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Stabilisation du système de Timochenko viscoélastique en présence d'un terme de retard nonlinéaire

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Introduction

Stabilization of evolution problems

Problems of global existence and stability in time of Partial Differential Equations made object, recently, of many work. In this thesis we were interested in study of the global existence and the stabilization of some evolution equations.

The purpose of stabilization is to attenuate the vibrations by feedback, thus it consists in guaranteeing the decrease of energy of the solutions to 0 in a more or less fast way by a mechanism of dissipation.

More precisely, the problem of stabilization consists in determining the asymptotic behaviour of the energy by $E(t)$, to study its limits in order to determine if this limit is null or not and if this limit is null, to give an estimate of the decay rate of the energy to zero.

This problem has been studied by many authors for various systems. In our study, we obtain severals type of stabilization

1) Strong stabilization: $E(t) \to 0$, as $t \to \infty$.

2) Logarithmic stabilization: $E(t) \leq c(\log(t))^{-\delta}, \forall t > 0, (c, \delta > 0)$.

3) polynomial stabilization: $E(t) \leq ct^{-\delta}, \forall t > 0, (c, \delta > 0)$

4) uniform stabilization: $E(t) \leq ce^{-\delta t}, \forall t > 0, (c, \delta > 0)$.

For wave equation with dissipation of the form $u'' - \Delta_x u + g(u') = 0$, stabilization problems have been investigated by many authors:
When $g: \mathbb{R} \to \mathbb{R}$ is continuous and increasing function such that $g(0) = 0$, global existence of solutions is known for all initial conditions (u_0, u_1) given in $H_0^1(\Omega) \times L^2(\Omega)$. This result is, for instance, a consequence of the general theory of nonlinear semi-groups of contractions generated by a maximal monotone operator (see Brézis **[6]**).

Moreover, if we impose on the control the condition $\forall \lambda \neq 0, g(\lambda) \neq 0$, then strong asymptotic stability of solutions occurs in $H_0^1(\Omega) \times L^2(\Omega)$, i.e.,

$$(u, u') \to (0, 0) \text{ strongly in } H_0^1(\Omega) \times L^2(\Omega),$$

without speed of convergence. These results follows, for instance, from the invariance principle of Lasalle (see for example C. M. Dafermos **[13]**, A. Haraux **[18]**, , F. Conrad, M. Pierre **[12]**). If the solution goes to 0 as time goes to ∞, how to get energy decay rates?

Dafermos has written in 1978 "Another advantage of this approach is that it is so simplistic that it requires only quite weak assumptions on the dissipative mechanism. The corresponding drawback is that the deduced information is also weak, never yielding, for example, decay rates of solutions."

Many authors have worked since then on energy decay rates. First results were obtained for linear stabilization, then for polynomial stabilization (see M. Nakao [**34**] A. Haraux [**18**], E. Zuazua [**41**] and V. Komornik [**23**]) and then extended to arbitrary growing feedbacks (close to 0). In the same time, geometrical aspects were considered.

By combining the multiplier method with the techniques of micro-local analysis, Lasiecka et al [**9**], [**14**], [**27**]-[**28**] have investigated different dissipative systems of partial differential equations (with Dirichlet and Neumann boundary conditions) under general geometrical conditions with nonlinear feedback without any growth restrictions near the origin or at infinity. The computation of decay rates is reduced to solving an appropriate explicitly given ordinary differential equation of monotone type. More precisely, the following explicit decay estimate of the energy is obtained:

$$E(t) \leq h(\frac{t}{t_0} - 1), \quad \forall t \geq t_0 \tag{1}$$

where $t_0 > 0$ and h is the solution of the following differential equation:

$$h'(t) + q(h(t)) = 0, \quad \forall t \geq 0 \quad \text{and} \quad h(0) = E(0) \tag{2}$$

and the function q is determined entirely from the behavior at the origin of the nonlinear feedback by proving that E satisfies

$$(Id - q)^{-1}\Big(E((m+1)t_0)\Big) \leq E(mt_0), \quad \forall m \in \mathbb{N}.$$

In this thesis, the main objective is to give a global existence and stabilization results.

This work consists in tree chapter,

Chapter 1: Global existence and energy decay of solution to a viscoelastic Timoshenko beam system with a nonlinear delay term

In this chapter, we consider the viscoelastic Timoshenko beam system with delay term in the weakly nonlinear internal feedback in a bounded domain:

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x + \psi)(x,t) + \mu_1 g_1(\psi_t(x,t)) \\ + \displaystyle\int_0^t h(t-s)\psi_{xx}(x,s)ds + \mu_2 g_2(\psi_t(x,t-\tau)) = 0. \end{cases}$$

We prove a global existence result using the energy method combined with the Faedo-Galerkin procedure under a condition between the delay term in the feedback and the weight of the term without delay.

Furthermore, we study the asymptotic behavior of solution using a perturbed energy method.
then next,

Chapter 2: Global existence and energy decay of solution for the Timoshenko beam system with a time varying delay term in the weakly nonlinear internal feedback

In this chapter, we consider the nonlinear Timoshenko system with a delay term in the weakly nonlinear internal feedback:

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x+\psi)(x,t) + \mu_1(t)g_1(\psi_t(x,t)) + \mu_2(t)g_2(\psi_t(x,t-\tau(t))) = 0, \end{cases}$$

in a bounded domain. Under appropriate conditions on μ_1, μ_2 and τ, we prove the global existence of solutions by the FaedoGalerkin method and establish a decay rate estimate for the energy using suitable Lyapunov functionals.
And the third is

Chapter 3: Global existence and energy decay of solution to a viscoelastic Timoshenko beam system with a time varying delay term in the weakly nonlinear internal feedback

In this chapter, we consider the viscoelastic Timoshenko beam system with a time varying delay term in the weakly nonlinear internal feedback in a bounded domain:

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x+\psi)(x,t) + \mu_1(t)g_1(\psi_t(x,t)) \\ \quad +\mu_2(t)g_2(\psi_t(x,t-\tau(t))) + \displaystyle\int_0^t h(t-s)\psi_{xx}(x,s)ds = 0. \end{cases}$$

We prove the global existence of its solutions in Sobolev spaces by means of the energy method combined with the Faedo-Galerkin procedure under a condition between the weight of the delay term in the feedback and the weight of the term without delay. Furthermore, we establish a decay rate estimate for the energy by introducing suitable Lyapunov functionals.

Preliminaries

0.1 Sobolev spaces

We denote by Ω an open domain in $\mathbb{R}^n, n \geq 1$, with a smooth boundary $\Gamma = \partial\Omega$. In general, some regularity of Ω will be assumed. We will suppose that either

$$\Omega \text{ is Lipschitz,}$$

i.e., the boundary Γ is locally the graph of a Lipschitz function, or

$$\Omega \text{ is of class } \mathcal{C}^r, r \geq 1,$$

i.e., the boundary Γ is a manifold of dimension $n \geq 1$ of class $\mathcal{C}^r$. In both cases we assume that Ω is totally on one side of Γ. These definitions mean that locally the domain Ω is below the graph of some function ψ, the boundary Γ is represented by the graph of ψ and its regularity is determined by that of the function ψ. Moreover, it is necessary to note that a domain with a continuous boundary is never on both sides of its boundary at any point of this boundary and that a Lipschitz boundary has almost everywhere a unit normal vector ν.

We will also use the following multi-index notation for partial differential derivatives of a function:

$$\partial_i^k u = \frac{\partial^k u}{\partial x_i^k} \text{ for all } k \in \mathbb{N} \text{ and } i = 1, ..., n,$$
$$D^\alpha u = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n} u = \frac{\partial^{\alpha_1 + \dots + \alpha_n} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}},$$
$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n, |\alpha| = \alpha_1 + \dots + \alpha_n.$$

We denote by $\mathcal{C}(D)$ (respectively $\mathcal{C}^k(D), k \in \mathbb{N}$ or $k = +\infty$) the space of real continuous functions on D (respectively the space of k times continuously differentiable functions on D), where D plays the role of Ω or its closure $\overline{\Omega}$. The space of real $\mathcal{C}^\infty$ functions on Ω with a compact support in Ω is denoted by $\mathcal{C}_0^\infty(\Omega)$ or $\mathcal{D}(\Omega)$ as in the distributions theory of Schwartz.The distributions space on Ω is denoted by $\mathcal{D}'(\Omega)$, i.e., the space of continuous linear form over $\mathcal{D}(\Omega)$.

For $1 \leq p \leq \infty$, we call $L^p(\Omega)$ the space of measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p dx \right)^{1/p} < +\infty \quad \text{for} \quad p < +\infty$$

$$\|f\|_{L^\infty(\Omega)} = \sup_\Omega |f(x)| < +\infty \quad \text{for} \quad p = +\infty$$

The space $L^p(\Omega)$ equipped with the norm $f \longrightarrow \|f\|_{L^p}$ is a Banach space: it is reflexive and separable for $1 < p < \infty$ (its dual is $L^{\frac{p}{p-1}}(\Omega)$), separable but not reflexive for $p = 1$ (its dual is $L^\infty(\Omega)$), and not separable, not reflexive for $p = \infty$ (its dual contains strictly $L^1(\Omega)$). In particular the space $L^2(\Omega)$ is a Hilbert space equipped with the scalar product defined by

$$(f, g)_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx.$$

We denote by $L^p_{loc}(\Omega)$ the space of functions which are L^p on any bounded sub-domain of Ω.

Similar space can be defined on any open set other than Ω, in particular, on the cylinder set $\Omega \times]a, b[$ or on the set $\Gamma \times]a, b[$, where $a, b \in \mathbb{R}$ and $a < b$.

Let U be a Banach space, $1 < p < +\infty$ and $-\infty \leq a < b \leq +\infty$, then $L^p(a, b; U)$ is the space of L^p functions f from (a, b) into U which is a Banach space for the norm

$$\|f\|_{L^p(a,b;U)} = \left(\int_a^b \|f(t, .)\|^p_U \, dt\right)^{1/p} < +\infty \quad \text{for} \quad p < +\infty$$

and for the norm

$$\|f\|_{L^\infty(a,b;U)} = \sup_{t\in(a,b)} \|f(t, .)\|_U < +\infty \quad \text{for} \quad p = +\infty$$

Similarly, for a Banach space $U, k \in \mathbb{N}$ and $-\infty < a < b < +\infty$, we denote by $C([a, b]; U)$ (respectively $C^k([a, b]; U)$) the space of continuous functions (respectively the space of k times continuously differentiable functions) f from $[a, b]$ into U, which are Banach spaces, respectively, for the norms

$$\|f\|_{\mathcal{C}(a,b;U)} = \sup_{t\in(a,b)} \|f(t, .)\|_U, \quad \|f\|_{\mathcal{C}^k(a,b;U)} = \sum_{i=0}^{k} \left\|\frac{\partial^i f}{\partial t^i}\right\|_{\mathcal{C}(a,b;U)}$$

0.1.1 Definition of Sobolev Spaces

Now, we will introduce the Sobolev spaces: The Sobolev space $W^{m,p}(\Omega)$ is defined to be the subset of L^p such that function f and its weak derivatives up to some order m have a finite L^p norm, for given $p \geq 1$.

$$W^{m,p}(\Omega) = \{f \in L^p(\Omega); D^\alpha f \in L^p(\Omega). \ \forall\alpha; |\alpha| \leq m\} \ ,$$

With this definition, the Sobolev spaces admit a natural norm,

$$f \longrightarrow \|f\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha|\leq m} \|D^\alpha f\|^p_{L^p(\Omega)}\right)^{1/p}, \text{ for } p < +\infty$$

and

$$f \longrightarrow \|f\|_{W^{m,\infty}(\Omega)} = \sum_{|\alpha|\leq m} \|D^\alpha f\|_{L^\infty(\Omega)} \ , \text{ for } p = +\infty$$

Space $W^{m,p}(\Omega)$ equipped with the norm $\| \, . \, \|_{W^{m,p}}$ is a Banach space. Moreover is a reflexive space for $1 < p < \infty$ and a separable space for $1 \leq p < \infty$. Sobolev spaces with $p = 2$ are especially important because of their connection with Fourier series and because they form a Hilbert space. A special notation has arisen to cover this case:

$$W^{m,2}(\Omega) = H^m(\Omega)$$

the H^m inner product is defined in terms of the L^2 inner product:

$$(f, g)_{H^m(\Omega)} = \sum_{|\alpha| \leq m} (D^\alpha f, D^\alpha g)_{L^2(\Omega)} \ .$$

The space $H^m(\Omega)$ and $W^{m,p}(\Omega)$ contain $\mathcal{C}^\infty(\overline{\Omega})$ and $\mathcal{C}^m(\overline{\Omega})$. The closure of $\mathcal{D}(\Omega)$ for the $H^m(\Omega)$ norm (respectively $W^{m,p}(\Omega)$ norm) is denoted by $H_0^m(\Omega)$ (respectively $W_0^{m,p}(\Omega)$).

Now, we introduce a space of functions with values in a space X (a separable Hilbert space).

The space $L^2(a, b; X)$ is a Hilbert space for the inner product

$$(f, g)_{L^2(a,b;X)} = \int_a^b (f(t), g(t))_X \, dt$$

We note that $L^\infty(a, b; X) = (L^1(a, b; X))'$.

Now, we define the Sobolev spaces with values in a Hilbert space X

For $m \in \mathbb{N}$, $p \in [1, \infty]$, we set:

$$W^{m,p}(a, b; X) = \left\{ v \in L^p(a, b; X); \frac{\partial^j v}{\partial t^j} \in L^p(a, b; X). \ \ \forall j \leq m \right\} \ ,$$

The Sobolev space $W^{m,p}(a, b; X)$ is a Banach space with the norm

$$\|f\|_{W^{m,p}(a,b;X)} = \left(\sum_{j=0}^m \left\| \frac{\partial^j f}{\partial t^j} \right\|^p_{L^p(a,b;X)} \right)^{1/p}, \ \text{for } p < +\infty$$

$$\|f\|_{W^{m,\infty}(a,b;X)} = \sum_{j=0}^m \left\| \frac{\partial^j f}{\partial t^j} \right\|_{L^\infty(a,b;X)}, \qquad \text{for } p = +\infty$$

The spaces $W^{m,2}(a, b; X)$ form a Hilbert space and it is noted $H^m(a, b; X)$. The $H^m(a, b; X)$ inner product is defined by:

$$(u, v)_{H^m(a,b;X)} = \sum_{j=0}^m \int_a^b \left(\frac{\partial^j u}{\partial t^j}, \frac{\partial^j v}{\partial t^j} \right)_X dt \ .$$

Theorem 0.1.1 *Let $1 \leq p \leq n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$$

where p^* is given by $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}$ (where $p = n, p^* = \infty$). Moreover there exists a constant $C = C(p, n)$ such that

$$\|u\|_{L^{p^*}} \leq C \|\nabla u\|_{L^p(\mathbb{R}^n)} \forall u \in W^{1,p}(\mathbb{R}^n).$$

Corollary 0.1.1 *Let $1 \leq p < n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \quad \forall q \in [p, p^*]$$

with continuous imbedding.

For the case $p = n$, we have

$$W^{1,n}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \quad \forall q \in [n, +\infty[$$

Theorem 0.1.2 *Let $p > n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$$

with continuous imbedding.

Corollary 0.1.2 *Let Ω a bounded domain in $\mathbb{R}^n$ of C^1 class with $\Gamma = \partial\Omega$ and $1 \leq p \leq \infty$. We have*

$$\begin{array}{ll} if & 1 \leq p < \infty, \ then\ W^{1,p}(\Omega) \subset L^{p^*}(\Omega) \ where\ \dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}. \\ if & p = n, \ then\ W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [p, +\infty[. \\ if & p > n, \ then\ W^{1,p}(\Omega) \subset L^\infty(\Omega) \end{array}$$

with continuous imbedding.
Moreover, if $p > n$, we have: $\forall u \in W^{1,p}(\Omega)$,

$$|u(x) - u(y)| \leq C|x - y|^\alpha \|u\|_{W^{1,p}(\Omega)} \ a.e\ x, y \in \Omega$$

with $\alpha = 1 - \dfrac{n}{p} > 0$ and C is a constant which depend on p, n and Ω. In particular $W^{1,p}(\Omega) \subset C(\overline{\Omega})$.

Corollary 0.1.3 *Let Ω a bounded domain in $\mathbb{R}^n$ of C^1 class with $\Gamma = \partial\Omega$ and $1 \leq p \leq \infty$. We have*

$$\begin{array}{ll} if & p < n, \ then\ \ W^{1,p}(\Omega) \subset L^q(\Omega), \quad \forall q \in [1, p^*[\ where\ \dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}. \\ if & p = n, \ then\ \ W^{1,p}(\Omega) \subset L^q(\Omega), \quad \forall q \in [p, +\infty[. \\ if & p > n, \ then\ \ W^{1,p}(\Omega) \subset C(\overline{\Omega}) \end{array}$$

with compact imbedding.

Remark 0.1.1 *We remark in particular that*

$$W^{1,p}(\Omega) \subset L^q(\Omega)$$

with compact imbedding for $1 \leq p \leq \infty$ and for $p \leq q < p^*$.

Corollary 0.1.4

$$\begin{array}{ll} if & \dfrac{1}{p} - \dfrac{m}{n} > 0, \ then\ \ W^{m,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \ where\ \dfrac{1}{q} = \dfrac{1}{p} - \dfrac{m}{n}. \\ if & \dfrac{1}{p} - \dfrac{m}{n} = 0, \ then\ \ W^{m,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \forall q \in [p, +\infty[. \\ if & \dfrac{1}{p} - \dfrac{m}{n} < 0, \ then\ \ W^{m,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n) \end{array}$$

with continuous imbedding.

0.2 Weak convergence

Let $(E; \|.\|_E)$ a Banach space and E' its dual space, i.e., the Banach space of all continuous linear forms on E endowed with the norm $\|.\|_{E'}$ defined by

$$\|f\|_{E'} =: \sup_{x \neq 0} \frac{|\langle f, x\rangle|}{\|x\|}$$

; where $\langle f, x\rangle$; denotes the action of f on$x$, i.e.$\langle f, x\rangle := f(x)$. In the same way, we can define the dual space of E' that we denote by E''. (The Banach space E'' is also called the bi-dual space of E). An element x of E can be seen as a continuous linear form on E' by setting $x(f) := \langle x, f\rangle$, which means that $E \subset E''$:

Definition 0.2.1 *The Banach space E is said to be reflexive if $E = E''$.*

Definition 0.2.2 *The Banach space E is said to be separable if there exists a countable subset D of E which is dense in E, i.e. $\overline{D} = E$.*

Theorem 0.2.1 *(Riesz). If $(H; \langle ., .\rangle)$ is a Hilbert space, $\langle ., .\rangle$ being a scalar product on H, then $H' = H$ in the following sense: to each $f \in H'$ there corresponds a unique $x \in H$ such that $f = \langle x, .\rangle$ and $\|f\|_{H'} = \|x\|_H$*

Remark : From this theorem we deduce that $H'' = H$. This means that a Hilbert space is reflexive.

Proposition 0.2.1 *If E is reflexive and if F is a closed vector subspace of E, then F is reflexive.*

Corollary 0.2.1 *The following two assertions are equivalent:*

i. E is reflexive;

ii. E' is reflexive.

0.2.1 Weak, weak star and strong convergence

Definition 0.2.3 *(Weak convergence in E). Let $x \in E$ and let $\{x_n\} \subset E$. We say that $\{x_n\}$ weakly converges to x in E, and we write $x_n \rightharpoonup x$ in E, if*

$$\langle f, x_n\rangle \to \langle f, x\rangle$$

for all $f \in E'$.

Definition 0.2.4 *(weak convergence in E'). Let $f \in E'$ and let $\{f_n\} \subset E'$. We say that $\{f_n\}$ weakly converges to f in E', and we write $f_n \rightharpoonup f$ in E', if*

$$\langle f_n, x\rangle \to \langle f, x\rangle$$

for all $x \in E''$.

Definition 0.2.5 *(weak star convergence). Let $f \in E'$ and let $\{f_n\} \subset E'$. We say that $\{f_n\}$ weakly star converges to f in E', and we write $f_n \rightharpoonup *f$ in E' if;*

$$\langle f_n, x\rangle \to \langle f, x\rangle$$

for all $x \in E$.

Remark : As $E \subset E''$ we have $f_n \rightharpoonup f$ in E' imply $f_n \rightharpoonup *f$ in E'. When E is reflexive, the last definitions are the same, i.e, weak convergence in E' and weak star convergence coincide.

Definition 0.2.6 *(strong convergence). Let $x \in E$ (resp. $f \in E'$) and let $\{x_n\} \subset E$ (resp $\{f_n\} \subset E'$). We say that $\{x_n\}$ (resp. $\{f_n\}$) strongly converges to x (resp. f), and we write $x_n \to x$ in E (resp. $f_n \to f$ in E'), if*

$$\underset{n}{lim} \|x_n - x\|_E = 0; (resp.\ \underset{n}{lim} \|f_n - f\|_{E'} = 0)$$

Proposition 0.2.2 *Let $x \in E$, let $\{x_n\} \subset E$, let $f \in E'$ and let $\{f_n\} \subset E'$.*

i. If $x_n \to x$ in E then $x_n \rightharpoonup x$ in E.

ii. If $x_n \rightharpoonup x$ in E then $\{x_n\}$ is bounded.

iii. If $x_n \rightharpoonup x$ in E then $\liminf_{n\to\infty} \|x_n\|_E \geq \|x\|_E$

iv. If $f_n \to f$ in E' then $f_n \rightharpoonup f$ in E' (and so $f_n \overset{*}{\rightharpoonup} f$ in E').

v. If $f_n \rightharpoonup f$ in E' then $\{f_n\}$ is bounded.

vi. If $f_n \rightharpoonup f$ in E' then then $\liminf_{n\to\infty} \|f_n\|_{E'} \geq \|f\|_{E'}$

Proposition 0.2.3 *(finite dimension). If $\dim E < \infty$ then strong, weak and weak star convergence are equivalent.*

0.2.2 Weak and weak star compactness

In finite dimension, i.e, $\dim E < \infty$, we have Bolzano-Weierstrass's theorem (which is a strong compactness theorem).

Theorem 0.2.2 *(Bolzano-Weierstrass). If $\dim E < \infty$ and if $\{x_n\} \subset E$) is bounded, then there exist $x \in E$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ strongly converges to x.*

The following two theorems are generalizations, in infinite dimension, of Bolzano- Weierstrass's theorem.

Theorem 0.2.3 *(weak star compactness, Banach-Alaoglu-Bourbaki). Assume that E is separable and consider $\{f_n\} \subset E'$) . If $\{x_n\}$ is bounded, then there exist $f \in E'$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}\}$ weakly star converges to f in E'.*

Theorem 0.2.4 *(weak compactness, Kakutani-Eberlein). Assume that E is reflexive and consider $\{x_n\} \subset E$). If $\{x_n\}$ is bounded, then there exist $x \in E$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ weakly converges to x in E.*

Weak, weak star convergence and compactness in $L^p(\Omega)$.

Definition 0.2.7 *(weak convergence in $L^p(\Omega)$ with $1 \le p < \infty$). Let Ω an open subset of $\mathbb{R}^n$ .We say that the sequence $\{f_n\}$ of $L^p(\Omega)$ weakly converges to $f \in L^p(\Omega)$, if*

$$\lim_n \int_\Omega f_n(x)g(x)dx = \int_\Omega f(x)g(x)dx \text{ for all } g \in L^q;\ (\frac{1}{p} + \frac{1}{q} = 1)$$

Definition 0.2.8 *(weak star convergence in $L^\infty(\Omega)$). We say that the sequence $\{f_n\} \subset L^\infty(\Omega)$ weakly star converges to $f \in L^\infty(\Omega)$, if*

$$\lim_n \int_\Omega f_n(x)g(x)dx = \int_\Omega f(x)g(x)dx \text{ for all } g \in L^1(\Omega)$$

Theorem 0.2.5 *(weak compactness in $L^p(\Omega)$) with $1 < p < \infty$. Given $\{f_n\} \subset L^p(\Omega)$, if $\{f_n\}$ is bounded, then there exist $f \in L^p(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_n \rightharpoonup f$ in $L^p(\Omega)$.*

Theorem 0.2.6 *(weak star compactness in $L^\infty(\Omega)$.
Given $\{f_n\} \subset L^\infty(\Omega)$, if $\{f_n\}$ is bounded, then there exist $f \in L^\infty(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_n \overset{*}{\rightharpoonup} f$ in $L^\infty(\Omega)$.*

Generalities. In what follows, Ω is a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary and $1 \le p \le \infty$.

Weak and weak star convergence in Sobolev spaces

For $1 \le p \le \infty$, $W^{1;p}(\Omega)$ is a Banach space. Denote the space of all restrictions to Ω of C^1-differentiable functions from $\mathbb{R}^N$ to $\mathbb{R}$ with compact support in R^N by $C^1(\overline{\Omega})$.

Theorem 0.2.7 *for every $1 \le p \le \infty$ $C^1(\overline{\Omega}) \subset W^{1;p}(\Omega) \subset L^p(\Omega)$, and, for $1 < p < \infty$, $C^1(\overline{\Omega})$ is dense in $W^{1;p}(\Omega)$.*

Definition 0.2.9 *(weak convergence in $W^{1;p}(\Omega)$ with $1 \le p < \infty$)).
We say the $\{f_n\} \subset W^{1;p}(\Omega)$ weakly converges to $f \in W^{1;p}(\Omega)$, and we write $f_n \rightharpoonup f$ in $W^{1;p}(\Omega)$, if $f_n \rightharpoonup f$ in $L^p(\Omega)$ and $\nabla f_n \rightharpoonup \nabla f$ in $L^p(\Omega;\ \mathbb{R}^N)$*

Definition 0.2.10 *(weak convergence in $W^{1;\infty}(\Omega)$
We say the $\{f_n\} \subset W^{1;\infty}(\Omega)$ weakly star converges to $f \in W^{1;\infty}(\Omega)$, and we write $f_n \overset{*}{\rightharpoonup} f$ in $W^{1;\infty}(\Omega)$, if $f_n \overset{*}{\rightharpoonup} f$ in $L^p(\Omega)$ and $\nabla f_n \overset{*}{\rightharpoonup} \nabla f$ in $L^\infty(\Omega;\ \mathbb{R}^N)$*

Theorem 0.2.8 *(Rellich). Let $1 \le p \le \infty$, $\{f_n\} \subset W^{1;p}(\Omega)$ and $f \in W^{1;p}(\Omega)$; if $f_n \rightharpoonup f$ in $W^{1;p}(\Omega)$ when $1 \le p < \infty$ (resp.$f_n \overset{*}{\rightharpoonup} f$ in $W^{1;\infty}(\Omega)$) when $p = \infty$) then $f_n \to f$ in $L^p(\Omega)$), which means that for every $1 \le p \le \infty$, the weak convergence in $W^{1;p}(\Omega)$ imply the strong convergence in $L^p(\Omega)$.*

Theorem 0.2.9 *Let $1 < p \leq \infty$ and let $\{f_n\} \subset W^{1;p}(\Omega)$. If $\{f_n\}$ is bounded, then there exist $f \in W^{1;p}(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \rightharpoonup f$ in $W^{1;p}(\Omega)$ when $1 < p < \infty$ (resp. $f_{n_k} \overset{*}{\rightharpoonup} f$ in $W^{1;\infty}(\Omega)$)*

As a consequence of this theorem we have

Corollary 0.2.2 *Let $1 < p \leq \infty$and let $\{f_n\} \subset W^{1;p}(\Omega)$. If $\{f_n\}$ is bounded, then there exist $f \in W^{1;p}(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \to f$ in $L^p(\Omega)$ and $\nabla f_{n_k} \rightharpoonup \nabla f$ in $L^p(\Omega)$ when $1 < p < \infty$ (resp.$\nabla f_{n_k} \overset{*}{\rightharpoonup} \nabla f$ in $L^\infty(\Omega)$)*

Theorem 0.2.10 *If $N < p \leq \infty$ and if $\{f_n\} \subset W^{1;p}(\Omega)$ is bounded, then there exist $f \in W^{1;p}(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}\}$ converges uniformly to f, and $\nabla f_{n_k} \rightharpoonup \nabla f$ in $W^{1;p}(\Omega)$ when $N < p < \infty$ (resp. $\nabla f_{n_k} \overset{*}{\rightharpoonup} \nabla f$ in $W^{1;\infty}$)*

0.3 Faedo-Galerkin method

We consider the Cauchy problem abstract's for a second order evolution equation in the separable Hilbert space with the inner product $\langle .,. \rangle$ and the associated norm $\|.\|$.

$$(P) \qquad \begin{cases} u''(t) + A(t)u(t) = f(t), \quad t \in [0,T] \\ (x,0) = u_0(x), \ u'(x,0) = u_1(x); \end{cases}$$

where u and f are unknown and given function, respectively, mapping the closed interval $[0,T] \subset \mathbb{R}$ into a real separable Hilbert space H ,$A(t)$ $(0 \leq t \leq T$) are linear bounded operators in H acting in the energy space $V \subset H$.
Assume that $\langle A(t)u(t), v(t) \rangle = a(t; u(t), v(t))$, for all $u, v \in V$; where $a(t;.,.)$ is a bilinear continuous in V.
The problem (P) can be formulated as: Found the solution $u(t)$ such that

$$(\widetilde{P}) \qquad \begin{cases} u \in C([0,T];V), u' \in C([0,T];H) \\ \langle u''(t), v \rangle + a(t; u(t), v) = \langle f, v \rangle \text{ in } D'(]0,T[) \\ u_0 \in V \ , \ u_1 \in H; \end{cases}$$

This problem can be resolved with the approximation process of Fadeo-Galerkin.

0.3.1 General method

Let V_m a sub-space of V with the finite dimension d_m, and let $\{w_{jm}\}$ one basis of V_m.
we define the solution u_m of the approximate problem

$$(P_m) \qquad \begin{cases} u_m(t) = \sum_{j=1}^{d_m} g_j(t) w_{jm} \\ u_m \in C([0,T];V_m), u'_m \in C([0,T];V_m) \quad , u_m \in L^2(0,T;V_m) \\ \langle u''_m(t), w_{jm} \rangle + a(t; u_m(t), w_{jm}) = \langle f, w_{jm} \rangle, \quad 1 \leq j \leq d_m \\ u_m(0) = \sum_{j=1}^{d_m} \xi_j(t) w_{jm} \ , \ u'_m(0) = \sum_{j=1}^{d_m} \eta_j(t) w_{jm} \end{cases}$$

where

$$\sum_{j=1}^{d_m} \xi_j(t) w_{jm} \longrightarrow u_0 \text{ in V as } m \longrightarrow \infty$$

$$\sum_{j=1}^{d_m} \eta_j(t) w_{jm} \longrightarrow u_1 \text{ in V as } m \longrightarrow \infty$$

By virtue of the theory of ordinary differential equations,the system (P_m) has unique local solution which is extend to a maximal interval $[0, t_m[$ by Zorn lemma since the non-linear terms have the suitable regularity. In the next step, we obtain a priori estimates for the solution, so that can be extended outside $[0, t_m[$, to obtain one solution defined for all $t > 0$.

0.3.2 A priori estimation and convergence

Using the following estimation

$$\|u_m\|^2 + \|u'_m\|^2 \leq C(\|u_m(0)\|^2 + \|u'_m(0)\|^2 + \int_0^T |f(s)|^2 ds) \ ; \quad 0 \leq t \leq T$$

and the Gronwall lemma we deduce that the solution u_m of the approximate problem (P_m) converges to the solution u of the initial problem (P).The uniqueness proves that u is the solution.

0.3.3 Gronwall lemma

Lemma 0.3.1 *Let $T > 0$, $g \in L^1(0,T)$, $g \geq 0$ a.e and c_1, c_2 are positives constants.Let $\varphi \in L^1(0,T)$ $\varphi \geq 0$ a.e such that $g\varphi \in L^1(0,T)$ and*

$$\varphi(t) \leq c_1 + c_2 \int_0^t g(s)\varphi(s)ds \quad a.e \ in \ (0,T).$$

then, we have

$$\varphi(t) \leq c_1 exp \ (c_2 \int_0^t g(s)ds) \quad a.e \ in \ (0,T).$$

0.4 Convex analysis

0.4.1 Fenchel conjugate functions

Let V be a topological vector space and let V' be its dual space with bilinear duality form $\langle .,. \rangle_{V',V}$.

Definition 0.4.1 *(Conjugate function)*
Let $F : V \longrightarrow \overline{\mathbb{R}}$ be an extend real valued function. The function $F^* : V' \longrightarrow \overline{\mathbb{R}}$ defined by

$$F^*(f) = \sup_{u \in V}(\langle f, u\rangle_{V',V} - F(u)), \quad \forall f \in V'$$

is said to by Fenchel (convex) conjugate or conjugate function of F.
The mapping $F \longrightarrow F^*$ is called the Legendre -Fenchel transformation.

Proposition 0.4.1 *Let $F : V \longrightarrow \overline{\mathbb{R}}$ be a given extend real valued function, the following statements are true*

- *i.$F^*(f) + F(u) \geq \langle f, u\rangle_{V',V}, \quad \forall f \in V', \ \forall u \in V$*
- *ii.Let f be in the dual $V'$of V and $\lambda \in \mathbb{R}$,the conjugate of affine function*

  $$u \longrightarrow (\langle f, u\rangle_{V',V} - \lambda$$

  is less than F if and only if

  $$F^*(f) \leq \lambda$$

- *iii.If F is identically equal to $+\infty$ then F^* is identically equal to $-\infty$ .Moreover ,if F is proper ,then the relation:*

  $$F^*(f) = \sup_{u \in V}(\langle f, u\rangle_{V',V} - F(u))$$

  may be restricted to the points u in the effective domain of F ($dom(F)$) .
- *iv. The function F^* is always in $\Gamma(V')$ (since F^* is the point-wise supremum of a family of affine continuous functions of V').Therefore, F^* is always o lower semi-continuous convex function on V'. Moreover, if F^*takes the value $-\infty$ then F^* is identically equal to $-\infty$.*

Proposition 0.4.2 (i) *Let F and G be tow given extend real valued functions of V into $\overline{\mathbb{R}}$, the following properties hold:*

1. *$F^*(0) = -\inf_{u \in V} F(u)$.*
2. *If F is less than G then G^* is less than F^*.*
3. *If $G(u) = F(\alpha u), \ \forall u \in V$, with $\alpha \neq 0$ then $G^*(f) = F^*(f/\alpha)$,$\forall f \in V'$.*
4. *$(\alpha F)^*(f) = \alpha F^*(f/\alpha), \ \forall f \in V', \ \ \forall \alpha > 0$.*
5. *$(F + \beta)^* = F^* - \beta, \ \ \forall \beta \in \mathbb{R}$.*

(ii) Given a family $(F_i)_{i\in J}$ *of functions from* V *into* $\overline{\mathbb{R}}$*, we have*

$$(\inf_{i\in J} F_i)^* = \sup_{i\in J} F_i^*$$

$$\sup_{i\in J} F_i^* \leq \inf_{i\in J}(F_i)^*$$

(iii)For every $a \in V$ *we denote by* F_a *the translated function (i.e,*$F_a(u) = F(u-a), \forall u \in V$*).Then* ${F_a}^*(f) = F^*(f) + \langle f, u\rangle_{V',V}, \quad \forall f \in V'$*.*

Theorem 0.4.1 *(Fenchel duality)Let* V *be a locally convex Hausdorff topological vector space over* $\mathbb{R}$ *with its dual* V'*. Let* F *and* G *be two power convex functions of* V *into* $\overline{\mathbb{R}}$ *.Assume that there exists* $u_0 \in dom(F) \cap dom(G)$ *such that* F *is continuous in* u_0 *.Then*

$$\inf_{u\in V}(F(u) + G(u)) = \sup_{f\in V'}(-F^*(-f) - G^*(f)).$$

Proof: From Fenchel inequality, we have for any function H

$$H^*(f) + H(u) \geq (\langle f, u\rangle_{V',V}, \quad \forall u \in V \ , \ \forall f \in V'$$

consequently, we have that

$$\inf_{u\in V}(F(u) + G(u)) \geq \sup_{f\in V'}(-F^*(-f) - G^*(f)).$$

(this fact is usually referred to as weak duality).
Denote $p := \inf_{u\in V}(F(u) + G(u))$, $q := \sup_{f\in V'}(-F^*(-f) - G^*(f))$ and $C := epiF$. To complete the proof, we show that $p \leq q$.
If $p = -\infty$ there is nothing to prove.Suppose now that $p \neq -\infty$.
It is clear that the interior of C : $intC$ is not empty (because F is continuous in u_0).
We introduce now the following sets:

$$A := intC,$$

$$B := \{(u, \lambda) \in V \times \mathbb{R} : \lambda \leq p - G(u)\}$$

The set A and B are convex (since F and G are convex) and disjoint (according to the definition of p), therefore, (because of Hahn-Banach's first geometric form) there exist a non zero continuous linear function $f \in V'$ and $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$H = \{(u, \lambda) \in V \times \mathbb{R} : (\langle f, u\rangle_{V',V} + \alpha\lambda = \beta\}$$

and

$$(3) \qquad \begin{array}{l} (\langle f, u\rangle_{V',V} + \alpha\lambda \geq \beta, \forall(u, \lambda) \in C, \\ (\langle f, u\rangle_{V',V} + \alpha\lambda \leq \beta, \forall(u, \lambda) \in B, \end{array}$$

By taking $u = u_0$ in the first part of the last inequality and by passing to the limit on $(\lambda \longrightarrow +\infty)$ we can deduce that $\alpha \geq 0$.

Prove now that $\alpha \neq 0$; for this we proceed by contradiction. Assume that $\alpha = 0$, then according to the last inequalities, we arrive at

$$\langle f, u\rangle_{V',V} \geq \beta, \forall u \in dom(F), \text{ and }, \langle f, u\rangle_{V',V} \leq \beta, \forall u \in dom(G).$$

In particular $\langle f, u_0\rangle_{V',V} = \beta$ (since $u_0 \in dom(F) \cap dom(G)$) and then $\langle f, u - u_0\rangle_{V',V} \geq 0$ for all u in $dom(F)$ .Consequently, $f = 0$ since $dom(F)$ is neighborhood of u_0. We thus have $\alpha > 0$.
According to

$$(4) \qquad \begin{array}{l} \langle f, u\rangle_{V',V} + \alpha\lambda \geq \beta, \forall (u, \lambda) \in C, \\ \langle f, u\rangle_{V',V} + \alpha\lambda \leq \beta, \forall (u, \lambda) \in B, \end{array}$$

and dividing by $\alpha > 0$, we obtain easily that

$$F^*(-f_\alpha) \leq -\beta_\alpha,$$

$$G^*(f_\alpha) \leq \beta_\alpha - p$$

and then $f_\alpha = f/\alpha$ and $\beta_\alpha = \beta/\alpha$.
Therefore ,$p \leq q$.This complete the proof.
Examples
1.Let C be a non-empty subset of topological vector space V and χ_C be its indicator function. Then the conjugate function ${\chi_C}^*$ is defined by

$${\chi_C}^*(f) = \sup_{u \in C} \langle f, u\rangle_{V',V}$$

and is called the support function of C. Moreover, if C is a closed and convex set, χ_C is closed and convex, and by the conjugacy theorem the conjugate of its support function is its indicator function.
2.Let $(V, \|.\|)$ be a Banach space, $(V', \|.\|_*)$ its dual, $\Psi_\alpha : t \in \mathbb{R} \longrightarrow |t|^\alpha/\alpha$ and $F_\alpha : V \longrightarrow \mathbb{R}$ such that $F_\alpha(u) = \Psi_\alpha(\|u\|)$, where $1 < \alpha < \infty$. Then

$$\begin{aligned} F^*_\alpha(f) &= \sup_{u \in V} (\langle f, u\rangle_{V',V} - F_\alpha(u)) \\ &= \sup_{\lambda \geq 0} \left(\|f\|_* \lambda - \frac{\lambda^\alpha}{\alpha} \right) \end{aligned}$$

Hence (by analyzing the function $r(\lambda) := \theta\lambda - \lambda^\alpha/\alpha$ where $\theta := \|f\|_*$ and $\lambda \in [0, +\infty[$, $F^*_\alpha(f) = \|f\|_*^{\alpha^*}/\alpha^*$ where $1/\alpha + 1/\alpha^* = 1$.Consequently

$$F^*_\alpha(f) = \Psi_{\alpha^*}(\|f\|_*)$$

3.We finish with an interesting example for the boundary valued problems in a lemma form.

Lemma 0.4.1 *Let $(V, \|.\|)$ be a Banach space, $(V', \|.\|_*)$ its dual and C be a non-empty closed and convex subset or V. Consider the convex and lower semi-continuous real-valued function F on V given by*

$$F(v) := \langle f, v\rangle_{V',V} + \chi_C(v - u) \quad \forall v \in V$$

where $u \in V$ and $f \in V'$ are given elements.
then the conjugate of F is

$$F^*(g) = \langle g - f, u\rangle_{V',V} + \chi_{C^*}(g - f) \quad \forall g \in V'$$

where $C^* = \{g \in V' : \ \langle g, v\rangle_{V',V} = 0 \ \ \forall v \in C\}$ (which is said to be the polar set of C)

Proof.Let $g \in V'$, we have

$$\begin{aligned} F^*(g) &= \sup_{v\in V}(\langle g, v\rangle_{V',V} - \langle f, v\rangle_{V',V} - \chi_C(v - u)) \\ &= \sup_{w\in C}\langle g - f, w + u\rangle_{V',V} \\ &= \langle g - f, u\rangle_{V',V} + \sup_{w\in C}\langle g - f, w\rangle_{V',V} \end{aligned}$$

This completes the proof (since $\sup_{w\in C}\langle g - f, w\rangle_{V',V} = \chi^*_C(g - f) = \chi_{C^*}(g - f)$).

