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par

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Publications

1. S. Abbas, M. Benchohra, N.Laledj and Y. Zhou, Exictence and Ulam Stability for implicit fractional q-difference equation, *Adv. Differ. Equ.* (2019), 480.

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3. S. Abbas, M. Benchohra, N. Laledj and Y. Zhou , Fractional q-Difference Equations on the Half Line, Archivum Mathematicum Vol.56 N 4 (2020),207-223.

Abstract

This Thesis deals with some existence, uniqueness and Ulam-Hyers-Rassias stability results for a class of implicit fractional q-difference equations. Some applications are made of some fixed point theorems in Banach spaces for the existence and uniqueness of solutions, next we prove that our problem is generalized Ulam-Hyers-Rassias stable. In this thesis we discuss the existence of weak solutions for a class of implicit fractional q-difference equations. The results are based on the fixed point theory and the concept of measure of weak noncompactness. The tools used include a generalization of the classical Darbo fixed point theorem for Fréchet spaces associated with the concept of measure of noncompactness. Also this thesis deals with some results about the existence of solutions and bounded solutions and the attractivity for a class of fractional q-difference equations. Some applications are made of Schauder fixed point theorem in Banach spaces and Darbo fixed point theorem in Fréchet spaces. We use some technics associated with the concept of measure of noncompactness and the diagonalization process. We study in this thesis a class of Caputo fractional q-difference inclusions in Banach spaces. We obtain some existence results by using the set-valued analysis, the measure of noncompactness, and the fixed point theory (Darbo and Mönch's fixed point theorems). in the last of this thesis we study some existence of weak solutions for a class of Caputo fractional q-difference inclusions and a coupled system of Caputo fractional q-difference inclusions by using the set-valued analysis and the Pettis integration.

Key words and phrases :

Fractional q-difference equation, implicit, solution, Ulam-Hyers-Rassias stability, measure of weak noncompactness, weak solution, measure of noncompactness, attractivity, diagonalization, bounded solution, Banach space, Fréchet space, fixed point, Fractional q-difference inclusion, Pettis q-difference inclusion, Caputo fractional derivative, coupled system.

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Contents

Introduction

Fractional differential equations have recently been applied in various areas of engineering, mathematics, physics, and other applied sciences [74]. For some fundamental results in the theory of fractional calculus and fractional differential equations we refer the reader to the monographs [9, 14, 13, 72, 52, 80], the paper [75] and the references therein. Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations and inclusions with Caputo fractional derivative; [14, 20]. Implicit fractional differential equations were analyzed by many authors see for instance [7, 9, 8, 11, 14, 28, 29, 30] and the references therein.

q-difference calculus or quantum calculus was first introduced by F.H Jackson in $1908 - 1910$ [47, 48],in these articls Jackson defined q-differential operator, q-calculus became a bridge between mathematics and physics, basic definitions and properties of q-difference calculus can be found in the book of V.Kac and P.Cheung [51]. Then the theory of q-calculs was extented to fractional q-difference calculus by W.A.Al-Salam [23] who introduced the concept of q-calculus, starting from the q-analogue of Cauchy's formula and Agrawal [16] who studied certain fractional q-integral operators and q-derivatives where he prouved the semi-group properties for Riemann-Liouville type fractional integral operators. Further, Rajkovic et al in [64, 65] generalized the notion of the left fractional q-integral operators and fractional q-derivatives by introducing variable lower limit and proved the semi-group properties.Also F.M.Atici and P.Eloe in [24] studied the fractional q-calculus on a time scale, they have developed some properties of fractional q-calculus of q-Laplace transform that they used to solve fractional q-difference equations. More recently maybe due to the explosion in research within the fractional differential calculus siting,new developments in this theory of fractional q-difference calculus were made,for example,q-analogue of the integral and differential fractional operators propreties such as Mittag Leffler function [64, 65], the q-Taylor's formula and the q-Laplace transform [24]. Due to its various applications in many subject,including quantum mechanics,particle physics and hypergeometric series,many researchers devoted their efforts to develop the theory in this field and many results were made. However,most of the work on the topic is based on Riemann-Liouville and Caputo type fractional differential equations (see Ahmad et al [17, 18, 19], q-analogue of this operators is defined in [23] and applications of this operators in investigated by M.H.Annaby [25]and T.Zhang [79].

During the last few years, the study of solutions for fractional boundary value problems as well as their its various applications in physics and engineering flourished many results were obtained by applying Caputo derivative and standard Riemann-Liouville fractional derivative (see,[35, 78, 80], and references therein), but for fractional q-difference boundary value problems are few.

Considerable attention has been given to the study of the Ulam stability of functional differential and integral equations; one can see the monographs [11, 49], the papers [1, 2, 3, 50, 66, 69, 70] and the references therein.

The measure of noncompactness which is one of the fundamental tools in the theory of nonlinear analysis was initiated by the pioneering articles of Alvàrez [22], Mönch [57] and was developed by Bana's and Goebel [27] and many researchers in the literature. The applications of the measure of noncompactness (for the weak case, the measure of weak noncompactenss developed by De Blasi [37]) can be seen in the wide range of applied mathematics: theory of differential equations (see [16, 59] and references therein). Recently, in [13, 22, 26, 27] the authors applied the measure of noncompactness to some classes of functional Riemann-Liouville or Caputo fractional differential equations in Banach spaces.

In [4, 5, 12], Abbas *et al.* presented some results on the local and global attractivity of solutions for some classes of fractional differential equations involving both the Riemann-Liouville and the Caputo fractional derivatives by employing some fixed point theorems.

Fractional differential equations and inclusions have attracted much more interest of mathematicians and physicists which provides an efficiency for the description of many practical dynamical arising in engineering, vulnerability of networks (fractional percolation on random graphs) and other applied sciences [9, 13, 75, 72, 52, 74, 80]. Recently, Riemann-Liouville and Caputo fractional differential equations with initial and boundary conditions are studied by many authors [14, 52]. In [18, 20, 77] the authors present some interesting results for classes of fractional differential inclusions.

Difference inclusions arise in the mathematical modeling of various problems in economics, optimal control, stochastic analysis, see for instance [45, 53, 73]. However q-difference inclusions are studied in few papers; see for example [18, 58].

Coupled differential and integro-differential equations appear in mathematical modeling of many biological phenomena and environmental issues. For further details on the utility of coupled systems, see [6, 46, 63], and references therein.

In the following we give an outline of our thesis organization, which consists of six chapters defining the contributed work.

Chapter 1: contains notation and preliminary results, definitions, theorems and other auxiliary results which will be needed in this thesis, in the first section we give some notations and definitions, in section 2, we present some properties of the theory of Weak solutions, in the third section, we give some properties of set-valued maps, in section 4, we present some properties of Measures of noncompactness, in the last section we cite some fixed point theorems.

Chapter 2: In chapter 2 we give some results of existence, uniqueness and Ulam-Hyers-Rassias stability results for a class of implicit fractional q-difference equations. Some applications are made of some fixed point theorems in Banach spaces for the existence and uniqueness of solutions, next we prove that our problem is generalized Ulam-Hyers-Rassias stable. Two illustrative examples are given in the last section. we discuss the existence, uniqueness and Ulam-Hyers-Rassias stability of solutions for the following implicit fractional q-difference equation

$$({}^cD_q^\alpha u)(t) = f(t, u(t), ({}^cD_q^\alpha u)(t)),\ t \in I := [0, T], \tag{1}$$

with the initiale condition

$$u(0) = u_0, \tag{2}$$

where $q \in (0, 1),\ \alpha \in (0, 1],\ T > 0,\ f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and ${}^cD_q^\alpha$ is the Caputo fractional q-difference derivative of order α.

Chapter 3: This chapter initiates the study of implicit Caputo fractional q-difference equations,we discuss some rasults about the existence of solutions and weak solutions for a class of implicit fractional q-difference equations. Some applications are made of Darbo fixed point theorem in Banach spaces.

In section 3.2 we discuss some existence results for a class of implicit fractional q-difference equations. The results are based on the fixed point theory in Banach spaces and the concept of measure of noncompactness (Kuratowski measure). An illustrative example is given in the last section.

we discuss the existence of solutions for the following implicit fractional q-difference equation

$$({}^cD_q^\alpha u)(t) = f(t, u(t), ({}^cD_q^\alpha u)(t)),\ t \in I := [0, T], \tag{3}$$

with the initiale condition

$$u(0) = u_0, \tag{4}$$

where $q \in (0, 1),\ \alpha \in (0, 1],\ T > 0,\ f : I \times E \times E \to E$ is a given continuous function, E is a real (or complex) Banach space with norm $\|\cdot\|$, and ${}^cD_q^\alpha$ is the Caputo fractional q-difference derivative of order α.

In Section 3.3 we discuss some results about the existence of weak solutions for a class of implicit fractional q-difference equations. The results are based on the fixed point theory and the concept of measure of weak noncompactness. An illustrative example is given in the last section.

we discuss in this section the existence of weak solutions for the following fractional q-difference equation

$$({}^cD_q^\alpha u)(t) = f(t, u(t), ({}^cD_q^\alpha u)(t)); \ t \in I := [0, T], \tag{5}$$

with the initial condition

$$u(0) = u_0, \tag{6}$$

where $q \in (0,1),\ \alpha \in (0,1],\ T > 0,\ f : I \times E \times E \to E$ is a given continuous function, E is a real (or complex) Banach space with norm $\|\cdot\|$, and ${}^cD_q^\alpha$ is the Caputo fractional q-difference derivative of order α.

Chapter 4: In this chapter we discuss some results about the existence of solutions and bounded solutions and the attractivity for a class of fractional q-difference equations. Some applications are made of Schauder fixed point theorem in Banach spaces and Darbo fixed point theorem in Fréchet spaces. We use some technics associated with the concept of measure of noncompactness and the diagonalization process. In this chapter we presented some results on the local and global attractivity of solutions for some classes of fractional differential equations involving both the Riemann-Liouville and the Caputo fractional derivatives by employing some fixed point theorems.
In this chapter we discuss the existence and the attractivity of solutions for the following functional fractional q-difference equation

$$({}^cD_q^\alpha u)(t) = f(t, u(t)); \ t \in \mathbb{R}_+ := [0, +\infty), \tag{7}$$

with the initiale condition

$$u(0) = u_0, \tag{8}$$

where $q \in (0,1),\ \alpha \in (0,1],\ f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and ${}^cD_q^\alpha$ is the Caputo fractional q-difference derivative of order α.

Next, by using a generalization of the classical Darbo fixed point theorem for Fréchet spaces associated with the concept of measure of noncompactness, we discuss the existence of solutions for the problem (7)-(8) in Fréchet spaces.

Finaly, we discuss the existence of bounded solutions for the problem (7)-(8) on $\mathbb{R}_+$, by applying Schauder's fixed point theorem associated with the diagonalization process.

This chapter initiates the study of Caputo fractional q-difference equations in Fréchet spaces, the attractivity and the boundedness of the solutions of fractional q-difference equations on the half line. Some illustrative examples are given in the last section.

Chapter 5: In this chapter, we study a class of Caputo fractional q-difference inclusions in Banach spaces. We obtain some existence results by using the set-valued analysis, the measure of noncompactness, and the fixed point theory (Darbo and Mönch's fixed point theorems). Finally we give an illustrative example in the last section. We initiate the study of fractional q-difference inclusions on infinite dimensional Banach spaces.

In this chapter we consider the Caputo fractional q-difference inclusion

$$({}^cD_q^\alpha u)(t) \in F(t, u(t)),\ t \in I := [0, T], \tag{9}$$

with the initial condition

$$u(0) = u_0 \in E, \tag{10}$$

where $(E, \|\cdot\|)$ is a real or complex Banach space, $q \in (0,1),\ \alpha \in (0,1],\ T > 0,\ F : I \times E \to \mathcal{P}(E)$ is a multivalued map, $\mathcal{P}(E) = \{Y \subset E : y \neq \emptyset\}$, and ${}^cD_q^\alpha$ is the Caputo fractional q-difference derivative of order α.

Chapter 6: This chapter deals with some existence of weak solutions for a class of Caputo fractional q-difference inclusions and a coupled system of Caputo fractional q-difference inclusions by using the set-valued analysis, and Mönch's fixed point theorem associated with the technique of measure of weak noncompactness. Two illustrative examples are given in the end.
In section 6.2 we discuss the existence of weak solutions for the following fractional q-difference inclusion

$$({}^cD_q^\alpha u)(t) \in F(t, u(t)),\ t \in I := [0, T], \tag{11}$$

with the initial condition

$$u(0) = u_0 \in E, \tag{12}$$

where E is a real (or complex) Banach space with norm $\|\cdot\|$ and dual E^*, such that E is the dual of a weakly compactly generated Banach space X, $q \in (0,1),\ \alpha \in (0,1],\ T > 0,\ F : I \times E \to \mathcal{P}(E)$ is a multivalued map, $\mathcal{P}(E)$ is the family of all nonempty subsets of E, ${}^cD_q^\alpha$ is the Caputo fractional q-difference derivative of order α.

Next in section 6.3 we consider the following coupled system of fractional q-difference inclusions

$$\begin{cases} ({}^cD_q^\alpha u)(t) \in F(t, v(t)) \\ ({}^cD_q^\alpha v)(t) \in G(t, u(t)) \end{cases} ;\ t \in I, \tag{13}$$

with the initial conditions

$$(u(0), v(0)) = (u_0, v_0) \in E \times E, \tag{14}$$

where $F, G : I \times E \to \mathcal{P}(E)$ are multivalued maps.

Finally we close our thesis with a conclusion and some perspectives

Chapter 1

Preliminaries

In this chapter we review some fundamental concepts, notations, definitions, fixed point theorems and properties required to establish our main results.

1.1 Notations and Definitions

Let $I := [0,T]; T > 0$. Consider the complete metric space $C(I) := C(I,\mathbb{R})$ of continuous functions from I into $\mathbb{R}$ equipped with the usual metric

$$d(u,v) := \max_{t\in I} |u(t) - v(t)|.$$

Notice that $C(I)$ is a Banach space with the supremum (uniform) norm

$$\|u\|_{\infty} := \sup_{t\in I} |u(t)|.$$

As usual, $L^1(I)$ denotes the space of measurable functions $v : I \to \mathbb{R}$ which are Lebesgue integrable with the norm

$$\|v\|_1 = \int_I |v(t)|dt.$$

Let us recall some definitions and properties of fractional q-calculus. For $a \in \mathbb{R}$, and $q \in (0,1)$ we set

$$[a]_q = \frac{1-q^a}{1-q}.$$

The q analogue of the power $(a-b)^n$ is

$$(a-b)^{(0)} = 1,\ (a-b)^{(n)} = \Pi_{k=0}^{n-1}(a-bq^k);\ a,b\in\mathbb{R},\ n\in\mathbb{N}.$$

In general,

$$(a-b)^{(\alpha)} = a^{\alpha}\Pi_{k=0}^{\infty}\left(\frac{a-bq^k}{a-bq^{k+\alpha}}\right);\ a,b,\alpha\in\mathbb{R}.$$

Definition 1.1 *[51] The q-gamma function is defined by*

$$\Gamma_q(\xi) = \frac{(1-q)^{(\xi-1)}}{(1-q)^{\xi-1}};\ \xi \in \mathbb{R} - \{0, -1, -2, \dots\}$$

Notice that the q-gamma function satisfies $\Gamma_q(1+\xi) = [\xi]_q \Gamma_q(\xi)$.

Definition 1.2 *[51] The q-derivative of order $n \in \mathbb{N}$ of a function $u : I \to \mathbb{R}$ is defined by*

$$(D_q u)(t) := (D_q^1 u)(t) = \frac{u(t) - u(qt)}{(1-q)t};\ t \neq 0, \quad (D_q u)(0) = \lim_{t\to 0}(D_q u)(t),$$

and

$$(D_q^n u)(t) = (D_q D_q^{n-1} u)(t);\ t \in I,\ n \in \{1, 2, \dots\}.$$

Set $I_t := \{tq^n : n \in \mathbb{N}\} \cup \{0\}$.

Definition 1.3 *[51] The q-integral of a function $u : I_t \to \mathbb{R}$ is defined by*

$$(I_q u)(t) = \int_0^t u(s) d_q s = \sum_{n=0}^{\infty} t(1-q) q^n f(tq^n),$$

provided that the series converges.

We note that $(D_q I_q u)(t) = u(t)$, while if u is continuous at 0, then

$$(I_q D_q u)(t) = u(t) - u(0).$$

Definition 1.4 *[65] The Riemann-Liouville fractional q-integral of order $\alpha \in \mathbb{R}_+ := [0, \infty)$ of a function $u : I \to \mathbb{R}$ is defined by $(I_q^0 u)(t) = u(t)$, and*

$$(I_q^\alpha u)(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} u(s) d_q s;\ t \in I.$$

Lemma 1.5 *[64] For $\alpha \in \mathbb{R}_+ := [0, \infty)$ and $\lambda \in (-1, \infty)$ we have*

$$(I_q^\alpha (t)^{(\lambda)})(t) = \frac{\Gamma_q(1+\lambda)}{\Gamma(1+\lambda+\alpha)} (t)^{(\lambda+\alpha)};\ 0 < t < T.$$

In particular,

$$(I_q^\alpha 1)(t) = \frac{1}{\Gamma_q(1+\alpha)} t^{(\alpha)}.$$

Definition 1.6 *[65] The Riemann-Liouville fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $u : I \to \mathbb{R}$ is defined by $(D_q^0 u)(t) = u(t)$, and*

$$(D_q^\alpha u)(t) = (D_q^{[\alpha]} I_q^{[\alpha]-\alpha} u)(t);\ t \in I,$$

where $[\alpha]$ is the integer part of α.

