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Chapter 1

Introduction

In recent years, fractional calculus has been increasingly applied in different fields of science. Physical phenomena related to electromagnetism, propagation of energy in dissipative systems, thermal stresses, relaxation vibrations, viscoelasticity and thermoelasticity are successfully described by fractional differential equations. Fractional calculus allows for the investigation of the nonlocal response of mechanical systems, this is the main advantage when compared to the classical calculus.

In the literature, a number of definitions of the fractional derivatives have been introduced, namely the Hadamard, Erdelyi-Kober, Riemann-Liouville, Riesz, Weyl, Grunwald-Letnikov, Jumarie and the Caputo representation.

In this thesis we were interested in study of the global existence and the stabilization of some non local evolution equations. More precisely, we will study a wave equations under dynamic boundary feedbacks of fractional derivative type.

The problem of stabilization for the initial boundary value problem

$$(P') \qquad \begin{cases} u_{tt} - \Delta u = 0 \text{ on } \Omega \times (0, +\infty), \\ u = 0 \text{ on } \Gamma_D \times (0, +\infty), \\ \dfrac{\partial u}{\partial \nu} + a(x)u_t = 0 \text{ on } \Gamma_N \times (0, +\infty), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) \text{ on } \Omega, \end{cases}$$

was investigated by several authors. In Haraux [**18**], Bardos, G. Lebeau and J. Rauch [**8**], Lebeau and Robbiano [**8**], Burq [**11**] and Xiaoyu Fu [**15**].

First, A. Haraux has shown that if $a \in L^\infty(\Gamma_N), a \not\equiv 0$, then any solution of (P') tends to 0 in $H^1(\Omega)$ strongly as $t \to +\infty$.

C. Bardos, G. Lebeau and J. Rauch [**8**] introduced a geometric control condition which is a necessary and sufficient condition for uniform exponential decay rate of the energy.

Moreover, Lebeau and Robbiano (see [**25**]) have shown that, in the case where the Neumann boundary condition is applied on the entire boundary, a weak condition on the feedback (which does not satisfy Geometric Control Condition) leads to logarithmic decay of regular solutions. The optimal result without geometrical hypothesis is given in [**11**]. We also recall the result by Fu [**15**], where the author proved a result similar to the one in [**25**] for less regular conditions ($\partial\Omega \in C^2$) by adopting the global Carleman estimate.

The purpose of stabilization is to attenuate the vibrations by feedback, it consists to guarantee the decay of the energy of solutions towards 0 in away, more or less fast. More precisely, we are interested to determine the asymptotic behavior of the energy denoted by $E(t)$ and to give an estimate of the decay rate of the energy. There are several types of stabilization

1)Strong stabilization : $E(t) \to 0$ as $t \to \infty$.

2)Logarithmic stabilization : $E(t) \leq c(lnt)^{-\delta}, c, \delta > 0$.

3)Polynomial stabilization : $E(t) \leq ct^{-\delta}, c, \delta > 0$.

4)Uniform stabilization : $E(t) \leq ce^{-\delta t}c, \delta > 0$.

The present thesis is devoted to the study of the global existence and asymptotic behaviour in time of solutions to wave equations. This work consists of essentially of three chapters:

Chapter 3: Optimal energy decay for a transmission problem of waves under a nonlocal boundary control

We considered the stabilization for the following wave equation with dynamic boundary control of fractional derivative type

$$(P) \qquad \begin{cases} \rho_1 u_{tt}(x,t) - \tau_1 u_{xx}(x,t) = 0 & \text{in } (0, l_0) \times (0, +\infty), \\ \rho_2 v_{tt}(x,t) - \tau_2 v_{xx}(x,t) = 0 & \text{in } (l_0, L) \times (0, +\infty), \end{cases}$$

We investigate the existence and decay properties of solutions for the initial boundary value problem and prove the global existence of its solutions in Sobolev spaces by means of the semigroup theory. To prove decay estimates, we use a technique based on a resolvent estimate and Borichev-Tomilov Theorem.

Chapter 4: Exponential Stability of Compactly Coupled Wave Equations with Time-Varying Delay Terms in the Boundary Feedbacks

We considered the stabilization for compactly coupled wave equations with boundary time-varying delay terms

$$\begin{cases} u_{tt} - \Delta u + l(u - v) = 0 & \text{in } \Omega \times (0, \infty) \\ v_{tt} - \Delta v + l(v - u) = 0 & \text{in } \Omega \times (0, \infty) \end{cases}$$

where $l, \alpha_1, \alpha_2, \beta_1$ and β_2 are positive real numbers, the time-varying delay $\tau(t)$ satisfies some conditions and the initial data are taken in suitable spaces.

We investigate the existence and decay properties of solutions for the initial boundary value problem and prove the global existence of its solutions in Sobolev spaces by means of the semigroup theory. To prove decay estimates, we introduce suitable energie and Lyapounov functionals.

Chapter 5: Blow-up for coupled nonlinear wave equations with fractional damping and source terms

We considered the following wave equation with dynamic boundary control of fractional derivative type

$$\begin{cases} u_{tt} + \partial_t^{1-\alpha} u = div(\rho_1(|\nabla u|^2)\nabla u) + f_1(u,v) \\ v_{tt} + \partial_t^{1-\beta} v = div(\rho_2(|\nabla v|^2)\nabla v) + f_2(u,v) \end{cases}$$

We investigate the existence and the uniqueness properties of solutions for the initial boundary value problem and prove an estimate of time of blow up of its solutions in Sobolev spaces by means of the concavity theorem of H. Levine.

Chapter 2

Preliminaries

In this chapter, we will recall the essential notions, and some basic results, concerning functional spaces, semi-groups and spectral analysis theories, These concepts and results represent an important tool for studying the following chapters.

2.1 Some functional spaces

Definition 2.1.1 *(L^p spaces) For $1 \leq p \leq \infty$, we call $L^p(\Omega)$ the space of measurable functions f on Ω such that*

$$\begin{aligned} \|f\|_{L^p(\Omega)} &= \left(\int_\Omega |f(x)|^p dx\right)^{1/p} < +\infty \quad \textit{for} \quad p < +\infty \\ \|f\|_{L^\infty(\Omega)} &= \sup_\Omega |f(x)| < +\infty \qquad\qquad \textit{for} \quad p = +\infty \end{aligned}$$

The space $L^p(\Omega)$ equipped with the norm $f \longrightarrow \|f\|_{L^p}$ is a Banach space. In particular the space $L^2(\Omega)$ is a Hilbert space equipped with the scalar product defined by

$$(f, g)_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx.$$

We denote by $L^p_{loc}(\Omega)$ the space of functions which are L^p on any bounded sub-domain of Ω.

Now, we will introduce the Sobolev spaces:

Definition 2.1.2 *The Sobolev space $W^{k,p}(\Omega)$ is defined to be the subset of L^p such that function f and its weak derivatives up to some order k have a finite L^p norm, for given $p \geq 1$.*

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega); D^\alpha f \in L^p(\Omega). \ \ \forall\alpha; |\alpha| \leq k\} \ ,$$

With this definition, the Sobolev spaces admit a natural norm,

$$f \longrightarrow \|f\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\leq m} \|D^\alpha f\|^p_{L^p(\Omega)}\right)^{1/p} , \ \textit{for } p < +\infty$$

and

$$f \longrightarrow \|f\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha|\leq m} \|D^\alpha f\|_{L^\infty(\Omega)} \ , \ for\ p = +\infty$$

Space $W^{k,p}(\Omega)$ equipped with the norm $\| \ . \ \|_{W^{k,p}}$ is a Banach space.

Definition 2.1.3 *Whene $p = 2$, we denote by*

$$W^{k,2}(\Omega) = H^k(\Omega)$$

the H^k inner product is defined in terms of the L^2 inner product:

$$(f, g)_{H^k(\Omega)} = \sum_{|\alpha|\leq k} (D^\alpha f, D^\alpha g)_{L^2(\Omega)} \ .$$

The space $H^m(\Omega)$ and $W^{k,p}(\Omega)$ contain $\mathcal{C}^\infty(\overline{\Omega})$ and $\mathcal{C}^m(\overline{\Omega})$. The closure of $\mathcal{D}(\Omega)$ for the $H^m(\Omega)$ norm (respectively $W^{m,p}(\Omega)$ norm) is denoted by $H_0^m(\Omega)$ (respectively $W_0^{k,p}(\Omega)$).

Theorem 2.1.1 *Let $k \in \mathbb{N}$ and $1 \leq p < \infty$. We then have*

$$W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$$

Moreover, the set $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.

Theorem 2.1.2 *(Sobolev, Morrey). Let $k \in \mathbb{N}$ and $1 \leq p < \infty$. We have the following embeddings.*

(i) Let $kp < n$, then

$$W^{k,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \ for\ all\ q \in [p, p^\star]$$

where p^* is given by $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{k}{n}$

(ii) Let $kp = n$, then

$$W^{k,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \ for\ all\ q \in [p, \infty]$$

(iii) Let $kp > n$, then there are $j \in \mathbb{N}_0$ and $0 < \beta < 1$ such that $k - \frac{n}{p} = j + beta$ or $k - \frac{n}{p} \in \mathbb{N}$. In the latter case we set $j = k - \frac{n}{p} - 1 \in \mathbb{N}_0$ and take any $0 < \beta < 1$, then

$$W^{k,p}(\mathbb{R}^n) \subset C_0^{j+\beta}(\mathbb{R}^n),$$

where

$$C_0^{j+\beta}(\mathbb{R}^n) = \{u \in C^j(\mathbb{R}^n) : \partial^\alpha u \leftarrow 0 \ as\ |x|_2 \leftarrow \infty \ and\ partial^\alpha u\ is$$
$$\beta\text{-}Holder\ continuous\ on\ \mathbb{R}^n,\ for\ all\ 0 \leq |\alpha| \leq j\}.$$

Corollary 2.1.1 *Let $k \in \mathbb{N}$, $j \in \mathbb{N}_0$, and $1 \leq p < \infty$. We have the following embeddings.*

(i) if $1 \leq p < \infty$ and $k - \frac{n}{p}\ geqj - \frac{n}{q}$, then

$$W^{k,p}(\mathbb{R}^n) \subset W^{j,p}(\mathbb{R}^n)$$

(ii) if $1 \leq p < \infty$ and $k - \frac{n}{p} = j$, then

$$W^{k,p}(\mathbb{R}^n) \subset W^{j,p}(\mathbb{R}^n)$$

(iii) if $0 < \beta < 1$ and $k - \frac{n}{p} = j + \beta$, then

$$W^{k,p}(\mathbb{R}^n) \subset C_0^{j+\beta}(\mathbb{R}^n)$$

2.2 Spectral theory

in this section we try to review the basic definitins and results on linear operators. Let $E,(\|.\|_E)$ and $(F,\|.\|_F)$ be two Banach spaces over $\mathbb{C}$, and H will always denote a Hilbert space equipped with the inner scalar product $<.,.>_H$ and the corresponding norm $\|.\|_H$.

2.2.1 Linear operators

A linear operator $A : E \mapsto F$ is a transformation which maps lineary E in F, that is

$$A(\alpha u + \beta v) = \alpha A(u) + \beta A(v), \quad \forall u, v \in E \quad and \quad \alpha, \beta \in \mathbb{C}$$

in this chapter we try to review the basic definitins and results on linear operators,

Definition 2.2.1 *A linear operator $A : E \mapsto F$ is said to be bounded there exists $C \geq 0$ such that*

$$\|Au\|_F < C\|u\|_E \quad \forall u \in E$$

The set of all bounded linear operators from E into F is denoted by $\mathcal{L}(E,F)$. Moreover, the set of all bounded linear operators from E into E is denoted by $\mathcal{L}(E)$

Definition 2.2.2 *A bounded operator $A \in \mathcal{L}(E,F)$ is said to be compact if for each sequence $(u)_{n\in\mathbb{N}} \in E$ with $\|u_n\|_E = 1$ for each $n \in \mathbb{N}$, the sequence $(Au_n)_{n\in\mathbb{N}}$ has a subsequence which converges in F.*

Definition 2.2.3 *An unbounded linear operator T from E into F is a pair (T,D(T)), consisting of a subspace $D(A) \subset E$ (called the domain of A) and a linear transformation*

$$T : D(A) \subset E \to F$$

In the case when $E = F$ then we say (A,D(A)) is an unbounded linear operator on E. If $D(A) = E$ then $A \in \mathcal{L}(E,F)$.

Definition 2.2.4 *Let $T : D(A) \subset E \to F$ be a linear operator.*

- *The range of T is defined by*

$$\mathcal{R}(A) = \{Tu : u \in D(A)\} \subset F$$

- *The kernel of T is defined by*

$$\ker(A) = \{u \in D(A) : Au = 0\} \subset E$$

Definition 2.2.5 *Let A be a linear operator from E to F . The operator A is called closed if for all $u_n \in D(A), n \in \mathbb{N}$, such that there exists $u = \lim x_n$ in E and $v = \lim Au_n$ in F , we have $u \in D(A)$ and $Au = v$.*

Definition 2.2.6 *Let A be a linear operator from E to F . The graph of A is given by*

$$G(A) = \{(u, Au) \in E \times F : u \in D(A)\}$$

The graph norm of A is defined by $\|u\|_A = \|u\|_E + \|Au\|_F$.

Theorem 2.2.1 *(Closed Graph Theorem). Let E and F be Banach spaces and A be a closed operator from E to F . Then A is bounded if and only if $D(A)$ is closed in E. In particular, a closed operator with $D(A) = E$ belongs to $\mathcal{L}(E, F)$.*

Theorem 2.2.2 *Let $(A, D(A))$be a closed linear operator on H then the space $(D(A), \|.\|_{D(A)})$ where $\|u\|_{D(A)} = \|Au\|_H + \|u\|_H \quad \forall u \in D(A)$ is Banach space.*

Definition 2.2.7 *Let $A : D(A) \subset E \to F$ be a closed linear operator.*

- *The resolvent set of A is defined by*

$$\rho(A) = \{\lambda \in \mathbb{C} : \lambda I - A \quad is \quad bijective \quad from \quad D(A) \quad onto \quad F\}$$

- *The resolvent of T is defined by*

$$\mathrm{R}(\lambda, A) = \{(\lambda I - A)^{-1} : \lambda \in \rho(T)\}$$

- *The spectrum set of A is the complement of the resolvent set in $\mathbb{C}$, denoted by*

$$\sigma(A) = \mathbb{C}/\rho(A)$$

Definition 2.2.8 *Let $A : D(A) \subset E \to F$ be a closed linear operator. we can split the spectrum $\sigma(A)$ of A into three disjoint sets, given by*

- *The ponctuel spectrum of A is define by*

$$\sigma_p(A) = \{\lambda \in \mathbb{C} : \ker(\lambda I - A) \neq \{0\}\}$$

  in this case λ is called an eignvalue of A

- *The continuous spectrum of A is define by*

$$\sigma_c(A) = \{\lambda \in \mathbb{C} : \ker(\lambda I - A) = 0, \overline{\mathcal{R}(\lambda I - A)} = F \quad and \quad (\lambda I - A)^{-1} \quad is \quad not \quad bounded\}.$$

- *The residual spectrum of A is define by*

$$\sigma_c(A) = \{\lambda \in \mathbb{C} : \ker(\lambda I - A) = 0 \quad and \quad \mathcal{R}(\lambda I - A) \quad is \quad not \quad dense \quad in \quad F\}$$

Definition 2.2.9 *Let $A : D(A) \subset E \to F$ be a closed unbounded linear operator and let λ be an eigenvalue of A. A non-zero element $e \in E$ is called a generalized eigenvalue of T associated with the eigenvalue λ, if there exists $n \in \mathbb{N}^*$ such that*

$$(\lambda I - A)^n e = 0 \quad and \quad (\lambda I - A)^{n-1} e \neq 0$$

. If $n = 1$, then is called an eigenvector.

Definition 2.2.10 *Let $A : D(A) \subset E \to F$ be a closed unbounded linear operator. We say that T has compact resolvent, if there exist $\lambda_0 \in \rho(A)$ such that $(\lambda_0 I - A)^{-1}$ is compact.*

Theorem 2.2.3 *Let $(A, D(A))$ be a closed unbounded linear operator on H then, $\rho(A)$ is an open set of* $\mathbb{C}$

Definition 2.2.11 *Let $(A, D(A))$ be a densely defined linear operator on H. The adjoint $A^\star$ of A is the operator defined by*

$$D(A^\star) = \{f \in H; \exists f^\star \in H; \langle f^\star, g\rangle = \langle f, Ag\rangle, \forall g \in D(A)\} \tag{2.1}$$

and $A^\star f = f^\star, \forall f \in D(A^\star)$.

Theorem 2.2.4 *Let A be a closed operator on H with dense domain. Then the following assertions hold.*

(i) $\sigma_r(A) = \sigma_p(A^\star)$
(ii) $\sigma(A) = \sigma(A^\star)$ *and* $R(\lambda, A)^\star = R(\lambda, A^\star), \forall \lambda \in \rho(A)$.

2.2.2 Semigroups

Now, we start to intoduce some basic concepts concerning the semigroups.

Definition 2.2.12 *A family $(S(t))_{t\geq 0}$ of bounded linear operations in X is called a semigroup if*

- $S(0) = I$ *(I is called identity operator on X).*
- $S(t+s) = S(t)S(s), \quad \forall t, s \geq 0$

Definition 2.2.13 *A semigroup of bounded linear operators, $(S(t))_{t\geq 0})$, is called*

- *Uniformly continuous semigroup if* $\lim_{t \leftarrow 0^+} \|S(t) - T\|_{\mathcal{L}(H)} = 0$
- *Strong continuous semigroup (in short, a C_0-semigroup) if for each $u \in H, S(t)u$ is continuous in t on $[0, +\infty[$*

Definition 2.2.14 *For a semigroup $(S(t))_{t\geq 0}$, we define an linear operator A with domain D(A) consisting of points u such that the limit*

$$Au := \lim_{t\to 0^+} \frac{S(t)u - u}{t}, \quad u \in D(A)$$

exists. Then A is called the infinitesimal generator of the semigroup in X.

Theorem 2.2.5 *Let $(S(t))_{t\geq 0}$ be a C_0-semigroup. Then there exist a constant $M \geq 1$ and $\omega \geq 0$ such that*

$$\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}, \forall t \geq 0$$

In the above theorem, if $\omega = 0$ then the corresponding semigroup is uniformly bounded. moreover, if $M = 1$ then $(S(t))_{t\geq 0}$ is said to be a C_0-semigroup of contractions.

Definition 2.2.15 *A linear operator (A,D(A)) on H, is said to be dissipative if*

$$\mathrm{R} < Au, u >_H \geq 0, \quad \forall u \in D(A)$$

Definition 2.2.16 *A linear operator $(A, D(A))$ on X, is said to be m-dissipative if*

- *A is dissipative operator*
- *A is maximal i.e. $\exists \lambda_0 > 0 \quad such \quad that \quad \mathcal{R}(\lambda_0 I - A) = X$*

Theorem 2.2.6 *(Lumer-Phillips) Let A be a linear operator with dense domain $D(A)$ in a Banach space X.*

(i) If A is dissipative and there exists $\lambda_0 > 0$ such that the range $\mathcal{R}(\lambda_0 I - A) = X$, then A generates a C_0 semigroup of contractions on X.

(ii) If A is the infinitesimal generator of a C_0-semigroup of contractions on X then $\mathcal{R}(\lambda I - A) = X$ for all $\lambda > 0$ and A is dissipative

Now we consider the abstract problem,

$$\begin{cases} U_t = \mathcal{A}U, \quad t > 0 \\ U(0) = U_0, \end{cases} \tag{2.2}$$

where A is the infinitesimal generator of C_0-semigroup $S(t)$ over a Hilbert space H.

Theorem 2.2.7 *(Hill-Yoshida) Let $(A, D(A))$ be a linear operator on H. Assume that A is the infinitesimal generator of C_0-semigroup of contractions$(S(t))_{t\geq 0}$*

1. *For $U_0 \in D(A)$, the problem (2.2) admits a unique strong solution*

$$U(t) = S(t)U_0 \in C^0(\mathbb{R}^+, D(A)) \cap C^1(\mathbb{R}^+, H)$$

.

2. *For $U_0 \in H$, the problem (2.2) admits a unique weak solution*

$$U(t) \in C^0(\mathbb{R}^+, H)$$

.

2.2.3 Stability of semigroups

We introduce some basic results about strong, exponential and polynomial stability of a C_0-semigroup.

Definition 2.2.17 *Assume that A is the operator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on X. We say that the C_0-semigroup $(S(t))_{t\geq 0}$ is*

- *Strongly (asymptotically) stable if*

$$\lim_{t\to+\infty} \|S(t)u\|_X = 0 \quad \forall u \in X$$

- *Exponentially (uniformly) stable if there exist two positive constants M and ϵ such that*

$$\|S(t)u\|_X \leq Me^{-\epsilon t}\|u\|_X \quad \forall t > 0, \forall u \in X$$

- *Polynomially stable if there exist two positive constants C and α such that*

$$\|S(t)u\|_X \leq Ct^{-\alpha}\|u\|_X \quad \forall t > 0, \forall u \in X$$

Example 2.2.1 *(Shift semigroup). Consider $H := L^2(\mathbb{R}_+, H_0)$ for a Hilbert space H_0 and $S()$ defined by $S(t)f(s) := f(s+t), f \in H, t, s \geq 0$. The semigroup $(S(t))_{t\geq 0}$ is called the left shift semigroup on H and is strongly stable.*

Now, we give the necessary conditions of strong stability of a C_0-semigroup.

Theorem 2.2.8 *(Arendt and Batty) Assume that A is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on a reflexive Banach space X. If*

- *A has no pure imaginary eigenvalues.*
- *$\sigma(A) \cap i\mathbb{R}$ is countable.*

Then S(t) is strongly stable.

The proof based on the Lyubich and Vu construction of the isometric limit semigroup (see [**13**]). An alternative proof given by Arendt and Batty (see [**4**]) using the Laplace transform.