0.4.2 Legendre transformation

In mathematics, the Legendre transformation or Legendre transform, named after Adrien-Marie Legendre, is an operation that transforms one real-valued function of a real variable into another. Specifically, the Legendre transform of a convex function F is the function F^* defined by

$$F^*(p) = \sup(px - F(x))$$

where "sup" represents the supremum. If F is differentiable, then $F^*(p)$ can be interpreted as the negative of the y-intercept of the tangent line to the graph of F that has slope p. In particular, the value of x that attains the maximum has the property :$F'(x) = p$

That is, the derivative of the function F becomes the argument to the function F^*. In particular, if F is convex (or concave up), then F^* satisfies the functional equation

$$F^*(F'(x)) = xF'(x) - F(x)$$

The Legendre transform is its own inverse. Like the familiar Fourier transform, the Legendre transform takes a function F and produces a function of a different variable p. However, while the Fourier transform consists of an integration with a kernel, the Legendre transform uses maximization as the transformation procedure. The transform is especially well behaved if F is a convex function. The Legendre transformation is an application of the duality relationship between points and lines. The functional relationship specified by F can be represented equally well as a set of (x, y) points, or as a set of tangent lines specified by their slope and intercept values. The Legendre transformation can be generalized to the Legendre-Fenchel transformation. It is commonly used in thermodynamics and in the Hamiltonian formulation of classical mechanics.

0.4.3 Jensen inequality

Let (Ω, A, μ) be a measure space, such that $\mu(\Omega) = 1$. If g is a real-valued function that is μ-integrable, and if φ is a convex function on the real line, then:

$$\varphi\left(\int_\Omega g\ d\mu\right) \leq \int_\Omega \varphi \circ g\ d\mu$$

In real analysis, we may require an estimate on $\varphi\left(\int_a^b g(x)\ dx\right)$ where a, b are real numbers, and g is a non-negative real-valued function that is Lebesgue-integrable. In this case, the Lebesgue measure of $[a, b]$ don't need to be unity. However, by integration by substitution, the interval can be rescaled so that it has measure unity. Then Jensen's inequality can be applied to get

$$\varphi\left(\int_a^b g(x)\ dx\right) \leq \frac{1}{b-a}\int_a^b \varphi((b-a)g(x))\ dx$$

0.5 Aubin -Lions lemma

The Aubin Lions lemma is a result in the theory of Sobolev spaces of Banach space-valued functions. More precisely, it is a compactness criterion that is very useful in the study of nonlinear evolutionary partial differential equations. The result is named after the French mathematicians Thierry Aubin and Jacques-Louis Lions. We complete the preliminaries by the useful inequalities of Gagliardo-Nirenberg and Sobolev-Poincaré.

Lemma 0.5.1 *Let X_0, X and X_1 be three Banach spaces with $X_0 \subseteq X \subseteq X_1$. Assume that X_0 is compactly embedded in X and that X is continuously embedded in X_1; assume also that X_0 and X_1 are reflexive spaces. For $1 < p, q < +\infty$, let*

$$W = \{u \in L^p([0,T]; X_0) / \quad \dot{u} \in L^q([0,T]; X_1)\}$$

Then the embedding of W into $L^p([0,T]; X)$ is also compact.

Lemma 0.5.2 (Gagliardo-Nirenberg) *Let $1 \leq r < q \leq +\infty$ and $p \leq q$. Then, the inequality*

$$\|u\|_{W^{m,q}} \leq C\|u\|^\theta_{W^{m,p}}\|u\|^{1-\theta}_r \quad \text{for} \quad u \in W^{m,p} \bigcap L^r$$

holds with some $C > 0$ and

$$\theta = \left(\frac{k}{n} + \frac{1}{r} - \frac{1}{q}\right)\left(\frac{m}{n} + \frac{1}{r} - \frac{1}{p}\right)^{-1}$$

provided that $0 < \theta \leq 1$ (we assume $0 < \theta < 1$ if $q = +\infty$).

Lemma 0.5.3 (Sobolev-Poincaré inequality) *Let q be a number with $2 \leq q < +\infty$ $(n = 1, 2)$ or $2 \leq q \leq 2n/(n-2)$ $(n \geq 3)$, then there is a constant $c_* = c(\Omega, q)$ such that*

$$\|u\|_q \leq c_*\|\nabla u\|_2 \quad \text{for} \quad u \in H^1_0(\Omega).$$

Chapter 1

GLOBAL EXISTENCE AND ENERGY DECAY OF SOLUTIONS TO A VISCOELASTIC TIMOSHENKO BEAM SYSTEM WITH A NONLINEAR DELAY TERM

1.1 Introduction

In this chapter, we investigate the existence and decay properties of solutions to the initial boundary value problem for the nonlinear Timoshenko beam system reading as

$$(P)\quad \begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) = 0 & \text{in }]0,1[\times]0,+\infty[, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x+\psi)(x,t) + \mu_1 g_1(\psi_t(x,t)) & \\ +\displaystyle\int_0^t h(t-s)\psi_{xx}(x,s)ds + \mu_2 g_2(\psi_t(x,t-\tau)) = 0 & \text{in }]0,1[\times]0,+\infty[, \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0 & t \geq 0, \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x) & x \in]0,1[, \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x) & x \in]0,1[, \\ \psi_t(x,t-\tau) = f_0(x,t-\tau) & \text{in }]0,1[\times]0,\tau[, \end{cases}$$

where $\tau > 0$ is a time delay, μ_1 and μ_2 are positive real numbers, and the initial data $(\varphi_0, \varphi_1, \psi_0, \psi_1, f_0)$ belong to a suitable function space. The remaining parameters will be discussed in what following.

A simple model describing the transverse vibration of a beam, which was developed in **[39]**, is given by a system of coupled hyperbolic equations of the form

$$\begin{cases} \rho u_{tt}(x,t) = (K(u_x-\phi))_x & \text{in }]0,L[\times]0,+\infty[, \\ \tilde{\rho}\phi_{tt}(x,t) = (EI\phi_x)_x + K(u_x-\phi) & \text{in }]0,L[\times]0,+\infty[, \end{cases}$$

where t denotes the time variable, x is the space variable along the beam of length L, in its equilibrium configuration, u is the transverse displacement and ϕ is the rotation angle of the filament of the beam w.r.t its the midline. The coefficients $\rho, \tilde{\rho}, E, I$ and K are, respectively, the mass density, the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section and the shear modulus.

In absence of the viscoelastic term (that is, if $h = 0$), problem (P) has been studied by many mathematicians. It is well known that in the further absence of a damping mechanism, the delay term causes instability of system (see, for instance [**15**]). In contrast, in absence of the delay term, the damping term assures global existence for arbitrary initial data and energy decay estimates depending on the rate of growth of g_1 (see [**5**], [**9**], [**10**], [**11**], [**21**], [**23**], [**31**], [**32**], [**33**] and [**36**]). In addition, we would like to mention the most recent work in this direction due to Alabau-Boussouira [**1**] which is the pioneer in establishing very general explicit decay rate estimates for solutions.

In recent years, PDEs with time delay effects have become an active area of research and arise in many practical problems (see, e.g.[**3**], [**38**], [**40**]). To stabilize a hyperbolic system involving delay terms, additional control terms are necessary (see [**7**], [**35**], [**29**]).

For instance, in [**35**] the authors studied the wave equation with a linear internal damping term with constant delay and determined suitable relations between μ_1 and μ_2, for which the stability or alternatively instability takes place. More precisely, they showed that the energy is exponentially decaying if $\mu_2 < \mu_1$ and they also found a sequence of delays for which the corresponding solution of (P) will be instable if $\mu_2 \geq \mu_1$. The main approach used in [**35**], is an observability inequality obtained with a Carleman estimate. Laskri and Said-Houari [**29**] examined problem (P) in the linear situation with $h \equiv 0$, $g_1(s) = g_2(s) = s$ for all $s \in \mathbb{R}$. Under the assumption $\mu_2 \leq \mu_1$ on the weights of the two feedbacks, they proved the well-posedness of the system. They also established for $\mu_2 < \mu_1$ an exponential decay result for the case of equal wave propagation speeds. Very recently, Benaissa and Bahlil [**7**] extended the result of [**29**] to the nonlinear case.

In the presence of the viscoelastic term ($h \neq 0$), Benaissa et al.[**8**] studied a viscoelastic wave equation in the presence of nonlinear delay term. They obtained an explicit decay rate. In Guesmia et al. [**17**] considered (P) for $g_2 \equiv 0$ and studied the influence of this dissipation on the decay rate of solutions. Precisely, they obtained an explicit and general decay rate, depending on g_1 and h, for the energy of solutions without any growth assumption on g_1 at the origin and under weaker conditions on the relaxation function h.

Our goal in this chapter is to prove purpose in this chapter is to give a global solvability in Sobolev spaces and energy decay estimates of the solutions to the problem (P) for a nonlinear damping and a delay term.

We should mention here that, to the best of our knowledge, there is no result concerning Timochenko beam system in the presence of viscoelastic and nonlinear degenerate delay terms.

To obtain global solutions to the problem (P), we use the argument combining the Galerkin approximation scheme (see [**8, 25**]) with the energy method.

To prove decay estimates, we use a perturbed energy method and some properties of convex functions. These arguments of convexity were introduced and developed by [**24**] and used by Liu and Zuazua [**26**] and Alabau-Boussouira [**1**].

1.2 Preliminaries and main results

For the relaxation function, the damping and the delay functions, we make the following hypotheses:
(H1) $(*)$ $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^2 function satisfying

$$h(0) = h_0 > 0, \qquad l = \int_0^{+\infty} h(s)ds < b.$$

$(**)$ There exists a nonincreasing differentiable function $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$h'(s) \leq -\zeta(s)h(s), \qquad \forall s \geq 0.$$

and

$$\int_0^{+\infty} \zeta(s)\, ds = +\infty.$$

(H2) $g_1 : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function of the class $C(\mathbb{R})$ such that there exist ϵ', c_1, $c_2 > 0$ and a convex and increasing function $H : \mathbb{R}_+ \to \mathbb{R}_+$ of the class $C^1(\mathbb{R}_+) \cap C^2(]0, \infty[)$ satisfying $H(0) = 0$, and H linear on $[0, \epsilon']$ or ($H'(0) = 0$ and $H'' > 0$ on $]0, \epsilon']$), such that

$$c_1|s| \leq |g_1(s)| \leq c_2|s| \quad \text{if } |s| \geq \epsilon', \tag{1.1}$$

$$g_1^2(s) \leq H^{-1}(sg_1(s)) \quad \text{if } |s| \leq \epsilon'. \tag{1.2}$$

$g_2 : \mathbb{R} \to \mathbb{R}$ is an odd nondecreasing function of the class $C^1(\mathbb{R})$ such that there exist $c_3, \alpha_1, \alpha_2 > 0$

$$|g_2'(s)| \leq c_3, \tag{1.3}$$

$$\alpha_1 \; sg_2(s) \leq G_2(s) \leq \alpha_2 \; sg_1(s), \tag{1.4}$$

where

$$G_2(s) = \int_0^s g_2(r)\, dr$$

and

$$\alpha_2\mu_2 < \alpha_1\mu_1. \tag{1.5}$$

(H3)

$$\frac{\rho_1}{K} = \frac{\rho_2}{b}.$$

In the following, we will need:

Lemma 1.2.1 (Sobolev-Poincaré's inequality) *Let q be a number with $2 \leq q < +\infty$ $(n = 1, 2)$ or $2 \leq q \leq 2n/(n-2)$ $(n \geq 3)$. Then there is a constant $c_* = c_*((0,1), q)$ such that*

$$\|\psi\|_q \leq c_* \|\nabla\psi\|_2 \quad \textit{for} \quad \psi \in H_0^1((0,1)).$$

We introduce as in **[35]** the new variable

$$z(x,\rho,t)=\psi_t(x,t-\tau\rho), x\in(0,1),\ \rho\in(0,1),\quad t>0. \tag{1.6}$$

Then, we have

$$\tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0,\quad \text{in } (0,1)\times(0,1)\times(0,+\infty). \tag{1.7}$$

Therefore, problem (P) is equivalent to:

$$\begin{cases} \rho_1\varphi_{tt}(x,t)-K(\varphi_x+\psi)_x(x,t)=0, & \text{in }]0,1[\times]0,+\infty[,\\ \rho_2\psi_{tt}(x,t)-b\psi_{xx}(x,t)+K(\varphi_x+\psi)(x,t) & \\ +\mu_1 g_1(\psi_t(x,t))+\displaystyle\int_0^t h(t-s)\psi_{xx}(x,s)ds & \\ +\mu_2 g_2(z(x,1,t))=0, & \text{in }]0,1[\times]0,+\infty[,\\ \tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0, & \text{in }]0,1[\times]0,1[\times]0,+\infty[,\\ \varphi(0,t)=\varphi(1,t)=\psi(0,t)=\psi(1,t)=0, & t\geq 0,\\ z(x,0,t)=\psi_t(x,t), & \text{on }]0,1[\times[0,+\infty[,\\ \psi(x,0)=\psi_0(x),\quad \psi_t(x,0)=\psi_1(x) & x\in]0,1[,\\ \varphi(x,0)=\varphi_0(x),\quad \varphi_t(x,0)=\varphi_1(x), & x\in]0,1[,\\ z(x,\rho,0)=f_0(x,-\rho\tau), & \text{in }]0,1[\times]0,1[. \end{cases} \tag{1.8}$$

Let ξ be a positive constant such that

$$\tau\frac{\mu_2(1-\alpha_1)}{\alpha_1}<\xi<\tau\frac{\mu_1-\alpha_2\mu_2}{\alpha_2}. \tag{1.9}$$

We define the energy associated to the solution of the problem (1.8) by the means of:

$$\begin{aligned} &E(t)=E(t,z,\varphi,\psi)=\frac{1}{2}\int_0^1\left\{\rho_1\varphi_t^2+\rho_2\psi_t^2+K|\varphi_x+\psi|^2\right\}dx\\ &+\frac{1}{2}\left(b-\int_0^t h(s)\,ds\right)\int_0^1\psi_x^2\,dx+\frac{1}{2}(h\circ\psi_x)(t)+\xi\int_0^1\int_0^1 G_2(z(x,\rho,t))\,d\rho\,dx. \end{aligned} \tag{1.10}$$

In the Sections 3 and 4 below, we prove the following theorem.

Theorem 1.2.1 *Let* $(\varphi_0,\varphi_1),(\psi_0,\psi_1)\in(H^2(0,1)\cap H_0^1(0,1))\times H_0^1(0,1)$, $f_0\in H_0^1((0,1);H^1(0,1))$ *satisfy the compatibility condition*

$$f_0(.,0)=\psi_1.$$

Assume that the hypothesis $(H1)$,$(H2)$ *and* $(H3)$ *hold. Then problem* (P) *admits a unique weak solution*

$$\begin{aligned} &\varphi\in L^\infty_{loc}(0,\infty;H^2(0,1)\cap H_0^1(0,1)),\quad \varphi_t\in L^\infty_{loc}(0,\infty;H_0^1(0,1)),\\ &\varphi_{tt}\in L^\infty_{loc}(0,\infty;L^2(0,1))\\ &\psi\in H^2(-\tau,0;H_0^1(0,1))\cap L^\infty_{loc}(-\tau,\infty;H^2(0,1)\cap H_0^1(0,1))\\ &\psi_t\in H^1(-\tau,0;H_0^1(0,1))\cap L^\infty_{loc}(-\tau,\infty;H_0^1(0,1))\\ &\psi_{tt}\in H^1(-\tau,0;H_0^1(0,1))\cap L^\infty_{loc}(0,\infty;L^2(0,1)) \end{aligned}$$

and, for some constants ω, ϵ_0 *we obtain the following decay property:*

$$E(t) \leq H_1^{-1}\left(\omega \int_0^t \zeta(s)\, ds\right), \quad \forall t > 0, \tag{1.11}$$

where

$$H_1(t) = \int_t^1 \frac{1}{H_2(s)}\, ds \tag{1.12}$$

and

$$H_2(t) = \begin{cases} t & \text{if } H \text{ is linear on } [0, \epsilon'], \\ tH'(\epsilon_0 t) & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0, \epsilon']. \end{cases}$$

Remark 1.2.1

(1) *the conditions of* $(H2)$*(with* $\epsilon_1 = 1$*) were introduced and employed by Lasiecka et al.* **[9, 24]** *in their study of the asymptotic behavior of solutions of nonlinear wave equations where they obtained decay estimates which depend e solution* $S(t)$ *of an explicit nonlinear ordinary differential equation*

$$S_t + r(S) = 0, \quad S(0) = E(0) = S_0.$$

where $r(s)$ *is a continuous, monotone increasing function which behaves as* $H(s)$ *at the origin. We mention here that Alabau-Boussouira* **[1]***) was e first who give an explicit expression of the function* r *at the origin.*

(2) *We need condition* $(H3)$ *only to establish a general decay estimate for the solutions of systems. Note that in general, the condition of equality of the speeds of wave propagation not have a physical sense, but from the mathematics point of view it is very important because system is weakly hyperbolic of constant multiplicity of order* 2*. We think that the interaction of the hyperbolicity (order of multiplicity) and the number of dissipative terms have an effect on the result.*

Example 1.2.1 *[**see**[8]]*

Let g *be given by* $g(s) = s^p(-\ln s)^q$*, where* $0 \leq p \leq 1$ *and* $q \in \mathbb{R}$ *on* $(0, \epsilon_1]$*. Then* $g'(s) = s^{p-1}(-\ln s)^{q-1}(p(-\ln s) - q)$ *which is an increasing function in a right neighborhood of* 0 *(if* $q = 0$ *we can take* $\epsilon_1 = 1$*). The function* H *is defined in the neighborhood of* 0 *by*

$$H(s) = \begin{cases} cs^{\frac{p+1}{2p}}(-\ln s)^{-\frac{q}{p}} & \text{if } 0 < p < 1, \quad q \in \mathbb{R} \\ cs(-\ln s)^{-q} & \text{if } p = 1, \quad q > 0 \\ c\sqrt{s}\ e^{-s^{\frac{1}{2q}}} & \text{if } p = 0, \quad q < 0 \end{cases}$$

and we have

$$H'(s) = \begin{cases} cs^{\frac{1-p}{2p}}(-\ln s)^{-\frac{p+q}{p}}\left(\dfrac{p+1}{2p}(-\ln s) + \dfrac{q}{p}\right) & \text{if } 0 < p \leq 1, \quad q \in \mathbb{R} \\ c\dfrac{1}{\sqrt{s}}\left(1 - \dfrac{1}{q}\ s^{\frac{1}{2q}}\right)e^{-s^{\frac{1}{2q}}} & \text{if } p = 0, \quad q < 0 \end{cases} \quad \text{when } s \text{ is near } 0.$$

Thus

$$H_2(s) = \begin{cases} cs^{\frac{p+1}{2p}} (-\ln s)^{-\frac{p+q}{p}} \left(\frac{p+1}{2p}(-\ln s) + \frac{q}{p}\right) & \text{if } 0 < p \leq 1, \quad q \in \mathbb{R} \\ c\sqrt{s}\left(1 - \frac{1}{q}\, s^{\frac{1}{2q}}\right) e^{-s^{\frac{1}{2q}}} & \text{if } p = 0, \quad q < 0 \end{cases} \quad \text{when } s \text{ is near } 0.$$

and

$$\begin{aligned} H_1(t) &= c\int_t^1 \frac{1}{s^{\frac{p+1}{2p}} (-\ln s)^{-\frac{p+q}{p}} \left(\frac{p+1}{2p}(-\ln s) + \frac{q}{p}\right)} ds \\ &= c\int_1^{\frac{1}{t}} \frac{z^{\frac{1-3p}{2p}}}{(\ln z)^{-\frac{p+q}{p}} \left(\frac{p+1}{2p}\ln z + \frac{q}{p}\right)} dz \ \text{ when } t \text{ is near } 0. \end{aligned}$$

and

$$\begin{aligned} H_1(t) &= c\int_t^1 \frac{1}{\sqrt{s}\left(1 - \frac{1}{q}\, s^{\frac{1}{2q}}\right) e^{-s^{\frac{1}{2q}}}} ds \\ &= c\int_1^{\frac{1}{t}} \frac{e^{(\frac{1}{z})^{\frac{1}{2q}}}}{z^{\frac{3}{2}}(1 - \frac{1}{q}\, (\frac{1}{z})^{\frac{1}{2q}})} dz, \quad p = 0, q < 0, \ \text{when } t \text{ is near } 0 \end{aligned}$$

We obtain in a neighborhood of 0

$$H_1(t) \equiv \begin{cases} ct^{\frac{p-1}{2p}} (-\ln t)^{\frac{q}{p}} & \text{if } 0 < p < 1, \quad q \in \mathbb{R} \\ c(-\ln t)^{1+q} & \text{if } p = 1,\ q > 0, \\ ct^{\frac{q-2}{2q}} e^{t^{\frac{1}{2q}}} & \text{if } p = 0,\ q < 0 \end{cases}$$

and then in a neighborhood of $+\infty$

$$H_1^{-1}(t) \equiv \begin{cases} ct^{-\frac{2p}{p-1}} (\ln t)^{-\frac{2q}{p-1}} & \text{if } 0 < p \leq 1, \quad q \in \mathbb{R}, \\ ce^{-t^{\frac{1}{1+q}}} & \text{if } p = 1,\ q > 0, \\ c(\ln t)^{2q} & \text{if } p = 0,\ q < 0. \end{cases}$$

Using the fact that $h(t) = t$ *as* t *goes to infinity, then*

$$E(t) \leq \begin{cases} c\tilde{\xi}(t)^{-\frac{2p}{p-1}} (\ln \tilde{\xi}(t))^{-\frac{2q}{p-1}} & \text{if } 0 < p \leq 1, \quad q \in \mathbb{R}, \\ ce^{-\tilde{\xi}(t)^{\frac{1}{1+q}}} & \text{if } p = 1,\ q < 1, \\ c(\ln \tilde{\xi}(t))^{2q} & \text{if } p = 0,\ q < 0, \\ ce^{-\tilde{\xi}(t)} & \text{if } p > 1 \text{ or } p = 1 \text{ and } q \leq 0. \end{cases}$$

where $\bar{\xi}(t) = \int_0^t \zeta(s) ds.$

We finish this section by giving an explicit upper bound for the derivative of energy.

Lemma 1.2.2 *Let (φ, ψ, z) be a solution of the problem (1.8). Then the energy functional defined by (1.10) satisfies*

$$
\begin{aligned}
E'(t) \;\le\; & -\left(\mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha_2\right)\int_0^1 \psi_t g_1(\psi_t)\,dx \\
& -\left(\frac{\xi}{\tau}\alpha_1 - \mu_2(1-\alpha_1)\right)\int_0^1 z(x,1,t)g_2(z(x,1,t))\,dx - \frac{1}{2}h(t)\|\psi_x(.,t)\|_2^2 \\
& +\frac{1}{2}(h'\circ\psi_x)(t) \\
\le\; & 0
\end{aligned}
\tag{1.13}
$$

***Proof*:** Multiplying the first equation in (1.8) by φ_t, the second equation by ψ_t, integrating over $(0,1)$ and using integration by parts, we get

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\left(\int_0^1 \left\{\rho_1\varphi_t^2 + \rho_2\psi_t^2 + K|\varphi_x+\psi|^2 + b\psi_x^2\right\}dx\right) + \mu_1\int_0^1 \psi_t g_1(\psi_t)\,dx \\
& = -\mu_2\int_0^1 \psi_t(x,t)g_2(z(x,1,t))\,dx + \int_0^1\int_0^t h(t-s)\psi_x(x,s)\psi_{tx}(x,t)\,ds\,dx.
\end{aligned}
\tag{1.14}
$$

The term in the right-hand side of (1.14) can be rewritten as follows

$$
\begin{aligned}
& \int_0^1\int_0^t h(t-s)\psi_x(x,s)\psi_{tx}(x,t)\;ds\;dx + \frac{1}{2}h(t)\;\|\psi_x(.,t)\|_2^2 \\
& = \frac{1}{2}\frac{d}{dt}\left[\int_0^t h(s)\,ds\|\psi_x(.,t)\|_2^2 - (h\circ\psi_x)(t)\right] + \frac{1}{2}(h'\circ\psi_x)(t).
\end{aligned}
$$

Consequently, equality (1.14) becomes

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\left(\int_0^1 \left\{\rho_1\varphi_t^2 + \rho_2\psi_t^2 + K|\varphi_x+\psi|^2\right\}dx\right) \\
& +\frac{1}{2}\frac{d}{dt}\left(\left(b - \int_0^t h(s)ds\right)\int_0^1 \psi_x^2 dx - (h\circ\psi_x)(t)\right) + \mu_1\int_0^1 \psi_t g_1(\psi_t)\,dx \\
& = -\mu_2\int_0^1 \psi_t(x,t)g_2(z(x,1,t))\,dx - \frac{1}{2}h(t)\|\psi_x(.,t)\|_2^2 + \frac{1}{2}(h'\circ\psi_x)(t).
\end{aligned}
\tag{1.15}
$$

We multiply the third equation in (1.8) by $\xi g_2(z(x,\rho,t))$ and integrate the result over $(0,1)\times(0,1)$ to obtain:

$$
\begin{aligned}
\xi\int_0^1\int_0^1 z' g_2(z(x,\rho,t))\,d\rho\,dx &= -\frac{\xi}{\tau}\int_0^1\int_0^1 \frac{\partial}{\partial\rho}G_2(z(x,\rho,t))\,d\rho\,dx \\
&= -\frac{\xi}{\tau}\int_0^1 (G_2(z(x,1,t)) - G_2(z(x,0,t)))\,dx.
\end{aligned}
\tag{1.16}
$$

Then

$$
\xi\frac{d}{dt}\int_0^1\int_0^1 G_2(z(x,\rho,t))\,d\rho\,dx = -\frac{\xi}{\tau}\int_0^1 G_2(z(x,1,t))\,dx + \frac{\xi}{\tau}\int_0^1 G_2(\psi_t)\,dx.
\tag{1.17}
$$

Using (1.14), (1.17) and Young's inequality, we get

$$
\begin{aligned}
E'(t) \le &-\left(\mu_1 - \frac{\xi\alpha_2}{\tau}\right)\int_0^1 \psi_t(x,t)g_1(\psi_t(x,t))\,dx - \frac{\xi}{\tau}\int_0^1 G_2(z(x,1,t))\,dx \\
&-\mu_2\int_0^1 \psi_t(x,t)g_2(z(x,1,t))\,dx - \frac{1}{2}h(t)\|\psi_x(.,t)\|_2^2 + \frac{1}{2}(h'\circ\psi_x)(t).
\end{aligned} \tag{1.18}
$$

Let G_2^* be the conjugate function of the convex function G_2, i.e., $G_2^*(s) = \sup_{t\in\mathbb{R}^+}(st - G_2(t))$. Then G_2^* is the Legendre transform of G_2, which is given by

$$
G_2^*(s) = s(G_2')^{-1}(s) - G_2[(G_2')^{-1}(s)], \quad \forall s \ge 0 \tag{1.19}
$$

(see Arnold [28], p.61-62) and satisfies the following inequality

$$
st \le G_2^*(s) + G_2(t), \quad \forall s,t \ge 0. \tag{1.20}
$$

Then, from the definition of G_2, we get

$$
G_2^*(s) = sg_2^{-1}(s) - G_2(g_2^{-1}(s)).
$$

Hence,

$$
\begin{aligned}
G_2^*(g_2(z(x,1,t))) &= z(x,1,t)g_2(z(x,1,t)) - G_2(z(x,1,t)) \\
&\le (1-\alpha_1)z(x,1,t)g_2(z(x,1,t)).
\end{aligned} \tag{1.21}
$$

Making use of (1.18), (1.20) and (1.21), we have

$$
\begin{aligned}
E'(t) \le &-\left(\mu_1 - \frac{\xi\alpha_2}{\tau}\right)\int_0^1 \psi_t g_1(\psi_t)\,dx - \frac{\xi}{\tau}\int_0^1 G_2(z(x,1,t))\,dx \\
&+\mu_2\int_0^1 \left(G_2(\psi_t) + G_2^*(g(z(x,1,t)))\right)\,dx - \frac{1}{2}h(t)\|\psi_x(.,t)\|_2^2 + \frac{1}{2}(h'\circ\psi_x)(t) \\
\le &-\left(\mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha_2\right)\int_0^1 \psi_t g_1(\psi_t)\,dx - \frac{\xi}{\tau}\int_0^1 G_2(z(x,1,t))\,dx \\
&+\mu_2\int_0^1 G_2^*(g_2(z(x,1,t)))\,dx - \frac{1}{2}h(t)\|\psi_x(.,t)\|_2^2 + \frac{1}{2}(h'\circ\psi_x)(t).
\end{aligned} \tag{1.22}
$$

Using (1.4) and (1.9), we obtain

$$
\begin{aligned}
E'(t) \le &-\left(\mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha_2\right)\int_0^1 \psi_t g_1(\psi_t)\,dx \\
&-\left(\frac{\xi}{\tau}\alpha_1 - \mu_2(1-\alpha_1)\right)\int_0^1 z(x,1,t)g_2(z(x,1,t))\,dx - \frac{1}{2}h(t)\|\psi_x(.,t)\|_2^2 + \frac{1}{2}(h'\circ\psi_x)(t) \\
\le &\ 0.
\end{aligned}
$$

1.3 Global Existence

We are now ready to prove Theorem 1.2.1 in the next two sections.

Throughout this section we assume $\varphi_0, \psi_0 \in H^2 \cap H_0^1(0,1), \varphi_1, \psi_1 \in H_0^1(0,1)$ and $f_0 \in H_0^1((0,1); H^1(0,1))$.

We employ the Galerkin method to construct a global solution. We follow the method in [**8**] with the necessary anges due to the presence of a coupled system of hyperbolic equations.

Let $T > 0$ be fixed and denote by V_k the space generated by $\{w_1, w_2, \ldots, w_k\}$ where the set $\{w_k, k \in \mathbb{N}\}$ is a basis of $H^2 \cap H_0^1$.

Now, we define for $1 \le j \le k$ the sequence $\phi_j(x, \rho)$ as follows:

$$\phi_j(., 0) = w_j.$$

Then, we can extend to an element of $H^2 \cap H_0^1((0,1); H^1(0,1))$ and denote with Z_k the linear space generated by $\{\phi_1, \phi_2, \ldots, \phi_k\}$.

We construct approximate solutions (φ_k, ψ_k, z_k), $k = 1, 2, 3, \ldots,$ in the form

$$\varphi_k(t) = \sum_{j=1}^{k} g_{jk} w_j, \quad \psi_k(t) = \sum_{j=1}^{k} \tilde{g}_{jk} w_j, \quad z_k(t) = \sum_{j=1}^{k} h_{jk} \phi_j,$$

where $g_{jk}, \tilde{g}_{jk}$ and $h_{jk}, j = 1, 2, \ldots, k$, are determined by the following ordinary integro-differential equations:

$$\rho_1(\varphi_k''(t), w_j) + K(\varphi_{kx}(t), w_{jx}) - k(\psi_{kx}(t), w_j) = 0, \quad 1 \le j \le k, \tag{1.23}$$

$$\varphi_k(0) = \varphi_{0k} = \sum_{j=1}^{k} (\varphi_0, w_j) w_j \to \varphi_0 \text{ in } H^2 \cap H_0^1 \text{ as } k \to +\infty, \tag{1.24}$$

$$\varphi_k'(0) = \varphi_{1k} = \sum_{j=1}^{k} (\varphi_1, w_j) w_j \to \varphi_1 \text{ in } H_0^1 \text{ as } k \to +\infty. \tag{1.25}$$

$$\begin{cases} \rho_2(\psi_k''(t), w_j) + b(\psi_{kx}(t), w_{jx}) + K((\varphi_{kx} + \psi_k)(t), w_j) + \mu_1(g_1(\psi_k'), w_j) \\ - \displaystyle\int_0^t h(t-s)(\psi_{kx}(s), w_{jx})\, ds + \mu_2(g_2(z_k(., 1)), w_j) = 0 \quad 1 \le j \le k, \\ z_k(x, 0, t) = \psi_k'(x, t) \end{cases} \tag{1.26}$$

$$\psi_k(0) = \psi_{0k} = \sum_{j=1}^{k} (\psi_0, w_j) w_j \to \psi_0 \text{ in } H^2 \cap H_0^1 \text{ as } k \to +\infty, \tag{1.27}$$

$$\psi_k'(0) = \psi_{1k} = \sum_{j=1}^{k} (\psi_1, w_j) w_j \to \psi_1 \text{ in } H_0^1 \text{ as } k \to +\infty. \tag{1.28}$$

and

$$(\tau z_{kt} + z_{k\rho}, \phi_j) = 0, \quad 1 \le j \le k, \tag{1.29}$$

$$z_k(\rho, 0) = z_{0k} = \sum_{j=1}^{k} (f_0, \phi_j) \phi_j \to f_0 \text{ in } H_0^1((0,1); H^1(0,1)) \text{ as } k \to +\infty. \tag{1.30}$$

By virtue of the theory of ordinary differential equations, the system (1.23)-(1.30) has a unique local solution which is extended to a maximal interval $[0, T_k[$ (with $0 < T_k \le +\infty$) by Zorn lemma since the nonlinear terms in (1.26) are locally Lipschitz continuous. Note that $(\varphi_k(t), \psi_k(t))$ is of the class C^2.

In the next step, we obtain a priori estimates for the solution such that it can be extended beyond $[0, T_k[$ to obtain a single solution defined for all $t > 0$.

We can utilize a standard compactness argument for the limiting procedure, for which it suffices to derive some a priori estimates for $(\varphi_k, \psi_k, z_k)_k$.

The first estimate. Since the sequences $(\varphi_{0k})_k$,$(\varphi_{1k})_k$,$(\psi_{0k})_k$, $(\psi_{1k})_k$ and $(z_{0k})_k$ converge, standard calculations, using (1.23)-(1.30), similar to those used to derive (1.13), yield a number C independent of k such that

$$\begin{aligned} E_k(t) + a_1 \int_0^t \int_0^1 \psi_k' g_1(\psi_k') dx\, ds + a_2 \int_0^t \int_0^1 z_k(x,1,s) g_2(z_k(x,1,s)) dx\, ds \\ + \frac{1}{2} \int_0^t h(s) \|\psi_{kx}(.,s)\|_2^2 ds - \frac{1}{2} \int_0^t (h' \circ \psi_{kx})(s) ds \le E_k(0) \le C, \end{aligned} \tag{1.31}$$

where

$$\begin{aligned} E_k(t) &= \frac{1}{2} \int_0^1 \{\rho_1 {\varphi_k'}^2 + \rho_2 {\psi_k'}^2 + K |\varphi_{kx} + \psi_k|^2\} dx \\ &+ \frac{1}{2} \left(b - \int_0^t h(s)\, ds \right) \int_0^1 \psi_{kx}^2\, dx + \frac{1}{2} (h \circ \psi_{kx})(t) + \xi \int_0^1 \int_0^1 G_2(z_k(x,\rho,t))\, d\rho\, dx. \end{aligned}$$

$$a_1 = \mu_1 - \frac{\xi}{\tau} \alpha_2 - \mu_2 \alpha_2 \text{ and } a_2 = \frac{\xi}{\tau} \alpha_1 - \mu_2 (1 - \alpha_1).$$

for some C independent of k. These estimates imply that the solution (φ_k, ψ_k, z_k) exists globally in $[0, +\infty[$.
Estimate (1.31) yields

$$\varphi_k, \psi_k \text{ are bounded in } L^\infty(0,T; H_0^1(0,1)) \tag{1.32}$$

$$\varphi_k', \psi_k' \text{ are bounded in } L^\infty(0,T; L^2(0,1)) \tag{1.33}$$

$$\psi_k'(t) g_1(\psi_k'(t)) \text{ is bounded in } L^1((0,1) \times (0,T)) \tag{1.34}$$

$$G_2(z_k(x,\rho,t)) \text{ is bounded in } L^\infty(0,T; L^1((0,1) \times (0,1))) \tag{1.35}$$

$$z_k(x,1,t) g_2(z_k(x,1,t)) \text{ is bounded in } L^1((0,1) \times (0,T)) \tag{1.36}$$

for any $T > 0$.
The second estimate. First, we estimate $\varphi_k''(0)$ and $\psi_k''(0)$. Testing (1.23) by $g_{jk}''(t)$, (1.26) by $\tilde{g}_{jk}''(t)$ and choosing $t = 0$ we obtain

$$\rho_1 \|\varphi_k''(0)\|_2 \le K(\|\varphi_{0kxx}\|_2 + \|\psi_{0kx}\|_2)$$

and

$$\rho_2 \|\psi_k''(0)\|_2 \le b \|\psi_{0kxx}\|_2 + K(\|\varphi_{0kx}\|_2 + \|\psi_{0k}\|_2) + \mu_1 \|g_1(\psi_{1k})\|_2 + \mu_2 \|g_2(z_{0k})\|_2.$$

Hence from (1.24), (1.25) and (1.30):

$$\|\varphi_k''(0)\|_2 \le C.$$

Since $g_1(\psi_{1k}), g_2(z_{0k})$ are bounded in $L^2(0,1)$ by (**H2**), (1.24), (1.27), (1.28) and (1.30) yield

$$\|\psi_k''(0)\|_2 \leq C.$$

Differentiating (1.23) and (1.26) with respect to t, we get

$$(\rho_1\varphi_k'''(t) - K\varphi_{kxx}'(t) - K\psi_{kx}'(t), w_j) = 0 \tag{1.37}$$

and

$$\begin{aligned}(\rho_2\psi_k'''(t) - b\psi_{kxx}'(t) + K\varphi_{kx}'(t) + K\psi_k'(t) + \frac{d}{dt}\left(\int_0^t h(t-s)\psi_{kxx}(s)\,ds\right)\\ +\mu_1{\psi''}_k(t)g_1'(\psi_k'(t)) + \mu_2 z_k'(x,1,t)g_2'(z_k(x,1,t)), w_j) = 0.\end{aligned} \tag{1.38}$$

Multiplying (1.37) by $g_{jk}''(t)$ and (1.38) by $\tilde{g}_{jk}''(t)$, summing over j from 1 to k, it follows that

$$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi_k''(t)\|_2^2\right) - K\int_0^1 (\varphi_{kx}' + \psi_k')_x\varphi_k''\,dx = 0 \tag{1.39}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\rho_2\|\psi_k''(t)\|_2^2 + b\|\psi_{kx}'(t)\|_2^2\right) + K\int_0^1 (\varphi_{kx}' + \psi_k')\psi_k''\,dx - h(0)\frac{d}{dt}(\psi_{kx}(t), \psi_{kx}'(t))\\ &+h(0)\|\psi_{kx}'(t)\|_2^2 - \frac{d}{dt}\int_0^t h'(t-s)(\psi_{kx}(s), \psi_{kx}'(t))\,ds + h'(0)(\psi_{kx}(t), \psi_{kx}'(t))\\ &+\int_0^t h''(t-s)(\psi_{kx}(s), \psi_{kx}'(t))\,ds + \mu_1\int_0^1 {\psi''}_k^2(t)g_1'(\psi_k'(t))\,dx\\ &\qquad+\mu_2\int_0^1 {\psi''}_k(t)z_k'(x,1,t)g_2'(z_k(x,1,t))\,dx = 0.\end{aligned} \tag{1.40}$$

Differentiating (1.29) with respect to t, we get

$$(\tau z_k''(t) + \frac{\partial}{\partial\rho}z_k', \phi_j) = 0.$$

Multiplying by $h_{jk}'(t)$, summing over j from 1 to k, it follows that

$$\frac{1}{2}\tau\frac{d}{dt}\|z_k'(t)\|_2^2 + \frac{1}{2}\frac{d}{d\rho}\|z_k'(t)\|_2^2 = 0. \tag{1.41}$$

Taking the sum of (1.39), (1.40) and (1.41), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi_k''(t)\|_2^2 + \rho_2\|\psi_k''(t)\|_2^2 + b\|\psi_{kx}'(t)\|_2^2 + K\|\varphi_{kx}'(t) + \psi_k'\|_2^2 + \tau\|z_k'(.,.,t)\|_{L^2((0,1)\times(0,1))}^2\right)\\ &+h(0)\|\psi_{kx}'(t)\|_2^2 + \mu_1\int_0^1 {\psi''}_k^2(t)g_1'(\psi_k'(t))\,dx + \frac{1}{2}\int_0^1 |z_k'(x,1,t)|^2\,dx\\ &= h(0)\frac{d}{dt}(\psi_{kx}(t), \psi_{kx}'(t)) + \frac{d}{dt}\int_0^t h'(t-s)(\psi_{kx}(s), \psi_{kx}'(t))\,ds - h'(0)(\psi_{kx}(t), \psi_{kx}'(t))\\ &-\int_0^t h''(t-s)(\psi_{kx}(s), \psi_{kx}'(t))\,ds - \mu_2\int_0^1 {\psi''}_k(t)z_k'(x,1,t)g_2'(z_k(x,1,t))\,dx + \frac{1}{2}\|\psi_k''(t)\|_2^2.\end{aligned}$$

