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MINISTERE DE L'ENSEIGNEMENT SUPERIEUR & DE LA RECHERCHE SCIENTIFIQUE

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THESE

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Pour obtenir le Diplôme de Doctorat

Spécialité : Mathématiques
Option : Equations différentielles théorie et applications

Intitulée

Structure de l'ensemble des solutions du problème aux limites non linéaire et la méthode itérative

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Acknowledgement

First and foremost, praises and thanks to the Allah the Almighty, for His showers of blessings throughout my research work to complete the research successfully.

I would like to my deep gratilude to Professor Samir Benaissa my research supervisor, for has guidance, encouragement and useful critiques of this indergraduate thesis.

Special thank to collaborator, Dr. Ziane Mohamed for their support and help.

I also wish to express my appreciation to my colleague, Dr. Mechab Boubaker for their help.

Also I thank my friends in Ecole normale supérieur Taleb Abderrahmen Laghouat, Aïdi Jemale, Hadra Toufik, Tahemi Nabile.

Besides my advisor, I would like to thanks the rest of my thesis committee: Professor Miloudi Mostafa and Dr. Tayeb Blouhi , for their encouragement, insightful comments, and hard questions.

I acknowledge with gratitude the support of university Djillali Liabés of Sidi Bel-Abbès and the specialized Depertment of mathematics and Laboratory of Statistics and Stochastic Processes.

Last but not the least, I would like to thank my family: my parents, for giving birth to me at the first place and supporting me spiritually throughout my life.

Notation

1. α : real number
2. X : Banach space
3. $\partial\Omega$: boundary of Ω
4. $co\Omega$: convex cover Ω
5. $\overline{co}\Omega$: closed convex Ω
6. $\mathcal{P}(X)$: set of exponent on X
7. H : Hilbert space
8. $C(J, X)$: Banach space of all continuous functions from J into X
9. $\|x\| = \sup\limits_{t\in J} |x(t)|$: norm of $C(J, X)$
10. $\rightharpoonup$: weak convergence
11. $\rightarrow$: strong convergence
12. l^p : classical seuence space with constant exponent p
13. $\bigtriangledown u = \left(\dfrac{\partial u}{\partial u_1}, \dfrac{\partial u}{\partial u_2}, \cdot, \dfrac{\partial u}{\partial u_N}\right)$: gradient of function u
14. $L^p(\Omega)$: Lebesgue space on Ω with constant exponent p
15. $C^k(\Omega)$: space of k times continuously differentialble functions on Ω, $k \geq 0$
16. $L^1(J, X)$: the space X-valued Bochner integrable functions on J
17. $E_{\alpha,\alpha}$: Mittag-Leffler functions
18. $\Gamma(\cdot)$: Gamma functions
19. $B(\cdot, \cdot)$: Beta functions
20. $f * g$: convolution product f of with g

21. ${}^{L}D^{\alpha}$: Riemann-Liouville derivative of order α

22. $I_{0^+}^{1-\alpha}$: Riemann-Liouville integral of order $1-\alpha$

23. ${}^{C}D^{\alpha}$: Caputo derivative of order α

24. $C_{1-\alpha}(J,X)$: weighted space of continuous functions

25. $\|x\|_{\alpha}$: norm of $C_{1-\alpha}(J,X)$

26. $\multimap$: multivalued map

27. $D(A)$: domain of a mapping A

28. $<\cdot,\cdot>$: inner product in L^2

Introduction

Many mathematicians show stong interest in fractional differential equations and inclusions so wonderful results have been obtained. The theory of fractional differential equations and inclusions is new and important branch of differential equation and inclusion theory, which has an extensive mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, electrical circuits, electro-analytical chemistry, biology.
The definitions of Riemann-Liouville fractional derivatives or integrals initial (local) conditions play an important role, in some pratical problems. Heymans and Podlubny, have demonstrated that it is possible to attribute physical meaning to initial coditions expressed in terms of Riemann-Liouville fractional derivatives or integrals on the field of the viscoelasticity. The nonlocal conditions has recently been used by Byszewski [19], Gaston [30], and they obtained the existence and uniqueness the solutions of nonlocal conditions Cauchy problems. In [27], Deng used the nonlocal conditions to describe the diffusion phenomenon of small amount of gas in a transparent tube.

The monotone iterative method based on lower and upper solutions is an effective and flexible mechanism. It yields monotone sequences of lower and upper approximate solutions that converge to the minimal and maximal solutions between the lower and upper solutions. For ordinary differential equations, many papers used the monotone iterative technique and the method of lower and upper solutions; see [[29, 53]] and monographs.

In order to make the thesis self-contrained, we devote the first chapter 1 to description of general information on fractional calculus, semigroups, space of functions, multivalued maps, measure of noncompactness.

In the second chapter 2, we first study monotone iterative method for weighted fractional differential equations in Banach space.
The suitable solutions of fractional Cauchy problems (IVP) with Riemann-Liouville derivative:

$$^{L}D^{\alpha}_{0^+}x(t) = f(t, x(t)), \;\; t \in J' := (0, b], \tag{1}$$

$$\lim_{t \to 0^+} t^{1-\alpha}x(t) = x_0. \tag{2}$$

We use a method of upper and lower solutions ou inequalite differential including monotone iterative technique to discuss the existence of solutions to the initial value problem (1)-(2) nonlinear in an ordered in infinite dimensional space. We give two successively iterative sequences to approximate the solutions are constructed, then (1)-(2) has minimal and maximal solutions. In the second part of this chapter, we discuss the existence and uniquencess of a solution to an initial value problem (IVP) for a class of nonlinear fractional involving Riemann-Liouville derivative of order $\alpha \in (0,1)$ withe the nonlocal initial conditions in a Banach space:

$$^{L}D^{\alpha}_{0^+}x(t) = f(t, x(t));\;\; t \in J' := (0, b], \tag{3}$$

$$(I^{1-\alpha}_{0^+}x)(0) + g(x) = x_0. \tag{4}$$

Then some sufficient conditions are established for the existence and uniqueness of solutions of (3)-(4) with nonlocal conditions. We prove our main result by introducing a regular measure of noncompactness in the weighted space of continuous functions and using Banach fixed point theory and Darbo fixed point theory. In the third part in chapter we investigate the topological structure of the solution set of an initial value problems (1)-(2) for nonlinear fractional differential equations in Banach space. In [69] Ziane studied on the solution set for weighted fractional differential equations in Banach space by the following assumptions, but motived in the thesis we give the other assumptions so that obtained good estimate. We prove that the solution set of the problem is nonempty, compact and, an R_δ-set by introducing a new regular measure of noncompactness in the weighted space of continuous functions.

The third chapter deals with fractional evolution equation with nonlocal conditions. We are considering the nonlocal Cauchy problems for a semilinear fractional differential equation in Banach space of the following form:

$$\begin{cases} ^{L}D^{\alpha}x(t) = Ax(t) + f(t, x(t));\;\; t \in (0, b], \\ I^{1-\alpha}_{0^+}x(t)\mid_{t=0} + g(x) = x_0. \end{cases} \tag{5}$$

We give existence two resuls of mild solutions in the problem (5).
In the second part of this chapter we investigete semilinear fractional evolution inclusion with nonlocal conditions involving a noncompact semigroup and souce term of multivalued type in Banach spaces:

$$\begin{cases} ^{L}D^{\alpha}_{0^+}x(t) \in Ax(t) + F(t, x(t)); \quad \text{a.e. } t \in (0, b];\;\; 0 < \alpha < 1, \\ I^{1-\alpha}_{0^+}x(t)\mid_{t=0} + g(x) = x_0 \in X, \end{cases} \tag{6}$$

First, a definition of integral solutions for fractional differential inclusions (6) is given. The existence mild solution of fractional evolution inclusion with nonlocal conditions by means a regular mearure of noncompactness and condensing map in the weighted space of continuous functions.

Chapter 1

Preliminaries

In this chapter, we introduce some notations on fractional calculus, semigroup, space of functions, measure of noncompactness, fixed point theorems, condensing maps , theory of multivalued analysis which are used throughout this thesis.

1.1 Notations and definitions

Let $J := [0, b]$, $b > 0$ and $(X, \|.\|)$ be a real Banach space. Denote $C(J, X)$ the space of X-valued continuous functions on J with the uniform norm

$$\|x\|_\infty = \sup\{\|x(t)\|,\ t \in J\}.$$

We denote by $L^p(J, X)$ and $1 \leq p \leq \infty$, the set of those Lebesgue measurable functions $f : J \to X$ for which $\|f\|_{L^p} < \infty$ where

$$\|f\|_{L^p} = \begin{cases} \left(\int_J \|f(t)\|^p dt\right)^{\frac{1}{p}},\ 1 \leq p < \infty, \\ ess \sup\limits_{t \in J} \|f(t)\|,\ p = \infty. \end{cases}$$

In particular, $L^1(J, X)$ is the Banach space of measurable functions $f : J \to X$ with the norm

$$\|f\|_{L^1} = \int_J \|f(t)\| dt,$$

and $L^\infty(J, X)$ is the Banach space of measurable functions $f : J \to X$ which are bounded, equipped with the norm

$$\|f\|_{L^\infty} = \inf\{c > 0, \|f(t)\| \leq c,\ \ a.e.\ t \in J\}.$$

Lemma 1.1 *(Hölder inequality) Assume that $p, q \geq 1$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. If $f \in L^p(J, \mathbb{R})$, $g \in L^q(J, \mathbb{R})$, then $fg \in L^1(J, \mathbb{R})$ and*

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Lemma 1.2 *(Arzela-Ascoli's Theorem) If a family* $F = \{f(t)\}$ *in* $C(J, X)$ *is uniformaly bounded and equicontinuous on* J *and, for any* $t^* \in J$, $\{f(t^*\}$ *is relatively compact, then* F *has a uniformly convergent subsequence* $\{f_n(t)\}_{n=1}^{\infty}$.

Lemma 1.3 *(Lebesgue's dominated convergence Theorem) Let* X *be a measurable set and let* $\{f_n\}$ *be a sequence of measurable functions such that* $\lim_{n\to\infty} f_n(x) = f(x)$ *a.e. in* X*, and for every* $n \in \mathbb{N}$, $\|f_n\| \leq g(x)$ *a.e. in* X *where* g *is integrable on* X*. Then*

$$\lim_{n\to\infty} \int_X f(x)dx = \int_X f(x)dx.$$

Theorem 1.4 *[68](Bochner's theorem) A measurable function* $f : (0, b) \to X$ *is Bochner integrable if* $\|f\|$ *is Lebesgue integrable.*

Definition 1.5 *[68] Let* X *be a Banach space. By a cone* $K \subset X$*, we understand a closed convex subset* K *such that* $\lambda K \subset K$ *for all* $\lambda \geq 0$ *and* $K \cap (-K) = \{0\}$*. We define a partial ordering* $\leq$ *with repect to* K *by* $x \leq y$ *if and only if* $(x - y) \in K$.

Definition 1.6 *[24] The gamma function* $\Gamma(z)$ *is defined by*

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}dt, \;\; Re(z) > 0,$$

where $t^{z-1} = e^{(z-1)log(t)}$*. This integral is convergent for all complex* $z \in \mathbb{C}$ $(Re(z) > 0)$*. For this function the reduction formula*

$$\Gamma(z + 1) = z\Gamma(z), \;\; Re(z) > 0$$

holds. In particular, if $z = n \in \mathbb{N}$*, then*

$$\Gamma(n + 1) = n!, \;\; n \in \mathbb{N},$$

with (as usual) $0! = 1$.

Lemma 1.7 *[24] Let* $\alpha, \beta \in \mathbb{R}_+$*. Then*

$$\int_0^1 t^{\alpha-1}(1 - t)^{\beta-1}dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)},$$

and hence

$$\int_0^x t^{\alpha-1}(x - t)^{\beta-1}dt = x^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}.$$

The integral in the first equation of Lemma is known as Beta function $B(\alpha, \beta)$.

We recall Gronwall's lemma for singular kernels.

Lemma 1.8 *[44] Let $v : J \to [0, \infty)$ be a real function and $\omega(\cdot)$ is a nonnegative, locally integrable function on J and there are constants λ and $0 < \alpha < 1$ such that*

$$v(t) \leq \omega(t) + \lambda \int_0^t (t-s)^{-\alpha} v(s) ds.$$

Then there exists a constant $K = K(\alpha)$ such that

$$v(t) \leq \omega(t) + K\lambda \int_0^t (t-s)^{-\alpha} \omega(s) ds,$$

for every $t \in J$.

1.2 Fractional integral and derivative.

In this section, we introduce some basic definitions and properties of fractional integrals and fractional derivatives, one can see [24, 40, 57, 68].

Definition 1.9 *(Left and right Riemann-Liouville fractional integrals) Let $J = [0, b]$, $(0 < b < \infty)$ be a finite interval of $\mathbb{R}$ the Riemann-Liouville fractional integrals $I_0^\alpha f$ and I_b^α of order $\alpha \in \mathbb{R}^+$ are defined by*

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,\ t > 0,\ \alpha > 0 \tag{1.1}$$

and

$$I_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) ds,\ t > 0,\ \alpha > 0, \tag{1.2}$$

respectively, provided the right side is point-wise defined on $[0, b]$.

Definition 1.10 *(Left and right Riemann-Liouville fractional derivatives) The left and right Riemann-Liouville derivative $D_0^\alpha f$ and D_b^α of order $\alpha\mathbb{R}^+$ are defined by:*

$$\begin{aligned} D_0^\alpha f(t) &= \frac{d^n}{dt^n} I_0^{(n-\alpha)} \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \left(\int_0^t (t-s)^{n-\alpha-1} f(s) ds \right),\ t > 0 \end{aligned}$$

Particular, when $0 < \alpha < 1$, then

$$D_0^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left(\int_0^t (t-s)^{-\alpha} f(s) ds \right)\ t > 0$$

and

$$D_b^\alpha f(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left(\int_t^b (s-t)^{-\alpha} f(s) ds \right),\ t < b.$$

Remark 1.11 *If $f \in C(J,X)$, it is obvious that Riemann-Liouville fractional integral of order $\alpha > 0$ exists on $[0,b]$. On the then hand following Kilbas et al. [40] we know that Riemann-Liouville fractional derivative of order $\alpha \in [n-1,n)$ exists almost every where on $[0,b]$ if $f \in C^n(J,X)$.*

Definition 1.12 *(Left and right Caputo fractional derivatives) The left and right Caputo fractional derivatives ${}^C D_0^\alpha f(t)$ and ${}^C D^\alpha f(t)$ of order $\alpha \in \mathbb{R}^+$ are defined by*

$$ {}^C D_0^\alpha f(t) = D_0^\alpha \left[f(t) - \sum_{k=0}^{n-1} \frac{(t)^k}{k!} f^{(k)}(0) \right] $$

and

$$ {}^C D_b^\alpha f(t) = D_b^\alpha \left(f(t) - \sum_{k=0}^{n-1} \frac{(b-t)^k}{k!} f^{(k)}(b) \right), $$

respectively, where $n = [\alpha] + 1$.

In particular, where $0 < \alpha < 1$, then

$$ {}^C D_0^\alpha f(t) = D_0^\alpha (f(t) - f(0)) $$

and

$$ {}^C D_b^\alpha f(t) = D_b^\alpha (f(t) - f(b)). $$

Riemann-Liouville fractional derivative and Caputo fractional derivative are counected with each orther by the following relations.

Proposition 1.13 *(i) If $f(t)$ is a functions for which Caputo fractional derivatives ${}^C D_0^\alpha f(t)$ and ${}^C D_b^\alpha f(t)$ of order $\alpha \in \mathbb{R}^+$ exist together with the Riemann-Liouville fractional derivatives $D_0^\alpha f(t)$ and $D_b^\alpha f(t)$, then*

$$ {}^C D_0^\alpha f(t) = D_0^\alpha f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\Gamma(k-\alpha+1)} (t)^k $$

and

$$ {}^C D_b^\alpha f(t) = D_b^\alpha f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{\Gamma(k-\alpha+1)} (b-t)^{k-\alpha}, $$

In particular, where $0 < \alpha < 1$, we have

$$ {}^C D_0^\alpha f(t) = D_0^\alpha f(t) - \frac{f(0)}{\Gamma(1-\alpha)} (t)^{-\alpha} $$

and

$$ {}^C D_b^\alpha f(t) = D_b^\alpha f(t) - \frac{f(b)}{\Gamma(1-\alpha)} (b-t)^{-\alpha}. $$

(ii) If $\alpha = n \in \mathbb{N}$ and the usual derivative $f^{(n)}(t)$ of order n exists, then ${}^{C}D_0^{\alpha} f$ and ${}^{C}D_b^{\alpha} f$ are represented by

$${}^{C}D_0^{\alpha} f(t) = f^{(n)}(t)$$

and

$${}^{C}D_0^{\alpha} f(t) = (-1)^n f^{(n)} f(t).$$

Proposition 1.14 *Let $\alpha \in \mathbb{R}^+$ and let n be given $n = [\alpha] + 1$.*
If $f(t) \in C^n(J, X)$, then Caputo fractional derivatives ${}^{C}D_0^{\alpha} f(t)$ and ${}^{C}D_b^{\alpha} f(t)$ exist almost every where on $[0, b]$, then

$${}^{C}D_0^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds \right)$$

and

$${}^{C}D_0^{\alpha} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\int_t^b (s-t)^{n-\alpha-1} f^{(n)}(s) ds \right).$$

In particular, when $0 < \alpha < 1$ and $f(t) \in C(J, X)$,

$${}^{C}D_0^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\int_0^t (t-s)^{-\alpha} f^{'}(s) ds \right)$$

and

$${}^{C}D_b^{\alpha} f(t) = -\frac{1}{\Gamma(1-\alpha)} \left(\int_t^b (s-t)^{-\alpha} f^{'}(s) ds \right).$$

Proposition 1.15 *If $\alpha > 0$ and $\beta > 0$ then the equation*

$$I_0^{\alpha} \left(I_0^{\beta} f(t) \right) = I_0^{\alpha+\beta} f(t) \tag{1.3}$$

are satisfied at almost every point $t \in [0, b]$ for $f(t) \in L^p([0, b], X)$ and $1 \leq p < \infty$. If $\alpha + \beta > 1$, then the relation in (1.3) at any point of $[0, b]$.

Proposition 1.16 *(i) If $\alpha > 0$ and $f(t) \in L^p([0, b], X)$ $(1 \leq p \leq \infty)$, then the following equality*

$$D_0^{\alpha} \left(I_0^{\alpha} f(t) \right) = f(t) \ \text{ and } \ D_b^{\alpha} \left(I_b^{\alpha} f(t) \right) = f(t),$$

hold almost every where on $[0, b]$.

(ii) If $\alpha > \beta > 0$ and $f(t) \in L^p([0, b], X)$, $(1 \leq p \leq \infty)$, then

$$D_0^{\beta} \left(I_0^{\alpha} f(t) \right) = I_0^{\alpha-\beta} f(t)$$

and

$$D_b^{\beta} \left(I_0^{\alpha} f(t) \right) = I_b^{\alpha-\beta} f(t),$$

hold almost every where on $[0, b]$.

To present the next property, we use the spaces of functions $I_0^\alpha(L^p)$ defined for $\alpha > 0$ and $1 \le p \le \infty$ by

$$I_0^\alpha(L^p) = \{f : \ f = I_0^\alpha \varphi, \ \varphi \in L^p([0,b], X)\}.$$

Then composition of the fractional integration operator I_0^α with the fractional differentiation operator D_0^α is given by the following result.

Proposition 1.17 *Set $\alpha > 0$, $n = [\alpha] + 1$. Let $f_{n-\alpha}(t) = I_0^{n-\alpha} f(t)$ be the fractional integral (1.2) of order $n - \alpha$,*

(i) If $1 \le p \le \infty$ and $f(t) \in I_0^\alpha(L^p)$, then

$$I_0^\alpha(D_0^\alpha f(t)) = f(t);$$

(ii) If $f(t) \in L^1([0,b], X)$ and $f_{n-\alpha}(t) \in C^n([0,b], X)$, then the equalety

$$I_0^\alpha({}^L D_0^\alpha f(t)) = f(t) - \sum_{j=1}^{n} \frac{f^{(n-j)}(0)}{\Gamma(\alpha - j + 1)} (t)^{\alpha - j},$$

holds almost every where on $[0, b]$.

Proposition 1.18 *Let $\alpha > 0$ and $n = [\alpha] + 1$. Also let $g_{n-\alpha}(t) = I_b^{n-\alpha} g(t)$ be the fractional integral (1.2) of order $n - \alpha$.*

(i) If $1 \le p \le \infty$ and $g(t) \in I_b^\alpha(L^p)$, then

$$I_b^\alpha(D_b^\alpha g(t)) = g(t);$$

(ii) If $g \in L^1(J, X)$ and $g_{n-\alpha}(t) \in C^n(J, X)$, then the equality

$$I_b^\alpha(D_b^\alpha g(t)) = g(t) - \sum_{j=1}^{n} \frac{(-1)^{n-j} g_{n-\alpha}^{(n-j)}(0)}{\Gamma(\alpha - j + 1)} (b - t)^{\alpha - j},$$

hold almost every where on $[0, b]$.

In particular, if $0 < \alpha < 1$, then

$$I_b^\alpha(D_0^\alpha g(t)) = g - \frac{g_{1-\alpha}(0)}{\Gamma(\alpha)} (b - t)^{\alpha - 1},$$

where $g_{1-\alpha}(t) = I_b^{1-\alpha} g(t)$.

