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Existence et comportement asymptotique de certains problèmes d'évolution avec des termes non locaux.

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Dédicace

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Existence et comportement asymptotique de certains problèmes d'évolution avec des termes non locaux

In this PhD thesis we study the global existence, asymptotic behavior in time of solutions to nonlinear evolutions equations of hyperbolic type. The decreasing of classical energy plays a crucial role in the study of global existence and in stabilization of various distributed systems.

In chapter 1, we consider a Euler-Bernoulli beam equation with a boundary control condition of fractional derivative type. We study stability of the system using the semigroup theory of linear operators and a result obtained by Borichev and Tomilov.

In chapter 2 we consider a Timoshenko system in bounded domain with a delay term in the nonlinear internal feedback and prove the global existence of its solutions in Sobolev spaces by means of the energy method combined with the Faedo-Galerkin procedure. under a condition between the weight of the delay term in the feedback and the weight of the term without delay. Furthermore, we establish a decay rate estimate for the energy by introducing suitable Lyapunov functionals.

Key words : evolutions equations, global existence, General decay, stabilization, delay term, Euler-Bernoulli beam equation, Timoshenko system.

Existence et comportement asymptotique de certains problèmes d'évolution avec des termes non locaux

Dans cette th‘ese nous ´etudions l'existence globale, comportement asymptotique en temps de solutions à des équations d'évolutions non linéaires de type hyperbolique. La décroissance de l'énergie classique joue un rôle crucial dans l'étude de l'existence globale et dans la stabilisation de divers systèmes distribués.

Dans le chapitre 1, nous considérons une équation de faisceau d'Euler-Bernoulli avec une condition de contrôle de limite de type dérivée fractionnaire. Nous étudions la stabilité du système en utilisant la théorie des semi-groupes des opérateurs linéaires et un résultat obtenu par Borichev et Tomilov.

Dans le chapitre 2, nous considérons un système de Timoshenko dans un domaine limité avec un délai dans le feedback interne non linéaire et prouvons l'existence globale de ses solutions dans les espaces de Sobolev à l'aide de la méthode d'énergie combinée avec la procédure de Faedo-Galerkin. dans une condition entre le poids du délai dans le feedback et le poids du terme sans délai. De plus, nous établissons une estimation du taux de décroissance de l'énergie en introduisant des fonctionnelles de Lyapunov appropriées.

Mots-clés : équations d'évolution, existence globale, taux de décroissance générale, stabilisation, terme de retard, équation de faisceau d'Euler-Bernoulli, système de Timoshenko.

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Introduction

Problems of global existence and stability in time of Partial Differential Equations have recently been the subject of much work. In this thesis we were interested in study of the global existence and the stabilization of some evolution equations.
The purpose of stabilization is to attenuate the vibrations by feedback, thus it consists in guaranteeing the decrease of energy of the solutions to 0 in a more or less fast way by a mechanism of dissipation.
More precisely, the problem of stabilization consists in determining the asymptotic behaviour of the energy by $E(t)$, to study its limits in order to determine if this limit is zero or not, to give an estimate of the decay rate of the energy to zero. This problem has been studied by many authors for various systems. In our study, we obtain several type of stabilization

1) Strong stabilization: $E(t) \to 0$, as $t \to \infty$.

2) Logarithmic stabilization: $E(t) \leq c(\log(t))^{-\delta}, \forall t > 0, (c, \delta > 0)$.

3) polynomial stabilization: $E(t) \leq ct^{-\delta}, \forall t > 0, (c, \delta > 0)$

4) uniform stabilization: $E(t) \leq ce^{-\delta t}, \forall t > 0, (c, \delta > 0)$. c which depends on initial data

For wave equation with dissipation of the form $u'' - \Delta_x u + g(u') = 0$, stabilization problems have been investigated by many authors:
When $g : \mathbb{R} \to \mathbb{R}$ is continuous and increasing function such that $g(0) = 0$, global existence of solutions is known for all initial conditions (u_0, u_1) given in $H_0^1(\Omega) \times L^2(\Omega)$. This result is, for instance, a consequence of the general theory of nonlinear semi-groups of contractions generated by a maximal monotone operator (see Brézis **[8]**). Moreover, if we impose on the control the condition $\forall \lambda \neq 0, g(\lambda) \neq 0$, then strong asymptotic stability of solutions occurs in $H_0^1(\Omega) \times L^2(\Omega)$, i.e.,

$$(u, u') \to (0, 0) \text{ strongly in } H_0^1(\Omega) \times L^2(\Omega),$$

without speed of convergence. These results follows, for instance, from the invariance principle of Lasalle (see for example A. Haraux **[20]**, F. Conrad, M. Pierre **[15]**). If the solution goes to 0 as time goes to ∞, how to get energy decay rates? Dafermos has written in 1978 "Another advantage of this approach is that it is so simplistic that it requires only quite weak assumptions on the dissipative mechanism. The corresponding drawback is that the deduced information is also weak, never yielding, for example, decay rates of solutions. Many authors have worked since then on energy decay rates. First results were obtained for linear stabilization, then for polynomial stabilization (see M. Nakao A. Haraux **[20]**, E. Zuazua and V. Komornik **[21]**) and then extended to arbitrary growing feedbacks (close to 0). In the same time, geometrical aspects were considered. By combining the multiplier method with the techniques of micro-local analysis, Lasiecka et al **[16]** have investigated different dissipative systems of partial differential equations (with Dirichlet and Neumann boundary conditions) under general geometrical conditions with nonlinear feedback without any growth restrictions near the origin or at infinity. The computation of decay rates is reduced to solving an appropriate explicitly given ordinary differential equation of monotone type. More precisely, the following explicit decay estimate of the energy is obtained:

$$E(t) \leq h(\frac{t}{t_0} - 1), \quad \forall t \geq t_0, \tag{1}$$

where $t_0 > 0$ and h is the solution of the following differential equation:

$$h'(t) + q(h(t)) = 0, \quad \forall t \geq 0 \quad \text{and} \quad h(0) = E(0) \tag{2}$$

and the function q is determined entirely from the behavior at the origin of the nonlinear feedback by proving that E satisfies

$$(Id - q)^{-1}\Big(E((m+1)t_0)\Big) \leq E(mt_0), \quad \forall m \in \mathbb{N}.$$

In this thesis, the main objective is to give a global existence and stabilization results.
This work consists in three chapters, the first, for Euler-Bernoulli equations with boundary dissipation of fractional derivative type.
the second, Well-posedness and exponential stability for a wave equation with nonlocal time-delay condition.
the third, Global existence and energy decay of solutions to Timoshenko beam system with a delay term.

- The **chapter 1**

  We investigate the existence and decay properties of solutions for the initial boundary value problem of the Euler Bernoulli beam equation of the type

$$(P) \qquad \varphi_{tt}(x,t) + \varphi_{xxxx}(x,t) = 0 \text{ in }]0,L[\times]0,+\infty[$$

where $(x,t) \in (0,L) \times (0,+\infty)$. This system is subject to the boundary conditions

$$\begin{cases} \varphi(0,t) = \varphi_x(0,t) = 0 & \text{in } (0,+\infty) \\ \varphi_{xx}(L,t) = 0 & \text{in } (0,+\infty) \\ \varphi_{xxx}(L,t) = \gamma \partial_t^{\alpha,\eta} \varphi(L,t) & \text{in } (0,+\infty) \end{cases}$$

where $\gamma > 0$. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α with respect to the time variable. It is defined as follows

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\, ds, \ \eta \geq 0.$$

The system is finally completed with initial conditions

$$\varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x)$$

where the initial data (φ_0, φ_1) belong to a suitable function space.

In this chapter, We prove the global existence to the solutions in the class $H^2 \cap H_0^1$ by means of the semi group theory.

We prove also the decay estimate of the energy using the multiplier method and Borivhev-Tomilov theorem.

- The **chapter 2**

  In this chapter we investigate the existence and decay properties of solutions for the initial boundary value problem of the nonlinear Timoshenko system of the type

$$\begin{cases} \rho_1 \varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) = 0 & \text{in }]0,1[\times]0,+\infty[, \\ \rho_2 \psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x + \psi)(x,t) + \mu_1(t)\psi_t(x,t) & \\ +\mu_2(t)\psi_t(x,t-\tau) = 0 & \text{in }]0,1[\times]0,+\infty[, \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0 & t \geq 0, \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x) & x \in]0,1[, \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x) & x \in]0,1[, \\ \psi_t(x,t-\tau) = f_0(x,t-\tau) & \text{in }]0,1[\times]0,\tau[, \end{cases}$$

where $\tau > 0$ is a time delay, the initial data (ψ_0, ψ_1, f_0) belong to a suitable function space. We prove the global existence of the solutions in Sobolev spaces by means of the energy method combined with the Faedo-Galerkin procedure,under a condition between the weight of the delay term in the feedback and the weight of the term without delay.Furthermore, we establish a decay rate estimate for the energy by introducing suitable Lyapunov functionals.

Chapter 1

Preliminaries

In this chapter, we will introduce and state without proofs some important materials needed in the proof of our results (See [**?**, **?**]),

1.1 Banach Spaces-Definition and properties

We first review some basic facts from calculus in the most important class of linear spaces " Banach spaces".

Definition 1.1.1. . *A Banach space is a complete normed linear space X. Its dual space X' is the linear space of all continuous linear functional $f : X \to \mathbb{R}$.*

Proposition 1.1.1. *X' equipped with the norm $\|.\|_{X'}$ defined by*

$$\|f\|_{X'} = \sup\{|f(u)| : \|u\| \leq 1\}, \tag{1.1}$$

is also a Banach space. We shall denote the value of $f \in X'$ at $u \in X$ by either $f(u)$ or $\langle f, u\rangle_{X',X}$.

Remark 1.1.1. *From X'we construct the bidual or second dual $X'' = (X')'$. Furthermore, with each $u \in X$ we can define $\varphi(u) \in X''$ by $\varphi(u)(f) = f(u), f \in X'$. This satisfies clearly $\|\varphi(x)\| \leq \|u\|$. Moreover, for each $u \in X$ there is an $f \in X'$ with $f(u) = \|u\|$ and $\|f\| = 1$. So it follows that $\|\varphi(x)\| = \|u\|$.*

Definition 1.1.2. . *Since φ is linear we see that*

$$\varphi : X \to X'',$$

is a linear isometry of X onto a closed subspace of X'', we denote this by

$$X \hookrightarrow X''.$$

Definition 1.1.3. *. If φ is onto X'' we say X is reflexive, $X \cong X''$.*

Theorem 1.1.1. *. Let X be Banach space. Then X is reflexive, if and only if,*

$$B_X = \{x \in X : \|x\| \leq 1\},$$

is compact with the weak topology $\sigma(X, X')$. (See the next subsection for the definition of $\sigma(X, X')$)

Definition 1.1.4. *. Let X be a Banach space, and let $(u_n)_{n\in N}$ be a sequence in X. Then u_n converges strongly to u in X if and only if*

$$\lim \|u_n - u\|_X = 0,$$

and this is denoted by $u_n \to u$, or $\lim_{n\to\infty} u_n = u$.

Definition 1.1.5. *The Banach space E is said to be separable if there exists a countable subset D of E which is dense in E, i.e. $\overline{D} = E$.*

Proposition 1.1.2. *If E is reflexive and if F is a closed vector subspace of E, then F is reflexive.*

Corollaire 1.1.1. *The following two assertions are equivalent:*
(i) E is reflexive;
(ii) E' is reflexive.

1.1.1 The weak and weak star topologies

Let X be a Banach space and $f \in X'$. Denote by

$$\begin{aligned} \varphi_f : X &\longrightarrow \mathbb{R} \\ x &\longmapsto \varphi_f(x), \end{aligned}$$

when f cover X', we obtain a family $(\varphi_f)_{f\in X'}$ of applications to X in $\mathbb{R}$.

Definition 1.1.6. *The weak topology on X, denoted by $\sigma(X, X')$, is the weakest topology on X for which every $(\varphi_f)_{f\in X'}$ is continuous.*

We will define the third topology on X', the weak star topology, denoted by $\sigma(X', X)$. For all $x \in X$. Denote by

$$\varphi_f : X' \longrightarrow \mathbb{R}$$
$$f \longmapsto \varphi_x(f) = \langle f, x\rangle_{X',X},$$

when x cover X, we obtain a family $(\varphi_x)_{x\in X'}$ of applications to X' in $\mathbb{R}$.

Definition 1.1.7. *. The weak star topology on X' is the weakest topology on X' for which every $(\varphi_x)_{x\in X'}$ is continuous.*

Remark 1.1.2. *Since $X \subset X''$, it is clear that, the weak star topology $\sigma(X', X)$ is weakest then the topology $\sigma(X', X'')$, and this later is weakest then the strong topology.*

Definition 1.1.8. *A sequence (u_n) in X is weakly convergent to x if and only if*

$$\lim_{n\to\infty} f(u_n) = f(u),$$

for every $f \in X'$, and this is denoted by $u_n \rightharpoonup u$

Remark 1.1.3. *1. If the weak limit exist, it is unique.*

2. *If $u_n \to u \in X$(strongly) then $u_n \rightharpoonup u$(weakly).*

3. *If $dimX < +\infty$, then the weak convergent implies the strong convergent.*

Proposition 1.1.3. *On the compactness in the three topologies in the Banach space X:*

1. *First, the unit ball*
$$B' \equiv \{x \in X : \|x\| \leq 1\}, \tag{1.2}$$
in X is compact if and only if $dim(X) < \infty$.

2. *Second, the unit ball B' in X'(The closed subspace of a product of compact spaces) is weakly compact in X' if and only if X is reflexive.*

3. *Third, B' is always weakly star compact in the weak star topology of X'.*

Proposition 1.1.4. *Let (f_n) be a sequence in X'. We have:*

1. $[f_n \rightharpoonup^* f \text{ in } \sigma(X', X)] \Leftrightarrow [f_n(x) \rightharpoonup^* f(x),\ \forall x \in X]$.

2. *If* $f_n \to f(strongly)$ *then* $f_n \rightharpoonup f$, *in* $\sigma(X', X'')$,
   If $f_n \rightharpoonup f$ *in* $\sigma(X', X'')$, *then* $f_n \rightharpoonup^* f$, *in* $\sigma(X', X)$.

3. *If* $f_n \rightharpoonup^* f$ *in* $\sigma(X', X)$ *then* $\|f_n\|$ *is bounded and* $\|f\| \leq \liminf \|f_n\|$.

4. *If* $f_n \rightharpoonup^* f$ *in* $\sigma(X', X)$ *and* $x_n \to x(strongly)$ *in* X, *then* $f_n(x_n) \to f(x)$.

1.1.2 Weak and weak star compactness

In finite dimension, i.e, $\dim E < \infty$, we have Bolzano-Weierstrass's theorem (which is a strong compactness theorem).

Theorem 1.1.2. *(Bolzano-Weierstrass). If* $\dim E < \infty$ *and if* $\{x_n\} \subset E)$ *is bounded, then there exist in* E *a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\{x_{n_k}\}$ *strongly converges to* x.

The following two theorems are generalizations, in infinite dimension, of Bolzano- Weierstrass's theorem.

Theorem 1.1.3. *(weak star compactness, Banach-Alaoglu-Bourbaki). Assume that E is separable and consider* $\{f_n\} \subset E')$ *. If* $\{x_n\}$ *is bounded, then there exist* $f \in E'$ *and a subsequence* $\{f_{n_k}\}$ *of* $\{f_n\}$ *such that* $\{f_{n_k}\}$ *weakly star converges to* f *in* E'.

Theorem 1.1.4. *(weak compactness, Kakutani-Eberlein). Assume that E is reflexive and consider* $\{x_n\} \subset E)$. *If* $\{x_n\}$ *is bounded, then there exist* $x \in E$ *and a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\{x_{n_k}\}$ *weakly converges to* x *in* E.

Weak, weak star convergence and compactness in $L^p(\Omega)$.

Definition 1.1.9. *(weak convergence in* $L^p(\Omega)$ *with* $1 \leq p < \infty$ *). Let* Ω *an open subset of* $\mathbb{R}^n$ *.We say that the sequence* $\{f_n\}$ *of* $L^p(\Omega)$ *weakly converges to* $f \in L^p(\Omega)$, *if*

$$\lim_n \int_\Omega f_n(x)g(x)dx = \int_\Omega f(x)g(x)dx \text{ for all } g \in L^q;\ (\frac{1}{p} + \frac{1}{q} = 1)$$

Definition 1.1.10. *(weak star convergence in* $L^\infty(\Omega)$ *). We say that the sequence* $\{f_n\} \subset L^\infty(\Omega)$ *weakly star converges to* $f \in L^\infty(\Omega)$ *, if*

$$\lim_n \int_\Omega f_n(x)g(x)dx = \int_\Omega f(x)g(x)dx \text{ for all } g \in L^1(\Omega)$$

Theorem 1.1.5. *(weak compactness in $L^p(\Omega)$) with $1 < p < \infty$. Given $\{f_n\} \subset L^p(\Omega)$, if $\{f_n\}$ is bounded, then there exist $f \in L^p(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_n \rightharpoonup f$ in $L^p(\Omega)$.*

Theorem 1.1.6. *(weak star compactness in $L^\infty(\Omega)$.*
Given $\{f_n\} \subset L^\infty(\Omega)$, if $\{f_n\}$ is bounded, then there exist $f \in L^\infty(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_n \overset{*}{\rightharpoonup} f$ in $L^\infty(\Omega)$.

Generalities. In what follows, Ω is a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary and $1 \leq p \leq \infty$.

Weak and weak star convergence in Sobolev spaces

For $1 \leq p \leq \infty$, $W^{1;p}(\Omega)$ is a Banach space. Denote the space of all restrictions to Ω of $\mathbb{C}^1$-differentiable functions from $\mathbb{R}^N$ to $\mathbb{R}$ with compact support in R^N by $\mathbb{C}(\overline{\Omega})$.

Theorem 1.1.7. *for every $1 \leq p \leq \infty$ $\mathbb{C}^1(\overline{\Omega}) \subset W^{1;p}(\Omega) \subset L^p(\Omega)$, and, for $1 < p < \infty$, $C^1(\overline{\Omega})$ is dense in $W^{1;p}(\Omega)$.*

Definition 1.1.11. *(weak convergence in $W^{1;p}(\Omega)$ with $1 \leq p < \infty$).)*
We say the $\{f_n\} \subset W^{1;p}(\Omega)$ weakly converges to $f \in W^{1;p}(\Omega)$, and we write $f_n \rightharpoonup f$ in $W^{1;p}(\Omega)$, if $f_n \rightharpoonup f$ in $L^p(\Omega)$ and $\nabla f_n \rightharpoonup \nabla f$ in $L^p(\Omega; \mathbb{R}^N)$

Definition 1.1.12. *(weak convergence in $W^{1;\infty}(\Omega)$*
. We say the $\{f_n\} \subset W^{1;\infty}(\Omega)$ weakly star converges to $f \in W^{1;\infty}(\Omega)$, and we write $f_n \overset{*}{\rightharpoonup} f$ in $W^{1;\infty}(\Omega)$, if $f_n \overset{*}{\rightharpoonup} f$ in $L^p(\Omega)$ and $\nabla f_n \overset{*}{\rightharpoonup} \nabla f$ in $L^\infty(\Omega; \mathbb{R}^N)$

Theorem 1.1.8. *(Rellich). Let $1 \leq p \leq \infty$, $\{f_n\} \subset W^{1;p}(\Omega)$ and $f \in W^{1;p}(\Omega)$; if $f_n \rightharpoonup f$ in $W^{1;p}(\Omega)$ when $1 \leq p < \infty$ (resp.$f_n \overset{*}{\rightharpoonup} f$ in $W^{1;\infty}(\Omega)$) when $p = \infty$) then $f_n \to f$ in $L^p(\Omega)$), which means that for every $1 \leq p \leq \infty$, the weak convergence in $W^{1;p}(\Omega)$ imply the strong convergence in $L^p(\Omega)$.*

Theorem 1.1.9. *Let $1 < p \leq \infty$ and let $\{f_n\} \subset W^{1;p}(\Omega)$. If $\{f_n\}$ is bounded, then there exist $f \in W^{1;p}(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \rightharpoonup f$ in $W^{1;p}(\Omega)$ when $1 < p < \infty$ (resp. $f_{n_k} \overset{*}{\rightharpoonup} f$ in $W^{1;\infty}(\Omega)$)*

As a consequence of this theorem we have

Propriété 1.1.1. *Let $1 < p \leq \infty$and let $\{f_n\} \subset W^{1;p}(\Omega)$. If $\{f_n\}$ is bounded, then there exist $f \in W^{1;p}(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \to f$ in $L^p(\Omega)$ and $\nabla f_{n_k} \rightharpoonup \nabla f$ in $L^p(\Omega)$ when $1 < p < \infty$ (resp.$\nabla f_{n_k} \overset{*}{\rightharpoonup} \nabla f$ in $L^\infty(\Omega)$)*

Theorem 1.1.10. *. If$N < p \leq \infty$ and if $\{f_n\} \subset W^{1;p}(\Omega)$ is bounded, then there exist $f \in W^{1;p}(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}\}$ converges uniformly to f, and $\nabla f_{n_k} \rightharpoonup \nabla f$ in $W^{1;p}(\Omega)$ when $N < p < \infty$ (resp. $\nabla f_{n_k} \overset{*}{\rightharpoonup} \nabla f$ in $W^{1;\infty}$)*

1.2 Functional Spaces

1.2.1 The $L^p(\Omega)$ spaces

Definition 1.2.1. *Let $1 \leq p \leq \infty$ and let Ω be an open domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Define the standard Lebesgue space $L^p(\Omega)$ by*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |f(x)|^p dx < \infty \right\}. \tag{1.3}$$

Notation 1.2.1. *If $p = \infty$, we have*

$$L^\infty(\Omega) = \{f : \Omega \to \mathbb{R} \text{ is measurable and there exists a constant } C \text{ such that } |f(x)| \leq C \text{ a.e } \in \Omega\}.$$

Also, we denote by

$$\|f\|_\infty = \inf\{C,\ |f(x)| \leq C \text{ a.e } \in \Omega\}. \tag{1.4}$$

Notation 1.2.2. *For $p \in \mathbb{R}$ and $1 \leq p \leq \infty$, we denote by q the conjugate of p i.e. $\frac{1}{p} + \frac{1}{q} = 1$.*

Theorem 1.2.1. *$L^p(\Omega)$ is a Banach space for all $1 \leq p \leq \infty$.*

Remark 1.2.1. *In particularly, when $p = 2$, $L^2(\Omega)$ equipped with the inner product*

$$\langle f, g \rangle_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx, \tag{1.5}$$

is a Hilbert space.

Theorem 1.2.2. *For $1 < p < \infty$, $L^p(\Omega)$ is a reflexive space.*

1.2.2 Some integral inequalities

We will give here some important integral inequalities. These inequalities play an important role in applied mathematics and also, it is very useful in our next chapters.

Theorem 1.2.3. *(Holder's inequality). Let $1 \leq p \leq \infty$. Assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $fg \in L^p(\Omega)$ and*

$$\int_\Omega |fg|dx \leq \|f\|_p \|g\|_q.$$

Lemma 1.2.1. *(Young's inequality). Let $f \in L^p(\mathbb{R})$ and $g \in L^g(\mathbb{R})$ with $1 < p < \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \geq 0$. Then $f * g \in L^r(\mathbb{R})$ and*

$$\|f * g\|_{L^r(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})}\|g\|_{L^q(\mathbb{R})}.$$

Lemma 1.2.2. *. Let $1 \leq p \leq r \leq q$, $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$, and $1 \leq \alpha \leq 1$. Then*

$$\|u\|_{L^r} \leq \|u\|_{L^p}^{\alpha}\|u\|_{L^q}^{1-\alpha}.$$

Lemma 1.2.3. *If $\mu(\Omega) < \infty$, $1 \leq p \leq q \leq \infty$, then $L^q \hookrightarrow L^p$ and*

$$\|u\|_{L^p} \leq \mu(\Omega)^{\frac{1}{p}-\frac{1}{q}}\|u\|_{L^q}.$$

1.2.3 The $W^{m,p}(\Omega)$ spaces

Proposition 1.2.1. *Let Ω be an open domain in $\mathbb{R}^N$. Then the distribution $T \in D'(\Omega)$ is in $L^p(\Omega)$ if there exists a function $f \in L^p(\Omega)$ such that*

$$\langle T, \varphi \rangle = \int_\Omega f(x)\varphi(x)dx, \text{ for all } \varphi \in D(\Omega),$$

where $1 \leq p \leq \infty$ and it's well-known that f is unique.

