, for $t \in [\theta, 1-\theta]$ and $s \in [0,1]$, we have*

$$g(t,s) \geq \min\{t, 1-t\} g(s,s) \geq \theta g(s,s).$$

Proof. For $0 \leq s \leq t \leq 1$, we have

$$0 \leq g(t,s) = s(1-t) \leq s(1-s) = g(s,s).$$

And for $0 \leq t \leq s \leq 1$, we have

$$g(t,s) = t(1-s) \leq s(1-s) = g(s,s).$$

Thus (i) holds.
If $s = 0$ or $s = 1$, we show that (ii) holds.
For $0 < s \leq t \leq 1$ and $s \neq 1$, we have

$$\frac{g(t,s)}{g(s,s)} = \frac{t(1-s)}{s(1-s)} = \frac{t}{s} \geq t \quad \forall t \in [0,1].$$

For $0 \leq t \leq s < 1$ and $s \neq 0$ we have

$$\frac{g(t,s)}{g(s,s)} = \frac{s(1-t)}{s(1-s)} = \frac{(1-t)}{(1-s)} \geq (1-t) \quad \forall t \in [0,1].$$

Then

$$g(t,s) \geq \min\{t, 1-t\} g(s,s).$$

Thus, there exist $\theta \in]0, \frac{1}{2}[$ such that

$$\frac{g(t,s)}{g(s,s)} \geq \theta, \quad \forall t \in [\theta, 1-\theta]$$

Thus (ii) holds.

Lemma 3.1.4. *The function G has the following properties*

(i) $G(t,s) \geqslant 0 \quad \forall t,s \in [0,1]$,

(ii) $G(t,s) \leqslant k_1 g(s,s) \quad \forall t,s \in [0,1]$ *and* $k_1 = 1 + \dfrac{\max\{\alpha,\beta\}}{(1-\alpha\eta)-\beta(1-\eta)}$,

(iii) $\min\limits_{\theta \leqslant t \leqslant 1-\theta} G(t,s) \geqslant k_2 g(s,s) \quad \forall t,s \in [0,1]$ *where* $\theta \in (0, \frac{1}{2})$ *and*

$$k_2 = \theta \left[1 + \frac{\beta + \min\{(\alpha-\beta)\theta, (\alpha-\beta)(1-\theta)\}}{(1-\alpha\eta)-\beta(1-\eta)} \right]$$

Proof.

(i) From (3.1.4) and (i) of Lemma 3.1.3, we get

$$G(t,s) \geq 0, \ \forall t,s \in [0,1]$$

(ii) By (3.1.4) and (i) of Lemma 3.1.3, we have

$$G(t,s) = g(t,s) + \frac{\beta + (\alpha-\beta)t}{(1-\alpha\eta)-\beta(1-\eta)} g(\eta,s)$$

$$\leq g(s,s) + \frac{\max(\alpha,\beta)}{(1-\alpha\eta)-\beta(1-\eta)} g(s,s) = k_1 g(s,s).$$

(ii) From (ii) of Lemma 3.1.3, for $t \in [\theta, 1-\theta]$ we have

$$\begin{aligned} G(t,s) &= g(t,s) + \frac{\beta + (\alpha-\beta)t}{(1-\alpha\eta)-\beta(1-\eta)} g(\eta,s) \\ &\geqslant \theta g(s,s) + \frac{\beta + \min\{(\alpha-\beta)\theta, (\alpha-\beta)(1-\theta)\}}{(1-\alpha\eta)-\beta(1-\eta)} \theta g(s,s) \\ &\geqslant \theta \left[1 + \frac{\beta + \min\{(\alpha-\beta)\theta, (\alpha-\beta)(1-\theta)\}}{(1-\alpha\eta)-\beta(1-\eta)} \right] g(s,s) = k_2 g(s,s). \end{aligned}$$

Lemma 3.1.5. *If $y \in C([0,1])$ and $y \geqslant 0$, then the unique solution u of the boundary value problem (3.1.1),(3.1.2) satisfies* $\min\limits_{\theta \leqslant t \leqslant 1-\theta} u(t) \geqslant \gamma \|u\|_0$ *where* $\|u\|_0 := \sup\{|u(t)|; \ 0 \leq t \leq 1\}$ *and* $\gamma := \dfrac{k_2}{k_1}$.

Proof. For any $t \in [0,1]$, by Lemma 3.1.4 we have

$$u(t) = \int_0^1 G(t,s)y(s)ds \leqslant k_1 \int_0^1 g(s,s)y(s)ds,$$

thus $||u||_0 \leqslant k_1 \int_0^1 g(s,s)y(s)ds$. Moreover, from (iii) of Lemma 3.1.4 for $t \in [\theta, 1-\theta]$, we have

$$u(t) = \int_0^1 G(t,s)y(s)ds \ \geqslant k_2 \int_0^1 g(s,s)y(s)ds \ \geqslant \frac{k_2}{k_1}||u||_0.$$

Therefore $\min\limits_{\theta \leqslant t \leqslant 1-\theta} u(t) \geqslant \gamma \|u\|_0$.

Lemma 3.1.6. *If u is a positive solution of (3.1.1),(3.1.2), then* $u(\eta) \geq \min\{\eta, 1-\eta\}\|u\|_0$.

Proof. By lemma (3.1.3) we have

$$\min\{\eta, 1-\eta\}g(s,s) \leq g(\eta,s) \leq g(s,s), \ \ \eta, s \in [0,1].$$

then, we have

$$\|u\|_0 \leq \lambda \int_0^1 g(s,s)y(s)ds + \frac{\lambda\alpha}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^1 g(\eta,s)y(s)ds$$

Multiplying both sides of the last inequality by $\min\{\eta, 1-\eta\}$, we get

$$\begin{aligned}\min\{\eta, 1-\eta\}\|u\|_0 &\leq \lambda \int_0^1 \min\{\eta, 1-\eta\}g(s,s)y(s)ds \\ &\quad + \frac{\lambda\alpha\min\{\eta, 1-\eta\}}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^1 g(\eta,s)y(s)ds \\ &\leq \lambda \int_0^1 g(\eta,s)y(s)ds \\ &\quad + \frac{\lambda(\beta + (\alpha-\beta)\eta)}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^1 g(\eta,s)y(s)ds = u(\eta).\end{aligned}$$

The proof of existence of positive solutions is based on application of the following theorem.

Theorem 3.1.1. *[19]*
Let X be a Banach space and $K \subset X$ be a cone. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K$ be a completely continuous operator such that either

1. $\| Tu \| \leqslant \| u \|, u \in K \cap \partial\Omega_1$, *and* $\| Tu \| \geqslant \| u \|, u \in K \cap \partial\Omega_2$ *; or*
2. $\| Tu \| \geqslant \| u \|, u \in K \cap \partial\Omega_1$, *and* $\| Tu \| \leqslant \| u \|, u \in K \cap \partial\Omega_2$.

Then T has a fixet point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Let $\tau_M = \max\{\tau_1, \tau_2\}$, $\tau_m = \min\{\tau_1, \tau_2\}$, and assume the following conditions are satisfied,

H1) $0 < \tau_M < \dfrac{1}{2}$;

H2) $a, b : (0,1) \longrightarrow [0,\infty)$ and $f, h : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ are continuous;

H3) $\displaystyle\int_0^1 g(s,s)(a(s)+b(s))ds < \infty$, $\theta \in [\tau_m, \dfrac{1}{2})$ such that $\displaystyle\int_{\theta+\tau_M}^{1-\theta+\tau_m} a(s)ds > 0$ and $\displaystyle\int_{\theta+\tau_M}^{1-\theta+\tau_m} b(s)ds > 0$.

Let $X_i = \{u \in C[-\tau_i, 1] : u(t) \geqslant 0, \forall t \in [-\tau_i, 1]; u(t) = \alpha u(\eta), \forall t \in [-\tau_i, 0];\ u(1) = \beta u(\eta)\}$, $(i = 1,2)$ with norme given by $\| u \|_i = \sup\{| u(t) | : -\tau_i \leqslant t \leqslant 1\}$ $(i = 1,2)$.
Then $(X_i, \| \cdot \|_i)$ $(i = 1,2)$ is a Banach space.
Define $E = X_1 \times X_2$ normed by $\| (.,.) \|$ where $\| (u,v) \| = \| u \|_1 + \| v \|_2$.
Then E is a Banach space.
Suppose that $(x(t), y(t))$ is a solution of (3.0.1), then it can be expressed as follows

$$(x(t), y(t)) = (A(x,y)(t), B(x,y)(t)), \tag{3.1.6}$$

where

$$A(x,y)(t) = \begin{cases} \alpha x(\eta), & -\tau_1 \le t \le 0, \\ \lambda \displaystyle\int_0^1 G(t,s)a(s)f(x(s-\tau_1), y(s-\tau_2))ds & 0 < t \le 1. \end{cases} \tag{3.1.7}$$

and

$$B(x,y)(t) = \begin{cases} \alpha y(\eta), & -\tau_2 \le t \le 0, \\ \mu \displaystyle\int_0^1 G(t,s)b(s)h(x(s-\tau_1), y(s-\tau_2))ds & 0 < t \le 1. \end{cases} \tag{3.1.8}$$

Let

$$K_i = \{u \in X_i : u(t) \geqslant 0, \forall t \in [0,1], \min_{\theta \leqslant t \leqslant 1-\theta} u(t) \geqslant \gamma \parallel u \parallel_i\} \ (i = 1, 2)$$

and $K = K_1 \times K_2$. Then K is a cone in E.
Define the operator T by

$$T(x,y)(t) = (A(x,y)(t), B(x,y)(t)) \tag{3.1.9}$$

for each $(x,y) \in K$. Obviously the solution of the boundary value problem (3.0.1) is the fixed point of operator T.

Lemma 3.1.7. *The fixed point of T is a solution of* (3.0.1) *and $T : K \longrightarrow K$ is completely continuous.*

Proof. One can find that $T : K \to K$. In fact, for each $(x,y) \in K$, we have $A(x,y)(t) \geq 0$ and $B(x,y)(t) \geq 0$. It is obvious that $||A(x,y)||_0 = ||A(x,y)||_1$, and

$$A(x,y)(t) \le k_1 \lambda \int_0^1 G(s,s)a(s)f(x(s-\tau_1), y(s-\tau_2))ds. \tag{3.1.10}$$

by (iii) of lemma (3.1.4).
Thus, we have

$$A(x,y)(t) \ge k_2 \lambda \int_0^1 G(s,s)a(s)f(x(s-\tau_1), y(s-\tau_2))ds \ge \gamma ||A(x,y)||_1, \text{for } t \in [0,1]. \tag{3.1.11}$$

Similarly, for each $(x, y) \in K$, we have $B(x, y)(t) \geq \gamma ||B(x, y)||_2$. That is $T(K) \subset K$. We shall show that T is completely continuous.

1. Let $\Omega \subset K$ be bounded. Thus there exist a positive constant M such that for all $(x, y) \in \Omega$, we have $||(x, y)|| \leq M$. Equivalently we have $||x||, ||y|| \leq M$. Put $L = \sup\{f(x, y) : \ \ ||x||_1, ||y||_2 \leq M\}$.
   So for $(x, y) \in \Omega$, we have
   $$||A(x,y)|| \leq \max\Big(\lambda \int_0^1 G(t,s)a(s)f(x(s-\tau_1), y(s-\tau_2))ds, \alpha x(\eta)\Big)$$
   $$\leq \max\Big(L\lambda \int_0^1 a(t)G(s,s)ds, \alpha M\Big).$$
   Hence $A(x, y)$ is uniformly bounded on Ω. In the similar way, we can show that $B(x, y)$ is uniformly bounded. Hence T is uniformly bounded.

2. Consider the sequence $\{(x_n, y_n)\} \subset \Omega$ where $\lim\limits_{n\to+\infty}(x_n, y_n) = (x, y) \in \Omega$. Since $\int_0^1 a(s)f(x(s), y(s))ds < \infty$, by Lebesgue Dominated Convergence theorem, we deduce that
   $$\int_0^1 a(s)f(x_n(s), y_n(s))ds - \int_0^1 a(s)f(x(s), y(s))ds \to 0, \text{ when } n \to +\infty$$
   Hence,
   $$||A(x_n, y_n) - A(x, y)||_1 \to 0, \text{ when } n \to +\infty.$$
   Thus, $A(x, y)$ is continuous on Ω. In a similar way, we can show that $B(x, y)$ is continuous. Hence, T is continuous.