Using (1.3), Cauchy-Schwarz and Young's inequalities, we obtain

$$|h'(0)(\psi_{kx}(t), \psi_{kx}'(t))| \leq \varepsilon\|\psi_{kx}(t)\|_2^2 + \frac{h'(0)^2}{4\varepsilon}\|\psi_{kx}'(t)\|_2^2,$$

$$\begin{aligned}\left|\int_0^t h''(t-s)(\psi_{kx}(s),\psi'_{kx}(t))\,ds\right| &\le \|\psi'_{kx}(t)\|_2\int_0^t |h''(t-s)|\|\psi_{kx}(s)\|_2\,ds\\ &\le \frac{1}{4\varepsilon}\|\psi'_{kx}(t)\|_2^2+\varepsilon\|h''\|_{L^1}\int_0^t |h''(t-s)|\|\psi_{kx}(s)\|_2^2\,ds,\end{aligned}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\rho_1\|\varphi''_k(t)\|_2^2+\rho_2\|\psi''_k(t)\|_2^2+b\|\psi'_{kx}(t)\|_2^2+K\|\varphi'_{kx}(t)+\psi'_k\|_2^2+\tau\|z'_k(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}\Big)\\ &+\mu_1\int_0^1 \psi''^2_k(t)g'_1(\psi'_k(t))\,dx+c\int_0^1|z'_k(x,1,t)|^2\,dx+h(0)\|\psi'_{kx}(t)\|_2^2\le c'\|\psi''_k(t)\|_2^2+\varepsilon\|\psi_{kx}(t)\|_2^2\\ &+\frac{h'(0)^2}{4\varepsilon}\|\psi'_{kx}(t)\|_2^2+\frac{1}{4\varepsilon}\|\psi'_{kx}(t)\|_2^2+\varepsilon\|h''\|_{L^1}\int_0^t|h''(t-s)|\|\psi_{kx}(s)\|_2^2\,ds\\ &+h(0)\frac{d}{dt}(\psi_{kx}(t),\psi'_{kx}(t))+\frac{d}{dt}\int_0^t h'(t-s)(\psi_{kx}(s),\psi'_{kx}(t))\,ds.\end{aligned}$$

Integrating the last inequality over $(0,t)$, we obtain

$$\begin{aligned}&\frac{1}{2}\Big(\rho_1\|\varphi''_k(t)\|_2^2+\rho_2\|\psi''_k(t)\|_2^2+b\|\psi'_{kx}(t)\|_2^2+K\|\varphi'_{kx}(t)+\psi'_k\|_2^2+\tau\|z'_k(.,.,t)\|^2_{L^2((0,1)\times(0,1))}\Big)\\ &+\mu_1\int_0^t\int_0^1 \psi''^2_k(s)g'_1(\psi'_k(s))\,dx\,ds+c\int_0^t\int_0^1|z'_k(x,1,s)|^2\,dx\,ds\\ &\le\frac{1}{2}\Big(\rho_1\|\varphi''_k(0)\|_2^2+\rho_2\|\psi''_k(0)\|_2^2+b\|\psi'_{kx}(0)\|_2^2+K\|\varphi'_{kx}(0)+\psi'_k(0)\|_2^2+\tau\|z'_k(.,.,0)\|^2_{L^2((0,1)\times(0,1))}\Big)\\ &+c'\int_0^t\|\psi''_k(s)\|_2^2\,ds+h(0)(\psi_{kx}(t),\psi'_{kx}(t))-h(0)(\psi_{kx}(0),\psi'_{kx}(0))\\ &+\int_0^t h'(t-s)(\psi_{kx}(s),\psi'_{kx}(t))\,ds+\left(\frac{1}{4\varepsilon}+\frac{h'(0)^2}{4\varepsilon}-h(0)\right)\int_0^t\|\psi'_{kx}(s)\|_2^2\,ds\\ &+(\varepsilon+\varepsilon\|h''\|^2_{L^1})\int_0^t\|\psi_{kx}(s)\|_2^2\,ds\end{aligned}$$

$$h(0)(\psi_{kx}(t),\psi'_{kx}(t))\le\varepsilon\|\psi'_{kx}(t)\|_2^2+\frac{h(0)^2}{4\varepsilon}\|\psi_{kx}(t)\|_2^2,$$

$$\int_0^t h'(t-s)(\psi_{kx}(s),\psi'_{kx}(t))\,ds\le\varepsilon\|\psi'_{kx}(t)\|_2^2+\frac{\zeta(0)\|h'\|_{L^1}\|h\|_{L^\infty}}{4\varepsilon}\int_0^t\|\psi_{kx}(s)\|_2^2\,ds.$$

$$\begin{aligned}&\rho_1\|\varphi''_k(t)\|_2^2+\|\psi''_k(t)\|_2^2+\|\psi'_{kx}(t)\|_2^2+K\|\varphi'_{kx}(t)+\psi'_k\|_2^2+\tau\|z'_k(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}\\ &+\mu_1\int_0^t\int_0^1\psi''^2_k(t)g'_1(\psi'_k(t))\,dx\,ds+c\int_0^t\int_0^1|z'_k(x,1,t)|^2\,dx\,ds\le M\end{aligned}$$

for all $t\in[0,T]$ and M is a positive constant independent of $k\in\mathbb{N}$. Therefore, we conclude that

(1.42) $$\varphi''_k,\psi''_k \text{ is bounded in } L^\infty(0,T;L^2)$$

(1.43) $$\varphi'_k,\psi'_k \text{ is bounded in } L^\infty(0,T;H_0^1)$$

(1.44) $$z'_k \text{ is bounded in } L^\infty(0,T;L^2((0,1)\times(0,1)))$$

The third estimate. Replacing w_j by $-w_{jxx}$ in (1.23) and (1.26), multiplying the result by $g'_{jk}(t)$ and $\tilde{g}'_{jk}(t)$, summing over j from 1 to k, it follows that

(1.45) $$\frac{1}{2}\frac{d}{dt}\Big(\rho_1\|\varphi'_{kx}(t)\|_2^2+\Big)+K\int_0^1(\varphi_{kx}+\psi_k)_x\varphi'_{kxx}\,dx=0.$$

$$(1.46)\quad \begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\rho_2\|\psi'_{kx}(t)\|_2^2 + b\|\psi_{kxx}(t)\|_2^2\right) - K\int_0^1(\varphi_{kx}+\psi_k)\psi'_{kxx}\,dx\\ &+\mu_1\int_0^1|\psi'_{kx}(t)|^2 g_1'(\psi'_k(t))\,dx - \int_0^t h(t-s)(\psi_{kxx}(s),\psi'_{kxx}(t))\,ds\\ &+\mu_2\int_0^1 \psi'_{kx}(t)z_{kx}(x,1,t)g_2'(z_k(x,1,t))\,dx = 0.\end{aligned}$$

$$\begin{aligned}&\int_0^t h(t-s)(\psi_{kxx}(s),\psi'_{kxx}(t))\,ds + \frac{1}{2}h(t)\|\psi_{kxx}(t)\|_2^2\\ &= \frac{1}{2}\frac{d}{dt}\left[\int_0^t h(s)\,ds\|\psi_{kxx}(t)\|_2^2 - (h\circ\psi_{kxx})(t)\right] + \frac{1}{2}(h'\circ\psi_{kxx})(t).\end{aligned}$$

Consequently, equality (1.46) becomes

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\rho_2\|\psi'_{kx}(t)\|_2^2 + (b-\int_0^t h(s)\,ds)\|\psi_{kxx}(t)\|_2^2 + (h\circ\psi_{kxx})(t)\right) - K\int_0^1(\varphi_{kx}+\psi_k)\psi'_{kxx}\,dx\\ &+\mu_1\int_0^1|\psi'_{kx}(t)|^2 g_1'(\psi'_k(t))\,dx + \frac{1}{2}h(t)\|\psi_{kxx}(t)\|_2^2 - \frac{1}{2}(h'\circ\psi_{kxx})(t)\\ &+\mu_2\int_0^1\psi'_{kx}(t)z_{kx}(x,1,t)g_2'(z_k(x,1,t))\,dx = 0.\end{aligned}\quad (1.47)$$

Replacing ϕ_j by $-\phi_{jxx}$ in (1.29), multiplying the resulting equation by $h_{jk}(t)$, summing over j from 1 to k, it follows that

$$(1.48)\quad \frac{1}{2}\tau\frac{d}{dt}\|z_{kx}(t)\|_2^2 + \frac{1}{2}\frac{d}{d\rho}\|z_{kx}(t)\|_2^2 = 0.$$

From (1.47), (1.46) and (1.48), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\rho_1\|\varphi'_{kx}(t)\|_2^2 + \rho_2\|\psi'_{kx}(t)\|_2^2 + K\|\varphi_{kxx}+\psi_{kx}(t)\|_2^2 + (b-\int_0^t h(s)\,ds)\|\psi_{kxx}(t)\|_2^2\\ &+\tau\|z_{kx}(x,\rho,t)\|^2_{L^2(0,1)\times(0,1))}\Big) + \frac{1}{2}h(t)\|\psi_{kxx}(t)\|_2^2 - \frac{1}{2}(h'\circ\psi_{kxx})(t) + \mu_1\int_0^1|\psi'_{kx}(t)|^2 g_1'(\psi'_k(t))\,dx\\ &+\frac{1}{2}\int_0^1|z_{kx}(x,1,t)|^2\,dx = -\mu_2\int_0^1\psi'_{kx}(t)z_{kx}(x,1,t)g_2'(z_k(x,1,t))\,dx\\ &\qquad +\frac{1}{2}\|\psi'_{kx}(t)\|_2^2.\end{aligned}$$

Using (1.3), Cauchy-Schwartz and Young's inequalities, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi'_{kx}(t)\|_2^2 + \rho_2\|\psi'_{kx}(t)\|_2^2 + K\|\varphi_{kxx}+\psi_{kx}(t)\|_2^2 + b\|\psi_{kx}(t)\|_2^2 + \tau\|z_{kx}(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}\right)\\ &+\mu_1\int_0^1|\psi'_{kx}(t)|^2 g_1'(\psi'_k(t))\,dx + c\int_0^1|z_{kx}(x,1,t)|^2\,dx \le c'\|\psi'_{kx}(t)\|_2^2.\end{aligned}$$

Integrating the last inequality over $(0,t)$ and using Gronwall's Lemma, we have

$$\begin{aligned}&\rho_1\|\varphi'_{kx}(t)\|_2^2 + \rho_2\|\psi'_{kx}(t)\|_2^2 + K\|\varphi_{kxx}+\psi_{kx}(t)\|_2^2 + b\|\psi_{kxx}(t)\|_2^2 + \tau\|z_{kx}(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))} \le\\ &e^{cT}\left(\rho_1\|\varphi'_{kx}(0)\|_2^2 + \rho_2\|\psi'_{kx}(0)\|_2^2 + K\|\varphi_{kxx}(0)+\psi_{kx}(0)\|_2^2 + b\|\psi_{kx}(0)\|_2^2 + \tau\|z_{kx}(x,\rho,0)\|^2_{L^2((0,1)\times(0,1))}\right)\end{aligned}$$

for all $t \in \mathbb{R}_+$, therefore, we conclude that

$$(1.49)\quad \varphi_k, \psi_k \text{ are bounded in } L^\infty(0,T;H^2\cap H_0^1(0,1)),$$

(1.50) $$z_k \text{ is bounded in } L^\infty(0,T;H_0^1(0,1;L^2(0,1))).$$

Applying Dunford-Petti's theorem, we conclude from (1.32), (1.33), (1.34), (1.35), (1.42), (1.43), (1.44), (1.49) and (1.50), after replacing the sequences φ_k, ψ_k and z_k with a subsequence if needed, that

(1.51) $$\begin{cases} \varphi_k \to \varphi \text{ weak-star in } L^\infty(0,T;H^2\cap H_0^1(0,1)) \\ \psi_k \to \psi \text{ weak-star in } L^\infty(0,T;H^2\cap H_0^1(0,1)), \end{cases}$$

$$\begin{cases} \varphi'_k \to \varphi' \text{ weak-star in } L^\infty(0,T;H_0^1(0,1)) \\ \psi'_k \to \psi' \text{ weak-star in } L^\infty(0,T;H_0^1(0,1)), \end{cases}$$

(1.52) $$\begin{cases} \varphi''{}_k \to \varphi'' \text{ weak-star in } L^\infty(0,T;L^2(0,1)) \\ \psi''{}_k \to \psi'' \text{ weak-star in } L^\infty(0,T;L^2(0,1)), \end{cases}$$

$$g_1(\psi'_k) \to \chi \text{ weak-star in } L^2((0,1)\times(0,T)),$$

$$z_k \to z \text{ weak-star in } L^\infty(0,T;H_0^1((0,1);L^2(0,1)),$$

(1.53) $$z'_k \to z' \text{ weak-star in } L^\infty(0,T;L^2((0,1)\times(0,1))),$$

$$g_2(z_k(x,1,t)) \to \psi \text{ weak-star in } L^2((0,1)\times(0,T))$$

for suitable functions $\varphi,\psi \in L^\infty(0,T;H^2\cap H_0^1(0,1)), z\in L^\infty(0,T;L^2((0,1)\times(0,1)))$, $\chi\in L^2((0,1)\times(0,T)), \psi\in L^2((0,1)\times(0,T))$ for all $T\geq 0$. We have to show that (φ,ψ,z) is a solution of (1.8).

From (1.32) and (1.33) we have that $(\psi'_k)_k$ is bounded in $L^\infty(0,T;H_0^1(0,1))$. Then $(\psi'_k)_k$ is bounded in $L^2(0,T;H_0^1)$. Since $(\psi''_k)_k$ is bounded in $L^\infty(0,T;L^2(0,1))$, then $(\psi''_k)_k$ is bounded in $L^2(0,T;L^2(0,1))$. Consequently $(\psi'_k)_k$ is bounded in $H^1(Q)$,where $Q=(0,1)\times(0,T)$.

Since the embedding $H^1(Q)\hookrightarrow L^2(Q)$ is compact, using Aubin-Lions theorem [25] we can extract a subsequence $(\psi_\nu)_\nu$ of $(\psi_k)_k$ such that

$$\psi'_\nu \to \psi' \text{ strongly in } L^2(Q).$$

Therefore

(1.54) $$\psi'_\nu \to \psi' \text{ strongly and a.e on } Q.$$

Similarly we obtain

(1.55) $$z_\nu \to z \text{ strongly and a.e on } Q.$$

Lemma 1.3.1 [8] *For each $T>0$, $g_1(\psi'), g_2(z(x,1,t))\in L^1(Q)$ and $\|g_1(\psi')\|_{L^1(Q)}, \|g_2(z(.,1,t))\|_{L^1(Q)}\leq K_1$, where K_1 is a constant independent of t.*

Lemma 1.3.2 *$g_1(\psi'_k)\to g_1(\psi')$ in $L^1((0,1)\times(0,T))$ and $g_2(z_k)\to g_2(z)$ in $L^1((0,1)\times(0,T))$.*

Hence

$$g_1(\psi'_k) \to g_1(\psi') \text{ weak star in } L^2(Q).$$

Similarly, we have

$$g_2(z'_k) \to g_2(z') \text{ weak star in } L^2(Q),$$

and this imply that

$$\int_0^T \int_0^1 g_1(\psi'_k) v\, dx\, dt \to \int_0^T \int_0^1 g_1(\psi') v\, dx\, dt \text{ for all } v \in L^2(0,T;H_0^1) \tag{1.56}$$

$$\int_0^T \int_0^1 g_2(z_k) v\, dx\, dt \to \int_0^T \int_0^1 g_2(z) v\, dx\, dt \text{ for all } v \in L^2(0,T;H_0^1) \tag{1.57}$$

as $k \to +\infty$.

It follows at once from (1.51), (1.52), (1.56), (1.57) and (1.53) that for each fixed $u, v \in L^2(0,T;H_0^1(0,1))$ and $w \in L^2(0,T;H_0^1((0,1)\times(0,1)))$

$$\begin{aligned}&\int_0^T \int_0^1 (\rho_1 \varphi''_k - K(\varphi_{kx} + \psi_k)_x) u\, dx\, dt \\ &\to \int_0^T \int_0^1 (\rho_1 \varphi'' - K(\varphi_x + \psi)_x) u\, dx\, dt\end{aligned}$$

$$\begin{aligned}&\int_0^T \int_0^1 (\rho_2 \psi''_k - b\psi_{kxx} + K(\varphi_{kx} + \psi_k) + + \int_0^t h(t-s)\psi_{kxx}(x,s)ds + \mu_1 g_1(\psi'_k) + \mu_2 g_2(z_k)) v\, dx\, dt \\ &\to \int_0^T \int_0^1 (\rho_2 \psi'' - b\psi_{xx} + K(\varphi_x + \psi) + \int_0^t h(t-s)\psi_{xx}(x,s)ds + \mu_1 g_1(\psi') + \mu_2 g_2(z)) v\, dx\, dt\end{aligned}$$

$$\int_0^T \int_0^1 \int_0^1 (\tau z'_k + \frac{\partial}{\partial \rho} z_k) w\, dx\, d\rho\, dt \to \int_0^T \int_0^1 \int_0^1 (\tau z' + \frac{\partial}{\partial \rho} z) w\, dx\, d\rho\, dt$$

as $k \to +\infty$. Hence,

$$\int_0^T \int_0^1 (\rho_1 \varphi'' - K(\varphi_x + \psi)_x) u\, dx\, dt = 0$$

$$\int_0^T \int_0^1 (\rho_2 \psi'' - b\psi_{xx} + K(\varphi_x + \psi) + \int_0^t h(t-s)\psi_{xx}(x,s)ds + \mu_1 g_1(\psi') + \mu_2 g_2(z)) v\, dx\, dt = 0$$

$$\int_0^T \int_0^1 \int_0^1 (\tau u' + \frac{\partial}{\partial \rho} z) w\, dx\, d\rho\, dt = 0, \qquad w \in L^2(0,T;H_0^1((0,1)\times(0,1))).$$

Thus the problem (P) admits a global weak solution (φ, ψ).

1.4 Asymptotic behavior

The method of the proof of decay rates heavily relies the introduction of suitable Lyapunov functionals see [**8**] and ideas introduced in [**29**] and [**1**], where convex analysis was exploited to obtain a precise description of the decay rates corresponding to the energy of the wave equations with dissipation that is not quantified at the origin.

We construct a Lyapunov functional L equivalent to E. For this, we define several functionals which allow us to obtain the estimates needed.

First we have the following estimate.

Lemma 1.4.1 *Let (φ, ψ, z) be the solution of (1.8). Then the functional F_1 defined by*

$$F_1(t) = -\int_0^1 (\rho_1 \varphi_t \varphi + \rho_2 \psi_t \psi)\, dx \tag{1.58}$$

satisfies, along the solution, the estimate

$$\begin{aligned}\frac{dF_1(t)}{dt} \leq & -\int_0^1 (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2) dx + K \int_0^1 |\varphi_x + \psi|^2 dx \\ & + c\int_0^1 \psi_x^2 dx + c h \circ \psi_x + c\int_0^1 g_1^2(\psi_t) dx + c\int_0^1 g_2^2(z(x,1,t)) dx.\end{aligned} \tag{1.59}$$

Proof. By taking the time derivative of (1.58)

$$\frac{dF_1(t)}{dt} = -\int_0^1 (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2) dx - \int_0^1 (\rho_1 \varphi_{tt} \varphi + \rho_2 \psi_{tt} \psi) dx.$$

Therefore, by using the first and the second equations in (1.8) and some integrations by parts, we obtain from the above inequality

$$\begin{aligned}\frac{dF_1(t)}{dt} = & -\int_0^1 (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2) dx + K \int_0^1 |\varphi_x + \psi|^2 dx - \int_0^1 \psi_x(x,t) \int_0^t h(t-s) \psi_x(x,s)\, ds\, dx \\ & + b\int_0^1 \psi_x^2 dx + \mu_1 \int_0^1 \psi g_1(\psi_t) dx + \mu_2 \int_0^1 \psi g_2(z(x,1,t)) dx.\end{aligned} \tag{1.60}$$

By exploiting Young's inequality and Poincaré's inequality, then (1.59) follows.

Lemma 1.4.2 *Let (φ, ψ, z) be the solution of (1.8). Assume that*

$$\frac{\rho_1}{K} = \frac{\rho_2}{b}. \tag{1.61}$$

Then the functional F_2 defined by

$$F_2(t) = \rho_2 \int_0^1 \psi_t (\varphi_x + \psi) dx + \rho_2 \int_0^1 \psi_x \varphi_t\, dx - \frac{\rho_1}{K} \int_0^1 \varphi_t \int_0^t h(t-s) \psi_x(s)\, ds\, dx. \tag{1.62}$$

satisfies, along the solution, the estimate

$$\begin{aligned}\frac{dF_2(t)}{dt} \leq & \left[(b\psi_x - \int_0^t h(t-s)\psi_x(s)\, ds) \varphi_x \right]_{x=0}^{x=1} - (K - \varepsilon) \int_0^1 (\varphi_x + \psi)^2 dx \\ & + \varepsilon \int_0^1 \varphi_t^2 dx - \frac{c}{\varepsilon} h' \circ \psi_x + \frac{c}{\varepsilon} \int_0^1 \psi_x^2 dx \\ & + \rho_2 \int_0^1 \psi_t^2 dx + \frac{c}{\varepsilon} \int_0^1 g_1^2(\psi_t) dx + \frac{c}{\varepsilon} \int_0^1 g_2^2(z(x,1,t)) dx\end{aligned} \tag{1.63}$$

for any $0 < \varepsilon < 1$.

Proof. Differentiating $F_2(t)$, with respect to t, we obtain

$$\begin{aligned}
\frac{dF_2}{dt}(t) &= \int_0^1 \rho_2\psi_{tt}(\varphi_x+\psi)dx + \int_0^1 \rho_2\psi_t(\varphi_x+\psi)_t dx + \rho_2\int_0^1 \psi_x\varphi_{tt}dx + \rho_2\int_0^1 \psi_{tx}\varphi_t dx \\
&\quad -\frac{\rho_1}{K}\int_0^1 \varphi_{tt}\int_0^t h(t-s)\psi_x(s)dsdx - \frac{\rho_1}{K}\int_0^1 \varphi_t\Big(\int_0^t h(t-s)\psi_x(s)ds\Big)' dx \\
&= \int_0^1 (\varphi_x+\psi)\Big[b\psi_{xx} - K(\varphi_x+\psi) - \int_0^t h(t-s)\psi_{xx}(s)\,ds - \mu_1 g_1(\psi_t) - \mu_2 g_2(z(x,1,t)\Big]dx \\
&\quad +\rho_2\int_0^1 \psi_t^2\,dx + \frac{\rho_2}{\rho_1}\int_0^1 K(\varphi_x+\psi)_x\psi_x\,dx - \int_0^1 (\varphi_x+\psi)_x\int_0^t h(t-s)\psi_x(s)\,ds\,dx \\
&\quad -\frac{\rho_1}{K}\int_0^1 \varphi_t\left(\int_0^t h(t-s)\psi_x(s)\,ds\right)' dx.
\end{aligned}$$

Then, by using Eqs.(1.8) and (1.61) we find

$$\begin{aligned}
\frac{dF_2(t)}{dt} &= \Big[(b\psi_x - \int_0^t h(t-s)\psi_x(s)\,ds)\varphi_x\Big]_{x=0}^{x=1} - K\int_0^1 (\varphi_x+\psi)^2 dx + \rho_2\int_0^1 \psi_t^2 dx \\
&\quad -\mu_1\int_0^1 (\varphi_x+\psi)g_1(\psi_t)dx - \mu_2\int_0^1 (\varphi_x+\psi)g_2(z(x,1,t))dx \\
&\quad -\frac{\rho_1}{K}h(t)\int_0^1 \varphi_t\psi_x\,dx - \frac{\rho_1}{K}\int_0^1 \varphi_t\int_0^t h'(t-s)\left(\psi_x(s)-\psi_x(t)\right)\,ds\,dx
\end{aligned}$$

By the Young inequality (1.63) is established.

Lemma 1.4.3 *Let $m \in C^1([0,1])$ be a function satisfying $m(0) = -m(1) = 2$. Then there exists $c > 0$ such that, for any $0 < \varepsilon < 1$, the functional F_3 defined by*

$$F_3(t) = \frac{1}{4\varepsilon}\int_0^1 \rho_2 m(x)\psi_t\Big(b\psi_x - \int_0^t h(t-s)\psi_x(s)\,ds\Big)\,dx + \frac{\varepsilon}{K}\int_0^1 \rho_1 m(x)\varphi_t\varphi_x\,dx$$

satisfies, along the solution, the estimate

$$\begin{aligned}
F_3'(t) &\le -\frac{1}{4\varepsilon}\Big[b\left(\psi_x(1,t) - \int_0^t h(t-s)\psi_x(1,s)\,ds\right)^2 \\
&\quad +\left(b\psi_x(0,t) - \int_0^t h(t-s)\psi_x(0,s)\,ds\right)^2\Big] \\
&\quad -\varepsilon\left((\varphi_x(1,t))^2 + (\varphi_x(0,t))^2\right) + (\frac{K}{4} + \frac{c}{K}\varepsilon)\int_0^1 (\psi+\varphi_x)^2\,dx + c\varepsilon\rho_1\int_0^1 \varphi_t^2\,dx \\
&\quad +\frac{c}{\varepsilon^2}\int_0^1 \psi_x^2\,dx + \frac{c}{\varepsilon^2}h\circ\psi_x + \frac{c}{\varepsilon}\int_0^1 \psi_t^2\,dx + c\int_0^1 g_1^2(\psi_t)\,dx \\
&\quad +c\int_0^1 g_2^2(z(x,1,t))\,dx - \frac{c}{\varepsilon}h'\circ\psi_x.
\end{aligned} \tag{1.64}$$

Proof. Using Eqs. (1.8) and integrating by parts, we obtain

$$
\begin{aligned}
F_3'(t) = {} & \frac{1}{4\varepsilon}\Big[-\bigg(\Big(b\psi_x(1,t)-\int_0^t h(t-s)\psi_x(1,s)\,ds\Big)^2 \\
& +\Big(b\psi_x(0,t)-\int_0^t h(t-s)\psi_x(0,s)\,ds\Big)^2\bigg) \\
& -\int_0^1 \frac{1}{2}m'(x)\Big(b\psi_x-\int_0^t h(t-s)\psi_x(s)\,ds\Big)^2\,dx \\
& -K\int_0^1 m(x)(\varphi_x+\psi)\Big(b\psi_x-\int_0^t h(t-s)\psi_x(s)\,ds\Big)\,dx \\
& -\int_0^1 m(x)\mu_1 g_1(\psi_t)\Big(b\psi_x-\int_0^t h(t-s)\psi_x(s)\,ds\Big)\,dx \\
& -\int_0^1 m(x)\mu_2 g_2(z(x,1,t))(b\psi_x-\int_0^t h(t-s)\psi_x(s)\,ds)\,dx-\int_0^1 \frac{\rho_2 b}{2}m'(x)(\psi_t)^2\,dx \\
& +\rho_2\int_0^1 m(x)\psi_t\int_0^t h'(t-s)(\psi_x(t)-\psi_x(s))\,ds)\,dx-\rho_2 h(t)\int_0^1 m(x)\psi_t\psi_x\,dx\Big] \\
& \frac{\varepsilon}{K}\Big[-K\Big((\varphi_x(1,t))^2+(\varphi_x(0,t))^2\Big)-\int_0^1\frac{K}{2}m'(x)\varphi_x^2\,dx+\int_0^1 Km(x)\psi_x\varphi_x\,dx \\
& -\int_0^1\frac{\rho_1}{2}m'(x)(\varphi_t)^2\,dx\Big].
\end{aligned}
\tag{1.65}
$$

Then exploiting Young's and Poincaré's inequalities and the fact that

$$\varphi_x^2 \le 2(\psi+\varphi_x)^2+2\psi^2$$

we obtain

$$
\begin{aligned}
F_3'(t) \le {} & \frac{1}{4\varepsilon}\Big[-\bigg(\Big(b\psi_x(1,t)-\int_0^t h(t-s)\psi_x(1,s)\,ds\Big)^2 \\
& +\Big(b\psi_x(0,t)-\int_0^t h(t-s)\psi_x(0,s)\,ds\Big)^2\bigg) \\
& +\frac{c}{\varepsilon}\int_0^1 \psi_x^2\,dx+\frac{c}{\varepsilon}h\circ\psi_x+\varepsilon K\int_0^1(\psi+\varphi_x)^2\,dx \\
& +\varepsilon\int_0^1 g_1^2(\psi_t)\,dx+\varepsilon\int_0^1 g_2^2(z(x,1,t))\,dx+c\int_0^1\psi_t^2\,dx-ch'\circ\psi_x\Big] \\
& +\frac{\varepsilon}{K}\Big[-K\Big((\varphi_x(1,t))^2+(\varphi_x(0,t))^2\Big)+c\int_0^1\psi_x^2\,dx+c\int_0^1(\psi+\varphi_x)^2\,dx \\
& +c\int_0^1\varphi_t^2\,dx\Big].
\end{aligned}
\tag{1.66}
$$

This gives (1.64).

Lemma 1.4.4 *Assume that $(H1)$ hold. Then, for sufficiently small ε, the functional F defined by*

$$F(t)=2c\varepsilon F_1(t)+F_2(t)+F_3(t)$$

satisfies, along the solution, the estimate

$$
\begin{aligned}
F'(t) \le {} & -\frac{k}{2}\int_0^1(\psi+\varphi_x)^2\,dx-\tau\int_0^1\varphi_t^2\,dx+c\int_0^1\psi_t^2\,dx+c\int_0^1\psi_x^2\,dx \\
& +c\int_0^1 g_1^2(\psi_t)\,dx+c\int_0^1 g_2^2(z(x,1,t))\,dx+ch\circ\psi_x-ch'\circ\psi_x
\end{aligned}
\tag{1.67}
$$

where $\tau = c\varepsilon\rho_1$.

***Proof*:** Using Lemmas 1.4.1, 1.4.2, 1.4.3 and the fact that

$$\begin{aligned}\Big[\Big(b\psi_x-\int_0^t h(t-s)\psi_x(s)ds\Big)\varphi_x\Big]_{x=0}^{x=1} \le \frac{1}{4\varepsilon}\Big[&\Big(b\psi_x(1,t)-\int_0^t h(t-s)\psi_x(1,s)ds\Big)^2\\ &+\Big(b\psi_x(0,t)-\int_0^t h(t-s)\psi_x(0,s)ds\Big)^2\Big]\\ &+\varepsilon\Big[\varphi_x^2(1)+\varphi_x^2(0)\Big].\end{aligned} \tag{1.68}$$

for any $0<\varepsilon<1$, we obtain (1.67).

Next, we introduce the following functional

$$I(t)=\int_0^1(\rho_2\psi_t\psi+\rho_1\varphi_t\omega)dx, \tag{1.69}$$

where w is the solution of

$$-\omega_{xx}=\psi_x, \qquad \omega(0)=\omega(1)=0. \tag{1.70}$$

Then we have the following estimate.

Lemma 1.4.5 *Let* (φ,ψ,z) *be the solution of (1.8), then for any* $\delta>0$, *we have the following estimate*

$$\begin{aligned}\frac{dI(t)}{dt}\le &-\frac{1}{2}\Big(b-\int_0^\infty h(s)\,ds\Big)\int_0^1\psi_x^2(x,t)dx+\frac{c}{\delta}\int_0^1\psi_t^2(x,t)dx\\ &+\delta\int_0^1\varphi_t^2(x,t)dx+c\int_0^1 g_1^2(\psi_t)dx+c\int_0^1 g_2^2(z(x,1,t))dx+c\ h\circ\psi_x.\end{aligned} \tag{1.71}$$

Proof. Using equation (1.8), we have

$$\begin{aligned}\frac{dI(t)}{dt}=&\Big(-b+\int_0^t h(s)\,ds\Big)\int_0^1\psi_x^2dx+\rho_2\int_0^1\psi_t^2dx-K\int_0^1\psi^2dx\\ &+\int_0^1\Big(\int_0^t h(t-s)(\psi_x(s)-\psi_x(t))\,ds\Big)\psi_x\,dx+K\int_0^1\omega_x^2dx+\rho_1\int_0^1\varphi_t\omega_t dx\\ &-\mu_1\int_0^1\psi g_1(\psi_t)dx-\mu_2\int_0^1\psi g_2(z(x,1,t))dx.\end{aligned} \tag{1.72}$$

Equation (1.70) trivially implies

$$\int_0^1\omega_x^2dx\le\int_0^1\psi^2dx\le\int_0^1\psi_x^2dx$$

$$\int_0^1\omega_t^2dx\le\int_0^1\omega_{tx}^2dx\le\int_0^1\psi_t^2dx \tag{1.73}$$

Using Young's inequality and Poincaré's inequality, the last two terms in (1.72) can be estimated as

$$
\begin{aligned}
&\mu_1\int_0^1 \psi g_1(\psi_t)dx + \int_0^1 (\int_0^t h(t-s)(\psi_x(s)-\psi_x(t))\,ds)\psi_x\,dx\\
&+\mu_2\int_0^1 \psi g_2(z(x,1,t))dx \le \frac{1}{2}\Big(b-\int_0^\infty h(s)ds\Big)\int_0^1 \psi_x^2 dx + c\int_0^1 g_1^2(\psi_t)dx\\
&\qquad +c\int_0^1 g_2^2(z(x,1,t))dx + ch\circ\psi_x.
\end{aligned} \tag{1.74}
$$

Consequently, from (1.72)-(1.74), we obtain (1.71). Now, let us introduce the following functional

$$
I_3(t) = \int_0^1\int_0^1 e^{-2\tau\rho} G_2(z(x,\rho,t))\,d\rho\,dx. \tag{1.75}
$$

Then the following result holds.

Lemma 1.4.6 *Let (φ,ψ,z) be the solution of (1.8). Then it holds*

$$
\frac{d}{dt}I_3(t) \le -\frac{\alpha_1 e^{-2\tau}}{\tau}\int_0^1 z(x,1,t)g_2(z(x,1,t))\,dx + \frac{\alpha_2}{\tau}\int_0^1 \psi_t(x,t)g_1(\psi_t(x,t))dx -2I_3(t). \tag{1.76}
$$

Proof. Differentiating (1.75) with respect to t and using the third equation in (1.8), we have

$$
\begin{aligned}
\frac{d}{dt}I_3(t) &= \int_0^1\int_0^1 e^{-2\tau\rho} z_t(x,\rho,t) g_2(z(x,\rho,t))\,d\rho\,dx\\
&= -\frac{1}{\tau}\int_0^1\int_0^1 e^{-2\tau\rho} z_\rho(x,\rho,t) g_2(z(x,\rho,t))\,d\rho\,dx\\
&= -\frac{1}{\tau}\int_0^1\int_0^1 e^{-2\tau\rho} \frac{d}{d\rho} G_2(z(x,\rho,t))\,d\rho\,dx\\
&= -\frac{1}{\tau}\int_0^1\int_0^1 \Big[\frac{d}{d\rho}\Big(e^{-2\tau\rho}G_2(z(x,\rho,t))\Big) + 2\tau e^{-2\tau\rho}G_2(z(x,\rho,t))dxd\rho\Big]\\
&= -\frac{1}{\tau}\int_0^1 \Big[e^{-2\tau}G_2(z(x,1,t)) - G_2(\psi_t(x,t))\Big]dx\\
&\quad -2\int_0^1\int_0^1 e^{-2\tau\rho}G_2(z(x,\rho,t))dxd\rho\\
&= -2\int_0^1\int_0^1 e^{-2\tau\rho}G_2(z(x,\rho,t))\,dx - \frac{1}{\tau}\int_0^t e^{-2\tau}G_2(z(x,1,t))dx\\
&\quad +\frac{1}{\tau}\int_0^1 G_2(\psi_t(x,t))dx\\
&= -\frac{e^{-2\tau}}{\tau}\int_0^1 G_2(z(x,1,t))\,dx + \frac{1}{\tau}\int_0^1 G_2(\psi_t(x,t))dx - 2I_3(t)\\
&\le \frac{\alpha_2}{\tau}\int_0^1 \psi_t(x,t)g_1(\psi_t(x,t))dx - \frac{\alpha_1 e^{-2\tau}}{\tau}\int_0^t z(x,1,t)g_2(z(x,1,t))dx\\
&\quad -2I_3(t).
\end{aligned}
$$

For $N_1, N_2 > 0$, let

$$
L(t) = N_1E(t) + N_2I(t) + F(t) + I_3(t). \tag{1.77}
$$

By combining (1.13), (1.67), (1.71), (1.76), we obtain

$$
\begin{aligned}
\frac{d}{dt}L(t) \leq & -\left(N_1 a_1 - \frac{\alpha_2}{\tau}\right)\int_0^1 \psi_t(x,t) g_1(\psi_t(x,t))\,dx \\
& -\left(N_1 a_2 + \alpha_1 \frac{e^{-2\tau}}{\tau} - (N_2 c + c)c_3\right)\int_0^1 z(x,1,t) g_2(z(x,1,t))\,dx \\
& -\left(\frac{N_2}{2}(b - \int_0^\infty h(s)\,ds) + \frac{1}{2}N_1 h(t) - c\right)\int_0^1 \psi_x^2\,dx \\
& -(\tilde{\tau} - N_2\delta)\int_0^1 \varphi_t^2\,dx + \left(N_2\frac{c}{\delta} + c\right)\int_0^1 \psi_t^2\,dx + (N_2 c + c)\int_0^1 g_1^2(\psi_t)\,dx \\
& -\frac{k}{2}\int_0^1 (\psi + \varphi_x)^2\,dx + (N_2 c + c) h \circ \psi_x + (\frac{N_1}{2} - c) h' \circ \psi_x - 2I_3.
\end{aligned} \tag{1.78}
$$

At this point, we have to choose our constants very carefully. First, let us choose N_2 sufficiently large so that

$$\left(\frac{N_2}{2}(b - \int_0^\infty h(s)\,ds) - c\right) > 0.$$

Next, we choose δ sufficiently small such that

$$(\tilde{\tau} - N_2\delta) > 0.$$

Then, we pick the constant $N_1 > 0$ sufficiently large such that

$$\left(N_1 a_1 - \frac{\alpha_2}{\tau}\right) > 0,$$

$$\left(N_1 a_2 + \alpha_1 \frac{e^{-2\tau}}{\tau} - (N_2 c + c')\right) > 0$$

and

$$\left(\frac{N_1}{2} - c\right) > 0.$$

Thus, (1.78) becomes

$$
\begin{aligned}
\frac{d}{dt}L(t) \leq & -d_1\int_0^1 \psi_x^2\,dx - d_2\int_0^1 \varphi_t^2\,dx - \frac{K}{2}\int_0^1 (\psi + \varphi_x)^2\,dx + d_3 h \circ \psi_x + d_4 h' \circ \psi_x \\
& -2I_3 + c\int_0^1 ((\psi_t)^2 + g_1^2(\psi_t))\,dx \\
\leq & -dE(t) + c\int_0^1 ((\psi_t)^2 + g_1^2(\psi_t))\,dx + c h \circ \psi_x.
\end{aligned} \tag{1.79}
$$

At this stage, we are in position to compare $L(t)$ with $E(t)$. We have the following Lemma.

Lemma 1.4.7 *For N_1 large enough, there exist two positive constants β_1 and β_2 depending on N_1, N_2 and ϵ, such that*

$$\beta_1 E(t) \leq L(t) \leq \beta_2 E(t) \qquad \forall t \geq 0. \tag{1.80}$$

Proof. We consider the functional

$$\mathcal{H}(t) = N_2 I(t) + F(t) + I_3(t)$$

and show that

$$|\mathcal{H}(t)| \leq \hat{C}E(t), \qquad \hat{C} > 0.$$

from (1.58),(1.69),(1.62) and (1.75), we obtain

$$\begin{aligned} |\mathcal{H}(t)| \leq & N_2 \left| \int_0^1 \rho_2 \psi_t \psi + \rho_1 \varphi_t \omega)(x,t)dx \right| + 2c\varepsilon \left| - \int_0^1 (\rho_1 \varphi_t \varphi + \rho_2 \psi_t \psi) dx \right| \\ & + \left| \rho_2 \int_0^1 \psi_t (\varphi_x + \psi) dx + \rho_2 \int_0^1 \psi_x \varphi_t \, dx - \frac{\rho_1}{k} \int_0^1 \varphi_t \int_0^t h(t-s) \psi_x(s) \, ds \, dx \right| \\ & + \left| \frac{1}{4\varepsilon} \int_0^1 \rho_2 m(x) \psi_t (b\psi_x - \int_0^t h(t-s)\psi_x(s) \, ds) \, dx + \frac{\varepsilon}{K} \int_0^1 \rho_1 m(x) \varphi_t \varphi_x \, dx \right| \\ & + \left| \int_0^1 \int_0^1 e^{-2\tau\rho} G_2(z(x,\rho,t)) \, d\rho \, dx \right|. \end{aligned} \tag{1.81}$$

By using (1.73),(1.70), the trivial relation

$$\int_0^1 \varphi^2(x,t)dx \leq 2 \int_0^1 (\varphi_x + \psi)^2(x,t)dx + 2 \int_0^1 \psi_x^2(x,t)dx,$$

together with Young's and Poincaré's inequalities, we get

$$\begin{aligned} |\mathcal{H}(t)| \leq & \alpha_1 \int_0^1 \varphi_t^2(x,t)dx + \alpha_2 \int_0^1 \psi_t^2(x,t)dx \\ & + \alpha_3 \int_0^1 (\varphi_x + \psi)^2(x,t)dx + \alpha_4 \int_0^1 \psi_x^2(x,t)dx \\ & + \alpha_5 h \circ \psi_x + \int_0^1 \int_0^1 G_2(z(x,\rho,t))dx \; d\rho \end{aligned} \tag{1.82}$$

where the positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 are determined as follows:

$$\begin{cases} \alpha_1 = \frac{N_2 \rho_1}{2} + \rho_2 + \frac{\epsilon \rho_1}{K}, \\ \alpha_2 = \frac{N_2 \rho_2}{2} + \rho_2 + \frac{\rho_2 b}{2\epsilon}, \\ \alpha_3 = \rho_1 + \frac{\rho_2}{2} + \frac{2\epsilon \rho_1}{K}, \\ \alpha_4 = \rho_2 + \frac{N_2}{2}\rho_2 + \rho_1 + \frac{\rho_2 b}{2\epsilon} + \frac{2\epsilon\rho_1}{K}, \\ \alpha_5 = \frac{\rho_2}{4\varepsilon} + \frac{\rho_1 \varepsilon}{K} \end{cases}$$

According to (1.82) , we have

$$|H(t)| \leq \hat{C}E(t)$$

for

$$\hat{C} = 2 \max \left\{ \frac{\alpha_1}{\rho_1}, \frac{\alpha_2}{\rho_2}, \frac{\alpha_3}{K}, \frac{\alpha_4}{b}, \frac{\alpha_5}{K}, \frac{1}{2\xi} \right\}.$$

Therefore, we obtain

$$|L(t) - N_1 E(t)| \leq \hat{C}E(t).$$

So, we can choose N_1 large enough such that $\beta_1 = N_1 - \hat{C} > 0, \beta_2 = N_1 + \hat{C} > 0$. Then (1.80) holds true. Therefore, (1.79) takes the form

$$\frac{d}{dt}L(t) \leq -C_3E(t) + C_4(h \circ \psi_x)(t) + C_5(\|\psi'\|_2^2 + \|g_1(\psi')\|_2^2), \tag{1.83}$$

where C_3, C_4 and C_5 are three positive constants.