Now, we will introduce the Sobolev spaces: The Sobolev space $W^{k,p}(\Omega)$ is defined to be the subset of L^p such that function f and its weak derivatives up to some order k have a finite L^p norm, for given $p \geq 1$.

$$W^{k,p}(\Omega) = \left\{ f \in L^p(\Omega); D^\alpha f \in L^p(\Omega).\ \ \forall\alpha; |\alpha| \leq k \right\}.$$

With this definition, the Sobolev spaces admit a natural norm:

$$f \longrightarrow \|f\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^p(\Omega)}^p \right)^{1/p}, \text{ for } p < +\infty$$

and

$$f \longrightarrow \|f\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^\infty(\Omega)}, \text{ for } p = +\infty$$

Space $W^{k,p}(\Omega)$ equipped with the norm $\| \ . \ \|_{W^{k,p}}$ is a Banach space. Moreover is a reflexive space for $1 < p < \infty$ and a separable space for $1 \leq p < \infty$. Sobolev spaces with $p = 2$ are especially important because of their connection with Fourier series and because they form a Hilbert space. A special notation has arisen to cover this case:

$$W^{k,2}(\Omega) = H^k(\Omega)$$

the H^k inner product is defined in terms of the L^2 inner product:

$$(f, g)_{H^k(\Omega)} = \sum_{|\alpha|\leq k} (D^\alpha f, D^\alpha g)_{L^2(\Omega)}.$$

The space $H^m(\Omega)$ and $W^{k,p}(\Omega)$ contain $\mathcal{C}^\infty(\overline{\Omega})$ and $\mathcal{C}^m(\overline{\Omega})$. The closure of $\mathcal{D}(\Omega)$ for the $H^m(\Omega)$ norm (respectively $W^{m,p}(\Omega)$ norm) is denoted by $H_0^m(\Omega)$ (respectively $W_0^{k,p}(\Omega)$). Now, we introduce a space of functions with values in a space X (a separable Hilbert space). The space $L^2(a, b; X)$ is a Hilbert space for the inner product

$$(f, g)_{L^2(a,b;X)} = \int_a^b (f(t), g(t))_X \, dt$$

We note that $L^\infty(a, b; X) = (L^1(a, b; X))'$. Now, we define the Sobolev spaces with values in a Hilbert space X. For $k \in N$, $p \in [1, \infty]$, we set:

$$W^{k,p}(a, b; X) = \left\{ v \in L^p(a, b; X); \frac{\partial v}{\partial x_i} \in L^p(a, b; X). \ \ \forall i \leq k \right\} ,$$

The Sobolev space $W^{k,p}(a, b; X)$ is a Banach space with the norm

$$\|f\|_{W^{k,p}(a,b;X)} = \left(\sum_{i=0}^{k} \| \frac{\partial f}{\partial x_i} \|_{L^p(a,b;X)}^p \right)^{1/p} , \quad \text{for } \ p < +\infty$$

and

$$\|f\|_{W^{k,\infty}(a,b;X)} = \sum_{i=0}^{k} \| \frac{\partial v}{\partial x_i} \|_{L^\infty(a,b;X)} , \quad \text{for } \ p = +\infty$$

The spaces $W^{k,2}(a, b; X)$ form a Hilbert space and it is noted $H^k(0, T; X)$. The $H^k(0, T; X)$ inner product is defined by:

$$(u, v)_{H^k(a,b;X)} = \sum_{i=0}^{k} \int_a^b \left(\frac{\partial u}{\partial x^i}, \frac{\partial v}{\partial x^i} \right)_X dt \ .$$

Theorem 1.2.4. *Let $1 \leq p \leq n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$$

where p^* is given by $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. Moreover there exists a constant $C = C(p, n)$ such that

$$\|u\|_{L^{p^*}} \leq C \|\nabla u\|_{L^p(\mathbb{R}^n)}, \quad \forall u \in W^{1,p}(\mathbb{R}^n).$$

Corollaire 1.2.1. *Let $1 \leq p < n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \quad \forall q \in [p, p^*]$$

with continuous imbedding.

For the case $p = n$, we have

$$W^{1,n}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \quad \forall q \in [n, +\infty[$$

Theorem 1.2.5. *Let $p > n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$$

with continuous imbedding.

Corollaire 1.2.2. *Let Ω a bounded domain in $\mathbb{R}^n$ of C^1 class with $\Gamma = \partial\Omega$ and $1 \leq p \leq \infty$. We have*

$$if\ 1 \leq p < \infty, \ \ then\ W^{1,p}(\Omega) \subset L^{p^*}(\Omega)\ where\ \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}.$$

$$if\ p = n, \quad then\ W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [p, +\infty[.$$

$$if\ p > n, \ \ then\ \ W^{1,p}(\Omega) \subset L^\infty(\Omega)$$

with continuous imbedding. Moreover, if $p > n$ we have:

$$\forall u \in W^{1,p}(\Omega), \quad |u(x) - u(y)| \leq C|x - y|^\alpha \|u\|_{W^{1,p}(\Omega)}\ a.e\ x, y \in \Omega$$

with $\alpha = 1 - \frac{n}{p} > 0$ and C is a constant which depend on p, n and Ω. In particular $W^{1,p}(\Omega) \subset C(\overline{\Omega})$.

Corollaire 1.2.3. *Let Ω a bounded domain in $\mathbb{R}^n$ of C^1 class with $\Gamma = \partial\Omega$ and $1 \leq p \leq \infty$. We have*

$$if\ p < n, \ then\ W^{1,p}(\Omega) \subset L^q(\Omega) \forall q \in [1, p^*[\ where\ \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}.$$

$$if\ p = n, \ then\ W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [p, +\infty[.$$

$$if\ p > n, \ then\ W^{1,p}(\Omega) \subset C(\overline{\Omega})$$

with compact imbedding.

Remark 1.2.2. *We remark in particular that*

$$W^{1,p}(\Omega) \subset L^q(\Omega)$$

with compact imbedding for $1 \leq p \leq \infty$ *and for* $p \leq q < p^*$.

Corollaire 1.2.4.

$$if \ \frac{1}{p} - \frac{m}{n} > 0, \ then \ W^{m,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \ where \ \frac{1}{q} = \frac{1}{p} - \frac{m}{n}.$$

$$if \ \frac{1}{p} - \frac{m}{n} = 0, \ then \ W^{m,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \forall q \in [p, +\infty[.$$

$$if \ \frac{1}{p} - \frac{m}{n} < 0, \ then \ W^{m,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$$

with continuous imbedding.

Lemma 1.2.4. *(Sobolev-Poincarés inequality)*

$$If \quad 2 \leq q \leq \frac{2n}{n-2}, n \geq 3 \quad and \quad q \geq 2, n = 1, 2,$$

then

$$\|u\|_q \leq C(q, \Omega) \|\nabla u\|_2, \quad \forall u \in H_0^1(\Omega).$$

Remark 1.2.3. *For all* $\varphi \in H^2(\Omega)$, $\Delta\varphi \in L^2(\Omega)$ *and for* Γ *sufficiently smooth, we have*

$$\|\varphi(t)\|_{H^2(\Omega)} \leq C \|\Delta\varphi(t)\|_{L^2(\Omega)}.$$

Proposition 1.2.2. *(Green's formula). For all* $u \in H^2(\Omega)$, $\upsilon \in H^1(\Omega)$ *we have*

$$-\int_\Omega \Delta u \upsilon dx = \int_\Omega \nabla u \nabla \upsilon dx - \int_{\partial\Omega} \frac{\partial u}{\partial \eta} \upsilon d\sigma,$$

where $\frac{\partial u}{\partial \eta}$ *is a normal derivation of* u *at* Γ.

1.2.4 Some Algebraic inequalities

Since our study based on some known algebraic inequalities, we want to recall few of them here.

Lemma 1.2.5. *(The Cauchy-Schwartz's inequality) Every inner product satisfies the Cauchy-Schwartz's inequality*

$$\langle x_1, x_2 \rangle \leq \|x_1\| \|x_2\|. \tag{1.6}$$

The equality sign holds if and only if x_1 *and* x_1 *are dependent.*

Lemma 1.2.6. *(Young's inequalities). For all $a, b \in \mathbb{R}^+$, we have*

$$ab \leq \alpha a^2 + \frac{1}{4\alpha} b^2 \tag{1.7}$$

where α is any positive constant.

Lemma 1.2.7. *For $a, b \geq 0$, the following inequality holds*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{1.8}$$

where, $\frac{1}{p} + \frac{1}{q} = 1$.

1.3 Existence Methods

1.3.1 Faedo-Galerkin's approximations

We consider the Cauchy problem abstract's for a second order evolution equation in the separable Hilbert space with the inner product $\langle .,. \rangle$ and the associated norm $\|.\|$.

$$\begin{cases} u''(t) + A(t)u(t) = f(t) & \text{t in } [0, T], \\ u(x, 0) = u_0(x), \ u'(x, 0) = u_1(x), \end{cases} \tag{1.9}$$

where u and f are unknown and given function, respectively, mapping the closed interval $[0, T] \subset \mathbb{R}$ into a real separable Hilbert space H. $A(t)$ $(0 \leq t \leq T\ )$ are linear bounded operators in H acting in the energy space $V \subset H$.
Assume that $\langle A(t)u(t), v(t) \rangle = a(t; u(t), v(t))$, for all $u, v \in V$; where $a(t; ., .)$ is a bilinear continuous in V. The problem (1.9) can be formulated as: Found the solution $u(t)$ such that

$$\begin{cases} u \in C([0, T]; V), u' \in C([0, T]; H) \\ \langle u''(t), v \rangle + a(t; u(t), v) = \langle f, v \rangle & \text{t in } D'(]0, T[), \\ u_0 \in V\ ,\ u_1 \in H, \end{cases} \tag{1.10}$$

This problem can be resolved with the approximation process of Fadeo-Galerkin.

Let V_m a sub-space of V with the finite dimension d_m, and let $\{w_{jm}\}$ one basis of V_m such that .

1. $V_m \subset V (\dim V_m < \infty), \forall m \in \mathbb{N}$

2. $V_m \to V$ such that, there exist a dense subspace ϑ in V and for all $v \in \vartheta$ we can get sequence $\{u_m\}_{m\in\mathbb{N}} \in V_m$ and $u_m \to u$ in V.

3. $V_m \subset V_{m+1}$ and $\overline{\cup_{m\in\mathbb{N}} V_m} = V$.

we define the solution u_m of the approximate problem

$$\begin{cases} u_m(t) = \sum_{j=1}^{d_m} g_j(t) w_{jm}, \\ u_m \in C([0,T];V_m), u'_m \in C([0,T];V_m),\ u_m \in L^2(0,T;V_m) \\ \langle u''_m(t), w_{jm}\rangle + a(t; u_m(t), w_{jm}) = \langle f, w_{jm}\rangle,\ 1 \le j \le d_m \\ u_m(0) = \sum_{j=1}^{d_m} \xi_j(t) w_{jm}, u'_m(0) = \sum_{j=1}^{d_m} \eta_j(t) w_{jm}, \end{cases} \tag{1.11}$$

where

$$\sum_{j=1}^{d_m} \xi_j(t) w_{jm} \longrightarrow u_0 \text{ in V as } m \longrightarrow \infty$$

$$\sum_{j=1}^{d_m} \eta_j(t) w_{jm} \longrightarrow u_1 \text{ in V as } m \longrightarrow \infty$$

By virtue of the theory of ordinary differential equations, the system (1.11) has unique local solution which is extend to a maximal interval $[0, t_m[$ by Zorn lemma since the non-linear terms have the suitable regularity. In the next step, we obtain a priori estimates for the solution, so that can be extended outside $[0, t_m[$ to obtain one solution defined for all $t > 0$.

1.3.2 A priori estimation and convergence

Using the following estimation

$$\|u_m\|^2 + \|u'_m\|^2 \le C\left\{\|u_m(0)\|^2 + \|u'_m(0)\|^2 + \int_0^T \|f(s)\|^2 ds\right\};\ 0 \le t \le T$$

and the Gronwall lemma we deduce that the solution u_m of the approximate problem (1.11) converges to the solution u of the initial problem (1.9). The uniqueness proves that u is the solution.

1.3.3 Gronwall's lemma

Lemma 1.3.1. *Let $T > 0$, $g \in L^1(0,T)$, $g \ge 0$ a.e and c_1, c_2 are positives constants. Let $\varphi \in L^1(0,T)$ $\varphi \ge 0$ a.e such that $g\varphi \in L^1(0,T)$ and*

$$\varphi(t) \le c_1 + c_2 \int_0^t g(s)\varphi(s) ds \quad a.e \text{ in } (0,T).$$

then, we have

$$\varphi(t) \leq c_1 exp \left(c_2 \int_0^t g(s)ds \right) \quad a.e \ in \ (0,T).$$

1.4 Aubin -Lions lemma

The Aubin Lions lemma is a result in the theory of Sobolev spaces of Banach space-valued functions. More precisely, it is a compactness criterion that is very useful in the study of nonlinear evolutionary partial differential equations. The result is named after the French mathematicians Thierry Aubin and Jacques-Louis Lions. We complete the preliminaries by the useful inequalities of Gagliardo-Nirenberg and Sobolev-Poincaré.

Lemma 1.4.1. *LetX_0,X and X_1 be three Banach spaces with $X_0 \subseteq X \subseteq X_1$. Assume that$X_0$ is compactly embedded in X and that X is continuously embedded in X_1; assume also that X_0 and X_1 are reflexive spaces. For $1 < p, q < +\infty$, let*

$$W = \{u \in L^p([0,T]; X_0) / \quad \dot{u} \in L^q([0,T]; X_1)\}$$

Then the embedding of W into $L^p([0,T]; X)$ is also compact.

Lemma 1.4.2 (Gagliardo-Nirenberg)**.** *Let $1 \leq r < q \leq +\infty$ and $p \leq q$. Then, the inequality*

$$\|u\|_{W^{m,q}} \leq C \|u\|^{\theta}_{W^{m,p}} \|u\|^{1-\theta}_r \quad for \quad u \in W^{m,p} \bigcap L^r$$

holds with some $C > 0$ and

$$\theta = \left(\frac{k}{n} + \frac{1}{r} - \frac{1}{q} \right) \left(\frac{m}{n} + \frac{1}{r} - \frac{1}{p} \right)^{-1}$$

provided that $0 < \theta \leq 1$ (we assume $0 < \theta < 1$ if $q = +\infty$).

Lemma 1.4.3 (Sobolev-Poincaré inequality)**.** *Let q be a number with $2 \leq q < +\infty$ $(n = 1, 2)$ or $2 \leq q \leq 2n/(n-2)$ $(n \geq 3)$, then there is a constant $c_* = c(\Omega, q)$ such that*

$$\|u\|_q \leq c_* \|\nabla u\|_2 \quad for \quad u \in H^1_0(\Omega).$$

1.5 Semigroup and spectral analysis theories

As the analysis done in this P.H.D thesis local on the semigroup and spectral analysis theories, we recall, in this chapter, some basic definitions and theorems which will be used in the following chapters.

1.5.1 Bounded and Unbounded linear operators

We start this chapter by give some well known results abound bounded and undounded operators. We are not trying to give a complete development, but rather review the basic definitions and theorems, mostly without proof. Let $(E, \|.\|_E)$ and $(E, \|.\|_E)$ be two Banach spaces over $\mathbb{C}$, and H will always denote a Hilbert space equipped with the scalar product $<.,.>_H$ and the corresponding norm $\|.\|_H$. A linear operator $T : E \longrightarrow F$ is a transformation which maps linearly E in F, that is

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v), \quad \forall u, v \in E and \alpha, \beta \in \mathbb{C}.$$

Definition 1.5.1. *A bounded operator $T : E \longrightarrow F$ is said to be bounded if there exists $C \geq 0$ such that*

$$\|Tu\|_F \leq C\|u\|_E \quad \forall u \in E.$$

The set of all bounded linear operators from E into F is denoted by $\mathcal{L}(E, F)$. Moreover, the set of all bounded linear operators from E into E is denoted by $\mathcal{L}(E)$.

Definition 1.5.2. *A bounded operator $T \in \mathcal{L}(E, F)$ is said to be compact if for each sequence $(x_n)_{n\in\mathbb{N}} \in E$ with $\|x_n\|_E = 1$ for each $n \in \mathbb{N}$, the sequence $(Tx_n)_{n\in\mathbb{N}} \in E$ has a subsequence which converges in F. The set of all compact operators from E into F is denoted by $\mathcal{K}(E, F)$. For simplicity one writes $\mathcal{K}(E, E) = \mathcal{K}(E)$.*

Definition 1.5.3. *Let $T \in \mathcal{L}(E, F)$, we define*

- *Range of T by*
$$\mathcal{R}(T) = \{Tu : \quad u \in E\} \subset F.$$
- *Kernel of T by*
$$ker(T) = \{u \in E : \quad Tu = 0\} \subset E.$$

Theorem 1.5.1. *(Fredholm alternative) if $T \in \mathcal{K}(E)$, then*

- *$ker(I - T)$ is finite dimension, (I is the identity operator on E).*
- *$\mathcal{R}(I - T)$ is closed.*
- *$ker(I - T) = 0 \Leftrightarrow \mathcal{R}(I - T) = E$.*

Definition 1.5.4. *Let $T : D(T) \subset E \longrightarrow F$ be an unbounded linear operator.*

- *The range of T is defined by*

$$\mathcal{R}(T) = \{Tu : \quad u \in D(T)\} \subset F.$$

- *The Kernel of T is defined by*

$$ker(T) = \{u \in D(T) : \quad Tu = 0\} \subset E.$$

- *The graph of T is defined by*

$$\mathcal{G}(T) = \{(u, Tu) : \quad u \in D(T)\} \subset E \times F.$$

Definition 1.5.5. *A map T is said to be closed if $\mathcal{G}(T)$ is closed in $E \times F$. The closedness of an unbounded linear operator T can be characterize as following if $u_n \in D(T)$ such that $u_n \longrightarrow u$ in E and $Tu_n \longrightarrow v$ in F, then $u \in D(T)$ and $Tu = v$.*

Definition 1.5.6. *Let $T : D(T) \subset E \longrightarrow F$ be a closed unbounded linear operator.*

- *The resolvent set of T is defined by*

$$\rho(T) = \{\lambda \in \mathbb{C} : \quad \lambda I - T \quad isbijective from \quad D(T) \quad onto \quad F\}.$$

- *The resolvent of T is defined by*

$$R(\lambda, T) = \{(\lambda I - T)^{-1} : \quad \lambda \in \rho(T)\}.$$

- *The spectrum set of T is the complement of the resolvent set in $\mathbb{C}$, denoted by*

$$\sigma(T) = \mathbb{C}/\rho(T).$$

Definition 1.5.7. *Let $T : D(T) \subset E \longrightarrow F$ be a closed unbounded linear operator. we can split the spectrum $\sigma(T)$ of T into three disjoint sets, given by*

- *The ponctuel spectrum of T is define by*

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : \quad ker(\lambda I - T) \neq 0\}$$

  in this case λ is called an eigenvalue of T.

- *The continuous spectrum of T is define by*

$$\sigma_c(T) = \{\lambda \in \mathbb{C} : \quad ker(\lambda I - T) = 0, \mathcal{R}(\lambda \bar{I} - T) = F \quad and (\lambda I - T)^{-1} is \quad not \quad bounded\}.$$

- *The residual spectrum of T is define by*

$$\sigma_r(T) = \{\lambda \in \mathbb{C} : \quad ker(\lambda I - T) = 0 and \mathcal{R}(\lambda I - T) \quad is \quad not \quad dense \quad in \quad F\}.$$

Definition 1.5.8. *Let $T : D(T) \subset E \longrightarrow F$ be a closed unbounded linear operator and let λ be an eigevalue of A. non-zero element $e \in E$ is called a generalized eigenvector of T associated with the eigenvalue value λ, if there exists $n \in^*$ such that*

$$(\lambda I - T)^n e = 0 \quad and \quad (\lambda I - T)^{n-1} e \neq 0.$$

if $n = 1$, then e is called an eigenvector.

Definition 1.5.9. *Let $T : D(T) \subset E \longrightarrow F$ be a closed unbounded linear operator. We say that T has compact resolvent, if there exist $\lambda_0 \in \rho(T)$ such that $(\lambda_0 I - T)^{-1}$ is compact.*

Theorem 1.5.2. *Let $(T, D(T))$ be a closed unbounded linear operator on H then the space $(D(T), \|.\|_{D(T)})$ where $\|u\|_{D(T)} = \|Tu\|_H + \|u\|_H \quad \forall u \in D(T)$ is banach space .*

Theorem 1.5.3. *Let $(T, D(T))$ be a closed unbounded linear operator on H then, $\rho(T)$ is an open set of $\mathcal{C}$.*

1.5.2 Semigroups, Existence and uniqueness of solution

In this section, we start by introducing some basic concepts concerning the semigroups. The vast majority of the evolution equations can be reduced to the form

$$\begin{cases} U_t = AU, \quad t > 0, \\ U(0) = U_0 \end{cases} \tag{1.12}$$

where A is the infinitesimal generator of a C_0-semigroup $S(t)$ over a Hilbert space H. Lets start by basic definitions and theorems. Let $(X, \|.\|_X)$ be a Banach space, and H be a Hilbert space equipped with the inner product $< .,. >_H$ and the induced norm $\|.\|_H$.

Definition 1.5.10. *A family $S(t)_{t \geq 0}$ of bounded linear operators in X is called a strong continous semigroup (in short, a C_0-semigroup) if*

i) *$S(0) = I_d$.*

ii) *$S(s+t) = S(s)S(t), \ \ \forall t \geq 0 \ \forall s \geq 0$.*

iii) *For each $u \in H$, $S(t)u$ is continous in t on $[0, +\infty[$.*

Sometimes we also denote $S(t)$ by e^{At}.

Definition 1.5.11. *For a semigroup $S(t)_{t\geq 0}$, we define an linear operator A with domain $D(A)$ consisting of points u such that the limit*

$$Au = \lim_{t \to 0^+} \frac{S(t)u - u}{t} \quad \forall u \in D(A)$$

exists. Then A is called the infinitesimal generator of the semigroup $S(t)_{t\geq 0}$.

Propriété 1.5.1. *Let $S(t)_{t\geq 0}$ be a C_0-semigroup in X. Then there exist a constant $M \geq 1$ and $\omega \geq 0$ such that*

$$\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}. \ \ \forall t \geq 0$$

If $\omega = 0$ then the corresponding semigroup is uniformly bounded. Moreover, if $M = 1$ then $S(t)_{t\geq 0}$ is said to be a C_0-semigroup of contractions.