3. Since G is continuous on $[0, 1] \times [0, 1]$, it is uniformly continuous.
   Then $\forall \varepsilon > 0$ there exists $\delta > 0$ such that $|t_1 - t_2| < \delta$ implies that $|G(t_2, s) - G(t_1, s)| < \varepsilon \ \forall s \in [0, 1]$. So, if $(x, y) \in \Omega$, we have
   $$|A(x(t_2), y(t_2)) - A(x(t_1), y(t_1))| \leq \lambda \int_0^1 |G(t_1,s) - G(t_2,s)|a(s)f(x(s-\tau_1), y(s-\tau_2))ds$$

$$\leq \lambda L\varepsilon \int_0^1 a(s)ds.$$

From the arbitrariness of ε, we get the equicontinuity of $A\Omega$. Similarly, we can show that B is equicontinuous. Thus, T is equicontinous.

The operator T is completely continuous by the mean of the Ascoli-Arzela theorem.

3.2 Main Results

For each $j \in C([0,\infty)\times[0,\infty),[0,\infty))$, we define

$$j_\rho = \underline{\lim}_{x+y\longrightarrow\rho} \frac{j(x,y)}{x+y}, \quad j^\rho = \overline{\lim}_{x+y\longrightarrow\rho} \frac{j(x,y)}{x+y} \quad (\rho = 0^+ \ or\ \infty).$$

Let

$$C_1 := \frac{1}{f_\infty \sup_{0\leqslant t\leqslant 1}(\alpha \min\{\eta, 1-\eta\} \int_0^{\tau_1} G(t,s)a(s)ds + \gamma \int_{\theta+\tau_1}^{1-\theta+\tau_1} G(t,s)a(s)ds)},$$

$$C_2 := \frac{\alpha}{k_1 f^0(\alpha \int_0^{\tau_M} g(s,s)a(s)ds + \int_{\tau_m}^1 g(s,s)a(s)ds)},$$

$$C_3 := \frac{1}{h_\infty \sup_{0\leqslant t\leqslant 1}(\alpha \min\{\eta, 1-\eta\} \int_0^{\tau_2} G(t,s)b(s)ds + \gamma \int_{\theta+\tau_2}^{1-\theta+\tau_2} G(t,s)b(s)ds)},$$

$$C_4 := \frac{1-\alpha}{k_1 h^0(\alpha \int_0^{\tau_M} g(s,s)b(s)ds + \int_{\tau_m}^1 g(s,s)b(s)ds)},$$

$$C_5 := \frac{1}{f_0 \sup_{0\leqslant t\leqslant 1}(\alpha \min\{\eta, 1-\eta\} \int_0^{\tau_1} G(t,s)a(s)ds + \gamma \int_{\theta+\tau_1}^{1-\theta+\tau_1} G(t,s)a(s)ds)},$$

$$C_6 := \frac{\alpha}{k_1 f^\infty \int_0^1 g(s,s)a(s)ds},$$

$$C_7 := \frac{1}{h_0 \sup_{0\leqslant t\leqslant 1}(\alpha \min\{\eta, 1-\eta\} \int_0^{\tau_2} G(t,s)b(s)ds + \gamma \int_{\theta+\tau_2}^{1-\theta+\tau_2} G(t,s)b(s)ds)},$$

$$C_8 := \frac{1-\alpha}{k_1 h^\infty \displaystyle\int_0^1 g(s,s)b(s)ds}.$$

Theorem 3.2.1. *Let* (H_1)*-*(H_3) *hold. Assume that* $f^0 < \infty$*,* $h^0 < \infty$ *and*

$$f_\infty > \frac{k_1 f^0(\alpha \int_0^{\tau_M} g(s,s)a(s)ds + \int_{\tau_m}^1 g(s,s)a(s)ds)}{\alpha f_\infty \sup_{0\leqslant t\leqslant 1}(\alpha \min\{\eta, 1-\eta\} \int_0^{\tau_1} G(t,s)a(s)ds + \gamma \int_{\theta+\tau_1}^{1-\theta+\tau_1} G(t,s)a(s)ds)},$$

$$h_\infty > \frac{k_1 h^0(\alpha \displaystyle\int_0^{\tau_M} g(s,s)b(s)ds + \int_{\tau_m}^1 g(s,s)b(s)ds)}{(1-\alpha)h_\infty \sup_{0\leqslant t\leqslant 1}(\alpha \min\{\eta, 1-\eta\} \int_0^{\tau_2} G(t,s)b(s)ds + \gamma \int_{\theta+\tau_2}^{1-\theta+\tau_2} G(t,s)b(s)ds)}.$$

Then, there exists at least one positive solution to (3.0.1) *for*

$$(\lambda, \mu) \in (C_1, C_2) \times (C_3, C_4). \tag{3.2.1}$$

Proof. By (3.2.1), there exist an $\varepsilon > 0$ such that

$$\frac{1}{(f_\infty - \varepsilon) \sup_{0\leqslant t\leqslant 1}(\alpha \min\{\eta, 1-\eta\} \int_0^{\tau_1} G(t,s)a(s)ds + \gamma \int_{\theta+\tau_1}^{1-\theta+\tau_1} G(t,s)a(s)ds)}$$
$$\leqslant \lambda \leqslant \frac{\alpha}{k_1(f^0+\varepsilon)(\alpha \displaystyle\int_0^{\tau_M} g(s,s)a(s)ds + \int_{\tau_m}^1 g(s,s)a(s)ds)} \tag{3.2.2}$$

and

$$\frac{1}{(h_\infty - \varepsilon) \sup_{0\leqslant t\leqslant 1}(\alpha \min\{\eta, 1-\eta\} \int_0^{\tau_2} G(t,s)b(s)ds + \gamma \int_{\theta+\tau_2}^{1-\theta+\tau_2} G(t,s)b(s)ds)}$$
$$\leqslant \mu \leqslant \frac{1-\alpha}{k_1(h^0+\varepsilon)(\alpha \displaystyle\int_0^{\tau_M} g(s,s)b(s)ds + \int_{\tau_m}^1 g(s,s)b(s)ds)}. \tag{3.2.3}$$

Let $\varepsilon > 0$ be fixed. By $f^0 < \infty$, $h^0 < \infty$, there exist $r_1 > 0$ such that

$$f(x,y) \leqslant (f^0 + \varepsilon)(x+y) \tag{3.2.4}$$

and

$$h(x,y) \leqslant (h^0+\varepsilon)(x+y) \tag{3.2.5}$$

for $0 < x, y \leqslant r_1$.

Let $\Omega_1 = \{(x,y) \in E :\parallel (x,y) \parallel < r_1\}$. Then for $(x,y) \in K \cap \partial\Omega_1$, by (3.2.2) and (3.2.4) we have

$$\begin{aligned}
\parallel A(x,y) \parallel_1 &\leqslant \max\Big(\lambda k_1 \int_0^1 g(s,s)a(s)f(x(s-\tau_1),y(s-\tau_2))ds, \alpha x(\eta)\Big)\\
&\leqslant \max\Big(\lambda k_1 (f^0+\varepsilon) \int_0^1 g(s,s)a(s)(x(s-\tau_1)+y(s-\tau_2))ds, \alpha x(\eta)\Big)\\
&= \max\Big(\lambda k_1 (f^0+\varepsilon)[\int_0^1 g(s,s)a(s)x(s-\tau_1) + \int_0^1 g(s,s)a(s)y(s-\tau_2))ds], \alpha x(\eta)\Big)\\
&= \max\Big(\lambda k_1 (f^0+\varepsilon)[\alpha x(\eta) \int_0^{\tau_1} g(s,s)a(s)ds + \int_{\tau_1}^1 g(s,s)a(s)x(s-\tau_1)ds\\
&\qquad +\alpha y(\eta) \int_0^{\tau_2} g(s,s)a(s)ds + \int_{\tau_2}^1 g(s,s)a(s)y(s-\tau_2)ds], \alpha x(\eta)\Big)\\
&\leqslant \max\Big(\lambda k_1 (f^0+\varepsilon)[\alpha \int_0^{\tau_1} g(s,s)a(s)ds||x||_1 + \int_{\tau_1}^1 g(s,s)a(s)ds||x||_1\\
&\qquad +\alpha \int_0^{\tau_2} g(s,s)a(s)ds||y||_2 + \int_{\tau_2}^1 g(s,s)a(s)ds||y||_2], \alpha x(\eta)\Big)\\
&\leqslant \max\Big(\lambda k_1 (f^0+\varepsilon)[\alpha \int_0^{\tau_M} g(s,s)a(s)ds \parallel x \parallel_1 + \int_{\tau_m}^1 g(s,s)a(s)ds \parallel x \parallel_1\\
&\qquad +\alpha \int_0^{\tau_M} g(s,s)a(s)ds \parallel y \parallel_2 + \int_{\tau_m}^1 g(s,s)a(s)ds \parallel y \parallel_2], \alpha x(\eta)\Big)\\
&= \max\Big(\lambda k_1 (f^0+\varepsilon)[\alpha \int_0^{\tau_M} g(s,s)a(s)ds + \int_{\tau_m}^1 g(s,s)a(s)ds](\parallel x \parallel_1 + \parallel y \parallel_2), \alpha x(\eta)\Big)\\
&\leq \max\Big(\alpha \parallel (x,y) \parallel, \alpha x(\eta)\Big) \leq \alpha \parallel (x,y) \parallel .
\end{aligned}$$

Similarly, by (3.2.3) and (3.2.5) we have

$$\begin{aligned}
\parallel B(x,y) \parallel_2 &\leqslant \max\Big(\mu k_1 (h^0+\varepsilon) \int_0^1 g(s,s)b(s)(x(s-\tau_1)+y(s-\tau_2))ds, \alpha y(\eta)\Big)\\
&\leq \max\Big(\mu k_1 (h^0+\varepsilon)[\alpha \int_0^{\tau_M} g(s,s)b(s)ds + \int_{\tau_m}^1 g(s,s)b(s)ds](\parallel x \parallel_1 + \parallel y \parallel_2), \alpha y(\eta)\Big)\\
&\leq (1-\alpha) \parallel (x,y) \parallel .
\end{aligned}$$

Conbining the above two inequalities, we get

$$\| T(x,y) \| = \| A(x,y) \|_1 + \| B(x,y) \|_2 \leqslant \alpha \| (x,y) \| + (1-\alpha) \| (x,y) \| = \| (x,y) \|$$

for $(x,y) \in K \cap \partial\Omega_1$.

By $f_\infty > 0$, $h_\infty > 0$, there exist $\overline{R}_1 > 0$ such that

$$f(x,y) \geqslant (f_\infty - \varepsilon)(x+y) \tag{3.2.6}$$

and

$$h(x,y) \geqslant (h_\infty - \varepsilon)(x+y) \tag{3.2.7}$$

for $x + y \geqslant \overline{R}_1$.