Proposition 1.19 *Let $\alpha > 0$ and let $y(t) \in L^\infty(J, X)$ or $y(t) \in C(J, X)$. Then*

$${}^C D_0^\alpha(I_0^\alpha y(t)) = y(t) \ \textit{ and } \ {}^C D_b^\alpha(I_b^\alpha y(t)) = y(t).$$

Proposition 1.20 *Let $\alpha > 0$ and $n = [\alpha] + 1$. If $y \in C^n(J, X)$, then*

$$I_0^\alpha(D_0^\alpha y(t)) = y(t) - \sum_0^{n-1} \frac{y^{(k)}(0)}{k!}(t)^k$$

and

$$I_b^\alpha\left({}^C D_b^\alpha y(t)\right) = y(t) - \sum_{k=0}^{n-1} \frac{(-1)^k y^k(b)}{k!}(b-t)^k.$$

In particular, if $0 < \alpha \leq 1$ and $y(t) \in C(J, X)$, then

$$I_0^\alpha\left({}^C D_0^\alpha y(t)\right) = y(t) - y(0) \text{ and } I_b^\alpha\left({}^C D_b^\alpha y(t)\right) = y(t) - y(b).$$

Lemma 1.21 *[60]*

(*i*) *Let $\xi, \eta \in \mathbb{R}$ be such that $\eta > -1$.*
If $t > 0$, then

$$\left(I_{0^+}^\xi \frac{s^\eta}{\Gamma(\eta+1)}\right)(t) = \begin{cases} \dfrac{t^{\xi+\eta}}{\Gamma(\xi+\eta+1)} & \text{if } \xi+\eta \neq -n \\ 0 & \text{if } \xi+\eta = -n \end{cases}, (n \in \mathbb{N}^+).$$

(*ii*) *let $\xi > 0$ and $\varphi \in L((0,b), X)$. Define*

$$G_\xi(t) = I_{0^+}^\xi \varphi \text{ for } t \in (0, b).$$

Then

$$\left(I_{0^+}^\eta G_\xi\right)(t) = \left(I_{0^+}^{\xi+\eta}\varphi\Big(t\right), \eta > 0 \text{ almost all } t \in [0, b].$$

Definition 1.22 *[14] Let $0 < \alpha < 1$. A function $x : J \to X$ has a fractional integral if the following integral*

$$I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{1-\alpha} x(s) ds, \tag{1.4}$$

is defined for $t \geq 0$, where $\Gamma(\cdot)$ is the gamma function.
The Riemann-Liouville derivative of x of order α is defined as

$${}^L D^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} x(s) ds = \frac{d}{dt} I^{1-\alpha} x(t), \tag{1.5}$$

provided it is well defined for $t \geq 0$. The previous integral is taken in Bochner sense. Let $\phi_\alpha(t) : \mathbb{R} \to \mathbb{R}$ defined by

$$\phi_\alpha(t) = \begin{cases} \dfrac{t^{1-\alpha}}{\Gamma(\alpha)}, & \text{if} \quad t > 0, \\ 0, & \text{if } t \leq 0. \end{cases}$$

Then

$$I^\alpha x(t) = (\phi_\alpha * x)(t),$$

and

$${}^L D^\alpha x(t) = \frac{d}{dt}(\phi_{1-\alpha} * x)(t).$$

1.2.1 Special functions

At the end of subsection, we present some properties of two special functions.

Definition 1.23 *[7] A function of the form*

$$E_{\alpha,\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)},\ \alpha, \beta > 0,\ z \in \mathbb{C},$$

is called the Mittag-Leffler function has the following asymptotic representation as $z \to \infty$:

$$E_{\alpha,\beta}(z) = \begin{cases} \frac{1}{\alpha} z^{\frac{1-\beta}{\alpha}} e^{z^{\frac{1}{\alpha}}} - \sum\limits_{n=1}^{N-1} \dfrac{z^{-n}}{\Gamma(\beta - \alpha n)} + O\left(|z|^{-N}\right),\ |\arg z| \leq \frac{1}{2}\pi\alpha, \\ -\sum\limits_{n=1}^{N-1} \dfrac{z^{-n}}{\Gamma(\beta - \alpha n)} + O\left(|z|^{-N}\right), \qquad |\arg(-z)| \leq (1 - \frac{1}{2}\alpha)\pi. \end{cases}$$

For short, set $E_\alpha(z) = E_{\alpha,1}$. Notice that second of the above formulae implies that in the case $z = \tau < 0$ and $0 < \alpha < 1$ we have

$$E_\alpha(\tau) \to 0 \text{ as } \tau \to -\infty.$$

Then Mittag-Leffler have the following properties:

Proposition 1.24 *[68] For $\alpha \in (0, 1)$ and $t \in \mathbb{R}^+$*

(i) $E_\alpha(t),\ E_{\alpha,\alpha}(t) > 0$;

(ii) $(E_\alpha(t))' = \frac{1}{\alpha} E_{\alpha,\alpha}(t)$;

(iii) $\lim\limits_{t \to -\infty} E_\alpha(t) = \lim\limits_{t \to -\infty} E_{\alpha,\alpha}(t) = 0$;

Definition 1.25 *[68] The Wright function Ψ_α is defined by*

$$\Psi_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-\theta)^n}{(n-1)!} \Gamma(n\alpha) \sin(n\pi\alpha),\ \theta \in \mathbb{C},$$

with $0 < \alpha < 1$.

Remark 1.26 *[68] If $\theta \in \mathbb{R}^+$, then*

$$\Psi_\alpha(\theta) = \frac{1}{\pi\alpha} \sum_{n=1}^{\infty} (-\theta)^{n-1} \frac{\Gamma(1 + \alpha n)}{n!} \sin(n\pi\alpha),\ \alpha \in (0, 1).$$

Proposition 1.27 *[68]*

(i) $\Psi_\alpha(t) \geq 0,\ t \in (0, \infty)$;

(ii) $\int_0^\infty \frac{\alpha}{t^{\alpha+1}}\Psi_\alpha(t^{-\alpha})e^{-\lambda t}dt = e^{-\lambda\alpha},\ Re(\lambda) \geq 0;$

(iii) $\int_0^\infty \Psi_\alpha(t)t^n dt = \frac{\Gamma(1+r)}{\Gamma(1+\alpha r)},\ r \in (-1,\infty);$

(iv) $\int_0^\infty \Psi_\alpha(t)e^{-zt}dt = E_\alpha(-z),\ z \in \mathbb{C};$

(v) $\int_0^\infty \alpha t\Psi_\alpha(t)e^{-zt}dt = E_{\alpha,\alpha}(-z),\ z \in \mathbb{C}.$

Lemma 1.28 *[58] For $0 < \alpha \leq 1$, the Mittag-Leffler type function $E_{\alpha,\alpha}(-\lambda t^\alpha)$ satisfies*

$$0 \leq E_{\alpha,\alpha}(-\lambda t^\alpha) \leq \frac{1}{\Gamma(\alpha)},\ \ t \in [0,\infty), \lambda \geq 0.$$

1.3 Some properties of set-valued maps

Now, we also introduce some basic definitions on multivalued maps. For more details, see [15, 32].

Let X, Y be two topological vector spaces. We denote by $\mathcal{P}(X) = \{A \subseteq X :\ A \neq \emptyset\}$ the family of all nonempty subsets of X;
$\mathcal{P}_{cp}(X) = \{A \in \mathcal{P}(X) : A \text{ is compact}\}$, $\mathcal{P}_b(X) = \{A \in \mathcal{P}(X) : A \text{ is bounded}\}$;
$\mathcal{P}_{cl}(X) = \{C \in \mathcal{P}(X) : A \text{ is closed}\}$, $\mathcal{P}_{cv}(X) = \{A \in \mathcal{P}(X) : A \text{ is convex}\}$;
$\mathcal{P}_{cp,cv}(X) = \mathcal{P}_{cp}(X) \cap \mathcal{P}_{cv}(X)$ denotes the collection of all non-empty compact and convex subsets of X.

Definition 1.29 *[15] A multivalued map (multimap) $\mathcal{F} : X \to \mathcal{P}(X)$ is called*

(i) upper semicontinuous (u.s.c.) if $\mathcal{F}^{-1}(V) = \{x \in X : \mathcal{F}(x) \subset V\}$ is an open subset of E for each open set $V \subset E$.

(ii) closed if its graph $\Gamma_{\mathcal{F}} = \{(x,y) \in X \times Y\ :\ y \in \mathcal{F}(x)\}$ is a closed subset of $X \times Y$,

(iii) compact if $\overline{\mathcal{F}}(X)$ is compact in Y.

(vi) quasicompact if its restriction of every compact subset $A \subset X$ is compact.

Lemma 1.30 *Let X and Y be two metric spaces and $\mathcal{F} : X \to \mathcal{P}_{cp}(Y)$ a closed quasi-compact multimap. Then $\mathcal{F}$ is u.s.c.*

Let (X,d) be a metric space induced from the normed space $(X,\|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$, given by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\},$$

where $d(a,B) = \inf_{b\in B} d(a,b)$, $d(b,A) = \inf_{a\in A} d(b,a)$. Then $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space.

Definition 1.31 *[36, 15] For a given $p \geq 1$, a multifunction $G : J \to \mathcal{P}_{cp,cv}(X)$ is called:*

(i) L^p-integrable if it admits an L^p-Bochner integrable selection, i.e., $t \in J$; a function $g \in L^p(J, X)$ such that $g(t) \in G(t)$ for a.e. $t \in J$.

(ii) L^p-integrably bounded if there exists a function $\xi \in L^p(J, \mathbb{R}_+)$ such that

$$\|G(t)\| \leq \xi(t) \text{ for a.e. } t \in J.$$

The set of all L^p-integrable selections of a multifunction $G : J \to \mathcal{P}_{cp,cv}(X)$ is denoted by S_G^P.

Definition 1.32 *The integral of an L^p-integrable multifunction $G : [0, b] \to \mathcal{P}_{cp,cv}(X)$ is defined in the following way*

$$\int_0^\tau G(s)ds = \left\{ \int_0^\tau f(s)ds : f \in S_G^p \right\},$$

for a.e. $\tau \in [0, b]$. In the sequel, we will need the following important property on the χ-estimation of the integral of multifunction.

Lemma 1.33 *[36] Let the space X be separable and the multifunction $\Phi : [0, b] \to \mathcal{P}_{cp,cv}(X)$ be integrable, integrably bounded and $\chi(\Phi(t)) \leq q(t)$ for a.a $t \in [0, b]$ where $q(\cdot) \in L^1([0, b]; \mathbb{R}^+)$. Then*

$$\chi\left(\int_0^\tau \Phi(s)ds\right) \leq \int_0^\tau q(s)ds, \quad \text{for all } \tau \in [0, b].$$

In particular, if the multifunction $\Phi : [0, b] \to \mathcal{P}_{cp,cv}(X)$ is measurable and integrably bounded then the function is $\chi(\Phi(\cdot))$ integrable and

$$\chi\left(\int_0^\tau \Phi(s)ds\right) \leq \int_0^\tau \chi(\Phi(s))ds, \quad \text{for all } \tau \in [0, b].$$

1.4 Semigroups

1.4.1 $\mathcal{C}_0$-Semigroups

Let X be a Banach space an $L(X)$ be the Banach space of linear bounded operators.

Definition 1.34 *[55] A One parameter family $\{T(t) \mid t \geq 0\} \subset L(X)$ satisfying the conditions:*

(i) $T(t)T(s) = T(t + s)$, for $t, s \geq 0$,

(ii) $T(0) = I$,

(iii) *the map* $t \to T(t)(x)$ *is strongly continuous, for each* $x \in X$, *i.e;*

$$\lim_{t\to 0^+} T(t)x = x,\ \forall x \in X.$$

A semigroup of bounded linear operators $T(t)$, *is uniformly continuous if*

$$\lim_{t\to 0^+} \|T(t) - I\|_{L(X)} = 0.$$

Here I *denotes the identity operator in* X.

Definition 1.35 *[55] Let* $\{T(t)\}_{t\geq 0}$ *be a* C_0*-semigroup defined on* X. *The linear operator* A *is the infinitesimal generator* $\{T(t)\}_{t\geq 0}$ *defined by*

$$A(x) = \lim_{t\to 0^+} \frac{T(h)x - x}{t}, \quad \textit{for } x \in D(A),$$

where $D(A) = \left\{ x \in X \mid \lim_{t\to 0^+} \frac{T(t)(x) - x}{t} \textit{ exists in } X \right\}$.

Remark 1.36 *(i) If there are* $M \geq 0$ *and* $\omega \in \mathbb{R}$ *such that* $\|T(t)\| \leq Me^{\omega t}$, *then*

$$(\lambda I - A)^{-1}x = \int_0^\infty e^{-\lambda t} T(t)x dt,\ Re(\lambda) > \omega.$$

(ii) A C_0*-semigroup* $\{T(t)\}_{t\geq 0}$ *is called exponentially stable if there exist constants* $M > 0$ *and* $\delta > 0$ *such that*

$$\|T(t)\| \leq Me^{-\delta t},\ t \geq 0.$$

(iii) The growth bound ω_0 *of* $\{T(t)\}_{t\geq 0}$ *is defined by*

$$\omega_0 = \inf\{\delta \in \mathbb{R} : \textit{ there exists } M_\delta > 0 \textit{ such that } \|T(t)\| \leq M_\delta e^{\delta t}, \forall t > 0\}.$$

Furthermore, ω_0 *can also be obtained by the following formula:*

$$\omega_0 = \lim_{t\to+\infty} \sup \frac{ln\|T(t)\|}{t}.$$

Definition 1.37 *[11] A* C_0*-semigroup* $\{T(t)\}_{t\geq 0}$ *is called uniformly bounded if there exists a constant* $M > 0$ *such that*

$$\|T(t)\| \leq M,\ t \geq 0.$$

Definition 1.38 *[55] A* C_0*-semigroup* $\{T(t)\}_{t\geq 0}$ *is called compact if* $T(t)$ *is compact for* $t > 0$.

Proposition 1.39 *[55] If $\{T(t)\}_{t\geq 0}$ is compact, then $\{T(t)\}$ is equicontinuous for $t > 0$.*

Definition 1.40 *A C_0-semigroup $\{T(t)\}_{t\geq 0}$ is called positive if $T(t)x \geq \theta$ for all $x \geq \theta$ and $t \geq 0$.*

Example 1.41 *[68] The linear operator $T_\alpha(t)$ is defined by*

$$T_\alpha(t) = \alpha \int_0^\infty \theta M_\alpha(\theta) T(t^\alpha \theta) d\theta,$$

and M_α is a probability density function which is defined by

$$M_\alpha(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)!\Gamma(1-\alpha n)}, \;\; 0 < \alpha < 1, \;\; \theta \in \mathbb{C}.$$

Lemma 1.42 *[68] The operator $T_\alpha(t)$ has the following properties:*

(i) For any fixed $t \geq 0$, $T_\alpha(t)$ is linear and bounded operators, i.e., for any $x \in X$,

$$\|T_\alpha(t)x\| \leq \frac{M}{\Gamma(\alpha)} \|x\|.$$

(ii) $T_\alpha(t)(t \geq 0)$ is strongly continuous.

(iii) For every $t > 0$, $T_\alpha(t)$ is also compact operator if $T(t)$ is compact.

1.5 Measure of noncompactness

We recall some definitions and properties of measure of noncompactness.

Definition 1.43 *[5] Let X be a Banach space, $\mathcal{P}(\mathcal{X})$ denote the collection of all nonempty subsets of X, and $(\mathcal{A}, \geq)$ a partially ordered set A map $\beta : \mathcal{P}(\mathcal{X}) \to \mathcal{A}$ is called a measure of noncompactness on X, MNC for short, if*

$$\beta(\overline{co}\Omega) = \beta(\Omega),$$

for every $\Omega \in \mathcal{P}(X)$, where $\overline{co}\Omega$ is the closure of convex hull of Ω.

Definition 1.44 *[36] A measure of noncompactness β is called*

(1) *monotone if $\Omega_0, \Omega_1 \in \mathcal{P}(X)$, $\Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$,*

(2) *nonsingular if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for every $a \in X$, $\Omega \in \mathcal{P}(X)$,*

(3) *invariant with respect to the union with compact sets if $\beta(K \cup \Omega) = \beta(\Omega)$ for every relatively compact set $K \subseteq X$ and $\Omega \in \mathcal{P}(X)$,*

(4) regular if the condition $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω,

(5) algebraically semi additive if $\beta(\gamma_1 + \gamma_2) \leq \beta(\gamma_1) + \beta(\gamma_2)$ where $\gamma_1 + \gamma_2 = \{x + y, x \in \gamma_1, y \in \gamma_2\}$,

(6) $\beta(\lambda\gamma) \leq |\lambda|\beta(\gamma)$ for any $\lambda \in \mathbb{R}$,

(7) if $\{W_n\}_{n=1}^{+\infty}$ is a decreasing sequence of bounded closed nonempty subset and $\lim_{n\to+\infty} \beta(W_n) = 0$, then $\cap_{n=1}^{+\infty} W_n$ is nonempty and compact.

We shall define the measure of noncompactness on $\mathcal{P}_b(X)$. Recall that a subset $A \subset X$ is relatively compact provided the closure $\overline{A}$ is compact.

1.5.1 Measure of noncompactness

Definition 1.45 *[61] Let X be a Banach space and $\mathcal{P}_b(X)$ the family of all bounded subsets of X. Then the function: $\alpha : \mathcal{P}_b(X) \to \mathbb{R}_+$ defined by:*

$$\alpha(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ admits a finite cover by sets of diameter } \leq \varepsilon\}$$

is called the Kuratowski measure of noncompactness, (the α-MNC for short).
Another function $\chi : \mathcal{P}_b(X) \to \mathbb{R}_+$ defined by:

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon - net\}.$$

is called the Hausdorff measure of noncompactness.

Definition 1.45 is very useful since α and χ have interesting properties, some of which are listed in the following

Proposition 1.46 *[61] Let X be a Banach space and $\gamma : \mathcal{P}_b(X) \to \mathbb{R}_+$ be either α or χ. Then:*

(a) $\gamma(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact)

(b) $\gamma(B) = 0 = \gamma(\overline{B}) = 0$

(c) $A \subset B \Rightarrow \gamma(A) \leq \gamma(B)$

(d) $\gamma(A + B) \leq \gamma(A) + \gamma(B)$

(e) $\gamma(c.B) \leq |c|\gamma(B)$

(f) $\gamma(coB) = \gamma(B)$.

(e) The function $\gamma : \mathcal{P}_b(X) \to \mathbb{R}_+$ is continuous with respect to the metric H_d on $\mathcal{P}_b(X)$.

Remark 1.47 *[61] For every $A \in \mathcal{P}_b(X)$, we have $\chi(A) \leq \alpha(A) \leq 2\chi(A)$.*

Now, we present the abstract definition of MNC. For more details, we refer to [5, 10, 36, 61, 68] and some references therein.

Definition 1.48 *Let $(\mathcal{A}, \geq)$ be a partially ordered set. A function $\beta : \mathcal{P}_b(E) \to \mathcal{A}$ is called a measure of noncompactness (MNC) in E if*

$$\beta(\overline{\text{co}}\Omega) = \beta(\Omega),$$

for every $\Omega \in \mathcal{P}_b(X)$.

Definition 1.49 *A measure of noncompactness β is called:*

(i) monotone if $\Omega_0, \Omega_1 \in \mathcal{P}_b(X)$, $\Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$

(ii) nonsingular if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for every $a \in X$, $\Omega \in \mathcal{P}_b(X)$;

(iii) regular if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

The following property of the Hausdorff MNC that can be easily verified. if $L : X \to X$ is a bounded linear operator then

$$\chi(L(\Omega)) = \|L\|\chi(\Omega) \tag{1.6}$$

Lemma 1.50 *Let $\{\Omega_n\}$ be a sequence of X such that $\Omega_n \supseteq \Omega_{n+1}$, $n \geq 1$ and $\beta(\Omega_n) \to 0$ as $n \to \infty$, where β is a monotone MNC in X. Then*

$$\beta\left(\bigcap_{n=1}^{\infty} \Omega_n\right) = 0.$$

In the following, several examples of useful measures of noncompactness in spaces of continuous functions are presented.

Example 1.51 *We consider general example of MNC in the space of continuous functions $C([0, b], X)$. For $\Omega \subset C([0, b], X)$ define*

$$\phi(\Omega) = \sup_{t \in [0,b]} \chi(\Omega(t)),$$

where χ is Hausdorff MNC in X and $\Omega(t) = \{y(t) : y \in \Omega\}$.

Example 1.52 *Consider another useful MNC in the space $C([0, b], X)$. For a bounded $\Omega \subset C([0, b], X)$, set*

$$\nu(\Omega) = \left(\sup_{t \in [0,b]} \chi(\Omega(t)), mod_C(\Omega)\right);$$

here, the modulus of equicontinuity of the set of functions $\Omega \subset C([0, b], X)$ has the following form:

$$mod_C(\Omega) = \lim_{\delta \to 0} \sup_{x \in \Omega} \max_{|t_1 - t_2| \leq \delta} \|x(t_1) - x(t_2)\|. \tag{1.7}$$

Example 1.53 *We consider one more MNC in the space $C([0,b],X)$. For a bounded $\Omega \subset C([0,b],X)$, set*

$$\nu(\Omega) = \max_{D\in\Delta(\Omega)} \left(\sup_{t\in[0,b]} e^{-Lt}\chi(D(t)), mod_C(D) \right);$$

where $\Delta(\Omega)$ is the collection of all denumerable subsets of Ω, L constant, and $mod_C(D)$ is given in formula 1.7.

For any $W \subset C(J,X)$, we define

$$\int_0^t W(s)ds = \left\{ \int_0^t u(s)ds : u \in W \right\}, \; for\, t \in [0,b],$$

where $W(s) = \{u(s) \in X : u \in W\}$.

Lemma 1.54 *[61] If $W \subset C(I,X)$ is bounded and equicontinuous, then $\beta(W(t))$ is continuous on J and*

$$\beta(W) = \max_{t\in J} \beta(H(t)) \;,\; \beta\left(\int_I x(t)dt, x\in W\right) \le \int_I \beta(W(t))dt$$

for $t \in [0,b]$.

Lemma 1.55 *[17, 36] If $\{u_n\}_{n=1}^{\infty} \subset L^1(J,X)$ satisfies $\|u_n(t)\| \le \kappa(t)$ a.e. on J for all $n \ge 1$ with some $\kappa \in L^1(J,\mathbb{R}_+)$. then the function $\chi(\{u_n\}_{n=1}^{\infty})$ be long to $L^1(J,\mathbb{R}_+)$ and*

$$\chi\left\{\left(\int_0^t u_n(s)ds : n \ge 1\right\}\right) \le 2\int_0^t \chi(u_n(s)ds : n \ge 1).$$

1.5.2 Condensing maps

Definition 1.56 *[5, 36] A multimap $\mathcal{F} : E \subseteq X \to \mathcal{P}_{cp}(X)$ is called condensing with respect to a MNC β (or β-condensing) if for each bounded set $\Omega \subseteq E$ that is not relatively compact, we have:*

$$\beta(F(\Omega)) \ngeq \beta(\Omega).$$

Lemma 1.57 *[36] For $1 \le p \le \infty$, a sequence of function $\{\xi_n\} \subset L^p([0,b],X)$ is called L^p-semicompact if it is L^p-integrably bounded, i.e.*

$$\|\xi_n(t)\| \le \mu(t) \, for\; a.e.\; t \in [0,b] \; and\; for\; all\, n = 1,2,....,$$

where $\mu \in L^p([0,b])$, and the set $\{\xi_n\}$ is relatively compact in X for a.e. $t \in [0,b]$.

Lemma 1.58 *[36] Every L^p-semicompact sequence $\{\xi_n\}$ is weakly compact in $L^1([0,b],X)$.*

Lemma 1.59 *[26] Let X be a Banach space, $C \subset X$ be closed and bounded an $F : C \to X$ a condensing map. Then $I - F$ is proper and $I - F$ maps closed subsets of C onto closed sets. Recall that the map $I - F$ is proper if is continuous and for every compact $K \subset X$, the set $(I - F)^{-1}(K)$ is compact.*

The application of the topological degree theory for condensing maps implies the following fixed point principle.

