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Devant le jury composé de :

Président :

Prof Abbes Benaissa UDL Sidi Bel Abbes

Examinateurs :

1-Prof Kacem Belghaba U Ahmed Benbella Oran1
2-Dr Abdelli Mama MCA Univ Mascara

Directeur de thèse : *Prof Ali Hakem UDL Sidi Bel Abbes*

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Dedication

In the Name of Allah, the Most Gracious, the Most Merciful. All praise be to Allah, the Lord of the worlds; and prayers and peace be upon Mohamed; His servant and messenger.

This thesis is dedicated to:

My great parents, who never stop giving of themselves in countless ways,

My dearest wife, who leads me through the valley of darkness with light of hope and support,

My beloved brothers (Nadir - Mustapha – Ahmed) and sisters (Khaidja - Aicha – Nacira); particularly my dearest brother, Ahmed, who stands by my side when things look bleak.

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All the people in my life who touch my heart,

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Contents

Chapitre 1

Introduction

1.1 Historic

In 1695, l'Hopital sent a letter to Leibniz. In his message, an important question about the order of the derivative emerged: What might be a derivative of order 1/2?

In an answer, Leibniz foresees the beginning of the area that nowadays is named fractional calculus (FC). In fact, FC is as old as the traditional calculus proposed independently by Newton and Leibniz.

In opposition to what occurs in the case of FC. This difference with classical calculus can be seen as a problem for the slow progress of FC up to 1900. After Leibniz, it was Euler (1738) that noticed the problem for a derivative of noninteger order.

Fourier (1822) suggested an integral representation in order to define the derivative, and his version can be considered the first definition for the derivative of arbitrary (positive) order.

Abel (1826) solved an integral equation associated with the tautochrone problem, which is considered to be the first application of FC.

Liouville (1832) suggested a definition based on the formula for differentiating the exponential function. This expression is known as the first Liouville definition. The second definition formulated by Liouville is presented in terms of an integral and is now called the version by Liouville for the integration of noninteger order.

After a series of works by Liouville, ten years after his death was published the most important paper by Riemann, independently, developed an approach to non-integer order derivatives in terms of a convenient convergent series.

Hadamard (1892) published a paper where the noninteger order derivative of an analytical function must be done in terms of its Taylor series.

Marchaud (1927) introduced a new definition for noninteger order of derivatives. This definition coincides with the Liouville version for “sufficiently good” functions.

Erdelyi-Kober (1940) presented a distinct definition for noninteger order of integration that is useful in applications involving integral and differential equations.

Caputo (1967) formulated a definition, more restrictive than the Riemann-Liouville but more appropriate to discuss problems involving a fractional differential equation with initial conditions.

After the first congress at the University of New Haven, in 1974, FC has developed and several applications emerged in many areas of scientific knowledge. As a consequence, distinct approaches to solving problems involving the derivative were proposed and distinct definitions of the fractional derivative are available in the literature, see [1].

In this work, we will look at fractional derivatives through fractional differential equations as well as trying to address the analytical and numerical solutions of this equation having the operator D^{α} define under specific conditions.

1.2 Thesis plan

We propose to study the differential equations of fractional order in the space C_{α}^{n}, and numerical solution of this equation.

In this spirit the thesis is composed of three chapters:

* In the first chapter, we introduce the tools of fractional analysis necessary to the realization of the problem of existence and uniqueness of differential fractional equations while passing by the theories of the operators and matrix, in addition to fractional local analysis tools and numerical method for the numeric solution.

* In the second chapter,

we show its existence by using the following steps: The first step:

We can reduce the equation in unknown u, $Pu = f$ to an equivalent fractional system,

$$D_t^{\beta} U = K(t)U + F, \tag{1.2.1}$$

where $0 < \beta \leq 1$, has a solution, with $\det K \neq 0$.

The second step:

Transform the system to a new system,

$$D_t^{\beta} U = \tilde{K}(t)U + \tilde{F}, \tag{1.2.2}$$

where $\tilde{K}$, is the diagonalized matrix of K, where can we write the system

$$\partial_t^{\beta} u_i = \theta_i u_i + f_i,$$

The third chapter :

The solution of $\partial_t^{\beta} u_i = \theta_i u_i + f_i$, we find $u_0 = u$, is the analytic solution of the fractional differential equation. We will explain this in an application supported by examples.

In the fourth chapter, we find the numerical solution u_i of the fractional differential equation by using the local fractional theory and numerical methods and then obtain the solution of the fractional differential equation by interpolation polynomial P_i.

This presents a systematic existence and formulations the solution of fractional differential equations.

The main difficulties of this research are:

The existence of many definitions of fractional derivative maks the difficulty for use of a specific definition, we should mention also that we can have several alternative expressions D^{α} for the same definition.

The difficulty decomposition of characteristic equation in IR.

1.3 Mathematical tools

The fractional differential calculus is a branch of mathematical analysis that examines the many different possibilities for the definition of real numerical powers or the complex number powers of the differential operator D^{α}. In this work, we will consider the different possibilities for solving differential equations.

The purpose of this part is to present the elements of the theory of fractional calculus described in this work.

Taylor's fractional formula

For k integer we have: $u(x) = \sum_{k\in\mathbb{Z}_+} \frac{u^{(k)}(x_0)}{\Gamma(k+1)} (x-x_0)^k$.

The generalization of this formula gives us: $u(x) = \sum_{k\in\mathbb{Z}_+} \frac{u^{(k\alpha)}(x_0)}{\Gamma(\alpha+1)} (x-x_0)^{k\alpha}$.

Leibniz's formula

For n integer we have:

$$\partial^n (uv) = \sum_{k\leq n} \binom{n}{k} \partial^{n-k} u \partial^k v.$$

The generalization of this formula gives us :

$$\partial^{n\alpha} (uv) = \sum_{0\leq k\leq n} \binom{n}{k} \partial^{(n-k)\alpha} u \partial^{k\alpha} v.$$

Proposition 1.3.1 *All operator P of ordre βm can be written in the form,*

$$\begin{aligned} P\left(t, D_t^{\beta m} u\right) &= \sum_{k\leq m} g_k \left(t, D^{\beta m} u\right) D_t^{\beta k} \\ &= D_t^{\beta m} u + \sum_{k\neq m} g_k D_t^{\beta k} u \\ &= P_m + P_{m-1}, \end{aligned}$$

P_m is the main part of the operator P.

Proposition 1.3.2 *P can be written in the form (Replace D_t with D_t^β in [1]),*

$$\begin{aligned}
P &= \left(D_t^\beta-\lambda_d\right)^{m_d}\circ....\circ\left(D_t^\beta-\lambda_2\right)^{m_2}\circ\left(D_t^\beta-\lambda_1\right)^{m_1}+R^{(0)}, \qquad (1.3.1)\\
R^{(0)} &= \sum_{j=0}^{m} r_j^{(0)} D_t^{\beta(m-j)},\ r_j^{(0)}=r_j^{(0)}(t),\\
r_j^{(0)} &\in C([-T,T]).
\end{aligned}$$

Proposition 1.3.3 *Let P an operator of the type* (1.3.1) *(Replace D_t with D_t^β in [1]), we have,*

$$\begin{aligned}
P &= P_d\circ....\circ P_2\circ P_1+R, \qquad R=\sum_{l=0}^{m} r_l D_t^{\beta(m-l)}, \qquad (1.3.2)\\
P_j &= \left(D_t^\beta-\lambda_j\right)^{m_j}+a_1^{(j)}\left(D_t^\beta-\lambda_j\right)^{m_j-1}+....+a_{m_j}^{(j)}=\left(D_t^\beta-\lambda_j\right)^{m_j}+\sum_{k=1}^{m_j} g_k^{(j)}\left(D_t^\beta-\lambda_j\right)^{m_j-k},\\
\lambda_j &\in C([-T,T])+C([-T,T]),g_k^{(j)}=g_k^{(j)}(t)\in C([-T,T]);r_l=r_l(t)\in C([-T,T]).
\end{aligned}$$

Proposition 1.3.4 *Let $a,\beta\in Q_+^*$ and $m\in\mathbb{Z}$, we define*

$$\begin{aligned}
\left[D^\alpha,D^\beta\right] &= D^\alpha D^\beta-D^\beta D^\alpha.\\
\left[D^\alpha,D^m\right] &= D^\alpha D^m-D^m D^\alpha.
\end{aligned}$$

Remark 1.3.5 *For $D^\beta D^\alpha=D^{\alpha+\beta}$, we have $\left[D^\alpha,D^\beta\right]=0$.*

1.3.1 Specific functions for fractional derivation

We present the Gamma functions of Euler and Mittag-Laffer, which will be used in the other chapters. These functions play a very important role in the theory of fractional calculus.

The Gamma function. One of the basic functions of fractional computing is the Euler Gamma function, which extends the factorial to non-integer values.

Definition 1.3.6 *The gamma function of Euler is defined by the following integral: for* $\mathrm{Re}(\alpha)>0$*, we defined $\Gamma(\alpha)$ by*

$$\Gamma(\alpha)=\int_0^{+\infty} t^{\alpha-1}e^{-t}dt, \qquad (1.3.3)$$

and incomplete Gamma function is

$$\Gamma(\alpha, x) = \int_{x}^{+\infty} t^{\alpha-1} e^{-t} dt. \quad (1.3.4)$$

An important property of the function $\Gamma(\alpha)$ is the following recursive relation:

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha).$$

We define the extension of $\Gamma(\alpha)$ for α negative as follows:

Suppose $-1 < \alpha < 0$ so $0 < \alpha+1 < 1$ and $\Gamma(\alpha)$ is well defined by Euler's formula, but not $\Gamma(\alpha)$.

We then agree to define $\Gamma(\alpha)$ by the relation $\Gamma(\alpha) = \frac{\Gamma(\alpha+1)}{\alpha}$, and the process is extended step by step.

Thus for $-(n+1) < \alpha < n$ (n positive integer or zero), we will have:

$$\Gamma(\alpha) = \frac{\Gamma(\alpha+n+1)}{\alpha(\alpha+1)...(\alpha+n)}.$$

The Bêta function. Euler's Beta function is defined by the following integral:

$$B(x;y) = \int_{0}^{1} t^{x-1}(1-t)^{y-1} dt \qquad (Rex > 0; Rey > 0). \quad (1.3.5)$$

The relationship between Euler's Beta function and Euler's Gamma is given by:

$$B(x;y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

The Mittag-Leffler function. The integral representation of the two-parameter Mittag-Leffler function is

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi} \int_{D} \frac{t^{\alpha-\beta} e^{t}}{t^{\alpha} - z} dt, \qquad z \in C, \quad Re(\alpha) > 0, \quad (1.3.6)$$

where the contour D is already defined.

And the corresponding integral representation of the one parameter Mittag-Leffler function

(in 1903), (for $\beta = 1$) is $E_\alpha(z) = \frac{1}{2\pi}\int\limits_D \frac{t^{\alpha-1}e^t}{t^\alpha - z}dt, \quad z \in C, \quad Re(\alpha) > 0$ where the contour D is already defined [1].

The two parameter Mittag-Leffler function [1] was defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + \alpha k)}, \qquad z, \beta \in C, Re(\alpha) > 0.$$

The one-parameter Mittag-Leffler function is denoted by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + \alpha k)}, \qquad z \in C, \quad Re(\alpha) > 0,$$

and $E_\alpha(t^\alpha)$ is defined by following series,

$$E_\alpha(t^\alpha) = \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{\Gamma(1 + \alpha k)}, \qquad z \in C, \quad Re(\alpha) > 0.$$

Some Properties Mittag-Leffler Function. Let $E_\alpha(at^\alpha) = \sum\limits_{k=0}^{\infty} \frac{a^k t^{k\alpha}}{\Gamma(1+\alpha k)}$ is the one parameter Mittag-Leffler function, with $0 < \alpha < 1$.

For $a \neq 0$

$$\begin{aligned} E_\alpha(at^\alpha)E_\alpha(bt^\alpha) &= E_\alpha((a+b)t^\alpha), \\ E_\alpha(at^\alpha)E_\alpha(-at^\alpha) &= 1, \\ D^\alpha(E_\alpha(at^\alpha)) &= aE_\alpha(at^\alpha), \\ D^\alpha(E_\alpha(at^\alpha)E_\alpha(bt^\alpha)) &= (a+b)E_\alpha(at^\alpha)E_\alpha(bt^\alpha), \qquad (1.3.7) \\ D^\alpha(E_\alpha(at^\alpha)E_\alpha(-at^\alpha)) &= D^\alpha 1 = 0. \end{aligned}$$

where D^α is Caputo or Jumarie derivative [1].

1.3.2 Fractional derivation

One long-standing problem of fractional calculus is that there exist too many definitions while lacking physical or geometric meanings.

There are many definitions of fractional derivative, that sammit be studied all or enumerated. We will mention some of them :

Caputo fractional derivative, Grunwald-Letnikov fractional derivative, Riemann-Liouville fractional derivative, Kolwanker-Gangal local fractional derivative, Jumarie modified fractional derivative, Grünwald–Letnikov derivative, Sonin–Letnikov derivative, Liouville derivative, Hadamard derivative, Marchaud derivative, Riesz derivative, Riesz–Miller derivative, Miller–Ross derivative, Weyl derivative, Erdélyi–Kober derivative; Machado derivative, Chen–Machado derivative, Coimbra derivative, Katugampola derivative, Caputo–Katugampola derivative, Hilfer derivative, Hilfer-Katugampola derivative, Davidson derivative, Chen derivative, Atangana–Baleanu derivative, Pichaghchi derivative,.......Unfortunately, most of these fractional derivatives have a lot of unusual properties.

There are attempts to unify the conditions and characteristics that must be achieved for a general definition of fractional derivatives.

Some definitions of Fractional Derivatives

We considered D_t^α in general as a fractional derivative.

Liouville derivative:

$$D^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{-\infty}^{x} (x-s)^{-\alpha} f(s)\, ds, \quad -\infty < x < +\infty. \tag{1.3.8}$$

Liouville left-sided derivative:

$$D_+^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{0}^{x} (x-s)^{-\alpha+n-1} f(s)\, ds, \quad 0 < x < +\infty. \tag{1.3.9}$$

Liouville right-sided derivative:

$$D_-^\alpha f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{x}^{+\infty} (x-s)^{-\alpha+n-1} f(s)\, ds, \quad x < +\infty. \tag{1.3.10}$$

Riemann-Liouville left-sided derivative:

$$D_{a+}^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{a}^{x} (x-s)^{-\alpha+n-1} f(s)\, ds, \quad a \leq x < +\infty. \tag{1.3.11}$$

Riemann-Liouville right-sided derivative:

$$D_{b-}^{\alpha} f\left(x\right) = \frac{\left(-1\right)^{n}}{\Gamma\left(n-\alpha\right)} \frac{d^{n}}{dx^{n}} \int\limits_{x}^{b} \left(x-s\right)^{-\alpha+n-1} f\left(s\right) ds, \quad x \leq b. \tag{1.3.12}$$

Modified Riemann-Liouville fractional derivative:

Caputo left-sided derivative:

$$D_{a+}^{\alpha} f\left(x\right) = \frac{1}{\Gamma\left(n-\alpha\right)} \int\limits_{a}^{x} \left(x-s\right)^{-\alpha+n-1} \left(\frac{d^{n}}{ds^{n}} f\left(s\right)\right) ds, \quad a \leq x < +\infty. \tag{1.3.13}$$

Caputo right-sided derivative:

$$D_{b-}^{\alpha} f\left(x\right) = \frac{\left(-1\right)^{n}}{\Gamma\left(n-\alpha\right)} \int\limits_{x}^{b} \left(x-s\right)^{-\alpha+n-1} \left(\frac{d^{n}}{ds^{n}} f\left(s\right)\right) ds, \quad x \leq b. \tag{1.3.14}$$

Grünwald-Letnikov left-sided derivative:

$${}_{a}^{G}D_{x}^{\alpha} f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(x-a)^{k-\alpha}}{\Gamma(n-\alpha+1)} + \frac{1}{\Gamma(n-\alpha)} \int\limits_{a}^{x} (x-s)^{n-\alpha-1} f^{(n)}(s) ds, \tag{1.3.15}$$

Grünwald-Letnikov right-sided derivative:

$${}_{x}^{G}D_{b}^{\alpha} f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(x-a)^{k-\alpha}}{\Gamma(n-\alpha+1)} + \frac{(-1)^{n}}{\Gamma(n-\alpha)} \int\limits_{x}^{b} (x-s)^{n-\alpha-1} f^{(n)}(s) ds, \tag{1.3.16}$$

Weyl derivative:

$${}_{x}D_{\infty}^{\alpha} f\left(x\right) = \frac{\left(-1\right)^{n}}{\Gamma\left(\alpha\right)} \frac{d^{n}}{dx^{n}} \left[\int\limits_{x}^{\infty} \left(s-x\right)^{\alpha-1} f\left(s\right) ds\right], \quad x < +\infty. \tag{1.3.17}$$

Marchaud derivative:

$$D^{\alpha} f\left(x\right) = \frac{\alpha}{\Gamma\left(1-\alpha\right)} \frac{d}{dx} \int\limits_{-\infty}^{x} \frac{f\left(x\right) - f\left(s\right)}{\left(x-s\right)^{1+\alpha}} ds, \quad -\infty < x < +\infty. \tag{1.3.18}$$

Marchaud left-sided derivative:

$$D_{+}^{\alpha}f\left(x\right)=\frac{\alpha}{\Gamma\left(1-\alpha\right)}\int_{0}^{x}\frac{f\left(x\right)-f\left(x-s\right)}{s^{1+\alpha}}ds,\quad 0<x<+\infty. \tag{1.3.19}$$

Marchaud right-sided derivative:

$$D_{-}^{\alpha}f\left(x\right)=\frac{1}{\Gamma\left(1-\alpha\right)}\int_{x}^{+\infty}\frac{f\left(x\right)-f\left(x+s\right)}{s^{1+\alpha}}ds,\quad x<+\infty. \tag{1.3.20}$$

Hadamard derivative:

$$D^{\alpha}f\left(x\right)=\frac{\alpha}{\Gamma\left(1-\alpha\right)}\int_{0}^{x}\frac{f\left(x\right)-f\left(s\right)}{\left(\ln(x/s)\right)^{1+\alpha}}ds,\quad 0<x<+\infty. \tag{1.3.21}$$

Chen left-sided derivative:

$$D_{c^{+}}^{\alpha}f\left(x\right)=\frac{1}{\Gamma\left(1-\alpha\right)}\frac{d}{dx}\int_{c}^{x}\left(x-s\right)^{-\alpha}f\left(s\right)ds,\quad c<x<+\infty. \tag{1.3.22}$$

Chen right-sided derivative:

$$D_{c^{-}}^{\alpha}f\left(x\right)=\frac{-1}{\Gamma\left(1-\alpha\right)}\frac{d}{dx}\int_{c}^{x}\left(x-s\right)^{-\alpha}f\left(s\right)ds,\quad x<c. \tag{1.3.23}$$

Davidson-Essex derivative:

$$D_{0^{+}}^{\alpha}f\left(x\right)=\frac{1}{\Gamma\left(1-\alpha\right)}\frac{d^{n+1-k}}{dx^{n+1-k}}\int_{0}^{x}\left(x-s\right)^{-\alpha+n-1}\left(\frac{d^{k}}{ds^{k}}f\left(s\right)\right)ds,\quad 0<x<+\infty. \tag{1.3.24}$$

Canavati derivative:

$${}_{a}D_{x}^{\alpha}f\left(x\right)=\frac{1}{\Gamma\left(1-n+\alpha\right)}\frac{d}{dx}\int_{a}^{x}\left(x-s\right)^{n-\alpha}\left(\frac{d^{n}}{ds^{n}}f\left(s\right)\right)ds,\quad n+1<\alpha\leq n\quad a<x<+\infty. \tag{1.3.25}$$

Jumarie derivative:

$$D_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-s)^{-\alpha+n-1} \left(\frac{d^n}{ds^n} f(s) \right) ds, \quad 0 < x < +\infty. \tag{1.3.26}$$

Riesz derivative:

$$D_x^{\alpha} f(x) = \frac{-1}{2\cos(\alpha\frac{\pi}{2})\Gamma(\alpha)} \frac{d^n}{dx^n} \left[\begin{array}{c} \int_{-\infty}^{x} (x-s)^{-\alpha+n-1} f(s)\, ds \\ + \int_{x}^{+\infty} (s-x)^{-\alpha+n-1} f(s)\, ds \end{array} \right], \ -\infty < x. \tag{1.3.27}$$

Cossar derivative:

$$D_{0+}^{\alpha} f(x) = \frac{-1}{\Gamma(1-\alpha)} \lim_{N\to\infty} \frac{d}{dx} \int_x^N (s-x)^{-\alpha} f(s)\, ds, \quad 0 < x < +\infty. \tag{1.3.28}$$

Local fractional derivative:

$$D_{0+}^{\alpha} f(t) = \lim_{\varepsilon\to\infty} \frac{f\left(te^{\varepsilon t^{-\alpha}}\right) - f(t)}{\varepsilon}. \tag{1.3.29}$$

Katugampola fractional derivative:

$$D_{0+}^{\alpha} f(t) = \lim_{\varepsilon\to\infty} \frac{f\left(te^{\varepsilon t^{-\alpha}}\right) - f(t)}{\varepsilon}. \tag{1.3.30}$$

Osler fractional derivative:

$${}_aD_z^{\alpha} f(z) = \frac{\Gamma(1+\alpha)}{2\pi i} \int_{D(a,z)} \frac{f(\xi)}{(\xi-z)^{\alpha+1}} ds. \tag{1.3.31}$$

We can see [5].

Remark 1.3.7 *For $0 < \alpha < 1, n = 1,$ in general the above definitions,*

left-sided:

$$D^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-s)^{-\alpha} f'(s)\, ds, \ -\infty \le a < x, \tag{1.3.32}$$

right-sided:

$$D^{\alpha} f\left(x\right) = \frac{-1}{\Gamma\left(1-\alpha\right)} \int\limits_{x}^{b} \left(s-x\right)^{-\alpha} f'\left(s\right) ds,\ \ x < b \le +\infty, \tag{1.3.33}$$

and

$$D^{\alpha}\left(D^{\beta} f\left(x\right)\right) = D^{\alpha}\left(D^{\beta} f\left(x\right)\right) = D^{\alpha+\beta} f\left(x\right).$$

Or

left-sided $D^{\alpha} f\left(x\right) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int\limits_{a}^{x} \left(x-s\right)^{-\alpha} f\left(s\right) ds, \ \ -\infty \le a < x,$

right-sided $D^{\alpha} f\left(x\right) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dx} \int\limits_{x}^{b} \left(s-x\right)^{-\alpha} f\left(s\right) ds, \ \ x < b \le +\infty,$

and

$$D^{\alpha}\left(D^{\beta} f\left(x\right)\right) = D^{\alpha+\beta} f\left(x\right) - \frac{f(a)(t-a)^{-\alpha}}{\Gamma(-\alpha)}$$

Some definitions of Fractional Integrals

Riemann-Liouville left-sided integral:

$$I_{a+}^{\alpha} f\left(x\right) = \frac{1}{\Gamma\left(\alpha\right)} \int\limits_{a}^{x} \left(x-s\right)^{\alpha-1} f\left(s\right) ds, \quad a \le x < +\infty. \tag{1.3.34}$$

Riemann-Liouville right-sided integral:

$$I_{b^-}^{\alpha} f\left(x\right) = \frac{1}{\Gamma\left(\alpha\right)} \int\limits_{x}^{b} \left(x-s\right)^{\alpha-1} f\left(s\right) ds, \quad x \le b. \tag{1.3.35}$$

$$I_{+}^{\alpha} f\left(x\right) = \frac{1}{\Gamma\left(\alpha\right)} \int\limits_{0}^{x} \left(x-s\right)^{-\alpha-1} f\left(s\right) ds, \quad 0 < x < +\infty. \tag{1.3.36}$$

Weyl integral:

$${}_{x}W_{\infty}^{\alpha} f\left(x\right) = \frac{1}{\Gamma\left(\alpha\right)} \int\limits_{x}^{\infty} \left(s-x\right)^{\alpha-1} f\left(s\right) ds, \quad x \le b. \tag{1.3.37}$$

Chen left-sided integral:

$$I_{c+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{c}^{x} (x-s)^{\alpha-1} f(s)\, ds, \quad x > c. \tag{1.3.38}$$

Chen right-sided integral:

$$I_{c-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{c} (s-x)^{\alpha-1} f(s)\, ds, \quad x < c. \tag{1.3.39}$$

Cossar integral:

$$I_{c}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{c}^{x} (x-s)^{\alpha-1} f(s)\, ds, \quad x > c. \tag{1.3.40}$$

Erdélyi left-sided integral:

$$I_{\sigma,\beta}^{\alpha} f(x) = \frac{\sigma x^{-\sigma(\alpha+\beta)}}{\Gamma(\alpha)} \int_{0}^{x} (x^{\sigma} - s^{\sigma})^{\alpha-1} s^{\sigma(\beta+1)-1} f(s)\, ds. \tag{1.3.41}$$

Erdélyi right-sided integral:

$$I_{\sigma,\beta}^{\alpha} f(x) = \frac{\sigma x^{\sigma\alpha}}{\Gamma(\alpha)} \int_{x}^{\infty} (s^{\sigma} - x^{\sigma})^{\alpha-1} s^{\sigma(1-\alpha-\beta)-1} f(s)\, ds. \tag{1.3.42}$$

Kober left-sided integral:

$$I_{1,\beta}^{\alpha} f(x) = \frac{x^{-(\alpha+\beta)}}{\Gamma(\alpha)} \int_{0}^{x} (x-s)^{\alpha-1} s^{\beta} f(s)\, ds. \tag{1.3.43}$$

Kober right-sided integral:

$$I_{1,\beta}^{\alpha} f(x) = \frac{x^{\alpha}}{\Gamma(\alpha)} \int_{x}^{\infty} (s-x)^{\alpha-1} s^{-\alpha-\beta} f(s)\, ds. \tag{1.3.44}$$

We can see [5].

Remark 1.3.8 *For $0 < \alpha < 1, n = 1,$ the above definitions in general,*

$$\textit{left-sided} \quad I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} (x-s)^{\alpha-1} f(s)\, ds, \quad -\infty \leq a < x,$$

$$\textit{right-sided} \quad I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{b} (s-x)^{\alpha-1} f(s)\, ds, \quad x < b \leq +\infty.$$

All definitions are attempted to satisfy the usual properties of the standard derivative. The only property inherited by all definitions of the fractional derivative is the linearity property. However, the following are the setbacks of one definition or another:

I) Most of the fractional derivatives do not satisfy $D^{\alpha}C = 0$.

II) Most of the fractional derivatives do not satisfy the known product rule

$$D^{\alpha} fg = gD^{\alpha} f + fD^{\alpha} g.$$

III) Most of the fractional derivatives do not satisfy the known quotient rule:

$$D^{\alpha} f/g = \frac{gD^{\alpha} f - fD^{\alpha} g}{g^2}.$$

IV) Most of the fractional derivatives do not satisfy the chain rule:

$$D^{\alpha} f(g(x)) = (g(x))^{\alpha} f_g^{(\alpha)}(g(x)).$$

V) Most of the fractional derivatives do not satisfy:

$$D^{\beta} D^{\alpha} f = D^{\beta+\alpha} f.$$

1.3.3 Some of the most famous expressions of fractional derivative :

Grunwald-Letnkov derivative.

Let the function $f(t)$ is integrable, is known as the Grunwald-Letnkov definition of fractional derivative :

Definition 1.3.9 *If the function $f(t) \in C^n([a,b])$, and $n-1 < \alpha < n$ then,*

$$ {}_{a}^{G}D_{t}^{\alpha} f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^{k-\alpha}}{\Gamma(n-\alpha+1)} + \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \tag{1.3.45}$$

where

$$ {}_a^G D_t^\alpha ({}_a^G D_t^\beta f(t)) = {}_a^G D_t^\beta ({}_a^G D_t^\alpha f(t)) = {}_a^G D_t^{\alpha+\beta} f(t). $$

For $0 < \alpha < 1$ expression is,

$$ {}_a^G D_t^\alpha f(t) = \frac{f(a)(t-a)^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds, \tag{1.3.46} $$

and

$$ {}_a^G D_t^\alpha C = \frac{C}{\Gamma(1-\alpha)} (t-a)^{-\alpha} \neq 0. $$

Riemann-Liouville derivative.

Let the function $f(t)$ is integrable expression as following defines,

Definition 1.3.10 *If the function $f(t) \in C^n([a,b])$, and $n-1 < \alpha < n$ then the integro-differential expression*

$$ {}_a^R D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s) ds = \frac{d^n}{dt^n} (I^{n-\alpha} f(t)), a < t. \tag{1.3.47} $$

$$ {}_t^R D_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s) ds = \frac{d^n}{dt^n} (I^{n-\alpha} f(t)), t < b. \tag{1.3.48} $$

Here the n is a positive integer number just greater than real number $n\alpha$.

For $0 < \alpha < 1$ expression is,

$$ {}_a^R D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left[\int_a^t (t-s)^{-\alpha} f(s) ds \right], a < t. \tag{1.3.49} $$

The above expression is known as the Riemann-Liouville definition of fractional derivative, with $n-1 \leq \alpha < n$.