Take $R_1 = \max\{\frac{\overline{R}_1}{\gamma}, 2r_1\}$ and set $\Omega_2 = \{(x,y) \in E : \| (x,y) \| < R_1\}$. Then for $(x,y) \in K \cap \partial\Omega_2$, by (3.2.2), (3.2.6) and lemma (3.1.6) we have

$$\begin{aligned}
\| A(x,y) \|_1 &= \max\Big(\lambda \sup_{0<t\leqslant 1} \int_0^1 G(t,s)a(s)f(x(s-\tau_1), y(s-\tau_2))ds, \alpha x(\eta)\Big) \\
&\geqslant \max\Big(\lambda \sup_{0<t\leqslant 1} \int_0^1 G(t,s)a(s)(f_\infty - \varepsilon)(x(s-\tau_1) + y(s-\tau_2))ds, \alpha x(\eta)\Big) \\
&\geqslant \max\Big(\lambda(f_\infty - \varepsilon) \sup_{0<t\leqslant 1} \int_0^1 G(t,s)a(s)x(s-\tau_1)ds, \alpha x(\eta)\Big) \\
&= \max\Big(\lambda(f_\infty - \varepsilon) \sup_{0<t\leqslant 1} [\int_0^{\tau_1} G(t,s)a(s)\alpha x(\eta)ds + \int_{\tau_1}^1 G(t,s)a(s)x(s-\tau_1)ds], \alpha x(\eta)\Big) \\
&\geqslant \max\Big(\lambda(f_\infty - \varepsilon) \sup_{0<t\leqslant 1} [\int_0^{\tau_1} G(t,s)a(s)\alpha \min\{\eta, 1-\eta\} \| x \|_1 ds \\
&+ \int_0^{1-\tau_1} G(t,s+\tau_1)a(s+\tau_1)x(s)ds], \alpha x(\eta)\Big) \\
&\geqslant \max\Big(\lambda(f_\infty - \varepsilon) \sup_{0<t\leqslant 1} [\alpha \min\{\eta, 1-\eta\} \int_0^{\tau_1} G(t,s)a(s)ds \| x \|_1 \\
&+ \int_\theta^{1-\theta} G(t,s+\tau_1)a(s+\tau_1)ds\gamma \| x \|_1], \alpha x(\eta)\Big)
\end{aligned}$$

$$= \max\Big(\lambda(f_\infty-\varepsilon)\sup_{0<t\leqslant 1}[\alpha\min\{\eta,1-\eta\}\int_0^{\tau_1}G(t,s)a(s)ds$$
$$+\gamma\int_\theta^{1-\theta}G(t,s+\tau_1)a(s+\tau_1)ds]\parallel x\parallel_1,\alpha x(\eta)\Big)$$
$$= \max\Big(\lambda(f_\infty-\varepsilon)\sup_{0<t\leqslant 1}[\alpha\min\{\eta,1-\eta\}\int_0^{\tau_1}G(t,s)a(s)ds$$
$$+\gamma\int_{\theta+\tau_1}^{1-\theta+\tau_1}G(t,s)a(s)ds]\parallel x\parallel_1,\alpha x(\eta)\Big)$$
$$\geqslant\parallel x\parallel_1 .$$

Similarly, by (3.2.3) and (3.2.7) we have

$$\parallel B(x,y)\parallel_2 \;\geqslant \max\Big(\mu(h_\infty-\varepsilon)\sup_{0<t\leqslant 1}\int_0^1 G(t,s)b(s)y(s-\tau_2)ds,\alpha y(\eta)\Big)$$
$$\geq \max\Big(\mu(h_\infty-\varepsilon)\sup_{0<t\leqslant 1}[\alpha\min\{\eta,1-\eta\}\int_0^{\tau_2}G(t,s)b(s)ds$$
$$+\gamma\int_{\theta+\tau_2}^{1-\theta+\tau_2}G(t,s)b(s)ds]\parallel y\parallel_2 ds,\alpha y(\eta)\Big)$$
$$\geqslant\parallel y\parallel_2 .$$

Thus, we get $\parallel T(x,y)\parallel=\parallel A(x,y)\parallel_1+\parallel B(x,y)\parallel_2\geqslant\parallel x\parallel_1+\parallel y\parallel_2=\parallel (x,y)\parallel$ for $(x,y)\in K\cap\partial\Omega_2$.

Therefore, by the first part of Theorem 3.1.1, T has a fixed point $(x,y)\in K\cap(\overline{\Omega}_2\setminus\Omega_1)$, and $(x(t),y(t))$ is a positive solution of (3.0.1). The proof is complete. □

Theorem 3.2.2. *Let (H_1)-(H_3) hold. Assume that $f^\infty<\infty$, $h^\infty<\infty$ and*

$$f_0 > \frac{k_1 f^\infty\int_0^1 g(s,s)a(s)ds}{\alpha\sup\limits_{0\leqslant t\leqslant 1}[\alpha\min\{\eta,1-\eta\}\int_0^{\tau_1}G(t,s)a(s)ds+\gamma\int_{\theta+\tau_1}^{1-\theta+\tau_1}G(t,s)a(s)ds]},$$

$$h_0 > \frac{k_1 h^\infty\int_0^1 g(s,s)b(s)ds}{(1-\alpha)\sup\limits_{0\leqslant t\leqslant 1}[\alpha\min\{\eta,1-\eta\}\int_0^{\tau_2}G(t,s)b(s)ds+\gamma\int_{\theta+\tau_2}^{1-\theta+\tau_2}G(t,s)b(s)ds]}.$$

Then there exist at least one positive solution of (3.0.1) *for*

$$(\lambda,\mu)\in(C_5,C_6)\times(C_7,C_8). \tag{3.2.8}$$

Proof. Suppose that (λ, μ) satisfies (3.2.8). Then, there exist $\varepsilon > 0$ such that

$$\frac{1}{(f_0-\varepsilon)\sup\limits_{0\leqslant t\leqslant 1}[\alpha\min\{\eta,1-\eta\}\int_0^{\tau_1}G(t,s)a(s)ds+\gamma\int_{\theta+\tau_1}^{1-\theta+\tau_1}G(t,s)a(s)ds]} \leqslant \lambda \leqslant \frac{\alpha}{k_1(f^\infty+\varepsilon)\int_0^1 g(s,s)a(s)ds}. \tag{3.2.9}$$

and

$$\frac{1}{(h_0-\varepsilon)\sup\limits_{0\leqslant t\leqslant 1}[\alpha\min\{\eta,1-\eta\}\int_0^{\tau_2}G(t,s)b(s)ds+\gamma\int_{\theta+\tau_2}^{1-\theta+\tau_2}G(t,s)b(s)ds]} \leqslant \mu \leqslant \frac{1-\alpha}{k_1(h^\infty+\varepsilon)\int_0^1 g(s,s)b(s)ds}. \tag{3.2.10}$$

By $f_0 > 0$ and $h_0 > 0$, there exist $H_1 > 0$ such that for $0 < x, y \leqslant H_1$, we have

$$f(x,y) \geqslant (f_0-\varepsilon)(x+y) \tag{3.2.11}$$

and

$$h(x,y) \geqslant (h_0-\varepsilon)(x+y). \tag{3.2.12}$$

Let $\Omega_1 = \{(x,y) \in E : \| (x,y) \| < H_1\}$. Then for $(x,y) \in K \cap \partial\Omega_1$, by (3.2.9) and (3.2.11) we have,

$$\begin{aligned}
\| A(x,y) \|_1 &= \max\Big(\lambda \sup_{0<t\leqslant 1}\int_0^1 G(t,s)a(s)f(x(s-\tau_1),y(s-\tau_2))ds, \alpha x(\eta)\Big)\\
&\geqslant \max\Big(\lambda \sup_{0<t\leqslant 1}\int_0^1 G(t,s)a(s)(f_0-\varepsilon)(x(s-\tau_1)+y(s-\tau_2))ds, \alpha x(\eta)\Big)\\
&\geqslant \max\Big(\lambda(f_0-\varepsilon) \sup_{0<t\leqslant 1}\int_0^1 G(t,s)a(s)x(s-\tau_1)ds, \alpha x(\eta)\Big)\\
&= \max\Big(\lambda(f_0-\varepsilon) \sup_{0<t\leqslant 1}[\int_0^{\tau_1} G(t,s)a(s)\alpha x(\eta)ds + \int_{\tau_1}^1 G(t,s)a(s)x(s-\tau_1)ds], \alpha x(\eta)\Big)\\
&\geqslant \max\Big(\lambda(f_0-\varepsilon) \sup_{0<t\leqslant 1}[\int_0^{\tau_1} G(t,s)a(s)\alpha\min\{\eta,1-\eta\} \| x \|_1 ds\\
&\quad + \int_0^{1-\tau_1} G(t,s+\tau_1)a(s+\tau_1) \| x \|_1 ds], \alpha x(\eta)\Big)
\end{aligned}$$

$$\geqslant \max\Big(\lambda(f_0-\varepsilon)\sup_{0<t\leqslant 1}[\alpha\min\{\eta,1-\eta\}\int_0^{\tau_1}G(t,s)a(s)ds \parallel x \parallel_1$$
$$+\int_\theta^{1-\theta}G(t,s+\tau_1)a(s+\tau_1)ds\gamma \parallel x \parallel_1 ds],\alpha x(\eta)\Big)$$
$$=\max\Big(\lambda(f_0-\varepsilon)\sup_{0<t\leqslant 1}[\alpha\min\{\eta,1-\eta\}\int_0^{\tau_1}G(t,s)a(s)ds$$
$$+\gamma\int_\theta^{1-\theta}G(t,s+\tau_1)a(s+\tau_1)ds] \parallel x \parallel_1,\alpha x(\eta)\Big)$$
$$=\max\Big(\lambda(f_0-\varepsilon)\sup_{0<t\leqslant 1}[\alpha\min\{\eta,1-\eta\}\int_0^{\tau_1}G(t,s)a(s)ds$$
$$+\gamma\int_{\theta+\tau_1}^{1-\theta+\tau_1}G(t,s)a(s)ds] \parallel x \parallel_1,\alpha x(\eta)\Big)$$
$$\geqslant \parallel x \parallel_1 .$$

Similarly, by (3.2.10) and (3.2.12) we have

$$\begin{aligned}\parallel B(x,y)\parallel_2 &\geqslant \max\Big(\mu(h_0-\varepsilon)\sup_{0<t\leqslant 1}\int_0^1 G(t,s)b(s)y(s-\tau_2)ds,\alpha y(\eta)\Big)\\ &=\max\Big(\mu(h_0-\varepsilon)\sup_{0<t\leqslant 1}[\alpha\min\{\eta,1-\eta\}\int_0^{\tau_2}G(t,s)b(s)ds\\ &\quad+\gamma\int_{\theta+\tau_2}^{1-\theta+\tau_2}G(t,s)b(s)ds] \parallel y \parallel_2 ds,\alpha y(\eta)\Big)\\ &\geqslant \parallel y \parallel_2 .\end{aligned}$$

Thus, we get $\parallel T(x,y)\parallel = \parallel A(x,y)\parallel_1 + \parallel B(x,y)\parallel_2 \geqslant \parallel x\parallel_1 + \parallel y\parallel_2 = \parallel (x,y)\parallel$ for $(x,y)\in K\cap\partial\Omega_1$.

Now, let $\widetilde{f}(w)=\max\limits_{0<x+y\leqslant w}f(x,y)$, $\widetilde{h}(w)=\max\limits_{0<x+y\leqslant w}h(x,y)$. Then $\widetilde{f}^\infty\leqslant f^\infty$, $\widetilde{h}^\infty\leqslant h^\infty$, and there exist $H_2>2H_1$, such that for $w\geqslant H_2$, we have

$$\widetilde{f}(w)\leqslant(\widetilde{f}^\infty+\varepsilon)w\leqslant(f^\infty+\varepsilon)w, \tag{3.2.13}$$

and

$$\widetilde{h}(w)\leqslant(\widetilde{h}^\infty+\varepsilon)w\leqslant(h^\infty+\varepsilon)w. \tag{3.2.14}$$

Take $\Omega_2=\{(x,y)\in E:\parallel (x,y)\parallel < H_2\}$. Then for $(x,y)\in K\cap\partial\Omega_2$, by (3.2.9) and (3.2.13) we have

$$
\begin{aligned}
\| A(x,y) \|_1 &\leqslant \max\left(k_1\lambda\int_0^1 g(s,s)a(s)f(x(s-\tau_1),y(s-\tau_2))ds, \alpha x(\eta)\right)\\
&\leqslant \max\left(k_1\lambda\int_0^1 g(s,s)a(s)\widetilde{f}(H_2)ds, \alpha x(\eta)\right)\\
&\leqslant \max\left(k_1\lambda(f^\infty+\varepsilon)\int_0^1 g(s,s)a(s)dsH_2, \alpha x(\eta)\right)\\
&\leqslant \alpha H_2.
\end{aligned}
$$

Similarly, by (3.2.10) and (3.2.14) we have

$$\| B(x,y) \|_2 \leqslant \max\left(k_1\mu(h^\infty+\varepsilon)\int_0^1 g(s,s)b(s)dsH_2, \alpha y(\eta)\right) \leqslant (1-\alpha)H_2.$$

Thus,

$$\| T(x,y) \| \leqslant \| A(x,y) \|_1 + \| B(x,y) \|_2 \leqslant \alpha H_2 + (1-\alpha)H_2 = H_2 = \| (x,y) \| .$$

Therefore, by the second part of Theorem 3.1.1, T has a fixed point $(x,y) \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, and it's a positive solution of (3.0.1). The proof is complete. □

Chapitre 4

Positive solutions for a nonlinear third-order boudary value problem with delay

In this chapter, the fixed-point index theorem in cones was used to discuss the third-order multi-point boundary value problem with delay

$$\begin{aligned} &u'''(t) + \lambda a(t) f(t, u(t-\tau)) = 0, \quad t \in [0,1], \\ &u(t) = \alpha u(\eta), \ \tau \leq t \leq 0, \\ &u'(0) = 0, \quad u(1) = \beta u(\eta), \end{aligned} \tag{4.0.1}$$

where α, β, η and τ are positive constants such that $\eta \in (0,1)$, $0 < \tau \leq \dfrac{1}{2}$, and λ is a positive real parameter.