Theorem 1.60 *[36] Let $V \subset X$ be a bounded open neighborhood of zero and $\Gamma : \overline{V} \to X$ a β-condensing map with respect to a monotone nonsingular MNC β in X. If Γ satisfies the boundary condition*

$$x \neq \lambda\Gamma(x),$$

for all $x \in \partial V$ and $0 < \lambda \leq 1$, then the fixed point set $\mathcal{F}ix\Gamma = \{x : \ x = \Gamma(x)\}$ is nonempty and compact.

Lemma 1.61 *[68] Let X and Y be metric spaces and $\mathcal{F} : X \to \mathcal{P}_{cp,cv}(Y)$ a closed quasi-compact multimap. Then $\mathcal{F}$ is u.s.c.*

Lemma 1.62 *[36] Let E be a closed subset of a Banach space X, β a monotone MNC in X, Λ a metric space and $G : \Lambda \times E \to \mathcal{P}_{cp}(X)$ a closed multimap which is β-condensing in second argument and such that the fixed point set $FixG(\lambda, \cdot) = \{x \in E : \ x \in G(\lambda, x)\}$ is non-empty for each $\lambda \in \Lambda$. Then the multimap $\mathcal{F} : \Lambda \to \mathcal{P}(X)$, where $\mathcal{F}(\lambda) = FixG(\lambda, \cdot)$ is u.s.c.*

Lemma 1.63 *[36] Let E be a closed subset of a Banach space X, β a monotone MNC in X and $F : X \to \mathcal{P}_{cp}(X)$ a closed multimap which is β-condensing on each bounded set. If the fixed point set $Fix\mathcal{F} := \{x, x \in \mathcal{F}(x)\}$ is bounded then it is compact.*

Theorem 1.64 *[36] Let $\mathcal{M}$ be a convex closed bounded subset of X and $\mathcal{F} : \mathcal{M} \to \mathcal{P}_{cp,cv}(\mathcal{M})$. a β-condensing multimap, where β is a monotone nonsingular MNC in X. Then the fixed point set $Fix\mathcal{F}$ is a non-empty compact set.*

1.5.3 R_δ-set

We recall some notions from geometric topology. Let (X, d), $(Y, d^{'})$ two metric spaces.

Definition 1.65 *$A \in \mathcal{P}(X)$ is a retract of X if there exists a continuous map $r : X \to A$ such that*

$$r(a) = a, \ \text{for every}\, a \in A.$$

Definition 1.66

(i) X is called an absolute retract (AR space) if for any metric space Y and any closed subset $D \subset X$, there exists a continuous function $h : D \to X$ which can be extended to a continuous function $\tilde{h} : Y \to X$.

(ii) X is called an absolute neighborhood retract (ANR space) if for any metric space Y, closed subset $D \subset Y$ and continuous function $h : D \to X$ there exists a neighborhood $D \subset U$ and a continuous extension $\tilde{h} : U \to X$ of h.
Obviously, if X is an AR space, then it is an ANR space.

Definition 1.67 *[23, 6] Let $A \in \mathcal{P}(X)$. Then set A is called a contractible space provided there exists a continuous homotopy $h : A \times [0,1] \to A$ and $x_0 \in A$ such that*

(i) $h(x, 0) = x$, for every $x \in A$;

(ii) $h(x, 1) = x_0$, for every $x \in A$,

i.e. if identity map is homotopic to a constant map (A is homotopically equivalent to a point).
Note that if $A \in \mathcal{P}_{cv,cl}(X)$, then A is contractible, but the class of contractible set is much larger than the class convex sets.

Definition 1.68 *[33] A compact metric space A is called an R_δ-set if there exists a decreasing sequence $\{A_n\}$ of compact contractible sets such that*

$$A = \bigcap_{n \geq 1} A_n.$$

Note that any R_δ-set is nonempty, compact, and connected.
Let as recall the well-know Lasota-Yorke approximation lemma.

Lemma 1.69 *[23] Let X be a normed space, E a metric space and $F : E \to X$ be a continuous map. Then, for each $\varepsilon > 0$, there is a locally Lipschitz map $F_\varepsilon : E \to X$ such that*

$$\|F(x) - F_\varepsilon(x)\| < \varepsilon, \text{ for every } x \in X$$

Theorem 1.70 *[16] Let (E, d) be a metric space, $(X, \|\cdot\|)$ a Banach space and $F : E \to X$ a proper map. Assume further that for each $\varepsilon > 0$ a proper map $F_\varepsilon : E \to X$ is given, and the following two conditions are satisfied.*

(i) $\|F_\varepsilon(x) - F(x)\| < \varepsilon$, for every $x \in E$,

(ii) for every $\varepsilon > 0$ and $u \in X$ in a neighborhood of the origin such that $\|u\| \leq \varepsilon$, the equation $F_\varepsilon(x) = u$ has exactly one solution x_ε.
Then the set $S = F^{-1}(0)$ is an R_δ-set.

1.6 Some fixed point theorems

In this section we give some fixed point theorems that will be used in the sequel.

Consider a mapping T of a set M into M (or into some set containing M).

Definition 1.71 *[59] Let T be a mapping of a metric space M into M. We say that T is a contraction mapping if there exists a number k such that $0 \leq k < 1$ and*

$$\rho(Tx, Ty) \leq k\rho(x, y)\ (\forall x, y \in M).$$

Theorem 1.72 *[59](Banach's fixed point Theorem) Any contraction mapping of a complete nonempty metric space M into M has a unique fixed point in M.*

Theorem 1.73 *[59](Schauder's fixed point Theorem) Let X be a Banach space and let $M \subseteq X$ be nonempty, convex, and closed, $T : M \to M$ is compact, then T has a fixed point.*

Corollary 1.74 *[59] Let T be a compact continuous mapping of M into M. Then T has a fixed point.*

Theorem 1.75 *[9](Krasnosel'skii). Let D be a closed convex and nonempty subset of a Banach space X. Let F_1, F_2 be two operators such that*

(i) $F_1x + F_2y \in D$ whenever $x, y \in D$;

(ii) F_1 is completely continuous;

(iii) F_2 is a contraction mapping.

Then there exists $z \in D$ such that $z = F_1z + F_2z$.

The following theorem is due to Mönch.

Theorem 1.76 *[49] Let X be a Banach space, U an open subset of X and $0 \in U$. Suppose that $N : U \to X$ is a continuous map which satisfies Mönch's condition (that is, if $D \subseteq \overline{U}$ is countable and $D \subseteq \overline{co}(\{0\} \cup N(D))$, then $\overline{D}$ is compact) and assume that*

$$x \neq \lambda N(x), \quad \textit{for } x \in \partial U \textit{ and } \lambda \in (0,1)$$

holds. Then Nhas a fixed point in $\overline{U}$.

Definition 1.77 *[54] The map $F : B \subset X \to X$ is said to be an β-contraction if there exists a positive constant $k < 1$ such that*

$$\beta(F(B_0) \leq k\beta(B_0),$$

for any bounded closed subset $B_0 \subseteq B$.

Theorem 1.78 *[54](Darbo-Sadovskii's fixed point theorem) If B is a bounded closed convex subset of a Banach space X, the continuous map $F : B \to B$ is an β-contraction, then the map F has at least one fixed point in B.*

Chapter 2

Weighted fractional differential equations in Banach spaces

Ce chapitre fait l'objet d'une publications dans Palestine Journal of Mathematics et Malaya Journal of Matematik.

MONOTONE ITERATIVE METHOD FOR WEIGHTED FRACTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACE

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Abstract: This paper deals with some existence results for a class of fractional differential equations involving Riemann-Liouville type fractional order derivative, by using the lower and upper solution method and the measure of noncompactness in in the weighted space of continuous functions, we prove the existence of maximal and minimal solutions. Finally, an example is also provided to illustrate our results.

MSC 2010 Classifications: Primary 34A08; Secondary 47H08
Keywords and phrases: Fractional initial value problem, Mittag-Leffler type function, measure of noncompactness, lower and upper solution method.

REMARKS ON THE FRACTIONAL ABSTRACT DIFFERENTIAL EQUATION WITH NONLOCAL CONDITIONS

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Abstract: In this paper, we study the existence and uniqueness of a solution to an initial value problem for a class of nonlineare fractional involving Riemann-Liouville derivative with nonlocal initial conditions in Banach spaces. We prove our main result by introducing a regular measure of noncompactness in the weighted space of continuous functions and using fixed point theory. Our result improve and complement several earlier related works. An example is given to illustrate the applications of the abstract result.

MSC 2010 Classifications: 26A33, 34A08, 34K30, 47H08, 47H10.
Keywords and phrases: Riemann-Liouville fractional derivative, Riemann-Liouville fractional integral, nonlocal initial conditions, point fixed, measure of noncompactness.

In this chapter, we first study existence of Cauchy problems cases to an initial value and nonlocal initial conditions for fractional equations. By using montone iterative method and fixed point theorems combined with Hausdorff measure of noncompactness, and condensing map, we discuss the existence and uniqueness of solutions for fractional equations with Riemann-Liouville derivative of order $\alpha \in (0, 1)$. Then the topological structure of solution sets is investigated.

2.1 Fractional differential equations

2.1.1 Monotone Iterative Method for weighted fractional differential equations in Banach spaces

The objective of the paper was discussed by using the lower and upper solution method, we prove the existence of iterative solutions for a class of fractional initial value problem (IVP):

$$ {}^{L}D^{\alpha}_{0^+}x(t) = f(t, x(t)), \quad t \in J' := (0, b], \tag{2.1} $$

$$ \lim_{t\to 0^+} t^{1-\alpha}x(t) = x_0, \tag{2.2} $$

where ${}^{L}D^{\alpha}_{0^+}$ is the Riemann-Liouville fractional derivative of order $\alpha \in (0, 1)$, moreover, we show the existence of maximal and minimal solutions. Let us recall the following definitions and results that will be used in the sequel. We consider the Banach space of continuous functions

$$ C_{1-\alpha}(J, X) = \{x \in C(J, X) : \lim_{t\to 0^+} t^{1-\alpha}x(t) \ \text{ exists } \}. $$

A norm in this space is given by

$$ \|x\|_\alpha = \sup_{t\in J}\{t^{1-\alpha}\|x\|_X\}. $$

For Ω a subset of the space $C_{1-\alpha}(J, X)$, define Ω_α by

$$ \Omega_\alpha = \{x_\alpha, x \in \Omega\}, $$

where

$$ x_\alpha(t) = \begin{cases} t^{1-\alpha}x(t), & \text{if } t \in (0, b], \\ \lim\limits_{t\to 0^+} t^{1-\alpha}x(t), & \text{if } t = 0. \end{cases} $$

It is clear that $x_\alpha \in C(J, X)$.

Lemma 2.1 *[45] A set $\Omega \subset C_{1-\alpha}(J, X)$ is relatively compact if and only if Ω_α is relatively compact in $C(J, X)$.*

Proof. See for instance [[18], Theorem 3].

Lemma 2.2 *[40] The linear initial value problem*

$$ {}^{L}D^{\alpha}_{0^+}x(t) + \lambda x(t) = p(t), \quad t \in (0, b], $$

$$ \lim_{t\to 0^+} t^{1-\alpha}x(t) = x_0, $$

where $\lambda \geq 0$ is a constant and $p \in L^1(J, X)$, has the following integral representation for a solution

$$ x(t) = \Gamma(\alpha)x_0t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha) + \int_0^t (t - s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t - s)^\alpha)p(s)ds. \tag{2.3} $$

Where $E_{\alpha,\alpha}(t)$ is a Mittag-Leffler function.

Lemma 2.3 *[35] Suppose that X is an ordered Banach space $u_0, y_0 \in X$, $u_0 \le y_0$, $D = [u_0, y_0]$, $N : D \to X$ is an increasing completely continuous operator and*

$$u_0 \le Nu_0, \;\; y_0 \ge Ny_0.$$

Then the operator N has a minimal fixed u^* and a maximal fixed y^*. If we let

$$u_n = Nu_{n-1} \;\; , \;\; y_n = Ny_{n-1}, \;\; n = 1, 2 \cdots ,$$

then

$$u_0 \le u_1 \le u_2 \le \cdots \le u_n \le \cdots \le y_n \le \cdots \le y_2 \le y_1 \le y_0,$$
$$u_n \to u^*, y_n \to y^*.$$

Definition 2.4 *A function $v(\cdot) \in C_{1-\alpha}(J, X)$ is called as a lower solution of* (2.11)-(2.12) *if it satisfies*

$$^L D^{\alpha}_{0+} v(t) \le f(t, v(t)), \;\; t \in (0, b], \tag{2.4}$$

$$\lim_{t \to 0^+} t^{1-\alpha} v(t) \le x_0. \tag{2.5}$$

Definition 2.5 *A function $w(\cdot) \in C_{1-\alpha}(J, X)$ is called as an upper solution of* (2.1)-(2.2) *if it satisfies*

$$^L D^{\alpha}_{0+} w(t) \ge f(t, w(t)), \;\; t \in (0, b], \tag{2.6}$$

$$\lim_{t \to 0^+} t^{1-\alpha} w(t) \ge x_0. \tag{2.7}$$

Before stating and proving the main results, we introduce following assumptions

(H_1) The map $f : [0, b] \times X \to X$ is continuous.

(H_2) There exists a constant $c > 0$ such that

$$\|f(t, x)\| \le c(1 + t^{1-\alpha}\|x\|) \;\text{ for all }\; t \in [0, b] \;\text{ and }\; x \in X.$$

(H_3) there exists a constant $c_1 > 0$, and let $F(t, x) = f(t, x) + \lambda x(t)$ such that for each nonempty, bounded set $\Omega \subset C_{1-\alpha}(J, X)$

$$\beta(F(t, \Omega(t)) \le c_1 \beta(\Omega(t)), \;\text{ for all }\; t \in [0, b],$$

where β is measure of noncompactness in X.

(H_4) Assume that $f : [0, b] \times X \to X$ the nonlinear term satisfies the monotoneity condition

$$f(t, x) - f(t, v) + \lambda(x - v) \ge 0, \;\; \forall t \in J, \quad \widehat{x}(t) \le v \le x \le \tilde{x}(t),$$

where $\lambda \ge 0$ is a constant and $\widehat{x}$, $\tilde{x}$ are lower and upper solutions of problem (2.1)- (2.2) respectively.

Theorem 2.6 *Suppose (H_1)-(H_3) holds. The function $x(\cdot) \in C_{1-\alpha}(J, X)$ solves problem* (2.1)-(2.2) *if and only if it a fixed point of the operator N defined by*

$$\begin{aligned} N(x)(t) &= \Gamma(\alpha) x_0 t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^{\alpha}) \\ &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^{\alpha})[f(s, x(s)) + \lambda x(s)] ds. \end{aligned}$$

Proof. It's clear that the operator N is well defined, i.e., for every $x \in C_{1-\alpha}(J,X)$ and $t>0$, the integral

$$\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^{\alpha})[f(s,x(s))+\lambda x(s)]ds,$$

belongs to $C_{1-\alpha}(J,X)$.

Let $D=[u_0,y_0]$, we define a mapping $N: D \to C_{1-\alpha}(J,X)$ by

$$\begin{aligned} N(x)(t) &= \Gamma(\alpha)x_0t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha}) \\ &\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^{\alpha})[f(s,x(s))+\lambda x(s)]ds. \end{aligned}$$

by lemma 2.2 $x \in D$ is solution of the problem(2.1)-(2.2) if and only if

$$x = Nx.$$

We will divid the proof in the several steps.

Step 1. N is continuous.
Let $\{x_n\}$ be a sequence such that $x_n \to x$ in D. Then

$$\begin{aligned} & t^{1-\alpha}\|N(x_n)(t)-N(x)(t)\| \\ & \le \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,x_n(s))-f(s,x(s))\|ds \\ & \quad + \frac{\lambda t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|x_n(s)-x(s)\|ds \end{aligned}$$

$$\begin{aligned} & \le \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}s^{1-\alpha}\|f(s,x_n(s))-f(s,x(s))\|ds \\ & \quad + \frac{\lambda t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}s^{1-\alpha}\|x_n(s)-x(s)\|ds \\ & \le \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}ds\|f(s,x_n(s))-f(s,x(s))\|_{\alpha} \\ & \quad + \frac{\lambda t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}ds\|x_n(s)-x(s)\|_{\alpha} \\ & \le \frac{b^{\alpha}}{\Gamma(\alpha)}B(\alpha,\alpha)\|f(\cdot,x_n(\cdot))-f(\cdot,x(\cdot))\|_{\alpha} + \frac{\lambda b^{\alpha}}{\Gamma(\alpha)}B(\alpha,\alpha)\|x_n(\cdot)-x(\cdot)\|_{\alpha} \end{aligned}$$

Using the hypothesis (H_2) we have

$$\|N(x_n)(t)-N(x)(t)\|_{\alpha} \longrightarrow 0 \ \text{ as } \ n \to +\infty.$$

Step 2. N maps bounded sets into bounded sets in D.
Indeed, it enough to show that there exists a positive constant l such that for each $x \in B_r = \{x \in D : \|x\|_{\alpha} \le r\}$ one has $\|N(x)\|_{\alpha} \le l$.

Let $x \in B_r \subset D$. Then for each $t \in (0, b]$, by (H_2) we have

$$
\begin{aligned}
t^{1-\alpha}\|Nx(t)\| &\leq \|x_0\| + \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, x(s))\| ds \\
&+ \frac{\lambda t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x(s)\| ds \\
&\leq \|x_0\| + \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} c(1 + s^{1-\alpha}\|x(s)\|) ds \\
&+ \frac{\lambda t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} s^{1-\alpha} \|x(s)\| ds \\
&\leq \|x_0\| + \frac{c t^{1-\alpha}}{\Gamma(\alpha)} \textstyle\int_0^t (t-s)^{\alpha-1}(1+r) ds \\
&+ \frac{\lambda t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} r ds \\
&\leq \|x_0\| + \frac{c b^{1-\alpha}(1+r)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\
&+ \frac{\lambda b^{1-\alpha} r}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} ds,
\end{aligned}
$$

$$
\|N(x)\|_\alpha \leq \|x_0\| + \frac{cb(1+r)}{\Gamma(\alpha+1)} + \frac{\lambda b^\alpha r \Gamma(\alpha)}{\Gamma(2\alpha)} := l.
$$

Step 3. N maps bounded sets into equicontinuous sets.
Let $t_1, t_2 \in (0, b]$, $t_1 \leq t_2$, let B_r be a bounded set in D as in step 2, and let $x \in B_r$, we have

$$
\begin{aligned}
&\|t_2^{1-\alpha} N(x)(t_2) - t_1^{1-\alpha} N(x)(t_1)\| \\
&\leq \Gamma(\alpha)\|x_0\| \left[E_{\alpha,\alpha}(-\lambda t_2^\alpha) - E_{\alpha,\alpha}(-\lambda t_1^\alpha)\right] \\
&+ \left(t_2^{1-\alpha} - t_1^{1-\alpha}\right) \left\| \int_0^{t_1} [(t_2-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t_2-s)^\alpha) - (t_1-s)^{\alpha-1} \right. \\
&\left. E_{\alpha,\alpha}(-\lambda(t_1-s)^\alpha)] f(s, x(s)) ds \right\| \\
&+ t_2^{1-\alpha} \left\| \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t_2-s)^\alpha) f(s, x(s)) ds \right\| \\
&+ \left(t_2^{1-\alpha} - t_1^{1-\alpha}\right) \left\| \int_0^{t_1} \left[(t_2-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t_2-s)^\alpha) - (t_1-s)^{\alpha-1} \right. \right. \\
&\left. \left. E_{\alpha,\alpha}(-\lambda(t_1-s)^\alpha)\right] (\lambda x(s)) ds \right\|
\end{aligned}
$$

$$
\begin{aligned}
&+t_2^{1-\alpha}\left\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t_2-s)^\alpha)(\lambda x(s))ds\right\| \\
&\leq I_1+\frac{t_2^{1-\alpha}-t_1^{1-\alpha}}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}](c(1+s^{1-\alpha}\|x(s)\|))ds \\
&+\frac{t_2^{1-\alpha}}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}(c(1+s^{1-\alpha}\|x(s)\|))ds \\
&+\frac{t_2^{1-\alpha}-t_1^{1-\alpha}}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}](\lambda s^{\alpha-1}s^{1-\alpha}\|x(s)\|)ds \\
&+\frac{t_2^{1-\alpha}}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}(\lambda s^{\alpha-1}s^{1-\alpha}\|x(s)\|)ds \\
&\leq I_1+\frac{t_2^{1-\alpha}-t_1^{1-\alpha}}{\Gamma(\alpha+1)}(c(1+r))\left[(t_2-t_1)^\alpha+t_1^\alpha-t_2^\alpha\right]+\frac{t_2^{1-\alpha}}{\Gamma(\alpha+1)}(c(1+r))\left[((t_2-t_1)^\alpha\right] \\
&+\frac{\Gamma(\alpha)(t_2^{1-\alpha}-t_1^{1-\alpha})}{\Gamma(2\alpha)}(\lambda r))\left[(t_2-t_1)^\alpha+t_1^\alpha-t_2^\alpha\right]+\frac{\Gamma(\alpha)t_2^{1-\alpha}}{\Gamma(2\alpha)}(\lambda r))\left[((t_2-t_1)^\alpha\right]
\end{aligned}
$$

where

$$I_1=\Gamma(\alpha)\|x_0\|[E_{\alpha,\alpha}(-\lambda t_2^\alpha)-E_{\alpha,\alpha}(-\lambda t_1^\alpha)].$$

Appling by the function $E_{\alpha,\alpha}(-\lambda t^\alpha)$ is uniformly continuous on $[0,b]$, we have I_1 tend to zero independently of $x\in B_r$ as $t_2\to t_1$.
Thus $\|t_2^{1-\alpha}N(x)(t_2)-t_1^{1-\alpha}N(x)(t_1)\|$ tend to zero independently of $x\in B_r$ as $t_2\to t_1$, which means that the set NB_r is equicontinuous.
Define $B_{r_0}=\{x\in D:\|x\|_\alpha\leq r_0\}$, where $r_0>0$ is taken so that

$$r_0\geq\left(\|x_0\|+\frac{cb}{\Gamma(\alpha+1)}\right)(1-L)^{-1},$$

such that

$$\frac{cb}{\Gamma(\alpha+1)}+\frac{\lambda b^\alpha\Gamma(\alpha)}{\Gamma(2\alpha)}\leq L<1.$$

Then B_{r_0} is closed convex bounded and hence $NB_{r_0}\subset B_{r_0}$.
Now we prove that there exists a compact subset $M\subset B_{r_0}\subset$ such that $NM\subset M$. We first costruct a series of sets $\{M_n\}\subset B_{r_0}$ by

$$M_0=B_{r_0},\ M_1=\overline{\text{conv}}NM_0,\ M_{n+1}=\overline{\text{conv}}NM_n,\ n=1,2\cdots.$$

From the above proof it is easy to see $M_{n+1}\subset M_n$ for $n=1,2\cdots$ and each $\widetilde{M}_n$ is equicontinuous. Further from Definition 1.49 and Lemma 1.54 we can derive that

$$
\begin{aligned}
&\beta\left(\widetilde{M}_{n+1}(t)\right)=\beta(t^{1-\alpha}M_{n+1}(t))=\beta(t^{1-\alpha}NM_n(t)) \\
&\leq\beta\left[\Gamma(\alpha)x_0E_{\alpha,\alpha}(-\lambda t^\alpha)+t^{1-\alpha}\int_0^t(t-s)^{\alpha-1}(E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)F(s,M_n(s))ds\right] \\
&\qquad\leq c_1\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\beta(M_n(s))ds.
\end{aligned}
$$

Define the function $F_n(t)=\beta(M_n(t))$ for $n=1,2\cdots$ we get

$$F_{n+1}(t)\leq c_1\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}F_n(s)ds, \tag{2.8}$$

for $n = 1, 2, \cdots$ the fact $M_{n+1} \subset M_n$.
Taking limit as $n \to \infty$ in (2.13) we get

$$F(t) \leq c_1 \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s) ds,$$

for all $t \in J$. An application of Lemma 1.8 yields $F(t) = 0$ for all $t \in J$.
Therefore, $\cap_{n=1}^{\infty} M_n = M$ is nonempty and compact in $D = [u_0, y_0]$ due to Definition 1.44, and $NM \subset M$ by definition of M_n.

