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Intitulée

Stabilisation de quelques problèmes d'évolution hyperbolique par des contrôles de type fractionnaire frontière

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Introduction

Problems of global existence and stability in time of Partial Differential Equations made object, recently, of many work. In this thesis, we study the existence and decay properties of solutions of certain hyperbolic systems in particular Bresse[1] system of the type

$$(P) \qquad \begin{cases} \rho_1\varphi_{tt} - Gh(\varphi_x + \psi + l\omega)_x - lEh(\omega_x - l\varphi) = 0 \\ \rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + \psi + l\omega) = 0 \\ \rho_1\omega_{tt} - Eh(\omega_x - l\varphi)_x + lGh(\varphi_x + \psi + l\omega) = 0 \end{cases}$$

where $(x,t) \in (0,L) \times (0,+\infty)$. This system is subject to the boundary conditions

$$\begin{array}{ll} \varphi(0,t) = 0, \quad \psi(0,t) = 0, \quad \omega(0,t) = 0 & \text{in } (0,+\infty) \\ Gh(\varphi_x + \psi + l\omega)(L,t) = -\gamma_1\partial_t^{\alpha,\eta}\varphi(L,t) & \text{in } (0,+\infty) \\ EI\psi_x(L,t) = -\gamma_2\partial_t^{\alpha,\eta}\psi(L,t) & \text{in } (0,+\infty) \\ Eh(\omega_x - l\varphi)(L,t) = -\gamma_3\partial_t^{\alpha,\eta}\omega(L,t) & \text{in } (0,+\infty) \end{array}$$

where $\gamma_1 \geq, \gamma_2 > 0, \gamma_3 > 0$. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α with respect to the time variable. It is defined as follows

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\, ds, \ \eta \geq 0.$$

These exponentially modified fractional integro-differential operators were first proposed in Choi and MacCamy [**15**]. In other words, we investigate three dissipative effects at the boundary (where $\gamma_1 > 0, \gamma_2 > 0, \gamma_3 > 0$) or two dissipative effects at the boundary (where $\gamma_1 = 0, \gamma_2 > 0, \gamma_3 > 0$).

Here $\rho_1 = \rho h$, $\rho_2 = \rho I$, $l = R^{-1}$, ρ is the density, E is the elastic modulus, G is the shear modulus, h is the cross-sectional area, I is the second moment of area of the cross-section and R is the radius of curvature. By ω, φ and ψ we are denoting the longitudinal, vertical

[1] Jacques Antoine Charles Bresse (9 October 1822, Vienne 22 May 1883) was a French civil engineer who specialized in the design and use of hydraulic motors.

and shear angle displacements (see Fig.1). We assume that all the above coefficients are positive constants.

The system is finally completed with initial conditions

$$\begin{cases} \varphi(x,0)=\varphi_0(x), \quad \varphi_t(x,0)=\varphi_1(x), \quad \psi(x,0)=\psi_0(x), \\ \psi_t(x,0)=\psi_1(x), \quad \omega(x,0)=\omega_0(x), \quad \omega_t(x,0)=\omega_1(x), \quad x\in(0,L) \end{cases}$$

where the initial data $(\varphi_0, \varphi_1, \psi_0, \psi_1, \omega_0, \omega_1)$ belong to a suitable function space.

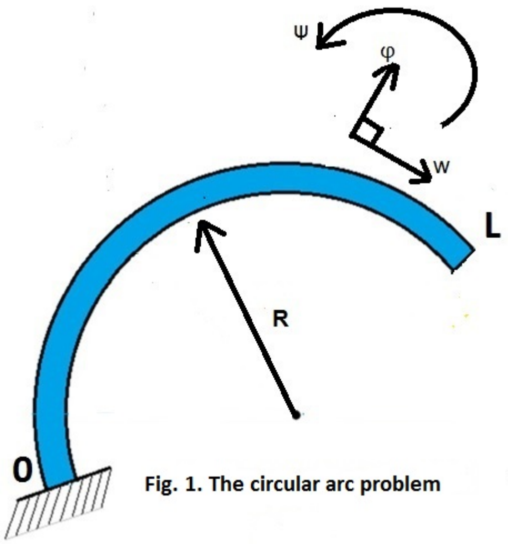

Fig. 1. The circular arc problem

Fig. 2. Example of circular arc problem

The original Bresse system, known as the **circular arch problem**, is given by the following equations (see [**8**] and [**24**]):

$$\begin{cases} \rho_1\varphi_{tt} &= Q_x + lN + F_1, \\ \rho_2\psi_{tt} &= M_x - Q + F_2, \\ \rho_1\omega_{tt} &= N_x - lQ + F_3, \end{cases}$$

where we use N, Q and M to denote the axial force, the shear force and the bending moment respectively. These forces are stress-strain relations for elastic behavior and given by

$$N = Eh(\omega_x - l\varphi), \quad Q = Gh(\varphi_x + \psi + l\omega) \quad \text{and} \quad M = EI\psi_x.$$

By the terms F_i we are denoting external forces (for our system, we consider $F_1 = F_2 = F_3 = 0$).