Definition 1.7 *[65] The Caputo fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $u : I \to \mathbb{R}$ is defined by $({}^C D_q^0 u)(t) = u(t)$, and*

$$({}^C D_q^\alpha u)(t) = (I_q^{[\alpha]-\alpha} D_q^{[\alpha]} u)(t);\ t \in I.$$

Lemma 1.8 *[65] Let $\alpha \in \mathbb{R}_+$. Then the following equality holds:*

$$(I_q^\alpha\ {}^C D_q^\alpha u)(t) = u(t) - \sum_{k=0}^{[\alpha]-1} \frac{t^k}{\Gamma_q(1+k)} (D_q^k u)(0).$$

In particular, if $\alpha \in (0,1)$, then

$$(I_q^\alpha\ {}^C D_q^\alpha u)(t) = u(t) - u(0).$$

Definition 1.9 *[21, 71] Let (M, d) be a metric space. A map $T : M \to M$ is said to be Lipschitzian if there exists a constant $k > 0$ (called Lipschitz constant) such that*

$$d(T(x), T(y)) \leq k d(x, y);\ for\ all\ x, y \in M.$$

A Lipschitzian mapping with a Lipschitz constant $k < 1$ is called a contraction.

Definition 1.10 *([21, 71]) A nondecreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a comparison function if it satisfies one of the following conditions:*

(1) *For any $t > 0$ we have*

$$\lim_{n\to\infty} \phi^{(n)}(t) = 0,$$

where $\phi^{(n)}$ denotes the n-th iteration of ϕ.

(2) *The function ϕ is right-continuous and satisfies*

$$\forall t > 0 : \phi(t) < t.$$

Remark 1.11 *The choice $\phi(t) = kt$ with $0 < k < 1$ gives the classical Banach contraction mapping principle.*

1.2 Theory of Weak Solutions

Consider the Banach space $C_E(I) := C(I, E)$ of continuous functions from I into E equipped with the usual supremum (uniform) norm

$$\|u\|_\infty := \sup_{t\in I} \|u(t)\|.$$

As usual, $L^1(I)$ denotes the space of measurable functions $v : I \to E$ which are Bochner integrable with the norm

$$\|v\|_1 = \int_0^T \|v(t)\| dt.$$

Let $(E, w) = (E, \sigma(E, E^*))$ be the Banach space E with its weak topology.

Definition 1.12 *[59, 61] A Banach space X is called weakly compactly generated (WCG, for short) if it contains a weakly compact set whose linear span is dense in X.*

Definition 1.13 *[59, 61] A function $h : E \to E$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to a weakly convergent sequence in E (i.e., for any (u_n) in E with $u_n \to u$ in (E, w) then $h(u_n) \to h(u)$ in (E, w)).*

Definition 1.14 *[59, 60] A function $F : Q \to \mathcal{P}_{cl,cv}(Q)$ has a weakly sequentially closed graph, if for any sequence $(x_n, y_n) \in Q \times Q, y_n \in F(x_n)$ for $n \in \{1, 2, \ldots\}$, with $x_n \to x$ in (E, ω), and $y_n \to y$ in (E, ω), then $y \in F(x)$.*

Definition 1.15 *[62] The function $u : I \to E$ is said to be Pettis integrable on I if and only if there is an element $u_J \in E$ corresponding to each $J \subset I$ such that $\phi(u_J) = \int_J \phi(u(s))ds$ for all $\phi \in E^*$, where the integral on the right hand side is assumed to exist in the sense of Bochner, (by definition, $u_J = \int_J u(s)ds$).*

Let $P(I, E)$ be the space of all $E-$valued Pettis integrable functions on I, and $L^1(I, E)$ be the Banach space of Lebesgue integrable functions $u : I \to E$. Define the class $P_1(I, E)$ by

$$P_1(I, E) = \{u \in P(I, E) : \varphi(u) \in L^1(I, \mathbb{R});\ for\ every\ \varphi \in E^*\}.$$

The space $P_1(I, E)$ is normed by

$$\|u\|_{P_1} = \sup_{\varphi \in E^*,\ \|\varphi\| \leq 1} \int_0^T |\varphi(u(x))| d\lambda x,$$

where λ stands for a Lebesgue measure on I.

The following result is due to Pettis (see [[62], Theorem 3.4 and Corollary 3.41]).

Proposition 1.16 *[62] If $u \in P_1(I, E)$ and h is a measurable and essentially bounded real-valued function, then $uh \in P_1(I, E)$.*

1.3 Some Properties of Set Valued Maps

We define the following subsets of $\mathcal{P}(E)$:

$P_{cl}(E) = \{Y \in \mathcal{P}(E) : Y \text{ is closed}\}$,

$P_{bd}(E) = \{Y \in \mathcal{P}(E) : Y \text{ is bounded}\}$,

$P_{cp}(E) = \{Y \in \mathcal{P}(E) : Y \text{ is compact}\}$,

$P_{cv}(E) = \{Y \in \mathcal{P}(E) : Y \text{ is convex}\}$,

$P_{cp,cv}(E) = P_{cp}(E) \cap P_{cv}(E)$.

Definition 1.17 *[38, 45] A multivalued map $G : E \to \mathcal{P}(E)$ is said to be convex (closed) valued if $G(x)$ is convex (closed) for all $x \in E$. A multivalued map G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in E for all $B \in P_b(E)$ (i.e. $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}$ exists).*

Definition 1.18 *[38, 45] A multivalued map $G : E \to \mathcal{P}(E)$ is called upper semi-continuous (u.s.c.) on E if $G(x_0) \in P_{cl}(E)$; for each $x_0 \in E$, and for each open set $N \subset E$ with $G(x_0) \in N$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subset N$. G is said to be completely continuous if $G(B)$ is relatively compact for every $B \in P_{bd}(E)$. An element $x \in E$ is a fixed point of G if $x \in G(x)$.*

We denote by $FixG$ the fixed point set of the multivalued operator G.

Lemma 1.19 *[38, 45] Let $G : X \to \mathcal{P}(E)$ be completely continuous with nonempty compact values. Then G is u.s.c. if and only if G has a closed graph, that is,*

$$x_n \to x_*, y_n \to y_*, y_n \in G(x_n) \Longrightarrow y_* \in G(x_*).$$

Definition 1.20 *[38, 45] A multivalued map $G : J \to P_{cl}(E)$ is said to be measurable if for every $y \in E$, the function*

$$t \to d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

Definition 1.21 *[45, 73] A multivalued map $F : I\times\mathbb{R} \to \mathcal{P}(E)$ is said to be Carathéodory if:*

(1) $t \to F(t, u)$ is measurable for each $u \in E$;

(2) $u \to F(t, u)$ is upper semicontinuous for almost all $t \in I$.

F is said to be L^1-Carathéodory if $(1), (2)$ and the following condition holds:

(3) For each $q > 0$, there exists $\varphi_q \in L^1(I, \mathbb{R}_+)$ such that

$$\|F(t, u)\|_{\mathcal{P}} = \sup\{|v| : v \in F(t, u)\} \leq \varphi_q \text{ for all } |u| \leq q \text{ and for a.e. } t \in I.$$

For each $u \in C_E(I)$, define the set of selections of F by

$$S_{F\circ u} = \{v \in L^1(I) : v(t) \in F(t, u(t)) \text{ a.e. } t \in I\}.$$

Let (E, d) be a metric space induced from the normed space $(E, |\cdot|)$. The function $H_d : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathbb{R}_+ \cup \{\infty\}$ given by:

$$H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\}.$$

is known as the Hausdorff-Pompeiu metric. For more details on multivalued maps see the books of Hu and Papageorgiou [45].

Lemma 1.22 *[61] Let F be a Carathéodory multivalued map and $\Theta : L^1(I) \to C_E(I)$; be a linear continuous map. Then the operator*

$$\Theta \circ S_{F\circ u} : C(I) \to \mathcal{P}_{cv,cp}(C(I)), \quad u \mapsto (\Theta \circ S_{F\circ u})(u) = \Theta(S_{F\circ u})$$

is a closed graph operator in $C_E(I) \times C_E(I)$.

Definition 1.23 *[45, 61] Let E be Banach space. A multivalued mapping $T : E \to \mathcal{P}_{cl,b}(E)$ is called $k-$set-Lipschitz if there exists a constant $k > 0$, such that $\mu(T(X)) \leq k\mu(X)$ for all $X \in \mathcal{P}_{cl,b}(E)$ with $T(X) \in \mathcal{P}_{cl,b}(E)$. If $k < 1$, then T is called a $k-$set-contraction on E.*

1.4 Measure of Noncompactness

Now let us recall some fundamental facts of the notion of measure of noncompactness.

1.4.1 Kuratowski measure of noncompactness.

Let $\mathcal{M}_X$ be the class of all bounded subsets of a metric space X.

Definition 1.24 *[27, 76] A function $\mu : \mathcal{M}_X \to [0,\infty)$ is said to be a measure of noncompactness on X if the following conditions are verified for all $B, B_1, B_2 \in \mathcal{M}_X$.*

(a) Regularity, i.e., $\mu(B) = 0$ if and only if B is precompact,

(b) invariance under closure, i.e., $\mu(B) = \mu(\overline{B})$,

(c) semi-additivity, i.e., $\mu(B_1 \cup B_2) = \max\{\mu(B_1), \mu(B_2)\}$.

Definition 1.25 *[27, 76] Let E be a Banach space and denote by Ω_E the family of bounded subsets of E. the map $\mu : \Omega_E \to [0,\infty)$ defined by*

$$\mu(M) = \inf\{\epsilon > 0 : M \subset \cup_{j=1}^m M_j, \operatorname{diam}(M_j) \leq \epsilon\},\ M \in \Omega_E,$$

is called the Kuratowski measure of noncompactness.

Theorem 1.26 *[?, 27] Let E be a Banach space. Let $C \subset L^1(I)$ be a countable set with $|u(t)| \leq h(t)$ for a.e. $t \in J$ and every $u \in C$, where $h \in L^1(I, \mathbb{R}_+)$. Then $\phi(t) = \mu(C(t)) \in L^1(I, \mathbb{R}_+)$ and verifies*

$$\mu\left(\left\{\int_0^T u(s)\,ds : u \in C\right\}\right) \leq 2\int_0^T \mu(C(s))\,ds,$$

where μ is the Kuratowski measure of noncompactness on the set E.

Some Properties.

(1) $\mu(M) = 0 \Leftrightarrow \overline{M}$ is compact (M is relatively compact).

(2) $\mu(M) = \mu(\overline{M})$.

(3) $M_1 \subset M_2 \Rightarrow \mu(M_1) \leq \mu(M_2)$.

(4) $\mu(M_1 + M_2) \leq \mu(M_1) + \mu(M_2)$.

(5) $\mu(cM) = |c|\mu(M),\ c \in \mathbb{R}$.

(6) $\mu(\operatorname{conv} M) = \mu(M)$.

1.4.2 Measure of Weak noncompactness.

Definition 1.27 *[37] Let E be a Banach space, Ω_E the bounded subsets of E and B_1 the unit ball of E. The De Blasi measure of weak noncompactness is the map $\beta : \Omega_E \to [0, \infty)$ defined by*

$$\beta(X) = \inf\{\epsilon > 0 : \textit{there exists a weakly compact } \Omega \subset E \text{ such that } X \subset \epsilon B_1 + \Omega\}.$$

Some Properties.

The De Blasi measure of weak noncompactness satisfies the following properties:

(a) $A \subset B \Rightarrow \beta(A) \leq \beta(B)$,

(b) $\beta(A) = 0 \Leftrightarrow$ A is weakly relatively compact,

(c) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$,

(d) $\beta(\overline{A}^{\omega}) = \beta(A)$, ($\overline{A}^{\omega}$ denotes the weak closure of A),

(e) $\beta(A + B) \leq \beta(A) + \beta(B)$,

(f) $\beta(\lambda A) = |\lambda|\beta(A)$,

(g) $\beta(conv(A)) = \beta(A)$,

(h) $\beta(\cup_{|\lambda| \leq h} \lambda A) = h\beta(A)$.

The next result follows directly from the Hahn-Banach theorem.

Proposition 1.28 *[37] Let E be a normed space, and $x_0 \in E$ with $x_0 \neq 0$. Then, there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$.*

For a given set V of functions $v : I \to E$ let us denote by

$$V(t) = \{v(t) : v \in V\};\ t \in I,$$

and

$$V(I) = \{v(t) : v \in V,\ t \in I\}.$$

Lemma 1.29 *[37] Let $H \subset C$ be a bounded and equicontinuous subset. Then the function $t \to \beta(H(t))$ is continuous on I, and*

$$\beta_C(H) = \max_{t\in I} \beta(H(t)),$$

and

$$\beta\left(\int_I u(s)ds\right) \leq \int_I \beta(H(s))ds,$$

where $H(s) = \{u(s) : u \in H,\ s \in I\}$, and β_C is the De Blasi measure of weak noncompactness defined on the bounded sets of C.

1.5 Some Properties in Fréchet Spaces

Let $X := C(\mathbb{R}_+, E)$ be the Fréchet space of all continuous functions v from $\mathbb{R}_+$ into a Banach space $(E, \|\cdot\|)$, equipped with the family of seminorms

$$\|v\|_n = \sup_{t\in[0,n]} \|v(t)\|;\ n \in \mathbb{N}^*,$$

and the distance

$$d(u,v) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|u-v\|_n}{1+\|u-v\|_n};\ u,v \in X.$$

Definition 1.30 *[40] A nonempty subset $B \subset X$ is said to be bounded if*

$$\sup_{v\in B} \|v\|_n < \infty;\ for\ n \in \mathbb{N}^*.$$

We recall the following definition of the notion of a sequence of measures of noncompactness in Fréchet space[40].

Definition 1.31 *[40] Let $\mathcal{M}_F$ be the family of all nonempty and bounded subsets of a Fréchet space F. A family of functions $\{\mu_n\}_{n\in\mathbb{N}}$ where $\mu_n : \mathcal{M}_F \to [0,\infty)$ is said to be a family of measures of noncompactness in the real Fréchet space F if it satisfies the following conditions for all $B, B_1, B_2 \in \mathcal{M}_F$:*

(a) $\{\mu_n\}_{n\in\mathbb{N}}$ is full, that is: $\mu_n(B) = 0$ for $n \in \mathbb{N}$ if and only if B is precompact,

(b) $\mu_n(B_1) \leq \mu_n(B_2)$ for $B_1 \subset B_2$ and $n \in \mathbb{N}$,

(c) $\mu_n(Conv B) = \mu_n(B)$ *for* $n \in \mathbb{N}$,

(d) If $\{B_i\}_{i=1,\cdots}$ *is a sequence of closed sets from* $\mathcal{M}_F$ *such that* $B_{i+1} \subset B_i$; $i = 1, \cdots$ *and if* $\lim\limits_{i\to\infty} \mu_n(B_i) = 0$, *for each* $n \in \mathbb{N}$, *then the intersection set* $B_\infty := \cap_{i=1}^{\infty} B_i$ *is nonempty.*

Example 1.32 *[26] For* $B \in \mathcal{M}_X$, $x \in B$, $n \in \mathbb{N}$ *and* $\epsilon > 0$, *let us denote by* $\omega^n(x,\epsilon)$ *the modulus of continuity of the function* x *on the interval* $[0,n]$; *that is,*

$$\omega^n(x,\epsilon) = \sup\{\|x(t) - x(s)\| : t, s \in [0,n], |t - s| \le \epsilon\}.$$

Further, let us put

$$\omega^n(B,\epsilon) = \sup\{\omega^n(x,\epsilon) : x \in B\},$$

$$\omega_0^n(B) = \lim_{\epsilon\to 0^+} \omega^n(B,\epsilon),$$

and

$$\mu_n(B) = \omega_0^n(B) + \sup_{t\in[0,n]} \mu(B(t)),$$

where μ *is a measure of noncompactness on the space* E.

The family of mappings $\{\mu_n\}_{n\in\mathbb{N}}$ *where* $\mu_n : \mathcal{M}_X \to [0,\infty)$, *satisfies the conditions (a)-(d) from Definition 1.31.*

Lemma 1.33 *[40] If* Y *is a bounded subset of a Fréchet space* F, *then for each* $\epsilon > 0$, *there is a sequence* $\{y_k\}_{k=1}^{\infty} \subset Y$ *such that*

$$\mu_n(Y) \le 2\mu_n(\{y_k\}_{k=1}^{\infty}) + \epsilon; \ \ for \ n \in \mathbb{N}.$$

Lemma 1.34 *[26, 76] If* $\{u_k\}_{k=0}^{\infty} \subset L^1([0,n])$ *is uniformly integrable, then* $\mu_n(\{u_k\}_{k=1}^{\infty})$ *is measurable for* $n \in \mathbb{N}^*$, *and*

$$\mu_n\left(\left\{\int_0^t u_k(s)ds\right\}_{k=1}^{\infty}\right) \le 2\int_0^t \mu_n(\{u_k(s)\}_{k=1}^{\infty})ds,$$

for each $t \in [0,n]$.

Definition 1.35 *[26, 76] Let* Ω *be a nonempty subset of a Fréchet space* F, *and let* $A : \Omega \to F$ *be a continuous operator which transforms bounded subsets of onto bounded ones. One says that* A *satisfies the Darbo condition with constants* $(k_n)_{n\in\mathbb{N}}$ *with respect to a family of measures of noncompactness* $\{\mu_n\}_{n\in\mathbb{N}}$, *if*

$$\mu_n(A(B)) \le k_n \mu_n(B)$$

for each bounded set $B \subset \Omega$ *and* $n \in \mathbb{N}$.
If $k_n < 1$; $n \in \mathbb{N}$ *then* A *is called a contraction with respect to* $\{\mu_n\}_{n\in\mathbb{N}}$.

1.6 Some Fixed Point Theorems

Theorem 1.36 *(Schauder fixed point theorem)[71, 44] Let X be a Banach space, D be a bounded closed convex subset of X and $T : D \to D$ be a compact and continuous map. Then T has at least one fixed point in D.*

Theorem 1.37 *(Mönch , [71, 57]) Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv} N(V) \text{ or } V = N(V) \cup \{0\} \Rightarrow \overline{V} \text{ is compact}, \tag{1.1}$$

holds for every subset V of D, then N has a fixed point.

Theorem 1.38 *[59, 57] Let Q be a nonempty, closed, convex and equicontinuous subset of a metrizable locally convex vector space $C(I, E)$ such that $0 \in Q$. Suppose $T : Q \to Q$ is weakly-sequentially continuous. If the implication*

$$\overline{V} = \overline{conv}(\{0\} \cup T(V)) \Rightarrow V \quad \text{is relatively weakly compact}, \tag{1.2}$$

holds for every subset $V \subset Q$, then the operator T has a fixed point.

