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Publications

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3. M. Benchohra and **K. Maazouz**, Global existence for nonlinear implicit fractional Differential Equations In Fréchet Spaces, submitted.

4. M. Benchohra and **K. Maazouz**, Ulam-Hyers stability for implicit fractional differential equation with anti-periodic conditions, submitted.

Abstract

In this thesis, we have considered the existence and uniqueness of solutions for a class of initial value problems ,boundary value problems , anti-periodic conditions problems and problems with delay for nonlinear implicit fractional differential equations with Caputo fractional derivative. The results will be obtained by means of fixed points theorems and by the technique of measures of noncompactness.

We discuss and establish the existence, uniqueness, Ulam-Hyers and Ulam-Hyers-Rassias stabilities of solutions for some classes of fractional differential equations in Banach and Fréchet spaces.

Key words and phrases :

Initial value problem, boundary value problems, finite delay, Green function, Caputo's fractional derivative, implicit fractional differential equations, Banach space, Fréchet space, fractional integral, existence and uniqueness, Gronwall's lemma, fixed point, measure of noncompactness, local and nonlocal conditions, integral boundary conditions Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability, generalized Ulam-Hyers-Rassias stability.

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Introduction

Fractional Calculus has its origin in the question of the extension of meaning. A well known example is the extension of meaning of real numbers to complex numbers, and another is the extension of meaning of factorials of integers to factorials of complex numbers. In generalized integration and differentiation the question of the extension of meaning is : Can the meaning of derivatives of integral order $\dfrac{d^n y}{dx^n}$ be extended to have meaning where n is any number (fractional, irrational or complex) ?

Fractional Differential equations are as old as the idea of the integer order ones is, they have been in the last decades when the use of fractional Differential equations has become more and more popular among many research areas. The theoretical and practical interest of this field is nowadays well established, and its applicability to science and engineering can be considered as emerging new topics. They are, in fact,useful tools for both the description of a more complex reality, and the enlargement of the practical applicability of the common integer order. Fractional Differential equations (fractional calculus) are specially interesting in automatic control and robotics.

Differential equations with fractional order are generalization of ordinary differential equations to non-integer order. In recent years, a great interest was devoted to study fractional differential equations, because of their appearance in various applications in Engineering and Physical Sciences, (see [75, 96, 105, 112, 117, 21, 22, 23]). Recently, there are several studies devoted to extend, if possible, results for fractional differential equations see ([6, 90, 92, 93, 94]). It is noted that the extension is not a straightforward process, due to the difficulties in the definition and the rules of fractional derivatives. Therefore, the theory of fractional differential equations is not established yet and there are still many open problems in this area. Unlike, the integer derivative, there are several definitions of the fractional derivative, which are not equivalent in general. However, the most popular ones are the Caputo and RiemannLiouville fractional derivatives.

In 1930, Kuratowski [89] introduced the concept of measure of noncompactness. Later, Banas̀ and Goebel [24] generalised this concept axiomatically, which is more convenient in applications. The tool of measure of noncompactness has been used in the theory of operator equations in Banach spaces. The fixed point theorems derived from them have many applications. There is considerable literature devoted to this subject (see, for example, ([15, 16, 24, 25, 26]). The principal application of measures of noncompactness in fixed point theory is through Darbo's fixed point theorem [24]. This yields a tool to investigate the existence and behaviour of solutions of many classes of integral equations such as those of Volterra, Fredholm and Uryson types .

Delay Equations have their origin in domains of applications, such as physics, engineering, biology, medicine and economics. They appear in connection with the fundamental problem to analyze a retarded process from the real world, to develop a corresponding mathematical model and to determine the future behavior (See [50]).

In the following we give an outline of our thesis organization, Consists of six chapters. We have organized this thesis as follows :

Chapter 1.

This chapter consists some preliminaries,some basic concepts, useful theorems and results ,notations, definitions, lemmas and fixed point theorems which are used throughout this thesis.

In **Chapter 2**, we discuss and establish the existence and uniqueness of solution for a class of initial value problem for nonlinear implicit fractional differential equations. In Section 2.1, we consider the following problem

$$^{c}D^{\alpha}y(t) = f(t, y(t), {}^{c}D^{\alpha}y(t)), \text{ for each } t \in J = [0,T], T > 0, \ 0 < \alpha < 1,$$

$$y(0) = y_0,$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function and $y_0 \in \mathbb{R}$. As application we present two illustrative examples.

And in Section 2.2 we consider the following problem :

$$^{c}D^{\beta}y(t) = f(t, y(t), {}^{c}D^{\beta}y(t)), \text{ for each } t \in J = [0,T], \ T > 0, \ 0 < \beta < 1,$$

$$y(0) = y_0,$$

where $^{c}D^{\beta}$ is the Caputo fractional derivative, $(E, ||.||)$ is a real Banach space, $f : J \times E \times E \to E$ is continuous function, and $y_0 \in E$.
two results are discussed ; the first is based on Darbo's fixed point theorem combined with the technique of measures of noncompactness, and the second on Mönch's fixed point theorems. At last, an example is given to illustrative the application of the main results.

In **Chapter 3**, we establish existence and uniqueness for the following problem

$$
\begin{aligned}
{}^cD^{\alpha}y(t) = f(t, y(t), {}^cD^{\alpha}y(t)), &\quad \text{for each } t \in J = [0,b],\ 0 < \alpha < 1, \\
y(0) + \lambda \int_0^b y(t)dt &= y(b),
\end{aligned}
$$

where ${}^cD^{\alpha}$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R}$ is a given function, and $\lambda \in (0, +\infty)$.
We present two results for the above problem. The first one is based on the Banach contraction principle, the second one on Schauder's fixed point.

In **Chapter 4**, we discuss and establish the existence, and uniqueness of solution for a class of boundary value problem.
In Section 4.1, we will give existence and uniqueness results for the followings problems of implicit fractional differential equations :

$$
{}^cD^{\alpha}y(t) = f(t, y(t), {}^cD^{\alpha}y(t)) = 0,\ \text{for each, } t \in J = [0,T],\ 0 < \alpha < 1,
$$

$$
ay(0) + by(T) = c,
$$

where ${}^cD^{\alpha}$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \to [0, \infty)$ is a given function and a, b, c are real constants with $a + b \neq 0$, , and

$$
\begin{aligned}
{}^cD^{\alpha}y(t) = f(t, y(t), {}^cD^{\alpha}y(t)), &\quad \text{for every } t \in J := [0,T], T > 0, \quad 0 < \alpha < 1 \\
y(0) + g(y) &= y_0
\end{aligned}
$$

where $g : C([0,T], \mathbb{R}) \longrightarrow \mathbb{R}$ a continuous function and y_0 a real constant. This type of non-local Cauchy problem was introduced by Byszewski [51]. The author observed that the non-local condition is more appropriate than the local condition (initial) to describe correctly some physics phenomenons [51], and proved the existence and the uniqueness of weak solutions and also classical solutions for this type of problems. We take an example of non-local conditions as follows :

$$
g(y) = \sum_{i=1}^{p} c_i y(t_i)
$$

where $c_i,\ i = 1, \ldots, p$ are constants and $0 < t_1 < \ldots < t_p \leqslant T$.

In **Chapter 5**, **T**his chapter consists of three sections. In section 5.1 we establish existence and uniqueness for the following problem

$$
{}^cD^{\alpha}y(t) = f(t, y_t, {}^cD^{\alpha}y(t)),\ t \in J = [0, +\infty) \quad 0 < \alpha < 1
$$

$$y(t) = \varphi(t), \quad t \in [-r, 0], \ r > 0$$

where ${}^cD^\alpha$ is the Caputo fractional derivative. $f : J \times C([-r, 0], \mathbb{R}) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given function such that $\varphi \in C([-r, 0], \mathbb{R})$.
For each function y defined on $[-r, \infty)$ and for any $t \in J$, we denote by y_t the element of $C([-r, 0], \mathbb{R})$ by :

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0].$$

$y_t(\cdot)$ represents the history of system state from time $t - r$ to time t.

Section 5.2 is devoted to fractional neutral functional differential equations,

$${}^cD^\alpha[y(t) - g(t, y_t)] = f(t, y_t, {}^cD^\alpha y(t)), \ t \in J = [0, +\infty) \quad 0 < \alpha < 1$$

$$y(t) = \varphi(t), \quad t \in [-r, 0], \ r > 0$$

where $g : J \times C([-r, 0], \mathbb{R}) \to \mathbb{R}$ is a given function such that $g(0, \varphi) = 0$.

We shall present uniqueness results, Our approach will be based upon a recent nonlinear alternative of Leray-Schauder type in Fréchet spaces due to Frigon and Granas [61].

Section 5.3 is devoted to discuss existence and uniqueness for the following problem

$${}^cD^\alpha y(t) = f(t, y(t), {}^cD^\alpha y(t)), \ t \in J = [0, +\infty)$$

$$y(0) = y_0,$$

where ${}^cD^\alpha$ is the Caputo fractional derivative. $f : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given function, $y_0 \in \mathbb{R}$.

We present results based on contractive maps in Fréchet spaces and the nonlinear alternative of Leray-Schauder type due to Frigon and Granas. At the end we illustrate the problem with an example.

In **Chapter 6**, The purpose of this chapter is to establish existence, uniqueness, Ulam-Hyers stability, generalized Ulam-Hyers stability, and Ulam-Hyers-Rassias stability for the following problems for implicit fractional order differential equation with anti-periodic condition, and fractional differential equation with finite delay.

$${}^cD^\alpha y(t) = f(t, y(t), {}^cD^\alpha y(t)), \ t \in J = [0, b] \quad 0 < \alpha < 1$$

$$y(0) = -y(b).$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R}$ is a given function.

$$^cD^{\alpha}\left[y(t)-g(t,y_t)\right]=f(t,y_t,{}^cD^{\alpha}y(t)),\ t\in J=[0,T]\,,\ T>0\ ,0<\alpha\leqslant 1$$

$$y(t)=\varphi(t),\ t\in[-r,0]\,,\ r>0$$

where $f: J\times C\left([-r,0],\mathbb{R}\right)\times\mathbb{R}\to\mathbb{R}$ and $g: J\times C\left([-r,0],\mathbb{R}\right)$ are two given functions such that $g(0,\varphi)=0$ and $\varphi\in C\left([-r,0],\mathbb{R}\right)$.

For each function y_t defined on $[-r,T]$ and for any $t\in J$, we denote by y_t the element of $C\left([-r,0],\mathbb{R}\right)$ defined by :

$$y_t(\theta)=y(t+\theta),\quad \theta\in[-r,0],$$

$y_t(.)$ represent the evolution history of system state from time $t-r$ to time t.

Chapitre 1

Preliminaries

1.1 Notations and definitions

In this chapter definitions and some auxiliary results are given regarding the main objects of the monograph : some notations and definitions of Fractional Calculus Theory, some definitions and properties of noncompactness measure, some fixed point theorems.

Let $J = [a, b]$ be an interval of $\mathbb{R}$ and $(E, |.|)$ be a real Banach space. Let $C(J, E)$ be the Banach space of continuous functions from J into E with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\}.$$

A measurable function $y : J \longrightarrow E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable.
Let $n \in \mathbb{N}$ and $J_0 = [0, n]$. By $C(J_0, \mathbb{R})$ we denote the space of continuous functions $y : J_0 \longrightarrow \mathbb{R}$ with the norm

$$\|y\|_n = \{\sup |y(t)| \quad t \in J_0\}.$$

Let $L^1(J_0, \mathbb{R})$ the space of Lebesgue-integrable functions $y : J_0 \to \mathbb{R}$, equipped with the norm

$$\|y\|_{L^1} = \int_{J_0} |y(t)|dt.$$

For each function y_t defined on $[-r, T]$ and for any $t \in J$, we denote by y_t the element of $C([-r, 0], \mathbb{R})$ defined by :

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0],$$

$y_t(.)$ represent the evolution history of system state from time $t - r$ to time t.

Theorem 1.1.1 *(The Dominated Convergence Theorem)([49]).*
Suppose that (f_n) is a sequence of integrable functions which converges almost everywhere to a function f and that there is a positive integrable function g satisfying

$$|f_n| < g \qquad for\, all\, n.$$

Then f is integrable and

$$\int f_n d\mu \longrightarrow \int f d\mu \qquad as\ n \longrightarrow \infty.$$

Theorem 1.1.2 *(Arzela-Ascoli)([60]) Let $A \in C(J,E)$, A is relatively compact (i.e. $\overline{A}$ is compact) if :*

1. *A is uniformly bounded i.e, there exists $M > 0$ such that*

$$\|f(t)\| < M \, for\, every\, f \in A \ and \ t \in J.$$

2. *A is equicontinuous i.e, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for each $t_1, t_2 \in J$, $|t_2 - t_1| \leqslant \delta$ implies $\|f(t_2) - f(t_1)\| \leqslant \varepsilon$, for every $f \in A$.*
3. *The set $\{f(t) : f \in A, t \in J\}$ is relatively compact in E.*

Definition 1.1.3 *An operator $T : E \longrightarrow E$ is called compact if the image of each bounded set $B \in E$ is relatively compact i.e ($\overline{T(B)}$ is compact). T is called completely continuous operator if it is continuous and compact.*

Definition 1.1.4 *([87, 107]). The fractional (arbitrary) order integral of the function $f \in L^1([0,T], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

where Γ is the gamma function.

Theorem 1.1.5 *[87]. For any $f \in C([a,b], \mathbb{R})$ the Riemann-Liouville fractional integral satisfies*

$$I^\alpha I^\beta f(t) = I^\beta I^\alpha f(t) = I^{\alpha+\beta} f(t),$$

for $\alpha, \beta > 0$.

Definition 1.1.6 *([86]). For a function f given on the interval $[0,T]$, the Caputo fractional-order derivative of order α of h, is defined by*

$$({}^cD^\alpha f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the real number α.

Lemma 1.1.7 *([97]) Let $\alpha \geqslant 0$ and $n = [\alpha] + 1$. Then*

$$I^\alpha({}^cD^\alpha f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k.$$

Remark 1.1.8 *([97])The Caputo derivative of a constant is equal to zero.*

Lemma 1.1.9 *([125]) Let $\alpha > 0$. Then the differential equation*

$$^{c}D^{\alpha} f(t) = 0$$

has solution $f(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [\alpha] + 1$.

Lemma 1.1.10 *([125]) Let $\alpha > 0$. Then*

$$I^{\alpha}(^{c}D^{\alpha} f(t)) = f(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [\alpha] + 1$.

Lemma 1.1.11 *([58]) Let $\alpha > 0$, $\alpha \notin \mathbb{N}$ and $m = [\alpha]$. Moreover assume that $f \in C^m[a,b]$. Then*

$$^{c}D_a^{\alpha} f \in C[a,b]$$

and

$$^{c}D_a^{\alpha} f(a) = 0.$$

Proposition 1.1.12 *[87] If $f \in L^1(J_0, \mathbb{R})$, then $\|I^{\alpha} f\|_{L_1} \leqslant \frac{n^{\alpha}}{\Gamma(\alpha+1)} \|f\|_{L_1}$.*

Generalization of Gronwall's Lemma .

Lemma 1.1.13 *([122]) Let $v : [0,T] \to [0,+\infty)$ be a real function and $w(.)$ is a nonnegative, locally integrable function on $[0,T]$ and there are constants $a > 0$ and $0 < \alpha < 1$ such that*

$$v(t) \leqslant w(t) + a \int_0^t (t-s)^{\alpha-1} v(s) ds,$$

Then, there exists a constant $K = K(\alpha)$ such that

$$v(t) \leqslant w(t) + Ka \int_0^t (t-s)^{\alpha-1} w(s) ds, \text{ for every } t \in [0,T].$$

1.2 Special Functions

Gamma Function (See [107]).
The gamma function $\Gamma(z)$ is one of the basic functions of the fractional calculus is defined by the integral

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt$$

which converges in the right half of the complex plane $Re(z) > 0$.
Let us recall some results on the gamma function.

One of the basic properties of the gamma function is that it satisfies the following functional equation :

$$\Gamma(z+1) = z\Gamma(z)$$

which can be easily proved by integrating by parts :

$$\Gamma(z+1) = \int_0^\infty e^{-t}t^z dt = \Big[-e^{-t}t^z\Big]_{t=0}^{t=\infty} + z\int_0^\infty e^{-t}t^{z-1}dt = z\Gamma(z).$$

In particular, if $z = n \in \mathbb{N}$, then

$$\Gamma(n+1) = n\Gamma(n) = n\cdot(n-1)! = n!,$$

with $0! = 1$.
The gamma function can be represented also by the limit

$$\Gamma(z) = \frac{n!n^z}{z(z+1)(z+2)\dots(z+n)}.$$

Beta Function (See [107]).
The beta function is usually defined by

$$\beta(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt, \qquad Re(z) > 0,\ Re(w) > 0.$$

We can express the beta function by :

$$\beta(w,z) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$$

from which it follows that

$$\beta(z,w) = \beta(w,z).$$

With the help of the beta function we can establish the following

$$\Gamma(z)\Gamma(1-z) = \beta(z,1-z) = \frac{\pi}{\sin(\pi z)}.$$

Taking $z = \frac{1}{2}$ we obtain a useful particular value

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

Mittag-Leffler Function(See [107]).
Mittag-Leffler function plays a very important role, in the theory of fractional differential equations which is denoted by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$$

A two-parameter function of the Mittag-Leffier type is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

It follows that

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z,$$

$$E_{1,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \frac{1}{z} \sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)!} = \frac{e^z - 1}{z},$$

$$E_{1,3}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+3)} = \frac{1}{z^2} \sum_{k=0}^{\infty} \frac{z^{k+2}}{(k+2)!} = \frac{e^z - 1 - z}{z^2}.$$

1.3 Fréchet Spaces

Definition 1.3.1 *([119]) A Fréchet space is a topological vector space with the following properties :*

1. *it is metrizable*
2. *it is complete*
3. *it is locally convex*

Remark 1.3.2 *([124]) A locally convex space is a normed space iff there exists a bounded neighborhood of zero.*
A locally convex space X is metrizable iff its topology is induced by an at most countable system $\{p_j\}$ of seminorms.

Examples (See [119]).
Hausdorff finite dimensional topological vector spaces, Hilbert spaces and Banach spaces are Fréchet spaces.
Some properties in Fréchet Spaces.
Let $E = (E, \|.\|_n)$ be a Fréchet space with a family of semi-norms $\{\|.\|_n\}_{n\in\mathbb{N}}$, we say that X is bounded if for every $n \in \mathbb{N}$, there exists $M_n > 0$ such that

$$\|x\|_n \leqslant M_n \quad for\ all\ x \in X.$$

To E we associate a sequence of Banach spaces $\{(E^n, \|\cdot\|_n)\}$ as follows : For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by : $x \sim_n y$ if and only if $\|x - y\|_n = 0$ for $x, y \in E$. We denote $E^n = (E|_{\sim_n}, \|\cdot\|_n)$ be the quotient space, the completion of E^n with respect to $\|\cdot\|_n$. To every $X \subset E$, we associate a sequence $\{X^n\}$ of subsets $X^n \subset E^n$ as follows : For every $x \in E$, we denote $[x]_n$ the equivalence class

of x of subset E^n and define $X^n = \{[x]_n : x \in X\}$. We denote by $\overline{X^n}$, $int_n(X^n)$ and $\partial_n X^n$, respectively, the closure, the interior and the boundary of X^n with respect to $\|\cdot\|_n$ in E^n.
We assume that the family of semi-norms $\{\|.\|_n\}_{n\in\mathbb{N}}$ verifies :

$$\|x\|_1 \leqslant \|x\|_2 \leqslant \|x\|_3 \leqslant \dots \ for\ every\ x \in X.$$

The following definition is the appropriate concept of contraction in E.
For more information about this subject see [61, 62].

Definition 1.3.3 *[61] A function $f : X \to X$ is said to be a contraction if for every $n \in \mathbb{N}$ there exists $k_n \in [0,1)$ such that :*

$$\|f(x) - f(y)\|_n \leqslant k_n \|x - y\|_n \ for\ all\ x, y \in X.$$

1.4 Measures of noncompactness

In this subsection we define the Kuratowski and Hausdorf measures of noncompactness (**MNCs** for short) and give their basic properties.

For a given set V of functions $v : J \longrightarrow E$, let us denote by

$$V(t) = \{v(t) : v \in V, t \in J\}$$

and

$$V(J) = \{v(t) : v \in V, t \in J\}.$$

Definition 1.4.1 *([15]) Let (X, d) be a complete metric space and $\mathcal{B}$ the family of bounded subsets of X. For every $\Omega \in \mathcal{B}$ the Kuratowski measure of noncompactness $\alpha(\Omega)$ of the set Ω is the infimum of the numbers $d > 0$ such that Ω admits a finite covering by sets of diameter smaller than d.*

Remark 1.4.2 *The diameter of a set B is the number* $\sup\{d(x,y) : x \in B, y \in B\}$ *denoted by diam(B), with diam$(\emptyset) = 0$.*
It is clear that $0 \leqslant \alpha(B) \leqslant diam(B) < +\infty$ for each nonempty bounded subset B of X and that $diam(B) = 0$ if and only if B is an empty set or consists of exactly one point.

Definition 1.4.3 *[24] Let E be a Banach space and Ω_E the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \to [0, \infty]$ defined by*

$$\alpha(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^n B_i \ and\ diam(B_i) \leq \epsilon\};\ here\ B \in \Omega_E,$$

where $diam(B_i) = sup\{||x - y|| : x, y \in B_i\}$.

In the definition of the Kuratowski measure we can consider balls instead of arbitrary sets. In this way we get the definition of the Hausdorff measure :

Definition 1.4.4 *[15] The Hausdorff measure of noncompactness $\chi(\Omega)$ of the set Ω is the infimum of the numbers φ such that Ω admits a finite covering by balls of radius smaller than φ.*

Properties of the Kuratowski and Hausdorf MNCs([15]).
Let A and B bounded sets.

(a) *regularity* : $\psi(A) = 0 \Leftrightarrow \overline{A}$ is compact , where $\overline{A}$ denotes the closure of A.
(b) *nonsingularity* : ψ is equal to zero on every one element-set.
(c) *monotonicity* : $A \subset B \Rightarrow \psi(A) \leqslant \psi(B)$.
(d) *semi-additivity* : $\psi(A \bigcup B) = max\{\psi(A), \psi(B)\}$.
(e) *semi-homogencity* : $\psi(\lambda A) = |\lambda|\psi(A)$; $\lambda \in \mathbb{R}$, where $\lambda(A) = \{\lambda x : x \in A\}$.
(f) *algebraic semi-additivity* : $\psi(A + B) \leqslant \psi(A) + \psi(B)$, where

$$A + B = \{x + y : x \in A, \quad y \in B\}.$$

(g) *invariance under translations* : $\psi(A + x_0) = \psi(A)$ for any $x_0 \in E$.
(h) $\psi(A) = \psi(\overline{A}) = \psi(convA)$, where $convA$ is the convex hull of A.
(i) $\psi(A \bigcap B) = min\{\psi(A), \psi(B)\}$.

Remark 1.4.5 *The measure of noncompactness was introduced by Kuratowski in order to generalize the Cantor intersection theorem*

Theorem 1.4.6 *[88] Let (X, d) be a complete metric space and $\{B_n\}$ be a decreasing sequence of nonempty, closed and bounded subsets of X and $\lim_{n\to\infty} \alpha(B_n) = 0$. Then the intersection B_∞ of all B_n is nonempty and compact.*

In [76], Horvath has proved the following generalization of the Kuratowski theorem :

Theorem 1.4.7 *[88] Let (X, d) be a complete metric space and $\{B_i\}_{i\in I}$ be a family of nonempty of closed and bounded subsets of X having the finite intersection property. If $\inf_{i\in I} \alpha(B_i) = 0$ then the intersection B_∞ of all B_i is nonempty and compact.*

Lemma 1.4.8 *([69]) If $V \subset C(J, E)$ is a bounded and equicontinuous set, then*
(i) the function $t \to \alpha(V(t))$ is continuous on J, and

$$\alpha_c(V) = \sup_{0\leqslant t\leqslant T} \alpha(V(t)).$$

(ii) $$\alpha\left(\int_0^T x(s)ds : x \in V\right) \leq \int_0^T \alpha(V(s))ds,$$
where $V(s) = \{x(s) : x \in V\},\ s \in J$.

Theorem 1.4.9 *([88]) Let $B(0, 1)$ be the unit ball in a Banach space X. Then*

$$\alpha(B(0, 1)) = \chi(B(0, 1)) = 0$$

if X is finite dimensional, and $\alpha(B(0, 1)) = 2$, $\chi(B(0, 1)) = 1$ otherwise.

Theorem 1.4.10 *([88]) Let $S(0,1)$ be the unit sphere in a Banach space X. Then $\alpha(S(0,1)) = \chi(S(0,1)) = 0$ if X is finite dimensional, and $\alpha(S(0,1)) = 2$, $\chi(S(0,1)) = 1$ otherwise.*

Theorem 1.4.11 *([88]) The Kuratowski and Hausdorff MNCs are related by the inequalities*

$$\chi(B) \leqslant \alpha(B) \leqslant 2\chi(B).$$

In the class of all infinite dimensional Banach spaces these inequalities are the best possible.

Example 1.4.12 *Let l^∞ be the space of all real bounded sequences with the supremum norm, and let A be a bounded set in l^∞. Then $\alpha(A) = 2\chi(A)$.*

For further facts concerning measures of noncompactness and their properties we refer to [15, 24, 26, 88, 118] and the references therein.