And,

$$ {}_a^R D_t^\alpha (I^\alpha f(t)) = f(t), $$

si $n-1 \leq \alpha < n$, $m-1 \leq \beta < m$, and

$$
\begin{aligned}
{}_a^R D_t^{\alpha}({}_a^R D_t^{\beta} f(t)) &= {}_a^R D_t^{\alpha+\beta} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^{k-\alpha-n}}{\Gamma(k-n-\alpha+1)}, \\
\frac{d^n}{dt^n}({}_a^R D_t^{\alpha} f(t)) &= {}_a^R D_t^{n+\alpha} f(t), \qquad (1.3.50) \\
{}_a^R D_t^{\alpha}(\frac{d^n}{dt^n} f(t)) &= {}_a^R D_t^{n+\alpha} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^{k-\alpha-n}}{\Gamma(k-n-\alpha+1)}. \qquad (1.3.51)
\end{aligned}
$$

So,

$$ {}_a^R D_t^{\alpha} f(t) =_a^G D_t^{\alpha} f(t), $$

$$ {}_a^R D_t^{\alpha}({}_a^R D_t^{\beta} f(t)) =_a^R D_t^{\beta}({}_a^r D_t^{\alpha} f(t)) =_a^R D_t^{\alpha+\beta} f(t), $$

and

$$ {}_a^R D_t^{\alpha} C = \frac{C}{\Gamma(1-\alpha)}(t-a)^{-\alpha} \neq 0. $$

But, another modification of the definition of (left /right) $R-L$ type fractional derivative of the function $f(x)$, in the interval $[a,b]$ was proposed by Jumarie [7] in the form described below,

$$
{}_a^j D_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int\limits_a^t (t-s)^{-\alpha-1} f(s) ds = I^{-\alpha} f(t), & \alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int\limits_0^t (t-s)^{-\alpha} f(s) ds = \frac{d}{dt}(I^{1-\alpha} f(t)), & 0 < \alpha < 1 \\ (f^{(\alpha-m)}(t))^{(m)}, \quad m < \alpha < m+1. \end{cases}
$$

Using the above definition Jumarie [7] proved,

$$ D_t^{\alpha}(f(t)g(t)) = (D_t^{\alpha} g(t)) f(t) + g(t) D_t^{\alpha}(f(t)). \qquad (1.3.52) $$

Again from the Jumarie definition of fractional derivative we have ${}_a^j D_t^{\alpha}(C) = 0$.

Caputo derivative

Definition of Caputo fractional derivative is given below.

Definition 1.3.11 *If the function* $f(t) \in C^n([a,b])$*, and* $n-1 < \alpha < n$ *then,*

$$ {}_a^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) dsn, $$

and

$$ \begin{aligned} {}_a^C D_t^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds = I^{n-\alpha}(\frac{d^n}{dt^n} f(t))_a^R D_t^{\alpha}(I^{\alpha} f(t)) \\ &= f(t) I^{\alpha}({}_a^C D_t^{\alpha} f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^k}{k!}. \end{aligned} \tag{1.3.53} $$

For $0 < \alpha < 1$ expression is,

$$ {}_a^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} f'(s) ds. $$

1.3.4 Fractional derivative of the usuel functions:

$f(t)$	C	$(t-a)^{\beta}$	e^{at}	$\sin(at)$
Grunwald-Letnkov ${}_a^G D_t^{\alpha} f(t)$	$\frac{C}{\Gamma(1-\alpha)}(t-a)^{-\alpha}$	$\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}$	$a^{\alpha} e^{at}$	$a^{\alpha} \sin(at + \alpha\frac{\pi}{2})$
Riemann-Liouville ${}_a^R D_t^{\alpha} f(t)$	$\frac{C}{\Gamma(1-\alpha)}(t-a)^{-\alpha}$	$\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}$	$a^{\alpha} e^{at}$	$a^{\alpha} \sin(at + \alpha\frac{\pi}{2})$
Caputo ${}_0^C D_t^{\alpha} f(t)$	0	$\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}$	$a^{\alpha} e^{at}$	$a^{\alpha} \sin(at + \alpha\frac{\pi}{2})$

(1.3.54)

1.3.5 Integral Transform

Let $f(t)$ be a function of t, the integral

$$ \int_{-\infty}^{+\infty} K(s,t) f(t) dt, \tag{1.3.55} $$

is defined as the integral transform (t) provided the integral is convergent, where $K(s,t)$ known as the kernel of transformation which is a function is a two variables s and t, s is a parameter independent of t.

Kernel $K(s,t)$ defines different types of transformations, some of them are given below:

i) We define Laplace transform of $f(t), L(f)$ if $K(s,t) = \begin{cases} e^{-st} & \text{when } t \geq 0 \\ 0 & \text{when } t < 0 \end{cases}$,

$$F(s) = L(f(t)) = \int_0^{+\infty} e^{-st} f(t)dt. \tag{1.3.56}$$

The following formula seems to be another useful property for the Laplace transform of the derivative of an integer order n of the function $f(t)$:

$$L(f^{(n)}(t)) = s^n F(s) - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0).$$

With the help of the inverse Laplace transform, the original $f(t)$ can be gained from the Laplace transform,

$$f(t) = L^{-1}(F(s)) = \frac{1}{2\pi i} \lim_{b \to +\infty} \int_{a-ib}^{a+ib} e^{st} F(s)ds, \quad a = Re(s). \tag{1.3.57}$$

Lemma 1.3.12 *For $\alpha \geq \beta > 0$, $a \in IR$ and $s^\alpha > |a|$ we have the following inverse Laplace transform formula*

$$L^{-1}\left[\frac{s^{\alpha-\beta}}{s^\alpha + a}\right] = t^{\beta-1} E_{\alpha,\beta}(-at^\alpha). \tag{1.3.58}$$

For $\alpha \geq \beta > 0, a \in IR$ and $s^{\alpha-\beta} > |a|$, we have the following inverse Laplace transform formula,

$$L^{-1}\left[\frac{1}{(s^\alpha + as^\beta)^{n+1}}\right] = t^{\alpha(n+1)-1} \sum_{k=0}^{+\infty} \frac{(-a)^k \begin{pmatrix} n+k \\ n \end{pmatrix}}{\Gamma(k(\alpha-\beta) + (n+1)\alpha)} t^{k(\alpha-\beta)} \tag{1.3.59}$$

Lemma 1.3.13 *For $\alpha \geq \beta > 0, \alpha > \delta, a \in IR$, $s^{\alpha-\beta} > |\alpha|$ and $\left|s^\alpha + as^\beta\right| > |b|$ we have,*

$$L^{-1}\left[\frac{s^{\delta}}{s^{\alpha}+as^{\beta}+b}\right]=t^{\alpha-\delta-1}\sum_{n=0}^{+\infty}\sum_{k=0}^{+\infty}\frac{(-b)^k(-a)^k\begin{pmatrix} n+k \\ n \end{pmatrix}}{\Gamma(k(\alpha-\beta)+(n+1)\alpha-\delta)}t^{k(\alpha-\beta)+n\alpha}. \qquad (1.3.60)$$

Laplace transform table of some basic fractional calculus :

$f(t)=L^{-1}(F(s))$	$F(s)=L(f(t))$	$f(t)=L^{-1}(F(s))$	$F(s)=L(f(t))$
$\frac{t^{\alpha-1}}{\Gamma(\alpha)}$	$\frac{1}{s^{\alpha}}$	$t^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})$	$\frac{1}{s^{\alpha}+a}$
$\frac{t^{\alpha-1}}{\Gamma(\alpha)}e^{-\alpha t}$	$\frac{1}{(s+a)^{\alpha}}$	$t^{\alpha}E_{1,1+\alpha}(at)$	$\frac{1}{s^{\alpha}(s-a)}$
$e^{-\alpha t}$		$t^{-\alpha}E_{1,1-\alpha}(at)$, $0<\alpha<1$	$\frac{s^{\alpha}}{s-a}$
$\frac{1}{\Gamma(\alpha)}\Gamma(\alpha,at)$	$\frac{a^{\alpha}}{s(s+a)^{\alpha}}$	$t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})$	$\frac{s^{\alpha-\beta}}{s^{\alpha}-a}$
$E_{\alpha}(-at^{\alpha})$	$\frac{s^{\alpha-1}}{(s+a)^{\alpha}}$	${}_1F_1(\alpha;1;at)$	$\frac{s^{\alpha-1}}{(s-a)^{\alpha}}$
$1-E_{\alpha}(-at^{\alpha})$	$\frac{a}{s(s^{\alpha}+a)}$	$\frac{t^{\beta-1}}{\Gamma(\beta)}{}_1F_1(\alpha;\beta;at)$	$\frac{s^{\alpha-\beta}}{(s-a)^{\alpha}}$

(1.3.61)

Laplace transform of some fractional operators with order α :

Derivative	Laplace Transform of
Riemann-Liouville integral ${}_0^RI_t^{\alpha}f(t)$	$L({}_0^RI_t^{\alpha}f(t))=s^{-\alpha}F(s)$
Riemann-Liouville derivative ${}_0^RD_t^{\alpha}f(t)$	$L({}_0^RD_t^{\alpha}f(t))=s^{\alpha}F(s)-{}_0^RD_t^{\alpha-1}f(t)_{t=0}$ $0\le\alpha<1$
Caputo derivative ${}_0^CD_t^{\alpha}f(t)$	$L({}_0^CD_t^{\alpha}f(t))=s^{\alpha}F(s)-f(0)$ $0\le\alpha<1$
Griinwald-Leitnikov derivative ${}_0^GD_t^{\alpha}f(t)$	$L({}_0^GD_t^{\alpha}f(t))=s^{\alpha}F(s)$ $0<\alpha<1$

(1.3.62)

We will take care only of the terms of the derivation that conform to the following :

As mentioned previously, there are a large number of definitions of fractional derivatives. We cannot make use of all the previous definitions and there are several attempts to generalize the definition of these fractional derivatives' forms. In this research, we will only care about definitions that satisfy the conditions, which are:

Condition 1.3.14 *Let* $\alpha \in [0,1]$. *An operator* D^{α} *is a fractional differential operator if it satisfies the following :*

1) Linearity: $D^{\alpha}(f+g) = D^{\alpha}(f) + D^{\alpha}(g)$ *for all* $f, g \in Dom(D^{\alpha})$.

2) $D^{0}(f) = f$ *for all functions* f, *and* $D^{1}(f) = f'$, *for all* $f \in Dom(D^{1})$.

3) The product law : $D^{\alpha} f(x)\, g(x) = f(x)\, D^{\alpha} g(x) + g(x)\, D^{\alpha} f(x)$.

Remark 1.3.15 *For* $0 < \alpha < 1, n = 1$, *the definitions,*(1.3.13),(1.3.14),(1.3.18),(1.3.19), (1.3.20), (1.3.21), (1.3.24), (1.3.25), (1.3.26) *the above conditions check.*

Chapitre 2

Existence of Solution

2.1 The existence of solutions for fractional differential equations

This part concerns the existence of solutions for fractional differential equations of the form :

$$P\left(t, D_t^{m'}u\right)u(t) = \sum_{k=0}^{k=n} g_k(t) D^{k\alpha_k} u(t) = f(t), \tag{2.1.1}$$

then $D^{k\alpha_k}, \alpha_k \in Q \neq \frac{1}{k}$ the (Riemann-Liouvillle or Caputo or .....) fractional derivative.

The coefficients $g_k(t) \in C([t_0, T])$, the function $f(t)$ are defined for $t \in [t_0, T]$.Their regularity must be such that for each $u \in C_\beta^{m'}([t_0, T]), m' = \max\limits_{k=0,..,n} \{k\alpha_k\}, g_{m'}(t) = 1$.

To prove the existence of the problem solution(2.1.1), we prove the following theorem:

Theorem 2.1.1 *The problem*(2.1.1)*, has a solution.*

Proof. We demonstrate the existence of a solution of(2.1.24) by converting it into an equivalent system that accepts the solution.

Let $\alpha_k = \frac{a_k}{b_k}; a_k \in IN, b_k \in IN, b = pgcm\{b_k\}, \beta = \frac{1}{b}, \frac{1}{b_k} = \frac{i_k}{b}$ and,

$$k\alpha_k = k\frac{a_k}{b_k} = \frac{1}{b}ki_k a_k = \beta k i_k a_k = \beta m_k,$$

with $m_k = ki_k a_k \in IN$,and $0 < \beta < 1$, the defined fractional operator by D^β,

$$\begin{array}{rcl} C([t_0,T]) & \longrightarrow & C_\beta([t_0,T]) \\ u(t) & \longmapsto & D^\beta u(t) \end{array}$$

we can transform the problem to,

$$\sum_{k=0}^{k=n} g_k(t) D^{\beta m_k} u(t) = f(t), \tag{2.1.2}$$

we have $\beta m_k = \beta + ... + \beta$ (m_k fois), we can write $D^{\beta m_k} = D^\beta ....D^\beta = (D^\beta)^{m_k}$, where $0 < \beta < 1$.

For each k there are m_k and d such that, if $k \leq n$ and $d \leq k_1$, so

$$\begin{array}{rcl} \sum_{k=0}^{k=n} g_k(t) D^{\beta m_k} & = & P = P_d \circ .... \circ P_2 \circ P_1 + R, \qquad R = \sum_{l=0}^{m} r_l D_t^{m-l} \\ P_i & = & \sum_{k=0}^{n_i} g_k^{(i)} (D_t^\beta - \lambda_j)^{n_i - k}, i = 1, ..d, \qquad \sum_{i=1}^{i=d} n_i = m_k \\ g_k^{(i)}(t) & \in & C([t_0,T]) \\ r_l & = & r_l(t) \in C([t_0,T]) \end{array} \tag{2.1.3}$$

where $\lambda_j, j = 1, 2, ...$, be continuous functions on an interval $[t_0, T)$, and

$$(D_t^\beta - \lambda_j) = (D_t^\beta - \lambda_j\,(t)\,.D_t^0).$$

The factorization of the operator P for the determination of the coefficients g_k, r_l and the calculation r, the number of roots multiplied.

By studying the case "$m' = m_k - r$" we extend the study for any order because of $m' = m - r < ...... < m$, and $C^{\alpha(m-1)} \subseteq C^{\alpha(m-r)}$.

From factorization(2.1.24), we can reduce the equation in unknown u,

$$\sum_{k=0}^{k=n} g_k(t) D^{\beta m_k} u = Pu = f, \tag{2.1.4}$$

to an equivalent system.

Without generality loss, but having only a simpler notation, consider the case $d = 2$ of an operator with two multiple characteristic roots, we both :

$$P = P_2 \circ P_1 + R, \tag{2.1.5}$$

and, by permutation,

$$P = \tilde{P}_1 \circ \tilde{P}_2 + \tilde{R}, \tag{2.1.6}$$

with

$$\begin{aligned} P_1 &= (D_t^\beta - \lambda_1)^{n_1} + \sum_{k=1}^{n_1} g_{j,k}^{(1)} (D_t^\beta - \lambda_1)^{n_1 - k}, && (2.1.7) \\ P_2 &= (D_t^\beta - \lambda_2)^{n_2} + \sum_{k=1}^{n_2} g_{j,k}^{(2)} (D_t^\beta - \lambda_2)^{n_2 - k}. && (2.1.8) \\ \tilde{P}_1 &= (D_t^\beta - \lambda_1)^{n_1} + \sum_{k=1}^{n_1} \tilde{g}_{j,k}^{(1)} (D_t^\beta - \lambda_1)^{n_1 - k}, && (2.1.9) \\ \tilde{P}_2 &= (D_t^\beta - \lambda_2)^{n_2} + \sum_{k=1}^{n_2} \tilde{g}_{j,k}^{(2)} (D_t^\beta - \lambda_2)^{n_2 - k}. && (2.1.10) \end{aligned}$$

The factors P_i and $\tilde{P}_i$ have the respective coefficients $g_{j,k}^{(i)}$ and $\tilde{g}_{j,k}^{(i)}$ of order 0 while the remainders R and $\tilde{R}$ have respective coefficients r_l and $\tilde{r}_l$ of the order $l - m_k$, and

$$P_2 \circ P_1 = P - R,$$

and

$$\tilde{P}_1 \circ \tilde{P}_2 = P - \tilde{R}.$$

For example : $n_1 = n_2 = 1$,

$$\begin{aligned} P_2 \circ P_1 &= (D_t^\beta - \lambda_2(t))(D_t^\beta - \lambda_1(t)) \\ &= D_t^{2\beta} - (\lambda_1(t) + \lambda_2(t)) D_t^\beta + \lambda_2(t)\lambda_1(t) + D_t^\beta \lambda_1(t) \\ &= D_t^{2\beta} - g_1(t) D_t^\beta + g_2(t), \end{aligned}$$

and

$$\begin{aligned} \tilde{P}_1 \circ \tilde{P}_2 &= (D_t^\beta - \lambda_1(t))(D_t^\beta - \lambda_2(t)) \\ &= D_t^{2\beta} - (\lambda_1(t) + \lambda_2(t)) D_t^\beta + \lambda_2(t)\lambda_1(t) + D_t^\beta \lambda_2(t) \\ &= D_t^{2\beta} - \tilde{g}_1(t) D_t^\beta + \tilde{g}_2(t). \end{aligned}$$

And $n_1 = 2$,

$$\begin{aligned} P_1 &= (D_t^\beta - \lambda_1(t))^2 = (D_t^\beta - \lambda_1(t))(D_t^\beta - \lambda_1(t)) \\ &= D_t^{2\beta} - 2\lambda_1(t) D_t^\beta + \lambda_1^2(t) + D_t^\beta \lambda_1(t) \\ &= D_t^{2\beta} - g_1(t) D_t^\beta + g_2(t). \end{aligned}$$

Then, given the function u, we define the vector

$$U = (u_0, ..., u_{m_k-1}, u_{m_k}, ...u_{2m_k-1})^t, \tag{2.1.11}$$

by

$$\Big[(D_t^{\beta}-\lambda_1)\curvearrowright(D_t^{\beta}-\lambda_2)\Big] \quad \Big[(D_t^{\beta}-\lambda_2)\curvearrowright(D_t^{\beta}-\lambda_1)\Big]$$

$$\begin{cases} u_0 = u, & u_n = u,\\ u_1 = (D_t^{\beta}-\lambda_1)u, & u_{n+1} = (D_t^{\beta}-\lambda_2)u,\\ . & .\\ . & .\\ u_{n_1-1} = (D_t^{\beta}-\lambda_1)^{n_1-1}u, & u_{n+n_2-1} = (D_t^{\beta}-\lambda_2)^{n_2-1}u\\ u_{n_1} = P_1u, & u_{n+n_2} = \tilde{P}_2u,\\ u_{n_1+1} = (D_t^{\beta}-\lambda_2)P_1u, & u_{n+n_2+1} = (D_t^{\beta}-\lambda_1)\tilde{P}_2u,\\ . & .\\ . & .\\ u_{n-1} = (D_t^{\beta}-\lambda_2)^{n_2-1}P_1u & u_{2n-1} = (D_t^{\beta}-\lambda_1)^{n_1-1}\tilde{P}_2u. \end{cases} \tag{2.1.12}$$

$, n_1+n_2 = m_k.$

The equation 2.1.4 is equivalent to,

$$\begin{cases} (D_t^{\beta}-\lambda_1)u_j = u_{j+1} \qquad (0\le j\le n_1-2)\\ (D_t^{\beta}-\lambda_1)u_{n_1-1} = u_{n_1} - \sum_{k=1}^{n_1} g^{(1)}_{n_1-k,k}u_{n_1-k}\\ (D_t^{\beta}-\lambda_2)u_j = u_{j+1} \qquad (n_1\le j\le m_k-2)\\ (D_t^{\beta}-\lambda_2)u_{m_k-1} = f - Ru - \sum_{k=1}^{n_2} g^{(2)}_{m_k-k,k}u_{m_k-k}\\ (D_t^{\beta}-\lambda_2)u_{m_k+j} = u_{m_k+j+1} \qquad (0\le j\le n_2-2)\\ (D_t^{\beta}-\lambda_2)u_{m_k+n_2-1} = u_{m_k+n_2} - \sum_{k=1}^{n_2} \tilde{g}^{(2)}_{m_k+n_2-k,k}u_{m_k+n_2-k}\\ (D_t^{\beta}-\lambda_1)u_{m_k+j} = u_{m_k+j+1} \qquad (n_2\le j\le m_k-2)\\ (D_t^{\beta}-\lambda_1)u_{2m_k-1} = f - \tilde{R}u - \sum_{k=1}^{n_2} \tilde{g}^{(1)}_{m_k-k,k}u_{2m_k-k} \end{cases} \tag{2.1.13}$$

Lemma 2.1.2 *Let u and $P\left(t, D_t^{\beta(m-r)}u\right)$ be as in proposition* (1.3.1) *and, given the function u, left the vector $u = (u_0, ..., u_{m_k-1}, u_{m_k}, ...u_{2m_k-1})^t$ the matrix defined in* (2.1.13)

This gives immediately

$$
\begin{cases}
Ru = \sum_{j=0}^{m_k-1} r_j(t)u_j, \\
\tilde{R}u = \sum_{j=0}^{m_k-1} \tilde{r}_j(t)u_{m_k+j}, \\
r_j, \tilde{r}_j \in C([0,T]),
\end{cases}
\tag{2.1.14}
$$

so,

$$
\begin{cases}
D_t^{\beta} u_j = \lambda_1 u_j + u_{j+1} \qquad (0 \le j \le n_1 - 2) \\
D_t^{\beta} u_{n_1-1} = \lambda_1 u_{n_1-1} + u_{n_1} - \sum_{k=1}^{n_1} g_{n_1-k,k}^{(1)} u_{n_1-k} \\
D_t^{\beta} u_j = \lambda_2 u_j + u_{j+1} \quad (n_1 \le j \le m_k - 2) \\
D_t^{\beta} u_{m_k-1} = \lambda_2 u_{m_k-1} + f - Ru - \sum_{k=1}^{n_2} g_{n_1-k,k}^{(2)} u_{m_k-k} \\
D_t^{\beta} u_{m_k+j} = \lambda_2 u_{m_k+j} + u_{m_k+j+1} \quad (0 \le j \le n_2 - 2) \\
D_t^{\beta} u_{m_k+n_2-1} = \lambda_2 u_{m_k+n_2-1} + u_{m_k+n_2} - \sum_{k=1}^{n_2} \tilde{g}_{n_1-k,k}^{(2)} u_{m_k+n_2-k} \\
D_t^{\beta} u_{m_k+j} = \lambda_1 u_{m_k+j} + u_{m_k+j+1} \quad (n_2 \le j \le m_k - 2) \\
D_t^{\beta} u_{2m_k-1} = \lambda_1 u_{2m_k-1} + f - \tilde{R}u - \sum_{k=1}^{n_2} \tilde{g}_{n_1-k,k}^{(1)} u_{2m_k-k},
\end{cases}
\tag{2.1.15}
$$

we have $u = u_0$, the rest R, $\tilde{R}$ is of order 0.

And we have

$$\begin{aligned}
\sum_{k=1}^{m_2} g_k^{(1)} u_{m-k} &= \sum_{j=0}^{m_1-1} g_{m_1-j}^{(1)} u_j \\
\sum_{k=1}^{m_2} g_k^{(2)} u_{m-k} &= \sum_{j=m_1}^{m-1} g_{m-j}^{(2)} u_j \\
\sum_{k=1}^{m_2} \tilde{g}_k^{(2)} u_{m+m_2-k} &= \sum_{j=m}^{m+m_2-1} \tilde{g}_{m+m_2-j}^{(2)} u_j \\
\sum_{k=1}^{m_2} \tilde{g}_k^{(1)} u_{2m-k} &= \sum_{j=m_1}^{m-1} \tilde{g}_{m-j}^{(1)} u_{m+j} \\
\sum_{j=0}^{m-1} b_j u_j &= \sum_{j=0}^{m_1-1} b_j u_j + \sum_{j=m_1}^{m-1} b_j u_j \\
\sum_{j=0}^{m-1} \tilde{b}_j u_{m+j} &= \sum_{j=0}^{m_1-1} \tilde{b}_j u_{m+j} + \sum_{j=m_1}^{m-1} \tilde{b}_j u_{m+j}
\end{aligned}$$

we have

$$\begin{cases}
\begin{cases} D_t^{\beta j} u = \sum_{l=1}^{j} g_l^{(j)}(t)\, u_{j-l} + u_j \\ \qquad 0 \leq j \leq m-1, \end{cases} \\
\begin{cases} D_t^{\beta j} u = \sum_{l=1}^{j} \tilde{g}_l^{(j)}(t)\, u_{m+j-l} + u_{m+j} \\ \qquad 0 \leq j \leq m-1. \end{cases}
\end{cases} \tag{2.1.16}$$

(2.1.14) and (2.1.13) give,

$$\begin{cases} D_t^{\beta} u_j - (\lambda_1 u_j + u_{j+1} \quad) = 0, \ (0 \leq j \leq m_1 - 2) \\ D_t^{\beta} u_{m_1-1} - \left(-\sum_{j=0}^{m_1-2} g_{m_1-j}^{(1)} u_j + \left(\lambda_1 - g_1^{(1)}\right) u_{m_1-1} + u_{m_1} \right) = 0 \\ D_t^{\beta} u_j - (\lambda_2 u_j + u_{j+1}) = 0, \ (m_1 \leq j \leq m-2) \\ D_t^{\beta} u_{m-1} - \left(-\sum_{j=0}^{m_1-1} b_j u_j - \sum_{j=m_1}^{m-1} \left(b_j + g_{m-j}^{(2)}\right) u_j + \lambda_2 u_{m-1} \right) = f \\ D_t^{\beta} u_{m+j} - (\lambda_2 u_{m+j} + u_{m+j+1} \quad) = 0, \ (0 \leq j \leq m_2 - 2) \\ D_t^{\beta} u_{m+m_2-1} - \left(- \sum_{j=m}^{m+m_2-2} \tilde{g}_{m+m_2-j}^{(2)} u_j + \left(\lambda_2 - \tilde{g}_1^{(2)}\right) u_{m+m_2-1} + u_{m+m_2} \right) = 0 \\ D_t^{\beta} u_{m+j} - (\lambda_1 u_{m+j} + u_{m+j+1} \quad) = 0, \ (m_2 \leq j \leq m-2) \\ D_t^{\beta} u_{2m-1} - \left(-\sum_{j=0}^{m_1-1} \tilde{b}_j u_{m+j} - \sum_{j=m_1}^{m-1} \left(\tilde{b}_j + \tilde{g}_{m-j}^{(1)}\right) u_{m+j} + \lambda_1 u_{2m-1} \right) = f \end{cases} \tag{2.1.17}$$

we put $U = (u_0, ..., u_{m-1}, u_m, ...u_{2m-1})^t$ and $F = (0, ..., 0, f, 0, ..., 0, f)^t$.

The problem(2.1.15) turns into an equivalent problem:

$$\partial_t^{\beta} U - K(t)U = F, \tag{2.1.18}$$

for a symmetric system $\partial_t^{\beta} - K$ of $dm_k - r \times \ dm_k - r$, $d = 2$ n, since $K = (\begin{matrix} A_1 & 0 \\ 0 & A_2 \end{matrix})$, is a real matrix λ_i. ■

2.1.1 Equivalent diagonal system

In this part, transform the system(2.1.18) to system,

$$\begin{cases} D_t^{\beta} U = \tilde{K}(t)U + \tilde{F}, \\ U(0) = G, \end{cases} \tag{2.1.19}$$

where $\tilde{K}$, is the diagonalized matrix of K.

We need to prove the following tools :

Proposition 2.1.3 *The system* (2.1.18) *has the solution,*

$$U \in C^{\beta}([0,T]),$$

such that,

$$\|U(t)\|_{C^{\beta}}^{2} \leq \left(1 + C_u |t|^{\beta}\right) \left[\int_{0}^{t} \|F(\tau)\|_{C}^{2} \, d\tau\right], \; t \in [0,T].$$

Proposition 2.1.4 *The solution of* (2.1.4),

$$u \in \bigcap_{j=0}^{m-r} C^{j\beta}([0,T]),$$

satisfies

$$\sum_{j=0}^{m-1} \left\|\partial_t^{j\beta} u(t)\right\|_{\mathcal{C}^{\beta(m-r-j)}}^{2} \leq C_u \left[\int_{0}^{t} |f(\tau)|^{2} \, d\tau\right], \; t \in [0,T].$$

Remark 2.1.5 *In both cases* C_u *depends on normes*

$$\left\|\partial_t^{\beta j} u(t,.)\right\|_{\mathcal{C}^{\beta(m-r-j)}}.$$

Proof. To resolve (2.1.18) in $\mathcal{C}^{\beta}$, it is necessary that $u(t) \in \mathcal{C}^{\beta(m-r)}$.