The Bresse system is more general than the well-known Timoshenko system where the longitudinal displacement ω is not considered (i.e. the curvature l is zero). There are a great number of publications concerning the stabilization of the Timoshenko system with different kinds of damping (see **[22]**, **[31]**, **[32]**, **[35]** and **[39]**). Raposo et al. **[39]** proved the exponential decay of the solution for the following linear system of Timoshenko-type beam equations with linear frictional dissipative terms:

$$\begin{aligned}&\rho_1\varphi_{tt} - Gh(\varphi_x + \psi)_x + \mu_1\varphi_t = 0\\&\rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + \psi) + \widetilde{\mu_1}\psi_t = 0.\end{aligned}$$

Messaoudi and Mustafa **[31]** (see also **[35]**) considered the stabilization for the following Timoshenko system with nonlinear internal feedbacks:

$$\begin{aligned}&\rho_1\varphi_{tt} - Gh(\varphi_x + \psi)_x + g_1(\psi_t) = 0\\&\rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + \psi) + g_2(\psi_t) = 0.\end{aligned}$$

Recently, Park and Kang **[35]** considered the stabilization of the Timoshenko system with weakly nonlinear internal feedbacks.

For the Bresse system, many damping terms have been considered by several authors (see **[41]**, **[42]**, **[26]**, **[1]**, **[17]**, **[34]**, **[25]**, **[45]**, **[14]** and **[2]**). In **[41]**, Soriano, Wenden Charles, Rodrigo Schulz considered a Bresse system with three internal feedbacks. They proved the exponential decay of the solution. The indefinite damping acting on the shear angle displacement was considered by Soriano et al. **[42]**.

In **[26]**, Liu and Rao considered a Bresse system coupled with two heat equations. The two wave equations for the longitudinal displacement and the shear angle displacement are effectively globally damped by the dissipation from the two heat equations. The wave equation about the vertical displacement is subject to a weak thermal damping and indirectly damped through the coupling. They establish exponential energy decay rate when the vertical and the longitudinal waves have the same speed of propagation. Otherwise, a polynomial-type decay is established.

The Bresse system with frictional damping was considered by Alabau-Boussouira et al. [**1**]. They showed that when the velocities are the same, the system is exponentially stable. If not, they proved that the solution goes to zero polynomially.
In [**17**] (see also [**34**]), the authors showed that the Bresse system is not exponentially stable but that there exists polynomial stability with rates that depend on the wave propagations and the regularity of the initial data.
In [**25**], Lima, Soufyane, Almeida Júnior considered a Bresse system with past history acting in the shear angle displacement. They proved the exponential decay of the solution if and only if the wave speeds are the same. Otherwise, a polynomial-type decay is established with optimal decay rate.
Wehbe and Youssef [**45**] considerd the elastic Bresse system with two locally distributed feedbacks and studied the exponentialy and polynomial stability.
Charles et al. [**14**] considered a Bresse system with nonlinear localized damping mechanisms acting in all the three wave equations. They proved an internal observability of the Bresse system by observing locally the vilocities.

However, for a Bresse system with the dissipative effect taking place at the boundary very little is known in the literature, more general and recent results in this direction were obtained in [**2**]. In this paper the authors established a result of exponential stability for a Bresse system with three dissipative effects concentrated at the boundary that is

$$\begin{array}{ll} \varphi(L,t)=0, \quad \psi(L,t)=0, \quad \omega(L,t)=0 & \text{in } (0,+\infty) \\ Gh(\varphi_x+\psi+l\omega)(0,t)=\gamma_1\partial_t\varphi(0,t) & \text{in } (0,+\infty) \\ EI\psi_x(0,t)=\gamma_2\partial_t\psi(0,t) & \text{in } (0,+\infty) \\ Eh(\omega_x-l\varphi)(0,t)=\gamma_3\partial_t\omega(0,t) & \text{in } (0,+\infty) \end{array}$$

where $\gamma_i>0, i=1,2,3$.