Up to now we have verified that there exists a nonempty bounded convex and compact subset $M \subset D$ such that $NM \subset M$. An employment of Schauder's fixed point theorem shows that there exists at least a fixed point x of N in M. Combining with the fact that $\lim\limits_{t \to 0^+} E_{\alpha,\alpha}(-\lambda t^\alpha) = E_{\alpha,\alpha}(0) = 1/\Gamma(\alpha)$ yields that $\lim\limits_{t \to 0^+} t^{1-\alpha}(Nx)(t) = x_0$. The proof is complete.

Theorem 2.7 *Assume* (H_1)-(H_4)*, hold, and* $v, w \in C_{1-\alpha}(J, X)$ *are lower and upper solutions of* (2.1)-(2.2) *respectively such that*

$$v(t) \leq w(t), \quad 0 \leq t \leq b.$$

Then, the fractional IVP (2.1)-(2.2) *has a minimal solution* u^* *and a maximal solution* y^* *such that*

$$u^* = \lim_n N^n v, \quad y^* = \lim_n N^n w.$$

Proof. Suppose that functions $v, w \in C_{1-\alpha}(J, X)$ are lower and upper solution of IVP (2.1)-(2.2). We consider in $C_{1-\alpha}(J, X)$ the order induced by the sector $D = [v, w]$ define $[v, w] = \{x \in C_{1-\alpha}(J, X) : v \leq x \leq w\}$, then there are $v \leq Nv$, $w \geq Nw$. In fact, by the definition of the lower solution, there exist $p(t) \geq 0$ and $\epsilon \geq 0$, we have

$$^L D_{0^+}^{\alpha} v(t) = f(t, v(t)) - p(t), \quad t \in (0, b],$$

$$\lim_{t \to 0^+} t^{1-\alpha} v(t) = x_0 - \epsilon.$$

Using Theorem 2.6 and Lemma 3.8 , one has

$$\begin{aligned} v(t) &= \Gamma(\alpha)(x_0 - \epsilon) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) \\ &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda (t-s)^\alpha) \left[f(s, v(s)) + \lambda v(s) - p(s) \right] ds \\ &\leq (Nv)(t). \end{aligned}$$

Similarly, there is $w \geq Nw$.
The operator $N : D \to C_{1-\alpha}(J, X)$ is increasing and completely continuous by the use of Lemma 2.10 the existence of u^*, y^* is obtained. The proof is complete.

2.2 An example

As an application of our results we consider the following fractional equation

$$^L D_{0^+}^{\alpha} x(t) = \frac{1}{e^{t^2} + 1} \left\{ \ln(|x_k| + 1) + \frac{1}{1 + k} \right\}_{k \in \mathbb{N}}, \quad t \in J = [0, 1], \tag{2.9}$$

$$\lim_{t \to 0^+} t^{1-\alpha} x(t) = x_0, \tag{2.10}$$

c_0 represents the space of all sequences converging to zero, which is a Banach space with respect to the norm

$$\|x\| = \sup_k |x_k|.$$

Let $t \in J$ and $x = \{x_k\}_k \in c_0$, we have

$$\begin{aligned}
\|f(t,x)\|_\infty &= \frac{1}{e^{t^2}+1}\| \ln(|x_k|+1) + \tfrac{1}{k+1}\|_\infty \\
&\leq \frac{1}{e^{t^2}+1}\left(\sup_k |x_k| + 1\right) \\
&\leq \frac{1}{e^{t^2}+1}(1 + \|x\|_\infty).
\end{aligned}$$

Hence condition $(H_1)-(H_2)$ are satisfied with $c = \dfrac{1}{2}$, for all $t \in [0,1]$.
So, that function F by defined

$$F(t,x(t)) = \frac{1}{e^{t^2}+1}\left\{\ln(|x_k|+1) + \frac{1}{k+1}\right\}_{k\in\mathbb{N}} + \lambda x(t), \quad \text{for all} \;\; t \in [0,1].$$

We recall that the measure of noncompactness β in space c_0 can be computed by means of the formula

$$\beta(\Omega) = \lim_{n\to+\infty} \sup_{x\in\Omega} \|(I-P_n)x\|_\infty.$$

Where Ω is a bounded subset in c_0 and P_n is the projection onto the linear span of n vectors, we get

$$\beta(F(t,\Omega)) \leq c_1\beta(\Omega(t)) \;\; \text{for all} \;\; t \in [0,1],$$

with $c = (e^{t^2}+1)^{-1}$. Therefore $\beta(F(t,\Omega(t)) \leq c_1\beta(\Omega(t))$, with $c_1 = \max(c,\lambda)$ due to (H_3) and definition . Then by Theorem 2.7 the problem (2.9)-(2.10) has a minimal and maximal solutions.

2.3 Fractional differential equations nonlocal conditions

Consider the following Cauchy problem for the nonlocal initial conditions fractional differential equation

$$^{L}D^{\alpha}_{0^+}x(t) = f(t,x(t)); \;\; t \in J' := (0,b], \tag{2.11}$$

$$(I^{1-\alpha}_{0^+}x)(0) + g(x) = x_0, \tag{2.12}$$

for instance we use

$$g(x) = \sum_{i=1}^{p} c_i x(t_i),$$

where $c_i(i = 1, 2 \cdot\cdot, p)$, are given constants such that $c_i \neq 0$ and $0 < t_1 < t_2 < \cdot < t_p \leq b$. To describe the diffusion phenomenon of a small amount in a transparent tube.

2.3.1 Existence and Uniqueness solutions.

We investigate in our the Cauchy problem for the fractional differential equation(2.11)-(2.12) above with the following assumptions.

(H_1) $f : [0, b] \times X \to X$ is continuous function.

(H_2) $\|f(t, x) - f(t, y)\| \le L\|x - y\|, \forall t \in [0, b], x, y \in X.$

(H_3) $g : C_{1-\alpha}([0, b], X) \to X$ is continuous and $\|g(x) - g(y)\| \le L_g\|x - y\|_\alpha$.

Theorem 2.8 *Under assumptions (H_1-H_3), if* $L_g < \dfrac{1}{2}$ *and* $L \le \dfrac{\Gamma(2\alpha)}{2b^\alpha\Gamma(\alpha)}$.
Then (2.11)-(2.12) *has a unique solution.*

Proof. Defined $T : C_{1-\alpha}(J, X) \to C_{1-\alpha}(J, X)$ by:

$$T(x)(t) = t^{\alpha-1}(x_0 - g(x)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s, x(s))ds.$$

Let $B_r = \{x \in C_{1-\alpha}(J, X), \|x\|_\alpha \le r\}$, where

$$r \ge 2\left[(\|x_0\| + g^*) + \frac{Mb}{\Gamma(\alpha + 1)}\right].$$

Then we can show that $T(B_r) \subset B_r$. So that $x \in B_r$ and set $g^* = \sup\limits_{x\in B_r} \|g(x)\|$, $M = \sup\limits_{t\in J} \|f(t, 0)\|$ then we get

$$t^{1-\alpha}\|T(x)(t)\| \le \|(x_0 - g(x))\| + \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}\|f(s, x(s))\|ds$$

$$\le (\|x_0\| + g^*) + \frac{b^{1-\alpha}}{\Gamma(\alpha)} \left[\int_0^t (t - s)^{\alpha-1} \left(\|f(s, x(s)) - f(s, 0)\| + \|f(s, 0)\|\right) ds\right]$$

$$\le (\|x_0\| + g^*) + \frac{Mb^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}ds$$

$$+\frac{b^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}s^{\alpha-1}s^{1-\alpha}\|f(s, x(s)) - f(s, 0)\|ds$$

$$\le (\|x_0\| + g^*) + \frac{Mb}{\Gamma(\alpha + 1)} + \frac{b^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}s^{\alpha-1}\|f(s, x(s)) - f(s, 0)\|_\alpha ds$$

$$\le (\|x_0\| + g^*) + \frac{Mb}{\Gamma(\alpha + 1)} + \frac{Lb^\alpha B(\alpha, \alpha)}{\Gamma(\alpha)} r \le r.$$

Now take $x, y \in C_{1-\alpha}(J, X)$, we get

$$
\begin{aligned}
& t^{1-\alpha}\|T(x)(t) - T(y)(t)\| \\
& \quad \leq \|g(x) - g(y)\| + \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, x(s)) - f(s, y(s))\| ds \\
& \quad \leq L_g \|x - y\|_\alpha + \frac{L b^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} s^{1-\alpha} \|x - y\| ds \\
& \quad \leq L_g \|x - y\|_\alpha + \frac{L b^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} \|x - y\|_\alpha ds \\
& \quad \leq L_g \|x - y\|_\alpha + \frac{L b^{\alpha} \Gamma(\alpha)}{\Gamma(2\alpha)} \|x - y\|_\alpha \\
& \quad \leq \left(L_g + \frac{L b^{\alpha} \Gamma(\alpha)}{\Gamma(2\alpha)} \right) \|x - y\|_\alpha \\
& \quad \leq \Omega_{L, L_g, b, \alpha} \|x - y\|_\alpha,
\end{aligned}
$$

where $\Omega_{L, L_g, b, \alpha} := \left(L_g + \frac{L b^{\alpha} \Gamma(\alpha)}{\Gamma(2\alpha)} \right)$, which depends only on the parameters involved in the problem. And since $\Omega_{L, L_g, b, \alpha} < 1$, then T is contraction mapping. Therefor, for by Banach's contraction principle T has a unique fixed point. It is clearly choose,

$$
b^* = \min \left(b, \left(\frac{\Gamma(2\alpha)}{2L\Gamma(\alpha)} \right)^{\frac{1}{\alpha}} \right).
$$

2.3.2 Existence Results

Our next result is based on the following the now assume

(H_4) there exists a constant $c_1 > 0$ such that

$$
\|f(t, x)\| \leq c_1 (1 + t^{1-\alpha} \|x(t)\|) \text{ for all } t \in [0, b], \text{and } x \in X.
$$

(H_5) there exists a constant $\widehat{L} > 0$ such that for each nonempty, bounded set $\Omega \subset C_{1-\alpha}(J, X)$

$$
\chi(f(t, \Omega)) \leq \widehat{L} \chi(\Omega(t)), \text{ for all } t \in J,
$$

where χ is the Hausdorff measure of noncompactness in E.

For brevity, let

$M_1 = \dfrac{c_1 b}{\Gamma(\alpha + 1)}$,

$M_2 = (\|x_0\| + g^*) + \dfrac{c_1 b}{\Gamma(\alpha + 1)}$.

Define an operator T on $C_{1-\alpha}(J, X)$ by

$$
(Tx)(t) = t^{\alpha-1}(x_0 - g(x)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds \; t \in (0, b],
$$

for any $x \in C_{1-\alpha}(J, X)$, let $(Tx)(t) = (T_1x)(t) + (T_2x)(t)$, where

$$(T_1x)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds,$$

$$(T_2x)(t) = t^{\alpha-1}(x_0 - g(x)).$$

Assume that $M_1 < 1$, and let
$B_r = \{x \in C_{1-\alpha}(J, X) : \|x\|_\alpha \leq r\}$, where $r \geq \dfrac{M_2}{1-M_1}$.

Lemma 2.9 *If the assumptions* (H_1),(H_4) *are satisfied with* $M_1 < 1$, *and* (H_5). *Then* $T_1(B_r)$ *is relatively compact set in* $C_{1-\alpha}(J, X)$.

Proof.

Using (H_4) we can easily prove that $T_1x \in C_{1-\alpha}(J, X)$ for any $x \in C_{1-\alpha}(J, X)$. Then T_1 is well defined on $C_{1-\alpha}(J, X)$. We divide the proof into a sequence of steps.

Step 1. T_1 is continuous.
Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $C_{1-\alpha}(J, X)$. Then

$$\begin{aligned}
&t^{1-\alpha}\|T_1(x_n)(t) - T_1(x)(t)\| \\
&\leq \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, x_n(s)) - f(s, x(s))\| ds \\
&\leq \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} s^{1-\alpha} \|f(s, x_n(s)) - f(s, x(s))\| ds \\
&\leq \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} \|f(\cdot, x_n(\cdot)) - f(\cdot, x(\cdot))\|_\alpha ds.
\end{aligned}$$

Using hypothesis (H_4) we have

$$\|T_1(x_n) - T_1(x)\|_\alpha \leq \frac{b^\alpha}{\Gamma(\alpha)} B(\alpha, \alpha) \|f(\cdot, x_n(\cdot)) - f(\cdot, x(\cdot))\|_\alpha.$$

Hence

$$\|T_1(x_n) - T_1(x)\|_\alpha \to 0, \quad \text{as } n \to +\infty.$$

Step 2. T_1 maps bounded sets into bounded sets in $C_{1-\alpha}(J, E)$.
Indeed, it is enough to show that there exists a positive constant l such that for each $x \in B_r = \{x \in C_{1-\alpha}(J, X) : \|x\|_\alpha \leq r\}$ one has $\|T_1(x)\|_\alpha \leq l$.

$$\begin{aligned}
t^{1-\alpha}\|T_1(x)(t)\| &\leq \frac{b^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, x(s))\| ds \\
&\leq \frac{b^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} c_1(1 + s^{1-\alpha}\|x(s)\|) ds \\
&\leq \frac{c_1 b^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (1+r) ds \\
&\leq \frac{c_1 b}{\Gamma(\alpha+1)} (1+r) := l.
\end{aligned}$$

Step 3. T_1 maps bounded sets into equicontinuous sets.

$$\|t_2^{1-\alpha}T_1(x)(t_2) - t_1^{1-\alpha}T_1(x)(t_1)\|$$

$$\leq \frac{(t_2^{1-\alpha} - t_1^{1-\alpha})}{\Gamma(\alpha)} \left\| \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] f(s, x(s)) ds \right\|$$

$$+ \frac{t_2^{1-\alpha}}{\Gamma(\alpha)} \left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} f(s, x(s)) ds \right\|$$

$$\leq \frac{(t_2^{1-\alpha} - t_1^{1-\alpha})}{\Gamma(\alpha)} \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] c_1(1 + s^{1-\alpha}\|x(s)\|) ds$$

$$+ \frac{t_2^{1-\alpha}}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} c_1(1 + s^{1-\alpha}\|x(s)\|) ds$$

$$\leq \frac{(t_2^{1-\alpha} - t_1^{1-\alpha})}{\Gamma(\alpha)} c_1(1 + r) \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] ds$$

$$+ \frac{t_2^{1-\alpha}}{\Gamma(\alpha)} c_1(1 + r) \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds.$$

Thus

$$\|t_2^{1-\alpha}T_1(x)(t_2) - t_1^{1-\alpha}T_1(x)(t_1)\|$$

$$\leq \frac{(t_2^{1-\alpha} - t_1^{1-\alpha})}{\Gamma(\alpha + 1)} c_1(1 + r)[(t_2 - t_1)^\alpha + (t_1^\alpha - t_2^\alpha)]$$

$$+ \frac{t_2^{1-\alpha}}{\Gamma(\alpha + 1)} c_1(1 + r)(t_2 - t_1)^\alpha.$$

As $t_2 \to t_1$, the right-hand side of above expression tends to zero. Then $T_1(B_r)$ is equicontinuous.
Step 4. T_1 is ν-condensing. We consider the measure of noncompactness defined in the following way. For every bounded subset $\Omega \subset C_{1-\alpha}(J, X)$.

$$\nu(\Omega) = \max_{\Omega \in \Delta(\Omega)} (\gamma(\Omega), \quad \text{mod}_{C_{1-\alpha}}(\Omega)), \tag{2.13}$$

$\Delta(\Omega)$ is the collection of all countable subsets of Ω and the maximum is taken in the sense of the partial order in the cone $\mathbb{R}_+^2$, γ is the damped modules of fiber noncompactness

$$\gamma(\Omega) = \sup_{t \in J} e^{-\mu t} \chi(\Omega_\alpha(t)), \ \mu \geq 0, \tag{2.14}$$

where $\Omega_\alpha(t) = \{x_\alpha(t) : x(t) \in \Omega\}$ and $\text{mod}_{C_{1-\alpha}}(\Omega)$ is the modulus of equicontinuity of the set of functions Ω given by formula

$$\text{mod}_{C_{1-\alpha}}(\Omega) = \lim_{\delta \to 0} \sup_{x \in \Omega} \max_{|t_1 - t_2| \leq \delta} \|x_\alpha(t_1) - x_\alpha(t_2)\|. \tag{2.15}$$

Let

$$\sigma(\mu) = \sup_{t \in J} \int_0^t (t - s)^{\alpha-1} s^{\alpha-1} e^{-\mu(t-s)} ds. \tag{2.16}$$

It is clear that

$$\sup_{t\in[0,b]}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}e^{-\mu(t-s)}ds \to 0 \ \text{ as } \ \mu\to+\infty.$$

We can choose μ such that

$$\overline{\sigma}=\frac{2\widehat{L}b^{1-\alpha}}{\Gamma(\alpha)}\sigma(\mu)<1. \tag{2.17}$$

From Lemma , the measure ν is well defined and give a monotone, nonsingular and regular measure of noncompactness in $C_{1-\alpha}(J,X)$.
Let $\Omega\subset C_{1-\alpha}(J,X)$ be a bounded subset such that

$$\nu(T_1(\Omega))\geq\nu(\Omega). \tag{2.18}$$

We will show that (3.28) implies that Ω is relatively compact. Let the maximum on the left-hand side of the inequality (3.28) be a chieved for the countable set $\{y^n\}_{n=1}^{+\infty}$ with

$$y^n(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f_n(s)ds,\ \{x^n\}_{n=1}^{+\infty}\subset\Omega, \tag{2.19}$$

and $f_n(t)=f(t,x^n(t))$.
We give now an upper estimate for $\gamma(\{y^n\}_{n=1}^{+\infty}$. By using (H_5) we have

$$\begin{aligned}
\chi(\{(t-s)^{\alpha-1}f_n(s)\}_{n=1}^{+\infty}) &\leq (t-s)^{\alpha-1}\widehat{L}\chi(\{x^n(s)\}_{n=1}^{+\infty})\\
&\leq \widehat{L}(t-s)^{\alpha-1}s^{\alpha-1}s^{1-\alpha}\chi(\{x^n(s)\}_{n=1}^{+\infty})\\
&= \widehat{L}(t-s)^{\alpha-1}s^{\alpha-1}\chi(\{x^n_\alpha(s)\}_{n=1}^{+\infty})\\
&\leq \widehat{L}(t-s)^{\alpha-1}s^{\alpha-1}e^{\mu s}\sup_{0\leq s\leq t}e^{-\mu s}\chi(\{x^n_\alpha(s)\}_{n=1}^{+\infty})\\
&= \widehat{L}(t-s)^{\alpha-1}s^{\alpha-1}e^{\mu s}\gamma(\{x^n\}_{n=1}^{+\infty}),
\end{aligned}$$

for all $t\in[0,b]$, $s\leq t$. Then applying Lemma 1.55, we obtain

$$\chi(\{y^n\}_{n=1}^{+\infty})\leq\frac{2\widehat{L}b^{1-\alpha}}{\Gamma(\alpha)}\sup_{t\in[0,b]}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}e^{\mu s}\gamma(\{x^n\}_{n=1}^{+\infty}).$$

Taking (2.17) and (2.19) into account, we derive

$$\gamma(\{y^n\}_{n=1}^{+\infty})\leq\overline{\sigma}\gamma(\{x^n\}_{n=1}^{+\infty}).$$

Combining the last inequality with (3.28), we have

$$\gamma(\{x^n\}_{n=1}^{+\infty})\leq\overline{\sigma}\gamma(\{x^n\}_{n=1}^{+\infty}).$$

Therefore

$$\gamma(\{x^n\}_{n=1}^{+\infty})=0.$$

Furthermore, from step 3, we know that
$\text{mod}\ _{C_{1-\alpha}}(T_1(\Omega))=0$ and (3.28) yields $\text{mod}\ _{C_{1-\alpha}}(\Omega)=0$. Finally,

$$\nu(\Omega)=(0,0),$$

which prove the relative compactness of set Ω.

Theorem 2.10 *Assume that (H_1), (H_3), (H_4), (H_5) hold, with $M_1<1$. Then (2.11)-(2.12) has least one solution.*

Proof.

Using (H_1), (H_4) can be prove that $Tx \in C_{1-\alpha}(J,X)$ for any $x \in C_{1-\alpha}(J,X)$. Then T is well defined on $C_{1-\alpha}(J,X)$. We will show that T satisfies all conditions of Theorem 2.8, the proof will be given in several steps.

For any $x \in B_r$ and $t \in J$, taking into account the imposed assumptions, we obtain

$$t^{1-\alpha}\|(Tx)(t)\| \leq (\|x_0\| + g^*) + \frac{b^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,x(s))\|ds$$

$$\leq (\|x_0\| + g^*) + \frac{b^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}c_1(1+s^{1-\alpha}\|x(s)\|)ds$$

$$\leq (\|x_0\| + g^*) + \frac{c_1 b^{1-\alpha}(1+r)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds$$

$$\leq (\|x_0\| + g^*) + \frac{c_1 b(1+r)}{\Gamma(\alpha+1)} \leq r.$$

Then T is maps B_r into B_r.

Next, we will show that T is continuous in B_r.

By (H_3), for $L_g < 1$ it is clear that T_2 is a contraction mapping.

This means that T is continuous in B_r.