Of the two proposition(1.3.3) and(2.1.16), and for k large enough, operators

$$q_j^{(k\beta)}(t) : \mathcal{C}^{\beta} \to \mathcal{C}^{\beta(m-r-k)}$$

continuous

and applying derivative fractional $\partial_t^{\beta} = \partial_t^{\beta}, 0 < \beta \leq 1$, (2.1.4) we get the problems for fractional derivatives $u^{(\beta)} = \partial_t^{\beta} u$,

$$\begin{aligned} &\partial_t^{\beta}(Pu) = f^{(\beta)} \\ &Pu^{(\beta)} + \partial_t^{\beta}(Pu) - Pu^{(\beta)} = Pu^{(\beta)} + \left[\partial^{\beta}, P\right] u = f^{(\beta)}, \end{aligned} \tag{2.1.20}$$

with

$$[\partial^{\beta}, P] = \partial_t^{\beta} P - P \partial_t^{\beta}.$$

We obtain the matrix of components $u_j^{(\beta)}$,

$$\tilde{U} = \left(u_j^{(\beta)}; 0 \leq j \leq 2m-1, 0 < \beta \leq 1\right), \tag{2.1.21}$$

obtained from (2.1.12) by replacing u by $u^{(\beta)}$, on the other hand,

$$\partial_t^{\beta}\left(Pu\right) = P^{(\beta)}u + Pu^{(\beta)} = Pu^{(\beta)} + \left[\partial^{\beta}, P\right]u,$$

so,

$$P^{(\beta)}u = \left[\partial^{\beta}, P\right]u.$$

■

Proposition 2.1.6 *Let the vector* $U = (u_0, ..., u_{m_k-1}, u_{m_k}, ...u_{2m_k-1})^t$ *defined by* (2.1.12). *For each* k *there are* m_k *and* r *such that, if* $k \leq n$*, there is then a matrix* $Q = Q(t)$. *Where,*

$$D^{\beta m'}u = QU, Q \in C([-T,T]) \tag{2.1.22}$$

Proof. We can suppose $|\lambda_j(t) - \lambda_i(t)| > 0, i \neq j$ and,

$$\rho(t) = \tau^{\beta} + \sum_{m_k} g_{\alpha}(t)\tau^{\beta},$$

we can write, $\rho_j(t) = (\frac{\tau^{\beta}(t) - \lambda_i(t)}{\lambda_j(t) - \lambda_i(t)}), j = 1, ..., d$, we have $\forall i \neq j \;\; \rho_j + \rho_i = 1$ et $\rho_j\lambda_i + \rho_i\lambda_j = \tau^{\beta}$, so $(m_k = m\ )$

$$\forall i \neq j \;\;, (\sum_{j=1}^{d}\sum_{\substack{i=1\\i\neq j}}^{d}(\rho_j + \rho_i))^{-1}(\sum_{j=1}^{d}\sum_{\substack{i=1\\i\neq j}}^{d}(\rho_j\lambda_i + \rho_i\lambda_j)) = \tau^{\beta},$$

therefore,

$$\tau^{\beta} = (2\binom{d}{2})^{1-m}(\sum_{j=1}^{d}\sum_{\substack{i=1\\i\neq j}}^{d}(\rho_j + \rho_i))^{m-1-k}(\sum_{j=1}^{d}\sum_{\substack{i=1\\i\neq j}}^{d}(\rho_j\lambda_i + \rho_i\lambda_j))^{k},$$

because,

$$(\tau^\beta)^{k-(m-1)} = (\sum_{j=1}^{d}\sum_{\substack{i=1\\i\neq j}}^{d}(\rho_j+\rho_i))^{m-1-k}(\sum_{j=1}^{d}\sum_{\substack{i=1\\i\neq j}}^{d}(\rho_j\lambda_i+\rho_i\lambda_j))^{k-(m-1)},$$

and,

$$\begin{aligned}
1 &= \frac{\tau^\beta}{\tau^\beta} = (\sum_{j=1}^{d}\sum_{\substack{i=1\\i\neq j}}^{d}(\rho_j+\rho_i))^{m-1-k}\frac{(\sum_{j=1}^{d}\sum_{\substack{i=1\\i\neq j}}^{d}(\rho_j\lambda_i+\rho_i\lambda_j))^k}{(\sum_{j=1}^{d}\sum_{\substack{i=1\\i\neq j}}^{d}(q_j\lambda_i+q_i\lambda_j))^{(m-1)}}\\
&= (\sum_{j=1}^{d}\sum_{\substack{i=1\\i\neq j}}^{d}(\rho_j+\rho_i))^{m-1-k}\frac{(\sum_{j=1}^{d}\sum_{\substack{i=1\\i\neq j}}^{d}(\rho_j\lambda_i+\rho_i\lambda_j))^k}{((2\binom{d}{2})\tau^\beta)^{(m-1)}}\\
&= \frac{((2\binom{d}{2}))^{(1-m)}}{\tau^\beta}(\sum_{j=1}^{d}\sum_{\substack{i=1\\i\neq j}}^{d}(\rho_j+\rho_i))^{m-1-k}(\sum_{j=1}^{d}\sum_{\substack{i=1\\i\neq j}}^{d}(\rho_j\lambda_i+\rho_i\lambda_j))^k
\end{aligned}$$

so,

$$\tau^\beta = (2\binom{d}{2})^{1-m}(\sum_{j=1}^{d}\sum_{\substack{i=1\\i\neq j}}^{d}(\rho_j+\rho_i))^{m-1-k}(\sum_{j=1}^{d}\sum_{\substack{i=1\\i\neq j}}^{d}(\rho_j\lambda_i+\rho_i\lambda_j))^k \tag{2.1.23}$$

For $d = 2$ we have,

$$\begin{aligned}
\tau^\beta &= (\frac{\tau^\beta-\lambda_1}{\lambda_2-\lambda_1}-\frac{\tau^\beta-\lambda_2}{\lambda_2-\lambda_1})^{m-1-k}\ (\frac{\tau^\beta-\lambda_1}{\lambda_2-\lambda_1}\lambda_2-\frac{\tau^\beta-\lambda_2}{\lambda_2-\lambda_1}\lambda_1)^k\\
&= (\frac{1}{\lambda_2-\lambda_1})^{m-1}((\tau^\beta-\lambda_1)+(-(\tau^\beta-\lambda_2)))^{m-1-k}\ ((\tau^\beta-\lambda_1)\lambda_2\\
&\quad +(-(\tau^\beta-\lambda_2))\lambda_1)^k,
\end{aligned}$$

and, by Newton's formula, we have

$$\tau^\beta = \sum_{k_1=0}^{m-1-k}\sum_{k_2=0}^{k}\left[(-1)^{m-1-k_1-k_2}(\lambda_2-\lambda_1)^{1-m}(\lambda_1)^{k-k_1}(\lambda_2)^{k_2}\right](\tau^\beta-\lambda_1)^{k_1+k_2}\ (\tau^\beta-\lambda_2)^{m-1-k_1-k_2}$$

,on put

$$g^{(k)}_{k_1,k_2} = (-1)^{m-1-k_1-k_2}(\lambda_2-\lambda_1)^{1-m}(\lambda_1)^{k-k_1}(\lambda_2)^{k_2} \tag{2.1.24}$$

we have

$$\tau^\beta = \sum_{k_1=0}^{m-1-k} \sum_{k_2=0}^{k} g^{(k)}_{k_1,k_2}(\tau^\beta - \lambda_1)^{k_1+k_2} \quad (\tau^\beta - \lambda_2)^{m-1-k_1-k_2} \tag{2.1.25}$$

where $ord\ g^{(k)}_{k_1,k_2} \leq k + 1 - m_k$.

Posed $k_3 = k_1 + k_2$,

$$\begin{aligned}
\tau^\beta &= \sum_{k_3=0}^{m-1} g^{(k)}_{k_3}(\tau^\beta - \lambda_1)^{k_3} \quad (\tau^\beta - \lambda_2)^{m-1-k_3} \\
&= \sum_{k_3=0}^{n_1+n_2-1} g^{(k)}_{k_3}(\tau^\beta - \lambda_1)^{k_3} \quad (\tau^\beta - \lambda_2)^{m-1-k_3} \\
&= \sum_{k_3=0}^{n_1-1} g^{(k)}_{k_3}(\tau^\beta - \lambda_1)^{k_3} \quad (\tau^\beta - \lambda_2)^{m-1-k_3} \\
&\quad + \sum_{k_3=n_1}^{n_1+n_2-1} g^{(k)}_{k_3}(\tau^\beta - \lambda_1)^{k_3} \quad (\tau^\beta - \lambda_2)^{m-1-k_3} \\
&= \sum_{k_3=0}^{n_1-1} g^{(k)}_{k_3}(\tau^\beta - \lambda_1)^{k_3} \quad (\tau^\beta - \lambda_2)^{m-1-k_3} \\
&\quad + \sum_{k_3=0}^{n_2-1} g^{(k)}_{k_3}(\tau^\beta - \lambda_1)^{k_3+n_1}(\tau^\beta - \lambda_2)^{n_2-1-k_3}
\end{aligned} \tag{2.1.26}$$

1) In this sum, for $k_3 = k_1 + k_2 < n_1$, let us write

$$\begin{aligned}
&(\tau^\beta - \lambda_1)^{k_3} \ (\tau^\beta - \lambda_2)^{n_2+n_1-1-k_3} = (\tau^\beta - \lambda_1)^{k_3} \ (\tau^\beta - \lambda_2)^{n_2}(\tau^\beta - \lambda_2)^{n_1-1-k_3} \\
= \ &(\tau^\beta - \lambda_1)^{k_3} \ (\tau^\beta - \lambda_2)^{n_2}[(\tau^\beta - \lambda_1) + (\lambda_1 - \lambda_2)]^{n_1-1-k_3}
\end{aligned}$$

and,

$$
\begin{aligned}
&(\tau^\beta - \lambda_1)^{k_3}\ (\tau^\beta - \lambda_2)^{n_2}[(\tau^\beta - \lambda_1) + (\lambda_1 - \lambda_2)]^{n_1-1-k_3} \\
= &\sum_{j=0}^{n_1-1-k_3} (\tau^\beta - \lambda_1)^{k_3+n_1-1-k_3-j}\ (\tau^\beta - \lambda_2)^{n_2}(\lambda_1 - \lambda_2)^j \\
= &\sum_{j=0}^{n_1-1-k_3} (\lambda_1 - \lambda_2)^j (\tau^\beta - \lambda_1)^{n_1-1-j}\ (\tau^\beta - \lambda_2)^{n_2}
\end{aligned}
\tag{2.1.27}
$$

2) and for $k_3 = k_1 + k_2 \geq n_1$,

$$
\begin{aligned}
(\tau^\beta - \lambda_1)^{k_3+n_1}(\tau^\beta - \lambda_2)^{n_2-1-k_3} &= (\tau^\beta - \lambda_1)^{n_1}(\tau^\beta - \lambda_1)^{k_3}(\tau^\beta - \lambda_2)^{n_2-1-k_3} \\
&= (\tau^\beta - \lambda_1)^{n_1}(\tau^\beta - \lambda_2)^{n_2-1-k_3}[(\tau^\beta - \lambda_2) + (\lambda_2 - \lambda_1)]^{k_3}
\end{aligned}
$$

and,

$$
\begin{aligned}
(\tau^\beta - \lambda_1)^{k_3}(\tau^\beta - \lambda_2)^{n_{1.}+n_2-1-k_3} &= (\tau^\beta - \lambda_1)^{n_1+k_3-n_1}(\tau^\beta - \lambda_2)^{n_{1.}+n_2-1-k_3} \\
&= (\tau^\beta - \lambda_1)^{n_1}(\tau^\beta - \lambda_1)^{k_3-n_1}(\tau^\beta - \lambda_2)^{n_{1.}+n_2-1-k_3} \\
&= (\tau^\beta - \lambda_1)^{n_1}(\tau^\beta - \lambda_2)^{n_{1.}+n_2-1-k_3}\left[(\tau^\beta - \lambda_2) + (\lambda_2 - \lambda_1)\right]^{k_3-n_1}
\end{aligned}
$$

so,

$$
\begin{aligned}
&(\tau^\beta - \lambda_1)^{n_1}(\tau^\beta - \lambda_2)^{n_{1.}+n_2-1-k_3}[(\tau^\beta - \lambda_2) + (\lambda_2 - \lambda_1)]^{k_3-n_1} \\
= &\sum_{j=0}^{k_3-n_1} (\tau^\beta - \lambda_1)^{n_1}(\tau^\beta - \lambda_2)^{n_{1.}+n_2-1-k_3+k_3-n_1-j}(\lambda_2 - \lambda_1)^j \\
= &\sum_{j=0}^{k_3-n_1} (\lambda_2 - \lambda_1)^j(\tau^\beta - \lambda_1)^{n_1}(\tau^\beta - \lambda_2)^{n_2-1-j}
\end{aligned}
\tag{2.1.28}
$$

and, If we have $n_1 + n_2 = m_k$ and $j' = j - n_1$ in the second sum, so

$$
\begin{aligned}
\tau^\beta &= \sum_{k_3=0}^{n_1+n_2-1} g_{k_3}^{(k)}(\tau^\beta-\lambda_1)^{k_3} \quad (\tau^\beta-\lambda_2)^{m-1-k_3} \\
&= \sum_{k_3=0}^{n_1-1} g_{k_3}^{(k)}(\tau^\beta-\lambda_1)^{k_3} \quad (\tau^\beta-\lambda_2)^{m-1-k_3} \\
&\quad + \sum_{k_3=n_1}^{n_1+n_2-1} g_{k_3}^{(k)}(\tau^\beta-\lambda_1)^{k_3} \quad (\tau^\beta-\lambda_2)^{m-1-k_3} \\
&= \sum_{k_3=0}^{n_1-1} g_{k_3}^{(k)}(\tau^\beta-\lambda_1)^{k_3} \quad (\tau^\beta-\lambda_2)^{m-1-k_3} \\
&\quad + \sum_{k_3=0}^{n_2-1} g_{k_3}^{(k)}(\tau^\beta-\lambda_1)^{k_3+n_1}(\tau^\beta-\lambda_2)^{n_2-1-k_3} \\
&= \sum_{k_3=0}^{n_1-1} g_{k_3}^{(k)} \sum_{j=0}^{n_1-1-k_3} (\tau^\beta-\lambda_1)^{n_1-1-j} \; (\tau^\beta-\lambda_2)^{n_2}(\lambda_1-\lambda_2)^j \\
&\quad + \sum_{k_3=0}^{n_2-1} g_{k_3}^{(k)}(\tau^\beta-\lambda_1)^{n_1} \sum_{j=0}^{k_3-n_1} (\lambda_2-\lambda_1)^j(\tau^\beta-\lambda_1)^{n_1}(\tau^\beta-\lambda_2)^{n_2-1-j}
\end{aligned}
$$

$$\sum_{j=0}^{n_1+n_2-1} g_{k_3,k_2}^{(k)}(\tau^\beta - \lambda_1)^j(\tau^\beta - \lambda_2)^{n_1+n_2-1-j}$$

$$= \sum_{j=0}^{n_1-1} g_{k_1,k_2}^{(k)}(\tau^\beta - \lambda_1)^j(\tau^\beta - \lambda_2)^{n_1+n_2-1-j}$$

$$+ \sum_{j=n_1}^{n_1+n_2-1} g_{k_1,k_2}^{(k)}(\tau^\beta - \lambda_1)^j(\tau^\beta - \lambda_2)^{n_1+n_2-1-j} \tag{2.1.29}$$

$$= \sum_{j=0}^{n_1-1} g_{k_1,k_2}^{(k)}(\tau^\beta - \lambda_1)^j(\tau^\beta - \lambda_2)^{n_1+n_2-1-j}$$

$$+ \sum_{j=0}^{n_2-1} g_{k_1,k_2}^{(k)}(\tau^\beta - \lambda_1)^{n_1+j}(\tau^\beta - \lambda_2)^{n_2-1-j} \tag{2.1.30}$$

$$= \sum_{j=0}^{n_1-1} g_{1,j}^{(k)}(\tau^\beta - \lambda_1)^j(\tau^\beta - \lambda_2)^{n_1+n_2-1-j}$$

$$+ \sum_{j=0}^{n_2-1} g_{2,j}^{(k)}(\tau^\beta - \lambda_1)^{n_1}(\tau^\beta - \lambda_2)^{n_2-1-j}(\tau^\beta - \lambda_1)^j \tag{2.1.31}$$

we have

$$\sum_{j=0}^{n_1+n_2-1} g_{k_1,k_2}^{(k)}(\tau^\beta - \lambda_1)^j(\tau^\beta - \lambda_2)^{n_1+n_2-1-j} = \sum_{j=0}^{n_1-1} g_{1,j}^{(k)}(\tau^\beta - \lambda_1)^j(\tau^\beta - \lambda_2)^{n_2} + \sum_{j=0}^{n_2-1} g_{2,j}^{(k)}(\tau^\beta - \lambda_2)^j(\tau^\beta - \lambda_1)^{n_1}$$

Applying again Newton's formula, we get

$$\tau^\beta = \sum_{j=0}^{n_1-1} b_{1,j}^{(k)}(\tau^\beta - \lambda_1)^j(\tau^\beta - \lambda_2)^{n_2} + \sum_{j=0}^{n_2-1} b_{2,j}^{(k)}(\tau^\beta - \lambda_2)^j(\tau^\beta - \lambda_1)^{n_1},$$

with *ord* $b_{i,j}^{(k)} \leq k + m_k - m_i - j \leq k' - m + r$.

From (2.1.12), this gives

$$\begin{aligned} D_t^{\beta k}u &= \sum_{j=0}^{n_1-1} b_{1,j}^{(k)}(\tau^\beta-\lambda_1)^j(\tau^\beta-\lambda_2)^{n_2}u + \sum_{j=0}^{n_2-1} b_{2,j}^{(k)}(\tau^\beta-\lambda_2)^j(\tau^\beta-\lambda_1)^{n_1}u \\ &= \sum_{j=0}^{n_1-1} b_{1,j}^{(k)}u_{m+n_2+j} + \sum_{j=0}^{n_2-1} b_{2,j}^{(k)}u_{m-1+j} + \sum_{l=0}^{m-1} r_l^{(k)} D_t^l u \end{aligned} \tag{2.1.32}$$

$ord\ r_l^{(k)} \le k-l-1 \le k-m \le k-m+r, ord b_{i,j}^{(k)} \le k-m+r.$

The third sum, we can substitute $D_t^l u$ with the expression given by (2.1.32) itself.

Repeating this process $k'+2m_k-1 = k'-(-2m_k+1)$ times, we get

$$\begin{cases} D_t^{\beta k'}u = \sum_{j=0}^{n_1-1} \tilde{b}_{1,j}^{(k')}u_{m_k+n_2+j} + \sum_{j=0}^{n_2-1} \tilde{b}_{2,j}^{(k)}u_{m_k-1+j} + \sum_{l=0}^{m-1} \tilde{r}_l^{(k)} D_t^l u \\ ord\ \tilde{b}_l^{(k)} \le k-m_k+r,\ ord\ \tilde{r}_l^{(k)} \le -2m_k+r+1 \end{cases} \tag{2.1.33}$$

Now, we use (2.1.16) for $D_t^{\beta l}$ in the third sum of (2.1.33) in order to obtain for each of k, $0 \le k \le m_k-1$

$$\begin{aligned} D_t^{k\beta}u &= \sum_{j=0}^{n_1-1} b_{1,j}^{(k)}u_{m+n_2+j} + \sum_{j=0}^{n_2-1} b_{2,j}^{(k)}u_{m_k-1+j} + \sum_{l=0}^{m_k-1} r_l^{(k)} D_t^{\beta l}u \\ &= \sum_{j=0}^{n_1-1} b_{1,j}^{(k)}u_{m_k+n_2+j} + \sum_{j=0}^{n_2-1} b_{2,j}^{(k)}u_{m_k-1+j} + \sum_{j=0}^{n_1+n_2-1} r_j^{(k)} D_t^{\beta j}u \\ &= \sum_{j=0}^{n_1-1} b_{1,j}^{(k)}u_{m_k+n_2+j} + \sum_{j=0}^{n_2-1} b_{2,j}^{(k)}u_{m_k-1+j} + \sum_{j=0}^{n_1+n_2-1} (\sum_{l=1}^{j} r_l^{(j)}u_{j-l} + u_j) \\ &= \sum_{j=0}^{n_1-1} b_{1,j}^{(k)}u_{m_k+n_2+j} + \sum_{j=0}^{n_2-1} b_{2,j}^{(k)}u_{m_k-1+j} + \sum_{j=0}^{n_1-1} (\sum_{l=1}^{j} r_l^{(j)}u_{j-l} + u_j) \\ &\quad + \sum_{j=0}^{n_2-1} (\sum_{l=1}^{j} r_l^{(j+n_1)}u_{j+n_1-l} + u_{j+n_1}) \\ &= \sum_{j=0}^{n_1-1} b_{1,j}^{(k)}u_{m_k+n_2+j} + \sum_{j=0}^{n_2-1} b_{2,j}^{(k)}u_{m_k-1+j} + \sum_{j=0}^{n_1-1} (\sum_{l=1}^{j} r_l^{(j)}u_{j-l} + u_j) \\ &\quad + \sum_{j=0}^{n_2-1} (\sum_{l=1}^{j} r_l^{(j+n_1)}u_{j+n_1-l} + u_{j+n_1}) \\ &= \sum_{j=0}^{n_1-1} b_{1,j}^{(k)}u_{m_k+n_2+j} + \sum_{j=0}^{n_2-1} b_{2,j}^{(k)}u_{m_k-1+j} + \sum_{j=0}^{n_1-1} r_{1,j}^{(k)}u_{m_k+n_2+j} + \sum_{j=0}^{n_2-1} r_{2,j}^{(k)}u_{m_k-1+j} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{n_1-1} b_{1,j}^{(k)} u_{m_k+n_2+j} + \sum_{j=0}^{n_2-1} b_{2,j}^{(k)} u_{m_k-1+j} + \sum_{j=0}^{n_1-1} r_{1,j}^{(k)} u_{m_k+n_2+j} + \sum_{j=0}^{n_2-1} r_{2,j}^{(k)} u_{m_k-1+j} \\
&= \sum_{j=0}^{n_1-1} (b_{1,j}^{(k)} + r_{1,j}^{(k)}) u_{m_k+n_2+j} + \sum_{j=0}^{n_2-1} (b_{2,j}^{(k)} + r_{2,j}^{(k)}) u_{m_k-1+j} \\
&= \sum_{j=0}^{n_1-1} q_{1,j}^{(k)} u_{m_k+n_2+j} + \sum_{j=0}^{n_2-1} q_{2,j}^{(k)} u_{m_k-1+j} \\
&= \sum_{j=0}^{2m_k-1} q_j^{(k)} u_j, \; ord\; q_j^{(k)} \leq \beta\left(k - m_k\right)
\end{aligned}
$$

$$
D_t^{\beta k} u = \sum_{j=0}^{2m_k-1} q_j^{(k)} u_j, \; ord\; q_j^{(k')} \leq \beta\left(k - m_k\right) \tag{2.1.34}
$$

We put $Q = (q_0, q_1, ..., q_{2m_k-1})$, so

$$
QU = (q_0, q_1, ..., q_{2m_k-1})(u_0, ..., u_{m_k-1}, u_{m_k}, ...u_{2m_k-1})^t = \sum_{j=0}^{2m_k-1} q_j u_j.
$$

It is sufficient to note that we have executed a finite number of products, depending only on m_k, n, and applying proposition (1.3.4) to fulfil the proof $D^{\beta m_k} u = QU$. ■

Proposition 2.1.7 *For every $0 \leq k \leq m - r$ there exists a matrix of functions F and a matrix $\tilde{Q}$ such that,*

$$
\begin{cases} \left[\partial^\beta, P\right] u = F\tilde{Q}\tilde{U}, 0 < \beta \leq 1 \\ F \in C^\beta\left[0, T\right]; \tilde{Q} \in C^\beta\left[0, T\right] \end{cases} \tag{2.1.35}
$$

Proof. The demonstration is achieved in steps :

First step: we define

$$
U^{(\delta)} = \left(u_j^{(\delta)}; 0 \leq j \leq 2m - 1\right)^t,
$$

then (2.1.21) becomes,

$$
\tilde{U} = \left(V^{(\delta)}; \delta \leq \beta\left(m - r\right)\right).
$$

By replacing in (2.1.22) u by $\partial^\delta u$ we obtain,

$$
D^{\beta(m-r)}\left(\partial^\beta u\right) = QU^{(\beta)}, \tag{2.1.36}
$$

from which the desired shape

$$\left[\partial^{\beta}, P\right] u = \sum_{\substack{|\alpha|\leq m \\ 0<\gamma\leq\beta}} f_{\beta,\gamma}\left(t; D^{\beta(m-r)}u\right) \partial^{\delta} D^{\beta(m-r)}\ u. \tag{2.1.37}$$

Second step :

■

Lemma 2.1.8 *Let* $\lambda_1(t) \neq \lambda_2(t)$,

have

(i) $\left[a, \left(D_t^{\beta} - \lambda_j\right)^{l}\right] = \sum_{1\leq k\leq l} a_{kj}\left(D_t^{\beta} - \lambda_j\right)^{l-k}$, $1 \leq l \leq n_j, ord\ a_{kj} = k\beta$

for any operator a *of order* h *;*

(ii) $\left[\left(D_t^{\beta} - \lambda_1\right)^{l}, \left(D_t^{\beta} - \lambda_2\right)^{d}\right] = \sum_{1\leq i\leq l, 1\leq j\leq d} a_{kj}\left(D_t^{\beta} - \lambda_1\right)^{l-i}\left(D_t^{\beta} - \lambda_2\right)^{d-j}$,

$1 \leq l \leq n_{1,} 1 \leq l \leq n_{2,} ord\ a_{kj} = \beta\left(i + j - 1\right)$

(iii) $\left[\left(D_t^{\beta} - \lambda_1\right)^{n_1-l}, \left(D_t^{\beta} - \lambda_2\right)^{n_2-d}\right] = \sum_{1\leq i\leq l} a_i\left(D_t^{\beta} - \lambda_1\right)^{n_1-i}\left(D_t^{\beta} - \lambda_2\right)^{n_2}$

$+ \sum_{1\leq i\leq d} b_i\left(D_t^{\beta} - \lambda_2\right)^{n_2-i}\left(D_t^{\beta} - \lambda_1\right)^{n_1}$

$+ \sum_{\substack{1\leq i\leq n_1-1 \\ 1\leq j\leq n_2-1}} c_{ij}\left(D_t^{\beta} - \lambda_1\right)^{i}\left(D_t^{\beta} - \lambda_2\right)^{j}$

$1 \leq l \leq n_{1,} 1 \leq l \leq n_2, l + d > 1,\ ord\left(a_i, b_i\right) = \beta\left(-l - d + i\right), ordc_{ij} = \beta\left(m - l - d - i - j - 1\right)$

Proof. (2.1.8)([6]) Let P, Q, R three operators. We have the identity:

$$\begin{aligned}
[P, QR] &= PQR - QRP \\
&= PQR - QPR + QPR - QRP \\
&= \left(PQ - QP\right)R + Q\left(PR - RP\right) \\
&= [P, Q]R + Q[P, R]
\end{aligned}$$

so,

$$[P, QR] = [P, Q]R + Q[P, R], \tag{2.1.38}$$

provided that all compositions are well defined.

By using (2.1.38), can prove (i) by induction on l, we put $\partial_j^\beta = D_t^\beta - \lambda_j$, and

$$\begin{aligned}
\left[a,\left(D_t^\beta-\lambda_j\right)^l\right] &= \left[a,\left(D_t^\beta-\lambda_j\right)^{l-1}\left(D_t^\beta-\lambda_j\right)^1\right]\\
&= \left[a,\left(D_t^\beta-\lambda_j\right)^{l-1}\right]\left(D_t^\beta-\lambda_j\right)^1+\left(D_t^\beta-\lambda_j\right)^{l-1}\left[a,\left(D_t^\beta-\lambda_j\right)^1\right]\\
&= \left[a,\left(D_t^\beta-\lambda_j\right)^{l-2}\left(D_t^\beta-\lambda_j\right)^1\right]\left(D_t^\beta-\lambda_j\right)^1\\
&\quad+\left(D_t^\beta-\lambda_j\right)^{l-1}\left[a,\left(D_t^\beta-\lambda_j\right)^1\right]\\
&= \left[a,\left(D_t^\beta-\lambda_j\right)^{l-2}\right]\left(D_t^\beta-\lambda_j\right)^2+\left(D_t^\beta-\lambda_j\right)^{l-2}\left[a,\left(D_t^\beta-\lambda_j\right)^1\right]\\
&\quad+\left(D_t^\beta-\lambda_j\right)^{l-1}\left[a,\left(D_t^\beta-\lambda_j\right)^1\right]\\
&= \left[a,\left(D_t^\beta-\lambda_j\right)^{1}\right]\left(D_t^\beta-\lambda_j\right)^{l-1}+\left(D_t^\beta-\lambda_j\right)^{l-2}\left[a,\left(D_t^\beta-\lambda_j\right)^1\right]\\
&\quad+\left(D_t^\beta-\lambda_j\right)^{l-1}\left[a,\left(D_t^\beta-\lambda_j\right)^1\right]\\
&\quad.\\
&\quad.\\
&= \sum_{1\le k\le l}\left(D_t^\beta-\lambda_j\right)^{l-k}\left[a,\left(D_t^\beta-\lambda_j\right)^1\right]\\
&= \sum_{1\le k\le l}\left(D_t^\beta-\lambda_j\right)^{l-k}\left(a\left(D_t^\beta-\lambda_j\right)^1-\left(D_t^\beta-\lambda_j\right)^1a\right)\\
&= \sum_{1\le k\le l}a_{kj}\left(D_t^\beta-\lambda_j\right)^{l-k}
\end{aligned}$$

(ii) $\left[\left(D_t^\beta-\lambda_1\right)^l,\left(D_t^\beta-\lambda_2\right)^d\right]=\sum_{1\le i\le l}a_i\left(D_t^\beta-\lambda_1\right)^{n_1-i}\left(D_t^\beta-\lambda_2\right)^{n_2}$,
by (i) and(2.1.38). We prove the equality (ii) with $l=1$,

$$\begin{aligned}
\left[\left(D_t^\beta-\lambda_1\right),\left(D_t^\beta-\lambda_2\right)^d\right] &= \left[\left(D_t^\beta-\lambda_1\right),\left(D_t^\beta-\lambda_2\right)\left(D_t^\beta-\lambda_2\right)^{d-1}\right]\\
&= \left(D_t^\beta-\lambda_2\right)\left[\left(D_t^\beta-\lambda_1\right),\left(D_t^\beta-\lambda_2\right)^{d-1}\right]\\
&\quad+\left[\left(D_t^\beta-\lambda_1\right),\left(D_t^\beta-\lambda_2\right)\right]\left(D_t^\beta-\lambda_2\right)^{d-1}
\end{aligned}$$

$$\begin{aligned}
\left[\left(D_t^\beta-\lambda_1\right),\left(D_t^\beta-\lambda_2\right)^d\right] &= \left[\left(D_t^\beta-\lambda_1\right),\left(D_t^\beta-\lambda_2\right)\left(D_t^\beta-\lambda_2\right)^{d-1}\right] \\
&= \left(D_t^\beta-\lambda_2\right)\left[\left(D_t^\beta-\lambda_1\right),\left(D_t^\beta-\lambda_2\right)^{d-1}\right] \\
&\quad +\left[\left(D_t^\beta-\lambda_1\right),\left(D_t^\beta-\lambda_2\right)\right]\left(D_t^\beta-\lambda_2\right)^{d-1} \\
\left[\left(D_t^\beta-\lambda_1\right),\left(D_t^\beta-\lambda_2\right)^{d-1}\right] &= \left[\left(D_t^\beta-\lambda_1\right),\left(D_t^\beta-\lambda_2\right)\left(D_t^\beta-\lambda_2\right)^{d-2}\right] \\
&= \left(D_t^\beta-\lambda_2\right)\left[\left(D_t^\beta-\lambda_1\right),\left(D_t^\beta-\lambda_2\right)^{d-2}\right] \\
&\quad +\left[\left(D_t^\beta-\lambda_1\right),\left(D_t^\beta-\lambda_2\right)\right]\left(D_t^\beta-\lambda_2\right)^{d-2}
\end{aligned}$$

$$\begin{aligned}
&\left[\left(D_t^\beta-\lambda_1\right),\left(D_t^\beta-\lambda_2\right)^d\right] \\
&=\left(D_t^\beta-\lambda_2\right)\begin{pmatrix} \left(D_t^\beta-\lambda_2\right)\left[\left(D_t^\beta-\lambda_1\right),\left(D_t^\beta-\lambda_2\right)^{d-2}\right] \\ +\left[\left(D_t^\beta-\lambda_1\right),\left(D_t^\beta-\lambda_2\right)\right]\left(D_t^\beta-\lambda_2\right)^{d-2}\end{pmatrix} \\
&\quad +\left[\left(D_t^\beta-\lambda_1\right),\left(D_t^\beta-\lambda_2\right)\right]\left(D_t^\beta-\lambda_2\right)^{d-1} \\
&=\left(D_t^\beta-\lambda_2\right)^2\left[\left(D_t^\beta-\lambda_1\right),\left(D_t^\beta-\lambda_2\right)^{d-2}\right] \\
&\quad +\left(D_t^\beta-\lambda_2\right)^1\left[\left(D_t^\beta-\lambda_1\right),\left(D_t^\beta-\lambda_2\right)\right]\left(D_t^\beta-\lambda_2\right)^{d-2} \\
&+\left[\left(D_t^\beta-\lambda_1\right),\left(D_t^\beta-\lambda_2\right)\right]\left(D_t^\beta-\lambda_2\right)^{d-1} \\
&=\sum_{1\le j\le d}\left(D_t^\beta-\lambda_2\right)^j\left[\left(D_t^\beta-\lambda_1\right),\left(D_t^\beta-\lambda_2\right)\right]\left(D_t^\beta-\lambda_2\right)^{d-j} \\
&=\sum_{1\le j\le d}\left(D_t^\beta-\lambda_2\right)^j\left(\left(D_t^\beta-\lambda_1\right)\left(D_t^\beta-\lambda_2\right)-\left(D_t^\beta-\lambda_2\right)\left(D_t^\beta-\lambda_1\right)\right)\left(D_t^\beta-\lambda_2\right)^{d-j} \\
&=\sum_{1\le j\le d}\left(D_t^\beta-\lambda_2\right)^j\left(D_t^\beta-\lambda_1\right)\left(D_t^\beta-\lambda_2\right)^{d-j+1}-\left(D_t^\beta-\lambda_2\right)^{j+1}\left(D_t^\beta-\lambda_1\right)\left(D_t^\beta-\lambda_2\right)^{d-j} \\
&=\sum_{1\le j\le d}a_{kj}\left(D_t^\beta-\lambda_2\right)^{d-j}.
\end{aligned}$$

By dint of (i), (ii) and(2.1.38), we prove (iii).