The boundary feedback under the consideration here are of fractional type and are described by the fractional derivatives

$$\partial_t^{\alpha,\eta}w(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}e^{-\eta(t-s)}\frac{dw}{ds}(s)\,ds,\ \eta\geq 0.$$

The order of our derivatives is between 0 and 1. Very little is known in the literature for this type of feedback. In addition, fractional derivatives involve singular and nonintegrable kernels ($t^{-\alpha}, 0<\alpha<1$). This leads to substantial mathematical difficulties since all the previous methods and techniques developed for convolution terms with regular and/or integrable kernels are no longer valid.

It has been shown (see [**30**]) that, as ∂_t, the fractional derivative ∂_t^α forces the system to become dissipative and the solution to approach the equilibrium state. Therefore, when applied on the boundary, we can consider them as controllers which help to reduce the vibrations.

In [**29**], B. Mbodje investigated the effect of the boundary feedback of fractional derivative type. He considered

$$\begin{cases} \varphi_{tt}(x,t) - \varphi_{xx}(x,t) = 0 & \text{in } (0,1)\times(0,+\infty) \\ \varphi(0,t) = 0 & \text{in } (0,+\infty) \\ \varphi_x(1,t) = -\gamma\partial_t^{\alpha,\eta}\varphi(1,t) & \text{in } (0,+\infty) \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x) & \text{in } (0,1) \end{cases}$$

and proved strong asymptotic stability of solutions when $\eta = 0$ and polynomial type decay rate of type $\frac{1}{t}$ when $\eta \neq 0$. He used energy method to prove polynomial decay rate.

Boundary dissipations of fractional order or, in general, of convolution type are not only important from the theoretical point of view but also for applications. They naturally arise in hereditary processes and fractal media to describe memory effects and anomalous phenomena (see [**37**]). Indeed, it has been observed by experiments that many concepts cannot be described in Newtonian terms. In other words, in many fields, phenomena with strange kinetics cannot be described within the framework of classical theory using integer-order derivatives. It could lead to a more adequate modeling and more robust control performance. For example, in viscoelasticity, due to the nature of the material microstructure, both elastic solid and viscous fluid like response qualities are involved. Using Boltzmann assumptions, we end up with a stress-strain relationship defined by a time convolution. More precisely, the stress at each point and at each instant does not depend only on the present value of the strain but also on the entire temporal prehistory of the motion from 0 up to time t. This is interpreted by a time convolution with a "relaxation function" as kernel. Viscoelastic response occurs in a variety of materials, such as soils, concrete, rubber, cartilage, biological tissue, glasses, and polymers (see [**4**], [**5**], [**6**] and [**28**]). In our case, the fractional dissipations may simply describe an active boundary viscoelastic damper designed for the purpose of reducing the vibrations.

Our purpose in this thesis is to give a global solvability in Sobolev spaces and energy decay estimates of the solutions to the problem (P) for linear damping. To obtain global solutions to the problem (P), we use the argument combining the semigroup theory ([**11**]) with the energy estimate method. To prove decay estimates, we use a frequency domain approach and a Theorem of A. Borichev and Y. Tomilov.

The thesis is organized as follows :

In **chapter 1**, we collect and present some concepts and results in semi-group theory, fractional derivatives and some results about unboubded linear operators, dissipative operators. In particular, Theorem 1.3.4, Theorem 1.3.5, Theorem 1.3.6, Theorem 1.3.7 and Theorem 1.3.9 will be frequently used throughout this thesis.

In **chapter 2**, we study the existence, uniqueness and stability of solutions for the Bresse system with three control on the boundary conditions given by fractional derivative type. Our main result is to show that the dissipative Bresse system is not exponentially stable when $\eta \geq 0$. Moreover, we show that the solution decays polynomially to zero when $\eta > 0$.

In **chapter 3**, we prove the existence, uniqueness and stability (strong stability) of solutions for the Bresse system with two boundary dissipations of fractional derivative type.

Chapter 1

PRELIMINARIES

In this chapitre, we recall some basic definitions and theorems which will be used in the following chapiters. We refer to [7, 9, 10, 11, 16, 21, 27, 36].