Theorem 1.39 *[59, 57] Let $X = C(I, E)$ be a Banach space with Q a nonempty, bounded, closed, convex and equicontinuous subset of a metrizable locally convex vector space C such that $0 \in Q$. Suppose $T : Q \to \mathcal{P}_{cl,cv}(Q)$ has weakly sequentially closed graph. If the implication*

$$\overline{V} \subset \overline{conv}(\{0\} \cup T(V)) \Rightarrow V \quad \text{is relatively weakly compact}, \tag{1.3}$$

holds for every subset $V \subset Q$, then the operator T has a fixed point.

Theorem 1.40 *[71] Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that*

$$d(T(x), T(y)) \leq \phi(d(x, y)),$$

where ϕ is a comparison function. Then T has a unique fixed point in X.

Now, we recall the set-valued versions of the Darbo and Mönch fixed point theorems.

Theorem 1.41 *(Darbo fixed point theorem) [39, 57] Let X be a bounded, closed and convex subset of a Banach space E and let $T : X \to \mathcal{P}_{cl,b}(X)$ be a closed and $k-$set-contraction. Then T has a fixed point.*

Theorem 1.42 *(Mönch fixed point theorem) [39, 40] Let E be Banach space and $K \subset E$ be a closed and convex set. Also, let U be a relatively open subset of K and $N : \overline{U} \to \mathcal{P}_{cv}(K)$. Suppose that N maps compact sets into relatively compact sets,* graph(N) *is closed and for some $x_0 \in U$, we have*

$$\text{conv}(x_0 \cup N(M)) \supset M \subset \overline{U} \text{ and } \overline{M} = \overline{U} \ (C \subset M \text{ countable}) \text{ imply } \overline{M} \text{ is compact} \quad (1.4)$$

and

$$x \notin (1-\lambda)x_0 + \lambda N(x) \quad \forall x \in \overline{U} \backslash U, \ \lambda \in (0,1). \quad (1.5)$$

Then there exists $x \in \overline{U}$ with $x \in N(x)$.

Theorem 1.43 *[39, 40] Let Ω be a nonempty, bounded, closed, and convex subset of a Fréchet space F and let $V : \Omega \to \Omega$ be a continuous mapping. Suppose that V is a contraction with respect to a family of measures of noncompactness $\{\mu_n\}_{n \in \mathbb{N}}$. Then V has at least one fixed point in the set Ω.*

Chapter 2

Existence and Ulam Stability for Implicit Fractional q-Difference Equations (1)

[1] [10] S. Abbas, M. Benchohra, N.Laledj and Y. Zhou, Exictence and Ulam Stability for implicit fractional q-difference equations, *Adv. Differ. Equ.* (2019), 480.

2.1 Introduction

In this chapter we discuss the existence, uniqueness and Ulam-Hyers-Rassias stability of solutions for the following implicit fractional q-difference equations.

$$({}^cD_q^\alpha u)(t) = f(t, u(t), ({}^cD_q^\alpha u)(t)),\ t \in I := [0, T], \tag{2.1}$$

with the initiale condition

$$u(0) = u_0, \tag{2.2}$$

where $q \in (0, 1)$, $\alpha \in (0, 1]$, $T > 0$, $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, and ${}^cD_q^\alpha$ is the Caputo fractional q-difference derivative of order α. This chapter deals with some existence, uniqueness and Ulam-Hyers-Rassias stability results for a class of implicit fractional q-difference equations. Some applications are made of some fixed point theorems in Banach spaces for the existence and uniqueness of solutions, next we prove that our problem is generalized Ulam-Hyers-Rassias stable. Two illustrative examples are given in the last section.

2.2 Existence Results

In this section, we are concerned with the existence and uniqueness of solutions of the problem (2.1)-(2.2).

[1](1)

Definition 2.1 *By a solution of the problem (2.1)-(2.2) we mean a continuous function $u \in C(I)$ with ${}^cD_q^\alpha u$ is continuous that satisfies the equation (2.1) on I and the initiale condition (2.2).*

From lemma 1.8, and in order to define the solution for the problem (2.1)-(2.2), we conclude the following lemma.

Lemma 2.2 *Let $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $f(\cdot, u, v) \in C(I)$, for each $u, v \in \mathbb{R}$. Then the problem (2.1)-(2.2) is equivalent to the problem of obtaining the solutions of the integral equation*

$$g(t) = f(t, u_0 + (I_q^\alpha g)(t), g(t)),$$

and if $g(\cdot) \in C(I)$, is the solution of this equation, then

$$u(t) = u_0 + (I_q^\alpha g)(t).$$

Proof: Let u be a solution of problem (2.1)-(2.2), and let $g(t) = ({}^cD_q^\alpha u)(t)$,for $t \in I$ we will prove that:$u(t) = u_0 + (I_q^\alpha g)(t)$ and satisfies the equation:

$$g(t) = f(t, u_0 + (I_q^\alpha g)(t), g(t)),$$

From lemma 1.8, we have:

$$u(t) = u_0 + (I_q^\alpha g)(t)$$

and it is easy to see that equation (2.1)implies $g(t) = f(t, u_0 + (I_q^\alpha g)(t), g(t))$, reciprocally, if u satisfies the integral equation $u(t) = u_0 + (I_q^\alpha g)(t)$,and if g satisfies equation $g(t) = f(t, u_0 + (I_q^\alpha g)(t), g(t))$,then u is a solution of the problem (2.1)-(2.2).

The following hypotheses will be used in the sequel.

(H_1) The function f satisfies the generalized Lipschitz condition:

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq \phi_1(|u_1 - u_2|) + \phi_2(|v_1 - v_2|),$$

for $t \in I$ and $u_1, u_2, v_1, v_2 \in \mathbb{R}$, where ϕ_1 and ϕ_2 are comparison functions.

(H_2) There exist functions $p, d, r \in C(I, [0, \infty))$ with $r(t) < 1$ such that

$$|f(t, u, v)| \leq p(t) + d(t)|u| + r(t)|v|, \text{ for each } t \in I \text{ and } u, v \in \mathbb{R}.$$

Set

$$p^* = \sup_{t \in I} p(t), \ d^* = \sup_{t \in I} d(t), \ r^* = \sup_{t \in I} r(t).$$

First, we prove an existence and uniqueness result for the problem (2.1)-(2.2).

Theorem 2.3 *Assume that the hypothesis (H_1) holds. Then there exist a unique solution of problem (2.1)-(2.2) on I.*

Proof. By using of lemma 2.2 , we transform the problem (2.1)- (2.2) into a fixed point problem. Consider the operator $N : C(I) \to C(I)$ defined by

$$(Nu)(t) = u_0 + (I_q^\alpha g)(t);\ t \in I, \tag{2.3}$$

where $g \in C(I)$ such that

$$g(t) = f(t, u(t), g(t)),\ or\ g(t) = f(t, u_0 + (I_q^\alpha g)(t), g(t)).$$

Let $u, v \in C(I)$. Then, for $t \in I$, we have

$$|(Nu)(t) - (Nv)(t)| \le \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |g(s) - h(s)| d_q s, \tag{2.4}$$

where $g, h \in C(I)$ such that

$$g(t) = f(t, u(t), g(t)),$$

and

$$h(t) = f(t, v(t), h(t)).$$

From (H_1), we obtain

$$|g(t) - h(t)| \le \phi_1(|u(t) - v(t|) + \phi_2(|g(t) - h(t|).$$

Thus

$$|g(t) - h(t)| \le (Id - \phi_2)^{-1} \phi_1(|u(t) - v(t|),$$

where Id is the identity function.
Set

$$L := \sup_{t \in I} \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s = \frac{T^\alpha}{\Gamma_q(\alpha)},$$

and $\phi := L(Id - \phi_2)^{-1}\phi_1$. From (2.4), we get

$$\begin{array}{rcl} |(Nu)(t) - (Nv)(t)| & \le & \phi(|u(t) - v(t)|) \\ & \le & \phi(d(u, v)). \end{array}$$

Hence, we get

$$d(N(u), N(v)) \le \phi(d(u, v)).$$

Consequently, from Theorem 1.40(Banach contraction), the operator N has a unique fixed point, which is the unique solution of the problem (2.1)-(2.2).

Theorem 2.4 *Assume that the hypothesis* (H_2) *holds. If*

$$r^* + Ld^* < 1,$$

then the problem (2.1)-(2.2) has at least one solution defined on I*.*

Proof. Let N be the operator defined in (2.3). Set

$$R \geq \frac{Lp^*}{1 - r^* - Ld^*},$$

and consider the closed and convex ball $B_R = \{u \in C(I) : \|u\|_\infty \leq R\}$.
Let $u \in B_R$. Then, for each $t \in I$, we have

$$|(Nu)(t)| \leq \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |g(s)| d_q s,$$

where $g \in C(I)$ such that

$$g(t) = f(t, u(t), g(t)).$$

By using (H_2), for each $t \in I$ we have

$$\begin{aligned} |g(t)| &\leq p(t) + d(t)|u(t)| + r(t)|g(t)| \\ &\leq p^* + d^* \|u\|_\infty + r^* |g(t)| \\ &\leq p^* + d^* R + r^* |g(t)|. \end{aligned}$$

Thus

$$|g(t)| \leq \frac{p^* + d^* R}{1 - r^*}.$$

Hence

$$\|N(u)\|_\infty \leq \frac{L(p^* + d^* R)}{1 - r^*},$$

which implies that

$$\|N(u)\|_\infty \leq R.$$

This proves that N maps the ball B_R into B_R. We shall show that the operator $N : B_R \to B_R$ is continuous and compact. The proof will be given in three steps.

Step1: N is continuous.
Let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence such that $u_n \to u$ in B_R. Then, for each $t \in I$, we have

$$|(Nu_n)(t) - (Nu)(t)| \leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |(g_n(s) - g(s))| d_q s$$

where $g_n, g \in C(I)$ such that

$$g_n(t) = f(t, u_n(t), g_n(t)),$$

and

$$g(t) = f(t, u(t), g(t)).$$

Since $u_n \to u$ as $n \to \infty$ and f is continuous function, we get

$$g_n(t) \to g(t) \text{ as } n \to \infty, \text{ for each } t \in I.$$

Hence

$$\|N(u_n) - N(u)\|_\infty \leq \frac{p^* + d^*R}{1 - r^*}\|g_n - g\|_\infty \to 0 \ \ as \ \ n \to \infty.$$

Step2: $N(B_R)$ is bounded. This is clear since $N(B_R) \subset B_R$ and B_R is bounded.

Step3: N maps bounded sets into equicontinuous sets in B_R.
Let $t_1, t_2 \in I$, such that $t_1 < t_2$ and let $u \in B_R$. Then, we have

$$\begin{array}{rcl} |(Nu)(t_1) - (Nu)(t_2)| & \leq & \int_0^{t_1} \frac{|(t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)}|g(s)|d_qs \\ & + & \int_{t_1}^{t_2} \frac{|(t_2-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)}|g(s)|d_qs. \end{array}.$$

where $g \in C(I)$ such that $g(t) = f(t, u(t), g(t))$. Hence

$$\begin{array}{rcl} |(Nu)(t_1) - (Nu)(t_2)| & \leq & \frac{p^*+d^*R}{1-r^*} \int_0^{t_1} \frac{|(t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)}d_qs \\ & + & \frac{p^*+d^*R}{1-r^*} \int_{t_1}^{t_2} \frac{|(t_2-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)}d_qs. \end{array}.$$

As $t_1 \to t_2$ and since g is continuous, the right-hand side of the above inequality tends to zero.
As a consequence of the above three steps with the Arzelá-Ascoli theorem, we can conclude that $N : B_R \to B_R$ is continuous and compact.
From an application of Theorem 1.36(Schauder's théorèm), we deduce that N has at least a fixed point which is a solution of problem (2.1)-(2.2).

2.3 Ulam Stability Results

In this section, we are concerned with the generalized Ulam-Hyers-Rassias stability results of the problem (2.1)-(2.2).

Now, we consider the Ulam stability for the problem (2.1)-(2.2). Let $\epsilon > 0$ and $\Phi : I \to \mathbb{R}_+$ be a continuous function. We consider the following inequalities

$$|({}^cD_q^\alpha u)(t) - f(t, u(t), ({}^cD_q^\alpha u)(t))| \leq \epsilon; \ t \in I. \tag{2.5}$$

$$|({}^cD_q^\alpha u)(t) - f(t, u(t), ({}^cD_q^\alpha u)(t))| \leq \Phi(t); \ t \in I. \tag{2.6}$$

$$|({}^cD_q^\alpha u)(t) - f(t, u(t), ({}^cD_q^\alpha u)(t))| \leq \epsilon\Phi(t); \ t \in I. \tag{2.7}$$

Definition 2.5 [69, 70] *The problem (2.1)-(2.2) is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that for each $\epsilon >$ and for each solution $u \in C(I)$ of the inequality (2.5) there exists a solution $v \in C(I)$ of (2.1)-(2.2) with*

$$|u(t) - v(t)| \leq \epsilon c_f; \ t \in I.$$

Definition 2.6 [69, 70] *The problem (2.1)-(2.2)is generalized Ulam-Hyers stable if there exists* $c_f : C(\mathbb{R}_+, \mathbb{R}_+)$ *with* $c_f(0) = 0$ *such that for each* $\epsilon > 0$ *and for each solution* $u \in C(I)$ *of the inequality (2.5) there exists a solution* $v \in C(I)$ *of (2.1)-(2.2) with*

$$|u(t) - v(t)| \leq c_f(\epsilon);\ t \in I.$$

Definition 2.7 [69, 70] *The problem (2.1)-(2.2) is Ulam-Hyers-Rassias stable with respect to* Φ *if there exists a real number* $c_{f,\Phi} > 0$ *such that for each* $\epsilon > 0$ *and for each solution* $u \in C(I)$ *of the inequality (2.7) there exists a solution* $v \in C(I)$ *of (2.1)-(2.2) with*

$$|u(t) - v(t)| \leq \epsilon c_{f,\Phi}\Phi(t);\ t \in I.$$

Definition 2.8 [69, 70] *The problem (2.1) is generalized Ulam-Hyers-Rassias stable with respect to* Φ *if there exists a real number* $c_{f,\Phi} > 0$ *such that for each solution* $u \in C(I)$ *of the inequality (2.6) there exists a solution* $v \in C(I)$ *of (2.1)-(2.2) with*

$$|u(t) - v(t)| \leq c_{f,\Phi}\Phi(t);\ t \in I.$$

Remark 2.9 *It is clear that*

(i) Definition (2.5) $\Rightarrow$ *Definition (2.6),*

(ii) Definition (2.7) $\Rightarrow$ *Definition (2.8),*

(iii) Definition (2.7) for $\Phi(\cdot) = 1 \ \Rightarrow$ *Definition (2.5).*

One can have similar remarks for the inequalities (2.5) and (2.7).

The following hypotheses will be used in the sequel.

(H_3) There exist functions $p_1, p_2, p_3 \in C(I, [0, \infty))$ with $p_3(t) < 1$ such that

$$(1+|u|+|v|)|f(t,u,v)| \leq p_1(t)\Phi(t)+p_2(t)\Phi(t)|u|+p_3(t)|v|, \text{ for each } t \in I \text{ and } u, v \in \mathbb{R},$$

(H_4) There exists $\lambda_\Phi > 0$ such that for each $t \in I$, we have

$$(I_q^\alpha \Phi)(t) \leq \lambda_\Phi \Phi(t).$$

Set $\Phi^* = \sup_{t\in I} \Phi(t)$ and

$$p_i^* = \sup_{t\in I} p_i(t),\ i \in \{1, 2, 3\}.$$

Theorem 2.10 *Assume that the hypotheses* (H_3) *and* (H_4) *hold. If*

$$p_3^* + Lp_2^*\Phi^* < 1,$$

then the problem (2.1)-(2.2) has at least one solution and it is generalized Ulam-Hyers-Rassias stable.

Proof. Consider the operator N defined in (2.3). We can see that Hypothesis (H_3) implies (H_2) with $p \equiv p_1\Phi,\ d \equiv p_2\Phi$ and $r \equiv p_3$.
Let u be a solution of the inequality (2.6), and let us assume that v is a solution of problem (2.1)-(2.2). Thus, we have

$$v(t) = u_0 + (I_q^\alpha h)(t),$$

where $h \in C(I)$ such that $h(t) = f(t, v(t), h(t))$.
From the inequality (2.6) for each $t \in I$, we have

$$|u(t) - u_0 - (I_q^\alpha g)(t)| \leq (I_q^\alpha \Phi)(t),$$

where $g \in C(I)$ such that $g(t) = f(t, u(t), g(t))$.
From the hypotheses (H_3) and (H_4), for each $t \in I$, we get

$$\begin{aligned}
|u(t) - v(t)| &\leq |u(t) - u_0 - (I_q^\alpha g)(t) + (I_q^\alpha (g-h))(t)| \\
&\leq (I_q^\alpha \Phi)(t) + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}(|(g(s)| + |h(s))|)d_q s \\
&\leq (I_q^\alpha \Phi)(t) + \frac{p_1^* + p_2^*}{1 - p_3^*}(I_q^\alpha \Phi)(t) \\
&\leq \lambda_\phi \Phi(t) + 2\lambda_\phi \frac{p_1^* + p_2^*}{1 - p_3^*}\Phi(t) \\
&\leq \left[1 + 2\frac{p_1^* + p_2^*}{1 - p_3^*}\right]\lambda_\phi \Phi(t) \\
&:= c_{f,\Phi}\Phi(t).
\end{aligned}$$

Hence, the problem (2.1)-(2.2) is generalized Ulam-Hyers-Rassias stable.

2.4 Examples

Example 1.
Consider the following problem of implicit fractional $\frac{1}{4}-$difference equations

$$\begin{cases} ({}^cD_{\frac{1}{4}}^{\frac{1}{2}}u)(t) = f(t, u(t), ({}^cD_{\frac{1}{4}}^{\frac{1}{2}}u)(t));\ t \in [0,1], \\ u(0) = 1, \end{cases} \tag{2.8}$$

where

$$f(t, u(t), ({}^cD_{\frac{1}{4}}^{\frac{1}{2}}u)(t)) = \frac{t^2}{1 + |u(t)| + |{}^cD_{\frac{1}{4}}^{\frac{1}{2}}u(t)|}\left(e^{-7} + \frac{1}{e^{t+5}}\right)u(t); \quad t \in [0,1].$$

The hypothesis (H_1) is satisfied with

$$\phi_1(t) = \phi_2(t) = t^2\left(e^{-7} + \frac{1}{e^{t+5}}\right)t.$$

Hence, Theorem 2.3 implies that our problem (2.8) has a unique solution defined on $[0,1]$.