According to Lemma 2.9, $T_1(B_r)$ is relatively compact in $C_{1-\alpha}(J,X)$, then $\chi(T_1(B_r)) = 0$. For any $x_1, x_2 \in B_r$, we have

$$t^{1-\alpha}\|T_2x_2(t) - T_2x_1(t)\| \leq \|g(x_2) - g(x_1)\|$$

which implies that

$$\|T_2x_2 - T_2x_1\|_\alpha \leq L_g\|x_2 - x_1\|_\alpha.$$

Hence

$$\beta(T_2(B_r)) \leq L_g\beta(B_r).$$

Therefore

$$\chi(T(B_r)) \leq \chi(T_1(B_r)) + \chi(T_2(B_r))$$

$$\leq L_g\chi(B_r).$$

Noting that $L_g < 1$, we find that the operator T is an χ-contraction in B_r. Then problem (2.11)-(2.12) has at least one solution in B_r. The proof is complet.

2.3.3 An example

In section, we discuss an example to illustrate our results. Let us consider the fractional differential equation nonlocal

$$^{L}D^{\alpha}_{0^+}x(t) = \frac{1}{e^{t^2}+1}\left\{\ln(|x_k|+1) + \frac{1}{k^2}\right\}_{k=1}^{\infty}, \quad t \in J = [0,1], \tag{2.20}$$

$$(I^{1-\alpha}_{0^+}x)(0) + g(x) = x_0, \tag{2.21}$$

c_0 represents the space of all sequences converging to zero, which is a Banach space with respect to the norm

$$\|x\| = \sup_k |x_k|.$$

Let $t \in J$ and $x = \{x_k\}_k \in c_0$, we have

$$\begin{aligned} \|f(t,x)\|_\infty &= \frac{1}{e^{t^2}+1}\|\ln(|x_k|+1)+\tfrac{1}{k^2}\|_\infty \\ &\le \frac{1}{e^{t^2}+1}\left(\sup_k |x_k| + 1\right) \\ &\le \frac{1}{e^{t^2}+1}(1+\|x\|_\infty). \end{aligned}$$

Hence conditions (H_1), (H_4) are satisfied with $p(t) = \dfrac{1}{e^{t^2}+1}$, for all $t \in [0,1]$.
We recall that the measure of noncompactness χ in space c_0 can be computed by means of the formula

$$\chi(\Omega) = \lim_{n\to+\infty} \sup_{x\in\Omega} \|(I-P_n)x\|_\infty.$$

Where Ω is a bounded subset in c_0 and P_n is the projection onto the linear span of n vectors, we get

$$\chi(f(t,\Omega)) \le \eta(t)\chi(\Omega(t)) \text{ for all } t \in [0,1],$$

with $\eta(t) = (e^{t^2}+1)^{-1}$. Hence (H_5) is satisfied.
Denote $g(x) = \sum\limits_{i=1}^{m} c_i x(t_i)$, then for any $x = \{x_k\}_k, y = \{y_k\}_k \in c_0$, one has

$$\|g(x)-g(y)\|_\infty \le \sum_{i=1}^{m} |c_i|\|x-y\|_\infty.$$

Clearly, $L_g = \sum\limits_{i=1}^{m} |c_i|$ and choose c_i such that $L_g < 1$.
Assume that (H_1), (H_3), (H_4), (H_5) is satisfied and $M_1 < 1$. Then by Theorem 2.10 the fractional problem (2.20)-(2.21) has least one solution.

2.4 Topological structure of solutions sets

Let X be a general Banach space and let $0 < \alpha < 1$. The objective to study the structure of solution sets of fractional diferential equations (2.1)-(2.2).

2.4.1 R_δ-set

Let $\sum_{x_0}^{f}[0,b]$ denote the set all solutions of the problem (2.1)-(2.2). We prove that the solution set of the problem (2.1)-(2.2) is nonempty, compact and, an R_δ-set by using the techniques of the theory of condensing maps combined with Browder-Gupta approach (see [18]), in general setting, namely when the function right-hand side has value in infinite dimensional Banach space. We need to make the following assumptions

(H_2') there exists function $\phi(t) \in L^{\frac{1}{q}}(J,\mathbb{R}^+)$, $q \in (0,\alpha)$ and a constant $c > 0$ such that

$$\|f(t,x(t))\| \le \phi(t) + ct^{1-\alpha}\|x(t)\|, \text{ for a.e. } t \in J \text{ and all } x \in C_{1-\alpha}(J,X).$$

(H_3') there exists a constant $c_1 > 0$ such that for each nonempty, bounded set $\Omega \subset C_{1-\alpha}([0,b],X)$,

$$\chi(f(t,\Omega)) \le c_1\chi(\Omega(t)), \text{ for all } t \in [0,b],$$

where χ is the Hausdorff measure of noncompactness in X

A ssume that

Theorem 2.11 *Assume that* (H_1)-$(H_3^{'})$ *are satisfied. Then the set* $\sum_{x_0}^{f}[0,b]$ *is an* R_δ*-set and hence is an acyclic space.*

Proof. Let $N : C_{1-\alpha}(J,X) \to C_{1-\alpha}(J,X)$ be defined by

$$N(x)(t) = t^{\alpha-1}x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x(s))ds \; t \in (0,b]. \tag{2.22}$$

Thus $\mathcal{F}ixN = \sum_{x_0}^{f}[0,b]$. Now, we show that $\sum_{x_0}^{f}[0,b] \neq \emptyset$. We divide the proof into a sequence of steps.
Step 1. N is continuous.
Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $C_{1-\alpha}(J,X)$. Then

$$t^{1-\alpha}\|N(x_n)(t) - N(x(t)\| \leq \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,x_n(s)) - f(s,x(s))\|ds$$

$$\leq \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}s^{1-\alpha}\|f(s,x_n(s)) - f(s,x(s))\|ds$$

$$\leq \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}\|f(\cdot,x_n(\cdot)) - f(\cdot,x(\cdot))\|_\alpha ds.$$

Using the hypothesis $(H_2^{'})$, we have

$$\|N(x_n) - N(x)\|_\alpha \leq \frac{b^\alpha}{\Gamma(\alpha)}B(\alpha,\alpha)\|f(\cdot,x_n(\cdot)) - f(\cdot,x(\cdot))\|_\alpha$$

Hence

$$\|N(x_n) - N(x)\|_\alpha \to 0 \;\text{ as }\; n \to +\infty.$$

Step 2. N maps bounded sets into bounded sets in $C_{1-\alpha}(J,X)$.
Indeed, it is enough to show that there exists a positive constant l such that for each $x \in B_r = \{x \in C_{1-\alpha}(J,X) : \|x\|_\alpha \leq r\}$ one has $\|N(x)\|_\alpha \leq l$.
Let $x \in B_r$. Then for each $t \in (0,b]$, by $(H_2^{'})$ we have

$$t^{1-\alpha}\|N(x)(t)\| \leq \|x_0\| + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,x(s))\|ds$$

$$\leq \|x_0\| + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\phi(s) + cs^{1-\alpha}\|x(s)\|)ds$$

$$\leq \|x_0\| + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi(s)ds + \frac{crt^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds$$

$$\leq \|x_0\| + \frac{1}{\Gamma(\alpha)}\left(\frac{1-q}{\alpha-q}b\right)^{1-q}\|\phi\|_{L^{\frac{1}{q}}(J,X)} + \frac{cbr}{\Gamma(\alpha+1)} := l.$$

Step 3. N maps bounded sets into equicontinuous sets.
Let $\tau_1, \tau_2 \in (0, b]$, $\tau_1 \leq \tau_2$, let B_r be a bounded set in $C_{1-\alpha}(J, X)$ as in step 2 an let $x \in B_r$, we have

$$
\begin{aligned}
&\|\tau_2^{1-\alpha} N(x)(\tau_2) - \tau_1^{1-\alpha} Nx(\tau_1)\| \\
&\leq \frac{\tau_2^{1-\alpha}}{\Gamma(\alpha)} \left\| \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} f(s, x(s)) ds \right\| \\
&+ \frac{1}{\Gamma(\alpha)} \left\| \int_0^{\tau_1} \left[\tau_2^{1-\alpha}(\tau_2 - s)^{\alpha-1} - \tau_1^{1-\alpha}(\tau_1 - s)^{\alpha-1}\right] f(s, x(s)) ds \right\| \\
&\leq \frac{b^{1-\alpha}}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} [\phi(s) + cr] ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^{\tau_1} \left[\tau_1^{1-\alpha}(\tau_1 - s)^{\alpha-1} - \tau_2^{1-\alpha}(\tau_2 - s)^{\alpha-1}\right] (\phi(s) + cr) ds \\
&\leq I_1 + I_2.
\end{aligned}
$$

Appling the absolute continuty of the Lebesgue integral, we have I_1 tend to zero independently of $x \in B_r$ as $\tau_2 \to \tau_1$.
Noting that

$$\left[\tau_1^{1-\alpha}(\tau_1 - s)^{\alpha-1} - \tau_2^{1-\alpha}(\tau_2 - s)^{\alpha-1}\right] (\phi(s) + cr) < \tau_1^{1-\alpha}(\tau_1 - s)^{\alpha-1}(\phi(s) + cr)$$

and $\displaystyle\int_0^{\tau_1} \tau_1^{1-\alpha}(\tau_1 - s)^{\alpha-1}(\phi(s) + cr) ds$ exists, then by the Lebesgue dominated convergence Theorem, we have I_2 tends to zero independently of $x \in B_r$ as $\tau_2 \to \tau_1$. Then $N(B_r)$ is equicontinuous.
Step 4. N is ν-condensing. We consider the measure of noncompactness defined in the following way. For every bounded subset $\Omega \subset C_{1-\alpha}(J, X)$,

$$\nu(\Omega) = \max_{\Omega \in \Delta(\Omega)} (\gamma(\Omega), mod_{C_{1-\alpha}}(\Omega)), \tag{2.23}$$

$\Delta(\Omega)$ is the collection of all countable subsets of Ω and the maximum is taken in the sense of the partial order in the cone $\mathbb{R}_+^2$. γ is the damped modulus of fibre noncompactness

$$\gamma(\Omega) = \sup_{t \in J} e^{-Lt} \chi(\Omega_\alpha(t)) \tag{2.24}$$

where $\Omega_\alpha(t) = \{x_\alpha(t) : x \in \Omega\}$. $mod_{C_{1-\alpha}}(\Omega)$ is the modulus of equicontinuity of the set of functions Ω given by the formula

$$mod_{C_{1-\alpha}}(\Omega) = \lim_{\delta \to 0} \sup_{x \in \Omega} \max_{|t_1 - t_2| \leq \delta} \|x_\alpha(t_1 - x_\alpha(t_2)\| \tag{2.25}$$

Let

$$q(L) = \sup_{t \in J} \int_0^t (t - s)^{\alpha-1} s^{\alpha-1} e^{-L(t-s)} ds.$$

It is clear that

$$\sup_{t \in J} \int_0^t (t - s)^{\alpha-1} s^{\alpha-1} e^{-L(t-s)} ds \to 0 \ \ as \ \ L \to +\infty.$$

We can choose L such that

$$\overline{q} := \frac{2cb^{1-\alpha}}{\Gamma(\alpha)} q(L) < 1. \tag{2.26}$$

From Lemma (2.1), the measure ν is well defined and give a monotone, nonsingular, semi-additive and regular measure of noncompactness in $C_{1-\alpha}(J,X)$.
Let $\Omega \subset C_{1-\alpha}(J,X)$ be a bounded subset such that

$$\nu(N(\Omega)) \geq \nu(\Omega). \tag{2.27}$$

We will show that (2.27) implies that Ω is relatively compact. Let the maximum on the left-hand side of the inequality (2.27) be achieved for the countable set $\{y^n\}_{n=1}^{+\infty}$ with

$$y^n(t) = t^{\alpha-1}x_0 + \mathcal{S}f_n(t), \quad \{x_n\}_{n=1}^{+\infty}, \tag{2.28}$$

with $\mathcal{S}f_n(t) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}f_n(s)ds$ and $f_n(t) = f(t, x^n(t))$. So that

$$\gamma(\{y^n\}_{n=1}^{+\infty}) = \gamma(\{\mathcal{S}f_n\}_{n=1}^{+\infty}). \tag{2.29}$$

We give now an upper estimate for $\gamma(\{y^n\}_{n=1}^{+\infty})$. By using $(H_3^{'})$ we have

$$\begin{aligned}
\chi(\{(t-s)^{\alpha-1}f_n(s)\}_{n=1}^{+\infty}) &\leq (t-s)^{\alpha-1}c_1\chi(\{x^n(s)\}_{n=1}^{+\infty}) \\
&\leq c_1(t-s)^{\alpha-1}s^{\alpha-1}s^{1-\alpha}\chi(\{x^n(s)\}_{n=1}^{+\infty}) \\
&= c_1(t-s)^{\alpha-1}s^{\alpha-1}\chi(\{x^n_\alpha(s)\}_{n=1}^{+\infty}) \\
&\leq c_1(t-s)^{\alpha-1}s^{\alpha-1}e^{Ls}\sup_{0\leq s\leq t} e^{-Ls}\chi(\{x^n_\alpha\}_{n=1}^{+\infty}) \\
&= c_1(t-s)^{\alpha-1}s^{\alpha-1}e^{Ls}\gamma(\{x^n\}_{n=1}^{+\infty}),
\end{aligned} \tag{2.30}$$

for all $t \in J$, $s \leq t$. Then applying Lemma 1.55 we obtain

$$\chi(\{\mathcal{S}f_n(t)\}_{n=1}^{+\infty}) \leq \frac{2c_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}e^{Ls}\gamma(\{x^n\}_{n=1}^{+\infty}).$$

Hence

$$t^{1-\alpha}\chi(\{\mathcal{S}f_n(t)\}_{n=1}^{+\infty}) \leq \frac{2c_1b^{1-\alpha}}{\Gamma(\alpha)}\sup_{t\in J}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}e^{Ls}\gamma(\{x^n\}_{n=1}^{+\infty}).$$

Taking (2.26) and (2.28) into account, we derive

$$\gamma(\{y^n\}_{n=1}^{+\infty}) \leq \overline{q}\gamma(\{x^n\}_{n=1}^{+\infty}). \tag{2.31}$$

Combining last inequality with (2.27), we have

$$\gamma(\{x^n\}_{n=1}^{+\infty}) \leq \overline{q}\gamma(\{x^n\}_{n=1}^{+\infty}).$$

Therefore

$$\gamma(\{x^n\}_{n=1}^{+\infty}) = 0.$$

Hence by (2.31) we get

$$\gamma(\{y^n\} = 0.$$

Furthermore, from step 3, we know that $mod_{C_{1-\alpha}}(N(\Omega)) = 0$ and (2.27) yields $mod_{C_{1-\alpha}}(\Omega) = 0$. Finally,

$$\nu(\Omega) = (0,0),$$

which prove the relative compactness of the set Ω.
Step 5. Apriori bounds.
Let $x = \lambda N(x)$ for some $0 < \lambda < 1$. This implies by $(H_2^{'})$

$$t^{1-\alpha}\|x(t)\| \le \|x_0\| + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,x(s))\|ds$$

$$\le \|x_0\| + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi(s)ds + \frac{ct^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{1-\alpha}\|x(s)\|ds$$

$$\le \|x_0\| + \frac{1}{\Gamma(\alpha)}\left(\frac{1-q}{\alpha-q}b\right)^{1-q}\|\phi\|_{L^{\frac{1}{q}}(J,X)} + \frac{ct^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{1-\alpha}\|x(s)\|ds.$$

From Lemma (1.8) there exists $K(\alpha) > 0$ such that

$$t^{1-\alpha}\|x(t)\| \le L + \frac{cb^{1-\alpha}K(\alpha)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Lds,$$

where

$$L := \|x_0\| + \frac{1}{\Gamma(\alpha)}\left(\frac{1-q}{\alpha-q}b\right)^{1-q}\|\phi\|_{L^{\frac{1}{q}}(J,X)}.$$

Then

$$\|x\|_\alpha \le L + \frac{cbLK(\alpha)}{\Gamma(\alpha+1)} =: \overline{M},$$

yielding the desired a priori boundedness.
By Theorem (1.60) $\mathcal{F}ixN$ is nonempty compact subset of $C_{1-\alpha}(J,X)$.
Given $\varepsilon_n \in (0,b)$ with $\varepsilon_n \to 0$ as $n \to +\infty$. By (H_1) according to Lemma (1.69), one can take a sequence $\{f_n\}$ of locally Lipschitz functions such that

$$\|f_n(t,x) - f(t,x)\| < \varepsilon_n, \text{ for all } t \in J \text{and } x \in X. \tag{2.32}$$

Making use of (2.32) and $(H_2^{'})$, we can assume that

$$\|f_n(t,x) - f(t,x)\| \le 1 + \phi(t) + ct^{1-\alpha}\|x\|, \ \ n \ge 1.$$

We define the approximation operator N_n by

$$N_n(x)(t) = t^{\alpha-1}x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f_n(s,x(s))ds, \ t \in (0,b]. \tag{2.33}$$

Since f_n is locally Lipschitz, the solution (2.33) is unique (see Theorem 5.1 in [23]).
Let

$$G(x) = (I-N)(x)$$

, by lemma (1.59) we can prove that the map $N_n : C_{1-\alpha}(J,X) \to C_{1-\alpha}(J,X)$ is condensing which allows us to define the condensing perturbation of identity $G_n(x) = (I - N_n)(x)$ which is a proper map. On the other hand, du to (2.32) we obtain that $\{G_n\}$ converges to G uniformly in $C_{1-\alpha}(J,X)$.

$$\|G_n(x)(t) - G(x)(t)\| \le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f_n(s,x(s)) - f(s,x(s))\|ds$$

$$\le \frac{b^\alpha}{\Gamma(\alpha+1)}\varepsilon_n, \ \text{ for } \ t \in (0,b],$$

and equation $G_n(x) = y$ has a unique solution for each $y \in C_{1-\alpha}(J,X)$ as well as equation (2.33). Therefore, all condition of Theorem (1.70) are met, then the solution set $G^{-1}(0)$ is an R_δ-set and hence an acyclic space.

2.4.2 An Example

As an application of our results we consider the following fractional differential equation

$$ {}^{L}D_{0^+}x_k(t) = \frac{1}{e^{t^2}+1} \mid x_k \mid + \frac{2}{3+e^t}, \tag{2.34} $$

$$ \lim_{t\to 0^+} t^{1-\alpha}x_k(t) = x_0, \tag{2.35} $$

l^∞ represents the space of all sequences bounded which is a Banach space with respect to the norm

$$ \|x\|_\infty = \sup_k \mid x_k \mid . $$

Let $t \in J$ and $x = \{x_k\}_k \in l^\infty$, and $f_k(t,x_k) = \frac{1}{e^{t^2}+1} \mid x_k \mid + \frac{2}{3+e^t}$ we have

$$ \begin{aligned} \|f(t,x)\|_\infty &\leq \frac{1}{e^{t^2}+1} \sup_k \mid x_k \mid + \frac{2}{3+e^t} \\ &\leq \frac{1}{e^{t^2}+1}\|x\|_\infty + \frac{2}{3+e^t} \end{aligned} \tag{2.36} $$

Hence condition (H_1)-$(H_2^{'})$ are satisfied with $p(t) = \frac{1}{e^{t^2}+1}$ and $\phi(t) = \frac{2}{3+e^t}$ for all $t \in [0,1]$. We recall that Hausdorff measure of noncompactness χ in the formula

$$ \chi(\Omega) = \lim_{n\to+\infty} \sup_{x\in\Omega} \|(I-P_n)x\|_\infty, $$

where Ω is the projection onto the linear span of the first n vectors in the standard basis [5],[61]. By (2.36) we get

$$ \chi(f(t,\Omega))\mu(t)\chi(\Omega(t)) \text{ for all } t \in [0,1], $$

with $\mu(t) = (e^{t^2}+1)^{-1}$. Hence $(H_3^{'})$ is satisfied. Therefore, from Theorem (2.11) the solution set of (2.34)-(2.35) is an R_δ-set.

Chapter 3

Weighted fractional evolution equations and inclusions in Banach spaces

Ce chapitre fait l'objet d'une publication dans Afrika Matematika journal.

EXISTENCE RESULTS FOR RIEMANN-LIOUVILLE FRACTIONAL EVOLUTION INCLUSIONS IN BANACH SPACES

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Abstract: The aim of this work is to study the existence of mild solution for semi-linear fractional evolution inclusions involving Riemann-Liouville derivative in Banach space. We prove our main result by introducing a regular measure of noncompactness in weighted space of continuous functions and using the condensing multivalued maps theory. Our result improve and complement several earlier related works. An example is given to illustrate the applications of the abstract result.

MSC 2010 Classifications:26A33; 34A60; 34G25; 47H04; 47H08; 47H10
Keywords and phrases: fractional evolution inclusions, Riemann-Liouville fractional derivatives, mild solutions, multivalued map, condensing map, measure of noncompactness.

3.1 Fractional evolution equation with nonlocal conditions

In this chapter, we study the existence of Cauchy problems for fractional evolution equation and inclusion with nonlocal conditions. The suitable mild solutions of fractional Cauchy problems with Riemann-Liouville derivative. We give two results, the first one is based on a Krasnosel'skii fixed point Theorem, and in the second approech we make use Mönch fixed point Theorem combined with the measure of noncompactness and condensing map.

3.1.1 Existence the mild solutions

We are considering the nonlocal Cauchy problems for a semilinear fractional differential equation in Banach space X of the following form:

$$\begin{cases} {}^{L}D^{\alpha}x(t) = Ax(t) + f(t, x(t)); \quad t \in (0, b], \\ I_{0^+}^{1-\alpha}x(t) \mid_{t=0} + g(x) = x_0, \end{cases} \tag{3.1}$$

where ${}^{L}D^{\alpha}$, $0 < \alpha < 1$, is the Riemann-Liouville fractional derivative, $f : [0, b] \times X \to X$ and $g : C_{1-\alpha}([0, b], X) \to X$ are given functions satisfying some assumptions, $A : D(A) \subseteq X \to X$ is a generator of a C_0-semigroup $\{T(t), t \geq 0\}$ on a Banach space X.

Lemma 3.1 *Cauchy problem* (3.1) *is equivalent to integral equations*

$$x(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0 - g(x)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(Ax(s) + f(s, x(s)))ds, \tag{3.2}$$

for $t \in (0, b]$ *provided that the integral in* (3.2) *exists.*

Proof. Suppose (3.2) is true. Then

$$\left(I_{0^+}^{1-\alpha}x\right)(t) = \left(I_{0^+}^{1-\alpha}\left[\frac{s^{\alpha-1}}{\Gamma(\alpha)}(x_0 - g(x)) + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\left[Ax(\tau) + f(\tau, x(\tau))d\tau\right]\right]\right)(t)$$

and applying Lemma 1.21, we obtain that

$$(I_{0^+}^{1-\alpha}x)(t) = x_0 - g(x) + \int_0^t \left[Ax(s) + f(s, x(s))\right] ds, \quad \textit{almost all } t \in [0, b].$$

This proves that $\left(I_{0^+}^{1-\alpha}x\right)(t)$ is absolutely continuous on $[0, b]$, then we have

$$\left({}^{L}D_{0^+}^{\alpha}x\right)(t) = \frac{d}{dt}(I_{0^+}^{1-\alpha}x)(t) = Ax(t) + f(t, x(t)), \quad \textit{almost all } t \in [0, b],$$

and

$$(I_{0^+}^{1-\alpha}x)(0) + g(x) = x_0,$$

(3.1) is true. Then

$$(I_{0^+}^{\alpha-1}\left({}^{L}D_{0^+}^{\alpha}x\right))(t) = \left(I_{0^+}^{\alpha-1}(Ax(s) + f(s, x(s))\right)(t)$$

since

$$(I_{0^+}^{\alpha-1}({}^{L}D_{0^+}^{\alpha}x))(t) = x(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)}(I_{0^+}^{1-\alpha}x)(0)$$

$$= x(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0 - g(x)), \quad \textit{for } t \in (0, b).$$

Then, we have

$$x(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0 - g(x)) + (I_{0^+}^{\alpha-1}(Ax(s) + f(s, x(s)))(t)$$

$$= \frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0 - g(x)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[Ax(s) + f(s, x(s))]ds$$

Before giving the definition of the mild solution of (3.1), we firstly prove the following lemma.