Definition 1.5.12. *An unbounded linear operator $(A, D(A))$ on H, is said to be dissipative if*

$$\Re < Au, u > \leq 0, \forall u \in D(A).$$

Definition 1.5.13. *An unbounded linear operator $(A, D(A))$ on X, is said to be m-dissipative if*

- *A is a dissipative operator.*
- *$\exists \lambda_O$ such that $\mathcal{R}(\lambda_0 I - A) = X$*

Theorem 1.5.4. *Let A be a m-dissipative operator, then*

- *$\mathcal{R}(\lambda_0 I - A) = X, \quad \forall \lambda > 0$*
- *$]0, \infty[\subseteq \rho(A)$.*

Theorem 1.5.5. *(Hille-Yosida)An unbounded linear operator $(A, D(A))$ on X, is the infinitesimal generator of a C_0-semigroup of contractions $S(t)_{t\geq 0}$ if and only if*

- *A is closed and $\bar{D(A)} = X$.*
- *The resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$, and for all $\lambda > 0$,*

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \leq \lambda^{-1}$$

Theorem 1.5.6. *(Lumer-Phillips) Let $(A, D(A))$ be an unbounded linear operator on X, with dense domain $D(A)$ in X. A is the infinitesimal generator of a C_0-semigroup of contractions if and only if it is a m-dissipative operator.*

Theorem 1.5.7. *Let $(A, D(A))$ be an unbounded linear operator on X. If A is dissipative with $\mathcal{R}(I - A) = X$, and X is reflexive then $\bar{D(A)} = X$.*

Propriété 1.5.2. *Let $(A, D(A))$ be an unbounded linear operator on H. A is the infinitesimal generator of a C_0-semigroup of contractions if and only if A is a m-dissipative operator.*

Theorem 1.5.8. *Let A be a linear operator with dense domain $D(A)$ in a Hilbert space H. If A is dissipative and $0 \in \rho(A)$ then A is the infinitesimal generator of a C_0-semigroup of contractions on H.*

Theorem 1.5.9. *(Hille-Yosida) Let $(A, D(A))$ be an unbounded linear operator on H. Assume that A is the infinitesimal generator of a C_0-semigroup of contractions $S(t)_{t\geq 0}$.*

1. *For $U_0 \in D(A)$, the problem (1.12) admits a unique strong solution*

$$U(t) = S(t)U_0 \in C^1([0, \infty[; H) \cap C([0, \infty[; D(A))$$

2. *For $U_0 \in D(A)$, the problem (1.12) admits a unique weak solution*

$$U(t) \in C^0([0, \infty[; H).$$

1.5.3 Stability of semigroup

In this section we start by itroducing some definion about strong, exponential and polynomial stability of a C_0-semigroup. Then we collect some results about the stability of C_0-semigroup. Let $(X, \|.\|_X$ be a Banach space, and H be a Hilbert space equipped with the inner product $< .,. >_H$ and the induced norm $\|.\|_H$.

Definition 1.5.14. *Assume that A is the generator of a strongly continuous semigroup of contractions $S(t)_{t\geq 0}$ on X. We say that the C_0-semigroup $S(t)_{t\geq 0}$ is*

- *Strongly stable if*

$$\lim_{t\to+\infty} \|S(t)u\|_X = 0, \quad \forall u \in X.$$

- *Uniformly stable if*

$$\lim_{t\to+\infty} \|S(t)\|_{\mathcal{L}(X)} = 0$$

- *Exponentially stable if there exist two positive constants M and ϵ such that*

$$\|S(t)u\|_X \leq Me^{-\epsilon t}\|u\|_X, \quad \forall t > 0, \quad \forall u \in X.$$

- *Polynomially stable if there exist two positive constants C and α such that*

$$\|S(t)u\|_X \leq Ct^{-\alpha}\|u\|_X, \quad \forall t > 0, \quad \forall u \in X.$$

Propriété 1.5.3. *Assume that A is the generator of a strongly continuous semigroup of contractions $S(t)_{t\geq 0}$ on X. The following statements are equivalent*

- *$S(t)_{t\geq 0}$ is uniformly stable.*

- *$S(t)_{t\geq 0}$ is exponentially stable.*

First, we look for the necessary conditions of strong stability of a C_0-semigroup. The result was obtained by Arendt and Batty.

Chapter 2

THE EULER-BERNOULLI BEAM EQUATION WITh BOUNDARY DISSIPATION OF FRACTIONAL DERIVATIVE TYPE

2.1 Introduction

In this chapter we investigate the existence and decay properties of solutions for the initial boundary value problem of the Euler Bernoulli beam equation of the type

$$(P) \qquad \varphi_{tt}(x,t) + \varphi_{xxxx}(x,t) = 0 \text{ in }]0,L[\times]0,+\infty[$$

where $(x,t) \in (0,L) \times (0,+\infty)$. This system is subject to the boundary conditions

$$\begin{cases} \varphi(0,t) = \varphi_x(0,t) = 0 & \text{in } (0,+\infty) \\ \varphi_{xx}(L,t) = 0 & \text{in } (0,+\infty) \\ \varphi_{xxx}(L,t) = \gamma \partial_t^{\alpha,\eta} \varphi(L,t) & \text{in } (0,+\infty) \end{cases} \tag{2.1}$$

where $\gamma > 0$. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α with respect to the time variable. It is defined as follows

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\, ds, \ \eta \geq 0.$$

The system is finally completed with initial conditions

$$\varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x)$$

where the initial data (φ_0, φ_1) belong to a suitable function space.
A simple model describing the transverse vibration of a system of non-homogeneous connected Euler-Bernoulli beams, which was developed in **[13]**, is given by a system of the form

$$\begin{cases} m\varphi_{tt}(x,t) + EI\varphi_{xxxx}(x,t) = 0 & \text{in }]0,L[\times]0,+\infty[\\ \varphi(0,t) = 0, & \\ \varphi_x(0,t) = 0, & \\ -EI\varphi_{xxx}(1,t) = -k_1^2\varphi_t(1,t), & k_1 \in \mathbb{R}, \\ -EI\varphi_{xx}(1,t) = k_2^2\varphi_{xt}(1,t), & k_2 \in \mathbb{R}, \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), & x \in (0,L) \end{cases} \tag{2.2}$$

where m denotes the mass density per unit length, El is the flexural rigidity coefficient, and the following variables have engineering meanings:

$$\begin{cases} \varphi = \text{ vertical displacement, } \varphi_t = \text{ velocity,} \\ \varphi_x = \text{ rotation, } \varphi_{xt} = \text{ angular velocity,} \\ -EI\varphi_{xx} = \text{ bending moment} \\ -EI\varphi_{xxx} = \text{ shear} \end{cases}$$

at a point x, at time t.
Boundary conditions $(EB)_2$ and $(EB)_3$ signify that the beam is clamped, at the left end, $x = 0$ while boundary conditions $(EB)_4$ and $(EB)_5$ at the right end, $x = 1$, respectively, signify

$$\begin{cases} \text{shear } (-El\varphi_{xxx}) \text{ is proportional to velocity } (\varphi_t) \\ \text{bending moment } (-EI\varphi_{xx}) \text{ is negatively proportional to angular velocity } (\varphi_{xt}) \end{cases}$$

Control of elastic systems is one of the main themes in control engineering. The case of the wave equation with linear and nonlinear boundary feedback has attracted a lot of attention in recent years. The bibliography of works in the direction is truly long (see **[4]**, **[10]**, **[11]**, **[12]**, **[20]**, **[21]**, **[27]**) and many energy estimates have been derived for arbitrary growing feedbacks (polynomial, exponential or logarithmic decay).

For plates, also with linear and nonlinear boundary feedback acting through shear forces and moments, we refer to **[24]**,**[25]**, for stabilization results and **[23]** for estimates of the decay. The more difficult case of control by moment only has been studied in **[26]**. All these papers are based on multiplied techniques.

The case of serially connected beams has also been considered, with linear feedback acting genuinely on the force at the nodes **[14]**. Exponential stability is proved in the case of nondecreasing density and nonincreasing flexural rigidity. The same result has been proved in the more difficult case of control by moment only, for single homogeneous beam **[13]**.

In **[33]** B. Mbodge studies the decay rate of the energy of the wave equation with a boundary fractional derivative control as in this paper. Using energy methods, she proves strong asymptotic stability under the condition $\eta = 0$ and a polynomial type decay rate $E(t) \leq C/t$ if $\eta \neq 0$.

The boundary feedback under the consideration are of fractional type and are described by the fractional derivatives

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\, ds$$

The order of our derivatives is between 0 and 1. Very little attention has been paid to this type of feedback. In addition to being nonlocal, fractional derivatives involve singular and nonintegrable kernels ($t^{-\alpha}, 0 < \alpha < 1$). This leads to substantial mathematical difficulties since all the previous methods developed for convolution terms with regular and/or integrable kernels are no longer valid.

It has been shown (see **[34]**) that, as ∂_t, the fractional derivative ∂_t^{α} forces the system to become dissipative and the solution to approach the equilibrium state. Therefore, when applied on the boundary, we can consider them as controllers which help to reduce the vibrations.

In recent years, the application of fractional calculus has become a new interest in research areas such as viscoelasticity, chaos, biology, wave propagation, fluid flow, electromagnetics, automatic control, and signal processing (see **[40]**). For example, in viscoelasticity, due to the nature of the material microstructure, both elastic solid and viscous fluid like response qualities are involved. Using Boltzmann assumptions, we end up with a stress-strain relationship defined by a time convolution. Viscoelastic response occurs in a variety of materials, such as soils, concrete, rubber, cartilage, biological tissue, glasses, and polymers (**[5]**, **[6]** and **[32]**). In our case, the fractional dissipations may come from a viscoelastic surface of the

beam or simply describe an active boundary viscoelastic damper designed for the purpose of reducing the vibrations (see **[33]**, **[34]**).
Our purpose in this chapter is to give a global solvability in Sobolev spaces and energy decay estimates of the solutions to the problem (P) with a boundary control of fractional derivative type. To obtain global solutions to the problem (P), we use the argument combining the semigroup theory (see **[8]**) with the energy estimate method. For decay estimates, Under the condition $\eta = 0$, using a spectral analysis, we prove non-uniform stability. On the other hand if $\eta \neq 0$, we also show a polynomial type decay rate using a frequency domain approach and a recent theorem of A. Borichev and Y. Tomilov.

2.2 Augmented model

This section is concerned with the reformulation of the model (P) into an augmented system. For that, we need the following claims.

Theorem 2.2.1 (see **[33]**). *Let μ be the function:*

$$\mu(\xi) = |\xi|^{(2\alpha-1)/2}, \quad -\infty < \xi < +\infty,\ 0 < \alpha < 1. \tag{2.3}$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t \phi(\xi, t) + (\xi^2 + \eta)\phi(\xi, t) - U(t)\mu(\xi) = 0, \quad -\infty < \xi < +\infty, \eta \geq 0, t > 0, \tag{2.4}$$

$$\phi(\xi, 0) = 0, \tag{2.5}$$

$$O(t) = (\pi)^{-1} \sin(\alpha\pi) \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi, t)\, d\xi \tag{2.6}$$

is given by

$$O = I^{1-\alpha,\eta} U = D^{\alpha,\eta} U \tag{2.7}$$

where

$$[I^{\alpha,\eta} f](t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{-\eta(t-\tau)} f(\tau)\, d\tau.$$

Lemma 2.2.1. *If $\lambda \in D = \{\lambda \in \mathbb{C} : Re\lambda + \eta > 0\} \cup \{\lambda \in \mathbb{C} : Im\lambda \neq 0\}$ then*

$$\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2}\, d\xi = \frac{\pi}{\sin \alpha\pi} (\lambda + \eta)^{\alpha-1}.$$

Proof Let us set

$$f_\lambda(\xi) = \frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}.$$

We have

$$\left|\frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\right| \leq \begin{cases} \frac{\mu^2(\xi)}{Re\lambda+\eta+\xi^2} \text{ or} \\ \frac{\mu^2(\xi)}{|Im\lambda|+\eta+\xi^2} \end{cases}$$

Then the function f_λ is integrable. Moreover

$$\left|\frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\right| \leq \begin{matrix} \frac{\mu^2(\xi)}{\eta_0+\eta+\xi^2} \text{ for all } Re\lambda \geq \eta_0 > -\eta \\ \frac{\mu^2(\xi)}{\tilde{\eta}_0+\xi^2} \text{for all} |Im\lambda| \geq \tilde{\eta}_0 > 0 \end{matrix}$$

From Theorem 1.16.1 in **[47]**, the function

$$f_\lambda : D \to \mathbb{C} \text{ is holomorphe.}$$

For a real number $\lambda > -\eta$, we have

$$\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\,d\xi = \int_{-\infty}^{+\infty} \frac{|\xi|^{2\alpha-1}}{\lambda+\eta+\xi^2}\,d\xi = \int_0^{+\infty} \frac{x^{\alpha-1}}{\lambda+\eta+x}\,dx \text{ (with } \xi^2 = x)$$

$$= (\lambda+\eta)^{\alpha-1}\int_1^{+\infty} y^{-1}(y-1)^{\alpha-1}\,dy \text{ (with } y = x/(\lambda+\eta)+1)$$

$$= (\lambda+\eta)^{\alpha-1}\int_0^1 z^{-\alpha}(1-z)^{\alpha-1}\,dz \text{ (with } z = 1/y)$$

$$= (\lambda+\eta)^{\alpha-1}B(1-\alpha,\alpha) = (\lambda+\eta)^{\alpha-1}\Gamma(1-\alpha)\Gamma(\alpha) = (\lambda+\eta)^{\alpha-1}\frac{\pi}{\sin\pi\alpha}$$

Both holomorphic functions f_λ and $\lambda \mapsto (\lambda+\eta)^{\alpha-1}\frac{\pi}{\sin\pi\alpha}$ coincide on the half line $]-\infty,-\eta[$, hence on D following the principe of isolated zeroes.
We are now in a position to reformulate system (P). Indeed, by using Theorem 2.2.1, system (P) may be recast into the augmented model:

$$\begin{cases} \varphi_{tt} + \varphi_{xxxx} = 0, \\ \partial_t\phi(\xi,t) + (\xi^2+\eta)\phi(\xi,t) - \varphi_t(L,t)\mu(\xi) = 0, \\ \varphi(0,t) = \varphi_x(0,t) = 0, \\ \varphi_{xx}(L,t) = 0, \\ \varphi_{xxx}(L,t) = \gamma(\pi)^{-1}\sin(\alpha\pi)\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi,t)\,d\xi, \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x). \end{cases} \tag{2.8}$$

We define the energy associated to the solution of the problem (P') by the following formula:

$$E(t) = \frac{1}{2}\|\varphi_t\|_2^2 + \frac{1}{2}\|\varphi_{xx}\|_2^2 + \frac{\gamma}{2}(\pi)^{-1}\sin(\alpha\pi)\int_{-\infty}^{+\infty}(\phi(\xi,t))^2\,d\xi. \tag{2.9}$$

Lemma 2.2.2. *Let (φ, ϕ) be a solution of the problem (P'). Then, the energy functional defined by (2.9) satisfies*

$$E'(t) = -(\pi)^{-1}\sin(\alpha\pi)\gamma \int_{-\infty}^{+\infty} (\xi^2 + \eta)(\phi(\xi, t))^2 \, d\xi \leq 0. \tag{2.10}$$

Proof

Multiplying the first equation in (P') by φ_t, integrating over $(0, L)$ and using integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}\|\varphi_t\|_2^2 + \int_0^L \varphi_{xxxx}\varphi_t dx = 0.$$

Then

$$\frac{d}{dt}\left(\frac{1}{2}\|\varphi_t\|_2^2 + \frac{1}{2}\|\varphi_{xx}\|_2^2\right) + \zeta\varphi_t(L, t) \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi, t) \, d\xi = 0. \tag{2.11}$$

Multiplying the second equation in (P') by $\gamma(\pi)^{-1}\sin(\alpha\pi)\phi_t$ and integrating over $(-\infty, +\infty)$, to obtain:

$$\frac{\zeta}{2}\frac{d}{dt}\|\phi\|_2^2 + \zeta \int_{-\infty}^{+\infty} (\xi^2 + \eta)(\phi(\xi, t))^2 \, d\xi - \zeta\varphi_t(L, t) \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi, t) \, d\xi = 0. \tag{2.12}$$

From (2.9), (2.11) and (2.12) we get

$$E'(t) = -\zeta \int_{-\infty}^{+\infty} (\xi^2 + \eta)(\phi(\xi, t))^2 \, d\xi$$

where $\zeta = (\pi)^{-1}\sin(\alpha\pi)\gamma$. This completes the proof of the lemma.

□

2.3 Global existence

In this section we will give well-posedness results for problem (2.8) using semigroup theory. Let us introduce the semigroup representation of the (2.8). Let $U = (\varphi, \varphi_t, \phi)^T$ and rewrite (2.8) as

$$\begin{cases} U' = \mathcal{A}U, \\ U(0) = (\varphi_0, \varphi_1, \phi_0), \end{cases} \tag{2.13}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix}\varphi\\ u\\ \phi\end{pmatrix}=\begin{pmatrix}u\\ -\varphi_{xxxx}\\ -(\xi^2+\eta)\phi+u(L)\mu(\xi)\end{pmatrix} \tag{2.14}$$

with domain

$$D(\mathcal{A})=\left\{\begin{array}{l}\varphi,u,\phi)^T \text{ in } \mathcal{H}:\varphi\in H^4(0,L)\cap H_L^2(0,L), u\in H_L^2(0,L),\\ -(\xi^2+\eta)\phi+u(L)\mu(\xi)\in L^2(-\infty,+\infty),\\ \varphi_{xx}(L)=0,\ \varphi_{xxx}(L)-\zeta\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)\,d\xi=0\\ |\xi|\phi\in L^2(-\infty,+\infty)\end{array}\right\}. \tag{2.15}$$

where the energy space $\mathcal{H}$ is defined as

$$\mathcal{H}=H_L^2(0,L)\times L^2(0,L)\times L^2(-\infty,+\infty)$$

where

$$H_L^2(0,L)=\{\varphi\in H^2(0,L):\varphi(0)=\varphi_x(0)=0\}$$

For $U=(\varphi,u,\phi)^T,\overline{U}=(\overline{\varphi},\overline{u},\overline{\phi})^T$, we define the following inner product in $\mathcal{H}$

$$\langle U,\overline{U}\rangle_{\mathcal{H}}=\int_0^L(u\overline{u}+\varphi_{xx}\overline{\varphi}_{xx})\,dx+\zeta\int_{-\infty}^{+\infty}\phi\overline{\phi}\,d\xi.$$

We show that the operator $\mathcal{A}$ generates a C_0- semigroup in $\mathcal{H}$. In this step, we prove that the operator $\mathcal{A}$ is dissipative. Let $U=(\varphi,u,\phi)^T$. Using (2.13), (2.10) and the fact that

$$E(t)=\frac{1}{2}\|U\|_{\mathcal{H}}^2, \tag{2.16}$$

we get

$$\langle\mathcal{A}U,U\rangle_{\mathcal{H}}=-\zeta\int_{-\infty}^{+\infty}(\xi^2+\eta)(\phi(\xi))^2\,d\xi. \tag{2.17}$$

Consequently, the operator $\mathcal{A}$ is dissipative. Now, we will prove that the operator $\lambda I-\mathcal{A}$ is surjective for $\lambda>0$. For this purpose, let $(f_1,f_2,f_3)^T\in\mathcal{H}$, we seek $U=(\varphi,u,\phi)^T\in D(\mathcal{A})$ solution of the following system of equations

$$\begin{cases}\lambda\varphi-u=f_1,\\ \lambda u+\varphi_{xxxx}=f_2,\\ \lambda\phi+(\xi^2+\eta)\phi-u(L)\mu(\xi)=f_3\end{cases} \tag{2.18}$$

Suppose that we have found φ. Therefore, the first equation in (2.18) gives

$$u = \lambda\varphi - f_1. \tag{2.19}$$

It is clear that $u \in H_L^2(0, L)$. Furthermore, by (2.18) we can find ϕ as

$$\phi = \frac{f_3(\xi) + \mu(\xi)u(L)}{\xi^2 + \eta + \lambda}. \tag{2.20}$$

By using (2.18) and (2.19) the function φ satisfying the following system

$$\lambda^2\varphi + \varphi_{xxxx} = f_2 + \lambda f_1. \tag{2.21}$$

Solving system (2.21) is equivalent to finding $\varphi \in H^4 \cap H_L^2(0, L)$ such that

$$\int_0^L (\lambda^2\varphi w + \varphi_{xxxx}w)\, dx = \int_0^L (f_2 + \lambda f_1)w\, dx, \tag{2.22}$$

for all $w \in H_L^2(0, L)$. By using (2.22) and (2.20) the function φ satisfying the following system

$$\begin{cases} \int_0^L (\lambda^2\varphi w + \varphi_{xx}w_{xx})\, dx + \tilde{\zeta}u(L)w(L) \\ = \int_0^L (f_2 + \lambda f_1)w\, dx - \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_3(\xi)\, d\xi \;\, w(L) \end{cases} \tag{2.23}$$

where $\tilde{\zeta} = \zeta \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2+\eta+\lambda}\, d\xi$. Using again (2.19), we deduce that

$$u(L) = \lambda\varphi(L) - f_1(L). \tag{2.24}$$

Inserting (2.24) into (2.23), we get

$$\begin{cases} \int_0^L (\lambda^2\varphi w + \varphi_{xx}w_{xx})\, dx + \tilde{\zeta}\lambda\varphi(L)w(L) = \int_0^L (f_2 + \lambda f_1)w\, dx \\ \qquad -\zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_3(\xi)\, d\xi \;\, w(L) + \tilde{\zeta} f_1(L)w(L) \end{cases} \tag{2.25}$$

Consequently, problem (2.25) is equivalent to the problem

$$a(\varphi, w) = L(w) \tag{2.26}$$

where the bilinear form $a : [H_L^2(0, L) \times H_L^2(0, L)] \to \mathbb{R}$ and the linear form
$L : H_L^2(0, L) \to \mathbb{R}$ are defined by

$$a(\varphi, w) = \int_0^L (\lambda^2\varphi w + \varphi_{xx}w_{xx})\, dx + \lambda\tilde{\zeta}\varphi(L)w(L)$$

and

$$L(w) = \int_0^L (f_2 + \lambda f_1) w \, dx - \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + \lambda} f_3(\xi) \, d\xi \; w(L) + \tilde{\zeta} f_1(L) w(L).$$

It is easy to verify that a is continuous and coercive, and L is continuous. So applying the Lax-Milgram theorem, we deduce that for all $w \in H_L^2(0, L)$ problem (2.26) admits a unique solution $\varphi \in H_L^2(0, L)$. Applying the classical elliptic regularity, it follows from (2.25) that $\varphi \in H^4(0, L)$. Therefore, the operator $\lambda I - \mathcal{A}$ is surjective for any $\lambda > 0$. Consequently, using HilleYosida theorem, we have the following results.

Theorem 2.3.1 (Existence and uniqueness)**.**

(1) If $U_0 \in D(\mathcal{A})$, then system (2.13) has a unique strong solution

$$U \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

(1) If $U_0 \in \mathcal{H}$, then system (2.13) has a unique weak solution

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

□

2.4 Lack of exponential stability

In order to state and prove our stability results, we need some lemmas.

Theorem 2.4.1 ([41])**.** *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on Hilbert space. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R}$$

and

$$\overline{\lim_{|\beta| \to \infty}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Theorem 2.4.2 ([9]). *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup on a Hilbert space. If*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \text{ and } \sup_{|\beta|\geq 1} \frac{1}{\beta^l}\|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < M$$

for some l, then there exist c such that

$$\|e^{\mathcal{A}t}U_0\|^2 \leq \frac{c}{t^{\frac{2}{l}}}\|U_0\|^2_{D(\mathcal{A})}.$$

Theorem 2.4.3 ([3]). *Let $\mathcal{A}$ be the generator of a uniformly bounded C_0. semigroup $\{S(t)\}_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. If:*

(i) $\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.