1.5 Fixed point theorems

We present some fixed point theorems which will be used in the following chapters

Theorem 1.5.1 (Banach's fixed point theorem [67]) *Let C be a non-empty closed subset of a Banach space X, then any contraction mapping T of C into itself has a unique fixed point.*

Theorem 1.5.2 (Schauder's fixed point theorem [67]) *. Let X be a Banach space. and C be a closed, convex and nonempty subset of X. Let $N : C \to C$ be a continuous mapping such that $N(C)$ is a relatively compact subset of X. Then N has at least one fixed point in C.*

Theorem 1.5.3 (Nonlinear Alternative of Leray-Schauder type [67]) *Let X be a Banach space with $C \subset X$ closed and convex. Assume U is a relatively open subset of C with $0 \in U$ and $N : \overline{U} \to C$ is a compact map. Then either,*
(i) N has a fixed point in $\overline{U}$; or
(ii) there is a point $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda N(u)$.

Theorem 1.5.4 (Darbo's Fixed Point Theorem [64, 67]) *Let X be a Banach space and C be a bounded, closed, convex and nonempty subset of X. Suppose a continuous mapping $N : C \to C$ is such that for all closed subsets D of C,*

$$\alpha(N(D)) \leqslant k\alpha(D), \tag{1.1}$$

where $0 \leqslant k < 1$, and α is the Kuratowski measure of noncompactness. Then N has a fixed point in C.

Remark 1.5.5 *Mappings satisfying the Darbo-condition (1.1) have subsequently been called k-set contractions.*

Theorem 1.5.6 (Mönch's Fixed Point Theorem [8, 98]) *Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv} N(V) \quad or \quad V = N(V) \cup \{0\} \Rightarrow \alpha(V) = 0$$

holds for every subset V of D, then N has a fixed point.
Here α is the Kuratowski measure of noncompactness.

Theorem 1.5.7 (Nonlinear alternative) *[61]. Let X be a closed subset of a Fréchet space E such that $0 \in X$ and $N : X \to E$ be a contraction map such that $N(X)$ is bounded. Then either*

(C1) T has a unique fixed point or

(C2) There exist $\lambda \in [0, 1)$, $n \in \mathbb{N}$ and $y \in \partial_n X^n$: $\|y - \lambda T(y)\|_n = 0$.

For more detail see [8, 17, 65, 67, 88, 123].

Chapitre 2

Existence results for Cauchy problems

2.1 Introduction

The purpose of this Chapter, is to establish existence and uniqueness results to the following problems :

$$^{c}D^{\alpha}y(t) = f(t, y(t), {}^{c}D^{\alpha}y(t)), \text{ for each, } t \in J = [0, T], T > 0, \; 0 < \alpha < 1, \tag{2.1}$$

$$y(0) = y_0, \tag{2.2}$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function and $y_0 \in \mathbb{R}$, and

$$^{c}D^{\beta}y(t) = f(t, y(t), {}^{c}D^{\beta}y(t)), \text{ for each } t \in J := [0, T], \; T > 0, \; 0 < \beta < 1, \tag{2.3}$$

$$y(0) = y_0, \tag{2.4}$$

where $^{c}D^{\beta}$ is the Caputo fractional derivative, $(E, ||.||)$ is a real Banach space, $f : J \times E \times E \to E$ is continuous function, and $y_0 \in E$.

Recently, fractional differential equations have been studied by Abbes *et al* [2, 3], Baleanu *et al* [21, 23], Diethelm [58], Kilbas and Marzan [86], Srivastava *et al* [87], Lakshmikantham *et al* [90], Samko *et al* [115]. More recently, some mathematicians have considered boundary value problems and boundary conditions for implicit fractional differential equations.

In [78], Hu and Wang investigated the existence of solution of the nonlinear fractional differential equation with integral boundary condition :

$$D^{\alpha}u(t) = f(t, u(t), D^{\beta}u(t)), \; t \in (0, 1), \; 1 < \alpha \leqslant 2, \; 0 < \beta < 1,$$

$$u(0) = u_0,\ u(1) = \int_0^1 g(s)u(s)ds,$$

where D^α is the Riemann-Liouville fractional derivative, $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, is continuous function and g be an integrable function.

In [116], by means of Schauder fixed-point theorem, Su and Liu studied the existence of nonlinear fractional boundary value problem involving Caputo's derivative :

$$^cD^\alpha u(t) = f(t, u(t), ^c D^\beta u(t)),\ \text{for each } t \in (0,1),\ 1 < \alpha \leqslant 2,\ 0 < \beta \leqslant 1,$$

$$u(0) = u'(1) = 0,\ \textit{or}\ u'(1) = u(1) = 0, \textit{or}\ u(0) = u(1) = 0,$$

where $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Many techniques have been developed for studying the existence and uniqueness of solutions of initial and boundary value problem for fractional differential equations. Several authors tried to develop a technique that depends on the Darbo or the Mönch fixed point theorems with the Hausdorff or Kuratowski measure of noncompactness. The notion of the measure of noncompactness was defined in many ways. In 1930, Kuratowski [89] defined the measure of non-compactness, $\alpha(A)$, of a bounded subset A of a metric space (X, d), and in 1955, Darbo [56] introduced a new type of fixed point theorem for noncompactness maps.

In Section 2.2, two results are discussed ; the first is based on Darbo's fixed point theorem combined with the technique of measures of noncompactness, the second on Mönch's fixed point theorem. At last, an example are included to show the applicability of the results. The content of this chapter is taken from ([42, 95]).

2.2 Existence results for NIFDEs

2.2.1 Existence of solutions

Let us defining what we mean by a solution of problem (2.1)-(2.2).

Definition 2.2.1 *A function $y \in C^1(J, \mathbb{R})$ is said to be a solution of the problem (2.1)-(2.2) if y satisfied equation (2.1) on J and condition (2.2) .*

For the existence of solutions for the problem (2.1)-(2.2), we need the following auxiliary lemma :

Lemma 2.2.2 *Let a function $f(t, u, v) : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous. Then the problem (2.1)-(2.2) is equivalent to the problem :*

$$y(t) = y_0 + I^\alpha \varphi(t) \tag{2.5}$$

where $\varphi \in C(J, \mathbb{R})$ satisfies the functional equation :

$$\varphi(t) = f(t, y_0 + I^\alpha(t), \varphi(t)).$$

Proof.
If ${}^cD^\alpha y(t) = \varphi(t)$ then $I^\alpha\, {}^cD^\alpha y(t) = I^\alpha \varphi(t)$. So we obtain $y(t) = y_0 + I^\alpha \varphi(t)$.
We are now in a position to state and prove the existence result for the problem (2.1)-(2.2) based on Banach's fixed point.

Theorem 2.2.3 *Assume that*

(H1) The function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H2) There exist constants $k > 0$ and $0 < l < 1$ such that

$$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \leqslant k|u - \bar{u}| + l|v - \bar{v}|$$

for any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in J$.

If

$$\frac{kT^\alpha}{(1 - l)\Gamma(\alpha + 1)} < 1, \tag{2.6}$$

then there exists a unique solution for the problem (2.1)-(2.2) on J.

Proof.
Transform the problem (2.1)-(2.2) into a fixed point problem. Define the operator $N : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ by :

$$N(y)(t) = y_0 + I^\alpha \varphi(t), \tag{2.7}$$

where $\varphi \in C(J, \mathbb{R})$ satisfies the functional equation

$$\varphi(t) = f(t, y(t), \varphi(t)).$$

Clearly, the fixed points of operator N are solutions of problem (2.1)-(2.2). Let $u, v \in C(J, \mathbb{R})$, then for $t \in J$, we have

$$(Nu)(t) - (Nv)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} (\varphi(s) - \psi(s)) ds,$$

where $\varphi, \psi \in C(J, \mathbb{R})$ be such that

$$\varphi(s) = f(s, u(s), \varphi(s)),$$

$$\psi(s) = f(s, v(s), \psi(s)).$$

Then, for $t \in J$

$$|(Nu)(t) - (Nv)(t)| \leqslant \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} |\varphi(s) - \psi(s)| ds. \tag{2.8}$$

By $(H2)$ we have

$$\begin{aligned} |\varphi(s) - \psi(s)| &= |f(s, u(s), \varphi(s)) - f(s, v(s), \psi(s))| \\ &\leqslant k|u(s) - v(s)| + l|\varphi(s) - \psi(s)|. \end{aligned}$$

Thus

$$|\varphi(s) - \psi(s)| \leqslant \frac{k}{1-l}|u(s) - v(s)|.$$

By (2.8) we have

$$\begin{aligned} |(Nu)(t) - (Nv)(t)| &\leqslant \frac{k}{(1-l)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|u(s) - v(s)|ds \\ &\leqslant \frac{kT^\alpha}{(1-l)\Gamma(\alpha+1)}\|u - v\|_\infty. \end{aligned}$$

Then

$$\|Nu - Nv\|_\infty \leqslant \frac{kT^\alpha}{(1-l)\Gamma(\alpha+1)}\|u - v\|_\infty.$$

By (2.6), the operator N is a contraction. Hence, by Banach's contraction principle, N has a unique fixed point which is a unique solution of the problem (2.1)-(2.2).

The next existence result is based on Schauder's fixed point theorem.

Theorem 2.2.4 *Assume (H1),(H2) and the following hypothesis holds.*
(H3) There exist $p, q, r \in C(J, \mathbb{R}_+)$ *with* $r^* = \sup_{t\in J} r(t) < 1$ *such that*

$$|f(t, u, w)| \leqslant p(t) + q(t)|u| + r(t)|w| \text{ for } t \in J \text{ and } u, w \in \mathbb{R}.$$

If

$$\frac{q^* T^\alpha}{(1-r^*)\Gamma(\alpha+1)} < 1, \tag{2.9}$$

where $p^* = \sup_{t\in J} p(t)$, *and* $q^* = \sup_{t\in J} q(t)$, *then the problem (2.1)-(2.2) has at least one solution.*

Proof. Let the operator N defined in (2.7). We shall show that N satisfies the assumptions of Schauder's fixed point theorem. The proof will be given in several steps.

Step 1 : N *is continuous.*
Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C(J, \mathbb{R})$. Then for each $t \in J$

$$|N(u_n)(t) - N(u)(t)| \leqslant \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|\varphi_n(s) - \varphi(s)|ds, \tag{2.10}$$

where $\varphi_n, \varphi \in C(J, \mathbb{R})$ such that

$$\varphi_n(s) = f(s, u_n(s), \varphi_n(s)),$$

and

$$\varphi(s) = f(s, u(s), \varphi(s)).$$

By $(H2)$ we have

$$\begin{array}{rcl} |\varphi_n(s) - \varphi(s)| & = & |f(s, u_n(s), \varphi_n(s)) - f(s, u(s), \varphi(s))| \\ & \leqslant & k|u_n(s) - u(s)| + l|\varphi_n(s) - \varphi(s)|. \end{array}$$

Then

$$|\varphi_n(s) - \varphi(s)| \leqslant \frac{k}{1-l}|u_n(s) - u(s)|.$$

Since $u_n \to u$, then we get $\varphi_n(s) \to \varphi(s)$ as $n \to \infty$ for each $s \in J$, and let $\eta > 0$ be such that, for each $s \in J$, we have $|\varphi_n(s)| \leqslant \eta$ and $|\varphi(s)| \leqslant \eta$, then, we have

$$\begin{array}{rcl} (t-s)^{\alpha-1}|\varphi_n(s) - \varphi(s)| & \leqslant & (t-s)^{\alpha-1}[|\varphi_n(s)| + |\varphi(s)|] \\ & \leqslant & 2\eta(t-s)^{\alpha-1}. \end{array}$$

For each $s \in J$, the function $s \to 2\eta(t-s)^{\alpha-1}$ is integrable on $[0, t]$, then the Lebesgue Dominated Convergence Theorem and (2.10) imply that

$$|N(u_n)(t) - N(u)(t)| \to 0 \; as \; n \to \infty,$$

hence

$$\|N(u_n) - N(u)\|_\infty \to 0 \text{ as } n \to \infty.$$

Consequently, N is continuous.
Let

$$\frac{M|y_0| + p^* T^\alpha}{M - q^* T^\alpha} \leqslant R,$$

where $M := (1 - r^*)\Gamma(\alpha + 1)$ and define

$$D_R = \{u \in C(J) : \|u\|_\infty \leqslant R\}.$$

It is clear that D_R is a bounded, closed and convex subset of $C(J, \mathbb{R})$.

Step 2 : $N(D_R) \subset D_R$.

Let $u \in D_R$ we show that $Nu \in D_R$.
We have, for each $t \in J$

$$|Nu(t)| \leqslant |y_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |\varphi(s)| ds. \tag{2.11}$$

By (H3) we have for each $s \in J$,

$$\begin{array}{rcl} |\varphi(s)| & = & |f(s, u(st), \varphi(s))| \\ & \leqslant & p^* + q^*R + r^*|\varphi(s)|. \end{array}$$

Then

$$|\varphi(s)| \leqslant \frac{p^* + q^*R}{1 - r^*}.$$

Thus (2.11) implies that

$$\begin{array}{rcl} |Nu(t)| & \leqslant & |y_0| + \dfrac{p^*T^\alpha}{(1 - r^*)\Gamma(\alpha + 1)} + \dfrac{q^*RT^\alpha}{M} \\ & \leqslant & |y_0| + \dfrac{p^*T^\alpha}{M} + \dfrac{q^*RT^\alpha}{M} \\ & \leqslant & R. \end{array}$$

Then $N(D_R) \subset D_R$.

Step 3 : $N(D_R)$ *is relatively compact.*

Let $t_1, t_2 \in J,\ t_1 < t_2$, and let $u \in D_R$. Then

$$\begin{array}{rcl} |N(u)(t_2) - N(u)(t_1)| & = & \left| \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}]\varphi(s)ds \right. \\ & & \left. + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t1}^{t_2} (t_2 - s)^{\alpha-1}\varphi(s)ds \right| \\ & \leqslant & \dfrac{M}{\Gamma(\alpha + 1)}(t_2^\alpha - t_1^\alpha + 2(t_2 - t_1)^\alpha). \end{array}$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero.

As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we conclude that $N : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ is continuous and compact. As a consequence of Schauder's fixed point theorem ([67]), we deduce that N has a fixed point which is a solution of the problem $(2.1) - (2.2)$.

The next existence result is based on Nonlinear alternative of Leray-Schauder type.

Theorem 2.2.5 *Assume (H1),(H2),(H3) hold. Then the problem (2.1)-(2.2) has at least one solution.*

Proof. Consider the operator N defined in (2.7). We shall show that N satisfies the assumptions of Leray-Schauder fixed point theorem. The proof will be given in several Steps.

Step 1 : N *is continuous.* See (Theorem 2.2.4 ,Step1)

Step 2 : N *maps bounded sets into bounded sets in* $C(J, \mathbb{R})$.

Indeed, it is enough to show that for any $\rho > 0$, there exist a positive constant ℓ such that for each $u \in B_\rho = \{u \in C(J, \mathbb{R}) : \|u\|_\infty \leqslant \rho\}$, we have $\|N(u)\|_\infty \leqslant \ell$.

For $u \in B_\rho$, we have, for each $t \in J$,

$$|Nu(t)| \leqslant |y_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |\varphi(s)| ds. \tag{2.12}$$

By (H3) we have for each $s \in J$,

$$\begin{aligned} |\varphi(s)| &= |f(s, u(s), \varphi(s))| \\ &\leqslant p(s) + q(s)\rho + r(s)|\varphi(s)| \\ &\leqslant p^* + q^*\rho + r^*|\varphi(s)|. \end{aligned}$$

Then

$$|\varphi(s)| \leqslant \frac{p^* + q^*\rho}{1 - r^*} := M^*.$$

Thus (2.12) implies that

$$|Nu(t)| \leqslant |y_0| + \frac{M^* T^\alpha}{\Gamma(\alpha+1)}.$$

Thus

$$\|Nu\|_\infty \leqslant |y_0| + \frac{M^* T^\alpha}{\Gamma(\alpha+1)} := l.$$

Then N maps bounded sets into bounded sets in $C(J, \mathbb{R})$.

Step 3 : *Clearly, N maps bounded sets into equicontinuous sets of $C(J, \mathbb{R})$.*

We conclude that $N : C(J, \mathbb{R}) \longrightarrow C(J, \mathbb{R})$ is completely continuous.

Last step : *A priori bounds.*

We now show there exists an open set $U \subseteq C(J, \mathbb{R})$ with $u \neq \lambda N(u)$, for $\lambda \in (0, 1)$ and $u \in \partial U$. Let $u \in C(J, \mathbb{R})$ and $u = \lambda N(u)$ for some $0 < \lambda < 1$. Thus for each $t \in J$, we have

$$u(t) = \lambda y_0 + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) ds.$$

This implies by (H2) that for each $t \in J$ we have

$$|u(t)| \leqslant |y_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |\varphi(s)| ds. \tag{2.13}$$

And, by (H3) we have for each $s \in J$,

$$|\varphi(s)| \leqslant p^* + q^*|u(s)| + r^*|\varphi(s)|.$$

Thus

$$|\varphi(s)| \leqslant \frac{1}{1-r^*}(p^* + q^*|u(s)|).$$

Hence

$$|u(t)| \leqslant |y_0| + \frac{p^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)} + \frac{q^*}{(1-r^*)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|u(s)|ds.$$

Then Lemma 1.1.13 implies that for each $t \in J$

$$|u(t)| \leqslant \left(|y_0| + \frac{p^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\right)\left(1 + \frac{kq^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\right).$$

Thus

$$\|u\|_\infty \leqslant \left(|y_0| + \frac{p^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\right)\left(1 + \frac{kq^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\right) := \overline{M}. \tag{2.14}$$

Let

$$U = \{u \in C(J) : \|u\|_\infty < \overline{M} + 1\}.$$

By the choice of U, there is no $u \in \partial U$ such that $u = \lambda N(u)$, for $\lambda \in (0,1)$. As a consequence of Leray-Schauder's theorem ([67]), we deduce that N has a fixed point u in $\overline{U}$ which is a solution to (2.1)-(2.2).

2.2.2 Examples

Example 1. Consider the following Cauchy problem

$$^cD^{\frac{1}{2}}y(t) = \frac{1}{2e^{t+1}(1+|y(t)|+|^cD^{\frac{1}{2}}y(t)|)}, \text{ for each, } t \in [0,1], \tag{2.15}$$

$$y(0) = 1. \tag{2.16}$$

Set

$$f(t,u,v) = \frac{1}{2e^{t+1}(1+|u|+|v|)}, \quad t \in [0,1], \ u, v \in \mathbb{R}.$$

Clearly, the function f is jointly continuous.
For any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in [0,1]$:

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \leqslant \frac{1}{2e}(|u-\bar{u}| + |v-\bar{v}|).$$

Hence condition (H2) is satisfied with $k = l = \frac{1}{2e}$.
It remains to show that condition (2.6) is satisfied. Indeed, we have

$$\frac{kT^{\alpha}}{(1-l)\Gamma(\alpha+1)} = \frac{1}{(2e-1)\Gamma(\frac{3}{2})} < 1.$$

It follows from Theorem 2.2.3 that the problem (2.15)-(2.16) has a unique solution.

Example 2. Consider the following Cauchy problem

$$^{c}D^{\frac{1}{2}}y(t) = \frac{(2+|y(t)|+|^{c}D^{\frac{1}{2}}y(t)|)}{2e^{t+1}(1+|y(t)|+|^{c}D^{\frac{1}{2}}y(t)|)}, \text{ for each, } t \in [0,1], \tag{2.17}$$

$$y(0) = 1. \tag{2.18}$$

Set

$$f(t,u,v) = \frac{(2+|u|+|v|)}{2e^{t+1}(1+|u|+|v|)}, \quad t \in [0,1], \ u,v \in \mathbb{R}.$$

Clearly, the function f is jointly continuous.
For any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in [0,1]$

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \leqslant \frac{1}{2e}(|u-\bar{u}| + |v-\bar{v}|).$$

Hence condition (H2) is satisfied with $k = l = \frac{1}{2e}$. Also, we have,

$$|f(t,u,v)| \leqslant \frac{1}{2e^{t+1}}(2+|u|+|v|).$$

Thus condition (H3) is satisfied with $p(t) = \frac{1}{e^{t+1}}$ and $q(t) = r(t) = \frac{1}{2e^{t+1}}$. And condition

$$\frac{q^{*}T^{\alpha}}{(1-r^{*})\Gamma(\alpha+1)} = \frac{1}{(2e-1)\Gamma(\frac{3}{2})} < 1,$$

is satisfied with $T = 1$, $\alpha = \frac{1}{2}$, and $q^{*} = r^{*} = \frac{1}{2e}$. It follows from Theorem 2.2.4 that the problem (2.17)-(2.18) has at least one solution.

2.3 Existence results for Cauchy problems in Banach space

2.3.1 Existence of solutions

Let us defining what we mean by a solution of problem (2.3)-(2.4).

Definition 2.3.1 *A function $u \in C^1(J, E)$ is said to be a solution of the problem (2.3)-(2.4) is u satisfied equation (2.3) and condition (2.4) on J.*

For the existence of solutions for the problem (2.3)-(2.4), we need the following auxiliary lemma :

Lemma 2.3.2 *Let a function $f(t, u, v) : J \times E \times E \to E$ be continuous. Then the problem (2.3)-(2.4) is equivalent to the problem :*

$$y(t) = y_0 + I^\beta \varphi(t) \tag{2.19}$$

where $\varphi \in C(J, E)$ satisfies the functional equation :

$$\varphi(t) = f(t, y_0 + I^\beta \varphi(t), \varphi(t)).$$

Proof. (See section 2.1)

We are now in a position to state and prove the existence result for the problem (2.3)-(2.4) based on concept of measures of noncompactness and Darbo's fixed point theorem.

Theorem 2.3.3 *Assume that*

(H4) The function $f : J \times E \times E \to E$ is continuous.

(H5) There exist constants $k > 0$ and $0 < l < 1$ such that

$$||f(t, u, v) - f(t, \bar{u}, \bar{v})|| \leqslant k||u - \bar{u}|| + l||v - \bar{v}||$$

for any $u, v, \bar{u}, \bar{v} \in E$ and $t \in J$.

(H6) There exist $p, q, r \in C(J, \mathbb{R}_+)$ with $r^* = \sup_{t \in J} r(t) < 1$ such that

$$||f(t, u, w)|| \leqslant p(t) + q(t)|u| + r(t)|v| \text{ for } t \in J \text{ and } u, v \in \mathbb{R}.$$

(H7) For any bounded sets $B_1, B_2 \subseteq E$,

$$\alpha\Big(f(t, B_1, B_2)\Big) \leqslant q(t)\alpha(B_1) + r(t)\alpha(B_2) \text{ for each } t \in J.$$

If

$$\frac{q^* T^\beta}{(1 - r^*)\Gamma(\beta + 1)} < 1, \tag{2.20}$$

where $q^* = \sup_{t \in J} q(t)$, then the problem (2.3)-(2.4) has at least one solution on J.

Proof. The proof will be given in several steps.

Transform the problem (2.3)-(2.4) into a fixed point problem. Define the operator

$N : C(J, E) \to C(J, E)$ by :

$$N(y)(t) = y_0 + I^\beta \varphi(t) \tag{2.21}$$

where $\varphi \in C(J, E)$ satisfies the functional equation

$$\varphi(t) = f(t, y(t), \varphi(t)).$$

Clearly, the fixed points of operator N are solutions of problem (2.3)-(2.4).

We shall show that N satisfies the assumptions of Darbo's fixed point Theorem. The proof will be given in several steps.

Step 1 : *N is continuous.*
Let $u, w \in C(J, E)$ and let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C(J, E)$. Then for each $t \in J$

$$||N(u_n)(t) - N(u)(t)|| \leqslant \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} ||\varphi_n(s) - \varphi(s)|| ds, \tag{2.22}$$

where $\varphi_n, \varphi \in C(J, E)$ such that

$$\varphi_n(t) = f(t, u_n(t), \varphi_n(t)),$$

and

$$\varphi(t) = f(t, u(t), \varphi(t)).$$

By $(H5)$ we have, for each $t \in J$,

$$\begin{array}{rcl} ||\varphi_n(t) - \varphi(t)|| & = & ||f(t, u_n(t), \varphi_n(t)) - f(t, u(t), \varphi(t))|| \\ & \leqslant & k||u_n(t) - u(t)|| + l||\varphi_n(t) - \varphi(t)||. \end{array}$$

Then

$$||\varphi_n(t) - \varphi(t)|| \leqslant \frac{k}{1-l} ||u_n(t) - u(t)||.$$

Since $u_n \to u$, then we get $\varphi_n(t) \to \varphi(t)$ as $n \to \infty$ for each $t \in J$.
Let a positive constant $\eta > 0$ be such that, for each $t \in J$, we have $||\varphi_n(t)|| \leqslant \eta$ and $||\varphi(t)|| \leqslant \eta$.
Then we have,

$$\begin{array}{rcl} (t-s)^{\beta-1} ||\varphi_n(s) - \varphi(s)|| & \leqslant & (t-s)^{\beta-1} [||\varphi_n(s)|| + ||\varphi(s)||] \\ & \leqslant & 2\eta (t-s)^{\beta-1}. \end{array}$$

For each $t \in J$, the function $s \longrightarrow 2\eta(t-s)^{\beta-1}$ is integrable on $[0,t]$, then by means of the Lebesgue Dominated Convergence Theorem and (2.22) has that

$$||N(u_n)(t) - N(u)(t)|| \to 0 \; as \; n \to \infty.$$

Then

$$\|| N(u_n) - N(u) \||_\infty \to 0 \; as \; n \to \infty.$$

Consequently, N is continuous.

Let

$$\frac{M|y_0| + p^* T^\alpha}{M - q^* T^\alpha} \leqslant R, \tag{2.23}$$

where $M := (1 - r^*)\Gamma(\alpha + 1)$, $p^* = \sup_{t \in J} p(t)$, and $q^* = \sup_{t \in J} q(t)$.
Define

$$D_R = \{u \in C(J) : \|u\|_\infty \leqslant R\}.$$

It is clear that D_R is a bounded, closed and convex subset of $C(J, E)$.

Step 2 : $N(D_R) \subset D_R$.