1.1 Sobolev spaces

In many problems of mathematical physics it is not sufficient to deal with the classical solutions of partial differential equations(PDE). It is necessary to introduce the notion of weak derivatives and to work in the so called Sobolev spaces.
We denote by Ω an open domain in $\mathbb{R}^n, n \geq 1$.
We will also use the following multi-index notation for partial differential derivatives of a function:

$$\partial_i^k u = \frac{\partial^k u}{\partial x_i^k} \text{ for all } k \in \mathbb{N} \text{ and } i = 1, ..., n,$$
$$D^\alpha u = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n} u = \frac{\partial^{\alpha_1 + \ldots + \alpha_n} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}},$$
$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n, |\alpha| = \alpha_1 + \ldots + \alpha_n.$$

Definition 1.1.1 *For $1 \leq p \leq \infty$, we call $L^p(\Omega)$ the space of measurable functions f on Ω such that*

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p dx \right)^{1/p} < +\infty \quad \textit{for} \quad p < +\infty$$

$$\|f\|_{L^\infty(\Omega)} = \sup_\Omega |f(x)| < +\infty \qquad \textit{for} \quad p = +\infty$$

The space $L^p(\Omega)$ equipped with the norm $f \longrightarrow \|f\|_{L^p}$ is a Banach space: it is reflexive and separable for $1 < p < \infty$ (its dual is $L^{\frac{p}{p-1}}(\Omega)$), separable but not reflexive for $p = 1$ (its dual is $L^\infty(\Omega)$), and not separable, not reflexive for $p = \infty$ (its dual contains strictly $L^1(\Omega)$). In particular the space $L^2(\Omega)$ is a Hilbert space equipped with the scalar product defined by

$$(f, g)_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx.$$

Definition 1.1.2 *The Sobolev space $W^{m,p}(\Omega)$ is defined to be the subset of L^p such that function f and its weak derivatives up to some order m have a finite L^p norm, for given $p \geq 1$.*

$$W^{m,p}(\Omega) = \{f \in L^p(\Omega); D^\alpha f \in L^p(\Omega).\ \ \forall\alpha; |\alpha| \leq m\} \ ,$$

With this definition, the Sobolev spaces admit a natural norm,

$$f \longrightarrow \|f\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha|\leq m} \|D^\alpha f\|^p_{L^p(\Omega)}\right)^{1/p} \ , \text{ for } p < +\infty$$

and

$$f \longrightarrow \|f\|_{W^{m,\infty}(\Omega)} = \sum_{|\alpha|\leq m} \|D^\alpha f\|_{L^\infty(\Omega)} \ , \text{ for } p = +\infty$$

The space $W^{m,p}(\Omega)$ equipped with the norm $\| \ . \ \|_{W^{m,p}}$ is a Banach space. Moreover is a reflexive space for $1 < p < \infty$ and a separable space for $1 \leq p < \infty$.

Remark 1.1.1 *Sobolev spaces $W^{m,p}(\Omega)$ with $p = 2$ are especially important because of their connection with Fourier series and because they form a Hilbert space. A special notation has arisen to cover this case:*

$$W^{m,2}(\Omega) = H^m(\Omega)$$

the H^m inner product is defined in terms of the L^2 inner product:

$$(f, g)_{H^m(\Omega)} = \sum_{|\alpha|\leq m} (D^\alpha f, D^\alpha g)_{L^2(\Omega)} \ .$$

1.2 M-Dissipative operators

In this section we introduce unbounded operators and put together some properties which will be frequently used.

1.2.1 Unboubded Linear Operators on Banach space

Let X and Y be two Banach spaces.

Definition 1.2.1 *An unbounded linear operator from X into Y is linear map $A : D(A) \subset X \to Y$ defined on a subspace $D(A) \subset X$ with values in Y. The set $D(A)$ is called the domain of the operateur A.*

If $X = Y$, we shall simply say that A is an unbounded linear operator on X.

Definition 1.2.2 *One says that A is bounded if $D(A) = X$ and if there is a constant $C \geq 0$ such that*

$$\|A\,x\|_Y \leq C\ \|x\|_X \quad \forall x \in X$$

The set of all bounded linear operators from X into Y is denoted by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$. Moreover, the set of all bounded linear operators from X into X is denoted by $\mathcal{L}(\mathcal{X})$.
The norm of a bounded linear operator is defind by

$$\|A\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} = \underset{x\neq 0}{Sup}\ \frac{\|Ax\|_Y}{\|x\|_X}$$

Definition 1.2.3 *Let $A : D(A) \subset X \to Y$ be an unbounded linear operator. We dfine*

Graph of A : $G(A) = \{(x\,,\,Ax)\ :\ x \in D(A)\} \subset X \times Y$,

Range of A : $R(A) = \{Ax\ :\ x \in D(A)\} \subset Y$,

Kernal of A : $N(A) = \{x \in D(A)\ :\ Ax\ = 0\} \subset X$.