(ii) The intersection of the spectrum $\sigma(\mathcal{A})$ with $i\mathbb{R}$ is at most a countable set,

then the semigroup $\{S(t)\}_{t\geq 0}$ is asymptotically stable, i.e, $\|S(t)z\|_{\mathcal{H}} \to 0$ as $t \to \infty$ for any $z \in \mathcal{H}$.

Theorem 2.4.4. *The semigroup generated by the operator $\mathcal{A}$ is not exponentially stable.*

Proof: We will examine two cases.
Case 1 $\eta = 0$: We shall show that $i\lambda = 0$ is not in the resolvent set of the operator $\mathcal{A}$. Indeed, noting that $(x\sin x, 0, 0)^T \in \mathcal{H}$, and denoting by $(\varphi, u, \phi)^T$ the image of $(x\sin x, 0, 0)^T$ by $\mathcal{A}^{-1}$, we see that $\phi(\xi) = |\xi|^{\frac{2\alpha-5}{2}} L\sin L$. But, then $\phi \notin L^2(-\infty, +\infty)$, since $\alpha \in]0, 1[$. And so $(\varphi, u, \phi)^T \notin D(\mathcal{A})$.
• **Case 2** $\eta \neq 0$: We aim to show that an infinite number of eigenvalues of $\mathcal{A}$ approach the imaginary axis which prevents the Euler-Bernoulli system (P) from being exponentially stable. Indeed we first compute the characteristic equation that gives the eigenvalues of $\mathcal{A}$. Let λ be an eigenvalue of $\mathcal{A}$ with associated eigenvector $U = (\varphi, u, \phi)^T$. Then $\mathcal{A}U = \lambda U$ is equivalent to

$$\begin{cases} \lambda\varphi - u = 0, \\ \lambda u + \varphi_{xxxx} = 0, \\ \lambda\phi + (\xi^2 + \eta)\phi - u(L)\mu(\xi) = 0 \end{cases} \tag{2.27}$$

From $(2.27)_1 - (2.27)_2$ for such λ, we find

$$\lambda^2\varphi + \varphi_{xxxx} = 0. \tag{2.28}$$

Using $(2.27)_3$ and $(2.27)_4$, we get

$$\begin{cases} \varphi(0) = 0, \varphi_x(0) = 0, \varphi_{xx}(L) = 0 \\ \varphi_{xxx}(L) - \zeta \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2+\lambda+\eta}\, d\xi u(L) \\ \qquad\qquad\qquad = \varphi_{xxx}(L) - \gamma\lambda(\lambda+\eta)^{\alpha-1}\varphi(L) = 0 \end{cases} \tag{2.29}$$

The caracteristics polynomiale of (2.28) is

$$s^4 + \lambda^2 = 0.$$

We find the roots

$$t_1(\lambda) = \frac{1}{\sqrt{2}}(1+i)\sqrt{\lambda}, \quad t_2(\lambda) = -t_1, t_3(\lambda) = it_1, t_4(\lambda) = -t_3.$$

Here and below, for simplicity we denote $t_i(\lambda)$ by t_i. The solution φ is given by

$$\varphi(x) = \sum_{i=1}^{4} c_i e^{t_i x}. \tag{2.30}$$

Thus the boundary conditions may be written as the following system:

$$M(\lambda)C(\lambda) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ t_1 & t_2 & t_3 & t_4 \\ t_1^2 e^{t_1 L} & t_2^2 e^{t_1 L} & t_3^2 e^{t_1 L} & t_4^2 e^{t_1 L} \\ h(t_1)e^{t_1 L} & h(t_2)e^{t_2 L} & h(t_1)e^{t_1 L} & h(t_1)e^{t_1 L} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2.31}$$

where we have set

$$h(r) = r^3 - \gamma\lambda(\lambda+\eta)^{\alpha-1}.$$

Hence a non-trivial solution φ exists if and only if the determinant of $M(\lambda)$ vanishes. Set $f(\lambda) = detM(\lambda)$, thus the characteristic equation is $f(\lambda) = 0$.

Our purpose in the sequel is to prove, thanks to Rouché's Theorem, that there is a subsequence of eigenvalues for which their real part tends to 0.

In the sequel, since $\mathcal{A}$ is dissipative, we study the asymptotic behavior of the large eigenvalues λ of $\mathcal{A}$ in the strip $-\alpha_0 \leq \mathcal{R}(\lambda) \leq 0$, for some $\alpha_0 > 0$ large enough and for such λ, we remark that $e^{t_i L}, i = 1, \ldots, 4$ remains bounded.

Lemma 2.4.1. *There exists $N \in \mathbb{N}]$ such that*

$$\{\lambda_k\}_{k\in\mathbb{Z}^*, |k|\geq N} \subset \sigma(\mathcal{A}) \tag{2.32}$$

where

$$\lambda_k = \frac{i}{4L^2}(2k+1)^2\pi^2 + \frac{\tilde{\alpha}}{k^{2(1-\alpha)}} + \frac{\beta}{|k|^{2(1-\alpha)}} + o\left(\frac{1}{k^{3-\alpha}}\right), |k| \geq N, \tilde{\alpha} \in i\mathbb{R}, \beta \in \mathbb{R}, \beta < 0.$$

Moreover for all $|k| \geq N$, the eigenvalues λ_k are simple.

Proof

$$\begin{aligned} f(\lambda) = -2t_1^4 e^{-iL\sqrt{2\lambda}} \Big(& 1 + e^{(1+i)L\sqrt{2\lambda}} + e^{2iL\sqrt{2\lambda}} + e^{(-1+i)L\sqrt{2\lambda}} + 4e^{iL\sqrt{2\lambda}} + (1-i)\tfrac{r}{t_1} \\ &+ (1+i)\tfrac{r}{t_1}e^{(1+i)L\sqrt{2\lambda}} - (1-i)\tfrac{r}{t_1}e^{2iL\sqrt{2\lambda}} - (1+i)\tfrac{r}{t_1}e^{(-1+i)L\sqrt{2\lambda}}\Big) \end{aligned} \tag{2.33}$$

Since all the eigenvalues locate on the open left-half complex plane, and since λ is symmetric with respect to the real axis, we need only to consider the case where $\pi/2 \leq \theta \leq \pi$. Since $\sqrt{\lambda} = \sqrt{|\lambda|}(\cos\frac{\theta}{2} + \sin\frac{\theta}{2})$, we see that

$$e^{-\sqrt{2\lambda}} = O(e^{-\mu}\sqrt{|\lambda|}), \quad e^{i\sqrt{2\lambda}} = O(e^{-\mu}\sqrt{|\lambda|}), \quad \mu > 0.$$

We set

$$\begin{aligned} \tilde{f}(\lambda) &= 1 + e^{(1+i)L\sqrt{2\lambda}} + e^{2iL\sqrt{2\lambda}} + e^{(-1+i)L\sqrt{2\lambda}} + (1-i)\tfrac{r}{t_1} + (1+i)\tfrac{r}{t_1}e^{(1+i)L\sqrt{2\lambda}} \\ &-(1-i)\tfrac{r}{t_1}e^{2iL\sqrt{2\lambda}} - (1+i)\tfrac{r}{t_1}e^{(-1+i)L\sqrt{2\lambda}} \quad (\text{ with } r = \gamma(\lambda+\eta)^{\alpha-1}) \\ &= f_0(\lambda) + \tfrac{f_1(\lambda)}{\lambda^{3/2-\alpha}} + o\left(\tfrac{1}{\lambda^{3/2-\alpha}}\right. \end{aligned} \tag{2.34}$$

where

$$f_0(\lambda) = 1 + e^{(1+i)L\sqrt{2\lambda}}, \tag{2.35}$$

$$f_1(\lambda) = \frac{\gamma(1-i)}{\sqrt{2}}\left(1 - i + (1+i)e^{(1+i)L\sqrt{2\lambda}}\right). \tag{2.36}$$

Note that f_0 and f_1 remain bounded in the strip $-\alpha_0 \leq \mathcal{R}(\lambda) \leq 0$.

Step 2. We look at the roots of f_0. From (2.35), f_0 has one familie of roots that we denote λ_k^0.

$$f_0(\lambda) = 0 \Leftrightarrow e^{(1+i)L\sqrt{2\lambda}} = -1.$$

Hence

$$(1+i)L\sqrt{2\lambda} = i(2k+1)\pi, \quad k \in \mathbb{Z},$$

i.e.,

$$\lambda_k^0 = \frac{i}{4L^2}(2k+1)^2\pi^2, \qquad k \in \mathbb{Z}.$$

Now with the help of Rouché's Theorem, we will show that the roots of $\tilde{f}$ are close to those of f_0. Changing in (2.34) the unknown λ by $u = (1+i)L\sqrt{2\lambda}$ then (2.34) becomes

$$\tilde{f}(u) = (e^u + 1) + O\left(\frac{1}{u}\right) = f_0(u) + O\left(\frac{1}{u}\right).$$

The roots of f_0 are $u_k = \frac{i}{4L^2}(2k+1)^2\pi^2, k \in \mathbb{Z}$, and setting $u = u_k + re^{it}, t \in [0, 2\pi]$, we can easily check that there exists a constant $C > 0$ independent of k such that $|e^u + 1| \geq Cr$ for r small enough. This allows to apply Rouché's Theorem. Consequently, there exists a subsequence of roots of $\tilde{f}$ which tends to the roots u_k of f_0. Equivalently, it means that there exists $N \in \mathbb{N}$ and a subsequence $\{\lambda_k\}_{|k|\geq N}$ of roots of $f(\lambda)$, such that $\lambda_k = \lambda_k^0 + o(1)$ which tends to the roots $\frac{i}{4L^2}(2k+1)^2\pi^2$ of f_0. Finally for $|k| \geq N, \lambda_k$ is simple since λ_k^0 is.

Step 3. From Step 2, we can write

$$\lambda_k = \frac{i}{4L^2}(2k+1)^2\pi^2 + \varepsilon_k. \tag{2.37}$$

Using (2.37), we get

$$e^{(1+i)L\sqrt{2\lambda}} = -1 - \frac{2L\varepsilon_k}{(2k+1)\pi} + o(\frac{\varepsilon_k}{k}). \tag{2.38}$$

Substituting (2.38) into (2.34), using that $\tilde{f}(\lambda_k) = 0$, we get:

$$\tilde{f}(\lambda_k) = -\frac{2L\varepsilon_k}{(2k+1)\pi} - \frac{4^{2-\alpha}L^{2-\alpha}\gamma}{((2k+1)\pi)^{2(1-\alpha)+1}i^{1-\alpha}} + o(\frac{\varepsilon_k}{k}) = 0 \tag{2.39}$$

and hence

$$\varepsilon_k = -\frac{4^{2-\alpha}L^{2-\alpha}\gamma}{2L((2k+1)\pi)^{2(1-\alpha)}}\left(\cos(1-\alpha)\frac{\pi}{2} - i\sin(1-\alpha)\frac{\pi}{2}\right). \tag{2.40}$$

From (2.40) we have in that case $|k|^{2(1-\alpha)}\mathcal{R}\lambda_k \sim \beta$, with

$$\beta = -\frac{2\gamma L^{1-\alpha}}{\pi^{2(1-\alpha)}}\cos(1-\alpha)\frac{\pi}{2}$$

The operator $\mathcal{A}$ has a non exponential decaying branch of eigenvalues. Thus the proof is complete.

Remark 2.4.1. *We can also show the lack of exponential stability by proving that the second condition in Theorem 2.4.1 does not hold. In particular, it can be shown that there is a sequence $\lambda_n \in \mathbb{R}$ diverging to ∞, and a bounded sequence $F_n \in \mathcal{H}$ such that*

$$\|(i\lambda_n - \mathcal{A})^{-1}F_n\| \to \infty \textit{ for all } n \textit{ large enough .}$$

We give details of the proof in the Appendix.

□

2.5 Asymptotic stability

Lemma 2.5.1. *$\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.*

Proof

We will argue by contraction. Let us suppose that there $\lambda \in \mathbb{R}, \lambda \neq 0$ and $U \neq 0$, such that $\mathcal{A}U = i\lambda U$. Then, we get

$$\begin{cases} i\lambda\varphi - u = 0, \\ i\lambda u + \varphi_{xxxx} = 0, \\ i\lambda\phi + (\xi^2 + \eta)\phi - u(L)\mu(\xi) = 0, \end{cases} \tag{2.41}$$

Then, from (2.17) we have

$$\phi \equiv 0. \tag{2.42}$$

From $(2.41)_3$, we have

$$u(L) = 0. \tag{2.43}$$

Hence, from $(2.41)_1$ we obtain

$$\varphi(L) = 0 \text{ and } \varphi_{xxx}(L) = 0. \tag{2.44}$$

From $(2.41)_1$ and $(2.41)_2$, we have

$$-\lambda^2\varphi + \varphi_{xxxx} = 0. \tag{2.45}$$

Now, we prove that $\varphi_x(L) = 0$. We have the following Lemma.

Lemma 2.5.2 ([15])**.** *Let $\varphi \in H^2(0, L)$ a solution of equation (2.45). Assume there exists $\zeta \in [0, L[$ such that $\varphi(\zeta), \varphi_x(\zeta), \varphi_{xx}(\zeta)$ are ≥ 0, and $\varphi(\zeta) + \varphi_x(\zeta) > 0$. Then $\varphi, \varphi_x, \varphi_{xx}$ are > 0 on $]\zeta, L]$.*

Proof We integrate equation (2.45) from $\zeta < x$ to x:

$$\begin{aligned} \varphi_{xxx}(x) &= \varphi_{xxx}(\zeta) + \textstyle\int_\zeta^x \varphi_{xxxx}(t)\,dt \\ &= \varphi_{xxx}(\zeta) + \lambda^2 \textstyle\int_\zeta^x \varphi(t)\,dt \end{aligned} \tag{2.46}$$

Integrating once more, we get

$$\begin{aligned} \varphi_{xx}(x) - \varphi_{xx}(\zeta) &= (x-\zeta)\varphi_{xxx}(\zeta) + \lambda^2 \textstyle\int_\zeta^x \int_\zeta^t \varphi(z)\,dz\,dt \\ &= (x-\zeta)\varphi_{xxx}(\zeta) + \lambda^2 \textstyle\int_\zeta^x (x-t)\varphi(t)\,dt. \\ \varphi_{xx}(x) &= \varphi_{xx}(\zeta) + (x-\zeta)\varphi_{xxx}(\zeta) + \lambda^2 \textstyle\int_\zeta^x (x-t)\varphi(t)\,dt. \end{aligned} \tag{2.47}$$

Since $\varphi(\zeta)+\varphi_x(\zeta)>0$ and $\varphi(\zeta)\geq 0$, there exists $\eta>0$ such that $\varphi>0$ on $]\zeta,\eta]$. Let $\eta\leq 1$ as large as possible, and suppose $\eta<1$, that is, $\varphi(\eta)=0$. By (2.47) and assumptions in Lemma 2.5.2, $\varphi_{xx}\geq 0$ on $[\zeta,\eta]$. Thus φ_x is nondecreasing, and therefore ≥ 0 on $[\zeta,\eta]$. Then φ is also nondecreasing on $[\zeta,\eta]$. But this contradicts $\varphi(\eta)=0$. Thus $\eta=1$ and φ is >0 on $]\zeta,\eta]$. The same is true for $\varphi_{xxx},\varphi_{xx}$ and φ_x.

Corollaire 2.5.1. *Soit $\varphi\in H^2(0,L)$ a solution of equation (2.45) such that $\varphi(L)\geq 0,\varphi_x(L)\leq 0,\varphi_{xx}(L)\geq 0$, and $\varphi(L)-\varphi_x(L)>0$. Then $\varphi>0$ on $[0,L[$.*

Proof We set $\psi(x)=\varphi(L-x)$. Then ψ satisfies (2.47). Then applying Lemme 2.5.2.
Now, as $\varphi(L)=0$, assume $\varphi_x(L)\neq 0$, for instance $\varphi_x(L)<0$, without restriction. By corollary 2.5.1, $\varphi>0$ on $[0,L[$, thus $\varphi(0)>0$, which is a contradiction. Therefore, $\varphi(L)=\varphi_x(L)=\varphi_{xx}(L)=\varphi_{xxx}(L)=0$.
Consider $X=(\varphi,\varphi_x,\varphi_{xx},\varphi_{xxx})$. Then we can rewrite (2.44) and (2.45) as the initial value problem

$$\begin{aligned}&\tfrac{d}{dx}X=\mathcal{B}X\\ &X(L)=0,\end{aligned}\tag{2.48}$$

, where

$$\mathcal{B}=\begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -\lambda^2 & 0 & 0 & 0\end{pmatrix}$$

By the Picard Theorem for ordinary differential equations the system (2.48) has a unique solution $X=0$. Therefore $\varphi=0$. It follows from (2.41), that $u=0$ and $\phi=0$, i.e., $U=0$. Consequently, $\mathcal{A}$ does not have purely imaginary eigenvalues, so the condition (i) of Theorem 2.4.3 holds. The condition (ii) of Theorem 2.4.3 will be satisfied if we show that $\sigma(\mathcal{A})\cap\{i\mathbb{R}$ is at most a countable set. We have the following lemma.

Lemma 2.5.3. *We have*

$$\begin{aligned}&i\mathbb{R}\subset\rho(\mathcal{A})\ \text{if}\ \eta\neq 0,\\ &i\mathbb{R}^*\subset\rho(\mathcal{A})\ \text{if}\ \eta=0\end{aligned}$$

where $\mathbb{R}^*=\mathbb{R}-\{0\}$.

Proof

Let $\lambda\in\mathbb{R}$. Let $F=(f_1,f_2,f_3)^T\in\mathcal{H}$ be given, and let $X=(\varphi,u,\phi)^T\in D(\mathcal{A})$ be such that

$$(i\lambda I-\mathcal{A})X=F.\tag{2.49}$$

Equivalently, we have

$$\begin{cases} i\lambda\varphi - u = f_1, \\ i\lambda u + \varphi_{xxxx} = f_2, \\ i\lambda\phi + (\xi^2+\eta)\phi - u(L)\mu(\xi) = f_3 \end{cases} \tag{2.50}$$

From $(2.50)_1$ and $(2.50)_2$, we have

$$-\lambda^2\varphi + \varphi_{xxxx} = (f_2 + i\lambda f_1) \tag{2.51}$$

Suppose that $\lambda \neq 0$. It is enough to consider $\lambda > 0$. Let $\lambda = \tau^2$. Taking into account the domain boundary conditions $\varphi(0) = \varphi_x(0) = 0$, implies that the general solution for (2.51) is of the form

$$\begin{aligned} \varphi(x) &= A(\cosh\tau x - \cos\tau x) + B(\sinh\tau x - \sin\tau x) \\ &+ \tfrac{1}{2\tau^3}\textstyle\int_0^x (f_2(\sigma) + i\tau^2 f_1(\sigma))(\sinh\tau(x-\sigma) - \sin\tau(x-\sigma))\,d\sigma \end{aligned} \tag{2.52}$$

. Hence

$$\begin{aligned} \varphi_x(x) &= \tau[A(\sinh\tau x + \sin\tau x) + B(\cosh\tau x - \cos\tau x)] \\ &+ \tfrac{1}{2\tau^2}\textstyle\int_0^x (f_2(\sigma) + i\tau^2 f_1(\sigma))(\cosh\tau(x-\sigma) - \cos\tau(x-\sigma))\,d\sigma, \end{aligned} \tag{2.53}$$

$$\begin{aligned} \varphi_{xx}(x) &= \tau^2[A(\cosh\tau x + \cos\tau x) + B(\sinh\tau x + \sin\tau x)] \\ &+ \tfrac{1}{2\tau}\textstyle\int_0^x (f_2(\sigma) + i\tau^2 f_1(\sigma))(\sinh\tau(x-\sigma) + \sin\tau(x-\sigma))\,d\sigma, \end{aligned} \tag{2.54}$$

$$\begin{aligned} \varphi_{xxx}(x) &= \tau^3[A(\sinh\tau x - \sin\tau x) + B(\cosh\tau x + \cos\tau x)] \\ &+ \tfrac{1}{2}\textstyle\int_0^x (f_2(\sigma) + i\tau^2 f_1(\sigma))(\cosh\tau(x-\sigma) + \cos\tau(x-\sigma))\,d\sigma. \end{aligned} \tag{2.55}$$

Taking the remaining boundary condition $\varphi_{xx}(L) = 0$, we obtain

$$\begin{aligned} &A(\cosh\tau L + \cos\tau L) + B(\sinh\tau L + \sin\tau L) \\ &= -\tfrac{1}{2\tau^3}\textstyle\int_0^L (f_2(\sigma) + i\tau^2 f_1(\sigma))(\sinh\tau(L-\sigma) + \sin\tau(L-\sigma))\,d\sigma. \end{aligned} \tag{2.56}$$

From $(2.50)_3$, we have

$$\phi(\xi) = \frac{u(L)\mu(\xi) + f_3(\xi)}{i\lambda + \xi^2 + \eta}.$$

Then

$$\varphi_{xxx}(L) = \zeta\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{i\lambda + \xi^2 + \eta}\,d\xi\, u(L) + \zeta\int_{-\infty}^{+\infty} \frac{\mu(\xi) f_3(\xi)}{i\lambda + \xi^2 + \eta}\,d\xi. \tag{2.57}$$

Since

$$\zeta\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{i\lambda + \xi^2 + \eta}\,d\xi = \gamma(i\lambda + \eta)^{\alpha-1}$$

and

$$u(L) = i\lambda\varphi(L) - f_1(L),$$

using (2.57), we get

$$\varphi_{xxx}(L) - i\tau^2\gamma(i\tau^2+\eta)^{\alpha-1}\varphi(L) = -\gamma(i\tau^2+\eta)^{\alpha-1}f_1(L) + \zeta\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_3(\xi)}{i\tau^2+\xi^2+\eta}\,d\xi.$$

Then

$$\begin{aligned}
&A[\tau^3(\sinh\tau L - \sin\tau L) - i\tau^2\gamma(i\tau^2+\eta)^{\alpha-1}(\cosh\tau L - \cos\tau L)]\\
&+B[\tau^3(\cosh\tau L + \cos\tau L) - i\tau^2\gamma(i\tau^2+\eta)^{\alpha-1}(\sinh\tau L - \sin\tau L)]\\
&= -\tfrac{1}{2}\int_0^L (f_2(\sigma) + i\tau^2 f_1(\sigma))(\cosh\tau(L-\sigma) + \cos\tau(L-\sigma))\,d\sigma\\
&+i\gamma(i\tau^2+\eta)^{\alpha-1}\tfrac{1}{2\tau}\int_0^L (f_2(\sigma) + i\tau^2 f_1(\sigma))(\sinh\tau(L-\sigma) - \sin\tau(L-\sigma))\,d\sigma\\
&-\gamma(i\tau^2+\eta)^{\alpha-1}f_1(L) + \zeta\int_{-\infty}^{+\infty}\tfrac{\mu(\xi)f_3(\xi)}{i\tau^2+\xi^2+\eta}\,d\xi.
\end{aligned} \tag{2.58}$$

Using (2.58) and (2.56), a linear system in A and B is obtained

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{pmatrix} \tag{2.59}$$

where

$$\begin{aligned}
m_{11} &= (\cosh\tau L + \cos\tau L),\\
m_{12} &= (\sinh\tau L + \sin\tau L),\\
m_{21} &= [\tau^3(\sinh\tau L - \sin\tau L) - i\tau^2\gamma(i\tau^2+\eta)^{\alpha-1}(\cosh\tau L - \cos\tau L)],\\
m_{22} &= [\tau^3(\cosh\tau L + \cos\tau L) - i\tau^2\gamma(i\tau^2+\eta)^{\alpha-1}(\sinh\tau L - \sin\tau L)].
\end{aligned}$$

$$\begin{aligned}
\tilde{C}_1 &= -\tfrac{1}{2\tau^3}\int_0^L (f_2(\sigma) + i\tau^2 f_1(\sigma))(\sinh\tau(L-\sigma) + \sin\tau(L-\sigma))\,d\sigma\\
\tilde{C}_2 &= -\tfrac{1}{2}\int_0^L (f_2(\sigma) + i\tau^2 f_1(\sigma))(\cosh\tau(L-\sigma) + \cos\tau(L-\sigma))\,d\sigma\\
&\quad +i\gamma(i\tau^2+\eta)^{\alpha-1}\tfrac{1}{2\tau}\int_0^L (f_2(\sigma) + i\tau^2 f_1(\sigma))(\sinh\tau(L-\sigma) - \sin\tau(L-\sigma))\,d\sigma\\
&\quad -\gamma(i\tau^2+\eta)^{\alpha-1}f_1(L) + \zeta\int_{-\infty}^{+\infty}\tfrac{\mu(\xi)f_3(\xi)}{i\lambda+\xi^2+\eta}\,d\xi
\end{aligned}$$