Let $u \in D_R$ we show that $Nu \in D_R$. We have, for each $t \in J$

$$||Nu(t)|| \leqslant ||y_0|| + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} ||\varphi(t)|| ds. \tag{2.24}$$

By (H6) we have for each $t \in J$,

$$||\varphi(t)|| \leqslant \frac{p^* + q^* R}{1 - r^*}.$$

Thus, (2.23) and (2.24) implies that

$$\begin{aligned} ||Nu(t)|| &\leqslant ||y_0|| + \frac{p^* T^\beta}{(1-r^*)\Gamma(\beta+1)} + \frac{q^* R T^\beta}{(1-r^*)\Gamma(\beta+1)} \\ &\leqslant ||y_0|| + \frac{p^* T^\beta}{M} + \frac{q^* R T^\beta}{M} \\ &\leqslant R. \end{aligned}$$

Consequently,

$$N(D_R) \subset D_R.$$

Step 3 : *$N(D_R)$ is bounded and equicontinuous.*

By **Step 2** we have $N(D_R)=\{N(u):u\in D_R\}\subset D_R$. Thus, for each $y\in D_R$ we have $\|N(u)\|_\infty \leqslant R$ which means that $N(D_R)$ is bounded.

Let $t_1,t_2\in J,\ t_1<t_2$, and let $u\in D_R$. Then

$$\begin{aligned} |N(u)(t_2)-N(u)(t_1)| &= \Big|\frac{1}{\Gamma(\beta)}\int_0^{t_1}[(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}]\varphi(s)ds \\ &\quad +\frac{1}{\Gamma(\beta)}\int_{t1}^{t_2}[(t_2-s)^{\beta-1}\varphi(s)ds\Big| \\ &\leqslant \frac{M}{\Gamma(\beta+1)}(t_2^\beta-t_1^\beta+2(t_2-t_1)^\beta). \end{aligned}$$

As $t_1\to t_2$, the right-hand side of the above inequality tends to zero. Then $N(D_R)$ is equicontinuous.

Last step : *The operator $N:D_R\to D_R$ is a strict set contraction.*

Let $V\subset D_R$ and $t\in J$, then we have,

$$\begin{aligned} \alpha(N(V)(t)) &= \alpha((Ny)(t),y\in V) \\ &\leqslant \frac{1}{\Gamma(\beta)}\Big\{\int_0^t (t-s)^{\beta-1}\alpha(\varphi(s))ds, y\in V\Big\}. \end{aligned}$$

Then $(H7)$ imply that, for each $s\in J$,

$$\begin{aligned} \alpha(\{\varphi(s),y\in V\}) &= \alpha(\{f(s,y(s),\varphi(s)),y\in V\}) \\ &\leqslant q(t)\alpha(\{y(s),y\in V\})+r(t)\alpha(\{\varphi(s),y\in V\}) \\ &\leqslant q^*\alpha(\{y(s),y\in V\})+r^*\alpha(\{\varphi(s),y\in V\}). \end{aligned}$$

Thus

$$\alpha\left(\{\varphi(s),y\in V\}\right) \leqslant \frac{q^*}{1-r^*}\alpha\{y(s),y\in V\}.$$

Then

$$\begin{aligned} \alpha(N(V)(t)) &\leqslant \frac{q^*}{(1-r^*)\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\{\alpha(y(s))ds,\},y\in V \\ &\leqslant \frac{q^*\alpha_c(V)}{(1-r^*)\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}ds \\ &\leqslant \frac{q^*T^\beta}{(1-r^*)\Gamma(\beta+1)}\alpha_c(V). \end{aligned}$$

Therefore

$$\alpha_c(NV) \leqslant \frac{q^* T^\beta}{(1-r^*)\Gamma(\beta+1)}\alpha_c(V).$$

So, by (2.20), the operator N is a set contraction. As a consequence of Theorem 1.5.4, we deduce that N has a fixed point which is solution to the problem (2.3)-(2.4).

The next existence result for the problem (2.3)-(2.4) is based on concept of measures of noncompactness and Mönch's fixed point theorem.

Theorem 2.3.4 *Assume (H4)-(H7) hold. Then the problem (2.3)-(2.4) has at least one solution.*

Proof. Consider the operator N defined in (2.21). We shall show that N satisfies the assumptions of Mönch's fixed point theorem. We know that $N : D_R \to D_R$ is bounded and continuous, we need to prove that there the implication

$$V = \overline{conv} N(V) \quad \text{or} \quad V = N(V) \cup \{0\} \Rightarrow \alpha(V) = 0$$

holds for every subset V of D_R.

Now let V be a subset of D_R such that $V \subset \overline{conv}(N(V) \cup \{0\})$. V is bounded and equicontinuous and therefore the function $t \to v(t) = \alpha(V(t))$ is continuous on J. By $(H7)$, Lemma 1.4.8 and the properties of the measure α we have for each $t \in J$

$$\begin{aligned}
v(t) &\leqslant \alpha(N(V)(t) \cup \{0\}) \\
&\leqslant \alpha(N(V)(t)) \\
&\leqslant \alpha\{(Ny)(t), y \in V\} \\
&\leqslant \frac{q^*}{(1-r^*)\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}\{\alpha(y(s))ds, y \in V\} \\
&\leqslant \frac{q^*}{(1-r^*)\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v(s)ds.
\end{aligned}$$

Lemma 1.1.13 implies that $v(t) = 0$ for each $t \in J$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzelà theorem, V is relatively compact in D_R. Applying now Theorem 1.5.6 we conclude that N has a fixed point $y \in D_R$. Hence N has a fixed point which is solution to the problem (2.3)-(2.4). This completes the proof.

2.3.2 Example

Consider the following infinite system

$$^{c}D^{\frac{1}{2}}y_n(t) = \frac{(3 + ||y_n(t)|| + ||^{c}D^{\frac{1}{2}}y_n(t)||)}{3e^{t+2}(1 + ||y_n(t)|| + ||^{c}D^{\frac{1}{2}}y_n(t)||)}, \text{ for each, } t \in [0, 1], \tag{2.25}$$

$$y_n(0) = 1. \tag{2.26}$$

Set

$$E = l^1 = \{y = (y_1, y_2, ..., y_n, ...), \sum_{n=1}^{\infty} |y_n| < \infty\},$$

and

$$f(t, u, v) = \frac{(3 + ||u|| + ||v||)}{3e^{t+2}(1 + ||u|| + ||v||)}, \quad t \in [0, 1], \ u, v \in E.$$

E is a Banach space with the norm $||y|| = \sum_{n=1}^{\infty} |y_n|$.
Clearly, the function f is jointly continuous.
For any $u, v, \bar{u}, \bar{v} \in E$ and $t \in [0, 1]$:

$$||f(t, u, v) - f(t, \bar{u}, \bar{v})|| \leqslant \frac{1}{3e^2}(||u - \bar{u}|| + ||v - \bar{v}||).$$

Hence condition (H5) is satisfied with $k = l = \frac{1}{3e^2}$.
Other,

$$||f(t, u, v)|| \leqslant \frac{1}{3e^{t+2}}(3 + ||u|| + ||v||).$$

Thus conditions (H6) and (H7) are satisfied with $p(t) = \frac{1}{e^{t+2}}$ and $q(t) = r(t) = \frac{1}{3e^{t+2}}$.
It follows from Theorem 2.3.4 that the problem (2.25)-(2.26) has at least one solution on J.

Chapitre 3

IFDEs with Integral Boundary Conditions (3)

[1]

3.1 Introduction

In this chapter, we are concerned with the existence of solutions for the following fractional differential equations with integral boundary conditions :

$$^cD^{\alpha}y(t) = f(t, y(t), ^cD^{\alpha}y(t)), \;\; \text{for each } t \in J = [0, b], \; 0 < \alpha < 1, \tag{3.1}$$

$$y(0) + \lambda \int_0^b y(t)dt = y(b), \tag{3.2}$$

where $^cD^{\alpha}$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R}$ is a given function, and $\lambda \in (0, +\infty)$.
we present two results for the problem (3.1)-(3.2). The first one is based on the Banach contraction principle, the second one on Schauder's fixed point theorem. Finally, we present two illustrative examples.

In [47] Benchohra and Mostefai studied the existence of weak solutions, for the boundary value problem, for fractional differential equations with mixed boundary conditions of the form

$$^cD^{\alpha}x(t) = f(t, x(t)), \;\; \text{for each } t \in I = [0, T], \; 0 < \alpha \leqslant 1,$$

$$x(0) + \mu \int_0^T x(t)dt = x(T),$$

[1(3)] [43] Benchohra and **K. Maazouz**, Existence and uniqueness results for implicit fractional differential equations with integral boundary conditions. *Commun. Appl. Anal.* **20** (2016), 355-366.

where ${}^cD^\alpha$ is the Caputo fractional derivative, $f : I \times E \longrightarrow E$ is a given function. E is a Banach space , and $\mu \in \mathbb{R}^*$. To investigate the existence of solutions of the problem above, they used Mönch's fixed point theorem combined with the technique of measures of weak noncompactness, which is an important method for seeking solutions of differential equations. This technique was mainly initiated in the monograph of Banas and Goebel [24] and subsequently developed and used in many papers ; see, for example, Banas et al. [25].

In [33] Hamani et al. studied the existence and uniqueness of solutions of the boundary value problem with fractional order differential inclusions and nonlinear integral conditions of the form

$$
{}^cD^\alpha y(t) \in F(t, y), \quad \text{for each } t \in I = [0, T],\ 1 < \alpha \leqslant 2,
$$

$$
y(0) - y'(0) = \int_0^T g(s, y)ds,
$$

$$
y(T) + y'(T) = \int_0^T h(s, y)ds,
$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $F : J \times \mathbb{R} \longrightarrow P(\mathbb{R})$ is a multivalued map, ($P(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$), and $g, h : J \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

In [48] Benchohra and Ouaar studied the existence of solutions, for following boundary value problem for fractional differential equations with mixed boundary conditions.

$$
{}^cD^\alpha y(t) = f(t, y(t)), \quad \text{for each } t \in J = [0, T],\ \alpha \in (0, 1],
$$

$$
y(0) + \mu \int_0^T y(s)ds = y(T),
$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given function, and $\mu \in \mathbb{R}^*$.

3.2 Existence of solutions

Let us defining what we mean by a solution of problem (3.1)-(3.2)

Definition 3.2.1 *A function $y \in C^1(J, \mathbb{R})$ is said to be a solution of (3.1)-(3.2) if y satisfies*
the equation ${}^cD^\alpha y(t) = f(t, y(t), {}^c D^\alpha y(t))$ on J, and the condition (3.2).

For the existence results for the problem (3.1)-(3.2) we need the following auxiliary lemmas.

Lemma 3.2.2 *Let $0 < \alpha < 1$ and let $h \in C(J, \mathbb{R})$ be a given function. Then the boundary value problem*

$$^cD^\alpha y(t) = h(t), \quad t \in J, \tag{3.3}$$

$$y(0) + \lambda \int_0^b y(t)dt = y(b) \tag{3.4}$$

has a unique solution given by

$$y(t) = \int_0^b G(t,s)h(s)ds$$

where $G(t,s)$ is the Green's function defined by

$$G(t,s) = \begin{cases} \dfrac{1}{b\Gamma(\alpha)}\Big(\dfrac{(b-s)^{\alpha-1}}{\lambda} + \dfrac{\alpha b(t-s)^{\alpha-1} - (b-s)^\alpha}{\alpha}\Big) & \textit{if} \quad 0 \leqslant s < t \\ \dfrac{1}{b\Gamma(\alpha)}\Big(\dfrac{(b-s)^{\alpha-1}}{\lambda} - \dfrac{(b-s)^\alpha}{\alpha}\Big) & \textit{if} \quad t \leqslant s < b. \end{cases} \tag{3.5}$$

Proof. By Lemma 1.1.10 we have

$$\begin{aligned} y(t) &= I^\alpha(^cD^\alpha y(t)) \\ &= I^\alpha h(t) - c_0 \qquad \text{for some constant } c_0 \in \mathbb{R}. \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds - c_0. \end{aligned}$$

We have by integration using Fubini's integral theorem

$$\begin{aligned} \int_0^b y(s)ds &= \int_0^b \Big(\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}h(\tau)d\tau - c_0\Big)ds \\ &= \int_0^b \Big(\frac{1}{\Gamma(\alpha)}\int_\tau^b (s-\tau)^{\alpha-1}ds\Big)h(\tau)d\tau - c_0 b \\ &= \frac{1}{\Gamma(\alpha+1)}\int_0^b (b-\tau)^\alpha h(\tau)d\tau - c_0 b. \end{aligned}$$

Applying the boundary condition (3.2), we have $y(0) = -c_0$

$$y(b) = \frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}h(s)ds - c_0$$

that is

$$c_0 = \frac{1}{b\Gamma(\alpha)}\int_0^b \left(\frac{(b-s)^\alpha}{\alpha} - \frac{(b-s)^{\alpha-1}}{\lambda}\right)(h(s)ds.$$

Therefore, the unique solution of (3.3)-(3.4) is

$$\begin{aligned}
y(t) &= \frac{1}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1}h(s)ds + \frac{1}{b}\int_0^b \Big(\frac{(b-s)^{\alpha-1}}{\lambda} - \frac{(b-s)^{\alpha}}{\alpha}\Big)h(s)ds\right] \\
&= \frac{1}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1} + \frac{1}{b}\Big(\frac{(b-s)^{\alpha-1}}{\lambda} - \frac{(b-s)^{\alpha}}{\alpha}\Big)\right]h(s)ds \\
&+ \frac{1}{b\Gamma(\alpha)}\int_t^b \left(\frac{(b-s)^{\alpha-1}}{\lambda} - \frac{(b-s)^{\alpha}}{\alpha}\right)h(s)ds \\
&= \int_0^b G(t,s)h(s)ds.
\end{aligned}$$

Remark 3.2.3 *The function* $t \in J \longmapsto \int_0^b G(t,s)ds$ *is continuous on* J*, and hence is bounded. Let*

$$G^* = \sup_{t\in J}\left\{\int_0^b |G(t,s)|ds\right\}.$$

Lemma 3.2.4 *A function* $y \in C^1(J,\mathbb{R})$ *is a solution of the problem (3.1)-(3.2) if and only if* $y \in C(J,\mathbb{R})$ *is a solution of the integral equation*

$$y(t) = \int_0^b G(t,s)\varphi(s)ds, \tag{3.6}$$

where $G(t,s)$ *is the Green's function given by (3.5) and* $\varphi \in C(J,\mathbb{R})$ *satisfies the implicit functional equation*

$$\varphi(s) = f(s, y(s), \varphi(s)).$$

Theorem 3.2.5 *Assume that*
(**H1**) $f : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ *is continuous.*
(**H2**) *There exist constants* $0 < l < 1$ *and* $k > 0$ *such that*

$$|f(t,x,y) - f(t,\overline{x},\overline{y})| \leqslant k|x-\overline{x}| + l|y-\overline{y}|$$

for any $x,\ y,\ \overline{x},\ \overline{y} \in \mathbb{R}$*, and* $t \in J$.
If

$$\frac{bkG^*}{1-l} < 1,$$

then there exists a unique solution for the problem (3.1)-(3.2).

Proof. We transform the problem (3.1)-(3.2) into fixed point problem. Consider the operator
$A : C(J, \mathbb{R}) \longrightarrow C(J, \mathbb{R})$ defined by

$$A(y)(t) = \int_0^b G(t,s)\varphi(s)ds, \tag{3.7}$$

where $G(t, s)$ is the Green's function given by (3.5) and $\varphi \in C(J, \mathbb{R})$ satisfies the implicit functional equation

$$\varphi(s) = f(s, y(s), \varphi(s)).$$

Clearly, from Lemmas 3.2.2 and 3.2.4 , the fixed points of A are solutions to the problem (3.1)-(3.2). We shall show that A is a contraction.
Let u, $v \in C(J, \mathbb{R})$. Then, for each $t \in J$, we have

$$(Au)(t) - (Av)(t) = \int_0^b G(t,s)\big(\varphi(s) - \psi(s)\big)ds,$$

where

$$\begin{aligned} \varphi(s) &= f(s, u(s), \varphi(s)\big), \\ \psi(s) &= f(s, v(s), \psi(s)), \end{aligned}$$

and

$$|\varphi(s) - \psi(s)| \leqslant k|u(s) - v(s)| + l|\varphi(s) - \psi(s)|.$$

Thus,

$$|\varphi(s) - \psi(s)| \leqslant \frac{k}{1-l}|u(s) - v(s)|.$$

Then,

$$\begin{aligned} \Big|(Au)(t) - (Av)(t)\Big| &\leqslant \int_0^b \Big|G(t,s)\big(\varphi(s) - \psi(s)\big)\Big|ds \\ &\leqslant \frac{k}{1-l}\int_0^b |G(t,s)||u(s) - v(s)|ds \\ &\leqslant \frac{bkG^*}{1-l}\|u - v\|_\infty. \end{aligned}$$

Thus

$$\|Au - Av\|_\infty \leqslant \frac{bkG^*}{1-l}\|u - v\|_\infty.$$

Since $\dfrac{bkG^*}{1-l} < 1$, the operator A is a contraction.
Then by Banach's fixed point theorem, the problem (3.1)-(3.2) has a unique solution on $[0, b]$.
Now we give an existence result based on Schauder's fixed point theorem.

Theorem 3.2.6 *Assume (**H1**) and (**H2**) hold. If*

$$1 - l - bkG^* > 0, \tag{3.8}$$

the problem (3.1)-(3.2) has at least one solution.

Proof. Let

$$D = \{y \in C(J, \mathbb{R}) : \|y\|_\infty \leqslant \gamma\}$$

where

$$\gamma > \frac{bf^*G^*}{1 - l - bkG^*}, \tag{3.9}$$

with $\quad f^* = \sup_{t \in J} |f(t, 0, 0)|$.

It is clear that D is a closed, convex subset of $C(J, \mathbb{R})$. Let the operator A be defined in (3.7). We shall show that A satisfies the assumptions of Schauder's fixed point Theorem. The proof will be given in several steps.

Step 1 : A is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \longrightarrow u$ in $C(J, \mathbb{R})$. Then for each $t \in J$, we have

$$\varphi_n(s) = f(s, u_n(s), \varphi_n(s))$$

and

$$\varphi(s) = f(s, u(s), \varphi(s)).$$

We have

$$|\varphi_n(s) - \varphi(s)| \leqslant k|u_n(s) - u(s)| + l|\varphi_n(s) - \varphi(s)|.$$

Thus,

$$|\varphi_n(s) - \varphi(s)| \leqslant \frac{k}{1 - l}|u_n(s) - u(s)|.$$

Then,

$$\begin{aligned} \Big|(Au_n)(t) - (Au)(t)\Big| &\leqslant \int_0^b \Big|G(t, s)\big(\varphi_n(s) - \varphi(s)\big)\Big| ds \\ &\leqslant \frac{k}{1 - l} \int_0^b |G(t, s)||u_n(s) - u(s)| ds. \end{aligned}$$

Since $u_n \longrightarrow u$, we get $\varphi_n \longrightarrow \varphi$, and the Lebesgue dominated convergence Theorem implies that

$$\|A(u_n) - A(u)\|_\infty \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Hence then A is continuous.

Step 2 : $A(D) \subset D$

Let $y \in D$. We will show that $Ay \in D$. For each $t \in J$, we have

$$|(Ay)(t)| \;=\; \Big|\int_0^b G(t, s)\varphi(s) ds\Big|$$

$$\leqslant \int_0^b |G(t,s)||\varphi(s)|ds.$$

By (**H2**) we have

$$\begin{aligned}
|\varphi(s)| &= |f(s,y(s),\varphi(s))| \\
&\leqslant |f(s,y(s),\varphi(s)) - f(s,0,0)| + |f(s,0,0)| \\
&\leqslant k|y(s)| + l|\varphi(s)| + |f(s,0,0)| \\
&\leqslant \frac{k|y(s)| + |f(s,0,0)|}{1-l} \\
&\leqslant \frac{k\|y\|_\infty + f^*}{1-l}.
\end{aligned}$$

Then,

$$\begin{aligned}
|(Ay)(t)| &\leqslant \frac{k\|y\|_\infty + f^*}{1-l}\int_0^b |G(t,s)|ds \\
&\leqslant \frac{k\|y\|_\infty + f^*}{1-l} bG^* \\
&\leqslant \frac{k\gamma + f^*}{1-l} bG^*.
\end{aligned}$$

By (3.9) we have

$$\|Ay\|_\infty \leqslant \gamma,$$

so $A(D) \subset D$.
Step 3 : A maps D into a equicontinuous set of $C(J,\mathbb{R})$.
Let $y \in D$, t_1, $t_2 \in J$, $t_1 < t_2$; then

$$\begin{aligned}
\Big|(Ay)(t_2) - (Ay)(t_1)\Big| &= \Big|\int_0^b G(t_2,s)\varphi(s))ds - \int_0^b G(t_1,s)\varphi(s))ds\Big| \\
&\leqslant \int_0^b \Big|G(t_2,s) - G(t_1,s)\Big||\varphi(s)|ds \\
&\leqslant \frac{k\|y\|_\infty + f^*}{1-l}\int_0^b \Big|G(t_2,s) - G(t_1,s)\Big|ds.
\end{aligned}$$

As $t_1 \longrightarrow t_2$ the right hand side of the above inequality tends to zero. By the Arzelà-Ascoli Theorem, A is completely continuous. Therefore, we deduce that A has a fixed point y which is a solution of problem (3.1)-(3.2).

3.2.1 Examples

In this section we give two examples to illustrate the usefulness of our main results.

Example 3.2.7 *Consider the boundary value problem*

$$^{c}D^{\frac{1}{2}}y(t) = \frac{|y(t)| + |^{c}D^{\frac{1}{2}}y(t)|}{10(1 + |y(t)| + |^{c}D^{\frac{1}{2}}y(t)|)}, \quad t \in J = [0,1], \tag{3.10}$$

$$y(0) + \int_0^1 y(t)dt = y(1). \tag{3.11}$$

Set

$$f(t,x,y) = \frac{x+y}{10(1+x+y)} \qquad (t,x,y) \in J \times [0,\infty) \times [0,\infty).$$

It is clear that f is continuous.
Let $x,\ y \in [0,\infty)$ and $t \in J$; then

$$\begin{aligned}
|f(t,x,y) - f(t,\overline{x},\overline{y})| &= \frac{1}{10}\left|\frac{x+y}{1+x+y} - \frac{\overline{x}+\overline{y}}{1+\overline{x}+\overline{y}}\right| \\
&= \frac{1}{10}\left|\frac{1}{1+\overline{x}+\overline{y}} - \frac{1}{1+x+y}\right| \\
&\leqslant \frac{1}{10}\left|x+y-\overline{x}-\overline{y}\right| \\
&\leqslant \frac{1}{10}\Big(|x-\overline{x}| + |y-\overline{y}|\Big).
\end{aligned}$$

*Then the condition (**H2**) holds with*

$$k = l = \frac{1}{10}.$$

From (3.5), G is given by

$$G(t,s) = \begin{cases} \dfrac{(1-s)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} + \dfrac{\frac{1}{2}(t-s)^{-\frac{1}{2}} - (1-s)^{\frac{1}{2}}}{\frac{1}{2}\Gamma(\frac{1}{2})} & \text{if} \quad 0 \leqslant s < t \\ \dfrac{(1-s)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} - \dfrac{(1-s)^{\frac{1}{2}}}{\frac{1}{2}\Gamma(\frac{1}{2})} & \text{if} \quad t \leqslant s < 1. \end{cases} \tag{3.12}$$

From (3.12) we have

$$\begin{aligned}
\int_0^1 G(t,s)ds &= \int_0^t G(t,s)ds + \int_t^1 G(t,s)ds \\
&= -\frac{(1-t)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} + \frac{1}{\Gamma(\frac{3}{2})} + \frac{t^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} \\
&\quad -\frac{1}{\Gamma(\frac{5}{2})} + \frac{(1-t)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}.
\end{aligned}$$

We can easily see that

$$G^* < \frac{4}{\Gamma(\frac{3}{2})} + \frac{1}{\Gamma(\frac{5}{2})} < \frac{10}{\sqrt{\pi}}.$$

Since $0 < \dfrac{bkG*}{1-l} < \dfrac{10}{9\sqrt{\pi}} < 1$, *Theorem 3.2.5 implies that the problem (3.10)-(3.11) has a unique solution.*

Example 3.2.8 *Consider the boundary value problem*

$$^cD^\alpha y(t) = \frac{3 + |y(t)| + |^cD^\alpha y(t)|}{(20+e^t)(1+|y(t)|+|^cD^\alpha y(t)|)} \quad t \in J = [0,1],\ \alpha \in (0,1), \tag{3.13}$$

$$y(0) + \int_0^1 y(t)dt = y(1). \tag{3.14}$$

Set

$$f(t,x,y) = \frac{3+|x|+|y|}{(20+e^t)(1+|x|+|y|)} \qquad (t,x,y) \in J \times \mathbb{R} \times \mathbb{R}.$$

It is clear that f *is continuous .*
Let $x,\ y \in \mathbb{R}$ *and* $t \in J$ *; then*

$$\begin{aligned}
|f(t,x,y) - f(t,\overline{x},\overline{y})| &= \frac{1}{(20+e^t)}\left|\frac{3+|x|+|y|}{1+|x|+|y|} - \frac{3+|\overline{x}|+|\overline{y}|}{1+|\overline{x}|+|\overline{y}|}\right| \\
&= \frac{2}{(20+e^t)}\left|\frac{1}{1+|\overline{x}|+|\overline{y}|} - \frac{1}{1+|x|+|y|}\right| \\
&\leqslant \frac{2}{(20+e^t)}\Big||x|+|y|-|\overline{x}|-|\overline{y}|\Big| \\
&\leqslant \frac{1}{10}\Big(|x-\overline{x}|+|y-\overline{y}|\Big).
\end{aligned}$$

*Then the assumption (**H2**) holds with*

$$k = l = \frac{1}{10}.$$

And we have

$$f^* = \sup_{t\in J}|f(t,0,0)| = \frac{1}{7}.$$

From (3.5), G is given by

$$G(t,s) = \begin{cases} \dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{\alpha(t-s)^{\alpha-1} - (1-s)^\alpha}{\alpha\Gamma(\alpha)} & if \quad 0 \leqslant s < t \\ \dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{(1-s)^\alpha}{\alpha\Gamma(\alpha)} & if \quad t \leqslant s < 1. \end{cases} \tag{3.15}$$

From (3.15) we have

$$\begin{aligned}\int_0^1 G(t,s)ds &= \int_0^t G(t,s)ds + \int_t^1 G(t,s)ds\\ &= -\frac{(1-t)^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)} + \frac{t^\alpha}{\Gamma(\alpha+1)}\\ &\quad -\frac{1}{\Gamma(\alpha+2)} + \frac{(1-t)^\alpha}{\Gamma(\alpha+1)}.\end{aligned}$$

We can easily see that

$$G^* < \frac{4}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+2)}$$

Condition (3.9) is satisfied for appropriate values of $\alpha \in (0,1)$. *Theorem 3.2.6 implies that the problem (3.13)-(3.14) has a at least one solution.*

Chapitre 4

NBVP for Implicit Fractional Differential Equations

4.1 Introduction

The purpose of this Chapter, is to establish existence and uniqueness results to the following problems for implicit fractional-order differential equations :

$$^cD^\alpha y(t) = f(t, y(t), ^cD^\alpha y(t)), \text{ for each } t \in J = [0, T], T > 0, \ 0 < \alpha < 1, \tag{4.1}$$

$$ay(0) + by(T) = c \tag{4.2}$$

where $^cD^\alpha$ is the fractional derivative of Caputo, $f : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ a continuous function, and a, b, c are real constants with $a + b \neq 0$.
and

$$^cD^\alpha y(t) = f(t, y(t), ^cD^\alpha y(t)), \text{ for each } t \in J = [0, T], T > 0, \ 0 < \alpha < 1, \tag{4.3}$$

$$y(0) + g(y) = y_0, \tag{4.4}$$

where $^cD^\alpha$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $g : C(J, \mathbb{R}) \to \mathbb{R}$ is a continuous function and $y_0 \in \mathbb{R}$. See ([41])

In [36], Benchohra *et al.* studied the existence of solutions for boundary value problems, for following implicit fractional-order differential equation :

$$^cD^\alpha y(t) = f(t, y(t)), \text{ for each } t \in J = [0, T], T > 0, \ 0 < \alpha \leqslant 1,$$

$$ay(0) + by(T) = c,$$

where $^cD^\alpha$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \to \mathbb{R}$ is a given function and a, b, c are real constants with $a + b \neq 0$.