Now, we estimate the last term in the right hand side of (1.83). We define

$$\Omega^+ = \{x \in (0,1) : |\psi'| \geq \varepsilon'\}, \quad \Omega^- = \{x \in (0,1) : |\psi'| \leq \varepsilon'\}.$$

From (1.1) and (1.2), it follows that

$$\int_{\Omega^+} (|\psi'|^2 + |g_1(\psi')|^2)\, dx \leq \mu_1 \int_{\Omega^+} \psi' g_1(\psi')\, dx \leq -\mu_1 E'(t). \tag{1.84}$$

Case 1: H is linear on $[0, \varepsilon']$. In this case one can easily check that there exists $\mu_1' > 0$, such that $|g_1(s)| \leq \mu_1'|s|$ for all $|s| \leq \varepsilon'$, and thus

$$\int_{\Omega^-} (|\psi'|^2 + |g_1(\psi')|^2)\, dx \leq \mu_1' \int_{\Omega^-} \psi' g_1(\psi')\, dx \leq -\mu_1' E'(t). \tag{1.85}$$

Plugging (1.84) and (1.85) into (1.83) gives

$$(L(t) + \mu E(t))' \leq -C_3 H_2(E(t)) + C_4(h \circ \psi_x)(t) \tag{1.86}$$

where $\mu = C_5(\mu_1 + \mu_1')$. Here and in the sequel, we take C_i to be a generic positive constant.
Case 2: $H'(0) = 0$ and $H'' > 0$ on $]0, \varepsilon']$.
Since H is convex and increasing, H^{-1} is concave and increasing. By the virtue of (1.1), the reversed Jensen's inequality for concave function, and (1.13), it follows that

$$\begin{aligned} \int_{\Omega^-} (|\psi'|^2 + |g_1(\psi')|^2)\, dx &\leq \int_{\Omega^-} H^{-1}(\psi' g_1(\psi'))\, dx \\ &\leq |\Omega| H^{-1}\left(\frac{1}{|\Omega|}\int_{\Omega^-} \psi' g_1(\psi')\, dx\right) \\ &\leq CH^{-1}(-C'E'(t)). \end{aligned} \tag{1.87}$$

A combination of (1.83), (1.84) and (1.87) yields

$$(L(t) + C_5\mu_1 E(t))' \leq -C_3E(t) + C_4(h \circ \psi_x)(t) + \tilde{C}_5 H^{-1}(-C'E'(t)), \quad t \geq 0. \tag{1.88}$$

Let us denote by H^* the conjugate function of the convex function H, i.e.,

$$H^*(s) = \sup_{t \in \mathbb{R}_+} (st - H(t)).$$

Then H^* is the Legendre transform of H, which is given by

$$H^*(s) = s(H')^{-1}(s) - H[(H')^{-1}(s)], \quad \forall s \geq 0 \tag{1.89}$$

and which satisfies the following inequality

$$st \leq H^*(s) + H(t), \quad \forall s, t \geq 0. \tag{1.90}$$

The relation (1.89) and the fact that $H'(0) = 0$ and $(H')^{-1}, H$ are increasing functions yield

$$H^*(s) \leq s(H')^{-1}(s), \quad \forall s \geq 0. \tag{1.91}$$

Making use of $E'(t) \leq 0, H''(t) \geq 0$, (1.88) and (1.91) we derive for $\varepsilon_0 > 0$ small enough

$$\begin{aligned}
&[H'(\varepsilon_0 E(t))\{L(t) + C_5\mu_1 E(t)\} + \tilde{C}_5 C' E(t)]' \\
&\quad = \varepsilon_0 E'(t) H''(\varepsilon_0 E(t))(L(t) + C_5\mu_1 E(t)) + \tilde{C}_5 C' E'(t) \\
&\qquad + H'(\varepsilon_0 E(t))(L'(t) + C_5\mu_1 E'(t)) \\
&\quad \leq -C_3 H'(\varepsilon_0 E(t))E(t) + C_4 H'(\varepsilon_0 E(t))(h \circ \psi_x)(t) + \tilde{C}_5 C' E'(t) \\
&\qquad + \tilde{C}_5 H'(\varepsilon_0 E(t)) H^{-1}(-C' E'(t)) \\
&\quad \leq -C_3 H'(\varepsilon_0 E(t))E(t) + \tilde{C}_5 H^*(H'(\varepsilon_0 E(t))) + C_4 H'(\varepsilon_0 E(0))(h \circ \psi_x)(t) \\
&\quad \leq -C_3 H'(\varepsilon_0 E(t))E(t) + \tilde{C}_5 H'(\varepsilon_0 E(t))\varepsilon_0 E(t) + C_4 H'(\varepsilon_0 E(0))(h \circ \psi_x)(t) \\
&\quad \leq -\tilde{C}_3 H'(\varepsilon_0 E(t))E(t) + C_4 H'(\varepsilon_0 E(0))(h \circ \psi_x)(t) \\
&\quad = -\tilde{C}_3 H_2(E(t)) + C_4 H'(\varepsilon_0 E(0))(h \circ \psi_x)(t).
\end{aligned} \tag{1.92}$$

We note that in the second inequality, we have used (1.90) and $0 \leq H'(\varepsilon_0 E(t)) \leq H'(\varepsilon_0 E(0))$. Let

$$\tilde{L}(t) = \begin{cases} L(t) + \mu E(t) & \text{if } H \text{ is linear on } [0, \varepsilon'] \\ H'(\varepsilon_0 E(t))\{L(t) + C_5\mu_1 E(t)\} & \\ \qquad + \tilde{C}_5 C' E(t) & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0, \varepsilon']. \end{cases} \tag{1.93}$$

From (1.86) and (1.92), it follows

$$\tilde{L}'(t) \leq -c_4 H_2(E(t)) + c_5 (h \circ \psi_x)(t), \quad \forall t \geq 0. \tag{1.94}$$

On the other hand, after choosing $M > 0$ larger if needed, we can observe from Lemma 1.4.7 that $L(t)$ is equivalent to $E(t)$. So, $\tilde{L}(t)$ is also equivalent to $E(t)$. Moreover, because $\zeta(t) \leq \zeta(0)$, there exists $\bar{\varepsilon} > 0$, such that

$$\zeta(t)\tilde{L}(t) + 2c_5 E(t) \leq \bar{\varepsilon} E(t), \quad \forall t \geq t_0. \tag{1.95}$$

Finally, let

$$\mathcal{L}(t) = \varepsilon(\zeta(t)\tilde{L}(t) + 2c_5 E(t)), \quad \text{for } 0 < \varepsilon < \frac{1}{\bar{\varepsilon}},$$

then we observe, from (1.94), (H1), (1.13) and (1.95), that

$$\begin{aligned}
\mathcal{L}'(t) &= \varepsilon(\zeta'(t)\tilde{L}(t) + \zeta(t)\tilde{L}'(t) + 2c_5 E'(t)) \\
&\leq -c_4\varepsilon\zeta(t) H_2(E(t)) + c_5\varepsilon\zeta(t)(h \circ \psi_x)(t) + 2c_5\varepsilon E'(t) \\
&\leq -c_4\varepsilon\zeta(t) H_2(E(t)) - c_5\varepsilon(h' \circ \psi_x)(t) + 2c_5\varepsilon E'(t) \\
&\leq -c_4\varepsilon\zeta(t) H_2(E(t)) \\
&\leq -c_4\varepsilon\zeta(t) H_2\left(\frac{1}{\bar{\varepsilon}}\left(\zeta(t)\tilde{L}(t) + 2c_5 E(t)\right)\right) \\
&\leq -c_4\varepsilon\zeta(t) H_2(\varepsilon(\zeta(t)\tilde{L}(t) + 2c_5 E(t))) = -c_4\varepsilon\zeta(t) H_2(\mathcal{L}(t)).
\end{aligned} \tag{1.96}$$

We have used the fact H_2 is increasing in the last two inequalities. Noting that $H_1' = -1/H_2$ (see (1.12)), we infer from (1.96)

$$\mathcal{L}'(t)H_1'(\mathcal{L}(t)) \geq c_4\varepsilon\zeta(t), \quad \forall t \geq t_0.$$

A simple Integration over (t_0, t) then yields

$$H_1(\mathcal{L}(t)) \geq H_1(\mathcal{L}(t_0)) + c_4\varepsilon \int_0^t \zeta(s)\, ds - c_4\varepsilon \int_0^{t_0} \zeta(s)\, ds.$$

Choose $\varepsilon > 0$ sufficiently small so that $H_1(\mathcal{L}(t_0)) - c_4\varepsilon \int_0^{t_0} \zeta(s)\, ds > 0$, then, thanks to the fact H_1^{-1} is decreasing, we infer

$$\begin{aligned} \mathcal{L}(t) &\leq H_1^{-1}\left(H_1(\mathcal{L}(t_0)) - c_4\varepsilon \int_0^{t_0} \zeta(s)\, ds + c_4\varepsilon \int_0^t \zeta(s)\, ds\right) \\ &\leq H_1^{-1}\left(c_4\varepsilon \int_0^t \zeta(s)\, ds\right). \end{aligned}$$

Consequently, the equivalence of $\mathcal{L}, \tilde{L}, L$ and E, yield

$$E(t) \leq H_1^{-1}\left(\omega \int_0^t \zeta(s)\, ds\right).$$

Chapter 2

GLOBAL EXISTENCE AND ENERGY DECAY OF SOLUTIONS FOR THE TIMOSHENKO BEAM SYSTEM WITH A TIME VARYING DELAY TERM IN THE WEAKLY NONLINEAR INTERNAL FEEDBACKS

2.1 Introduction

In this chapter we investigate the existence and decay properties of solutions for the initial boundary value problem of a nonlinear Timoshenko system of the form

$$(P)\quad \begin{cases} \rho_1\varphi_{tt}(x,t)-K(\varphi_x+\psi)_x(x,t)=0 & \text{in }]0,1[\times]0,+\infty[,\\ \rho_2\psi_{tt}(x,t)-b\psi_{xx}(x,t)+K(\varphi_x+\psi)(x,t)+\mu_1(t)g_1(\psi_t(x,t)) & \\ \qquad +\mu_2(t)g_2(\psi_t(x,t-\tau(t)))=0 & \text{in }]0,1[\times]0,+\infty[,\\ \varphi(0,t)=\varphi(1,t)=\psi(0,t)=\psi(1,t)=0 & t\geq 0,\\ \psi(x,0)=\psi_0(x),\quad \psi_t(x,0)=\psi_1(x) & x\in]0,1[,\\ \varphi(x,0)=\varphi_0(x),\quad \varphi_t(x,0)=\varphi_1(x) & x\in]0,1[,\\ \psi_t(x,t-\tau(0))=f_0(x,t-\tau(0)) & \text{in }]0,1[\times]0,\tau(0)[, \end{cases}$$

where $\tau(t)>0$ is a time varying delay, and the initial data (ψ_0,ψ_1,f_0) belong to a suitable function space.

A simple model describing the transverse vibration of a beam, which was developed in **[39]**, is given by a system of coupled hyperbolic equations of the form

$$\begin{cases} \rho u_{tt}(x,t)=(K(u_x-\phi))_x & \text{in }]0,L[\times]0,+\infty[,\\ \tilde{\rho}\phi_{tt}(x,t)=(EI\phi_x)_x+K(u_x-\phi) & \text{in }]0,L[\times]0,+\infty[, \end{cases}$$

where t denotes the time variable, x is the space variable along the beam of length L, in its equilibrium configuration, u is the transverse displacement of the beam and ϕ is the rotation angle of the filament of the beam. The coefficients $\rho, \tilde{\rho}, E, I$ and K are respectively the density (the mass per unit length), the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus.

In absence of delay ($\mu_2 = 0$), the problem of existence and energy decay have been extensively studied by several authors (see [**1**], [**21**], [**31**], [**32**], [**33**] and [**36**]) and many energy estimates have been derived for arbitrary growing feedbacks (polynomial, exponential or logarithmic decay). The decay rate of the energy (when t goes to infinity) depends on the function μ_1 and on the rate of growth of g_1.

Time delay is the property of a physical system by which the response to an applied force is delayed in its effect (see [**37**]). Whenever material, information or energy is physically transmitted from one place to another, there is a delay associated with the transmission. Time delays so often arise in many physical, chemical, biological and economical phenomena, see for example [**3**], [**38**], [**40**] and the references therein. The presence of delay may be a source of instability. Indeed, in [**15**], the authors showed that a small delay may destabilize a system which is uniformly stable in the absence of delay. For instance in [**35**] the authors studied the wave equation with linear internal damping term with constant delay (g_i linear, $\mu_1(t) \equiv 1, \tau(t) = const$ in the problem (P)). They determined suitable relations between μ_1 and μ_2, for which the stability or alternatively instability takes place. More precisely, they showed that the energy is exponentially stable if $\mu_2 < \mu_1$ and they also found a sequence of delays for which the corresponding solution of (P) will be instable if $\mu_2 \geq \mu_1$. The main approach used in [**35**], is an observability inequality obtained with a Carleman estimate.

The case of time varying delay in the wave equation has been studied recently by Nicaise, Pignotti and Valein [**35**] in the linear case (g_i linear, $\mu_1(t) \equiv 1$ in problem (P)) and proved an exponential stability result under the condition

$$\mu_2 < \sqrt{1-d}\mu_1$$

where the constant d satisfies

$$\tau'(t) \leq d < 1, \quad \forall t > 0.$$

In [**7**] Benaissa et all, extended the above result to nonlinear situation and established a decay rate estimate .

Our purpose in this chapter is to give a global solvability in Sobolev spaces and energy decay estimates of the solutions to the problem (P) for a nonlinear damping and a delay term. We should mention here that, to the best of our knowledge, there is no result concerning Timochenko beam system with the presence of nonlinear degenerate delay term.

To obtain global solutions to the problem (P), we use the argument combining the Galerkin approximation scheme (see [**25**]) with the energy estimate method. The technic based on the theory of nonlinear semigroups used in [**35**] and the variable norm of Kato in Refs. [**20**] and [**22**] do not seem to be applicable in the nonlinear case.

To prove decay estimates, we use a perturbed energy method and some properties of convex functions. These arguments of convexity were introduced and developed by [**14**] and [**24**] and used by Liu and Zuazua [**26**] , Eller et al [**16**] and Alabau-Boussouira [**1**].

2.2 Preliminaries and main results

In order to state and prove our results, we need some assumptions, as well as, some lemmas.

First assume the following hypotheses:

(H1) $\mu_1 : \mathbb{R}_+ \to]0, +\infty[$ is a non-increasing function of class $C^1(\mathbb{R}_+)$ satisfying

$$\int_0^{+\infty} \mu_1(\tau)\, d\tau = +\infty, \tag{2.1}$$

$$|\mu_1'(t)| \leq c\mu_1(t). \tag{2.2}$$

(H2) $\mu_2 : \mathbb{R}_+ \to \mathbb{R}$ is a function of class $C^1(\mathbb{R}_+)$, which is not necessarily positive or monotone, such that

$$|\mu_2(t)| \leq \beta\mu_1(t), \tag{2.3}$$

$$|\mu_2'(t)| \leq \tilde{c}\mu_1(t), \tag{2.4}$$

for some $0 < \beta < 1$ and $\tilde{c} > 0$.

(H3) $g_1 : \mathbb{R} \to \mathbb{R}$ is a non-decreasing function of the class $C(\mathbb{R})$ such that there exist ϵ_1, c_1, $c_2 > 0$ and a convex and increasing function $H : \mathbb{R}_+ \to \mathbb{R}_+$ of the class $C^1(\mathbb{R}_+) \cap C^2(]0, \infty[)$ satisfying $H(0) = 0$, and H linear on $[0, \epsilon_1]$ or ($H'(0) = 0$ and $H'' > 0$ on $]0, \epsilon_1]$), such that

$$c_1|s| \leq |g_1(s)| \leq c_2|s| \quad \text{if } |s| \geq \epsilon_1, \tag{2.5}$$

$$s^2 + g_1^2(s) \leq H^{-1}(sg_1(s)) \quad \text{if } |s| \leq \epsilon_1. \tag{2.6}$$

$g_2 : \mathbb{R} \to \mathbb{R}$ is an odd non-decreasing function of the class $C^1(\mathbb{R})$ such that there exist $c_3, \alpha_1, \alpha_2 > 0$

$$|g_2'(s)| \leq c_3 \tag{2.7}$$

$$\alpha_1\, sg_2(s) \leq G_2(s) \leq \alpha_2\, sg_1(s), \tag{2.8}$$

where

$$G_2(s) = \int_0^s g_2(r)\, dr$$

(H4) τ is a function such that

$$\tau \in W^{2,\infty}([0, T]), \forall T > 0 \tag{2.9}$$

$$0 < \tau_0 \leq \tau(t) \leq \tau_1, \quad \forall t > 0 \tag{2.10}$$

where the constant d satisfies

$$\tau'(t) \leq d < 1, \quad \forall t > 0, \tag{2.11}$$

where τ_0 and τ_1 are two positive constants.

(H5) The weight of dissipation and the delay satisfy:

$$\beta < \frac{\alpha_1(1-d)}{\alpha_2(1-\alpha_1 d)}. \tag{2.12}$$

We now state some Lemmas needed later.

Lemma 2.2.1 (Sobolev-Poincaré's inequality) *Let q be a number with $2 \leq q < +\infty$ ($n = 1, 2$) or $2 \leq q \leq 2n/(n-2)$ ($n \geq 3$). Then there is a constant $c_* = c_*((0,1), q)$ such that*

$$\|\psi\|_q \leq c_* \|\nabla \psi\|_2 \quad \textit{for} \quad \psi \in H_0^1((0,1)).$$

We introduce as in **[35]** the new variable

$$z(x,\rho,t) = \psi_t(x, t - \tau(t)\rho), x \in (0,1), \ \rho \in (0,1), \quad t > 0. \tag{2.13}$$

Then, we have

$$\tau(t) z'(x,\rho,t) + (1 - \tau'(t)\rho) z_\rho(x,\rho,t) = 0, \ \text{ in } (0,1) \times (0,1) \times (0,+\infty). \tag{2.14}$$

Therefore, problem (P) is equivalent to:

$$\begin{cases} \rho_1 \varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) = 0 & \text{in }]0,1[\times]0,+\infty[, \\ \rho_2 \psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x + \psi)(x,t) & \\ \quad +\mu_1(t) g_1(\psi_t(x,t)) + \mu_2(t) g_2(z(x,1,t)) = 0 & \text{in }]0,1[\times]0,+\infty[, \\ \tau(t) z'(x,\rho,t) + (1 - \tau'(t)\rho) z_\rho(x,\rho,t) = 0, & \text{in }]0,1[\times]0,1[\times]0,+\infty[, \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0 & t \geq 0, \\ z(x,0,t) = \psi_t(x,t) & \text{on }]0,1[\times[0,+\infty[, \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), & x \in]0,1[, \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), & x \in]0,1[, \\ z(x,\rho,0) = f_0(x, -\rho\tau(0)), & \text{in }]0,1[\times]0,1[. \end{cases} \tag{2.15}$$

Let $\bar{\xi}$ be a positive constant such that

$$\frac{\beta(1-\alpha_1)}{\alpha_1(1-d)} < \bar{\xi} < \frac{1 - \alpha_2\beta}{\alpha_2}. \tag{2.16}$$

We define the energy associated to the solution of the problem (2.15) by the following formula:

$$\begin{aligned} E(t) = E(t,z,\varphi,\psi) = & \frac{1}{2} \int_0^1 \left\{ \rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + K|\varphi_x + \psi|^2 + b\psi_x^2 \right\} dx \\ & + \xi(t)\tau(t) \int_0^1 \int_0^1 G_2(z(x,\rho,t))\, d\rho\, dx. \end{aligned} \tag{2.17}$$

where

$$\xi(t) = \bar{\xi}\mu_1(t).$$

We have the following theorem.

Theorem 2.2.1 *Let $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in (H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1)$, $f_0 \in H_0^1((0,1); H^1(0,1))$ satisfy the compatibility condition*

$$f_0(.,0) = \psi_1.$$

Assume that the hypothesis $(\mathbf{H1}) - (\mathbf{H2}) - (\mathbf{H3})$ *holds. Then the problem* (P) *admits a unique weak solution*

$$\psi, \varphi \in L^\infty([0,\infty); H^2(0,1) \cap H_0^1(0,1)), \quad \psi_t, \varphi_t \in L^\infty([0,\infty); H_0^1(0,1)),$$
$$\psi_{tt}, \varphi_{tt} \in L^\infty([0,\infty); L^2(0,1))$$

and, for some constants ω_1, ω_2 *and* ω_3, ϵ_0 *we obtain the following decay property:*

$$E(t) \leq \omega_1 H_1^{-1}\left(\omega_2 \tilde{\mu}_1(t) + \omega_3\right), \quad \forall t > 0, \tag{2.18}$$

where $\tilde{\mu}_1(t) = \int_0^t \mu_1(\tau)\, d\tau$ *and*

$$H_1(t) = \int_t^1 \frac{1}{H_2(s)}\, ds \tag{2.19}$$

and

$$H_2(t) = \begin{cases} t & \text{if } H \text{ is linear on } [0,\epsilon'], \\ tH'(\epsilon_0 t) & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0,\epsilon']. \end{cases}$$

Remark 2.2.1 *The proof will be given in section 3 and section 4.*

Remark 2.2.2 *1.By the mean value Theorem for integrals and the monotonicity of* g_2*, we find that*

$$G_2(s) = \int_0^s g_2(r)\, dr \leq s g_2(s).$$

Then, $\alpha_1 \leq \alpha_2 \leq 1$.

2. We need the condition (2.7) only to prove global existence, so if we study the energy decay, we can replace the linear growth order of the function $g_2(s)$ *for large* $|s|$ *by nonlinear polynomial growth.*

3. In this chapter, we study the interaction between damping and delay term acting on the domain Ω*. In a forthcoming paper, we will consider the following problem*

$$(P') \quad \begin{cases} \rho_1 \varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) = 0 & \text{in }]0,1[\times]0,+\infty[, \\ \rho_2 \psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x + \psi)(x,t) = 0 & \text{in }]0,1[\times]0,+\infty[, \\ \dfrac{\partial \psi(x,t)}{\partial \nu} = \mu_1(t) g_1(\psi_t(x,t)) & \\ \qquad\qquad + \mu_2(t) g_2(\psi_t(x,t-\tau(t))) & \text{on }]0,1[\times]0,+\infty[, \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0 & t \geq 0, \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x) & x \in]0,1[, \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x) & x \in]0,1[, \\ \psi_t(x,t-\tau(0)) = f_0(x,t-\tau(0)) & \text{in }]0,1[\times]0,\tau(0)[, \end{cases}$$

and adress the questions of global existence and energy decay and, moreover, the optimality of the energy decay. Indeed, without delay term Martinez and Vancostenoble **[30]** *proved the optimality. The more interesting study is the application of the technique in the paper of Martinez and Vancostenoble to obtain an optimal result following the interaction between the damping and the delay term.*

Example. Let g_1 be given by $g_1(s) = s^p(-\ln s)^q$, where $p \geq 1$ and $q \in \mathbb{R}$ on $(0, \epsilon_1]$. Then $g_1'(s) = s^{p-1}(-\ln s)^{q-1}(p(-\ln s) - q)$ which is an increasing function in a right neighborhood of 0 (if $q = 0$ we can take $\epsilon_1 = 1$). The function H is defined in the neighborhood of 0 by

$$H(s) = cs^{\frac{p+1}{2}}(-\ln\sqrt{s})^q.$$

We have

$$H'(s) = cs^{\frac{p-1}{2}}\left(-\ln\sqrt{s}\right)^{q-1}\left(\frac{p+1}{2}\left(-\ln\sqrt{s}\right) - \frac{q}{2}\right) \text{ when } s \text{ is near } 0.$$

Thus

$$H_2(s) = cs^{\frac{p+1}{2}}\left(-\ln\sqrt{s}\right)^{q-1}\left(\frac{p+1}{2}\left(-\ln\sqrt{s}\right) - \frac{q}{2}\right) \text{ when } s \text{ is near } 0.$$

and

$$\begin{aligned} H_1(t) &= c\int_t^1 \frac{1}{s^{\frac{p+1}{2}}(-\ln\sqrt{s})^{q-1}\left(\frac{p+1}{2}\left(-\ln\sqrt{s}\right) - \frac{q}{2}\right)}\,ds \\ &= c\int_1^{\frac{1}{\sqrt{t}}} \frac{z^{p-2}}{(\ln z)^{q-1}\left(\frac{p+1}{2}\ln z - \frac{q}{2}\right)}\,dz \text{ when } t \text{ is near } 0. \end{aligned}$$

We obtain in a neighborhood of 0

$$H_1(t) \equiv \begin{cases} c\dfrac{1}{t^{\frac{p-1}{2}}\left(-\ln t\right)^{q-1}} & \text{if } p > 1, \\ c\left(-\ln t\right)^{1-q} & \text{if } p = 1,\ q < 1, \\ c\left(\ln(-\ln t)\right) & \text{if } p = 1,\ q = 1. \end{cases}$$

and then in a neighborhood of $+\infty$

$$H_1^{-1}(t) \equiv \begin{cases} ct^{-\frac{2}{p-1}}(\ln t)^{-\frac{2q}{p-1}} & \text{if } p > 1, \\ ce^{-t^{\frac{1}{1-q}}} & \text{if } p = 1,\ q < 1, \\ ce^{-e^t} & \text{if } p = 1,\ q = 1. \end{cases}$$

Then

$$E(t) \leq \begin{cases} c\tilde{\mu}_1(t)^{-\frac{2}{p-1}}(\ln\tilde{\mu}_1(t))^{-\frac{2q}{p-1}} & \text{if } p > 1, \\ ce^{-\tilde{\mu}_1(t)^{\frac{1}{1-q}}} & \text{if } p = 1,\ q < 1, \\ ce^{-e^{\tilde{\mu}_1(t)}} & \text{if } p = 1,\ q = 1. \end{cases}$$

We finish this section by giving an explicit upper bound for the derivative of the energy.

Lemma 2.2.2 *Let (φ, ψ, z) be a solution of the problem (2.15). Then, the energy functional defined by (2.17) satisfies*

$$\begin{aligned} E'(t) \;\leq\; & -\mu_1(t)\left(1 - \bar{\xi}\alpha_2 - \beta\alpha_2\right)\int_0^1 \psi_t g_1(\psi_t)\,dx \\ & -\mu_1(t)\left(\bar{\xi}(1-\tau'(t))\alpha_1 - \beta(1-\alpha_1)\right)\int_0^1 z(x,1,t)g_2(z(x,1,t))\,dx \\ \leq\; & 0 \end{aligned} \tag{2.20}$$

***Proof*:** Multiplying the first equation in (2.15) by φ_t, the second equation by ψ_t, integrating over $(0,1)$ and using integration by parts, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\int_0^1\left\{\rho_1\varphi_t^2+\rho_2\psi_t^2+K|\varphi_x+\psi|^2+b\psi_x^2\right\}dx\right)\\ =-\mu_1(t)\int_0^1\psi_t g_1(\psi_t)\,dx-\mu_2(t)\int_0^1\psi_t(x,t)g_2(z(x,1,t))\,dx.\end{aligned}\tag{2.21}$$

We multiply the third equation in (2.15) by $\xi g_2(z(x,\rho,t))$ and integrate the result over $(0,1)\times(0,1)$, to obtain:

$$\begin{aligned}\xi(t)\tau(t)\int_0^1\int_0^1 z' g_2(z(x,\rho,t))\,d\rho\,dx\\ =-\xi(t)\int_0^1\int_0^1(1-\tau'(t)\rho)\frac{\partial}{\partial\rho}G_2(z(x,\rho,t))\,d\rho\,dx.\end{aligned}\tag{2.22}$$

Consequently,

$$\begin{aligned}&\frac{d}{dt}\left(\xi(t)\tau(t)\int_0^1\int_0^1 G_2(z(x,\rho,t))\,d\rho\,dx\right)\\ &\quad=-\xi(t)\int_0^1\int_0^1\frac{\partial}{\partial\rho}\Big((1-\tau'(t)\rho)G_2(z(x,\rho,t))\Big)\,d\rho\,dx\\ &\quad\quad+\xi'(t)\tau(t)\int_0^1\int_0^1 G_2(z(x,\rho,t)))\,dxd\rho\\ &\quad=\xi(t)\tau'(t)\int_0^1 G_2(z(x,1,t)))\,dx+\xi(t)\int_0^1(G_2(z(x,0,t))-G_2(z(x,1,t)))\,dx\\ &\quad\quad+\xi'(t)\tau(t)\int_0^1\int_0^1 G_2(z(x,\rho,t)))\,dxd\rho.\end{aligned}\tag{2.23}$$

From (2.21), (2.23) and using Young inequality we get

$$\begin{aligned}E'(t)\le&-(\mu_1(t)-\xi(t)\alpha_2)\int_0^1\psi_t g_1(\psi_t)\,dx-\xi(t)(1-\tau'(t))\int_0^1 G_2(z(x,1,t))\,dx\\ &-\mu_2(t)\int_0^1\psi_t(t)g_2(z(x,1,t))\,dx.\end{aligned}\tag{2.24}$$

Let us denote G_2^* to be the conjugate function of the convex function G_2, i.e., $G_2^*(s)=\sup_{t\in\mathbb{R}^+}(st-G_2(t))$. Then G_2^* is the Legendre transform of G_2, which is given by (see Arnold [**4**], p. 61-62, and Lasiecka [**14**])

$$G_2^*(s)=s(G_2')^{-1}(s)-G_2[(G_2')^{-1}(s)],\quad\forall s\ge 0\tag{2.25}$$

and satisfies the following inequality

$$st\le G_2^*(s)+G_2(t),\quad\forall s,t\ge 0.\tag{2.26}$$

Then, from the definition of G_2, we get

$$G_2^*(s)=sg_2^{-1}(s)-G_2(g_2^{-1}(s)).$$

Hence

$$G_2^*(g_2(z(x,1,t))) = z(x,1,t)g_2(z(x,1,t)) - G_2(z(x,1,t)) \leq (1-\alpha_1)z(x,1,t)g_2(z(x,1,t)). \tag{2.27}$$

Making use of (2.24), (2.26) and (2.27), we have

$$\begin{aligned} E'(t) \leq & -(\mu_1(t) - \xi(t)\alpha_2) \int_0^1 \psi_t g_1(\psi_t)\,dx - \xi(t)(1-\tau'(t)) \int_0^1 G_2(z(x,1,t))\,dx \\ & + \mu_2(t) \int_0^1 \left(G_2(\psi_t) + G_2^*(g_2(z(x,1,t)))\right)\,dx \\ \leq & -(\mu_1(t) - \xi(t)\alpha_2 - \mu_2(t)\alpha_2) \int_0^1 \psi_t g_1(\psi_t)\,dx - \xi(t)(1-\tau'(t)) \int_0^1 G_2(z(x,1,t))\,dx \\ & + |\mu_2(t)| \int_0^1 G_2^*(g_2(z(x,1,t)))\,dx. \end{aligned} \tag{2.28}$$

Using (2.8) and (2.16), we obtain

$$\begin{aligned} E'(t) \leq & -(\mu_1(t) - \xi(t)\alpha_2 - |\mu_2(t)|\alpha_2) \int_0^1 \psi_t g_1(\psi_t)\,dx \\ & -(\xi(t)(1-\tau'(t))\alpha_1 - |\mu_2(t)|(1-\alpha_1)) \int_0^1 z(x,1,t)g_2(z(x,1,t))\,dx. \end{aligned}$$

Using (2.3), we have

$$\begin{aligned} E'(t) \leq & -\mu_1(t)\left(1 - \bar{\xi}\alpha_2 - \beta\alpha_2\right) \int_0^1 \psi_t g_1(\psi_t)\,dx \\ & -\mu_1(t)\left(\bar{\xi}(1-\tau'(t))\alpha_1 - \beta(1-\alpha_1)\right) \int_0^1 z(x,1,t)g_2(z(x,1,t))\,dx \\ \leq & \ 0 \end{aligned}$$

Then, by using (2.11) and (2.16), our conclusion follows.

2.3 Global Existence

We are now ready to prove Theorem 2.2.1 in the next two sections.

Throughout this section we assume $\varphi_0, \psi_0 \in H^2 \cap H_0^1(0,1), \varphi_1, \psi_1 \in H_0^1(0,1)$ and $f_0 \in H_0^1((0,1); H^1(0,1))$.

We employ the Galerkin method to construct a global solution. Let $T > 0$ be fixed and denote by V_k the space generated by $\{w_1, w_2, \ldots, w_k\}$ where the set $\{w_k, k \in \mathbb{N}\}$ is a basis of $H^2 \cap H_0^1$.

Now, we define for $1 \leq j \leq k$ the sequence $\phi_j(x,\rho)$ as follows:

$$\phi_j(x,0) = w_j.$$

Then, we may extend $\phi_j(x,0)$ by $\phi_j(x,\rho)$ over $L^2((0,1)\times(0,1))$ and denote Z_k the space generated by $\{\phi_1, \phi_2, \ldots, \phi_k\}$.

We construct approximate solutions $(\varphi_k, \psi_k, z_k), k = 1, 2, 3, \ldots$, in the form

$$\varphi_k(t) = \sum_{j=1}^{k} g_{jk} w_j, \quad \psi_k(t) = \sum_{j=1}^{k} \tilde{g}_{jk} w_j, \quad z_k(t) = \sum_{j=1}^{k} h_{jk}\phi_j,$$

where $g_{jk}, \tilde{g}_{jk}$ and $h_{jk}, j = 1, 2, \ldots, k$, are determined by the following ordinary integro-difference-differential equations:

$$\rho_1(\varphi_k''(t), w_j) - K(\varphi_{kx}(t), w_{jx}) - k(\psi_{kx}(t), w_j) = 0, \quad 1 \leq j \leq k, \tag{2.29}$$

$$\varphi_k(0) = \varphi_{0k} = \sum_{j=1}^{k} (\varphi_0, w_j) w_j \to \varphi_0 \text{ in } H^2 \cap H_0^1 \text{ as } k \to +\infty, \tag{2.30}$$

$$\varphi_k'(0) = \varphi_{1k} = \sum_{j=1}^{k} (\varphi_1, w_j) w_j \to \varphi_1 \text{ in } H_0^1 \text{ as } k \to +\infty. \tag{2.31}$$

$$\begin{cases} \rho_2(\psi_k''(t), w_j) + b(\psi_{kx}(t), w_{jx}) + K((\varphi_{kx} + \psi_k)(t), w_j) + \mu_1(t)(g_1(\psi_k'), w_j) \\ +\mu_2(t)(g_2(z_k(.,1)), w_j) = 0 \quad 1 \leq j \leq k, \\ z_k(x, 0, t) = \psi_k'(x, t) \end{cases} \tag{2.32}$$

$$\psi_k(0) = \psi_{0k} = \sum_{j=1}^{k} (\psi_0, w_j) w_j \to \psi_0 \text{ in } H^2 \cap H_0^1 \text{ as } k \to +\infty, \tag{2.33}$$

$$\psi_k'(0) = \psi_{1k} = \sum_{j=1}^{k} (\psi_1, w_j) w_j \to \psi_1 \text{ in } H_0^1 \text{ as } k \to +\infty. \tag{2.34}$$

and

$$\begin{cases} (\tau(t) z_{kt} + (1 - \tau'(t)\rho) z_{k\rho}, \phi_j) = 0, \\ 1 \leq j \leq k, \end{cases} \tag{2.35}$$

$$z_k(\rho, 0) = z_{0k} = \sum_{j=1}^{k} (f_0, \phi_j) \phi_j \to f_0 \text{ in } H_0^1((0,1); H^1(0,1)) \text{ as } k \to +\infty. \tag{2.36}$$

By virtue of the theory of ordinary differential equations, the system (2.29)-(2.36) has a unique local solution which is extended to a maximal interval $[0, T_k[$ (with $0 < T_k \leq +\infty$) by Zorn lemma since the nonlinear terms in (2.32) are locally Lipschitz continuous. Note that $(\varphi_k(t), \psi_k(t))$ is from the class C^2.

In the next step we obtain a priori estimates for the solution, such that it can be extended outside $[0, T_k[$ to obtain one solution defined for all $t > 0$.

We can utilize a standard compactness argument for the limiting procedure and it suffices to derive some a priori estimates for (φ_k, ψ_k, z_k).