In this thesis, exponential stability results are obtained using different methods : multipliers method, frequency domain approach obtained by Huang-Pruss, Riez basis approach.

Theorem 2.2.9 *(Huang-Pruss) Assume that A is the generator of strongly continuous semi-group of contractions $(S(t))_{t\geq 0}$ on H. $S(t)$ is uniformly stable if and only if*

- $i\mathbb{R} \subset \rho(A)$.
- $\sup_{\beta\in\mathbb{R}} \|(i\beta I - A)^{-1}\|_{\mathcal{L}(H)} < +\infty$

Definition 2.2.18 *Assume that A is the infinitesimal generator of C_0-semigroup of contractions $(S(t))_{t\geq 0}$ on a Hilbert space H.*

- *The growth bound of A is define by*

$$\omega_0(A) := \inf\{\omega \in \mathbb{R}; \exists N_\omega \in \mathbb{R} : \|S(t)\| \leq N_\omega e^{\omega t}, \quad \forall t \geq 0\}$$

- *The spectral bound of A is define by*

$$s(A) = \sup\{\mathrm{R}(\lambda) : \lambda \in \sigma(A)\}$$

Notice that $s(A) \leq \omega_0(A)$ for any infinitesimal generator of a strongly continuous semigroup.

In the case when the C_0-semigroup is not exponentially stable we look for a polynomial one.

Theorem 2.2.10 *(Batty, A.Borichev and Y.Tomilov, Z.Liu and B.Rao) Assume that A is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on H. If $i\mathbb{R} \subset \rho(A)$, then for a fixed $l > 0$ the following conditions are equivalent*

1. $\lim\limits_{|\lambda|\to+\infty} \sup \dfrac{1}{\lambda^l} \|(\lambda I - A)^{-1}\|_{\mathcal{L}(H)} < +\infty$
2. $\|S(t)U_0\|_H \leq \dfrac{C}{t^{l-1}} \|U_0\|_{D(A)} \quad \forall t > 0, \quad U_0 \in D(A), \quad for \quad some \quad C > 0$

2.3 Fractional calculus

The purpose of fractional calculation is to extend the fractional derivation or integration using not only intrger order but also non-integer orders. In this section we are interested in two approaches to the most practical fractional derivations. The first is the Riemman-Liouville approach, which is based on integration In order to define the non integer derivation, the second is Caputo's approach. In fact, both approaches start from the Riemann-Liouville fractional integral defined as follow,

Definition 2.3.1 *The fractional integral of order $\alpha > 0$, in sense Rieamann-Liouville is given by*

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad t > a$$

In analogy of the ordinary case, we define the fractional derivative of order $\alpha > 0$, in the sense of Rieamann-Liouville as the left inverse of the Rieamann-Liouville integral of order $\alpha > 0$,

Definition 2.3.2 *The fractional derivative of order $\alpha > 0$, in sens of Rieamann-Liouville of a function f defined on the interval $[a, b]$ is given by*

$$D^{\alpha}_{RL,t}(t) = D^n_t I^{n-\alpha}_t f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s) ds, \quad n = [\alpha] + 1, t > a$$

In particular, if $\alpha = 0$, then

$$D^0_{RL,a} f(t) = I^0_a f(t) = f(t)$$

If $\alpha = n \in \mathbb{N}$, then

$$D^0_{RL,a} f(t) = f^{(n)}(t).$$

moreover, if $0 < \alpha < 1$, then $n = 1$, then

$$D^{\alpha}_{RL,a} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{-\alpha} f(s) ds, \quad t > a$$

Now, the fractional derivatives in the sense of Caputo is defined.

Definition 2.3.3 *The fractional derivative of order $\alpha > 0$, in sense of Caputo is given by*

$$D^{\alpha}_{C,t} f(t) = I^{n-\alpha}_t D^n_t f(t)$$

$$D^{\alpha}_{C,a} f(t) = D^{\alpha}_{RL,a} \left(f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^k \right).$$

where

$$n = \begin{cases} [\alpha] + 1 & if \quad \alpha \notin \mathbb{N}, \\ \alpha & if \quad \alpha \in \mathbb{N}^*, \end{cases}$$

In particular, where $0 < \alpha < 1$, the relation take the form

$$D^{\alpha}_{C,a} f(t) = D^{\alpha}_{C,a}([f(t) - f(a)]) = I^{1-\alpha}_a f\prime(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-a)^{-\alpha} f\prime(s) ds.$$

If $\alpha \in \mathbb{N}$, then $D^{\alpha}_{Ca} f(t) = f^n(t)$.

For $\alpha \to (n-1)^+$ the behavior of the tow derivatives

$$\begin{cases} D^{\alpha}_{RL,t} f(t) \to D^n_t I^1_t = D^{n-1}_t f(t), \\ D^{\alpha}_{C,t} f(t) \to I^1_t D^n_t = D^{n-1}_t f(t) - D^{n-1}_t f(0^+) \end{cases}$$

1. **Relation with Reimann-Liouville derivitive**
   Let $\alpha > 0$ with $(n-1) < \alpha < n$ et $n \in \mathbb{N}^*$, suppose that f is a function such that the derivative in the sense of Caputo and Reimann-Liouville exist then::

$$D^{\alpha}_{C,t} f(t) = D^{\alpha}_{RL,t} \left[f(t) - \sum_{j=0}^{n-1} \frac{t^j}{j!} f^{(j)}(0^+) \right] \tag{2.3}$$

   The derivation of order α of $f(x)$ in the sense of Caputo is equal to that of Reimann-Liouville if:

$$f^{(j)}(a) = 0 \text{ pour } j = 0, 1, 2, ..., n-1$$

2. **Composition with the Fractional Integration Operator**
   If f is a continuous function we have:

$$D^{\alpha}_{C,t}(I^{\alpha}_{t} f(t)) = f(t) \tag{2.4}$$

   and

$$I^{\alpha}_{t}(D^{\alpha}_{C,t} f(t)) = f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a)(t-a)^{j}}{j!} \tag{2.5}$$

3. **Laplace Transform of Caputo's Fractional Derivative**
   If f a function continues then:

$$\mathcal{L}(D^{\alpha}_{C} f)(s) = s^{\alpha}(\mathcal{L}f(s)) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0^{+}) \tag{2.6}$$

Chapter 3

Optimal energy decay for a transmission problem of waves under a nonlocal boundary control

3.1 Introduction

In this chapter we study a transmission wave system with boundary control of nonlocal type given by

$$(P) \qquad \begin{cases} \rho_1 u_{tt}(x,t) - \tau_1 u_{xx}(x,t) = 0 & \text{in } (0,l_0) \times (0,+\infty), \\ \rho_2 v_{tt}(x,t) - \tau_2 v_{xx}(x,t) = 0 & \text{in } (l_0,L) \times (0,+\infty), \end{cases}$$

where ρ_1, ρ_2, τ_1 and τ_2 are positive constants that represent the densities and tensions of the strings u and v, respectively, and the initial conditions are

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x). \tag{3.1}$$

The transmission condition is

$$u(l_0,t) = v(l_0,t), \quad \rho_2\tau_1 u_x(l_0,t) = \rho_1\tau_2 v_x(l_0,t) \quad \forall t \in (0,+\infty), \tag{3.2}$$

followed by the boundary conditions

$$u(0,t) = 0, \quad \tau_2 v_x(L,t) + \gamma\rho_2 \partial_t^{\alpha,\eta} v(L,t) = 0 \quad \forall t \in (0,+\infty) \tag{3.3}$$

and conditions of compatibility

$$u_0(l_0) = v_0(l_0), u_1(l_0) = v_1(l_0), \quad , \rho_2\tau_1 u_{0x}(l_0) = \rho_1\tau_2 v_{0x}(l_0), \tag{3.4}$$

where $\gamma > 0$, the initial data (u_0, u_1, v_0, v_1) belong to a suitable function space. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α, $0 < \alpha < 1$, with respect to the time variable (see Choi and MacCamy [**12**] and E. Blanc, G. Chiavassa, and B. Lombard [**9**]). It is defined as follows

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\, ds, \quad \eta \geq 0.$$

Very little attention has been paid to this type of feedback. Moreover, fractional derivatives involve singular and nonintegrable kernels ($t^\alpha, 0 < \alpha < 1$). This leads to substantial mathematical difficulties such as numerical approximation.

In [**29**], B. Mbodje investigates the decay rate of the energy of the wave equation with a boundary nonlocal control, that is,

$$(PBF) \qquad \begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0 & \text{in } (0,L)\times(0,+\infty), \\ u(0,t) = 0 & \text{on } (0,+\infty), \\ u_x(L,t) + \gamma \partial_t^{\alpha,\eta} u_t(L,t) = 0 & \text{on } (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } (0,L). \end{cases}$$

Using energy methods, he proves strong asymptotic stability under the condition $\eta = 0$ and a polynomial type decay rate $E(t) \leq c/t$ if $\eta \neq 0$.

Very recently, In [**1**], Benaissa and al. considered the Euler-Bernoulli beam equation with boundary dissipation of nonlocal type defined by

$$(PEF) \qquad \begin{cases} u_{tt}(x,t) + u_{xxxx}(x,t) = 0 & \text{in } (0,L)\times(0,+\infty), \\ u(0,t) = u_x(0,t) = 0 & \text{on } (0,+\infty), \\ u_{xx}(L,t) = 0 & \text{on } (0,+\infty), \\ u_{xxx}(L,t) - \gamma \partial_t^{\alpha,\eta} u_t(L,t) = 0 & \text{on } (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } (0,L). \end{cases}$$

They proved, under the condition $\eta = 0$, by a spectral analysis, the non uniform stability. On the other hand, for $\eta > 0$, they also proved that the energy of system (PEF) decays as time goes to infinity as $t^{-1/(1-\alpha)}$.

The question we are interested in this paper is what are the stability properties of our system $((P), (3.1) - (3.4))$. Indeed, this system involves two wave equations coupled at interface with only one nonlocal control acting on a part of the boundary of the second equation. So, From the mathematical point of view, it is important to study the stability of an equation of 1D waves with discontinuous coefficients in a bounded domain. Moreover, this system happens frequently in applications where the domain is occupied by two different types of materials, that is, while one of them is simply elastic, the other is subject to the action of an external force. Let us mention here that the case $\alpha = 1$ corresponds to a static boundary control, that is,

$$\tau_2 v_x(L,t) + \gamma \rho_2 v_t(L,t) = 0 \quad \forall t \in (0,+\infty).$$

It is well known that the energy of the solution decays exponentially under the conditions (see [**38**]).

$$\chi = \frac{\tau_1}{\rho_1} - \frac{\tau_2}{\rho_2} > 0.$$

Nowadays, fractional calculus is not only important from the theoretical point of view but also for applications. The main reason for the diffusion of fractional calculus is that it actually provides a more accurate tool to describe several physical systems. For instance, phenomena such as heat conduction through a semi-infinite solid, water flowing through a porous dyke or infinite lossy transmission lines are indeed fractional. In many industrial and research fields, fractional calculus can be conveniently used. Among these, relevant research topics are electrical circuits, chemical processes, signal processing, viscoelasticity, chaos theory, and obviously control systems (see [**5**], [**6**], [**7**], [**27**], [**36**] and [**40**]). In our case, the fractional dissipations may simply describe an active boundary viscoelastic damper designed for the purpose of reducing the vibrations (see [**29**] [**30**]).

The organization of this paper is as follows. In section **2**, first we show that the system (P) can be replaced by an augmented model by coupling the transmission wave system with a suitable diffusion equation that can be reformulate into classical input output dynamic systems and we deduce the well-posedness property of the problem by the semigroup approach. Second, using a criteria of Arendt-Batty [**4**] we show that the augmented model is strongly stable in the absence of compactness of the resolvent. In section **3**, we show the lack of exponential stability by spectral analysis. In section **4**, we show an optimal energy decay rate depending on the parameter α. The proof heavily relies on a precise estimate of the resolvent of the generator associated to the semi-group and Borichev-Tomilov Theorem.

3.2 Well-Posedness and Strong Stability

This section is concerned with the reformulation of the model (P) into an augmented system. For that, we need the following claims.

Theorem 3.2.1 (see [29]) *Let μ be the function:*

$$\mu(\xi) = |\xi|^{(2\alpha-1)/2}, \quad -\infty < \xi < +\infty,\ 0 < \alpha < 1. \tag{3.5}$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t \phi(\xi, t) + (\xi^2 + \eta)\phi(\xi, t) - U(t)\mu(\xi) = 0, \quad -\infty < \xi < +\infty, \eta \geq 0, t > 0, \tag{3.6}$$

$$\phi(\xi, 0) = 0, \tag{3.7}$$

$$O(t) = (\pi)^{-1}\sin(\alpha\pi)\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi, t)\, d\xi \tag{3.8}$$

is given by

$$O = I^{1-\alpha,\eta}U \tag{3.9}$$

where

$$[I^{\alpha,\eta}f](t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}e^{-\eta(t-\tau)}f(\tau)\, d\tau.$$

Lemma 3.2.1 (see [1]) *If* $\lambda \in D_\eta = \mathbb{C} \backslash]-\infty, -\eta]$ *then*

$$F(\lambda) = \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2}\, d\xi = \frac{\pi}{\sin \alpha\pi} (\lambda + \eta)^{\alpha - 1}.$$

3.2.1 Well-Posedness

We are now in a position to reformulate system (P). Indeed, by using Theorem 3.2.1, system (P) becomes

$$(P') \qquad \begin{cases} \rho_1 u_{tt}(x,t) - \tau_1 u_{xx}(x,t) = 0 & \text{in } (0, l_0) \times (0, +\infty), \\ \rho_2 v_{tt}(x,t) - \tau_2 v_{xx}(x,t) = 0 & \text{in } (l_0, L) \times (0, +\infty), \\ \partial_t \phi(\xi, t) + (\xi^2 + \eta)\phi(\xi, t) - v_t(L,t)\mu(\xi) = 0 & \text{in } (-\infty, \infty) \times (0, +\infty), \\ u(l_0, t) = v(l_0, t), \quad \rho_2 \tau_1 u_x(l_0, t) = \rho_1 \tau_2 v_x(l_0, t) & \text{on } (0, +\infty), \\ u(0,t) = 0 & \text{on } (0, +\infty) \\ \tau_2 v_x(L,t) + \zeta \rho_2 \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi, t)\, d\xi = 0 & \text{on } (0, +\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{on } (0, l_0), \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x) & \text{on } (l_0, L), \end{cases}$$

where $\zeta = (\pi)^{-1} \sin(\alpha\pi)\gamma$. For a solution (u, v, ϕ) of (P'), we define the energy

$$E(t) = \frac{1}{2} \int_0^{l_0} \left(|u_t|^2 + \frac{\tau_1}{\rho_1} |u_x|^2 \right) dx + \frac{1}{2} \int_{l_0}^{L} \left(|v_t|^2 + \frac{\tau_2}{\rho_2} |v_x|^2 \right) dx + \frac{\zeta}{2} \int_{-\infty}^{+\infty} |\phi(\xi, t)|^2\, d\xi. \quad (3.10)$$

Lemma 3.2.2 *Let* (u, v, ϕ) *be a regular solution of the problem* (P')*. Then, the energy functional defined by (3.10) satisfies*

$$E'(t) = -\zeta \int_{-\infty}^{+\infty} (\xi^2 + \eta) |\phi(\xi, t)|^2\, d\xi \leq 0. \quad (3.11)$$

Proof

Multiplying the first equation in (P') by $\overline{u}_t$, integrating by parts over $(0, l_0)$, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_0^{l_0} \left(|u_t|^2 + \frac{\tau_1}{\rho_1} |u_x|^2 \right) dx - \frac{\tau_1}{\rho_1} \Re u_x(l_0) \overline{u}_t(l_0) = 0. \quad (3.12)$$

Multiplying the second equation in (P') by $\overline{v}_t$, integrating by parts over (l_0, L), we obtain

$$\frac{1}{2} \frac{d}{dt} \int_0^{l_0} \left(|v_t|^2 + \frac{\tau_2}{\rho_2} |v_x|^2 \right) dx + \frac{\tau_2}{\rho_2} \Re v_x(l_0) \overline{v}_t(l_0) - \frac{\tau_2}{\rho_2} \Re v_x(L) \overline{v}_t(L, t) = 0. \quad (3.13)$$

Adding the two equations above, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_0^{l_0} \left(|u_t|^2 + \frac{\tau_1}{\rho_1} |u_x|^2 \right) dx + \frac{1}{2} \frac{d}{dt} \int_0^{l_0} \left(|v_t|^2 + \frac{\tau_2}{\rho_2} |v_x|^2 \right) dx - \frac{\tau_2}{\rho_2} \Re v_x(L) \overline{v}_t(L, t) = 0.$$

From the boundary condition $(P')_6$, we have

$$\frac{1}{2}\frac{d}{dt}\left[\int_0^{l_0}\left(|u_t|^2+\frac{\tau_1}{\rho_1}|u_x|^2\right)dx+\int_0^{l_0}\left(|v_t|^2+\frac{\tau_2}{\rho_2}|v_x|^2\right)dx\right] \\ +\zeta\overline{v}_t(L,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi,t)\,d\xi=0. \tag{3.14}$$

Multiplying the third equation in (P') by $\zeta\overline{\phi}_t$ and integrating over $(-\infty,+\infty)$, to obtain:

$$\frac{\zeta}{2}\frac{d}{dt}\|\phi\|_2^2+\zeta\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,t)|^2\,d\xi-\zeta\Re v_t(L,t)\int_{-\infty}^{+\infty}\mu(\xi)\overline{\phi}(\xi,t)\,d\xi=0. \tag{3.15}$$

Consequently, it is resulted from (3.10), (3.14) and (3.15) that

$$E'(t)=-\zeta\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,t)|^2\,d\xi.$$

This completes the proof of the lemma.

We now discuss the well-posedness of (P'). For this purpose, we introduce the following space:

$$H_*^1(0,l_0)=\{u\in H^1(0,l_0):u(0)=0\}.$$

We then reformulate (P') into a semigroup setting. Let $\tilde{u}=u_t,\tilde{v}=v_t$, and set

$$\mathcal{H}=\{H_*^1(0,l_0)\times L^2(0,l_0)\times H^1(l_0,L)\times L^2(l_0,L)\times L^2(-\infty,+\infty)\backslash u(l_0)=v(l_0)\}$$

equipped with the inner product

$$\langle U,U_1\rangle_{\mathcal{H}}=\int_0^{l_0}\left(\tilde{u}\overline{\tilde{u}}_1+\frac{\tau_1}{\rho_1}u_x\overline{u}_{1x}\right)dx+\int_{l_0}^{L}\left(\tilde{v}\overline{\tilde{v}}_1+\frac{\tau_2}{\rho_2}v_x\overline{v}_{1x}\right)dx+\zeta\int_{-\infty}^{+\infty}\phi\overline{\phi}_1\,d\xi$$

for any $U=(u,\tilde{u},v,\tilde{v},\phi)^T$ and $U_1=(u_1,\tilde{u}_1,,v_1,\tilde{v}_1,\phi_1)^T$.

Let $U=(u,\tilde{u},v,\tilde{v},\phi)^T$ and rewrite (P') as

$$\begin{cases}U'=\mathcal{A}U,\\ U(0)=U_0=(u_0,u_1,v_0,v_1,\phi_0),\end{cases} \tag{3.16}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix}u\\ \tilde{u}\\ v\\ \tilde{v}\\ \phi\end{pmatrix}=\begin{pmatrix}\tilde{u}\\ \frac{\tau_1}{\rho_1}u_{xx}\\ \tilde{v}\\ \frac{\tau_2}{\rho_2}v_{xx}\\ -(\xi^2+\eta)\phi+\tilde{v}(L)\mu(\xi)\end{pmatrix}. \tag{3.17}$$

The domain of $\mathcal{A}$ is

$$D(\mathcal{A})=\left\{\begin{array}{l}(u,\tilde{u},v,\tilde{v},\phi)^T \text{ in } \mathcal{H}:u\in H^2(0,L)\cap H_*^1(0,l_0),\tilde{u}\in H_*^1(0,l_0),\\ v\in H^2(l_0,L),\tilde{v}\in H^1(l_0,L),u(l_0)=v(l_0),\rho_2\tau_1u_x(l_0)=\rho_1\tau_2v_x(l_0),\\ \tilde{u}(l_0)=\tilde{v}(l_0),-(\xi^2+\eta)\phi+\tilde{v}(L)\mu(\xi)\in L^2(-\infty,+\infty),\\ \tau_2v_x(L)+\zeta\rho_2\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)\,d\xi=0,\\ |\xi|\phi\in L^2(-\infty,+\infty)\end{array}\right\}. \tag{3.18}$$

The well-posedness of problem (P') is ensured by the following theorem.