In [35], Benchohra and Hamani studied the existence of solutions for boundary value problems, for fractional order differential inclusions :

$$^{c}D^{\alpha}y(t) \in F(t, y(t)) = 0, \text{ for each } t \in J = [0, T], \ 0 < \alpha \leqslant 1,$$

$$ay(0) + by(T) = c,$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative, $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a given multivalued function and a, b, c are real constants with $a + b \neq 0$.

In [84], Karthikeyan and Trujillo studied the existence of nonlinear fractional boundary value problem :

$$^{c}D^{\alpha}y(t) = \varphi(t)f(t, y(t), (Sy)(t)), \text{ for each } t \in J = [0, T], \ 0 < \alpha < 1,$$

$$ay(0) + by(T) = c,$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative of order α, $f : J \times X \times X \to X$ is a given function, X is a Banach spaces and a, b, c are real constants with $a + b \neq 0$, and S is a nonlinear integral operator given by

$$(Sy)(t) = \int_0^t k(t, s)y(s)ds.$$

where $k \in C(J \times J, \mathbb{R}^+)$.

Fractional differential equations with nonlocal conditions have been discussed in [4, 9, 59, 70, 101, 102] and references therein. Nonlocal conditions were initiated by Byszewski [53] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski [51, 52], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena. For example, in [57], the author used

$$g(y) = \sum_{i=1}^{p} c_i y(\tau_i) \tag{4.5}$$

where $c_i, \ i = 1, \ldots, p$, are given constants and $0 < \tau_1 < ... < \tau_p \leqslant T$, to describe the diffusion phenomenon of a small amount of gas in a transparent tube. In this case, (4.5) allows the additional measurements at $\tau_i, \ i = 1, \ldots, p$.

4.2 Existence of solutions

Let us defining what we mean by a solution of problem (4.1)-(4.2).

Definition 4.2.1 *A function $u \in C(J,\mathbb{R})$ is said to be a solution of the problem (4.1)-(4.2) if u satisfies equation (4.1) on J and condition (4.2).*

For the existence of solutions for the problem (4.1)-(4.2), we need the following auxiliary lemma :

Lemma 4.2.2 *([35]) Let $0 < \alpha < 1$ and $g : J \to \mathbb{R}$ be continuous. Then the fractional boundary value problem*

$$\begin{gathered} {}^cD^\alpha y(t) = \varphi(t), \ \textit{for each}, \ t \in J, \ 0 < \alpha < 1, \\ ay(0) + b(T) = c, \end{gathered}$$

where a, b, c are real constants with $a + b \neq 0$, has a unique solution which is given by

$$\begin{aligned} y(t) &= \frac{c}{a+b} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) ds \\ &- \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \varphi(s) ds. \end{aligned}$$

We are now in a position to state and prove the existence result for the problem (4.1)-(4.2) based on Banach's fixed point.

Theorem 4.2.3 *Assume that*

(H1) The function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H2) There exist constants $k > 0$ and $0 < l < 1$ such that

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \leqslant k|u-\bar{u}| + l|v-\bar{v}| \ \textit{for any} \ u, v, \bar{u}, \bar{v} \in \mathbb{R} \ \textit{and} \ t \in J.$$

If

$$\frac{kT^\alpha}{(1-l)\Gamma(\alpha+1)} \left(1 + \frac{|b|}{|a+b|}\right) < 1, \tag{4.6}$$

then there exists a unique solution for the problem (4.1)-(4.2) on J.

Proof. Transform the problem (4.1)-(4.2) into a fixed point problem. Define the operator $N : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ by

$$\begin{aligned} N(y)(t) &= \frac{c}{a+b} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) ds \\ &- \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \varphi(s) ds, \end{aligned} \tag{4.7}$$

where $\varphi \in C(J,\mathbb{R})$ satisfies the functional equation

$$\varphi(t) = f(t, y(t), \varphi(t)).$$

Clearly, the fixed points of operator N are solutions of problem (4.1)-(4.2). Let $u, w \in C(J, \mathbb{R})$. Then for $t \in J$, we have

$$\begin{aligned}(Nu)(t) - (Nw)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(\varphi(s) - \psi(s))ds \\ &- \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}(\varphi(s) - \psi(s))ds,\end{aligned}$$

where $\varphi, \psi \in C(J, \mathbb{R})$ be such that

$$\varphi(t) = f(t, u(t), \varphi(t)),$$

and

$$\psi(t) = f(t, w(t), \psi(t)).$$

Then, for $t \in J$

$$\begin{aligned}|(Nu)(t) - (Nw)(t)| &\leqslant \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|\varphi(s) - \psi(s)|ds \\ &+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}|\varphi(s) - \psi(s)|ds.\end{aligned} \tag{4.8}$$

By $(H2)$ we have

$$\begin{aligned}|\varphi(t) - \psi(t)| &= |f(t, u(t), \varphi(t)) - f(t, w(t), \psi(t))| \\ &\leqslant k|u(t) - w(t)| + l|\varphi(t) - \psi(t)|.\end{aligned}$$

Thus

$$|\varphi(t) - \psi(t)| \leqslant \frac{k}{1-l}|u(t) - w(t)|.$$

By (4.8) we have, for $t \in J$

$$\begin{aligned}|(Nu)(t) - (Nw)(t)| &\leqslant \frac{k}{(1-l)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|u(s) - w(s)|ds \\ &+ \frac{|b|k}{|a+b|(1-l)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}|u(s) - w(s)|ds \\ &\leqslant \frac{kT^\alpha}{(1-l)\Gamma(\alpha+1)}\left(1 + \frac{|b|}{|a+b|}\right)\|u - w\|_\infty.\end{aligned}$$

Then,

$$\|Nu - Nw\|_\infty \leq \frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)}\left(1 + \frac{|b|}{|a+b|}\right)\|u - w\|_\infty.$$

By (4.6), the operator N is a contraction. Hence, by Banach's Contraction Principle, N has a unique fixed point which is a solution of the problem (4.1)-(4.2).

The next existence result is based on Schauder's fixed point Theorem.

Theorem 4.2.4 *Assume (H1),(H2) and the following hypothesis holds.*

(H3) There exist $p, q, r \in C(J, \mathbb{R}_+)$ with $r^* = \sup_{t\in J} r(t) < 1$ such that

$$|f(t, u, w)| \leqslant p(t) + q(t)|u| + r(t)|w| \text{ for } t \in J \text{ and } u, w \in \mathbb{R}.$$

If

$$q^* M\left(1 + \frac{|b|}{|a+b|}\right) < 1, \tag{4.9}$$

where $q^* = \sup_{t\in J} q(t)$, and $M = \frac{T^\alpha}{(1-r^*)\Gamma(\alpha+1)}$, then the problem (4.1)-(4.2) has at least one solution.

Proof. Let the operator N defined in (4.7). We shall show that N satisfies the assumptions of Schauder's fixed point Theorem. The proof will be given in several steps.

Step 1 : N *is continuous.*

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C(J, \mathbb{R})$. Then for each $t \in J$

$$\begin{aligned} |N(u_n)(t) - N(u)(t)| &\leqslant \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |\varphi_n(s) - \varphi(s)| ds \\ &+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |\varphi_n(s) - \varphi(s)| ds, \end{aligned} \tag{4.10}$$

where $\varphi_n, \varphi \in C(J, \mathbb{R})$ such that

$$\varphi_n(t) = f(t, u_n(t), \varphi_n(t)),$$

and

$$\varphi(t) = f(t, u(t), \varphi(t)).$$

By $(H2)$ we have

$$\begin{aligned} |\varphi_n(t) - \varphi(t)| &= |f(t, u_n(t), \varphi_n(t)) - f(t, u(t), \varphi(t))| \\ &\leqslant k|u_n(t) - u(t)| + l|\varphi_n(t) - \varphi(t)|. \end{aligned}$$

Then

$$|\varphi_n(t) - \varphi(t)| \leqslant \frac{k}{1-l} |u_n(t) - u(t)|.$$

Since $u_n \to u$, then we get $\varphi_n(t) \to \varphi(t)$ as $n \to \infty$ for each $t \in J$, and let $\eta > 0$ be such that, for each $t \in J$, $|\varphi_n(t)| \leqslant \eta$ and $|\varphi(t)| \leqslant \eta$, then, we have

$$\begin{aligned} (t-s)^{\alpha-1} |\varphi_n(s) - \varphi(s)| &\leqslant (t-s)^{\alpha-1} [|\varphi_n(s)| + |\varphi(s)|] \\ &\leqslant 2\eta (t-s)^{\alpha-1}. \end{aligned}$$

For each $t \in J$, the function $s \to 2\eta(t-s)^{\alpha-1}$ is integrable on $[0,t]$, then the Lebesgue Dominated Convergence Theorem and (4.10) imply that

$$|N(u_n)(t) - N(u)(t)| \to 0 \; as \; n \to \infty,$$

and hence

$$\|N(u_n) - N(u)\|_\infty \to 0 \text{ as } n \to \infty.$$

Consequently, N is continuous.

Let $p^* = \sup_{t\in J} p(t)$, and

$$\frac{\frac{|c|}{|a+b|} + \left(1 + \frac{|b|}{|a+b|}\right) p^* M}{1 - \left(1 + \frac{|b|}{|a+b|}\right) q^* M} \leqslant R,$$

and define the set

$$D_R = \{u \in C(J,\mathbb{R}) : \|u\|_\infty \leqslant R\}.$$

It is clear that D_R is a bounded, closed and convex subset of $C(J,\mathbb{R})$.

Step 2 : $N(D_R) \subset D_R$.

Let $u \in D_R$ we show that $Nu \in D_R$. We have, for each $t \in J$

$$\begin{aligned} |Nu(t)| &\leqslant \frac{|c|}{|a+b|} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |\varphi(s)| ds \\ &+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |\varphi(s)| ds. \end{aligned} \tag{4.11}$$

By (H3) we have for each $t \in J$,

$$\begin{aligned} |\varphi(t)| &= |f(t, u(t), \varphi(t))| \\ &\leqslant p(t) + q(t)|u(t)| + r(t)|\varphi(t)| \\ &\leqslant p(t) + q(t)R + r(t)|\varphi(t)| \\ &\leqslant p^* + q^*R + r^*|\varphi(t)|. \end{aligned}$$

Then

$$|\varphi(t)| \leqslant \frac{p^* + q^*R}{1 - r^*} := M_1.$$

Thus (4.11) implies that

$$|Nu(t)| \leqslant \frac{|c|}{|a+b|} + \frac{(p^* + q^*R)T^\alpha}{(1-r^*)\Gamma(\alpha+1)} + \frac{|b|(p^* + q^*R)T^\alpha}{|a+b|(1-r^*)\Gamma(\alpha+1)}$$

$$\begin{aligned}
&\leqslant \frac{|c|}{|a+b|}+(p^*+q^*R)M+\frac{|b|(p^*+q^*R)M}{|a+b|}\\
&\leqslant \frac{|c|}{|a+b|}+p^*M\left(1+\frac{|b|}{|a+b|}\right)+q^*M\left(1+\frac{|b|}{|a+b|}\right)R\\
&\leqslant R.
\end{aligned}$$

Then $N(D_R)\subset D_R$.

Step 3 : *$N(D_R)$ is relatively compact.*

Let $t_1,t_2\in J,\ t_1<t_2$, and let $u\in D_R$. Then

$$\begin{aligned}
|N(u)(t_2)-N(u)(t_1)| &= \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\varphi(s)ds\right.\\
&\quad\left.+\frac{1}{\Gamma(\alpha)}\int_{t1}^{t_2}[(t_2-s)^{\alpha-1}\varphi(s)ds\right|\\
&\leqslant \frac{M_1}{\Gamma(\alpha+1)}(t_2^\alpha-t_1^\alpha+2(t_2-t_1)^\alpha).
\end{aligned}$$

As $t_1\longrightarrow t_2$, the right-hand side of the above inequality tends to zero.

As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we conclude that $N:C(J,\mathbb{R})\rightarrow C(J,\mathbb{R})$ is continuous and compact. As a consequence of Schauder's fixed point theorem ([67]), we deduce that N has a fixed point which is a solution of the problem (4.1)-(4.2).

The next existence result is based on Nonlinear alternative of Leray-Schauder type.

Theorem 4.2.5 *Assume (H1)-(H3) and* (4.9) *hold. Then the problem (4.1)-(4.2) has at least one solution.*

Proof. Consider the operator N defined in (4.7). We shall show that N satisfies the assumptions of Leray-Schauder fixed point theorem. The proof will be given in several steps.

Step 1 : *Clearly N is continuous.*

Step 2 : *N maps bounded sets into bounded sets in $C(J,\mathbb{R})$.*

Indeed, it is enough to show that for any $\rho>0$, there exist a positive constant ℓ such that for each $u\in B_\rho=\{u\in C(J,\mathbb{R}):\|u\|_\infty\leqslant\rho\}$, we have $\|N(u)\|_\infty\leqslant\ell$.

For $u\in B_\rho$, we have, for each $t\in J$,

$$\begin{aligned}
|Nu(t)| &\leqslant \frac{|c|}{|a+b|}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|\varphi(s)|ds\\
&+ \frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}|\varphi(s)|ds.
\end{aligned}\tag{4.12}$$

By (H3) we have for each $t \in J$,

$$\begin{aligned} |\varphi(t)| &= |f(t, u(t), \varphi(t))| \\ &\leqslant p(t) + q(t)|u(t)| + r(t)|\varphi(t)| \\ &\leqslant p(t) + q(t)\rho + r(t)|\varphi(t)| \\ &\leqslant p^* + q^*\rho + r^*|\varphi(t)|. \end{aligned}$$

Then

$$|\varphi(t)| \leqslant \frac{p^* + q^*\rho}{1 - r^*} := M^*.$$

Thus (4.12) implies that

$$|Nu(t)| \leqslant \frac{|c|}{|a+b|} + \frac{M^* T^\alpha}{\Gamma(\alpha+1)} + \frac{|b| M^* T^\alpha}{|a+b|\Gamma(\alpha+1)}.$$

Consequently,

$$\|Nu\|_\infty \leqslant \frac{|c|}{|a+b|} + \frac{M^* T^\alpha}{\Gamma(\alpha+1)} + \frac{|b| M^* T^\alpha}{|a+b|\Gamma(\alpha+1)} := l.$$

Step 3 : *Clearly, N maps bounded sets into equicontinuous sets of $C(J, \mathbb{R})$.*

We conclude that $N : C(J, \mathbb{R}) \longrightarrow C(J, \mathbb{R})$ is completely continuous.

Last step : *A priori bound.*

We now show there exists an open set $U \subseteq C(J, \mathbb{R})$ with $u \neq \lambda N(u)$, for $\lambda \in (0, 1)$ and $u \in \partial U$. Let $u \in C(J, \mathbb{R})$ and $u = \lambda N(u)$ for some $0 < \lambda < 1$. Thus for each $t \in J$, we have

$$u(t) = \lambda \frac{c}{a+b} + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) ds + \frac{\lambda b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |\varphi(s)| ds.$$

This implies by (H2) that for each $t \in J$

$$\begin{aligned} |u(t)| &\leq \frac{|c|}{|a+b|} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) ds \\ &+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |\varphi(s)| ds. \end{aligned} \tag{4.13}$$

And, by (H3) we have, for each $t \in J$,

$$|\varphi(t)| = |f(t, u(t), \varphi(t))|$$

$$\begin{aligned} &\leqslant p(t)+q(t)|u(t)|+r(t)|\varphi(t)| \\ &\leqslant p^*+q^*|u(t)|+r^*|\varphi(t)|. \end{aligned}$$

Thus

$$|\varphi(t)| \leqslant \frac{1}{1-r^*}(p^*+q^*|u(t)|).$$

Hence

$$\begin{aligned} |u(t)| &\leqslant \frac{|c|}{|a+b|}+\frac{p^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\left(1+\frac{|b|}{|a+b|}\right) \\ &+ \frac{q^*}{(1-r^*)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|u(s)|ds \\ &+ \frac{|b|q^*}{(1-r^*)|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}|u(s)|ds \\ &\leqslant \frac{|c|}{|a+b|}+\frac{p^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\left(1+\frac{|b|}{|a+b|}\right) \\ &+ \frac{q^*\|u\|_\infty}{(1-r^*)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\ &+ \frac{|b|q^*\|u\|_\infty}{(1-r^*)|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}ds \\ &\leqslant \frac{|c|}{|a+b|}+\frac{p^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\left(1+\frac{|b|}{|a+b|}\right) \\ &+ \frac{q^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\left(1+\frac{|b|}{|a+b|}\right)\|u\|_\infty. \end{aligned}$$

Then

$$\begin{aligned} \|u\|_\infty &\leqslant \tfrac{|c|}{|a+b|}+\frac{p^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\left(1+\frac{|b|}{|a+b|}\right) \\ &+ \frac{q^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\left(1+\frac{|b|}{|a+b|}\right)\|u\|_\infty. \end{aligned}$$

Thus

$$\|u\|_\infty\left[1-\left(1+\frac{|b|}{|a+b|}\right)q^*M\right] \leqslant \frac{|c|}{|a+b|}+\left(1+\frac{|b|}{|a+b|}\right)p^*M.$$

Consequently,

$$\|u\|_\infty \leqslant \frac{\frac{|c|}{|a+b|}+\left(1+\frac{|b|}{|a+b|}\right)p^*M}{1-\left(1+\frac{|b|}{|a+b|}\right)q^*M} := \overline{M}. \tag{4.14}$$

Let

$$U=\{u\in C(J,\mathbb{R}) : \|u\|_\infty < \overline{M}+1\}.$$

By the choice of U, there is no $u\in\partial U$ such that $u=\lambda N(u)$, for $\lambda\in(0,1)$. As a consequence of Leray-Schauder's theorem ([67]), we deduce that N has a fixed point u in $\overline{U}$ which is a solution to (4.1)-(4.2).

4.2.1 Examples

Example 1. Consider the following boundary value problem

$$^cD^{\frac{1}{2}}y(t) = \frac{1}{10e^{t+2}(1+|y(t)|+|^cD^{\frac{1}{2}}y(t)|)}, \text{ for each, } t \in [0,1], \tag{4.15}$$

$$y(0)+y(1)=0. \tag{4.16}$$

Set

$$f(t,u,v) = \frac{1}{10e^{t+2}(1+|u|+|v|)}, \quad t \in [0,1], \ u,v \in \mathbb{R}$$

Clearly, the function f is jointly continuous.
For any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in [0,1]$

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \leqslant \frac{1}{10e^2}(|u-\bar{u}|+|v-\bar{v}|).$$

Hence condition (H2) is satisfied with $k = l = \frac{1}{10e^2}$.
Thus condition

$$\frac{kT^\alpha}{(1-l)\Gamma(\alpha+1)}\left(1+\frac{|b|}{|a+b|}\right) = \frac{3}{2(10e^2-1)\Gamma(\frac{3}{2})} = \frac{3}{(10e^2-1)\sqrt{\pi}} < 1,$$

is satisfied with $a = b = T = 1$, $c = 0$, and $\alpha = \frac{1}{2}$. It follows from Theorem 4.2.3 that the problem (4.15)-(4.16) as a unique solution on J.

Example 2. Consider the following boundary value problem

$$^cD^{\frac{1}{2}}y(t) = \frac{(2+|y(t)|+|^cD^{\frac{1}{2}}y(t)|)}{12e^{t+9}(1+|y(t)|+|^cD^{\frac{1}{2}}y(t)|)}, \text{ for each, } t \in [0,1], \tag{4.17}$$

$$\frac{1}{2}y(0)+\frac{1}{2}y(1)=1. \tag{4.18}$$

Set

$$f(t,u,v) = \frac{(2+|u|+|v|)}{12e^{t+9}(1+|u|+|v|)}, \quad t \in [0,1], \ u,v \in \mathbb{R}.$$

Clearly, the function f is continuous.
For any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in [0,1]$

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \leqslant \frac{1}{12e^9}(|u-\bar{u}|+|v-\bar{v}|).$$

Hence condition (H2) is satisfied with $k = l = \frac{1}{12e^9}$. Also, we have,

$$|f(t,u,v)| \leqslant \frac{1}{12e^{t+9}}(2+|u|+|v|).$$

Thus condition (H3) is satisfied with $p(t) = \frac{1}{6e^{t+9}}$ and $q(t) = r(t) = \frac{1}{12e^{t+9}}$. And condition

$$q^*M\left(1 + \frac{|b|}{|a+b|}\right) = \frac{3}{2(12e^9-1)\Gamma(\frac{3}{2})} = \frac{3}{(12e^9-1)\sqrt{\pi}} < 1,$$

is satisfied with $a = b = \frac{1}{2}$, $c = T = 1$, $\alpha = \frac{1}{2}$, and $q^* = r^* = \frac{1}{12e^9}$.

It follows from Theorem 4.2.4 that the problem (4.17)-(4.18) has at least one solution on J.

4.3 Nonlocal problems

4.3.1 Existence of solutions

Let us defining what we mean by a solution of problem $(4.3) - (4.4)$.

Definition 4.3.1 *A function $u \in C^1(J, \mathbb{R})$ is said to be a solution of the problem $(4.3) - (4.4)$ is u satisfied equation (4.3) on J and condition (4.4).*

For the existence of solutions for the problem $(4.3) - (4.4)$, we need the following auxiliary lemma :

Lemma 4.3.2 *Let $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Then the problem $(4.3) - (4.4)$ is equivalent to the problem :*

$$y(t) = y_0 - \varphi(y) + I^\alpha g(t) \tag{4.19}$$

where $g \in C(J, \mathbb{R})$ satisfies the functional equation :

$$g(t) = f(t, y_0 - \varphi(y) + I^\alpha g(t), g(t)).$$

Proof. If ${}^cD^\alpha y(t) = g(t)$ then $I^\alpha\ {}^cD^\alpha y(t) = I^\alpha g(t)$. So we obtain $y(t) = y_0 - \varphi(y) + I^\alpha g(t)$.

We are now in a position to state and prove the existence result for the problem $(4.3) - (4.4)$ based on Banach's fixed point.

Theorem 4.3.3 *Assume*

(H1) The function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H2) There exist constants $K > 0$ and $0 < L < 1$ such that

$$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \leqslant K|u - \bar{u}| + L|v - \bar{v}|, \text{ for any } u, v, \bar{u}, \bar{v} \in \mathbb{R}, t \in J.$$

(H3) There exists a constant $0 < \gamma < 1$ *such that*

$$|\varphi(u) - \varphi(\bar{u})| \leqslant \gamma |u - \bar{u}|, \text{ for any } u, \bar{u} \in C(J, \mathbb{R}).$$

If

$$C := \gamma + \frac{KT^{\alpha}}{(1-L)\Gamma(\alpha+1)} < 1, \tag{4.20}$$

then there exists a unique solution for the problem (4.3-(4.4) on J.