The first estimate. Since the sequences $(\varphi_{0k})_k, (\varphi_{1k})_k, (\psi_{0k})_k, (\psi_{1k})_k$ and $(z_{0k})_k$ converge, standard calculations, using (2.29)-(2.36), similar to those used to derive (2.20), yield a number C independent of k such that

$$\begin{aligned} &E_k(t) + \int_0^t \int_0^1 a_1(s) \psi_k' g_1(\psi_k') dx\, ds \\ &+ \int_0^t \int_0^1 a_2(s) z_k(x, 1, t) g_2(z_k(x, 1, t)) dx\, ds \leq E_k(0) \leq C, \end{aligned} \tag{2.37}$$

where

$$E_k(t) = \frac{1}{2}\int_0^1 \{\rho_1 {\varphi'_k}^2 + \rho_2 {\psi'_k}^2 + K|\varphi_{kx} + \psi_k|^2 + b\psi_{kx}^2\}dx$$
$$+\xi(t)\tau(t)\int_0^1\int_0^1 G_2(z_k(x,\rho,t))\,d\rho\,dx.$$

$$a_1(t) = \mu_1(t)\left(1 - \bar{\xi}\alpha_2 - \beta\alpha_2\right) \text{ and } a_2(t) = \mu_1(t)\left(\bar{\xi}(1-\tau'(t))\alpha_1 - \beta(1-\alpha_1)\right).$$

These estimates imply that the solution (φ_k, ψ_k, z_k) exists globally in $[0, +\infty[$.
Estimate (2.37) yields

$$\varphi_k, \psi_k \text{ are bounded in } L^\infty(0,T;H_0^1(0,1)) \tag{2.38}$$

$$\varphi'_k, \psi'_k \text{ are bounded in } L^\infty(0,T;L^2(0,1)) \tag{2.39}$$

$$\mu_1(t)\psi'_k(t)g_1(\psi'_k(t)) \text{ is bounded in } L^1((0,1)\times(0,T)) \tag{2.40}$$

$$G_2(z_k(x,\rho,t)) \text{ is bounded in } L^\infty(0,T;L^1((0,1)\times(0,1))) \tag{2.41}$$

$$\mu_1(t)z_k(x,1,t)g_2(z_k(x,1,t)) \text{ is bounded in } L^1((0,1)\times(0,T)) \tag{2.42}$$

for y $T > 0$.
The second estimate. First, we estimate $\varphi''_k(0)$ and $\psi''_k(0)$. Testing (2.29) by $g''_{jk}(t)$, (2.32) by $\tilde{g}''_{jk}(t)$ and choosing $t = 0$ we obtain

$$\rho_1\|\varphi''_k(0)\|_2 \le K(\|\varphi_{0kxx}\|_2 + \|\psi_{0kx}\|_2)$$

and

$$\rho_2\|\psi''_k(0)\|_2 \le b\|\psi_{0kxx}\|_2 + K(\|\varphi_{0kx}\|_2 + \|\psi_{0k}\|_2) + \mu_1(0)\|g_1(\psi_{1k})\|_2 + \mu_2(0)\|g_2(z_{0k})\|_2.$$

Hence from (2.30), (2.31) and (2.36):

$$\|\varphi''_k(0)\|_2 \le C.$$

Since $(g_1(\psi_{1k}))_k, (g_2(z_{0k}))_k$ are bounded in $L^2(0,1)$ by (**H3**), (2.30), (2.33), (2.34) and (2.36) yield

$$\|\psi''_k(0)\|_2 \le C.$$

Differentiating (2.29) and (2.32) with respect to t, we get

$$(\rho_1\varphi'''_k(t) - K\varphi'_{kxx}(t) - K\psi'_{kx}(t), w_j) = 0 \tag{2.43}$$

and

$$\begin{aligned}(\rho_2\psi'''_k(t) - b\psi'_{kxx}(t) + K\varphi'_{kx}(t) + K\psi'_k(t) + \mu_1(t){\psi''}_k(t)g'_1(\psi'_k(t)) + \mu'_1(t)g_1(\psi'_k(t))\\ +\mu_2(t)z'_k(x,1,t)g'_2(z_k(x,1,t)) + \mu'_2(t)g_2(z_k(x,1,t)), w_j) = 0.\end{aligned} \tag{2.44}$$

Multiplying (2.43) by $g''_{jk}(t)$ and (2.44) by $\tilde{g}''_{jk}(t)$, summing over j from 1 to k, it follows that

$$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi''_k(t)\|_2^2\right) - K\int_0^1 (\varphi'_{kx} + \psi'_k)_x\varphi''_k\,dx = 0 \tag{2.45}$$

$$\frac{1}{2}\frac{d}{dt}\left(\rho_2\|\psi''_k(t)\|_2^2+b\|\psi'_{kx}(t)\|_2^2\right)+K\int_0^1(\varphi'_{kx}+\psi'_k)\psi''_k\,dx$$
$$(2.46)\qquad +\mu_1(t)\int_0^1 \psi''^2_k(t)g_1'(\psi'_k(t))\,dx+\mu_1'(t)\int_0^1\psi''_k(t)g_1(\psi'_k(t))\,dx$$
$$+\mu_2(t)\int_0^1\psi''_k(t)z'_k(x,1,t)g_2'(z_k(x,1,t))\,dx+\mu_2'(t)\int_0^1\psi''_k(t)g_2(z_k(x,1,t))\,dx=0$$

Differentiating (2.35) with respect to t, we get

$$\left(\left(\frac{\tau(t)}{1-\tau'(t)\rho}\right)'z'_k+\frac{\tau(t)}{1-\tau'(t)\rho}z''_k(t)+\frac{\partial}{\partial\rho}z'_k,\phi_j\right)=0.$$

Multiplying by $h'_{jk}(t)$, summing over j from 1 to k, it follows that

$$(2.47)\qquad \left(\frac{\tau(t)}{1-\tau'(t)\rho}\right)'\|z'_k(t)\|_2^2+\frac{1}{2}\frac{\tau(t)}{1-\tau'(t)\rho}\frac{d}{dt}\|z'_k(t)\|_2^2+\frac{1}{2}\frac{d}{d\rho}\|z'_k(t)\|_2^2=0.$$

Then, we have

$$(2.48)\qquad \frac{1}{2}\left(\frac{\tau(t)}{1-\tau'(t)\rho}\right)'\|z'_k(t)\|_2^2+\frac{1}{2}\frac{d}{dt}\left(\frac{\tau(t)}{1-\tau'(t)\rho}\|z'_k(t)\|_2^2\right)+\frac{1}{2}\frac{d}{d\rho}\|z'_k(t)\|_2^2=0.$$

Taking the sum of (2.45), (2.46) and (2.47), we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi''_k(t)\|_2^2+\rho_2\|\psi''_k(t)\|_2^2+b\|\psi'_{kx}(t)\|_2^2+K\|(\varphi'_{kx}+\psi'_k)(t)\|_2^2+\int_0^1\frac{\tau(t)}{1-\tau'(t)\rho}\|z'_k(.,\rho,t)\|^2_{L^2(0,1)}\,d\rho\right)$$
$$+\mu_1(t)\int_0^1\psi''^2_k(t)g_1'(\psi'_k(t))\,dx+\frac{1}{2}\int_0^1|z'_k(x,1,t)|^2\,dx$$
$$=-\frac{1}{2}\int_0^1\left(\frac{\tau(t)}{1-\tau'(t)\rho}\right)'\|z'_k(.,\rho,t)\|_2^2\,d\rho-\mu_2(t)\int_0^1\psi''_k(t)z'_k(x,1,t)g_2'(z_k(x,1,t))\,dx$$
$$-\mu_1'(t)\int_0^1\psi''_k(t)g_1(\psi'_k(t))\,dx-\mu_2'(t)\int_0^1\psi''_k(t)g_2(z_k(x,1,t))\,dx+\frac{1}{2}\|\psi''_k(t)\|_2^2.$$

Using (**H3**),(2.7), Cauchy-Schwarz and Young's inequalities, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi''_k(t)\|_2^2+\rho_2\|\psi''_k(t)\|_2^2+b\|\psi'_{kx}(t)\|_2^2+K\|(\varphi'_{kx}+\psi'_k)(t)\|_2^2+\int_0^1\frac{\tau(t)}{1-\tau'(t)\rho}\|z'_k(.,\rho,t)\|^2_{L^2(0,1)}\,d\rho\right)$$
$$+\mu_1(t)\int_0^1\psi''^2_k(t)g_1'(\psi'_k(t))\,dx+c\int_0^1|z'_k(x,1,t)|^2\,dx$$
$$\le c'\|\psi''_k(t)\|_2^2+c''\int_0^1\frac{\tau(t)}{1-\tau'(t)\rho}\|z'_k(.,\rho,t)\|^2_{L^2(0,1)}\,d\rho$$
$$+|\mu_1'(t)|\int_0^1|\psi''_k(t)||g_1(\psi'_k(t))|\,dx+|\mu_2'(t)|\int_0^1|\psi''_k(t)||g_2(z_k(x,1,t))|\,dx.$$

Now, we estimate the last two terms of the above inequality

$$\begin{aligned}
&\int_0^1 |\psi''_k(t)||g_1(\psi'_k(t))|\,dx\\
&\qquad\le \frac{1}{2}\int_0^1 |\psi''_k(t)|^2\,dx + \frac{1}{2}\int_0^1 |g_1(\psi'_k(t))|^2\,dx\\
&\qquad\le \frac{1}{2}\int_0^1 |\psi''_k(t)|^2\,dx + \frac{1}{2}\int_{|\psi'_k|\ge 1} |g_1(\psi'_k(t))|^2\,dx + \frac{1}{2}\int_{|\psi'_k|\le 1} |g_1(\psi'_k(t))|^2\,dx\\
&\qquad\le \frac{1}{2}\int_0^1 |\psi''_k(t)|^2\,dx + c\int_{|\psi'_k|\ge 1} \psi'_k(t)g_1(\psi'_k(t))\,dx + \frac{1}{2}\int_0^1 H^{-1}(\psi'_k g_1(\psi'_k))\,dx\\
&\qquad\le \frac{1}{2}\int_0^1 |\psi''_k(t)|^2\,dx + c\int_{|\psi'_k|\ge 1} \psi'_k(t)g_1(\psi'_k(t))\,dx + cH^{-1}(\int_0^1 (\psi'_k g_1(\psi'_k))\,dx)\\
&\qquad\le \frac{1}{2}\int_0^1 |\psi''_k(t)|^2\,dx + c\int_{|\psi'_k|\ge 1} \psi'_k(t)g_1(\psi'_k(t))\,dx + c'H^*(1) + c''\int_0^1 \psi'_k g_1(\psi'_k)\,dx.\\
&\qquad\le \frac{1}{2}\int_0^1 |\psi''_k(t)|^2\,dx + c'H^*(1) + c''\int_0^1 \psi'_k g_1(\psi'_k)\,dx.
\end{aligned}$$

Hence

$$\begin{aligned}
|\mu'_1(t)|\int_0^1 |\psi''_k(t)||g_1(\psi'_k(t))|\,dx &\le c\|\psi''_k(t)\|_2^2 + c'|\mu'_1(t)|H^*(1) + c''|\mu'_1(t)|\int_0^1 \psi'_k g_1(\psi'_k)\,dx.\\
&\le c\|\psi''_k(t)\|_2^2 + c'H^*(1)|\mu'_1(t)| + c''(-E').
\end{aligned}$$

From (2.7) (that is $|g_2(s)| \le c|s|\ \forall s \in \mathbb{R}$) we obtain

$$\begin{aligned}
\int_0^1 |\psi''_k(t)||g_2(z_k(x,1,t))|\,dx &\le \frac{1}{2}\int_0^1 |\psi''_k(t)|^2\,dx + \frac{1}{2}\int_0^1 |g_2(z_k(x,1,t))|^2\,dx\\
&\le \frac{1}{2}\int_0^1 |\psi''_k(t)|^2\,dx + c\int_0^1 z_k(x,1,t)g_2(z_k(x,1,t))\,dx.
\end{aligned}$$

Hence

$$\begin{aligned}
|\mu'_1(t)|\int_0^1 |\psi''_k(t)||g_2(z_k(x,1,t))|\,dx &\le c\|\psi''_k(t)\|_2^2 + c'|\mu'_1(t)|\int_0^1 z_k(x,1,t)g_2(z_k(x,1,t))\,dx.\\
&\le c\|\psi''_k(t)\|_2^2 + c'(-E').
\end{aligned}$$

Integrating the last inequality over $(0,t)$ and using Gronwall's Lemma, we obtain

$$\begin{aligned}
&\rho_1\|\varphi''_k(t)\|_2^2 + \rho_2\|\psi''_k(t)\|_2^2 + b\|\psi'_{kx}(t)\|_2^2 + K\|(\varphi'_{kx}+\psi'_k)(t)\|_2^2 + \int_0^1 \frac{\tau(t)}{1-\tau'(t)\rho}\|z'_k(.,\rho,t)\|^2_{L^2(0,1)}\,d\rho\\
&\le e^{cT}(\rho_1\|\varphi''_k(0)\|_2^2 + \rho_2\|\psi''_k(0)\|_2^2 + b\|\psi'_{kx}(0)\|_2^2 + K\|\varphi'_{kx}(0)+\psi'_k(0)\|_2^2\\
&\quad + \int_0^1 \frac{\tau(0)}{1-\tau'(0)\rho}\|z'_k(.,\rho,0)\|^2_{L^2(0,1)}\,d\rho + c\mu_1(0) + E(0))
\end{aligned}$$

for all $t \in \mathbb{R}_+$, therefore, we conclude that

$$\varphi''_k,\psi''_k \text{ is bounded in } L^\infty(0,T;L^2) \tag{2.49}$$

(2.50) $$\varphi'_k, \psi'_k \text{ is bounded in } L^\infty(0,T;H_0^1)$$

(2.51) $$\tau(t)z'_k \text{ is bounded in } L^\infty(0,T;L^2((0,1)\times(0,1)))$$

The third estimate. Replacing w_j by $-w_{jxx}$ in (2.29) and (2.32), multiplying the result by $g'_{jk}(t)$ and $\tilde{g}'_{jk}(t)$, summing over j from 1 to k, it follows that

(2.52) $$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi'_{kx}(t)\|_2^2 +\right) + K\int_0^1 (\varphi_{kx}+\psi_k)_x\varphi'_{kxx}\,dx = 0.$$

$$\frac{1}{2}\frac{d}{dt}\left(\rho_2\|\psi'_{kx}(t)\|_2^2 + b\|\psi_{kxx}(t)\|_2^2\right) - K\int_0^1 (\varphi_{kx}+\psi_k)\psi'_{kxx}\,dx + \mu_1(t)\int_0^1 |\psi'_{kx}(t)|^2 g'_1(\psi'_k(t))\,dx$$
$$+\mu_2(t)\int_0^1 \psi'_{kx}(t)z_{kx}(x,1,t)g'_2(z_k(x,1,t))\,dx = 0.$$
(2.53)

Replacing ϕ_j by $-\phi_{jxx}$ in (2.35), multiplying the resulting equation by $h_{jk}(t)$, summing over j from 1 to k, it follows that

(2.54) $$\frac{\tau(t)}{2(1-\tau'(t)\rho)}\frac{d}{dt}\|z_{kx}(t)\|_2^2 + \frac{1}{2}\frac{d}{d\rho}\|z_{kx}(t)\|_2^2 = 0.$$

Then

(2.55) $$\frac{1}{2}\frac{d}{dt}\left(\frac{\tau(t)}{1-\tau'(t)\rho}\|z_{kx}(t)\|_2^2\right) - \frac{1}{2}\left(\frac{\tau(t)}{1-\tau'(t)\rho}\right)'\|z_{kx}(t)\|_2^2 + \frac{1}{2}\frac{d}{d\rho}\|z_{kx}(t)\|_2^2 = 0.$$

From (2.52), (2.53) and (2.54),(2.55), we have

$$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi'_{kx}(t)\|_2^2 + \rho_2\|\psi'_{kx}(t)\|_2^2 + K\|\varphi_{kxx}+\psi_{kx}(t)\|_2^2 + b\|\psi_{kxx}(t)\|_2^2 + \int_0^1 \frac{\tau(t)}{1-\tau'(t)\rho}\|z_{kx}(x,\rho,t)\|_{L^2(0,1)}^2\,d\rho\right)$$
$$+\mu_1(t)\int_0^1 |\psi'_{kx}(t)|^2 g'_1(\psi'_k(t))\,dx + \frac{1}{2}\int_0^1 |z_{kx}(x,1,t)|^2\,dx = -\mu_2(t)\int_0^1 \psi'_{kx}(t)z_{kx}(x,1,t)g'_2(z_k(x,1,t))\,dx$$
$$+\frac{1}{2}\int_0^1\left(\frac{\tau(t)}{1-\tau'(t)\rho}\right)'\|z_{kx}(x,\rho,t)\|_2^2\,d\rho + \frac{1}{2}\|\psi'_{kx}(t)\|_2^2.$$

Using (2.7), Cauchy-Schwartz and Young's inequalities, we obtain

(2.56) $$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big[\rho_1\|\varphi'_{kx}(t)\|_2^2 + \rho_2\|\psi'_{kx}(t)\|_2^2 + K\|\varphi_{kxx}+\psi_{kx}(t)\|_2^2\\ &\quad + b\|\psi_{kx}(t)\|_2^2 + \int_0^1 \frac{\tau(t)}{1-\tau'(t)\rho}\|z_{kx}(x,\rho,t)\|_{L^2(0,1)}^2\,d\rho\Big]\\ &\quad + \mu_1(t)\int_0^1 |\psi'_{kx}(t)|^2 g'_1(\psi'_k(t))\,dx + c\int_0^1 |z_{kx}(x,1,t)|^2\,dx\\ &\le c'\|\psi'_{kx}(t)\|_2^2 + c''\int_0^1 \frac{\tau(t)}{1-\tau'(t)\rho}\|z_{kx}(x,\rho,t)\|_{L^2(0,1)}^2\,d\rho.\end{aligned}$$

Integrating the last inequality over $(0,t)$ and using Gronwall's Lemma, we have

(2.57) $$\begin{aligned}&\rho_1\|\varphi'_{kx}(t)\|_2^2 + \rho_2\|\psi'_{kx}(t)\|_2^2 + K\|\varphi_{kxx}+\psi_{kx}(t)\|_2^2\\ &\quad + b\|\psi_{kxx}(t)\|_2^2 + \int_0^1 \frac{\tau(t)}{1-\tau'(t)\rho}\|z_{kx}(x,\rho,t)\|_{L^2(0,1)}^2 d\rho\\ &\le e^{cT}\Big[\rho_1\|\varphi'_{kx}(0)\|_2^2 + \rho_2\|\psi'_{kx}(0)\|_2^2 + K\|\varphi_{kxx}(0)+\psi_{kx}(0)\|_2^2\Big]\\ &\quad + b\|\psi_{kx}(0)\|_2^2 + \int_0^1 \frac{\tau(0)}{1-\tau'(0)\rho}\|z_{kx}(x,\rho,0)\|_{L^2(0,1)}^2 d\rho\Big],\end{aligned}$$

for all $t \in \mathbb{R}_+$, therefore, we conclude that

$$\varphi_k, \psi_k \text{ are bounded in } L^\infty(0,T;H^2 \cap H_0^1(0,1)), \tag{2.58}$$

$$z_k \text{ is bounded in } L^\infty(0,T;H_0^1(0,1;L^2(0,1))). \tag{2.59}$$

Applying Dunford-Petti's theorem we conclude from (2.38), (2.39), (2.40), (2.41), (2.49), (2.50), (2.51), (2.58) and (2.59), after replacing the sequences φ_k, ψ_k and z_k with a subsequence if needed, that

$$\begin{cases} \varphi_k \to \varphi \quad \text{weak-star in} \quad L^\infty(0,T;H^2 \cap H_0^1(0,1)) \\ \psi_k \to \psi \quad \text{weak-star in} \quad L^\infty(0,T;H^2 \cap H_0^1(0,1)), \end{cases} \tag{2.60}$$

$$\begin{cases} \varphi'_k \to \varphi' \text{ weak-star in} \quad L^\infty(0,T;H_0^1(0,1)) \\ \psi'_k \to \psi' \quad \text{weak-star in} \quad L^\infty(0,T;H_0^1(0,1)), \end{cases}$$

$$\begin{cases} \varphi''_k \to \varphi'' \text{ weak-star in} \quad L^\infty(0,T;L^2(0,1)) \\ \psi''_k \to \psi'' \quad \text{weak-star in} \quad L^\infty(0,T;L^2(0,1)), \end{cases} \tag{2.61}$$

$$g_1(\psi'_k) \to \chi \quad \text{weak-star in} \quad L^2((0,1)\times(0,T);\mu_1(t)),$$

$$z_k \to z \quad \text{weak-star in} \quad L^\infty(0,T;H_0^1((0,1);L^2(0,1)),$$

$$z'_k \to z' \quad \text{weak-star in} \quad L^\infty(0,T;L^2((0,1)\times(0,1))), \tag{2.62}$$

$$g_2(z_k(x,1,t)) \to \psi \quad \text{weak-star in} \quad L^2((0,1)\times(0,T);\mu_1(t))$$

for suitable functions $\varphi, \psi \in L^\infty(0,T;H^2 \cap H_0^1(0,1)), z \in L^\infty(0,T;L^2((0,1)\times(0,1)))$, $\chi \in L^2((0,1)\times(0,T);\mu_1(t)), \psi \in L^2((0,1)\times(0,T);\mu_1(t))$ for all $T \geq 0 \Big(L^2((0,1)\times(0,T);\mu_1)$ is the space of square-summable functions with weight $\mu_1 \Big)$. We have to show that (φ,ψ,z) is a solution of (2.15).

From (2.38) and (2.39) we have (ψ'_k) is bounded in $L^\infty(0,T;H_0^1(0,1))$. Then (ψ'_k) is bounded in $L^2(0,T;H_0^1)$. Since (ψ''_k) is bounded in $L^\infty(0,T;L^2(0,1))$, then (ψ''_k) is bounded in $L^2(0,T;L^2(0,1))$. Consequently (ψ'_k) is bounded in $H^1(Q)$,where $Q=(0,1)\times(0,T)$.

Since the embedding $H^1(Q) \hookrightarrow L^2(Q)$ is compact, using Aubin-Lions theorem [25] we can extract a subsequence (ψ_ν) of (ψ_k) such that

$$\psi'_\nu \to \psi' \text{ strongly in } \quad L^2(Q).$$

Therefore

$$\psi'_\nu \to \psi' \text{ strongly and a.e on } Q. \tag{2.63}$$

Similarly we obtain

$$z_\nu \to z \text{ strongly and a.e on } Q. \tag{2.64}$$

Lemma 2.3.1 *For each $T>0$, $g_1(\psi'), g_2(z(x,1,t)) \in L^1(Q)$ and $\|g_1(\psi')\|_{L^1(Q)}, \|g_2(z(x,1,t))\|_{L^1(Q)} \leq K_1$, where K_1 is a constant independent of t.*

Proof. By **(H3)** and (2.63) we have

$$g_1(\psi'_k(x,t)) \to g_1(\psi'(x,t)) \text{ a.e. in } Q,$$

$$0 \le g_1(\psi'_k(x,t))\psi'_k(x,t) \to g_1(\psi'(x,t))\psi'(x,t) \text{ a.e. in } Q$$

Hence, by (2.40) and Fatou's lemma we have

$$\int_0^T \int_0^1 \mu_1(t)\psi'(x,t)g_1(\psi'(x,t))\,dx\,dt \le K \text{ for } T > 0. \tag{2.65}$$

By Cauchy-Schwarz inequality and using (2.65), we have

$$\begin{aligned}\int_0^T \int_0^1 \mu_1(t)|g_1(\psi'(x,t))|\,dx\,dt &\le c|Q|^{\frac{1}{2}} \left(\int_0^T \int_0^1 \mu_1(t)\psi' g_1(\psi')\,dx\,dt\right)^{\frac{1}{2}} \\ &\le c|Q|^{\frac{1}{2}} K^{\frac{1}{2}} \equiv K_1\end{aligned}$$

Lemma 2.3.2 *$g_1(\psi'_k) \to g_1(\psi')$ in $L^1((0,1)\times(0,T);\mu_1(t))$ and $g_2(z_k) \to g_2(z)$ in $L^1((0,1)\times(0,T);\mu_1(t))$.*

Proof. Let $E \subset (0,1) \times [0,T]$ and set

$$E_1 = \left\{(x,t) \in E;\ g_1(\psi'_k(x,t)) \le \frac{1}{\sqrt{|E|}}\right\}, \quad E_2 = E \setminus E_1,$$

where $|E|$ is the measure of E. If $M(r) := \inf\{|s|;\ s \in \mathbb{R} \text{ and } |g_1(s)| \ge r\}$,

$$\int_E \mu_1(t)|g_1(\psi'_k)|\,dxdt \le \sqrt{|E|} + \left(M\left(\frac{1}{\sqrt{|E|}}\right)\right)^{-1} \int_{E_2} \mu_1(t)|\psi'_k g_1(\psi'_k)|\,dxdt.$$

Applying (2.40) we deduce that $\sup_k \int_E |g_1(\psi'_k)|\,dxdt \to 0$ as $|E| \to 0$. From Vitali's convergence theorem we deduce that $g_1(\psi'_k) \to g_1(\psi')$ in $L^1((0,1)\times(0,T);\mu_1(t))$, hence

$$g_1(\psi'_k) \to g_1(\psi') \text{ weak star in } L^2(Q;\mu_1(t)).$$

Similarly, we have

$$g_2(z'_k) \to g_2(z') \text{ weak star in } L^2(Q;\mu_1(t)),$$

and this imply that

$$\int_0^T \int_0^1 \mu_1(t)g_1(\psi'_k)v\,dx\,dt \to \int_0^T \int_0^1 \mu_1(t)g_1(\psi')v\,dx\,dt, \quad \text{for all } v \in L^2(0,T;H_0^1(0,1);\mu_1(t)) \tag{2.66}$$

$$\int_0^T \int_0^1 g_2(z_k)v\,dx\,dt \to \int_0^T \int_0^1 g_2(z)v\,dx\,dt, \qquad \text{for all } v \in L^2(0,T;H_0^1) \tag{2.67}$$

as $k \to +\infty$. It follows at once from (2.60), (2.61), (2.66), (2.67) and (2.62) that for each fixed $u, v \in L^2(0,T; H_0^1(0,1); \mu_1(t))$ and $w \in L^2(0,T; H_0^1((0,1)\times(0,1)); \mu_1(t))$

$$\int_0^T \int_0^1 (\rho_1 \varphi_k'' - K(\varphi_{kx} + \psi_k)_x) u \, dx \, dt \to \int_0^T \int_0^1 (\rho_1 \varphi'' - K(\varphi_x + \psi)_x) u \, dx \, dt$$

$$\int_0^T \int_0^1 (\rho_2 \psi_k'' - b\psi_{kxx} + K(\varphi_{kx} + \psi_k) + +\mu_1(t) g_1(\psi_k') + \mu_2(t) g_2(z_k)) v \, dx \, dt \to \int_0^T \int_0^1 (\rho_2 \psi'' - b\psi_{xx} + K(\varphi_x + \psi) + \mu_1(t) g_1(\psi') + \mu_2(t) g_2(z)) v \, dx \, dt$$

$$\int_0^T \int_0^1 \int_0^1 (\tau(t) z_k' + (1-\tau'(t)\rho)\frac{\partial}{\partial \rho} z_k) w \, dx \, d\rho \, dt \to \int_0^T \int_0^1 \int_0^1 (\tau(t) z' + (1-\tau'(t)\rho)\frac{\partial}{\partial \rho} z) w \, dx \, d\rho \, dt$$

as $k \to +\infty$. Hence

$$\int_0^T \int_0^1 (\rho_1 \varphi'' - K(\varphi_x + \psi)_x) u \, dx \, dt = 0$$

$$\int_0^T \int_0^1 (\rho_2 \psi'' - b\psi_{xx} + K(\varphi_x + \psi) + \mu_1(t) g_1(\psi') + \mu_2(t) g_2(z)) v \, dx \, dt = 0$$

and

$$\int_0^T \int_0^1 \int_0^1 (\tau(t) z' + (1-\tau'(t)\rho)\frac{\partial}{\partial \rho} z) w \, dx \, d\rho \, dt = 0.$$

Thus the problem (P) admits a global weak solution (φ, ψ).

Uniqueness. Let (φ_1, ψ_1, z_1) and (φ_2, ψ_2, z_2) be two solutions of problem (2.15). Then $(w, \tilde{w}, \tilde{\tilde{w}}) = (\varphi_1, \psi_1, z_1) - (\varphi_2, \psi_2, z_2)$ verifies

$$(2.68)\quad \begin{cases} \rho_1 w_{tt}(x,t) - K(w_x + \tilde{w})_x(x,t) = 0 & \text{in }]0,1[\times]0,+\infty[, \\ \rho_2 \tilde{w}''(x,t) - b\tilde{w}_{xx}(x,t) + K(w_x + \tilde{w}) & \\ \quad +\mu_1(t) g_1(\psi_1'(x,t)) - \mu_1(t) g_1(\psi_2'(x,t)) & \\ \quad +\mu_2(t) g_2(z_1(x,1,t)) - \mu_2(t) g_2(z_2(x,1,t)) = 0, & \text{in }]0,1[\times]0,+\infty[, \\ \tau(t)\tilde{w}'(x,\rho,t) + (1-\tau'(t)\rho)\tilde{w}_\rho(x,\rho,t) = 0, & \text{in } (0,1)\times]0,1[\times]0,+\infty[\\ w(0,t) = w(1,t) = \tilde{w}(0,t) = \tilde{w}(1,t) = 0, & t \geq 0 \\ \tilde{\tilde{w}}(x,0,t) = \psi_1'(x,t) - \psi_2'(x,t), & \text{on }]0,1[\times[0,+\infty[\\ w(x,0) = w'(x,0) = \tilde{w}(x,0) = \tilde{w}'(x,0) = 0, & \text{in }]0,1[\\ \tilde{\tilde{w}}(x,\rho,0) = 0 & \text{in }]0,1[\times]0,1[\end{cases}$$

Multiplying the first and the second equation in (2.68) by w', integrating over $(0,1)$ and using an integration by parts, we get

$$(2.69)\quad \frac{1}{2}\frac{d}{dt}(\rho_1 \|w'\|_2^2) + K\int_0^1 (w_x + \tilde{w})_x w' \, dx = 0$$

$$(2.70)\quad \frac{1}{2}\frac{d}{dt}(\rho_2 \|\tilde{w}'\|_2^2 + b\|\tilde{w}_x\|_2^2) + K\int_0^1 (w_x + \tilde{w})\tilde{w}' \, dx + \mu_1(t)(g_1(\psi_1') - g_1(\psi_2'), \tilde{w}') + \mu_2(t)(g_2(z_1(x,1,t)) - g_2(z_2(x,1,t)), \tilde{w}') = 0.$$

Multiplying the second equation in (2.68) by $\tilde{\tilde{w}}$, integrating over $(0,1)\times(0,1)$, we get

$$\frac{1}{2}\frac{\tau(t)}{1-\tau'(t)\rho}\frac{d}{dt}\|\tilde{w}\|_2^2+\frac{1}{2}\frac{d}{d\rho}\|\tilde{w}\|_2^2=0. \tag{2.71}$$

Then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1\left(\frac{\tau(t)}{1-\tau'(t)\rho}\|\tilde{w}\|_2^2\right)d\rho-\frac{1}{2}\int_0^1\left(\frac{\tau(t)}{1-\tau'(t)\rho}\right)'\|\tilde{w}\|_2^2\,d\rho&\\ +\frac{1}{2}(\|\tilde{w}(x,1,t)\|_2^2-\|w'\|_2^2)=0.&\end{aligned} \tag{2.72}$$

From (2.69), (2.70), (2.72 and using Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big[\rho_1\|w'\|_2^2+\rho_2\|\tilde{w}'\|_2^2+b\|\tilde{w}_x\|_2^2+K\|w_x+\tilde{w}\|_2^2\\ &\quad+\int_0^1\frac{\tau(t)}{1-\tau'(t)\rho}\|\tilde{\tilde{w}}'\|_2^2\,d\rho\Big]+\mu_1(t)(g_1(\psi_1')-g_1(\psi_2'),\tilde{w}')\\ &\quad+\frac{1}{2}\|\tilde{\tilde{w}}(x,1,t)\|_2^2=-\mu_2(t)(g_2(z_1(x,1,t))-g_2(z_2(x,1,t)),\tilde{w}')+\frac{1}{2}\|\tilde{w}'\|_2^2\\ &\qquad\qquad\leq\frac{1}{2}\|\tilde{w}'\|_2^2+\mu_2(t)||g_2(z_1(x,1,t))-g_2(z_2(x,1,t))||_2||\tilde{w}'||_2.\end{aligned}$$

Using condition (2.7) and Young's inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|w'\|_2^2+\rho_2\|\tilde{w}'\|_2^2+b\|\tilde{w}_x\|_2^2+K\|w_x+\tilde{w}\|_2^2+\int_0^1\frac{\tau(t)}{1-\tau'(t)\rho}\|\tilde{\tilde{w}}'\|_2^2\,d\rho\right)\leq c\|\tilde{w}'\|_2^2,$$

where c is a positive constant. Then integrating over $(0,t)$, using Gronwall's lemma, we conclude that

$$\rho_1\|w'\|_2^2+\rho_2\|\tilde{w}'\|_2^2+b\|\tilde{w}_x\|_2^2+K\|w_x+\tilde{w}\|_2^2+\int_0^1\frac{\tau(t)}{1-\tau'(t)\rho}\frac{\tau(t)}{1-\tau'(t)\rho}\|\tilde{\tilde{w}}'\|_2^2\,d\rho=0.$$

Hence, uniqueness follows.

2.4 Asymptotic behavior

Now we construct a Lyapunov functional L equivalent to E. For this, we define several functionals which allow us to obtain the needed estimates.

Then we have the following estimate.

Lemma 2.4.1 *Let (φ,ψ,z) be the solution of (2.15). Then the functional F_1 defined by*

$$F_1(t)=-\int_0^1(\rho_1\varphi_t\varphi+\rho_2\psi_t\psi)\,dx \tag{2.73}$$

satisfies, along the solution, the estimate

$$\begin{aligned}\frac{dF_1(t)}{dt}&\leq-\int_0^1(\rho_1\varphi_t^2+\rho_2\psi_t^2)dx+K\int_0^1|\varphi_x+\psi|^2dx\\ &\quad+c\int_0^1\psi_x^2dx+c\int_0^1g_1^2(\psi_t)dx+c|\mu_2(t)|\int_0^1g_2^2(z(x,1,t))dx.\end{aligned} \tag{2.74}$$

Proof. By taking the time derivative of (2.73)

$$\frac{dF_1(t)}{dt} = -\int_0^1 (\rho_1\varphi_t^2 + \rho_2\psi_t^2)dx - \int_0^1 (\rho_1\varphi_{tt}\varphi + \rho_2\psi_{tt}\psi)dx.$$

Therefore, by using the first and the second equations in (2.15) and some integrations by parts, we obtain from the above inequality

$$\begin{aligned}\frac{dF_1(t)}{dt} &= -\int_0^1 (\rho_1\varphi_t^2 + \rho_2\psi_t^2)dx + K\int_0^1 |\varphi_x + \psi|^2 dx \\ &+ b\int_0^1 \psi_x^2 dx + \mu_1(t)\int_0^1 \psi g_1(\psi_t)dx + \mu_2(t)\int_0^1 \psi g_2(z(x,1,t))dx.\end{aligned} \tag{2.75}$$

By exploiting Young's inequality and Poincaré's inequality, then (2.74) holds.

Lemma 2.4.2 *Let (φ, ψ, z) be the solution of (2.15). Assume that*

$$\frac{\rho_1}{K} = \frac{\rho_2}{b}. \tag{2.76}$$

Then the functional F_2 defined by

$$F_2(t) = \rho_2\int_0^1 \psi_t(\varphi_x + \psi)dx + \rho_2\int_0^1 \psi_x\varphi_t\, dx. \tag{2.77}$$

satisfies, along the solution, the estimate

$$\begin{aligned}\frac{dF_2(t)}{dt} &\le \Big[b\varphi_x\psi_x\Big]_{x=0}^{x=1} - (K-\varepsilon)\int_0^1 (\varphi_x + \psi)^2 dx \\ &+ \rho_2\int_0^1 \psi_t^2 dx + \frac{c}{\varepsilon}\int_0^1 g_1^2(\psi_t)dx + \frac{c}{\varepsilon}|\mu_2(t)|\int_0^1 g_2^2(z(x,1,t))dx\end{aligned} \tag{2.78}$$

for any $0 < \varepsilon < 1$.

Proof. Differentiating $F_2(t)$, with respect to t, we obtain

$$\begin{aligned}\frac{dF_2(t)}{dt} &= \int_0^1 \rho_2\psi_{tt}(\varphi_x + \psi)dx + \int_0^1 \rho_2\psi_t(\varphi_x + \psi)_t dx + \rho_2\int_0^1 \psi_x\varphi_{tt}dx + \rho_2\int_0^1 \psi_{tx}\varphi_t dx. \\ &= \int_0^1 (\varphi_x + \psi)\Big[b\psi_{xx} - K(\varphi_x + \psi) - \mu_1(t)g_1(\psi_t) - \mu_2(t)g_2(z(x,1,t)\Big]dx + \rho_2\int_0^1 \psi_t^2\, dx \\ &+ \frac{\rho_2}{\rho_1}\int_0^1 K(\varphi_x + \psi)_x\psi_x\, dx.\end{aligned}$$

Then, by using Eqs.(2.15) and (2.76) we find

$$\begin{aligned}\frac{dF_2(t)}{dt} &= \Big[b\varphi_x\psi_x\Big]_{x=0}^{x=1} - K\int_0^1 (\varphi_x + \psi)^2 dx + \rho_2\int_0^1 \psi_t^2 dx \\ &- \mu_1(t)\int_0^1 (\varphi_x + \psi)g_1(\psi_t)dx - \mu_2(t)\int_0^1 (\varphi_x + \psi)g_2(z(x,1,t))dx.\end{aligned}$$

By the Young inequality (2.78) is established.

Lemma 2.4.3 *Let $m \in C^1([0,1])$ be a function satisfying $m(0) = -m(1) = 2$. Then there exists $c > 0$ such that, for any $0 < \varepsilon < 1$, the functional F_3 defined by*

$$F_3(t) = \frac{b}{4\varepsilon}\int_0^1 \rho_2 m(x)\psi_t\psi_x\,dx + \frac{\varepsilon}{k}\int_0^1 \rho_1 m(x)\varphi_t\varphi_x\,dx$$

satisfies, along the solution, the estimate

$$\begin{aligned} F_3'(t) \le & -\frac{b^2}{4\varepsilon}((\psi_x(1,t))^2 + (\psi_x(0,t))^2) - \varepsilon((\varphi_x(1,t))^2 + (\varphi_x(0,t))^2) \\ & +(\frac{k}{4} + \frac{c}{k}\varepsilon)\int_0^1 (\psi + \varphi_x)^2\,dx + c\varepsilon\rho_1\int_0^1 \varphi_t^2\,dx + \frac{c}{\varepsilon^2}\int_0^1 \psi_x^2\,dx \\ & +\frac{c}{\varepsilon}\int_0^1 \psi_t^2\,dx + c\int_0^1 g_1^2(\psi_t)\,dx + c|\mu_2(t)|\int_0^1 g_2^2(z(x,1,t))\,dx \end{aligned} \tag{2.79}$$

Proof. Using Eqs. (2.15) and integrating by parts, obtain

$$\begin{aligned} F_3'(t) = & \frac{b}{4\varepsilon}\Big[-b((\psi_x(1,t))^2 + (\psi_x(0,t))^2) - \int_0^1 \frac{b}{2}m'(x)\psi_x^2\,dx - K\int_0^1 m(x)\psi_x(\varphi_x + \psi)\,dx \\ & - \int_0^1 m(x)\mu_1(t)g_1(\psi_t)\psi_x\,dx - \int_0^1 m(x)\mu_2(t)g_2(z(x,1,t))\psi_x\,dx - \int_0^1 \frac{\rho_2}{2}m'(x)(\psi_t)^2\,dx\Big] \\ & +\frac{\varepsilon}{k}\Big[-k((\varphi_x(1,t))^2 + (\varphi_x(0,t))^2) - \int_0^1 \frac{K}{2}m'(x)\varphi_x^2\,dx + \int_0^1 Km(x)\psi_x\varphi_x\,dx \\ & - \int_0^1 \frac{\rho_1}{2}m'(x)(\varphi_t)^2\,dx\Big] \end{aligned}$$

Then by the Young and Poincaré inequalities and the fact that

$$\varphi_x^2 \le 2(\psi + \varphi_x)^2 + 2\psi^2$$

we obtain

$$\begin{aligned} F_3'(t) \le & \frac{b}{4\varepsilon}\Big[-b((\psi_x(1,t))^2 + (\psi_x(0,t))^2) + \frac{c}{\varepsilon}\int_0^1 \psi_x^2\,dx + \varepsilon\frac{K}{b}\int_0^1 (\psi + \varphi_x)^2\,dx + \varepsilon\int_0^1 g_1^2(\psi_t)\,dx \\ & +\varepsilon|\mu_2(t)|\int_0^1 g_2^2(z(x,1,t))\,dx + c\int_0^1 \psi_t^2\,dx\Big] + \frac{\varepsilon}{k}\Big[-k((\varphi_x(1,t))^2 + (\varphi_x(0,t))^2) + c\int_0^1 \psi_x^2\,dx \\ & +c\int_0^1 (\psi + \varphi_x)^2\,dx + c\int_0^1 \varphi_t^2\,dx\Big] \end{aligned}$$

This gives (2.79).

Lemma 2.4.4 *Assume that $(H1)$ hold. Then, for sufficiently small ε, the functional F defined by*

$$F(t) = 2c\varepsilon F_1(t) + F_2(t) + F_3(t)$$

satisfies, along the solution, the estimate

$$\begin{aligned} F'(t) \le & -\frac{k}{2}\int_0^1 (\psi + \varphi_x)^2\,dx - \tau\int_0^1 \varphi_t^2\,dx + c\int_0^1 \psi_t^2\,dx + c\int_0^1 \psi_x^2\,dx \\ & +c\int_0^1 g_1^2(\psi_t)\,dx + c|\mu_2(t)|\int_0^1 g_2^2(z(x,1,t))\,dx, \end{aligned} \tag{2.80}$$

where $\tau = c\varepsilon\rho_1$.

Proof. Using Lemmas 2.4.1, 2.4.2, 2.4.3 and the fact that

$$[b\varphi_x\psi_x]_{x=0}^{x=1} \leq \varepsilon[\varphi_x^2(1)+\varphi_x^2(0)] + \frac{b^2}{4\varepsilon}[\psi_x^2(1)+\psi_x^2(0)] \tag{2.81}$$

for any $0 < \varepsilon < 1$, we obtain (2.80).

Next, we introduce the following functional

$$I(t) = \int_0^1 (\rho_2\psi_t\psi + \rho_1\varphi_t\omega)dx, \tag{2.82}$$

where w is the solution of

$$-\omega_{xx} = \psi_x, \qquad \omega(0) = \omega(1) = 0. \tag{2.83}$$

Then we have the following estimate.

Lemma 2.4.5 *Let (φ, ψ, z) be the solution of (2.15), then for any $\delta > 0$, we have the following estimate*

$$\begin{aligned} \frac{dI(t)}{dt} \leq & \frac{-b}{2}\int_0^1 \psi_x^2(x,t)dx + \frac{c}{\delta}\int_0^1 \psi_t^2(x,t)dx \\ & +\delta\int_0^1 \varphi_t^2(x,t)dx + c\int_0^1 g_1^2(\psi_t)dx + c|\mu_2(t)|\int_0^1 g_2^2(z(x,1,t))dx. \end{aligned} \tag{2.84}$$

Proof. Using Eqs. (2.15), we have

$$\begin{aligned} \frac{dI(t)}{dt} = & -b\int_0^1 \psi_x^2 dx + \rho_2\int_0^1 \psi_t^2 dx - K\int_0^1 \psi^2 dx \\ & +K\int_0^1 \omega_x^2 dx + \rho_1\int_0^1 \psi_t\omega_t dx - \mu_1(t)\int_0^1 \psi g_1(\psi_t)dx \\ & -\mu_2(t)\int_0^1 \psi g_2(z(x,1,t))dx. \end{aligned} \tag{2.85}$$

It is clear that, from (2.83), we have

$$\begin{aligned} & \int_0^1 \omega_x^2 dx \leq \int_0^1 \psi^2 dx \leq \int_0^1 \psi_x^2 dx \\ & \int_0^1 \omega_t^2 dx \leq \int_0^1 \omega_{tx}^2 dx \leq \int_0^1 \psi_t^2 dx \end{aligned} \tag{2.86}$$

By using Young's inequality and Poincaré's inequality, the last two terms in (2.85) can be estimated as

$$\begin{aligned} \mu_1(t)\int_0^1 \psi g_1(\psi_t)dx + \mu_2(t)\int_0^1 \psi g_2(z(x,1,t))dx \leq & \frac{b}{2}\int_0^1 \psi_x^2 dx \\ & +c\int_0^1 g_1^2(\psi_t)dx + c|\mu_2(t)|\int_0^1 g_2^2(z(x,1,t))dx. \end{aligned} \tag{2.87}$$

Consequently, from (2.85)-(2.87), we obtain (2.84).