Theorem 3.2.2 (Existence and uniqueness)

(1) If $U_0 \in D(\mathcal{A})$, then system (3.16) has a unique strong solution

$$U \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

(2) If $U_0 \in \mathcal{H}$, then system (3.16) has a unique weak solution

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

Proof of Theorem 3.2.2. We show that $\mathcal{A}$ is monotone maximal. First, it is easy to see that we have

$$\Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -E'(t) = -\zeta \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi(\xi)|^2 \, d\xi. \tag{3.19}$$

For the maximality, let $F = (f_1, f_2, f_3, f_4, f_5)^T \in \mathcal{H}$ and look for $U = (u, \tilde{u}, v, \tilde{v}, \phi)^T \in D(\mathcal{A})$ satisfying $\lambda U - \mathcal{A}U = F$ for $\lambda > 0$, that is,

$$\begin{cases} \lambda u - \tilde{u} = f_1, \\ \lambda \tilde{u} - \dfrac{\tau_1}{\rho_1} u_{xx} = f_2, \\ \lambda v - \tilde{v} = f_3, \\ \lambda \tilde{v} - \dfrac{\tau_2}{\rho_2} v_{xx} = f_4, \\ \lambda \phi + (\xi^2 + \eta)\phi - \tilde{v}(L)\mu(\xi) = f_5. \end{cases} \tag{3.20}$$

Assume that with the suitable regularity we have found u and v, then

$$\begin{aligned} \tilde{u} &= \lambda u - f_1, \\ \tilde{v} &= \lambda v - f_3. \end{aligned} \tag{3.21}$$

It is clear that $\tilde{u} \in H^1_*(0, l_0)$ and $\tilde{v} \in H^1(l_0, L)$. Furthermore, by (3.20) we can find ϕ as

$$\phi = \frac{f_5(\xi) + \mu(\xi)\tilde{v}(L)}{\xi^2 + \eta + \lambda}. \tag{3.22}$$

From (3.20) and (3.21) one can see that the functions u and v satisfy the following system

$$\begin{aligned} \lambda^2 u - \tfrac{\tau_1}{\rho_1} u_{xx} &= f_2 + \lambda f_1, \\ \lambda^2 v - \tfrac{\tau_2}{\rho_2} v_{xx} &= f_4 + \lambda f_3. \end{aligned} \tag{3.23}$$

Solving system (3.23) is equivalent to finding $u \in H^2 \cap H^1_*(0, l_0)$ and $v \in H^2(l_0, L)$ such that

$$\begin{aligned} \int_0^{l_0} (\lambda^2 u \overline{w} - \frac{\tau_1}{\rho_1} u_{xx} \overline{w}) \, dx &= \int_0^{l_0} (f_2 + \lambda f_1) \overline{w} \, dx, \\ \int_{l_0}^{L} (\lambda^2 v \overline{\chi} - \frac{\tau_2}{\rho_2} v_{xx} \overline{\chi}) \, dx &= \int_{l_0}^{L} (f_4 + \lambda f_3) \overline{\chi} \, dx, \end{aligned} \tag{3.24}$$

for all $w \in H^1_*(0, l_0)$ and $\chi \in H^1(l_0, L)$. From (3.24) and (3.22) one can see that the functions u and v satisfy the following system

$$\begin{cases} \displaystyle\int_0^{l_0} (\lambda^2 u\overline{w} + \frac{\tau_1}{\rho_1} u_x \overline{w}_x)\, dx + \int_{l_0}^{L} (\lambda^2 v\overline{\chi} + \frac{\tau_2}{\rho_2} v_x \overline{\chi}_x)\, dx + \tilde{\zeta}\lambda v(L)\overline{\chi}(L) \\ \displaystyle= \int_0^{l_0} (f_2 + \lambda f_1)\overline{w}\, dx + \int_{l_0}^{L} (f_4 + \lambda f_3)\overline{\chi}\, dx - \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + \lambda} f_5(\xi)\, d\xi\ \overline{\chi}(L) + \tilde{\zeta} f_3(L)\overline{\chi}(L), \end{cases} \tag{3.25}$$

where $\tilde{\zeta} = \zeta \displaystyle\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2 + \eta + \lambda}\, d\xi$. Consequently, problem (3.25) is equivalent to the problem

$$a((u, v), (w, \chi)) = L(w, \chi), \tag{3.26}$$

where the bilinear form $a : [H^1_*(0, l_0) \times H^1(l_0, L)]^2 \to \mathbb{R}$ and the linear form $L : H^1_*(0, l_0) \times H^1(l_0, L) \to \mathbb{R}$ are defined by

$$a((u, v), (w, \chi)) = \int_0^{l_0} (\lambda^2 u\overline{w} + \frac{\tau_1}{\rho_1} u_x \overline{w}_x)\, dx + \int_{l_0}^{L} (\lambda^2 v\overline{\chi} + \frac{\tau_2}{\rho_2} v_x \overline{\chi}_x)\, dx + \tilde{\zeta}\lambda v(L)\overline{\chi}(L)$$

and

$$L(w, \chi) = \int_0^{l_0} (f_2 + \lambda f_1)\overline{w}\, dx + \int_{l_0}^{L} (f_4 + \lambda f_3)\overline{\chi}\, dx - \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + \lambda} f_5(\xi)\, d\xi\ \overline{\chi}(L) + \tilde{\zeta} f_3(L)\overline{\chi}(L).$$

It is easy to verify that a is continuous and coercive, and L is continuous. Applying the Lax-Milgram Theorem, we infer that for all $(w, \chi) \in H^1_*(0, l_0) \times H^1(l_0, L)$ problem (3.26) has a unique solution $(u, v) \in H^1_*(0, l_0) \times H^1(l_0, L)$. Applying the classical elliptic regularity, it follows from (3.25) that $(u, v) \in H^2(0, l_0) \times H^2(l_0, L)$. Therefore, the operator $\lambda I - \mathcal{A}$ is surjective for any $\lambda > 0$. At last, the result of Theorem 3.2.2 follows from the Hille-Yosida theorem.

3.2.2 Strong stability of the system

Because of the unboundedness of the ξ-domain for the diffusive equation, the resolvent of $\mathcal{A}$ is not compact, then the classical methods such as LaSalle's invariance principle or the spectrum decomposition theory of Benchimol are not applicable in this case. We use a general criteria of Arendt-Batty (see **[4]** or **[26]**), following which a C_0-semigroup of contractions $e^{t\mathcal{A}}$ in a Banach space is strongly stable, if $\mathcal{A}$ has no pure imaginary eigenvalues and $\sigma(\mathcal{A}) \cap i\mathbb{R}$ contains only a countable number of elements. Our main result is the following theorem.

Theorem 3.2.3 *The C_0-semigroup $e^{t\mathcal{A}}$ is strongly stable in $\mathcal{H}$; i.e, for all $U_0 \in \mathcal{H}$, the solution of (3.16) satisfies*

$$\lim_{t\to\infty} \|e^{t\mathcal{A}} U_0\|_{\mathcal{H}} = 0.$$

For the proof of Theorem 3.2.3, we need the following two lemmas.

Lemma 3.2.3 *$\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.*

Proof
We make a distinction between $i\lambda = 0$ and $i\lambda \neq 0$.
Step 1. Solving for $\mathcal{A}U = 0$ leads to $U = 0$, thanks to the boundary conditions in (3.18). Hence, $i\lambda = 0$ is not an eigenvalue of $\mathcal{A}$.
Step 2. We will argue by contradiction. Let us suppose that there $\lambda \in \mathbb{R}, \lambda \neq 0$ and $U \neq 0$, such that $\mathcal{A}U = i\lambda U$. Then, we get

$$\begin{cases} i\lambda u - \tilde{u} = 0, \\ i\lambda \tilde{u} - \dfrac{\tau_1}{\rho_1} u_{xx} = 0, \\ i\lambda v - \tilde{v} = 0, \\ i\lambda \tilde{v} - \dfrac{\tau_2}{\rho_2} v_{xx} = 0, \\ i\lambda \phi + (\xi^2 + \eta)\phi - \tilde{v}(L)\mu(\xi) = 0. \end{cases} \tag{3.27}$$

Then, from (3.19) we have

$$\phi \equiv 0. \tag{3.28}$$

From $(3.27)_5$, we have

$$\tilde{v}(L) = 0. \tag{3.29}$$

Hence, from $(3.27)_3$ and $(3.18)_4$ we obtain

$$v(L) = 0 \text{ and } v_x(L) = 0. \tag{3.30}$$

Inserting $(3.27)_3$ into $(3.27)_4$, we get

$$-\lambda^2 v - \frac{\tau_2}{\rho_2} v_{xx} = 0. \tag{3.31}$$

The solution of the equation (3.31) is given by

$$v(x) = c_1 \cos \frac{\lambda}{r_2} x + c_2 \sin \frac{\lambda}{r_2} x, \quad r_2 = \sqrt{\frac{\tau_2}{\rho_2}}.$$

From boundary conditions (3.30), we deduce that

$$v \equiv 0.$$

Now, from the boundary transmission conditions, we get

$$u(l_0) = u_x(l_0) = 0.$$

Similarly, we deduce that

$$u \equiv 0.$$

Therefore $U = 0$. Consequently, $\mathcal{A}$ does not have purely imaginary eigenvalues.

Lemma 3.2.4
If $\lambda \neq 0$, the operator $i\lambda I - \mathcal{A}$ is surjective.
If $\lambda = 0$ and $\eta \neq 0$, the operator $i\lambda I - \mathcal{A}$ is surjective.

Proof
Case 1: $\lambda \neq 0$. Let $F = (f_1, f_2, f_3, f_4, f_5)^T \in \mathcal{H}$ be given, and let $X = (u, \tilde{u}, v, \tilde{v}, \phi)^T \in D(\mathcal{A})$ be such that

$$(i\lambda I - \mathcal{A})X = F. \tag{3.32}$$

Equivalently, we have

$$\begin{cases} i\lambda u - \tilde{u} = f_1, \\ i\lambda\tilde{u} - \dfrac{\tau_1}{\rho_1}u_{xx} = f_2, \\ i\lambda v - \tilde{v} = f_3, \\ i\lambda\tilde{v} - \dfrac{\tau_2}{\rho_2}v_{xx} = f_4, \\ i\lambda\phi + (\xi^2+\eta)\phi - \tilde{v}(L)\mu(\xi) = f_5. \end{cases} \tag{3.33}$$

We divide the proof into three steps, as follows:
Step 1. Inserting $(3.33)_1, (3.33)_3$ into $(3.33)_2$ and $(3.33)_4$, we get

$$\begin{cases} -\lambda^2 u - r_1 u_{xx} = (f_2 + i\lambda f_1), \\ -\lambda^2 v - r_2 v_{xx} = (f_4 + i\lambda f_3). \end{cases} \tag{3.34}$$

Solving system (3.34) is equivalent to finding $(u, v) \in H^2 \cap H^1_*(0, l_0) \times H^2(l_0, L)$ such that

$$\begin{cases} \displaystyle\int_0^{l_0} (-\lambda^2 u\overline{w} - r_1 u_{xx}\overline{w})\, dx = \int_0^{l_0} (f_2 + i\lambda f_1)\overline{w}\, dx, \\ \displaystyle\int_{l_0}^{L} (-\lambda^2 v\overline{\chi} - r_2 v_{xx}\overline{\chi})\, dx = \int_{l_0}^{L} (f_4 + i\lambda f_3)\overline{\chi}\, dx \end{cases} \tag{3.35}$$

for all $(w, \chi) \in H^1_*(0, l_0) \times H^1(l_0, L)$. By using $(3.33)_3$ and $(3.33)_5$ the functions u and v satisfying the following system

$$\begin{cases} \displaystyle\int_0^{l_0} (-\lambda^2 u\overline{w} + \frac{\tau_1}{\rho_1}u_x\overline{w}_x)\, dx + \int_{l_0}^{L} (-\lambda^2 v\overline{\chi} + \frac{\tau_2}{\rho_2}v_x\overline{\chi}_x)\, dx + \tilde{\zeta} i\lambda v(L)\overline{\chi}(L) \\ \displaystyle = \int_0^{l_0} (f_2 + i\lambda f_1)\overline{w}\, dx + \int_{l_0}^{L} (f_4 + i\lambda f_3)\overline{\chi}\, dx - \zeta\int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+i\lambda} f_5(\xi)\, d\xi\ \overline{\chi}(L) \\ \qquad\qquad + \tilde{\zeta} f_3(L)\overline{\chi}(L). \end{cases} \tag{3.36}$$

We can rewrite (3.36) as

$$-(L_\lambda U, V)_{H^1_R} + (U, V)_{H^1_R} = l(V), \tag{3.37}$$

where

$$H^1_R(0, L) = \{(u, v) \in H^1_*(0, l_0) \times H^1(l_0, L) \backslash u(l_0) = v(l_0)\}$$

with the inner product defined by

$$(U, V)_{H^1_R} = \frac{\tau_1}{\rho_1}\int_0^{l_0} u_x\overline{w}_x\, dx + \frac{\tau_2}{\rho_2}\int_{l_0}^{L} v_x\overline{\chi}_x\, dx - i\tilde{\zeta}\lambda v(L)\overline{\chi}(L).$$

$$(L_\lambda U, V)_{H^1_R} = \int_0^{l_0} \lambda^2 u\overline{w}\, dx + \int_{l_0}^{L} \lambda^2 v\overline{\chi}\, dx.$$

Using the compactness embedding from $(L^2(0,l_0)\times L^2(l_0,L))$ into $(H^1_R(0,L))'$ and from $H^1_R(0,L)$ into $L^2(0,l_0)\times L^2(l_0,L)$ we deduce that the operator L_λ is compact from $L^2(0,l_0)\times L^2(l_0,L)$ into $L^2(0,l_0)\times L^2(l_0,L)$. Consequently, by Fredholm alternative, proving the existence of U solution of (3.37) reduces to proving that 1 is not an eigenvalue of L_λ. Indeed if 1 is an eigenvalue, then there exists $U\neq 0$, such that

$$(L_\lambda U,V)_{H^1_R}=(U,V)_{H^1_R}\quad \forall V\in H^1_R. \tag{3.38}$$

In particular for $V=U$, it follows that

$$\lambda^2\left[\|u\|^2_{L^2(0,l_0)}+\|v\|^2_{L^2(l_0,L)}\right]-i\lambda\tilde{\zeta}|v(L)|^2=\|u_x\|^2_{L^2(0,l_0)}+\|v_x\|^2_{L^2(l_0,L)}.$$

Hence, we have

$$v(L)=0. \tag{3.39}$$

From (3.38), we obtain

$$v_x(L)=0 \tag{3.40}$$

and

$$\begin{cases}-\lambda^2u-r_1u_{xx}=0,\\ -\lambda^2v-r_2v_{xx}=0.\end{cases} \tag{3.41}$$

The general solutions for (3.41) are of the form

$$\begin{array}{l}u(x)=c_1\cos\frac{\lambda}{\sqrt{r_1}}x+c_2\sin\frac{\lambda}{\sqrt{r_1}}x,\\ v(x)=c_3\cos\frac{\lambda}{\sqrt{r_2}}x+c_4\sin\frac{\lambda}{\sqrt{r_2}}x.\end{array} \tag{3.42}$$

Taking into account the boundary conditions $u(0)=0$ and $v(L)=v_x(L)=0$, we get

$$c_1=c_3=c_4=0.$$

Moreover, taking into account the boundary transmission conditions $u(l_0)=v(l_0)$ and $r_1u_x(l_0)=r_2v_x(l_0)$ we deduce that $c_2=0$. Then $U=0$.

Hence $i\lambda-\mathcal{A}$ is surjective for all $\lambda\in\mathbb{R}^*$.

Case 2: $\lambda=0$ **and** $\eta\neq 0$.
The system (3.33) is reduced to the following

$$\begin{cases}-\tilde{u}=f_1,\\ -\dfrac{\tau_1}{\rho_1}u_{xx}=f_2,\\ -\tilde{v}=f_3,\\ -\dfrac{\tau_2}{\rho_2}v_{xx}=f_4,\\ (\xi^2+\eta)\phi-\tilde{v}(L)\mu(\xi)=f_5.\end{cases} \tag{3.43}$$

With the second and third equations of (3.43), we get

$$\begin{cases}u(x)=-\frac{1}{r_1}\displaystyle\int_0^x\int_0^s f_2(r)\,dr\,ds+Cx,\\ v(x)=-\frac{1}{r_2}\displaystyle\int_{l_0}^x\int_{l_0}^s f_4(r)\,dr\,ds+C'x+C''.\end{cases}$$

From $(3.43)_3$ and $(3.43)_5$, we have

$$-\gamma\eta^{\alpha-1}f_3(L) + r_2 v_x(L) + \zeta\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_5(\xi)}{\xi^2+\eta}\,d\xi = 0.$$

We find

$$C' = \frac{1}{r_2}\left[\int_{l_0}^{L} f_4(r)\,dr + \gamma\eta^{\alpha-1}f_3(L) - \zeta\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_5(\xi)}{\xi^2+\eta}\,d\xi\right].$$

From boundary transmission conditions, we find

$$u(l_0) = v(l_0) \Rightarrow l_0C - C'' = \frac{1}{r_1}\int_0^{l_0}\int_0^{s} f_2(r)\,dr\,ds + C'l_0.$$

$$r_1u_x(l_0) = r_2v_x(l_0) \Rightarrow Cr_1 = \int_0^{l_0}\int_0^{s} f_2(r)\,dr + C'r_2.$$

We find

$$C = \frac{1}{r_1}\left[\int_0^{l_0}\int_0^{s} f_2(r)\,dr + C'r_2\right],$$
$$C'' = l_0(C - C') - \frac{1}{r_1}\int_0^{l_0}\int_0^{s} f_2(r)\,dr.$$

Hence $\mathcal{A}$ is surjective. The proof is thus complete.
Proof of Theorem 3.2.3. By Lemma 3.2.3, the operator $\mathcal{A}$ has no pure imaginary eigenvalues and by Lemma 3.2.4 $R(i\lambda - \mathcal{A}) = \mathcal{H}$ for all $\lambda \in \mathbb{R}^*$ and $R(i\lambda - \mathcal{A}) = \mathcal{H}$ for $\lambda = 0$ and for all $\eta > 0$. Therefore, the closed graph theorem of Banach implies that $\sigma(\mathcal{A})\cap i\mathbb{R} = \emptyset$ if $\eta > 0$ and $\sigma(\mathcal{A})\cap i\mathbb{R} = \{0\}$ if $\eta = 0$.

3.3 Lack of Exponential Stability

Our goal in this section is to show that system (P) is not exponentially stable. We need the following well known theorem.

Theorem 3.3.1 ([19]-[37]) *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on Hilbert space. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R}$$

and

$$\overline{\lim_{|\beta|\to\infty}}\|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Our main result is

Theorem 3.3.2 *The semigroup generated by the operator $\mathcal{A}$ is not exponentially stable.*

Proof: We will examine two cases.
Case 1 $\eta = 0$: We shall show that $i\lambda = 0$ is not in the resolvent set of the operator $\mathcal{A}$. Indeed, noting that $(-x\sin x, 0, -x\sin x, 0, 0)^T \in \mathcal{H}$, and denoting by $(u, \tilde{u}, v, \tilde{u}, \phi)^T$ the image of $(-x\sin x, 0, -x\sin x, 0, 0)^T$ by $\mathcal{A}^{-1}$, we see that $\phi(\xi) = |\xi|^{\frac{2\alpha-5}{2}} L\sin L$. But $\phi \notin L^2(-\infty, +\infty)$, since $\alpha \in]0, 1[$ and so $(u, \tilde{u}, v, \tilde{u}, \phi)^T \notin D(\mathcal{A})$.
• **Case 2** $\eta \neq 0$: We aim to show that an infinite number of eigenvalues of $\mathcal{A}$ approach the imaginary axis which prevents the wave system (P) from being exponentially stable. Indeed We first compute the characteristic equation that gives the eigenvalues of $\mathcal{A}$. Let λ be an eigenvalue of $\mathcal{A}$ with associated eigenvector $U = (u, \tilde{u}, v, \tilde{v}, \phi)^T$. Then $\mathcal{A}U = \lambda U$ is equivalent to

$$\begin{cases} \lambda u - \tilde{u} = 0, \\ \lambda\tilde{u} - \dfrac{\tau_1}{\rho_1} u_{xx} = 0, \\ \lambda v - \tilde{v} = 0, \\ \lambda\tilde{v} - \dfrac{\tau_2}{\rho_2} v_{xx} = 0, \\ \lambda\phi + (\xi^2 + \eta)\phi - \tilde{v}(L)\mu(\xi) = 0. \end{cases} \tag{3.44}$$

Inserting $(3.44)_1, (3.44)_3$ into $(3.44)_2, (3.44)_4$ and $(3.44)_5$, we get

$$\begin{cases} \lambda^2 u - \dfrac{\tau_1}{\rho_1} u_{xx} = 0 \text{ in } (0, l_0), \\ \lambda^2 v - \dfrac{\tau_2}{\rho_2} v_{xx} = 0 \text{ in } (l_0, L). \\ (\lambda + \xi^2 + \eta)\phi - \lambda v(L)\mu(\xi) = 0. \end{cases} \tag{3.45}$$

Using equation $(3.45)_3$, Lemma 3.2.1 and the boundary conditions we have

$$\frac{\tau_2}{\rho_2} v_x(L) + \gamma\lambda(\lambda + \eta)^{\alpha-1} v(L) = 0. \tag{3.46}$$

Finally, using the fact $u(0) = 0, u(l_0) = v(l_0), \tau_1\rho_2 u_x(l_0) = \tau_2\rho_1 v_x(l_0)$ and (3.46) we get the following system

$$\begin{cases} \lambda^2 u - \dfrac{\tau_1}{\rho_1} u_{xx} = 0 \text{ in } (0, l_0), \\ \lambda^2 v - \dfrac{\tau_2}{\rho_2} v_{xx} = 0 \text{ in } (l_0, L). \\ u(0) = 0, \quad u(l_0) = v(l_0), \quad \tau_1\rho_2 u_x(l_0) = \tau_2\rho_1 v_x(l_0), \\ \dfrac{\tau_2}{\rho_2} v_x(L) + \gamma\lambda(\lambda + \eta)^{\alpha-1} v(L) = 0. \end{cases} \tag{3.47}$$

The general solutions of equations $(3.47)_1$ and $(3.47)_2$ are given by

$$\begin{cases} u(x) = \displaystyle\sum_{i=1}^{2} c_i e^{t_i x}, \\ v(x) = \displaystyle\sum_{i=3}^{4} c_i e^{t_i x}, \end{cases} \tag{3.48}$$

where $t_1 = \sqrt{\rho_1/\tau_1}\lambda,\ t_2 = -t_1,\ t_3 = \sqrt{\rho_2/\tau_2}\lambda,\ t_4 = -t_3$.

Thus the boundary conditions may be written as the following system

$$M(\lambda)C(\lambda) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ e^{t_1 l_0} & e^{-t_1 l_0} & -e^{t_3 l_0} & -e^{-t_3 l_0} \\ \frac{\tau_1}{\rho_1} t_1 e^{t_1 l_0} & -\frac{\tau_1}{\rho_1} t_1 e^{-t_1 l_0} & -\frac{\tau_2}{\rho_2} t_3 e^{t_3 l_0} & \frac{\tau_2}{\rho_2} t_3 e^{-t_3 l_0} \\ 0 & 0 & h(t_3) e^{t_3 L} & h(-t_3) e^{-t_3 L} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{3.49}$$

where

$$h(r) = \frac{\tau_2}{\rho_2} r + \gamma\lambda(\lambda+\eta)^{\alpha-1}.$$

Hence a non-trivial solution φ exists if and only if the determinant of $M(\lambda)$ vanishes. Set $f(\lambda) = det M(\lambda)$, thus the characteristic equation is $f(\lambda) = 0$.