Example 2.
Consider now the following problem of implicit fractional $\frac{1}{4}-$difference equations

$$\begin{cases} ({}^cD_{\frac{1}{4}}^{\frac{1}{2}}u)(t) = f(t,u(t),({}^cD_{\frac{1}{4}}^{\frac{1}{2}}u)(t));\ t\in[0,1], \\ u(0)=2, \end{cases} \tag{2.9}$$

where

$$\begin{cases} f(t,x,y) = \frac{t^2}{1+|x|+|y|}\left(e^{-7}+\frac{1}{e^{t+5}}\right)(t^2+xt^2+y);\ t\in(0,1], \\ f(0,x,y)=0. \end{cases}$$

The hypothesis (H_3) is satisfied with $\Phi(t)=t^2$ and $p_i(t)=\left(e^{-7}+\frac{1}{e^{t+5}}\right)t;\ i\in\{1,2,3\}$. Hence, Theorem 2.4 implies that our problem (2.9) has at least a solution defined on $[0,1]$. Also, the hypothesis (H_4) is satisfied. Indeed, for each $t\in(0,1]$, there exists a real number $0<\epsilon<1$ such that $\epsilon<t\le 1$, and

$$\begin{aligned} (I_q^\alpha\Phi)(t) &\le \frac{t^2}{\epsilon^2(1+q+q^2)} \\ &\le \frac{1}{\epsilon^2}\Phi(t) \\ &= \lambda_\Phi\Phi(t). \end{aligned}$$

Consequently, Theorem 2.10 implies that the problem (2.9) is generalized Ulam-Hyers-Rassias stable.

Chapter 3

Implicit Fractional q-Difference Equations in Banach Spaces (2)

[1] [11] S. Abbas, M. Benchohra, J. Henderson and N. Laledj, Existence Theory for Implicite Fractional q-Difference Equations in Banach Spaces. *Studia Universitatis Babes-Bolyai Mathematica.* (to appear)

3.1 Introduction

In this chapter, we establish, in Section 3.2 some existence results for a class of implicit fractional q-difference equations. The results are based on the fixed point theory in Banach spaces and the concept of measure of noncompactness to some classes of functional Riemann-Liouville or Caputo fractional differential equations in Banach spaces. An illustrative example is given in the last section.

we discuss the existence of solutions for the following implicit fractional q-difference equation:

$$({}^cD_q^\alpha u)(t) = f(t, u(t), ({}^cD_q^\alpha u)(t)),\ t \in I := [0, T], \tag{3.1}$$

with the initiale condition

$$u(0) = u_0, \tag{3.2}$$

where $q \in (0,1),\ \alpha \in (0,1],\ T > 0,\ f : I \times E \times E \to E$ is a given continuous function, E is a real (or complex) Banach space with norm $\|\cdot\|$, and ${}^cD_q^\alpha$ is the Caputo fractional q-difference derivative of order α.

In section 3.3 we discuss the existence of weak solutions for a class of implicit fractional q-difference equations. The results are based on the fixed point theory and the concept of measure of weak noncompactness. An illustrative example is given in the last this

[1](2)

section. we discuss the existence of weak solutions for the following fractional q-difference equation:

$$({}^cD_q^\alpha u)(t) = f(t, u(t), ({}^cD_q^\alpha u)(t)); \ t \in I := [0, T], \tag{3.3}$$

with the initial condition

$$u(0) = u_0, \tag{3.4}$$

where $q \in (0,1)$, $\alpha \in (0,1]$, $T > 0$, $f : I \times E \times E \to E$ is a given continuous function, E is a real (or complex) Banach space with norm $\|\cdot\|$, and ${}^cD_q^\alpha$ is the Caputo fractional q-difference derivative of order α.

3.2 Existence Results for Implicit Fractional q-Difference Equations in Banach Spaces

3.2.1 Main Results

In this section, we are concerned with the existence results of the problem(3.1)-(3.2).

Definition 3.1 *By a solution of the problem (3.1)-(3.2) we mean a continuous function u that satisfies the equation (3.1) on I and the initial condition (3.2).*

From lemma 1.8, and in order to define the solution for the problem (3.1)-(3.2), we conclude the following lemma.

Lemma 3.2 *Let $f : I \times E \times E \to E$ such that $f(\cdot, u, v) \in C(I)$, for each $u, v \in E$. Then the problem (3.1)-(3.2) is equivalent to the problem of obtaining the solutions of the integral equation*

$$g(t) = f(t, u_0 + (I_q^\alpha g)(t), g(t)),$$

and if $g(\cdot) \in C(I)$, is the solution of this equation, then

$$u(t) = u_0 + (I_q^\alpha g)(t).$$

The following hypotheses will be used in the sequel.

(H_1) The function $f : I \times E \times E \to E$ is continuous.

(H_2) There exists a continuous function $p \in C(I, \mathbb{R}_+)$, such that

$$\|f(t, u, v)\| \le p(t); \ for \ t \in I, \ and \ u, v \in E,$$

(H_3) For each bounded and measurable set $B \subset E$ and for each $t \in I$, we have

$$\mu(f(t, B, {}^C D_q^\alpha B)) \le p(t)\mu(B),$$

where ${}^C D_q^\alpha B = \{{}^C D_q^\alpha w : w \in B\}$, and μ is a measure of noncompactness on E.

Set

$$p^* = \sup_{t\in I} p(t), \; and \; L := \sup_{t\in I} \int_0^T \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s.$$

Theorem 3.3 *Assume that the hypotheses $(H_1)-(H_3)$ hold. If*

$$\ell := Lp^* < 1, \tag{3.5}$$

then the problem (3.1)-(3.2) has at least one solution defined on I.

Proof. By using of lemma 3.2, we transform the problem (3.1)- (3.2) into a fixed point problem. Consider the operator $N : C_E(I) \to C_E(I)$ defined by

$$(Nu)(t) = u_0 + (I_q^\alpha g)(t); \; t \in I, \tag{3.6}$$

where $g \in C_E(I)$ such that

$$g(t) = f(t, u(t), g(t)), \; or \; g(t) = f(t, u_0 + (I_q^\alpha g)(t), g(t)).$$

For any $u \in C_E(I)$ and each $t \in I$, we have

$$\begin{aligned} \|(Nu)(t)\| &\leq \|u_0\| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |g(s)| d_q s \\ &\leq \|u_0\| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) d_q s \\ &\leq \|u_0\| + p^* \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s \\ &\leq \|u_0\| + Lp^* \\ &:= R. \end{aligned}$$

Thus

$$\|N(u)\|_\infty \leq R. \tag{3.7}$$

This proves that N transforms the ball $B_R := B(0, R) = \{w \in C : \|w\|_\infty \leq R\}$ into itself. We shall show that the operator $N : B_R \to B_R$ satisfies all the assumptions of Monch Theorem 1.37. The proof will be given in three steps.

Step 1. $N : B_R \to B_R$ *is continuous.*
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_R. Then, for each $t \in I$, we have

$$\|(Nu_n)(t) - (Nu)(t)\| \; \leq \; \int_0^t \tfrac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \|(g_n(s) - g(s))\| d_q s,$$

where $g_n, g \in C(I)$ such that

$$g_n(t) = f(t, u_n(t), g_n(t)),$$

and

$$g(t) = f(t, u(t), g(t)).$$

Since $u_n \to u$ as $n \to \infty$ and f is continuous, we get

$$g_n(t) \to g(t) \text{ as } n \to \infty, \text{ for each } t \in I.$$

Hence

$$\|N(u_n) - N(u)\|_\infty \leq L\|g_n - g\|_\infty \to 0 \;\; as \;\; n \to \infty.$$

Step 2. *$N(B_R)$ is bounded and equicontinuous.*
Since $N(B_R) \subset B_R$ and B_R is bounded, then $N(B_R)$ is bounded.
Next, let $t_1, t_2 \in I$, $t_1 < t_2$ and let $u \in B_R$. Thus, we have

$$\|(Nu)(t_2) - (Nu)(t_1)\| \leq \left\| \int_0^{t_2} \frac{(t_2 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s) d_q s - \int_0^{t_1} \frac{(t_1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s) d_q s \right\|.$$

where $g \in C_E(I)$ such that

$$g(t) = f(t, u(t), g(t)).$$

Hence, we get

$$\begin{aligned}
\|(Nu)(t_2) - (Nu)(t_1)\| &\leq \int_{t_1}^{t_2} \frac{(t_2 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) d_q s \\
&+ \int_0^{t_1} \left| \frac{(t_2 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} - \frac{(t_1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \right| d_q s \\
&\leq p^* \int_{t_1}^{t_2} \frac{(t_2 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) d_q s \\
&+ p^* \int_0^{t_1} \left| \frac{(t_2 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} - \frac{(t_1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \right| d_q s.
\end{aligned}$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero.

Step 3. *The implication (1.1) holds.*
Now let V be a subset of B_R such that $V \subset \overline{N(V)} \cup \{0\}$. V is bounded and equicontinuous and therefore the function $t \to v(t) = \mu(V(t))$ is continuous on I. By (H_3) and the properties of the measure μ, for each $t \in I$, we have

$$\begin{aligned}
v(t) &\leq \mu((NV)(t) \cup \{0\}) \\
&\leq \mu((NV)(t)) \\
&\leq \int_0^t \frac{(tq - s)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) \mu(V(s)) d_q s \\
&\leq \int_0^t \frac{(tq - s)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) v(s) d_q s \\
&\leq Lp^* \|v\|_\infty.
\end{aligned}$$

Thus
$$\|v\|_\infty \leq \ell\|v\|_\infty.$$
From (3.5), we get $\|v\|_\infty = 0$, that is, $v(t) = \mu(V(t)) = 0$, for each $t \in I$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzelà theorem, V is relatively compact in B_R. Applying now Theorem 1.37, we conclude that N has a fixed point which is a solution of the problem (3.1)-(3.2).

3.2.2 An Example

Let
$$l^1 = \left\{ u = (u_1, u_2, \dots, u_n, \dots) : \sum_{n=1}^{\infty} |u_n| < \infty \right\}$$
be the Banach space with the norm
$$\|u\|_{l^1} = \sum_{n=1}^{\infty} |u_n|.$$
Consider the following problem of implicit fractional $\frac{1}{4}-$ difference equations
$$\begin{cases} ({}^cD_{\frac{1}{4}}^{\frac{1}{2}} u_n)(t) = f_n(t, u(t), ({}^cD_{\frac{1}{4}}^{\frac{1}{2}} u)(t)); \ t \in [0,1], \\ u(0) = (0, 0, \dots, 0, \dots), \end{cases} \tag{3.8}$$
where
$$\begin{cases} f_n(t,u,v) = \dfrac{t^{\frac{-1}{4}}(2^{-n} + u_n(t))\sin t}{64L(1 + \|u\|_{l^1} + \sqrt{t})(1 + \|u\|_{l^1} + \|v\|_{l^1})}, \ t \in (0,1], \\ \qquad f_n(0,u,v) = 0, \end{cases} .$$
with
$$f = (f_1, f_2, \dots, f_n, \dots), \ \text{and } u = (u_1, u_2, \dots, u_n, \dots).$$
For each $t \in (0,1]$, we have
$$\begin{aligned} \|f(t,u(t))\|_{l^1} &= \sum_{n=1}^{\infty} |f_n(s, u_n(s))| \\ &\leq \frac{t^{\frac{-1}{4}}|\sin t|}{64L(1 + \|u\|_{l^1} + \sqrt{t})(1 + \|u\|_{l^1} + \|v\|_{l^1})}(1 + \|u\|_{l^1}) \\ &\leq \frac{t^{\frac{-1}{4}}}{64L}. \end{aligned}$$
Thus, the hypothesis (H_2) is satisfied with
$$\begin{cases} p(t) = \dfrac{t^{\frac{-1}{4}}|\sin t|}{64L}; \ t \in (0,1], \\ \quad p(0) = 0. \end{cases}$$

So; we have $p^* \leq \frac{1}{64L}$, and then

$$Lp^* = \frac{1}{16} < 1.$$

A simple computation shows that all conditions of Theorem 3.3 are satisfied. Hence, the problem (3.8) has at least one solution defined on $[0, 1]$.

3.3 Weak Solutions for Implicit Fractional q-Difference Equations

3.3.1 Existence of weak solutions

In this section, we are concerned with the existence results of the problem(3.3)-(3.4).

Definition 3.4 *By a weak solution of the problem (3.3)-(3.4) we mean a measurable and continuous function u that satisfies the equation (3.3) on I and the initial condition (3.4).*

From lemma 1.8, and in order to define the solution for the problem (3.3)-(3.4), we conclude the following lemma.

Lemma 3.5 *Let $f : I \times E \times E \to E$ such that $f(\cdot, u, v) \in C_E(I)$, for each $u, v \in E$. Then the problem (3.3)-(3.4) is equivalent to the problem of obtaining the solutions of the integral equation*

$$g(t) = f(t, u_0 + (I_q^\alpha g)(t), g(t)),$$

and if $g(\cdot) \in C_E(I)$, is the solution of this equation, then

$$u(t) = u_0 + (I_q^\alpha g)(t).$$

Remark 3.6 *Let $g \in P_1([I, E)$. For every $\varphi \in E^*$, we have*

$$\varphi(I_q^\alpha g)(t) = (I_q^\alpha \varphi g)(t); \ for \ a.e. \ t \in I.$$

The following hypotheses will be used in the sequel.

In the sequel, we use the following hypotheses:

(H_1) For a.e. $t \in I$, the function $v \to f(t, u, v)$ is weakly sequentially continuous,

(H_2) For each $u, v \in E$, the function $t \to f(t, u, v)$ is Pettis integrable a.e. on I,

(H_3) There exists $p \in C(I, \mathbb{R}_+)$ such that for all $\varphi \in E^*$, we have

$$|\varphi(f(t, u, v))| \leq p(t); \ for \ a.e. \ t \in I, \ and \ each \ u, v \in E,$$

(H_4) For each bounded and measurable set $B \subset E$ and for each $t \in I$, we have

$$\beta(f(t, B, (^cD_q^\alpha B))) \leq p(t)\beta(B),$$

where $^cD_q^\alpha B = \{^cD_q^\alpha w : w \in B\}$.

Set

$$p^* = \sup_{t \in I} p(t).$$

Theorem 3.7 *Assume that the hypotheses $(H_1) - (H_4)$ hold. If*

$$\ell := \frac{p^* T^{(\alpha)}}{\Gamma_q(1+\alpha)} < 1, \tag{3.9}$$

then the problem (3.3)-(3.4) has at least one weak solution defined on I.

Proof. Consider the operator $N : C_E(I) \to C_E(I)$ defined by:

$$(Nu)(t) = u_0 + (I_q^\alpha g)(t);\ t \in I, \tag{3.10}$$

where $g \in C_E(I)$ such that

$$g(t) = f(t, u(t), g(t)),\ \textit{or}\ g(t) = f(t, u_0 + (I_q^\alpha g)(t), g(t)).$$

First notice that, the hypotheses imply that for each $u \in C_E(I)$, the function

$$f(t, u(t), (^cD_q^\alpha u)(t)),\ \textit{for a.e.}\ t \in I,$$

is Pettis integrable. Thus, the operator N is well defined. Let $R > 0$ be such that $R > \frac{p^* T^{(\alpha)}}{\Gamma_q(1+\alpha)}$, and consider the set

$$Q = \Big\{ u \in C_E(I) : \|u\|_\infty \leq R \text{ and } \|u(t_2) - u(t_1)\| \leq p^* \int_{t_1}^{t_2} \frac{(t_2 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_qs$$
$$+ p^* \int_0^{t_1} \left| \frac{(t_2 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} - \frac{(t_1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \right| d_qs \Big\}.$$

Clearly, the subset Q is closed, convex end equicontinuous. We shall show that the operator N satisfies all the assumptions of Theorem 1.38(Monch generalised). The proof will be given in several steps.

Step 1. *N maps Q into itself.*
Let $u \in Q$, $t \in I$ and assume that $(Nu)(t) \neq 0$. Then there exists $\varphi \in E^*$ such that $\|(Nu)(t)\| = |\varphi((Nu)(t))|$. Thus

$$\|(Nu)(t)\| = \varphi((Nu)(t) = u_0 + (I_q^\alpha g)(t)),$$

where $g \in C_E(I)$ such that

$$g(t) = f(t, u(t), g(t)).$$

Thus

$$\begin{aligned}
\|(Nu)(t)\| &\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |\varphi(g(s))| d_q s \\
&\leq p^* \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s \\
&\leq \frac{p^* T^{(\alpha)}}{\Gamma_q(1+\alpha)} \\
&\leq R.
\end{aligned}$$

Next, let $t_1, t_2 \in I$ such that $t_1 < t_2$ and let $u \in Q$, with

$$(Nu)(t_2) - (Nu)(t_1) \neq 0.$$

Then there exists $\varphi \in E^*$ such that

$$\|(Nu)(t_2) - (Nu)(t_1)\| = |\varphi((Nu)(t_2) - (Nu)(t_1))|,$$

and $\|\varphi\| = 1$. Then

$$\begin{aligned}
&\|(Nu)(t_2) - (Nu)(t_1)\| = |\varphi((Nu)(t_2) - (Nu)(t_1))| \\
&\leq \varphi\left(\int_0^{t_2} \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s) d_q s - \int_0^{t_1} \frac{(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s) d_q \right),
\end{aligned}$$

where $g \in C_E(I)$ such that

$$g(t) = f(t, u(t), g(t)).$$

Thus, we get

$$\begin{aligned}
\|(Nu)(t_2) - (Nu)(t_1)\| &\leq \int_{t_1}^{t_2} \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |\varphi(p(s))| d_q s \\
&+ \int_0^{t_1} \left| \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} - \frac{(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \right| |\varphi(g(s))| d_q s \\
&\leq p^* \int_{t_1}^{t_2} \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s \\
&+ p^* \int_0^{t_1} \left| \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} - \frac{(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \right| d_q s.
\end{aligned}$$

Hence $N(Q) \subset Q$.