Proof. Transform the problem (4.3-(4.4) into a fixed point problem. Define the operator $N : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ by :

$$N(y)(t) = y_0 - \varphi(y) + I^{\alpha} g(t), \tag{4.21}$$

where $g \in C(J, \mathbb{R})$ satisfies the functional equation

$$g(t) = f(t, y(t), g(t)).$$

Clearly, the fixed points of operator N are solutions of problem (4.3-(4.4). Let $u, w \in C(J, \mathbb{R})$. Then for $t \in J$, we have

$$\begin{aligned} (Nu)(t) - (Nw)(t) &= \varphi(w) - \varphi(u) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (g(s) - h(s)) ds, \end{aligned}$$

where $g, h \in C(J, \mathbb{R})$ be such that

$$g(t) = f(t, u(t), g(t)),$$

$$h(t) = f(t, w(t), h(t)).$$

Then, for $t \in J$

$$\begin{aligned} |(Nu)(t) - (Nw)(t)| &\leqslant |\varphi(u) - \varphi(w)| \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |g(s) - h(s)| ds. \end{aligned} \tag{4.22}$$

By $(H2)$ we have

$$\begin{aligned} |g(t) - h(t)| &= |f(t, u(t), g(t)) - f(t, w(t), h(t))| \\ &\leqslant K|u(t) - w(t)| + L|g(t) - h(t)|. \end{aligned}$$

Thus

$$|g(t) - h(t)| \leqslant \frac{K}{1-L} |u(t) - w(t)|.$$

By (4.22) and $(H3)$ we have

$$\begin{aligned}
|(Nu)(t)-(Nw)(t)| &\leqslant \gamma|u(t)-w(t)| \\
&+ \frac{K}{(1-L)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|u(s)-w(s)|ds \\
&\leqslant \gamma\|u-w\|_\infty \\
&+ \sup_{0\leqslant t\leqslant T}|u(t)-w(t)|\frac{K}{(1-L)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\
&\leqslant \gamma\|u-w\|_\infty + \frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)}\|u-w\|_\infty.
\end{aligned}$$

Then

$$\|Nu-Nw\|_\infty \leqslant \left[\gamma + \frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)}\right]\|u-w\|_\infty.$$

By (4.20), the operator N is a contraction. Hence, by Banach's contraction principle, N has a unique fixed point which is a solution of the problem (4.3-(4.4).

4.3.2 An example

Consider the following problem with nonlocal conditions

$$^cD^{\frac{1}{2}}y(t) = \frac{1}{2e^{t+1}(1+|y(t)|+|^cD^{\frac{1}{2}}y(t)|)}, \text{ for each } t\in[0,1], \tag{4.23}$$

$$y(0)+\varphi(y)=1, \tag{4.24}$$

where

$$\varphi(y)=\frac{|y|}{10+|y|}.$$

Set

$$f(t,u,v)=\frac{1}{2e^{t+1}(1+|u|+|v|)},\quad t\in[0,1],\ u,v\in\mathbb{R}.$$

Clearly, the function f is continuous.
For any $u,v,\bar{u},\bar{v}\in\mathbb{R}$ and $t\in[0,1]$:

$$|f(t,u,v)-f(t,\bar{u},\bar{v})|\leqslant\frac{1}{2e}(|u-\bar{u}|+|v-\bar{v}|).$$

Hence condition (H2) is satisfied with $K=L=\frac{1}{2e}$.
Let

$$\varphi(u)=\frac{u}{10+u},\quad u\in[0,\infty).$$

Let $u,v\in[0,\infty)$. Then we have

$$|\varphi(u)-\varphi(v)| = \left|\frac{u}{10+u}-\frac{v}{10+v}\right| = \frac{10|u-v|}{(10+u)(10+v)}$$

$$\leqslant \frac{1}{10}|u-v|.$$

Thus condition

$$C=\gamma+\frac{KT^{\alpha}}{(1-L)\Gamma(\alpha+1)}<1,$$

is satisfied with $T=1$, $\gamma=\frac{1}{10}$ and $\alpha=\frac{1}{2}$. It follows from Theorem 4.3.3 that the problem (4.23)-(4.24) has a unique solution on $[0,1]$.

Chapitre 5

Nonlinear Implicit FDEs in Fréchet Spaces (5)

[1]

5.1 Introduction

The purpose of this Section, is to establish existence and uniqueness results to the following implicit fractional differential equation with delay in Fréchet Spaces.

$$^{c}D^{\alpha}y(t) = f(t, y_t, {}^{c}D^{\alpha}y(t)), \ t \in J = [0, +\infty) \quad 0 < \alpha < 1 \tag{5.1}$$

$$y(t) = \varphi(t), \quad t \in [-r, 0], \ r > 0 \tag{5.2}$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative. $f : J \times C([-r, 0], \mathbb{R}) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given function such that $\varphi \in C([-r, 0], \mathbb{R})$.
For each function y defined on $[-r, \infty)$ and for any $t \in J$, we denote by y_t the element of $C([-r, 0], \mathbb{R})$ by :

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0].$$

$y_t(\cdot)$ represents the history of system state from time $t - r$ to time t.

Section 5.2 is devoted to fractional neutral functional differential equations,

$$^{c}D^{\alpha}[y(t) - g(t, y_t)] = f(t, y_t, {}^{c}D^{\alpha}y(t)), \ t \in J = [0, +\infty) \quad 0 < \alpha < 1 \tag{5.3}$$

$$y(t) = \varphi(t), \quad t \in [-r, 0], \ r > 0 \tag{5.4}$$

where $g : J \times C([-r, 0], \mathbb{R}) \to \mathbb{R}$ is a given function such that $g(0, \varphi) = 0$.

[1(5)] [44] M. Benchohra and K. Maazouz, Existence and uniqueness results for nonlinear implicit fractional differential equations with delay in Fréchet spaces, *Commun. Appl. Nonl. Anal.* **23**(2016), 48-59.

We shall present uniqueness results for the problems (5.1)-(5.2) and (5.3)-(5.4). Our approach will be based upon a recent nonlinear alternative of Leray-Schauder type in Fréchet spaces due to Frigon and Granas [62].

Very recently Baghli and Benchohra considered in [20] a class of partial functional evolution equations in Fréchet spaces.

In [28] Belarbi et al. discussed the existence of solutions for following initial value problem with infinite delay on unbounded interval

$$D^{\alpha}y(t) = f(t, y_t) \;\; t \in J = [0, \infty), \;\; 0 < \alpha < 1,$$

$$y(t) = \phi(t), \qquad t \in (-\infty, 0],$$

where $f : J \times B \to \mathbb{R}^n$ is a given function, and $\phi \in B$, B is the phase space, and y_t is the element of B defined by

$$y_t(\theta) = y(t + \theta), \qquad \theta \in (-\infty, 0].$$

The notion of the phase space B plays an important role in the study of both qualitative and quantitative theory. A usual choice is a semi-normed space satisfying suitable axioms which was introduced by Hale and Kato [71].

In [39] Benchohra et al. discussed the existence of solutions for following initial value problem with infinite delay

$$D^{\alpha}y(t) = f(t, y_t) \;\; t \in J = [0, b], \;\; 0 < \alpha < 1,$$

$$y(t) = \phi(t), \qquad t \in (-\infty, 0],$$

where is the Riemman-Liouville fractional derivative, $f : J \times B \to \mathbb{R}$ is a given function, $\phi \in B$, $\phi(0) = 0$, and B is phase space.

5.2 Nonlinear Implicit FDEs with Delay in Fréchet Spaces

5.2.1 IFDEs of fractional order

Let us start by defining what we mean by a solution of problem (5.1)-(5.2). Let the space

$$\Omega = \{y : [-r, \infty) \to \mathbb{R}; y \in C([-r, 0], \mathbb{R}) \textit{ and } y \in C^1(J, \mathbb{R})\}.$$

Definition 5.2.1 *A function $y \in \Omega$ is called solution of the problem (5.1)-(5.2) if it satisfies the equation ${}^cD^{\alpha}y(t) = f(t, y_t, {}^cD^{\alpha}y(t))$, on J and the condition $y(t) = \varphi(t)$ on $[-r, 0]$.*

Lemma 5.2.2 *Let $0<\alpha<1$ and $h:[0,\infty)\to\mathbb{R}$ be a continuous function. Then the problem*

$$\begin{cases} {}^cD^\alpha y(t)=h(t), & t\in[0,\infty) \\ y(t)=\varphi(t), & t\in[-r,0] \end{cases}$$

has a unique solution which is given by

$$y(t)=\begin{cases} \varphi(0)+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}h(s)ds, & t\in[0,\infty); \\ \varphi(t), & t\in[-r,0]. \end{cases}$$

Lemma 5.2.3 *Let $f(t,u,v):\mathbb{R}_+\times C([-r,0],\mathbb{R})\times\mathbb{R}\to\mathbb{R}$ be a continuous function, then the problem (5.1)-(5.2) is equivalent to the problem*

$$y(t)=\begin{cases} \varphi(0)+I^\alpha x(t), & t\in[0,\infty) \\ \varphi(t), & t\in[-r,0] \end{cases} \tag{5.5}$$

where $x(t)\in C(\mathbb{R}_+,\mathbb{R})$ satisfies the functional equation

$$x(t)=f(t,y_t,x(t)).$$

Proof. Let y be a solution of the problem (5.5), show that y is solution of (5.1)-(5.2). We have

$$y(t)=\begin{cases} \varphi(0)+I^\alpha x(t), & t\in[0,\infty) \\ \varphi(t), & t\in[-r,0]. \end{cases}$$

for $t\in[-r,0]$, we have $y(t)=\varphi(t)$, so the condition (5.2) is satisfied.
On the other hand, for $t\in J$, we have

$${}^cD^\alpha y(t)=x(t)=f(t,y_t,x(t)).$$

So

$${}^cD^\alpha y(t)=f(t,y_t,{}^cD^\alpha y(t)).$$

Then, y is a solution of the problem (5.1)-(5.2).

Theorem 5.2.4 *Assume that*
(H1) *$f:J\times C([-r,0],\mathbb{R})\times\mathbb{R}\to\mathbb{R}$ is a continuous function.*
(H2) *For each $n\in\mathbb{N}$ there exist $k_n,\ l_n\in C(J_0,\mathbb{R}_+)$ with $l_n^*=\sup\limits_{t\in J_0} l_n(t)<1$ such that*

$$|f(t,u,v)-f(t,\overline{u},\overline{v})|\leqslant k_n(t)|u-\overline{u}|+l_n(t)|v-\overline{v}|$$

for any $u,\ \overline{u}\in C([-r,0],\mathbb{R})$, $v,\ \overline{v}\in\mathbb{R}$, and $t\in J_0$.
If

$$\frac{k_n^*n^\alpha}{(1-l_n^*)\Gamma(\alpha+1)}<1, \tag{5.6}$$

where $k_n^*=\sup\limits_{t\in J_0} k_n(t)$, then the problem (5.1)-(5.2) has a unique solution.

Proof. For every $n \in \mathbb{N}$, we define in $C(J, \mathbb{R})$ the seminorms by

$$\|y\|_n := \sup\{|y(t)| : t \in J_0 = [0, n]\}.$$

Then $C([-r, \infty), \mathbb{R})$ is a Fréchet space with the family of semi-norms $\{\|.\|_n\}_{n \in \mathbb{N}}$.
Transform the problem (5.1)-(5.2) into a fixed point problem. Consider the operator $T : C([-r, \infty), \mathbb{R}) \to C([-r, \infty), \mathbb{R})$ defined by

$$Ty(t) = \begin{cases} \varphi(0) + I^\alpha x(t), & t \in [0, \infty) \\ \varphi(t), & t \in [-r, 0]. \end{cases}$$

It is clear that fixed points of T are solutions of the problem (5.1)-(5.2).
Let y be a possible solution of the problem (5.1)-(5.2). Given $n \in \mathbb{N}$ and $t \leqslant n$, such that

$$y(t) = \lambda T y(t), \quad t \in [-r, n], \quad n \in \mathbb{N}, \quad \lambda \in (0, 1)$$

$$y(t) = \begin{cases} \lambda\big[\varphi(0) + I^\alpha x(t)\big], & t \in [0, n], \\ \lambda \varphi(t), & t \in [-r, 0]. \end{cases}$$

$$\begin{aligned}
|y(t)| &= \lambda\Big|\varphi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s, x(s))\Big| \\
&\leqslant |\varphi(0)| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big| f(s, y_s, x(s)) - f(s, 0, 0)\Big| ds + I^\alpha |f(s, 0, 0)| \\
&\leqslant |\varphi(0)| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big(k_n^* |y_s| + l_n^* |x(s)|\Big) ds + I^\alpha |f(s, 0, 0)|.
\end{aligned}$$

But by (H2) we have

$$|x(s)| \leqslant k_n^* |y_s| + l_n^* |x(s)| \leqslant \frac{k_n^* |y_s|}{1 - l_n^*}.$$

Then

$$\begin{aligned}
|y(t)| &\leqslant |\varphi(0)| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{k_n^*}{1 - l_n^*} |y_s| ds + I^\alpha |f(s, 0, 0)| \\
&\leqslant |\varphi(0)| + \frac{k_n^*}{1 - l_n^*} \|y\|_n \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + I^\alpha |f(s, 0, 0)| \\
&\leqslant |\varphi(0)| + \frac{k_n^* n^\alpha}{(1 - l_n^*)\Gamma(\alpha+1)} \|y\|_n + \frac{n^\alpha f_n^*}{\Gamma(\alpha+1)}
\end{aligned}$$

where $f_n^* = \sup\limits_{t \in J_0} |f(t, 0, 0)|$.
Then

$$\|y\|_n \leqslant \frac{|\varphi(0)| + \dfrac{n^\alpha f_n^*}{\Gamma(\alpha+1)}}{1 - \dfrac{k_n^* n^\alpha}{(1 - l_n^*)\Gamma(\alpha+1)}} := M_n.$$

Now, set
$$\Omega = \{y \in C([-r,\infty) : \|y\|_n \leqslant M_n + 1\}.$$
Clearly, Ω is a closed subset of $C(J,\mathbb{R})$. We shall show that T is a contraction operator. Indeed, consider y, $\overline{y} \in \Omega$, if $t \in [-r,0]$, then
$$|Ty(t) - T\overline{y}(t)| = 0.$$
For $t \in J$, we have
$$|Ty(t) - T\overline{y}(t)| = |I^\alpha x(t) - I^\alpha \overline{x}(t)| \leqslant I^\alpha |x(t) - \overline{x}(t)|. \tag{5.7}$$
For any $t \in J$,
$$\begin{aligned} |x(t) - \overline{x}(t)| &\leqslant |f(t, y_t, x(t)) - f(t, \overline{y}_t, \overline{x}(t))| \\ &\leqslant k_n(t)|y_t - \overline{y}_t| + l_n(t)|x(t) - \overline{x}(t)| \\ &\leqslant k_n^*|y_t - \overline{y}_t| + l_n^*|x(t) - \overline{x}(t)|. \end{aligned}$$
Thus
$$|x(t) - \overline{x}(t)| \leqslant \frac{k_n^*}{1 - l_n^*}|y_t - \overline{y}_t|$$
then (5.7) becomes
$$\begin{aligned} |Ty(t) - T\overline{y}(t)| &\leqslant \frac{k_n^*}{(1 - l_n^*)\Gamma(\alpha)} \int_0^t (t-s)|y_s - \overline{y}_s| ds \\ &\leqslant \frac{k_n^* n^\alpha}{(1 - l_n^*)\Gamma(\alpha+1)} \|y_t - \overline{y}_t\|_n \\ &\leqslant \frac{k_n^* n^\alpha}{(1 - l_n^*)\Gamma(\alpha+1)} \|y - \overline{y}\|_n. \end{aligned}$$
Then
$$\|Ty(t) - T\overline{y}(t)\|_n \leqslant \frac{k_n^* n^\alpha}{(1 - l_n^*)\Gamma(\alpha+1)} \|y - \overline{y}\|_n.$$
By (6.7) the operator T is a contraction for all $n \in \mathbb{N}$.
From the choice of Ω, there is no $y \in \partial\Omega$ such that $y = \lambda T y$, for some $\lambda \in (0,1)$. Then the second statement (C2) in Theorem 1.5.7 does not hold. The nonlinear alternative of Frigon-Granas shows that the first statement (C1) holds. Thus we deduce that the operator T has a unique fixed-point $y \in \Omega$ which is a unique solution of problem (5.1)-(5.2). This completes the proof.

5.2.2 IFDEs of neutral type

In this section we give existence results for problem (5.3)-(5.4).

Definition 5.2.5 *A function $y \in \Omega$ is called solution of the problem (5.3)-(5.4) if it satisfies the equation* ${}^cD^\alpha\Big[y(t) - g(t, y_t)\Big] = f(t, y_t, {}^c D^\alpha y(t))$ *on* J *and the condition* $y(t) = \varphi(t)$ *on* $[-r, 0]$.

Lemma 5.2.6 *Let* $0 < \alpha < 1$ *and* $h : [0, \infty) \to \mathbb{R}$ *be a continuous function. Then the problem*

$$\begin{cases} {}^cD^\alpha[y(t) - g(t, y_t)] = h(t), & t \in [0, \infty) \\ y(t) = \varphi(t), & t \in [-r, 0], \end{cases}$$

has a unique solution which is given by

$$y(t) = \begin{cases} \varphi(0) + g(t, y_t) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} h(s) ds, & t \in [0, \infty); \\ \varphi(t), & t \in [-r, 0]. \end{cases}$$

Lemma 5.2.7 *Let* $f(t, u, v) : \mathbb{R}_+ \times C([-r, 0], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ *be a continuous function, then the problem (5.3)-(5.4) is equivalent to the problem*

$$y(t) = \begin{cases} \varphi(0) + I^\alpha x(t), & t \in [0, \infty) \\ \varphi(t), & t \in [-r, 0]. \end{cases} \tag{5.8}$$

where $x(t) \in C(\mathbb{R}_+, \mathbb{R})$ *satisfies the functional equation*

$$x(t) = f(t, y_t, x(t)) + {}^c D^\alpha g(t, y_t).$$

Proof. Let y be a solution of the problem (5.8). We show that y is solution of (5.3)-(5.4). We have

$$y(t) = \begin{cases} \varphi(0) + I^\alpha x(t), & t \in [0, \infty) \\ \varphi(t), & t \in [-r, 0]. \end{cases}$$

For $t \in [-r, 0]$, we have $y(t) = \varphi(t)$, so the condition (5.4) is satisfied.
On the other hand, for $t \in J$, we have

$${}^cD^\alpha y(t) = x(t) = f(t, y_t, x(t)) + {}^c D^\alpha g(t, y_t).$$

So

$${}^cD^\alpha[y(t) - g(t, y_t)] = f(t, y_t, {}^c D^\alpha y(t)).$$

Then, y is a solution of the problem (5.3)-(5.4).
In this work, we study the existence of the unique solution defined on the semi-infinite positive real interval $[0, \infty)$ for a neutral differential equations of fractional order by the use the nonlinear alternative of Frigon-Granas type for contraction maps in Fréchet space. This is the subject of the following theorem.

Theorem 5.2.8 *Assume the conditions (H1),(H2) and the following conditions :*
(H3) $g : J \times C([-r, 0], \mathbb{R})$ *is continuous.*
(H4) *There exists* $m_n \in C(J_0, \mathbb{R}_+)$ *such that*

$$|g(t, u) - g(t, \overline{u})| \leqslant m_n(t)|u - \overline{u}|.$$

If

$$\frac{k_n^* n^\alpha}{(1 - l_n^*)\Gamma(\alpha + 1)} + \frac{m_n^*}{(1 - l_n^*)} < 1, \tag{5.9}$$

where $m_n^* = \sup_{t \in J_0} m_n(t)$, *then the problem (5.3)-(5.4) has a unique solution.*

Proof. For every $n \in \mathbb{N}$, we define in $C(J, \mathbb{R})$ the seminorms by

$$\|y\|_n := \sup\{|y(t)| : t \in J_0 = [0, n]\}.$$

Then $C([-r, \infty), \mathbb{R})$ is a Fréchet space with the family of semi-norms $\{\|.\|_n\}_{n \in \mathbb{N}}$.
Transform the problem (5.3)-(5.4) into a fixed point problem. Consider the operator $T : C([-r, \infty), \mathbb{R}) \to C([-r, \infty), \mathbb{R})$ defined by

$$Ty(t) = \begin{cases} \varphi(0) + I^\alpha x(t), & t \in [0, \infty) \\ \varphi(t), & t \in [-r, 0]. \end{cases}$$

It is clear that fixed points of T are solutions of the problem (5.3)-(5.4).
Let y be a possible solution of the problem (5.3)-(5.4). Given $n \in \mathbb{N}$ and $t \leqslant n$, such that

$$y(t) = \lambda T y(t), \quad t \in [-r, n], \quad n \in \mathbb{N}, \quad \lambda \in (0, 1)$$

$$y(t) = \begin{cases} \lambda\big[\varphi(0) + I^\alpha x(t)\big], & t \in [0, n], \\ \lambda \varphi(t), & t \in [-r, 0]. \end{cases}$$

$$\begin{aligned} |y(t)| &= \lambda\Big[\varphi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1}\big(f(s, y_s, x(s)) +^c D^\alpha g(s, y_s)\big) ds\Big] \\ &\leqslant |\varphi(0)| + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} |f(s, y_s, x(s))| ds + |g(t, y_t)| + |g(0, y_0)| \\ &\leqslant |\varphi(0)| + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} \Big| f(s, y_s, x(s)) - f(s, 0, 0) \Big| ds + I^\alpha |f(s, 0, 0)| + |g(t, y_t)|. \end{aligned}$$

But by **(H4)** we have

$$\begin{aligned} |g(t, y_t)| &= |g(t, y_t) - g(t, 0) + g(t, 0)| \\ &\leqslant m_n(t)|y_t| + |g(t, 0)| \\ &\leqslant m_n^* \|y\|_n + g_n^* \end{aligned}$$

where $g_n^* = \sup_{t\in J_0}|g(t,0)|$.
Then

$$|y(t)| \leqslant |\varphi(0)| + \frac{n^\alpha k_n^*}{(1-l_n^*)\Gamma(\alpha+1)}\|y\|_n + \frac{n^\alpha f_n^*}{\Gamma(\alpha+1)} + m_n^*\|y\|_n + g_n^*.$$

Finally

$$\|y\|_n \leqslant \frac{|\varphi(0)| + g_n^* + \dfrac{n^\alpha f_n^*}{\Gamma(\alpha+1)}}{1 - m_n^* - \dfrac{n^\alpha k_n^*}{(1-l_n^*)\Gamma(\alpha+1)}} := M_n'.$$

Now, set

$$\Omega' = \{y \in C([-r,\infty) : \|y\|_n \leqslant M_n' + 1\}.$$

Clearly, Ω' is a closed subset of $C(J,\mathbb{R})$. We shall show that T is a contraction operator. Indeed, consider y, $\overline{y} \in \Omega'$. If $t \in [-r,0]$, then

$$|Ty(t) - T\overline{y}(t)| = 0.$$

For $t \in J$, we have

$$|Ty(t) - T\overline{y}(t)| = |I^\alpha x(t) - I^\alpha \overline{x}(t)| \leqslant I^\alpha|x(t) - \overline{x}(t)|. \tag{5.10}$$

For any $t \in J$,

$$\begin{aligned} |x(t) - \overline{x}(t)| &\leqslant |f(t,y_t,x(t)) - f(t,\overline{y}_t,\overline{x}(t))| +^c D^\alpha|g(t,y_t) - g(t,\overline{y}_t)| \\ &\leqslant k_n(t)|y_t - \overline{y}_t| + l_n(t)|x(t) - \overline{x}(t)| +^c D^\alpha|g(t,y_t) - g(t,\overline{y}_t)| \\ &\leqslant k_n^*|y_t - \overline{y}_t| + l_n^*|x(t) - \overline{x}(t)| +^c D^\alpha|g(t,y_t) - g(t,\overline{y}_t)|. \end{aligned}$$

Thus

$$|x(t) - \overline{x}(t)| \leqslant \frac{k_n^*}{1-l_n^*}|y_t - \overline{y}_t| + \frac{1}{1-l_n^*}{}^cD^\alpha|g(t,y_t) - g(t,\overline{y}_t)|.$$

Then (5.10) becomes

$$\begin{aligned} |Ty(t) - T\overline{y}(t)| &\leqslant \frac{k_n^*}{(1-l_n^*)\Gamma(\alpha)}\int_0^t (t-s)|y_s - \overline{y}_s|ds + \frac{1}{1-l_n^*}I^\alpha\left[{}^cD^\alpha|g(t,y_t) - g(t,\overline{y}_t)|\right] \\ &\leqslant \frac{k_n^* n^\alpha}{(1-l_n^*)\Gamma(\alpha+1)}\|y_t - \overline{y}_t\|_n + \frac{1}{1-l_n^*}\Big[|g(t,y_t) - g(t,\overline{y}_t)| + |g(0,y_0) - g(0,\overline{y}_0)|\Big] \\ &\leqslant \frac{k_n^* n^\alpha}{(1-l_n^*)\Gamma(\alpha+1)}\|y_t - \overline{y}_t\|_n + \frac{1}{1-l_n^*}|g(t,y_t) - g(t,\overline{y}_t)| \\ &\leqslant \frac{k_n^* n^\alpha}{(1-l_n^*)\Gamma(\alpha+1)}\|y - \overline{y}\|_n + \frac{m_n^*}{1-l_n^*}\|y_t - \overline{y}_t\|_n \\ &\leqslant \left[\frac{k_n^* n^\alpha}{(1-l_n^*)\Gamma(\alpha+1)} + \frac{m_n^*}{1-l_n^*}\right]\|y - \overline{y}\|_n. \end{aligned}$$

Then

$$\|Ty(t) - T\overline{y}(t)\|_n \leqslant \Big[\frac{k_n^* n^\alpha}{(1-l_n^*)\Gamma(\alpha+1)} + \frac{m_n^*}{1-l_n^*}\Big]\|y-\overline{y}\|_n.$$

By (5.9) the operator T is a contraction for all $n \in \mathbb{N}$.
From the choice of Ω', there is no $y \in \partial\Omega'$ such that $y = \lambda T y$, for some $\lambda \in (0,1)$. Then the second statement (C2) in Theorem 1.5.7 does not hold. The nonlinear alternative of Frigon-Granas shows that the first statement (C1) holds. Thus we deduce that the operator T has a unique fixed-point $y \in \Omega'$ which is a unique solution of problem (5.3)-(5.4). This completes the proof.