Let the determinant of the linear system given in (2.59) be denoted by D. Then the following is obtained:

$$\begin{aligned}
D &= m_{11}m_{22} - m_{12}m_{21}\\
&= \tau^3(\cosh\tau L + \cos\tau L)^2 - \tau^3(\sinh\tau L + \sin\tau L)(\sinh\tau L - \sin\tau L)\\
&+2i\gamma(i\tau^2+\eta)^{\alpha-1}[\cosh\tau L\sin\tau L - \sinh\tau L\cos\tau L]\\
&= 2\tau^3(1+\cosh\tau L\cos\tau L) + 2i\gamma\tau^2(i\tau^2+\eta)^{\alpha-1}[\cosh\tau L\sin\tau L - \sinh\tau L\cos\tau L]\\
&= 2\tau^3(1+\cosh\tau L\cos\tau L) + 2\gamma\tau^2(\tau^4+\eta^2)^{\frac{\alpha-1}{2}}\sin(1-\alpha)\theta\ [\cosh\tau L\sin\tau L - \sinh\tau L\cos\tau L]\\
&+2i\gamma\tau^2(\tau^4+\eta^2)^{\frac{\alpha-1}{2}}\sin(1-\alpha)\theta\ [\cosh\tau L\sin\tau L - \sinh\tau L\cos\tau L]
\end{aligned}$$

where $\theta \in]-\pi/2, \pi/2[$ such that

$$\begin{array}{l} \cos\theta = \frac{\eta}{\sqrt{\lambda^2+\eta^2}} \\ \sin\theta = \frac{\lambda}{\sqrt{\lambda^2+\eta^2}}. \end{array}$$

The roots of

$$[\cosh\varpi \sin\varpi - \sinh\varpi \cos\varpi] = 0$$

are of the form $\varpi_k = \delta_k + k\pi,\ \delta_k < \pi/4,\ k \in \mathbb{N}^*$. Hence

$$1 + \cosh\varpi_k \cos\varpi_k \neq 0 \quad \forall k \in \mathbb{N}^*.$$

Then

$$D \neq 0 \quad \forall \lambda \in \mathbb{R}^*.$$

Hence $i\lambda - \mathcal{A}$ is surjective for all $\lambda \in \mathbb{R}^*$. Now, if $\lambda = 0$ and $\eta \neq 0$, the system (2.50) is reduced to the following system

$$\begin{array}{l} u = -f_1, \\ \varphi_{xxxx} = f_2, \\ (\xi^2+\eta)\phi - u(L)\mu(\xi) = f_3, \end{array} \tag{2.60}$$

We deduce from $(2.60)_2$

$$\varphi_{xxx}(x) = \int_0^x f_2(s)\,ds + C.$$

$$\varphi_{xx}(x) = \int_0^x \int_0^s f_2(r)\,dr\,ds + Cx + C'.$$

$$\varphi_x(x) = \int_0^x \int_0^s \int_0^r f_2(z)\,dz\,dr\,ds + \frac{C}{2}x^2 + C'x + C''.$$

$$\varphi(x) = \int_0^x \int_0^s \int_0^r \int_0^z f_2(w)\,dw\,dz\,dr\,ds + \frac{C}{6}x^3 + \frac{C'}{2}x^2 + C''x + C'''.$$

As $\varphi(0) = \varphi_x(0) = 0$, we find $C'' = C''' = 0$.

From $(2.60)_1$ and $(2.60)_3$, we have

$$\begin{array}{rl} \varphi_{xxx}(L) = & \zeta \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2+\eta}\,d\xi u(L) + \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi) f_3(\xi)}{\xi^2+\eta}\,d\xi \\ = & -\gamma\eta^{\alpha-1} f_1(L) + \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi) f_3(\xi)}{\xi^2+\eta}\,d\xi. \end{array}$$

We find

$$C = -\int_0^L f_2(r)\,dr - \gamma\eta^{\alpha-1} f_1(L) - \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi) f_3(\xi)}{\xi^2+\eta}\,d\xi.$$

Because $\varphi_{xx}(L) = 0$, we find

$$C' = -CL - \int_0^L \int_0^s f_2(r)\, dr\, ds.$$

Hence $\mathcal{A}$ is surjective.

Lemma 2.5.4. *Let $\mathcal{A}$ be defined by (2.14). Then*

$$\mathcal{A}^* \begin{pmatrix} \varphi \\ u \\ \phi \end{pmatrix} = \begin{pmatrix} -u \\ \varphi_{xxxx} \\ -(\xi^2+\eta)\phi - u(L)\mu(\xi) \end{pmatrix} \tag{2.61}$$

with domain

$$D(\mathcal{A}^*) = \left\{ \begin{array}{l} (\varphi, u, \phi)^T \ \text{in}\ \mathcal{H} : \varphi \in H^4(0,L) \cap H^2_L(0,L), u \in H^2_L(0,L), \\ -(\xi^2+\eta)\phi - u(L)\mu(\xi) \in L^2(-\infty,+\infty), \\ \varphi_{xx}(L) = 0,\ \ \varphi_{xxx}(L) - \zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi = 0 \\ |\xi|\phi \in L^2(-\infty,+\infty) \end{array} \right\}. \tag{2.62}$$

Proof

Let $U = (\varphi, u, \phi)^T$ and $V = (\tilde{\varphi}, \tilde{u}, \tilde{\phi})^T$. We have $< \mathcal{A}U, V >_{\mathcal{H}} = < U, \mathcal{A}^* V >_{\mathcal{H}}$.

$$\begin{aligned} < \mathcal{A}U, V >_{\mathcal{H}} &= \int_0^L u_{xx}\tilde{\varphi}_{xx}\, dx - \int_0^L \tilde{u}\varphi_{xxxx}\, dx + \zeta \int_{-\infty}^{+\infty} [-(\xi^2+\eta)\phi + u(L)\mu(\xi)]\tilde{\phi}\, d\xi \\ &= \int_0^L u\tilde{\varphi}_{xxxx}\, dx - \int_0^L \tilde{u}_{xx}\varphi_{xx}\, dx - \tilde{\varphi}_{xxx}(L)u(L) + \tilde{\varphi}_{xx}(L)u_x(L) - \varphi_{xxx}(L)\tilde{u}(L) \\ &\quad -\zeta \int_{-\infty}^{+\infty} \phi[(\xi^2+\eta)\tilde{\phi}]\, d\xi + \zeta u(L) \int_{-\infty}^{+\infty} \mu(\xi)\tilde{\phi}\, d\xi. \end{aligned}$$

As $\varphi_{xxx}(L) = \zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi\, d\xi$ and if we set $\tilde{\varphi}_{xx}(L) = 0$ and $\tilde{\varphi}_{xxx}(L) = \zeta \int_{-\infty}^{+\infty} \mu(\xi)\tilde{\phi}\, d\xi$, we find

$$< \mathcal{A}U, V >_{\mathcal{H}} = \int_0^L u\tilde{\varphi}_{xxxx}\, dx - \int_0^L \tilde{u}_{xx}\varphi_{xx}\, dx - \zeta \int_{-\infty}^{+\infty} \phi[(\xi^2+\eta)\tilde{\phi} + \mu(\xi)\tilde{u}(L)]\, d\xi.$$

Theorem 2.5.1. *$\sigma_r(\mathcal{A}) = \emptyset$, where $\sigma_r(\mathcal{A})$ denotes the set of residual spectrum of $\mathcal{A}$.*

Proof

Since $\lambda \in \sigma_r(\mathcal{A}), \overline{\lambda} \in \sigma_p(\mathcal{A}^*)$ the proof will be accomplished if we can show that $\sigma_p(\mathcal{A}) = \sigma_p(\mathcal{A}^*)$. This is because obviously the eigenvalues of A are symmetric on the real axis. From

(2.61), the eigenvalue problem $\mathcal{A}^*Z = \lambda Z$ for $\lambda \in \mathbb{C}$ and $0 \neq Z = (\varphi, u, \phi, v) \in D(\mathcal{A}^*)$ we have

$$\begin{cases} \lambda\varphi + u = 0, \\ \lambda u - \varphi_{xxxx} = 0, \\ \lambda\phi + (\xi^2+\eta)\phi + u(L)\mu(\xi) = 0, \end{cases} \tag{2.63}$$

From $(2.63)_1$ and $(2.63)_2$, we find

$$\lambda^2\varphi + \varphi_{xxxx} = 0, \tag{2.64}$$

As $\varphi_{xxx}(L) = \zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\,d\xi$, we deduce from $(2.63)_3$ and $(2.63)_1$ that

$$\begin{aligned} \varphi_{xxx}(L) = \zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\,d\xi &= -\zeta u(L) \int_{-\infty}^{+\infty} \tfrac{\mu(\xi)^2}{\lambda+\xi^2+\eta}\,d\xi \\ = \gamma\lambda(\lambda+\eta)^{\alpha-1}\varphi(L). \end{aligned} \tag{2.65}$$

with the following conditions

$$\varphi(0) = 0,\ \varphi_x(0) = 0,\ \varphi_{xx}(L) = 0. \tag{2.66}$$

System (2.64)-(2.66) is the same as (2.28) and (2.33). Hence $\mathcal{A}^*$ has the same eigenvalues with $\mathcal{A}$. The proof is complete.

□

•**Case2** $\eta \neq 0$:

Theorem 2.5.2. *The semigroup $S_{\mathcal{A}}(t)_{t\geq 0}$ is polynomially stable and*

$$\|S_{\mathcal{A}}(t)U_0\|_{\mathcal{H}} \leq \frac{1}{t^{1/2(1-\alpha)}}\|U_0\|_{D(\mathcal{A})}.$$

Proof

We will need to study the resolvent equation $(i\lambda - \mathcal{A})U = F$, for $\lambda \in \mathbb{R}$, namely

$$\begin{cases} i\lambda\varphi - u = f_1, \\ i\lambda u + \varphi_{xxxx} = f_2, \\ i\lambda\phi + (\xi^2+\eta)\phi - u(L)\mu(\xi) = f_3, \end{cases} \tag{2.67}$$

where $F = (f_1, f_2, f_3)^T$. Taking inner product in $\mathcal{H}$ with U and using (2.17) we get

$$|Re\langle \mathcal{A}U, U\rangle| \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{2.68}$$

This implies that

$$\zeta \int_{-\infty}^{+\infty} (\xi^2+\eta)(\phi(\xi,t))^2\, d\xi \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{2.69}$$

and, applying $(2.67)_1$, we obtain

$$||\lambda||\varphi(L)| - |f_1(L)||^2 \leq |u(L)|^2.$$

We deduce that

$$|\lambda|^2|\varphi(L)|^2 \leq c|f_1(L)|^2 + c|u(L)|^2.$$

Moreover, from $(2.67)_4$, we have

$$\varphi_{xxx}(L) = \zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi.$$

Then

$$\begin{aligned} |\varphi_{xxx}(L)|^2 &\leq \zeta^2 \left|\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi\right|^2 \\ &\leq \zeta^2 \left(\int_{-\infty}^{+\infty} (\xi^2+\eta)^{-1}|\mu(\xi)|^2\, d\xi\right) \int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi)|^2\, d\xi \\ &\leq c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \end{aligned} \tag{2.70}$$

From $(2.67)_3$, we obtain

$$u(L)\mu(\xi) = (i\lambda + \xi^2 + \eta)\phi - f_3(\xi). \tag{2.71}$$

By multiplying $(2.71)_1$ by $(i\lambda+\xi^2+\eta)^{-1}\mu(\xi)$, we get

$$(i\lambda+\xi^2+\eta)^{-1}u(L)\mu^2(\xi) = \mu(\xi)\phi - (i\lambda+\xi^2+\eta)^{-1}\mu(\xi)f_3(\xi). \tag{2.72}$$

Hence, by taking absolute values of both sides of (2.72), integrating over the interval $]-\infty,+\infty[$ with respect to the variable ξ and applying Cauchy-Schwartz inequality, we obtain

$$S|u(L)| \leq U\left(\int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi|^2\, d\xi\right)^{\frac{1}{2}} + V\left(\int_{-\infty}^{+\infty} |f_3(\xi)|^2\, d\xi\right)^{\frac{1}{2}} \tag{2.73}$$

where

$$S = \int_{-\infty}^{+\infty} (|\lambda|+\xi^2+\eta)^{-1}|\mu(\xi)|^2\, d\xi$$

$$U = \left(\int_{-\infty}^{+\infty} (\xi^2+\eta)^{-1}|\mu(\xi)|^2\, d\xi\right)^{\frac{1}{2}}$$

$$V = \left(\int_{-\infty}^{+\infty} (|\lambda|+\xi^2+\eta)^{-2}|\mu(\xi)|^2\, d\xi\right)^{\frac{1}{2}}.$$

Thus, by using again the inequality $2PQ \leq P^2 + Q^2, P \geq 0, Q \geq 0$, we get

$$S^2|u(L)|^2 \leq 2U^2 \left(\int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi|^2 \, d\xi \right) + 2V^2 \left(\int_{-\infty}^{+\infty} |f_3(\xi)|^2 \, d\xi \right). \tag{2.74}$$

We deduce that

$$|u(L)|^2 \leq c|\lambda|^{2-2\alpha} \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + c\|F\|_{\mathcal{H}}^2. \tag{2.75}$$

Let us introduce the following notation

$$\mathcal{I}_\varphi(\alpha) = |u(\alpha)|^2 + |\varphi_{xx}(\alpha)|^2$$

$$\mathcal{E}_\varphi(L) = \int_0^L \mathcal{I}_\varphi(s) \, ds.$$

Lemma 2.5.5. *Let $q \in H^1(0, L)$. We have that*

$$\int_0^L q_x[|u(x)|^2 + 3|\varphi_{xx}(x)|^2] \, dx + 2\int_0^L q_{xx}\varphi_{xx}\overline{\varphi}_x \, dx \leq [q\mathcal{I}_\varphi]_0^L - 2L\varphi_{xxx}(L)\varphi_x(L) + R \tag{2.76}$$

where R satisfies

$$|R| \leq C\|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}}.$$

for a positive constant C.

Proof

To get (2.76), let us multiply the equation $(2.67)_2$ by $q\overline{\varphi}_x$ Integrating on $(0, L)$ we obtain

$$i\lambda \int_0^L uq\overline{\varphi}_x \, dx + \int_0^L \varphi_{xxxx} q\overline{\varphi}_x \, dx = \int_0^L f_2 q\overline{\varphi}_x \, dx$$

or

$$-\int_0^L uq(\overline{i\lambda\varphi_x}) \, dx + \int_0^L q\varphi_{xxxx}\overline{\varphi}_x \, dx = \int_0^L f_2 q\overline{\varphi}_x \, dx.$$

Since $i\lambda\varphi_x = u_x + f_{1x}$ taking the real part in the above equality results in

$$-\frac{1}{2}\int_0^L q\frac{d}{dx}|u|^2 \, dx - \frac{1}{2}\int_0^L q\frac{d}{dx}|\varphi_{xx}|^2 \, dx + [\varphi_{xxx}\varphi_x q]_0^L + \int_0^L q_{xx}\varphi_{xx}\overline{\varphi}_x \, dx + \int_0^L q_x|\varphi_{xx}|^2 \, dx$$
$$= Re\int_0^L f_2 q\overline{\varphi}_x \, dx + Re\int_0^L uq\overline{f}_{1x} \, dx.$$

Performing an integration by parts we get

$$\int_0^L q_x[|u(x)|^2 + 3|\varphi_{xx}(x)|^2] \, dx + 2\int_0^L q_{xx}\varphi_{xx}\overline{\varphi}_x \, dx = [q\mathcal{I}_\varphi]_0^L - 2q(L)\varphi_{xxx}(L)\varphi_x(L) + R$$

where

$$R = 2Re\int_0^L f_2 q\overline{\varphi}_x \, dx + 2Re\int_0^L uq\overline{f}_{1x} \, dx.$$

It is clear that

$$|R| \leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{2.77}$$

If we take $q(x) = x$ in Lemma 2.5.5 we arrive at

$$\mathcal{E}_\varphi(L) \leq L\mathcal{I}_\varphi(L) - 2L\varphi_{xxx}(L)\varphi_x(L) + R. \tag{2.78}$$

Using the continuous embeddings from $H^2(0,L)$ into $C^1([0,L])$ we deduce

$$|\varphi_x(L)| \leq C\|\varphi\|_{H^2(0,L)} \leq C'\|\varphi_{xx}\|_{L^2(0,L)} \leq C'\|U\|_{\mathcal{H}}.$$

Using inequalities (2.78) and (2.77) we conclude that there exists a positive constant C such that

$$\int_0^L \mathcal{I}_\varphi(s)\,ds \leq L\mathcal{I}_\varphi(L) + C(\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}})^{\frac{1}{2}}\|U\|_{\mathcal{H}} + C'\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{2.79}$$

Since that

$$\int_{-\infty}^{+\infty} (\phi(\xi))^2\,d\xi \leq C\int_{-\infty}^{+\infty} (\xi^2+\eta)(\phi(\xi))^2\,d\xi \leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

Substitution of inequalities (2.75) into (2.79) we get that

$$\|U\|^2_{\mathcal{H}} \leq C(|\lambda|^{2-2\alpha}+1)\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C'(\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}})^{\frac{1}{2}}\|U\|_{\mathcal{H}} + C''\|F\|^2_{\mathcal{H}}.$$

So we have

$$\|U\|_{\mathcal{H}} \leq C|\lambda|^{2-2\alpha}\|F\|_{\mathcal{H}}.$$

The conclusion then follows by applying the Theorem 2.4.2.

$\square$

Chapter 3

GLOBAL EXISTENCE AND ENERGY DECAY OF SOLUTIONS TO TIMOSHENKO BEAM SYSTEM WITH A DELAY TERM

3.1 Introduction

In this chapter we study the boundary stabilization of the Timoshenko systen in unbounded interval $(0, +\infty)$. The system is given by the two coupled hyperbolic equations.

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) = 0 & \text{in }]0,1[\times]0,+\infty[, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x + \psi)(x,t) + \mu_1(t)\psi_t(x,t) & \\ \qquad\qquad +\mu_2(t)\psi_t(x,t-\tau) = 0 & \text{in }]0,1[\times]0,+\infty[. \end{cases} \tag{3.1}$$

where t denotes the time variable and x is the space variable along the beam of lenght 1 in its equilibrium configuration the unknowns $\varphi = \varphi(x,t)$ and $\psi = \psi(x,t)$ represent respectively, the transverse displacement of the beam and the rotation angle of the filament of the beam. In (3.7) $\rho_1 = \rho, \rho_2 = I_\rho, b = EI$, where ρ, I_ρ, E, I and K are, respectively, the density (the mass per unit length), the polar moment of inertia of the cross-section, Young's modulus of elasticity, the moment of inertia of a cross-section and the shear modulus.

System (3.1) is subjected to the following boundary condition:

$$\begin{cases} \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0 \quad t \geq 0. \end{cases} \tag{3.2}$$

where $t \in (0, +\infty)$ and parameters a, k are positive constants.
Also we consider the following initial conditions:

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \ \varphi_t(x,0) = \varphi_1(x) \\ \psi(x,0) = \psi_0(x), \ \psi_t(x,0) = \psi_1(x), \ x \in (0,1) \\ \psi_t(x, t-\tau) = f_0(x, t-\tau), \ x \in (0,1), \ t \in (0,\tau). \end{cases} \tag{3.3}$$

Where $\tau > 0$ is the time delay. The initial data $(\varphi_0, \psi_0, \varphi_1, \psi_1, f_0)$ belongs to suitable functional space. Delay effects arise in many applications and practical problems and it is well-known that an arbitrarily smal delay may destabilize a system wich is uniformly asymptotically stable in the absence of delay see R. Datko(1991).
In the absence of the delay in system (3.1), that is for $\tau = 0$, a large amount of literature is available on this model, addressing problems of the existence, uniqueness and asymptotic behaviour in time when some damping effects are considered, such as: fractional damping, viscoelastic damping and thermal dissipation.
Namely Soufiane (1999) showed the exponential stability of the uniform Timoshenko beam by using one distributed feedback. Shi and Feng (2001) considered the case of the uniform Timoshenko beam under two locally distributed feedback and proved an exponential stability result, other wise, only the asymptotic stability has been proved. Xu and Yung (2003) proved an exponential stability of the uniform Timoshenko beam by two pointwise control.
Concerning the Timoshenko system with memory, we refer to Alves et al (2011), Amar-khodja et al (2003), Muñoz Revera and Fernandez Sare (2008) and referenes there in.
In the presence of a delay term in (3.1), a few works are available, Said- Houari and Laskri (2010) have considered the following Timoshenko system with a delay term in the internal feedback:

$$\begin{cases} \rho_1 \varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) = 0 & \text{in } (0,L) \times (0,+\infty), \\ \rho_2 \psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x + \psi)(x,t) + \mu_1 \psi_t(x,t) & \\ \qquad\qquad + \mu_2 \psi_t(x, t-\tau) = 0 & \text{in } (0,L) \times (0,+\infty). \end{cases} \tag{3.4}$$

Under the assumption $\mu_1 \geq \mu_2$ on the weights of the two feedbacks, they proved the well-posedness of the system. They also established an exponential decay result for the case equal-speed wave propagation.

Subsequently, the work in Said-Houari and Laskri (2010) has been extended to the case of time varying delay of the form $\psi_t(x, t-\tau(t))$ by Kirane et al.(2011), by using the variable norm technique of Kato and under some restriction on the parameters μ_1, μ_2 and on the delay function $\tau(t)$,the hypothesis between the weight of the delay term in the feedback, an exponential decay result of the total energy has been proved.
Ammari et al (2010) have treated the N-dimentional wave equation

$$\begin{cases} u_{tt} - \Delta u(x,t) + a u_t(x, t-\tau) = 0 & \text{x } \in \Omega, t > 0 \\ u(x,0) = 0 & \text{x } \in \Gamma_0, t > 0, \\ \frac{\partial u}{\partial \nu}(x,t) = -ku(x,t) & \text{x } \in \Gamma_1, t > 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), u_t(x,t) = g(x,t) & \text{x } \in \Omega, t \in (-\tau, 0). \end{cases} \tag{3.5}$$

Where Ω is an open bounded domain of $\mathbb{R}^N$,$N \geq 2$ with boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$,$\Gamma_0 \cap \Gamma_1 = \emptyset$. Under the usual geometric condition on the domain Ω, they showed an exponential stability result, provided that the delay coefficient a is sufficiently small.
When both the damping and the delay in (3.5) are acting in the boundary that is if $(3.5)_3$ remplaced by

$$\frac{\partial u}{\partial \nu}(x,t) = -ku(x,t) - a u_t(x, t-\tau), x \in \Gamma_1, t > 0, \tag{3.6}$$

Nicaise and Pignotti (2006) investigated this problem and showed an exponential decay rate of the total energy under the assumption

$$a < k$$

on the contrary if (3.6) does not hold, they found a sequence of delays for which the corresponding solution of (3.5) will be unstable. The analysis in Nicaise and Pignotti (2006) is based on a observability inequality obtained with a Carleman estimate. The result presented here extends the one in Ammari et al(2010) to the Timoshenko system. Our purpose in this paper is to give a global solvability in Sobolev spaces and energy decay estimates of the solutions to the problem 3.1 for a nonlinear damping and a delay term. We should mention here that, to the best of our knowledge, there is no result concerning Timochenko beam system with the presence of nonlinear degenerate delay term. To obtain global solutions to the problem 3.1, we use the argument combining the Galerkin approximation scheme (see **[29]**) with the energy estimate method. The technic based on the theory of nonlinear semigroups used in **[39]** does not seem to be applicable in the nonlinear case. To prove decay estimates, we use a perturbed energy method and some properties of convex functions. These arguments of convexity were introduced and developed by **[16]** and **[27]** and used by Liu and Zuazua **[30]** and Alabau-Boussouira **[4]**.