Step 2. *N is weakly-sequentially continuous.*
Let (u_n) be a sequence in Q and let $(u_n(t)) \to u(t)$ in (E, ω) for each $t \in I$. Fix $t \in I$,

since f satisfies the assumption (H_1), we have $f(t, u_n(t), ({}^cD_q^\alpha u_n)(t))$ converges weakly uniformly to $f(t, u(t), ({}^cD_q^\alpha u)(t))$. Hence the Lebesgue dominated convergence theorem for Pettis integral implies $(Nu_n)(t)$ converges weakly uniformly to $(Nu)(t)$ in (E, ω), for each $t \in I$. Thus, $N(u_n) \to N(u)$. Hence, $N : Q \to Q$ is weakly-sequentially continuous.

Step 3. *The implication (1.2) holds.*
Let V be a subset of Q such that $\overline{V} = \overline{conv}(N(V) \cup \{0\})$. Obviously

$$t \in I : V(t) \subset \overline{conv}(NV)(t)) \cup \{0\}).$$

Further, as V is bounded and equicontinuous, by Lemma 3 in [34] the function $t \to v(t) = \beta(V(t))$ is continuous on I. From (H_3), (H_4), Lemma 1.29 and the properties of the measure β, for any $t \in I$, we have

$$\begin{aligned} v(t) &\leq \beta((NV)(t) \cup \{0\}) \\ &\leq \beta((NV)(t)) \\ &\leq (I_q^\alpha g)(t), \end{aligned}$$

where $g \in C_E(I)$ such that

$$g(t) = f(t, u(t), g(t)).$$

Then, we have

$$\begin{aligned} v(t) &\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s)\beta(V(s))d_q s \\ &\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s)v(s)d_q s \\ &\leq \frac{p^* T^{(\alpha)}}{\Gamma_q(1+\alpha)} \|v\|_\infty. \end{aligned}$$

Thus

$$\|v\|_\infty \leq \ell \|v\|_\infty.$$

From (3.9), we get $\|v\|_\infty = 0$, that is $v(t) = \beta(V(t)) = 0$, for each $t \in I$, and then by Theorem 2 in [56], V is weakly relatively compact in $C_E(I)$. Applying now Theorem 1.38, we conclude that N has a fixed point, which is a weak solution of the problem (3.3)-(3.4).

3.3.2 An Example

Let

$$l^1 = \left\{ u = (u_1, u_2, \ldots, u_n, \ldots) : \sum_{n=1}^{\infty} |u_n| < \infty \right\}$$

be the Banach space with the norm

$$\|u\|_{l^1} = \sum_{n=1}^{\infty} |u_n|.$$

Consider the following problem of implicit fractional $\frac{1}{4}-$difference equations

$$\begin{cases}({}^cD_{\frac{1}{4}}^{\frac{1}{2}}u_n)(t) = f_n(t, u(t), ({}^cD_{\frac{1}{4}}^{\frac{1}{2}}u)(t)); \ t \in [0,1], \\ u(0) = (0, 0, \ldots, 0, \ldots),\end{cases} \tag{3.11}$$

where

$$\begin{cases} f_n(t,u,v) = \dfrac{\Gamma_{\frac{1}{4}}(\frac{3}{2})\ t^{\frac{-1}{4}}(2^{-n} + u_n(t))\sin t}{64(1 + \|u\|_{l^1} + \sqrt{t})(1 + \|u\|_{l^1} + \|v\|_{l^1})}; \ t \in (0,1], \\ f_n(0,u,v) = 0,\end{cases}$$

with

$$f = (f_1, f_2, \ldots, f_n, \ldots), \ and \ u = (u_1, u_2, \ldots, u_n, \ldots).$$

For each $t \in (0,1]$, we have

$$\begin{aligned} \|f(t,u(t))\|_{l^1} &= \sum_{n=1}^{\infty} |f_n(s, u_n(s))| \\ &\leq \frac{\Gamma_{\frac{1}{4}}(\frac{3}{2})\ t^{\frac{-1}{4}}|\sin t|}{64(1 + \|u\|_{l^1} + \sqrt{t})(1 + \|u\|_{l^1} + \|v\|_{l^1})}(1 + \|u\|_{l^1}) \\ &\leq \frac{\Gamma_{\frac{1}{4}}(\frac{3}{2})\ t^{\frac{-1}{4}}|\sin t|}{64}. \end{aligned}$$

Thus, the hypothesis (H_2) is satisfied with

$$\begin{cases} p(t) = \dfrac{\Gamma_{\frac{1}{4}}(\frac{3}{2})\ t^{\frac{-1}{4}}|\sin t|}{64}; \ t \in (0,1], \\ p(0) = 0.\end{cases}$$

Since $|\sin t| < t$; for each $t \in (0,1]$, then $t^{\frac{-1}{4}}|\sin t| < t^{\frac{3}{4}} \leq 1$. So, we have $p^* \leq \frac{\Gamma_{\frac{1}{4}}(\frac{3}{2})}{64}$, and then $\ell \leq \frac{1}{64} < 1$. Hence all conditions of Theorem 3.7 are satisfied. Thus, the problem (3.11) has at least one solution defined on $[0,1]$.

Chapter 4

Fractional q-Difference Equations on the Half Line $^{(3)}$

[1] [12] S. Abbas, M. Benchohra, N. Laledj and Y. Zhou , Fractional q-Difference Equations on the Half Line, Archivum Mathematicum Vol.56 N 4 (2020),207-223.

4.1 Introduction

The aim of this chapter is to study some rasults about the existence of solutions and bounded solutions and the attractivity for a class of fractional q-difference equations. Some applications are made of Schauder fixed point theorem in Banach spaces and Darbo fixed point theorem in Fréchet spaces. We use some technics associated with the concept of measure of noncompactness and the diagonalization process. Some illustrative examples are given in the last section. in this chapter we discuss the existence and the attractivity of solutions for the following functional fractional q-difference equation

$$({}^{c}D_{q}^{\alpha}u)(t) = f(t, u(t));\ t \in \mathbb{R}_{+} := [0, +\infty), \tag{4.1}$$

with the initiale condition

$$u(0) = u_0, \tag{4.2}$$

where $q \in (0,1),\ \alpha \in (0,1],\ f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a given function, and ${}^{c}D_{q}^{\alpha}$ is the Caputo fractional q-difference derivative of order α.

Next, by using a generalization of the classical Darbo fixed point theorem for Fréchet spaces associated with the concept of measure of noncompactness, we discuss the existence of solutions for the problem (4.1)-(4.2) in Fréchet spaces.

Finally, we discuss the existence of bounded solutions for the problem (4.1)-(4.2) on $\mathbb{R}_+$, by applying Schauder's fixed point theorem associated with the diagonalization process.

[1] (3)

This chapter initiates the study of Caputo fractional q-difference equations in Fréchet spaces, the attractivity and the boundedness of the solutions of fractional q-difference equations on the half line.

4.2 Existence and attractivity results

By BC we denote the Banach space of all bounded and continuous functions from $\mathbb{R}_+$ into $\mathbb{R}$ equipped with the norm

$$\|u\|_{BC} := \sup_{t\in\mathbb{R}_+} |u(t)|.$$

Let $\emptyset \neq \Omega \subset BC$, and let $G : \Omega \to \Omega$, and consider the solutions of the equation

$$(Gu)(t) = u(t). \tag{4.3}$$

We introduce the following concept of attractivity of solutions for equation (4.3).

Definition 4.1 *Solutions of equation (4.3) are locally attractive if there exists a ball $B(u_0, \eta)$ in the space BC such that, for arbitrary solutions $v = v(t)$ and $w = w(t)$ of equations (4.3) belonging to $B(u_0, \eta) \cap \Omega$, we have*

$$\lim_{t\to\infty}(v(t) - w(t)) = 0. \tag{4.4}$$

When the limit (4.4) is uniform with respect to $B(u_0, \eta) \cap \Omega$, solutions of equation (4.3) are said to be uniformly locally attractive (or equivalently that solutions of (4.3) are locally asymptotically stable).

Lemma 4.2 *([36], p. 62). Let $D \subset BC$. Then D is relatively compact in BC if the following conditions hold:*
(a) D is uniformly bounded in BC,
(b) The functions belonging to D are almost equicontinuous on $\mathbb{R}_+$,
i.e. equicontinuous on every compact of $\mathbb{R}_+$,
(c) The functions from D are equiconvergent, that is, given $\epsilon > 0$ there exists $T(\epsilon) > 0$ such that $|u(t) - \lim_{t\to\infty} u(t)| < \epsilon$ for any $t \geq T(\epsilon)$ and $u \in D$.

Definition 4.3 *By a solution of the problem (4.1)-(4.2) we mean a function $u \in BC$ that satisfies the equation (4.1) on I and the initiale condition (4.2).*

From the lemma 1.8, and in order to define the solution for the problem (4.1)-(4.2), we conclude the following lemma.

Lemma 4.4 *Let $f : I \times \mathbb{R} \to \mathbb{R}$ such that $f(\cdot, u) \in C(I)$, for each $u \in \mathbb{R}$. Then the problem (4.1)-(4.2) is equivalent to the problem of obtaining the solutions of the integral equation*

$$u(t) = u_0 + (I_q^\alpha f(\cdot, u(\cdot)))(t).$$

The following hypotheses will be used in the sequel.

(H_1) The function $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H_2) There exists a continuous function $p : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t,u)| \leq p(t),\ \ for\ \ t \in \mathbb{R}_+,\ \ and\ each\ u \in \mathbb{R},$$

and

$$\lim_{t\to\infty} (I_q^\alpha p)(t) = 0.$$

Set

$$p^* = \sup_{t\in\mathbb{R}_+} (I_q^\alpha p)(t).$$

Now, we present a theorem concerning the existence and the attractivity of solutions of our problem (4.1)-(4.2).

Theorem 4.5 *Assume that the hypotheses* (H_1) *and* (H_2) *hold. Then the problem (4.1)-(4.2) has at least one solution defined on* $\mathbb{R}_+$. *Moreover, solutions of problem (4.1)-(4.2) are uniformly locally attractive.*

Proof. Consider the operator N such that, for any $u \in BC$,

$$(Nu)(t) = u_0 + I_q^\alpha f(\cdot, u(\cdot))(t). \tag{4.5}$$

The operator N maps BC into BC Indeed the map $N(u)$ is continuous on $\mathbb{R}_+$ for any $u \in BC$, and for each $t \in \mathbb{R}_+$, we have

$$\begin{aligned} |(Nu)(t)| &\leq |u_0| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,u(s))| d_q s \\ &\leq |u_0| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) d_q s \\ &\leq |u_0| + p^* \\ &= R. \end{aligned}$$

Thus

$$\|N(u)\|_{BC} \leq R. \tag{4.6}$$

Hence, $N(u) \in BC$, and the operator N maps the ball

$$B_R := B(0,R) = \{w \in BC : \|w\|_{BC} \leq R\}$$

into itself.

From Lemma 4.4, the solutions of the problem (4.1)-(4.2) are the fixed points of the operator N. We shall show that the operator $N : B_R \to B_R$ satisfies all the assumptions of Theorem 1.36. The proof will be given in several steps.

Step 1. *N is continuous.*
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_R. Then, for each $t \in \mathbb{R}_+$, we have

$$|(Nu_n)(t) - (Nu)(t)| \leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,u_n(s)) - f(s,u(s))| d_q s. \tag{4.7}$$

Case 1. If $t \in [0,T], T > 0$, then, since $u_n \to u$ as $n \to \infty$ and f is continuous, by the Lebesgue dominated convergence theorem, equation (4.7) implies

$$\|N(u_n) - N(u)\|_{BC} \to 0 \quad \text{as } n \to \infty.$$

Case 2. If $t \in (T,\infty), T > 0$, then from the hypotheses and (4.7), we get

$$|(Nu_n)(t) - (Nu)(t)| \leq 2\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) d_q s. \tag{4.8}$$

Since $u_n \to u$ as $n \to \infty$ and $(I_q^\alpha p)(t) \to 0$ as $t \to \infty$, then (4.8) gives

$$\|N(u_n) - N(u)\|_{BC} \to 0 \quad \text{as } n \to \infty.$$

Step 2. *$N(B_R)$ is uniformly bounded.*
This is clear since $N(B_R) \subset B_R$ and B_R is bounded.

Step 3. *$N(B_R)$ is equicontinuous on every compact subset $[0,T]$ of $\mathbb{R}_+$; $T > 0$.*
Let $t_1, t_2 \in [0,T]$, $t_1 < t_2$, and let $u \in B_R$. Set $p_* = \sup\limits_{t\in[0,T]} p(t)$. Then we have

$$\begin{aligned}
|(Nu)(t_2) - (Nu)(t_1)| &\leq \int_0^{t_1} \frac{|(t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} |f(s,u(s))| d_q s \\
&+ \int_{t_1}^{t_2} \frac{|(t_2-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} |f(s,u(s))| d_q s \\
&\leq p_* \int_0^{t_1} \frac{|(t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} d_q s \\
&+ p_* \int_{t_1}^{t_2} \frac{|(t_2-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} d_q s.
\end{aligned}$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero.

Step 4. *$N(B_R)$ is equiconvergent.*
Let $t \in \mathbb{R}_+$ and $u \in B_R$. Then we have

$$\begin{aligned} |(Nu)(t)| &\leq |u_0| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,u(s))| d_q s \\ &\leq |u_0| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) d_q s \\ &\leq |u_0| + (I_q^\alpha p)(t). \end{aligned}$$

Since $(I_q^\alpha p)(t) \to 0,\ as\ t \to +\infty$, we get

$$|(Nu)(t)| \to |u_0|,\ as\ t \to +\infty.$$

Hence,

$$|(Nu)(t) - (Nu)(+\infty)| \to 0,\ as\ t \to +\infty.$$

As a consequence of Steps 1 to 4, together with the Lemma 4.2, we can conclude that $N : B_R \to B_R$ is continuous and compact. From an application of Schauder's theorem (Theorem 1.36), we deduce that N has a fixed point u which is a solution of the problem (4.1)-(4.2) on $\mathbb{R}_+$.

Step 5. *The uniform local attractivity of solutions.*
Let us assume that u_0 is a solution of problem (4.1)-(4.2) with the conditions of this theorem. Taking $u \in B(u_0, 2p^*)$, we have

$$\begin{aligned} |(Nu)(t) - u_0(t)| &= |(Nu)(t) - (Nu_0)(t)| \\ &\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,u(s)) - f(s,u_0(s))| d_q s \\ &\leq 2\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) d_q s \\ &\leq 2p^*. \end{aligned}$$

Thus, we get

$$\|N(u) - u_0\|_{BC} \leq 2p^*.$$

Hence, we obtain that N is a continuous function such that

$$N(B(u_0, 2p^*)) \subset B(u_0, 2p^*).$$

Moreover, if u is a solution of problem (4.1)-(4.2), then

$$\begin{aligned} |u(t) - u_0(t)| &= |(Nu)(t) - (Nu_0)(t)| \\ &\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,u(s)) - f(s,u_0(s))| ds \\ &\leq 2(I_q^\alpha p)(t). \end{aligned}$$

Thus

$$|u(t) - u_0(t)| \leq 2(I_q^\alpha p)(t) \to\ 0\ as\ t \to \infty.$$

Consequently, all solutions of problem (4.1)-(4.2) are uniformly locally attractive.

4.3 Existence Results in Fréchet Spaces

Let $X := C(\mathbb{R}_+, E)$ be the Fréchet space of all continuous functions v from $\mathbb{R}_+$ into a Banach space $(E, \|\cdot\|)$, equipped with the family of seminorms

$$\|v\|_n = \sup_{t\in[0,n]} \|v(t)\|;\ n \in \mathbb{N}^*,$$

and the distance

$$d(u,v) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|u-v\|_n}{1+\|u-v\|_n};\ u,v \in X.$$

In this section, we are concerned with the existence of solutions of our problem (4.1)-(4.2).

Definition 4.6 *By a solution of the problem (4.1)-(4.2) we mean a continuous function $u \in X$ that satisfies the equation (4.1) on $\mathbb{R}_+$ and the initiale condition (4.2).*

The following hypotheses will be used in the sequel.

(H_{01}) The function $t \mapsto f(t,u)$ is measurable on I for each $u \in E$, and the function $u \mapsto f(t,u)$ is continuous on E for a.e. $t \in \mathbb{R}_+$,

(H_{02}) There exists a continuous function $p : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|f(t,u)\| \le p(t)(1+\|u\|);\ for\ a.e.\ t \in I,\ and\ each\ u \in E,$$

(H_{03}) For each bounded and measurable set $B \subset E$, and for each $t \in \mathbb{R}_+$, we have

$$\mu(f(t,B)) \le p(t)\mu(B),$$

where μ is a measure of noncompactness on the Banach space E.

For $n \in \mathbb{N}^*$, let

$$p_n^* = \sup_{t\in[0,n]} p(t),$$

and define on X the family of measure of noncompactness by

$$\mu_n(D) = \omega_0^n(D) + \sup_{t\in[0,n]} \mu(D(t)),$$

where $D(t) = \{v(t) \in \mathbb{R} : v \in D\};\ t \in [0,n]$.

Theorem 4.7 *Assume that the hypotheses $(H_{01} - (H_{03})$ Hold.*
If

$$\frac{4n^\alpha p_n^*}{\Gamma_q(1+\alpha)} < 1; \tag{4.9}$$

for each $n \in \mathbb{N}^*$, then the problem (4.1)-(4.2) has at least one solution in X.

Proof. Consider the operator $N : X \to X$ defined by (4.5) Clearly, the fixed points of the operator N are solution of the problem (4.1)—(4.2).