5.2.3 An example

In this section we give an example to illustrate our main results. Let us consider the fractional functional differential equation with finite delay,

$$^cD^\alpha\Big[y(t) - \frac{1}{c_n(2+t)(1+|y_t|)}\Big] = \frac{1+|y_t|+|^cD^\alpha y(t)|}{c_n(2+t)}, \qquad t \in J = [0,\infty),\ \alpha \in (0,1) \tag{5.11}$$

$$y(t) = \varphi(t),\ t \in [-r,0],\ r > 0 \tag{5.12}$$

where $\varphi \in C([-r,0],\mathbb{R})$.
Set

$$g(t,w) = \frac{1}{c_n(2+t)(1+|w|)}, \qquad t \in J,\ w \in \mathbb{R}$$

and

$$f(t,u,v) = \frac{1+|u|+|v|}{c_n(2+t)}, \qquad (t,u,v) \in J \times \mathbb{R} \times \mathbb{R}$$

where

$$c_n = \frac{\Gamma(\alpha+1) + 8(n^\alpha + \Gamma(\alpha+1))}{2\Gamma(\alpha+1)}, \qquad n \in \mathbb{N}.$$

It is clear that f is continuous . Then, let $u,\ v,\ \overline{u},\ \overline{v} \in \mathbb{R}$ and $t \in J$

$$\begin{aligned}
\Big|f(t,u,v) - f(t,\overline{u},\overline{v})\Big| &= \left|\frac{1+|u|+|v|}{c_n(2+t)} - \frac{1+|\overline{u}|+|\overline{v}|}{c_n(2+t)}\right| \\
&= \frac{1}{c_n(2+t)}\Big||u|+|v|-|\overline{u}|-|\overline{v}|\Big| \\
&\leqslant \frac{1}{c_n(2+t)}\left(\Big||u|-|\overline{u}|\Big| + \Big||v|-|\overline{v}|\Big|\right) \\
&\leqslant \frac{1}{c_n(2+t)}\Big(|u-\overline{u}| + |v-\overline{v}|\Big).
\end{aligned}$$

Then the assumption (**H2**) holds with

$$l_n(t) = k_n(t) = \frac{1}{c_n(2+t)}.$$

Then

$$l_n^* = k_n^* = \frac{1}{2c_n}.$$

It is clear that g is continuous . Then, let $w,\ \overline{w} \in \mathbb{R}$ and $t \in J$

$$\begin{aligned}
\Big|g(t,w) - g(t,\overline{w})\Big| &= \frac{1}{c_n(2+t)}\Big|\frac{1}{1+|w|} - \frac{1}{1+|\overline{w}|}\Big| \\
&= \frac{1}{c_n(2+t)}\Big|\frac{1+|\overline{w}|-1-|w|}{(1+|w|)(1-|\overline{w}|)}\Big| \\
&\leqslant \frac{1}{c_n(2+t)}\Big||w|-|\overline{w}|\Big| \\
&\leqslant \frac{1}{c_n(2+t)}|w-\overline{w}|.
\end{aligned}$$

Then the assumption (**H4**) holds with

$$m_n(t) = \frac{1}{c_n(2+t)}.$$

Then

$$m_n^* = \frac{1}{2c_n}.$$

Finally we shall check that condition (5.9) is satisfied. Indeed we have

$$\frac{k_n^* n^\alpha}{(1-l_n^*)\Gamma(\alpha+1)} + \frac{m_n^*}{1-l_n^*} = \frac{1}{4} < 1.$$

Then by Theorem 5.2.8 the problem (5.11)-(5.12) has a unique solution on $[0,\infty)$.

5.3 Global Existence for NIFDEs In Fréchet Spaces (5)

[2]

$$^cD^\alpha y(t) = f(t, y(t), {}^cD^\alpha y(t)),\ t \in J = [0,+\infty) \tag{5.13}$$

$$y(0) = y_0, \tag{5.14}$$

[2(5)] [45] M. Benchohra and K. Maazouz, Global Existence for Nonlinear Implicit Fractional Differential Equations In Fréchet Spaces, (submitted).

where ${}^cD^\alpha$ is the Caputo fractional derivative. $f : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given function, $y_0 \in \mathbb{R}$.

We present results for the problem (5.13)-(5.14), based on contractive maps in Fréchet spaces and the nonlinear alternative of Leray-Schauder type due to Frigon and Granas. At the end we illustrate the problem with an example.

5.3.1 Existence of solutions

Let us start by defining what we mean by a solution of the problem (5.13)-(5.14).

Definition 5.3.1 . *A function $y \in C^1(\mathbb{R}_+, \mathbb{R})$ is said to be a solution of the problem (5.13)-(5.14) if y satisfies the equation ${}^cD^\alpha y(t) = f(t, y(t), {}^cD^\alpha y(t))$ on J, and the condition $y(0) = y_0$.*

For the existence of solutions for the problem (5.13)-(5.14), we need the following auxiliary lemma.

Lemma 5.3.2 *The solution of the problem (5.13)-(5.14) can be expressed by the integral equation*

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds,$$

where x is the solution of the functional integral equation

$$x(t) = f(t, y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds, x(t))$$

Proof. Let ${}^cD^\alpha y(t) = x(t))$ in equation (5.13), then

$$x(t) = f(t, y(t), x(t))$$

and

$$\begin{aligned} y(t) &= y_0 + I^\alpha x(t)) \\ &= y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds. \end{aligned}$$

Let us introduce the following assumptions :

(H1) $f : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous .

(H2) For each $n \in \mathbb{N}$, there exist $\ell_n, k_n \in C(J_0, \mathbb{R}_+)$ such that for each $t \in J_0$

$$|f(t, y, z) - f(t, \overline{y}, \overline{z})| \leqslant \ell_n(t)|y - \overline{y}| + k_n(t)|z - \overline{z}|, \ for\ each\ y, \overline{y}, z,\ \overline{z} \in \mathbb{R}.$$

Set

$$\ell_n^* = \sup_{t \in J_0} \ell_n(t), \quad k_n^* = \sup_{t \in J_0} k_n(t).$$

Theorem 5.3.3 *Assume that the assumptions* $(\mathbf{H1})-(\mathbf{H2})$ *are satisfied. If*

$$\frac{\ell_n^* n^{2\alpha}}{\Gamma^2(\alpha+1)}+\frac{k_n^* n^{\alpha}}{\Gamma(\alpha+1)}<1, \tag{5.15}$$

then the problem $(5.13)-(5.14)$ *has a unique solution.*

Proof. For every $n\in\mathbb{N}$, we define in $C(J,\mathbb{R})$ the semi norms by

$$\|y\|_n:=\sup\{|y(t)|:\ t\in J_0=[0,n]\}.$$

Then $C(J,\mathbb{R})$ is a Fréchet space with the family of semi-norms $\{\|.\|_n\}_{n\in\mathbb{N}}$.
Transform the problem (5.13)-(5.14) into a fixed point problem. Consider the operator $\mathcal{T}:C(J,\mathbb{R})\longrightarrow C(J,\mathbb{R})$ define by :

$$(\mathcal{T}x)(t)=y_0+I^{\alpha}x(t),$$

where

$$x(t)=f(t,y_0+I^{\alpha}x(t),x(t)).$$

Clearly, the fixed points of the operator $\mathcal{T}$ are solutions of the problem (5.13)-(5.14).
Let y be a possible solution of the problem $(5.13)-(5.14)$. Given $n\in\mathbb{N}$ and $t\leqslant n$, then with the view of $(\mathbf{H1})$, $(\mathbf{H2})$, for $x=\lambda\mathcal{T}x,\ \lambda\in(0,1)$ we have

$$\begin{aligned}
|x(t)| &= \lambda|y_0+I^{\alpha}x(t)|\\
&\leqslant |y_0|+|I^{\alpha}x(t)|\\
&\leqslant |y_0|+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|x(s))|ds\\
&= |y_0|+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|f(s,y_0+I^{\alpha}x(s),x(s))|ds\\
&= |y_0|+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|f(s,y_0+I^{\alpha}x(s),x(s))-f(s,0,0)+f(s,0,0)|ds\\
&\leqslant |y_0|+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big[\ell_n(s)|y_0+I^{\alpha}x(s)|+k_n(s)|x(s)|\Big]ds+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,0,0)|ds\\
&\leqslant |y_0|+|y_0|\frac{\ell_n^*}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds+\frac{\ell_n^*}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}I^{\alpha}|x(s)|ds\\
&\quad+\frac{k_n^*}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|x(s)|ds+\frac{f_n}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds
\end{aligned}$$

Therefore,

$$\|x\|_n \leqslant |y_0|+\frac{|y_0|\ell_n^*n^{\alpha}}{\Gamma(\alpha+1)}+\frac{\ell_n^*n^{2\alpha}}{\Gamma^2(\alpha+1)}\|x\|_n+\frac{k_n^*n^{\alpha}}{\Gamma(\alpha+1)}\|x\|_n+\frac{f_nn^{\alpha}}{\Gamma(\alpha+1)}$$

$$\leqslant \quad |y_0| + \frac{|y_0|\ell_n^* n^\alpha}{\Gamma(\alpha+1)} + \frac{f_n n^\alpha}{\Gamma(\alpha+1)} + \left(\frac{\ell_n^* n^{2\alpha}}{\Gamma^2(\alpha+1)} + \frac{k_n^* n^\alpha}{\Gamma(\alpha+1)}\right)\|x\|_n$$

where $\qquad f_n = \sup_{t\in J_0} |f(t,0,0)|.$

Then

$$\|x\|_n \leqslant \frac{|y_0| + \frac{|y_0|\ell_n^* n^\alpha + f_n n^\alpha}{\Gamma(\alpha+1)}}{1 - \left(\frac{\ell_n^* n^{2\alpha}}{\Gamma^2(\alpha+1)} + \frac{k_n^* n^\alpha}{\Gamma(\alpha+1)}\right)} := M_n.$$

Now, set

$$\Omega = \{x \in C(J,\mathbb{R}) : \|x\|_n \leqslant M_n + 1 \ for\ all\ n \in \mathbb{N}\}.$$

Clearly, Ω is a closed subset of $C(J,\mathbb{R})$ we shall show that $\mathcal{T}$ is a contraction operator. Indeed, consider $x, \overline{x} \in \Omega$, for each $t \in [0,n]$ and $n \in \mathbb{N}$, from (**H2**) we have

$$\begin{aligned}
|(\mathcal{T}x)(t) - (\mathcal{T}\overline{x})(t)| &= |I^\alpha x(t) - I^\alpha \overline{x}(t)| \\
&= |I^\alpha (x(t) - \overline{x}(t))| \\
&= |I^\alpha\left(f(t, y_0 + I^\alpha x(t), x(t)) - f(t, y_0 + I^\alpha \overline{x}(t), \overline{x}(t))\right)| \\
&\leqslant \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left(\ell_n(s)|I^\alpha(x(s) - \overline{x}(s))| + k_n(s)|x(s) - \overline{x}(s)|\right)\right| ds \\
&\leqslant \ell_n^* \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} I^\alpha |x(s) - \overline{x}(s)| ds + k_n^* \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x(s) - \overline{x}(s)| ds.
\end{aligned}$$

Therefore,

$$\|\mathcal{T}x - \mathcal{T}\overline{x}\|_n \leqslant \left(\frac{\ell_n^* n^{2\alpha}}{\Gamma^2(\alpha+1)} + \frac{k_n^* n^\alpha}{\Gamma(\alpha+1)}\right)\|x - \overline{x}\|_n.$$

By (5.15) hence the operator $\mathcal{T}$ is a contraction for all $n \in \mathbb{N}$. From the choice of Ω there is no $y \in \partial\Omega$ such that $x = \lambda \mathcal{T}x$, for some $\lambda \in (0,1)$. Then the second statement (C2) in Theorem 1.5.7 does not holds.The nonlinear alternative of Frigon-Granas shows that the first statement (C1) holds. Thus, we deduce that the operator $\mathcal{T}$ has a unique fixed-point x in Ω which is a unique solution of the problem (5.13)-(5.14). This completes the proof.

5.3.2 Example.

In this section we give an example to illustrate the usefulness of our main results. Let us consider the following fractional initial value problem,

$$^cD^\alpha y(t) = \frac{3 + |y(t)| + |^cD^\alpha y(t)|}{c_n(20 + e^t)(1 + |y(t)| + |^cD^\alpha y(t)|)} \quad, \ t \in J := [0,+\infty),\ \alpha \in (0,1), \quad (5.16)$$

$$y(0) = y_0 \in \mathbb{R}, \quad (5.17)$$

where

$$c_n = \frac{5\left(n^{2\alpha} + \Gamma(\alpha+1)n^{\alpha}\right)}{\Gamma^2(\alpha+1)},\ n \in \mathbb{N}^*$$

We set

$$f(t,y,z) = \frac{3+|y|+|z|}{c_n(20+e^t)(1+|y|+|z|)} \qquad (t,y,z) \in J \times \mathbb{R} \times \mathbb{R}.$$

Let us show that conditions (**H1**)-(**H2**) hold.
It is clear that f is continuous .
Then let $y,\ \overline{y},\ z,\ \overline{z} \in \mathbb{R}$ then for each $n \in \mathbb{N}^*$ and $t \in J_0$.
We have

$$\begin{aligned} |f(t,y,z) - f(t,\overline{y},\overline{z})| &= \frac{2}{c_n(20+e^t)}\left|\frac{1}{1+|\overline{y}|+|\overline{z}|} - \frac{1}{1+|y|+|z|}\right| \\ &\leqslant \frac{2}{c_n(20+e^t)}\Big||y|+|z|-|\overline{y}|-|\overline{z}|\Big| \\ &\leqslant \frac{2}{c_n(20+e^t)}\Big(|y-\overline{y}|+|z-\overline{z}|\Big) \end{aligned}$$

Then the assumption (**H2**) holds with

$$\ell_n(t) = k_n(t) = \frac{2}{c_n(20+e^t)}.$$

Since

$$\ell_n^* = k_n^* = \sup\left\{\frac{2}{c_n(20+e^t)},\ t \in J_0\right\} \leqslant \frac{1}{10c_n}.$$

Finally we shall check that condition (5.15) is satisfied.
Indeed,

$$\frac{\ell_n^* n^{2\alpha}}{\Gamma^2(\alpha+1)} + \frac{k_n^* n^{\alpha}}{\Gamma(\alpha+1)} = \frac{1}{50} < 1$$

is satisfied for $\alpha \in (0,1)$. Then by Theorem 5.3.3 the problem (5.16)-(5.17) has a unique solution on $[0,+\infty)$.

Chapitre 6

Ulam-Hyers and Ulam-Hyers-Rassias stabilities (6)

[1]

6.1 Introduction

We adopt the definition in Rus [109] : Ulam-Hyers ,generalized Ulam-Hyers , Ulam-Hyers-Rassias and generalized Ulam-Hyers-Rassias stabilities for the equation, for the implicit fractional-order differential equation (6.1).

In [13], the authors studied a Caputo-type anti-periodic fractional boundary value problem of the form :

$$\begin{gathered} {}^cD^{\alpha}y(t) = f(t, y(t)), \quad t \in [0, T],\ 1 < \alpha \leqslant 2, \\ y(0) = -y(T), \quad {}^cD^{\beta}y(0) = -{}^cD^{\beta}(T),\ 0 < \beta < 1. \end{gathered}$$

In [14], the authors investigates a new class of anti-periodic studied a Caputo-type anti-periodic fractional boundary boundary value problems of higher order fractional differential equations :

$$\begin{gathered} {}^cD^{\alpha}y(t) = f(t, y(t)), \quad t \in [0, T],\ 2 < \alpha \leqslant 3, \\ y(0) = -y(T), \\ {}^cD^{\beta}y(0) = -{}^cD^{\beta}(T), \quad {}^cD^{\beta+1}y(0) = -{}^cD^{\beta+1}(T), \quad 0 < \beta < 1. \end{gathered}$$

[1(6)] [46] M. Benchohra and K. Maazouz, Ulam-Hyers stability for implicit fractional differential equation with anti-periodic conditions, (submitted).

6.2 IFDE with anti-periodic condition

The purpose of this section is to establish existence, uniqueness, Ulam-Hyers stability, generalized Ulam-Hyers stability, and Ulam-Hyers-Rassias stability for the following problem for implicit fractional order differential equation with anti-periodic condition.

$$^{c}D^{\alpha}y(t) = f(t, y(t), ^{c}D^{\alpha}y(t)),\ t \in J = [0, b] \quad 0 < \alpha < 1 \tag{6.1}$$

$$y(0) = -y(b). \tag{6.2}$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R}$ is a given function.

Definition 6.2.1 *The equation* (6.1) *is Ulam-Hyers stable if there exists a real number* $c_f > 0$ *such that for each* $\epsilon > 0$ *and for each solution* $z \in C^1(J, \mathbb{R})$ *of the inequality*

$$|^{c}D^{\alpha}z(t) - f(t, z_t, ^{c}D^{\alpha}z(t))| \leqslant \epsilon,\ t \in J,$$

there exists a solution $y \in C^1(J, \mathbb{R})$ *of equation* (6.1) *with*

$$|z(t) - y(t)| \leqslant c_f\epsilon,\ t \in J.$$

Definition 6.2.2 *The equation* (6.1) *is generalized Ulam-Hyers stable if there exists* $\psi_f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\psi_f(0) = 0$, *such that for each solution* $z \in C^1(J, \mathbb{R})$ *of the inequality*

$$|^{c}D^{\alpha}z(t) - f(t, z_t, ^{c}D^{\alpha}z(t))| \leq \epsilon,\ t \in J,$$

there exists a solution $y \in C^1(J, \mathbb{R})$ *of the equation* (6.1) *with*

$$|z(t) - y(t)| \leq \psi_f(\epsilon),\ t \in J.$$

Definition 6.2.3 *The equation* (6.1) *is Ulam-Hyers-Rassias stable with respect to* $\phi \in C(J, \mathbb{R}_+)$ *if there exists a real number* $c_f > 0$ *such that for each* $\epsilon > 0$ *and for each solution* $z \in C^1(J, \mathbb{R})$ *of the inequality*

$$|^{c}D^{\alpha}z(t) - f(t, z_t, ^{c}D^{\alpha}z(t))| \leqslant \epsilon\phi(t),\ t \in J,$$

there exists a solution $y \in C^1(J, \mathbb{R})$ *of equation* (6.1) *with*

$$|z(t) - y(t)| \leqslant c_f\epsilon\phi(t),\ t \in J.$$

Definition 6.2.4 *The equation* (6.1) *is generalized Ulam-Hyers-Rassias stable with respect to* $\phi \in C(J, \mathbb{R}_+)$ *if there exists a real number* $c_{f,\phi} > 0$ *such that for each solution* $z \in C^1(J, \mathbb{R})$ *of the inequality*

$$|^{c}D^{\alpha}z(t) - f(t, z_t, ^{c}D^{\alpha}z(t))| \leqslant \phi(t),\ t \in J,$$

there exists a solution $y \in C^1(J, \mathbb{R})$ *of equation* (6.1) *with*

$$|z(t) - y(t)| \leqslant c_{f,\phi}\phi(t),\ t \in J.$$

Remark 6.2.5 *A function $z \in C^1(J, \mathbb{R})$ is a solution of the inequality*

$$|{}^cD^\alpha z(t) - f(t, z_t, {}^cD^\alpha z(t))| \leqslant \epsilon, \ t \in J,$$

if and only if there exists a function $h \in C(J, \mathbb{R})$ (which depends on y) such that

i) $|h(t)| \leqslant \epsilon$, $\forall t \in J$.

ii) ${}^cD^\alpha z(t) = f(t, z_t, {}^cD^\alpha z(t)) + h(t)$, $t \in J$.

Remark 6.2.6 *Clearly,*

i) *Definition* (6.2.1) $\Rightarrow$ *Definition* (6.2.2)

ii) *Definition* (6.2.3) $\Rightarrow$ *Definition* (6.2.4) .

Remark 6.2.7 *A solution of the implicit differential equation*

$$|{}^cD^\alpha z(t) - f(t, z_t, {}^cD^\alpha z(t))| \leqslant \epsilon, \ t \in J,$$

with fractional order is called an fractional $\epsilon-$solution of the implicit fractional differential equation (6.1).

6.2.1 Existence of solutions

Let us start by defining what we mean by a solution of problem (6.1)-(6.2).

Definition 6.2.8 *a function $y \in C(J, \mathbb{R})$ is said to be a solution of the problem (6.1)-(6.2) if y satisfied equation ${}^cD^\alpha y(t) = f(t, y(t), {}^cD^\alpha y(t))$ on J and condition (6.2).*

For the existence results for the problem (6.1)-(6.2) we need the following auxiliary lemmas.

Lemma 6.2.9 *Let $h : J \longrightarrow \mathbb{R}$ be a continuous function. Then the problem*

$$ {}^cD^\alpha y(t) = h(t), \ t \in J \quad 0 < \alpha < 1 \tag{6.3}$$

$$ y(0) = -y(b), \tag{6.4}$$

has a unique solution which is given by

$$ y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds - \frac{1}{2\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} h(s) ds. \tag{6.5}$$

Proof By Lemma 1.1.10 we have

$$\begin{aligned} y(t) &= I^\alpha({}^cD^\alpha y(t)) \\ &= I^\alpha h(t) - c_0 \qquad \text{for some constant } c_0 \in \mathbb{R}. \end{aligned}$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds - c_0.$$

By condition (6.4)

$$y(0) = -c_0 = -y(b) = \frac{-1}{\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} h(s)ds + c_0.$$

Then

$$c_0 = \frac{1}{2\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} h(s)ds,$$

and

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds - \frac{1}{2\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} h(s)ds.$$

Lemma 6.2.10 *For $f : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function, the problem (6.1)-(6.2) can be expressed by the integral equation*

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s)ds - \frac{1}{2\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} \varphi(s)ds, \tag{6.6}$$

where $\varphi \in C(J, \mathbb{R})$ satisfies the functional equation

$$\varphi(t) = f(t, y(t), ^c D^\alpha \varphi(t)).$$

Proof Let y be a solution of (6.6). We shall show that y is solution of (6.1)-(6.2). We have

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s)ds - \frac{1}{2\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} \varphi(s)ds.$$

Then

$$\begin{aligned} ^c D^\alpha y(t) &= {}^c D^\alpha \left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s)ds - \frac{1}{2\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} \varphi(s)ds \right] \\ &= {}^c D^\alpha \left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s)ds \right] \\ &= \varphi(t) \\ &= f((t, y(t), ^c D^\alpha \varphi(t)). \end{aligned}$$

Finally we have

$$^c D^\alpha y(t) = f((t, y(t), ^c D^\alpha \varphi(t)).$$

On the other hand by (6.6)
we have

$$y(0) = -\frac{1}{2\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} \varphi(s)ds,$$

and

$$y(b) = \frac{1}{2\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} \varphi(s) ds.$$

Then

$$y(0) = -y(b).$$

Consequently y is solution of problem (6.1)-(6.2).
Conversely if ${}^cD^\alpha y(t) = \varphi(t)$ then $I^\alpha({}^cD^\alpha y(t)) = I^\alpha \varphi(t)$. So we obtain

$$\begin{aligned} y(t) &= y(0) + I^\alpha \varphi(t) \\ &= -y(b) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) ds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) ds - \frac{1}{2\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} \varphi(s) ds. \end{aligned}$$

Theorem 6.2.11 *Assume that*
*(***H1***)* $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a continuous function.*
*(***H2***) there exist* $0 < l < 1$ *and* k *such that*

$$|f(t,x,y) - f(t,\overline{x},\overline{y})| \leqslant k|x-\overline{x}| + l|y-\overline{y}|$$

for each $t \in J$ *and* x, $\overline{x}$, y, $\overline{y} \in \mathbb{R}$.
If

$$k < \frac{2(1-l)\Gamma(\alpha+1)}{3b^\alpha}, \tag{6.7}$$

then the problem (6.1)-(6.2) has a unique solution.

Proof Transform the problem (6.1)-(6.2) into fixed point problem. Consider the operator
$A : C(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R})$ defined by

$$Ay(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) ds - \frac{1}{2\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} \varphi(s) ds. \tag{6.8}$$

Clearly, from Lemma 6.2.10 the fixed points of A are solutions to the problem (6.1)-(6.2). We shall show that A is a contraction.
Let u, $v \in C(J,\mathbb{R})$. Then, for each $t \in J$ we have

$$(Au)(t) - (Av)(t) = \frac{1}{2\Gamma(\alpha)} \left[2\int_0^t (t-s)^{\alpha-1} \big(\varphi(s) - \psi(s)\big) ds - \int_0^b (b-s)^{\alpha-1} \big(\varphi(s) - \psi(s)\big) ds \right],$$

where

$$\varphi(s) = f\big(s, u(s), \varphi(s)\big),$$

and

$$\psi(s) = f(s, v(s), \psi(s)).$$

By condition (**H2**) we have

$$\begin{aligned} |\varphi(s) - \psi(s)| &\leqslant k|u(s) - v(s)| + l|\varphi(s) - \psi(s)| \\ &\leqslant \frac{k}{1-l}|u(s) - v(s)|. \end{aligned}$$

Then we obtain

$$\begin{aligned} \Big|(Au)(t) - (Av)(t)\Big| &\leqslant \frac{1}{2\Gamma(\alpha)}\left[2\int_0^t (t-s)^{\alpha-1}\Big|\varphi(s) - \psi(s))\Big|ds + \int_0^b (b-s)^{\alpha-1}\Big|\varphi(s) - \psi(s))\Big|ds\right] \\ &\leqslant \frac{k}{2(1-l)\Gamma(\alpha)}\left[2\int_0^t (t-s)^{\alpha-1}|u(s) - v(s)|ds + \int_0^b (b-s)^{\alpha-1}|u(s) - v(s)|ds\right] \\ &\leqslant \frac{3b^\alpha k}{2(1-l)\Gamma(\alpha+1)}\|u - v\|_\infty. \end{aligned}$$

Finally

$$\|Au - Av\|_\infty \leqslant \frac{3b^\alpha k}{2(1-l)\Gamma(\alpha+1)}\|u - v\|_\infty.$$

By (6.7), the operator A is a contraction.
Then by Banach's fixed point theorem the problem (6.1)-(6.2) have a unique solution.