3.2 Preliminaries and main results

First assume the following hypotheses: **(H1)** $\mu_1 : \mathbb{R}+ \to]0, +\infty[$ is a non-increasing function of the class $C^1(\mathbb{R}+)$ satisfying

$$\left|\frac{\mu_1'(t)}{\mu_1(t)}\right| \leq M \tag{3.7}$$

(H2) $\mu_2 : \mathbb{R}+ \to \mathbb{R}$ is a function of class $C^1(\mathbb{R}+)$ wich is not necessarily positive or monotone, such that

$$|\mu_2(t)| \leq \beta\mu_1(t) \tag{3.8}$$

$$|\mu_2'(t)| \leq M'\mu_1(t) \tag{3.9}$$

We first state some Lemmas which will be needed later.

Lemma 3.2.1 (Sobolev-Poincaré's inequality)**.** *Let q be a number with $2 \leq q < +\infty$ $(n = 1, 2)$ or $2 \leq q \leq 2n/(n-2)$ $(n \geq 3)$. Then there is a constant $c_* = c_*((0,1), q)$ such that*

$$\|\psi\|_q \leq c_*\|\nabla\psi\|_2 \quad \textit{for} \quad \psi \in H_0^1((0,1)).$$

We introduce as in **[39]** the new variable

$$z(x, \rho, t) = \psi_t(x, t - \tau\rho), x \in (0,1), \ \rho \in (0,1), \quad t > 0. \tag{3.10}$$

Then, we have

$$\tau z'(x, \rho, t) + z_\rho(x, \rho, t) = 0, \ \text{ in } (0,1) \times (0,1) \times (0, +\infty). \tag{3.11}$$

Therefore, problem (P) is equivalent to:

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) = 0 & \text{in }]0,1[\times]0,+\infty[, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x+\psi)(x,t) + \mu_1(t)\psi_t(x,t) & \\ \qquad\qquad +\mu_2(t)z(x,1,t) = 0 & \text{in }]0,1[\times]0,+\infty[, \\ \tau z'(x,\rho,t) + z_\rho(x,\rho,t) = 0 & \text{in }]0,1[\times]0,1[\times]0,+\infty[, \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0 & t \geq 0, \\ z(x,0,t) = \psi_t(x,t) & \text{on }]0,1[\times[0,+\infty[, \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x) & x \in]0,1[, \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x) & x \in]0,1[, \\ z(x,\rho,0) = f_0(x,-\rho\tau) & \text{in }]0,1[\times]0,1[\end{cases} \tag{3.12}$$

Let ξ_1 be a positive constant such that

$$\tau\beta < \xi_1 < \tau(2-\beta). \tag{3.13}$$

We define the energy associated to the solution of the problem (3.12) by the following formula:

$$\begin{array}{c} E(t) = E(t,z,\varphi,\psi) = \frac{1}{2}\int_0^1 \left\{\rho_1\varphi_t^2 + \rho_2\psi_t^2 + K|\varphi_x+\psi|^2 + b\psi_x^2\right\} dx \\ +\frac{\xi(t)}{2}\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho\,dx. \end{array} \tag{3.14}$$

We have the following theorem.

Theorem 3.2.1. *Let* $(\varphi_0,\varphi_1),(\psi_0,\psi_1) \in (H^2(0,1)\cap H_0^1(0,1))\times H_0^1(0,1), f_0 \in H_0^1((0,1);H^1(0,1))$ *satisfy thecompatibility condition*

$$f_0(.,0) = \psi_1.$$

Assume that the hypothesis $(H1)$ holds. Then the problem (P) admits a unique weak solution

$$\begin{array}{l} \psi,\varphi \in L^\infty_{loc}((-\tau,\infty);H^2(0,1)\cap H_0^1(0,1)), \quad \psi_t,\varphi_t \in L^\infty_{loc}((-\tau,\infty);H_0^1(0,1)), \\ \psi_{tt},\varphi_{tt} \in L^\infty_{loc}((-\tau,\infty);L^2(0,1)) \end{array} \tag{3.15}$$

and, for some constants ω_1,ω_2 and ω_3,ϵ_0 we obtain the following decay property:

$$E(t) \leq \omega_1 e^{-\omega_2 t}, \quad \forall t > 0, \tag{3.16}$$

We finish this section by giving an explicit upper bound for the derivative of the energy.

Lemma 3.2.2. *Let (φ, ψ, z) be a solution of the problem (3.12). Then, the energy functional defined by (3.14) satisfies*

$$\begin{aligned} E'(t) &\leq -\left(\mu_1(t) - \tfrac{\xi(t)}{2\tau} - \tfrac{|\mu_2(t)|}{2}\right) \textstyle\int_0^1 \psi_t^2\,dx - \left(\tfrac{\xi(t)}{2\tau} - \tfrac{|\mu_2(t)|}{2}\right) \int_0^1 z^2(x,1,t)\,dx \\ &\leq 0 \end{aligned} \tag{3.17}$$

Proof. Multiplying the first equation in (3.12) by φ_t, the second equation by ψ_t, integrating over $(0,1)$ and using integration by parts, we get

$$\begin{aligned} \tfrac{1}{2}\tfrac{d}{dt}\left(\textstyle\int_0^1 \{\rho_1\varphi_t^2 + \rho_2\psi_t^2 + K|\varphi_x+\psi|^2 + b\psi_x^2\}\,dx\right) = \\ -\mu_1(t)\textstyle\int_0^1 \psi_t^2(x,)\,dx - \mu_2(t)\int_0^1 \psi_t(x,t)z(x,1,t)\,dx. \end{aligned} \tag{3.18}$$

We multiply the third equation in (3.12) by $\xi(t)z(x,\rho,t)$ and integrate the result over $(0,1)\times(0,1)$, to obtain:

$$\xi(t)\tau \textstyle\int_0^1\int_0^1 z_t(x,\rho,t)z(x,\rho,t)\,d\rho\,dx + \xi(t)\int_0^1\int_0^1 z_\rho(x,\rho,t)z(x,\rho,t)\,d\rho\,dx = 0 \tag{3.19}$$

This yields

$$\frac{\xi(t)\tau}{2}\frac{d}{dt}\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho\,dx + \frac{\xi(t)}{2}\int_0^1\int_0^1 \frac{d}{d\rho}z^2(x,\rho,t)\,d\rho\,dx \tag{3.20}$$

Which gives

$$\begin{aligned} \tfrac{\tau}{2}\left[\tfrac{d}{dt}\left(\xi(t)\textstyle\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho\,dx\right) - \xi'(t)\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho\,dx\right] \\ +\tfrac{\xi(t)}{2}\textstyle\int_0^1 z^2(x,\rho,t)\,d\rho\,dx - \tfrac{\xi(t)}{2}\int_0^1 \psi_t^2(x,t)dx = 0. \end{aligned} \tag{3.21}$$

Consequently,

$$\begin{aligned} \tfrac{\tau}{2}\tfrac{d}{dt}\left(\xi(t)\textstyle\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho\,dx\right) = \tfrac{\xi'(t)}{2}\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho\,dx - \tfrac{\xi(t)}{2}\int_0^1 z^2(x,\rho,t)\,d\rho\,dx \\ +\tfrac{\xi(t)}{2}\textstyle\int_0^1 \psi_t^2(x,t)dx = 0. \end{aligned} \tag{3.22}$$

Combination of (12) and (16) leads to

$$\begin{aligned} \tfrac{1}{2}\tfrac{d}{dt}\left(\textstyle\int_0^1 \left\{\rho_1\varphi_t^2 + \rho_2\psi_t^2 + K|\varphi_x+\psi|^2 + b\psi_x^2 + \xi(t)\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho\,dx\right\}dx\right) \\ = -\mu_1(t)\textstyle\int_0^1 \psi_t^2(x,t)\,dx - \mu_2(t)\int_0^1 \psi_t(x,t)z(x,1,t)\,dx \\ +\tfrac{\xi'(t)}{2}\textstyle\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho\,dx - \tfrac{\xi(t)}{2\tau}\int_0^1 z^2(x,\rho,t)\,d\rho\,dx + \tfrac{\xi(t)}{2\tau}\int_0^1 \psi_t^2(x,t)dx. \end{aligned} \tag{3.23}$$

Recalling the definition of $E(t)$ in (8) we arrive at

$$E'(t) = -\mu_1(t)\int_0^1 \psi_t^2(x,t)\,dx - \mu_2(t)\int_0^1 \psi_t(x,t)z(x,1,t)\,dx \\ +\tfrac{\xi'(t)}{2}\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho\,dx - \tfrac{\xi(t)}{2\tau}\int_0^1 z^2(x,\rho,t)\,d\rho\,dx + \tfrac{\xi(t)}{2\tau}\int_0^1 \psi_t^2(x,t)dx. \tag{3.24}$$

$$E'(t) \le -\left(\mu_1(t) - \tfrac{\xi(t)}{2\tau}\right)\int_0^1 \psi_t^2(x,t)\,dx \\ -\mu_2(t)\int_0^1 \psi_t(x,t)z(x,1,t)\,dx - \tfrac{\xi(t)}{2\tau}\int_0^1 z^2(x,\rho,t)\,dx. \tag{3.25}$$

Due to Young's inequality, we have

$$\int_0^1 \psi_t(x,t)z(x,1,t)\,dx \le \frac{1}{2}||\psi_t(x,t)||_2^2 + \frac{1}{2}||z(x,1,t)||_2^2. \tag{3.26}$$

Inserting (20) in (19), we obtain

$$E'(t) \le -\left(\mu_1(t) - \frac{\xi(t)}{2\tau} - \frac{|\mu_2(t)|}{2}\right)\int_0^1 \psi_t^2(x,t)\,dx - \left(\frac{\xi(t)}{2\tau} - \frac{|\mu_2(t)|}{2}\right)\int_0^1 z^2(x,\rho,t)\,dx. \tag{3.27}$$

$$\le -\mu_1(t)\left(1 - \frac{\xi_1(t)}{2\tau} - \frac{\beta}{2}\right)\int_0^1 \psi_t^2(x,t)\,dx - \mu_1(t)\left(\frac{\xi_1(t)}{2\tau} - \frac{\beta}{2}\right)\int_0^1 z^2(x,\rho,t)\,dx \le 0. \tag{3.28}$$

This completes the proof of lemma.

3.3 Global Existence

We are now ready to prove Theorem 3.2.1 in the next two sections. Throughout this section we assume $\varphi_0, \psi_0 \in H^2 \cap H_0^1(0,1), \varphi_1, \psi_1 \in H_0^1(0,1)$ and $f_0 \in H_0^1((0,1); H^1(0,1))$.

We employ the Galerkin method to construct a global solution. Let $T > 0$ be fixed and denote by V_k the space generated by $\{w_1, w_2, \ldots, w_k\}$ where the set $\{w_k, k \in \mathbb{N}\}$ is a basis of $H^2 \cap H_0^1$.

Now, we define for $1 \le j \le k$ the sequence $\phi_j(x,\rho)$ as follows:

$$\phi_j(x,0) = w_j.$$

Then, we may extend $\phi_j(x,0)$ by $\phi_j(x,\rho)$ over $L^2((0,1)\times(0,1))$ and denote Z_k the space generated by $\{\phi_1, \phi_2, \ldots, \phi_k\}$.

We construct approximate solutions $(\varphi_k, \psi_k, z_k), k = 1, 2, 3, \ldots,$ in the form

$$\varphi_k(t) = \sum_{j=1}^k g_{jk} w_j, \quad \psi_k(t) = \sum_{j=1}^k \tilde{g}_{jk} w_j, \quad z_k(t) = \sum_{j=1}^k h_{jk}\phi_j,$$

where $g_{jk}, \tilde{g}_{jk}$ and $h_{jk}, j = 1, 2, \ldots, k$, are determined by the following ordinary differential equations:

$$\rho_1(\varphi_k''(t), w_j) + K(\varphi_{kx}(t), w_{jx}) - K(\psi_{kx}(t), w_j) = 0, \quad 1 \leq j \leq k, \tag{3.29}$$

$$\varphi_k(0) = \varphi_{0k} = \sum_{j=1}^{k} (\varphi_0, w_j) w_j \to \varphi_0 \text{ in } H^2 \cap H_0^1 \text{ as } k \to +\infty, \tag{3.30}$$

$$\varphi_k'(0) = \varphi_{1k} = \sum_{j=1}^{k} (\varphi_1, w_j) w_j \to \varphi_1 \text{ in } H_0^1 \text{ as } k \to +\infty. \tag{3.31}$$

$$\begin{cases} \rho_2(\psi_k''(t), w_j) + b(\psi_{kx}(t), w_{jx}) + K((\varphi_{kx} + \psi)(t), w_j) + \mu_1(t)(\psi_k', w_j) \\ +\mu_2(t)(z_k(.,1), w_j) = 0 \quad 1 \leq j \leq k, \\ z_k(x, 0, t) = \psi_k'(x, t) \end{cases} \tag{3.32}$$

$$\psi_k(0) = \psi_{0k} = \sum_{j=1}^{k} (\psi_0, w_j) w_j \to \psi_0 \text{ in } H^2 \cap H_0^1 \text{ as } k \to +\infty, \tag{3.33}$$

$$\psi_k'(0) = \psi_{1k} = \sum_{j=1}^{k} (\psi_1, w_j) w_j \to \psi_1 \text{ in } H_0^1 \text{ as } k \to +\infty. \tag{3.34}$$

and

$$(\tau z_{kt} + z_{k\rho}, \phi_j) = 0, \quad 1 \leq j \leq k, \tag{3.35}$$

$$z_k(\rho, 0) = z_{0k} = \sum_{j=1}^{k} (f_0, \phi_j) \phi_j \to f_0 \text{ in } H_0^1((0,1); H^1(0,1)) \text{ as } k \to +\infty. \tag{3.36}$$

By virtue of the theory of ordinary differential equations, the system (3.29)-(3.36) has a unique local solution which is extended to a maximal interval $[0, T_k[$ (with $0 < T_k \leq +\infty$) by Zorn lemma since the nonlinear terms in (3.32) are locally Lipschitz continuous. Note that $(\varphi_k(t), \psi_k(t))$ is from the class C^2.

In the next step we obtain a priori estimates for the solution, such that it can be extended outside $[0, T_k[$ to obtain one solution defined for all $t > 0$.

We can utilize a standard compactness argument for the limiting procedure and it suffices to derive some a priori estimates for (φ_k, ψ_k, z_k).

The first estimate. Since the sequences $\varphi_{0k}, \varphi_{1k}, \psi_{0k}, \psi_{1k}$ and z_{0k} converge, then standard calculations, using (3.29)-(3.36), similar to those used to derive (3.17), yield C independent of k such that

$$\begin{aligned} &E_k(t) + \int_0^t \int_0^1 a_1(s)(\psi_k')^2 dx\ ds \\ &+ \int_0^t \int_0^1 a_2(t) z_k^2(x,1,t) dx\ ds \le E_k(0) \le C, \end{aligned} \tag{3.37}$$

where

$$\begin{aligned} E_k(t) = \tfrac{1}{2}\int_0^1 \{\rho_1 {\varphi_k'}^2 + \rho_2 {\psi_k'}^2 + K|\varphi_{kx} + \psi_k|^2 + b\psi_{kx}^2\} dx \\ + \tfrac{\xi(t)}{2} \int_0^1 \int_0^1 z_k^2(x,\rho,t)\, d\rho\, dx. \end{aligned} \tag{3.38}$$

$$a_1(t) = \mu_1(t)\left(1 - \frac{\xi_1}{2\tau} - \frac{\beta}{2}\right) \text{ and } a_2(t) = \mu_1(t)\left(\frac{\xi_1}{2\tau} - \frac{\beta}{2}\right).$$

for some C independent of k. These estimates imply that the solution (φ_k, ψ_k, z_k) exists globally in $[0, +\infty[$.
Estimate (3.37) yields

$$\varphi_k, \psi_k \text{ are bounded in } L^\infty_{loc}(0,\infty; H_0^1(0,1)) \tag{3.39}$$

$$\varphi_k', \psi_k' \text{ are bounded in } L^\infty_{loc}(0,\infty; L^2(0,1)) \tag{3.40}$$

$$\mu_1(t)(\psi_k')^2(t) \text{ is bounded in } L^1((0,1)\times(0,T)) \tag{3.41}$$

$$\mu_1(t) z_k^2(x,\rho,t) \text{ is bounded in } L^\infty_{loc}(0,\infty; L^1((0,1)\times(0,1))) \tag{3.42}$$

$$\mu_1(t) z_k^2(x,1,t) \text{ is bounded in } L^1((0,1)\times(0,T)) \tag{3.43}$$

The second estimate. First, we estimate $\varphi_k''(0)$ and $\psi_k''(0)$. Testing (3.29) by $g_{jk}''(t)$, (3.32) by $\tilde{g}_{jk}''(t)$ and choosing $t = 0$ we obtain

$$\rho_1 \|\varphi_k''(0)\|_2 \le K(\|\varphi_{0kxx}\|_2 + \|\psi_{0kx}\|_2)$$

and

$$\rho_2 \|\psi_k''(0)\|_2 \le b\|\psi_{0kxx}\|_2 + K(\|\varphi_{0kx}\|_2 + \|\psi_{0k}\|_2) + \mu_1(0)\|g_1(\psi_{1k})\|_2 + \mu_2(0)\|g_2(z_{0k})\|_2.$$

Hence from (3.30), (3.31) and (3.36):

$$\|\varphi_k''(0)\|_2 \le C.$$

Since $g_1(\psi_{1k}), g_2(z_{0k})$ are bounded in $L^2(0,1)$ by (**H1**), (3.30), (3.33), (3.34) and (3.36) yield

$$\|\psi_k''(0)\|_2 \le C.$$

Differentiating (3.29) and (3.32) with respect to t, we get

$$(\rho_1 \varphi_k'''(t) - K\varphi_{kxx}'(t) - K\psi_{kx}'(t), w_j) = 0 \tag{3.44}$$

and

$$\begin{aligned}&(\rho_2\psi'''_k(t) - b\psi'_{kxx}(t) + K\varphi'_{kx}(t) + K\psi'_k(t) + \mu_1(t)\psi''{}_k(t) + \mu'_1(t)\psi'{}_k(t)\\ &+\mu_2(t)z'_k(x,1,t) + \mu'_2(t)z_k(x,1,t), w_j) = 0.\end{aligned} \tag{3.45}$$

Multiplying (3.44) by $g''_{jk}(t)$ and (3.45) by $\tilde{g}''_{jk}(t)$, summing over j from 1 to k, it follows that

$$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi''_k(t)\|_2^2\right) - K\int_0^1 (\varphi'_{kx} + \psi'_k)_x\varphi''_k\,dx = 0 \tag{3.46}$$

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\left(\rho_2\|\psi''_k(t)\|_2^2 + b\|\psi'_{kx}(t)\|_2^2\right) + K\textstyle\int_0^1(\varphi'_{kx} + \psi'_k)\psi''_k\,dx + \mu_1(t)\int_0^1 \psi''^2_k(t)\,dx\\ &\quad +\mu'_1(t)\textstyle\int_0^1 \psi''{}_k(t)\psi'{}_k(t)\,dx + \mu_2(t)\int_0^1 \psi''{}_k(t)z'_k(x,1,t)\,dx + \mu'_2(t)\int_0^1 \psi''{}_k(t)z_k(x,1,t)\,dx = 0.\end{aligned} \tag{3.47}$$

Differentiating (3.35) with respect to t, we get

$$(\tau z''_k(t) + \frac{\partial}{\partial\rho}z'_k, \phi_j) = 0.$$

Multiplying by $h'_{jk}(t)$, summing over j from 1 to k, it follows that

$$\frac{1}{2}\tau\frac{d}{dt}\|z'_k(t)\|_2^2 + \frac{1}{2}\frac{d}{d\rho}\|z'_k(t)\|_2^2 = 0. \tag{3.48}$$

Taking the sum of (3.46), (3.47) and (3.48), we obtain

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\Big(\rho_1\|\varphi''_k(t)\|_2^2 + \rho_2\|\psi''_k(t)\|_2^2 + b\|\psi'_{kx}(t)\|_2^2 + K\|\varphi'_{kx}(t) + \psi'_k\|_2^2 + \tau\|z'_k(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}\Big)\\ &\qquad +\mu_1(t)\textstyle\int_0^1 \psi''^2_k(t)\,dx + \frac{1}{2}\int_0^1 |z'_k(x,1,t)|^2\,dx\\ &= -\mu_2(t)\textstyle\int_0^1 \psi''{}_k(t)z'_k(x,1,t)\,dx - \mu'_1(t)\int_0^1 \psi''{}_k(t)\psi'{}_k(t)\,dx - \mu'_2(t)\int_0^1 \psi''{}_k(t)z_k(x,1,t)\,dx + \frac{1}{2}\|\psi''_k(t)\|_2^2.\end{aligned}$$

Using **(H1)**, **(H2)**, Cauchy-Schwarz and Young's inequalities, we obtain

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\Big(\rho_1\|\varphi''_k(t)\|_2^2 + \rho_2\|\psi''_k(t)\|_2^2 + b\|\psi'_{kx}(t)\|_2^2 + K\|\varphi'_{kx}(t) + \psi'_k\|_2^2 + \tau\|z'_k(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}\Big)\\ &\qquad +\mu_1(t)\textstyle\int_0^1 \psi''^2_k(t)\,dx + \frac{1}{2}\int_0^1 |z'_k(x,1,t)|^2\,dx\\ &\leq |\mu_2(t)|\|\psi''_k(t)\|_2\|z'_k(x,1,t)\|_2 + |\mu'_1(t)|\|\psi''_k(t)\|_2\|\psi'_k(t)\|_2 + |\mu'_2(t)|\|\psi''_k(t)\|_2\|z_k(x,1,t)\|_2 + \tfrac{1}{2}\|\psi''_k(t)\|_2^2.\\ &\leq \tfrac{|\mu_2(t)|^2}{2}\|\psi''_k(t)\|_2^2 + \tfrac{1}{2}\|z'_k(x,1,t)\|_2^2 + \tfrac{|\mu'_1(t)|}{4}\|\psi''_k(t)\|_2^2 + |\mu'_1(t)|\|\psi'_k(t)\|_2^2 + \tfrac{|\mu'_2(t)|}{2}\|\psi''_k(t)\|_2^2\\ &+|\mu'_2(t)|\|z_k(x,1,t)\|_2^2 + \tfrac{1}{2}\|\psi''_k(t)\|_2^2.\\ &\leq c'\|\psi''_k(t)\|_2^2 + |\mu'_1(t)|\|\psi'_k(t)\|_2^2 + \tfrac{1}{2}\|z'_k(x,1,t)\|_2^2 + |\mu'_2(t)|\|z_k(x,1,t)\|_2^2.\\ &\leq c'\|\psi''_k(t)\|_2^2 + M\mu_1(t)\|\psi'_k(t)\|_2^2 + M_1\mu_1(t)\|z_k(x,1,t)\|_2^2 + \tfrac{1}{2}\|z'_k(x,1,t)\|_2^2.\end{aligned}$$