Lemma 3.2 *If*

$$x(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0 - g(x)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[Ax(s) + f(s, x(s))]ds, \quad \textit{for } t > 0, \tag{3.3}$$

holds, then we have

$$x(t) = t^{\alpha-1}T_\alpha(t)(x_0 - g(x)) + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s, x(s))ds, \quad \textit{for } t > 0,$$

where

$$T_\alpha(t) = \alpha \int_0^{+\infty} \theta M_\alpha(\theta)T(t^\alpha\theta)d\theta.$$

Proof. Let $\lambda > 0$. Applying the Laplace transform

$$\nu(\lambda) = \int_0^\infty e^{-\lambda s}x(s)ds \;\; \textit{and} \;\; \omega(\lambda) = \int_0^\infty e^{-\lambda s}(f(s, x(s))ds, \quad \textit{for } \lambda > 0$$

to (3.3), we have

$$\begin{aligned}\nu(\lambda) &= \frac{1}{\lambda^\alpha}(x_0 - g(x)) + \frac{1}{\lambda^\alpha}A\nu(\lambda) + \frac{1}{\lambda^\alpha}\omega(\lambda) \\ &= (\lambda^\alpha I - A)^{-1}(x_0 - g(x)) + (\lambda^\alpha I - A)^{-1}\omega(\lambda) \\ &= \int_0^\infty e^{-\lambda^\alpha s}Q(s)(x_0 - g(x))ds + \int_0^\infty e^{-\lambda^\alpha s}Q(s)\omega(\lambda)ds,\end{aligned} \tag{3.4}$$

provided that the integrals in (3.4) exist, where I is the indentity operator defined on X.
Set

$$\psi_\alpha(\theta) = \frac{\alpha}{\theta^{\alpha+1}}M_\alpha(\theta^{-\alpha}),$$

whose Laplace transform is given by

$$\int_0^\infty e^{-\lambda\theta}\psi_\alpha(\theta)d\theta = e^{-\lambda^\alpha}, \quad \textit{where } \alpha \in (0, 1). \tag{3.5}$$

Using (3.5), we get

$$\begin{aligned}\int_0^\infty e^{-\lambda^\alpha s}Q(s)(x_0 - g(x))ds &= \int_0^\infty \alpha t^{\alpha-1}e^{-(\lambda t)^\alpha}Q(t^\alpha)(x_0 - g(x))dt \\ &= \int_0^\infty \int_0^\infty \alpha\psi_\alpha(\theta)e^{-(\lambda t\theta)}Q(t^\alpha)t^{\alpha-1}(x_0 - g(x))d\theta dt\end{aligned} \tag{3.6}$$

Before stating and proving the main results, we introduce the following conditions.

(H_1) $T(t)(t>0)$ is continuous in the uniform operator topology for $t>0$.

(H_2) for almost all $t \in J$, the function $f(t,\cdot): X \to X$ is continuous and for each $z \in X$, the function $f(\cdot, z): J \to X$ is strongly measurable.

(H_3) there exists a constant $L>0$ such that

$$\|f(t,x)\| \leq L(1+t^{1-\alpha}\|x\|) \text{for all } t \in J, \text{and all } x \in C_{1-\alpha}(J,X).$$

(H_4) $g: C_{1-\alpha}(J,X) \to X$ is continuous and there exists a constant $L_g>0$ such that

$$\|g(x)-g(y)\| \leq L_g\|x-y\|_\alpha, \text{for all } x,y \in C_{1-\alpha}(J,X).$$

With

$$1-\frac{M}{\Gamma(\alpha)}L_g+\frac{Mb}{\Gamma(\alpha+1)}L>0.$$

Theorem 3.3 *Assume that* $(H_1)-(H_4)$ *hold, and*

$$\frac{M}{\Gamma(\alpha)}L_g<1.$$

Then the problem (3.1) *has at least one mild solution.*

Proof. Define an operator F on $C_{1-\alpha}(J,X)$ by

$$(Fx)(t)=t^{\alpha-1}T_\alpha(t)(x_0-g(x))+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)f(s,x(s))ds,\ t \in J.$$

For brevity, let

$$M_1=\frac{M}{\Gamma(\alpha)}L_g+\frac{Mb}{\Gamma(\alpha+1)}L,$$

$$M_2=\frac{M}{\Gamma(\alpha)}(\|x_0\|+\|g(0)\|)+\frac{Mb}{\Gamma(\alpha+1)}L.$$

For any $x \in C_{1-\alpha}(J,X)$, let $(Fx)(t)=(F_1x)(t)+(F_2x)(t)$, where

$$(F_1x)(t)=\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)f(s,x(s))ds,\ t \in J,$$

$(F_2x)(t)=t^{\alpha-1}T_\alpha(t)(x_0-g(x)),\ t \in J$. Assume that $M_1<1$, and let

$B_r=\{x \in C_{1-\alpha}(J,X): \|x\|_\alpha \leq r\}$, where $r \geq \dfrac{M_2}{1-M_1}$. We divide the proof inton sequence steps.

step1. For any $x \in B_r$, we prove that $Fx=F_1x+F_2x \in B_r$.

$$\begin{aligned}
&t^{1-\alpha}\|(F_1x+F_2x)(t)\| \\
&\quad = t^{1-\alpha}\|t^{\alpha-1}T_\alpha(t)(x_0-g(x))+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)f(s,x(s))ds\| \\
&\quad \leq \frac{M}{\Gamma(\alpha)}(\|x_0\|+\|g(x)\|)+\frac{Mt^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x(s))ds\| \\
&\quad \leq \frac{M}{\Gamma(\alpha)}(\|x_0\|+L_g\|x\|+\|g(0)\|)+\frac{Mt^{1-\alpha}}{\Gamma(\alpha}\int_0^t(t-s)^{\alpha-1}L(1+s^{1-\alpha}\|x(s)\|) \\
&\quad \leq \frac{M}{\Gamma(\alpha)}(\|x_0\|+L_gr+\|g(0)\|)+\frac{Mb}{\Gamma(\alpha+1)}L(1+r).
\end{aligned}$$

Hence $Fx = F_1x + F_2x \in B_r$.
step2. F_2 is contraction on B_r.
For $x, y \in B_r$ and for $t \in J$, we have

$$
\begin{aligned}
t^{1-\alpha}\|(F_2x)(t) - (F_2y)(t)\| &= \|T_\alpha(t)(g(x) - g(y))\| \\
\|(F_2x)(t) - (F_2y)(t)\|_\alpha &\leq \frac{M}{\Gamma(\alpha)} L_g \|x - y\|_\alpha,
\end{aligned}
$$

which implies that F_2 is a contraction.
step3. We show that F_1 is continuous.

Let $\{x_n\}$ be sequence such that $x_n \to x$ in $C_{1-\alpha}(J, X)$. For each $t \in J$, we have

$$
\begin{aligned}
& t^{1-\alpha}\|(F_1x_n)(t) - (F_1x)(t)\| \\
&= t^{1-\alpha}\left\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s, x_n(s))ds - \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s, x(s))ds\right\| \\
&\quad \leq \frac{Mt^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}s^{1-\alpha}\|f(s, x_n(s)) - f(s, x(s))\|ds \\
&\quad \leq \frac{Mb^\alpha}{\Gamma(\alpha)} B(\alpha, \alpha)\|f(\cdot, x_n(\cdot)) - f(\cdot, x(\cdot))\|_\alpha.
\end{aligned}
$$

Using the hypothesis (H_1) and (H_3) we have

$$\|F_1x_n - F_1x\|_\alpha \to 0 \text{ as } n \to +\infty.$$

Thus F_1 is continuous.

step4. F_1 maps bounbed sets into bounded sets in $C_{1-\alpha}(J, X)$.
Indeed, it is enough to show that there exists positive constant l such that for each $x \in B_r$ one has $\|F_1x\| \leq l$.

$$
\begin{aligned}
t^{1-\alpha}\|(F_1x)(t)\| &= t^{1-\alpha}\left\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s, x(s))ds\right\| \\
&\leq \frac{Mt^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(t, x(s))\|ds \\
&\leq \frac{MLb}{\Gamma(\alpha+1)}(1+r) := l.
\end{aligned}
$$

step5. F_1 maps bounded sets into equicontinuous sets.

Let $x \in B_r$ and $t_1, t_2 \in J$ with $0 < t_1 < t_2 \leq b$, If we take $t_1, t_2 \in J'$ such that $0 < t_1 \leq t_2$, then for arbitrary f_n we will have

$$\begin{aligned}
&\|t_2^{1-\alpha} F_1(x)(t_2)) - t_1^{1-\alpha} F_1(x)(t_1))\| \\
&= \left\| t_2^{1-\alpha} \int_0^{t_2} (t_2 - s)^{\alpha-1} T_\alpha(t_2 - s) f(s, x(s)) ds \right. \\
&\quad \left. - t_1^{1-\alpha} \int_0^{t_1} (t_1 - s)^{\alpha-1} T_\alpha(t_1 - s) f(s, x(s)) ds \right\| \\
&\leq \left\| t_2^{1-\alpha} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} T_\alpha(t_2 - s) f(s, x(s)) ds \right\| \\
&+ \left\| \int_0^{t_1} \left[t_1^{1-\alpha}(t_1 - s)^{\alpha-1} - t_2^{1-\alpha}(t_2 - s)^{\alpha-1} \right] T_\alpha(t_2 - s) f(s, x(s)) ds \right\| \\
&+ \left\| t_1^{1-\alpha} \int_0^{t_1} (t_1 - s)^{\alpha-1} \left[T_\alpha(t_2 - s) - T_\alpha(t_1 - s) \right] f(s, x(s)) ds \right\| \\
&\leq \frac{Mb^{1-\alpha}}{\Gamma(\alpha)} L(1+r) \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds \\
&\quad + \frac{M}{\Gamma(\alpha)} L(1+r) \int_0^{t_1} \left[t_1^{1-\alpha}(t_1 - s)^{\alpha-1} - t_2^{1-\alpha}(t_2 - s)^{\alpha-1} \right] \\
&+ \| \int_0^{t_1-\epsilon} t_1^{1-\alpha}(t_1 - s)^{\alpha-1} \left[T_\alpha(t_2 - s) - T_\alpha(t_1 - s) \right] f(s, x(s)) ds \| \\
&+ \| \int_{t_1-\epsilon}^{t_1} t_1^{1-\alpha}(t_1 - s)^{\alpha-1} \left[T_\alpha(t_2 - s) - T_\alpha(t_1 - s) \right] f(s, x(s)) ds \| \\
&\leq I_1 + I_2 + I_3 + I_4
\end{aligned}$$

where

$$I_1 = \frac{Mb^{1-\alpha}}{\Gamma(\alpha)} L(1+r) \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds,$$

$$I_2 = \frac{M}{\Gamma(\alpha+1)} L(1+r) \left[(t_2 - t_1) + (t_2 - t_1)^\alpha \right],$$

$$I_3 = \sup_{s \in [0, t_1 - \epsilon]} \|T_\alpha(t_2 - s) - T_\alpha(t_1 - s)\| \left[\frac{bL(1+r)}{\alpha} \right],$$

$$I_4 = \frac{2ML(1+r)b^{1-\alpha}}{\Gamma(\alpha)} \int_{t_1-\epsilon}^{t_1} (t_1 - s)^{\alpha-1} ds.$$

Applying the absolute continuity of the Lebesgue integral we have I_1, I_2, I_4 tend to zero independently of $x \in \Omega$ as $t_2 \to t_1$. The continuity of $(T_\alpha(t), t \geq 0)$ in t in the uniform operator topology, it is easy to see that I_3 tends to zero independently of $x \in \Omega$ as $t_2 \to t_1$. Since the set $F_1(B_r)$ is equicontinuous. As a consequence of the Arzela-Ascoli Theorem, we can conclude that $F_1 : B_r \to B_r$, is completely continuous. An employment of Krasnoselskii's fixed point Theorem shows that there exists at least a fixed point x of $F = F_1 + F_2$, which is a mild solution to problem (3.26).

The following result is based on Mönch's fixed point Theorem combining with the measure of noncompactness.

Let $L^p(J,X)$ the space of X-valued Bochner integrable functions on J with the norm

$$\|f\|_{L^p} = \left(\int_J |f|(t)|^p dt\right)^{\frac{1}{p}}.$$

Lemma 3.4 *[37] Let a sequence $\{\eta_n\} \subset L^{\frac{1}{p}}(J,X)$, where $p \in (0,\alpha)$, be bounded and $\eta_n \rightharpoonup \eta$ in $L^1(J,X)$. Then $S(\eta_n) \rightharpoonup S(\eta)$ in $C(J,X)$.*

Consider the map defined by

$$S : L^\infty(J,X) \to C_{1-\alpha}(J,X)$$

$$S(\phi)(t) = \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)\phi(s)ds.$$

Lemma 3.5 *The operator S have the following properties:*

(i) If $\frac{1}{\alpha} < p < \infty$, then there exists a constant $C > 0$ such that

$$\|S(\xi)(t) - S(\eta)(t)\|^p \leq C^p \int_0^t \|\xi(s) - \eta(s)\|^p ds \quad \forall \xi, \eta \in L^p(J,X).$$

(ii) For each compact set $K \subset X$ and sequence $\{\eta_n\} \subset L^\infty(J,X)$ such that $\eta_n \subset K$ for a.e $t \in J$, the weak convergence $\eta_n \rightharpoonup \eta$ in $L^1(J,X)$ implies the convergence $S(\eta_n) \to S(\eta)$ in $C_{1-\alpha}(J,X)$.

Proof. (i) By using the Hölder inequality, we get:

$$\begin{aligned} & \|S(\xi)(t) - S(\eta)(t)\|_X \\ \leq & \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)\|(\xi(s) - \eta(s))\|_X ds \\ \leq & \frac{M}{\Gamma(\alpha)} \left[\int_0^t (t-s)^{\frac{(\alpha-1)p}{p-1}} ds\right]^{\frac{p-1}{p}} \left[\int_0^t \|\xi(s) - \eta(s)\|_X^p ds\right]^{\frac{1}{p}}. \end{aligned}$$

Then

$$\|S(\xi)(t) - S(\eta)(t)\|_X^p \leq C^p \int_0^t \|\xi(s) - \eta(s)\|_X^p ds,$$

where

$$C = \left[\frac{p-1}{\alpha p - 1}\right]^{\frac{p-1}{p}} \frac{M b^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)}.$$

(ii) Applying Lemma 1.33 and Lemma 1.42, we obtain:

$$\begin{aligned} (\{S(\eta_n)(t)\}) &\leq \int_0^t (t-s)^{\alpha-1} \chi(\{T_\alpha(t-s)\eta_n\}) ds \\ &\leq \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \chi(\{\eta_n\}) ds = 0. \end{aligned}$$

This means that sequence $\{S(\eta_n)(t)\}_{n=1}^{\infty} \subset X$ is relatively compact for each $t \in J$.
From the other side, if we take $t_1, t_2 \in J'$ such that $0 < t_1 \leq t_2$ then we have

$$\begin{aligned}
&\|t_2^{1-\alpha} S(\eta_n(t_2)) - t_1^{1-\alpha} S(\eta_n(t_1))\| \\
= &\left\| t_2^{1-\alpha} \int_0^{t_2} (t_2-s)^{\alpha-1} T_\alpha(t_2-s)\eta_n(s)ds - t_1^{1-\alpha} \int_0^{t_1} (t_1-s)^{\alpha-1} T_\alpha(t_1-s)\eta_n(s)ds \right\| \\
\leq &\left\| t_2^{1-\alpha} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} T_\alpha(t_2-s)\eta_n(s)ds \right\| \\
&+ \left\| \int_0^{t_1} \left[t_1^{1-\alpha}(t_1-s)^{\alpha-1} - t_2^{1-\alpha}(t_2-s)^{\alpha-1} \right] T_\alpha(t_2-s)\eta_n(s)ds \right\| \\
&+ \left\| t_1^{1-\alpha} \int_0^{t_1} (t_1-s)^{\alpha-1} \left[T_\alpha(t_2-s) - T_\alpha(t_1-s) \right] \eta_n(s)ds \right\| \\
\leq &\ Z_1 + Z_2 + Z_3,
\end{aligned}$$

where

$$Z_1 = \left\| t_2^{1-\alpha} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} T_\alpha(t_2-s)\eta_n(s)ds \right\|,$$

$$Z_2 = \left\| \int_0^{t_1} \left[t_1^{1-\alpha}(t_1-s)^{\alpha-1} - t_2^{1-\alpha}(t_2-s)^{\alpha-1} \right] T_\alpha(t_2-s)\eta_n(s)ds \right\|,$$

$$Z_3 = \left\| t_1^{1-\alpha} \int_0^{t_1} (t_1-s)^{\alpha-1} \left[T_\alpha(t_2-s) - T_\alpha(t_1-s) \right] \eta_n(s)ds \right\|.$$

By using Lemma 1.42 and condition (H_4), for each $\epsilon_1 > 0$, we can choose $\delta_1 > 0$ such that $|t_2 - t_1| < \delta_1$ implies the following estimate:

$$Z_1 \leq \frac{M\|\omega_K\|_\infty b^{1-\alpha}(t_2-t_1)}{\Gamma(\alpha+1)} < \epsilon_1.$$

To estimate Z_2, take constant $d > 0$, for which we have

$$\begin{aligned}
Z_2 \leq &\left\| \int_0^{t_1-d} \left[t_1^{1-\alpha}(t_1-s)^{\alpha-1} - t_2^{1-\alpha}(t_2-s)^{\alpha-1} \right] T_\alpha(t_2-s)\eta_n(s)ds \right\| \\
&+ \left\| \int_{t_1-d}^{t_1} \left[t_1^{1-\alpha}(t_1-s)^{\alpha-1} - t_2^{1-\alpha}(t_2-s)^{\alpha-1} \right] T_\alpha(t_2-s)\eta_n(s)ds \right\|
\end{aligned}$$

where

$$I_1 = \left\| \int_0^{t_1-d} \left[t_1^{1-\alpha}(t_1-s)^{\alpha-1} - t_2^{1-\alpha}(t_2-s)^{\alpha-1} \right] T_\alpha(t_2-s)\eta_n(s)ds \right\|,$$

$$I_2 = \left\| \int_{t_1-d}^{t_1} \left[t_1^{1-\alpha}(t_1-s)^{\alpha-1} - t_2^{1-\alpha}(t_2-s)^{\alpha-1} \right] T_\alpha(t_2-s)\eta_n(s)ds \right\|.$$

Consider the function $u : [d, b] \to \mathbb{R}$, $u(\tau) = \tau^{\alpha-1}(\tau - s)^{1-\alpha}$, where $\tau = t - s$ the given function is continuous on the interval $[d, b]$, hence by the Dini theorem, it is uniformly continuous on this interval, i.e., for each $\zeta > 0$ there exists δ_2, such that $|\tau_2 - \tau_1| < \delta_2 < d$, $\tau_1, \tau_2 \in [d, b]$ implies

$$|u(\tau_2) - u(\tau_1)| < \zeta,$$

we get:

$$I_1 \le \frac{M\|\omega_K\|_\infty \zeta(t_1 - d)}{\Gamma(\alpha)} < \epsilon_2.$$

By direct integration, for I_2 we obtain:

$$\frac{M\|\omega_K\|_\infty t_2^{1-\alpha} d^\alpha (1 + 2^\alpha)}{\Gamma(\alpha + 1)} < \epsilon_3.$$

The family of operators $T_\alpha(t), t > 0$ is strongly continuous for $x \in K$, i.e., for each $\zeta_1 > 0$ there exists $\delta_3 > 0$ such that $|t_2 - t_1| < \delta_3$ implies

$$\|T_\alpha(t_2 - s)x - T_\alpha(t_1 - s)x\| < \zeta_1, \ \{\eta_n(t)\} \in K,$$

we get the following estimate:

$$Z_3 \le \zeta_1 b < \epsilon_4.$$

Therefore, for each $\epsilon > 0$ we may choose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ such that

$$\|S(\eta_n(t_2)) - S(\eta_n(t_1))\| \le Z_1 + Z_2 + Z_3 < \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 < \epsilon.$$

So, the sequence $\{S(\eta_n)\}$ is equicontinuous. From Lemma 2.1, we conclude that the sequence $\{S(\eta_n)\} \subset C_{1-\alpha}(J, X)$ is relatively compact. From Lemma 3.4, we know that the weak convergence $\eta_n \rightharpoonup \eta$ implies $S(\eta_n) \rightharpoonup S(\eta)$. Since the sequence $\{S(\eta_n)\}$ is relatively compact, so that $S(\eta_n) \to S(\eta)$ in $C_{1-\alpha}(J, X)$.

For the forthcoming analysis, we need the following hypothesis.

(G_1) there exists a function $\zeta \in L^\infty_+(J)$ such that

$$\|f(t, x(t))\| \le \zeta(t)(1 + t^{1-\alpha}\|x\|) \text{ for all } t \in J \text{ and all } x \in X.$$

(G_2) there exists a function $\mu \in L^\infty_+(J)$ such that for every nonempty, bounded set $\Omega \subset X$ we have

$$\chi(f(t, x(t)) \le \mu(t)\chi(\Omega),$$

for a.e. $t \in J$, where χ is the Hausdorff MNC in X.