For any $n \in \mathbb{N}^*$, we set

$$R_n \geq \frac{|u_0|\Gamma_q(1+\alpha) + p_n^* n^\alpha}{\Gamma_q(1+\alpha) - p_n^* n^\alpha},$$

and we consider the ball

$$B_{R_n} := B(0, R_n) = \{w \in X : \|w\|_n \leq R_n\}.$$

For any $n \in \mathbb{N}^*$, and each $u \in B_{R_n}$ and $t \in [0, n]$ we have

$$\begin{aligned}
\|(Nu)(t)\| &\leq |u_0| + \int_0^t \frac{(tq-s)^{(\alpha-1)}}{\Gamma_q(\alpha)} \|f(s, u(s))\| d_q s \\
&\leq \|u_0\| + \int_0^t \frac{(tq-s)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s)(1 + \|u(s)\|) d_q s \\
&\leq \|u_0\| + p_n^*(1 + R_n) \int_0^t \frac{(tq-s)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s \\
&\leq \|u_0\| + \frac{n^\alpha p_n^*}{\Gamma_q(1+\alpha)}(1 + R_n) \\
&\leq R_n.
\end{aligned}$$

Thus

$$\|N(u)\|_n \leq R_n. \tag{4.10}$$

This proves that N transforms the ball B_{R_n} into itself. We shall show that the operator $N : B_{R_n} \to B_{R_n}$ satisfies all the assumptions of Theorem 1.43. The proof will be given in several steps.

Step 1. *$N : B_{R_n} \to B_{R_n}$ is continuous.*
Let $\{u_k\}_{k \in \mathbb{N}}$ be a sequence such that $u_k \to u$ in B_{R_n}. Then, for each $t \in [0, n]$, we have

$$\|(Nu_k)(t) - (Nu)(t)\| \leq \int_0^t \frac{(tq-s)^{(\alpha-1)}}{\Gamma_q(\alpha)} \|f(s, u_k(s)) - f(s, u(s))\| d_q s.$$

Since $u_k \to u$ as $k \to \infty$, the Lebesgue dominated convergence theorem implies that

$$\|N(u_k) - N(u)\|_n \to 0 \quad \text{as } k \to \infty.$$

Step 2. *$N(B_{R_n})$ is bounded.*
Since $N(B_{R_n}) \subset B_{R_n}$ and B_{R_n} is bounded, then $N(B_{R_n})$ is bounded.

Step 3. *For each bounded and equicontinuous subset D of B_{R_n}, $\mu_n(N(D)) \leq \ell_n\mu_n(D)$.* From Lemmas 1.33 and 1.34, for any $D \subset B_{R_n}$ and any $\epsilon > 0$, there exists a sequence $\{u_k\}_{k=0}^{\infty} \subset D$, such that for all $t \in [0, n]$, we have

$$\begin{aligned}
\mu((ND)(t)) &= \mu\left(\left\{u_0 + \int_0^t \frac{(tq-s)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,u(s))d_qs;\ u \in D\right\}\right)\\
&\leq 2\mu\left(\left\{\int_0^t \frac{(tq-s)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,u_k(s))d_qs\right\}_{k=1}^{\infty}\right) + \epsilon\\
&\leq 4\int_0^t \frac{(tq-s)^{(\alpha-1)}}{\Gamma_q(\alpha)} \mu(\{f(s,u_k(s))\}_{k=0}^{\infty} d_qs + \epsilon\\
&\leq 4\int_0^t \frac{(tq-s)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s)\mu(\{u_k(s)\}_{k=1}^{\infty})d_qs + \epsilon\\
&\leq \frac{4n^\alpha p_n^*}{\Gamma_q(1+\alpha)}\mu_n(D) + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, then

$$\mu((ND)(t)) \leq \frac{4n^\alpha p_n^*}{\Gamma_q(1+\alpha)}\mu_n(D).$$

Thus

$$\mu_n(N(D)) \leq \frac{4n^\alpha p_n^*}{\Gamma_q(1+\alpha)}\mu_n(D).$$

As a consequence of steps 1 to 3 together with Theorem 1.43, we can conclude that N has at least one fixed point in B_{R_n} which is a solution of problem (4.1)-(4.2).

4.4 Existence of bounded solutions

In this section, we are concerned with the existence of bounded solutions of our problem

$$\begin{cases} ({}^cD_q^\alpha u)(t) = f(t,u(t));\ t \in \mathbb{R}_+,\\ \\ u(0) = u_0 \in \mathbb{R},\quad u \textit{ is bounded on } \mathbb{R}_+, \end{cases} \tag{4.11}$$

Definition 4.8 *By a bounded solution of the problem (4.11) we mean a measurable and bounded function u on $\mathbb{R}_+$ such that $u(0) = u_0$, and u satisfies the fractional q-difference equation $({}^cD_q^\alpha u)(t) = f(t,u(t))$ on $\mathbb{R}_+$.*

The following hypotheses will be used in the sequel.

(H_{11}) The function $t \mapsto f(t,u)$ is measurable on $I_n := [0,n];\ n \in \mathbb{N}$ for each $u \in \mathbb{R}$, and the function $u \mapsto f(t,u)$ is continuous for a.e. $t \in I_n$,

(H_{12}) There exists a continuous function $p_n : I_n \to \mathbb{R}_+$ such that

$$|f(t,u)| \leq p_n(t),\ \ for\ a.e.\ t \in I_n,\ and\ each\ u \in \mathbb{R}.$$

Set

$$p_n^* = \sup_{t\in I_n} p_n(t).$$

Theorem 4.9 *Assume that the hypotheses* (H_{11}) *and* (H_{12}) *hold. Then the problem (4.11) has at least one bounded solution defined on* $\mathbb{R}_+$.

Proof. The proof will be given in two parts. Fix $n \in \mathbb{N}$ and consider the problem

$$\begin{cases} ({}^C D_q^\alpha u)(t) = f(t,u(t));\ t \in I_n, \\ \\ u(0) = u_0. \end{cases} \tag{4.12}$$

Part 1. We begin by showing that (4.12) has a solution $u_n \in C(I_n)$ with

$$\|u_n\|_\infty \leq R_n := \frac{n^\alpha p_n^*}{\Gamma_q(1+\alpha)}.$$

Consider the operator $N : C(I_n) \to C(I_n)$ defined by (4.5) Clearly, the fixed points of the operator N are solution of the problem (4.12).
For any $u \in C(I_n)$, and each $t \in I_n$ we have

$$\begin{aligned} |(Nu)(t)| &\leq |u_0| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,u(s))| d_q s \\ &\leq |u_0| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p_n(s) d_q s \\ &\leq |u_0| + p_n^* \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s \\ &\leq \frac{n^\alpha p_n^*}{\Gamma_q(1+\alpha)}. \end{aligned}$$

Thus

$$\|N(u)\|_\infty \leq R_n. \tag{4.13}$$

This proves that N transforms the ball $B_{R_n} := B(0,R_n) = \{w \in C(I_n) : \|w\|_\infty \leq R_n\}$ into itself. We shall show that the operator $N : B_{R_n} \to B_{R_n}$ satisfies all the assumptions of Theorem 1.36. The proof will be given in several steps.

Step 1. $N : B_{R_n} \to B_{R_n}$ *is continuous.*
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_{R_n}. Then, for each $t \in I_n$, we have

$$|(Nu_n)(t) - (Nu)(t)|$$

$$\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u_n(s))-f(s,u(s))|d_qs. \tag{4.14}$$

Since $u_n \to u$ as $n \to \infty$ and (H_{11}), the by the Lebesgue dominated convergence theorem, equation (4.14) implies

$$\|N(u_n)-N(u)\|_\infty \to 0 \quad \text{as } n \to \infty.$$

Step 2. $N(B_{R_n})$ *is uniformly bounded.*
This is clear since $N(B_{R_n}) \subset B_{R_n}$ and B_{R_n} is bounded.

Step 3. $N(B_{R_n})$ *is equicontinuous.*
Let $t_1, t_2 \in I_n$, $t_1 < t_2$ and let $u \in B_{R_n}$. Thus we have

$$\begin{aligned}
&|(Nu)(t_2)-(Nu)(t_1)| \\
&\leq \int_0^{t_1} \frac{|(t_2-qs)^{(\alpha-1)}-(t_1-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)}|f(s,u(s))|d_qs \\
&+ \int_{t_1}^{t_2} \frac{|(t_2-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)}|f(s,u(s))|d_qs \\
&\leq p_n^* \int_0^{t_1} \frac{|(t_2-qs)^{(\alpha-1)}-(t_1-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)}d_qs \\
&+ p_n^* \int_{t_1}^{t_2} \frac{|(t_2-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)}d_qs.
\end{aligned}$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero.

As a consequence of steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that N is continuous and compact. From an application of Schauder's theorem (Theorem 1.36), we deduce that N has a fixed point u which is a solution of the problem (4.12).

Part 2. The diagonalization process.
Now, we use the following diagonalization process. For $k \in \mathbb{N}$ let

$$\begin{cases} w_k(t) = u_{n_k}(t);\ t \in [0,n_k], \\ w_k(t) = u_{n_k}(n_k);\ t \in [n_k,\infty). \end{cases}$$

Here $\{n_k\}_{k\in\mathbb{N}^*}$ is a sequence of numbers satisfying

$$0 < n_1 < n_2 < \ldots n_k < \ldots \uparrow \infty.$$

Let $S = \{w_k\}_{k=1}^{\infty}$ Notice that

$$|w_{n_k}(t)| \leq R_n : \ for\ t \in [0,n_1],\ k \in \mathbb{N}$$

Also, if $k \in \mathbb{N}$ and $t \in [0, n_1]$, we have

$$w_{n_k}(t) = u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, w_{n_k}(s)) d_q s.$$

Thus, for $k \in \mathbb{N}$ and $t, x \in [0, n_1]$, we have

$$|w_{n_k}(t) - w_{n_k}(x)| \leq \int_0^{n_1} \frac{|(t-qs)^{(\alpha-1)} - (x-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} |f(s, w_{n_k}(s))| d_q s.$$

Hence

$$|w_{n_k}(t) - w_{n_k}(x)| \leq p_1^* \int_0^{n_1} \frac{|(t-qs)^{(\alpha-1)} - (x-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} d_q s.$$

The Arzelà-Ascoli Theorem guarantees that there is a subsequence $\mathbb{N}_1^*$ of $\mathbb{N}$ and a function $z_1 \in C([0, n_1], \mathbb{R})$ with $u_{n_k} \to z_1$ as $k \to \infty$ in $C([0, n_1], \mathbb{R})$ through $\mathbb{N}_1^*$. Let $\mathbb{N}_1 = \mathbb{N}_1^* - \{1\}$. Notice that

$$|w_{n_k}(t)| \leq R_n : \ for\ t \in [0, n_2],\ k \in \mathbb{N}$$

Also, if $k \in \mathbb{N}$ and $t, x \in [0, n_2]$, we have

$$|w_{n_k}(t) - w_{n_k}(x)| \leq p_2^* \int_0^{n_2} \frac{|(t-qs)^{(\alpha-1)} - (x-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} d_q s.$$

The Arzelà-Ascoli Theorem guarantees that there is a subsequence $\mathbb{N}_2^*$ of $\mathbb{N}_1$ and a function $z_2 \in C([0, n_2], \mathbb{R})$ with $u_{n_k} \to z_2$ as $k \to \infty$ in $C([0, n_2], \mathbb{R})$ through $\mathbb{N}_2^*$. Note that $z_1 = z_2$ on $[0, n_1]$ since $\mathbb{N}_2^* \subset \mathbb{N}_1$. Let $\mathbb{N}_2 = \mathbb{N}_2^* - \{2\}$. Proceed inductively to obtain for $m = 3, 4, \ldots$ a subsequence $\mathbb{N}_m^*$ of $\mathbb{N}_{m-1}$ and a function $z_m \in C([0, n_m], \mathbb{R})$ with $u_{n_k} \to z_m$ as $k \to \infty$ in $C([0, n_m], \mathbb{R})$ through $\mathbb{N}_m^*$. Let $\mathbb{N}_m = \mathbb{N}_m^* - \{m\}$.

Define a function y as follows. Fix $t \in (0, \infty)$ and let $m \in \mathbb{N}$ with $t \leq n_m$. Then define $u(t) = z_m(t)$. Thus $u \in C((0, \infty, \mathbb{R})),\ u(0) = u_0$ and $|u(t)| \leq R_n : \ for\ t \in [0, \infty)$. Again fix $t \in (0, \infty)$ and let $m \in \mathbb{N}$ with $t \leq n_m$. Then for $n \in \mathbb{N}_m$ we have

$$u_{n_k}(t) = u_0 + \int_0^{n_m} \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, w_{n_k}(s)) d_q s.$$

Let $n_k \to \infty$ through $\mathbb{N}_m$ to obtain

$$z_m(t) = u_0 + \int_0^{n_m} \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, z_m(s)) d_q s.$$

We can use this method for each $x \in [0, n_m]$ and for each $m \in \mathbb{N}$. Thus

$$({}^C D_q^\alpha u)(t) = f(t, u(t));\ for\ t \in [0, n_m]$$

for each $m \in \mathbb{N}$ and the constructed function u is a solution of problem (4.11).

4.5 Some Examples

Example 1. Consider the following problem of fractional $\frac{1}{4}-$difference equations

$$\begin{cases} ({}^cD^{\frac{1}{2}}_{\frac{1}{4}}u)(t) = f(t, u(t));\ t \in \mathbb{R}_+, \\ u(0) = 1, \end{cases} \tag{4.15}$$

where

$$\begin{cases} f(t,u) = \dfrac{t^{\frac{-1}{4}} \sin t}{(1+\sqrt{t})(1+|u|)};\ t \in (0,\infty),\ u \in \mathbb{R}, \\ f(0,u) = 0; \qquad\qquad\qquad\qquad u \in \mathbb{R}. \end{cases}$$

Clearly, the function f is continuous.
The hypothesis (H_2) is satisfied with

$$\begin{cases} p(t) = \dfrac{t^{\frac{-1}{4}} |\sin t|}{1+\sqrt{t}};\ t \in (0,\infty), \\ p(0) = 0. \end{cases}$$

All conditions of Theorem 4.5 are satisfied. Hence, the problem (4.15) has at least one solution defined on $\mathbb{R}_+$, and solutions of this problem are uniformly locally attractive.

Example 2. Let

$$l^1 = \left\{ u = (u_1, u_2, \dots, u_k, \dots) : \sum_{k=1}^{\infty} |u_n| < \infty \right\}$$

be the Banach space with the norm

$$\|u\|_{l^1} = \sum_{k=1}^{\infty} |u_k|,$$

and $F := C(\mathbb{R}_+, l^1)$ be the Fréchet space of all continuous functions v from $\mathbb{R}_+$ into l^1, equipped with the family of seminorms

$$\|v\|_n = \sup_{t\in[0,n]} \|v(t)\|_{l^1};\ n \in \mathbb{N}^*.$$

Consider the following problem of fractional $\frac{1}{4}-$difference equations

$$\begin{cases} ({}^cD^{\frac{1}{2}}_{\frac{1}{4}}u_k)(t) = f_k(t, u(t));\ t \in \mathbb{R}_+, \\ u_k(0) = 0;\ t \in \mathbb{R}_+,\ k \in \mathbb{N}, \end{cases} \tag{4.16}$$

where

$$f_k(t,u) = \frac{c_n(2^{-k}+u_k)t^{\frac{5}{4}}\sin t}{128(1+\sqrt{t})};\ u \in l^1,$$

for each $t \in [0,n];\ n \in \mathbb{N}^*$, with

$$c_n = 16n^{-\frac{7}{4}}\Gamma_{\frac{1}{4}}\left(\frac{3}{2}\right);\ n \in \mathbb{N}^*,$$

$$f = (f_1, f_2, \dots, f_k, \dots),\ \textit{and}\ u = (u_1, u_2, \dots, u_k, \dots).$$

Since

$$\|f(t,u)\|_{l^1} = \sum_{k=1}^{\infty} |f_k(s,u)| \le \frac{t^{\frac{5}{4}}c_n}{128}(1+\|u\|_{l^1});\ t \in [0,n],\ n \in \mathbb{N}^*,$$

then the hypothesis (H_{02}) is satisfied with

$$p(t) = \frac{t^{\frac{5}{4}}c_n}{64};\ t \in [0,n],\ n \in \mathbb{N}^*.$$

So; for any $n \in \mathbb{N}^*$, we have

$$p_n^* = \frac{n^{\frac{5}{4}}c_n}{128}.$$

The condition (4.9) is satisfied. Indeed;

$$\frac{4n^{\frac{1}{2}}p_n^*}{\Gamma_q(1+\alpha)} = 16n^{-\frac{7}{4}}\Gamma_{\frac{1}{4}}\left(\frac{3}{2}\right)\frac{n^{\frac{5}{4}}}{128}\,\frac{4n^{\frac{1}{2}}}{\Gamma_{\frac{1}{4}}\left(\frac{3}{2}\right)} = \frac{1}{2} < 1.$$

Therefore all conditions of Theorem 4.7 are satisfied. Hence, the problem (4.16) has at least one solution defined on $\mathbb{R}_+$.

Example 3. Consider the following problem of fractional $\frac{1}{4}$−difference equations

$$\begin{cases} ({}^C D_{\frac{1}{4}}^{\frac{1}{2}}u)(t) = f(t,u(t));\ t \in \mathbb{R}_+, \\ u(0) = 2,\ u \text{ is bounded on } \mathbb{R}_+, \end{cases} \tag{4.17}$$

where

$$f(t,u) = \frac{e^{t+1}}{1+|u|}(1+u);\quad t \in \mathbb{R}_+.$$

The hypothesis (H_{12}) is satisfied with $p_n(t) = e^{t+1}$. So, $p_n^* = e^{n+1}$. Simple computations show that all conditions of Theorem 4.9 are satisfied. It follows that the problem (4.17) has at least one bounded solution defined on $\mathbb{R}_+$.

Chapter 5

Fractional q-Difference Inclusions in Banach Spaces

5.1 Introduction

The aim of this chapter is to study a class of Caputo fractional q-difference inclusions in Banach spaces. We obtain some existence results by using the set-valued analysis, the measure of noncompactness, and the fixed point theory (Darbo and Mönch's fixed point theorems). Finally we give an illustrative example in the last section. We initiate the study of fractional q-difference inclusions on infinite dimensional Banach spaces. In this chapter we consider the Caputo fractional q-difference inclusion

$$({}^{c}D_q^{\alpha}u)(t) \in F(t, u(t)),\ t \in I := [0, T], \tag{5.1}$$

with the initial condition

$$u(0) = u_0 \in E, \tag{5.2}$$

where $(E, \|\cdot\|)$ is a separable Banach space, $q \in (0,1),\ \alpha \in (0,1],\ T > 0,\ F : I \times E \to \mathcal{P}(E)$ is a multivalued map, $\mathcal{P}(E) = \{Y \subset E : Y \neq \emptyset\}$, and ${}^{c}D_q^{\alpha}$ is the Caputo fractional q-difference derivative of order α.