6.2.2 Ulam-Hyers stability

Theorem 6.2.12 *Assume that the assumptions (***H1***), (***H2***) and condition (6.7) hold. Then the equation (6.1) is Ulam-Hyers stable.*

Proof. Let $x \in C(J, \mathbb{R})$ be a solution of the inequation

$$|^cD^\alpha x(t) - f(t, x(t),^c D^\alpha x(t))| \leqslant \varepsilon,\ t \in J. \tag{6.9}$$

Let us denote by $y \in C(J, \mathbb{R})$ the unique solution of the problem

$$^cD^\alpha y(t) = f(t, y(t),^c D^\alpha y(t)),\ for\ each\ t \in J,\ 0 < \alpha < 1$$

$$y(0) = -x(b).$$

By Lemma 6.2.10 we have

$$y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_y(s)ds - \frac{1}{2\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\varphi_y(s)ds,$$

where $\varphi_y \in C(J,\mathbb{R})$ satisfies the functional equation

$$\varphi_y(t) = f(t, y(t), ^c D^\alpha \varphi_y(t)).$$

By formula (6.9) we obtain

$$\left|x(t) + \frac{1}{2\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\varphi_x(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_x(s)ds\right| \leqslant \frac{\varepsilon b^\alpha}{\Gamma(\alpha+1)}$$

where

$$\varphi_x(t) = f(t, x(t), ^c D^\alpha \varphi_x(t)).$$

On the other hand, we have for each $t \in J$

$$\begin{aligned}
|x(t)-y(t)| &= \left|x(t) + \frac{1}{2\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\varphi_x(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_y(s)ds\right| \\
&= \left|x(t) + \frac{1}{2\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\varphi_x(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_x(s)ds\right. \\
&+ \left.\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\varphi_x(s)-\varphi_y(s))ds\right| \\
&\leqslant \left|x(t) + \frac{1}{2\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\varphi_x(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_x(s)ds\right| \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big|\varphi_x(s)-\varphi_y(s)\Big|ds.
\end{aligned}$$

By (**H2**) we have

$$\begin{aligned}
|\varphi_x(s)-\varphi_y(s)| &= |f(s, x(s), ^c D^\alpha\varphi_x(s)) - f(s, y(s), ^c D^\alpha\varphi_y(s))| \\
&\leqslant k|x(s)-y(s)| + l|\varphi_x(s)-\varphi_y(s)| \\
&\leqslant \frac{k}{1-l}|x(s)-y(s)|.
\end{aligned}$$

Then

$$|x(t)-y(t)| \leqslant \frac{\varepsilon b^\alpha}{\Gamma(\alpha+1)} + \frac{k}{(1-l)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|x(s)-y(s)|ds.$$

By Lemma 1.1.13 we have

$$|x(t)-y(t)| \leqslant \frac{\varepsilon b^\alpha}{\Gamma(\alpha+1)}\left[1 + \frac{\gamma k b^\alpha}{(1-l)\Gamma(\alpha+1)}\right] := c,\varepsilon,$$

where $\gamma = \gamma(\alpha)$ is a constant. So the problem (6.1)-(6.2) is Ulam-Hyers stable.

Remark 6.2.13 *By putting $\psi(\varepsilon) = c,\varepsilon, \quad \psi(0) = 0$ yields that the problem (6.1)-(6.2) is generalized Ulam-Hyers stable.*

6.2.3 Ulam-Hyers-Rassias stability

Theorem 6.2.14 *Assume that (**H1**), (**H2**) and (**H3**) The function $\psi \in C(J, \mathbb{R}_+)$ is increasing and there exists $\lambda_\psi > 0$ such that for each $t \in J$*

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \psi(s) ds \leqslant \lambda_\psi \psi(t).$$

Then the problem (6.1)-(6.2) is Ulam-Hyers-Rassias stable with respect to ψ.

Proof. Let $x \in C(J, \mathbb{R})$ be a solution of the inequation

$$|{}^c D^\alpha x(t) - f(t, x(t), {}^c D^\alpha x(t))| \leqslant \varepsilon \psi(t), \; t \in J, \; \varepsilon > 0. \tag{6.10}$$

Let us denote by $y \in C(J, \mathbb{R})$ the unique solution of the problem

$${}^c D^\alpha y(t) = f(t, y(t), {}^c D^\alpha y(t)), \; for\ each\, t \in J, \; 0 < \alpha < 1$$

$$y(0) = -x(b).$$

By formula (6.10) we obtain

$$\begin{aligned} \left| x(t) + \frac{1}{2\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} \varphi_x(s) ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi_x(s) ds \right| &\leqslant \varepsilon I^\alpha \psi(t) \\ &\leqslant \varepsilon \lambda_\psi \psi(t). \end{aligned}$$

On the other hand we have for each $t \in J$

$$\begin{aligned} |x(t) - y(t)| &= \left| x(t) + \frac{1}{2\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} \varphi_x(s) ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi_y(s) ds \right| \\ &= \left| x(t) + \frac{1}{2\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} \varphi_x(s) ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi_x(s) ds \right. \\ &+ \left. \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (\varphi_x(s) - \varphi_y(s)) ds \right| \\ &\leqslant \left| x(t) + \frac{1}{2\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} \varphi_x(s) ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi_x(s) ds \right| \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big| \varphi_x(s) - \varphi_y(s) \Big| ds \\ &\leqslant \varepsilon \lambda_\psi \psi(t) + \frac{k}{(1-l)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |x(s) - y(s)| ds \\ &\leqslant \varepsilon \lambda_\psi \psi(t) + \frac{\gamma_1 k \varepsilon \lambda_\psi}{(1-l)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \psi(s) ds \end{aligned}$$

$$\leqslant \left[\lambda_\psi + \frac{\gamma_1 k \lambda_\psi^2}{1-l}\right]\varepsilon\psi(t) := c\varepsilon\psi(t)$$

where $\gamma_1 = \gamma_1(\alpha)$ is a constant.

Then, for each $t \in J$

$$|x(t) - y(t)| \leqslant c\varepsilon\psi(t).$$

So the problem (6.1)-(6.2) is Ulam-Hyers-Rassias stable.

6.2.4 Example

Consider the following problem

$$^cD^{\frac{1}{2}}y(t) = \frac{3 + |y(t)| + |^cD^{\frac{1}{2}}y(t)|}{(20+e^t)(1+|y(t)| + |^cD^{\frac{1}{2}}y(t)|)} \quad t \in J = [0,1], \tag{6.11}$$

$$y(0) = -y(1). \tag{6.12}$$

Set

$$f(t,x,y) = \frac{3+|x|+|y|}{(20+e^t)(1+|x|+|y|)} \qquad (t,x,y) \in J \times \mathbb{R} \times \mathbb{R}.$$

It is clearly that f is continuous.
Then, let x, $\overline{x}$, y, $\overline{y} \in \mathbb{R}$ and $t \in J$

$$\begin{aligned} |f(t,x,y) - f(t,\overline{x},\overline{y})| &= \frac{2}{(20+e^t)}\left|\frac{1}{1+|\overline{x}|+|\overline{y}|} - \frac{1}{1+|x|+|y|}\right| \\ &\leqslant \frac{2}{(20+e^t)}\Big||x|+|y|-|\overline{x}|-|\overline{y}|\Big| \\ &\leqslant \frac{1}{10}\Big(|x-\overline{x}| + |y-\overline{y}|\Big) \end{aligned}$$

Then the assumption (**H2**) holds with

$$k = l = \frac{1}{10}$$

Thus condition

$$\frac{3b^\alpha k}{2(1-l)\Gamma(\alpha+1)} = \frac{\frac{3}{10}}{2(1-\frac{1}{10})\Gamma(\frac{3}{2})} = \frac{1}{6\Gamma(\frac{3}{2})} = \frac{1}{3\sqrt{\pi}} < 1$$

is satisfied. It follows from Theorem 6.2.11 that the problem (6.11)-(6.12) has a unique solution on J. And it follows from Theorem 6.2.12 that the problem (6.11)-(6.12) is Ulam-Hyers stable.

6.3 NIDFE with finite delay

The purpose of this section, is to establish four types of Ulam-Hyers, generalized Ulam-Hyers, Ulam-Hyers-Rassias and generalized Ulam-Hyers-Rassias stabilities for the equation, for the following problem of implicit fractional-order differential equation :

$$ {}^{c}D^{\alpha}\left[y(t)-g(t,y_t)\right]=f(t,y_t,{}^{c}D^{\alpha}y(t)),\ t\in J=[0,T]\,,\ T>0\ ,0<\alpha<1 \tag{6.13}$$

$$y(t)=\varphi(t),\ t\in[-r,0]\,,\ r>0 \tag{6.14}$$

where $f:J\times C\left([-r,0]\,,\mathbb{R}\right)\times\mathbb{R}\rightarrow\mathbb{R}$ and $g:J\times C\left([-r,0]\,,\mathbb{R}\right)$ are two given functions such that $g(0,\varphi)=0$ and $\varphi\in C\left([-r,0]\,,\mathbb{R}\right)$.
For each function y defined on $[-r,T]$ and for any $t\in J$, we denote by y_t the element of $C\left([-r,0]\,,\mathbb{R}\right)$ defined by :

$$y_t(\theta)=y(t+\theta),\quad \theta\in[-r,0],$$

$y_t(.)$ represent the evolution history of system state from time $t-r$ to time t.

The present results initiate the Ulam stability of such class of problems See ([30]).

6.3.1 Existence of solutions

Set

$$\Omega=\{y:[-r,T]\rightarrow\mathbb{R}\ :\ y\in C([-r,0],\mathbb{R})\ and\ y\in C^1(J,\mathbb{R})\}.$$

Definition 6.3.1 *A function $y\in\Omega$ is called solution of the problem (6.13)-(6.14) if it satisfies the equation (6.13) on J and the condition (6.14) on $[-r,0]$.*

Lemma 6.3.2 *Let $0<\alpha<1$ and $h:[0,T]\rightarrow\mathbb{R}$ be a continuous function. Then the linear problem*

$$ {}^{c}D^{\alpha}\left[y(t)-g(t,y_t)\right]=h(t),\ t\in J$$

$$y(t)=\varphi(t),\ t\in[-r,0]$$

has a unique solution which is given by

$$y(t)=\begin{cases}\varphi(0)+g(t,y_t)+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t(t-s)^{\alpha-1}h(s)ds,\ t\in J\\ \varphi(t),\quad t\in[-r,0]\,.\end{cases}$$

Lemma 6.3.3 *Let $f(t,u,v):J\times C\left([-r,0]\,,\mathbb{R}\right)\times\mathbb{R}\rightarrow\mathbb{R}$ be a continuous function, then the problem (6.13)-(6.14) is equivalent to the problem*

$$y(t)=\begin{cases}\varphi(0)+I^{\alpha}K_y(t),\ t\in J\\ \varphi(t),\ t\in[-r,0]\end{cases} \tag{6.15}$$

where $K_y\in C\left(J,\mathbb{R}\right)$ satisfies the functional equation

$$K_y(t)=f(t,y_t,K_y(t))+{}^{c}D^{\alpha}g(t,y_t).$$

Proof. Let y solution of the problem (6.15), show that y is solution of (6.13)-(6.14).
We have

$$y(t)=\begin{cases}\varphi(0)+I^{\alpha}K_y(t), & t\in J\\ \varphi(t), & t\in[-r,0]\end{cases}$$

for $t\in[-r,0]$, we have $y(t)=\varphi(t)$, so the condition (6.14) is satisfied.
On the other hand, for $t\in J$, we have

$$^{c}D^{\alpha}y(t)=K_y(t)=f(t,y_t,K_y(t))+^{c}D^{\alpha}g(t,y_t).$$

So

$$^{c}D^{\alpha}\left[y(t)-g(t,y_t)\right]=f(t,y_t,^{c}D^{\alpha}y(t)).$$

Then, y is well solution of the problem (6.13)-(6.14).

Theorem 6.3.4 *Assume that the assumptions*

(H1) $f:J\times C([-r,0],\mathbb{R})\times\mathbb{R}\to\mathbb{R}$ *is a continuous function.*

(H2) there exist $K>0$ *and* $0<\overline{K}<1$ *such that :*

$$|f(t,u,v)-f(t,\bar{u},\bar{v})|\leqslant K\left\|u-\bar{u}\right\|_C+\overline{K}|v-\bar{v}|$$

for any $u,\bar{u}\in C([-r,0],\mathbb{R}),v,\bar{v}\in\mathbb{R}$ *and* $t\in J$.

(H3) there exists $L>0$ *such that :*

$$|g(t,u)-g(t,v)|\leqslant L\left\|u-v\right\|_C$$

for any $u,v\in C([-r,0],\mathbb{R})$ *and* $t\in J$*, hold*
If

$$\frac{KT^{\alpha}}{\left(1-\overline{K}\right)\Gamma(\alpha+1)}+\frac{L}{\left(1-\overline{K}\right)}<1, \tag{6.16}$$

then, the problem (6.13)-(6.14) *has a unique solution.*

Proof. Let the operator $N:C([-r,T],\mathbb{R})\to C([-r,T],\mathbb{R})$ defined by

$$Ny(t)=\begin{cases}\varphi(0)+I^{\alpha}K_y(t), & t\in J\\ \varphi(t), & t\in[-r,0].\end{cases} \tag{6.17}$$

By Lemma 6.3.3, it is clear that the fixed points of N are solutions of the problem (6.13)-(6.14).
Let $y,\tilde{y}\in C([-r,T],\mathbb{R})$. If $t\in[-r,0]$, then

$$|Ny(t)-N\tilde{y}(t)|=0,$$

For $t\in J$, we have

$$|Ny(t)-N\tilde{y}(t)|=|I^{\alpha}K_y(t)-I^{\alpha}K_{\tilde{y}}(t)|\leq I^{\alpha}\left|K_y(t)-K_{\tilde{y}}(t)\right|. \tag{6.18}$$

For any $t \in J$

$$\begin{aligned} |K_y(t) - K_{\tilde{y}}(t)| &\leqslant |f(t, y_t, K_y(t)) - f(t, \tilde{y}_t, K_{\tilde{y}}(t))| \\ &\quad + {}^cD^\alpha |g(t, y_t) - g(t, \tilde{y}_t)| \\ &\leqslant K \|y_t - \tilde{y}_t\|_C + \overline{K} |K_y(t) - K_{\tilde{y}}(t)| \\ &\quad + {}^cD^\alpha |g(t, y_t) - g(t, \tilde{y}_t)| \,. \end{aligned}$$

Thus

$$|K_y(t) - K_{\tilde{y}}(t)| \leqslant \frac{K}{1 - \overline{K}} \|y_t - \tilde{y}_t\|_C + \left(\frac{1}{1 - \overline{K}} \right) {}^cD^\alpha |g(t, y_t) - g(t, \tilde{y}_t)| \,. \tag{6.19}$$

By replacing (6.19) in the inequality (6.18) , we find

$$\begin{aligned} |Ny(t) - N\tilde{y}(t)| &\leqslant \frac{K}{\left(1 - \overline{K}\right) \Gamma(\alpha)} \int_0^t (t-s)^{\alpha - 1} \|y_s - \tilde{y}_s\|_C \, ds \\ &\quad + \frac{1}{1 - \overline{K}} I^\alpha \, {}^cD^\alpha |g(t, y_t) - g(t, \tilde{y}_t)| \\ &\leqslant \frac{KT^\alpha}{\left(1 - \overline{K}\right) \Gamma(\alpha + 1)} \|y - \tilde{y}\|_\infty \\ &\quad + \frac{1}{1 - \overline{K}} \left(|g(t, y_t) - g(t, \tilde{y}_t)| + |g(0, y_0) - g(0, \tilde{y}_0)| \right) \\ &\leqslant \frac{KT^\alpha}{\left(1 - \overline{K}\right) \Gamma(\alpha + 1)} \|y - \tilde{y}\|_\infty + \frac{L}{1 - \overline{K}} \|y_t - \tilde{y}_t\|_C \\ &\leqslant \left[\frac{KT^\alpha}{\left(1 - \overline{K}\right) \Gamma(\alpha + 1)} + \frac{L}{1 - \overline{K}} \right] \|y - \tilde{y}\|_\infty \,, \end{aligned}$$

then

$$\|Ny - N\tilde{y}\|_\infty \leqslant \left[\frac{KT^\alpha}{\left(1 - \overline{K}\right) \Gamma(\alpha + 1)} + \frac{L}{\left(1 - \overline{K}\right)} \right] \|y - \tilde{y}\|_\infty \,.$$

From (6.16), it follows that N admits a unique fixed point which is solution of the problem (6.13)-(6.14).

6.3.2 Ulam-Hyers Stability Results

Theorem 6.3.5 *Assume that* $(H1)$-$(H3)$, (6.16) *are satisfied. If*

$$\overline{K} + L < 1, \tag{6.20}$$

then the problem (6.13)-(6.14) *is Ulam-Hyers stable.*

Proof. Let $\epsilon > 0$ and $z \in \Omega$ be a function satisfying the inequality

$$\left|{}^{c}D^{\alpha}z(t) - f(t, z_t, {}^{c}D^{\alpha}z(t)) - {}^{c}D^{\alpha}g(t, z_t)\right| \leqslant \epsilon \text{ for each } t \in J,$$

which is equivalent to

$$\left|{}^{c}D^{\alpha}z(t) - K_z(t)\right| \leqslant \epsilon \tag{6.21}$$

and let $y \in C([-r, T], \mathbb{R})$ be the unique solution of the problem

$$\begin{cases} {}^{c}D^{\alpha}\left[y(t) - g(t, y_t)\right] = f(t, y_t, {}^{c}D^{\alpha}y(t)),\ t \in J \\ z(t) = y(t) = \varphi(t),\ t \in [-r, 0]. \end{cases}$$

By integration of the inequality (6.21), we obtain

$$|z(t) - I^{\alpha}K_z(t)| \leqslant \frac{\epsilon T^{\alpha}}{\Gamma(\alpha+1)}.$$

We consider the function γ_1 defined by

$$\gamma_1(t) = \sup\{|z(s) - y(s)| : -r \leqslant s \leqslant t\}, 0 \leq t \leqslant T,$$

then, there exists $t^* \in [-r, T]$ such that $\gamma_1(t) = |z(t^*) - y(t^*)|$.
If $t^* \in [-r, 0]$, then $\gamma_1(t) = 0$.
If $t^* \in [0, T]$, then

$$\begin{aligned} \gamma_1(t) &\leqslant |z(t) - I^{\alpha}K_z(t)| + I^{\alpha}|K_z(t) - K_y(t)| \\ &\leqslant \frac{\epsilon T^{\alpha}}{\Gamma(\alpha+1)} + I^{\alpha}|K_z(t) - K_y(t)|. \end{aligned} \tag{6.22}$$

On the other hand, we have

$$\begin{aligned} |K_z(t) - K_y(t)| &\leqslant |f(t, z_t, K_z(t)) - f(t, y_t, K_y(t)| \\ &\quad + {}^{c}D^{\alpha}|g(t, z_t) - g(t, y_t)| \\ &\leqslant K\gamma_1(t) + \overline{K}|K_z(t) - K_y(t)| \\ &\quad + {}^{c}D^{\alpha}|g(t, z_t) - g(t, y_t)|, \end{aligned}$$

then

$$|K_z(t) - K_y(t)| \leqslant \frac{K}{1-\overline{K}}\gamma_1(t) + \frac{1}{1-\overline{K}}\ {}^{c}D^{\alpha}|g(t, z_t) - g(t, y_t)|. \tag{6.23}$$

By replacing (6.23) in the inequality (6.22), we get

$$\begin{aligned} \gamma_1(t) &\leq \frac{\epsilon T^{\alpha}}{\Gamma(\alpha+1)} + \frac{K}{\left(1-\overline{K}\right)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\gamma_1(s)ds \\ &\quad + \frac{1}{1-\overline{K}}|g(t, z_t) - g(t, y_t)| \end{aligned}$$

$$\leq \frac{\epsilon T^{\alpha}}{\Gamma(\alpha+1)} + \frac{K}{\left(1-\overline{K}\right)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \gamma_1(s) ds + \frac{L}{1-\overline{K}} \gamma_1(t),$$

then

$$\gamma_1(t) \leqslant \frac{\epsilon T^{\alpha}\left(1-\overline{K}\right)}{\left[1-\left(\overline{K}+L\right)\right]\Gamma(\alpha+1)} + \frac{K}{\left[1-\left(\overline{K}+L\right)\right]\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \gamma_1(s) ds,$$

and by the Gronwall's Lemma, we get

$$\gamma_1(t) \leqslant \frac{\epsilon T^{\alpha}\left(1-\overline{K}\right)}{\left[1-\left(\overline{K}+L\right)\right]\Gamma(\alpha+1)} \left[1 + \frac{K T^{\alpha} \sigma_1}{\left[1-\left(\overline{K}_1+L\right)\right]\Gamma(\alpha+1)}\right] := c\epsilon,$$

where $\sigma_1 = \sigma_1(\alpha)$ a constant, which completes the proof of the theorem. Moreover, if we set $\psi(\epsilon) = c\psi; \psi(0) = 0$, then the problem (6.13)-(6.14) is generalized Ulam-Hyers stable.

Theorem 6.3.6 *Assume that* $(H1)$-$(H3)$, (6.16), (6.20) *and*

(H4) there exists an increasing function $\phi \in C(J, \mathbb{R}_+)$ *and there exists* $\lambda_\phi > 0$ *such that for any* $t \in J$:

$$I^{\alpha}\phi(t) \leq \lambda_\phi \phi(t)$$

are satisfied. Then, the problem (6.13)-(6.14) *is Ulam-Hyers-Rassias stable.*

Proof. Let $z \in \Omega$ be solution of the following inequality

$$\left|{}^c D^{\alpha} z(t) - f(t, z_t, {}^c D^{\alpha} z(t)) - {}^c D^{\alpha} g(t, z_t)\right| \leqslant \epsilon\phi(t), \ t \in J, \ \epsilon > 0.$$

The above inequality is equivalent to

$$\left|{}^c D^{\alpha} z(t) - K_z(t)\right| \leq \epsilon\phi(t) \tag{6.24}$$

and let $y \in C([-r, T], \mathbb{R})$ be the unique solution of Cauchy problem

$$\begin{cases} {}^c D^{\alpha}\left[y(t) - g(t, y_t)\right] = f(t, y_t, {}^c D^{\alpha} y(t)), \ t \in J \\ z(t) = y(t) = \varphi(t), \ t \in [-r, 0]. \end{cases}$$

By integration of (6.24), we obtain for any $t \in J$

$$\left|z(t) - I^{\alpha} K_z(t)\right| \leq \epsilon I^{\alpha}\phi(t) \leqslant \epsilon\lambda_\phi\phi(t).$$

Using the function γ_1 which is defined in the proof of Theorem 6.3.5, we get :
if $t^* \in [-r, 0]$ then $\gamma_1(t) = 0$.
If $t^* \in [0, T]$, then we have

$$\gamma_1(t) \leqslant \left|z(t) - I^{\alpha} K_z(t)\right| + I^{\alpha}\left|K_z(t) - K_y(t)\right|$$

$$\leqslant \epsilon\lambda_{\phi}\phi\left(t\right)+I^{\alpha}\left|K_{z}(t)-K_{y}(t)\right| \tag{6.25}$$

from which it follows that

$$\left|K_{z}(t)-K_{y}(t)\right|\leqslant\frac{K}{1-\overline{K}}\gamma_{1}\left(t\right)+\frac{1}{1-\overline{K}}\ {}^{c}D^{\alpha}\left|g(t,z_{t})-g(t,y_{t})\right|. \tag{6.26}$$

By replacing (6.26) in the inequality (6.25), we obtain

$$\begin{aligned}\gamma_{1}\left(t\right) &\leqslant \epsilon\lambda_{\phi}\phi\left(t\right)+\frac{K}{\left(1-\overline{K}\right)\Gamma\left(\alpha\right)}\int_{0}^{t}\left(t-s\right)^{\alpha-1}\gamma_{1}\left(s\right)ds\\ &\quad+\frac{1}{1-\overline{K}}\left|g(t,z_{t})-g(t,y_{t})\right|\\ &\leqslant \epsilon\lambda_{\phi}\phi\left(t\right)+\frac{K}{\left(1-\overline{K}\right)\Gamma\left(\alpha\right)}\int_{0}^{t}\left(t-s\right)^{\alpha-1}\gamma_{1}\left(s\right)ds+\frac{L}{1-\overline{K}}\gamma_{1}\left(t\right),\end{aligned}$$

then

$$\gamma_{1}\left(t\right)\leqslant\frac{\left(1-\overline{K}\right)\epsilon\lambda_{\phi}\phi\left(t\right)}{1-\left(\overline{K}+L\right)}+\frac{K}{\left[1-\left(\overline{K}+L\right)\right]\Gamma\left(\alpha\right)}\int_{0}^{t}\left(t-s\right)^{\alpha-1}\gamma_{1}\left(s\right)ds,$$

by the Gronwall's Lemma, we get

$$\begin{aligned}\gamma_{1}(t) &\leqslant \frac{\left(1-\overline{K}\right)\epsilon\lambda_{\phi}\phi\left(t\right)}{1-\left(\overline{K}+L\right)}\left[1+\frac{KT^{\alpha}\sigma_{2}}{\left[1-\left(\overline{K}+L\right)\right]\Gamma\left(\alpha+1\right)}\right]\\ &\leqslant \left[\frac{\left(1-\overline{K}\right)\lambda_{\phi}}{1-\left(\overline{K}+L\right)}\left(1+\frac{KT^{\alpha}\sigma_{2}}{\left[1-\left(\overline{K}+L\right)\right]\Gamma\left(\alpha+1\right)}\right)\right]\epsilon\phi\left(t\right)=c\epsilon\phi\left(t\right),\end{aligned}$$

where $\sigma_{2}=\sigma_{2}\left(\alpha\right)$ a constant. Then the problem (6.13)-(6.14) is Ulam-Hyers-Rassias stable.