(G_3) there exists a constant $C_g > 0$ such that

$$\chi(g(\Omega)) \le C_g \chi(\Omega_\alpha), \text{ for all } \Omega \subset C_{1-\alpha}(J, X).$$

(G_4) if $\Omega \subset C_{1-\alpha}(J, X)$ is a bounded set, then

$$mod_{C_{1-\alpha}} T_\alpha(\cdot) g(\Omega) = 0.$$

Theorem 3.6 *Assume that hypotheses* $(G_1) - (G_4)$, (H_1)-(H_2) *and* (H_4) *holds, if* $M_1 < 1$. *Then the nonlocal problem* (3.1) *has at least one mild solution.*

Proof. Consider the operator

$$N : C_{1-\alpha}(J, X) \to C_{1-\alpha}(J, X),$$

defined by

$$N(x)(t) = t^{\alpha-1} T_\alpha(t)(x_0 - g(x)) + \int_0^t (t - s)^{\alpha-1} T_\alpha(t - s) f(s, x(s)) ds. \tag{3.7}$$

We break the proof into a sequence of steps.
step1. N is a continuous.
It is easily prove the continuity of N, because the same reasoning as in the previous result concerning the continuous of F_1.

step2. N is a ν-condensing.
We consider the measure of noncompactness defined in the following way. For every bounded subset $\Omega \subset C_{1-\alpha}(J,X)$

$$\nu(\Omega) = \max_{\Omega \in \Delta(\Omega)} (\gamma(\Omega), mod_{C_{1-\alpha}}(\Omega)), \tag{3.8}$$

$\Delta(\Omega)$ is the collection of all countable subsets of Ω and maximum is taken in the sense of the partial order in the cone $\mathbb{R}^2_+$. γ is the damped modulus of fiber noncompactness

$$\gamma(\Omega) = \sup_{t \in [0,b]} e^{-Lt} \chi(\Omega_\alpha(t)); \tag{3.9}$$

where

$$mod_{C_{1-\alpha}}(\Omega) = \lim_{\delta \to 0} \sup_{x \in \Omega} \max_{|t_1 - t_2| \le \delta} \|x_\alpha(t_1) - x_\alpha(t_2)\|. \tag{3.10}$$

We can choose L such that

$$q := \sup_{t \in J} \left(2 \frac{b^{1-\alpha} M \|\mu\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-L(t-s)} ds + \frac{M}{\Gamma(\alpha)} C_g \right) < 1. \tag{3.11}$$

From the Arzela-Ascoli Theorem, the measure ν given a nonsingular and regular measure of noncompactness in $C_{1-\alpha}(J,X)$.
Let $\Omega \subset C_{1-\alpha}(J,X)$ be a bounded subset such that

$$\nu(N(\Omega)) \ge \nu(\Omega). \tag{3.12}$$

We will show that (3.12) implies that Ω is relatively compact.
Let the maximum on the left-hand side of the inequality (3.12) be achieved for the countable set $\{y^n\}_{n=1}^{+\infty}$ with

$$y^n(t) = t^{\alpha-1} T_\alpha(t)[x_0 - g(x^n)] + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, x^n(s)) ds, \ \{x^n\}_{n=1}^{+\infty} \subset \Omega. \tag{3.13}$$

We define the operators

$$\Upsilon : L^1(J,X) \to C_{1-\alpha}(J,X),$$

by

$$\Upsilon(f_n)(t) = \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, x^n(s)) ds, \tag{3.14}$$

and

$$\Upsilon^*(x)(t) = t^{\alpha-1} T_\alpha(t)[x_0 - g(x)]. \tag{3.15}$$

From the constraction of Υ, Υ^*, we have

$$y^n = \Upsilon^*(x^n) + \Upsilon(f_n), \tag{3.16}$$

where

$$\Upsilon^*(x^n)(t) = t^{\alpha-1} T_\alpha(t)[x_0 - g(x^n)], \tag{3.17}$$

with

$$\Upsilon(f_n)(t) = \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f_n(s) ds \text{ and } f_n(t) = f(t, x^n(t)).$$

We give an upper estimate for $\gamma(\{y^n\}_{n=1}^{+\infty})$. By using (G_2), we have

$$\begin{aligned}
\chi(\{(t-s)^{\alpha-1} f_n(s)\}_{n=1}^{+\infty}) &\le (t-s)^{\alpha-1} \mu(s) \chi(\{x^n(s)\}_{n=1}^{+\infty}) \\
&\le \mu(s)(t-s)^{\alpha-1} s^{\alpha-1} s^{1-\alpha} \chi(\{x^n(s)\}_{n=1}^{+\infty}) \\
&= \mu(s)(t-s)^{\alpha-1} s^{\alpha-1} \chi(\{x^n_\alpha(s)\}_{n=1}^{+\infty}) \\
&\le \mu(s)(t-s)^{\alpha-1} s^{\alpha-1} e^{Ls} \sup_{0 \le s \le t} e^{-Ls} \chi(\{x^n_\alpha(s)\}_{n=1}^{+\infty}) \\
&\le \mu(s)(t-s)^{\alpha-1} s^{\alpha-1} e^{Ls} \gamma(\{x^n\}_{n=1}^{+\infty}),
\end{aligned} \tag{3.18}$$

for all $t \in J$, $s \leq t$. Then appling Lemma 1.55 end (3.8) we obtain

$$\chi(\{\Upsilon f_n(t)\}_{n=1}^{+\infty}) \leq 2\frac{M\|\mu\|_\infty}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}e^{Ls}ds\right)\gamma(\{x^n\}_{n=1}^{+\infty}). \tag{3.19}$$

Using (G_3), we have

$$\begin{aligned}\chi(\{\Upsilon^* x^n(s)\}_{n=1}^{+\infty}) &= \chi(t^{\alpha_1}T_\alpha(t)(x_0 - g(x^n)\}_{n=1}^{+\infty}) \\ &\leq \chi(t^{\alpha-1}T_\alpha(t)\{g(x^n)\}_{n=1}^{+\infty}) \\ &\leq \frac{Mt^{\alpha-1}}{\Gamma(\alpha)}C_g\chi(\{x^n\}_{n=1}^{+\infty}) \\ &\leq \frac{Mt^{\alpha-1}}{\Gamma(\alpha)}C_g e^{Lt}\gamma(\{x^n\}_{n=1}^{+\infty}).\end{aligned} \tag{3.20}$$

Thus, we get from (3.16), (3.19) and (3.37), we obtain that

$$\chi(\{y^n\}_{n=1}^{+\infty}) \leq \chi(\{\Upsilon f_n\}_{n=1}^{+\infty}) + \chi(\{\Upsilon^* x^n\}_{n=1}^{+\infty})$$

Hence

$$\begin{aligned}t^{1-\alpha}\chi(\{y^n\}_{n=1}^{+\infty}) &\leq 2\frac{Mb^{1-\alpha}\|\mu\|_\infty}{\Gamma(\alpha)}\sup_{t\in J}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}e^{Ls}ds\gamma(\{x^n\}_{n=1}^{+\infty}) \\ &+\frac{M}{\Gamma(\alpha)}C_g e^{Lt}\gamma(\{x^n\}_{n=1}^{+\infty}).\end{aligned}$$

Taking by (3.16),(3.11) into account, we derive

$$\gamma(\{y^n\}_{n=1}^{+\infty}) \leq q\gamma(\{x^n\}_{n=1}^{+\infty}). \tag{3.21}$$

Combining the last inequality with (3.12), we have

$$\gamma(\{x^n\}_{n=1}^{+\infty}) \leq q\gamma(\{x^n\}_{n=1}^{+\infty}). \tag{3.22}$$

Therefore

$$\gamma(\{x^n\}_{n=1}^{+\infty}) = 0. \tag{3.23}$$

Hence by (3.21), we get

$$\gamma(\{y^n\}_{n=1}^{+\infty}) = 0. \tag{3.24}$$

By using (3.8),(3.23) and assumption (G_2) we can prove that set $\{f_n\}_{n=1}^{+\infty}$ is semicompact. Now, by appling Lemma (3.7), we can conclude that the set $\{\Upsilon f_n\}_{n=1}^{+\infty}$ is relatively compact in $C_{1-\alpha}(J, X)$.
We know that $mod_{C_{1-\alpha}}(N(\Omega)) = 0$ and (3.12) yields $mod_{C_{1-\alpha}}(\Omega) = 0$. Finally

$$\nu(\Omega) = (0, 0),$$

which prove the relative compactness of the set Ω.
Let $\Omega \subset \overline{U}$ be countable, bounded and $\Omega \subseteq \overline{co}(\{0\} \cup N(\Omega))$. Since ν is a monotone nonsingular, regular MNC, one has

$$\nu(\Omega) \leq \nu(\overline{co}(\{0\} \cup N(\Omega))) \leq \nu(N(\Omega)).$$

Therefore $\nu(\Omega) = (0, 0)$, then Ω is relatively compact set.
step5. A priori bounds.

We will demonstrate that the solutions set is a priori bounded. Indeed, let $x = \lambda Nx$ and $\lambda \in (0,1)$. For every $t \in J$, we have

$$\begin{aligned}
&t^{1-\alpha}\|x(t)\| \le \frac{M}{\Gamma(\alpha)}\|x_0 - g(x)\| + \frac{Mt^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,x(s))\|ds \\
&\le \frac{M}{\Gamma(\alpha)}(\|x_0\| + L_g\|x\|_\alpha + \|g(0)\|) + \frac{Mt^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\zeta(s)(1+s^{1-\alpha}\|x(s)\|))ds \\
&\le \frac{M}{\Gamma(\alpha)}(\|x_0\| + L_g\|x\| + \|g(0)\|) + \frac{Mb\|\zeta\|_\infty}{\Gamma(\alpha+1)} + \frac{Mb^{1-\alpha}\|\zeta\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|x(s)\|_\alpha ds.
\end{aligned}$$

From Lemma 1.8, there exists $K(\alpha) > 0$ such that

$$\|x(t)\|_\alpha \le L + \frac{Mb^{1-\alpha}\|\zeta\|_\infty K(\alpha)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Lds,$$

where

$$L := \frac{M}{\Gamma(\alpha)}(\|x_0\| + L_g\|x\| + \|g(0)\|) + \frac{Mb\|\zeta\|_\infty}{\Gamma(\alpha+1)}.$$

Then

$$\|x\|_\alpha \le L + \frac{MLb\|\zeta\|_\infty K(\alpha)}{\Gamma(\alpha+1)} =: \overline{M},$$

yielding the desired a priori boundedness. So, there exists $M^* > 0$ such that $\|x\|_\alpha \neq M^*$. Set $U = \{x \in \Omega : \|x\|_\alpha < M^*\}$.
From the choice of U there is no $x \in \partial U$ such that $x = \lambda Nx$ for some $\lambda \in (0,1)$.
Thus, we get a fixed point of N in $\overline{U}$ due to the Mönch's Theorem.

3.1.2 An example

let $X = L^2([0,\pi],\mathbb{R})$As an application of our results we consider the following partial differential equation with nonlocal conditions of the form:

$$\begin{cases}
{}^L D^\alpha_{0^+} z(t,x) = \dfrac{\partial^2}{\partial x^2} z(t,x) + \varphi(t)\sin z(t,x), \quad x \in [0,\pi],\ t \in J := [0,1], \\
z(t,0) = z(t,\pi) = 0, \quad t \in (0,1], \\
I^{1-\alpha}_{0^+} z(t,x) + \sum\limits_{i=1}^m c_i z(t_i,x) = z_0(x), \quad x \in [0,\pi],\ t_i \in (0,1),\ i = 1,\cdots,m.
\end{cases} \tag{3.25}$$

Where $x(t) = z(t,\cdot)$, that is, $z(t,x) = z(t)(x)$. Define the operator A by $Az = z''$, with the domain, $D(A) = \{v(\cdot) \in X,\ v, v' \textit{ are absolutely continuous}, v'' \in X,\ z(t,0) = z(t,\pi) = 0\}$. Then A can be written as

$$Az = \sum_{n=1}^\infty (-n^2) < z, e_n > e_n,\ z \in D(A),$$

where $e_n(x) = \sqrt{\frac{2}{\pi}}\sin nx$, $0 \le x \le \pi$, $n = 1, 2, \cdot\cdot$, is an orthonormal basis of X. It is well know that A is the infinitesimal generator of a strongly continuous semigroup $T(t)$ on X, give by

$$T(t)z = \sum_{n=1}^\infty e^{-n^2 t} < x, e_n > e_n, z \in X, \textit{and}\ \|T(t)\| \le 1 = M,\ t \ge 0.$$

To write system (3.25) in the form we define $f : J \times X \longrightarrow X$, $g : C_{1-\alpha}(J,X) \longrightarrow X$ defined by

$$f(t,x(t)) = \varphi(t)\sin z(t,x),$$

$$g(z(t,x)) = \sum_{i=1}^{m} c_i z(t_i, x),$$

note that f is Carathéodory function which yields condition (H_2),

$$L = \sup_{t\in J} |\varphi(t)|, \quad L_g = \sum_{i=1}^{m} |c_i|,$$

and choose c_i such that

$$\frac{1}{\Gamma(\alpha)} L_g < 1, \quad 1 - \left(\frac{L_g}{\Gamma(\alpha)} + \frac{L}{\Gamma(\alpha+1)} \right) > 0.$$

An easy computation allow us to verify condition (H_3), and from the choose of $\{c_i\}_{i=1}^{n}$ it follows condition (H_4). Since the condition $(G_1) - (G_4)$ of the Theorem 3.3 are satisfied, the problem (3.25) has at least one mild solution.

3.2 Fractional evolution inclusion with nonlocal conditions

The aim of this work is to study the existence of mild solution for semilinear fractional order differential inclusions with nonlocal conditions involving Riemann-Liouville derivative in Banach space.

3.2.1 Existence results

study the existence for Riemann-Liouville fractional evolution inclusions with nonlocal conditions

$$\begin{cases} {}^{L}D_{0^+}^{\alpha} x(t) \in Ax(t) + F(t, x(t)); \quad \text{a.e. } t \in (0, b]; \ 0 < \alpha < 1, \\ I_{0^+}^{1-\alpha} x(t) \mid_{t=0} + g(x) = x_0 \in X, \end{cases} \tag{3.26}$$

where ${}^{L}D^{\alpha}$ is the Riemann-Liouville fractional derivative of order $0 < \alpha < 1$, $I_{0^+}^{1-\alpha}$ is Riemann-Liouville integral of order $1-\alpha$,and $g : C_{1-\alpha}([0, b], X) \to X$ are given functions, $A : D(A) \subseteq X \to X$ is a generator of a C_0-semigroup $\{T(t), t \geq 0\}$ on a Banach space X. $F : [0, b] \times X \to \mathcal{P}(X) := 2^X \backslash \{\emptyset\}$ is a multivalued map satisfying some assumptions.

Let us mention that the fractional evolution inclusion of the type (3.26) was investigated by Huang et al [31] in the case when A generates a compact semigroup. The principal goal of this paper is to extend such results to the case when the semigroup generated by A is noncompact. Our approach is employing the fixed point theory technique for multivalued condensing maps under compactness type conditions on the nonlinearity term. We define in the sequel a suitable measure of noncompactness in the weighted space of continuous functions, prove that the solution multioperator is condensing with respect to this measure of noncompactness.

Let a multimap $F : [0, b] \times X \to \mathcal{P}_{cp,cv}(X)$ be such that:

(H_1) A is a defined linear operator in X, generating an equicontinuous semigroup $\{T(t)\}_{t\geq 0}$.

(H_2) For each $x \in X$ the multifunction $F(\cdot, x) : J \to \mathcal{P}_{,}(X)$ is measurable.

(H_3) For a.e. $t \in J$ the multimap $F(t, \cdot) : X \to \mathcal{P}_{,}(X)$ is u.s.c.

(H_4) For each $r > 0$ there exists a function $\omega_r \in L^{\infty}(J, \mathbb{R}_+)$ such that for each $x \in X$ with $\|x\| \leq r$, we have

$$\|F(t, x)\| \leq \omega_r(t)$$

for a.e $t \in J$.

(H_5) There exists a function $\mu \in L^\infty(J, \mathbb{R}_+)$ such that for each bounded set $\Omega \subset X$ we have

$$\chi(F(t,\Omega)) \leq \mu(t)\chi(\Omega),$$

for a.e. $t \in J$, where χ is the Hausdorff MNC in X.

(H_6) $g : C_{1-\alpha}(J,X) \to X$ is a continuous function and there exists a constant $g_1 > 0$ such that

$$\|g(x)\|_X \leq g_1(1 + t^{1-\alpha}\|x\|), \text{ for all } \Omega \subset C_{1-\alpha}(J,X).$$

(H_7) there exists a constant $C_g > 0$ such that

$$\chi(g(\Omega)) \leq C_g \chi(\Omega_\alpha), \text{ for all } \Omega \subset C_{1-\alpha}(J,X).$$

(H_8) if $\Omega \subset C_{1-\alpha}(J,X)$ is a bounded set, then

$$mod_{C_{1-\alpha}} T_\alpha(\cdot) g(\Omega) = 0.$$

Before stating and proving the main results, we introduce the following assumptions : For $x \in C_{1-\alpha}(J,X)$, consider the multifunction:

$$\Phi_F : J \to \mathcal{P}_{cp,cv}(X), \quad \Phi_F(t) = F(t, x(t)).$$

To solve our problem we will use the superposition multioperator $P_F^\infty : C_{1-\alpha}(J,X) \to \mathcal{P}(L^\infty(J,X))$ defined in the following way

$$P_F^\infty(x) = S_{\Phi_F}^\infty.$$

To search for mild solutions of problem (3.26), consider the map

$$S : L^\infty(J,X) \to C_{1-\alpha}(J,X)$$

$$S(\phi)(t) = \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)\phi(s)ds$$

and the function $g_\alpha(x)(t) = t^{\alpha-1}T_\alpha(t)[x_0 - g(x)]$ for each $(t,x) \in J' \times X$. Consider the multioperator $G : C_{1-\alpha}(J,X) \to \mathcal{P}(C_{1-\alpha}(J,X))$, given in the following way

$$G(x) = g_\alpha(x) + S \circ P_F^\infty(x), \quad t \in J'.$$

Since F has convex values, so does P_F^∞. This implies that G has convex values as well. On the other hand, x is a mild solution of (3.26) if it is a fixed point of the solution operator G.

Lemma 3.7 *The operator S have the following properties:*

(i) If $p \in (0,\alpha)$, then there exists a constant $C > 0$ such that

$$\|S(\xi)(t) - S(\eta)(t)\|^{\frac{1}{p}} \leq C^{\frac{1}{p}} \int_0^t \|\xi(s) - \eta(s)\|^{\frac{1}{p}} ds \ \ \forall \xi, \eta \in L^{\frac{1}{p}}(J,X).$$

(ii) For each compact set $K \subset X$ and sequence $\{\eta_n\} \subset L^\infty(J,X)$ such that $\eta_n \subset K$ for a.e $t \in J$, the weak convergence $\eta_n \rightharpoonup \eta$ in $L^1(J,X)$ implies the convergence $S(\eta_n) \to S(\eta)$ in $C_{1-\alpha}(J,X)$.

Proof. (i) By using the Hölder inequality, we get:

$$\begin{aligned}
& t^{1-\alpha}\|S(\xi)(t)-S(\eta)(t)\| \\
\leq\ & t^{1-\alpha}\left\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)s^{\alpha-1}s^{1-\alpha}(\xi(s)-\eta(s))ds\right\| \\
\leq\ & \frac{Mt^{1-\alpha}}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\frac{\alpha-1}{1-p}}s^{\frac{\alpha-1}{1-p}}ds\right]^{1-p}\left[\int_0^t \|\xi(s)-\eta(s)\|_\alpha^{\frac{1}{p}}ds\right]^p.
\end{aligned}$$

Then

$$\|S(\xi)(t)-S(\eta)(t)\|_\alpha^{\frac{1}{p}} \leq C^{\frac{1}{p}}\int_0^t \|\xi(s)-\eta(s)\|_\alpha^{\frac{1}{p}}ds,$$

where

$$C=\left[B\left(\frac{\alpha-p}{1-p};\frac{\alpha-p}{1-p}\right)\right]^{1-p}\frac{Mb^{\alpha(1+p)}}{\Gamma(\alpha)}.$$

To prove that the multioperator G is condensing, define the vector measure of noncompactness in the space $C_{1-\alpha}(J,X)$

$$\nu:\mathcal{P}(C_{1-\alpha}(J,X))\to\mathbb{R}^2_+.$$

With the values in the cone $\mathbb{R}^2_+$ defined by

$$\nu(\Omega)=(\gamma(\Omega),mod_{C_{1-\alpha}}(\Omega)),$$

where $\Delta(\Omega)$ denotes the collection of all countable subsets of Ω. γ is the damped modulus of fiber noncompactness

$$\gamma(\Omega)=\sup_{t\in J}e^{-Lt}\chi(\Omega_\alpha(t)), \tag{3.27}$$

where $\Omega_\alpha(t)=\{x_\alpha(t):x\in\Omega\}$. mod $_{C_{1-\alpha}}$ is the modulus of equicontinuity of the set of function Ω given by the formula

$$\text{mod }_{C_{1-\alpha}}(\Omega)=\lim_{\delta\to 0}\sup_{x\in\Omega}\max_{|t_1-t_2|\leq\delta}\|x_\alpha(t_1)-x_\alpha(t_2)\|. \tag{3.28}$$

We can choose L such that

$$\overline{\sigma}=\sup_{t\in J}\left(2\frac{b^{1-\alpha}M\|\mu\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{-L(t-s)}ds+\frac{M}{\Gamma(\alpha)}C_g\right)<1. \tag{3.29}$$

Lemma 3.8 *The operator G is ν-condensing.*

Proof. Let $\Omega\subset C_{1-\alpha}(J,X)$ be a bounded subset such that

$$\nu(G(\Omega))\geq\nu(\Omega), \tag{3.30}$$

where the inequality is taken in the sense of the order in $\mathbb{R}^2$ induced by the cone $\mathbb{R}^2_+$.