Now, let us introduce the following functional

$$I_3(t) = \xi(t)\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}G_2(z(x,\rho,t))\,d\rho\,dx. \tag{2.88}$$

Then the following result holds.

Lemma 2.4.6 *Let (φ,ψ,z) be the solution of (2.15). Then it holds*

$$\frac{d}{dt}I_3(t)\leq -2I_3(t)-\xi(t)(1-\tau'(t))e^{-2\tau(t)}\alpha_1\int_0^1 z(x,1,t)g_2(z(x,1,t))dx \\ +\xi(t)\alpha_2\int_0^1 \psi_t(x,t)g_1(\psi_t(x,t))\,dx. \tag{2.89}$$

Proof. Differentiating (2.88) with respect to t and using the third equation in (2.15), we have

$$\begin{aligned}\frac{d}{dt}I_3(t) &= (\xi(t)\tau(t))'\int_0^1\int_0^1 e^{-2\tau(t)\rho}G_2(z(x,\rho,t))\,d\rho\,dx\\ &\quad -2\xi(t)\tau(t)\tau'(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}\rho G_2(z(x,\rho,t))d\rho\,dx\\ &\quad +\xi(t)\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}z_t g_2(z)\,d\rho\,dx\end{aligned} \tag{2.90}$$

By using the third equation in (2.15), the last term in (2.90) can be rewritten as follows

$$\begin{aligned}&\xi(t)\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}z_t g_2(z)\,d\rho\,dx\\ &=\xi(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}(\tau'(t)\rho-1)z_\rho g_2(z)\,d\rho\,dx\end{aligned} \tag{2.91}$$

Also, one can see that

$$\begin{aligned}\xi(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}(\tau'(t)\rho-1)z_\rho g_2(z)\,d\rho\,dx &= \xi(t)\int_0^1\int_0^1\frac{d}{d\rho}\left(e^{-2\tau(t)\rho}(\tau'(t)\rho-1)G_2(z)\right)\,d\rho\,dx\\ &\quad +2\xi(t)\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}(\tau'(t)\rho-1)G_2(z)\,d\rho\,dx\\ &\quad -\xi(t)\tau'(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}G_2(z)\,d\rho\,dx\end{aligned} \tag{2.92}$$

Using (2.92) and (2.91), Eq. (2.90) takes the form

$$\begin{aligned}\frac{d}{dt}I_3(t) &= -2I_3(t)+\xi'(t)\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}G_2(z(x,\rho,t))dx\\ &\quad -\xi(t)(1-\tau'(t))e^{-2\tau(t)}\int_0^1 G_2(z(x,1,t))dx+\xi(t)\int_0^1 G_2(\psi_t(x,t))\,dx.\\ &\leq -2I_3(t)-\xi(t)(1-\tau'(t))e^{-2\tau(t)}\alpha_1\int_0^1 z(x,1,t)g_2(z(x,1,t))dx\\ &\quad +\xi(t)\alpha_2\int_0^1\psi_t(x,t)g_1(\psi_t(x,t))\,dx.\end{aligned} \tag{2.93}$$

For $N_1,N_2>0$, let

$$L(t)=N_1E(t)+N_2I(t)+F(t)+I_3(t). \tag{2.94}$$

By combining (2.20), (2.80), (2.84), (2.89), we obtain

$$
\begin{aligned}
\frac{d}{dt}L(t) \le & -\mu_1(t)\left(N_1 a_1 - \bar{\xi}\alpha_2\right)\int_0^1 \psi_t g_1(\psi_t(x,t))\,dx - \left(N_2\frac{b}{2} - c\right)\int_0^1 \psi_x^2\,dx \\
& -\mu_1(t)\Big(N_1 a_2 + \bar{\xi}\alpha_1(1-d)e^{-2\tau_0} - \beta(N_2 c + c)c_3\Big)\int_0^1 z(x,1,t)g_2(z(x,1,t))\,dx \\
& -(\tau - N_2\delta)\int_0^1 \varphi_t^2\,dx + \left(N_2\frac{c}{\delta} + c\right)\int_0^1 \psi_t^2\,dx \\
& -\frac{K}{2}\int_0^1 (\psi+\varphi_x)^2\,dx + (N_2 c + c)\int_0^1 g_1^2(\psi_t)\,dx.
\end{aligned} \tag{2.95}
$$

At this point, we have to choose our constants very carefully. First, let us choose N_2 sufficiently large so that

$$\left(N_2\frac{b}{2} - c\right) > 0.$$

Next, we choose δ sufficiently small such that

$$(\tau - N_2\delta) > 0.$$

Then, we pick the constant $N_1 > 0$ sufficiently large such that

$$\left(N_1 a_1 - \bar{\xi}\alpha_2\right) > 0$$

and

$$\Big(N_1 a_2 + \alpha_1(1-d)e^{-2\tau_0} - \beta(N_2 c + c)c_3\Big) > 0.$$

Thus, (2.95) becomes

$$
\begin{aligned}
\frac{d}{dt}L(t) \le & -d_1\int_0^1 \psi_x^2\,dx - d_2\int_0^1 \varphi_t^2\,dx - \frac{K}{2}\int_0^1 (\psi+\varphi_x)^2\,dx \\
& + c\mu_1(t)\int_0^1 ((\psi_t)^2 + g_1^2(\psi_t))\,dx \\
\le & -dE(t) + c\mu_1(t)\int_0^1 ((\psi_t)^2 + g_1^2(\psi_t))\,dx.
\end{aligned} \tag{2.96}
$$

At this stage, we are in position to compare $L(t)$ with $E(t)$. We have the following Lemma.

Lemma 2.4.7 *For N_1 large enough, there exist two positive constants β_1 and β_2 depending on N_1, N_2 and ϵ, such that*

$$\beta_1 E(t) \le L(t) \le \beta_2 E(t) \qquad \forall t \ge 0. \tag{2.97}$$

Proof. We consider the functional

$$\mathcal{H}(t) = N_2 I(t) + F(t) + I_3(t)$$

and show that

$$|\mathcal{H}(t)| \leq \hat{C}E(t), \qquad C > 0.$$

from (2.73),(2.82),(2.77) and (2.88), we obtain

$$\begin{aligned}|\mathcal{H}(t)| \leq N_2 \left|\int_0^1 \rho_2\psi_t\psi + \rho_1\varphi_t\omega)(x,t)dx\right| + \left|-\int_0^1 (\rho_1\varphi_t\varphi + \rho_2\psi_t\psi)dx\right| \\ + \left|\rho_2\int_0^1 \psi_t(\varphi_x+\psi)dx + \rho_2\int_0^1 \psi_x\varphi_t\,dx\right| \\ + \left|\xi(t)\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}G_2(z(x,\rho,t))\,d\rho\,dx\right| \\ + \left|\frac{b}{4\varepsilon}\int_0^1 \rho_2 m(x)\psi_t\psi_x\,dx + \frac{\varepsilon}{k}\int_0^1 \rho_1 m(x)\varphi_t\varphi_x\,dx\right|.\end{aligned} \tag{2.98}$$

By using (2.86),(2.83), the trivial relation

$$\int_0^1 \varphi^2(x,t)dx \leq 2\int_0^1 (\varphi_x+\psi)^2(x,t)dx + 2\int_0^1 \psi_x^2(x,t)dx,$$

Young's and Poincaré's inequalities, we get

$$\begin{aligned}|\mathcal{H}(t)| \leq \alpha_1\int_0^1 \varphi_t^2(x,t)dx + \alpha_2\int_0^1 \psi_t^2(x,t)dx \\ +\alpha_3\int_0^1 (\varphi_x+\psi)^2(x,t)dx + \alpha_4\int_0^1 \psi_x^2(x,t)dx \\ +\left|\xi(t)\tau(t)\int_0^1\int_0^1 G_2(z(x,\rho,t))dx\ d\rho\right|\end{aligned} \tag{2.99}$$

where the positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 are determined as follows:

$$\begin{cases}\alpha_1 = \frac{N_2\rho_1}{2} + \rho_2 + \frac{\epsilon\rho_1}{K}, \\ \alpha_2 = \frac{N_2\rho_2}{2} + \rho_2 + \frac{\rho_2 b}{2\epsilon}, \\ \alpha_3 = \rho_1 + \frac{\rho_2}{2} + \frac{2\epsilon\rho_1}{K}, \\ \alpha_4 = \rho_2 + \frac{N_2}{2}\rho_2 + \rho_1 + \frac{\rho_2 b}{2\epsilon} + \frac{2\epsilon\rho_1}{K},\end{cases}$$

According to (2.99) , we have

$$|H(t)| \leq \hat{C}E(t)$$

for

$$\hat{C} = 2\max\left\{\frac{\alpha_1}{\rho_1}, \frac{\alpha_2}{\rho_2}, \frac{\alpha_3}{k}, \frac{\alpha_4}{b}\right\}.$$

Therefore, we obtain

$$|L(t) - N_1E(t)| \leq \hat{C}E(t).$$

So, we can choose N_1 large enough so that $\beta_1 = N_1 - \hat{C} > 0, \beta_2 = N_1 + \hat{C} > 0$. Then (2.97) holds true. Therefore, (2.96) takes the form

$$\frac{d}{dt}L(t) \leq -C_3E(t) + C_5\mu_1(t)\Big(\|\psi_t\|_2^2 + \|g_1(\psi_t)\|_2^2\Big), \tag{2.100}$$

where C_3 and C_5 are two positive constants.

Now, we estimate the last term in the right hand side of (2.100). We define

$$\Omega^+ = \{x \in (0,1) : |\psi'| \geq \varepsilon'\}, \quad \Omega^- = \{x \in (0,1) : |\psi'| \leq \varepsilon'\}.$$

From (2.5) and (2.6), it follows that

$$\begin{aligned} \mu_1(t) \int_{\Omega^+} (|\psi'|^2 + |g_1(\psi')|^2)\, dx &\leq c_2 \mu_1(t) \int_{\Omega^+} \psi' g_1(\psi')\, dx \\ &\leq -c_2 E'(t). \end{aligned} \tag{2.101}$$

Case 1: H is linear on $[0, \varepsilon']$. In this case one can easily check that there exists $c_1' > 0$, such that $|g_1(s)| \leq c_1'|s|$ for all $|s| \leq \varepsilon'$, and thus

$$\begin{aligned} \mu_1(t) \int_{\Omega^-} (|\psi'|^2 + |g_1(\psi')|^2)\, dx &\leq c_1' \mu_1(t) \int_{\Omega^-} \psi' g_1(\psi')\, dx \\ &\leq -c_1' E'(t). \end{aligned} \tag{2.102}$$

Substitution of (2.101) and (2.102) into (2.100) gives

$$(\mu_1 L + cE)'(t) \leq -C\mu_1(t) H_2(E(t)) \tag{2.103}$$

where $c = C_5(c_2 + c_1')$ and here and in the sequel we take C_i to be a generic positive constant.
Case 2: $H'(0) = 0$ and $H'' > 0$ on $]0, \varepsilon']$.
Since H is convex and increasing, H^{-1} is concave and increasing. By the virtue of (2.5), the reversed Jensen's inequality for concave function, and (2.20), it follows that

$$\begin{aligned} \mu_1(t) \int_{\Omega^-} (|\psi'|^2 + |g_1(\psi')|^2)\, dx &\leq \mu_1(t) \int_{\Omega^-} H^{-1}(\psi' g_1(\psi'))\, dx \\ &\leq \mu_1(t) |\Omega| H^{-1}\left(\frac{1}{|\Omega|} \int_{\Omega^-} \psi' g_1(\psi')\, dx\right) \\ &\leq C\mu_1(t) H^{-1}(-C' E'(t)). \end{aligned} \tag{2.104}$$

A combination of (2.100), (2.101) and (2.104) yields

$$\begin{aligned} (\mu_1 L + C_5 c_2 E)'(t) &\leq -C_3 \mu_1(t) E(t) + C C_5 \mu_1(t) H^{-1}(-C' E'(t)) \\ &\leq -\tilde{C}_3 \mu_1(t) E(t) + \tilde{C}_5 \mu_1(t) H^{-1}(-C' E'(t)), \quad t \geq 0. \end{aligned} \tag{2.105}$$

Let us denote by H^* the conjugate function of the convex function H, i.e.,

$$H^*(s) = \sup_{t \in \mathbb{R}_+} (st - H(t)).$$

Then H^* is the Legendre transform of H, which is given by

$$H^*(s) = s(H')^{-1}(s) - H[(H')^{-1}(s)], \quad \forall s \geq 0 \tag{2.106}$$

and which satisfies the following inequality

$$st \leq H^*(s) + H(t), \quad \forall s, t \geq 0. \tag{2.107}$$

The relation (2.106) and the fact that $H'(0)=0$ and $(H')^{-1}, H$ are increasing functions yield

$$H^*(s) \le s(H')^{-1}(s), \quad \forall s \ge 0. \tag{2.108}$$

Making use of $E'(t)\le 0, H''(t)\ge 0$, (2.105) and (2.108) we derive for $\varepsilon_0>0$ small enough

$$\begin{array}{rl} & [H'(\varepsilon_0 E)\{\mu_1 L + C_5 c_2 E\} + \tilde{C}_5 C' E]'(t) \\ = & \varepsilon_0 E'(t) H''(\varepsilon_0 E(t))[\mu_1(t)L(t) + C_5 c_2 E(t)] + H'(\varepsilon_0 E(t))(\mu_1 L + C_5 c_2 E)'(t) \\ & +\tilde{C}_5 C' E'(t) \\ \le & -\tilde{C}_3 \mu_1(t) H'(\varepsilon_0 E(t))E(t) + \tilde{C}_5 \mu_1(t) H'(\varepsilon_0 E(t)) H^{-1}(-C'E'(t)) + \tilde{C}_5 C' E'(t) \\ \le & -\tilde{C}_3 \mu_1(t) H'(\varepsilon_0 E(t))E(t) + \tilde{C}_5 H^*(\mu_1(t) H'(\varepsilon_0 E(t))) \\ \le & -\tilde{C}_3 \mu_1(t) H'(\varepsilon_0 E(t))E(t) + \tilde{C}_5 \mu_1(t) H'(\varepsilon_0 E(t))(H')^{-1}(\mu_1(t) H'(\varepsilon_0 E(t))) \\ \le & -\tilde{C}_3 \mu_1(t) H'(\varepsilon_0 E(t))E(t) + \tilde{C}'_5 \mu_1(t) H'(\varepsilon_0 E(t))\varepsilon_0 E(t) \\ \le & -\tilde{C}_4 \mu_1(t) H_2(E(t)). \end{array} \tag{2.109}$$

We note that in the second inequality, we have used (2.107).
Let

$$\tilde{L}(t) = \begin{cases} \mu_1(t)L(t) + cE(t) & \text{if } H \text{ is linear on } [0,\varepsilon'] \\ H'(\varepsilon_0 E(t))\{\mu_1(t)L(t) + C_5 c_1 E(t)\} + \tilde{C}_5 C' E(t) & \text{if } H'(0)=0 \text{ and } H''>0 \text{ on }]0,\varepsilon']. \end{cases} \tag{2.110}$$

From (2.103) and (2.109), it follows

$$\tilde{L}'(t) \le -c_4 \mu_1(t) H_2(E(t)), \quad \forall t \ge 0. \tag{2.111}$$

On the other hand, after choosing $M>0$ larger if needed, we can observe from Lemma 2.4.7 that $L(t)$ is equivalent to $E(t)$. So, $\tilde{L}(t)$ is also equivalent to $E(t)$. By the fact that H_2 is increasing, we obtain

$$\tilde{L}'(t) \le -\tilde{c}_4 \mu_1(t) H_2(\tilde{L}(t)), \quad \forall t \ge 0. \tag{2.112}$$

Noting that $H_1' = -1/H_2$ (see (2.19)), we infer from (2.112)

$$\tilde{L}'(t) H_1'(\tilde{L}(t)) \ge \tilde{c}_4 \mu_1(t), \quad \forall t \ge 0.$$

A simple Integration over $(0,t)$ yields

$$H_1(\tilde{L}(t)) \ge H_1(\tilde{L}(0)) + \tilde{c}_4 \int_0^t \mu_1(\tau)d\tau.$$

Then, exploiting the fact that H_1^{-1} is decreasing, we infer

$$\tilde{L}(t) \le H_1^{-1}\left(H_1(\tilde{L}(0)) + \tilde{c}_4 \int_0^t \mu_1(\tau)d\tau\right)$$

Consequently, the equivalence of $L, \tilde{L}$ and E, yields the estimate

$$E(t) \le \omega_1 H_1^{-1}\left(\omega_2 \tilde{\mu_1}(t) + \omega_3\right)$$

where

$$\tilde{\mu_1}(t) = \int_0^t \mu_1(\tau)d\tau.$$

Chapter 3

GLOBAL EXISTENCE AND ENERGY DECAY OF SOLUTIONS TO A VISCOELASTIC NONLINEAR TIMOSHENKO BEAM SYSTEM WITH A TIME VARYING DELAY TERM IN THE WEAKLY NONLINEAR INTERNAL FEEDBACK

3.1 Introduction

In this chapter we investigate the existence and decay properties of solutions for the initial boundary value problem of the nonlinear Timoshenko system of the type

$$(P)\quad \begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) = 0 & \text{in }]0,1[\times]0,+\infty[,\\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x+\psi)(x,t) + \mu_1(t)g_1(\psi_t(x,t)) & \\ \quad +\mu_2(t)g_2(\psi_t(x,t-\tau(t))) + \displaystyle\int_0^t h(t-s)\psi_{xx}(x,s)ds = 0, & \text{in }]0,1[\times]0,+\infty[,\\ \varphi(0,t)=\varphi(1,t)=\psi(0,t)=\psi(1,t)=0 & t\geq 0,\\ \psi(x,0)=\psi_0(x),\quad \psi_t(x,0)=\psi_1(x) & x\in]0,1[,\\ \varphi(x,0)=\varphi_0(x),\quad \varphi_t(x,0)=\varphi_1(x) & x\in]0,1[,\\ \psi_t(x,t-\tau(0)) = f_0(x,t-\tau(0)) & \text{in }]0,1[\times]0,\tau(0)[, \end{cases}$$

where $\tau > 0$ is a time delay, μ_1 and μ_2 are positive real numbers, and the initial data (ψ_0,ψ_1,f_0) belong to a suitable function space.

A simple model describing the transverse vibration of a beam, which was developed in [**39**], is given by a system of coupled hyperbolic equations of the form

$$\begin{cases} \rho u_{tt}(x,t) = (K(u_x - \phi))_x & \text{in }]0,L[\times]0,+\infty[, \\ \tilde{\rho}\phi_{tt}(x,t) = (EI\phi_x)_x + K(u_x - \phi) & \text{in }]0,L[\times]0,+\infty[, \end{cases}$$

where t denotes the time variable, x is the space variable along the beam of length L, in its equilibrium configuration, u is the transverse displacement of the beam and ϕ is the rotation angle of the filament of the beam. The coefficients $\rho, \tilde{\rho}, E, I$ and K are respectively the density (the mass per unit length), the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus.

In the absence of delay ($\mu_2 = 0$), the damping term assures global existence for arbitrary initial data and energy decay estimates depending on the rate of growth of g_1 (see [**1**], [**21**], [**31**], [**32**], [**33**] and [**36**]). In addition, we would like to mention the most recent work in this direction due to Cavalcanti et al. [**9**] which is the pioneer in establishing very general explicit decay rate estimates for solutions to a wave equation with boundary damping-source.

In recent years, PDEs with time delay effects have become an active area of research and arise in many practical problems (see, for example, [**3**], [**38**]). The presence of delay may be a source of instability. For example, it was proved in [**15**] that an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay. To stabilize a hyperbolic system involving input delay terms, additional control terms are necessary (see [**?**], [**19**]). For instance, in [**?**] the authors studied the wave equation with a linear internal damping term with constant delay and determined suitable relations between μ_1 and μ_2, for which the stability or alternatively instability takes place. More precisely, they showed that the energy is exponentially stable if $\mu_2 < \mu_1$ and they also found a sequence of delays for which the corresponding solution of (P) will be instable if $\mu_2 \geq \mu_1$. The main approach used in [**?**], is an observability inequality obtained with a Carleman estimate. Laskri and Said-Houari [**29**] examined problem (P) in the linear situation (that is $g_1(s) = g_2(s) = s$ for all $s \in \mathbb{R}$). Under the assumption $\mu_2 \leq \mu_1$ on the weights of the two feedbacks, they proved the well-posedness of the system. They also established for $\mu_2 < \mu_1$ an exponential decay result for the case of equal speed wave propagation. We also recall the result by Han and Xu [**19**], where the authors proved a result similar to the one in [**29**] for the case when both the damping and the delay act on the boundary and for the one-space dimension by adopting the spectral analysis approach.

Our purpose in this paper is to give a global solvability in Sobolev spaces and energy decay estimates of the solutions to the problem (P) for a nonlinear damping and a delay term. We should mention here that, to the best of our knowledge, there is no result concerning Timochenko beam system with the presence of nonlinear degenerate delay term.

To obtain global solutions to the problem (P), we use the argument combining the Galerkin approximation scheme (see [**25**]) with the energy estimate method. The technic based on the theory of nonlinear semigroups used in [**?**] does not seem to be applicable in the nonlinear case.

To prove decay estimates, we use a perturbed energy method and some properties of convex functions. These arguments of convexity were introduced and developed by [**14**] and

[**24**] and used by Liu and Zuazua [**26**] and Alabau-Boussouira [**1**].

3.2 Preliminaries and main results

In the first assume the following hypotheses on the relaxation function, the damping and the delay functions,
(H1) $(\star)$ $h:\ \mathbb{R}_+ \to \mathbb{R}_+$ is a C^2 function satisfying

$$h(0) = h_0 > 0, \quad \int_0^{+\infty} h(s)ds < b.$$

$(\star\star)$ There exists a non-increasing differentiable function $\zeta:\ \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$h'(s) \le -\zeta(s)h(s), \quad \forall s \ge 0.$$

and

$$\int_0^{+\infty} \zeta(s)ds = +\infty.$$

(H2) $\mu_1:\ \mathbb{R}_+ \to]0, +\infty[$ is a non-increasing function of class $C^1(\mathbb{R}_+)$ satisfying

$$\int_0^{+\infty} \mu_1(\tau)\, d\tau = +\infty, \tag{3.1}$$

$$|\mu_1'(t)| \le c\mu_1(t). \tag{3.2}$$

(H3) $\mu_2:\ \mathbb{R}_+ \to \mathbb{R}$ is a function of class $C^1(\mathbb{R}_+)$, which is not necessarily positive or monotone, such that

$$|\mu_2(t)| \le \beta\mu_1(t), \tag{3.3}$$

$$|\mu_2'(t)| \le \tilde{c}\mu_1(t), \tag{3.4}$$

for some $0 < \beta < 1$ and $\tilde{c} > 0$.
(H4) $g_1:\ \mathbb{R} \to \mathbb{R}$ is a non-decreasing function of the class $C(\mathbb{R})$ such that there exist ϵ_1, c_1, $c_2 > 0$ and a convex and increasing function $H:\ \mathbb{R}_+ \to \mathbb{R}_+$ of the class $C^1(\mathbb{R}_+) \cap C^2(]0, \infty[)$ satisfying $H(0) = 0$, and H linear on $[0, \epsilon_1]$ or ($H'(0) = 0$ and $H'' > 0$ on $]0, \epsilon_1]$), such that

$$c_1|s| \le |g_1(s)| \le c_2|s| \quad \text{if } |s| \ge \epsilon_1, \tag{3.5}$$

$$s^2 + g_1^2(s) \le H^{-1}(sg_1(s)) \quad \text{if } |s| \le \epsilon_1. \tag{3.6}$$

$g_2:\ \mathbb{R} \to \mathbb{R}$ is an odd non-decreasing function of the class $C^1(\mathbb{R})$ such that there exist $c_3, \alpha_1, \alpha_2 > 0$

$$|g_2'(s)| \le c_3 \tag{3.7}$$

$$\alpha_1\ sg_2(s) \le G_2(s) \le \alpha_2\ sg_1(s), \tag{3.8}$$

where

$$G_2(s) = \int_0^s g_2(r)\, dr$$

(H5) τ is a function such that

$$\tau \in W^{2,\infty}([0,T]), \forall T > 0 \tag{3.9}$$

$$0 < \tau_0 \le \tau(t) \le \tau_1, \quad \forall t > 0 \tag{3.10}$$

where the constant d satisfies

$$\tau'(t) \le d < 1, \quad \forall t > 0, \tag{3.11}$$

where τ_0 and τ_1 are two positive constants.

(H6) The weight of dissipation and the delay satisfy:

$$\beta < \frac{\alpha_1(1-d)}{\alpha_2(1-\alpha_1 d)}. \tag{3.12}$$

We first state some Lemmas which will be needed later.

Lemma 3.2.1 (Sobolev-Poincaré's inequality) *Let q be a number with $2 \le q < +\infty$ $(n = 1, 2)$ or $2 \le q \le \dfrac{2n}{(n-2)}$ $(n \ge 3)$. Then there is a constant $c_* = c_*((0,1), q)$ such that*

$$\|\psi\|_q \le c_* \|\nabla_x \psi\|_2 \quad \text{for} \quad \psi \in H_0^1((0,1)).$$

We introduce as in **[35]** the new variable

$$z(x,\rho,t) = \psi_t(x, t-\tau(t)\rho), x \in (0,1), \ \rho \in (0,1), \quad t > 0. \tag{3.13}$$

Then, we have

$$\tau(t) z_t(x,\rho,t) + (1-\tau'(t)\rho) z_\rho(x,\rho,t) = 0, \ \text{ in } (0,1)\times(0,1)\times(0,+\infty). \tag{3.14}$$

Therefore, problem (P) is equivalent to:

$$\begin{cases}
\rho_1 \varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) = 0 & \text{in }]0,1[\times]0,+\infty[, \\
\rho_2 \psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x + \psi)(x,t) & \\
+\mu_1(t) g_1(\psi_t(x,t)) + \mu_2(t) g_2(z(x,1,t)) & \\
\qquad + \displaystyle\int_0^t h(t-s)\psi_{xx}(x,s)ds = 0 & \text{in }]0,1[\times]0,+\infty[, \\
\tau(t) z_t(x,\rho,t) + (1-\tau'(t)\rho) z_\rho(x,\rho,t) = 0 & \text{in }]0,1[\times]0,1[\times]0,+\infty[, \\
\varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0 & t \ge 0, \\
z(x,0,t) = \psi_t(x,t) & \text{on }]0,1[\times[0,+\infty[, \\
\psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x) & x \in]0,1[, \\
\varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x) & x \in]0,1[, \\
z(x,\rho,0) = f_0(x, -\rho\tau(0)) & \text{in }]0,1[\times]0,1[.
\end{cases} \tag{3.15}$$

We define the energy associated to the solution of the problem (3.15) by the following formula:

$$\begin{aligned}
E(t) &= E(t,z,\varphi,\psi) = \frac{1}{2}\int_0^1 \left\{\rho_1\varphi_t^2 + \rho_2\psi_t^2 + K|\varphi_x+\psi|^2\right\} dx \\
&+ \frac{1}{2}\left(b - \int_0^t h(s)ds\right)\|\psi_x(t)\|_2^2 + \frac{1}{2}(h\circ\psi_x)(t) + \xi(t)\tau(t)\int_0^1\int_0^1 G_2(z(x,\rho,t))\,d\rho\,dx.
\end{aligned} \tag{3.16}$$

Let $\bar{\xi}$ be a positive constant such that

$$\frac{\beta(1-\alpha_1)}{(1-d)\alpha_1} < \bar{\xi} < \frac{1-\alpha_2\beta}{\alpha_2}. \tag{3.17}$$

where

$$\xi(t) = \bar{\xi}\mu_1(t).$$

and

$$(h\circ v)(t) = \int_0^1\int_0^t h(t-s)\left(v(t)-v(s)\right)^2 dsdx, \quad \forall v \in L^2([0,1]), \tag{3.18}$$

We have the following theorem.

Theorem 3.2.1 *Let* $(\varphi_0,\varphi_1),(\psi_0,\psi_1) \in (H^2(0,1)\cap H_0^1(0,1))\times H_0^1(0,1)$, $f_0 \in H_0^1((0,1);H^1(0,1))$ *satisfy the compatibility condition*

$$f_0(.,0) = \psi_1.$$

Assume that the hypothesis $(H1)$,$(H2)$ *and* $(H3)$ *holds. Then the problem* (P) *admits a unique weak solution*

$$\begin{aligned}
&\varphi \in L^\infty_{loc}(0,\infty;H^2(0,1)\cap H_0^1(0,1)), \quad \varphi_t \in L^\infty_{loc}(0,\infty;H_0^1(0,1)),\\
&\varphi_{tt} \in L^\infty_{loc}(0,\infty;L^2(0,1))\\
&\psi \in H^2(-\tau(0),0;H_0^1(0,1)) \cap L^\infty_{loc}(-\tau(0),\infty;H^2(0,1)\cap H_0^1(0,1))\\
&\psi_t \in H^1(-\tau(0),0;H_0^1(0,1)) \cap L^\infty_{loc}(-\tau(0),\infty;H^1(0,1))\\
&\psi_{tt} \in H^1(-\tau(0),0;H_0^1(0,1)) \cap L^\infty_{loc}(0,\infty;L^2(0,1))
\end{aligned}$$

and, for some constants ω,ϵ_0 *we obtain the following decay property:*

$$E(t) \le H_1^{-1}\left(\omega\int_0^t \zeta(s)\mu_1(s)ds\right), \quad \forall t > 0, \tag{3.19}$$

where

$$H_1(t) = \int_t^1 \frac{1}{H_2(s)}\,ds \tag{3.20}$$

and

$$H_2(t) = \begin{cases} t & \text{if } H \text{ is linear on } [0,\epsilon'],\\ tH'(\epsilon_0 t) & \text{if } H'(0)=0 \text{ and } H''>0 \text{ on }]0,\epsilon']. \end{cases}$$

Lemma 3.2.2 *Let (φ, ψ, z) be a solution of the problem (3.15). Then, the energy functional defined by (3.16) satisfies*

$$
\begin{aligned}
E'(t) &\leq -\left(1-\alpha_2\bar{\xi}-\alpha_2\beta\right)\mu_1(t)\int_0^1 \psi_t(x,t)g_1(\psi_t(x,t))\,dx \\
&- \left(\left(1-\tau'(t)\right)\bar{\xi}\alpha_1-\left(1-\alpha_1\right)\beta\right)\mu_1(t)\int_0^1 z(x,1,t)g_2(z(x,1,t))dx \\
&+ \frac{1}{2}\left(h'\circ\psi_x\right)(t)-\frac{1}{2}h(t)\|\psi_x(.,t)\|_2^2 \\
&\leq 0.
\end{aligned}
\tag{3.21}
$$

Proof. Multiplying the first equation in (3.15) by φ_t, the second equation by ψ_t, integrating over $(0,1)$ and using integration by parts, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\int_0^1\left\{\rho_1\varphi_t^2+\rho_2\psi_t^2+K|\varphi_x+\psi|^2+b\psi_x^2\right\}dx\right) \\
&\quad+\mu_1(t)\int_0^1 g_1(\psi_t(x,t))\psi_t(x,t)\,dx+\mu_2(t)\int_0^1\psi_t(x,t)g_2(z(x,1,t))\,dx \\
&=\frac{d}{2dt}\left[\left(\int_0^t h(s)ds\right)\|\psi_x(.,t)\|_2^2-\left(h\circ\psi_x\right)(t)\right]+\frac{1}{2}\left(h'\circ\psi_x\right)(t) \\
&\quad-\frac{1}{2}h(t)\|\psi_x(.,t)\|_2^2.
\end{aligned}
\tag{3.22}
$$

We multiply the third equation in (3.15) by $\xi(t)g_2(z(x,\rho,t))$, and integrate the result over $(0,1)\times(0,1)$, to obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\xi(t)\tau(t)\int_0^1\int_0^1 G_2(z(x,\rho,t))dxd\rho\right)=\xi'(t)\tau(t)\int_0^1\int_0^1 G_2(z(x,\rho,t))dxd\rho \\
&\quad-\xi(t)\left(1-\tau'(t)\right)\int_0^1 G_2(z(x,1,t))dx+\xi(t)\int_0^1 G_2(z(x,0,t))dx
\end{aligned}
\tag{3.23}
$$

From (3.21),(3.22) we can get

$$
\begin{aligned}
&E'(t)=-\mu_1(t)\int_0^1 g_1(\psi_t(x,t))\psi_t(x,t)\,dx-\mu_2(t)\int_0^1\psi_t(x,t)g_2(z(x,1,t))\,dx \\
&+\frac{1}{2}\left(h'\circ\psi_x\right)(t)-\frac{1}{2}h(t)\|\psi_x(.,t)\|_2^2+\xi'(t)\tau(t)\int_0^1\int_0^1 G_2(z(x,\rho,t))dxd\rho \\
&\quad-\xi(t)\left(1-\tau'(t)\right)\int_0^1 G_2(z(x,1,t))dx+\xi(t)\int_0^1 G_2(\psi_t(x,t))dx
\end{aligned}
\tag{3.24}
$$

By recalling (3.8), we arrive at

$$
\begin{aligned}
E'(t)&\leq-\left(\mu_1(t)-\alpha_2\xi(t)\right)\int_0^1\psi_t(x,t)g_1(\psi_t(x,t))\,dx+\mu_2(t)\int_0^1\psi_t(x,t)g_2(z(x,1,t))\,dx \\
&-\xi(t)\left(1-\tau'(t)\right)\int_0^1 G_2(z(x,1,t))dx+\frac{1}{2}\left(h'\circ\psi_x\right)(t)-\frac{1}{2}h(t)\|\psi_x(.,t)\|_2^2
\end{aligned}
\tag{3.25}
$$

Let us denote G_2^* to be the conjugate function of the convex function G_2, i.e.,

$$
G_2^*(s)=\sup_{t\in\mathbb{R}^+}\left(st-G_2(t)\right).
$$

Then G_2^* is the Legendre transform of G_2, which is given by (see Arnold **[4]**, p. 61-62, and Lasiecka **[14]**)

$$G_2^*(s) = s(G_2')^{-1}(s) - G_2[(G_2')^{-1}(s)], \quad \forall s \geq 0 \tag{3.26}$$

and satisfies the following inequality

$$st \leq G_2^*(s) + G_2(t), \quad \forall s, t \geq 0. \tag{3.27}$$

Then, from the definition of G_2, we get

$$G_2^*(s) = sg_2^{-1}(s) - G_2(g_2^{-1}(s)).$$

Hence

$$\begin{aligned} G_2^*(g_2(z(x,1,t))) &= z(x,1,t)g_2(z(x,1,t)) - G_2(z(x,1,t)) \\ &\leq (1-\alpha_1)z(x,1,t)g_2(z(x,1,t)). \end{aligned} \tag{3.28}$$

Making use of (3.27) and (3.28), we have

$$\begin{aligned} E'(t) &\leq -(\mu_1(t) - \alpha_2\xi(t)) \int_0^1 \psi_t(x,t)g_1(\psi_t(x,t))\,dx + \mu_2(t)\int_0^1 [G_2(\psi_t(x,t)) + G_2^*(g_2(z(x,1,t)))]\,dx \\ &- \xi(t)(1-\tau'(t))\int_0^1 G_2(z(x,1,t))dx + \frac{1}{2}(h'\circ\psi_x)(t) - \frac{1}{2}h(t)\|\psi_x(.,t)\|_2^2 \end{aligned} \tag{3.29}$$

Using (3.8) and (3.28), we obtain

$$\begin{aligned} E'(t) &\leq -\left(1 - \alpha_2\bar{\xi} - \alpha_2\beta\right)\mu_1(t)\int_0^1 \psi_t(x,t)g_1(\psi_t(x,t))\,dx \\ &- \left((1-\tau'(t))\,\bar{\xi}\alpha_1 - (1-\alpha_1)\,\beta\right)\mu_1(t)\int_0^1 z(x,1,t)g_2(z(x,1,t))dx \\ &+ \frac{1}{2}(h'\circ\psi_x)(t) - \frac{1}{2}h(t)\|\psi_x(.,t)\|_2^2 \\ &\leq 0. \end{aligned} \tag{3.30}$$

3.3 Global Existence

We are now ready to prove Theorem 3.2.1 in the next two sections.

Throughout this section we assume $\varphi_0, \psi_0 \in H^2 \cap H_0^1(0,1), \varphi_1, \psi_1 \in H_0^1(0,1)$ and $f_0 \in H_0^1((0,1); H^1(0,1))$.

We employ the Galerkin method to construct a global solution. Let $T > 0$ be fixed and denote by V_k the space generated by $\{w_1, w_2, \ldots, w_k\}$ where the set $\{w_k, k \in \mathbb{N}\}$ is a basis of $H^2 \cap H_0^1$.

Now, we define for $1 \leq j \leq k$ the sequence $\phi_j(x,\rho)$ as follows:

$$\phi_j(x,0) = w_j.$$

Then, we may extend $\phi_j(x,0)$ by $\phi_j(x,\rho)$ over $L^2((0,1)\times(0,1))$ and denote Z_k the space generated by $\{\phi_1, \phi_2, \ldots, \phi_k\}$.

We construct approximate solutions $(\varphi_k, \psi_k, z_k), k = 1, 2, 3, \ldots,$ in the form

$$\varphi_k(t) = \sum_{j=1}^{k} g_{jk} w_j, \quad \psi_k(t) = \sum_{j=1}^{k} \tilde{g}_{jk} w_j, \quad z_k(t) = \sum_{j=1}^{k} h_{jk} \phi_j,$$

where $g_{jk}, \tilde{g}_{jk}$ and $h_{jk}, j = 1, 2, \ldots, k$, are determined by the following ordinary differential equations:

$$\rho_1(\varphi_k''(t), w_j) + K(\varphi_{kx}(t), w_{jx}) - k(\psi_{kx}(t), w_j) = 0, \quad 1 \leq j \leq k, \tag{3.31}$$

$$\varphi_k(0) = \varphi_{0k} = \sum_{j=1}^{k} (\varphi_0, w_j) w_j \to \varphi_0 \text{ in } H^2 \cap H_0^1 \text{ as } k \to +\infty, \tag{3.32}$$

$$\varphi_k'(0) = \varphi_{1k} = \sum_{j=1}^{k} (\varphi_1, w_j) w_j \to \varphi_1 \text{ in } H_0^1 \text{ as } k \to +\infty. \tag{3.33}$$

$$\begin{cases} \rho_2(\psi_k''(t), w_j) + b(\psi_{kx}(t), w_{jx}) + K((\varphi_{kx} + \psi_k)(t), w_j) + \mu_1(t)(g_1(\psi_k'), w_j) \\ +\mu_2(t)(g_2(z_k(.,1)), w_j) - \int_0^t h(t-s)(\psi_{kx}(s), w_{jx}) ds = 0 \quad 1 \leq j \leq k, \\ z_k(x, 0, t) = \psi_k'(x, t) \end{cases} \tag{3.34}$$

$$\psi_k(0) = \psi_{0k} = \sum_{j=1}^{k} (\psi_0, w_j) w_j \to \psi_0 \text{ in } H^2 \cap H_0^1 \text{ as } k \to +\infty, \tag{3.35}$$

$$\psi_k'(0) = \psi_{1k} = \sum_{j=1}^{k} (\psi_1, w_j) w_j \to \psi_1 \text{ in } H_0^1 \text{ as } k \to +\infty. \tag{3.36}$$

and

$$(\tau(t) z_{kt} + (1 - \tau'(t)\rho) z_{k\rho}, \phi_j) = 0, \quad 1 \leq j \leq k, \tag{3.37}$$

$$z_k(\rho, 0) = z_{0k} = \sum_{j=1}^{k} (f_0, \phi_j) \phi_j \to f_0 \text{ in } H_0^1((0,1); H^1(0,1)) \text{ as } k \to +\infty. \tag{3.38}$$

By virtue of the theory of ordinary differential equations, the system (3.31)-(3.38) has a unique local solution which is extended to a maximal interval $[0, T_k[$ (with $0 < T_k \leq +\infty$) by Zorn lemma since the nonlinear terms in (3.34) are locally Lipschitz continuous. Note that $(\varphi_k(t), \psi_k(t))$ is from the class C^2.

In the next step we obtain a priori estimates for the solution, such that it can be extended outside $[0, T_k[$ to obtain one solution defined for all $t > 0$.

We can utilize a standard compactness argument for the limiting procedure and it suffices to derive some a priori estimates for (φ_k, ψ_k, z_k).