Our purpose in the sequel is to prove, thanks to Rouché's Theorem, that there is a subsequence of eigenvalues for which their real part tends to 0.

In the sequel, since $\mathcal{A}$ is dissipative, we study the asymptotic behavior of the large eigenvalues λ of $\mathcal{A}$ in the strip $-\alpha_0 \leq \Re(\lambda) \leq 0$, for some $\alpha_0 > 0$ large enough and for such λ, we remark that $e^{t_i}, i = 1, 2$ remains bounded.

Case 1 $\frac{\tau_1}{\rho_1} = \frac{\tau_2}{\rho_2}$

Lemma 3.3.1 *There exists $N \in \mathbb{N}$ such that*

$$\{\lambda_k\}_{k \in \mathbf{Z}^*, |k| \geq N} \subset \sigma(\mathcal{A}) \tag{3.50}$$

where

$$\lambda_k = i\frac{1}{rL}\left(k + \frac{1}{2}\right)\pi + \frac{\tilde{\alpha}}{k^{1-\alpha}} + \frac{\beta}{|k|^{1-\alpha}} + o\left(\frac{1}{k^{3-\alpha}}\right), k \geq N, \tilde{\alpha} \in i\mathbb{R}, \beta \in \mathbb{R}, \beta < 0,\ r = \sqrt{\frac{\rho_1}{\tau_1}}.$$

$$\lambda_k = \overline{\lambda_{-k}} \text{ if } k \leq -N.$$

Moreover for all $|k| \geq N$, the eigenvalues λ_k are simple.

Proof

$$\begin{aligned} f(\lambda) &= -2t_1^2 r_1 (e^{t_1 L} + e^{-t_1 L}) - 2\gamma t_1 \lambda(\lambda+\eta)^{\alpha-1}(e^{t_1 L} - e^{-t_1 L}) \\ &= -2t_1^2 r_1 \left((e^{t_1 L} + e^{-t_1 L}) + \frac{\gamma}{\sqrt{\tau_1/\rho_1}} \frac{e^{t_1 L} - e^{-t_1 L}}{\lambda^{1-\alpha}} + o\left(\frac{1}{\lambda^{1-\alpha}}\right)\right). \\ &= -2r_1 t_1^2 e^{-t_1 L} \left((e^{2t_1 L} + 1) + \frac{\gamma}{\sqrt{\tau_1/\rho_1}} \frac{e^{2t_1 L} - 1}{\lambda^{1-\alpha}} + o\left(\frac{1}{\lambda^{1-\alpha}}\right)\right). \end{aligned} \tag{3.51}$$

We set

$$\begin{aligned} \tilde{f}(\lambda) &= (e^{2t_1 L} + 1) + \frac{\gamma}{\sqrt{\tau_1/\rho_1}} \frac{e^{2t_1 L} - 1}{\lambda^{1-\alpha}} + o\left(\frac{1}{\lambda^{1-\alpha}}\right) \\ &= f_0(\lambda) + \tfrac{f_1(\lambda)}{\lambda^{1-\alpha}} + o\left(\tfrac{1}{\lambda^{1-\alpha}}\right) \end{aligned} \tag{3.52}$$

where

$$f_0(\lambda) = e^{2t_1 L} + 1, \tag{3.53}$$

$$f_1(\lambda) = +\frac{\gamma}{\sqrt{\tau_1/\rho_1}}(e^{t_1 L} - 1), \tag{3.54}$$

Note that f_0 and f_1 remain bounded in the strip $-\alpha_0 \leq \mathcal{R}(\lambda) \leq 0$.

Step 2. We look at the roots of f_0. From (3.53), f_0 has one familie of roots that we denote λ_k^0.

$$f_0(\lambda) = 0 \Leftrightarrow e^{2\sqrt{\frac{\rho_1}{\tau_1}}\lambda L} = -1.$$

Hence

$$2r\lambda L = i(2k+1)\pi, \quad k \in \mathbf{Z}, r = \sqrt{\frac{\rho_1}{\tau_1}}$$

i.e.,

$$\lambda_k^0 = \frac{i(2k+1)\pi}{2rL}, \quad k \in \mathbf{Z}.$$

Now with the help of Rouché's Theorem, we will show that the roots of $\tilde{f}$ are close to those of f_0. Changing in (3.52) the unknown λ by $u = 2\sqrt{\frac{\rho_1}{\tau_1}}\lambda L$ then (3.52) becomes

$$\tilde{f}(u) = (e^u + 1) + O\left(\frac{1}{u^{(1-\alpha)}}\right) = f_0(u) + O\left(\frac{1}{u^{(1-\alpha)}}\right)$$

The roots of f_0 are $u_k = \frac{i\left(k+\frac{1}{2}\right)}{rL}\pi, k \in \mathbf{Z}$, and setting $u = u_k + re^{it}, t \in [0, 2\pi]$, we can easily check that there exists a constant $C > 0$ independent of k such that $|e^u + 1| \geq Cr$ for r small enough. This allows to apply Rouché's Theorem. Consequently, there exists a subsequence of roots of $\tilde{f}$ which tends to the roots u_k of f_0. Equivalently, it means that there exists $N \in \mathbb{N}$ and a subsequence $\{\lambda_k\}_{|k|\geq N}$ of roots of $f(\lambda)$, such that $\lambda_k = \lambda_k^0 + o(1)$ which tends to the roots $\frac{i\left(k+\frac{1}{2}\right)}{rL}\pi$ of f_0. Finally for $|k| \geq N, \lambda_k$ is simple since λ_k^0 is.

Step 3. From Step 2, we can write

$$\lambda_k = i\frac{1}{rL}\left(k + \frac{1}{2}\right)\pi + \varepsilon_k. \tag{3.55}$$

Using (3.55), we get

$$e^{2r\lambda_k L} = -1 - 2rL\varepsilon_k - 2rL^2\varepsilon_k^2 + o(\varepsilon_k^2). \tag{3.56}$$

Substituting (3.56) into (3.52), using the fact that $\tilde{f}(\lambda_k) = 0$, we get:

$$\tilde{f}(\lambda_k) = -2rL\varepsilon_k - \frac{2\gamma}{\sqrt{\tau_1/\rho_1}}\frac{1}{(\frac{i(2k+1)\pi}{2rL})^{1-\alpha}} + o(\varepsilon_k) = 0 \tag{3.57}$$

and hence

$$\begin{aligned} \varepsilon_k &= -\frac{\gamma r^{1-\alpha}}{L^\alpha((k+\frac{1}{2})i\pi)^{1-\alpha}} + o\left(\frac{1}{k^{1-\alpha}}\right) \\ &= -\frac{\gamma r^{1-\alpha}}{L^\alpha((k+\frac{1}{2})\pi)^{1-\alpha}}\left(\cos(1-\alpha)\frac{\pi}{2} - i\sin(1-\alpha)\frac{\pi}{2}\right) + o\left(\frac{1}{k^{1-\alpha}}\right) \text{ for } k \succeq 0 \end{aligned} \tag{3.58}$$

From (3.58) we have in that case $|k|^{1-\alpha}\mathcal{R}\lambda_k \sim \beta$, with

$$\beta = -\frac{\gamma r^{1-\alpha}}{L^\alpha \pi^{1-\alpha}} \cos(1-\alpha)\frac{\pi}{2}.$$

Case 2 $\dfrac{\tau_1}{\rho_1} \neq \dfrac{\tau_2}{\rho_2}$

Lemma 3.3.2 *There exists $N \in \mathbb{N}$ such that*

$$\{\lambda_k\}_{k\in\mathbf{Z}^*,|k|\geq N} \subset \sigma(\mathcal{A}) \tag{3.59}$$

where

$$\lambda_k = i\mu_k + \frac{\tilde{\alpha}}{k^{1-\alpha}} + \frac{\beta}{|k|^{1-\alpha}} + o\left(\frac{1}{k^{3-\alpha}}\right), k \geq N, \tilde{\alpha} \in i\mathbb{R}, \beta \in \mathbb{R}, \beta < 0.$$

$$\lambda_k = \overline{\lambda_{-k}} \text{ if } k \leq -N.$$

Moreover for all $|k| \geq N$, the eigenvalues λ_k are simple.

Proof

$$\begin{aligned}
&f(\lambda) = r_2^2 t_3^2 (e^{(-t_1 l + t_3 l - t_3 L)} - e^{(-t_1 l - t_3 l + t_3 L)} - e^{(t_1 l + t_3 l - t_3 L)} + e^{(t_1 l - t_3 l + t_3 L)})\\
&-r_2 t_3 d(e^{(-t_1 l + t_3 l - t_3 L)} + e^{(-t_1 l - t_3 l + t_3 L)} - e^{(t_1 l + t_3 l - t_3 L)} - e^{(t_1 l - t_3 l + t_3 L)})\\
&-r_1 t_1 d(e^{(-t_1 l + t_3 l - t_3 L)} - e^{(-t_1 l - t_3 l + t_3 L)} - e^{(t_1 l - t_3 l + t_3 L)} + e^{(t_1 l + t_3 l - t_3 L)})\\
&+r_1 r_2 t_1 t_3 (e^{(-t_1 l + t_3 l - t_3 L)} + e^{(-t_1 l - t_3 l + t_3 L)} + e^{(t_1 l + t_3 l - t_3 L)} + e^{(t_1 l - t_3 l + t_3 L)})\\
&= \sqrt{r_2}\lambda^2 \Bigg[\sqrt{r_2}(e^{t_1 l} - e^{-t_1 l})(e^{(L-l)t_3} - e^{-(L-l)t_3}) + \sqrt{r_1}(e^{t_1 l} + e^{-t_1 l})(e^{(L-l)t_3} + e^{-(L-l)t_3})\\
&+\gamma \frac{(e^{t_1 l} - e^{-t_1 l})(e^{(L-l)t_3} + e^{-(L-l)t_3}) + \frac{\sqrt{r_1}}{\sqrt{r_1}}(e^{t_1 l} + e^{-t_1 l})(e^{(L-l)t_3} - e^{-(L-l)t_3})}{(\lambda+\eta)^{\alpha-1}} \Bigg]\\
&= \sqrt{r_2}\lambda^2 \Bigg[\sqrt{r_2}(e^{t_1 l} - e^{-t_1 l})(e^{(L-l)t_3} - e^{-(L-l)t_3}) + \sqrt{r_1}(e^{t_1 l} + e^{-t_1 l})(e^{(L-l)t_3} + e^{-(L-l)t_3})\\
&+\gamma \frac{(e^{t_1 l} - e^{-t_1 l})(e^{(L-l)t_3} + e^{-(L-l)t_3}) + \frac{\sqrt{r_1}}{\sqrt{r_1}}(e^{t_1 l} + e^{-t_1 l})(e^{(L-l)t_3} - e^{-(L-l)t_3})}{\lambda^{\alpha-1}}\\
&+o\left(\frac{1}{\lambda^{1-\alpha}}\right) \Bigg].
\end{aligned} \tag{3.60}$$

We set

$$\begin{aligned}
&\tilde{f}(\lambda) = \sqrt{r_2}(e^{t_1 l} - e^{-t_1 l})(e^{(L-l)t_3} - e^{-(L-l)t_3}) + \sqrt{r_1}(e^{t_1 l} + e^{-t_1 l})(e^{(L-l)t_3} + e^{-(L-l)t_3})\\
&+\gamma \frac{(e^{t_1 l} - e^{-t_1 l})(e^{(L-l)t_3} + e^{-(L-l)t_3}) + \frac{\sqrt{r_1}}{\sqrt{r_1}}(e^{t_1 l} + e^{-t_1 l})(e^{(L-l)t_3} - e^{-(L-l)t_3})}{\lambda^{\alpha-1}} + o\left(\frac{1}{\lambda^{1-\alpha}}\right)\\
&= f_0(\lambda) + \tfrac{f_1(\lambda)}{\lambda^{1-\alpha}} + o\left(\tfrac{1}{\lambda^{1-\alpha}}\right),
\end{aligned} \tag{3.61}$$

where

$$f_0(\lambda) = \sqrt{r_2}(e^{t_1 l} - e^{-t_1 l})(e^{(L-l)t_3} - e^{-(L-l)t_3}) + \sqrt{r_1}(e^{t_1 l} + e^{-t_1 l})(e^{(L-l)t_3} + e^{-(L-l)t_3}), \quad (3.62)$$

$$f_1(\lambda) = \gamma\left((e^{t_1 l} - e^{-t_1 l})(e^{(L-l)t_3} + e^{-(L-l)t_3}) + \frac{\sqrt{r_1}}{\sqrt{r_1}}(e^{t_1 l} + e^{-t_1 l})(e^{(L-l)t_3} - e^{-(L-l)t_3})\right). \quad (3.63)$$

We look at the roots of f_0. From (3.62), f_0 has one familie of roots that we denote λ_k^0. Indeed, $f_0(\lambda) = 0$ corresponds to the eigenvalues problem to the conservative problem associated with (P'):

$$(P_0) \qquad \begin{cases} \rho_1 u_{tt}(x,t) - \tau_1 u_{xx}(x,t) = 0 & \text{in } (0,l_0)\times(0,+\infty),\\ \rho_2 v_{tt}(x,t) - \tau_2 v_{xx}(x,t) = 0 & \text{in } (l_0,L)\times(0,+\infty),\\ u(l_0,t) = v(l_0,t), \quad \rho_2\tau_1 u_x(l_0,t) = \rho_1\tau_2 v_x(l_0,t) & \text{on } (0,+\infty),\\ u(0,t) = 0 & \text{on } (0,+\infty)\\ v_x(L,t) = 0 & \text{on } (0,+\infty),\\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{on } (0,l_0),\\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x) & \text{on } (l_0,L). \end{cases}$$

The abstract formulation of (P_0) is:

$$\mathcal{A}_0 \begin{pmatrix} u\\ \tilde u\\ v\\ \tilde v \end{pmatrix} = \begin{pmatrix} \tilde u\\ \dfrac{\tau_1}{\rho_1} u_{xx}\\ \tilde v\\ \dfrac{\tau_2}{\rho_2} v_{xx} \end{pmatrix}. \quad (3.64)$$

The domain of $\mathcal{A}_0$ is

$$D(\mathcal{A}_0) = \left\{ \begin{array}{l} (u,\tilde u,v,\tilde v)^T \text{ in } \mathcal{H}_0 : u \in H^2(0,L)\cap H^1_*(0,l_0), \tilde u \in H^1_*(0,l_0),\\ v \in H^2(l_0,L), \tilde v \in H^1(l_0,L), u(l_0) = v(l_0), \rho_2\tau_1 u_x(l_0) = \rho_1\tau_2 v_x(l_0),\\ \tilde u(l_0) = \tilde v(l_0),\ v_x(L) = 0, \end{array} \right\}, \quad (3.65)$$

where

$$\mathcal{H}_0 = \{H^1_*(0,l_0)\times L^2(0,l_0)\times H^1(l_0,L)\times L^2(l_0,L)\backslash u(l_0) = v(l_0)\}.$$

$\mathcal{A}_0$ is clearly a skew adjoint operator with a compact resolvent, then there is an orthonormal system of eigenvectors of $\mathcal{A}_0$ which is complete in $\mathcal{H}_0$. All eigenvalues of $\mathcal{A}_0$ are of the form $i\mu_k, \mu_k \in \mathbb{R}$. Now

$$\begin{aligned} f_0(i\mu_k) = 0 \quad &\Leftrightarrow \quad \tan\left(\sqrt{\tfrac{\rho_1}{\tau_1}} l\mu_k\right)\tan\left(\sqrt{\tfrac{\rho_2}{\tau_2}}(L-l)\mu_k\right) = \frac{\frac{\rho_1}{\tau_1}}{\frac{\rho_2}{\tau_2}}\\ &\Leftrightarrow \quad \tan\left(\sqrt{\tfrac{\rho_1}{\tau_1}} l\mu_k\right) = \frac{\frac{\rho_1}{\tau_1}}{\frac{\rho_2}{\tau_2}}\cot\left(\sqrt{\tfrac{\rho_2}{\tau_2}}(L-l)\mu_k\right). \end{aligned}$$

By representation of graph of the functions tan and cot, we easily have $\mu_k \sim ck$ for large k and a constant c depending on parameters $\rho_1, \tau_1, \rho_2, \tau_2, l$ and L. Moreover, the algebraic multiplicity of μ_k is one. Then, we follow exactly as the case $\tau_1/\rho_1 = \tau_2/\rho_2$.

The operator $\mathcal{A}$ has a non exponential decaying branche of eigenvalues. Thus the proof is complete.

3.3.1 Residual spectrum of $\mathcal{A}$

Lemma 3.3.3 *Let $\mathcal{A}$ be defined by (3.17). Then*

$$\mathcal{A}^* \begin{pmatrix} u \\ \tilde{u} \\ v \\ \tilde{v} \\ \phi \end{pmatrix} = \begin{pmatrix} -\tilde{u} \\ -\frac{\tau_1}{\rho_1} u_{xx} \\ -\tilde{v} \\ -\frac{\tau_2}{\rho_2} v_{xx} \\ -(\xi^2+\eta)\phi - \tilde{v}(L)\mu(\xi) \end{pmatrix} \tag{3.66}$$

with domain

$$D(\mathcal{A}^*) = \left\{ \begin{array}{l} (u,\tilde{u},v,\tilde{v},\phi)^T \text{ in } \mathcal{H} : u \in H^2(0,l_0) \cap H^1_*(0,l_0), \tilde{u} \in H^1_*(0,l_0), \\ v \in H^2(l_0,L), \tilde{v} \in H^1(l_0,L), u(l_0) = v(l_0), \tilde{u}(l_0) = \tilde{v}(l_0), \rho_2\tau_1 u_x(l_0) = \rho_1\tau_2 v_x(l_0), \\ -(\xi^2+\eta)\phi + \tilde{v}(L)\mu(\xi) \in L^2(-\infty,+\infty), \\ \tau_2 v_x(L) + \zeta\rho_2 \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,t)\,d\xi = 0, \\ |\xi|\phi \in L^2(-\infty,+\infty) \end{array} \right\}. \tag{3.67}$$

Proof
Let $U = (u,\tilde{u},v,\tilde{u},\phi)^T$ and $V = (u_1,\tilde{u}_1,v_1,\tilde{v}_1,\phi_1)^T$. We have $< \mathcal{A}U, V >_{\mathcal{H}} = < U, \mathcal{A}^*V >_{\mathcal{H}}$.

$$\begin{aligned} < \mathcal{A}U, V >_{\mathcal{H}} &= \int_0^{l_0} (\frac{\tau_1}{\rho_1}\tilde{u}_x\overline{u}_{1x} + \frac{\tau_1}{\rho_1}\overline{\tilde{u}}_1 u_{xx})\,dx + \int_{l_0}^{L} (\frac{\tau_2}{\rho_2}\tilde{v}_x\overline{v}_{1x} + \frac{\tau_2}{\rho_2}\overline{\tilde{v}}_1 v_{xx})\,dx \\ &\quad +\zeta\int_{-\infty}^{+\infty} [-(\xi^2+\eta)\phi + \tilde{v}(L)\mu(\xi)]\overline{\phi}_1\,d\xi \\ &= -\int_0^{l_0} (\frac{\tau_1}{\rho_1}\tilde{u}\overline{u}_{1xx} + \frac{\tau_1}{\rho_1}\overline{\tilde{u}}_{1x} u_x)\,dx - \int_{l_0}^{L} (\frac{\tau_2}{\rho_2}\tilde{v}\overline{v}_{1xx} + \frac{\tau_2}{\rho_2}\overline{\tilde{v}}_{1x} v_x)\,dx \\ &\quad +\tfrac{\tau_1}{\rho_1}u_x(l_0)\overline{\tilde{u}}_1(l_0) - \tfrac{\tau_2}{\rho_2}v_x(l_0)\overline{\tilde{v}}_1(l_0) + \tfrac{\tau_1}{\rho_1}\tilde{u}(l_0)\overline{u}_{1x}(l_0) - \tfrac{\tau_2}{\rho_2}\tilde{v}(l_0)\overline{v}_{1x}(l_0) \\ &\quad +\tfrac{\tau_2}{\rho_2}v_x(L)\overline{\tilde{v}}_1(L) - \zeta\int_{-\infty}^{+\infty} \phi[(\xi^2+\eta)\overline{\phi}_1]\,d\xi \\ &\quad +\tfrac{\tau_2}{\rho_2}\tilde{v}(L)\overline{v}_{1x}(L) + \zeta\tilde{v}(L)\int_{-\infty}^{+\infty} \mu(\xi)\overline{\phi}_1\,d\xi \end{aligned}$$

As $\frac{\tau_1}{\rho_1}u_x(l_0) = \frac{\tau_2}{\rho_2}v_x(l_0), \tilde{u}(l_0) = \tilde{v}(l_0), \frac{\tau_2}{\rho_2}v_x(L) = -\zeta\int_{-\infty}^{+\infty}\mu(\xi)\phi\,d\xi$ and if we set $\frac{\tau_1}{\rho_1}u_{1x}(l_0) = \frac{\tau_2}{\rho_2}v_{1x}(l_0)$ and $\tilde{u}_1(l_0) = \tilde{v}_1(l_0)$ and $\frac{\tau_2}{\rho_2}v_{1x}(L) = -\zeta\int_{-\infty}^{+\infty}\mu(\xi)\phi_1\,d\xi$, we find

$$\begin{aligned} < \mathcal{A}U, V >_{\mathcal{H}} &= -\int_0^{l_0} (\frac{\tau_1}{\rho_1}\tilde{u}\overline{u}_{1xx} + \frac{\tau_1}{\rho_1}\overline{\tilde{u}}_{1x} u_x)\,dx - \int_{l_0}^{L} (\frac{\tau_2}{\rho_2}\tilde{v}\overline{v}_{1xx} + \frac{\tau_2}{\rho_2}\overline{\tilde{v}}_{1x} v_x)\,dx \\ &\quad -\zeta\int_{-\infty}^{+\infty} \phi[(\xi^2+\eta)\overline{\phi}_1 + \mu(\xi)\overline{\tilde{v}}_1(L)]\,d\xi. \end{aligned}$$

Theorem 3.3.3 $\sigma_r(\mathcal{A}) = \emptyset$, *where $\sigma_r(\mathcal{A})$ denotes the set of residual spectrum of $\mathcal{A}$.*

Proof
Since $\lambda \in \sigma_r(\mathcal{A}), \overline{\lambda} \in \sigma_p(\mathcal{A}^*)$ the proof will be accomplished if we can show that $\sigma_p(\mathcal{A}) = \sigma_p(\mathcal{A}^*)$.