We have the representation of the identity operator

$$1=q\left(D_t^\beta-\lambda_1\right)-q\left(D_t^\beta-\lambda_2\right)+r,$$

with $q(t)=\frac{1}{\lambda_2(t)-\lambda_1(t)}$ and $r(t)$ d'ordre -1,

we prove (iii) by induction on $l+d$ the use of (i), (ii),(2.1.38) and identity in

$$\left(D_t^\beta-\lambda_1\right)^{n_1-l}\left(D_t^\beta-\lambda_2\right)^{n_2-d}=\left(D_t^\beta-\lambda_1\right)^{n_1-l}\left[q\left(D_t^\beta-\lambda_1\right)-q\left(D_t^\beta-\lambda_2\right)+r\right]\left(D_t^\beta-\lambda_2\right)^{n_2-d}.$$

Since one must only perform compositions, depending on l,d, and n.

End of the proof of the lemma(2.1.8).

Third step :

Let δ such that $\beta(M-r+1)<\delta\leq\beta M$ $(r\geq 1), M\leq m$ after (1.3.2) we have $P=P_2P_1+R$.

Only the larger order terms that $\beta(M+m-r)$ can appear in $[\partial^\delta,P]$,

$$[\partial^\delta,P_2P_1]=[\partial^\delta,P_2]P_1+P_2[\partial^\delta,P_1].$$

Let's use again(2.1.38)and (1.3.2), We have

$$\begin{cases}[\partial^\delta,P_2]P_1=[\partial^\delta,\left(D_t^\beta-\lambda_2\right)^{n_2}]\left(D_t^\beta-\lambda_1\right)^{n_1}\\ \qquad+\sum\limits_{i,j}[\partial^\delta,a_j^{(2)}]\left(D_t^\beta-\lambda_2\right)^{n_2-j}a_i^{(1)}\left(D_t^\beta-\lambda_1\right)^{n_1-i}\\ \qquad+\sum\limits_{i,j}a_j^{(2)}[\partial^\delta,\left(D_t^\beta-\lambda_2\right)^{n_2-j}]a_i^{(1)}\left(D_t^\beta-\lambda_1\right)^{n_1-i}\\ 1\leq i\leq n_{1,}1\leq j\leq n_2,i+j>1,\end{cases}\tag{2.1.39}$$

the factors of the second member in equality(2.1.39) are the compositions of the operators $a,[\partial^\delta,q],\left(D_t^\beta-\lambda_1\right)^{n_1-i},\left(D_t^\beta-\lambda_2\right)^{n_2-j}$ with $ord(a)=0$, $ord(q)\leq 1$.

Repeated use of(2.1.38) given

$$[\partial^\delta,\left(D_t^\beta-\lambda_j\right)^{n_j-d}]=\sum_{1\leq h\leq n_j-d}\left(D_t^\beta-\lambda_j\right)^{n_j-d-h}[\partial^\delta,\left(D_t^\beta-\lambda_j\right)]\left(D_t^\beta-\lambda_j\right)^{h-1}\tag{2.1.40}$$

Because

$$
\begin{aligned}
[\partial^{\delta}, \left(D_t^{\beta}-\lambda_j\right)^{n_j-d}] &= \\
&= [\partial^{\delta}, \left(D_t^{\beta}-\lambda_j\right)^{n_j-d-1}\left(D_t^{\beta}-\lambda_j\right)^{1}] \\
&= \left(D_t^{\beta}-\lambda_j\right)^{n_j-d-1}[\partial^{\delta}, \left(D_t^{\beta}-\lambda_j\right)^{1}] + [\partial^{\delta}, \left(D_t^{\beta}-\lambda_j\right)^{n_j-d-1}]\left(D_t^{\beta}-\lambda_j\right)^{1} \\
& . \\
& . \\
&= \sum_{1\le h\le n_j-d}\left(D_t^{\beta}-\lambda_j\right)^{n_j-d-h}[\partial^{\delta}, \left(D_t^{\beta}-\lambda_j\right)]\left(D_t^{\beta}-\lambda_j\right)^{h-1}
\end{aligned}
$$

where, we have

$$[\partial^{\delta}, q] = \sum_{\substack{0\le\gamma\le\delta \\ |\gamma|<r}}\binom{\delta}{\gamma}q_{(\gamma)}\partial^{\delta-\gamma} = \sum_{\substack{0\le\gamma\le\delta \\ |\gamma|<r}}a_{\gamma}\partial^{\delta-\gamma}, \quad ord\ a_{\gamma}\le 0,$$

the rest contains all the order terms $\le \beta\left(|\delta|+m-r\right)$ in (2.1.39).

On the other hand, (2.1.40) and (i) of the lemma (2.1.8), and all order compositions $>$ $\beta\left(|\delta|+m-r\right)$ in (2.1.39), entrain

$$
\begin{cases}
\sum\limits_{i,j,\gamma}a_{i,j,\gamma}\left(D_t^{\beta}-\lambda_2\right)^{n_2-j}\left(D_t^{\beta}-\lambda_1\right)^{n_1-i}\partial^{\delta-\gamma}, \\
1\le i\le n_1, 1\le j\le n_2, i+j<r \\
0\le\gamma<\delta,\ \ |\gamma|\le r-i-j \\
\ ord\ a_{i,j,\gamma}\le 0
\end{cases}
\tag{2.1.41}
$$

for $i+j\ge 2$,

$$
\begin{cases}
[\partial^{\delta}, P_2]P_1 = \sum\limits_{l,\gamma_1}b_{l,\gamma_1}\left(D_t^{\beta}-\lambda_1\right)^{n_1-l}\left(D_t^{\beta}-\lambda_2\right)^{n_2}\partial^{\delta-\gamma}+ \\
\sum\limits_{d,\gamma_2}b_{d,\gamma_2}\left(D_t^{\beta}-\lambda_2\right)^{n_2-d}\left(D_t^{\beta}-\lambda_1\right)^{n_1}\partial^{\delta-\gamma}+R \\
1\le l,d\le r-1, 0\le\gamma_1,\gamma_2<\delta,\ \ |\gamma_1|\le r-l, |\gamma_2|\le r-d \\
\ ord\ \left(b_{l,\gamma_1}, b_{d,\gamma_2}\right)\le 0, ord\ \left(R\right)\le|\delta|+n-r.
\end{cases}
\tag{2.1.42}
$$

R can be replaced by $\sum_{|\alpha|\leq n-r, |\delta|\leq|\delta|} c_{\alpha,\gamma}\partial^{\alpha+\delta}$,$c_{\alpha,\gamma}$ ordre 0.

In the same way, we get $P_2[\partial^\delta, P_1]$.

Apply the(2.1.42) operator to function u, we get

$$\sum_{\substack{|\alpha|\leq n \\ 0<\gamma\leq\delta}} f^{(\delta)}_{\alpha,\gamma} q^{(\delta)}_{\gamma,j} u^{(\gamma)}_{\gamma j}, ord q^{(\delta)}_{\gamma,j} \leq 0,$$

than,

$$\tilde{Q} = \left(q_0^{(\delta)}, q_1^{(\delta)}, ..., q_{2m-1}^{(\delta)}\right)^t, \tilde{U} = \left(u_0^{(\gamma)}, u_1^{(\gamma)}, ..., u_{2m-1}^{(\gamma)}\right)^t,$$

and,

$$F = (f_0, f_1, ..., f_{2m-1}),$$

so,

$$\begin{aligned} F\tilde{Q}\tilde{U} &= \left(f_0^{(\delta)}, f_1^{(\delta)}, ..., f_{2m-1}^{(\delta)}\right)\left(q_0^{(\delta)}, q_1^{(\delta)}, ..., q_{2m-1}^{(\delta)}\right)^t\left(u_0^{(\gamma)}, u_1^{(\gamma)}, ..., u_{2m-1}^{(\gamma)}\right)^t \\ &= \sum_{\substack{|\alpha|\leq m \\ 0<\gamma\leq\delta}} f^{(\delta)}_{\alpha,\gamma} q^{(\delta)}_{\gamma,j} u^{(\gamma)}_{\gamma j} \end{aligned}$$

$$\begin{aligned} \left[\partial^\delta, P\right] u &= \sum_{\substack{|\alpha|\leq m \\ 0<\gamma\leq\delta}} f^{(\delta)}_{\alpha,\gamma} q^{(\delta)}_{\gamma,j} u^{(\gamma)}_{\gamma j} \\ &= F\tilde{Q}\tilde{U}\ . \end{aligned}$$

Let (2.1.16) (2.1.35), allows to write

$$\begin{aligned} F\left(t, D^{\beta(M+m-r)}u\right) &= \tilde{F}_0 + \tilde{F}\left(t, D^{\beta(M+m-r)}u\right). \\ F\left(t, D^{\beta(M+m-r)}\right)\tilde{Q}\tilde{U} &= \tilde{F}_0\tilde{Q}\tilde{U} + \tilde{F}\left(t, D^{\beta(M+m-r)}u\right)\tilde{Q}\tilde{U}, \end{aligned}$$

with

$$\tilde{F}_0 = (0, ..., 0, f, 0, ..., 0, f),$$

the systems (2.1.13) and (2.1.14) prove that the equation (2.1.20) is equivalent to the system

$$\partial_t^\beta \tilde{U} - \tilde{K}\ \tilde{U} - \tilde{F}\ \tilde{Q}\ \tilde{U} = \tilde{F}_0$$

where

$$\partial_t^\beta \tilde{U} - \left(\tilde{K}\ + \tilde{F}\ \tilde{Q}\ \right) \tilde{U} = \tilde{F}_0,$$

$$\partial_t^\beta \tilde{U} - \bar{K}\ \tilde{U} = \tilde{F}_0, \tag{2.1.43}$$

with,

$$\begin{aligned}
\bar{K} &= \tilde{K}\ + \tilde{F}\ \tilde{Q} \\
\tilde{K} &= \tilde{K}\left(t, D^{\beta(M+m-r)}\right), \tilde{Q} = \tilde{Q}\left(t, D^{\beta(M+m-r)}u\right), \\
\tilde{F} &= \tilde{F}\left(t, D^{\beta(M+m-r)}u\right), \tilde{F}_0 = \tilde{F}_0\left(t\right).
\end{aligned}$$

The operator $\partial_t^\beta - \bar{K}$ is symmetrical and $\bar{K}$ a real diagonal matrix.

So far, taking the proposition (2.1.6) into consideration, we have proved the following result of posedness for equivalent problems(2.1.4) and (2.1.18) : equivalent problem: $\partial_t^\beta U - \bar{K}(t)U = \tilde{F}_0$ is a simple form

$$\partial_j^\beta u_i(t) = \left(D_t^\beta - \lambda_j(t)\right) u_i(t) = f_i(t), j = 1, 2; i = 0, ..2m - 1. \tag{2.1.44}$$

Which means that if equations (2.1.45) is an accepted solution, the solution of equation (2.1.45) is $u = u_0$. ■

Remark 2.1.9 *In case $\beta = 1$, in the same way as before, we get an equivalent system,*

$$\partial_t U - \bar{K}(t)U = \tilde{F}, \tag{2.1.45}$$

and

$$\left(D_t - \lambda_j(t)\right) u_i(t) = f_i(t), j = 1, 2; i = 0, ..2m - 1, \tag{2.1.46}$$

and (2.1.45) *accepts the solution in accordance with the terms of the first-order system equations solution.*

Chapitre 3

Analytic solution

The existence of the solution does not mean that it can be set analytically. In this chapter, we will use some techniques to find analytical solutions?

We're going to take a look at fractional differential equations of the form :

$$P\left(t, D_t^{m'} u\right) u(t) = \sum_{k=0}^{k=n} g_k(t) D^{k\alpha_k} u(t) = f(t), \tag{3.0.1}$$

with

$$P = P_2 \circ P_2 \circ ... P_m, \tag{3.0.2}$$

so

$$\left(P_1 \circ P_2 \circ ... P_m\right) u(t) = \sum_{k=0}^{k=n} g_k(t) D^{k\alpha_k} u(t) = f(t), g_n(t) = 1,$$

and,

$$P_i = (D_t^{\beta} - \lambda_i)^{n_i} = (D_t^{\beta} - \lambda_i)(D_t^{\beta} - \lambda_i)..(D_t^{\beta} - \lambda_i), \sum_{i=1}^{i=m} n_i = n. \tag{3.0.3}$$

So, in this chapter we're also going to have an example dealing with fractional differential equations with Laplace transforms as well as a discussion of some larger systems of differential equations.

Here is a brief listing of the topics in this chapter.

In this section, we'll start the chapter off with a quick look at some of the basic ideas behind solving fractional differential equations. Included will be updated definitions/facts for the Principle of Superposition,

Homogeneous Differential Equations – In this section, we will extend the ideas behind solving α order, homogeneous differential equations to higher order.

We are concerned with solution $u(t)$ of equation(2.1.44), which depends on the solution of equations,

$$D_t^{\alpha} u_i(t) - \lambda_i(t) u_i(t) = g_i(t). \tag{3.0.4}$$

Which can be write into the system,

$$\begin{aligned} D_t^{\alpha} u_{i,H}(t) - \lambda_i(t) u_{i,H}(t) &= 0, & (3.0.5) \\ D_t^{\alpha} u_{i,p}(t) - \lambda_i(t) u_{i,p}(t) &= g_i(t). & (3.0.6) \end{aligned}$$

Assuming that Equations (3.0.4) accept the solution $u_i(t) = u_{i,p}(t) + u_{i,H}(t)$, where (3.0.5) is called homogeneous equations, and (3.0.6) non-homogeneous equation.

Where $u_{i,H}(t)$ is called homogeneous solution of equation (3.0.4), and $u_{i,p}(t)$ the particular equation(3.0.4), whereas $u_0(t) = u(t)$.

For $\alpha = 1$, the differential equation of the first order is famous.

For $\alpha \neq 1$, the fractional differential equation :

First, $\lambda(t) = \lambda$ is constant and $D_t^{\alpha} u(t) - \lambda u(t) = g(t)$.

We look for the general solution of homogeneous equation $D_t^{\alpha} u(t) - \lambda(t) u(t) = 0$.

I will be interested in solving the differential equation (3.0.4) by solving the homogeneous equation (3.0.4) or solving the differential equation(3.0.6), for which the particular solution is known,

Considering the derivative of function $f(t) > 0$ with order $0 < \alpha \leq 1$,we suggest defining the fractal logarithm

$Ln_{\alpha}(f(t))$, which are subject to properties $D_t^{\alpha}(\ Ln_{\alpha}(f(t))) = \frac{D_t^{\alpha} f(t)}{f(t)}$ and $Ln_{\alpha}(e^{a}) = \frac{(a)^{\alpha}}{\Gamma(\alpha+1)}$.

The following properties can be inferred

1- if $a = 0, Ln_\alpha(e^a) = Ln_\alpha(1) = 0.$

2- $Ln_\alpha(A.B) = Ln_\alpha(A) + Ln_\alpha(B) ; Ln_\alpha(A/B) = Ln_\alpha(A) - Ln_\alpha(B)$

3- for $A = 1, Ln_\alpha(1/B) = -Ln_\alpha(B)$

For $\alpha \neq 1, \lambda(t) = \lambda$, is the fractional differential equation $D_t^\alpha u(t) - \lambda u(t) = g(t)$.

3.0.2 The general solution of the nonhomogeneous fractional differentials equations

Let the fractional differential equation

$$D_t^\alpha u(t) - \lambda(t) u(t) = g(t) \tag{3.0.7}$$

Now, we propose some techniques for solving the (3.0.7) fractional differential equation

I) $\lambda(t)$ is contunus function :

A) If we can write $\lambda(t)$ in the form

$$-\lambda(t) = \frac{D_t^\alpha \theta(t)}{\theta(t)} \tag{3.0.8}$$

with $\theta(t) \neq 0$.

The general solution of the homogeneous fractional differential equation is

$$\begin{aligned} D_t^\alpha u_h(t) + \frac{D_t^\alpha \theta(t)}{\theta(t)} u_h(t) &= 0, \\ \frac{\theta(t) D_t^\alpha u_h(t) + D_t^\alpha \theta(t)}{\theta(t)} &= 0 \\ D_t^\alpha [\theta(t) u_h(t)] &= 0 \end{aligned}$$

$$u_h(t) = \frac{c}{\theta(t)} \tag{3.0.9}$$

And the particular solution, remplacing (3.0.8) in (3.0.7) ,we get

$$\theta(t) D_t^\alpha u_p(t) + D_t^\alpha \theta(t) u_p(t) = \theta(t) g(t)$$

so,

$$D_t^{\alpha}\left[\theta\left(t\right)u_p(t)\right]=\theta\left(t\right)g(t)$$

we have

$$u_p(t)=\frac{I_t^{\alpha}\left(\theta\left(t\right)g(t)\right)}{\theta\left(t\right)}$$

Or by using the value of a constant in (3.0.9),

$$u_p(t)=\frac{c\left(t\right)}{\theta\left(t\right)}$$

we have

$$D_t^{\alpha}u_p(t)-\lambda\left(t\right)u_p(t)=u_H(t)D_t^{\alpha}c\left(t\right)+c\left(t\right)D_t^{\alpha}u_H\left(t\right)-\lambda\left(t\right)c\left(t\right)u_H(t)=g(t)$$

so,

$$u_H(t)D_t^{\alpha}c\left(t\right)=g(t)$$

is giving

$$c\left(t\right)=I_t^{\alpha}\left(\theta\left(t\right)g(t)\right)$$

output

$$u_p(t)=\frac{I_t^{\alpha}\left(\theta\left(t\right)g(t)\right)}{\theta\left(t\right)}.$$

Outcome

$$u(t)=\frac{c+I_t^{\alpha}\left(\theta\left(t\right)g(t)\right)}{\theta\left(t\right)}.$$

B) If we can write $\lambda\left(t\right)$ in the form

$$\lambda\left(t\right)=\frac{D_t^{\alpha}\theta\left(t\right)}{\theta\left(t\right)}\tag{3.0.10}$$

with $\theta\left(t\right)\neq0$.

The general solution of the homogeneous fractional differential equation is

$$\begin{aligned} D_t^{\alpha} u_h(t) - \frac{D_t^{\alpha}\theta\,(t)}{\theta\,(t)} u_h(t) &= 0 \\ \frac{\theta\,(t)\, D_t^{\alpha} u_h(t) - D_t^{\alpha}\theta\,(t)}{\theta^2\,(t)} u_h(t) &= 0 \\ D_t^{\alpha}\left[\frac{u_h(t}{\theta\,(t)})\right] &= 0 \end{aligned}$$

$$u_h(t) = c\theta\,(t) \tag{3.0.11}$$

And the particular solution, remplaceing(3.0.8) in(3.0.7) ,we get

$$\begin{aligned} \theta\,(t)\, D_t^{\alpha} u_p(t) - D_t^{\alpha}\theta\,(t)\, u_p(t) &= \theta\,(t)\, g(t) \\ \frac{\theta\,(t)\, D_t^{\alpha} u_p(t) - D_t^{\alpha}\theta\,(t)\, u_p(t)}{\theta^2\,(t)} &= \frac{g(t)}{\theta\,(t)}, \end{aligned}$$

so,

$$D_t^{\alpha}\left[\frac{u_p(t)}{\theta\,(t)}\right] = \frac{g(t)}{\theta\,(t)},$$

we have

$$u_p(t) = \theta\,(t)\, I_t^{\alpha}\left(\frac{g(t)}{\theta\,(t)}\right).$$

Or by using the value of a constant in(3.0.9) ,

$$u_p(t) = c\,(t)\,\theta\,(t)\,,$$

we have

$$D_t^{\alpha} u_p(t) - \lambda\,(t)\, u_p(t) = u_H(t) D_t^{\alpha} c\,(t) + c\,(t)\, D_t^{\alpha} u_H\,(t) - \lambda\,(t)\, c\,(t)\, u_H(t) = g(t),$$

so,

$$u_H(t) D_t^{\alpha} c\,(t) = g(t)$$

is giving

$$c\,(t) = I_t^{\alpha}\left(\frac{g(t)}{\theta\,(t)}\right),$$

output

$$u_p(t) = \theta(t)\, I_t^{\alpha}\left(\theta(t)\, g(t)\right).$$

Outcome

$$u(t) = \theta(t)\left(c + I_t^{\alpha}\left(\theta(t)\, g(t)\right)\right).$$

C) **Using the fractional derivative of chain rule** :

Let $y > 0$ and $f(y) = \ln(y)$, $f'(y) = \frac{1}{y}$, by usig the chain rule [7]

$$D_t^{\alpha} f\left(g(t)\right) = f_g'\left(g(t)\right) D_t^{\alpha} g(t),$$

we have

$$D_t^{\alpha} \ln\left(u(t)\right) = \frac{D_t^{\alpha} u(t)}{u(t)}.$$

The general solution of the homogeneous fractional differential equation

$$D_t^{\alpha} u(t) - \lambda(t)\, u(t) = 0,$$

can be written as

$$\frac{D_t^{\alpha} u(t)}{u(t)} = \lambda(t)$$

So,

$$\begin{aligned} Ln_{\alpha}\left(u(t)\right) &= I_t^{\alpha}\lambda(t) \\ u_H(t) &= c e^{I_t^{\alpha}\lambda(t)} \end{aligned} \tag{3.0.12}$$

And the particular solution, by using the value of a constant in (3.0.12),

$$u_p(t) = c(t)\, e^{I_t^{\alpha}\lambda(t)},$$

we have

$$D_t^\alpha u_p(t) - \lambda(t)\, u_p(t) = u_H(t) D_t^\alpha c(t) + c(t)\, D_t^\alpha u_H(t) - \lambda(t)\, c(t)\, u_H(t) = g(t),$$

so,

$$u_H(t) D_t^\alpha c(t) = g(t)$$

is giving

$$c(t) = I_t^\alpha \left(g(t) e^{-I_t^\alpha \lambda(t)} \right)$$

output

$$u_p(t) = e^{I_t^\alpha \lambda(t)} I_t^\alpha \left(g(t) e^{-I_t^\alpha \lambda(t)} \right)$$

outcome

$$u(t) = e^{I_t^\alpha \lambda(t)} \left(c + I_t^\alpha \left(g(t) e^{-I_t^\alpha \lambda(t)} \right) \right)$$

II) $\lambda(t) = \lambda$ is constant :

A) **Using the Laplace transform**

Proposition 3.0.10 *Let the fractional differential equation* $D_t^\alpha u(t) - \lambda u(t) = g(t)$ *Using the Laplace transform,* $F(s)$ *is obtained* $u(t) = L^{-1}(\frac{G(s)}{s^\alpha - \lambda})$*with* $G(s) = L(g(t))$ *and* $F(s) = L(u(t))$.

Proof. we have

$$\begin{aligned} L(D_t^\alpha u(t) + \lambda u(t)) &= L(D_t^\alpha u(t)) + \lambda L(u(t)) = L(g(t)) \\ s^\alpha F(s) + \lambda F(s) &= G(s) \\ F(s) &= \frac{G(s)}{s^\alpha + \lambda} \end{aligned}$$

So $u(t) = L^{-1}(\frac{G(s)}{s^\alpha + \lambda})$ ■

Example 3.0.11 $g(t) = 1 + t$, $G(s) = L(1 + t) = \frac{1}{s} + \frac{1}{s^2} = \frac{s-1}{s^2}$

$$\begin{aligned} u(t) &= L^{-1}(\frac{s-1}{s^2(s^\alpha + \lambda)}) = L^{-1}(\frac{1}{s(s^\alpha + \lambda)}) - L^{-1}(\frac{1}{(s^\alpha + \lambda)}) \\ &= \frac{1}{\lambda} L^{-1}(\frac{\lambda}{s(s^\alpha + \lambda)}) - L^{-1}(\frac{1}{(s^\alpha + \lambda)}) = \frac{1}{\lambda}(1 - E_\alpha(-at^\alpha)) - t^{\alpha-1} E_{\alpha,\alpha}(-at^\alpha) \end{aligned}$$

B) **Using the Mittag-leffler function**

Proposition 3.0.12 *the fractional differential equation* $D_t^{\alpha}u(t)-\lambda u(t)=0$ *has solution in the form* $u(t)=E_{\alpha}(\lambda t^{\alpha})$, *where* E_{α} *is Mittag-leffler function.*

Proof. let $u(t)=E_{\alpha}(\lambda t^{\alpha})$, we have

$$D_t^{\alpha}u(t)=D_t^{\alpha}E_{\alpha}(\lambda t^{\alpha})=\lambda E_{\alpha}(\lambda t^{\alpha})=\lambda u(t).$$

■

And the particular solution, by useing the value of a constant in (3.0.12) ,

$$u_p(t)=c\,(t)\,E_{\alpha}(\lambda t^{\alpha})$$

we have

$$D_t^{\alpha}u_p(t)-\lambda\,(t)\,u_p(t)=u_H(t)D_t^{\alpha}c\,(t)+c\,(t)\,D_t^{\alpha}u_H\,(t)-\lambda\,(t)\,c\,(t)\,u_H(t)=g(t)$$

so,

$$E_{\alpha}(\lambda t^{\alpha})D_t^{\alpha}c\,(t)=g(t)$$

is giving

$$c\,(t)=I_t^{\alpha}\left(\frac{g(t)}{E_{\alpha}(\lambda t^{\alpha})}\right)$$

output

$$u_p(t)=E_{\alpha}(\lambda t^{\alpha})I_t^{\alpha}\left(\frac{g(t)}{E_{\alpha}(\lambda t^{\alpha})}\right)$$

outcome

$$u(t)=E_{\alpha}(\lambda t^{\alpha})\left(c+I_t^{\alpha}\left(\frac{g(t)}{E_{\alpha}(\lambda t^{\alpha})}\right)\right)$$

C) **Using the exponential form**

Proposition 3.0.13 *the fractional differential equation* $D_t^{\alpha}u(t)-\lambda u(t)=0$, *has it solution in the form* $u(t)=ce^{\lambda^{\frac{1}{\alpha}}t}$.

Proof. Let $u(t) = ce^{\lambda^{\frac{1}{\alpha}} t}$, we have

$$\begin{aligned} D_t^\alpha u(t) &= D_t^\alpha c e^{\lambda^{\frac{1}{\alpha}} t} = c D_t^\alpha e^{\lambda^{\frac{1}{\alpha}} t} + e^{\lambda^{\frac{1}{\alpha}} t} D_t^\alpha c \\ &= c(\lambda^{\frac{1}{\alpha}})^\alpha e^{\lambda^{\frac{1}{\alpha}} t} + e^{\lambda^{\frac{1}{\alpha}} t} D_t^\alpha c = c\lambda e^{\lambda^{\frac{1}{\alpha}} t} \\ &= \lambda u(t). \end{aligned}$$

■

And the particular solution, by using the value of a constant in (3.0.12) ,

$$u_p(t) = c\,(t)\, e^{\lambda^{\frac{1}{\alpha}} t}$$

we have

$$D_t^\alpha u_p(t) - \lambda\,(t)\, u_p(t) = u_H(t) D_t^\alpha c\,(t) + c\,(t)\, D_t^\alpha u_H\,(t) - \lambda\,(t)\, c\,(t)\, u_H(t) = g(t)$$

so,

$$e^{\lambda^{\frac{1}{\alpha}} t} D_t^\alpha c\,(t) = g(t)$$

is giving

$$c\,(t) = I_t^\alpha \left(\frac{g(t)}{e^{\lambda^{\frac{1}{\alpha}} t}} \right)$$

output

$$u_p(t) = e^{\lambda^{\frac{1}{\alpha}} t} I_t^\alpha \left(\frac{g(t)}{e^{\lambda^{\frac{1}{\alpha}} t}} \right)$$

outcome

$$u(t) = e^{\lambda^{\frac{1}{\alpha}} t} \left(c + I_t^\alpha \left(\frac{g(t)}{e^{\lambda^{\frac{1}{\alpha}} t}} \right) \right).$$

Without generality loss, but having only a simpler notation, consider the case $n_i = 1$ and $i = 1, 2$ of an operator with two characteristic roots λ_1, λ_2.