The first estimate. Since the sequences $\varphi_{0k}, \varphi_{1k}, \psi_{0k}, \psi_{1k}$ and z_{0k} converge, then standard calculations, using (3.31)-(3.38), similar to those used to derive (3.21), yield C independent of k such that

$$\begin{aligned} &E_k(t) + a_1 \int_0^t \int_0^1 \psi_k' g_1(\psi_k') dx\, ds \\ &+ a_2 \int_0^t \int_0^1 z_k(x, 1, t) g_2(z_k(x, 1, t)) dx\, ds \leq E_k(0) \leq C, \end{aligned} \tag{3.39}$$

where

$$E_k(t) = \frac{1}{2}\int_0^1 \{\rho_1 {\varphi_k'}^2 + \rho_2 {\psi_k'}^2 + K|\varphi_{kx} + \psi_k|^2\}dx + \frac{1}{2}\left(b - \int_0^t h(s)ds\right)\|\psi_{kx}(.,t)\|_2^2$$
$$+\xi(t)\tau(t)\int_0^1\int_0^1 G_2(z_k(x,\rho,t))\,d\rho\,dx.$$

$$a_1 = \mu_1(t) - \alpha_2\xi(t) - \alpha_2\mu_2(t) \text{ and } a_2 = \alpha_1\xi(t)(1-\tau'(t)) - (1-\alpha_1)\mu_2(t).$$

for some C independent of k. These estimates imply that the solution (φ_k, ψ_k, z_k) exists globally in $[0, +\infty[$.
Estimate (3.39) yields

$$\varphi_k, \psi_k \text{ are bounded in } L^\infty(0,T; H_0^1(0,1)) \tag{3.40}$$

$$\varphi_k', \psi_k' \text{ are bounded in } L^\infty(0,T; L^2(0,1)) \tag{3.41}$$

$$\psi_k'(t)g_1(\psi_k'(t)) \text{ is bounded in } L^1((0,1)\times(0,T)) \tag{3.42}$$

$$G_2(z_k(x,\rho,t)) \text{ is bounded in } L^\infty(0,T; L^1((0,1)\times(0,1))) \tag{3.43}$$

$$z_k(x,1,t)g_2(z_k(x,1,t)) \text{ is bounded in } L^1((0,1)\times(0,T)) \tag{3.44}$$

The second estimate. First, we estimate $\varphi_k''(0)$ and $\psi_k''(0)$. Testing (3.31) by $g_{jk}''(t)$, (3.34) by $\tilde{g}_{jk}''(t)$ and choosing $t = 0$ we obtain

$$\rho_1\|\varphi_k''(0)\|_2 \le K(\|\varphi_{0kxx}\|_2 + \|\psi_{0kx}\|_2)$$

and

$$\rho_2\|\psi_k''(0)\|_2 \le b\|\psi_{0kxx}\|_2 + K(\|\varphi_{0kx}\|_2 + \|\psi_{0k}\|_2) + \mu_1\|g_1(\psi_{1k})\|_2 + \mu_2\|g_2(z_{0k})\|_2.$$

Hence from (3.32), (3.33) and (3.38):

$$\|\varphi_k''(0)\|_2 \le C.$$

Since $g_1(\psi_{1k}), g_2(z_{0k})$ are bounded in $L^2(0,1)$ by (**H1**), (3.32), (3.35), (3.36) and (3.38) yield

$$\|\psi_k''(0)\|_2 \le C.$$

Differentiating (3.31) and (3.34) with respect to t, we get

$$(\rho_1\varphi_k'''(t) - K\varphi_{kxx}'(t) - K\psi_{kx}'(t), w_j) = 0 \tag{3.45}$$

and

$$\begin{aligned}(\rho_2\psi_k'''(t) - b\psi_{kxx}'(t) + K\varphi_{kx}'(t) + K\psi_k'(t) + \mu_1'(t)g_1(\psi_k'(t)) + \mu_1(t)\psi_k''(t)g_1'(\psi_k'(t))\\ +\mu_2'(t)g_2(z_k(x,1,t)) + \mu_2(t)z_k'(x,1,t)g_2'(z_k(x,1,t))\\ +\frac{d}{dt}\left(\int_0^t h(t-s)\psi_{kxx}(s)ds\right), w_j) = 0.\end{aligned} \tag{3.46}$$

Multiplying (3.45) by $g''_{jk}(t)$ and (3.46) by $\tilde{g}''_{jk}(t)$, summing over j from 1 to k, it follows that

$$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi''_k(.,t)\|_2^2\right) - K\int_0^1 (\varphi'_{kx} + \psi'_k)_x\varphi''_k\,dx = 0 \tag{3.47}$$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\rho_2\|\psi''_k(.,t)\|_2^2 + b\|\psi'_{kx}(.,t)\|_2^2\right) + K\int_0^1 (\varphi'_{kx}(t) + \psi'_k(t))\psi''_k(t)\,dx\\
&+\mu'_1(t)\int_0^1 \psi''_k(t)g_1(\psi'_k(t))\,dx + \mu_1(t)\int_0^1 \psi''^2_k(t)g'_1(\psi'_k(t))\,dx\\
&+\mu'_2(t)\int_0^1 \psi''_k(t)g_2(z_k(x,1,t))dx + \mu_2(t)\int_0^1 \psi''_k(t)z'_k(x,1,t)g'_2(z_k(x,1,t))dx\\
&-h(0)\frac{d}{dt}(\psi_{kx}(t),\psi'_{kx}(t)) + h(0)\|\psi'_{kx}(.,t)\|_2^2 + h'(0)(\psi_{kx}(t),\psi'_{kx}(t))\\
&-\frac{d}{dt}\int_0^t h'(t-s)(\psi_{kx}(s),\psi'_{kx}(t))ds + \int_0^t h''(t-s)(\psi_{kx}(s),\psi'_{kx}(t))ds = 0.
\end{aligned} \tag{3.48}$$

Differentiating (3.37) with respect to t, we get

$$(\left(\frac{\tau(t)}{1-\tau'(t)\rho}\right)' z'_k(t) + \frac{\tau(t)}{1-\tau'(t)\rho}z''_k(t) + \frac{\partial}{\partial\rho}z'_k(t),\phi_j) = 0.$$

Multiplying by $h'_{jk}(t)$, summing over j from 1 to k, it follows that

$$\left(\frac{\tau(t)}{1-\tau'(t)\rho}\right)'\|z'_k(t)\|_2^2 + \frac{1}{2}\frac{\tau(t)}{1-\tau'(t)\rho}\frac{d}{dt}\|z'_k(t)\|_2^2 + \frac{1}{2}\frac{d}{d\rho}\|z'_k(t)\|_2^2 = 0. \tag{3.49}$$

Then, we have

$$\frac{1}{2}\left(\frac{\tau(t)}{1-\tau'(t)\rho}\right)'\|z'_k(t)\|_2^2 + \frac{1}{2}\frac{d}{dt}\left(\frac{\tau(t)}{1-\tau'(t)\rho}\|z'_k(t)\|_2^2\right) + \frac{1}{2}\frac{d}{d\rho}\|z'_k(t)\|_2^2 = 0, \tag{3.50}$$

integrate the result over $(0,1)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_0^1\left(\frac{\tau(t)}{1-\tau'(t)\rho}\right)'\|z'_k(t)\|_2^2 d\rho + \frac{1}{2}\frac{d}{dt}\int_0^1\left(\frac{\tau(t)}{1-\tau'(t)\rho}\|z'_k(t)\|_2^2\right)d\rho\\
&+\frac{1}{2}\|z'_k(x,1,t)\|_2^2 = \frac{1}{2}\|\psi''(.,t)\|_2^2,
\end{aligned} \tag{3.51}$$

Taking the sum of (3.47), (3.48) and (3.51), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi''_k(t)\|_2^2 + \rho_2\|\psi''_k(t)\|_2^2 + b\|\psi'_{kx}(t)\|_2^2 + K\|\varphi'_{kx}(t) + \psi'_k(t)\|_2^2\right.\\
&\left.+\int_0^1\frac{\tau(t)}{1-\tau'(t)\rho}\|z'_k(.,\rho,t)\|^2_{L^2(0,1)}d\rho\right) + \mu_1(t)\int_0^1 \psi''^2_k(t)g'_1(\psi'_k(t))\,dx\\
&+\frac{1}{2}\int_0^1 |z'_k(x,1,t)|^2\,dx + h(0)\|\psi'_{kx}(t)\|_2^2\\
&= -\mu'_1(t)\int_0^1 \psi''_k(t)g_1(\psi'_k(t))dx - \mu_2(t)\int_0^1 \psi''_k(t)z'_k(x,1,t)g'_2(z_k(x,1,t))dx\\
&-\mu'_2(t)\int_0^1 \psi''_k(t)g_2(z_k(x,1,t))dx + \frac{1}{2}\|\psi''_k(t)\|_2^2 + h(0)\frac{d}{dt}(\psi_{kx}(t),\psi'_{kx}(t))\\
&-h'(0)(\psi_{kx}(t),\psi'_{kx}(t)) + \frac{d}{dt}\int_0^t h'(t-s)(\psi_{kx}(s),\psi'_{kx}(t))ds\\
&-\int_0^t h''(t-s)(\psi_{kx}(s),\psi'_{kx}(t))ds - \frac{1}{2}\int_0^1\left(\frac{\tau(t)}{1-\tau'(t)\rho}\right)'\|z'_k(.,\rho,t)\|_2^2 d\rho.
\end{aligned}$$

Using (3.7), Cauchy-Schwarz and Young's inequalities, we obtain

$$|h'(0)(\psi_{kx}(t),\psi'_{kx}(t))| \le \epsilon\|\psi_{kx}(t)\|_2^2 + \frac{[h'(0)]^2}{4\epsilon}\|\psi'_{kx}(t)\|_2^2,$$

$$\begin{aligned}\left|\int_0^t h''(t-s)(\psi_{kx}(s),\psi'_{kx}(t))ds\right| &\le \|\psi'_{kx}(t)\|_2\int_0^t |h''(t-s)|\,\|\psi_{kx}(s)\|_2 ds\\ &\le \frac{1}{4\epsilon}\|\psi'_{kx}(t)\|_2^2 + \epsilon\|h''\|_{L^1}\int_0^t |h''(t-s)|\,\|\psi_{kx}(s)\|_2^2 ds,\end{aligned}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\rho_1\|\varphi''_k(t)\|_2^2 + \rho_2\|\psi''_k(t)\|_2^2 + b\|\psi'_{kx}(t)\|_2^2 + K\|\varphi'_{kx}(t)+\psi'_k(t)\|_2^2\\ &+\int_0^1 \frac{\tau(t)}{1-\tau'(t)\rho}\|z'_k(.,\rho,t)\|^2_{L^2(0,1)}d\rho\Big) + \mu_1(t)\int_0^1 \psi''^2_k(t)g'_1(\psi'_k(t))\,dx\\ &+c\int_0^1 |z'_k(x,1,t)|^2\,dx + h(0)\|\psi'_{kx}(t)\|_2^2\\ &\le c'\|\psi''_k(t)\|_2^2 + \epsilon\|\psi_{kx}(t)\|_2^2 + \frac{[h'(0)]^2}{4\epsilon}\|\psi'_{kx}(t)\|_2^2 + \epsilon\|h''\|_{L^1}\int_0^t |h''(t-s)|\,\|\psi_{kx}(s)\|_2^2 ds\\ &+\frac{1}{4\epsilon}\|\psi'_{kx}(t)\|_2^2 + \frac{d}{dt}\int_0^t h'(t-s)(\psi_{kx}(s),\psi'_{kx}(t))ds\\ &+c''\int_0^1 \frac{\tau(t)}{1-\tau'(t)\rho}\|z'_k(.,\rho,t)\|^2_{L^2(0,1)}d\rho + h(0)\frac{d}{dt}(\psi_{kx}(t),\psi'_{kx}(t))\\ &+|\mu'_1(t)|\int_0^1 |\psi''_k(t)|\,|g_1(\psi'_k(t))|\,dx + |\mu'_2(t)|\int_0^1 |\psi''_k(t)|\,|g_2(z_k(x,1,t))|\,dx\end{aligned}$$

Now, we estimate the last two terms of the above inequality

$$\int_0^1 |\psi''_k(t)|\,|g_1(\psi'_k(t))|\,dx \le \frac{1}{2}\int_0^1 |\psi''_k(t)|^2\,dx + \frac{1}{2}\int_0^1 |g_1(\psi'_k(t))|^2\,dx$$

$$\begin{aligned}&\le \frac{1}{2}\int_0^1 |\psi''_k(t)|^2\,dx + \frac{1}{2}\int_{|\psi'_k|\ge 1} |g_1(\psi'_k(t))|^2\,dx + \frac{1}{2}\int_{|\psi'_k|\le 1} |g_1(\psi'_k(t))|^2\,dx\\ &\le \frac{1}{2}\int_0^1 |\psi''_k(t)|^2\,dx + c\int_{|\psi'_k|\ge 1}\psi'_k(t)g_1(\psi'_k(t))dx + \frac{1}{2}\int_0^1 H^{-1}\left(\psi'_k(t)g_1(\psi'_k(t))\right)dx\\ &\le \frac{1}{2}\int_0^1 |\psi''_k(t)|^2\,dx + c\int_{|\psi'_k|\ge 1}\psi'_k(t)g_1(\psi'_k(t))dx + cH^{-1}\left(\int_0^1 \psi'_k(t)g_1(\psi'_k(t))dx\right)\\ &\le \frac{1}{2}\int_0^1 |\psi''_k(t)|^2\,dx + c\int_{|\psi'_k|\ge 1}\psi'_k(t)g_1(\psi'_k(t))dx + c'H^\star(1) + c''\int_0^1 \psi'_k(t)g_1(\psi'_k(t))dx\\ &\le \frac{1}{2}\int_0^1 |\psi''_k(t)|^2\,dx + c'H^\star(1) + c''\int_0^1 \psi'_k(t)g_1(\psi'_k(t))dx\end{aligned}$$

Hence

$$\begin{aligned}|\mu'_1(t)|\int_0^1 |\psi''_k(t)|\,|g_1(\psi'_k(t))|\,dx &\le c\|\psi''_k(t)\|^2 + c'|\mu'_1(t)|H^\star(1) + c''|\mu'_1(t)|\int_0^1 \psi'_k(t)g_1(\psi'_k(t))dx\\ &\le c\|\psi''_k(t)\|^2 + c'|\mu'_1(t)|H^\star(1) + c''(-E').\end{aligned}$$

From (3.7) (that is $|g_2(s)| \le c|s| \ \ \forall s \in \mathbb{R}$) we obtain

$$\begin{aligned}\int_0^1 |\psi''_k(t)|\,|g_2(z_k(x,1,t))|\,dx &\le \frac{1}{2}\int_0^1 |\psi''_k(t)|^2\,dx + \frac{1}{2}\int_0^1 |g_2(z_k(x,1,t))|^2\,dx\\ &\le \frac{1}{2}\int_0^1 |\psi''_k(t)|^2\,dx + c\int_0^1 z_k(x,1,t)g_2(z_k(x,1,t))dx\end{aligned}$$

Hence

$$|\mu_2'(t)|\int_0^1 |\psi_k''(t)|\,|g_2(z_k(x,1,t))|\,dx \leq c\|\psi_k''(t)\|_2^2 + c'|\mu_2'(t)|\int_0^1 z_k(x,1,t)g_2(z_k(x,1,t))dx$$
$$\leq c\|\psi_k''(t)\|_2^2 + c'(-E').$$

So, we can obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\rho_1\|\varphi_k''(t)\|_2^2 + \rho_2\|\psi_k''(t)\|_2^2 + b\|\psi_{kx}'(t)\|_2^2 + K\|\varphi_{kx}'(t) + \psi_k'(t)\|_2^2 \\
&+ \int_0^1 \frac{\tau(t)}{1-\tau'(t)\rho}\|z_k'(.,\rho,t)\|_{L^2(0,1)}^2 d\rho\Big) + \mu_1(t)\int_0^1 {\psi''}_k^2(t)g_1'(\psi_k'(t))\,dx \\
&+c\int_0^1 |z_k'(x,1,t)|^2\,dx + h(0)\|\psi_{kx}'(t)\|_2^2 \\
&\leq c'\|\psi_k''(t)\|_2^2 + \epsilon\|\psi_{kx}(t)\|_2^2 + \frac{[h'(0)]^2}{4\epsilon}\|\psi_{kx}'(t)\|_2^2 + \epsilon\|h''\|_{L^1}\int_0^t |h''(t-s)|\,\|\psi_{kx}(s)\|_2^2 ds \\
&+\frac{1}{4\epsilon}\|\psi_{kx}'(t)\|_2^2 + \frac{d}{dt}\int_0^t h'(t-s)(\psi_{kx}(s),\psi_{kx}'(t))ds \\
&+c''\int_0^1 \frac{\tau(t)}{1-\tau'(t)\rho}\|z_k'(.,\rho,t)\|_{L^2(0,1)}^2 d\rho + h(0)\frac{d}{dt}(\psi_{kx}(t),\psi_{kx}'(t)) + c'(-E') + c'|\mu_1'(t)|H^\star(1)
\end{aligned}$$

Integrating the last inequality over $(0,t)$, we deduce that

$$\begin{aligned}
&\rho_1\|\varphi_k''(t)\|_2^2 + \rho_2\|\psi_k''(t)\|_2^2 + b\|\psi_{kx}'(t)\|_2^2 + K\|\varphi_{kx}'(t) + \psi_k'(t)\|_2^2 \\
&+\int_0^1 \frac{\tau(t)}{1-\tau'(t)\rho}\|z_k'(.,\rho,t)\|_{L^2(0,1)}^2 d\rho + \int_0^t\int_0^1 \mu_1(t){\psi''}_k^2(t)g_1'(\psi_k'(t))\,dxds \\
&+c\int_0^t\int_0^1 |z_k'(x,1,t)|^2\,dxds \leq \rho_1\|\varphi_k''(0)\|_2^2 + \rho_2\|\psi_k''(0)\|_2^2 + b\|\psi_{kx}'(0)\|_2^2 \\
&\qquad +K\|\varphi_{kx}'(0) + \psi_k'(0)\|_2^2 + h(0)(\psi_{kx}(t),\psi_{kx}'(t)) \\
&\qquad -h(0)(\psi_{kx}(0),\psi_{kx}'(0)) + c'\int_0^t \|\psi_k''(s)\|_2^2 ds \\
&\qquad +\int_0^1 \frac{\tau(0)}{1-\tau'(0)\rho}\|z_k'(.,\rho,0)\|_{L^2(0,1)}^2 d\rho \\
&\qquad +\int_0^t h'(t-s)(\psi_{kx}(s),\psi_{kx}'(t))ds \\
&\qquad +\left(\frac{1}{4\epsilon} + \frac{h'(0)^2}{4\epsilon} - h(0)\right)\int_0^t \|\psi_{kx}'(s)\|_2^2 ds \\
&\qquad +(\epsilon + \epsilon\|h''\|_{L^1}^2)\int_0^t \|\psi_{kx}(s)\|_2^2 ds \\
&\qquad +c''\int_0^t\int_0^1 \frac{\tau(s)}{1-\tau'(s)\rho}\|z_k'(.,\rho,s)\|_{L^2(0,1)}^2 d\rho ds \\
&\qquad +c_1(E(0)) + c_2\mu_1(0),
\end{aligned} \tag{3.52}$$

$$h(0)(\psi_{kx}(t),\psi_{kx}'(t)) \leq \epsilon\|\psi_{kx}'(t)\|_2^2 + \frac{h(0)^2}{4\epsilon}\|\psi_{kx}(t)\|_2^2.$$

$$\int_0^t h'(t-s)(\psi_{kx}(s),\psi_{kx}'(t))ds \leq \epsilon\|\psi_{kx}'(t)\|_2^2 + \frac{\zeta(0)\|h'\|_{L^1}\|h'\|_{L^\infty}}{4\epsilon}\int_0^t \|\psi_{kx}(s)\|_2^2 ds.$$

Then from (3.52), after choosing ϵ small enough and using Gronwall's lemma, we obtain

$$
\begin{aligned}
&\rho_1\|\varphi''_k(t)\|_2^2 + \rho_2\|\psi''_k(t)\|_2^2 + b\|\psi'_{kx}(t)\|_2^2 + K\|\varphi'_{kx}(t) + \psi'_k(t)\|_2^2 \\
&+\int_0^1 \frac{\tau(t)}{1-\tau'(t)\rho}\|z'_k(.,\rho,t)\|^2_{L^2(0,1)}d\rho + \int_0^t\int_0^1 \mu_1(t){\psi''}_k^2(t)g'_1(\psi'_k(t))\,dxds \\
&+c\int_0^t\int_0^1 |z'_k(x,1,t)|^2\,dxds \le M,
\end{aligned}
\tag{3.53}
$$

for all $t \in [0,T]$, where M is a positive constant independent of $k \in \mathbb{N}$. Therefor, we conclude that

$$\varphi''_k, \psi''_k \text{ is bounded in } L^\infty(0,T;L^2) \tag{3.54}$$

$$\varphi'_k, \psi'_k \text{ is bounded in } L^\infty(0,T;H_0^1) \tag{3.55}$$

$$\tau(t)z'_k \text{ is bounded in } L^\infty(0,T;L^2((0,1)\times(0,1))) \tag{3.56}$$

The third estimate. Replacing w_j by $-w_{jxx}$ in (3.31) and (3.34), multiplying the result by $g'_{jk}(t)$ and $\tilde{g}'_{jk}(t)$, summing over j from 1 to k, it follows that

$$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi'_{kx}(t)\|_2^2+\right) + K\int_0^1(\varphi_{kx}+\psi_k)_x(t)\varphi'_{kxx}(t)\,dx = 0. \tag{3.57}$$

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\rho_2\|\psi'_{kx}(t)\|_2^2 + b\|\psi_{kxx}(t)\|_2^2\right) - K\int_0^1(\varphi_{kx}+\psi_k)\psi'_{kxx}\,dx \\
&+\mu_1(t)\int_0^1[\psi'_{kx}(t)]^2 g'_1(\psi'_k(t))dx + \mu_2(t)\int_0^1 {\psi'}_{kx}(t)z_{kx}(x,1,t)g'_2(z_k(x,1,t))\,dx \\
&\qquad - \int_0^t h(t-s)(\psi_{kxx}(s),\psi'_{kxx}(t))ds = 0.
\end{aligned}
\tag{3.58}
$$

$$
\begin{aligned}
&\int_0^t h(t-s)(\psi_{kxx}(s),\psi'_{kxx}(t))ds + \frac{1}{2}h(t)\|\psi_{kxx}(t)\|_2^2 \\
&= \frac{1}{2}\frac{d}{dt}\left[\int_0^t h(s)ds\|\psi_{kxx}(t)\|_2^2 - (h\circ\psi_{kxx})(t)\right] + \frac{1}{2}(h'\circ\psi_{kxx})(t).
\end{aligned}
\tag{3.59}
$$

Consequently, equation (3.58) becomes

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\rho_2\|\psi'_{kx}(t)\|_2^2 + \left(b - \int_0^t h(s)ds\right)\|\psi_{kxx}(t)\|_2^2 + (h\circ\psi_{kxx})(t)\right) \\
&+\frac{1}{2}h(t)\|\psi_{kxx}(t)\|_2^2 - \frac{1}{2}(h'\circ\psi_{kxx})(t) + K\int_0^1(\varphi_{kxx}(t)+\psi_{kx}(t))\psi'_{kx}(t)dx \\
&+\mu_1(t)\int_0^1|\psi'_{kx}(t)|^2 g'_1(\psi'_k(t))dx\,dx \\
&+\mu_2(t)\int_0^1 {\psi'}_{kx}(t)z_{kx}(x,1,t)g'_2(z_k(x,1,t))\,dx = 0.
\end{aligned}
\tag{3.60}
$$

Replacing ϕ_j by $-\phi_{jxx}$ in (3.37), multiplying the resulting equation by $h_{jk}(t)$, summing over j from 1 to k, it follows that

$$\frac{1}{2}\frac{d}{dt}\left(\frac{\tau(t)}{1-\tau'(t)\rho}\|z_{kx}(t)\|_2^2\right) - \frac{1}{2}\left(\frac{\tau(t)}{1-\tau'(t)\rho}\right)'\|z_{kx}(t)\|_2^2 + \frac{1}{2}\frac{d}{d\rho}\|z_{kx}(t)\|_2^2 = 0. \tag{3.61}$$

Integrating the last equality over $(0,1)$, we deduce that

$$\begin{aligned}&\frac{d}{2dt}\int_0^1\left[\frac{\tau(t)}{1-\tau'(t)\rho}\left\|z_{kx}(.,\rho,t)\right\|^2_{L^2(0,1)}\right]d\rho\\ &-\frac{1}{2}\int_0^1\left(\frac{\tau(t)}{1-\tau'(t)\rho}\right)'\left\|z_{kx}(.,\rho,t)\right\|^2_{L^2(0,1)}d\rho+\frac{1}{2}\left\|z_{kx}(.,1,t)\right\|^2_{L^2(0,1)}\\ &=\frac{1}{2}\left\|\psi'_{kx}(.,t)\right\|^2_{L^2(0,1)}.\end{aligned}\tag{3.62}$$

From (3.57), (3.60) and (3.62), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big[\rho_1\|\varphi'_{kx}(t)\|_2^2+\rho_2\|\psi'{}_{kx}(t)\|_2^2+K\|\varphi_{kxx}(t)+\psi_{kx}(t)\|_2^2\\ &+\left(b-\int_0^t h(s)ds\right)\|\psi_{kxx}(t)\|_2^2\\ &+(h\circ\psi_{kxx})(t)+\int_0^1\frac{\tau(t)}{1-\tau'(t)\rho}\|z_{kx}(.,\rho,t)\|^2_{L^2(0,1)\times(0,1)}d\rho\Big]\\ &+\frac{1}{2}h(t)\|\psi_{kxx}(t)\|_2^2-\frac{1}{2}(h'\circ\psi_{kxx})(t)\\ &+\mu_1(t)\int_0^1|\psi'_{kx}(t)|^2g_1'(\psi'_k(t))\,dx+\frac{1}{2}\int_0^1|z_{kx}(x,1,t)|^2\,dx\\ &=-\mu_2(t)\int_0^1\psi'_{kx}(t)z_{kx}(x,1,t)g_2'(z_k(x,1,t))\,dx\\ &\quad+\frac{1}{2}\int_0^1\left(\frac{\tau(t)}{1-\tau'(t)\rho}\right)'\|z_{kx}(.,\rho,t)\|_2^2d\rho+\frac{1}{2}\|\psi'_{kx}(.,t)\|_2^2.\end{aligned}\tag{3.63}$$

Using (3.7), Cauchy-Schwartz and Young's inequalities, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big[\rho_1\|\varphi'_{kx}(t)\|_2^2+\rho_2\|\psi'{}_{kx}(t)\|_2^2+K\|\varphi_{kxx}(t)+\psi_{kx}(t)\|_2^2\\ &+\left(b-\int_0^t h(s)ds\right)\|\psi_{kxx}(t)\|_2^2\\ &+(h\circ\psi_{kxx})(t)+\int_0^1\frac{\tau(t)}{1-\tau'(t)\rho}\|z_{kx}(.,\rho,t)\|^2_{L^2(0,1)\times(0,1)}d\rho\Big]\\ &+\frac{1}{2}h(t)\|\psi_{kxx}(t)\|_2^2-\frac{1}{2}(h'\circ\psi_{kxx})(t)+|\mu_1(t)|\int_0^1|\psi'_{kx}(t)|^2g_1'(\psi'_k(t))\,dx\\ &+c\int_0^1|z_{kx}(x,1,t)|^2\,dx\le c'\|\psi'_{kx}(t)\|_2^2+c''\int_0^1\frac{\tau(t)}{1-\tau'(t)\rho}\|z_{kx}(.,\rho,t)\|_2^2d\rho.\end{aligned}\tag{3.64}$$

Integrating the last inequality over $(0,t)$ and using Gronwall's Lemma, we have

$$\begin{aligned}&\rho_1\|\varphi'_{kx}(t)\|_2^2+\rho_2\|\psi'{}_{kx}(t)\|_2^2+K\|\varphi_{kxx}(t)+\psi_{kx}(t)\|_2^2+\left(b-\int_0^t h(s)ds\right)\|\psi_{kxx}(t)\|_2^2\\ &\quad+(h\circ\psi_{kxx})(t)+\int_0^1\frac{\tau(t)}{1-\tau'(t)\rho}\|z_{kx}(.,\rho,t)\|^2_{L^2(0,1)\times(0,1)}d\rho\\ &\le e^{cT}\Big(\rho_1\|\varphi'_{kx}(0)\|_2^2+\rho_2\|\psi'{}_{kx}(0)\|_2^2+K\|\varphi_{kxx}(0)+\psi_{kx}(0)\|_2^2+b\|\psi_{kxx}(0)\|_2^2\\ &\quad+\int_0^1\frac{\tau(0)}{1-\tau'(0)\rho}\|z_{kx}(.,\rho,0)\|^2_{L^2(0,1)\times(0,1)}d\rho\Big)\end{aligned}\tag{3.65}$$

for all $t \in \mathbb{R}_+$, therefore, we conclude that

$$\varphi_k, \psi_k \text{ are bounded in } L^\infty(0,T; H^2 \cap H_0^1(0,1)), \tag{3.66}$$

$$z_k \text{ is bounded in } L^\infty(0,T; H_0^1(0,1; L^2(0,1))). \tag{3.67}$$

Applying Dunford-Petti's theorem we conclude from (3.40), (3.41), (3.42), (3.43), (3.54), (3.55), (3.56), (3.66) and (3.67), after replacing the sequences φ_k, ψ_k and z_k with a subsequence if needed, that

$$\begin{cases} \varphi_k \to \varphi \quad \text{weak-star in } \ L^\infty(0,T; H^2 \cap H_0^1(0,1)) \\ \psi_k \to \psi \quad \text{weak-star in } \ L^\infty(0,T; H^2 \cap H_0^1(0,1)), \end{cases} \tag{3.68}$$

$$\begin{cases} \varphi'_k \to \varphi' \text{ weak-star in } \ L^\infty(0,T; H_0^1(0,1)) \\ \psi'_k \to \psi' \ \text{ weak-star in } \ L^\infty(0,T; H_0^1(0,1)), \end{cases}$$

$$\begin{cases} \varphi''_k \to \varphi'' \text{ weak-star in } \ L^\infty(0,T; L^2(0,1)) \\ \psi''_k \to \psi'' \ \text{ weak-star in } \ L^\infty(0,T; L^2(0,1)), \end{cases} \tag{3.69}$$

$$g_1(\psi'_k) \to \chi \ \text{ weak-star in } \ L^2((0,1)\times(0,T)),$$

$$z_k \to z \ \text{ weak-star in } \ L^\infty(0,T; H_0^1((0,1); L^2(0,1)),$$

$$z'_k \to z' \ \text{ weak-star in } \ L^\infty(0,T; L^2((0,1)\times(0,1))), \tag{3.70}$$

$$g_2(z_k(x,1,t)) \to \psi \ \text{ weak-star in } \ L^2((0,1)\times(0,T))$$

for suitable functions $\varphi, \psi \in L^\infty(0,T; H^2 \cap H_0^1(0,1)), z \in L^\infty(0,T; L^2((0,1)\times(0,1)))$, $\chi \in L^2((0,1)\times(0,T)), \psi \in L^2((0,1)\times(0,T))$ for all $T \geq 0$. We have to show that (φ, ψ, z) is a solution of (3.15).

From (3.40) and (3.41) we have (ψ'_k) is bounded in $L^\infty(0,T; H_0^1(0,1))$. Then (ψ'_k) is bounded in $L^2(0,T; H_0^1)$. Since (ψ''_k) is bounded in $L^\infty(0,T; L^2(0,1))$, then (ψ''_k) is bounded in $L^2(0,T; L^2(0,1))$. Consequently (ψ'_k) is bounded in $H^1(Q)$,where $Q = (0,1)\times(0,T)$.

Since the embedding $H^1(Q) \hookrightarrow L^2(Q)$ is compact, using Aubin-Lions theorem [25] we can extract a subsequence (ψ_ν) of (ψ_k) such that

$$\psi'_\nu \to \psi' \text{ strongly in } \ L^2(Q).$$

Therefore

$$\psi'_\nu \to \psi' \text{ strongly and a.e on } Q. \tag{3.71}$$

Similarly we obtain

$$z_\nu \to z \text{ strongly and a.e on } Q. \tag{3.72}$$

Lemma 3.3.1 *For each $T > 0$, $g_1(\psi'), g_2(z(x,1,t)) \in L^1(Q)$ and $\|g_1(\psi')\|_{L^1(Q)}, \|g_2(z(x,1,t))\|_{L^1(Q)} \leq K_1$, where K_1 is a constant independent of t.*

Proof. By (H1) and (3.71) we have

$$g_1(\psi_k'(x,t)) \to g_1(\psi'(x,t)) \text{ a.e. in } Q,$$

$$0 \le g_1(\psi_k'(x,t))\psi_k'(x,t) \to g_1(\psi'(x,t))\psi'(x,t) \text{ a.e. in } Q$$

Hence, by (3.42) and Fatou's lemma we have

$$\int_0^T \int_0^1 u'(x,t) g_1(\psi'(x,t))\,dx\,dt \le K \text{ for } T > 0. \tag{3.73}$$

By Cauchy-Schwarz inequality and using (3.73), we have

$$\begin{aligned}\int_0^T \int_0^1 |g_1(\psi'(x,t))|\,dx\,dt &\le c|Q|^{\frac{1}{2}} \left(\int_0^T \int_0^1 \psi' g_1(\psi')\,dx\,dt \right)^{\frac{1}{2}} \\ &\le c|Q|^{\frac{1}{2}} K^{\frac{1}{2}} \equiv K_1\end{aligned}$$

Lemma 3.3.2 *$g_1(\psi_k') \to g_1(\psi')$ in $L^1((0,1)\times(0,T))$ and $g_2(z_k) \to g_2(z)$ in $L^1((0,1)\times(0,T))$.*

Proof. Let $E \subset (0,1) \times [0,T]$ and set

$$E_1 = \left\{ (x,t) \in E;\ g_1(\psi_k'(x,t)) \le \frac{1}{\sqrt{|E|}} \right\}, \quad E_2 = E \setminus E_1,$$

where $|E|$ is the measure of E. If $M(r) := \inf\{|s|;\ s \in \mathbb{R} \text{ and } |g_1(s)| \ge r\}$,

$$\int_E |g_1(\psi_k')|\,dxdt \le \sqrt{|E|} + \left(M\left(\frac{1}{\sqrt{|E|}} \right) \right)^{-1} \int_{E_2} |\psi_k' g_1(\psi_k')|\,dxdt.$$

Applying (3.42) we deduce that $\sup_k \int_E |g_1(\psi_k')|\,dxdt \to 0$ as $|E| \to 0$. From Vitali's convergence theorem we deduce that $g_1(\psi_k') \to g_1(\psi')$ in $L^1((0,1)\times(0,T))$, hence

$$g_1(\psi_k') \to g_1(\psi') \text{ weak star in } L^2(Q).$$

Similarly, we have

$$g_2(z_k') \to g_2(z') \text{ weak star in } L^2(Q),$$

and this imply that

$$\int_0^T \int_0^1 g_1(\psi_k') v\,dx\,dt \to \int_0^T \int_0^1 g_1(\psi') v\,dx\,dt \text{ for all } v \in L^2(0,T;H_0^1) \tag{3.74}$$

$$\int_0^T \int_0^1 g_2(z_k) v\,dx\,dt \to \int_0^T \int_0^1 g_2(z) v\,dx\,dt \text{ for all } v \in L^2(0,T;H_0^1) \tag{3.75}$$

as $k \to +\infty$. It follows at once from (3.68), (3.69), (3.74), (3.75) and (3.70) that for each fixed $u, v \in L^2(0,T; H_0^1(0,1))$ and $w \in L^2(0,T; H_0^1((0,1)\times(0,1)))$

$$\int_0^T \int_0^1 (\rho_1 \varphi_k'' - K(\varphi_{kx} + \psi_k)_x) u \, dx \, dt \\ \to \int_0^T \int_0^1 (\rho_1 \varphi'' - K(\varphi_x + \psi)_x) u \, dx \, dt$$

$$\int_0^T \int_0^1 (\rho_2 \psi_k'' - b\psi_{kxx} + K(\varphi_{kx} + \psi_k) + +\mu_1 g_1(\psi_k') + \mu_2 g_2(z_k)) v \, dx \, dt \\ \to \int_0^T \int_0^1 (\rho_2 \psi'' - b\psi_{xx} + K(\varphi_x + \psi) + \mu_1 g_1(\psi') + \mu_2 g_2(z)) v \, dx \, dt$$

$$\int_0^T \int_0^1 \int_0^1 (\tau(t) z_k' + (1-\tau'(t)\rho)\frac{\partial}{\partial \rho} z_k) w \, dx \, d\rho \, dt \to \int_0^T \int_0^1 \int_0^1 (\tau(t) z' + (1-\tau'(t)\rho)\frac{\partial}{\partial \rho} z) w \, dx \, d\rho \, dt$$

as $k \to +\infty$. Hence

$$\int_0^T \int_0^1 (\rho_1 \varphi'' - K(\varphi_x + \psi)_x) u \, dx \, dt = 0$$

$$\int_0^T \int_0^1 (\rho_2 \psi'' - b\psi_{xx} + K(\varphi_x + \psi) + \mu_1 g_1(\psi') + \mu_2 g_2(z)) v \, dx \, dt = 0$$

$$\int_0^T \int_0^1 \int_0^1 (\tau(t) z' + (1-\tau'(t)\rho)\frac{\partial}{\partial \rho} z) w \, dx \, d\rho \, dt = 0, \qquad w \in L^2(0,T; H_0^1((0,1)\times(0,1))).$$

Thus the problem (P) admits a global weak solution (φ, ψ).

Uniqueness. Let (φ_1, ψ_1, z_1) and (φ_2, ψ_2, z_2) be two solutions of problem (3.15). Then $(w, \tilde{w}, \tilde{\tilde{w}}) = (\varphi_1, \psi_1, z_1) - (\varphi_2, \psi_2, z_2)$ verifies

$$(3.76) \quad \begin{cases} \rho_1 w_{tt}(x,t) - K(w_x + \tilde{w})_x(x,t) = 0, & \text{in }]0,1[\times]0,+\infty[, \\ \rho_2 \tilde{w}''(x,t) - b\tilde{w}_{xx}(x,t) + K(w_x + \tilde{w}) & \\ +\mu_1(t) g_1(\psi_1'(x,t)) - \mu_1(t) g_1(\psi_2'(x,t)) & \\ +\mu_2(t) g_2(z_1(x,1,t)) - \mu_2(t) g_2(z_2(x,1,t)) & \\ + \int_0^t h(t-s)\tilde{w}_{xx}(x,s) ds = 0, & \text{in }]0,1[\times]0,+\infty[, \\ \tau(t)\tilde{\tilde{w}}_t(x,\rho,t) + (1-\tau'(t)\rho)\tilde{\tilde{w}}_\rho(x,\rho,t) = 0, & \text{in } (0,1)\times]0,1[\times]0,+\infty[, \\ w(0,t) = w(1,t) = \tilde{w}(0,t) = \tilde{w}(1,t) = 0, & t \geq 0, \\ \tilde{\tilde{w}}(x,0,t) = \psi_1'(x,t) - \psi_2'(x,t), & \text{on }]0,1[\times[0,+\infty[, \\ w(x,0) = w_0(x) = 0, \quad w'(x,0) = w_1(x) = 0, & \text{in }]0,1[, \\ \tilde{w}(x,0) = \tilde{w}_0(x) = 0, \quad \tilde{w}_t(x,0) = \tilde{w}_1(x) = 0, & \text{in }]0,1[, \\ \tilde{\tilde{w}}(x,\rho,0) = 0 & \text{in }]0,1[\times]0,1[. \end{cases}$$

Multiplying the first equation in (3.76) by w', the second equation in (3.76) by $\tilde{w}'$, integrating over $(0,1)$ and using an integration by parts, we get

$$(3.77) \qquad \frac{1}{2}\frac{d}{dt}(\rho_1 \|w'\|_2^2) + K \int_0^1 (w_x + \tilde{w})_x w' \, dx = 0$$

$$\begin{aligned}(3.78)\quad &\frac{1}{2}\frac{d}{dt}(\rho_2\|\tilde{w}'\|_2^2+b\|\tilde{w}_x\|_2^2)+K\int_0^1(w_x+\tilde{w})\tilde{w}'\,dx+\mu_1(t)(g_1(\psi_1')-g_1(\psi_2'),\tilde{w}')\\ &+\mu_2(t)(g_2(z_1(x,1,t))-g_2(z_2(x,1,t)),\tilde{w}')\\ &-\int_0^t h(t-s)(\tilde{w}_x(x,s),\tilde{w}_x')ds=0\end{aligned}$$

Multiplying the third equation in (3.76) by $\tilde{\tilde{w}}$, integrating over $(0,1)\times(0,1)$, we get

$$\frac{1}{2}\frac{\tau(t)}{1-\tau'(t)\rho}\frac{d}{dt}\|\tilde{\tilde{w}}\|_2^2+\frac{1}{2}\frac{d}{d\rho}\|\tilde{\tilde{w}}\|_2^2=0. \tag{3.79}$$

Then

$$\begin{aligned}(3.80)\quad &\frac{1}{2}\frac{d}{dt}\int_0^1\left(\frac{\tau(t)}{1-\tau'(t)\rho}\|\tilde{\tilde{w}}\|_2^2\right)d\rho-\frac{1}{2}\int_0^1\left(\frac{\tau(t)}{1-\tau'(t)\rho}\right)'\|\tilde{\tilde{w}}\|_2^2d\rho\\ &+\frac{1}{2}\left(\|\tilde{\tilde{w}}(x,1,t)\|_2^2-\|\tilde{w}'\|_2^2\right)=0\end{aligned}$$

We can write

$$\begin{aligned}(3.81)\quad &\int_0^t h(t-s)(\tilde{w}_x(s),\tilde{w}_x(t))ds+\frac{1}{2}h(t)\|\tilde{w}_x(t)\|_2^2\\ &=\frac{1}{2}\frac{d}{dt}\left[\int_0^t h(s)ds\|\tilde{w}_x(t)\|_2^2-(h\circ\tilde{w}_x)(t)\right]+\frac{1}{2}(h'\circ\tilde{w}_x)(t).\end{aligned}$$

From (3.77), (3.78), (3.80), (3.81) , we obtain

$$\begin{aligned}(3.82)\quad &\frac{1}{2}\frac{d}{dt}\Big[\rho_1\|w'\|_2^2+\rho_2\|\tilde{w}'\|_2^2+\left(b-\int_0^t h(s)ds\right)\|\tilde{w}_x\|_2^2+K\|w_x+\tilde{w}\|_2^2\\ &+\int_0^1\frac{\tau(t)}{1-\tau'(t)\rho}\|\tilde{\tilde{w}}\|_2^2\,d\rho+(h\circ\tilde{w}_x)(t)\Big]+\mu_1(t)(g_1(\psi_1'(t))-g_1(\psi_2'(t)),\tilde{w}')\\ &+\frac{1}{2}\|\tilde{\tilde{w}}(x,1,t)\|_2^2=-\mu_2(t)(g_2(z_1(x,1,t))-g_2(z_2(x,1,t)),\tilde{w}')\\ &\qquad+\frac{1}{2}\|\tilde{w}'\|_2^2+\frac{1}{2}\int_0^1\left(\frac{\tau(t)}{1-\tau'(t)\rho}\right)'\|\tilde{\tilde{w}}\|_2^2d\rho\\ &\qquad+\frac{1}{2}(h'\circ\tilde{w}_x)(t)+\frac{1}{2}h(t)\|\tilde{w}_x(t)\|_2^2\end{aligned}$$

Using (3.82), and Cauchy-Schwartz, Young's inequalities, we obtain

$$\begin{aligned}(3.83)\quad &\frac{1}{2}\frac{d}{dt}\Big(\rho_1\|w'\|_2^2+\rho_2\|\tilde{w}'\|_2^2+\left(b-\int_0^t h(s)ds\right)\|\tilde{w}_x\|_2^2+K\|w_x+\tilde{w}\|_2^2\\ &+\int_0^1\frac{\tau(t)}{1-\tau'(t)\rho}\|\tilde{\tilde{w}}\|_2^2\,d\rho+(h\circ\tilde{w}_x)(t)\Big)+\mu_1(t)(g_1(\psi_1'(t))-g_1(\psi_2'(t)),\tilde{w}')\\ &+\frac{1}{2}\|\tilde{\tilde{w}}(x,1,t)\|_2^2\le c'\int_0^1\left(\frac{\tau(t)}{1-\tau'(t)\rho}\right)\|\tilde{\tilde{w}}_x\|_2^2d\rho+c''\|\tilde{w}_x'(t)\|_2^2.\end{aligned}$$

Then integrating over $(0,t)$, using Gronwall's lemma, we conclude that

$$\begin{aligned}(3.84)\quad &\rho_1\|w'\|_2^2+\rho_2\|\tilde{w}'\|_2^2+\left(b-\int_0^t h(s)ds\right)\|\tilde{w}_x\|_2^2+K\|w_x+\tilde{w}\|_2^2\\ &+\int_0^1\frac{\tau(t)}{1-\tau'(t)\rho}\|\tilde{\tilde{w}}\|_2^2\,d\rho+(h\circ\tilde{w}_x)(t)=0.\end{aligned}$$

Hence, uniqueness follows.