We assume the following hypotheses :

(H_3) $\alpha \in (0,1)$, $0 < \beta < \dfrac{1}{\eta^2}$,

(H_1) $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous,

(H_2) $a : [0,1] \to [0,\infty)$ is continuous and does not vanish identically on any subinterval.

4.1 Preliminaries

In this section, we give some preliminaries needed for the rest of this work.

Definition 4.1.1. *A function $u \in C([-\tau, 1])$ is called a solution of (4.0.1) if it satisfies the following properties*

1. $u(t) \geq 0 \ \forall t \in [-\tau, 1]$,
2. $u(t) = \alpha u(\eta) \ \forall t \in [-\tau, 0]$, $u(1) = \beta u(\eta)$, $u'(0) = 0$,
3. $u \in C^3([0,1])$ *and* $u'''(t) = -\lambda a(t) f(t, u(t-\tau)) \ \forall t \in [0,1]$.

Furthermore, u is a positive solution of (4.0.1) if it is a solution of (4.0.1) with $u(t) > 0 \ \forall t \in (0,1)$.

Lemma 4.1.1.
For any $y \in C([0,1])$ the problem

$$u'''(t) + y(t) = 0, \qquad t \in [0,1], \tag{4.1.1}$$

$$u(0) = \alpha u(\eta), \qquad u'(0) = 0, \qquad u(1) = \beta u(\eta), \tag{4.1.2}$$

has a unique solution $u(t) = \int_0^1 G(t,s) y(s) ds$, where

$$G(t,s) = g(t,s) + \frac{\beta t^2 + \alpha (1 - t^2)}{(1 - \beta \eta^2) - \alpha (1 - \eta^2)} g(\eta, s). \tag{4.1.3}$$

and

$$g(t,s) = \begin{cases} \dfrac{t^2(1-s)^2 - (t-s)^2}{2}, & 0 \leqslant s \leqslant t \leqslant 1, \\ \dfrac{t^2(1-s)^2}{2}, & 0 \leqslant t < s \leqslant 1. \end{cases} \tag{4.1.4}$$

Proof.

From (4.1.1), we have

$$u(t) = u(0)+u'(0)t+\frac{1}{2}u''(0)t^2-\frac{1}{2}\int_0^t (t-s)^2y(s)ds := A+Bt+Ct^2-\frac{1}{2}\int_0^t (t-s)^2y(s)ds.$$

with

$$\begin{aligned}
&u'(t) = B + 2Ct - \int_0^t (t-s)y(s)ds,\\
&u'(0) = B = 0.\\
&u(0) = A,\\
&u(\eta) = A + C\eta^2 - \frac{1}{2}\int_0^\eta (\eta - s)^2 y(s)ds,\\
&u(1) = A + C - \frac{1}{2}\int_0^1 (1-s)^2 y(s)ds,
\end{aligned}.$$

From $u(0) = \alpha u(\eta)$, we have

$$(1-\alpha)A - \alpha\eta^2 C = -\frac{1}{2}\alpha\int_0^\eta (\eta - s)^2 y(s)ds.$$

From $u(1) = \beta u(\eta)$, we have

$$(1-\beta)A + (1-\beta\eta^2)C = \frac{1}{2}\int_0^1 (1-s)^2 y(s)ds - \frac{1}{2}\beta\int_0^\eta (\eta - s)^2 y(s)ds.$$

Therefore,

$$A = \frac{\alpha}{(1-\beta\eta^2) - \alpha(1-\eta^2)}\left[\frac{\eta^2}{2}\int_0^1 (1-s)^2 y(s)ds - \frac{1}{2}\int_0^\eta (\eta - s)^2 y(s)ds\right]$$

and

$$C = \frac{\beta - \alpha}{(1-\beta\eta^2) - \alpha(1-\eta^2)}\left[\frac{\eta^2}{2}\int_0^1 (1-s)^2 y(s)ds - \frac{1}{2}\int_0^\eta (\eta - s)^2 y(s)ds\right]$$

From which it follows that

$$
\begin{aligned}
u(t) &= \frac{\alpha}{(1-\beta\eta^2)-\alpha(1-\eta^2)}\left[\frac{\eta^2}{2}\int_0^1 (1-s)^2 y(s)ds - \frac{1}{2}\int_0^\eta (\eta-s)^2 y(s)ds\right] \\
&\quad + \frac{t^2}{(1-\beta\eta^2)-\alpha(1-\eta^2)}\left[\frac{1-\alpha}{2}\int_0^1 (1-s)^2 y(s)ds - \frac{\beta-\alpha}{2}\int_0^\eta (\eta-s)^2 y(s)ds\right] \\
&\quad - \frac{1}{2}\int_0^t (t-s)^2 y(s)ds \\
&= \frac{\alpha\eta^2+(1-\alpha)t^2}{(1-\beta\eta^2)-\alpha(1-\eta^2)}\int_0^1 \frac{(1-s)^2}{2} y(s)ds - \frac{\beta t^2+\alpha(1-t^2)}{(1-\beta\eta^2)-\alpha(1-\eta^2)}\int_0^\eta \frac{(\eta-s)^2}{2} y(s)ds \\
&\quad - \frac{1}{2}\int_0^t (t-s)^2 y(s)ds \\
&= \int_0^1 \frac{t^2(1-s)^2}{2} y(s)ds + \frac{\beta t^2+\alpha(1-t^2)}{(1-\beta\eta^2)-\alpha(1-\eta^2)}\int_0^1 \frac{\eta^2(1-s)^2}{2} y(s)ds \\
&\quad - \frac{\beta t^2+\alpha(1-t^2)}{(1-\beta\eta^2)-\alpha(1-\eta^2)}\int_0^\eta \frac{(\eta-s)^2}{2} y(s)ds - \frac{1}{2}\int_0^t (t-s)^2 y(s)ds \\
&= \int_0^t \frac{t^2(1-s)^2}{2} y(s)ds + \int_t^1 \frac{t^2(1-s)^2}{2} y(s)ds \\
&\quad + \frac{\beta t^2+\alpha(1-t^2)}{(1-\beta\eta^2)-\alpha(1-\eta^2)}\int_0^\eta \frac{\eta^2(1-s)^2}{2} y(s)ds + \frac{\beta t^2+\alpha(1-t^2)}{(1-\beta\eta^2)-\alpha(1-\eta^2)}\int_\eta^1 \frac{\eta^2(1-s)^2}{2} y(s)ds \\
&\quad - \frac{\beta t^2+\alpha(1-t^2)}{(1-\beta\eta^2)-\alpha(1-\eta^2)}\int_0^\eta \frac{(\eta-s)^2}{2} y(s)ds - \frac{1}{2}\int_0^t (t-s)^2 y(s)ds \\
&= \int_0^t \frac{t^2(1-s)^2-(t-s)^2}{2} y(s)ds + \int_t^1 \frac{t^2(1-s)^2}{2} y(s)ds \\
&\quad + \frac{\beta t^2+\alpha(1-t^2)}{(1-\beta\eta^2)-\alpha(1-\eta^2)}\int_0^\eta \frac{\eta^2(1-s)^2-(\eta-s)^2}{2} y(s)ds \\
&\quad + \frac{\beta t^2+\alpha(1-t^2)}{(1-\beta\eta^2)-\alpha(1-\eta^2)}\int_\eta^1 \frac{\eta^2(1-s)^2}{2} y(s)ds \\
&= \int_0^1 g(t,s)y(s)ds + \frac{\beta t^2+\alpha(1-t^2)}{(1-\beta\eta^2)-\alpha(1-\eta^2)}\int_0^1 g(\eta,s)y(s)ds = \int_0^1 G(t,s)y(s)ds.
\end{aligned}
$$

The function u presented above is the unique solution to the problem (4.1.1) (4.1.2)

Lemma 4.1.2.

The function g has the following properties

1. $0 \le g(t,s) \le s(1-s)^2 \quad \forall t,s \in [0,1]$,
2. $g(t,s) \ge \Phi(t)s(1-s)^2 \quad \forall t,s \in [0,1]$, *where*

$$\Phi(t)=\begin{cases}\dfrac{t^2}{2}, & t\in[0,\frac{1}{2}],\\ \dfrac{t(1-t)}{2}, & t\in[\frac{1}{2},1].\end{cases}\tag{4.1.5}$$

Proof.

It is obvious that g is positive. Moreover, for $0\le s\le t\le 1$,
$g(t,s)=\dfrac{1}{2}[t^2(1-s)^2-(t-s)^2]=\dfrac{1}{2}s(1-t)[t(1-s)+(t-s)]\le\dfrac{1}{2}[2s(1-s)^2]=s(1-s)^2.$
For $0\le t\le s\le 1$, $g(t,s)=\dfrac{1}{2}t^2(1-s)^2\le\dfrac{1}{2}s^2(1-s)^2\le s(1-s)^2$. Thus (1) holds.
If $s=0$ or $s=1$, we easily see that (2) holds. If $s\in]0,1[$ and $t\in[0,\dfrac{1}{2}]$, we have for $0<s\le t\le\dfrac{1}{2}$

$$\begin{aligned}\frac{g(t,s)}{s(1-s)^2}=&\ \frac{t^2(1-s)^2-(t-s)^2}{2s(1-s)^2}=\frac{s(1-t)[t(1-s)+(t-s)]}{2s(1-s)^2},\\ &\ge\frac{s(1-t)t(1-s)}{2s(1-s)^2}\ge\frac{t(1-t)}{2}\ge\frac{t^2}{2},\quad \forall t\in[0,\frac{1}{2}].\end{aligned}.$$

For $\dfrac{1}{2}\le t\le s<1$, we have,

$$\frac{g(t,s)}{s(1-s)^2}=\frac{t^2(1-s)^2}{2s(1-s)^2}=\frac{t^2}{2s}\ge\frac{t^2}{2}\ge\frac{t(1-t)}{2},\quad \forall t\in[\frac{1}{2},1].$$

Thus (2) holds.

Lemma 4.1.3.

The function G has the following properties

1. $G(t,s)\geqslant 0\quad \forall t,s\in[0,1]$,
2. $G(t,s)\leqslant M_1 s(1-s)^2\quad \forall t,s\in[0,1]$ *and* $M_1=\dfrac{\max\left(1+\alpha\eta^2,1+\beta(1-\eta^2)\right)}{(1-\beta\eta^2)-\alpha(1-\eta^2)}$,
3. $\min\limits_{t\in\left[\sigma,\frac{1}{2}\right]}G(t,s)\geqslant M_2 s(1-s)^2\quad \forall t,s\in[0,1]$ *where* $M_2=\dfrac{\sigma^2}{2}+\dfrac{\left(\beta\sigma^2+\frac{3}{4}\alpha\right)\Phi(\eta)}{(1-\beta\eta^2)-\alpha(1-\eta^2)}$ *and* $\sigma\in\left(0,\dfrac{1}{2}\right)$.

Proof.

It is clear that (1) holds. Two cases will be considered for the proof of (2).

Case(1) For $0 \leq t \leq \eta$; by Lemma 4.1.2 (1) we have,

$$G(t,s) \leq s(1-s)^2 + \frac{\beta\eta^2+\alpha}{(1-\beta\eta^2)-\alpha(1-\eta^2)}s(1-s)^2 \leq M_1 s(1-s)^2.$$

Case(2) For $\eta \leq t \leq 1$, we have,

$$G(t,s) \leq s(1-s)^2 + \frac{\beta+\alpha(1-\eta^2)}{(1-\beta\eta^2)-\alpha(1-\eta^2)}s(1-s)^2 \leq M_1 s(1-s)^2.$$

Then we have (2).