Proposition 3.0.14 *Let* $\lambda_1\,(t)\,, \lambda_2\,(t)$, $0 < \beta \leq 1$ *and the fractional differential equation:*

$$\left(D_t^{2\alpha} - (\lambda_1\,(t) + \lambda_2\,(t))\, D_t^\alpha + (\lambda_1\,(t)\, \lambda_2\,(t) - D_t^\alpha \lambda_2\,(t)) \right) u = f(t) \tag{3.0.13}$$

Equation (3.0.13) *has the solutions.*

Proof. Let the fractional differential equation

$$\left(D_t^{\alpha}-\lambda_1\left(t\right)\right)u_1\left(t\right)=f(t) \tag{3.0.14}$$

with the solutions of(3.0.14) is $u_1\left(t\right)=u_{1,h}\left(t\right)+u_{1,p}\left(t\right).$

And

$$\left(D_t^{\alpha}-\lambda_2\left(t\right)\right)u_2\left(t\right)=u_{1,p}\left(t\right) \tag{3.0.15}$$

with the solutions of (3.0.15) are $u_2\left(t\right)=u_{2,h}\left(t\right)+u_{2,p}\left(t\right).$

We have

$$\begin{aligned}
& \left(D_t^{2\alpha}-\left(\lambda_1\left(t\right)+\lambda_2\left(t\right)\right)D_t^{\alpha}+\left(\lambda_1\left(t\right)\lambda_2\left(t\right)-D_t^{\alpha}\lambda_2\left(t\right)\right)\right)u_{2,h}\left(t\right) \\
= & \left(D_t^{2\alpha}u_{2,h}\left(t\right)-\left(\lambda_1\left(t\right)+\lambda_2\left(t\right)\right)D_t^{\alpha}u_{2,h}\left(t\right)+\left(\lambda_1\left(t\right)\lambda_2\left(t\right)-D_t^{\alpha}\lambda_2\left(t\right)\right)\right)u_{2,h}\left(t\right) \\
= & D_t^{\alpha}\left(D_t^{\alpha}-\lambda_2\left(t\right)\right)u_{2,h}\left(t\right)-\lambda_1\left(t\right)\left(D_t^{\alpha}-\lambda_2\left(t\right)\right)u_{2,h}\left(t\right) \\
= & D_t^{\alpha}\left(0\right)-\lambda_1\left(t\right)\left(0\right)=0,
\end{aligned}$$

so $u_{2,h}\left(t\right)$ is the general homogeneous solution of (3.0.13).

And

$$\begin{aligned}
& \left(D_t^{2\alpha}-\left(\lambda_1\left(t\right)+\lambda_2\left(t\right)\right)D_t^{\alpha}+\left(\lambda_1\left(t\right)\lambda_2\left(t\right)-D_t^{\alpha}\lambda_2\left(t\right)\right)\right)u_{2,p}\left(t\right) \\
= & \left(D_t^{2\alpha}u_{2,p}\left(t\right)-\left(\lambda_1\left(t\right)+\lambda_2\left(t\right)\right)D_t^{\alpha}u_{2,p}\left(t\right)+\left(\lambda_1\left(t\right)\lambda_2\left(t\right)-D_t^{\alpha}\lambda_2\left(t\right)\right)\right)u_{2,p}\left(t\right) \\
= & D_t^{\alpha}\left(D_t^{\alpha}-\lambda_2\left(t\right)\right)u_{2,p}\left(t\right)-\lambda_1\left(t\right)\left(D_t^{\alpha}-\lambda_2\left(t\right)\right)u_{2,p}\left(t\right) \\
= & D_t^{\alpha}u_{1,p}\left(t\right)-\lambda_1\left(t\right)u_{1,p}\left(t\right)=\left(D_t^{\alpha}-\lambda_1\left(t\right)\right)u_1\left(t\right)=f(t),
\end{aligned}$$

so $u_{2,h}\left(t\right)$ is the particular solution of (3.0.13).

Outcome $u\left(t\right)=u_{2,h}\left(t\right)+u_{2,p}\left(t\right)$ is the solution of (3.0.13). ■

Remark 3.0.15 *In the same way for case* $\lambda_1\left(t\right)=\lambda_2\left(t\right),n_1\neq 1.$

We can generalize this proposition,

Proposition 3.0.16 *Let $\lambda_i(t), i = 1,\ 0 < \beta \leq 1$ and the fractional differential equation :*

$$\left((D_t^\beta - \lambda_1)^{n_1}(D_t^\beta - \lambda_2)^{n_2}.....(D_t^\beta - \lambda_m)^{n_m}\right) u(t) = f(t). \tag{3.0.16}$$

The general solution of equation (3.0.16) *of the form,*

$$u(t) = C_i t^{(k-1)\alpha} u_{i,H}(t) + u_{0,P}(t),\ \ k = 1,..,n_i.$$

With, $u_{i,H}(t)$ is the solution of homogeneous equation $(D_t^\beta - \lambda_i(t))u(t) = 0$, and we get $u_{0,P}(t)$ of the solution,

$$\begin{cases} (D_t^\beta - \lambda_{m-1}(t))u_{m-1}(t) = f(t) \\ (D_t^\beta - \lambda_i(t))u_{i-1,P}(t) = u_{i,P}(t),\ \ i = 1,...,m-1. \end{cases}$$

Proof. Let the fractional differential equation

$$(D_t^\alpha - \lambda_1(t))\, u_1(t) = f(t), \tag{3.0.17}$$

with the solution of (3.0.14) are $u_1(t) = u_{1,h}(t) + u_{1,p}(t)$.

And

$$(D_t^\alpha - \lambda_i(t))\, u_{k+1}(t) = u_{k,p}(t), k = 1,.., \sum_{i=1}^{i=m} n_i = n, \tag{3.0.18}$$

with the solution of (3.0.15) is $u_{k+1}(t) = u_{k+1,h}(t) + u_{k+1,p}(t)$.

We prove by the recurrence

$$P(m) : u(t) = C_i t^{(k-1)} u_{m,H}(t) + u_{0,P}(t),\ \ k = 1,..,n_i, \tag{3.0.19}$$

is the solution of (3.0.16).

For $P(n = 1)$ is true, proved in the precedent proposition.

Assume $P(n = l)$ is true, then $P(n = l + 1)$ also is true.

Let $P\left(n=l\right):u_{l}\left(t\right)=u_{l,h}\left(t\right)+u_{l,p}\left(t\right)$ be the solution, and let $u_{l+1}\left(t\right)=u_{l+,h}\left(t\right)+u_{l+,p}\left(t\right)$

$$\begin{aligned}
&\left((D_t^{\beta}-\lambda_1)^{n_1}(D_t^{\beta}-\lambda_2)^{n_2}.....(D_t^{\beta}-\lambda_l)^{n_l}(D_t^{\beta}-\lambda_{l+1})^{n_{l+1}}\right)u_{l+1}\left(t\right)\\
=\;&(D_t^{\beta}-\lambda_1)^{n_1}(D_t^{\beta}-\lambda_2)^{n_2}.....(D_t^{\beta}-\lambda_l)^{n_l}\left((D_t^{\beta}-\lambda_{l+1})^{n_{l+1}}u_{l+1}\left(t\right)\right)\\
=\;&\left((D_t^{\beta}-\lambda_1)^{n_1}(D_t^{\beta}-\lambda_2)^{n_2}.....(D_t^{\beta}-\lambda_l)^{n_l}\right)u_{l,p}\left(t\right)=f(t),
\end{aligned}$$

thus $P\left(n=l+1\right)$ is true.

Therefore, we finish the proof with the conclusion that n is true. ■

In the nexte we try all cas possiblle,

3.1 The differential equations

The decomposition of $P=P_1\circ P_2+R$,with $(R=0$ Or $R\neq 0)$ remainder (existence or nonexistence).

We shall apply with differential equation, with $\beta=1$.

3.1.1 The differential equations $(\beta=1)$ with remainder nonexistence R=0:

Example 3.1.1 *In this example, let the differential equation with* $\beta=1$,*and* ,$R=0$,

$$u^{(\prime\prime)}-5u^{(\prime)}+6u=2e^{t}, \tag{3.1.1}$$

write $D_t^2u-5D_tu+6u=2e^t$, *is clear* $m_k=2$, *and the function characteristic*

$$\begin{aligned}
D_t^2u-5D_tu+6u &= 2e^t\\
(D_t-3)(D_t-2)u &= 2e^t\\
P_2P_1u+Ru &= f,
\end{aligned}$$

so,

$$\lambda_1=3,\lambda_2=2,$$

and $d = 2,\ n_1 = 1, n_2 = 1,$ *so*

$$D_t\lambda_1 = D_t\lambda_2 = 0,$$

we have

$$P_1P_2 = (D_t - 3)(D_t - 2)u = (D_t - 3)(D_t - 2)u = P_2P_1 = 2e^t,$$

and,

$$Ru = \tilde{R}u = 0$$

The system equivalent with $\beta = 1$ *is,*

$$\begin{aligned} D_t^{\beta}u_{n_1-1} &= \lambda_1 u_{n_1-1} + u_{n_1} - \sum_{k=1}^{n_1} g^{(1)}_{n_1-k}u_{n_1-k} \\ (D_t^{\beta} - \lambda_2)u_{m_k-1} &= f - Ru - \sum_{k=1}^{n_2} g^{(2)}_{n_1-k}u_{m_k-k} \end{aligned}$$

So

$$\begin{aligned} (D_t - \lambda_1 + g_1^{(1)})u_0 &= u_1 \\ (D_t - \lambda_2 + b_1^{(1)} + g_1^{(2)})u_1 &= -b_0^{(1)}u_0 + f \end{aligned}$$

and

$$\begin{aligned} Pu &= (P_2 \circ P_1 + R)u = f \\ &= ((D-3) + g_1^{(1)})((D-2) + g_1^{(2)})u + Ru = f \\ &= (D-3)(D-2)u + g_1^{(1)}(D-3)u + g_1^{(2)}(D-2)u + g_1^{(1)}g_1^{(2)}u + Ru = f \\ &= (D_t-3)(D_t-2)u + Ru = f \end{aligned}$$

After compensation, we get with $n_1 = 1, n_2 = 1,$

then $g_1^{(1)} = g_2^{(1)} = 0, g_1^{(2)} = g_2^{(2)} = 0, , b_0^{(1)} = -2, b_0^{(2)} = -3, b_1^{(2)} = b_1^{(1)} = -1,$

so,

$$\begin{aligned} (D_t - 2)(D_t - 3)u + Ru &= f \\ (D_t - 3)(D_t - 2)u + \tilde{R}\ u &= f \end{aligned}$$

and

$$\begin{aligned}(D_t-3)u_0 &= u_1\\ (D_t-2)\,u_1 &= e^t\\ (D_t-2)u_2 &= u_3\\ (D_t-3)\,u_3 &= e^t,\end{aligned}$$

so

$$\begin{aligned}D_tu_0 &= 3u_0+u_1\\ D_tu_1 &= 2u_1+e^t\\ D_tu_2 &= 2u_2+u_3\\ D_tu_3 &= 3u_3+e^t\end{aligned}$$

the system is produced

$$\partial_t U - K(t)U = F \tag{3.1.2}$$

with

$$K=\begin{pmatrix}3&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&3\end{pmatrix}, \text{ and } F=\begin{pmatrix}0\\e^t\\0\\e^t\end{pmatrix}$$

, $\det K \neq 0$, *so the system*(3.1.2) *has a solution.*

We need only the system,

$$\begin{cases} u_0 = u\\ (D_t-3)u_0 = u_1\\ (D_t-2))u_1 = 2e^t\\ (D_t-2)u_2 = u_3\\ (D_t-3)\,u_3 = e^t\end{cases} \tag{3.1.3}$$

and just solve the differentials equations of the first degree

$$(D_t - g_i(t))u_i = f_i(t)$$

The solution of(3.1.3) *is :*

The solution of $(D_t - 2)u_1 = 2e^t$, *is* $u_1 = c_1 e^{2t} - e^t$,

so $(D_t - 3)u_0 = -e^t$, *is* $u_0(t) = c_2 e^{3t} - 2e^t$.

So, the solution of $(D_t - 3)u_2 = e^t$, *is* $u_3 = c_3 e^{2t} - e^t$,

so $(D_t - 2)u_2 = -e^t$, *is* $u_2(t) = c_2 e^{3t} - \frac{1}{2}e^t$.

So the solution of (3.1.1) *is :* $u(t) = c_2 e^{3t} + c_1 e^{2t} + 2e^t$;

Verify that the general solution satisfies the differential equation (3.1.1)

$$\begin{aligned}
u''(t) &= 9c_2 e^{3t} - 4c_1 e^{2t} + \frac{1}{2}e^t \\
-5u'(t) &= -5\left(3c_2 e^{3t} - 2c_1 e^{2t} + \frac{1}{2}e^t\right) \\
6u(t) &= 6c_2 e^{3t} - 6c_1 e^{2t} + \frac{6}{2}e^t \\
u''(t) - 5u'(t) + 6u(t) &= (9 + 6 - 15)\, c_2 e^{3t} + (10 - 4 - 6)\, c_1 e^{2t} + \left(\frac{1}{2} + \frac{6}{2} - \frac{1}{2}\right) e^t = e^t.
\end{aligned}$$

Example 3.1.2 *In this example, let the differential equation with* $\beta = 1$,*and* , $R = 0$,

$$u^{(4)} - 5u^{(3)} + 9u^{(2)} - 7u^{(1)} + 2u = 8e^{3t}, \tag{3.1.4}$$

write $D_t^4 u - 5D_t^3 u + 9D_t^2 u - 7D_t u + 2u = 8e^{3t}$, *is clear* $m_k = 4$, *and the function characteristic*

$$r^4 - 5r^3 + 9r^2 - 7r + 2 = (r - 1)^3 (r - 2) = 0$$

so,

$$\begin{aligned}(D_t-1)^3(D_t-2)u &= 8e^{3t}, \lambda_1 = 1, \lambda_2 = 2,\\ P_2P_1u + Ru &= f,\end{aligned}$$

and $d = 2$, $n_1 = 3, n_2 = 1$, *so*

$$D_t\lambda_1 = D_t\lambda_2 = 0,$$

we have

$$P_1P_2 = (D_t-1)^3(D_t-2)u = (D_t-2)(D_t-1)^3u = P_2P_1 = 8e^{3t},$$

and,

$$Ru = \tilde{R}u = 0$$

The system equivalent with $\beta = 1$ *is,*

$$\begin{aligned}D_t^{\beta}u_{n_1-1} = \lambda_1 u_{n_1-1} + u_{n_1} - \sum_{k=1}^{n_1} g_{n_1-k}^{(1)} u_{n_1-k}\\ (D_t^{\beta} - \lambda_2)u_{m_k-1} = f - Ru - \sum_{k=1}^{n_2} g_{n_1-k}^{(2)} u_{m_k-k}\end{aligned}$$

So

$$\begin{aligned}(D_t - \lambda_1 + g_1^{(1)})u_0 = u_1\\ (D_t - \lambda_2 + b_1^{(1)} + g_1^{(2)})u_1 = -b_0^{(1)}u_0 + f\end{aligned}$$

and

$$\begin{aligned}Pu &= (P_2 \circ P_1 + R)u = f\\ &= ((D-1) + g_1^{(1)})((D-1) + g_1^{(1)})((D-1) + g_1^{(1)})((D-2) + g_1^{(2)})u + Ru = f\\ &= (D_t-1)^3(D_t-2)u + Ru = f\end{aligned}$$

After compensation, we get with $n_1 = 3, n_2 = 1$,

then $g_1^{(1)} = g_2^{(1)} = 0, g_1^{(2)} = g_2^{(2)} = 0, , b_0^{(1)} = -5, b_0^{(2)} = 9, b_1^{(2)} = -7, b_1^{(1)} = 2,$

so,

$$
\begin{aligned}
(D_t-1)^3(D_t-2)u+Ru &= f \\
(D_t-2)(D_t-1)^3u+\tilde{R}\ u &= f
\end{aligned}
$$

and

$$
\begin{aligned}
(D_t-1)u_0 &= u_1 \\
(D_t-1)u_1 &= u_2 \\
(D_t-1)u_2 &= u_3 \\
(D_t-2)\,u_3 &= 8e^{3t} \\
(D_t-2)u_4 &= u_5 \\
(D_t-1)u_5 &= u_6 \\
(D_t-1)u_6 &= u_7 \\
(D_t-1)\,u_7 &= 8e^{3t},
\end{aligned}
$$

so

$$
\begin{aligned}
D_tu_0 &= u_0+u_1 \\
D_tu_1 &= u_1+u_2 \\
D_tu_2 &= u_2+u_3 \\
D_tu_3 &= 2u_3+8e^{3t} \\
D_tu_4 &= 2u_4+u_3 \\
D_tu_5 &= u_5+u_4 \\
D_tu_6 &= u_6+u_5 \\
D_tu_7 &= u_5+8e^{3t},
\end{aligned}
$$

the system is produced

$$\partial_t U - K(t)U = F \tag{3.1.5}$$

with

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \text{ and } F = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 8e^{3t} \\ 0 \\ 0 \\ 0 \\ 8e^{3t} \end{pmatrix}$$

, $\det K = 4 \neq 0$, *so the system*(3.1.5) *has a solution.*

We need only the system,

$$\begin{cases} (D_t - 1))u_3 = 8e^{3t} \\ (D_t - 1)u_2 = u_{3,p} \\ (D_t - 1)u_1 = u_{2,p} \\ (D_t - 2)u_0 = u_{1,p} \end{cases} \tag{3.1.6}$$

and just solve the differentials equations of the first degree

$$(D_t - g_i(t))u_i = f_i(t)$$

The solution of (3.1.6) *is :*

The general solution of the homogeneous equation (3.1.4) *is*

$$u_h(t) = \left(c_1 + tc_2 + t^2c_3\right)e^t + c_4e^{2t}.$$

The particular solution of (3.1.4) *:*

The solution of $(D_t - 1)u_3 = 8e^{3t}$, *is* $u_3 = c_1e^t + 4e^{3t}$,

so $(D_t - 1)u_2 = 4e^{3t}$, *is* $u_2(t) = c_2e^t + 2e^{3t}$,

and the solution of $(D_t - 1)u_1 = 2e^{3t}$, *is* $u_3 = c_3e^t + e^{3t}$,

$so\ (D_t-2)u_0=e^{3t},\ is\ u_0\left(t\right)=c_4e^{2t}+e^{3t}.$

So the solution of(3.1.4) *is* $u_p\left(t\right)=u_{0,p}\left(t\right)=e^{3t}$

So the solution of(3.1.4) *is :* $u\left(t\right)=\left(c_1+tc_2+t^2c_3\right)e^t+c_4e^{2t}+e^{3t};$

Verify that the general solution satisfies the differential equation(3.1.4) ,

$$\begin{aligned}
2u\left(t\right) &= 2\left[\left(c_1+tc_2+t^2c_3\right)e^t+c_4e^{2t}+e^{3t}\right]\\
-7u^{(1)}\left(t\right) &= -7\left[\left(c_1+(1+t)\,c_2+\left(t^2+2t\right)c_3\right)e^t+2c_4e^{2t}+3e^{3t}\right]\\
9u^{(2)}\left(t\right) &= 9\left[\left(c_1+(2+t)\,c_2+\left(t^2+4t+2\right)c_3\right)e^t+4c_4e^{2t}+9e^{3t}\right]\\
-5u^{(3)}\left(t\right) &= -5\left[\left(c_1+(3+t)\,c_2+\left(t^2+6t+6\right)c_3\right)e^t+8c_4e^{2t}+27e^{3t}\right]\\
u^{(4)}\left(t\right) &= \left(c_1+(4+t)\,c_2+\left(t^2+8t+12\right)c_3\right)e^t+16c_4e^{2t}+81e^{3t},
\end{aligned}$$

so

$$u^{(4)}-5u^{(3)}+9u^{(2)}-7u^{(1)}+2u=8e^{3t}.$$

Example 3.1.3 *Let the differential equation* $(\beta=1,\ R=0)$.

$$u^{(\prime\prime)}+\frac{5}{t}u^{(\prime)}+\frac{3}{t^2}u=\frac{1}{t^2}, \tag{3.1.7}$$

write $D_t^2u+(\frac{2}{t}+\frac{3}{t})D_tu+\left(\frac{6}{t^2}-\frac{3}{t^2}\right)u=\frac{1}{t^2}$ *is clear* $m_k=2,$ *and the function characteristic*

$$\begin{aligned}
D_t^2u+(\frac{2}{t}+\frac{3}{t})D_tu+\left(\frac{6}{t^2}+\left(\frac{3}{t}\right)'\right)u &= \frac{1}{t^2}\\
(D_t+\frac{2}{t})(D_t+\frac{3}{t})u &= \frac{1}{t^2}
\end{aligned}$$

and let $\lambda_i(t), i=1,2$

$$\lambda_1(t)=\frac{3}{t},\lambda_2(t)=\frac{2}{t}$$

and so $d=2,\ n_1=1,n_2=1$

$$
\begin{aligned}
P_j &= (D-\lambda_j)^{n_j} + \sum_{i=1}^{n_j} g_i^{(j)} (D-\lambda_j)^{n_j-i}, j = 1, 2..., d. \\
P_1 &= (D-\lambda_1) + g_1^{(1)}, P_2 = (D-\lambda_2) + g_1^{(2)} \\
P_2 P_1 &= P_1 P_2,
\end{aligned}
$$

and

$$
\begin{aligned}
Ru &= \sum_{j=0}^{m_k-1=1} b_j^{(1)}(t) u_j = b_0^{(1)} u_0 + b_1^{(1)} u_1 \\
u_1 &= (D-\lambda_1) u_0, u = u_0
\end{aligned}
$$

The system equivalent with $\beta = 1$ *is,*

$$
\begin{aligned}
D_t^{\beta} u_{n_1-1} = \lambda_1 u_{n_1-1} + u_{n_1} - \sum_{k=1}^{n_1} g_{n_1-k}^{(1)} u_{n_1-k} \\
(D_t^{\beta} - \lambda_2) u_{m_k-1} = f - Ru - \sum_{k=1}^{n_2} g_{n_1-k}^{(2)} u_{m_k-k}.
\end{aligned}
$$

So,

$$
\begin{aligned}
(D_t - \lambda_1 + g_1^{(1)}) u_0 = u_1 \\
(D_t - \lambda_2 + b_1^{(1)} + g_1^{(2)}) u_1 = -b_0^{(1)} u_0 + f
\end{aligned}
$$

and,

$$
\begin{aligned}
Pu &= (P_2 \circ P_1 + R) u = f \\
&= ((D-\lambda_1) + g_1^{(1)})((D-\lambda_2) + g_1^{(2)}) u + Ru - f \\
&= (D-\lambda_1)(D-\lambda_2) u + g_1^{(1)} (D-\lambda_2) u + g_1^{(2)} (D-\lambda_1) u + g_1^{(1)} g_1^{(2)} u + Ru - f \\
&= (D_t - t)(D_t - e^t) u + te^t (D_t - e^t) u = f.
\end{aligned}
$$

After compensation, we get

$$\begin{aligned} Ru &= \sum_{j=0}^{m_k-1=1} b_j^{(1)}(t)u_j = b_0^{(1)}u_0 + b_1^{(1)}u_1 = te^t(D_t - e^t)u \\ &= te^t u_1 = \tilde{R}u \end{aligned}$$

with $n_1 = 1, n_2 = 1$, *then* $g_1^{(1)} = g_2^{(1)} = 0, g_1^{(2)} = g_2^{(2)} = 0, b_1^{(1)} = b_2^{(1)} = 0, b_1^{(1)} = b_2^{(2)} = 0,$

the system is produced

$$\partial_t U - K(t)U = F \tag{3.1.8}$$

with

$$F = \begin{pmatrix} 0 \\ \frac{1}{t^2} \\ 0 \\ \frac{1}{t^2} \end{pmatrix}, \text{ and } K = \begin{pmatrix} \frac{3}{t} & 0 & 0 & 0 \\ 0 & \frac{2}{t} & 0 & 0 \\ 0 & 0 & \frac{2}{t} & 0 \\ 0 & 0 & 0 & \frac{3}{t} \end{pmatrix}$$

, $\det K \neq 0$, *so the system* (3.1.8) *has a solution.*

We can write(3.1.8) *,of the form,*

$$\begin{cases} u_0 = u = u_2 \\ (D_t + \frac{1}{t})u_0 = u_1 \\ (D_t + \frac{2}{t})u_1 = \frac{1}{t^2} \\ (D_t + \frac{2}{t})u_2 = u_3 \\ (D_t + \frac{2}{t})u_3 = -\frac{1}{t^2}u_3 + \frac{1}{t^2} \end{cases} \tag{3.1.9}$$

we need only the system,

$$\begin{cases} u_0 = u \\ (D_t + \frac{3}{t})u_0 = u_1 \\ (D_t + \frac{2}{t})u_1 = \frac{1}{t^2} \end{cases} \tag{3.1.10}$$

and just solve the differentials equations of the first degree

$$(D_t - g_i(t))u_i = f_i(t)$$

the solution of(3.1.10) *is :*

The solution of $(D_t + \frac{2}{t})u_1 = \frac{1}{t^2}$, *is* $u_1 = c\frac{1}{2t} + \frac{1}{t}$,

so $(D_t + \frac{3}{t})u_0 = \frac{1}{t}$, *is* $u_0(t) = u(t) = c_2\frac{1}{t^3} + \frac{1}{3}$.

So the solution of (3.1.7) *is :* $u(t) = c_2\frac{1}{t^3} + c_1\frac{1}{2t} + \frac{1}{3}$,

Verify that the general solution satisfies the differential equation(3.1.7) ,

$$\begin{aligned}
u''(t) &= c_2\frac{12}{t^5} + c_1\frac{1}{t^3} \\
\frac{5}{t}u'(t) &= +\frac{5}{t}\left(c_2\frac{-3}{t^4} - c_1\frac{1}{2t^2}\right) \\
\frac{3}{t^2}u(t) &= \frac{3}{t^2}\left(c_2\frac{1}{t^3} + c_1\frac{1}{2t} + \frac{1}{3}\right)
\end{aligned}$$

so

$$u^{(\prime\prime)} + \frac{5}{t}u^{(\prime)} + \frac{3}{t^2}u = (12 + 3 - 15)\, c_2\frac{1}{t^5} + \left(1 - \frac{5}{2} + \frac{3}{2}\right) c_1\frac{1}{t^3} + \frac{1}{t^2} = \frac{1}{t^2}.$$

3.1.2 The differential equations with remainder existence R≠0:

Example 3.1.4 *In this example, let the differential equation with* $\beta = 1$, *and* $R \neq 0$,

$$u^{(\prime\prime)} - 3u^{(\prime)} + 3u = e^t, \tag{3.1.11}$$

write $D_t^2 u - 3D_t u + 3u = e^t$, *is clear* $m_k = 2$, *and the function characteristic*

$$D_t^2 u - 3D_t u + 3u = (D_t - \lambda_1)(D_t - \lambda_2)u = (D_t - \lambda_2)(D_t - \lambda_1)u = e^t$$

so,

$$\lambda_1 + \lambda_2 = 3, \lambda_1\lambda_2 = 3; \lambda_1 = \frac{3}{2} + \frac{\sqrt{3}}{2}i = \bar{\lambda}_2$$

and so $d = 2$, $n_1 = 1, n_2 = 1$, *and*

$$D_t\lambda_1 = D_t\lambda_2 = 0.$$

We have

$$P_2P_1 = P_1P_2.$$

And,

$$Ru = \tilde{R}u \neq 0$$

and,

The equivalent system with $\beta = 1$ is,

$$D_t^\beta u_{n_1-1} = \lambda_1 u_{n_1-1} + u_{n_1} - \sum_{k=1}^{n_1} g_{n_1-k}^{(1)} u_{n_1-k}$$
$$(D_t^\beta - \lambda_2) u_{m_k-1} = f - \sum_{k=1}^{n_2} g_{n_1-k}^{(2)} u_{m_k-k}$$

So

$$(D_t - \lambda_1 + g_1^{(1)}) u_0 = u_1$$
$$(D_t - \lambda_2 + b_1^{(1)} + g_1^{(2)}) u_1 = -b_0^{(1)} u_0 + f$$

After compensation, we get with $n_1 = 1, n_2 = 1$,

then $g_1^{(1)} = g_2^{(1)} = 0, g_1^{(2)} = g_2^{(2)} = 0, , b_0^{(1)} = -2, b_0^{(2)} = -3, b_1^{(2)} = b_1^{(1)} = -1,$

so,

$$\begin{aligned}(D_t - \lambda_1)(D_t - \lambda_2)u &= f \\ (D_t - \lambda_2)(D_t - \lambda_1)u &= f\end{aligned}$$

and

$$\begin{aligned}(D_t - \lambda_2)u_0 &= u_1 \\ (D_t - \lambda_1)\, u_1 &= e^t \\ (D_t - \lambda_1)u_2 &= u_3 \\ (D_t - \lambda_2)\, u_3 &= e^t\end{aligned}$$

the system is produced

$$\partial_t U - K(t)U = F \tag{3.1.12}$$

with

$$F = \begin{pmatrix} 0 \\ e^t \\ 0 \\ e^t \end{pmatrix}, \text{ and } K = \begin{pmatrix} \lambda_2 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}$$

, $\det K \neq 0$, *so the system* (3.1.12) *has a solution.*

We need only the system,

$$\begin{cases} u_0 = u \\ (D_t - \lambda_2)u_0 = u_1 \\ (D_t - \lambda_1))u_1 = e^t \\ (D_t - \lambda_1)u_2 = u_3 \\ (D_t - \lambda_2)u_3 = e^t \end{cases} \tag{3.1.13}$$

and just solve the differential equation of the first degree

$$(D_t - g_i(t))u_i = f_i(t)$$

The solution of(3.1.13) *is :*

The solution of $(D_t - \lambda_1)u_1 = e^t$, *is* $u_1 = c_1 e^{\lambda_1 t} + \frac{1}{1-\lambda_1} e^t$,

so $(D_t - \lambda_2)u_0 = \frac{1}{1-\lambda_1} e^t$, *is* $u_0(t) = c_2 e^{\lambda_2 t} + \frac{1}{(1-\lambda_1)(1-\lambda_2)} e^t = c_2 e^{\lambda_2 t} + e^t$.