This is because obviously the eigenvalues of $\mathcal{A}$ are symmetric on the real axis. From (3.66), the eigenvalue problem $\mathcal{A}^*Z = \lambda Z$ for $\lambda \in \mathbb{C}$ and $0 \neq Z = (u, \tilde{u}, v, \tilde{v}, \phi) \in D(\mathcal{A}^*)$ we have

$$\begin{cases} \lambda u + \tilde{u} = 0, \\ \lambda \tilde{u} + \frac{\tau_1}{\rho_1} u_{xx} = 0, \\ \lambda v + \tilde{v} = 0, \\ \lambda \tilde{v} + \frac{\tau_2}{\rho_2} v_{xx} = 0, \\ \lambda \phi + (\xi^2 + \eta)\phi + \tilde{v}(L)\mu(\xi) = 0. \end{cases} \tag{3.68}$$

Inserting $(3.68)_1$, $(3.68)_3$ into $(3.68)_2, (3.68)_4$ and $(3.68)_5$, we find

$$\begin{cases} \lambda^2 u - \frac{\tau_1}{\rho_1} u_{xx} = 0, \\ \lambda^2 v - \frac{\tau_2}{\rho_2} v_{xx} = 0, \\ \lambda \phi + (\xi^2 + \eta)\phi - \lambda v(L)\mu(\xi) = 0. \end{cases} \tag{3.69}$$

Using equation $(3.69)_3$, we easily have

$$\gamma(\lambda + \eta)^{\alpha - 1} \lambda v(L) + \frac{\tau_2}{\rho_2} \varphi_x(L) = 0 \tag{3.70}$$

with the following conditions

$$u(0) = 0, u(l_0) = v(l_0), \frac{\tau_1}{\rho_1} u_x(l_0) = \frac{\tau_2}{\rho_2} v_x(l_0). \tag{3.71}$$

System (3.69)-(3.71) is the same as (3.47). Hence $\mathcal{A}^*$ has the same eigenvalues with $\mathcal{A}$. The proof is complete.

3.4 Polynomial Stability and Optimality(for $\eta \neq 0$)

In the previous section,we have shown that the transmission wave system is not exponentially stable. In this section, we prove that it is polynomially stable with an optimal rate of decay when $\eta > 0$. To achieve this, we use a recent result by Borichev and Tomilov [**10**]. Accordingly, if we consider a bounded C_0-semigroup $S(t) = e^{\mathcal{A}t}$ on a Hilbert space. If

$$i\mathbb{R} \subset \rho(\mathcal{A}) \text{ and } \overline{\lim}_{|\beta| \to \infty} \frac{1}{\beta^\delta} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$$

for some $\delta > 0$, then there exists c such that

$$\|e^{\mathcal{A}t} U_0\|^2 \leq \frac{c}{t^{\frac{2}{\delta}}} \|U_0\|^2_{D(\mathcal{A})}.$$

Our main result is as follows.

Theorem 3.4.1 *The semigroup $S_{\mathcal{A}}(t)_{t \geq 0}$ is polynomially stable and*

$$E(t) = \|S_{\mathcal{A}}(t) U_0\|^2_{\mathcal{H}} \leq \frac{1}{t^{2/(1-\alpha)}} \|U_0\|^2_{D(\mathcal{A})}.$$

Moreover, the rate of energy decay $t^{-2/(1-\alpha)}$ is optimal for any initial data in $D(\mathcal{A})$.

Proof
We will need to study the resolvent equation $(i\lambda - \mathcal{A})U = F$, for $\lambda \in \mathbb{R}$, namely

$$\begin{cases} i\lambda u - \tilde{u} = f_1, \\ i\lambda\tilde{u} - \dfrac{\tau_1}{\rho_1}u_{xx} = f_2, \\ i\lambda v - \tilde{v} = f_3, \\ i\lambda\tilde{v} - \dfrac{\tau_2}{\rho_2}v_{xx} = f_4, \\ i\lambda\phi + (\xi^2+\eta)\phi - \tilde{v}(L)\mu(\xi) = f_5. \end{cases} \tag{3.72}$$

We divide the proof into three steps, as follows:
Step 1. Inserting $(3.72)_1, (3.72)_3$ into $(3.72)_2$ and $(3.72)_4$, we get

$$\begin{aligned} \lambda^2 u + r_1 u_{xx} &= -(f_2 + i\lambda f_1), \\ \lambda^2 v + r_2 v_{xx} &= -(f_4 + i\lambda f_3), \end{aligned}$$

where $r_1 = \tau_1/\rho_1, r_2 = \tau_2/\rho_2$. As $u(0) = 0$, then

$$\begin{aligned} u(x) &= c_1 \sin\tfrac{\lambda}{\sqrt{r_1}}x - \frac{1}{\sqrt{r_1}\lambda}\int_0^x (f_2(\sigma) + i\lambda f_1(\sigma))\sin\frac{\lambda}{\sqrt{r_1}}(x-\sigma)\,d\sigma, \\ v(x) &= v(l_0)\cos\tfrac{\lambda}{\sqrt{r_2}}(x-l_0) + v_x(l_0)\tfrac{\sqrt{r_2}}{\lambda}\sin\tfrac{\lambda}{\sqrt{r_2}}(x-l_0) \\ &\quad - \frac{1}{\sqrt{r_2}\lambda}\int_{l_0}^x (f_4(\sigma) + i\lambda f_3(\sigma))\sin\frac{\lambda}{\sqrt{r_2}}(x-\sigma)\,d\sigma \end{aligned} \tag{3.73}$$

and hence

$$\begin{aligned} u_x(x) &= c_1\tfrac{\lambda}{\sqrt{r_1}}\cos\tfrac{\lambda}{\sqrt{r_1}}x - \tfrac{1}{r_1}\int_0^x (f_2(\sigma) + i\lambda f_1(\sigma))\cos\frac{\lambda}{\sqrt{r_1}}(x-\sigma)\,d\sigma, \\ v_x(x) &= -v(l_0)\tfrac{\lambda}{\sqrt{r_2}}\sin\tfrac{\lambda}{\sqrt{r_2}}(x-l_0) + v_x(l_0)\cos\tfrac{\lambda}{\sqrt{r_2}}(x-l_0) \\ &\quad - \frac{1}{r_2}\int_{l_0}^x (f_4(\sigma) + i\lambda f_3(\sigma))\cos\frac{\lambda}{\sqrt{r_2}}(x-\sigma)\,d\sigma. \end{aligned} \tag{3.74}$$

Step 2. With the fifth equation of (3.72), we get

$$\phi(\xi) = \frac{\tilde{v}(L)\mu(\xi) + f_5(\xi)}{i\lambda + \xi^2 + \eta}. \tag{3.75}$$

Inserting (3.75) in the boundary condition $(P')_6$, we deduce that

$$\begin{aligned} &r_2 v_x(L) + i\gamma\lambda(i\lambda+\eta)^{\alpha-1}v(L) \\ &\qquad = \gamma(i\lambda+\eta)^{\alpha-1}f_3(L) - \zeta\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_5(\xi)}{i\lambda+\xi^2+\eta}\,d\xi. \end{aligned} \tag{3.76}$$

Then

$$
\begin{aligned}
&v(l_0)\left[-r_2\tfrac{\lambda}{\sqrt{r_2}}\sin\tfrac{\lambda}{\sqrt{r_2}}(L-l_0)+d\cos\tfrac{\lambda}{\sqrt{r_2}}(L-l_0)\right]\\
&\qquad\qquad +v_x(l_0)\left[r_2\cos\tfrac{\lambda}{\sqrt{r_2}}(L-l_0)+d\tfrac{\sqrt{r_2}}{\lambda}\sin\tfrac{\lambda}{\sqrt{r_2}}(L-l_0)\right]\\
&=\gamma(i\lambda+\eta)^{\alpha-1}f_3(L)-\zeta\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_5(\xi)}{i\lambda+\xi^2+\eta}\,d\xi+\int_{l_0}^{L}(f_4(\sigma)+i\lambda f_3(\sigma))\cos\frac{\lambda}{\sqrt{r_2}}(L-\sigma)\,d\sigma\\
&\qquad\qquad +\frac{d}{\sqrt{r_2}\lambda}\int_{l_0}^{L}(f_4(\sigma)+i\lambda f_3(\sigma))\sin\frac{\lambda}{\sqrt{r_2}}(L-\sigma)\,d\sigma,
\end{aligned}
\tag{3.77}
$$

where $d=\gamma\lambda(i\lambda+\eta)^{\alpha-1}$. Using the transmission conditions $v(l_0)=u(l_0)$ and $r_2v_x(l_0)=r_1u_x(l_0)$, we get

$$
\begin{aligned}
v(l_0) &= c_1\sin\tfrac{\lambda}{\sqrt{r_1}}l_0-\frac{1}{\sqrt{r_1}\lambda}\int_0^{l_0}(f_2(\sigma)+i\lambda f_1(\sigma))\sin\frac{\lambda}{\sqrt{r_1}}(l_0-\sigma)\,d\sigma\\
v_x(l_0) &= \frac{r_1}{r_2}\left(c_1\frac{\lambda}{\sqrt{r_1}}\cos\frac{\lambda}{\sqrt{r_1}}l_0-\frac{1}{r_1}\int_0^{l_0}(f_2(\sigma)+i\lambda f_1(\sigma))\cos\frac{\lambda}{\sqrt{r_1}}(l_0-\sigma)\,d\sigma\right).
\end{aligned}
\tag{3.78}
$$

Using (3.78), we can rewrite (3.77) as an equation in the unknown c_1

$$
\begin{aligned}
&c_1\Big[\sin\tfrac{\lambda}{\sqrt{r_1}}l_0\left(-\sqrt{r_2}\lambda\sin\tfrac{\lambda}{\sqrt{r_2}}(L-l_0)+d\cos\tfrac{\lambda}{\sqrt{r_2}}(L-l_0)\right)\\
&+\cos\tfrac{\lambda}{\sqrt{r_1}}l_0\left(\sqrt{r_1}\lambda\cos\tfrac{\lambda}{\sqrt{r_2}}(L-l_0)+d\tfrac{\sqrt{r_1}}{\sqrt{r_2}}\sin\tfrac{\lambda}{\sqrt{r_2}}(L-l_0)\right)\Big]\\
&=\gamma(i\lambda+\eta)^{\alpha-1}f_3(L)-\zeta\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_5(\xi)}{i\lambda+\xi^2+\eta}\,d\xi\\
&+\int_{l_0}^{L}(f_4(\sigma)+i\lambda f_3(\sigma))\cos\frac{\lambda}{\sqrt{r_2}}(L-\sigma)\,d\sigma+\frac{d}{\sqrt{r_2}\lambda}\int_{l_0}^{L}(f_4(\sigma)+i\lambda f_3(\sigma))\sin\frac{\lambda}{\sqrt{r_2}}(L-\sigma)\,d\sigma\\
&+\left[\int_0^{l_0}(f_2(\sigma)+i\lambda f_1(\sigma))\sin\frac{\lambda}{\sqrt{r_1}}(l_0-\sigma)\,d\sigma\right]\left[-\tfrac{\sqrt{r_2}}{\sqrt{r_1}}\sin\tfrac{\lambda}{\sqrt{r_2}}(L-l_0)+\frac{d}{\sqrt{r_1}\lambda}\cos\frac{\lambda}{\sqrt{r_2}}(L-l_0)\right]\\
&+\left[\int_0^{l_0}(f_2(\sigma)+i\lambda f_1(\sigma))\cos\frac{\lambda}{\sqrt{r_1}}(l_0-\sigma)\,d\sigma\right]\left[\cos\tfrac{\lambda}{\sqrt{r_2}}(L-l_0)+\tfrac{d}{\sqrt{r_2}\lambda}\sin\tfrac{\lambda}{\sqrt{r_2}}(L-l_0)\right]
\end{aligned}
\tag{3.79}
$$

Step 3. We set

$$
\begin{aligned}
g(\lambda)=&\Big[\sin\tfrac{\lambda}{\sqrt{r_1}}l_0\left(-\sqrt{r_2}\lambda\sin\tfrac{\lambda}{\sqrt{r_2}}(L-l_0)+d\cos\tfrac{\lambda}{\sqrt{r_2}}(L-l_0)\right)\\
&+\cos\tfrac{\lambda}{\sqrt{r_1}}l_0\left(\sqrt{r_1}\lambda\cos\tfrac{\lambda}{\sqrt{r_2}}(L-l_0)+d\tfrac{\sqrt{r_1}}{\sqrt{r_2}}\sin\tfrac{\lambda}{\sqrt{r_2}}(L-l_0)\right)\Big]\\
=&\Big[\lambda\left(\sqrt{r_1}\cos\tfrac{\lambda}{\sqrt{r_1}}l_0\cos\tfrac{\lambda}{\sqrt{r_2}}(L-l_0)-\sqrt{r_2}\sin\tfrac{\lambda}{\sqrt{r_1}}l_0\sin\tfrac{\lambda}{\sqrt{r_2}}(L-l_0)\right)\\
&+d\left(\sin\tfrac{\lambda}{\sqrt{r_1}}l_0\cos\tfrac{\lambda}{\sqrt{r_2}}(L-l_0)+\tfrac{\sqrt{r_1}}{\sqrt{r_2}}\cos\tfrac{\lambda}{\sqrt{r_1}}l_0\sin\tfrac{\lambda}{\sqrt{r_2}}(L-l_0)\right)\Big]
\end{aligned}
\tag{3.80}
$$

As $f_1\in H^1_*(0,l_0)$ and $f_3\in H^1(l_0,L)$, we have

$$
\left|\int_{l_0}^{L}(f_4(\sigma)+i\lambda f_3(\sigma))\cos\frac{\lambda}{\sqrt{r_2}}(L-\sigma)\,d\sigma\right|\le c(\|f_4\|_{L^2(l_0,L)}+\|f_3\|_{H^1(l_0,L)}).
$$

$$
\left|\int_{l_0}^{L}(f_4(\sigma)+i\lambda f_3(\sigma))\sin\frac{\lambda}{\sqrt{r_2}}(L-\sigma)\,d\sigma\right|\le c(\|f_4\|_{L^2(l_0,L)}+\|f_3\|_{H^1(l_0,L)}).
$$

$$\left| \int_0^{l_0} (f_2(\sigma) + i\lambda f_1(\sigma)) \sin \frac{\lambda}{\sqrt{r_1}} (l_0 - \sigma)\, d\sigma \right| \leq c(\|f_2\|_{L^2(0,l_0)} + \|f_1\|_{H^1(0,l_0)}).$$
$$\left| \int_0^{l_0} (f_2(\sigma) + i\lambda f_1(\sigma)) \cos \frac{\lambda}{\sqrt{r_1}} (l_0 - \sigma)\, d\sigma \right| \leq c(\|f_2\|_{L^2(0,l_0)} + \|f_1\|_{H^1(0,l_0)}).$$

If $r_1 = r_2$. Then

$$g(\lambda) = \sqrt{r_1}\lambda \cos \frac{\lambda}{\sqrt{r_1}} L + d \sin \frac{\lambda}{\sqrt{r_1}} L.$$

We can easily prove that

$$|g(\lambda)| \geq c|\lambda|^{\alpha} \text{ for } \lambda \text{ large.}$$

Hence

$$|c_1| \leq c|\lambda|^{-\alpha} \text{ for } \lambda \text{ large.}$$

Then, we deduce that

$$\|u_x\|_{L^2(0,l_0)} \leq c|\lambda|^{1-\alpha} \text{ for } \lambda \text{ large.}$$

Moreover, as $v(l_0) = u(l_0)$ and $r_2 v_x(l_0) = r_1 u_x(l_0)$, we have

$$|v(l_0)| \leq c|\lambda|^{-\alpha}, |v_x(l_0)| \leq c|\lambda|^{1-\alpha} \text{ for } \lambda \text{ large.}$$

Hence

$$\|v_x\|_{L^2(l_0,L)} \leq c|\lambda|^{1-\alpha} \text{ for } \lambda \text{ large.}$$

From $(3.72)_1, (3.72)_3$ and (3.73), we have

$$\|\tilde{u}\|_{L^2(0,l_0)}, \quad \|\tilde{v}\|_{L^2(l_0,L)} \leq c|\lambda|^{1-\alpha} \text{ for } \lambda \text{ large.}$$

From (3.75), we get

$$\begin{aligned} \|\phi\|_{L^2(-\infty,\infty)} &\leq |\tilde{v}(L)| \left\| \frac{\mu(\xi)}{i\lambda + \xi^2 + \eta} \right\|_{L^2(-\infty,\infty)} + \left\| \frac{f_5(\xi)}{i\lambda + \xi^2 + \eta} \right\|_{L^2(-\infty,\infty)} \\ &\leq c|\lambda|^{-\alpha/2} \left(\|f_1\|_{H^1(0,l_0)} + \|f_2\|_{L^2(l_0,L)} \right) + c\frac{1}{|\lambda|} \|f_5\|_{L^2(-\infty,\infty)}. \end{aligned}$$

Thus, we conclude that

$$\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{H}} \leq c|\lambda|^{1-\alpha} \quad \text{as } |\lambda| \to \infty. \tag{3.81}$$

If $r_1 \neq r_2$. Then from (3.80), system

$$\begin{cases} \left(\sqrt{r_1} \cos \frac{\lambda}{\sqrt{r_1}} l_0 \cos \frac{\lambda}{\sqrt{r_2}}(L - l_0) - \sqrt{r_2} \sin \frac{\lambda}{\sqrt{r_1}} l_0 \sin \frac{\lambda}{\sqrt{r_2}}(L - l_0)\right) &= 0 \\ \left(\sin \frac{\lambda}{\sqrt{r_1}} l_0 \cos \frac{\lambda}{\sqrt{r_2}}(L - l_0) + \frac{\sqrt{r_1}}{\sqrt{r_2}} \cos \frac{\lambda}{\sqrt{r_1}} l_0 \sin \frac{\lambda}{\sqrt{r_2}}(L - l_0)\right) &= 0 \end{cases}$$

is equivalent to

$$\begin{cases} \tan \frac{\lambda}{\sqrt{r_1}} l_0 \ \tan \frac{\lambda}{\sqrt{r_2}}(L - l_0) &= \sqrt{\frac{r_1}{r_2}} \\ \dfrac{\tan \frac{\lambda}{\sqrt{r_1}} l_0}{\tan \frac{\lambda}{\sqrt{r_2}}(L - l_0)} &= -\sqrt{\frac{r_1}{r_2}} \end{cases}$$

which is impossible. Therefore, in all cases, we have

$$|g(\lambda)| \geq c|\lambda|^{\alpha} \text{ for } \lambda \text{ large.}$$

Similarly to the case $r_1 = r_2$, we obtain the estimation (3.81).

Besides, we prove that the decay rate is optimal. Indeed, the decay rate is consistent with the asymptotic expansion of eigenvalues which shows a behavior of the real part like $k^{-(1-\alpha)}$.

Chapter 4

Exponential Stability of Compactly Coupled Wave Equations with Time-Varying Delay Terms in the Boundary Feedbacks

4.1 Introduction

In recent years, linear system of compactly coupled wave equations under various internal and boundary feedbacks has been studied (**[23]**,**[32]**,**[39]**).

Motivated by Nicaise and Pignotti [**34**], see also [**33**], we study in this chapter the stability problem for compactly coupled wave equations with boundary time-varying delay terms. These results are obtained by introducing suitable energie and Lyapounov functionals.

let $\Omega \subset \mathbb{R}^n$ be an open bounded domain of with boundary Γ of class C^2 which is devided into two parts Γ_1 and Γ_2 , i.e. $\Gamma = \Gamma_1 \cup \Gamma_2$, with $\bar{\Gamma_1} \cap \bar{\Gamma_2} = \emptyset$.

$$\begin{cases} u_{tt} - \Delta u + l(u - v) = 0 & \text{in } \Omega \times (0, \infty) \\ v_{tt} - \Delta v + l(v - u) = 0 & \text{in } \Omega \times (0, \infty) \\ u(x,t) = v(x,t) = 0 & \text{on } \Gamma_1 \times (0, \infty) \\ \frac{\partial u(x,t)}{\partial \nu} = -\alpha_1 u_t(x,t) - \alpha_2 u_t(x, t - \tau(t)) & \text{on } \Gamma_2 \times (0, \infty) \\ \frac{\partial u(x,t)}{\partial \nu} = -\beta_1 u_t(x,t) - \beta_2 u_t(x, t - \tau(t)) & \text{on } \Gamma_2 \times (0, \infty) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x) & \text{in } \Omega \\ u_t(x, t - \tau) = f_0(x, t - \tau(0)) & \text{on } \Gamma_2 \times (0, \tau(0)) \\ v_t(x, t - \tau) = g_0(x, t - \tau(0)) & \text{on } \Gamma_2 \times (0, \tau(0)) \end{cases} \tag{4.1}$$

where $l, \alpha_1, \alpha_2, \beta_1$ and β_2 are positive real numbers, and the time-varying delay $\tau(t)$ satisfies

$$\tau'(t) < 1, \quad \forall t > 0, \tag{4.2}$$

and

$$\exists M > 0; \quad 0 < \tau_0 \leq \tau(t) \leq M, \quad \forall t > 0, \tag{4.3}$$

Moreover, we assume that

$$\tau \in W^{2,\infty}([0,T]), \quad \forall T > 0. \tag{4.4}$$

We assume that there exists $x_0 \in \mathbb{R}^N$ such that denoting by m the standard multiplier

$$m(x) := x - x_0$$

we have

$$m(x) \cdot \nu(x) \leq 0 \qquad \text{on } \Gamma_1 \tag{4.5}$$

and, for some positive constant δ

$$m(x) \cdot \nu(x) \geq \delta \qquad \text{on } \Gamma_2 \tag{4.6}$$

We give an exponential stability result, under the conditions

$$\alpha_2 < \sqrt{1-d}\alpha_1, \qquad \beta_2 < \sqrt{1-d}\beta_1 \tag{4.7}$$

where d is a constant such that

$$\tau'(t) \leq d < 1, \qquad \forall t > 0. \tag{4.8}$$

The above problem, with both $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ and a constant delay τ, has been studied by S. Rebiai, F. Sidiali **[39]**, they investigate the uniform exponential stability under the assumptions,

$$\alpha_1 > \alpha_2, \beta_1 > \beta_2,$$

the result is obtained by introducing a suitable energy function and by using an observability estimate. Recently in **[17]** the authors introduced the same problem with distributed delay terms in the boundary or internal feedbacks. They investigate the stability of solutions, under some assumptions.