5.2 Existence Results

First, we state the definition of a solution of the problem (5.1)-(5.2).

Definition 5.1 *By a solution of the problem (5.1)-(5.2) we mean a function $u \in C_E(I)$ that satisfies the initial condition (5.2) and the equation $({}^{C}D_q^{\alpha}u)(t) = v(t)$ on I, where $v \in S_{F\circ u}$.*

In the sequel, we need the following hypotheses.

(H_1) The multivalued map $F : I \times E \to \mathcal{P}_{cp,cv}(E)$ is Carathéodory,

(H_2) There exists a function $p \in L^\infty(I, \mathbb{R}_+)$ such that

$$\|F(t,u)\|_{\mathcal{P}} = \sup\{\|v\|_C : v(t) \in F(t,u)\} \leq p(t);$$

for a.e. $t \in I$, and each $u \in E$,

(H_3) For each bounded set $B \subset C_E(I)$ and for each $t \in I$, we have

$$\mu(F(t, B(t)) \leq p(t)\mu(B(t)),$$

where $B(t) = \{u(t) : u \in B\}$,

(H_4) The function $\phi \equiv 0$ is the unique solution in $C_E(I)$ of the inequality

$$\Phi(t) \leq 2p^*(I_q^\alpha \Phi)(t),$$

where p is the function defined in (H_3), and

$$p^* = esssup_{t \in I} p(t).$$

Remark 5.2 *In (H_3), μ is the Kuratowski measure of noncompactness on the space E.*

Theorem 5.3 *If the hypotheses $(H_1) - (H_3)$ and the condition*

$$L := \frac{p^* T^{(\alpha)}}{\Gamma_q(1+\alpha)} < 1$$

hold, then the problem (5.1)-(5.2) has at least one solution defined on I.

Proof. Consider the multivalued operator $N : C_E(I) \to \mathcal{P}(C_E(I))$ defined by:

$$N(u) = \Big\{ h \in C_E(I) : h(t) = \mu_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v(s) d_q s; \; v \in S_{F \circ u} \Big\}. \tag{5.3}$$

From Lemma 1.8, the fixed points of N are solutions of the problem (5.1)-(5.2). Set

$$R := \|u_0\| + \frac{p^* T^{(\alpha)}}{\Gamma_q(1+\alpha)},$$

and let $B_R := \{u \in C_E(I) : \|u\|_\infty \leq R\}$ be the bounded, closed and convex ball of $C_E(I)$. We shall show in three steps that the multivalued operator $N : B_R \to \mathcal{P}_{cl,b}(C_E(I))$ satisfies all assumptions of Theorem 1.41.

Step 1. $N(B_R) \in \mathcal{P}(B_R)$.
Let $u \in B_R$, and $h \in N(u)$. Then for each $t \in I$ we have

$$h(t) = u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v(s) d_q s,$$

for some $v \in S_{F\circ u}$. On the other hand,

$$\begin{aligned}
\|h(t)\| &\leq \|u_0\| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \|v(s)\| d_q s \\
&\leq \|u_0\| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) d_q s \\
&\leq \|u_0\| + esssup_{t\in I} p(t) \int_0^T \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s \\
&= \|u_0\| + \frac{p^* T^{(\alpha)}}{\Gamma_q(1+\alpha)}.
\end{aligned}$$

Hence $\|h\|_\infty \leq R$, and so $N(B_R) \in \mathcal{P}(B_R)$.

Step 2. $N(u) \in \mathcal{P}_{cl}(B_R)$ *for each* $u \in B_R$.
Let $\{u_n\}_{n\geq 0} \in N(u)$ such that $u_n \longrightarrow \tilde{u}$ in $C_E(I)$. Then, $\tilde{u} \in B_R$ and there exists $f_n(\cdot) \in S_{F\circ u}$ be such that, for each $t \in I$, we have

$$u_n(t) = u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f_n(s) d_q s.$$

Hrom (H_1), and since F has compact values, then we may pass to a subsequence if necessary to get that $f_n(\cdot)$ converges to f in $L^1(I)$, and then $f \in S_{F\circ u}$. Thus, for each $t \in I$, we get

$$u_n(t) \longrightarrow \tilde{u}(t) = u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s) d_q s.$$

Hence $\tilde{u} \in N(u)$.

Step 3. *N satisfies the Darbo condition.*
Let $U \subset B_R$, then for each $t \in I$, we have

$$\mu((NU)(t)) = \mu(\{(Nu)(t) : u \in U\}).$$

Let $h \in N(u)$. Then, there exists $f \in S_{F\circ u}$ such that for each $t \in I$, we have

$$h(t) = u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s) d_q s.$$

From Theorem 1.26 and since $U \subset B_R \subset C(I)$, then

$$\mu((NU)(t)) \leq 2\int_0^t \mu\left(\left\{\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s) : u \in U\right\}\right) d_qs.$$

Now, since $f \in S_{F\circ u}$ and $u(s) \in U(s)$, we have

$$\mu(\{(t-qs)^{(\alpha-1)}f(s)\}) = (t-qs)^{(\alpha-1)}p(s)\mu(U(s)).$$

Then

$$\mu((NU)(t)) \leq 2\int_0^t \mu\left(\left\{\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s)\right\}\right) d_qs.$$

Thus

$$\mu((NU)(t)) \leq 2p^*\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\mu(U(s))d_qs.$$

Hence

$$\mu((NU)(t)) \leq \frac{2p^*T^{(\alpha)}}{\Gamma_q(1+\alpha)}\mu(U).$$

Therefore,

$$\mu(N(U)) \leq L\mu(U),$$

which implies the N is a $L-$set-contraction.
As a consequence of Theorem 1.41, we deduce that N has a fixed point that is a solution of the problem (5.1)-(5.2).

Now, we prove an other existence result by applying Theorem 1.42.

Theorem 5.4 *If the hypotheses $(H_1)-(H_4)$ hold, then there exists at least one solution of problem (5.1)-(5.2).*

Proof. Consider the multivalued operator $N : C_E(I) \to \mathcal{P}(C_E(I))$ defined in (5.3). We shall show in five steps that the multivalued operator N satisfies all assumptions of Theorem 1.42.

Step 1. *$N(u)$ is convex for each $u \in C_E(I)$.*
Let $h_1, h_2 \in N(u)$, then there exist $v_1, v_2 \in S_{F\circ u}$ such that

$$h_i(t) = \mu_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}v_i(s)d_qs;\ t \in I,\ i = 1, 2.$$

Let $0 \leq \lambda \leq 1$. Then, for each $t \in I$, we have

$$(\lambda h_1 + (1-\lambda)h_2)(t) = \mu_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}(\lambda v_1(s) + (1-\lambda)v_2(s))d_qs.$$

Since $S_{F\circ u}$ is convex (because F has convex values), we have $\lambda h_1 + (1-\lambda)h_2 \in N(u)$.

Step 2. *For each compact $M \subset C_E(I)$, $N(M)$ is relatively compact.*
Let (h_n) by any sequence in $N(M)$, where $M \subset C_E(I)$ is compact. We show that (h_n) has a convergent subsequence from Arzéla-Ascoli compactness criterion in $C(IE)$. Since $h_n \in N(M)$ there are $u_n \in M$ and $v_n \in S_{F\circ u_n}$ such that

$$h_n(t) = \mu_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v_n(s) d_q s.$$

Using Theorem 1.26 and the properties of the measure μ, we have

$$\mu(\{h_n(t)\}) \leq 2\int_0^t \mu\left(\left\{\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v_n(s)\right\}\right) d_q s. \tag{5.4}$$

On the other hand, since M is compact, the set $\{v_n(s) : n \geq 1\}$ is compact. Consequently, $\mu(\{v_n(s) : n \geq 1\}) = 0$ for a.e. $s \in I$. Furthermore

$$\mu(\{(t-qs)^{(\alpha-1)} v_n(s)\}) = (t-qs)^{(\alpha-1)} \mu(\{v_n(s) : n \geq 1\}) = 0.$$

for a.e. $t, s \in I$. Now (5.4) implies that $\{h_n(t) : n \geq 1\}$ is relatively compact for each $t \in I$. In addition, for each $t_1, t_2 \in I$; with $t_1 < t_2$, we have

$$\begin{aligned}
&\|h_n(t_2) - h_n(t_1)\| \\
&\leq \left\| \int_0^{t_2} \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) d_q s - \int_0^{t_1} \frac{(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) d_q s \right\| \\
&\leq \int_{t_1}^{t_2} \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) d_q s \\
&\quad + \int_0^{t_1} \frac{|(t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} p(s) d_q s \\
&\leq \frac{p^* T^\alpha}{\Gamma_q(1+\alpha)} (t_2-t_1)^\alpha \\
&\quad + p^* \int_0^{t_1} \frac{|(t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} d_q s \\
&\to 0 \ as \ t_1 \longrightarrow t_2.
\end{aligned} \tag{5.5}$$

This shows that $\{h_n : n \geq 1\}$ is equicontinuous. Consequently, $\{h_n : n \geq 1\}$ is relatively compact in $C_E(I)$.

Step 3. *The graph of N is closed.*
Let $(u_n, h_n) \in \text{graph}(N)$, $n \geq 1$, with $(\|u_n - u\|, \|h_n - h\|) \to (0.0)$, as $n \to \infty$. We have to show that $(u, h) \in \text{graph}(N)$. $(u_n, h_n) \in \text{graph}(N)$ means that $h_n \in N(u_n)$, which implies that there exists $v_n \in S_{F\circ u_n}$, such that for each $t \in I$,

$$h_n(t) = u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v_n(s) d_q s.$$

Consider the continuous linear operator $\Theta : L^\infty(I) \to C_E(I)$,

$$\Theta(v)(t) \mapsto h(t) = u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v(s) d_q s.$$

Clearly, $\|h_n(t) - h(t)\| \to 0$ as as $n \to \infty$. From Lemma 1.22 it follows that $\Theta \circ S_F$ is a closed graph operator. Moreover, $h_n(t) \in \Theta(S_{F\circ u_n})$. Since $u_n \to u$, Lemma 1.22 implies

$$h(t) = u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v(s) d_q s.$$

for some $v \in S_{F\circ u}$.

Step 4. *M is relatively compact in $C_E(I)$.*
Let $M \subset \overline{U}$; with $M \subset \operatorname{conv}(\{0\} \cup N(M))$, and let $\overline{M} = \overline{C}$; for some countable set $C \subset M$. the set $N(M)$ is equicontinuous from (5.5). Therefore,

$$M \subset \operatorname{conv}(\{0\} \cup N(M)) \Longrightarrow M \textit{ is equicontinuous}.$$

By applying the Arzéla-Ascoli theorem; the set $M(t)$ is relatively compact for each $t \in I$. Since $C \subset M \subset \operatorname{conv}(\{0\} \cup N(M))$, then there exists a countable set $H = \{h_n : n \geq 1\} \subset N(M)$ such that $C \subset \operatorname{conv}(\{0\} \cup H)$. Thus, there exist $u_n \in M$ and $v_n \in S_{F\circ u_n}$ such that

$$h_n(t) = u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v_n(s) d_q s.$$

From Theorem 1.26, we get

$$M \subset \overline{C} \subset \overline{conv}(\{0\} \cup H)) \Longrightarrow \mu(M(t)) \leq \mu(\overline{C}(t)) \leq \mu(H(t)) = \mu(\{h_n(t) \, : n \geq 1\}).$$

Using now the inequality (5.4) in step 2, we obtain

$$\mu(M(t)) \leq 2 \int_0^t \mu \left(\left\{ \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v_n(s) \right\} \right) d_q s.$$

Since $v_n \in S_{F\circ u_n}$ and $u_n(s) \in M(s)$, we have

$$\mu(M(t)) \leq 2 \int_0^t \mu \left(\left\{ \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v_n(s) : n \geq 1 \right\} \right) d_q s.$$

Also, since $v_n \in S_{F\circ u_n}$ and $u_n(s) \in M(s)$, then from (H_3) we get

$$\mu(\{(t-qs)^{(\alpha-1)} v_n(s); \, n \geq 1\}) = (t-qs)^{(\alpha-1)} p(s) \mu(M(s)).$$

Hence

$$\mu(M(t)) \leq 2p^* \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \mu(M(s)) d_q s.$$

Consequently, from (H_4), the function Φ given by $\Phi(t) = \mu(M(t))$ satisfies $\Phi \equiv 0$; that is, $\mu(M(t)) = 0$ for all $t \in I$. Finally, the Arzéla-Ascoli theorem implies that M is relatively compact in $C_E(I)$.

Step 5. *The priori estimate.*
Let $u \in C(IE)$ such that $u \in \lambda N(u)$ for some $0 < \lambda < 1$. Then

$$u(t) = \lambda u_0 + \lambda \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v(s) d_q s,$$

for each $t \in I$, where $v \in S_{F \circ u}$. On the other hand,

$$\begin{aligned} \|u(t)\| &\leq \|u_0\| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \|v(s)\| d_q s \\ &\leq \|u_0\| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) d_q s \\ &\leq \|u_0\| + \frac{p^* T^{(\alpha)}}{\Gamma_q(1+\alpha)}. \end{aligned}$$

Then

$$\|u\| \leq \|u_0\| + \frac{p^* T^{(\alpha)}}{\Gamma_q(1+\alpha)} := d.$$

Set

$$U = \{u \in C_\gamma : \|u\| < 1 + d\}.$$

Hence, the condition (1.5) is satisfied. Finally, Theorem 1.42 implies that N has at least one fixed point $u \in C_E(I)$ which is a solution of problem (5.1)-(5.2).

5.3 An Example

Let

$$E = l^1 = \left\{ u = (u_1, u_2, \ldots, u_n, \ldots), \sum_{n=1}^{\infty} |u_n| < \infty \right\}$$

be the Banach space with the norm

$$\|u\|_E = \sum_{n=1}^{\infty} |u_n|.$$

Consider now the following problem of fractional $\frac{1}{4}-$ difference inclusion

$$\begin{cases} ({}^c D_{\frac{1}{4}}^{\frac{1}{2}} u_n)(t) \in F_n(t, u(t)); \ t \in [0, e], \\ u(0) = (1, 0, \ldots, 0, \ldots), \end{cases} \tag{5.6}$$

where
$$F_n(t,u(t)) = \frac{t^2e^{-4-t}}{1+\|u(t)\|_E}[u_n(t)-1,u_n(t)];\ t\in[0,e],$$
with $u=(u_1,u_2,\dots,u_n,\dots)$. Set $\alpha=\frac{1}{2}$, and $F=(F_1,F_2,\dots,F_n,\dots)$.
For each $u\in E$ and $t\in[0,e]$, we have
$$\|F(t,u)\|_{\mathcal{P}} \le p(t),$$
with $p(t)=t^2e^{-t-4}$. Hence, the hypothesis (H_2) is satisfied with $p^*=e^{-2}$. A simple computation shows that conditions of Theorem 5.4 are satisfied. Hence, the problem (5.6) has at least one solution defined on $[0,e]$.

Chapter 6

Weak Solutions for Caputo Pettis Fractional q-Difference Inclusions

6.1 Introduction

This chapter deals with some existence of weak solutions for a class of Caputo fractional q-difference inclusions and a coupled system of Caputo fractional q-difference inclusions by using the set-valued analysis, and Mönch's fixed point theorem associated with the technique of measure of weak noncompactness. Two illustrative examples are given in the end. In Section 6.2 we discuss the existence of weak solutions for the following fractional q-difference inclusion

$$({}^cD_q^\alpha u)(t) \in F(t,u(t)),\ t \in I := [0,T], \tag{6.1}$$

with the initial condition

$$u(0) = u_0 \in E, \tag{6.2}$$

where E is a real (or complex) Banach space with norm $\|\cdot\|$ and dual E^*, such that E is the dual of a weakly compactly generated Banach space X, $q \in (0,1),\ \alpha \in (0,1],\ T > 0,\ F : I \times E \to \mathcal{P}(E)$ is a multivalued map, $\mathcal{P}(E)$ is the family of all nonempty subsets of E, ${}^cD_q^\alpha$ is the Caputo fractional q-difference derivative of order α.

Next in Section 6.3 we consider the following coupled system of fractional q-difference inclusions

$$\begin{cases} ({}^cD_q^\alpha u)(t) \in F(t,v(t)) \\ ({}^cD_q^\alpha v)(t) \in G(t,u(t)) \end{cases} ;\ t \in I, \tag{6.3}$$

with the initial conditions

$$(u(0),v(0)) = (u_0,v_0) \in E \times E, \tag{6.4}$$

where $F,G : I \times E \to \mathcal{P}(E)$ are multivalued maps.

6.2 Caputo-Pettis Fractional q-Difference Inclusions

Let us start by defining what we mean by a weak solution of the problem (6.1)-(6.2).

Definition 6.1 *By a weak solution of the problem (6.1)-(6.2) we mean a function $u \in C_E(I)$ that satisfies*

$$u(t) = u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v(s) d_q s,$$

where $v \in S_{F \circ u}$.

Corollary 6.2 *Let $F : I \times E \to \mathcal{P}(E)$ be such that $S_{F \circ u} \subset C_E(I)$ for any $u \in C_E(I)$. Then problem (6.1)-(6.2) is equivalent to the problem of the solutions of the integral equation*

$$u(t) = u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v(s) d_q s,$$

where $v \in S_{F \circ u}$.