So the solution of (3.1.11) *is :* $u(t) = c_2 e^{\lambda_2 t} + c_1 e^{\lambda_1 t} + e^t$;

Verify that the general solution satisfies the differential equation(3.1.11) ,

$$\begin{aligned} u''(t) &= c_2 \lambda_2^2 e^{\lambda_2 t} + c_1 \lambda_1^2 e^{\lambda_1 t} + e^t = c_2 \left(\frac{3}{2} - \frac{3\sqrt{3}}{2} i \right) e^{\lambda_2 t} + c_1 \left(\frac{3}{2} + \frac{3\sqrt{3}}{2} i \right) e^{\lambda_1 t} + e^t \\ -3u'(t) &= -3c_2 \lambda_2 e^{\lambda_2 t} - 3c_1 \lambda_1 e^{\lambda_1 t} - 3e^t = c_2 \left(-\frac{9}{2} + \frac{3\sqrt{3}}{2} i \right) e^{\lambda_2 t} + c_1 \left(-\frac{9}{2} - \frac{3\sqrt{3}}{2} i \right) e^{\lambda_1 t} \\ &\quad -3e^t \\ 3u(t) &= 3c_2 e^{\lambda_2 t} + 3c_1 e^{\lambda_2 t} + 3e^t, \end{aligned}$$

so

$$u''(t) - 3u'(t) + 3u(t) = c_2(0) c_2 e^{\lambda_2 t} + c_1(0) e^{\lambda_2 t} + (3 - 3 + 1) e^t = e^t.$$

Example 3.1.5 *In this example, let the differential equation with $\beta = 1$, and $R \neq 0$,*

$$u^{(\prime')} - 4tu^{(\prime)} + \left(4t^2 + 2\right) u = \left(4t^2 - 4t + 3\right) e^t, \tag{3.1.14}$$

write $D_t^2 u - 4tD_t u + \left(4t^2 + 4\right) u = \left(4t^2 - 4t + 5\right) e^t$, *is clear* $m_k = 2$, *and the function characteristic*

$$\begin{aligned} D_t^2 u - 4tD_t u + \left(4t^2 + 4\right) u &= (D_t - \lambda_1)(D_t - \lambda_2)u \\ &= D_t^2 u - \left(\lambda_1(t) + \lambda_2(t)\right) D_t u + \left(\lambda_1(t)\lambda_2(t) - D_t\lambda_2(t)\right) u \\ &= \left(4t^2 - 4t + 5\right) e^t, \end{aligned}$$

we have

$$\lambda_1(t) = \alpha(t) + i\beta(t) = \bar{\lambda}_2(t)$$

so,

$$\lambda_1(t) + \lambda_2(t) = 2\alpha(t) = 4t, \lambda_1(t)\lambda_2(t) - D_t\lambda_2(t) = 4t^2 + 4; \lambda_1(t) = 2t + 2i = \bar{\lambda}_2(t)$$

and so $d = 2$, $n_1 = 1, n_2 = 1$, *and*

$$D_t\lambda_1 = D_t\lambda_2 = 2.$$

We have

$$P_2P_1 = P_1P_2.$$

And,

$$\tilde{R}u \neq Ru \neq 0.$$

The equivalent system with $\beta = 1$ is,

$$\begin{gathered} D_t^\beta u_{n_1-1} = \lambda_1 u_{n_1-1} + u_{n_1} - \sum_{k=1}^{n_1} g_{n_1-k}^{(1)} u_{n_1-k} \\ (D_t^\beta - \lambda_2)u_{m_k-1} = f - \sum_{k=1}^{n_2} g_{n_1-k}^{(2)} u_{m_k-k}, \end{gathered}$$

So

$$(D_t - \lambda_1 + g_1^{(1)})u_0 = u_1$$
$$(D_t - \lambda_2 + b_1^{(1)} + g_1^{(2)})u_1 = -b_0^{(1)}u_0 + f$$

After compensation, we get with $n_1 = 1, n_2 = 1,$

then $g_1^{(1)} = g_2^{(1)} = 0, g_1^{(2)} = g_2^{(2)} = 0, , b_0^{(1)} = -2, b_0^{(2)} = -3, b_1^{(2)} = b_1^{(1)} = -1,$

so,

$$\begin{aligned}(D_t - \lambda_1(t))(D_t - \lambda_2(t))u &= f \\ (D_t - \lambda_2(t))(D_t - \lambda_1(t))u &= f\end{aligned}$$

and

$$\begin{aligned}(D_t - \lambda_2(t))u_0 &= u_1 \\ (D_t - \lambda_1(t))\,u_1 &= e^t \\ (D_t - \lambda_1(t))u_2 &= u_3 \\ (D_t - \lambda_2(t))\,u_3 &= e^t\end{aligned}$$

the system is produced

$$\partial_t U - K(t)U = F \tag{3.1.15}$$

with

$$F = \begin{pmatrix} 0 \\ \left(4t^2 - 4t + 5\right) e^t \\ 0 \\ \left(4t^2 - 4t + 5\right) e^t \end{pmatrix}, \text{ and } K = \begin{pmatrix} \lambda_2(t) & 0 & 0 & 0 \\ 0 & \lambda_1(t) & 0 & 0 \\ 0 & 0 & \lambda_1(t) & 0 \\ 0 & 0 & 0 & \lambda_2(t) \end{pmatrix}$$

, $\det K \neq 0$, *so the system* (3.1.15) *has a solution.*

We need only the system,

$$\begin{cases} u_0 = u \\ (D_t - \lambda_2(t))u_0 = u_1 \\ (D_t - \lambda_1(t)))u_1 = \left(4t^2 - 4t + 5\right) e^t \\ (D_t - \lambda_1(t))u_2 = u_3 \\ (D_t - \lambda_2(t))u_3 = \left(4t^2 - 4t + 5\right) e^t \end{cases} \tag{3.1.16}$$

and just solve the differential equation of the first degree

$$(D_t - g_i(t))u_i = f_i(t)$$

The solution of (3.1.16) *is :*

The solution of $(D_t - \lambda_1)u_1 = e^t$, *is* $u_1 = c_1 e^{t^2+2ti} + (1-\lambda_2)e^t$,

so $(D_t - \lambda_2)u_0 = (1-\lambda_2)e^t$, *is* $u_0(t) = c_2 e^{t^2-2ti} + e^t$.

So the solution of(3.1.14) *is :* $u(t) = c_2 e^{t^2+2ti} + c_1 e^{t^2-2ti} + e^t$;

Verify that the general solution satisfies the differential equation (3.1.14) ,

$$\begin{aligned} u''(t) &= c_2\left(4t^2 - 4 + 8ti\right)e^{t^2+2ti} + c_1\left(4t^2 - 4 - 8ti\right)e^{t^2-2ti} + e^t \\ -4tu'(t) &= c_2\left(-4t^2 - 8ti\right)e^{t^2+2ti} + c_1\left(-4t^2 + 8ti\right)e^{t^2-2ti} - 4te^t \\ \left(4t^2+4\right)u(t) &= c_2\left(4t^2+4\right)e^{\lambda_2 t} + c_1\left(4t^2+4\right)e^{\lambda_2 t} + \left(4t^2+4\right)e^t \end{aligned}$$

$$\begin{aligned} u''(t) - 4tu'(t) + \left(4t^2+4\right)u(t) &= c_2(0)\,c_2 e^{\lambda_2 t} + c_1(0)\,e^{\lambda_2 t} + \left(4t^2 + 4 - 4t + 1\right)e^t \\ &= \left(4t^2 - 4t + 5\right)e^t. \end{aligned}$$

Example 3.1.6 *In this example, let the differential equation with* $\beta = 1$, *and* $R \neq 0$,

$$u^{(\prime\prime')} - (4t+1)\,u^{(\prime\prime)} + \left(4t^2 + 4t - 2\right)u^{(\prime)} - \left(4t^2 - 8t + 2\right)u = (8t-4)\,e^t, \tag{3.1.17}$$

write $D_t^3 u - (4t+1)\,D_t^2 u + \left(4t^2 + 4t - 2\right)D_t u - \left(4t^2 - 8t + 2\right)u = (8t-4)\,e^t$, *is clear* $m_k = 3$, *we put*

$$\lambda_1 ; \lambda_2(t) = \alpha(t) + i\beta(t) = \bar{\lambda}_3(t)$$

and the function characteristic

$$\begin{aligned}
& D_t^3 u - (4t+1) D_t^2 u + \left(4t^2 + 4t - 2\right) D_t u - \left(4t^2 - 8t + 2\right) u \\
= \;& (D_t - \lambda_1)(D_t - \lambda_2)(D_t - \lambda_3) u \\
= \;& (D_t - \lambda_1)\left[D_t^2 u - (\lambda_2(t) + \lambda_3(t)) D_t u + (\lambda_2(t)\lambda_3(t) - D_t\lambda_3(t)) u\right] \\
= \;& D_t^3 u - (\lambda_1(t) + \lambda_2(t) + \lambda_3(t)) D_t^2 u + \begin{pmatrix} D_t(\lambda_2(t) + \lambda_3(t)) + \lambda_2(t)\lambda_3(t) \\ -D_t\lambda_3(t) + \lambda_1(t)\lambda_2(t)\lambda_3(t) \end{pmatrix} D_t u \\
& + (D_t(\lambda_2(t)\lambda_3(t) - D_t\lambda_3(t)) + \lambda_2(t)\lambda_3(t) - D_t\lambda_3(t)) u \\
= \;& (8t - 4) e^t,
\end{aligned}$$

so,

$$\begin{aligned}
\lambda_2(t) + \lambda_3(t) &= \lambda_1(t) + 2\alpha(t) = 4t + 1, \lambda_2(t)\lambda_3(t) - D_t\lambda_2(t) = 4t^2 + 2; \\
4t^2 - 8t + 2 &= (D_t(\lambda_2(t)\lambda_3(t) - D_t\lambda_3(t)) + \lambda_2(t)\lambda_3(t) - D_t\lambda_3(t)), \\
\lambda_2(t) &= 2t + 2i = \bar{\lambda}_{23}(t), \lambda_1 = 1,
\end{aligned}$$

and so $d = 2,\ n_1 = 1, n_2 = 1,$ *and*

$$D_t\lambda_1 = 0, D_t\lambda_2 = D_t\lambda_3 = 2.$$

The equivalent system with $\beta = 1$ *is,*

$$\begin{gathered}
D_t^\beta u_{n_1-1} = \lambda_1 u_{n_1-1} + u_{n_1} - \sum_{k=1}^{n_1} g_{n_1-k}^{(1)} u_{n_1-k} \\
(D_t^\beta - \lambda_2) u_{m_k-1} = f - \sum_{k=1}^{n_2} g_{n_1-k}^{(2)} u_{m_k-k}
\end{gathered}$$

So

$$\begin{gathered}
(D_t - \lambda_1 + g_1^{(1)}) u_0 = u_1 \\
(D_t - \lambda_2 + b_1^{(1)} + g_1^{(2)}) u_1 = -b_0^{(1)} u_0 + f
\end{gathered}$$

After compensation, we get with $n_1 = 1, n_2 = 1$,

then $g_1^{(1)} = g_2^{(1)} = 0, g_1^{(2)} = g_2^{(2)} = 0, , b_0^{(1)} = -2, b_0^{(2)} = -3, b_1^{(2)} = b_1^{(1)} = -1,$

so,

$$(D_t - \lambda_1(t))(D_t - \lambda_2(t))(D_t - \lambda_3(t))u = f$$

and

$$\begin{aligned} (D_t - \lambda_3(t))u_0 &= u_1 \\ (D_t - \lambda_2(t))u_1 &= u_2 \\ (D_t - \lambda_1(t))\,u_2 &= (8t-4)\,e^t \end{aligned}$$

the system is produced

$$\partial_t U - K(t)U = F \tag{3.1.18}$$

with

$$F = \begin{pmatrix} 0 \\ 0 \\ (8t-4)\,e^t \\ 0 \\ 0 \\ (8t-4)\,e^t \end{pmatrix}, \text{ and } K = \begin{pmatrix} \lambda_3(t) & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2(t) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3(t) \end{pmatrix}$$

, $\det K \neq 0$, *so the system* (3.1.18) *has a solution.*

We need only the system,

$$\begin{cases} u_0 = u \\ (D_t - \lambda_1(t))\,u_2 = (8t-4)\,e^t \\ (D_t - \lambda_2(t))u_1 = u_2 \\ (D_t - \lambda_3(t))u_0 = u_1 \end{cases} \tag{3.1.19}$$

and just solve the differential equation of the first degree

$$(D_t - g_i(t))u_i = f_i(t)$$

The solution of (3.1.19) *is :*

The solution of $(D_t - \lambda_1(t))\, u_2 = (8t-4)\, e^t,$ *is* $u_2 = c_1 e^t + \left(4t^2 - 4t + 5\right) e^t,$

and $(D_t - \lambda_2) u_2 = \left(4t^2 - 4t + 5\right) e^t,$ *is* $u_0(t) = c_2 e^{t^2+2ti} + (1-\lambda_3)\, e^t,$

so $(D_t - \lambda_3) u_0 = (1-\lambda_3)\, e^t,$ *is* $u_0(t) = c_3 e^{t^2-2ti} + (1-\lambda_2)\, e^t.$

So the solution of (3.1.17) *is :* $u(t) = c_2 e^{t^2+2ti} + c_3 e^{t^2-2ti} + c_1 e^t + e^t;$

Verify that the general solution satisfies the differential equation(3.1.17) ,

$$\begin{aligned}
u^{('')} &= c_2(4+2t+2i)\, e^{t^2+2ti} + c_1(4+2t-2i)\, e^{t^2-2ti} + c_3 e^t + e^t \\
-(4t+1)\, u''(t) &= c_2\left(4t^2 - 4 + 8ti\right) e^{t^2+2ti} + c_1\left(4t^2 - 4 - 8ti\right) e^{t^2-2ti} + c_3 e^t \\
&\quad -(4t+1)\, e^t \\
\left(4t^2 + 4t - 2\right) u'(t) &= c_2\left(-4t^2 - 8ti\right) e^{t^2+2ti} + c_1\left(-4t^2 + 8ti\right) e^{t^2-2ti} + c_3 e^t \\
&\quad +\left(4t^2 + 4t - 2\right) e^t \\
-\left(4t^2 - 8t + 2\right) u(t) &= -c_2\left(4t^2 - 8t + 2\right) e^{t^2+2ti} + c_1\left(4t^2 + 4\right) e^{t^2-2ti} + c_3 e^t \\
&\quad -\left(4t^2 + 4\right) e^t
\end{aligned}$$

so,

$$\begin{aligned}
& u^{('')} - (4t+1)\, u^{('')} + \left(4t^2 + 4t - 2\right) u^{(')} - \left(4t^2 - 8t + 2\right) u \\
= \;& c_2(0)\, e^{t^2+2ti} + c_1(0)\, e^{t^2-2ti} + c_3(0)\, e^t + (4t + 4t - 4 - 1 + 1)\, e^t \\
= \;& (8t-4)\, e^t.
\end{aligned}$$

3.2 The fractional differential equations.

The fractional differential equations is related to the definition of fractional derivative, which can distinguish between them : $D_t^{\alpha} C = \frac{f(a)(t-a)^{-\alpha}}{\Gamma(1-\alpha)} \neq 0$ or $D_t^{\alpha} C = 0$ and $D_t^{\alpha} D_t^{\beta} = D_t^{\alpha+\beta}$or $D_t^{\alpha} D_t^{\alpha} = D_t^{2\alpha} + \frac{f(a)(t-a)^{-\alpha}}{\Gamma(1-\alpha)}$.

Example: Let $f(a) \neq 0$, and $\alpha \in]0,1[$.

$f(t)$	${}_a^G D_t^\alpha f(t)C$	${}_a^R D_t^\alpha f(t)$	${}_0^C D_t^\alpha f(t)$
$D_t^\alpha C$	$\frac{C}{\Gamma(1-\alpha)}(t-a)^{-\alpha} \neq 0$	$\frac{C}{\Gamma(1-\alpha)}(t-a)^{-\alpha} \neq 0$	0
$D_t^\alpha D_t^\alpha$	${}_a^G D_t^{2\alpha}$	${}_a^R D_t^{2\alpha} - \frac{f(a)(t-a)^{-\alpha}}{\Gamma(1-\alpha)} \neq {}_a^R D_t^{2\alpha}$	${}_a^C D_t^{2\alpha}$

If we have $f(a) = 0$, we will get $D_t^\alpha C = 0$ and $D_t^\alpha D_t^\alpha = D_t^{2\alpha}$.

We randomly select the following examples:

3.2.1 The fractional differential equations with remainder nonexistence R=0.

Example 3.2.1 *Let the fractional differential equation with* $D_t^\alpha C = 0$ *and* $D_t^\alpha D_t^\alpha = D_t^{2\alpha}$,

$$u^{(\frac{1}{3})} - 2u^{(\frac{1}{6})} + u = e^t \left(t^{\frac{1}{3}} - \frac{2}{3} t^{\frac{1}{6}} + \Gamma\left(\frac{4}{3}\right)\right) \tag{3.2.1}$$

the function characteristic

$$\begin{aligned} \tau^{\frac{1}{3}} - 2\tau^{\frac{1}{6}} + 1 &= (\tau^{\frac{1}{6}})^2 - 3(\tau^{\frac{1}{6}})^1 + 2 \\ &= (\tau^{\frac{1}{6}} - 1)^2 \end{aligned}$$

we can write, $D_t^{(\frac{1}{3})}u - 2D_t^{(\frac{1}{6})}u + D_t^{(0)}u = (D_t^{(\frac{1}{6})} - D_t^{(0)})(D_t^{(\frac{1}{6})} - D_t^{(0)})u = (D_t^{(\frac{1}{6})} - 1)^2 u$,*is clear* $\beta = \frac{1}{6}, m_k = 2,$ *and* $\lambda_1(t) = 1$,

also $d = 2$ *and* $m_k = 3$, $n_1 = 2, n_2 = 1 \Rightarrow r = 1, m' = m_k - r, r = 1$,

and,

$$\begin{aligned} P_j &= (D - \lambda_j)^{n_j} + \sum_{i=1}^{n_j} g_i^{(j)} (D - \lambda_j)^{n_j - i}, j = 1, 2..., d. \\ P_1 &= (D - \lambda_1)^2 + g_1^{(1)} + g_2^{(1)}(D - \lambda_1), P_2 = (D - \lambda_2) + g_1^{(2)} \\ P_2 P_1 &= P_1 P_2 \end{aligned}$$

we can write

$$P_1 = (D_t^{\frac{1}{6}} - 1)^2,$$

and

$$\begin{aligned} Ru &= \sum_{j=0}^{m_k-1=1} b_j^{(1)}(t)u_j = b_0^{(1)}u_0 + b_1^{(1)}u_1 \\ u_1 &= (D - \lambda_1)u_0, u = u_0 \end{aligned}$$

So

$$(D_t - \lambda_1 + g_1^{(1)})u_0 = u_1$$
$$(D_t - \lambda_2 + b_1^{(1)} + g_1^{(2)})u_1 = -b_0^{(1)}u_0 + f$$

we can write $n_1 = 2, n_2 = 2,$

The equivalent system is,

$$\begin{cases} (D_t^\beta - \lambda_1)u_0 = u_1 - g_1^{(1)}u_0 \\ (D_t^\beta - \lambda_1)u_1 = u_2 \\ (D_t^\beta - \lambda_1)u_2 = f - Ru - (g_1^{(2)}u_2 + g_0^{(2)}u_1) \\ (D_t^\beta - \lambda_1)u_3 = u_4 \\ (D_t^\beta - \lambda_2)u_4 = u_5 - (\ \tilde{g}_1^{(2)}u_4 + \tilde{g}_0^{(2)}u_3) \\ (D_t^\beta - \lambda_1)u_5 = f - \tilde{R}u - (\tilde{g}_1^{(1)}u_5 + \tilde{g}_0^{(1)}u_4) \end{cases}$$

then $\quad g_1^{(1)} = g_2^{(1)} = 0, g_1^{(2)} = g_2^{(2)} = 0, b_1^{(1)} = b_2^{(1)} = 0, b_1^{(1)} = b_2^{(2)} = 0,$

the system is produced

$$\begin{cases} (D_t^\beta - \lambda_1)u_0 = u_1 \\ (D_t^\beta - \lambda_2)u_1 = u_2 \\ (D_t^\beta - \lambda_2)u_2 = 0 \\ (D_t^\beta - \lambda_2)u_3 = u_4 \\ (D_t^\beta - \lambda_2)u_4 = u_5 \\ (D_t^\beta - \lambda_1)u_5 = 0 \\ \lambda_1 = 2, \lambda_2 = 1 \end{cases}$$

$$\partial_t^{\beta} U = KU + F \tag{3.2.2}$$

with

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \textit{and} \quad F = \begin{pmatrix} 0 \\ e^t \left(t^{\frac{1}{3}} - 2t^{\frac{1}{6}} \right) \\ 0 \\ e^t \left(t^{\frac{1}{3}} - 2t^{\frac{1}{6}} \right) \end{pmatrix},$$

$\det K = 4 \neq 0$, *so the system* (3.2.2) *has a solution.*

We can write(3.2.2) ,*of the form,*

$$\begin{aligned} D_t^{(\frac{1}{3})} u - 2 D_t^{(\frac{1}{6})} u + D_t^{(0)} u &= e^t \left(t^{\frac{1}{3}} - 2 \ t^{\frac{1}{6}} \right) \\ (D_t^{\frac{1}{6}} - 1)(D_t^{\frac{1}{6}} - 1) u &= e^t \left(t^{\frac{1}{3}} - 2 \ t^{\frac{1}{6}} \right) \\ (D_t^{\frac{1}{6}} - 1) u_1 &= e^t \left(t^{\frac{1}{3}} - 2 \ t^{\frac{1}{6}} \right) \\ (D_t^{\frac{1}{6}} - 1) u_0 &= u_1 \\ u_0 &= u \end{aligned}$$

we need only the system,

$$\begin{aligned} D_t^{\frac{1}{6}} u_1(t) &= u_1(t) \\ u_2(t) &= c_1 e^t \\ (D_t^{\frac{1}{6}} - 1) u_1(t) = u_2(t) &= D_t^{\frac{1}{6}} u_1(t) - u_1(t) = c_1 e^t \\ u_1(t) &= -e^t \\ (D_t^{\frac{1}{6}} - 1) u_0(t) = u_1(t) &= D_t^{\frac{1}{6}} u_0(t) - u_0 = -e^t \\ u_0(t) &= e^t \end{aligned} \tag{3.2.3}$$

and just solve the differentials equations of the fractional degree

$$(D_t^{\beta} - \lambda_i) u_i = f_i(t)$$

the solution of (3.2.3) *is :*

The solution of $(D_t^{\frac{1}{6}}-1))u_1 = e^t\left(2t^{\frac{1}{3}} - 2\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{7}{6}\right)}t^{\frac{1}{6}} + \Gamma\left(\frac{4}{3}\right)\right)$, *is*

$$u_1(t) = e^t\left(c_1 + 2\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{3}{2}\right)}\ t^{\frac{1}{2}} - 2\ t^{\frac{1}{3}} + \frac{1}{\Gamma\left(\frac{7}{6}\right)}t^{\frac{1}{6}}\right),$$

so $(D_t^{\frac{1}{6}}-1)u_0 = e^t\left(2\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{3}{2}\right)}\ t^{\frac{1}{2}} - 2\ t^{\frac{1}{3}} + \frac{1}{\Gamma\left(\frac{7}{6}\right)}t^{\frac{1}{6}}\right)$, *is* $u_0(t) = e^t\left(c_2 +\ t^{\frac{1}{3}}\right)$.
So the solution of (3.2.1) *is :*

$$u(t) = e^t\left(c_1 + t^{\frac{1}{6}}c_2 + t^{\frac{1}{3}}\right).$$

Verify that the general solution satisfies the differential equation (3.2.1),

$$\begin{aligned}
D_t^{(\frac{1}{3})}u &= e^t\left(c_1 + t^{\frac{1}{6}}c_2 + t^{\frac{1}{3}} + \Gamma\left(\frac{4}{3}\right)\right)\\
-2D_t^{(\frac{1}{6})}u &= -2e^t\left[c_1 + t^{\frac{1}{6}}c_2 + t^{\frac{1}{3}} + \frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{7}{6}\right)}t^{\frac{1}{6}}\right]\\
u(t) &= e^t\left(c_1 + t^{\frac{1}{6}}c_2 + t^{\frac{1}{3}}\right)\\
D_t^{(\frac{1}{3})}u - 2D_t^{(\frac{1}{6})}u + D_t^{(0)}u &= e^t\begin{pmatrix} \left(c_1 + c_2\ t^{\frac{1}{6}} + t^{\frac{1}{3}}\right)(1-2+1) \\ +2t^{\frac{1}{3}} + \Gamma\left(\frac{4}{3}\right) - 2\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{7}{6}\right)}t^{\frac{1}{6}} \end{pmatrix}\\
&= e^t\left(2t^{\frac{1}{3}} - 2\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{7}{6}\right)}t^{\frac{1}{6}} + \Gamma\left(\frac{4}{3}\right)\right).
\end{aligned}$$

Example 3.2.2 *Let the fractional differential equation, with* $D_t^{\alpha}C = 0$ *and* $D_t^{\alpha}D_t^{\alpha} = D_t^{2\alpha}$

$$\begin{aligned}
& u^{(\frac{1}{3})}(t) - t^{-\frac{1}{6}}\left(\frac{1}{\Gamma\left(\frac{11}{6}\right)} + \Gamma\left(\frac{7}{6}\right)\right)u^{(\frac{1}{6})}(t) + t^{-\frac{1}{3}}\left(\frac{\Gamma\left(\frac{7}{6}\right)}{\Gamma\left(\frac{11}{6}\right)} - \Gamma\left(\frac{7}{6}\right)\frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{4}{6}\right)}\right)u(t)\\
= & \left(\frac{1}{\Gamma\left(\frac{11}{6}\right)} - \frac{\Gamma\left(\frac{11}{6}\right)}{\Gamma\left(\frac{10}{6}\right)}\right)t^{\frac{5}{6}} \qquad (3.2.4)
\end{aligned}$$

we can write,

$$\left[D_t^{\frac{1}{3}} - t^{-\frac{1}{6}}\left(\frac{1}{\Gamma\left(\frac{11}{6}\right)} + \Gamma\left(\frac{7}{6}\right)\right)D_t^{\frac{1}{6}} + t^{-\frac{1}{3}}\left(\frac{\Gamma\left(\frac{7}{6}\right)}{\Gamma\left(\frac{11}{6}\right)} - \Gamma\left(\frac{7}{6}\right)\frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{4}{6}\right)}\right)\right]u\left(t\right)$$
$$= \left(\frac{1}{\Gamma\left(\frac{11}{6}\right)} - \frac{\Gamma\left(\frac{11}{6}\right)}{\Gamma\left(\frac{10}{6}\right)}\right)t^{\frac{5}{6}}$$

is clear $\beta = \frac{1}{6}, m_k = 2,$ *and the function characteristic*

$$\left(D_t^{\frac{1}{6}} - \frac{1}{\Gamma\left(\frac{11}{6}\right)}t^{-\frac{1}{6}}\right)\left(D_t^{\frac{1}{6}} - \Gamma\left(\frac{7}{6}\right)t^{\frac{-1}{6}}\right)u\left(t\right) = \left(\frac{1}{\Gamma\left(\frac{11}{6}\right)} - \frac{\Gamma\left(\frac{11}{6}\right)}{\Gamma\left(\frac{10}{6}\right)}\right)t^{\frac{5}{6}}$$

and let $\lambda_i(t), i = 1, 2,$

$$\lambda_1(t) = \frac{1}{\Gamma\left(\frac{11}{6}\right)}t^{-\frac{1}{6}}, \lambda_2(t) = \Gamma\left(\frac{7}{6}\right)t^{\frac{-1}{6}}$$

also $d = 2$ *and* $m_k = 2,\ n_1 = 1, n_2 = 1 \Rightarrow r = 1, m' = m_k - r, r = 1,$

$$\begin{aligned} P_j &= (D - \lambda_j)^{n_j} + \sum_{i=1}^{n_j} g_i^{(j)}(D - \lambda_j)^{n_j - i}, j = 1, 2..., d. \\ P_1 &= (D - \lambda_1)^2 + g_1^{(1)} + g_2^{(1)}(D - \lambda_1), P_2 = (D - \lambda_2) + g_1^{(2)}, \end{aligned}$$

we can write

$$P_2P_1u = (D_t^{\frac{1}{6}} - \lambda_1(t))(D_t^{\frac{1}{6}} - \lambda_2(t))u = \left(\frac{1}{\Gamma\left(\frac{11}{6}\right)} - \frac{\Gamma\left(\frac{11}{6}\right)}{\Gamma\left(\frac{10}{6}\right)}\right)t^{\frac{5}{6}} = f\left(t\right),$$

so

$$\begin{aligned} Ru &= 0 \\ u &= u_0 \\ u_1 &= (D_t^{\frac{1}{6}} - \lambda_2(t))u_0 \\ (D_t^{\frac{1}{6}} - \lambda_1(t))u_1 &= \left(\frac{1}{\Gamma\left(\frac{11}{6}\right)} - \frac{\Gamma\left(\frac{11}{6}\right)}{\Gamma\left(\frac{10}{6}\right)}\right)t^{\frac{5}{6}}. \end{aligned}$$

And

$$\begin{aligned} P_1P_2u &= (D_t^{\frac{1}{6}} - \lambda_2(t))(D_t^{\frac{1}{6}} - \lambda_1(t))u = \left(\frac{1}{\Gamma\left(\frac{11}{6}\right)} - \frac{\Gamma\left(\frac{11}{6}\right)}{\Gamma\left(\frac{10}{6}\right)}\right)t^{\frac{5}{6}} \\ &= f(t) - D_t^{\frac{1}{6}}\left(\lambda_2(t) - \lambda_1(t)\right) \\ &= f(t) - \left(\Gamma\left(\frac{7}{6}\right) - \frac{1}{\Gamma\left(\frac{11}{6}\right)}\right)\frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{4}{6}\right)}t^{-\frac{1}{3}} \end{aligned}$$

so,

$$\begin{aligned} \tilde{R} &= \left(\Gamma\left(\frac{7}{6}\right) - \frac{1}{\Gamma\left(\frac{11}{6}\right)}\right)\frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{4}{6}\right)}t^{-\frac{1}{3}} \\ u &= u_2 \\ u_3 &= (D_t^{\frac{1}{6}} - \lambda_1(t))u_2 \\ (D_t^{\frac{1}{6}} - \lambda_1(t))u_3 &= f(t) - \tilde{R}. \end{aligned}$$