4.2 Well-Posedness of the System

We introduce the auxilliary variables

$$y(x,\rho,t) = u_t(x, t-\tau(t)\rho), \quad z(x,\rho,t) = v_t(x, t-\tau\rho(t)), \quad (x,\rho,t) \in \Gamma_2 \times (0,1) \times (0,\infty) \tag{4.9}$$

Then

$$\tau(t)y_t(x,\rho,t)+(1-\rho'(t)\rho)y_\rho(x,\rho,t) = 0, \quad \tau(t)z_t(x,\rho,t)+(1-\rho'(t)\rho)z_\rho(x,\rho,t) = 0, \quad (x,\rho,t) \in \Gamma_2\times(0,1) \tag{4.10}$$

Thus , system (4.1) becomes

$$\begin{array}{ll}
u_{tt} - \Delta u + l(u-v) = 0 \qquad \text{in } \Omega \times (0,\infty) \\
v_{tt} - \Delta v + l(v-u) = 0 \qquad \text{in } \Omega \times (0,\infty) \\
\tau(t) y_t(x,\rho,t) + (1-\rho'(t)\rho) y_\rho(x,\rho,t) = 0, \quad \text{in } \Gamma_2 \times (0,1) \times (0,\infty) \\
\tau(t) z_t(x,\rho,t) + (1-\rho'(t)\rho) z_\rho(x,\rho,t) = 0, \quad \text{in } \Gamma_2 \times (0,1) \times (0,\infty) \\
u(x,t) = v(x,t) = 0 \qquad \text{on } \Gamma_1 \times (0,\infty) \\
\frac{\partial u}{\partial \nu}(x,t) = -\alpha_1 u_t(x,t) - \alpha_2 y(x,1,t) \qquad \text{on } \Gamma_2 \times (0,\infty) \\
\frac{\partial u}{\partial \nu}(x,t) = -\beta_1 u_t(x,t) - \beta_2 z(x,1,t) \qquad \text{on } \Gamma_2 \times (0,\infty) \\
y(x,0,t) = u_t(x,t), \quad z(x,0,t) = v_t(x,t), \quad \text{in } \Gamma_2 \times (0,\infty) \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \qquad \text{in } \Omega \\
v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x) \qquad \text{in } \Omega \\
y(x,\rho,0) = f_0(x,,-\rho\tau(0)), \quad z(x,1,t) = g_0(x,,-\rho\tau(0)), \quad \text{in } \Gamma_2 \times (0,1)
\end{array} \tag{4.11}$$

Denote by $\mathcal{H}$ the Hilbert space

$$\mathcal{H} := (H^1_{\Gamma_1}(\Omega))^2 \times (L^2(\Omega))^2 \times (L^2(\Gamma_2 \times (0,1)))^2 \tag{4.12}$$

we define the inner product in $\mathcal{H}$

$$\begin{aligned}
\left\langle \begin{pmatrix} u \\ v \\ \varphi \\ \psi \\ y \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\varphi} \\ \tilde{\psi} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} \right\rangle_{\mathcal{H}} &= \int_\Gamma [\nabla u(x) \nabla \tilde{u}(x) + \varphi(x)\tilde{\varphi}(x)]dx + \int_{\Gamma_2}\int_0^1 y(x,\rho)\tilde{y}(x,\rho)d\rho d\Gamma \\
&\int_\Gamma [\nabla v(x) \nabla \tilde{v}(x) + \psi(x)\tilde{\psi}(x)]dx + \int_{\Gamma_2}\int_0^1 z(x,\rho)\tilde{z}(x,\rho)d\rho d\Gamma \\
&+ l \int_\Omega (u(x)-v(x))(\tilde{u}(x)-\tilde{v}(x))dx
\end{aligned} \tag{4.13}$$

Let $U(t) = (u, v, \varphi, \psi, y, z)^T$, then U satisfies the problem

$$\begin{cases} U_t = \mathcal{A}U(t), & t > 0 \\ U(0) = U_0 \end{cases} \tag{4.14}$$

where $U_0 := (u_0, v_0, u_1, v_1, f_0(.,-\tau), g_0(.,-\tau), \theta_0, \eta_0)$, and the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ \varphi \\ \psi \\ y \\ z \end{pmatrix} = \begin{pmatrix} \varphi \\ \psi \\ a\Delta u - l(u-v) \\ b\Delta v - l(v-u) \\ \frac{\tau'(t)\rho - 1}{\tau(t)} y_\rho \\ \frac{\tau'(t)\rho - 1}{\tau(t)} z_\rho \end{pmatrix}$$

The domain of $\mathcal{A}$ is

$$D(\mathcal{A}) = \left\{ \begin{array}{c} (u,v,\varphi,\psi,y,z)^T \in \mathcal{H} : u, v \in E(\Delta, L^2(\Omega)) \cap H^1_{\Gamma_1}(\Omega), \varphi, \psi \in H^1_{\Gamma_1}(\Omega), \\ y, z \in L^2(\Gamma_2; H^1(0,1)) : \frac{\partial u}{\partial \nu} = -\alpha_1 v - \alpha_2 y(\cdot,1) \text{ on } \Gamma_2; \varphi = y(\cdot,0) \text{ on } \Gamma_2, \\ \frac{\partial v}{\partial \nu} = -\beta_1 v - \beta_2 z(\cdot,1) \text{ on } \Gamma_2; \quad \psi = z(\cdot,0) \text{ on } \Gamma_2 \end{array} \right\} \tag{4.15}$$

Where

$$E(\Delta, L^2(\Omega)) = \Big\{ w \in H^1(\Omega) : \Delta w \in L^2(\Omega) \Big\}$$

Since the domain of the operator $\mathcal{A}(t)$ is independent of the time t, we can write,

$$D(\mathcal{A}(t)) = D(\mathcal{A}(0)), \qquad t > 0. \tag{4.16}$$

Theorem 4.2.1 *(see* **[20]**,**[21]***)*
Assume that
(i) $Y = D(\mathcal{A}(0))$ *is a dense subset of* $\mathcal{H}$.
(ii) (4.16) *holds.*
(iii) for all $t \in [0, T]$, $\mathcal{A}(t)$ *generates a strongly continuous semigroup on* $\mathcal{H}$ *and the family* $\mathcal{A} = \{\mathcal{A}(t) : t \in [0, T]\}$ *is stable with stability constants* C *and* m *independent of* t
(iv) $\partial_t \mathcal{A}$ *belongs to* $L^\infty_\star([0, T], B(Y, \mathcal{H}))$, *the space of equivalent classes of essentially bounded, strongly measurable functions from* $[0, T]$ *into the set* $B(Y, \mathcal{H})$ *of bounded operators from* Y *into* $\mathcal{H}$.
Then, problem (4.14) *has a unique solution* $U \in C([0, T], Y) \cap C^1([0, T], \mathcal{H}))$ *for any initial datum in* Y.

Lemma 4.2.1 $D(\mathcal{A}(0))$ *is dense in* $\mathcal{H}$.

Proof.
Let $(\hat{u}, \hat{v}, \hat{\varphi}, \hat{\psi}, \hat{y}, \hat{z}) \in \mathcal{H}$ such that,

$$0 = \left\langle \begin{pmatrix} u \\ v \\ \varphi \\ \psi \\ y \\ z \end{pmatrix}, \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{\varphi} \\ \hat{\psi} \\ \hat{y} \\ \hat{z} \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_\Gamma [\nabla u(x) \nabla \hat{u}(x) + \varphi(x)\hat{\varphi}(x)]dx + \int_{\Gamma_2} \int_0^1 y(x, \rho)\tilde{y}(x, \rho) d\rho d\Gamma$$

$$\int_\Gamma [\nabla v(x) \nabla \tilde{v}(x) + \psi(x)\hat{\psi}(x)]dx + \int_{\Gamma_2} \int_0^1 z(x, \rho)\hat{z}(x, \rho) d\rho d\Gamma$$

$$+l \int_\Omega (u(x) - v(x))(\hat{u}(x) - \hat{v}(x))dx \tag{4.17}$$

for all $(u, v, \varphi, \psi, y, z)^T \in D(\mathcal{A}(0))$.

- $u = v = \varphi = \psi = y = 0$ and $z \in D(\Gamma_2 \times (0, 1))$. As $(0, 0, 0, 0, 0, z)^T \in D(\mathcal{A}(0))$, from (4.17) we get,

  $$\int_{\Gamma_2} \int_0^1 z(x, \rho)\hat{z}(x, \rho) d\rho d\Gamma = 0.$$

  Since $D(\Gamma_2 \times (0, 1))$ is dense in $L^2(\Gamma_2 \times (0, 1))$, we deduce that $\hat{z} = 0$.

  Similary for

- $u = v = \varphi = \psi = z = 0$ and $y \in D(\Gamma_2 \times (0, 1))$. Then $\hat{y} = 0$.

- $u = v = \varphi = y = z = 0$ and $\psi \in D(\Omega)$. Then $\hat{\psi} = 0$.
- $u = v = \psi = y = z = 0$ and $\varphi \in D(\Omega)$.Then $\hat{\varphi} = 0$.

  For the case

- $v = \varphi = \psi = y = z = 0$ and $u \in D(\Delta) := \{\varphi \in E(\Delta, L^2(\Omega)) \cap V : \frac{\partial \varphi}{\partial n} = 0 \text{ on } \Gamma_2\}$, we get from (4.17),

$$\int_\Gamma \nabla u(x) \nabla \hat{u}(x) dx = 0.$$

  Since $D(\Delta)$ is dense in $H^1_{\Gamma_1}$, then $\hat{u} = 0$.

  And similary for

- $u = \varphi = \psi = y = z = 0$ and $v \in D(\Delta)$, then $\hat{v} = 0$.

Theorem 4.2.2 *For every $U_0 \in \mathcal{H}$ the problem* (4.14) *has a unique solution $U \in C([0, \infty), \mathcal{H})$ In addition, if we assume $U_0 \in D(\mathcal{A}(0))$, then we have*

$$U \in C([0, \infty), D(\mathcal{A}(0)) \cap C^1([0, \infty), \mathcal{H})). \tag{4.18}$$

Proof. We define the time dependent inner product in $\mathcal{H}$

$$\left\langle \begin{pmatrix} u \\ v \\ \varphi \\ \psi \\ y \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\varphi} \\ \tilde{\psi} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} \right\rangle_t = \int_\Omega [\nabla u(x) \nabla \tilde{u}(x) + \varphi(x) \tilde{\varphi}(x)] dx + \mu \tau(t) \int_{\Gamma_2} \int_0^1 y(x, \rho) \tilde{y}(x, \rho) d\rho d\Gamma$$

$$\int_\Omega [\nabla v(x) \nabla \tilde{v}(x) + \psi(x) \tilde{\psi}(x)] dx + \xi \tau(t) \int_{\Gamma_2} \int_0^1 z(x, \rho) \tilde{z}(x, \rho) d\rho d\Gamma$$

$$+ l \int_\Omega (u(x) - v(x))(\tilde{u}(x) - \tilde{v}(x)) dx \tag{4.19}$$

we calculate $\langle \mathcal{A}(t)U, U \rangle_t$, for a fixed t.

$$\langle \mathcal{A}(t)U, U \rangle_t = \left\langle \begin{pmatrix} \varphi \\ \psi \\ a\Delta u - l(u - v) \\ b\Delta v - l(v - u) \\ \frac{\tau'(t)\rho - 1}{\tau(t)} y_\rho \\ \frac{\tau'(t)\rho - 1}{\tau(t)} z_\rho \end{pmatrix}, \begin{pmatrix} u \\ v \\ \varphi \\ \psi \\ y \\ z \end{pmatrix} \right\rangle_t$$

$$= \int_\Omega \nabla u(x) \nabla u(x) + \varphi(x) \Delta u(x) dx + \mu \int_{\Gamma_2} \int_0^1 (1 - \tau'(t)\rho) y_\rho(x, \rho) y(x, \rho) d\rho d\Gamma$$

$$+ \int_\Omega [\nabla v(x) \nabla v(x) + \psi(x) \Delta v(x) dx + \xi \int_{\Gamma_2} \int_0^1 (1 - \tau'(t)\rho) z_\rho(x, \rho) z(x, \rho) d\rho d\Gamma$$

$$+ l \int_\Omega (u(x) - v(x))^2 dx \tag{4.20}$$

using Cauchy-Schwarz's inequality

$$\begin{aligned}\langle \mathcal{A}(t)U,U\rangle_t \leq &\left(-\alpha_1+\frac{\alpha_2}{2\sqrt{1-d}}+\frac{\mu}{2}\right)\int_{\Gamma_1}\varphi^2(x)d\Gamma+\int_{\Gamma_1}y^2(x,1)d\Gamma\\ &+\left(-\beta_1+\frac{\beta_2}{2\sqrt{1-d}}+\frac{\xi}{2}\right)\int_{\Gamma_1}\psi^2(x)d\Gamma+\int_{\Gamma_1}z^2(x,1)d\Gamma\\ &+\kappa(t)\,\langle U,U\rangle_t\end{aligned} \tag{4.21}$$

where,

$$\kappa(t)=\frac{(\tau'(t)^2+1)^{\frac{2}{2}}}{2\tau(t)} \tag{4.22}$$

Then,

$$\langle \mathcal{A}(t)U,U\rangle_t-\kappa(t)\,\langle ,U,U\rangle_t\leq 0, \tag{4.23}$$

which means that the operator $\tilde{\mathcal{A}}(t)=\mathcal{A}(t)-\kappa(t)I$ is dissipative.

Now, we will show that $\lambda I-\mathcal{A}(t)$ is surjective for fixed $t>0$ and $\lambda>0$.

$$(\lambda I-\mathcal{A}(t))\begin{pmatrix}u\\ v\\ \varphi\\ \psi\\ y\\ z\end{pmatrix}=\begin{pmatrix}\tilde{u}\\ \tilde{v}\\ \tilde{\varphi}\\ \tilde{\psi}\\ \tilde{y}\\ \tilde{z}\end{pmatrix} \tag{4.24}$$

that is verifying

$$\begin{cases}\lambda u-\varphi=\hat{u}\\ \lambda v-\psi\hat{v}\\ \lambda\varphi-a\Delta u+l(u-v)=\hat{\varphi}\\ \lambda\psi-b\Delta v+l(v-u)=\hat{\psi}\\ \lambda y-\frac{\tau'(t)\rho-1}{\tau(t)}y_\rho=\hat{y}\\ \lambda z-\frac{\tau'(t)\rho-1}{\tau(t)}z_\rho=\hat{z}\end{cases} \tag{4.25}$$

$$\begin{aligned}\varphi&=\lambda u-\hat{u}\\ \psi&=\lambda v-\hat{v}\end{aligned} \tag{4.26}$$

$$y(x,0)=\varphi(x),\qquad z(x,0)=\psi(x),\quad x\in\Gamma_2. \tag{4.27}$$

$$\begin{aligned}\lambda y(x,\rho)+\tfrac{1-\tau'(t)\rho}{\tau(t)}y_\rho(x,\rho)=\hat{y}\\ \lambda z(x,\rho)+\tfrac{1-\tau'(t)\rho}{\tau(t)}z_\rho(x,\rho)=\hat{z}\end{aligned} \tag{4.28}$$

We can easiely check (see **[34]**), that if $\tau'(t)=0$,

$$\begin{aligned}y(x,\rho)=\varphi(x)e^{-\lambda\rho(t)}+\tau(t)e^{-\lambda\rho(t)}\textstyle\int_0^\rho\hat{y}(x,\sigma)d\sigma,\\ z(x,\rho)=\psi(x)e^{-\lambda\rho(t)}+\tau(t)e^{-\lambda\rho(t)}\textstyle\int_0^\rho\hat{z}(x,\sigma)d\sigma,\end{aligned}$$

therefore,

$$\begin{aligned}
&\textstyle\int_{\Omega}(\lambda^2uw+\nabla u\nabla w)dx+\int_{\Omega}(\lambda^2v\chi+\nabla v\nabla\chi)dx+\int_{\Gamma_2}(\alpha_1+\alpha_2e^{-\lambda\tau(t)})\lambda uwd\Gamma\\
&+\textstyle\int_{\Gamma_2}(\beta_1+\beta_2e^{-\lambda\tau(t)}\lambda v\chi d\Gamma=\int_{\Gamma_2}(\hat{\varphi}+\lambda\hat{u})wdx+\int_{\Gamma_2}(\alpha_1u-\alpha_2y_0)wd\Gamma\\
&+\textstyle\int_{\Gamma_2}(\hat{\psi}+\lambda\hat{v})\chi dx+\int_{\Gamma_2}(\beta_1v-\beta_2z_0)wd\Gamma
\end{aligned}\tag{4.29}$$

Similary for $\tau'(t)\neq 0$,

$$\begin{aligned}
&\textstyle\int_{\Omega}(\lambda^2uw+\nabla u\nabla w)dx+\int_{\Omega}(\lambda^2v\chi+\nabla v\nabla\chi)dx+\int_{\Gamma_2}(\alpha_1+\alpha_2e^{-\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t))})\lambda uwd\Gamma\\
&+\textstyle\int_{\Gamma_2}(\beta_1+\beta_2e^{-\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t))})\lambda v\chi d\Gamma=\int_{\Gamma_2}(\hat{\varphi}+\lambda\hat{u})wdx+\int_{\Gamma_2}(\alpha_1u-\alpha_2y_0)wd\Gamma\\
&+\textstyle\int_{\Gamma_2}(\hat{\psi}+\lambda\hat{v})\chi dx+\int_{\Gamma_2}(\beta_1v-\beta_2z_0)wd\Gamma
\end{aligned}\tag{4.30}$$

It is easy to verify that left-hand side of (4.29) and (4.30) is continuous and coercive, and right hand-side of them is continuous. By applying the Lax-Milgram Theorem, (4.29) and (4.30) has a unique solution $(u,v)\in H^1_{\Gamma_1}(\Omega)\times H^1_{\Gamma_1}(\Omega)$.

We find that $(u,v,\varphi,\psi,y,z)^T\in D(\mathcal{A}(t))$. Again as $\kappa(t)>0$, this proves that

$$\lambda I-\tilde{\mathcal{A}}(t)=(\lambda+\kappa(t))I-\mathcal{A}(t)\text{ is surjective}\tag{4.31}$$

for any $\lambda>0$ and $t>0$.

Finaly we can easiely check (see **[34]**) that,

$$\frac{\|\phi\|_t}{\|\phi\|_s}\leq e^{\frac{c}{\tau_0}|t-s|};\forall t,s\in[0,T]\tag{4.32}$$

where $\phi=(u,\varphi,v,\psi,y,z)^T$ and c is a positive constant and,

$$\frac{d}{dt}\tilde{\mathcal{A}}(t)\in L^\infty_\star([0,T],B(D(\mathcal{A})(0),\mathcal{H}))\tag{4.33}$$

the space of equivalence classes of essentially bounded, strongly measurable functions from $[0,T]$ into $B(D(\mathcal{A})(0),\mathcal{H})$.

Then, by Proposition 1.1 from **[21]**, (4.32), (4.23) and (4.31) we conclude that the family $\tilde{\mathcal{A}}=\{\tilde{\mathcal{A}}(t):t\in[0,T]\}$ is a stable family of generators in $\mathcal{H}$ with stability constants independent of t. Therefore, the assumptions (i)-(iv) of Theorem 4.2.1 are verified by Lemma 4.2.1, (4.16), (4.32), (4.23), (4.33), and (4.31). Thus, the problem

$$\begin{cases}\tilde{U}_t=\tilde{\mathcal{A}}\tilde{U}(t), & t>0\\ \tilde{U}(0)=\tilde{U}_0\end{cases}\tag{4.34}$$

has a unique solution $\tilde{U}\in C([0,\infty),D(\mathcal{A}(0))\cap C^1([0,\infty),\mathcal{H}))$ for $U_0\in D(\mathcal{A}(0)$.

Then the solution of (4.14) is given by,

$$U(t)=e^{\beta(t)}\tilde{U}(t)$$

with $\beta(t)=\int_0^t\kappa(s)ds$

4.3 Exponential Stability

For a solution of (4.1), we define the energy

$$\begin{aligned} E(t) &= \frac{1}{2}\int_\Omega [u_t^2(x,t)+|\nabla u(x,t)|^2+v_t^2(x,t)+|\nabla v(x,t)|^2+l(u-v)^2]dx \\ &+\frac{\mu}{2}\tau(t)\int_{\Gamma_2}\int_0^1 u_t^2(x,t-\tau(t)\rho)d\rho d\Gamma+\frac{\xi}{2}\tau(t)\int_{\Gamma_2}\int_0^1 v_t^2(x,t-\tau(t)\rho)d\rho d\Gamma \end{aligned} \tag{4.35}$$

where,

$$\frac{\alpha_2}{\sqrt{1-d}}<\mu<2\alpha_1-\frac{\alpha_2}{\sqrt{1-d}}, \tag{4.36}$$

$$\frac{\beta_2}{\sqrt{1-d}}<\xi<2\beta_1-\frac{\beta_2}{\sqrt{1-d}} \tag{4.37}$$

Theorem 4.3.1 *Assume* (4.7). *There exist positive constants K_1, K_2 such that for any solution of problem*

$$E(t)\le K_1E(0)e^{-K_2t}. \tag{4.38}$$

Proof. We proceed in several steps.