Integrating the last inequality over $(0,t)$, we obtain

$$\begin{aligned}&\rho_1\|\varphi''_k(t)\|_2^2 + \rho_2\|\psi''_k(t)\|_2^2 + b\|\psi'_{kx}(t)\|_2^2 + K\|\varphi'_{kx}(t) + \psi'_k\|_2^2 + \tau\|z'_k(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}\\ &\leq \Big(\rho_1\|\varphi''_k(0)\|_2^2 + \rho_2\|\psi''_k(0)\|_2^2 + b\|\psi'_{kx}(0)\|_2^2 + K\|\varphi'_{kx}(0) + \psi'_k(0)\|_2^2 + \tau\|z'_k(x,\rho,0)\|^2_{L^2((0,1)\times(0,1))}\Big)\\ &+2M\textstyle\int_0^t \mu_1(s)\|\psi'_k(s)\|_2^2\,ds + 2M_1\int_0^t \mu_1(s)\|z_k(x,1,t)\|_2^2\,ds + \int_0^t \|\psi''_k(s)\|_2^2\,ds\end{aligned}$$

Using the Gronwall's lemma, we deduce that

$$\begin{aligned}&\rho_1\|\varphi_k''(t)\|_2^2+\rho_2\|\psi_k''(t)\|_2^2+b\|\psi_{kx}'(t)\|_2^2+K\|\varphi_{kx}'(t)+\psi_k'\|_2^2+\tau\|z_k'(x,\rho,t)\|_{L^2((0,1)\times(0,1))}^2\\&\leq Ce^{c't}\end{aligned}$$

for all $t\in\mathbb{R}_+$, therefore, we conclude that

$$\varphi_k'',\psi_k'' \text{ is bounded in } L^\infty_{loc}(0,\infty;L^2) \tag{3.49}$$

$$\varphi_k',\psi_k' \text{ is bounded in } L^\infty_{loc}(0,\infty;H_0^1) \tag{3.50}$$

$$z_k' \text{ is bounded in } L^\infty_{loc}(0,\infty;L^2((0,1)\times(0,1))) \tag{3.51}$$

The third estimate. Replacing w_j by $-w_{jxx}$ in (3.29) and (3.32), multiplying the result by $g_{jk}'(t)$ and $\tilde{g}_{jk}'(t)$, summing over j from 1 to k, it follows that

$$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi_{kx}'(t)\|_2^2+\right)+K\int_0^1(\varphi_x+\psi)_x\varphi_{kxx}'\,dx=0. \tag{3.52}$$

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}\left(\rho_2\|\psi_{kx}'(t)\|_2^2+b\|\psi_{kxx}(t)\|_2^2\right)-K\textstyle\int_0^1(\varphi_x+\psi)\psi_{kxx}'\,dx+\mu_1(t)\int_0^1|\psi_{kx}'(t)|^2\,dx\\+\mu_2(t)\textstyle\int_0^1\psi'{}_{kx}(t)z_{kx}(x,1,t)\,dx=0.\end{aligned} \tag{3.53}$$

Replacing ϕ_j by $-\phi_{jxx}$ in (3.35), multiplying the resulting equation by $h_{jk}(t)$, summing over j from 1 to k, it follows that

$$\frac{1}{2}\tau\frac{d}{dt}\|z_{kx}(t)\|_2^2+\frac{1}{2}\frac{d}{d\rho}\|z_{kx}(t)\|_2^2=0. \tag{3.54}$$

From (3.52), (3.53) and (3.54), we have

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\Big(\rho_1\|\varphi_{kx}'(t)\|_2^2+\rho_2\|\psi_{kx}'(t)\|_2^2+K\|\varphi_{kxx}+\psi_{kx}(t)\|_2^2+b\|\psi_{kxx}(t)\|_2^2+\tau\|z_{kx}(x,\rho,t)\|_{L^2(0,1)\times(0,1))}^2\Big)\\&+\mu_1(t)\textstyle\int_0^1|\psi_{kx}'(t)|^2\,dx+\frac{1}{2}\int_0^1|z_{kx}(x,1,t)|^2\,dx=-\mu_2(t)\int_0^1\psi_{kx}'(t)z_{kx}(x,1,t)\,dx\\&+\tfrac{1}{2}\|\nabla\psi_k'(t)\|_2^2.\end{aligned}$$

Using **(H2)**, Cauchy-Schwartz and Young's inequalities, we obtain

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\Big(\rho_1\|\varphi_{kx}'(t)\|_2^2+\rho_2\|\psi_{kx}'(t)\|_2^2+K\|\varphi_{kxx}+\psi_{kx}(t)\|_2^2+b\|\psi_{kx}(t)\|_2^2+\tau\|z_{kx}(x,\rho,t)\|_{L^2((0,1)\times(0,1))}^2\Big)\\&+\mu_1(t)\textstyle\int_0^1|\psi_{kx}'(t)|^2\,dx+c\int_0^1|z_{kx}(x,1,t)|^2\,dx\leq c'\|\psi_{kx}'(t)\|_2^2.\end{aligned}$$

Integrating the last inequality over $(0,t)$ and using Gronwall's Lemma, we have

$$\begin{aligned}&\rho_1\|\varphi_{kx}'(t)\|_2^2+\rho_2\|\psi_{kx}'(t)\|_2^2+K\|\varphi_{kxx}+\psi_{kx}(t)\|_2^2+b\|\psi_{kxx}(t)\|_2^2+\tau\|z_{kx}(x,\rho,t)\|_{L^2((0,1)\times(0,1))}^2\leq\\&e^{cT}\Big(\rho_1\|\varphi_{kx}'(0)\|_2^2+\rho_2\|\psi_{kx}'(0)\|_2^2+K\|\varphi_{kxx}(0)+\psi_{kx}(0)\|_2^2+b\|\psi_{kx}(0)\|_2^2+\tau\|z_{kx}(x,\rho,0)\|_{L^2((0,1)\times(0,1))}^2\Big)\end{aligned}$$

for all $t \in \mathbb{R}_+$, therefore, we conclude that

$$\varphi_k, \psi_k \text{ are bounded in } L^\infty_{loc}(0,\infty; H^2 \cap H^1_0(0,1)), \tag{3.55}$$

$$z_k \text{ is bounded in } L^\infty_{loc}(0,\infty; H^1_0(0,1; L^2(0,1))). \tag{3.56}$$

Applying Dunford-Petti's theorem we conclude from (3.39), (3.40), (3.41), (3.42), (3.49), (3.50), (3.51), (3.55) and (3.56), after replacing the sequences φ_k, ψ_k and z_k with a subsequence if needed, that

$$\begin{cases} \varphi_k \to \varphi \quad \text{weak-star in } \ L^\infty_{loc}(0,\infty; H^2 \cap H^1_0(0,1)) \\ \psi_k \to \psi \quad \text{weak-star in } \ L^\infty_{loc}(0,\infty; H^2 \cap H^1_0(0,1)) \end{cases} \tag{3.57}$$

$$\begin{cases} \varphi'_k \to \varphi' \text{ weak-star in } \ L^\infty_{loc}(0,\infty; H^1_0(0,1)) \\ \psi_k \to \psi' \quad \text{weak-star in } \ L^\infty_{loc}(0,\infty; H^1_0(0,1)) \end{cases} \tag{3.58}$$

$$\begin{cases} \varphi''{}_k \to \varphi'' \text{ weak-star in } \ L^\infty_{loc}(0,\infty; L^2(0,1)) \\ \psi''{}_k \to \psi'' \quad \text{weak-star in } \ L^\infty_{loc}(0,\infty; L^2(0,1)) \end{cases} \tag{3.59}$$

$$z_k \to z \quad \text{weak-star in } \ L^\infty_{loc}(0,\infty; H^1_0((0,1); L^2(0,1)),$$

$$z'_k \to z' \quad \text{weak-star in } \ L^\infty_{loc}(0,\infty; L^2((0,1) \times (0,1))), \tag{3.60}$$

for suitable functions $\varphi, \psi \in L^\infty(0,T; H^2 \cap H^1_0(0,1)), z \in L^\infty(0,T; L^2((0,1) \times (0,1)))$, $\chi \in L^2((0,1) \times (0,T)), \psi \in L^2((0,1) \times (0,T))$ for all $T \geq 0$. We have to show that (φ, ψ, z) is a solution of (3.12).
From (3.39) and (3.40) we have (ψ'_k) is bounded in $L^\infty(0,T; H^1_0(0,1))$. Then (ψ'_k) is bounded in $L^2(0,T; H^1_0)$. Since (ψ''_k) is bounded in $L^\infty(0,T; L^2(0,1))$, then (ψ''_k) is bounded in $L^2(0,T; L^2(0,1))$. Consequently (ψ'_k) is bounded in $H^1(Q)$,where $Q = (0,1) \times (0,T)$.

Since the embedding $H^1(Q) \hookrightarrow L^2(Q)$ is compact, using Aubin-Lions theorem [29] we can extract a subsequence (ψ_ν) of (ψ_k) such that

$$\psi'_\nu \to \psi' \text{ strongly in } \ L^2(Q).$$

Therefore

$$\psi'_\nu \to \psi' \text{ strongly and a.e on } Q. \tag{3.61}$$

Similarly we obtain

$$z_\nu \to z \text{ strongly and a.e on } Q. \tag{3.62}$$

It follows at once from (3.57), (3.59), (??), (??) and (3.60) that for each fixed $u, v \in L^2(0,T;H_0^1(0,1))$ and $w \in L^2(0,T;H_0^1((0,1)\times(0,1)))$

$$\begin{aligned}&\textstyle\int_0^T\int_0^1(\rho_1\varphi_k'' - K(\varphi_{kx}+\psi_k)_x)u\,dx\,dt\\ &\to \textstyle\int_0^T\int_0^1(\rho_1\varphi'' - K(\varphi_x+\psi)_x)u\,dx\,dt\end{aligned}$$

$$\begin{aligned}&\textstyle\int_0^T\int_0^1(\rho_2\psi_k'' - b\psi_{kxx} + K(\varphi_{kx}+\psi_k) + +\mu_1(t)\psi_k' + \mu_2(t)z_k)v\,dx\,dt\\ &\to \textstyle\int_0^T\int_0^1(\rho_2\psi'' - b\psi_{xx} + K(\varphi_x+\psi) + \mu_1(t)\psi' + \mu_2(t)zv\,dx\,dt\end{aligned}$$

$$\int_0^T\int_0^1\int_0^1(\tau z_k' + \frac{\partial}{\partial\rho}z_k)w\,dx\,d\rho\,dt \to \int_0^T\int_0^1\int_0^1(\tau z' + \frac{\partial}{\partial\rho}z)w\,dx\,d\rho\,dt$$

as $k \to +\infty$. Hence

$$\int_0^T\int_0^1(\rho_1\varphi'' - K(\varphi_x+\psi)_x)u\,dx\,dt = 0$$

$$\int_0^T\int_0^1(\rho_2\psi'' - b\psi_{xx} + K(\varphi_x+\psi) + \mu_1(t)\psi' + \mu_2(t)zv\,dx\,dt = 0$$

$$\int_0^T\int_0^1\int_0^1(\tau u' + \frac{\partial}{\partial\rho}z)w\,dx\,d\rho\,dt = 0, \qquad w \in L^2(0,T;H_0^1((0,1)\times(0,1))).$$

Thus the problem (P) admits a global weak solution (φ,ψ).

Uniqueness. Let (φ_1,ψ_1,z_1) and (φ_2,ψ_2,z_2) be two solutions of problem (3.12). Then $(w,\tilde{w},\tilde{\tilde{w}}) = (\varphi_1,\psi_1,z_1) - (\varphi_2,\psi_2,z_2)$ verifies

$$\begin{cases}\rho_1 w_{tt}(x,t) - K(w_x+\tilde{w})_x(x,t) = 0 & \text{in }]0,1[\times]0,+\infty[,\\ \rho_2\tilde{w}''(x,t) - b\tilde{w}_{xx}(x,t) + K(w_x+\tilde{w}) + \mu_1(t)\tilde{w}'(x,t) & \\ \quad +\mu_2(t)\tilde{\tilde{w}}(x,1,t), & \text{in }]0,1[\times]0,+\infty[,\\ \tau\tilde{\tilde{w}}'(x,\rho,t) + \tilde{\tilde{w}}_\rho(x,\rho,t) = 0, & \text{in } (0,1)\times]0,1[\times]0,+\infty[\\ w(0,t) = w(1,t) = \tilde{w}(0,t) = \tilde{w}(1,t) = 0, & t \geq 0\\ \tilde{\tilde{w}}(x,0,t) = \psi_1'(x,t) - \psi_2'(x,t) & \text{on }]0,1[\times[0,+\infty[\\ w(x,0) = w'(x,0) = \tilde{w}(x,0) = \tilde{w}'(x,0) = 0, & \text{in }]0,1[\\ \tilde{\tilde{w}}(x,\rho,0) = 0 & \text{in }]0,1[\times]0,1[\end{cases} \tag{3.63}$$

Multiplying the first equation by w' and the second by $\tilde{w}'$ in (3.63), integrating over $(0,1)$ and using an integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}(\rho_1\|w'\|_2^2) + K\int_0^1(w_x+\tilde{w})_x w'\,dx = 0 \tag{3.64}$$

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}(\rho_2\|\tilde{w}'\|_2^2 + b\|\tilde{w}_x\|_2^2) + K\textstyle\int_0^1(w_x+\tilde{w})\tilde{w}'\,dx + \mu_1(t)\|\tilde{w}'\|_2^2\\ &+\mu_2(t)(\tilde{\tilde{w}}(x,1,t),\tilde{w}') = 0.\end{aligned} \tag{3.65}$$

Multiplying the third equation in (3.63) by $\tilde{\tilde{w}}$, integrating over $(0,1)\times(0,1)$, we get

$$\frac{\tau}{2}\frac{d}{dt}\|\tilde{\tilde{w}}\|_2^2+\frac{1}{2}\frac{d}{d\rho}\|\tilde{w}\|_2^2 \tag{3.66}$$

$$\tau\frac{1}{2}\frac{d}{dt}\int_0^1\|\tilde{\tilde{w}}'\|_2^2\,d\rho+\frac{1}{2}(\|\tilde{\tilde{w}}(x,1,t)\|_2^2-\|\tilde{w}'\|_2^2)=0. \tag{3.67}$$

From (3.64), (3.65), (3.67) and using Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\left(\rho_1\|w'\|_2^2+\rho_2\|\tilde{w}'\|_2^2+b\|\tilde{w}_x\|_2^2+K\|w_x+\tilde{w}\|_2^2+\tau\int_0^1\|\tilde{\tilde{w}}'\|_2^2\,d\rho\right)\\&+\mu_1(t)\|\tilde{w}\|_2^2+\tfrac{1}{2}\|\tilde{\tilde{w}}(x,1,t)\|_2^2=-\mu_2(t)(\tilde{\tilde{w}}(x,1,t),\tilde{w}')+\tfrac{1}{2}\|\tilde{w}'\|_2^2\\&\leq\tfrac{1}{2}\|\tilde{w}'\|_2^2+|\mu_2(t)|\|\tilde{\tilde{w}}(x,1,t)\|_2\|\tilde{w}'\|_2.\end{aligned}$$

Using Young's inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|w'\|_2^2+\rho_2\|\tilde{w}'\|_2^2+b\|\tilde{w}_x\|_2^2+K\|w_x+\tilde{w}\|_2^2+\tau\int_0^1\|\tilde{\tilde{w}}'\|_2^2\,d\rho\right)\leq c\|\tilde{w}'\|_2^2,$$

where c is a positive constant. Then integrating over $(0,t)$, using Gronwall's lemma, we conclude that

$$\rho_1\|w'\|_2^2+\rho_2\|\tilde{w}'\|_2^2+b\|\tilde{w}_x\|_2^2+K\|w_x+\tilde{w}\|_2^2+\tau\int_0^1\|\tilde{\tilde{w}}'\|_2^2\,d\rho=0.$$

Hence, uniquness follows.

3.4 Asymptotic behavior

Now we construct a Lyapunov functional L equivalent to E. For this, we define several functionals which allow us to obtain the needed estimates.
Then we have the following estimate.

Lemma 3.4.1. *Let (φ,ψ,z) be the solution of (3.12). Then the functional F_1 defined by*

$$F_1(t)=-\int_0^1(\rho_1\varphi_t\varphi+\rho_2\psi_t\psi)\,dx \tag{3.68}$$

satisfies, along the solution, the estimate

$$\begin{aligned}\frac{dF_1(t)}{dt}\leq-\int_0^1(\rho_1\varphi_t^2+(\rho_2-c)\psi_t^2)dx+K\int_0^1|\varphi_x+\psi|^2dx\\+c\int_0^1\psi_x^2dx+c|\mu_2(t)|\int_0^1\psi z(x,1,t)dx.\end{aligned} \tag{3.69}$$

Proof. By taking the time derivative of (3.68)

$$\frac{dF_1(t)}{dt} = -\int_0^1 (\rho_1\varphi_t^2 + \rho_2\psi_t^2)dx - \int_0^1 (\rho_1\varphi_{tt}\varphi + \rho_2\psi_{tt}\psi)dx.$$

Therefore, by using the first and the second equations in (3.12) and some integrations by parts, we obtain from the above inequality

$$\begin{aligned}\tfrac{dF_1(t)}{dt} &= -\textstyle\int_0^1(\rho_1\varphi_t^2+\rho_2\psi_t^2)dx + K\int_0^1|\varphi_x+\psi|^2dx \\ &\quad +b\textstyle\int_0^1\psi_x^2dx + \mu_1(t)\int_0^1\psi\psi_t dx + \mu_2(t)\int_0^1\psi z(x,1,t)dx.\end{aligned} \tag{3.70}$$

By exploiting Young's inequality and Poincaré's inequality, then (3.69) holds.

□

Lemma 3.4.2. *Let (φ,ψ,z) be the solution of (3.12). Assume that*

$$\frac{\rho_1}{K} = \frac{\rho_2}{b}. \tag{3.71}$$

Then the functional F_2 defined by

$$F_2(t) = \rho_2\int_0^1 \psi_t(\varphi_x+\psi)dx + \rho_2\int_0^1 \psi_x\varphi_t\, dx. \tag{3.72}$$

satisfies, along the solution, the estimate

$$\begin{aligned}&\tfrac{dF_2(t)}{dt} = [b\varphi_x\psi_x]_{x=0}^{x=1} - (K-\varepsilon)\textstyle\int_0^1(\varphi_x+\psi)^2dx + (\rho_2+\varepsilon)\int_0^1\psi_t^2dx \\ &-\tfrac{c}{\varepsilon}|\mu_2(t)|\textstyle\int_0^1(\varphi_x+\psi)z(x,1,t)dx.\end{aligned} \tag{3.73}$$

for any $0<\varepsilon<1$.

Proof. Differentiating $F_2(t)$, with respect to t, we obtain

$$\begin{aligned}&\tfrac{dF_2(t)}{dt} = \textstyle\int_0^1 \rho_2\psi_{tt}(\varphi_x+\psi)dx + \int_0^1\rho_2\psi_t(\varphi_x+\psi)_t dx + \rho_2\int_0^1\psi_x\varphi_{tt}dx + \rho_2\int_0^1\psi_{tx}\varphi_t dx. \\ &= \textstyle\int_0^1(\varphi_x+\psi)[b\psi_{xx} - k(\varphi_x+\psi) - \mu_1(t)\psi_t - \mu_2(t)z(x,1,t)]dx + \rho_2\int_0^1\psi_t^2\,dx + \frac{\rho_2}{\rho_1}\int_0^1 k(\varphi_x+\psi)_x\psi_x\,dx.\end{aligned}$$

Then, by using Eqs.(3.12) and (3.71) we find

$$\begin{aligned}\tfrac{dF_2(t)}{dt} &= [b\varphi_x\psi_x]_{x=0}^{x=1} - K\textstyle\int_0^1(\varphi_x+\psi)^2dx + \rho_2\int_0^1\psi_t^2dx \\ &\quad -\mu_1(t)\textstyle\int_0^1(\varphi_x+\psi)\psi_t dx - \mu_2(t)\int_0^1(\varphi_x+\psi)z(x,1,t)dx.\end{aligned} \tag{3.74}$$

□

Lemma 3.4.3. *Let $m \in C^1([0,1])$ be a function satisfying $m(0) = -m(1) = 2$. Then there exists $c > 0$ such that, for any $0 < \varepsilon < 1$, the functional F_3 defined by*

$$F_3(t) = \frac{b}{4\varepsilon}\int_0^1 \rho_2 m(x)\psi_t\psi_x\,dx + \frac{\varepsilon}{k}\int_0^1 \rho_1 m(x)\varphi_t\varphi_x\,dx$$

satisfies, along the solution, the estimate

$$\begin{aligned}F_3'(t) &\le -\tfrac{b^2}{4\varepsilon}((\psi_x(1,t))^2 + (\psi_x(0,t))^2) - \varepsilon((\varphi_x(1,t))^2 + (\varphi_x(0,t))^2)\\ &+(\tfrac{k}{4} + \tfrac{c}{k}\varepsilon)\textstyle\int_0^1(\psi+\varphi_x)^2\,dx + c\varepsilon\rho_1\int_0^1\varphi_t^2\,dx + \frac{c}{\varepsilon^2}\int_0^1\psi_x^2\,dx\\ &+\tfrac{c}{\varepsilon}\textstyle\int_0^1\psi_t^2\,dx + c|\mu_2(t)|\int_0^1\|z(x,1,t)\|^2\,dx\end{aligned} \tag{3.75}$$

Proof. Using Eqs. (3.12) and integrating by parts, obtain

$$\begin{aligned}F_3'(t) = &\tfrac{b}{4\varepsilon}\Big[-b((\psi_x(1,t))^2 + (\psi_x(0,t))^2) - \textstyle\int_0^1 \frac{b}{2}m'(x)\psi_x^2\,dx - k\int_0^1 m(x)\psi_x(\varphi_x+\psi)\,dx\\ &- \textstyle\int_0^1 m(x)\mu_1(t)\psi_t\psi_x\,dx - \int_0^1 m(x)\mu_2(t)z(x,1,t)\psi_x\,dx - \int_0^1\frac{\rho_2}{2}m'(x)(\psi_t)^2\,dx\Big]\\ &\tfrac{\varepsilon}{k}\Big[-k((\varphi_x(1,t))^2 + (\varphi_x(0,t))^2) - \textstyle\int_0^1\frac{k}{2}m'(x)\varphi_x^2\,dx + \int_0^1 km(x)\psi_x\varphi_x\,dx - \int_0^1\frac{\rho_1}{2}m'(x)(\varphi_t)^2\,dx\Big]\end{aligned}$$

Then by the Young and Poincaré inequalities and the fact that

$$\varphi_x^2 \le 2(\psi+\varphi_x)^2 + 2\psi^2$$

we obtain

$$\begin{aligned}F_3'(t) \le &\tfrac{b}{4\varepsilon}\big[-b((\psi_x(1,t))^2 + (\psi_x(0,t))^2)\\ &+\tfrac{c}{\varepsilon}\textstyle\int_0^1\psi_x^2\,dx + \varepsilon\frac{k}{b}\int_0^1(\psi+\varphi_x)^2\,dx + \varepsilon\int_0^1 g_1^2(\psi_t)\,dx + \varepsilon\int_0^1 g_2^2(z(x,1,t))\,dx + c\int_0^1\psi_t^2\,dx\Big]\\ &\tfrac{\varepsilon}{k}\Big[-k((\varphi_x(1,t))^2 + (\varphi_x(0,t))^2) + c\textstyle\int_0^1\psi_x^2\,dx + c\int_0^1(\psi+\varphi_x)^2\,dx + c\int_0^1\varphi_t^2\,dx\Big]\end{aligned}$$

This gives (3.75).