3.4 Asymptotic behavior

Now we construct a Lyapunov functional L equivalent to E. For this, we define several functionals which allow us to obtain the needed estimates. Then we have the following estimate.

Lemma 3.4.1 *Let (φ, ψ, z) be the solution of (3.15). Then the functional F_1 defined by*

$$F_1(t) = -\int_0^1 (\rho_1 \varphi_t \varphi + \rho_2 \psi_t \psi)\, dx \tag{3.85}$$

satisfies, along the solution, the estimate

$$\begin{aligned} \frac{dF_1(t)}{dt} \leq & -\int_0^1 (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2) dx + K \int_0^1 |\varphi_x + \psi|^2 dx \\ & + c\int_0^1 \psi_x^2 dx + c' \int_0^1 g_1^2(\psi_t) dx + c'' \int_0^1 g_2^2(z(x,1,t)) dx + c(h \circ \psi_x)(t). \end{aligned} \tag{3.86}$$

Proof. By taking the time derivative of (3.85)

$$\frac{dF_1(t)}{dt} = -\int_0^1 (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2) dx - \int_0^1 (\rho_1 \varphi_{tt} \varphi + \rho_2 \psi_{tt} \psi) dx.$$

Therefore, by using the first and the second equations in (3.15) and some integrations by parts, we obtain from the above inequality

$$\begin{aligned} \frac{dF_1(t)}{dt} = & -\int_0^1 (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2) dx + K \int_0^1 |\varphi_x + \psi|^2 dx \\ & + b\int_0^1 \psi_x^2 dx + \mu_1(t) \int_0^1 \psi g_1(\psi_t) dx + \mu_2(t) \int_0^1 \psi g_2(z(x,1,t)) dx \\ & - \int_0^1 \psi_x(t) \left(\int_0^t h(t-s) \psi_x(s) ds \right) dx \\ = & -\int_0^1 (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2) dx + K \int_0^1 |\varphi_x + \psi|^2 dx + \mu_1(t) \int_0^1 \psi g_1(\psi_t) dx \\ & + \mu_2(t) \int_0^1 \psi g_2(z(x,1,t)) dx + \left(b - \int_0^t h(s) ds \right) \|\psi_x(t)\|_2^2 \\ & - \int_0^1 \psi_x(t) \left(\int_0^t h(t-s)(\psi_x(s) - \psi_x(t)) ds \right) dx \end{aligned} \tag{3.87}$$

By exploiting Young's inequality, we get

$$\begin{aligned} & \int_0^1 \psi_x(t) \left(\int_0^t h(t-s)(\psi_x(t) - \psi_x(s)) ds \right) dx \\ & \leq \delta_1 \int_0^1 |\psi_x(t)|^2 dx + \delta_2 \int_0^1 \left(\int_0^t h(t-s) |\psi_x(t) - \psi_x(s)| ds \right)^2 dx \\ & \leq c' \int_0^1 |\psi_x(t)|^2 dx + c(h \circ \psi_x)(t). \end{aligned} \tag{3.88}$$

By exploiting Young's inequality and Poincaré's inequality, , then (3.86) holds.

Lemma 3.4.2 *Let (φ, ψ, z) be the solution of (3.15). Assume that*

$$\frac{\rho_1}{K} = \frac{\rho_2}{b}. \tag{3.89}$$

Then the functional F_2 defined by

$$F_2(t) = \rho_2 \int_0^1 \psi_t(\varphi_x + \psi)dx + \rho_2 \int_0^1 \psi_x \varphi_t \, dx - \frac{\rho_1}{k} \int_0^1 \varphi_t \int_0^t h(t-s)\psi_x(s)dsdx. \tag{3.90}$$

satisfies, along the solution, the estimate

$$\begin{aligned} \frac{dF_2}{dt}(t) \quad \leq \quad & \left[\left(b\psi_x - \int_0^t h(t-s)\psi_x(s)ds\right)\varphi_x\right]_{x=0}^{x=1} - (K-\varepsilon)\int_0^1 (\varphi_x + \psi)^2 dx \\ & + \varepsilon \int_0^1 \varphi_t^2 dx - \frac{c}{\varepsilon}(h' \circ \psi_x)(t) + \frac{c}{\varepsilon}\int_0^1 \psi_x^2 dx + \frac{c}{\varepsilon}\int_0^1 g_1^2(\psi_t)dx \\ & + \rho_2 \int_0^1 \psi_t^2 dx + \frac{c}{\varepsilon}\int_0^1 g_2^2(z(x,1,t))dx \end{aligned} \tag{3.91}$$

for any $0 < \varepsilon < 1$.

Proof. Differentiating $F_2(t)$, with respect to t, we obtain

$$\begin{aligned} \frac{dF_2}{dt}(t) = & \int_0^1 \rho_2 \psi_{tt}(\varphi_x + \psi)dx + \int_0^1 \rho_2 \psi_t(\varphi_x + \psi)_t dx + \rho_2 \int_0^1 \psi_x \varphi_{tt} dx + \rho_2 \int_0^1 \psi_{tx}\varphi_t dx \\ & - \frac{\rho_1}{K}\int_0^1 \varphi_{tt}\int_0^t h(t-s)\psi_x(s)dsdx - \frac{\rho_1}{K}\int_0^1 \varphi_t \Big(\int_0^t h(t-s)\psi_x(s)ds\Big)' dx \\ = & \int_0^1 (\varphi_x + \psi)\Big[b\psi_{xx} - k(\varphi_x + \psi) - \mu_1(t)g_1(\psi_t) - \mu_2(t)g_2(z(x,1,t) - \int_0^t h(t-s)\psi_{xx}(s)ds\Big]dx \\ & + \rho_2 \int_0^1 \psi_t^2 \, dx + \frac{\rho_2}{\rho_1}\int_0^1 K(\varphi_x + \psi)_x \psi_x \, dx - \int_0^1 (\varphi_x + \psi)_x \int_0^t h(t-s)\psi_x(s)dsdx \\ & - \frac{\rho_1}{k}\int_0^1 \varphi_t \Big(\int_0^t h(t-s)\psi_x(s)ds\Big)' dx. \end{aligned}$$

Then, by using Eqs.(3.15) and (3.89) we find

$$\begin{aligned} \frac{dF_2(t)}{dt} \quad = \quad & \left[\left(b\psi_x - \int_0^t h(t-s)\psi_x(s)ds\right)\varphi_x\right]_{x=0}^{x=1} - K\int_0^1 (\varphi_x + \psi)^2 dx + \rho_2 \int_0^1 \psi_t^2 dx \\ & - \mu_1(t)\int_0^1 (\varphi_x + \psi)g_1(\psi_t)dx - \mu_2(t)\int_0^1 (\varphi_x + \psi)g_2(z(x,1,t))dx \\ & - \frac{\rho_1}{k}h(t)\int_0^1 \varphi_t \psi_x dx - \frac{\rho_1}{K}\int_0^1 \varphi_t \int_0^t h'(t-s)\left(\psi_x(s) - \psi_x(t)\right)dsdx. \end{aligned}$$

By the Young inequality and using (3.88), (3.91) is established.

Lemma 3.4.3 *Let $m \in C^1([0,1])$ be a function satisfying $m(0) = -m(1) = 2$. Then there exists $c > 0$ such that, for any $0 < \varepsilon < 1$, the functional F_3 defined by*

$$F_3(t) = \frac{1}{4\varepsilon}\int_0^1 \rho_2 m(x)\psi_t\Big(b\psi_x - \int_0^t h(t-s)\psi_x(s)ds\Big)dx + \frac{\varepsilon}{k}\int_0^1 \rho_1 m(x)\varphi_t \varphi_x \, dx$$

satisfies, along the solution, the estimate

$$
\begin{aligned}
\frac{dF_3(t)}{dt} \leq & -\frac{1}{4\varepsilon}\Big[\Big(b\psi_x(1,t) - \int_0^t h(t-s)\psi_x(1,s)ds\Big)^2 \\
& + \Big(b\psi_x(0,t) - \int_0^t h(t-s)\psi_x(0,s)ds)^2\Big] - \varepsilon\Big((\varphi_x(1,t))^2 + (\varphi_x(0,t))^2\Big) \\
& + (\frac{k}{4} + \frac{c}{k}\varepsilon)\int_0^1 (\psi+\varphi_x)^2\,dx + c\varepsilon\rho_1\int_0^1 \varphi_t^2\,dx + \frac{c}{\varepsilon^2}\int_0^1 \psi_x^2\,dx \\
& + \frac{c}{\varepsilon}\int_0^1 \psi_t^2\,dx + c\int_0^1 g_1^2(\psi_t)\,dx + c\int_0^1 g_2^2(z(x,1,t))\,dx \\
& + \frac{c}{\varepsilon^2}(h\circ\psi_x) - \frac{c}{\varepsilon}h'\circ\psi_x.
\end{aligned} \tag{3.92}
$$

Proof. Using Eqs. (3.15) and integrating by parts, we obtain

$$
\begin{aligned}
\frac{dF_3(t)}{dt} = & \frac{1}{4\varepsilon}\Big[-\Big((b\psi_x(1,t) - \int_0^t h(t-s)\psi_x(1,s)ds)^2 + (b\psi_x(0,t) - \int_0^t h(t-s)\psi_x(0,s)ds)^2\Big) \\
& -\frac{1}{2}\int_0^1 m'(x)\Big(b\psi_x - \int_0^t h(t-s)\psi_x(s)ds\Big)^2 dx - k\int_0^1 m(x)(\varphi_x+\psi)(b\psi_x - \int_0^t h(t-s)\psi_x(s)ds)\,dx \\
& - \int_0^1 m(x)\mu_1(t)g_1(\psi_t)(b\psi_x - \int_0^t h(t-s)\psi_x(s)ds)dx - \frac{b\rho_2}{2}\int_0^1 m'(x)\psi_t^2dx - \rho_2 h(t)\int_0^1 m(x)\psi_t\psi_x dx \\
& - \int_0^1 m(x)\mu_2(t)g_2(z(x,1,t))(b\psi_x - \int_0^t h(t-s)\psi_x(s)ds)dx \\
& + \rho_2\int_0^1 m(x)\psi_t\int_0^t h'(t-s)(\psi_x(t)-\psi_x(s))dsdx\Big] + \frac{\varepsilon}{k}\Big[-k\Big((\varphi_x(1,t))^2 + (\varphi_x(0,t))^2\Big) \\
& - \int_0^1 \frac{k}{2}m'(x)\varphi_x^2\,dx + \int_0^1 km(x)\psi_x\varphi_x\,dx - \int_0^1 \frac{\rho_1}{2}m'(x)(\varphi_t)^2\,dx\Big]
\end{aligned}
$$

Then by the Young and Poincaré inequalities and the fact that

$$\varphi_x^2 \leq 2(\psi+\varphi_x)^2 + 2\psi^2$$

we obtain

$$
\begin{aligned}
\frac{dF_3(t)}{dt} \leq & \frac{1}{4\varepsilon}\Big[-\Big((b\psi_x(1,t) - \int_0^t h(t-s)\psi_x(1,s)ds)^2 + (b\psi_x(0,t) - \int_0^t h(t-s)\psi_x(0,s)ds)^2\Big) \\
& + \frac{c}{\varepsilon}\int_0^1 \psi_x^2\,dx + \varepsilon k\int_0^1 (\psi+\varphi_x)^2\,dx + \varepsilon\int_0^1 g_1^2(\psi_t)\,dx + \varepsilon\int_0^1 g_2^2(z(x,1,t))\,dx + c\int_0^1 \psi_t^2\,dx \\
& + \frac{c}{\varepsilon}(h\circ\psi_x) - c(h'\circ\psi_x)\Big] + \frac{\varepsilon}{k}\Big[-k((\varphi_x(1,t))^2 + (\varphi_x(0,t))^2) + c\int_0^1 \psi_x^2\,dx + c\int_0^1 (\psi+\varphi_x)^2\,dx \\
& + c\int_0^1 \varphi_t^2\,dx\Big]
\end{aligned}
$$

This gives (3.92).

Lemma 3.4.4 *Assume that* $(H1)$ *hold. Then, for sufficiently small* ε*, the functional* F *defined by*

$$F(t) = 2c\varepsilon F_1(t) + F_2(t) + F_3(t)$$

satisfies, along the solution, the estimate

$$\begin{aligned} \frac{dF}{dt}(t) &\le -\frac{k}{2}\int_0^1 (\psi+\varphi_x)^2\,dx - \tau\int_0^1 \varphi_t^2\,dx + c\int_0^1 \psi_t^2\,dx + c\int_0^1 \psi_x^2\,dx \\ &+c\int_0^1 g_1^2(\psi_t)\,dx + c\int_0^1 g_2^2(z(x,1,t))\,dx + c(h\circ\psi_x)(t) - c(h'\circ\psi_x)(t), \end{aligned} \tag{3.93}$$

where $\tau = c\varepsilon\rho_1$.

Proof. Using Lemmas 3.4.1, 3.4.2, 3.4.3 and the fact that

$$\begin{aligned} \Big[\Big(b\psi_x - \int_0^t h(t-s)\psi_x(s)ds\Big)\varphi_x\Big]_{x=0}^{x=1} &\le \frac{1}{4\varepsilon}\Big[\Big(b\psi_x(1,t) - \int_0^t h(t-s)\psi_x(s)ds\Big)^2 \\ +\Big(b\psi_x(0,t) - \int_0^t h(t-s)\psi_x(0,s)ds\Big)^2\Big] &+ \varepsilon\Big[\varphi_x^2(1)+\varphi_x^2(0)\Big] \end{aligned} \tag{3.94}$$

for any $0<\varepsilon<1$, we obtain (3.93).

Next, we introduce the following functional

$$I(t) = \int_0^1 (\rho_2\psi_t\psi + \rho_1\varphi_t\omega)dx, \tag{3.95}$$

where ω is the solution of

$$-\omega_{xx} = \psi_x, \qquad \omega(0)=\omega(1)=0. \tag{3.96}$$

Then we have the following estimate.

Lemma 3.4.5 *Let* (φ,ψ,z) *be the solution of (3.15), then for any* $\delta>0$*, we have the following estimate*

$$\begin{aligned} \frac{dI(t)}{dt} &\le -\frac{1}{2}\Big(b-\int_0^\infty h(s)ds\Big)\int_0^1 \psi_x^2(x,t)dx + \frac{c}{\delta}\int_0^1 \psi_t^2(x,t)dx + \delta\int_0^1 \varphi_t^2(x,t)dx \\ &+c\int_0^1 g_1^2(\psi_t)dx + c\int_0^1 g_2^2(z(x,1,t))dx + c(h\circ\psi_x). \end{aligned} \tag{3.97}$$

Proof. Using Eqs. (3.15), we have

$$\begin{aligned} \frac{dI(t)}{dt} &= \Big(-b+\int_0^t h(s)ds\Big)\int_0^1 \psi_x^2dx + \rho_2\int_0^1 \psi_t^2dx - K\int_0^1 \psi^2dx \\ &+\int_0^1\Big(\int_0^t h(t-s)(\psi_x(s)-\psi_x(t))ds\Big)\psi_xdx + K\int_0^1 \omega_x^2dx + \rho_1\int_0^1 \varphi_t\omega_tdx \\ &-\mu_1(t)\int_0^1 \psi g_1(\psi_t)dx - \mu_2(t)\int_0^1 \psi g_2(z(x,1,t))dx. \end{aligned} \tag{3.98}$$

It is clear that, from (3.96), we have

$$\int_0^1 \omega_x^2dx \le \int_0^1 \psi^2dx \le \int_0^1 \psi_x^2dx$$

$$\int_0^1 \omega_t^2 dx \le \int_0^1 \omega_{tx}^2 dx \le \int_0^1 \psi_t^2 dx \tag{3.99}$$

By using Young's inequality and Poincaré's inequality, the last two terms in (3.98) can be estimated as

$$\begin{aligned}
&\mu_1(t)\int_0^1 \psi g_1(\psi_t)dx + \mu_2(t)\int_0^1 \psi g_2(z(x,1,t))dx \\
&+\int_0^1 \Big(\int_0^t h(t-s)(\psi_x(s)-\psi_x(t))ds\Big)\psi_x dx \le \frac{1}{2}\Big(b-\int_0^\infty h(s)ds\Big)\int_0^1 \psi_x^2 dx \\
&+c\int_0^1 g_1^2(\psi_t)dx + c\int_0^1 g_2^2(z(x,1,t))dx + c(h\circ\psi_x).
\end{aligned} \tag{3.100}$$

Consequently, from (3.98)-(3.100), we obtain (3.97).

Now, let us introduce the following functional

$$I_3(t) = \xi(t)\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho} G_2(z(x,\rho,t))\, d\rho\, dx. \tag{3.101}$$

Then the following result holds.

Lemma 3.4.6 *Let (φ,ψ,z) be the solution of (3.15). Then it holds*

$$\begin{aligned}
\frac{d}{dt}I_3(t) &\le -2I_3(t) - \xi(t)(1-\tau'(t))e^{-2\tau(t)}\alpha_1\int_0^1 z(x,1,t)g_2(z(x,1,t))dx \\
&\quad +\xi(t)\alpha_2\int_0^1 \psi_t(x,t)g_1(\psi_t(x,t))dx
\end{aligned} \tag{3.102}$$

Proof. Differentiating (3.101) with respect to t and using the third equation in (3.15), we have

$$\begin{aligned}
\frac{d}{dt}I_3(t) &= \xi'(t)\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}G_2(z(x,\rho,t))\,d\rho dx + \xi(t)\tau'(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}G_2(z(x,\rho,t))\,d\rho dx \\
&\quad -2\xi(t)\tau(t)\tau'(t)\int_0^1\int_0^1 \rho e^{-2\tau(t)\rho}G_2(z(x,\rho,t))d\rho dx \\
&\quad +\xi(t)\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}z_t(x,\rho,t)g_2(z(x,\rho,t))\,d\rho dx \\
&= \xi'(t)\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}G_2(z(x,\rho,t))\,d\rho dx + \xi(t)\tau'(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}G_2(z(x,\rho,t))\,d\rho dx \\
&\quad -2\xi(t)\tau(t)\tau'(t)\int_0^1\int_0^1 \rho\tau'(t)e^{-2\tau(t)\rho}G_2(z(x,\rho,t))dxd\rho \\
&\quad -\xi(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}\left(1-\tau'(t)\rho\right)z_\rho(x,\rho,t)g_2(z(x,\rho,t))\,d\rho\ dx \\
&= \xi'(t)\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}G_2(z(x,\rho,t))\,d\rho dx + \xi(t)\tau'(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}G_2(z(x,\rho,t))\,d\rho dx \\
&\quad -2\xi(t)\tau(t)\tau'(t)\int_0^1\int_0^1 \rho\tau'(t)e^{-2\tau(t)\rho}G_2(z(x,\rho,t))dxd\rho \\
&\quad -\xi(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}\left(1-\tau'(t)\rho\right)\frac{d}{d\rho}G_2(z(x,\rho,t))\,d\rho\ dx \\
&= -2I_3(t) + \xi'(t)\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}G_2(z(x,\rho,t))\,d\rho dx \\
&\quad -\xi(t)(1-\tau'(t))e^{-2\tau(t)}\int_0^1 G_2(z(x,1,t))dx + \xi(t)\int_0^1 G_2(z(x,0,t))dx
\end{aligned}$$

$$\leq -2I_3(t) - \xi(t)(1-\tau'(t))e^{-2\tau(t)}\alpha_1 \int_0^1 z(x,1,t)g_2(z(x,1,t))dx$$
$$+\xi(t)\alpha_2 \int_0^1 \psi_t(x,t)g_1(\psi_t(x,t))dx$$

For $N_1, N_2 > 0$, let

$$L(t) = N_1E(t) + N_2I(t) + F(t) + I_3(t).$$

By combining (3.21), (3.93), (3.97), (3.102), we obtain

$$\begin{aligned}
\frac{dL}{dt}(t) \leq & -\mu_1(t)\Big(N_1a_1 - \frac{\alpha_2}{\tau(t)}\Big)\int_0^1 \psi_t g_1(\psi_t(x,t))\,dx - \Big(N_2\frac{b}{2} - c\Big)\int_0^1 \psi_x^2\,dx \\
& -\mu_1(t)\Big(N_1a_2 + \alpha_1\frac{(1-\tau'(t))e^{-2\tau(t)}}{\tau(t)} - (N_2c+c)c_2\Big)\int_0^1 z(x,1,t)g_2(z(x,1,t))\,dx \\
& -(\tau - N_2\delta)\int_0^1 \varphi_t^2\,dx + \Big(N_2\frac{c}{\delta} + c\Big)\int_0^1 \psi_t^2\,dx + (cN_2+c)(h\circ\psi_x) \\
& -\frac{k}{2}\int_0^1 (\psi+\varphi_x)^2\,dx + (N_2c+c)\int_0^1 g_1^2(\psi_t)\,dx + (\frac{1}{2}N_1 - c)(h'\circ\psi_x).
\end{aligned}$$

$$\begin{aligned}
\text{(3.103)} \quad \leq & -\mu_1(t)\Big(N_1a_1 - \frac{\alpha_2}{\bar\tau}\Big)\int_0^1 \psi_t g_1(\psi_t(x,t))\,dx - \Big(N_2\frac{b}{2} - c\Big)\int_0^1 \psi_x^2\,dx \\
& -\mu_1(t)\Big(N_1a_2 + \alpha_1\frac{(d-1)e^{-2\tau_0}}{\bar\tau} - (N_2c+c)c_2\Big)\int_0^1 z(x,1,t)g_2(z(x,1,t))\,dx \\
& -(\tau - N_2\delta)\int_0^1 \varphi_t^2\,dx + \Big(N_2\frac{c}{\delta} + c\Big)\int_0^1 \psi_t^2\,dx \\
& -\frac{k}{2}\int_0^1 (\psi+\varphi_x)^2\,dx + (N_2c+c)\int_0^1 g_1^2(\psi_t)\,dx + (cN_2+c)(h\circ\psi_x) \\
& +(\frac{1}{2}N_1 - c)(h'\circ\psi_x).
\end{aligned}$$

At this point, we have to choose our constants very carefully. First, let us choose N_2 sufficiently large so that

$$\Big(N_2\frac{b}{2} - c\Big) > 0.$$

Next, we choose δ sufficiently small such that

$$(\tau - N_2\delta) > 0.$$

Then, we pick the constant $N_1 > 0$ sufficiently large such that

$$\Big(N_1a_1 - \frac{\alpha_2}{\bar\tau}\Big) > 0,$$

and

$$\Big(N_1a_2 + \alpha_1\frac{e^{-2\tau_0}}{\bar\tau} - (N_2c+c)c_3\Big) > 0.$$

Thus, (3.103) becomes

$$\begin{aligned}
\frac{d}{dt}L(t) \leq & -d_1\int_0^1 \psi_x^2\,dx - d_2\int_0^1 \varphi_t^2\,dx - \frac{k}{2}\int_0^1(\psi+\varphi_x)^2\,dx \\
\text{(3.104)} \quad & +c\mu_1(t)\int_0^1 ((\psi_t)^2 + g_1^2(\psi_t))\,dx + d_3h\circ\psi_x + d_4h'\circ\psi_x \\
\leq & -dE(t) + c\mu_1(t)\int_0^1 ((\psi_t)^2 + g_1^2(\psi_t))\,dx + ch\circ\psi_x(t).
\end{aligned}$$

At this stage, we are in position to compare $L(t)$ with $E(t)$. We have the following Lemma.

Lemma 3.4.7 *For N_1 large enough, there exist two positive constants β_1 and β_2 depending on N_1, N_2 and ϵ, such that*

$$\beta_1 E(t) \leq L(t) \leq \beta_2 E(t) \qquad \forall t \geq 0. \tag{3.105}$$

Proof. We consider the functional

$$\mathcal{H}(t) = N_2 I(t) + F(t) + I_3(t)$$

and show that

$$|\mathcal{H}(t)| \leq \hat{C}E(t), \qquad C > 0.$$

from (3.85),(3.95),(3.90) and (3.101), we obtain

$$\begin{aligned}|\mathcal{H}(t)| \leq N_2 &\left|\int_0^1 (\rho_2\psi_t\psi + \rho_1\varphi_t\omega)dx\right| + \left|2c\varepsilon\int_0^1 (\rho_1\varphi_t\varphi + \rho_2\psi_t\psi)dx\right| \\ &+ \left|\frac{b}{4\varepsilon}\int_0^1 \rho_2 m(x)\psi_t\psi_x\,dx + \frac{\varepsilon}{k}\int_0^1 \rho_1 m(x)\varphi_t\varphi_x\,dx\right| \\ &+ \left|\rho_2\int_0^1 \psi_t(\varphi_x+\psi)dx + \rho_2\int_0^1 \psi_x\varphi_t\,dx\right| + \left|\xi(t)\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}G_2(z(x,\rho,t))\,d\rho\,dx\right| \\ &+ \left|\frac{\rho_1}{k}\int_0^1 \varphi_t\int_0^t h(t-s)\psi_x(s)dsdx\right| + \left|\frac{1}{4\varepsilon}\int_0^1 \rho_2 m(x)\psi_t\int_0^t h(t-s)\psi_x(s)dsdx\right| \quad .\end{aligned} \tag{3.106}$$

By using (3.99),(3.96), the trivial relation

$$\int_0^1 \varphi^2(x,t)dx \leq 2\int_0^1 (\varphi_x+\psi)^2(x,t)dx + 2\int_0^1 \psi_x^2(x,t)dx,$$

Young's and Poincaré's inequalities, we get

$$\begin{aligned}|\mathcal{H}(t)| \leq \alpha_1 &\int_0^1 \varphi_t^2(x,t)dx + \alpha_2\int_0^1 \psi_t^2(x,t)dx \\ &+\alpha_3\int_0^1 (\varphi_x+\psi)^2(x,t)dx + \alpha_4\int_0^1 \psi_x^2(x,t)dx \\ &+\alpha_5(h\circ\psi_x) + \left|\xi(t)\tau(t)\int_0^1\int_0^1 G_2(z(x,\rho,t))dx\;d\rho\right|\end{aligned} \tag{3.107}$$

where the positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are determined as follows:

$$\begin{cases}\alpha_1 = \frac{N_2\rho_1}{2} + \rho_2 + \frac{\epsilon\rho_1}{K}, \\ \alpha_2 = \frac{N_2\rho_2}{2} + \rho_2 + \frac{\rho_2 b}{2\epsilon}, \\ \alpha_3 = \rho_1 + \frac{\rho_2}{2} + \frac{2\epsilon\rho_1}{K}, \\ \alpha_4 = \rho_2 + \frac{N_2}{2}\rho_2 + \rho_1 + \frac{\rho_2 b}{2\epsilon} + \frac{2\epsilon\rho_1}{K}, \\ \alpha_5 = \frac{\rho_1\varepsilon}{k} + \frac{\rho_2}{4\varepsilon},\end{cases}$$

According to (3.107) , we have

$$|\mathcal{H}(t)| \leq \hat{C}E(t)$$

for

$$\hat{C} = 2\max\left\{\frac{\alpha_1}{\rho_1}, \frac{\alpha_2}{\rho_2}, \frac{\alpha_3}{k}, \frac{\alpha_4}{b}, \alpha_5, \frac{1}{2\bar{\xi}\tau_0}\right\}.$$

Therefore, we obtain

$$|L(t) - N_1E(t)| \leq \hat{C}E(t).$$

So, we can choose N_1 large enough so that $\beta_1 = N_1 - \hat{C} > 0, \beta_2 = N_1 + \hat{C} > 0$. Then (3.105) holds true. Therefore, (3.104) takes the form

$$\frac{d}{dt}L(t) \leq -C_3E(t) + C_4(h \circ \psi_x)(t) + C_5\mu_1(t)\Big(\|\psi_t\|_2^2 + \|g_1(\psi_t)\|_2^2\Big), \tag{3.108}$$

where C_3, C_4 and C_5 are three positive constants. Now, we estimate the last term in the right hand side of (3.108). We define

$$\Omega^+ = \{x \in (0,1) : |\psi_t| \geq \varepsilon'\}, \quad \Omega^- = \{x \in (0,1) : |\psi_t| \leq \varepsilon'\}.$$

From (3.5) and (3.6), it follows that

$$\mu_1(t)\int_{\Omega^+}(|\psi_t|^2 + |g_1(\psi_t)|^2)\,dx \leq c_2\mu_1(t)\int_{\Omega^+}\psi_t g_1(\psi_t)\,dx \leq -c_2E'(t). \tag{3.109}$$

Case 1: H is linear on $[0, \varepsilon']$. In this case one can easily check that there exists $\mu_1' > 0$, such that $|g_1(s)| \leq \mu_1'|s|$ for all $|s| \leq \varepsilon'$, and thus

$$\mu_1(t)\int_{\Omega^-}(|\psi_t|^2 + |g_1(\psi_t)|^2)\,dx \leq c_1'\mu_1(t)\int_{\Omega^-}\psi_t g_1(\psi_t)\,dx \leq -c_1'E'(t). \tag{3.110}$$

Substitution of (3.109) and (3.110) into (3.108) gives

$$\begin{aligned}(\mu_1 L + cE)'(t) &\leq -c_3\mu_1(t)E(t) + c_4'(h \circ \psi_x)(t) \\ &\leq -c_3\mu_1(t)H_2(E(t)) + c_4'(h \circ \psi_x)(t)\end{aligned} \tag{3.111}$$

where $c = C_5(c_2 + c_1')$ and here and in the sequel we take C_i to be a generic positive constant.
Case 2: $H'(0) = 0$ and $H'' > 0$ on $]0, \varepsilon']$.
Since H is convex and increasing, H^{-1} is concave and increasing. By the virtue of (3.5), the reversed Jensen's inequality for concave function, and (3.21), it follows that

$$\begin{aligned}\mu_1(t)\int_{\Omega^-}(|\psi'|^2 + |g_1(\psi')|^2)\,dx &\leq \mu_1(t)\int_{\Omega^-}H^{-1}(\psi' g_1(\psi'))\,dx \\ &\leq \mu_1(t)|\Omega|H^{-1}\left(\frac{1}{|\Omega|}\int_{\Omega^-}\psi' g_1(\psi')\,dx\right) \\ &\leq C\mu_1(t)H^{-1}(-C'E'(t)).\end{aligned} \tag{3.112}$$

A combination of (3.108), (3.109) and (3.112) yields

$$(3.113)\qquad \begin{array}{rcl} (\mu_1 L + C_5 c_2 E)'(t) & \leq & -C_3\mu_1(t)E(t) + c_4'(h\circ\psi_x)(t) \\ & \leq & +CC_5\mu_1(t)H^{-1}(-C'E'(t)) \\ & \leq & -\tilde{C}_3\mu_1(t)E(t) + c_4'(h\circ\psi_x)(t) \\ & & +\tilde{C}_5\mu_1(t)H^{-1}(-C'E'(t)), \quad t\geq 0. \end{array}$$

Let us denote by H^* the conjugate function of the convex function H, i.e.,

$$H^*(s) = \sup_{t\in \mathbb{R}_+} (st - H(t)).$$

Then H^* is the Legendre transform of H, which is given by

$$(3.114)\qquad H^*(s) = s(H')^{-1}(s) - H[(H')^{-1}(s)], \quad \forall s\geq 0$$

and which satisfies the following inequality

$$(3.115)\qquad st \leq H^*(s) + H(t), \quad \forall s,t\geq 0.$$

The relation (3.114) and the fact that $H'(0)=0$ and $(H')^{-1}, H$ are increasing functions yield

$$(3.116)\qquad H^*(s) \leq s(H')^{-1}(s), \quad \forall s\geq 0.$$

Making use of $E'(t)\leq 0, H''(t)\geq 0$, (3.113) and (3.116) we derive for $\varepsilon_0>0$ small enough

$$(3.117)\qquad \begin{array}{rl} & [H'(\varepsilon_0 E)\{\mu_1 L + C_5 c_2 E\} + \tilde{C}_5 C'E]'(t) \\ = & \varepsilon_0 E'(t)H''(\varepsilon_0 E(t))[\mu_1(t)L(t) + C_5c_2E(t)] + \tilde{C}_5C'E'(t) \\ & +H'(\varepsilon_0E(t))(\mu_1 L + C_5c_2E)'(t) \\ \leq & -\tilde{C}_3\mu_1(t)H'(\varepsilon_0E(t))E(t) + \tilde{C}_5C'E'(t) + c_4'H'(\varepsilon_0E(t))(h\circ\psi_x)(t) \\ & +\tilde{C}_5\mu_1(t)H'(\varepsilon_0E(t))H^{-1}(-C'E'(t)) \\ \leq & -\tilde{C}_3\mu_1(t)H'(\varepsilon_0E(t))E(t) + \tilde{C}_5H^*(\mu_1(t)H'(\varepsilon_0E(t))) \\ & +c_4'H'(\varepsilon_0E(t))(h\circ\psi_x)(t) \\ \leq & -\tilde{C}_3\mu_1(t)H'(\varepsilon_0E(t))E(t) + \tilde{C}_5'\varepsilon_0\mu_1(t)H'(\varepsilon_0E(t))E(t) \\ & +c_4'H'(\varepsilon_0E(t))(h\circ\psi_x)(t) \\ \leq & -\tilde{C}_4\mu_1(t)H'(\varepsilon_0E(t))E(t) + c_4'H'(\varepsilon_0E(t))(h\circ\psi_x)(t) \\ \leq & -\tilde{C}_4\mu_1(t)H_2(E(t)) + C_4'(h\circ\psi_x)(t). \end{array}$$

We note that in the second inequality, we have used (3.115) and $0\leq H'(\varepsilon_0E(t))\leq H'(\varepsilon_0E(0))$. Let

$$(3.118)\qquad \tilde{L}(t) = \begin{cases} \mu_1(t)L(t) + cE(t) & \text{if } H \text{ is linear on } [0,\varepsilon'] \\ H'(\varepsilon_0E(t))\{\mu_1(t)L(t) + C_5c_1E(t)\} & \\ \quad +\tilde{C}_5C'E(t) & \text{if } H'(0)=0 \text{ and } H''>0 \\ & \text{on }]0,\varepsilon']. \end{cases}$$

From (3.111) and (3.117), it follows

$$(3.119)\qquad \tilde{L}'(t) \leq -c_4\mu_1(t)H_2(E(t)) + c_5(h\circ\psi_x)(t), \quad \forall t\geq 0.$$

On the other hand, after choosing $M > 0$ larger if needed, we can observe from Lemma 3.4.7 that $L(t)$ is equivalent to $E(t)$. So, $\tilde{L}(t)$ is also equivalent to $E(t)$. Moreover, because $\zeta(t) \le \zeta(0)$, there exoists $\bar{\varepsilon}$, such that

$$\zeta(t)\tilde{L}(t) + 2c_5 E(t) \le \bar{\varepsilon} E(t), \quad \forall t \ge t_0. \tag{3.120}$$

Finally, let

$$\mathcal{L}(t) = \varepsilon(\zeta(t)\tilde{L}(t) + 2c_5 E(t)), \qquad \text{for } 0 < \varepsilon < \frac{1}{\bar{\varepsilon}},$$

then we observe, from (3.119), (H1), (3.21) and (3.120), that

$$\begin{aligned}
\mathcal{L}'(t) &= \varepsilon(\zeta'(t)\tilde{L}(t) + \zeta(t)\tilde{L}'(t) + 2c_5 E'(t)) \\
&\le -c_4\varepsilon\zeta(t)\mu_1(t)H_2(E(t)) + c_5\varepsilon\zeta(t)(h \circ \psi_x)(t) + 2c_5\varepsilon E'(t) \\
&\le -c_4\varepsilon\zeta(t)\mu_1(t)H_2(E(t)) - c_5\varepsilon(h' \circ \psi_x)(t) + 2c_5\varepsilon E'(t) \\
&\le -c_4\varepsilon\zeta(t)\mu_1(t)H_2(E(t)) \\
&\le -c_4\varepsilon\zeta(t)\mu_1(t)H_2\left(\frac{1}{\bar{\varepsilon}}\left(\zeta(t)\tilde{L}(t) + 2c_5 E(t)\right)\right) \\
&\le -c_4\varepsilon\zeta(t)\mu_1(t)H_2(\varepsilon(\zeta(t)\tilde{L}(t) + 2c_5 E(t))) = -c_4\varepsilon\zeta(t)\mu_1(t)H_2(\mathcal{L}(t)).
\end{aligned} \tag{3.121}$$

We have used the fact H_2 is increasing in the last two inequalities. Noting that $H_1' = -1/H_2$ (see (3.20)), we infer from (3.121)

$$\mathcal{L}'(t)H_1'(\mathcal{L}(t)) \ge c_4\varepsilon\zeta(t)\mu_1(t), \quad \forall t \ge t_0.$$

A simple Integration over (t_0, t) then yields

$$H_1(\mathcal{L}(t)) \ge H_1(\mathcal{L}(t_0)) + c_4\varepsilon \int_0^t \zeta(s)\mu_1(s)\,ds - c_4\varepsilon \int_0^{t_0} \zeta(s)\mu_1(s)\,ds.$$

Choose $\varepsilon > 0$ sufficiently small so that $H_1(\mathcal{L}(t_0)) - c_4\varepsilon \int_0^{t_0} \zeta(s)\mu_1(s)\,ds > 0$, then, thanks to the fact H_1^{-1} is decreasing, we infer

$$\begin{aligned}
\mathcal{L}(t) &\le H_1^{-1}\left(H_1(\mathcal{L}(t_0)) - c_4\varepsilon \int_0^{t_0} \zeta(s)\mu_1(s)\,ds + c_4\varepsilon \int_0^t \zeta(s)\mu_1(s)\,ds\right) \\
&\le H_1^{-1}\left(c_4\varepsilon \int_0^t \zeta(s)\mu_1(s)\,ds\right).
\end{aligned}$$

Consequently, the equivalence of $\mathcal{L}, \tilde{L}, L$ and E, yield

$$E(t) \le H_1^{-1}\left(\omega \int_0^t \zeta(s)\mu_1(s)\,ds\right).$$

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