6.3.3 Examples

Example 1. Consider the problem of neutral fractional differential equation

$${}^{c}D^{\frac{1}{2}}\left[y(t)-\frac{te^{-t}\|y_{t}\|_{C}}{\left(9+e^{t}\right)\left(1+\|y_{t}\|_{C}\right)}\right]=\frac{2+\|y_{t}\|_{C}+|{}^{c}D^{\frac{1}{2}}y(t)|}{12e^{t+9}(1+\|y_{t}\|_{C}+|{}^{c}D^{\frac{1}{2}}y(t)|)},\quad t\in[0,1] \tag{6.27}$$

$$y(t)=\varphi(t);\quad t\in\left[-r,0\right],\quad r>0 \tag{6.28}$$

where $\varphi\in C\left(\left[-r,0\right],\mathbb{R}\right)$.
Set

$$g(t,w)=\frac{te^{-t}w}{\left(9+e^{t}\right)\left(1+w\right)},\quad (t,w)\in[0,1]\times[0,+\infty)$$

and

$$f(t,u,v)=\frac{2+u+v}{12e^{t+9}(1+u+v)},\quad (t,u,v)\in[0,1]\times[0,+\infty)\times[0,+\infty).$$

Notice that $g(0,w)=0$, for any $w\in[0,+\infty)$.
Clearly, the function f is continuous. Hence, $(H1)$ is satisfied.
We have,

$$|f(t,u,v)-f(t,\bar{u},\bar{v})|\leqslant\frac{1}{12e^{9}}\left(\|u-\bar{u}\|_C+|v-\bar{v}|\right)$$

$$|g(t,u)-g(t,\bar{u})|\leq\frac{1}{10}\|u-\bar{u}\|_C$$

for any $u,\bar{u}\in C([-r,0],\mathbb{R}),\ v,\bar{v}\in\mathbb{R}\ and\ t\in[0,1]$.
Hence, conditions $(H2)$ and $(H3)$ are satisfied with $K=\overline{K}=\dfrac{1}{12e^{9}}$ and $L=\dfrac{1}{10}$.
Also, condition

$$\frac{KT^{\alpha}}{(1+\overline{K})\Gamma(\alpha+1)}+\frac{L}{(1-\overline{K})}=\frac{20+12e^{9}\sqrt{\pi}}{10\sqrt{\pi}(12e^{9}-1)}<1,$$

is satisfied with $T=1$, $\alpha=\dfrac{1}{2}$.
By Lemma 6.3.4, the problem (6.27)-(6.28) admits a unique solution.
Since

$$\overline{K}+L=\frac{10+12e^{9}}{120e^{9}}<1,$$

then, by Theorem 6.3.5, the problem (6.27)-(6.28) is Ulam-Hyers stable.
Example 2. Consider the problem of neutral fractional differential equation :

$$^{c}D^{\frac{1}{2}}\left[y(t)-\frac{t}{5e^{t+2}\left(1+\|y_t\|_C\right)}\right]=\frac{e^{-t}}{7+e^{t}}\left[\frac{\|y_t\|_C}{1+\|y_t\|_C}-\frac{|^{c}D^{\frac{1}{2}}y(t)|}{1+|^{c}D^{\frac{1}{2}}y(t)|}\right],t\in[0,1]\tag{6.29}$$

$$y(t)=\varphi(t),\quad t\in[-r,0],\quad r>0,\tag{6.30}$$

where $\varphi\in C([-r,0],\mathbb{R})$.
Set

$$g(t,w)=\frac{t}{5e^{t+2}(1+w)},\quad (t,w)\in[0,1]\times[0,+\infty),$$

and

$$f(t,u,v)=\frac{e^{-t}}{(7+e^{t})}\left(\frac{u}{1+u}-\frac{v}{1+v}\right),\quad (t,u,v)\in[0,1]\times[0,+\infty)\times[0,+\infty).$$

Notice that $g(0,w)=0$, for any $w\in[0,+\infty)$.
Clearly, the function f is continuous. Hence, $(H1)$ is satisfied.

$$|f(t,u,v)-f(t,\bar{u},\bar{v})|\leqslant\frac{1}{8}\|u-\bar{u}\|_C+\frac{1}{8}|v-\bar{v}|$$

$$\left|g(t,u)-g(t,\bar{u})\right| \leqslant \frac{1}{5e^2}\left\|u-\bar{u}\right\|_C$$

for any $u, \bar{u} \in C([-r,0], \mathbb{R}),\ v, \bar{v} \in \mathbb{R}\ and\ t \in [0,1]$.
Hence, conditions $(H2)$ and $(H3)$ are satisfied with $K = \overline{K} = \dfrac{1}{8}$ and $L = \dfrac{1}{5e^2}$.
We have

$$\frac{KT^\alpha}{(1+\overline{K})\Gamma(\alpha+1)} + \frac{L}{(1-\overline{K})} = \frac{10e^2+8\sqrt{\pi}}{35e^2\sqrt{\pi}} < 1.$$

By Lemma 6.3.4, the problem (6.29)-(6.30) admits a unique solution.
Since

$$\overline{K} + L = \frac{5e^2+8}{40e^2} < 1,$$

then, by Theorem 6.3.5, the problem (6.29)-(6.30) is Ulam-Hyers stable.

Conclusion and Perspectives

In this thesis, we have considered the following nonlinear implicit fractional differential equation

$$^{c}D^{\alpha}y(t) = f(t, y(t), ^{c}D^{\alpha}y(t))$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative, and $0 < \alpha < 1$.
to subject of boundary value problem , local and, non-local conditions.

We discussed and established the existence, uniqueness and the stability of the solutions for implicit fractional differential equation with anti-periodic conditions , then with finite delay.

It would be interesting, for a future research, to look for problems with infinite delay , asymptotic stability, and using Hadamard derivative.

Bibliographie

[1] S. Abbas and M. Benchohra, On the generalized Ulam-Hyers-Rassias stability for Darboux problem for partial fractional implicit differential equations. *Applied Mathematics E-Notes*, **14** (2014), 20-28.

[2] S. Abbas, M. Benchohra and G M. N'Guérékata, *Topics in Fractional Differential Equations*, Springer-Verlag, New York, 2012.

[3] S. Abbas, M. Benchohra and A N. Vityuk, On fractional order derivatives and Darboux problem for implicit differential equations. *Frac. Calc. Appl. Anal.* **15**, (2012), 168-182.

[4] S.A. Abd-Salam, A.M.A. El-Sayed, On the stability of a fractional-order differential equation with nonlocal initial condition, *Electron. J. Qual. Theory Differ. Equat.* 29 (2008), 1-8.

[5] R.P. Agarwal and B. Ahmad, Existence theory for anti-periodic boundary value problems of fractional differential equations and inclusions, *Comput. Math. Appl.* **62** (2011), 1200-1214.

[6] R.P. Agarwal, M. Benchohra, S. Hamani, A survey on existing results for boundary value problems of nonlinear fractional differential equations and inclusions. *Acta Appl. Math*, (109), 973-1033, 2010.

[7] R.P Agarwal, M. Benchohra and S. Hamani, Boundary value problems for fractional differential equations, *Adv. Stud. Contemp. Math.* **16** (2) (2008), 181-196.

[8] R. P. Agarwal, M. Meehan and D. O'Regan, *Fixed Point Theory and Applications*, Cambridge University Press, Cambridge, 2001.

[9] B. Ahmad, J.R. Graef, Coupled systems of nonlinear fractional differential equations with nonlocal boundary conditions. *Panamer. Math. J.* **19** (2009), 29–39.

[10] B. Ahmad and J.J. Nieto, Existence of solutions for nonlocal boundary value problems of higher-order nonlinear fractional differential equations. *Abstrac. Appl. Anal.* Volume 2009 (2009), Article ID 494720, 9 pages.

[11] B. Ahmad and J.J. Nieto, Existence of solutions for impulsive anti-periodic boundary value problems of fractional order, *Taiwaness J. Math.*, **15** (3) (2011), 981-993.

[12] B. Ahmad and J.J. Nieto, Anti-periodic fractional boundary value problems with nonlinear term depending on lower order derivative, *Fract. Calc. Appl. Anal.*, **15** (2012), 451-462.

[13] B. Ahmad and J.J. Nieto, Anti-periodic fractional boundary value problems, *Comput. Math. Appl*, **62** (3) (2011), 1150-1156.

[14] B. Ahmad, J.J. Nieto, A. Alsaedi and N. Mohamad, On a New Class of Anti-periodic Fractional Boundary Value Problems, *Abst. Appl. Anal*, (2013), 7 p.

[15] K. K. Akhmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodkina and B. N. Sadovskii, *Measures of noncompactness and condensing operators*, Birkhäuser Verlag, Basel, Bostn, Berlin, 1992.

[16] J. C. Alvàrez, Measure of noncompactness and fixed points of nonexpansive condensing mappings in locally convex spaces, *Rev. Real. Acad. Cienc. Exact. Fis. Natur. Madrid* **79** (1985), 53-66.

[17] J. Andres and L. Górniewicz, *Topological Fixed Point Principles for Boundary Value Problems,* Kluwer Academin Publishers, Dordrecht (2003).

[18] J. Appell, Implicit Functions, Nonlinear Integral Equations, and the Measure of Noncompactness of the superposition Operator. *J. Math. Anal. Appl.* **83**, (1981), 251-263.

[19] A. Arara, M. Benchohra, N. Hamidi and J.J. Nieto, Fractional order differential equations on an unbounded domain, *Nonlinear Anal. TMA* **72** (2010), 580-586.

[20] S. Baghli and M. Benchohra, Uniqueness results for partial functional differential equations in Fréchet spaces, *Fixed Point Theory* **9**(2)(2008), 395-406.

[21] D. Baleanu, K. Diethelm, E. Scalas, and J.J. Trujillo, *Fractional Calculus Models and Numerical Methods*, World Scientific Publishing, New York, 2012.

[22] D. Baleanu, Z.B. Güvenç and J.A.T. Machado, *New Trends in Nanotechnology and Fractional Calculus Applications*, Springer, New York, 2010.

[23] D. Baleanu, J.A.T. Machado and A.C.-J. Luo *Fractional Dynamics and Control*, Springer, 2012.

[24] J. Banaѐs and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Marcel Dekker, New York, 1980.

[25] J. Banaѐs, K. Sadarangani, On some measures of noncompactness in the space of continuous functions, *Nonlinear Anal.* **68** (2008), 377-383.

[26] J. Banaѐs and L. Olszowy, Measures of noncompactness related to monotonicity, *Comment. Math. (Prace Mat.)* **41** (2001), 13-23.

[27] Banaѐs, J, Zając, T : A new approach to the theory of functional integral equations of fractional order. J. Math. Anal. Appl. **375**, 375-387 (2011)

[28] A. Belarbi, M. Benchohra, and A. Ouahab Uniqueness results for fractional functional differential equations with infinite delay in Fréchet spaces. *Applicable. Anal* **85**(2006), 1459-1470

[29] M. Benchohra, F. Berhoun, N. Hamidi and J.J. Nieto, Fractional differential inclusions with anti-periodic boundary conditions, *Nonlinear Anal. Forum* **19** (2014), 27-35.

[30] M. Benchohra, S. Bouriah, Existence and stability results for neutral functional differential equations of fractional order with delay. *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* **23**(4) (2016), 295-307.

[31] M. Benchohra, N. Hamidi and J. Henderson, Fractional differential equations with anti-periodic boundary conditions *Numer. Funct. Anal. Optim.* **34** (4) (2013), 404-414.

[32] M. Benchohra, A. Cabada and D. Seba, An existence result for non-linear fractional differential equations on Banach spaces, *Boundary Value Problems.* Volume 2009 (2009), Article ID 628916, 11 pages.

[33] M. Benchohra, J.R. Graef and S. Hamani, Existence results for boundary value problems with nonlinear fractional differential inclusions and integral conditions, (2010), 1-16.

[34] M. Benchohra, J.R. Graef and S. Hamani, Existence results for boundary value problems with nonlinear fractional differential equations, *Appl. Anal.* **87** (7) (2008), 851-863.

[35] M. Benchohra, and S. Hamani, Boundary Value Problems for Differential Inclusions with Fractional Order, *Discus. Mathem. Diff. Incl., Contr. and Optim.* **28** (2008), 147-164.

[36] M. Benchohra, S. Hamani and S.K. Ntouyas, Boundary value problems for differential equations with fractional order, *Surv. Math. Appl.* **3** (2008), 1-12.

[37] M. Benchohra, S. Hamani, J. J. Nieto and B. A. Slimani, Existence of solutions to differential inclusions with fractional order and impulses, *Electron. J. Differential Equations* Vol. 2010 (2010), No. 80, 1-18.

[38] M. Benchohra, J. Henderson and S. K. Ntouyas, *Impulsive Differential Equations and Inclusions*, Hindawi Publishing Corporation, Vol. 2, New York, 2006.

[39] M. Benchohra, J. Henderson, S.K. Ntouyas and A. Ouahab, Existence results for fractional order functional differential equations with infinite delay, *J. Math. Anal. Appl.* **338** (2) (2008), 1340-1350.

[40] M. Benchohra and J. E. Lazreg, Nonlinear fractional implicit differential equations. *Commun. Appl. Anal.* **17** (2013), 471-482.

[41] M. Benchohra and J. E. Lazreg, Existence and uniqueness results for nonlinear implicit fractional differential equations with boundary conditions. *Romanian J. Math. Comput. Sc.* **4** (1) (2014), 60-72.

[42] M. Benchohra and J. E. Lazreg, Existence results for nonlinear implicit fractional differential equations. *Surv. Math. Appl.* **9** (2014), 79-92.

[43] M. Benchohra and K. Maazouz, Existence and uniqueness results for implicit fractional differential equations with integral boundary conditions. *Commun. Appl. Anal.* **20** (2016), 355-366.

[44] M. Benchohra and K. Maazouz, Existence and uniqueness results for nonlinear implicit fractional differential equations with delay in Fréchet spaces, *Commun. Appl. Nonl. Anal.* **23**(2016), 48-59.

[45] M. Benchohra and K. Maazouz, Global Existence for Nonlinear Implicit Fractional Differential Equations In Fréchet Spaces, submitted.

[46] M. Benchohra and K. Maazouz, Ulam-Hyers stability for implicit fractional differential equation with anti-periodic conditions, submitted.

[47] M. Benchohra and F. Mostefai, Weak solutions for nonlinear fractional differential equations with integral boundary conditions in Banach spaces, *Opuscula Math* **32**(2012), 31-40.

[48] M. Benchohra and F. Ouaar, Existence results for nonlinear fractional differential equations with integral boundary conditions *Bull. Math. Anal. Appl.* **2**(2010), 7-15.

[49] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer (2010).

[50] S. Busenberg M. Martelli, *Delay Differential Equations and Dynamical Systems*, Springer-Verlag, 1991.

[51] L. Byszewski, Theorems about existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, *J. Math. Anal. Appl.* **162** (1991), 494-505.

[52] L. Byszewski, Existence and uniqueness of mild and classical solutions of semilinear functional-differential evolution nonlocal Cauchy problem. *Selected problems of mathematics*, 25-33, 50th Anniv. Cracow Univ. Technol. Anniv. Issue, 6, Cracow Univ. Technol., Krakow, 1995

[53] L. Byszewski and V. Lakshmikantham, Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space, *Appl. Anal.* **40** (1991), 11-19.

[54] A. Chen and Y. Chen, Existence of Solutions to Anti-Periodic Boundary value problem for nonlinear Fractional Differential Equations with Impulses. *Adv. Difference. Equat.* (2011), 17 pages.

[55] Y. J. Cho , Th. M. Rassias and R. Saadati, *Stability of Functional Equations in Random Normed Spaces*, Science-Business Media, Springer 52, 2013.

[56] G. Darbo, Punti uniti in transformazioni a condominio non compatto, *Rend Sem. Mat. Univ. Padova* **24** (1955), 84-92.

[57] K. Deng, Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions, *J. Math. Anal. Appl.* **179** (1993), 630-637.

[58] K. Diethelm, *The Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics, 2010.

[59] A. El-Sayed, F. Gaafar, Stability of a nonlinear non-autonomous fractional order systems with different delays and non-local conditions. *Adv. Difference Equations* **47** (1) (2011), 12 pp.

[60] A. Friedman, Foundations of modern analysis, *Dover Publications,Inc*, New York,1982.

[61] M. Frigon and A. Granas, Résultats de type Leray-Schauder pour des contractions multivoques, *Topol. Meth. Nonl. Anal.* **4** (1994), 197-208.

[62] M. Frigon and A. Granas, Résultats de type Leray-Schauder pour des contractions sur des espaces de Fréchet, *Ann. Sci. Math. Québec* **22** (2) (1998), 161-168.

[63] P. Gavruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.* **184** (1994), 431-436.

[64] K. Goebel, *Concise Course on Fixed Point Theorems*, Yokohama Publishers, Japan, 2002.

[65] L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings*, Mathematics and its Applications, 495, Kluwer Academic Publishers, Dordrecht, 1999.

[66] J. R. Graef, J. Henderson and A. Ouahab, *Impulsive Differential Inclusions. A Fixed Point Approch*, De Gruyter, Berlin/Boston, 2013.

[67] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[68] A. Guezane-Lakoud, R. Khaldi, Solvability of a fractional boundary value problem with fractional integral condition, *Nonlinear Anal.* **75** (2012), 2692-2700.

[69] D.J. Guo, V. Lakshmikantham, X. Liu, *Nonlinear Integral Equations in Abstract Spaces,* Kluwer Academic Publishers, Dordrecht, 1996.

[70] Z. Guo, M. Liu, Existence and uniqueness of solutions for fractional order integrodifferential equations with nonlocal initial conditions. *Panamer. Math. J.* **21** (2011), 51-61.

[71] J. Hale, J. Kato, Phase space for retarded equations with infinite delay, *Funkcial. Ekvac.* **21**(1978) 11-14.

[72] J. Henderson and A. Ouahab, A Filippov's theorem, some existence results and the compactness of solution sets of impulsive fractional order differential inclusions, submitted.

[73] R. Hermann, *Fractional Calculus : An Introduction For Physicists*, World Scientific Publishing Co. Pte. Ltd. 2011.

[74] N. Heymans and I. Podlubny, Pysical interpretation of initial conditions for fractional differential equations with Riemnn-Liouville fractional derivatives, *Rheol Acta* **45** (2006), 765-771.

[75] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[76] Ch. Horvath, Measure of Non-compactness and multivalued mappings in complete metric topological spaces, *J. Math. Anal. Appl.* **108** (1985), 403-408.

[77] D. H. Hyers, On the stability of the linear functional equation, *Proc. Natl. Acad.Sci. U.S.A.* 27 (1941), 222-224.

[78] M. Hu and L. Wang, Existence of solutions for a nonlinear fractional differential equation with integral boundary condition, *Int. J. Math. Comp. Sc.*, **7**(1) (2011).

[79] Rabha W. Ibrahim, Stability for Univalent Solutions of Complex Fractional Differential Equations, *Proc. Pakistan. Acad. Sci.*, Vol. 49 **3**, 2012, 227-232.

[80] K.W. Jun and H.M. Kim, On the stability of an n-dimensional quadratic and additive functional equation, *Math. Inequal. Appl.* **19** (9) (2006), 854-858.

[81] S.M. Jung, On the Hyers-Ulam stability of the functional equations that have the quadratic property, *J. Math. Anal. Appl.* **222** (1998), 126-137.

[82] S.M. Jung, Hyers-Ulam stability of linear differential equations of first order, II *Appl. Math. Lett.* **19** (2006), 854-858.

[83] S.M. Jung, K.S. Lee, Hyers-Ulam stability of first order linear partial differential equations with constant coefficients *Math. Inequal. Appl.* **10** (2007), 261-266.

[84] K. Karthikeyan and J.J. Trujillo, Existence and uniqueness results for fractional integrodifferential equations with boundary value conditions, *Commun. Nonlinear Sci. Numer. Simulat.* **17** (2012) 4037-4043.

[85] G.H. Kim, On the stability of functional equations with square-symmetric operation, *Math. Inequal. Appl* **17** (4) (2001), 257-266.

[86] A.A. Kilbas and S. A. Marzan, Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions, *Diff. Equat.* **41** (2005), 84-89.

[87] A.A. Kilbas, Hari M. Srivastava, and Juan J. Trujillo, *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[88] W.A. Kirk and B. Sims, *Handbook of Metric Fixed Point Theory*, Springer, 2001.

[89] K. Kuratowski, Sur les espaces complets, *Fund. Math.* **15** (1930), 301-309.

[90] V. Lakshmikantham, S. Leela and J. Vasundhara, *Theory of Fractional Dynamic Systems*, Cambridge Academic Publishers, Cambridge, 2009.

[91] V. Lakshmikantham, and J. Vasundhara Devi. Theory of fractional differential equations in a Banach space. *Eur. J. Pure Appl. Math.* **1** (2008), 38-45.

[92] V. Lakshmikantham and A.S. Vatsala, Basic theory of fractional differential equations, *Nonlinear Anal.* **69** (2008), 2677-2682.

[93] V. Lakshmikantham and A.S. Vatsala, General uniqueness and monotone iterative technique for fractional differential equations *Appl. Math. Lett.*, **21** (2008) 828-834.

[94] V. Lakshmikantham and A.S. Vatsala, Theory of fractional differential inequalities and applications, *Commun. Appl. Anal.*, **11**, No. 3-4 (2007), 395-402.

[95] J. E. Lazreg, Une contribution dans l'étude des équations différentielles fractionnaires avec impulsion, Thèse de doctorat, Univ. Sidi Bel Abbès 2014.

[96] F. Mainardi, *Fractional Calculus and waves in Linear Viscoelasticity*. Imperial College Press, London 2010.

[97] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.

[98] H. Mönch, Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces, *Nonlinear Anal.* **4** (1980), 985-999.

[99] S. A. Murad and S. Hadid, An existence and uniqueness theorem for fractional differential equation with integral boundary condition, *J. Frac. Calc. Appl.* **3** (6), (2012), 1-9.

[100] Q. A. Naqvi and M. Abbas, Complex and Higher Order Fractional Curl Operator in Electromagnetics, *Optics Communications* **241**, (2004), 349-355.

[101] G M. N'Guérékata, A Cauchy problem for some fractional abstract differential equation with non local conditions, *Nonlinear Anal.* **70** (5) (2009), 1873-1876.

[102] G M. N'Guérékata, Corrigendum : A Cauchy problem for some fractional differential equations, *Commun. Math. Anal.* **7** (2009), 11-11.

[103] M. Obloza, Hyers stability of the linear differential equation, *Rocznik Nauk-Dydakt. Prace Mat.* **13** (1993), 259-270.

[104] K.B. Oldham and J. Spanier, *The Fractional Calculus : theory and application of differentiation and integration to arbitrary order*, Academic Press, New York, London, 1974.

[105] M. D. Ortigueira, *Fractional Calculus for Scientists and Engineers.* Lecture Notes in Electrical Engineering, 84. Springer, Dordrecht, 2011.

[106] I. Petras, *Fractional-Order Nonlinear Systems : Modeling, Analysis and Simulation* Springer, 2011.

[107] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[108] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.* **72** , (1978), 297-300.

[109] J. M. Rassias, *Functional Equations, Difference Inequalities and Ulam Stability Notions (F.U.N.)*, Nova Science Publishers, Inc. New York, 2010.

[110] Th. M. Rassias and J. Brzdek, *Functional Equations in Mathematical Analysis*, Springer 2012.

[111] S. S. Ray, A new approach for the application of Adomian decomposition method for the solution of fractional space diffusion equation with insulated ends, *Appl. Math. Comput* **202** , (2) (2008), 544-549.

[112] B. Ross, *Fractional Calculus and Its Applications*, Proceedings of the International Conference, New Haven, Springer-Verlag, New York (1974)

[113] I. A. Rus, Ulam stabilities of ordinary differential equations in a Banach space, *Carpathian J. Math.* **26** (2010), 103-107.

[114] P. Sahoo, T. Barman and J. P. Davim, *Fractal Analysis in Machining*, Springer, London, 2011.

[115] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[116] X. Su and L. Liu, Existence of solution for boundary value problem of nonlinear fractional differential equation, *Appl. Math.* **22** (3) (2007) 291-298.

[117] V.E. Tarasov, *Fractional Dynamics : Application of Fractional Calculus to Dynamics of particles, Fields and Media*, Springer, Heidelberg ; Higher Education Press, Beijing, 2010.

[118] J. M. Ayerbe Toledano, T. Dominguez Benavides and G. Lopez Acedo, *Measures of Noncompactness in Metric Fixed Point Theory*, BirkhÄauser, Basel, 1997.

[119] F. Treves, *Topological vector spaces, distributions and kernels*, Academic press New York, London, 1967.

[120] F. Wang and Z. Liu, Anti-periodic fractional boundary value problems for nonlinear differential equations of fractional order, *Adv. in Difference Equat.*, (2012), 12 pages.

[121] J. Wang, L. Lv and Y. Zhou, Ulam stability and data dependence for fractional differential equations with Caputo derivative, *Electron. J. Qual. Theory Differ. Equat.* **63** (2011), 1-10.

[122] H. Ye, J. Gao, and Y. Ding, A generalized Gronwall inequality and its application to a fractional differential equation, *J. Math. Anal. Appl.* **328** (2007), 1075-1081.

[123] K. Yosida, *Fonctional Analysis,* 6th edn. Springer-Verlag, Berlin, 1980.

[124] E. Zeidler, *Nonlinear Functional Analysis and Its Applications I*, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1985.

[125] S. Zhang, Positive solutions for boundary-value problems of nonlinear fractional diffrential equations, *Electron. J. Differential Equations* 2006, No. 36, pp. 1-12.