From (2) of Lemma 4.1.2 we have,

$$\begin{aligned}\min_{\sigma\leq t\leq\frac{1}{2}} G(t,s) &\geqslant s(1-s)^2\left[\Phi(t) + \frac{\beta t^2+\alpha(1-t^2)}{(1-\beta\eta^2)-\alpha(1-\eta^2)}\Phi(\eta)\right] \\ &\geq s(1-s)^2\left[\frac{\sigma^2}{2} + \frac{\beta\sigma^2+\frac{3}{4}\alpha}{(1-\beta\eta^2)-\alpha(1-\eta^2)}\Phi(\eta)\right] \\ &= M_2 s(1-s)^2.\end{aligned}$$

Thus (3) holds.

Lemma 4.1.4.

If $y \in C([0,1])$ and $y \geqslant 0$, then the unique solution u of the boundary value problem (4.1.1),(4.1.2) satisfies $\min_{\sigma\leqslant t\leqslant\frac{1}{2}} u(t) \geqslant \rho\|u\|_1$ where $\|u\|_1 := \sup\{|u(t)|;\ 0 \leq t \leq 1\}$ and $\rho := \dfrac{M_2}{M_1}$.

Proof.

For any $t \in [0,1]$, by Lemma 4.1.3 we have,

$$u(t) = \int_0^1 G(t,s)y(s)ds \leq M_1\int_0^1 s(1-s)^2 y(s)ds,$$

thus $\| u \|_1 \leq M_1\int_0^1 s(1-s)^2y(s)ds$. Moreover, from Lemma 4.1.3 for $t \in \left[\sigma, \frac{1}{2}\right]$ and $\sigma \in \left(0, \frac{1}{2}\right)$ we have,

$$u(t) = \int_0^1 G(t,s)y(s)ds \geq M_2\int_0^1 s(1-s)^2y(s)ds \geq \frac{M_2}{M_1} \| u \|_1 .$$

Lemma 4.1.5.
If u is a positive solution of (4.1.1),(4.1.2), then $u(\eta) \geqslant \gamma||u||$, where $\gamma = \dfrac{\gamma_1}{\gamma_2}$, $\gamma_1 = \min\left\{\dfrac{\eta^2}{2}, \dfrac{\eta(1-\eta)}{2}\right\}$ and $\gamma_2 = \max\left(1, \dfrac{\gamma_1 \max(\alpha,\beta)}{\beta\eta^2+\alpha(1-\eta^2)}\right)$.

Proof.
For every positive solution u of (4.0.1) we have,

$$u(t) = \lambda\int_0^1 g(t,s)a(s)f(s,u(s-\tau))ds + \lambda\frac{\beta t^2+\alpha(1-t^2)}{(1-\beta\eta^2)-\alpha(1-\eta^2)}\int_0^1 g(\eta,s)a(s)f(s,u(s-\tau))ds.$$

From (1) of Lemma 4.1.2 we have,

$$\| u \| = \lambda\int_0^1 s(1-s)^2 a(s)f(s,u(s-\tau))ds + \lambda\frac{\omega}{(1-\beta\eta^2)-\alpha(1-\eta^2)}\int_0^1 g(\eta,s)a(s)f(s,u(s-\tau))ds$$

where $\omega = \max(\alpha,\beta)$. Then

$$\begin{aligned}\min\left\{\frac{\eta^2}{2}, \frac{\eta(1-\eta)}{2}\right\} \| u \| \leq{}& \lambda\int_0^1 \min\left\{\frac{\eta^2}{2}, \frac{\eta(1-\eta)}{2}\right\} s(1-s)^2 a(s)f(s,u(s-\tau))ds \\ &+\lambda\frac{\omega\min\left\{\frac{\eta^2}{2}, \frac{\eta(1-\eta)}{2}\right\}}{(1-\beta\eta^2)-\alpha(1-\eta^2)}\int_0^1 g(\eta,s)a(s)f(s,u(s-\tau))ds\end{aligned}$$

From (2) of Lemma 4.1.2 we have,

$$\begin{aligned}\min\left\{\frac{\eta^2}{2}, \frac{\eta(1-\eta)}{2}\right\} \| u \| \leq{}& \lambda\int_0^1 g(\eta,s)a(s)f(s,u(s-\tau))ds \\ &+\lambda\omega'\frac{\beta\eta^2+\alpha(1-\eta^2)}{(1-\beta\eta^2)-\alpha(1-\eta^2)}\int_0^1 g(\eta,s)a(s)f(s,u(s-\tau))ds\end{aligned}$$

where $\omega' = \dfrac{\omega\gamma_1}{\beta\eta^2+\alpha(1-\eta^2)}$. Then we deduce that

$$\gamma_1 \| u \| \leq \max(1,\omega)u(\eta) = \gamma_2 u(\eta).$$

Let define,

(C_1) $f^0 = \limsup_{u\longrightarrow 0^+} \max_{t\in[0,1]} \dfrac{f(t,u)}{u}$,

(C_2) $f_0 = \liminf_{u \longrightarrow 0^+} \min_{t\in[0,1]} \dfrac{f(t,u)}{u}$,

(C_3) $f^\infty = \lim \sup_{u \longrightarrow \infty} \max_{t\in[0,1]} \dfrac{f(t,u)}{u}$,

(C_4) $f_\infty = \liminf_{u \longrightarrow \infty} \min_{t\in[0,1]} \dfrac{f(t,u)}{u}$,

and

$$A = [\max_{0\le t\le 1}(\alpha \int_0^\tau G(t,s)a(s)ds + \int_\tau^1 G(t,s)a(s)ds)]^{-1},$$

$$B = [\alpha \int_\sigma^\tau G(t,s)a(s)ds + \int_\sigma^{\frac{1}{2}} G(t,s)a(s)ds]^{-1}.$$

The proof of our main results is based upon an application of the fixed-point index theorem in cones.

Lemma 4.1.6. *Let Ω be a bounded open subset of X Banach space, with $\theta \in \Omega$, and let $T : K \cap \overline{\Omega} \longrightarrow K$ be a completely continuous mapping. If $\mu T u \neq u$ for every $u \in K \cap \partial\Omega$ and $0 < \mu \le 1$, then $i(T, K \cap \Omega, K) = 1$.*

Lemma 4.1.7. *Let Ω be a bounded open subset of X Banach space, and let $T : K \cap \overline{\Omega} \longrightarrow K$ be a completely continuous mapping. If there exists an $v \in K \setminus \{\theta\}$, such that $u - Tu \neq \zeta v$ for every $u \in K \cap \partial\Omega$ and $\zeta \ge 0$, then $i(T, K \cap \Omega, K) = 0$.*

4.2 Main Results

Let $X = C[-\tau, 1]$ be a Banach space with norm $\| u \| = sup\{| u(t) |: -\tau \le t \le 1\}$, and K be a cone in X, defined by $K = \{u \in C[-\tau, 1], \min_{t\in[\sigma,\frac{1}{2}]} u(t) \ge \rho \| u \|\}$. Define an operator $T : C[-\tau, 1] \longrightarrow C[-\tau, 1]$ by

$$Tu(t) = \begin{cases} \alpha u(\eta), & -\tau \le t \le 0; \\ \lambda \displaystyle\int_0^1 G(t,s)a(s)f(s, u(s-\tau))ds, & 0 \leqslant t \leqslant 1. \end{cases} \tag{4.2.1}$$

Lemma 4.2.1.

The fixed point of T is a solution of (4.0.1) and $T : \mathrm{K} \to \mathrm{K}$ is completely continuous.

Proof. We have

$$\begin{aligned}
&(Tu)'''(t) + \lambda a(t) f(t, u(t-\tau)) = 0, \qquad t \in J = [0,1],\\
&(Tu)(t) = \alpha (Tu)(\eta), \quad -\tau \leqslant t \leqslant 0\\
&(Tu)(1) = \beta (Tu)(\eta), \quad (Tu)'(0) = 0,
\end{aligned}$$

Therefore, the fixed point of T is a solution of (4.0.1).

It is easy to check that $T : \mathrm{K} \to \mathrm{K}$ is well defined. By Lemma 4.1.4, we now that $T(\mathrm{K}) \subset \mathrm{K}$.

Next, we shall show T is completely continuous. Suppose $u_n \to u$ $(n \to \infty)$ and $u_n \in K$, $\forall n \in \mathbb{N}$, then there exists $M > 0$ such that $\|u_n\| \leq M$. Since f is continuous on $[0,1] \times [0, M]$, it is uniformly continuous. Therefore, $\forall \varepsilon > 0$, there exists a $\delta > 0$ such that $|x - y| < \delta$ implies $|f(s,x) - f(s,y)| < \epsilon$, $\forall s \in [0,1],\ x, y \in [0, M]$. For this $\delta > 0$, there exists a N such that $\|u_n - u\| < \delta$ for $n > N$, so $|f(s, u_n(s-\tau)) - f(s, u(s-\tau))| < \varepsilon$, for $n > N$ and $s \in [0,1]$. This implies that

$$\begin{aligned}
|Tu_n(t) - Tu(t)| &= |\lambda \int_0^1 G(t,s) a(s) (f(s, u_n(s-\tau)) - f(s, u(s-\tau))) ds|\\
&\leq \lambda \int_0^1 G(t,s) a(s) |f(s, u_n(s-\tau)) - f(s, u(s-\tau)| ds\\
&\leq \epsilon \lambda \int_0^1 G(t,s) a(s) ds.
\end{aligned}$$

Therefore, T is continuous.

Let Ω be any bounded subset of K, then there exists $\gamma > 0$, such that $||u|| \leqslant \gamma$, for all $u \in \Omega$. Since f is continuous on $[0,1] \times [0, \gamma]$, there exists a $L > 0$, such that $|f(t, v)| < L$, for $\forall t \in [0,1]$, $v \in [0, \gamma]$. Consequently, for all $u \in \Omega$, $t \in [0.1]$,

$$|Tu(t)| = \left| \lambda \int_0^1 G(t,s) a(s) f(s, u(s-\tau)) ds \right| \leqslant \lambda M_1 L \int_0^1 s(1-s)^2 a(s) ds.$$

Which implies the boundedness of $T\Omega$.
Since $G(t,s)$ is continuous on $[0,1]\times[0,1]$, it is uniformly continuous. Then for $\forall\epsilon>0$, there exists a $\delta>0$, such that $|t_1-t_2|<\delta$ implies that $|G(t_1,s)-G(t_2,s)|<\epsilon$, $\forall s\in[0,1]$. So if $u\in\Omega$

$$|Tu(t_1)-Tu(t_2)|\le\lambda\int_0^1|G(t_1,s)-G(t_2,s)|a(s)f(s,u_n(s-\tau))ds\le\lambda L\epsilon\int_0^1 a(s)ds$$

From the arbitrariness of ϵ, we get $T\Omega$ equicontinuity. The operator T is completely continuous by the mean of the Ascoli-Arzela theorem.