The equivalent system is

$$\left\{\begin{array}{c} (D_t^{\beta} - \lambda_1)u_0 = u_1 - g_1^{(1)}u_0 \\ (D_t^{\beta} - \lambda_2)u_1 = u_2 \\ (D_t^{\beta} - \lambda_2)u_2 = f - Ru - (g_1^{(2)}u_2 + g_0^{(2)}u_1) \\ (D_t^{\beta} - \lambda_2)u_3 = u_4 \\ (D_t^{\beta} - \lambda_2)u_4 = u_5 - (\ \tilde{g}_1^{(2)}u_4 + \tilde{g}_0^{(2)}u_3) \\ (D_t^{\beta} - \lambda_1)u_5 = f - \tilde{R}u - (\tilde{g}_1^{(1)}u_5 + \tilde{g}_0^{(1)}u_4) \end{array}\right. \quad t$$

hen $g_1^{(1)} = g_2^{(1)} = g_1^{(2)} = g_2^{(2)} = 0$

$$\left\{\begin{array}{c} (D_t^{\beta} - \lambda_1)u_0 = u_1 \\ (D_t^{\beta} - \lambda_2)u_1 = f(t) \\ (D_t^{\beta} - \lambda_2)u_2 = u_3 \\ (D_t^{\beta} - \lambda_2)u_3 = f(t) - \left(\Gamma\left(\frac{7}{6}\right) - \frac{1}{\Gamma\left(\frac{11}{6}\right)}\right)\frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{4}{6}\right)}t^{-\frac{1}{3}} \end{array}\right.$$

$$\partial_t^{\beta}U = K(t)U + F \tag{3.2.5}$$

$$with\ \ K=\begin{pmatrix} \frac{1}{\Gamma\left(\frac{11}{6}\right)}t^{-\frac{1}{6}} & 1 & 0 & 0 \\ 0 & \Gamma\left(\frac{7}{6}\right)t^{-\frac{1}{6}} & 0 & 0 \\ 0 & 0 & \Gamma\left(\frac{7}{6}\right)t^{-\frac{1}{6}} & 1 \\ 0 & 0 & 0 & \frac{1}{\Gamma\left(\frac{11}{6}\right)}t^{-\frac{1}{6}} \end{pmatrix},$$

$$and\ \ F=\begin{pmatrix} 0 \\ f(t) \\ 0 \\ f(t)-\left(\Gamma\left(\frac{7}{6}\right)-\frac{1}{\Gamma\left(\frac{11}{6}\right)}\right)\frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{4}{6}\right)}t^{-\frac{1}{3}} \end{pmatrix}.$$

$\det K=\left(\frac{\Gamma\left(\frac{7}{6}\right)}{\Gamma\left(\frac{11}{6}\right)}t^{-\frac{1}{3}}-1\right)\left(\frac{\Gamma\left(\frac{7}{6}\right)}{\Gamma\left(\frac{11}{6}\right)}t^{-\frac{1}{3}}\right)\neq 0$ *if* $t>\left(\frac{\Gamma\left(\frac{7}{6}\right)}{\Gamma\left(\frac{11}{6}\right)}\right)^3$, *so the system* (3.2.5) *has a solution.*

We can write (3.2.5) ,*of the form,*

$$\left(D_t^{\frac{1}{3}}-t^{-\frac{1}{6}}\left(\frac{1}{\Gamma\left(\frac{10}{6}\right)}+\Gamma\left(\frac{7}{6}\right)\right)D_t^{\frac{1}{6}}+t^{\frac{-1}{3}}\left(\frac{\Gamma\left(\frac{7}{6}\right)}{\Gamma\left(\frac{10}{6}\right)}-\Gamma\left(\frac{7}{6}\right)\frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{4}{6}\right)}t^{\frac{-1}{6}}\right)\right)u(t)=\left(\frac{\Gamma\left(\frac{11}{6}\right)-1}{\Gamma\left(\frac{10}{6}\right)}\right)t^{\frac{4}{6}}$$

$$\left(D_t^{\frac{1}{6}}-\frac{1}{\Gamma\left(\frac{10}{6}\right)}t^{-\frac{1}{6}}\right)\left(D_t^{\frac{1}{6}}-\Gamma\left(\frac{7}{6}\right)t^{\frac{-1}{6}}\right)u(t)=\left(\frac{\Gamma\left(\frac{11}{6}\right)-1}{\Gamma\left(\frac{10}{6}\right)}\right)t^{\frac{4}{6}}$$

$$\begin{aligned} \left(D_t^{\frac{1}{6}}-\frac{1}{\Gamma\left(\frac{10}{6}\right)}t^{-\frac{1}{6}}\right)u_1&=At^{\frac{4}{6}} \\ \left(D_t^{\frac{1}{6}}-\Gamma\left(\frac{7}{6}\right)t^{\frac{-1}{6}}\right)u_0&=u_1 \\ u_0&=u \end{aligned} \tag{3.2.6}$$

and just solve the differentials equations of the fractional degree

$$(D_t^{\beta}-\lambda_i)u_i=f_i(t)$$

The solution of(3.2.6) *is :*

The general solution of the homogeneous solutions of

$$\frac{D_t^{\frac{1}{6}}u_{1,h}(t)}{u_{1,h}(t)}=\frac{1}{\Gamma\left(\frac{11}{6}\right)}t^{-\frac{1}{6}}=\frac{\frac{\Gamma(2)}{\Gamma\left(\frac{11}{6}\right)}t^{1-\frac{1}{6}}}{t}=\frac{D_t^{\frac{1}{6}}t}{t}$$

$$u_{1,h}(t)=c_1t$$

The particular solution of

$$\left(D_t^{\frac{1}{6}} - \frac{1}{\Gamma\left(\frac{10}{6}\right)} t^{-\frac{1}{6}} \right) u_{1,p}(t) = \left(\frac{1}{\Gamma\left(\frac{10}{6}\right)} - \frac{1}{\Gamma^2\left(\frac{11}{6}\right)} - \Gamma\left(\frac{7}{6}\right) \frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{4}{6}\right)} \right) t^{\frac{4}{6}} = A t^{\frac{4}{6}},$$

$$\begin{aligned} u_{1,p}(t) &= c_1(t)\, t \\ c_1(t) &= D_t^{-\frac{1}{6}} A t^{\frac{-2}{6}} = A \frac{\Gamma\left(\frac{4}{6}\right)}{\Gamma\left(\frac{5}{6}\right)} t^{\frac{-1}{6}} \end{aligned}$$

so

$$u_{1,p}(t) = u_{1,p}(t) = A \frac{\Gamma\left(\frac{10}{6}\right)}{\Gamma\left(\frac{11}{6}\right) - 1} t^{\frac{5}{6}},$$

, is $u_1(t) = c_1 t + A \frac{\Gamma\left(\frac{10}{6}\right)}{\Gamma\left(\frac{11}{6}\right)-1} t^{\frac{5}{6}}$,

$$\begin{aligned} \left(D_t^{\frac{1}{6}} - \frac{1}{\Gamma\left(\frac{10}{6}\right)} t^{-\frac{1}{6}} \right) u_{1,p}(t) &= \left(D_t^{\frac{1}{6}} - \frac{1}{\Gamma\left(\frac{10}{6}\right)} t^{-\frac{1}{6}} \right) \left(c_1 t + A \frac{\Gamma\left(\frac{10}{6}\right)}{\Gamma\left(\frac{11}{6}\right) - 1} t^{\frac{5}{6}} \right) \\ &= D_t^{\frac{1}{6}} \left(c_1 t + A \frac{\Gamma\left(\frac{4}{6}\right)}{\Gamma\left(\frac{5}{6}\right)} t^{\frac{5}{6}} \right) - \frac{1}{\Gamma\left(\frac{10}{6}\right)} t^{-\frac{1}{6}} \begin{pmatrix} A \frac{\Gamma\left(\frac{10}{6}\right)}{\Gamma\left(\frac{11}{6}\right)-1} t^{\frac{5}{6}} \\ + c_1 t \end{pmatrix} \\ &= c_1 \left(D_t^{\frac{1}{6}} t - \frac{1}{\Gamma\left(\frac{10}{6}\right)} t^{\frac{5}{6}} \right) + A \frac{\Gamma\left(\frac{10}{6}\right)}{\Gamma\left(\frac{11}{6}\right) - 1} \left(D_t^{\frac{1}{6}} t^{\frac{5}{6}} - \frac{1}{\Gamma\left(\frac{10}{6}\right)} t^{\frac{4}{6}} \right) \\ &= c_1 \left(\frac{1}{\Gamma\left(\frac{10}{6}\right)} t^{\frac{5}{6}} - \frac{1}{\Gamma\left(\frac{10}{6}\right)} t^{\frac{5}{6}} \right) \\ &\quad + A \frac{\Gamma\left(\frac{10}{6}\right)}{\Gamma\left(\frac{11}{6}\right) - 1} \left(\frac{\Gamma\left(\frac{10}{6}\right)}{\Gamma\left(\frac{10}{6}\right)} t^{\frac{4}{6}} - \frac{1}{\Gamma\left(\frac{10}{6}\right)} t^{\frac{4}{6}} \right) \\ &= A \left(\frac{\Gamma\left(\frac{10}{6}\right)}{\Gamma\left(\frac{11}{6}\right) - 1} \right) \left(\frac{\Gamma\left(\frac{11}{6}\right) - 1}{\Gamma\left(\frac{10}{6}\right)} \right) t^{\frac{4}{6}} = A t^{\frac{4}{6}} \end{aligned}$$

and $\left(D_t^{\frac{1}{6}} - \Gamma\left(\frac{7}{6}\right) t^{\frac{-1}{6}} \right) u_0(t) = A \frac{\Gamma\left(\frac{10}{6}\right)}{\Gamma\left(\frac{11}{6}\right)-1} t^{\frac{5}{6}}$ *is :*
and the particular solution of $\left(D_t^{\frac{1}{6}} - \Gamma\left(\frac{7}{6}\right) t^{\frac{-1}{6}} \right) u_0(t) = A \frac{\Gamma\left(\frac{10}{6}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{11}{6}\right)-1} t^{\frac{5}{6}} = u_{0,p_1}(t)$, *so*

$$\begin{aligned} D_t^{\frac{1}{6}} \left(\frac{u_{0,h}(t)}{t^{\frac{1}{6}}} \right) &= 0 \\ u_{0,h}(t) &= c_2 t^{\frac{1}{6}}, \end{aligned}$$

so

$$\begin{aligned} u_{0,p}(t) &= c_2(t)\,t^{\frac{1}{6}} \\ c_2(t) &= D_t^{\frac{-1}{6}}\left(t^{\frac{-1}{6}}\left(A\frac{\Gamma\left(\frac{10}{6}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{11}{6}\right)-1}t^{\frac{5}{6}}\right)\right) \\ &= D_t^{\frac{-1}{6}}\left(A\frac{\Gamma\left(\frac{10}{6}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{11}{6}\right)-1}t^{\frac{4}{6}}\right) \\ &= A\frac{\Gamma\left(\frac{10}{6}\right)\Gamma\left(\frac{5}{6}\right)}{\left(\Gamma\left(\frac{11}{6}\right)-1\right)}t^{\frac{5}{6}}, \end{aligned}$$

$$u_0(t) = c_2 t^{\frac{1}{6}} + A\frac{\Gamma\left(\frac{10}{6}\right)\Gamma\left(\frac{5}{6}\right)}{\left(\Gamma\left(\frac{11}{6}\right)-1\right)}t,$$

$$\begin{aligned} \left(D_t^{\frac{1}{6}} - \Gamma\left(\frac{7}{6}\right)t^{\frac{-1}{6}}\right)u_{0,p}(t) &= \left(D_t^{\frac{1}{6}} - \Gamma\left(\frac{7}{6}\right)t^{\frac{-1}{6}}\right)\begin{pmatrix} A\frac{\Gamma\left(\frac{10}{6}\right)\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{11}{6}\right)}{\left(\Gamma\left(\frac{11}{6}\right)-1\right)\left(1-\Gamma\left(\frac{11}{6}\right)\Gamma\left(\frac{7}{6}\right)\right)}t \\ +c_2 t^{\frac{1}{6}} \end{pmatrix} \\ &= c_2\left(D_t^{\frac{1}{6}} - \Gamma\left(\frac{7}{6}\right)t^{\frac{-1}{6}}\right)t \\ &\quad +A\frac{\Gamma\left(\frac{10}{6}\right)\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{11}{6}\right)}{\left(\Gamma\left(\frac{11}{6}\right)-1\right)\left(1-\Gamma\left(\frac{11}{6}\right)\Gamma\left(\frac{7}{6}\right)\right)}\left(D_t^{\frac{1}{6}}t - \Gamma\left(\frac{7}{6}\right)t^{\frac{5}{6}}\right) \\ &= c_2\left(\Gamma\left(\frac{7}{6}\right) - \Gamma\left(\frac{7}{6}\right)\right) \\ &\quad +A\frac{\Gamma\left(\frac{10}{6}\right)\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{11}{6}\right)}{\left(\Gamma\left(\frac{11}{6}\right)-1\right)\left(1-\Gamma\left(\frac{11}{6}\right)\Gamma\left(\frac{7}{6}\right)\right)}\left(\frac{1}{\Gamma\left(\frac{11}{6}\right)} - \Gamma\left(\frac{7}{6}\right)\right)t^{\frac{5}{6}} \\ &= A\frac{\Gamma\left(\frac{10}{6}\right)\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{11}{6}\right)}{\left(\Gamma\left(\frac{11}{6}\right)-1\right)\left(1-\Gamma\left(\frac{11}{6}\right)\Gamma\left(\frac{7}{6}\right)\right)}\frac{1-\Gamma\left(\frac{11}{6}\right)\Gamma\left(\frac{7}{6}\right)}{\Gamma\left(\frac{11}{6}\right)}t^{\frac{5}{6}} \\ &= A\frac{\Gamma\left(\frac{10}{6}\right)\Gamma\left(\frac{5}{6}\right)}{\left(\Gamma\left(\frac{11}{6}\right)-1\right)}t^{\frac{5}{6}} \end{aligned}$$

we can write,

$$u(t) = c_2 t^{\frac{1}{6}} + c_1\frac{1}{\Gamma\left(\frac{7}{6}\right)}t^{\frac{7}{6}} + t,$$

Verify that the general solution satisfies the fractional differential equation (3.2.4),

$$\left(D_t^{\frac{1}{3}} - t^{-\frac{1}{6}}\left(\frac{1}{\Gamma\left(\frac{11}{6}\right)} + \Gamma\left(\frac{7}{6}\right)\right)D_t^{\frac{1}{6}} + t^{\frac{-1}{3}}\left(\frac{\Gamma\left(\frac{7}{6}\right)}{\Gamma\left(\frac{11}{6}\right)} - \Gamma\left(\frac{7}{6}\right)\frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{4}{6}\right)}\right)\right)u(t) = At^{\frac{4}{6}}$$

$$
\begin{aligned}
& c_2\left(D_t^{\frac{1}{3}}-t^{-\frac{1}{6}}\left(\frac{1}{\Gamma\left(\frac{11}{6}\right)}+\Gamma\left(\frac{7}{6}\right)\right)D_t^{\frac{1}{6}}+t^{\frac{-1}{3}}\left(\frac{\Gamma\left(\frac{7}{6}\right)}{\Gamma\left(\frac{11}{6}\right)}-\Gamma\left(\frac{7}{6}\right)\frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{4}{6}\right)}t^{\frac{-1}{6}}\right)\right)t^{\frac{1}{6}} \\
= & c_2\left(D_t^{\frac{1}{3}}t^{\frac{1}{6}}-t^{-\frac{1}{6}}\left(\frac{1}{\Gamma\left(\frac{11}{6}\right)}+\Gamma\left(\frac{7}{6}\right)\right)D_t^{\frac{1}{6}}t^{\frac{1}{6}}+t^{\frac{-1}{3}}\left(\frac{\Gamma\left(\frac{7}{6}\right)}{\Gamma\left(\frac{11}{6}\right)}-\Gamma\left(\frac{7}{6}\right)\frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{4}{6}\right)}\right)\right)t^{\frac{1}{6}} \\
= & c_2\left(\frac{1}{\Gamma\left(\frac{5}{6}\right)}-\left(\frac{1}{\Gamma\left(\frac{11}{6}\right)}+\Gamma\left(\frac{7}{6}\right)\right)\Gamma\left(\frac{7}{6}\right)+\left(\frac{\Gamma\left(\frac{7}{6}\right)}{\Gamma\left(\frac{11}{6}\right)}-\Gamma\left(\frac{7}{6}\right)\frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{4}{6}\right)}\right)\right)t^{\frac{-1}{6}} \\
= & c_2\left(\frac{1}{\Gamma\left(\frac{5}{6}\right)}-\Gamma\left(\frac{7}{6}\right)\Gamma\left(\frac{7}{6}\right)-\Gamma\left(\frac{7}{6}\right)\frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{4}{6}\right)}\right)t^{\frac{-1}{6}} \\
= & 0,
\end{aligned}
$$

so

$$
\begin{aligned}
& \left(D_t^{\frac{1}{3}}-t^{-\frac{1}{6}}\left(\frac{1}{\Gamma\left(\frac{11}{6}\right)}+\Gamma\left(\frac{7}{6}\right)\right)D_t^{\frac{1}{6}}+t^{\frac{-1}{3}}\left(\frac{\Gamma\left(\frac{7}{6}\right)}{\Gamma\left(\frac{11}{6}\right)}-\Gamma\left(\frac{7}{6}\right)\frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{4}{6}\right)}\right)\right)t \\
= & \left(\frac{1}{\Gamma\left(\frac{10}{6}\right)}-\left(\frac{1}{\Gamma\left(\frac{11}{6}\right)}+\Gamma\left(\frac{7}{6}\right)\right)\frac{1}{\Gamma\left(\frac{11}{6}\right)}+\left(\frac{\Gamma\left(\frac{7}{6}\right)}{\Gamma\left(\frac{11}{6}\right)}-\Gamma\left(\frac{7}{6}\right)\frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{4}{6}\right)}\right)\right)t^{\frac{5}{6}} \\
= & \left(\frac{1}{\Gamma\left(\frac{10}{6}\right)}-\left(\frac{1}{\Gamma\left(\frac{11}{6}\right)}\right)\frac{1}{\Gamma\left(\frac{11}{6}\right)}-\Gamma\left(\frac{7}{6}\right)\frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{4}{6}\right)}\right)t^{\frac{5}{6}} \\
= & At^{\frac{5}{6}}
\end{aligned}
$$

Example 3.2.3 *Let the fractional differential equation ,with* $D_t^{\alpha}C \neq 0$ *and* $D_t^{\alpha}D_t^{\alpha} \neq D_t^{2\alpha}$*,with* $u(a) = c \neq 0$ *is non-constant,*

$$
u^{(1)}-\left(\frac{2}{\Gamma\left(\frac{1}{2}\right)}+\Gamma\left(\frac{1}{2}\right)\right)t^{-\frac{1}{2}}u^{\left(\frac{1}{2}\right)}-\frac{1}{2}t^{-1}\left(\Gamma^2\left(\frac{1}{2}\right)+1\right)u=-\frac{u(a)}{\Gamma\left(\frac{1}{2}\right)}t^{-\frac{1}{2}} \qquad (3.2.7)
$$

we can write,

$$\begin{aligned} & D_t u-\left(\frac{2}{\Gamma\left(\frac{1}{2}\right)}+\Gamma\left(\frac{1}{2}\right)\right) t^{-\frac{1}{2}} D_t^{\frac{1}{2}} u-\frac{1}{2} t^{-1}\left(\Gamma^2\left(\frac{1}{2}\right)+1\right) u=-\frac{u(a)}{\Gamma\left(\frac{1}{2}\right)} t^{-\frac{1}{2}}, \\ & D_t^{\frac{1}{2}+\frac{1}{2}} u+\frac{u(a)}{\Gamma\left(\frac{1}{2}\right)} t^{-\frac{1}{2}}-2\left(\frac{1}{\Gamma\left(\frac{1}{2}\right)}+\Gamma\left(\frac{3}{2}\right)\right) t^{-\frac{1}{2}} D_t^{\frac{1}{2}} u-\frac{1}{2} t^{-1}\left(\Gamma^2\left(\frac{1}{2}\right)+1\right) u \\ = & D_t^{\frac{1}{2}}\left(D_t^{\frac{1}{2}} u\right)-2 \lambda_1 D_t^{\frac{1}{2}} u+\left(\lambda_1 \lambda_1-D_t^{\frac{1}{2}} \lambda_1\right) u \\ = & \left(D_t^{\frac{1}{2}}-\lambda_1\right)\left(D_t^{\frac{1}{2}}-\lambda_1\right) u=0 \end{aligned}$$

is clear $\beta=\frac{1}{2}, m_k=1$,*with*

$$\lambda_1=\lambda_2=\left(\frac{1}{\Gamma\left(\frac{1}{2}\right)}+\Gamma\left(\frac{3}{2}\right)\right) t^{-\frac{1}{2}}=\left(\frac{1}{\Gamma\left(\frac{1}{2}\right)}+\frac{1}{2} \Gamma\left(\frac{1}{2}\right)\right) t^{-\frac{1}{2}},$$

and

$$\begin{aligned} \lambda_1 \lambda_1-D_t^{\frac{1}{2}} \lambda_1 & =t^{-1}\left(\frac{1}{\Gamma^2\left(\frac{1}{2}\right)}+1+\frac{1}{4} \Gamma^2\left(\frac{1}{2}\right)-\frac{1}{\Gamma^2\left(\frac{1}{2}\right)}-\frac{1}{2} \Gamma^2\left(\frac{1}{2}\right)-\frac{3}{2}\right) \\ & =-\frac{1}{2} t^{-1}\left(\Gamma^2\left(\frac{1}{2}\right)+1\right), \end{aligned}$$

also $d=2$ *and* $m_k=3$, $n_1=2$, *and* $R=0$.

So

$$\begin{cases} (D_t^{\beta}-\lambda_1)u_0=u_1 \\ (D_t^{\beta}-\lambda_1)u_1=0 \\ (D_t^{\beta}-\lambda_1)u_2=u_3 \\ (D_t^{\beta}-\lambda_1)u_3=0 \end{cases}$$

The equivalent system is,

$$\partial_t^{\frac{1}{2}} U=KU \tag{3.2.8}$$

with

$$K=\begin{pmatrix}\lambda_1 & 0 & 0 & 0\\ 0 & \lambda_1 & 0 & 0\\ 0 & 0 & \lambda_1 & 0\\ 0 & 0 & 0 & \lambda_1\end{pmatrix}, \textit{and} \quad F=\begin{pmatrix}0\\0\\0\\0\end{pmatrix},$$

$\det K = 4 \neq 0$, *so the system* (3.2.2) *has a solution.*

We can write (3.2.2) ,*of the form,*

$$D_t^{2\frac{1}{2}}u-\left(\frac{2}{\Gamma\left(\frac{1}{2}\right)}+\Gamma\left(\frac{1}{2}\right)\right)t^{-\frac{1}{2}}D_t^{\frac{1}{2}}u-\frac{1}{2}t^{-1}\left(\Gamma^2\left(\frac{1}{2}\right)+1\right)u=$$

$$u^{(1)}-\left(\tfrac{2}{\Gamma\left(\frac{1}{2}\right)}-\Gamma\left(\tfrac{1}{2}\right)\right)t^{-\frac{1}{2}}u^{\left(\frac{1}{2}\right)}-\tfrac{1}{2}t^{-1}\left(\Gamma^2\left(\tfrac{1}{2}\right)+1\right)u=$$
$$\left(D_t^{\frac{1}{2}}-\lambda_1\right)\left(D_t^{\frac{1}{2}}-\lambda_1\right)u=0$$
$$(D_t^{\frac{1}{2}}-\lambda_1)u_1=0$$
$$\left(D_t^{\frac{1}{2}}-\lambda_1\right)u_0=u_1$$
$$u_0=u$$

we need only the system,

$$\begin{aligned} D_t^{\frac{1}{2}}u_1(t)&=\lambda_1u_1(t)\\ D_t^{\frac{1}{2}}u_0(t)&=\lambda_1u_0(t)\end{aligned} \tag{3.2.9}$$

and just solve the differentials equations of the fractional degree

$$(D_t^{\frac{1}{2}}-\lambda_1)u_i=0$$

the solution of(3.2.9) *is :*

The solution of $(D_t^{\frac{1}{2}}-\lambda_1)^2u_1=0$, *whit* $\lambda_1=\left(\frac{1}{\Gamma\left(\frac{1}{2}\right)}+\Gamma\left(\frac{3}{2}\right)\right)t^{-\frac{1}{2}}$, *we have* $\frac{D_t^{\alpha}t^{\beta}}{t^{\beta}}=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{-\alpha}$, *and* $\frac{D_t^{\alpha}\left(t^{\frac{1}{2}}+1\right)}{\left(t^{\frac{1}{2}}+1\right)}=\lambda_1$, *we get* $u_1(t)=c_1\left(t^{\frac{1}{2}}+1\right)$ *and for* $n_1=2$, $u_2(t)=c_2t\left(t^{\frac{1}{2}}+1\right)$.

So the solution of (3.2.7) *is :*

$$u(t) = \left(t^{\frac{1}{2}} + 1\right)\left(t^{\frac{1}{2}} c_2 + c_1\right).$$

Verify that the general solution satisfies the differential equation (3.1.17),

$$\begin{aligned}
& D_t c t^{\frac{1}{2}} - \left(\frac{2}{\Gamma\left(\frac{1}{2}\right)} - \Gamma\left(\frac{1}{2}\right)\right) t^{-\frac{1}{2}} D_t^{\frac{1}{2}} c t^{\frac{1}{2}} - \frac{1}{2} t^{-1}\left(-\frac{2}{\Gamma\left(\frac{1}{2}\right)} + \Gamma^2\left(\frac{1}{2}\right) + 1\right) c t^{\frac{1}{2}} \\
= \; & c t^{-\frac{1}{2}}\left[\frac{1}{2} - \left(\frac{2}{\Gamma\left(\frac{1}{2}\right)} + \Gamma\left(\frac{1}{2}\right)\right)\left(\frac{1}{\Gamma\left(\frac{1}{2}\right)} - \frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right) + \frac{2}{\Gamma\left(\frac{1}{2}\right)} - \frac{1}{2}\Gamma^2\left(\frac{1}{2}\right) - \frac{1}{2}\right] \\
= \; & c t^{-\frac{1}{2}}\left[\frac{1}{2} - \left(\frac{2}{\Gamma\left(\frac{1}{2}\right)} + \Gamma\left(\frac{1}{2}\right)\right)\left(\frac{1}{\Gamma\left(\frac{1}{2}\right)} - \frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right) + \frac{2}{\Gamma\left(\frac{1}{2}\right)} - \frac{1}{2}\Gamma^2\left(\frac{1}{2}\right) - \frac{1}{2}\right] \\
= \; & c t^{-\frac{1}{2}}[0] + \frac{2}{\Gamma\left(\frac{1}{2}\right)} t^{-\frac{1}{2}}(c - u(a)) = 0
\end{aligned}$$

3.2.2 The fractional differential equations with remainder existence R≠0:

Example 3.2.4 *In this example, let the differential equation with* $D_t^{\alpha} C = 0$, *and* $R \neq 0$,

$$u^{\left(\frac{1}{2}\right)} - 2u^{\left(\frac{1}{4}\right)} + 2u = \sqrt{2}, \tag{3.2.10}$$

$$\begin{aligned}
u^{\left(\frac{1}{2}\right)} - 2u^{\left(\frac{1}{4}\right)} + 2u &= \sqrt{2} \\
D_t^{2\frac{1}{4}} u - 2D_t^{\frac{1}{2}} u + 2u &= \sqrt{2} \\
\left(D_t^{\frac{1}{4}} - \lambda_1\right)\left(D_t^{\frac{1}{4}} - \lambda_2\right) u &= \sqrt{2}
\end{aligned}$$

write $D_t^{2\frac{1}{4}} u - 2D_t^{\frac{1}{2}} u + 2u = \sqrt{2}$, *is clear* $m_k = 2$, *we put*

$$\lambda_1(t) = \alpha(t) + i\beta(t) = \bar{\lambda}_2(t)$$

so

$$\begin{aligned} \lambda_1 &= 1+i=\bar{\lambda}_2(t), \\ \lambda_1+\lambda_2 &= 2, \\ \lambda_1\lambda_2 &= (1+i)(1-i)=2, \end{aligned}$$

and the function characteristic

$$\begin{aligned} & D_t^{2\frac{1}{4}}u-2D_t^{\frac{1}{2}}u+2u \\ = & \left(D_t^{\frac{1}{4}}-\lambda_1\right)\left(D_t^{\frac{1}{4}}-\lambda_2\right)u=\sqrt{2} \end{aligned}$$

and so $d=2$, $n_1=1, n_2=1$, *and*

$$D_t^{\frac{1}{4}}\lambda_1=0,$$

so,

$$(D_t^{\frac{1}{4}}-\lambda_1(t))(D_t^{\frac{1}{4}}-\lambda_2(t))u=f$$

and

$$\begin{aligned} (D_t^{\frac{1}{4}}-\lambda_2(t))u_0 &= u_1 \\ \left(D_t^{\frac{1}{4}}-\lambda_1(t)\right)u_1 &= \sqrt{2} \end{aligned}$$

the system is produced

$$\partial_t^{\frac{1}{4}}U-K(t)U=F \tag{3.2.11}$$

with

$$F=\begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \\ \sqrt{2} \end{pmatrix}, \text{ and } K=\begin{pmatrix} \lambda_1(t) & 0 & 0 & 0 \\ 0 & \lambda_2(t) & 0 & 0 \\ 0 & 0 & \lambda_2(t) & 0 \\ 0 & 0 & 0 & \lambda_1(t) \end{pmatrix}$$

, $\det K \neq 0$, *so the system*(3.2.11) *has a solution.*

We need only the system,

$$\begin{cases} u_0 = u \\ \left(D_t^{\frac{1}{4}} - \lambda_1(t)\right) u_1 = \sqrt{2} \\ (D_t^{\frac{1}{4}} - \lambda_2(t))u_0 = u_1 \end{cases} \tag{3.2.12}$$

and just solve the differential equation of the first degree

$$(D_t^{\frac{1}{4}} - g_i(t))u_i = f_i(t)$$

The solution of (3.2.12) *is : We have* $D_t^{\alpha} e^{\lambda t} = \lambda^{\alpha} e^{\lambda t}$

The solution of $\left(D_t^{\frac{1}{4}} - \lambda_1\right) u_1 = \sqrt{2}$, $\frac{D_t^{\frac{1}{4}} e^{\lambda t}}{e^{\lambda t}} = \lambda^{\frac{1}{4}} = \lambda_1$, *we get* $\lambda = \lambda_1^4 = -4$, *and* $D_t^{\frac{1}{4}} c_1(t) = \sqrt{2} e^{4t} = D_t^{\frac{1}{4}} e^{4t}$,

so $u_1(t) = c_1 e^{-4t} + c_1(t) e^{-4t} = c_1 e^{-4t} + 1$,

and $(D_t^{\frac{1}{4}} - \lambda_2)u_0 = 1$, *we have* $D_t^{\frac{1}{4}} c_2(t) = e^{4t} = \frac{1}{\sqrt{2}} D_t^{\frac{1}{4}} e^{4t}$, *so* $u_0(t) = c_2 e^{-4t} + \frac{1}{\sqrt{2}}$,

So the solution of(3.2.10) *is :* $u(t) = ce^{-4t} + \frac{1}{\sqrt{2}}$;

Verify that the general solution satisfies the differential equation(3.2.10) ,

$$\begin{aligned}
& D_t^{\frac{1}{2}}\left(ce^{-4t} + \frac{1}{\sqrt{2}}\right) - 2D_t^{\frac{1}{4}}\left(ce^{-4t} + \frac{1}{\sqrt{2}}\right) + 2\left(ce^{-4t} + \frac{1}{\sqrt{2}}\right) \\
= \; & D_t^{\frac{1}{2}}\left(ce^{-4t} + \frac{1}{\sqrt{2}}\right) - 2D_t^{\frac{1}{4}}\left(ce^{-4t} + \frac{1}{\sqrt{2}}\right) + 2\left(ce^{-4t} + \frac{1}{\sqrt{2}}\right) \\
= \; & D_t^{\frac{1}{2}}\left(ce^{-4t}\right) - 2D_t^{\frac{1}{4}}\left(ce^{-4t}\right) + 2\left(ce^{-4t}\right) \\
& + D_t^{\frac{1}{2}}\left(\frac{1}{\sqrt{2}}\right) - 2D_t^{\frac{1}{4}}\left(\frac{1}{\sqrt{2}}\right) + 2\left(\frac{1}{\sqrt{2}}\right) \\
= \; & ce^{-4t}\left[(1+i)^2 - 2(1+i) + 2\right] + \sqrt{2} \\
= \; & ce^{-4t}[0] + \sqrt{2} = \sqrt{2}
\end{aligned}$$

Chapitre 4

Numerical solution of fractional differential equations

If we cannot set the general solution analytically, we will assign a numerical solution in the vicinity of t using the interpolations in the vicinity of t_0 to a solution in the polynomial form.