□

Lemma 3.4.4. *Assume that $(H1)$ hold. Then, for sufficiently small ε, the functional F defined by*

$$F(t) = 2c\varepsilon F_1(t) + F_2(t) + F_3(t)$$

satisfies, along the solution, the estimate

$$F'(t) \le -\tfrac{k}{2}\textstyle\int_0^1(\psi+\varphi_x)^2\,dx - \tau\int_0^1\varphi_t^2\,dx + c\int_0^1\psi_t^2\,dx + c\int_0^1\psi_x^2\,dx + c\int_0^1 z(x,1,t)^2\,dx, \tag{3.76}$$

where $\tau = c\varepsilon\rho_1$.

Proof. Using Lemmas 3.4.1, 3.4.2, 3.4.3 and the fact that

$$[b\varphi_x\psi_x]_{x=0}^{x=1} \leq \varepsilon[\varphi_x^2(1)+\varphi_x^2(0)] + \frac{b^2}{4\varepsilon}[\psi_x^2(1)+\psi_x^2(0)] \tag{3.77}$$

for any $0<\varepsilon<1$, we obtain (3.76).

□

Next, we introduce the following functional

$$I(t) = \int_0^1 (\rho_2\psi_t\psi + \rho_1\varphi_t\omega)dx, \tag{3.78}$$

where w is the solution of

$$-\omega_{xx} = \psi_x, \qquad \omega(0)=\omega(1)=0. \tag{3.79}$$

Then we have the following estimate.

Lemma 3.4.5. *Let (φ,ψ,z) be the solution of (3.12), then for any $\delta>0$, we have the following estimate*

$$\begin{aligned} \tfrac{dI(t)(t)}{dt} \leq \tfrac{-b}{2}\textstyle\int_0^1 \psi_x^2(x,t)dx + \tfrac{c}{\delta}\int_0^1 \psi_t^2(x,t)dx \\ +\delta \textstyle\int_0^1 \varphi_t^2(x,t)dx + c|\mu_2(t)|\int_0^1 z(x,1,t)^2dx. \end{aligned} \tag{3.80}$$

Proof. Using Eqs. (3.12), we have

$$\begin{aligned} \tfrac{dI(t)}{dt} &= -b\textstyle\int_0^1 \psi_x^2dx + \rho_2\int_0^1 \psi_t^2dx - K\int_0^1 \psi^2dx \\ &+K\textstyle\int_0^1 \omega_x^2dx + \rho_1\int_0^1 \psi_t\omega_tdx - \mu_1\int_0^1 \psi g_1(\psi_t)dx - \mu_2\int_0^1 \psi g_2(z(x,1,t))dx. \end{aligned} \tag{3.81}$$

It is clear that, from (3.79), we have

$$\begin{aligned} \int_0^1 \omega_x^2dx \leq \int_0^1 \psi^2dx \leq \int_0^1 \psi_x^2dx \\ \int_0^1 \omega_t^2dx \leq \int_0^1 \omega_{tx}^2dx \leq \int_0^1 \psi_t^2dx \end{aligned} \tag{3.82}$$

By using Young's inequality and Poincaré's inequality we obtain (3.80).

□

Now, let us introduce the following functional

$$I_3(t) = \int_0^1\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)\,d\rho\,dx. \tag{3.83}$$

Then the following result holds.

Lemma 3.4.6. *Let* (φ, ψ, z) *be the solution of (3.12). Then it holds*

$$\frac{d}{dt}I_3(t) \leq -I_3(t) - \frac{c}{2\tau}\int_0^1 z(x,1,t)^2 dx + \frac{1}{2\tau}\int_0^1 \psi_t^2(x,t)\,dx. \tag{3.84}$$

Proof. Differentiating (3.83) with respect to t and using the third equation in (3.12), we have

$$\begin{aligned}
&\tfrac{d}{dt}\left(\int_0^1\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)\,d\rho\,dx\right) = -\tfrac{1}{\tau}\int_0^1\int_0^1 e^{-2\tau\rho}zz_\rho(x,\rho,t)\,d\rho\;dx\\
&= -\int_0^1\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)\,d\rho\;dx - \tfrac{1}{2\tau}\int_0^1\int_0^1 \tfrac{\partial}{\partial\rho}e^{-2\tau\rho}z^2(x,\rho,t)\,d\rho\;dx.
\end{aligned}$$

The above formula implies that there exists a positive constant c such that (75) holds. □

Proof of theorem2.1. To finalize the proof of Theorem 2.1, we define the lyapunov functional $\mathcal{L}$ as follows For $N_1, N_2 > 0$, let

$$\mathcal{L}(t) = N_1E(t) + N_2I(t) + F(t) + I_3(t), \tag{3.85}$$

whereN_1 andN_2 are positive real numbers which will be chosen later. By combining (3.17), (3.76), (3.80), (3.84), we obtain

$$\begin{aligned}
&\tfrac{d}{dt}\mathcal{L}(t) \leq -\left(N_1a_1(t) - N_2(t)\tfrac{c}{\delta} - c - \tfrac{c}{\delta}\right)\int_0^1 \psi_t^2\,dx\\
&-\left(N_1a_2 - N_2c|\mu_2(t)| - c - c|\mu_2(t)|\right)\int_0^1 z^2(x,1,t)\,dx - \left(N_2\tfrac{b}{2} - c + \tfrac{b}{2}\right)\int_0^1 \psi_x^2\,dx\\
&-\left(N_2\delta + \tau - \delta\right)\int_0^1 \varphi_t^2\,dx\\
&-\tfrac{k}{2}\int_0^1(\psi+\varphi_x)^2\,dx
\end{aligned} \tag{3.86}$$

At this point, we have to choose our constants very carefully. First, let us choose N_2 sufficiently large so that

$$\left(N_2\frac{b}{2} - c + \frac{b}{2}\right) > 0.$$

Next, we choose δ sufficiently small such that

$$\left(N_2\delta + \tau - \delta\right) > 0.$$

Then, we pick the constant $N_1 > 0$ sufficiently large such that

$$\left(N_1a_1(t) - N_2(t)\frac{c}{\delta} - c - \frac{c}{\delta}\right)$$

and

$$\left(N_1a_2 - N_2c|\mu_2(t)| - c - c|\mu_2(t)|\right).$$

Thus, (3.86) becomes

$$\begin{aligned}&\tfrac{d}{dt}L(t) \leq -d_1 \int_0^1 \psi_x^2\,dx - d_2 \int_0^1 \varphi_t^2\,dx - \tfrac{k}{2}\int_0^1 (\psi+\varphi_x)^2\,dx \\ &+c\int_0^1 z(x,1,t)^2 (\leq -dE(t) + c\int_0^1 z^2(x,1,t)\,dx.\end{aligned} \tag{3.87}$$

which implies by (3.14), that there exists also $\eta > 0$

$$\frac{d}{dt}\mathcal{L}(t) \leq -\eta E(t). \tag{3.88}$$

At this stage, we are in position to compare $\mathcal{L}(t)$ with $E(t)$. We have the following Lemma.

Lemma 3.4.7. *For N_1 large enough, there exist two positive constants β_1 and β_2 depending on N_1, N_2 and ϵ, such that*

$$\beta_1 E(t) \leq \mathcal{L}(t) \leq \beta_2 E(t) \qquad \forall t \geq 0. \tag{3.89}$$

Proof. We consider the functional

$$\mathcal{H}(t) = N_2 I(t) + F(t) + I_3(t)$$

and show that

$$|\mathcal{H}(t)| \leq \hat{C}E(t), \qquad C > 0.$$

from (3.68),(3.78),(3.72) and (3.83), we obtain

$$\begin{aligned}&|\mathcal{H}(t)| \leq N_2\left|\int_0^1 \rho_2\psi_t\psi + \rho_1\varphi_t\omega)(x,t)dx\right| + \left|-\int_0^1(\rho_1\varphi_t\varphi + \rho_2\psi_t\psi)dx\right| + \\ &\left|\rho_2\int_0^1 \psi_t(\varphi_x+\psi)dx + \rho_2\int_0^1 \psi_x\varphi_t\,dx\right| + \left|\tfrac{b}{4\varepsilon}\int_0^1 \rho_2 m(x)\psi_t\psi_x\,dx + \tfrac{\varepsilon}{k}\int_0^1 \rho_1 m(x)\varphi_t\varphi_x\,dx\right| \\ &+\left|\int_0^1\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)\,d\rho\,dx\right|.\end{aligned} \tag{3.90}$$

By using (3.82),(3.79), the trivial relation

$$\int_0^1 \varphi^2(x,t)dx \leq 2\int_0^1 (\varphi_x+\psi)^2(x,t)dx + 2\int_0^1 \psi_x^2(x,t)dx,$$

Young's and Poincaré's inequalities, we get

$$\begin{aligned}|\mathcal{H}(t)| \leq\ &\alpha_1\int_0^1 \varphi_t^2(x,t)dx + \alpha_2\int_0^1 \psi_t^2(x,t)dx \\ &+\alpha_3\int_0^1(\varphi_x+\psi)^2(x,t)dx + \alpha_4\int_0^1 \psi_x^2(x,t)dx \\ &+\int_0^1\int_0^1 z^2(x,\rho,t)dx\,d\rho\end{aligned} \tag{3.91}$$

where the positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are determined as follows:

$$\begin{cases} \alpha_1 = \frac{N_2\rho_1}{2} + \rho_2 + \frac{\epsilon\rho_1}{K}, \\ \alpha_2 = \frac{N_2\rho_2}{2} + \rho_2 + \frac{\rho_2 b}{2\epsilon}, \\ \alpha_3 = \rho_1 + \frac{\rho_2}{2} + \frac{2\epsilon\rho_1}{K}, \\ \alpha_4 = \rho_2 + \frac{N_2}{2}\rho_2 + \rho_1 + \frac{\rho_2 b}{2\epsilon} + \frac{2\epsilon\rho_1}{K} \end{cases}$$

According to (3.91) , we have

$$|H(t)| \leq \hat{C}E(t)$$

for

$$\hat{C} = 2\max\left\{\frac{\alpha_1}{\rho_1}, \frac{\alpha_2}{\rho_2}, \frac{\alpha_3}{k}, \frac{\alpha_4}{b}, \frac{1}{2\xi}\right\}.$$

Therefore, we obtain

$$|L(t) - N_1 E(t)| \leq \hat{C}E(t).$$

So, we can choose N_1 large enough so that $\beta_1 = N_1 - \hat{C} > 0, \beta_2 = N_1 + \hat{C} > 0$. Then (3.89) holds true.

□

Combining and (79) and (80), we conclude

$$\frac{d}{dt}\mathcal{L}(t) \leq -\lambda\mathcal{L}(t) \qquad \forall t \geq 0. \tag{3.92}$$

A simple integration of (83) leads to

$$\frac{d}{dt}\mathcal{L}(t) \leq \mathcal{L}(0)e^{-\lambda t} \qquad \forall t \geq 0. \tag{3.93}$$

Again, the use of (80) (84) yields the desired result (9). This completes the proof of Theorem 2.1.

Remark 3.4.1. *According to the result of the paper [20], where a simple wave equation has been treated, it might be possible to prove the result of Therem 2.1 by using a suitable observability estimateof the forme*

$$E(0) \leq C_0 \int_0^T \int_0^1 \left(u_t^2(x,t) + u_t^2(x,t-\tau)\right) dx\, dt. \tag{3.94}$$

Where C_0 is positive constant. Once (85) holds, then we can obtain easily

$$E(T) \leq \zeta E(0), \tag{3.95}$$

With $\zeta < 1$. *Since our system (1) is invariant by translation and the energy is non-increasing, then applying this argument on* $[(m-1)T, mT], form = 1, 2, ...$ *we arrive at*

$$E(mT) \leq \zeta E((m-1)T) \leq ... \leq \zeta^m E(0), m = 1, 2, ... \tag{3.96}$$

This last inequality implies

$$E(mT) \leq e^{-wmT} E(0), m = 1, 2, ... \tag{3.97}$$

with $w = \frac{1}{T} ln \frac{1}{\zeta}$.*So for arbitrary positive* t*, there exists* m *such that* $(m-1)T < t \leq mT$ *and since* $E(t)$ *is non-increasing function, we conclude*

$$E(t) \leq E((m-1)T) \leq e^{-w(m-1)T} E(0) \leq \frac{1}{\zeta} e^{-wt} E(0) \tag{3.98}$$

see [20].

Appendix

We will show the lack of exponential stabilty by frequency domain method.
We show the existence of a sequence $(\lambda_\mu) \subset \mathbb{R}$ with $\lim_{\mu\to\infty} |\lambda_\mu| = \infty$ and $(U_\mu) \subset D(\mathcal{A})$ to $F_\mu \subset \mathcal{H}$ such that $(i\lambda_\mu I - \mathcal{A})U_\mu = F_\mu$ is bounded in $\mathcal{H}$ and $\lim_{\mu\to\infty} \|U_\mu\|_{\mathcal{H}} = \infty$. Let $F = F_\mu = (f_1, f_2, 0)^T$ with $U_\mu = (\varphi_\mu, u_\mu, \phi_\mu)^T$.
Now, introducing following notations

$$\begin{aligned}
I_1 &= \int_0^L e^{-\tau\sigma}(if_{1xx}(\sigma) + f_2(\sigma))\, d\sigma,\\
I_2 &= e^{-\tau L}\int_0^L e^{\tau\sigma}(if_{1xx}(\sigma) + f_2(\sigma))\, d\sigma,\\
I_3 &= \int_0^L \sin\tau(L-\sigma)(-if_{1xx}(\sigma) + f_2(\sigma))\, d\sigma,\\
I_4 &= \int_0^L \cos\tau(L-\sigma)(-if_{1xx}(\sigma) + f_2(\sigma))\, d\sigma.
\end{aligned}$$

Note that

$$\begin{aligned}
I_1 = I_2 &= \mathcal{O}(\tau^{-\frac{1}{2}}(\|f_{1xx}\| + \|f_2\|)),\\
I_3 = I_4 &= \mathcal{O}(\|f_{1xx}\| + \|f_2\|).
\end{aligned}$$

From (2.54) and (2.59), we have

$$\begin{aligned}
\varphi_{xx}(x) = {} & \tau^2[A(\cosh\tau x + \cos\tau x) + B(\sinh\tau x + \sin\tau x)]\\
& + \tfrac{1}{2\tau}\int_0^x (f_2(\sigma) + i\tau^2 f_1(\sigma))(\sinh\tau(x-\sigma) + \sin\tau(x-\sigma))\, d\sigma,
\end{aligned} \tag{3.99}$$

where

$$\begin{aligned}
A &= \tfrac{1}{D}\left(m_{22}\tilde{C}_1 - m_{12}\tilde{C}_2\right)\\
B &= \tfrac{1}{D}\left(m_{11}\tilde{C}_2 - m_{21}\tilde{C}_1\right).
\end{aligned}$$

Using the fact that

$$\begin{aligned}
&\int_0^L f_1(\sigma)(\sinh\tau(x-\sigma)+\sin\tau(x-\sigma))\,d\sigma\\
&\qquad=\tfrac{1}{\tau^2}\int_0^L f_{1xx}(\sigma)(\sinh\tau(x-\sigma)-\sin\tau(x-\sigma))\,d\sigma\\
&\int_0^L f_1(\sigma)(\cosh\tau(x-\sigma)+\cos\tau(x-\sigma))\,d\sigma\\
&\qquad=\tfrac{1}{\tau^2}\int_0^L f_{1xx}(\sigma)(\cosh\tau(x-\sigma)-\cos\tau(x-\sigma))\,d\sigma\\
&\int_0^L f_1(\sigma)(\sinh\tau(x-\sigma)-\sin\tau(x-\sigma))\,d\sigma\\
&\qquad=-\tfrac{1}{\tau^2}f_1(L)+\tfrac{1}{\tau^2}\int_0^L f_{1xx}(\sigma)(\sinh\tau(x-\sigma)+\sin\tau(x-\sigma))\,d\sigma,
\end{aligned}$$

we deduce that

$$\tilde{C}_1=\frac{1}{\tau^3}\left(-\frac{1}{4}e^{\tau L}I_1+\frac{1}{4}I_2-\frac{1}{2}I_3\right)$$

$$\begin{aligned}
\tilde{C}_2=-\tfrac{1}{4}e^{\tau L}I_1-\tfrac{1}{4}I_2-\tfrac{1}{2}I_4-i\gamma\tfrac{(i\tau^2+\eta)^{\alpha-1}}{2\tau}I_3+i\gamma\tfrac{(i\tau^2+\eta)^{\alpha-1}}{4\tau}e^{\tau L}I_1\\
-i\gamma\tfrac{(i\tau^2+\eta)^{\alpha-1}}{4\tau}I_2+\xi\int_{-\infty}^{\infty}\tfrac{\mu(\xi)f_3(\xi)}{i\tau^2+\xi^2+\eta}\,d\xi.
\end{aligned}$$

The second derivative of the solution is of the form:

$$\begin{aligned}
\varphi_{xx}(x)&=\tau^2[A(\cosh\tau x+\cos\tau x)+B(\sinh\tau x+\sin\tau x)]\\
&\quad+\tfrac{1}{4\tau}e^{\tau x}I_1+\mathcal{O}(\tau^{-\frac{1}{2}}(\|f_{1xx}\|+\|f_2\|))\\
&=C_1e^{\tau x}+C_2e^{-\tau x}+C_3\cos\tau x+C_4\sin\tau x+\mathcal{O}(\tau^{-\frac{1}{2}}(\|f_{1xx}\|+\|f_2\|))
\end{aligned}$$

where

$$\begin{aligned}
C_1&=\tfrac{1}{2}\left((A+B)\tau^2+\tfrac{1}{2\tau}I_1\right)\\
C_2&=\tfrac{1}{2}(A-B)\tau^2\\
C_3&=A\tau^2\\
C_4&=B\tau^2
\end{aligned}$$

Considering only the dominant terms of τ, the following is obtained:

$$\begin{aligned}
C_1D&=\tfrac{1}{4}(I_1\sin\tau L+I_2-I_3(\sin\tau L+\cos\tau L)+I_4(\sin\tau L-\cos\tau L))\tau^2+\mathcal{O}(\tau^{2\alpha-\frac{3}{2}}\|F\|)\\
C_2D&=\tfrac{1}{4}(I_1\sin\tau L+I_2-I_3+I_4)\tau^2e^{\tau L}+\mathcal{O}(\tau^{2\alpha-1}e^{\tau L}\|F\|)\\
C_3D&=\tfrac{1}{4}(I_1(\sin\tau L-\cos\tau L)+I_2-I_3+I_4)\tau^2e^{\tau L}+\mathcal{O}(\tau^{2\alpha-1}e^{\tau L}\|F\|)\\
C_4D&=\tfrac{1}{4}(-I_1(\sin\tau L+\cos\tau L)-I_2+I_3-I_4)\tau^2e^{\tau L}+\mathcal{O}(\tau^{2\alpha-1}e^{\tau L}\|F\|).
\end{aligned}$$

it follows that

$$\begin{aligned}
\|\varphi_{xx}\|&=\tfrac{1}{D}\sqrt{\tfrac{L}{2}(|C_3D|^2+|C_4D|^2)+\mathcal{O}(\tau^3e^{2\tau L}\|F\|^2_{\mathcal{H}})}+\mathcal{O}(\tau^{-1}\|F\|_{\mathcal{H}})\\
&=\tfrac{\tau^2}{4D}\sqrt{L|I_4-I_3|^2e^{2\tau L}+\mathcal{O}(\tau^{-\frac{1}{2}}e^{2\tau L}\|F\|^2_{\mathcal{H}})}+\mathcal{O}(\tau^{-1}\|F\|_{\mathcal{H}})\\
&=\tfrac{\tau^2e^{\tau L}}{4D}\sqrt{L|I_4-I_3|^2+\mathcal{O}(\tau^{-\frac{1}{2}}\|F\|^2_{\mathcal{H}})}+\mathcal{O}(\tau^{-1}\|F\|_{\mathcal{H}})
\end{aligned}$$

For every τ large enough, a function $F = (f_1, 0, 0)$ can be chosen with $f_1 \in H_0^2$ such that

$$\sqrt{L|I_4 - I_3|^2 + \mathcal{O}(\tau^{-\frac{1}{2}}\|F\|_{\mathcal{H}}^2)} \geq K\|F\|_{\mathcal{H}}, \tag{3.100}$$

where constant K does not depend on τ.

For this purpose, let f_1 be defined with

$$f_1(x) = -\frac{1}{\tau^2}\sin\left(\tau(L-x) - \frac{\pi}{4}\right) - \frac{x}{\tau}\cos\left(\tau L - \frac{\pi}{4}\right) + \frac{1}{\tau^2}\sin\left(\tau L - \frac{\pi}{4}\right).$$

Then

$$f_{1xx}(x) = \sin\left(\tau(L-x) - \frac{\pi}{4}\right),$$

$$\|f_{1xx}\|_2^2 = \frac{L}{2} + \mathcal{O}(\tau^{-1}). \tag{3.101}$$

Hence

$$\|F\|_{\mathcal{H}}^2 = \frac{L}{2} + \mathcal{O}(\tau^{-1}). \tag{3.102}$$

which implies that for all τ large enough, $\|F\|_{\mathcal{H}}$ is bounded by some constant independent of τ. There holds:

$$\begin{aligned} I_4 - I_3 &= i\int_0^L (\sin\tau(L-\sigma) - \cos\tau(L-\sigma)) f_{1xx}(\sigma)\, d\sigma \\ &= i\sqrt{2}\int_0^L \sin(\tau(L-\sigma) - \pi/4)^2\, d\sigma \\ &= i\sqrt{2}\|f_{1xx}\|_2^2. \end{aligned}$$

Therefore (3.100) follows easily from (3.101) and (3.102) for τ large enough. Moreover

$$\|\varphi_{xx}\|_2 \geq \frac{\tau^2 e^{\tau L}}{4D}\|F\|_{\mathcal{H}} + \mathcal{O}(\tau^{-1}\|F\|_{\mathcal{H}}).$$

for all τ large enough. Now, Choose $\tau = \tau_n = \frac{\pi}{L}(n + \frac{1}{2})$. Hence, a suffciently large n can always be found so that

$$D \leq S\tau^{2\alpha}e^{\tau L},$$

where constant $S > 0$ does not depend on τ. For such τ, there holds:

$$\|\varphi_{xx}\|_2 \geq \frac{1}{4S}\tau^{(2-2\alpha)}\|F\|_{\mathcal{H}} + \mathcal{O}(\tau^{-1}\|F\|_{\mathcal{H}}).$$

This implies that there exists some constant $\tilde{M} > 0$ independent of τ such that

$$\|(i\tau^2 - \mathcal{A})^{-1}\|_{\mathcal{H}} \geq \tilde{M}\tau^{(2-2\alpha)}.$$

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Summary

In this PhD thesis we study stability and asymptotic behavior in time of solutions to nonlinear evolutions equations of hyperbolic type. We prove the global existence and we establish a decay rate estimate for the energy by means of the semi group
theory of linear operators and the energy method combined with the Faedo-Galerkin procedure.

Resumé

Dans cette thèse nous étudions la stabilité et comportement asymptotique en temps de solutions des équations d’évolutions non linéaires de type hyperbolique.
Nous montrons l’existence globale et nous établissons une estimation du taux de décroissance de l’énergie à l’aide de la théorie des semi-groupes des opérateurs linéaires et la méthode d’énergie combinée avec la procédure de Faedo-Galerkin.

ملخص

في هذه الأطروحة ندرس الوجود الشامل والسلوك التقاربي
لحلول معادلات التطورات غير الخطية للنوع الزائدي
بالنسبة لزمن

باستخدام نظرية أشباه الموصلات الخطية وطريقة الطاقة جنبًا
إلى جنب مع إجراء Faedo-Galerkin