Step1

Differentiating $E(t)$ with respect to t,

$$\begin{aligned} \frac{d}{dt}E(t) &= \int_\Omega [u_{tt}u_t+\nabla u\nabla u_t+v_{tt}v_t+\nabla v\nabla v_t+l(u-v)(u_t-v_t)]dx \\ &+\frac{\mu}{2}\tau'(t)\int_{\Gamma_2}\int_0^1 u_t^2(x,t-\tau(t)\rho)d\rho d\Gamma+\frac{\xi}{2}\tau'(t)\int_{\Gamma_2}\int_0^1 v_t^2(x,t-\tau(t)\rho)d\rho d\Gamma \\ &+\mu\tau(t)\int_{\Gamma_2}\int_0^1 u_t(x,t-\tau(t)\rho)u_{tt}(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho d\Gamma \\ &+\xi\tau(t)\int_{\Gamma_2}\int_0^1 v_t(x,t-\tau(t)\rho)v_{tt}(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho d\Gamma \end{aligned} \tag{4.39}$$

appliying Green's formula,

$$\begin{aligned} \frac{d}{dt}E(t) &= \int_{\Gamma_2}\frac{\partial u}{\partial\nu}u_t d\Gamma+\frac{\mu}{2}\tau'(t)\int_{\Gamma_2}\int_0^1 u_t^2(x,t-\tau(t)\rho)d\rho d\Gamma \\ &+\mu\tau(t)\int_{\Gamma_2}\int_0^1 u_t(x,t-\tau(t)\rho)u_{tt}(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho d\Gamma \\ &+\int_{\Gamma_2}\frac{\partial v}{\partial\nu}v_t d\Gamma+\frac{\xi}{2}\tau'(t)\int_{\Gamma_2}\int_0^1 v_t^2(x,t-\tau(t)\rho)d\rho d\Gamma \\ &+\xi\tau(t)\int_{\Gamma_2}\int_0^1 v_t(x,t-\tau(t)\rho)v_{tt}(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho d\Gamma \end{aligned} \tag{4.40}$$

Now, we have

$$u_t(x,t-\tau(t)\rho)=\tau^{-1}(t)u_\rho(x,t-\tau(t)\rho),$$

$$v_t(x,t-\tau(t)\rho)=\tau^{-1}(t)v_\rho(x,t-\tau(t)\rho),$$

which lead to

$$u_{tt}(x,t-\tau(t)\rho)=\tau^{-2}(t)u_{\rho\rho}(x,t-\tau(t)\rho),$$
$$v_{tt}(x,t-\tau(t)\rho)=\tau^{-2}(t)v_{\rho\rho}(x,t-\tau(t)\rho),$$

Therefore,

$$\begin{aligned}\int_0^1 u_t(x,t-\tau(t)\rho)u_{tt}(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho &= -\frac{1}{2}\tau'(t)\tau^{-1}(t)\int_0^1 u_t^2(x,t-\tau(t)\rho)d\rho\\ &\quad -\frac{\tau^{-1}(t)}{2}u_t^2(x,t-\tau(t))(1-\tau'(t))\\ &\quad +\frac{\tau^{-1}(t)}{2}u_t^2(x,t). \qquad (4.41)\end{aligned}$$

Similary,

$$\begin{aligned}\int_0^1 v_t(x,t-\tau(t)\rho)v_{tt}(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho &= -\frac{1}{2}\tau'(t)\tau^{-1}(t)\int_0^1 v_t^2(x,t-\tau(t)\rho)d\rho\\ &\quad -\frac{\tau^{-1}(t)}{2}v_t^2(x,t-\tau(t))(1-\tau'(t))\\ &\quad +\frac{\tau^{-1}(t)}{2}v_t^2(x,t). \qquad (4.42)\end{aligned}$$

Using (4.40), (4.41), (4.42) and the boundary conditions $(4.1)_4$,$(4.1)_5$, we have

$$\begin{aligned}\frac{d}{dt}E(t) &= -\alpha_1\int_{\Gamma_2}u_t^2(x,t)d\Gamma-\alpha_2\int_{\Gamma_2}u_t(x,t)u_t(x,t-\tau(t))d\Gamma\\ &\quad +\frac{\mu}{2}\int_{\Gamma_2}u_t^2(x,t)d\Gamma-\frac{\mu}{2}\int_{\Gamma_2}u_t^2(x,t-\tau(t))(1-\tau'(t))d\Gamma\\ &\quad -\beta_1\int_{\Gamma_2}v_t^2(x,t)d\Gamma-\beta_2\int_{\Gamma_2}v_t(x,t)v_t(x,t-\tau(t))d\Gamma\\ &\quad +\frac{\xi}{2}\int_{\Gamma_2}v_t^2(x,t)d\Gamma-\frac{\xi}{2}\int_{\Gamma_2}v_t^2(x,t-\tau(t))(1-\tau'(t))d\Gamma \qquad (4.43)\end{aligned}$$

applying Cauchy-Schwarz?s inequality, we obtain

$$\begin{aligned}\frac{d}{dt}E(t) &\leq -\alpha_1\int_{\Gamma_2}u_t^2(x,t)d\Gamma+\frac{1}{\sqrt{1-d}}\frac{\alpha_2}{2}\int_{\Gamma_2}u_t^2(x,t)d\Gamma\\ &\quad +\sqrt{1-d}\frac{\alpha_2}{2}\int_{\Gamma_2}u_t^2(x,t-\tau(t))d\Gamma-\frac{\mu}{2}(1-\tau'(t))\int_{\Gamma_2}u_t^2(x,t-\tau(t))d\Gamma\\ &\quad +\frac{\xi}{2}\int_{\Gamma_2}v_t^2(x,t)d\Gamma-\beta_1\int_{\Gamma_2}v_t^2(x,t)d\Gamma+\frac{1}{\sqrt{1-d}}\frac{\beta_2}{2}\int_{\Gamma_2}v_t^2(x,t)d\Gamma\\ &\quad +\sqrt{1-d}\frac{\beta_2}{2}\int_{\Gamma_2}v_t^2(x,t-\tau(t))d\Gamma-\frac{\xi}{2}(1-\tau'(t))\int_{\Gamma_2}v_t^2(x,t-\tau(t))d\Gamma\\ &\quad +\frac{\xi}{2}\int_{\Gamma_2}v_t^2(x,t)d\Gamma \qquad (4.44)\end{aligned}$$

Then, for some positive constant C,,

$$\frac{d}{dt}E(t) \leq -C\left(\int_{\Gamma_2}[u_t^2(x,t)+u_t^2(x,t-\tau(t))+v_t^2(x,t)+v_t^2(x,t-\tau(t))d\Gamma\right) \tag{4.45}$$

Step 2.

Now, let us introduce the Lyapounov functional

$$L(t)=E(t)+\gamma\left\{\int_\Omega(2m\cdot\nabla u+(n-1)u)u_t dx+\int_\Omega(2m\cdot\nabla v+(n-1)v)v_t dx+\mathcal{E}(t)\right\} \tag{4.46}$$

where γ is a positive small constant that we will choose later on and $\mathcal{E}(t)$ is defined by

$$\mathcal{E}(t)=\frac{\mu}{2}\tau(t)\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}u_t^2(x,t-\tau(t)\rho)d\rho d\Gamma+\frac{\xi}{2}\tau(t)\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}v_t^2(x,t-\tau(t)\rho)d\rho d\Gamma \tag{4.47}$$

Moreover, we denote by $E_S(t)$ the standard energy for the system without delay, that is

$$E_S(t)=\frac{1}{2}\int_\Omega[u_t^2(x,t)+|\nabla u(x,t)|^2+v_t^2(x,t)+|\nabla v(x,t)|^2+l(u-v)^2]dx \tag{4.48}$$

First we have,

$$\begin{aligned}
&\frac{d}{dt}\left\{\int_\Omega(2m\cdot\nabla u+(n-1)u)u_t dx+\int_\Omega(2m\cdot\nabla v+(n-1)v)v_t dx\right\}\\
&\quad=-\int_\Omega[u_t^2(x,t)+|\nabla u(x,t)|^2 dx-\int_\Omega v_t^2(x,t)+|\nabla v(x,t)|^2 dx\\
&\quad+\int_{\Gamma_2}(m\cdot\nu)[u_t^2(x,t)-|\nabla u(x,t)|^2 d\Gamma+\int_{\Gamma_2}(m\cdot\nu)[v_t^2(x,t)-|\nabla v(x,t)|^2 d\Gamma\\
&\quad+\int_{\Gamma_2}(2m\cdot\nabla u+(n-1)u)\frac{\partial u}{\partial\nu}d\Gamma+\int_{\Gamma_2}(2m\cdot\nabla v+(n-1)v)\frac{\partial v}{\partial\nu}d\Gamma
\end{aligned} \tag{4.49}$$

Young's inequality,

$$\begin{aligned}
&\frac{d}{dt}\left\{\int_\Omega(2m\cdot\nabla u+(n-1)u)u_t dx+\int_\Omega(2m\cdot\nabla v+(n-1)v)v_t dx\right\}\\
&\quad\leq-\int_\Omega[u_t^2(x,t)+|\nabla u(x,t)|^2 dx-\int_\Omega v_t^2(x,t)+|\nabla v(x,t)|^2 dx\\
&\quad+\int_{\Gamma_2}(m\cdot\nu)u_t^2(x,t)d\Gamma-\delta\int_{\Gamma_2}|\nabla u(x,t)|^2 d\Gamma+\int_{\Gamma_2}(m\cdot\nu)[v_t^2(x,t)d\Gamma\\
&\quad-\delta\int_{\Gamma_2}|\nabla v(x,t)|^2 d\Gamma+\frac{c}{\varepsilon}\int_{\Gamma_2}\left(\frac{\partial u}{\partial\nu}\right)^2 d\Gamma+\frac{c}{\varepsilon}\int_{\Gamma_2}\left(\frac{\partial v}{\partial\nu}\right)^2 d\Gamma\\
&\quad+\varepsilon\int_\Omega[u_t^2(x,t)+|\nabla u(x,t)|^2 dx+\varepsilon\int_\Omega v_t^2(x,t)+|\nabla v(x,t)|^2 dx
\end{aligned} \tag{4.50}$$

for some positive constants ε, c. For ε small enough we deduce

$$\begin{aligned}
&\frac{d}{dt}\left\{\int_\Omega(2m\cdot\nabla u+(n-1)u)u_t dx+\int_\Omega(2m\cdot\nabla v+(n-1)v)v_t dx\right\}\leq C_0E_S(t)\\
&\quad+C\int_{\Gamma_2}\left\{u_t^2(x,t)+\left(\frac{\partial u}{\partial\nu}\right)^2+v_t^2(x,t)+\left(\frac{\partial v}{\partial\nu}\right)^2\right\}d\Gamma
\end{aligned} \tag{4.51}$$

Step 3.

Differentiating (4.47) we have

$$
\begin{aligned}
\frac{d}{dt}\mathcal{E}(t) &= \mu\tau'(t)\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}u_t^2(x,t-\tau(t)\rho)d\rho d\Gamma - 2\mu\tau'(t)\tau(t)\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\rho u_t^2(x,t-\tau(t)\rho)d\rho d\Gamma\\
&+2\mu\tau(t)\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}u_t(x,t-\tau(t)\rho)u_{tt}(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho d\Gamma\\
&+\xi\tau'(t)\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}v_t^2(x,t-\tau(t)\rho)d\rho d\Gamma - 2\xi\tau'(t)\tau(t)\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\rho v_t^2(x,t-\tau(t)\rho)d\rho d\Gamma\\
&+2\xi\tau(t)\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}v_t(x,t-\tau(t)\rho)v_{tt}(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho d\Gamma
\end{aligned}
\tag{4.52}
$$

Integrating by parts the third and the last terms in (4.52). We obtain

$$
\begin{aligned}
&\int_0^1 e^{-2\tau(t)\rho}u_t(x,t-\tau(t)\rho)u_{tt}(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho\\
&\qquad= -\frac{1}{2}\tau'(t)\tau^{-1}(t)\int_0^1 e^{-2\tau(t)\rho}u_t^2(x,t-\tau(t)\rho)d\rho\\
&\qquad\quad -\int_0^1 e^{-2\tau(t)\rho}u_t^2(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho\\
&-\frac{\tau^{-1}(t)}{2}e^{-2\tau(t)}u_t^2(x,t-\tau(t))(1-\tau'(t)) + \frac{\tau^{-1}(t)}{2}u_t^2(x,t).
\end{aligned}
\tag{4.53}
$$

Similary,

$$
\begin{aligned}
&\int_0^1 e^{-2\tau(t)\rho}v_t(x,t-\tau(t)\rho)v_{tt}(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho\\
&\qquad= -\frac{1}{2}\tau'(t)\tau^{-1}(t)\int_0^1 e^{-2\tau(t)\rho}v_t^2(x,t-\tau(t)\rho)d\rho\\
&\qquad\quad -\int_0^1 e^{-2\tau(t)\rho}v_t^2(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho\\
&-\frac{\tau^{-1}(t)}{2}e^{-2\tau(t)}v_t^2(x,t-\tau(t))(1-\tau'(t)) + \frac{\tau^{-1}(t)}{2}v_t^2(x,t).
\end{aligned}
\tag{4.54}
$$

and so,

$$
\begin{aligned}
\frac{d}{dt}\mathcal{E}(t) &= -2\mu\tau(t)\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}u_t^2(x,t-\tau(t)\rho)d\rho d\Gamma\\
&\quad -2\mu e^{-2\tau(t)\rho}\int_{\Gamma_2}u_t^2(x,t-\tau(t)\rho)(1-\rho'(t))d\Gamma + \mu\int_{\Gamma_2}u_t^2(x,t)d\Gamma\\
&\quad -2\xi\tau(t)\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}v_t^2(x,t-\tau(t)\rho)d\rho d\Gamma\\
&\quad -2\xi e^{-2\tau(t)\rho}\int_{\Gamma_2}v_t^2(x,t-\tau(t)\rho)(1-\rho'(t))d\Gamma + \xi\int_{\Gamma_2}v_t^2(x,t)d\Gamma
\end{aligned}
\tag{4.55}
$$

Then,

$$
\frac{d}{dt}\mathcal{E}(t) \le -2\mathcal{E}(t) + \mu\int_{\Gamma_2}u_t^2(x,t)d\Gamma + \xi\int_{\Gamma_2}v_t^2(x,t)d\Gamma \tag{4.56}
$$

Step 4.

From (4.45), (4.51) and (4.56), we have

$$\frac{d}{dt}L(t) \leq -C\int_{\Gamma_2}\left\{u_t^2(x,t)+u_t^2(x,t-\tau(t))+v_t^2(x,t)+v_t^2(x,t-\tau(t))\right\}d\Gamma \\ +\gamma\left(-C_0E_S(t)-2\mathcal{E}(t)\right. \\ \left.+C_1\int_{\Gamma_2}\left\{u_t^2(x,t)+u_t^2(x,t-\tau(t))+v_t^2(x,t)+v_t^2(x,t-\tau(t))\right\}d\Gamma\right) \tag{4.57}$$

Therefore,

$$\frac{d}{dt}L(t) \leq -\gamma C_0E_S(t)-2\gamma\mathcal{E}(t) \leq -CL(t). \tag{4.58}$$

This implies

$$L(t) \leq e^{-Ct}L(0)$$

and so,

$$E(t) \leq K_1E(0)e^{-K_2t},$$

for suitable constants $K_1, K_2 > 0$.

Chapter 5

Blow-up for coupled nonlinear wave equations with fractional damping and source terms

5.1 Introduction

In this chapter, We consider the following system

$$u_{tt} + \partial_t^{1-\alpha} u = div(\rho_1(|\nabla u|^2)\nabla u) + f_1(u, v) \tag{5.1}$$

$$v_{tt} + \partial_t^{1-\beta} v = div(\rho_2(|\nabla v|^2)\nabla v) + f_2(u, v) \tag{5.2}$$

with boundary conditions

$$u = v = 0 \tag{5.3}$$

and initial data

$$u(x, 0) = u_0, \quad u_t(x, 0) = u_1, \quad v(x, 0) = v_0, \quad v_1(x, 0) = v_1, \tag{5.4}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$, $n = 1, 2, 3$.

Let $F(u, v) = a|u + v|^{p+1} + 2b|uv|^{\frac{p+1}{2}}$ with $a, b > 0$, $p \geq 3$ if $n = 1, 2$ and $p = 3$ if $n = 3$; $f_1(u, v) = \frac{\partial F}{\partial u}$ and $f_2(u, v) = \frac{\partial F}{\partial v}$ One can easily verify that

$$uf_1(u, v) + vf_2(u, v) = (p + 1)F(u, v) \tag{5.5}$$

There exist two positive constants c_0 and c_1 such that

$$c_0(|u|^{p+1} + |v|^{p+1}) \leq F(u, v) \leq c_1(|u|^{p+1} + |v|^{p+1}) \tag{5.6}$$

Problems of this type arise in material science and physics, which have been studied by many authors (**[14]**,**[35]**,**[2]**,**[31]**).

Concerning a single wave equation with $\rho_1, \rho_2 = 1$, Kirane and Tatar [**22**] introduce the equation of the form

$$u_{tt} - \partial_t^{\alpha} u = \Delta u + a|u|^{p-1}u$$

they prove that solutions growth exponentially for sufficiently large initial data. In [**41**], using an argument involving Fourier transforms and the Hardy-Littlewood-Sobolev inequality, authors prove a finite time blow up of solutions. Alaimia and Tatar [**3**] prove finite time blow up without the dependence of the initial data on the time variable T, for this goal, they introduce a new functional which controle some undesirable terms that appear while using the Georgiev and Todorova [**16**] argument. This problem (with a = 0) has been studied by Matignon et al. in [**28**]. They obtained some results on well posedness and asymptotic stability by transforming the problem into a standard one.

Throughout this chapter, we consider the coupled system (5.1)-(5.4), where

$$\rho_i(s) = b_1 + b_2 s^{q_i}, \quad q_i > 0, \quad b_1 + b_2 > 0 \tag{5.7}$$

In the following, we establish that the solutions of this problem blow up in finite time for sufficiently large initial data using the same techniques as in [**24**].

Now, we state the local existence and uniqueness of the solution of problem

Theorem 5.1.1 *(see* [**41**]*). Assume* (5.6) *holds. Then for any initial data* $u_0 \in W_0^{1,2q_1+2}(\Omega) \cap L^{p+1}(\Omega), v_0 \in W_0^{1,2q_2+2}(\Omega) \cap L^{p+1}(\Omega)$ *and* $u_1, v_1 \in L^2(\Omega)$*, problem* (5.1) – (5.4) *admits a unique local weak solution* (u, v)*:*

$$u \in \mathcal{C}\left([0,T); W_0^{1,2q_1+2}(\Omega) \cap L^{p+1}(\Omega)\right) \cap \mathcal{C}^1\left([0,T); L^2(\Omega)\right),$$

$$v \in \mathcal{C}\left([0,T); W_0^{1,2q_2+2}(\Omega) \cap L^{p+1}(\Omega)\right) \cap \mathcal{C}^1\left([0,T); L^2(\Omega)\right),$$

for some $T > 0$.