Theorem 4.2.1. *[17]*
Assume that (H_1)*-*(H_3) *hold and* $f_\infty>0$*,* $f^0<\infty$*,* $\frac{B}{f_\infty}<\frac{A}{f^0}$*, then for any* $\lambda\in(\frac{B}{f_\infty},\frac{A}{f^0})$ *the problem* 4.0.1 *has at least one positive solution.*

Proof.
For every $\lambda\in\big(\dfrac{B}{f_\infty},\dfrac{A}{f^0}\big)$. By the condition of $f^0<\dfrac{A}{\lambda}$ and the definition of f^0, there exist constants R_0, $\varepsilon_0>0$, when $0<u\le R_0$, we have $\dfrac{f(t,u)}{u}\le\dfrac{1}{\lambda}(A-\varepsilon_0)$, namely,

$$f(t,u)\le\frac{1}{\lambda}(A-\varepsilon_0)u,\qquad 0<u\le R_0.$$

Let $\Omega_0=\{u\in C[-\tau,1]:\parallel u\parallel<R_0\}$, for $u\in K\cap\partial\Omega_0$, from (4.2.1) we get

$$\begin{aligned}
\parallel Tu\parallel &= \max_{0\le t\le1}\lambda\int_0^1 G(t,s)a(s)f(s,u(s-\tau))ds,\\
&\le \frac{1}{\lambda}(A-\varepsilon_0)\max_{0\le t\le1}\lambda\int_0^1 G(t,s)a(s)u(s-\tau)ds,\\
&= (A-\varepsilon_0)\max_{0\le t\le1}\int_0^1 G(t,s)a(s)u(s-\tau)ds,\\
&= (A-\varepsilon_0)\max_{0\le t\le1}[\alpha\int_0^\tau G(t,s)a(s)u(\eta)ds+\int_\tau^1 G(t,s)a(s)u(s-\tau)ds],\\
&\le (A-\varepsilon_0)\max_{0\le t\le1}[\alpha\int_0^\tau G(t,s)a(s)ds+\int_\tau^1 G(t,s)a(s)ds]R_0,\\
&\le R_0-\varepsilon_0R_0\max_{0\le t\le1}[\alpha\int_0^\tau G(t,s)a(s)ds+\int_\tau^1 G(t,s)a(s)ds]<R_0.
\end{aligned}$$

Therefore, for every $u \in K \cap \partial\Omega_0$, $0 < \mu \leq 1$, when $\| u \| = R_0$ we have $\mu Tu \neq u$. In fact, if there exist $u_0 \in K \cap \partial\Omega_0$ and $0 < \mu_0 \leq 1$, such that $\mu_0 Tu_0 = u_0$, then $\| Tu_0 \| = \dfrac{1}{\mu_0} \| u_0 \| \geq \| u_0 \| = R_0$; this is a contradiction. Hence T satisfies the condition of Lemma (4.1.6) in $K \cap \partial\Omega_0$. By Lemma 4.1.6, we have

$$i(T, K \cap \Omega_0, K) = 1 \tag{4.2.2}$$

On the other hand, by the condition of $f_\infty > \frac{B}{\lambda}$, and the definition of f_∞, there exist $R_1' > \rho R_0, \varepsilon_1 > 0$, and for $u \geq R_1'$, we have

$$f(t,u) \geq \frac{1}{\lambda}(B + \varepsilon_1)u.$$

Let $R_1 = \dfrac{R_1'}{\rho} > R_0$, and $\Omega_1 = \{u \in C[-\tau, 1] : \| u \| < R_1\}$, then $\min\{u(t) : t \in [\sigma, \frac{1}{2}]\} \geq \rho \| u \| = R_1'$, $\forall u \in K \cap \partial\Omega_1$.

Choose $v(t) = 1, v \in K \setminus \{\theta\}$. We prove that T satisfies the condition of Lemma 4.1.7 in $K \cap \partial\Omega_1$; namely, $u - Tu \neq \zeta v$ for every $u \in K \cap \partial\Omega_1$ and $\zeta \geq 0$. If it is not true, there exist $u_0 \in K \cap \partial\Omega_1$ and $\zeta_0 \geq 0$, such that $u_0 - Tu_0 = \zeta_0 v$. Let $\gamma = \min\{u_0(t) : t \in [\sigma, \frac{1}{2}]\}$, $\gamma \geq \rho \| u_0 \| = \rho R_1 = R_1'$, for $t \in [\sigma, \frac{1}{2}]$; we have

$$\begin{aligned}
u_0(t) &= \lambda \int_0^1 G(t,s)a(s)f(s, u(s-\tau))ds + \zeta_0, \\
&\geq \frac{1}{\lambda}(B + \varepsilon_1)\lambda \int_0^1 G(t,s)a(s)u_0(s-\tau)ds + \zeta_0, \\
&\geq (B + \varepsilon_1) \int_\sigma^{\frac{1}{2}} G(t,s)a(s)u_0(s-\tau)ds + \zeta_0, \\
&= (B + \varepsilon_1)[\int_\sigma^\tau G(t,s)a(s)u_0(s-\tau)ds + \int_\tau^{\frac{1}{2}} G(t,s)a(s)u_0(s-\tau)ds] + \zeta_0, \\
&\geq (B + \varepsilon_1)[\alpha \int_\sigma^\tau G(t,s)a(s)u_0(\eta)ds + \int_\tau^{\frac{1}{2}} G(t,s)a(s)u_0(s-\tau)ds] + \zeta_0, \\
&\geq (B + \varepsilon_1)\gamma[\alpha \int_\sigma^\tau G(t,s)a(s)ds + \gamma \int_\sigma^{\frac{1}{2}} G(t,s)a(s)ds] + \zeta_0 > \gamma, \\
&= \gamma + \varepsilon_1\gamma[\alpha \int_\sigma^\tau G(t,s)a(s)ds + \gamma \int_\sigma^{\frac{1}{2}} G(t,s)a(s)ds] + \zeta_0 > \gamma,
\end{aligned}$$

and this is a contradiction. Thus T satisfies the condition of Lemma 4.1.7 in $K \cap \partial\Omega_1$. By Lemma 4.1.7 we have

$$i(T, K \cap \Omega_1, K) = 0 \tag{4.2.3}$$

Now, from (4.2.2) and (4.2.3) it follows that

$$i(T, K \cap (\Omega_1 \setminus \overline{\Omega_0}), K) = i(T, K \cap \Omega_1, K) - i(T, K \cap \Omega_0, K) = -1. \tag{4.2.4}$$

Therefore, T has a fixed point in $K \cap (\Omega_1 \setminus \overline{\Omega_0})$, which is a positive solution of the problem 4.0.1.

Theorem 4.2.2. *[17]*
Assume that (H_1)-(H_3) *hold and* $f_0 > 0$, $f^\infty < \infty$, $\dfrac{B}{f_0} < \dfrac{A}{f^\infty}$; *then for any* $\lambda \in (\dfrac{B}{f_0}, \dfrac{A}{f^\infty})$ *the problem* 4.0.1 *has at least one positive solution.*

Proof.
Let $\lambda \in (\dfrac{B}{f_0}, \dfrac{A}{f^\infty})$. By the condition of $f^\infty < \dfrac{A}{\lambda}$ and the definition of f^∞, there exist constants $R_2', \varepsilon_2 > 0$, when $u > R_2'$, we have $f(t,u) \leq \dfrac{1}{\lambda}(A - \varepsilon_2)u$.
Let $N = max\{f(t,u) : 0 \leq u \leq R_2'\}$, then

$$f(t,u) \leq N + \frac{1}{\lambda}(A - \varepsilon_2)u, \qquad u \in [0, \infty].$$

Choose $R_2 > \dfrac{AN}{\varepsilon_2 f^\infty}$; let $\Omega_2 = \{u \in C[-\tau, 1] :\| u \|< R_2\}$, then for every $u \in K \cap \partial\Omega_2$; we get

$$\begin{aligned}
\| Tu \| &= \max_{0\le t\le 1} \lambda \int_0^1 G(t,s)a(s)f(s,u(s-\tau))ds,\\
&\le \max_{0\le t\le 1} \lambda \int_0^1 G(t,s)a(s)[N+\frac{1}{\lambda}(A-\varepsilon_2)u(s-\tau)]ds,\\
&\le A \| u \| \max_{0\le t\le 1}[\alpha \int_0^\tau G(t,s)a(s)ds + \int_\tau^1 G(t,s)a(s)ds]\\
&\quad +(N-\frac{1}{\lambda}\varepsilon_2 \| u \|)\lambda \max_{0\le t\le 1}\int_0^1 G(t,s)a(s)u(s-\tau)ds,\\
&= R_2 + (N-\frac{1}{\lambda}\varepsilon_2 R_2)\lambda \max_{0\le t\le 1}\int_0^1 G(t,s)a(s)u(s-\tau)ds,\\
&< R_2.
\end{aligned}$$

Thus, for $u \in K\cap\partial\Omega_2$ and $0<\mu\le 1$, when $\| u \| = R_2$, we have $\mu Tu \ne u$. In fact, if there exist $u_0 \in K\cap\partial\Omega_2$ and $0<\mu_0\le 1$, such that $\mu_0 Tu_0 = u_0$, then $\| Tu_0 \| = \frac{1}{\mu_0} \| u_0 \| \ge \| u_0 \| = R_2$; this is a contradiction. Hence T satisfies the condition of Lemma 4.1.6 in $K\cap\partial\Omega_2$. By Lemma 4.1.6, we have

$$i(T, K\cap\partial\Omega_2, K) = 1. \tag{4.2.5}$$

On the other hand, by the condition of $f_0 > \frac{B}{\lambda}$, and the definition of f_0, there exist $R_3' > \rho R_3, \varepsilon_3 > 0$, and for $u \ge R_3'$, we have

$$f(t,u) \ge \frac{1}{\lambda}(B+\varepsilon_3)u.$$

Let $R_3 = \frac{R_3'}{\rho} > R_2$, and $\Omega_3 = \{u \in C[-\tau,1] : \| u \| < R_3\}$, then $\min\{u(t) : t \in [\sigma,\frac{1}{2}]\} \ge \rho \| u \| = R_3'$, for all $u \in K\cap\partial\Omega_3$.

Choose $v(t)=1, v\in K\setminus\{\theta\}$. We prove that T satisfies the condition of Lemma 4.1.7 in $K\cap\partial\Omega_3$; namely, $u - Tu \ne \zeta v$ for every $u \in K\cap\partial\Omega_3$ and $\zeta \ge 0$. If it is not true, there exist $u_0 \in K\cap\partial\Omega_3$ and $\zeta_0 \ge 0$, such that $u_0 - Tu_0 = \zeta_0 v$. Let $\gamma = \min\{u_0(t) : t\in[\sigma,\frac{1}{2}]\}$, $\gamma \ge \rho \| u_0 \| = \rho R_3 = R_3'$, for $t \in [\sigma,\frac{1}{2}]$; we have

$$
\begin{aligned}
u_0(t) &= \lambda \int_0^1 G(t,s)a(s)f(s,u(s-\tau))ds + \zeta_0, \\
&\geq \frac{1}{\lambda}(B+\varepsilon_3)\lambda \int_0^1 G(t,s)a(s)u_0(s-\tau)ds + \zeta_0, \\
&\geq (B+\varepsilon_3) \int_\sigma^{\frac{1}{2}} G(t,s)a(s)u_0(s-\tau)ds + \zeta_0, \\
&= (B+\varepsilon_3)[\int_\sigma^\tau G(t,s)a(s)u_0(s-\tau)ds + \int_\tau^{\frac{1}{2}} G(t,s)a(s)u_0(s-\tau)ds] + \zeta_0, \\
&\geq (B+\varepsilon_3)[\alpha \int_\sigma^\tau G(t,s)a(s)u_0(\eta)ds + \int_{\tau+\sigma}^{\frac{1}{2}} G(t,s)a(s)u_0(s-\tau)ds] + \zeta_0, \\
&\geq (B+\varepsilon_3)[\alpha\gamma \int_\sigma^\tau G(t,s)a(s)ds + \gamma \int_\sigma^{\frac{1}{2}-\tau} G(t,s)a(s)ds] + \zeta_0, \\
&= (1+\frac{\varepsilon_3}{B})\gamma + \zeta_0 > \gamma,
\end{aligned}
$$

and this is a contradiction. Thus T satisfies the condition of Lemma 4.1.7 in $K \cap \partial\Omega_3$. By Lemma 4.1.7 we get

$$
i(T, K \cap \Omega_3, K) = 0 \tag{4.2.6}
$$

Now, from (4.2.5), (4.2.6), it follows that

$$
i(T, K \cap (\Omega_2 \setminus \overline{\Omega_3}), K) = i(T, K \cap \Omega_2, K) - i(T, K \cap \Omega_3, K) = 1. \tag{4.2.7}
$$

Therefore, T has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega_3})$, which is a positive solution of the problem 4.0.1.

Chapitre 5

Positive solutions for a quadratic mixed type of delay differential equation with eigenvalues

In this chapter, we consider the existence of positive solutions for the following quadratic mixed type of delay differential equation with eigenvalues

$$\begin{aligned} & u''(t) + \lambda p(t) f(t, u(t-\tau), \int_0^t k(t,s)u(s)ds) = 0, \quad t \in (0,1), \\ & u(t) = \alpha u(\eta), \quad -\tau \leq t \leq 0, \\ & u(1) = \beta u(\eta). \end{aligned} \tag{5.0.1}$$

Let $v(t) = \int_0^t k(t,s)u(s)ds$, we suppose that $f(t,u,v)$ is neither superlinear or sub-linear. We study the existence of positive solutions for a quadratic mixed type of delay differential equation with eigenvalues by the mean of Krasnosel'skii fixed point theorem 1.2.2 on cone.