We introduce the following hypotheses:

(H_1) $F : I \times E \to \mathcal{P}_{cp,cl,cv}(E)$ has weakly sequentially closed graph;

(H_2) For each $u \in C_E(I)$, there exists a function $v \in S_{F \circ u}$ wich is measurable a.e. on I and Pettis integrable on I;

(H_3) There exists a function $p \in L^\infty(I, \mathbb{R}_+)$ such that for all $\varphi \in E^*$, we have

$$\|F(t,u)\|_{\mathcal{P}} = \sup_{v \in S_{F \circ u}} |\varphi(v)| \leq p(t); \;\; for\ a.e.\ t \in I,\ and\ each\ u \in E;$$

(H_4) For each bounded and measurable set $B \subset E$ and for each $t \in I$, we have

$$\beta(F(t,B)) \leq p(t)\beta(B).$$

Set

$$p^* = \operatorname*{ess\,sup}_{t \in I} p(t),$$

Theorem 6.3 *Assume that the hypotheses $(H_1) - (H_4)$ hold. If*

$$L := \frac{p^* T^\alpha}{\Gamma_q(1+\alpha)} < 1, \tag{6.5}$$

then the problem (6.1)-(6.2) has a weak solution defined on I.

Proof. Consider the multi-valued map $N : C_E(I) \to \mathcal{P}_{cl}(C_E(I))$ defined by:

$$(Nu)(t) = \left\{ h \in C_E(I) : h(t) = u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v(s) d_q s;\ v \in S_{F\circ u} \right\}. \tag{6.6}$$

Our hypotheses imply that for each $u \in C_E(I)$, there exists a Pettis integrable function $v \in S_{F\circ u}$, and for each $s \in [0,t]$, the function

$$t \mapsto (t-qs)^{\alpha-1} v(s),\ \ for\ a.e.\ t \in I,$$

is Pettis integrable. Thus, N is well defined. Let $R > 0$ be such that

$$R > \frac{p^* T^\alpha}{\Gamma_q(1+\alpha)},$$

and consider the set

$$\begin{aligned} Q = \Big\{ & u \in C_E(I) : \|u\|_\infty \le R \text{ and } \|u(t_2) - u(t_1)\| \\ & \le \frac{p^* T^\alpha}{\Gamma_q(1+\alpha)}(t_2 - t_1)^\alpha + \frac{p^*}{\Gamma_q(\alpha)} \int_0^{t_1} |(t_2 - qs)^{\alpha-1} - (t_1 - qs)^{\alpha-1}| d_q s \Big\}. \end{aligned}$$

The set Q is closed, convex and equicontinuous. We shall show in several steps that N satisfies the assumptions of Theorem 1.39.

Step 1. *$N(u)$ is convex for each $u \in Q$.*
For that, let $h_1, h_2 \in N(u)$. Then there exist $v_1, v_2 \in S_{F\circ u}$ such that, for each $t \in I$, and for any $i = 1, 2$, we have

$$h_i(t) = u_0 + \int_0^t (t-s)^{\alpha-1} \frac{v_i(s)}{\Gamma_q(\alpha)} ds.$$

Let $0 \le \lambda \le 1$. Then, for each $t \in I$, we have

$$[\lambda h_1 + (1-\lambda) h_2](t) \ = \ u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} (\lambda v_1(s) + (1-\lambda) v_2(s)) d_q s.$$

Since F has convex values, $S_{F\circ u}$ is convex. Hence, it follows that

$$\lambda h_1 + (1-\lambda) h_2 \in N(u).$$

Step 2. *N maps Q into itself.*
Let $h \in N(Q)$, then there exists $u \in Q$ such that $h \in N(u)$, and there exists a Pettis integrable function $v \in S_{F\circ u}$. Assume that $h(t) \neq 0$. Then there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ such that $\|h(t)\| = |\varphi(h(t))|$. Thus

$$\|h(t)\| = \varphi\left(u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v(s) d_q s \right).$$

Hence

$$\begin{aligned}\|h(t)\| &\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|\varphi(v(s))|d_qs \\ &\leq \frac{p^*}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{\alpha-1}d_qs \\ &\leq \frac{p^*T^\alpha}{\Gamma_q(1+\alpha)} \\ &\leq R.\end{aligned}$$

Next, let $t_1, t_2 \in I$ such that $t_1 < t_2$ and let $h \in N(u)$, with

$$h(t_2) - h(t_1) \neq 0.$$

Then there exists $\varphi \in E^*$ such that

$$\|h(t_2) - h(t_1)\| = |\varphi(h(t_2) - h(t_1))|,$$

and $\|\varphi\| = 1$. Then, we have

$$\begin{aligned}\|h(t_2) - h(t_1)\| &= |\varphi(h(t_2) - h(t_1))| \\ &\leq \varphi\left(\int_0^{t_2}(t_2-qs)^{\alpha-1}\frac{v(s)}{\Gamma_q(\alpha)}d_qs - \int_0^{t_1}(t_1-qs)^{\alpha-1}\frac{v(s)}{\Gamma_q(\alpha)}d_qs\right).\end{aligned}$$

Thus, we get

$$\begin{aligned}\|h(t_2) - h(t_1)\| &\leq \int_{t_1}^{t_2}(t_2-qs)^{\alpha-1}\frac{|\varphi(v(s))|}{\Gamma_q(\alpha)}d_qs \\ &\quad + \int_0^{t_1}|(t_2-qs)^{\alpha-1} - (t_1-qs)^{\alpha-1}|\frac{|\varphi(v(s))|}{\Gamma_q(\alpha)}d_qs \\ &\leq \int_{t_1}^{t_2}(t_2-qs)^{\alpha-1}\frac{p(s)}{\Gamma_q(\alpha)}d_qs \\ &\quad + \int_0^{t_1}|(t_2-qs)^{\alpha-1} - (t_1-qs)^{\alpha-1}|\frac{p(s)}{\Gamma_q(\alpha)}d_qs.\end{aligned}$$

Hence, we obtain

$$\|h(t_2) - h(t_1)\| \leq \frac{p^*T^\alpha}{\Gamma_q(1+\alpha)}(t_2-t_1)^\alpha + \frac{p^*}{\Gamma_q(\alpha)}\int_0^{t_1}|(t_2-qs)^{\alpha-1} - (t_1-qs)^{\alpha-1}|d_qs.$$

This implies that $h \in Q$. Hence $N(Q) \subset Q$.

Step 3. *N has weakly-sequentially closed graph.*
Let (u_n, w_n) be a sequence in $Q \times Q$, with $u_n(t) \to u(t)$ in (E, ω) for each $t \in I$, $w_n(t) \to$

$w(t)$ in (E,ω) for each $(t \in I$, and $w_n \in N(u_n)$ for $n \in \{1,2,\ldots\}$.
We show that $w \in \Omega(u)$. Since $w_n \in \Omega(u_n)$, there exists $v_n \in S_{F\circ u_n}$ such that

$$w_n(t) = u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v_n(s) d_q s.$$

We show that there exists $v \in S_{F\circ u}$ such that, for each $t \in I$,

$$w(t) = u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v(s) d_q s.$$

From the fact that $F(\cdot,\cdot)$ has compact values, there exists a Pettis integrable subsequence v_{n_m}; such that

$$v_{n_m}(t) \in F(t, u_n(t)) \text{ a.e. } t \in I,$$

and

$$v_{n_m}(\cdot) \to v(\cdot) \text{ in } (E,\omega) \text{ as } m \to \infty.$$

As $F(t,\cdot)$ has weakly sequentially closed graph, $v(t) \in F(t, u(t))$. Then by the Lebesgue Dominated Convergence Theorem for the Pettis integral, we obtain

$$\varphi(w_n(t)) \to \varphi\left(u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v_n(s) d_q s\right),$$

i.e. $w_n(t) \to (Nu)(t)$ in (E,ω). Since this holds, for each $t \in I$, we get $w \in N(u)$.

Step 4. *The condition (1.3) of Theorem 1.39 holds.*
Let $V \subset Q$, such that $\overline{V} = \overline{conv}(\Omega(V) \cup \{0\})$.
For each$t \in I,\ V(t) \subset \overline{conv}(\Omega(V(t)) \cup \{0\})$. Since V is bounded and equicontinuous, the function $t \to v(t) = \beta(V(t))$ is continuous on I. By (H_4) and the properties of β, for any $t \in I$ we have

$$\begin{aligned}
v(t) &\leq \beta((NV)(t) \cup \{0\}) \\
&\leq \beta((NV)(t)) \\
&\leq \beta\{(Nu)(t) : u \in V\} \\
&\leq \beta\left\{\int_0^t (t-qs)^{\alpha-1} \frac{v(s)}{\Gamma_q(\alpha)} d_q s : v(t) \in S_{F\circ u},\ u \in V\right\} \\
&\leq \beta\Big\{\int_0^t (t-qs)^{\alpha-1} \frac{F(s,V(s))}{\Gamma_q(\alpha)} d_q s\Big\} \\
&\leq \int_0^t (t-qs)^{\alpha-1} \frac{\beta(V(s))}{\Gamma_q(\alpha)} d_q s \\
&\leq \int_0^t (t-qs)^{\alpha-1} \frac{p(s)v(s)}{\Gamma_q(\alpha)} d_q s \\
&\leq \frac{p^* T^\alpha}{\Gamma_q(1+\alpha)} \|v\|_\infty \\
&= L\|v\|_\infty.
\end{aligned}$$

In particular,

$$\|u\|_\infty \leq L\|v\|_\infty.$$

By (6.5) it follows that $\|v\|_\infty = 0$, that is, $v(t) = \beta(V(t)) = 0$ for each $t \in I$, and then V is weakly relatively compact in $C_E(I)$. Applying now Theorem 1.39, we conclude that N has a fixed point which is a weak solution of the problem (6.1)-(6.2).

6.3 Coupled Systems of Caputo-Pettis Fractional q-Difference Inclusions

$\mathcal{C} := C_E(I) \times C_E(I)$ is a Banach space with the norm:

$$\|(u,v)\|_{\mathcal{C}} = \|u\|_\infty + \|v\|_\infty.$$

Definition 6.4 *By a weak solution of the problem (6.3)-(6.4) we mean a coupled measurable functions $(u,v) \in \mathcal{C}$ that satisfies*

$$\begin{cases} u(t) = u_0 + \displaystyle\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} w(s) d_q s, \\ v(t) = v_0 + \displaystyle\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} z(s) d_q s, \end{cases}$$

where $w \in S_{F\circ v}$, and $z \in S_{F\circ u}$.

Consider the following:

(H_{01}) $F, G : I \times E \to \mathcal{P}_{cp,cl,cv}(E)$ have weakly sequentially closed graph,

(H_{02}) For all continuous functions $u, v : I \to E$, there exist measurable functions $w \in S_{F\circ v}$, $z \in S_{F\circ u}$, a.e. on I and w, z are Pettis integrable on I,

(H_{03}) There exist $p, d \in L^\infty(I, \mathbb{R}_+)$ such that for all $\varphi \in E^*$, we have

$$\|F(t,v)\|_{\mathcal{P}} \leq p(t), \text{ for a.e. } t \in I, \text{ and each } v \in E,$$

$$\|G(t,u)\|_{\mathcal{P}} \leq d(t), \text{ for a.e. } t \in I, \text{ and each } u \in E.$$

(H_{04}) For each bounded and measurable set $B \subset E$ and for each $t \in I$, we have

$$\beta(F(t,B) \leq p(t)\beta(B), \quad \textit{and} \quad \beta(G(t,B) \leq d(t)\beta(B).$$

Set

$$p^* = \operatorname*{ess\,sup}_{t\in I} p(t), \; d^* = \operatorname*{ess\,sup}_{t\in I} d(t).$$

Theorem 6.5 *Assume that the hypotheses* $(H_{01}) - (H_{04})$ *hold. If*

$$\frac{p^* T^\alpha}{\Gamma_q(1+\alpha)} < 1, \text{ and } \frac{d^* T^\alpha}{\Gamma_q(1+\alpha)} < 1, \tag{6.7}$$

then the problem (6.3)-(6.4) has at least one weak solution defined on I.

Proof. Consider the multi-valued map $N : \mathcal{C} \to \mathcal{P}_{cl}(\mathcal{C})$ defined by:

$$(N(u,v))(t) = ((N_1 u)(t), (N_2 v)(t)),$$

where $N_1, N_2 : C(I) \to \mathcal{P}_{cl}()$ with

$$(N_1 u)(t) = \left\{ h \in C_E(I) : h(t) = u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} w(s) d_q s;\ w \in S_{F \circ v} \right\}, \tag{6.8}$$

and

$$(N_2 v)(t) = \left\{ h \in C(I) : h(t) = v_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} z(s) d_q s;\ z \in S_{F \circ u} \right\}. \tag{6.9}$$

For each $(u,v) \in \mathcal{C}$, there exist Pettis integrable functions $w \in S_{F \circ v},\ z \in S_{F \circ u}$, and for each $s \in [0,t]$, the functions

$$t \mapsto (t-qs)^{\alpha-1} w(s), \quad \text{and} \quad t \mapsto (t-qs)^{\alpha-1} z(s);\ \text{for a.e. } t \in I,$$

are Pettis integrable. Thus, the multi-function N is well defined. Let $R > 0$ be such that

$$R > \max\left\{ \frac{p^* T^\alpha}{\Gamma_q(1+\alpha)}, \frac{d^* T^\alpha}{\Gamma_q(1+\alpha)} \right\}$$

and consider the set

$$\Lambda = \Big\{ (u,v) \in \mathcal{C} : \|(u,v)\|_{\mathcal{C}} \le R \text{ and } \|u(t_2) - u(t_1)\| \le \frac{p^* T^\alpha}{\Gamma_q(1+\alpha)} (t_2 - t_1)^\alpha$$
$$+ \frac{p^*}{\Gamma_q(\alpha)} \int_0^{t_1} |(t_2 - qs)^{\alpha-1} - (t_1 - qs)^{\alpha-1}| d_q s, \text{ and } \|v(t_2) - v(t_1)\|$$
$$\le \frac{d^* T^\alpha}{\Gamma_q(1+\alpha)} (t_2 - t_1)^\alpha + \frac{d^*}{\Gamma_q(\alpha)} \int_0^{t_1} |(t_2 - qs)^{\alpha-1} - (t_1 - qs)^{\alpha-1}| d_q s \Big\}.$$

The subset Λ of $\mathcal{C}$ is closed, convex end equicontinuous. As in the proof of Theorem 6.3, we can show that $N(u,v)$ is convex for each $(u,v) \in \Lambda,\ N(\Lambda) \subset \Lambda,\ N$ has weakly-sequentially closed graph, and the Theorem 1.39 condition (1.3) holds. Hence, the operator N satisfies all the assumptions of Theorem 1.39. Therefore; we conclude that N has a fixed point which is a weak solution of the problem (6.3)-(6.4).

6.4 Examples

Let

$$E = l^1 = \left\{ u = (u_1, u_2, \ldots, u_n, \ldots) : \sum_{n=1}^{\infty} |u_n| < \infty \right\}$$

be the Banach space with the norm

$$\|u\|_E = \sum_{n=1}^{\infty} |u_n|.$$

Example 1. Consider the following problem of fractional $\frac{1}{4}$−difference inclusion

$$\begin{cases} ({}^C D_{\frac{1}{4}}^{\frac{1}{2}} u_n)(t) \in F_n(t, u(t)); \ t \in [0,1], \\ u(0) = (1, 0, \ldots, 0, \ldots), \end{cases} \tag{6.10}$$

where

$$F_n(t, u(t)) = \frac{ct^2 e^{-4-t}}{1 + \|u(t)\|_E} [u_n(t) - 1, u_n(t)]; \ t \in [0,1],$$

with

$$u = (u_1, u_2, \ldots, u_n, \ldots), \ and \ c := \frac{e^4}{4} \Gamma_{\frac{1}{4}} \left(\frac{1}{2} \right).$$

and F is closed and convex valued. Set

$$F = (F_1, F_2, \ldots, F_n, \ldots).$$

For each $u \in E$ and $t \in [0,1]$, we have

$$\|F(t, u(t))\|_{\mathcal{P}} \leq ct^2 \frac{1}{e^{t+4}}.$$

Hence, the hypothesis (H_3) is satisfied with $p^* = ce^{-4}$. We shall show that condition (6.5) holds with $T = 1$. Indeed,

$$L = \frac{ce^{-4}}{\Gamma_{\frac{1}{4}}(\frac{1}{2})} = \frac{1}{4} < 1.$$

Simple computations show that all conditions of Theorem 6.3 are satisfied. Hence, the problem (6.10) has at least one weak solution defined on $[0,1]$.

Example 2. We consider now the following coupled system of fractional $\frac{1}{4}$−difference inclusions

$$\begin{cases} ({}^C D_{\frac{1}{4}}^{\frac{1}{2}} u_n)(t) \in F_n(t, v(t)) \\ ({}^C D_{\frac{1}{4}}^{\frac{1}{2}} v_n)(t) \in G_n(t, u(t)) \\ u(0) = (1, 0, \ldots, 0, \ldots), \ v(0) = (0, 1, 0, \ldots, 0, \ldots) \end{cases} ; \ t \in [0,1], \tag{6.11}$$

where

$$F_n(t, v(t)) = \frac{ct^2 e^{-4-t}}{1 + \|u(t)\|_E}[v_n(t) - 1, v_n(t)],$$

$$G_n(t, u(t)) = \frac{ct^2 e^{-4-t}}{1 + \|u(t)\|_E}[u_n(t), 1 + u_n(t)];\ t \in [0, 1],$$

with

$$u = (u_1, u_2, \ldots, u_n, \ldots),\ v = (v_1, v_2, \ldots, v_n, \ldots),\ \textit{and}\ c := \frac{e^4}{4}\Gamma_{\frac{1}{4}}\left(\frac{1}{2}\right).$$

Set

$$F = (F_1, F_2, \ldots, F_n, \ldots),\ G = (G_1, G_2, \ldots, G_n, \ldots).$$

Simple computations show that all conditions of Theorem 6.5 are satisfied. Hence, the problem (6.11) has at least one weak solution (u, v) defined on $[0, 1]$.

Conclusion and Perspective

In this thesis, we have presented some results to the theory of existence uniqueness and Ulam-Hyers-Rassias stability results for a class of implicit fractional q-difference equations, and we have discused some results about the existence of weak solutions . Also we have presented in this thesis some existence results for a class of Caputo fractional q-difference inclusions in Banach spaces and the existence of weak solutions for the semilar class. This results are obtained by using the fixed point theory and the notion of measure of noncompactness. Such notion requires the use of the set-valued analysis conditions on the right-hand side, and the concept of Pettis integration and an appropriate fixed point theorem.

We plan to study in the future research, problems for nonlinear implicit fractional q-difference equations and inclusions with finite and infinite delay in Banach spaces.

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