For $\eta > 0$, we define the fractional derivative in the sense of Caputo as follows:

$$\partial_t^\eta \omega(t) := \frac{1}{\Gamma(1-\eta)} \int_0^t (t-s)^{-\eta} \omega_s(s) ds.$$

Let us define

$$E(t) = \int_\Omega \left\{ \frac{1}{2}(u_t^2 + v_t^2) + \frac{1}{2}[P_1(|\nabla u|^2) + P_2(|\nabla v|^2)] - F(u,v) \right\} dx \tag{5.8}$$

where $P_i(s) = \int_0^s \rho_i(\xi) d\xi, \quad s \geq 0, i = 1, 2$. It follows that

$$\frac{dE(t)}{dt} = -\frac{1}{\Gamma(\alpha)} \int_\Omega u_t(t) \int_0^t (t-s)^{-\alpha} u_t(s) ds dx - \frac{1}{\Gamma(\beta)} \int_\Omega v_t(t) \int_0^t (t-s)^{-\beta} v_t(s) ds dx. \tag{5.9}$$

Integrating from 0 to t, we obtain

$$\begin{aligned} &E(t) - E(0) = -\tfrac{1}{\Gamma(\alpha)} \textstyle\int_0^t \int_\Omega u_t(s) \int_0^s (s-z)^{-\alpha} u_t(z) dz dx ds \\ &-\tfrac{1}{\Gamma(\beta)} \textstyle\int_0^t \int_\Omega v_t(s) \int_0^s (s-z)^{-\beta} v_t(z) dz dx ds. \end{aligned} \tag{5.10}$$

5.2 Blow up of solutions

Theorem 5.2.1 *Suppose that (u, v) is the solution of the system (5.1)-(5.4). Then, for any $T > 0$, there exist $T^\star \leq T$ and sufficiently large initial data for wich (u, v) blows uo at $T^\star$.*

Proof Let

$$H(t) = -\int_0^t E(s)ds + (dt + l)\int_\Omega (u_0^2 + v_0^2)dx \tag{5.11}$$

where d and l are positive contants to be determined later. Obviously,

$$H'(t) = d\int_\Omega (u_0^2 + v_0^2)dx - E(t) \geq d\int_\Omega (u_0^2 + v_0^2)dx - E(0) \tag{5.12}$$

It suffices to choose d such that

$$d\int_\Omega (u_0^2 + v_0^2)dx - E(0) = H'(0) > 0 \tag{5.13}$$

Then $H'(t) > 0$, and by (5.10) and (5.12), we have

$$H'(t) - H'(0) = E(t) - E(0) = -\frac{1}{\Gamma(\alpha)}\int_0^t \int_\Omega u_t(s)\int_0^s (s-z)^{-\alpha}u_t(z)dzdxds \tag{5.14}$$

$$-\frac{1}{\Gamma(\beta)}\int_0^t \int_\Omega v_t(s)\int_0^s (s-z)^{-\beta}v_t(z)dzdxds \leq 0. \tag{5.15}$$

Considering the functional

$$\Psi(t) = H^{1-\gamma} + \frac{\varepsilon}{2}\left(\int_\Omega (u^2 + v^2)dx - \int_\Omega (u_0^2 + v_0^2)dx\right)$$

where $\varepsilon > 0$, and $0 < \gamma < \frac{p-1}{2(p+1)}$. We have

$$\Psi(0) = H^{1-\gamma}(0) = \left(l\int_\Omega (u_0^2 + v_0^2)dx\right)^{1-\gamma}$$

and

$$\Psi'(t) = (1-\gamma)H^{-\gamma}(t)H'(t) + \varepsilon\int_\Omega (uu_t + vv_t)dx \tag{5.16}$$

A differentiation of (5.16) followed by an integration gives

$$\Psi'(t) = (1-\gamma)H^{-\gamma}(t)H'(t) + \varepsilon\int_\Omega (u_0u_1 + v_0v_1)dx + \varepsilon\int_0^t \int_\Omega (u_t^2 + v_t^2)dxds$$

$$+\varepsilon\int_0^t \int_\Omega (uu_{tt} + vv_{tt})dxds. \tag{5.17}$$

We multiply equation (5.1) (resp. (5.2)) by u (resp. v) and integrate over $\Omega \times (0, t)$, we obtain

$$\begin{aligned}
\int_0^t \int_\Omega (uu_{tt} + vv_{tt})dxds \quad = \quad & -\int_0^t \int_\Omega [\rho_1(|\nabla u|^2)|\nabla u|^2 + \rho_2(|\nabla v|^2)|\nabla v|^2]dxds \\
& +(p+1)\int_0^t \int_\Omega F(u, v)dxds \\
& -\frac{1}{\Gamma(\alpha)}\int_0^t \int_\Omega u\int_0^s (s-z)^{-\alpha}u_t(z)dzdxds \\
& -\frac{1}{\Gamma(\beta)}\int_0^t \int_\Omega v\int_0^s (s-z)^{-\beta}v_t(z)dzdxds
\end{aligned} \tag{5.18}$$

since

$$\int_0^t\int_\Omega[\rho_1(|\nabla u|^2)|\nabla u|^2+\rho_2(|\nabla v|^2)|\nabla v|^2]dxds = b_1\int_0^t\int_\Omega(|\nabla u|^2+|\nabla v|^2)dxds$$
$$+b_2\int_0^t\int_\Omega(|\nabla u|^{2(q_1+1)}+|\nabla v|^{2(q_2+1)})dxds \qquad (5.19)$$

for $q=\max\{q_1,q_2\}$, it follows from the definition of $H(t)$ that,

$$\begin{aligned}-b_2\int_0^t\int_\Omega(|\nabla u|^{2(q_1+1)}+|\nabla v|^{2(q_2+1)})dxds \ \geq\ & 2(q+1)H(t)-2(q+1)(dt+l)\int_\Omega(u_0^2+v_0^2)dx\\ &+(q+1)\int_0^t\int_\Omega(u_t^2+v_t^2)dxds\\ &+(q+1)b_1\int_0^t\int_\Omega(|\nabla u|^2+|\nabla v|^2)dxds\\ &-2(q+1)\int_0^t\int_\Omega F(u,v)dxds \qquad (5.20)\end{aligned}$$

Hence, taking into account (5.17), (5.18), (5.19) and (5.20), we get

$$\begin{aligned}\Psi'(t)\ \geq\ & (1-\gamma)H^{-\gamma}(t)H'(t)+\varepsilon\int_\Omega(u_0u_1+v_0v_1)dx+\varepsilon\int_0^t\int_\Omega(u_t^2+v_t^2)dxds\\ &-\varepsilon b_1\int_0^t\int_\Omega(|\nabla u|^2+|\nabla v|^2)dxds+2\varepsilon(q+1)H(t)-2\varepsilon(q+1)(dt+l)\int_\Omega(u_0^2+v_0^2)dx\\ &+\varepsilon(q+1)\int_0^t\int_\Omega(u_t^2+v_t^2)dxds+\varepsilon(q+1)b_1\int_0^t\int_\Omega(|\nabla u|^2+|\nabla v|^2)dxds\\ &-2\varepsilon(q+1)\int_0^t\int_\Omega F(u,v)dxds+\varepsilon(p+1)\int_0^t\int_\Omega F(u,v)dxds\\ &-\frac{1}{\Gamma(\alpha)}\int_0^t\int_\Omega u\int_0^s(s-z)^{-\alpha}u_t(z)dzdxds\\ &-\frac{1}{\Gamma(\beta)}\int_0^t\int_\Omega v\int_0^s(s-z)^{-\beta}v_t(z)dzdxds \qquad (5.21)\end{aligned}$$

Let us define the extension operators to the whole domain as follows:

$$L\omega(\tau)=\begin{cases}\omega(\tau) \text{ if } \tau\in[0,t],\\ 0 \text{ if } \tau\in\mathbb{R}\backslash[0,t]\end{cases}$$

and

$$Lk_\alpha(\tau)=\begin{cases}k_\alpha(\tau) \text{ if } \tau>0,\\ 0 \text{ if } \tau\leq 0\end{cases}$$

Employing the Parseval theorem, we can rewrite

$$\begin{aligned}\frac{1}{\Gamma(\alpha)}\int_0^t u(s)\int_0^s(s-z)^{-\alpha}u_t(z)dzds &= \int_{-\infty}^{+\infty}Lu(s)\int_{-\infty}^{+\infty}Lk_\alpha(s-z)(Lu_t)(z)dzds\\ &= \int_{-\infty}^{+\infty}F(Lu)(\sigma)\overline{F(Lk_\alpha\star Lu_t)}(\sigma)d\sigma\end{aligned}$$

where $F(f)$ denotes the usual Fourier transform of f.

We use the same method as in [24], we have

$$\int_{-\infty}^{+\infty} Lu(s)\int_{-\infty}^{+\infty} Lk_\alpha(s-z)(Lu_t)(z)dzds \le \frac{\delta}{\cos(\alpha\pi/2)}\int_{-\infty}^{+\infty} Lu_t(s)(L(k_\alpha)\star Lu_t)(s)ds + \frac{1}{4\delta\cos(\alpha\pi/2)}\int_{-\infty}^{+\infty} Lu(s)(L(k_\alpha)\star Lu_t)(s)ds \quad (5.22)$$

Similarly,

$$\int_{-\infty}^{+\infty} Lv(s)\int_{-\infty}^{+\infty} Lk_\beta(s-z)(Lv_t)(z)dzds \le \frac{\delta}{\cos(\beta\pi/2)}\int_{-\infty}^{+\infty} Lv_t(s)(L(k_\beta)\star Lv_t)(s)ds + \frac{1}{4\delta\cos(\beta\pi/2)}\int_{-\infty}^{+\infty} Lv(s)(L(k_\beta)\star Lv_t)(s)ds \quad (5.23)$$

Inserting the estimates (5.22) and (5.23) in (5.21), we find

$$\begin{aligned}\Psi'(t) \ \ge\ & (1-\gamma)H^{-\gamma}(t)H'(t) + \varepsilon\int_\Omega(u_0u_1+v_0v_1)dx + 2\varepsilon(q+1)H(t)\\ & -2\varepsilon(q+1)(dt+l)\int_\Omega(u_0^2+v_0^2)dx + \varepsilon(q+2)\int_0^t\int_\Omega(u_t^2+v_t^2)dxds\\ & +\varepsilon qb_1\int_0^t\int_\Omega(|\nabla u|^2+|\nabla v|^2)dxds + \varepsilon(p-2q-1)\int_0^t\int_\Omega F(u,v)dxds\\ & -\frac{\varepsilon\delta}{\cos(\alpha\pi/2)}\int_{-\infty}^{+\infty} Lu_t(s)(L(k_\alpha)\star Lu_t)(s)ds\\ & -\frac{\varepsilon}{4\delta\cos(\alpha\pi/2)}\int_{-\infty}^{+\infty} Lu(s)(L(k_\alpha)\star Lu_t)(s)ds\\ & -\frac{\varepsilon\delta}{\cos(\beta\pi/2)}\int_{-\infty}^{+\infty} Lv_t(s)(L(k_\beta)\star Lv_t)(s)ds\\ & -\frac{\varepsilon}{4\delta\cos(\beta\pi/2)}\int_{-\infty}^{+\infty} Lv(s)(L(k_\beta)\star Lv_t)(s)ds \qquad (5.24)\end{aligned}$$

If we set $K=\min\{\cos(\alpha\pi/2),\cos(\beta\pi/2)\}$, from (5.14), we see that

$$\frac{\varepsilon\delta}{K}[H'(0)-H'(t)] \le -\frac{\varepsilon\delta}{\cos(\alpha\pi/2)}\int_{-\infty}^{+\infty} Lu_t(s)(L(k_\alpha)\star Lu_t)(s)ds - \frac{\varepsilon\delta}{\cos(\beta\pi/2)}\int_{-\infty}^{+\infty} Lv_t(s)(L(k_\beta)\star Lv_t)(s)ds \quad (5.25)$$

From (see [24]), we have the estimates

$$\int_{-\infty}^{+\infty} Lu(s)(L(k_\alpha)\star Lu_t)(s)ds \le \frac{t^\beta}{\Gamma(1+\alpha)}\int_0^t\int_\Omega|u|^2dxds \quad (5.26)$$

Similary,

$$\int_{-\infty}^{+\infty} Lv(s)(L(k_\beta)\star Lv_t)(s)ds \le \frac{t^\alpha}{\Gamma(1+\beta)}\int_0^t\int_\Omega|v|^2dxds \quad (5.27)$$

Substituting relations (5.25), (5.26) and (5.27) in (5.24), we find

$$\begin{aligned}\Psi'(t) \geq\ & \left[(1-\gamma)H^{-\gamma}(t) - \frac{\varepsilon\delta}{K}\right]H'(t) + \frac{\varepsilon\delta}{K}H'(0) + \varepsilon\int_\Omega (u_0u_1 + v_0v_1)dx \\ & +2\varepsilon(q+1)H(t) - 2\varepsilon(q+1)(dt+l)\int_\Omega (u_0^2+v_0^2)dx \\ & +\varepsilon(q+2)\int_0^t\int_\Omega (u_t^2+v_t^2)dxds + \varepsilon q b_1\int_0^t\int_\Omega (|\nabla u|^2+|\nabla v|^2)dxds \\ & +\varepsilon(p-2q-1)\int_0^t\int_\Omega F(u,v)dxds \\ & -\frac{\varepsilon}{4\delta K\Gamma(\alpha+1)}t^\alpha\int_0^t\int_\Omega |u|^2dxds - \frac{\varepsilon}{4\delta K\Gamma(\beta+1)}t^\beta\int_0^t\int_\Omega |v|^2dxds \end{aligned} \tag{5.28}$$

Taking $\delta = MKH^{-\gamma}(t)$ for some $M>0$, and since $F(u,v)\geq c_0(|u|^{p+1}+|v|^{p+1})$, we get

$$\begin{aligned}\Psi'(t) \geq\ & [(1-\gamma) - M\varepsilon]\,H^{-\gamma}(t)H'(t) + \varepsilon M H^{-\gamma}(t)H'(0) + \varepsilon\int_\Omega (u_0u_1+v_0v_1)dx \\ & +2\varepsilon(q+1)H(t) - 2\varepsilon(q+1)(dt+l)\int_\Omega (u_0^2+v_0^2)dx \\ & +\varepsilon(q+2)\int_0^t\int_\Omega (u_t^2+v_t^2)dxds + \varepsilon q b_1\int_0^t\int_\Omega (|\nabla u|^2+|\nabla v|^2)dxds \\ & +\varepsilon(p-2q-1)c_0\int_0^t\int_\Omega (|u|^{p+1}+|v|^{p+1})dxds \\ & -\frac{\varepsilon}{4MK^2\Gamma(\alpha+1)}t^\alpha H^\gamma(t)\int_0^t\int_\Omega |u|^2dxds \\ & -\frac{\varepsilon}{4MK^2\Gamma(\beta+1)}t^\beta H^\gamma(t)\int_0^t\int_\Omega |v|^2dxds \end{aligned} \tag{5.29}$$

Hôlder's inequality implies

$$\begin{aligned}\int_0^t\int_\Omega |u|^2dxds \leq\ & |\Omega|^{\frac{p-1}{p+1}}t^{\frac{p-1}{p+1}} \times \left(\int_0^t\int_\Omega |u|^{p+1}dxds\right)^{\frac{2}{p+1}} \\ \leq\ & B_1\left(\int_0^t\int_\Omega (|u|^{p+1}+|v|^{p+1})dxds\right)^{\frac{2}{p+1}} \end{aligned} \tag{5.30}$$

where $B_1 = |\Omega|^{\frac{p-1}{p+1}}t^{\frac{p-1}{p+1}}$. Similary,

$$\int_0^t\int_\Omega |v|^2dxds \leq B_1\left(\int_0^t\int_\Omega (|u|^{p+1}+|v|^{p+1})dxds\right)^{\frac{2}{p+1}} \tag{5.31}$$

from the definition of $H(t)$, we see that

$$\begin{aligned} J :=\ & H^\gamma(t)\left(\int_0^t\int_\Omega (|u|^{p+1}+|v|^{p+1})dxds\right)^{\frac{2}{p+1}} \\ \leq\ & \left[\int_0^t\int_\Omega F(u,v)dxds + (dt+l)\int_\Omega (u_0^2+v_0^2)dx\right]^\gamma \times \left(\int_0^t\int_\Omega (|u|^{p+1}+|v|^{p+1})dxds\right)^{\frac{2}{p+1}} \end{aligned}$$

since $F(u,v) \le c_1(|u|^{p+1} + |v|^{p+1})$, we get

$$\begin{aligned} J &\le \left[c_1^\gamma \left(\int_0^t \int_\Omega (|u|^{p+1} + |v|^{p+1}) dx ds\right)^\gamma + (dt+l)^\gamma \left(\int_\Omega (u_0^2 + v_0^2) dx\right)^\gamma\right] \\ &\quad \times \left(\int_0^t \int_\Omega (|u|^{p+1} + |v|^{p+1}) dx ds\right)^{\frac{2}{p+1}} \end{aligned}$$

or

$$\begin{aligned} J &\le c_1^\gamma \left(\int_0^t \int_\Omega (|u|^{p+1} + |v|^{p+1}) dx ds\right)^{\gamma + \frac{2}{p+1}} \\ &\quad + (dt+l)^\gamma \left(\int_\Omega (u_0^2 + v_0^2) dx\right)^\gamma \left(\int_0^t \int_\Omega (|u|^{p+1} + |v|^{p+1}) dx ds\right)^{\frac{2}{p+1}} \end{aligned}$$

As $\gamma + \frac{2}{p+1} < 1$, we obtain

$$\begin{aligned} J &\le c_1^\gamma \left[1 + \int_0^t \int_\Omega (|u|^{p+1} + |v|^{p+1}) dx ds\right] \\ &\quad + 2(dT+l)^\gamma \left(\int_\Omega (u_0^2 + v_0^2) dx\right)^\gamma \left[1 + \int_0^t \int_\Omega (|u|^{p+1} + |v|^{p+1}) dx ds\right] \end{aligned} \tag{5.32}$$

Substituting (5.18), (5.29) and (5.32) in (5.29) we get

$$\begin{aligned} \Psi'(t) &\ge [(1-\gamma) - M\varepsilon] H^{-\gamma}(t) H'(t) + \varepsilon M H^{-\gamma}(t) H'(0) + \varepsilon \int_\Omega (u_0 u_1 + v_0 v_1) dx \\ &\quad + 2\varepsilon(q+1) H(t) - 2\varepsilon(q+1)(dt+l) \int_\Omega (u_0^2 + v_0^2) dx \\ &\quad + \varepsilon(q+2) \int_0^t \int_\Omega (u_t^2 + v_t^2) dx ds + \varepsilon q b_1 \int_0^t \int_\Omega (|\nabla u|^2 + |\nabla v|^2) dx ds \\ &\quad + \varepsilon(p - 2q - 1) c_0 \int_0^t \int_\Omega (|u|^{p+1} + |v|^{p+1}) dx ds \\ &\quad - \frac{\varepsilon A}{M K^2} \left[1 + \int_0^t \int_\Omega (|u|^{p+1} + |v|^{p+1}) dx ds\right] \end{aligned}$$

with $A = \frac{1}{4}\left[\frac{B_1 T^\alpha}{\Gamma(\alpha+1)} + \frac{B_1 T^\beta}{\Gamma(\beta+1)}\right] [c_1^\gamma + (dT+l)^\gamma (\int_\Omega (u_0^2 + v_0^2) dx)^\gamma]$

Choosing $\varepsilon > 0$ such that $\varepsilon \le \frac{1-\gamma}{M}$, it follows that

$$\begin{aligned} \Psi'(t) &\ge 2\varepsilon(q+1) H(t) + \varepsilon(q+2) \int_0^t \int_\Omega (u_t^2 + v_t^2) dx ds \\ &\quad + \varepsilon \left(\int_\Omega (u_0 u_1 + v_0 v_1) dx - 2(q+1)(dT+l) \int_\Omega (u_0^2 + v_0^2) dx - \frac{2A}{M}\right) \\ &\quad + \varepsilon \left((p - 2q - 1) c_0 - \frac{2A}{M}\right) \int_0^t \int_\Omega (|u|^{p+1} + |v|^{p+1}) dx ds \end{aligned}$$

We choose u_0, u_1, v_0 and v_1 such that

$$\int_\Omega (u_0 u_1 + v_0 v_1) dx - 2(q+1)(dT+l) \int_\Omega (u_0^2 + v_0^2) dx - \frac{2A}{M} \ge 0$$

Next, choose B so that $0 < B \leq (p-2q-1)c_0 - \frac{2A}{M}$. We obtain,

$$\Psi'(t) \geq 2\varepsilon(q+1)H(t) + \varepsilon(q+2)\int_0^t\int_\Omega (u_t^2+v_t^2)dxds + \varepsilon B\int_0^t\int_\Omega (|u|^{p+1}+|v|^{p+1})dxds \quad (5.33)$$

On the other hand, we have

$$\Psi^{\frac{1}{1-\gamma}}(t) \leq 2^{\frac{1}{1-\gamma}}\left[H(t) + \epsilon^{\frac{1}{1-\gamma}}\left(\int_0^t\int_\Omega (uu_t+vv_t)dxds\right)^{\frac{1}{1-\gamma}}\right] \quad (5.34)$$

$$\left(\int_0^t\int_\Omega (uu_t+vv_t)dxds\right)^{\frac{1}{1-\gamma}} \leq \left(\int_0^t\int_\Omega uu_t dxds\right)^{\frac{1}{1-\gamma}} + \left(\int_0^t\int_\Omega vv_t dxds\right)^{\frac{1}{1-\gamma}}$$

Applying Cauchy-Schwarz, Hôlder and Young inequalities (see [24]), we obtain

$$\left(\int_0^t\int_\Omega (uu_t+vv_t)dxds\right)^{\frac{1}{1-\gamma}} \leq \lambda\left[\int_0^t\int_\Omega (u_t^2+v_t^2)dxds + T^\mu\int_0^t\int_\Omega (|u|^{p+1}+|v|^{p+1})dxds\right] \quad (5.35)$$

for some $\lambda > 0$ and $\mu = \frac{p-1}{(p+1)(1-2\gamma)}$.

From (5.33), (5.34) and (5.35), we have

$$\Psi^{\frac{1}{1-\gamma}}(t) \leq R\Psi'(t) \quad (5.36)$$

where R is a positive constant. A simple computation yields

$$\Psi^{\frac{1}{1-\gamma}}(t) \geq \frac{1}{\Psi^{-\frac{1}{1-\gamma}}(0) - \frac{\gamma t}{R(1-\gamma)}}$$

which implies that $\Psi(t)$ blows up at some time $T^\star \leq \frac{R(1-\gamma)}{\gamma\Psi^{\frac{1}{1-\gamma}}(0)}$.

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Abstract

In this thesis we considered some evolution problems with the presence of dissipation of fractional derivative type. In particular, we consider transmission system which consist of two coupled wave equations and coupled wave equations with sourse terms. Under assumptions on initial data and boundary conditions, we focused our study on the global existence and asymptotic behavior and blow up of solutions where we obtained several results.

Résume

Dans cette thèse, nous avons considéré quelques problèmes d’évolution hyperbolique avec la présence des termes dissipatifs de type fractionnaires. En particulier on considère le système onde-onde qui est constitué de deux équations des ondes couplées. Sous quelques hypothèses sur les données initiales et aux bords, nous avons concentré notre étude sur l'existence globale et le comportement asymptotique ainsi que l’explosion des solutions où nous avons obtenu plusieurs résultats sur les propriétés de l’énergie.

ملخص

في هذه الأطروحة اقترحنا بعض المسائل الرياضية لمعادلات و جمل معادلات بوجود آليات للتبديد ذات أشكال كسرية من زوايا مختلفة. ندرس خاصة جمل موجات الإرسال تحت بعض الفرضيات على الشروط الابتدائية و الشروط الحدية، ركزنا دراستنا على وجود الحلول ودراسة السلوك المقارب للحلول الموجودة عند اللانهاية الزمنية أين توصلنا لإيجاد عدة نتائج حول طريقة تناقص الطاقة والإنفجار في وقت محدود