We will show that Ω is a relatively compact in $C_{1-\alpha}(J,X)$. By definition of ν, there exists a sequence $\{y^n\}_{n=1}^{+\infty}\subset G(\Omega)$ such that

$$\nu(G(\Omega))=(\gamma(\{y^n\}_{n=1}^{+\infty}),mod_{C_{1-\alpha}}(\{y^n\}_{n=1}^{+\infty})). \tag{3.31}$$

with

$$y^n=g_\alpha(x^n)+S(f_n),\ \ \{x^n\}_{n=1}^{+\infty}\subset\Omega, \tag{3.32}$$

where

$$\begin{aligned}
& g_\alpha(x^n)(t)=t^{\alpha-1}T_\alpha(t)[x_0-g(x^n)],\ t\in J'; \\
& Sf_n(t)=\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f_n(s)ds \text{ and } f_n\in P_F^\infty(x^n).
\end{aligned} \tag{3.33}$$

We give now an upper estimate for $\gamma\{y^n\}_{n=1}^{+\infty}$. By using assumption (H_4), we have

$$\begin{aligned}
\chi(\{(t-s)^{\alpha-1}f_n(s)\}_{n=1}^{+\infty}) &\leq (t-s)^{\alpha-1}\mu(s)s^{\alpha-1}s^{1-\alpha}\chi(\{x^n(s)\}_{n=1}^{+\infty}) \\
&= (t-s)^{\alpha-1}\mu(s)s^{\alpha-1}\chi(\{x_\alpha^n(s)\}_{n=1}^{+\infty}) \\
&\leq \mu(s)(t-s)^{\alpha-1}e^{Ls}\sup_{0\leq s\leq t} e^{-Ls}\chi(\{x_\alpha^n(s)\}_{n=1}^{+\infty}) \\
&\leq \mu(s)(t-s)^{\alpha-1}s^{\alpha-1}e^{Ls}\gamma(\{x^n\}_{n=1}^{+\infty}),
\end{aligned} \tag{3.34}$$

for all $t\in J$, $s\leq t$. Then applying Lemma 1.55, we obtain

$$\chi(\{Sf_n(t)\}_{n=1}^{+\infty}) \leq \frac{2M\|\mu\|_\infty}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}e^{Ls}ds\right)\gamma(\{x^n\}_{n=1}^{+\infty}). \tag{3.35}$$

Noting that

$$g_\alpha(x^n)(t) = t^{\alpha-1}T_\alpha(t)x_0 - t^{\alpha-1}T_\alpha(t)g(x^n),\ t\in J'; \tag{3.36}$$

Using (H_7) and estimate (3.36), we have

$$\begin{aligned}
\chi(\{g_\alpha(x^n)(s)\}_{n=1}^{+\infty}) &= \chi(\{t^{\alpha-1}T_\alpha(t)x_0 - t^{\alpha-1}T_\alpha(t)g(x^n)\}_{n=1}^{+\infty}) \\
&\leq \chi(t^{\alpha-1}T_\alpha(t)\{g(x^n)\}_{n=1}^{+\infty}) \\
&\leq \frac{Mt^{\alpha-1}}{\Gamma(\alpha)}C_g\chi(\{x^n\}_{n=1}^{+\infty}) \\
&\leq \frac{Mt^{\alpha-1}}{\Gamma(\alpha)}C_g e^{Lt}\gamma(\{x^n\}_{n=1}^{+\infty}).
\end{aligned} \tag{3.37}$$

Combining (3.32),(3.35) and (3.37), we get

$$\gamma\left(\{y^n\}_{n=1}^{+\infty}\right) \leq \overline{\sigma}\gamma\left(\{x^n\}_{n=1}^{+\infty}\right). \tag{3.38}$$

Then inequality (3.30) implies that

$$\gamma\left(\{x^n\}_{n=1}^{+\infty}\right) \leq \overline{\sigma}\gamma\left(\{x^n\}_{n=1}^{+\infty}\right),$$

and therefore

$$\gamma\left(\{x^n\}_{n=1}^{+\infty}\right) = 0. \tag{3.39}$$

But then (3.38) implies

$$\gamma(\{y^n\}_{n=1}^{+\infty}) = 0. \tag{3.40}$$

Putting (3.39) together with (3.34), we obtain that set $\{f_n\}_{n=1}^{+\infty}$ is semicompact.

Now, prove that

$$\text{mod}_{\,C_{1-\alpha}}(\{Sf_n\}_{n=1}^{\infty}) = 0.$$

To do it, let us show that the set

$$\left\{\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f_n ds : f_n\in P_F^\infty(x_n)\right\}$$

is equicontinuous. If we take $t_1, t_2 \in J'$ such that $0 < t_1 \leq t_2$, then for arbitrary f_n we will have

$$
\begin{aligned}
& \|t_2^{1-\alpha} S(f_n(t_2)) - t_1^{1-\alpha} S(f_n(t_1))\| \\
= \; & \left\| t_2^{1-\alpha} \int_0^{t_2} (t_2 - s)^{\alpha-1} T_\alpha(t_2 - s) f_n(s) ds - t_1^{1-\alpha} \int_0^{t_1} (t_1 - s)^{\alpha-1} T_\alpha(t_1 - s) f_n(s) ds \right\| \\
\leq \; & \left\| t_2^{1-\alpha} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} T_\alpha(t_2 - s) f_n(s) ds \right\| \\
& + \left\| \int_0^{t_1} \left[t_1^{1-\alpha} (t_1 - s)^{\alpha-1} - t_2^{1-\alpha} (t_2 - s)^{\alpha-1} \right] T_\alpha(t_2 - s) f_n(s) ds \right\| \\
& + \left\| t_1^{1-\alpha} \int_0^{t_1} (t_1 - s)^{\alpha-1} \left[T_\alpha(t_2 - s) - T_\alpha(t_1 - s) \right] f_n(s) ds \right\| \\
\leq \; & \frac{M\|\omega_K\| b^{1-\alpha}}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds + \frac{M\|\omega_K\|}{\Gamma(\alpha)} \int_0^{t_1} \left[t_1^{1-\alpha} (t_1 - s)^{\alpha-1} - t_2^{1-\alpha} (t_2 - s)^{\alpha-1} \right] \\
& + \| \int_0^{t_1 - \epsilon} t_1^{1-\alpha} (t_1 - s)^{\alpha-1} \left[T_\alpha(t_2 - s) - T_\alpha(t_1 - s) \right] f_n ds \| \\
& + \| \int_{t_1 - \epsilon}^{t_1} t_1^{1-\alpha} (t_1 - s)^{\alpha-1} \left[T_\alpha(t_2 - s) - T_\alpha(t_1 - s) \right] f_n ds \| \\
\leq \; & I_1 + I_2 + I_3 + I_4
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \frac{M\|\omega_K\| b^{1-\alpha}}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds, \\
I_2 &= \frac{M\|\omega_K\|}{\Gamma(\alpha+1)} \left[(t_2 - t_1) + (t_2 - t_1)^\alpha \right], \\
I_3 &= \sup_{s \in [0, t_1 - \epsilon]} \|T_\alpha(t_2 - s) - T_\alpha(t_1 - s)\| \left[\frac{b\|\omega_K\|}{\alpha} \right], \\
I_4 &= \frac{2M\|\omega_K\| b^{1-\alpha}}{\Gamma(\alpha)} \int_{t_1 - \epsilon}^{t_1} (t_1 - s)^{\alpha-1} ds.
\end{aligned}
$$

Applying the absolute continuity of the Lebesgue integral we have I_1, I_2, I_4 tend to zero independently of $x \in \Omega$ as $t_2 \to t_1$. By (H_1), it is easy to see that I_3 tends to zero independently of $x \in \Omega$ as $t_2 \to t_1$. Since the set $\{Sf_n\}_n^{+\infty}$ is equicontinuous, we can conclude that the set $\{Sf_n\}_{n=1}^{+\infty}$ is relatively compact in $C_{1-\alpha}(J, X)$.

This yields

$$mod_{C_{1-\alpha}}(\{S(f_n)\}_{n=1}^{+\infty}) = 0. \tag{3.41}$$

By (H_8), we have

$$mod_{C_{1-\alpha}}(\{g_\alpha(x^n)\}_{n=1}^{+\alpha}) = 0. \tag{3.42}$$

Taking (3.32) into account again, we obtain

$$mod_{C_{1-\alpha}}(\{y^n\}_{n=1}^{+\infty}) = 0. \tag{3.43}$$

Now it follows from (3.40)-(3.43) that

$$\nu(\Omega) = (0, 0). \tag{3.44}$$

So, we conclude that Ω is relatively compact set yielding that the multioperator G is condensing w.r.t. to the MNC ν.

Lemma 3.9 *The multioperator G is u.s.c.*

Proof. Since the family $(T_\alpha(t), t \geq 0)$ is strongly continuous, it is sufficient to prove the assertion for the multioperator $S \circ P_F^\infty$. The proof is proceeded in two steps.

Step 1. G has a closed graph with compact values.
Suppose that $\{\upsilon_n\} \subset G(Q)$. Then there exists a sequence $\{x_n\} \subset Q$ such that

$$\upsilon_n = g_\alpha(x_n) + S \circ P_F^\infty(x_n), \quad t \in J'.$$

Let $x_n \to x$ in X and $\upsilon_n \in G(x_n)$, $\upsilon_n \to \upsilon$ in $C_{1-\alpha}(J, X)$. We claim that $\upsilon \in G(x)$. The first observed that

$$g_\alpha(x_n) \to g_\alpha(x), \tag{3.45}$$

in $C_{1-\alpha}(J, X)$ in accordance with (H_6). In addition, since for each sequence $f_n \in P_F^\infty(x_n), n \geq 1$ for a.e. $t \in J$, according to the hypothesis (H_5), the set $\{f_n(t)\}_{n=1}^\infty$, is relatively compact in X, hence the sequence $\{f_n\}_{n=1}^\infty$ is L^1-semicompact. Consequently $\{f_n\}_{n=1}^\infty$ is weakly compact in $L^1(J, X)$, so we can assume that $f_n \rightharpoonup f$. Due to Lemma 2.1, $\{Sf_n\}_{n=1}^\infty$ is relatively compact. By applying Lemma 3.7 and (3.45) we have the convergence $G(x_n) \to G(x)$.
With the same technique, we obtain that G has compact values.

Step 2. The multioperator G is u.s.c. In view of Lemma 1.30, it suffices to check that G is quasicompact multimap.

Let Q be a compact set. We prove that $G(Q)$ is relatively compact of $C_{1-\alpha}$. Assume that $\{\upsilon_n\} \subset G(Q)$. Then

$$\upsilon_n = g_\alpha(x_n) + S \circ P_F^\infty(x_n), \quad t \in J',$$

where $\{f_n\} \in P_F^\infty$, for certain sequence $\{x_n\} \subset Q$. Hypothesis (H_4) and (H_5) yield the fact that $\{f_n\}$ is semicompact and then weakly compact sequence in L^1. Similar arguments as in the previous proof of the closeness imply that $\{\upsilon_n\}$ is relatively compact in $C_{1-\alpha}(J, X)$. Thus, $\{\upsilon_n\}$ converges in $C_{1-\alpha}(J, X)$, so the multioperator G is u.s.c.

Let us prove the global existence result for Lipschitz assumption for the function from nonlocal condition.

Theorem 3.10 *Under conditions (H_1)-(H_3), (H_5), (H_8) and the following sub-linear growth condition*

(H_4') *there exists a function $\phi \in L_+^\infty(J)$ and a constant $c > 0$ such that*

$$\|F(t, x)\| \leq \phi(t)(1 + ct^{1-\alpha}\|x\|), \quad \textit{for a.e. } t \in J.$$

(H_7') *there exists a constant $L_g > 0$ such that*

$$\|g(x) - g(y)\|_X \leq L_g \|x - y\|_\alpha.$$

The set of all mild solutions to Cauchy problem (3.26), $\Theta^F_{x_0}(J)$ is a nonempty compact subset of the space $C_{1-\alpha}(J,X)$.

Proof. In the space $C_{1-\alpha}(J,X)$ with the norm $\|\cdot\|_\alpha$ consider the ball

$$\overline{B}_r(0)=\{x\in C_{1-\alpha}(J,X),\|x\|_\alpha\leq r\},$$

where $r>0$ is taken so that

$$r\geq\left(\frac{M}{\Gamma(\alpha)}[\|x_0\|+\|g(0)\|]+\frac{Mb\|\phi\|_\infty}{\Gamma(\alpha+1)}\right)(1-N)^{-1}$$

such that

$$\frac{M}{\Gamma(\alpha)}L_g+\frac{cMb\|\phi\|_\infty}{\Gamma(\alpha+1)}\leq N<1$$

Let us prove now that the multioperator G transforms the ball $\overline{B}_r(0)$ into itself.
In fact, if we take $x\in\overline{B}_r(0)$ and $y\in G(x)$, and for any $f\in P^\infty_F$, we have:

$$\begin{aligned}
&t^{1-\alpha}\|y(t)\|\\
\leq\;&\frac{M}{\Gamma(\alpha)}(\|x_0\|+\|g(x)\|)+\frac{Mt^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|f(s)\|ds\\
\leq\;&\frac{M}{\Gamma(\alpha)}(\|x_0\|+L_g\|x\|_\alpha+\|g(0)\|)\\
&+\frac{Mb^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\phi(s)(1+cs^{1-\alpha}\|x(s)\|)ds\\
\leq\;&\frac{M}{\Gamma(\alpha)}(\|x_0\|+L_gr+\|g(0)\|)+\frac{Mb\|\phi\|_\infty}{\Gamma(\alpha+1)}+\frac{cMb\|\phi\|_\infty}{\Gamma(\alpha+1)}r.
\end{aligned}$$

Thus $\|x\|_\alpha\leq r$.

Form Lemmas 3.8 and 3.9, we know that the multioperator G is u.s.c. and ν-condensing. From Theorem 3.19, we obtain that the set $\Theta^F_{x_0}(J)$ is nonempty.

Now, we can show that the set $\Theta^F_{x_0}(J)$ is a priori bounded.

In fact, from the above estimate it follows that for $x\in\Theta^F_{x_0}(J)$ and $f\in P^\infty_F(x)$

$$x(t)=t^{\alpha-1}T_\alpha(t)[x_0-g(x)]+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)f(s)ds,\ \ t\in J'.$$

Moreover, let $g^*>0$ is constant such that $g^*=\sup\limits_{x\in C_{1-\alpha}(J,X)}\|g(x)\|$, we have

$$\begin{aligned}
t^{1-\alpha}\|x(t)\|&\leq\frac{M}{\Gamma(\alpha)}\|(x_0-g(x))\|+\frac{Mt^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|f(s)\|ds\\
&\leq\frac{M}{\Gamma(\alpha)}(\|x_0\|+g^*)+\frac{Mt^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\phi(s)(1+cs^{1-\alpha}\|x(s)\|)ds\\
&\leq\frac{M}{\Gamma(\alpha)}(\|x_0\|+g^*)+\frac{Mb\|\phi\|_\infty}{\Gamma(\alpha+1)}+\frac{cMb^{1-\alpha}\|\phi\|_\infty}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|x(s)\|_\alpha ds.
\end{aligned}$$

Let

$$v(t)=t^{1-\alpha}\|x(t)\|,\ \ \omega(t)=\frac{M}{\Gamma(\alpha)}(\|x_0\|+g^*)+\frac{Mb\|\phi\|_\infty}{\Gamma(\alpha+1)},$$

from Lemma 1.8 , we conclude that there exists a constant $K = K(\alpha)$ such that

$$\begin{aligned} t^{1-\alpha}\|x(t)\| &\leq \omega(t) + \frac{cb^{1-\alpha}MK}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\omega(s)ds \\ &\leq \left(1+\frac{cbMK}{\Gamma(\alpha+1)}\right)\left[\frac{M}{\Gamma(\alpha)}(\|x_0\|+g^*)+\frac{Mb\|\phi\|_\infty}{\Gamma(\alpha+1)}\right] := \overline{M}. \end{aligned}$$

Hence

$$\|x\|_\alpha \leq \overline{M}.$$

Applying Lemma 1.63, we obtain that the set $\Theta^F_{x_0}(J)$ is compact.

3.2.2 An example

Let Ω be a nonempty bounded open set in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Denote $X = L^p(\Omega)$, with $1 \leq p < \infty$ and $a \in \mathbb{R}^n$. Consider the fractional partial (transport) differential inclusion

$$\begin{cases} {}^LD^\alpha u(t,x) \in a\cdot\nabla u(t,x) + F(t,u(t,x)), & [0,b]\times\Omega, \\ u(t,x) = 0, & [0,b]\times\partial\Omega \\ I^{1-\alpha}_{0^+}u(t,x) + g(u(t,x)) = u_0(x), & \{0\}\times\Omega, \end{cases} \tag{3.46}$$

where the partial derivatives are taken in the sense of distributions over Ω, and

$$F(t,u) = F_1(t,u) + F_2(t,u), \quad \text{for each } (t,u) \in [0,b]\times\Omega. \tag{3.47}$$

To model (3.46), we assume that :

(A1) For each $u \in \Omega$, the multifunctions $F_i(\cdot,u) : [0,b]\times\Omega \to \mathcal{P}_{cp,cv}(\mathbb{R}^n)$, $i = 1,2$ are measurable.

(A2) There exists $k(\cdot) \in L^\infty(J,\mathbb{R}_+)$ such that the multifunction $F_1(t,\cdot)$ is $k(\cdot)$-Lipschitz w.r.t the Hausdorff metric for each $t \in [0,b]$, i.e.

$$H(F_1(t,u),F_1(t,v)) \leq k(t)\|u-v\|, \quad \text{for any } \ u,v \in \Omega.$$

(A3) The multifunction $F_2(t,\cdot)$ is compact.

Denote

$$\begin{cases} D(A) = \{u \in X; \quad a\cdot\nabla u \in X\} \\ Au = a\cdot\nabla u \end{cases}$$

$$\begin{aligned} &u(t)(x) = u(t,x), \\ &F_i(t,u(t,x)) = F_i(t,u(t))(x),\ i = 1,2. \end{aligned}$$

From [62, Theorem 4.4.1], A generates a noncompact semigroup $T(t)$ given by

$$T(t)u = u(x-ta), \quad \text{for each } u \in X,\ t \in \mathbb{R}.$$

Clearly, the semigroup $T(t)$ is continuous in the uniform operator topology (it is isometry).
We define $g : C_{1-\alpha}(J,X) \to X$ such that $g(z(x,t)) = \sum\limits_{i=1}^m c_i z(x,t_i)$, where

$$\|g(x(t)) - g(y(t))\| \leq \sum_{i=1}^m |c_i|\|x-y\|_\alpha,$$

$L_g = \sum_{i=1}^{m} |c_i|$, and choose c_i such that $\dfrac{M}{\Gamma(\alpha)} L_g + \dfrac{cMb\|k\|_\infty}{\Gamma(\alpha+1)} \leq N < 1$, and

$$r \geq \left(\frac{M}{\Gamma(\alpha)} [\|x_0\| + \|g(0)\|] + \frac{Mb\|\phi\|_\infty}{\Gamma(\alpha+1)} \right) (1-N)^{-1}$$

The system (3.46) can be written in the abstract form given by (3.26). All assumptions in Theorem 3.10 are satisfied (see for instance [36, Corollary 2.21]). Then, the problem (3.46) has at least one mild solution.

Perspective

The purpose of the first future works is to investigate the topological structure of the solution set for fractional differential equation nonlocal conditions (2.11)-(2.12) in Banach spaces.

We study the topological stucture of the solution set for fractional nonlocal evolution inclusions (3.26). We prove that the solution set for all problem is nonempty, compact and R_δ-set.

We study existence of fractional optimal controls governed by semilinear fractional nonlocal evolution equations via a continuous semigroup in Banach space

$$\begin{cases} {}^L D_{0^+}^{\alpha} x(t) = Ax(t) + f(t, x(t)) + B(t)u(t), \\ \\ (I_{0^+}^{1-\alpha} x)(0) + g(x) = x_0, \end{cases} \tag{3.48}$$

where ${}^L D_{0^+}^{\alpha}$ denotes Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$ and $I_{0^+}^{1-\alpha}$ is Riemann-Liouville fractional integral of order $1-\alpha$, A is the generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space X, f is E-value function, u takes value from another Banach space X, B is a linear operator from X into E.

In this paper, we study semilinear functional fractional evolution equations with infinite delay in the Banach spaces X

$${}^L D_{0^+}^{\alpha} x(t) = Ax(t) + f(t, x_t); \quad t \in J' := (0, b], \tag{3.49}$$

$$\tilde{x}_0 = \phi \in \mathcal{B}, \tag{3.50}$$

where ${}^L D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$, $\tilde{x}(t) = t^{1-\alpha} x(t)$, $f : J \times \mathcal{B} \longrightarrow \mathcal{X}$ is a given function satisfying some assumptions, and $\mathcal{B}$ the phase space that will specified later, $A : D(A) \subset X \longrightarrow X$ is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$, and $\phi(0) \neq 0$. The principal goal of this paper is to extend such results to the case when the semigroup generated by A is noncompact.

In finally, We consider the following fractional stochastic evolution inclusions with nonlocal conditions

$$\begin{cases} {}^L D_{0^+}^{\alpha} x(t) \in Ax(t) + f(t, x(t)) + \Sigma(t, x(t)) \dfrac{dW(t)}{dt}, \\ \\ (I_{0^+}^{1-\alpha} x)(0) + g(x) = x_0, \end{cases} \tag{3.51}$$

where ${}^L D_{0^+}^{\alpha}$ denotes Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$ and $I_{0^+}^{1-\alpha}$ is Riemann-Liouville fractional integral of order $1-\alpha$, A is the generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ in a Hilbert space H, with inner product $\langle \cdot, \cdot \rangle$ and state $x(\cdot)$ takes value in H, $f : J \times H \longrightarrow H$, $g : H \longrightarrow H$. As an example of g, the following function can be considered:

$$g(x) = \sum_{i=1}^{p} K_i x(t_i)$$

where $K_i : H \longrightarrow H$ are given linear operators, the operators K_i can be given by

$$K_i x(t_i, y) = \int_0^b K_i(\xi, y) x(t_i, \xi) d\xi,$$

where $K_i = (i = 1, 2, \cdots, p)$ are continuous kernel functions and $\Sigma : J \times H \multimap H$ is a nonempty, bounded, closed, and convex multivalued map $\{W(t)\}_{t\geq 0}$ is a given K-valued Browinian motion or Wiener process with a finite trace nuclear covariance operator $Q \geq 0$, here K is a Hilbert space with inner product $(\cdot, \cdot)_K$ and norm $|\cdot|_K$.

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الملخص (بالعربية) :

الهدف من هذه الأطروحة هو دراسة مسألة الوجود و الوحدانية لحلول بعض أنواع المعادلات التفاضلية, التفاضلية التطورية و كذلك التطورية الإحتوائية ذات رتبة ناطقة بمفهوم ريمان- ليوفيل بشروط محلية و غير محلية.
باستعمال تقنية التراجعية الرتيبة أثباتنا وجود الحل الأقصى و الحل الأدنى , النظريات النقطة الصامدة ل شاودر مونك مقترنة بالقياس غير المتراص ل كراوشكي والقياس غير المتراص ل هزدورف مع نظرية التكثيف لإثبات الوجود للحلول التطورية الإحتوائية ذات رتبة ناطقة بمفهوم ريمان- ليوفيل بشروط غير محلية.

Résumé (Français et/ou Anglais) :

Résumé

Cette thèse est consacrée, l'étude de l'existence et unicité pour certaine classes d'équations différentielles fractionnaire, évolution fractionnaire et même tempe inclusions évolution fractionnaire au sens de Riemann-Liouville avec des conditions locales et non locales dans des espaces de Banach de dimension infinie.

Ces résultats ont été obtenue par l'utilisation du méthode monotone itérative pour l'estimée les solutions minimale et maximale est aussi du théorèmes point fixe de Schauder et de Mönch combiné avec les mesures des non compacités de Kuratowski, de Hausdorff et théorèmes de condensée pour résoudre de problème inclusions évolution fractionnaire.

Summary

In this thesis, we present existence and uniqueness results for certain classes of fractional differential equations and fractional evolution equations, moreover fractional evolution inclusions in the sense of Riemann-Liouville with nonlocal and local conditions on Banach spaces of infinite dimension.

These results were obtained by using the fixed point theorems the Schauder's and Mönch's combined with the measure of non compactness of Kuratowski, Hausdorff and condensing maps theory.