Solving differential equations of fractional order in an accurate, reliable and efficient way is much more difficult than in the standard integer-order case; moreover, the majority of the computational tools do not provide built-in functions for this kind of problem. The increasing interest in applications of fractional calculus has motivated the development and the investigation of numerical methods specifically devised to solve fractional differential equations (FDEs). Finding analytical solutions of FDEs is, indeed, even more, difficult than solving standard ordinary differential equations (ODEs), in the majority of cases, it is only possible to provide a numerical approximation of the solution.

Although several computing environments (such as, for instance, Maple, Mathematica, MATLAB and Python) provide robust and easy-to-use codes for numerically solving ODEs, the solution of FDEs still seems not to have been addressed by almost all computational tools, and usually, researchers have to write codes by themselves for the numerical treatment of FDEs.

In this section we will address both :

1- The approximate numerical solution (y_i) of differential equations $F\left(t, D_t^{\beta} u\right) = 0$ using local fractional

2- Using the previous values (y_i) we will build a solution $u(t) = cP_{\alpha}(t)$ using interpolation fractional polynomials.

4.0.3 Numerical methods

Our main concern here is to give a numerical scheme to solve a fractional differential equation $(0 < \alpha \leqslant 1)$.

The use of local fractional orders differential and integral operators in mathematical models has become increasingly widespread in recent years (see ([11]) and ([13])). Several forms of local fractional differential equations have been proposed in standard models, and there has been significant interest in developing numerical schemes for their solution.

We shall be concerned with the construction and the analysis of numerical methods for :

Calculating of the values local fractional integral by some method to approximate

$$\begin{cases} {}_{t_0}D_t^{\alpha}u(t) - \lambda(t)u(t) = g(t), \\ u \mid_{t_0} = a_0, \end{cases} \tag{4.0.1}$$

with $0 < \alpha \leq 1$.

Local fractional derivatives

We use the following local fractional derivatives definitions [11],

$$ {}_{x_0}D_x^{\alpha} y\left(x\right) = y^{(\alpha)}(x_0) = \frac{d^{\alpha} y\left(x\right)}{dx^{\alpha}} \lfloor_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^{\alpha}\left(y\left(x\right) - y\left(x_0\right)\right)}{\left(x - x_0\right)^{\alpha}}, \tag{4.0.2}$$

with

$$\Delta^{\alpha}\left(y\left(x\right) - y\left(x_0\right)\right) = \Gamma\left(\alpha + 1\right)\Delta\left(y\left(x\right) - y\left(x_0\right)\right), \quad 0 < \alpha \leq 1. \tag{4.0.3}$$

** We propose a new practical definition for the local derivative

$$ {}_{x_0}D_x^{\alpha} y\left(x\right) = y^{(\alpha)}(x_0) = \frac{d^{\alpha} y\left(x\right)}{dx^{\alpha}} \lfloor_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^{\alpha}\left(y\left(x\right) - y\left(x_0\right)\right)}{x^{\alpha} - x_0^{\alpha}}, \tag{4.0.4}$$

with

$$\Delta^{\alpha}\left(y\left(x\right) - y\left(x_0\right)\right) = \Gamma\left(\alpha + 1\right)\Delta\left(y\left(x\right) - y\left(x_0\right)\right), dx^{\alpha} = x^{\alpha} - x_0^{\alpha}, \quad 0 < \alpha \leq 1. \tag{4.0.5}$$

And

$$ {}_{x}D_{x_n}^{\alpha} y\left(x\right) = y^{(\alpha)}(x_n) = \frac{d^{\alpha} y\left(x\right)}{dx^{\alpha}} \lfloor_{x=x_n} = \lim_{x \to x_n} \frac{\nabla^{\alpha}\left(y\left(x_n\right) - y\left(x\right)\right)}{x_n^{\alpha} - x^{\alpha}}, \tag{4.0.6}$$

with

$$\nabla^{\alpha}\left(y\left(x_{n}\right)-y\left(x\right)\right)=\Gamma\left(\alpha+1\right)\nabla\left(y\left(x_{n}\right)-y\left(x\right)\right),\quad 0<\alpha\leq 1. \tag{4.0.7}$$

And

$$_{x}D_{x_{i}}^{\alpha}y\left(x\right)=y^{(\alpha)}(x_{i})=\frac{d^{\alpha}y\left(x\right)}{dx^{\alpha}}\lfloor_{x=x_{i}}=\lim_{x\to x_{i}}\frac{\rhd^{\alpha}\left(y\left(x\right)-y\left(x_{i}\right)\right)}{x^{\alpha}-x_{i}^{\alpha}}, \tag{4.0.8}$$

with

$$\rhd^{\alpha}\left(y\left(x_{i}\right)-y\left(x\right)\right)=\Gamma\left(\alpha+1\right)\left(y\left(x+x_{i+1}-x_{i}\right)-y\left(x+x_{i-1}-x_{i}\right)\right),\quad 0<\alpha\leq 1. \tag{4.0.9}$$

4.0.4 Finite Difference

Let the arguments x_i each having a mate $Y_i = y(x_i)$ and supposing the arguments spaced, so that ${x_{i+1}}^{\alpha} - {x_i}^{\alpha} = h^{\alpha}$, the following 3 types of fractional differences of the Y_i values are very useful :

1) The fractional **forward** differences (4.0.5), $\triangle^{\alpha}Y_i = \Gamma(\alpha+1)(Y_{i+1} - Y_i)$.

2) The fractional **backward** differences (4.0.7), $\nabla^{\alpha}Y_i = \Gamma(\alpha+1)(Y_i - Y_{i-1})$.

3) The fractional **central** differences (4.0.9), $\rhd^{\alpha} Y_i = \Gamma(\alpha+1)(Y_{i+1} - Y_{i-1})$.

The fractional differences of these differences in general are denoted,

$$\triangle^{k\alpha}Y_{i}=\Gamma^{k}\left(\alpha+1\right)\triangle^{k}Y_{i}=\Gamma^{k}\left(\alpha+1\right)\sum_{m=0}^{k}(-1)^{m}\begin{pmatrix}k\\ m\end{pmatrix}y\left(x_{i+k-m}\right),k=2,...,n \tag{4.0.10}$$

defines the k th differences.

Some operations on extended finite difference ([11]) :

The differences of a constant function,

$$\triangle^{\alpha}C=0,$$

1. and for a constant another function,

$$\triangle^{\alpha}CY_{k}=C\triangle^{\alpha}Y_{k},$$

where C denotes a constant.

The difference of a sum of two functions :

$$\triangle^{\alpha}(Y_k + Z_k) = \triangle^{\alpha} Y_k + \triangle^{\alpha} Z_k.$$

The linearity property :

$$\triangle^{\alpha}(C_1 Y_k + C_2 Z_k) = C_1 \triangle^{\alpha} Y_k + C_2 \triangle^{\alpha} Z_k,$$

where C_1 and C_2 are constants.

The differences of a product :

$$\triangle^{\alpha}(Y_k.Z_k) = Z_k \triangle^{\alpha} Y_k + Y_{k+1} \triangle^{\alpha} Z_k,$$

in which the argument $k+1$ should be noted.

The differences of a quotient :

$$\triangle^{\alpha}\left(\frac{Y_k}{Z_k}\right) = \frac{Z_k \triangle^{\alpha} Y_k - Y_k \triangle^{\alpha} Z_k}{Z_{k+1} Z_k},$$

and again the argument $k+1$ should be noted.

Remark 4.0.5 *In the same concept, we can rewrite the relationships of the fractional (**backward/central**) differences.*

4.0.5 The Fractional Polynomial

Given the $n+1$ data points $(x_i, y_i), i = 0, 1, 2, ..., n$, $P_{\alpha}(x)$ the polynomial of the fractional degree such that interpolation conditions are satisfied (Generalization and improve) $P_{\alpha}(x_i) = y_i$.

Theorem 4.0.6 *Suppose data points x_i; $i = 0, 1, .., n$ are distinct numbers in the interval $[a, b]$ and y_i; $i = 0, 1, .., n$, there exists a unique fractional polynomial $P_{\alpha}(x)$ of degree $\leq n\alpha$, such that the interpolation conditions $P_{\alpha}(x_i) = y_i$ are satisfied.*

Imagine that there is an unknown function $f(x)$ for which someone supplies you with its (exact) values at $(n+1)$ distinct points $x_0 < x_1 < ... < x_n$, i.e. $f(x_i) = y_i, i = 0, 1, .., n$ are given.

Newton's interpolating polynomials

We begin by recalling the Newton interpolating polynomial. This polynomial is a polynomial of degree $n\alpha$ ([11])

Proposition 4.0.7

$$P_\alpha(x) = a_0 + \sum_{k=1}^{n} a_k \prod_{i=0}^{k-1} (x^\alpha - x_i^\alpha), \tag{4.0.11}$$

satisfit $P_\alpha(x_j) = y_j, j = 0, 1, ...n,$ *with* $a_k = \frac{\Delta^k f(x_0)}{x_k^\alpha - x_0^\alpha} = \frac{\Delta^k y_0}{x_k^\alpha - x_0^\alpha},$

and

$$\triangle^k [y(x_0)] = \triangle y(x_0, \ldots x_k) = \sum_{i=0}^{k} (-1)^i \begin{pmatrix} k \\ i \end{pmatrix} y(x_{k-i}).$$

We can write the fractional forward Newton polynomial (4.0.11) by,

$$P_\alpha(x) = y_0 + \sum_{k=1}^{n} \Delta^k{}_\alpha y_0 (\prod_{i=0}^{k-1} (x^\alpha - x_i^\alpha)).$$

Lagrange Interpolating Polynomials

We can represent the function $f(x)$ by a polynomial of degree $P_\alpha(x) = A_0 + A_1 x^\alpha + ... + A_n x^{n\alpha}$.

Can be rewritten as:

$$P_\alpha(x) = \sum_{i=0}^{i=n} y_i \prod_{k=0,k\neq i}^{n} \frac{(x^\alpha - x_k^\alpha)}{(x_i^\alpha - x_k^\alpha)} = \sum_{i=0}^{i=n} y_i L_{i,\alpha}(x), \tag{4.0.12}$$

with $P_\alpha(x_i) = y_i$,

$$L_{i,\alpha}(x) = \prod_{k=0,k\neq i}^{n} \frac{x^\alpha - x_k^\alpha}{x_i^\alpha - x_k^\alpha},$$

satisfy, $L_{i,\alpha}(x_i) = 1$, and $L_{i,\alpha}(x_k) = 0$.

4.0.6 Approximate of fractional integrals

Let's suppose that we are given an initial-value problem of the form

$$\begin{cases} {}_{t_0}D_t^{\alpha} y\left(t\right) = \lambda\left(t\right) y(t) + f(t) \\ {}_{x_0}D_t^{\alpha-1} y \mid_{t=0} = a_0 \end{cases} \tag{4.0.13}$$

In integrals method for one equation, the approximate value of the unknown function at the next point $\Delta_{\alpha} x_i = x_{i+1}^{\alpha} - x_i^{\alpha}$,

$$\begin{aligned} {}_{x_0}D_x^{\alpha} y(t) &= \lambda(t) y(t) + g(t) \\ y(t) &= {}_{t_0}I_t^{\alpha}\left(\lambda(t) y(t) + g(t)\right) \\ &= \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha}\left(\lambda(s) y(s) + g(s)\right) ds \end{aligned}$$

We structure a local fractional iteration procedure, as

$$y(t_n) = \frac{1}{\Gamma(\alpha)} \sum_{i=0}^{i=n} (t_{i+1} - s_i)^{\alpha}\left[\left(\lambda(s_i) y(s_i) + g(s_i)\right)\right]\left(\left(s_{i+1} - s_i\right)\right),$$

where

$$s_i = (1-k)\, t_i + k t_{i+1}, k \in [0, 1[\,,$$

and

$$\begin{aligned} s_{i+1} - s_i &= (1-k)\, t_{i+1} + k t_{i+2} - (1-k)\, t_i - k t_{i+1} \\ &= (1-k)\left(t_{i+1} - t_i\right) + k\left(t_{i+2} - t_{i+1}\right) = t_{i+1} - t_i \end{aligned}$$

so,

$$y(t_n) = \frac{1}{\Gamma(\alpha)} \sum_{i=0}^{i=n} (1-k)^{\alpha-1} (h)^{\alpha+1}\left[\left(\lambda(s_i) y(s_i) + g(s_i)\right)\right], \tag{4.0.14}$$

By assuming that the value of $y\left(x_i\right)$ is exact, we find that the application of to compute $y\left(x_{i+1}\right)$ creates an error in the value of $y\left(x_{i+1}\right)$. This error is called the local truncation error,

e_{i+1}^{α}. Defining the local solution, $y(x)$, by

$$e_{i+1}^{\alpha} = y\left(x_{i+1}\right) - y_{i+1}$$

can be obtained by comparing the formula for y_{i+1} with Taylor's series expansion of the local solution about the point x_i. Since,

$$\begin{aligned} y\left(x_{i+1}\right) &= y\left(x_i + h\right) = y\left(x_i\right) + \frac{(h)^{\alpha}}{\Gamma\left(\alpha+1\right)} f(x_i, y(x_i)) + \frac{\left(\triangle x_i\right)^{2\alpha}}{\Gamma\left(2\alpha+1\right)} f^{(2\alpha)}(\xi(x)), \\ e_{i+1}^{\alpha} &= \theta\left(\left(\triangle x_i\right)^{2\alpha}\right). \end{aligned}$$

In general, we are interested by approximating the problem (4.0.13) using local fractional iteration procedure

$$\begin{aligned} y_{i+1} &= y_i + \frac{(\triangle x_i)^{\alpha}}{\Gamma(\alpha+1)} \left[(1-\theta)\left(\lambda(x_i)y(x_i) + g(x_i)\right) + \theta f(x_{i+1}, y(x_{i+1}))\right] \\ &= y_i + \frac{(x_{i+1}^{\alpha} - x_i^{\alpha})}{\Gamma(\alpha+1)} (\lambda(x_i)y(x_i) + g(x_i)). \end{aligned} \tag{4.0.15}$$

Example 4.0.8 *Let*

$$\left(D_t^{\frac{1}{6}} - \Gamma\left(\frac{7}{6}\right) t^{\frac{-1}{6}} \right) u\left(t\right) = -\left(\frac{1}{\Gamma\left(\frac{5}{6}\right)} + \Gamma\left(\frac{7}{6}\right)\right) t^{\frac{-7}{6}}, y\left(1\right) = 1, \tag{4.0.16}$$

and the particular solution is $y\left(t\right) = \frac{1}{t}$, *in* $[1, 2]$, *and* $u(2) = \frac{1}{2}, h = 0, 1$.

There are a lot of algorithms that can give approximation to the solution. We propose this algorithm to find sequential approximations of the solution in $[1, 2]$,

$$\begin{aligned} y_{i+1} &= \Gamma(\alpha+1)\left(y_i + (t_{i+1}^{\alpha} - t_i^{\alpha})(\lambda(t_i)y(t_i) + g(t_i))\right) \\ &= \Gamma(\alpha+1)\left(1 + (t_{i+1}^{\alpha} - t_i^{\alpha}) t_i^{\frac{-1}{6}}\right) y(x_i) - \left(\frac{1}{\Gamma\left(\frac{5}{6}\right)} + \Gamma\left(\frac{7}{6}\right)\right) t_i^{\frac{-7}{6}}, \end{aligned}$$

the Successive Approximation y_i *for the Solution* $y(t_i)$ *and error of fractional Differential Equa-*

tions (4.0.16) ,

t	1.1	1.2	1.3	1.4	1.5
y_i	$0,914559615$	$0,903082586$	$0,829761612$	$0,763002806$	$0,702019015$
$y(exact)$	$0,909090909$	$0,833333333$	$0,769230769$	$0,714285714$	$0,666666667$
$Error$	$0,005468706$	$0,069749253$	$0,060530843$	$0,048717092$	$0,035352348$
t	1.6	1.7	1.8	1.9	2
y_i	$0,646171422$	$0,5949291$	$0,547842084$	$0,504522883$	$0,464633428$
$y(exact)$	$0,625$	$0,588235294$	$0,555555556$	$0,526315789$	$0,5$
$Error$	$0,021171422$	$0,006693806$	$0,007713471$	$0,021792906$	$0,035366572$

After that, we use the values y_i to construct fractional polynomial.

Let the valuesy$_i$ are calculated previously, given the n data points $(x_i, y_i), i = 1, 2, ..., n$, $P_\alpha(x) \simeq y(x)$ the polynomial of the fractional degree such that interpolation conditions are satisfied $P_\alpha(x_i) = y_i$, where $P_\alpha(x)$:

$$P_\alpha(x) = a_0 + \sum_{k=1}^{n} a_k \prod_{i=0}^{k-1} (x^\alpha - x_i^\alpha) = y_0 + \sum_{k=1}^{n} \Delta^k{}_\alpha y_0 (\prod_{i=0}^{k-1} (x^\alpha - x_i^\alpha)$$

Newton Divided Difference Table(see [11]).

$a_0 = y_0$	$a_1 = \Delta_{\frac{1}{6}} y_i$	$a_2 = \Delta^2_{\frac{1}{6}} y_0$	$a_3 = \Delta^3_{\frac{1}{6}} y_0$	$a_4 = \Delta^4_{\frac{1}{6}} y_0$
1	$-5,677606897$	$18,57656704$	$-45,78928702$	$94,23257323$
$a_5 = \Delta^5_{\frac{1}{6}} y_0$	$a_6 = \Delta^6_{\frac{1}{6}} y_0$	$a_7 = \Delta^7_{\frac{1}{6}} y_0$	$a_8 = \Delta^8_{\frac{1}{6}} y_0$	$a_9 = \Delta^9_{\frac{1}{6}} y_0$
$-170,6852705$	$280,7785447$	$-428,1122615$	$613,6262944$	$-5043,330472$

And $P_\alpha(2) = 0,499999968$ *and* $y(2) = 0,5$

$$e_\alpha = |P_\alpha(2) - y(2)| = 0,0000000032.$$

So, we can write the general solution approximation of fractional differential equation (4.0.16) ,

$$y(t) = P_\alpha(t) = a_0 + \sum_{k=1}^{n} a_k \prod_{i=0}^{k-1} (t^\alpha - t_i^\alpha).$$

The table of error is $e_\alpha = |P_\alpha(t) - y(t)|$.

t	0.25	0.5	0.75	1.25	1.5
e_α	$0,003114461$	$2,42553E-05$	$1,49861E-07$	$3,4317E-13$	0
t	2.5	3	3.5	4	4.5
e_α	$8,29072E-08$	$2,07035E-06$	$1,67773E-05$	$7,72705E-05$	$2,5387E-04$

Remark 4.0.9 *Outside interval* $[x_0, x_n]$ *the error is near zero.*

Example 4.0.10 *Let*

$$u^{(\frac{1}{2})} - \frac{t^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}u = t^{\frac{1}{4}}\left(3\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} - \frac{t}{\Gamma\left(\frac{1}{2}\right)}\right), \qquad u(1) = 2, \tag{4.0.17}$$

and the exact solution is $u(t) = \frac{c}{t} + t^{\frac{3}{4}}$, *in* $[1,2]$, $c = 1, h = 0,1$.

We propose this algorithm to find sequential approximations of the solution in $[1,2]$,

$$y_{i+1} = y_i + \Gamma(\alpha+1)\left((t_{i+1}^{\alpha} - t_i^{\alpha})\frac{t_i^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}y_i + t_i^{\frac{1}{4}}\left(3\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} - \frac{t_i}{\Gamma\left(\frac{1}{2}\right)}\right)\right),$$

Successive Approximation y_i *for the Solution* $y(t_i)$ *and error of fractional Differential Equations* (4.0.17) ,

t_i	1.1	1.2	1.3	1.4	1.5
y_i	$2,371089999$	$2,869720027$	$3,35793816$	$3,871554393$	$4,426155877$
t_i	1.6	1.7	1.8	1.9	2
y_i	$5,032361535$	$5,69930395$	$6,43584207$	$7,251113082$	$8,154845485$

by the best estimate of y_i *given the* n *data points* $(x_i, y_i), i = 1,2,...,n,$ $P_\alpha(x) \simeq y(t)$ *the polynomial of the fractional degree such that interpolation conditions are satisfied* $P_\alpha(x_i) = y_i$, *where* $P_\alpha(x) = \sum_{i=0}^{i=n} y_i \prod_{k=0,k\neq i}^{n} \frac{(x^\alpha - x_k^\alpha)}{(x_i^\alpha - x_k^\alpha)} = \sum_{i=0}^{i=n} y_i L_{i,\alpha}(x)$ *Lagrange polynomial.*

Lagrange polynomial Table ,

$L_0(2)$	$L_1(2)$	$L_2(2)$	$L_3(2)$	$L_4(2)$
$-0,213820032$	$2,592433201$	$-13,98439932$	$44,26051715$	$-91,12563169$
$L_5(2)$	$L_6(2)$	$L_7(2)$	$L_8(2)$	$L_9(2)$
$127,6453775$	$-123,2969363$	$81,14888088$	$-34,8476974$	$8,821275993$

And $P_\alpha(2) = 2,181792803,\ y(2) = \ 2,181792831$ *and*

$$e_\alpha = |P_\alpha(2) - y(2)| = 0,0000000027$$

So, we can write the general solution approximation of fractional differential equation $(4.0.17)$,

$$y(t) = P_\alpha(t) = \sum_{i=0}^{i=n} a_i \prod_{k=0,k\neq i}^{n} (t^\alpha - t_k^\alpha) = \sum_{i=0}^{i=n} \frac{y_i}{\prod_{k=0,k\neq i}^{n}(t_i^\alpha - t_k^\alpha)} \prod_{k=0,k\neq i}^{n} (t^\alpha - t_k^\alpha).$$

The table of error is $e_\alpha = |P_\alpha(t) - y(t)|$.

t	0.25	0.5	0.75	0.9	$1,25$
e_α	$0,084868941$	$0,001415033$	$1,42653E-05$	$3,0634E-07$	$6,50999E-11$
t	1.5	1.75	2.25	2.5	$3,5$
e_α	0	$6,95288E-11$	$2,66508E-06$	$3,96307E-05$	$0,001302527$

Remark 4.0.11 *Outside interval* $[x_0, x_n]$ *the error is near zero.*

Example 4.0.12 *Let*

$$u^{(\frac{1}{3})} - 2u^{(\frac{1}{6})} + u = e^t\left(2t^{\frac{1}{3}} - 2\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{7}{6}\right)}t^{\frac{1}{6}} + \Gamma\left(\frac{4}{3}\right)\right), \qquad u(0) = 1, u^{(\frac{1}{6})}(0) = \Gamma\left(\frac{7}{6}\right), \quad (4.0.18)$$

and the exact solution is $u(t) = e^t\left(c_1 + t^{\frac{1}{6}}c_2 + t^{\frac{1}{3}}\right)$, *in* $[0,1]$, $c_1 = c_2 = 1, h = 0,1$.

We need only the system,

$$\begin{cases} \left(D_t^{\frac{1}{6}} - 1\right)v = e^t\left(2t^{\frac{1}{3}} - 2\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{7}{6}\right)}t^{\frac{1}{6}} + \Gamma\left(\frac{4}{3}\right)\right) = f(t) \\ \left(D_t^{\frac{1}{6}} - 1\right)y = v \end{cases}$$

and just solve the differential equation of the first degree

$$(D_t^{\frac{1}{6}}-1)u_i=f_i\,(t)$$

We propose this algorithm to find sequential approximations of the solution in $[0,1]$,

$$\begin{aligned} v_{i+1} &= \left(v_i+\Gamma(\alpha+1)\left((t_{i+1}^{\alpha}-t_i^{\alpha})v_i+f\,(t_i)\right)\right) \\ y_{i+1} &= \left(y_i+\Gamma(\alpha+1)\left((t_{i+1}^{\alpha}-t_i^{\alpha})y_i+v_{i+1}\right)\right), \end{aligned}$$

where

$$\left(D_t^{\frac{1}{6}}-1\right)y_0=v_{0,}\;y_0=u\,(0)=1.$$

Successive Approximation y_i *for the Solution* $y(t_i)$ *and error of fractional Differential Equations* $(4.0.18)$,

t_i	0.1	0.2	0.3	0.4	0.5
y_i	$2,371089999$	$2,869720027$	$3,35793816$	$3,871554393$	$4,426155877$
t_i	0.6	0.7	0.8	0.9	1
y_i	$5,032361535$	$5,69930395$	$6,43584207$	$7,251113082$	$8,154845485$

by the best estimate of y_i *given the* n *data points* $(x_i,y_i), i=1,2,...,n,\ P_\alpha(x)\simeq y\,(t)$ *the polynomial of the fractional degree such that interpolation conditions are satisfied* $P_\alpha(x_i)=y_i$, *where* $P_\alpha(x)=\sum_{i=0}^{i=n} y_i\prod_{k=0,k\neq i}^{n}\frac{(x^\alpha-x_k^\alpha)}{(x_i^\alpha-x_k^\alpha)}=\sum_{i=0}^{i=n} y_iL_{i,\alpha}(x)$ *Lagrange polynomial.*

Lagrange polynomial Table ,

$L_0\,(1)$	$L_1\,(1)$	$L_2\,(1)$	$L_3\,(1)$	$L_4\,(1)$
$-2,36091E-09$	$0,001301852$	$-0,056271884$	$0,667273078$	$-3,639139146$
$L_5\,(1)$	$L_6\,(1)$	$L_7\,(1)$	$L_8\,(1)$	$L_9\,(1)$
$11,03149473$	$-20,18248322$	$22,88603303$	$-15,77504964$	$6,066841211$

And $P_\alpha(1)=8,154783766$, $y\,(1)=8,154845485$ *and*

$$e_\alpha=|P_\alpha(1)-y\,(1)|=0,000006171$$

So, we can write the general solution approximation of fractional differential equation (4.0.18) ,

$$y(t) = P_\alpha(t) = \sum_{i=0}^{i=n} a_i \prod_{k=0,k\neq i}^{n} (t^\alpha - t_k^\alpha) = \sum_{i=0}^{i=n} \frac{y_i}{\prod_{k=0,k\neq i}^{n} (t_i^\alpha - t_k^\alpha)} \prod_{k=0,k\neq i}^{n} (t^\alpha - t_k^\alpha).$$

The table of error is $e_\alpha = |P_\alpha(t) - y(t)|$.

t	0.25	0.55	0.75	0.95	1
e_α	$1,42572E-05$	$2,79195E-07$	$3,84522E-07$	$1,38373E-05$	$6,17191E-05$
t	1.3	1.5	1.7	1.9	2
e_α	$0,005071895$	$0,026746732$	$0,094529825$	$0,261766421$	$0,408147583$

Remark 4.0.13 *Outside interval* $[t_0, t_n]$ *the error is near zero.*

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END

Abstract

In this work, we have proved the existence of the solution to the differential equations of one variable by transforming them into a system of equations that accept the solution and can be solved analytically. Whenever we have a problem to find the exact solution, we can calculate the numerical solution of the equations that are difficult to be solved analytically, and create the solution by using the interpolation. Some examples are provided to illustrate that.

مــلخـــــــــص

في هذا العمل قمنا باثبات وجود الحل للمعادلات التفاضليه الكسريه ذات متغير واحد بتحويلها الى جملة معادلات قطرية تقبل الحل و يمكن حلها تحليليا، و لماتكون لدينا مشكل ايجاد الحل الدقيق يمكن حساب الحل الرقمي للمعادلات التي يصعب حلها تحليليا وانشاء الحل باستخدام الاستقطاب مدعمين ذالك بامثلة ،

Résumé

Dans ce travail, nous avons prouvé l'existence de la solution des équations différentielles factionnaire d'une variable en transformé en un système d'équations acceptant la solution et pouvant être résolue analytiquement. Une solution approchée est calculée en utilisant la solution numérique d'équations difficiles à résoudre analytiquement et créer la solution en utilisant l'interpolation soutenue par des exemples.