Assume the following conditions are satisfied,

H_1) $f : J \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$. is continuous function, $J = (0, 1)$,

H_2) $0 < \alpha \leqslant 1$, $0 < \beta < \frac{1-\alpha\eta}{1-\eta}$, $0 < \eta < 1$ and $0 < \tau < 1$.

H_3) $p(s) \in C(J_1, \mathbb{R}^+)$, $J_1 = [0, 1]$.

H_4) $D = \{(t, s) \in J_1 \times J_1 : t > s\}$, $k(t, s) \in C(D, \mathbb{R}^+)$,
$m_1 = \min\{k(s, t) : (s, t) \in D\}$, $M_1 = \max\{k(s, t) : (s, t) \in D\}$.

H_5) $u \in C^2[-\tau, 1]$, $u(t) \geq 0, t \in [-\tau, 1]$.

5.1 Preliminaries

In this section, we present some definitions and some lemmas.

Definition 5.1.1. *$u(t)$ is the positive solution of (5.0.1) if and only if it satisfies the following conditions :*

1. *For an arbitrary $t \in [-\tau, 1], u(t) \geq 0$ and $u(t)$ is continuous.*
2. *When $t \in [-\tau, 0], u(t) = \alpha u(\eta)$ and $u(1) = \beta u(\eta)$ $(0 < \eta < 1)$.*
3. *$u''(t) = -\lambda p(s) f(t, u(t-\tau), v(t))$, for all $t \in [0, 1]$.*

Lemma 5.1.1. *If $u(t)$ is the solution of (5.0.1) then $u(t)$ can be represented as*

$$u(t) = \begin{cases} \lambda \int_0^1 G(t, s)p(s)f(s, u, v)ds + \dfrac{\lambda[\beta + (\alpha - \beta)t]}{(1 - \alpha\eta) - \beta(1 - \eta)} \int_0^1 G(\eta, s)p(s)f(s, u, v)ds, & 0 \leqslant t \leqslant 1 \\ \alpha u(\eta), \qquad -\tau \leqslant t \leqslant 0, & \end{cases} \tag{5.1.1}$$

where

$$G(t, s) = \begin{cases} t(1 - s), & 0 \leqslant t \leqslant s \leqslant 1, \\ s(1 - t), & 0 \leqslant s \leqslant t \leqslant 1. \end{cases} \tag{5.1.2}$$

Lemma 5.1.2. *The function G has the following properties*

1. *$0 \leq G(t, s) \leq G(s, s), \forall t, s \in [0, 1]$.*

2. Let $\gamma \in [0, \frac{1}{2}]$*. Then for* $t \in J_\gamma = [\gamma, 1-\gamma]$, $s \in [0,1]$ *we have*

$$G(t,s) \geq \min\{t, 1-t\} G(s,s) \geq \gamma G(s,s).$$

Let

1) $L_1 = \sup_{t \in J} \Big[\alpha \min\{\eta, 1-\eta\} \int_0^\tau G(t,s)p(s)ds + \gamma\gamma_1 \int_{\gamma+\tau}^{1-\gamma} G(s,s)p(s)(1+m_1 s)ds \Big]$,
   $\gamma_1 = \dfrac{k_2}{k_1}$, $0 < \gamma < \frac{1-\tau}{2} \leq 1-\gamma \leq 1-\tau$.
2) $L_2 = k_1 \Big[\alpha \int_0^\tau G(s,s)p(s) + \int_0^1 G(s,s)p(s)(1+M_1 s)ds \Big]$.

We denote the space X as

$$X = \{u \in C[-\tau, 1] : u(t) \geq 0, \text{ for } t \in [-\tau, 1],\ u(t) = \alpha u(\eta),\ \ -\tau \leq t \leq 0, u(1) = \beta u(\eta)\}$$

normed by $\| u \| = \sup\{| u(t) | : -\tau \leqslant t \leqslant 1\}$. Then $(X, \| \, . \, \|)$ is a Banach space.
We also define a cone K in the space X where

$$K = \{u \in X : u(t) \geq 0, t \in [0,1], \min_{\gamma \leqslant t \leqslant 1-\gamma} u(t) \geq \gamma_1 \| u \|\}.$$

Define the operator T by

$$Tu(t) = \begin{cases} \lambda \displaystyle\int_0^1 G(t,s)p(s)f(s,u,v)ds + \frac{\lambda[\beta + (\alpha-\beta)t]}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^1 G(\eta,s)p(s)f(s,u,v)ds, & 0 \leqslant t \leqslant 1 \\ \alpha u(\eta), \quad -\tau \leqslant t \leqslant 0, \end{cases} \tag{5.1.3}$$

Lemma 5.1.3. *The fixed point of* T *is a solution of* (5.0.1).

Proof. From 5.1.3, we have
$(Tu)"(t) + \lambda p(t) f(s,u,v) = 0, \quad t \in [0,1],$
$(Tu)(t) = \alpha (Tu)(\eta), \ -\tau \leq t \leq 0.$
$(Tu)(1) = \beta (Tu)(\eta)$

Thus, the fixed point of T is the solution of the equation(5.0.1).

Lemma 5.1.4. *$T : K \longrightarrow K$ is completely continuous operator.*

Proof.
We have $\| Tu \| = \| Tu \|_{[0,1]}$, $Tu(t) \geq 0$ for all $u(t) \in K$, and $Tu(t) \geq \gamma_1 \| Tu \|$.
Then $T : K \longrightarrow K$. we can conclude that T is a completely continuous operator by Arzela-Ascoli Theorem. (see Chap.2)

5.2 Main Results

Let

$$f_0 = \lim_{u,v \longrightarrow 0} \min_{t \in [-\tau,1]} \frac{f(t,u,v)}{\sqrt{u^2+v^2}}, \quad f_\infty = \lim_{u,v \longrightarrow \infty} \min_{t \in [-\tau,1]} \frac{f(t,u,v)}{\sqrt{u^2+v^2}} \quad .$$

$$f^0 = \overline{\lim_{u,v \longrightarrow 0}} \sup_{t \in [-\tau,1]} \frac{f(t,u,v)}{\sqrt{u^2+v^2}}, \quad f^\infty = \overline{\lim_{u,v \longrightarrow \infty}} \sup_{t \in [-\tau,1]} \frac{f(t,u,v)}{\sqrt{u^2+v^2}}.$$

$$k_1 = 1 + \frac{\max\{\alpha, \beta\}}{(1-\alpha\eta) - \beta(1-\eta)}, \quad k_2 = \gamma \left[1 + \frac{\beta + \min\{(\alpha-\beta)\gamma, (\alpha-\beta)(1-\gamma)\}}{(1-\alpha\eta) - \beta(1-\eta)} \right]$$

Theorem 5.2.1. *[18]*
Assume that (H_1)-(H_4) hold and $0 < f_\infty < +\infty$, $0 < f^0 < +\infty$, then there exists at least one positive solution the problem to (5.0.1) for

$$\lambda \in (\frac{1}{L_1 f_\infty}, \frac{1}{L_2 f^0}). \tag{5.2.1}$$

Proof. Let $\lambda \in (\frac{1}{L_1 f_\infty}, \frac{1}{L_2 f^0})$, then there exists an $\varepsilon > 0$ such that

$$\frac{1}{L_1(f_\infty - \varepsilon)} \leq \lambda \leq \frac{1}{L_2(f^0 + \varepsilon)}. \tag{5.2.2}$$

By $0 < f^0 < +\infty$, there exists $r_1 > 0$ such that

$$f(t,u,v) \leq (f^0 + \varepsilon)\sqrt{u^2+v^2} \quad \text{for} \quad 0 < \sqrt{u^2+v^2} \leq r_1.$$

Let $\Omega_1 = \{u \in X : \| \sqrt{u^2+v^2} \| < r_1\}$, then for $u, v \in K \cap \partial\Omega_1$, we have

$$
\begin{aligned}
\| Tu \| &\leq \lambda \int_0^1 G(s,s)p(s)f(s,u,v)ds + \frac{\lambda \max\{\alpha,\beta\}}{(1-\alpha\eta)-\beta(1-\eta)} \int_0^1 G(s,s)p(s)f(s,u,v)ds \\
&\leq \lambda(f^0+\varepsilon)k_1 \int_0^1 G(s,s)p(s)\sqrt{u^2+v^2}ds \\
&\leq \lambda(f^0+\varepsilon)k_1 \int_0^1 G(s,s)p(s)(u+v)ds \\
&\leq \lambda(f^0+\varepsilon)k_1 \Big[\int_0^1 G(s,s)p(s)u(s-\tau)ds + \int_0^1 G(s,s)p(s) \int_0^s k(t,s)u(t)dtds \Big] \\
&\leq \lambda(f^0+\varepsilon)k_1 \Big[\int_0^\tau G(s,s)p(s)\alpha u(\eta)ds + \int_\tau^1 G(s,s)p(s)u(s-\tau)ds \\
&\quad + M_1 \int_0^1 G(s,s)p(s) \int_0^s u(t)dtds \Big] \\
&\leq \lambda(f^0+\varepsilon)k_1 \Big[\alpha \int_0^\tau G(s,s)p(s)ds + \int_0^1 G(s,s)p(s)(1+M_1 s)ds\Big] \| u \| \\
&= \lambda(f^0+\varepsilon)L_2 \| u \| \leq \| u \| .
\end{aligned}
$$

For the same above $\varepsilon > 0$, by $0 < f_\infty < \infty$ there exists $r_2 > r_1$ such that

$$
f(t,u,v) > (f_\infty - \varepsilon)\sqrt{u^2+v^2} \text{ for } \sqrt{u^2+v^2} \geq r_2. \tag{5.2.3}
$$

Since $\gamma < \dfrac{1-\tau}{2}$ then $\gamma + \tau < 1 - \gamma$.
Let $\Omega_2 = \{u \in X : \| \sqrt{u^2+v^2} \| < r_2\}$, then for $u, v \in K \cap \partial\Omega_2$ and by lemma 3.1.6, we have

$$
\begin{aligned}
\| Tu \| &\geq \lambda \sup_{t\in J} \int_0^1 G(t,s)p(s)f(s,u,v)ds \\
&\geq \lambda(f_\infty - \varepsilon) \sup_{t\in J} \int_0^1 G(t,s)p(s)\sqrt{u^2+v^2}ds \\
&\geq \lambda(f_\infty - \varepsilon) \sup_{t\in J} \int_0^1 G(t,s)p(s)\frac{u+v}{2}ds \\
&= \frac{\lambda(f_\infty - \varepsilon)}{2} \sup_{t\in J} \Big[\int_0^\tau G(t,s)p(s)\alpha u(\eta)ds + \int_\tau^1 G(t,s)p(s)u(s-\tau)ds \\
&\quad + \int_0^1 G(t,s)p(s) \int_0^s k(t,s)u(t)dtds \Big]
\end{aligned}
$$

$$\geq \frac{\lambda(f_\infty-\varepsilon)}{2}\sup_{t\in J}\Big[\int_0^\tau G(t,s)p(s)\alpha\min\{\eta,1-\eta\}||u||ds+\int_0^{1-\tau}G(t,s+\tau)p(s+\tau)u(s)ds$$
$$+m_1\int_0^1 G(t,s)p(s)\int_0^s u(t)dtds\Big]$$
$$\geq \frac{\lambda(f_\infty-\varepsilon)}{2}\sup_{t\in J}\Big[\alpha\min\{\eta,1-\eta\}\int_0^\tau G(t,s)p(s)ds||u||+\int_\gamma^{1-\gamma}G(t,s+\tau)p(s+\tau)\gamma_1\parallel u\parallel ds$$
$$+m_1\int_\gamma^{1-\gamma}G(t,s)p(s)s\gamma_1\parallel u\parallel dtds\Big]$$
$$\geq \frac{\lambda(f_\infty-\varepsilon)}{2}\sup_{t\in J}\Big[\alpha\min\{\eta,1-\eta\}\int_0^\tau G(t,s)p(s)ds+\gamma_1\int_{\gamma+\tau}^{1-\gamma+\tau}G(t,s)p(s)ds$$
$$+m_1\gamma_1\int_\gamma^{1-\gamma}G(t,s)p(s)sds\Big]\parallel u\parallel$$
$$\geq \frac{\lambda(f_\infty-\varepsilon)}{2}\sup_{t\in J}\Big[\alpha\min\{\eta,1-\eta\}\int_0^\tau G(t,s)p(s)ds+\gamma\gamma_1\int_{\gamma+\tau}^{1-\gamma}G(s,s)p(s)(1+m_1s)ds\Big]\parallel u\parallel$$
$$=\lambda(f_\infty-\varepsilon)L_1\geq\parallel u\parallel.$$

Therefore, by the first part of Theorem (1.2.2), T has a fixed point $u\in K\cap(\overline{\Omega}_2\backslash\Omega_1)$, and $u(t)$ is a positive solution of (5.0.1). The proof is complete.

Theorem 5.2.2. *[18]*
Assume that (H_1)-(H_4) hold and $0<f_0<\infty$, $0<f^\infty<\infty$, then there exists at least one positive solution to the problem (5.0.1) for

$$\lambda\in(\frac{1}{L_1f_0},\frac{1}{L_2f^\infty}). \tag{5.2.4}$$

Proof. by (5.2.4), there exists an $\varepsilon>0$ such that

$$\frac{1}{L_1(f_0-\varepsilon)}\leq\lambda\leq\frac{1}{L_2(f^\infty+\varepsilon)}. \tag{5.2.5}$$

By $0<f_0<+\infty$, there exists $R_1>0$ such that

$$f(t,u,v)\geq(f_0-\varepsilon)\sqrt{u^2+v^2}\text{ for }\sqrt{u^2+v^2}\leq R_1. \tag{5.2.6}$$

Let $\Omega_1=\{u\in X:\parallel\sqrt{u^2+v^2}\parallel<R_1\}$, then for $u,v\in K\cap\partial\Omega_1$, we have

$$
\begin{aligned}
\| Tu \| & \geq \lambda \sup_{t\in J} \int_0^1 G(t,s)p(s)f(s,u,v)ds \\
& \geq \lambda(f_0-\varepsilon) \sup_{t\in J} \int_0^1 G(t,s)p(s)\sqrt{u^2+v^2}ds \\
& \geq \lambda(f_0-\varepsilon) \sup_{t\in J} \int_0^1 G(t,s)p(s)\frac{u+v}{2}ds \\
& = \frac{\lambda(f_0-\varepsilon)}{2} \sup_{t\in J} \Big[\int_0^\tau G(t,s)p(s)\alpha u(\eta)ds + \int_\tau^1 G(t,s)p(s)u(s-\tau)ds \\
& \quad + \int_0^1 G(t,s)p(s) \int_0^s k(t,s)u(t)dtds \Big] \\
& \geq \frac{\lambda(f_0-\varepsilon)}{2} \sup_{t\in J} \Big[\int_0^\tau G(t,s)p(s)\alpha \min\{\eta, 1-\eta\} ||u||ds + \int_0^{1-\tau} G(t,s+\tau)p(s+\tau)u(s)ds \\
& \quad + m_1 \int_0^1 G(t,s)p(s) \int_0^s u(t)dtds \Big] \\
& \geq \frac{\lambda(f_0-\varepsilon)}{2} \sup_{t\in J} \Big[\alpha \min\{\eta, 1-\eta\} \int_0^\tau G(t,s)p(s)ds||u|| + \int_\gamma^{1-\gamma} G(t,s+\tau)p(s+\tau)\gamma_1 \| u \| ds \\
& \quad + m_1 \int_\gamma^{1-\gamma} G(t,s)p(s)s\gamma_1 \| u \| dtds \Big] \\
& \geq \frac{\lambda(f_0-\varepsilon)}{2} \sup_{t\in J} \Big[\alpha \min\{\eta, 1-\eta\} \int_0^\tau G(t,s)p(s)ds + \gamma_1 \int_{\gamma+\tau}^{1-\gamma+\tau} G(t,s)p(s)ds \\
& \quad + m_1\gamma_1 \int_\gamma^{1-\gamma} G(t,s)p(s)sds \Big] \| u \| \\
& \geq \frac{\lambda(f_0-\varepsilon)}{2} \sup_{t\in J} \Big[\alpha \min\{\eta, 1-\eta\} \int_0^\tau G(t,s)p(s)ds + \gamma\gamma_1 \int_{\gamma+\tau}^{1-\gamma} G(s,s)p(s)(1+m_1 s)ds \Big] \| u \| \\
& = \lambda(f_0-\varepsilon)L_1 \geq \| u \| .
\end{aligned}
$$

For the same above $\varepsilon > 0$, by $0 < f^\infty < \infty$ there exists $R_2 > R_1$ such that

$$
f(t,u,v) > (f^\infty + \varepsilon)\sqrt{u^2+v^2} \text{ for } \sqrt{u^2+v^2} \geq R_2. \tag{5.2.7}
$$

Let $\Omega_2 = \{u \in X : \| \sqrt{u^2+v^2} \| < R_2\}$ for $u, v \in K \cap \partial\Omega_2$, we get

$$
\begin{aligned}
\| Tu \| &\leq \lambda \int_0^1 G(s,s)p(s)f(s,u,v)ds + \frac{\lambda \max\{\alpha,\beta\}}{(1-\alpha\eta)-\beta(1-\eta)} \int_0^1 G(s,s)p(s)f(s,u,v)ds \\
&\leq \lambda(f^{\infty}+\varepsilon)k_1 \int_0^1 G(s,s)p(s)\sqrt{u^2+v^2}ds \\
&\leq \lambda(f^{\infty}+\varepsilon)k_1 \int_0^1 G(s,s)p(s)(u+v)ds \\
&\leq \lambda(f^{\infty}+\varepsilon)k_1 \Big[\int_0^1 G(s,s)p(s)u(s-\tau)ds + \int_0^1 G(s,s)p(s) \int_0^s k(t,s)u(t)dtds \Big] \\
&\leq \lambda(f^{\infty}+\varepsilon)k_1 \Big[\int_0^{\tau} G(s,s)p(s)\alpha u(\eta)ds + \int_{\tau}^1 G(s,s)p(s)u(s-\tau)ds \\
&\quad + M_1 \int_0^1 G(s,s)p(s) \int_0^s u(t)dtds \Big] \\
&\leq \lambda(f^{\infty}+\varepsilon)k_1 \Big[\alpha \int_0^{\tau} G(s,s)p(s)ds + \int_0^1 G(s,s)p(s)(1+M_1 s)ds\Big] \| u \| \\
&= \lambda(f^{\infty}+\varepsilon)L_2 \| u \| \leq \| u \| .
\end{aligned}
$$

Therefore, by the second part of theorem (1.2.2), T has a fixed point $u \in K \cap (\overline{\Omega}_2 \backslash \Omega_1)$, and $u(t)$ is a positive solution of (5.0.1). The proof is complete.

Theorem 5.2.3. *[18]*
Assume that (H_1)-(H_4) *hold and* $f_{\infty} = \infty$, $f^0 = 0$, *then there exists two positive numbers* λ_1, λ_2 *when* $\lambda_1 \leq \lambda \leq \lambda_2$, *the problem* (5.0.1) *has at least a positive solution.*

Proof. Since $f_{\infty} = \infty$, we can choose a positive constant $M > 0$ such that $f(t,u,v) \geq M = aR$ $(a > 0)$ for any $\sqrt{u^2+v^2} \geq R$, $t \in J$. Let

$$
\Omega_1 = \{u \in X : \| \sqrt{u^2+v^2} \| < R\}, \quad \lambda_1 = \Big[a\gamma \int_{\gamma}^{1-\gamma} G(s,s)p(s)ds\Big]^{-1}
$$

for $u, v \in K \cap \partial\Omega_1$, $\lambda \geq \lambda_1$, we have

$$
\begin{aligned}
\| Tu \| &\geq \lambda \sup_{t\in J} \int_0^1 G(t,s)p(s)f(s,u,v)ds \\
&\geq \lambda M \sup_{t\in J} \int_0^1 G(t,s)p(s)ds \\
&\geq \lambda M \sup_{t\in J} \int_\gamma^{1-\gamma} G(t,s)p(s)ds \\
&\geq \lambda M \gamma \int_\gamma^{1-\gamma} G(s,s)p(s)ds \\
&= \lambda a R \gamma \int_\gamma^{1-\gamma} G(s,s)p(s)ds = \frac{\lambda}{\lambda_1} R \\
&= \| \sqrt{u^2+v^2} \| \geq \| u \| .
\end{aligned}
$$

Because $f^0 = 0$, we choose a value small enough for $\varepsilon > 0$, so that

$$
\lambda_2 = \Big[\varepsilon k_1 \alpha \int_0^\tau G(s,s)p(s)ds + \int_0^1 G(s,s)p(s)(1+M_1 s)ds\Big]^{-1} > \lambda
$$

and there exists $0 < r < R$ such that $f(t,u,v) \leq \varepsilon\sqrt{u^2+v^2}$ for any $\sqrt{u^2+v^2} \leq r$.
Let

$$
\Omega_2 = \{u \in X : \| \sqrt{u^2+v^2} \| < r\}
$$

for $u, v \in K \cap \partial\Omega_2$, we have

$$
\begin{aligned}
\| Tu \| &\leq \lambda \int_0^1 G(s,s)p(s)f(s,u,v)ds + \frac{\lambda \max\{\alpha,\beta\}}{(1-\alpha\eta)-\beta(1-\eta)} \int_0^1 G(s,s)p(s)f(s,u,v)ds \\
&\leq \lambda \varepsilon k_1 \int_0^1 G(s,s)p(s)\sqrt{u^2+v^2}ds \\
&\leq \lambda \varepsilon k_1 \int_0^1 G(s,s)p(s)(u+v)ds
\end{aligned}
$$

$$
= \lambda \varepsilon k_1 \Big[\int_0^1 G(s,s)p(s)u(s-\tau)ds + \int_0^1 G(s,s)p(s) \int_0^s k(t,s)u(t)dtds\Big]
$$

$$
\leq \lambda \varepsilon k_1 \Big[\int_0^\tau G(s,s)p(s)\alpha u(\eta)ds + \int_\tau^1 G(s,s)p(s)u(s-\tau)ds + M_1 \int_0^1 G(s,s)p(s) \int_0^s u(t)dtds\Big]
$$

$$
\leq \lambda \varepsilon k_1 \Big[\alpha \int_0^\tau G(s,s)p(s)ds + \int_0^1 G(s,s)p(s)(1+M_1 s)ds\Big] \| u \| = \frac{\lambda}{\lambda_2} \| u \| \leq \| u \| .
$$

Therefore, by the second part of theorem (1.2.2), T has a fixed point $u \in K \cap (\overline{\Omega}_1 \backslash \Omega_2)$, and $u(t)$ has at least positive solution of (5.0.1). The proof is complete.

Exemple 5.2.1. *Consider the equation*

$$\begin{cases} -u''(t) = \dfrac{1}{t}\sqrt{t^2+1} \times \dfrac{[u^2(t-\frac{1}{6}) + v^2(t)][1 + u(t-\frac{1}{6}) + v(t)]}{2 + u(t-\frac{1}{6}) + v(t)}, & 0 < t < 1, \\ u(t) = \frac{1}{2}u(\frac{1}{2}), & -\tau \leqslant t \leqslant 1 \\ u(1) = \dfrac{1}{2}u(\dfrac{1}{2}), \end{cases} \tag{5.2.8}$$

where

$$f(t,u,v) = \sqrt{t^2+1} \times \frac{[u^2(t-\frac{1}{6}) + v^2(t)][1 + u(t-\frac{1}{6}) + v(t)]}{2 + u(t-\frac{1}{6}) + v(t)},$$

$$v(t) = \int_0^1 (t+s+1)u(s)ds, \qquad \textit{and} \qquad k(t,s) = t+s+1.$$

Then $k_1 = 3$, $p(t) = \dfrac{1}{t}$ *and* $t = 0$ *is its singularity. Here we have* $f_\infty = \infty$, $f^0 = 0$. *If we choose* $M = 100$, $\gamma = \dfrac{1}{4}$, $a = \dfrac{100}{11}$ *then when* $\sqrt{u^2+v^2} \geq 11$ *and* $f(t,u,v) \geq M$, *we can calculate that* $\lambda_1 = 1.76$. *If we choose* $\varepsilon = 0.01$ *calculations show that* $\lambda_2 \simeq 46.451$
From Theorem 5.2.3, we have that if $1.76 < \lambda < 46.451$, *the problem (5.0.1) has at least one positive solution.*

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