Definition 1.2.4 *An unbounded linear operator A is a closed operateur if its graph $G(A)$ is closed in $X \times Y$*

Definition 1.2.5 *Let $A : D(A) \subset X \to Y$ be an unbounded linear operator. We say that A is a densely defined operateur in X, or A is an operateur with dense domaine in X, if $D(A)$ is dense in X,i.e., $\overline{D(A)} = X$.*

Definition 1.2.6 *Let $A : D(A) \subset X \to Y$ be a densely defined operator in X. The adjoint operator of A is the operator $A^* : D(A^*) \subset Y' \to X'$ defined by*

$$D(A^*) = \{y \in Y'\ :\ \exists\, C \geq 0\, such\, that\, \langle Ax\,,\,y\rangle_{Y\times Y'} \leq C\,\|x\|_X\, for\, all\, x \in D(A)\,\}$$

and

$$\langle x, A^*y\rangle_{X\times X'} = \langle Ax, y\rangle_{Y\times Y'} \quad \forall x \in D(A), \quad \forall y \in D(A^*)$$

Definition 1.2.7 *A bounded linear operator $A : X \to Y$ is said to be compact if $T(B_X)$ has compact closure in Y.*

The set of all compact operators from X into Y is denoted by $K(X, Y)$. Moreover, the set $K(X, X)$ is denoted by $K(X)$.

Theorem 1.2.1 (Fredholm alternative) . *Let $A \in K(X)$. Then:*

1. *$N(I - A)$ is finite-dimensionel,*
2. *$N(I - A)$ is closed and $R(I - A) = N(I - A^*)^{\perp}$,*
3. *$N(I - A) = 0 \Leftrightarrow R(I - A) = X$,*
4. *$\dim N(I - A) = \dim N(I - A^*)$.*

Remark 1.2.1 *The Fredholm alternative deals with solvability of the equation $u - Au = f$.*

1.2.2 The Resolvent set and the Spectrum of Linear Operators

Let X be a Banach space, and A be a closed unbounded operator on X.

Definition 1.2.8 *The resolvent set of A is given by*

$$\rho(A) = \{\lambda \in \mathbb{C};\ \lambda I - A : D(A) \to X \textit{ is bijective}\}$$

and its spectrum by

$$\sigma(A) = \mathbb{C} \backslash \rho(A)$$

If $\lambda \in \rho(A)$, then $R(\lambda, A) = (\lambda I - A)^{-1}$ is called the resolvent of A (at λ).

Remark 1.2.2 *The numbers in $\rho(A)$ are called regular values of A.*

Theorem 1.2.2 *The sets $\rho(A)$ and $\sigma(A)$ are open and closed, respectively.*

Definition 1.2.9 *The point spectrum or ponctuel spectrum of A is defined by*

$$\begin{aligned}\sigma_p(A) &= \{\lambda \in \mathbb{C} : \textit{ there exists some } v \in D(A)\backslash\{0\} \textit{ with } Av = \lambda v\} \\ &= \{\lambda \in \mathbb{C} : N(\lambda I - A) \neq \{0\}\} \subset \sigma(A)\end{aligned}$$

Remark 1.2.3 *If $\lambda \in \sigma_p(A)$, then there exists a vecteur $v \neq 0$ such that $(\lambda I - A)v = 0$, i.e., $Av = \lambda v$. Such a vector is called un eigenvector of A and the corresponding number λ an eigenvalue of A.*

Definition 1.2.10 *The continuous spectrum $\sigma_c(A)$ is the set of all numbers $\lambda \in \mathbb{C}$ such that $N(\lambda I - A) = 0$, $R(\lambda I - A) \neq X$, but $\overline{R(\lambda I - A)} = X$.*

Definition 1.2.11 *The residual spectrum $\sigma_r(A)$ is the set of all numbers $\lambda \in \mathbb{C}$ such that $N(\lambda I - A) = 0$ and $\overline{R(\lambda I - A)} \neq X$.*

Remark 1.2.4 *It is apparent that the sets $\sigma_p(A)$, $\sigma_c(A)$, $\sigma_r(A)$ are disjoint, and that*

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A)$$

Proposition 1.2.1 (Spectrum of the adjoint operator) *Let H be a Hilbert space, and $A \in \mathcal{L}(\mathcal{H})$. Then:*

(i) $\lambda \in \rho(A) \Leftrightarrow \overline{\lambda} \in \rho(A^*)$,

(ii) $\lambda \in \sigma_p(A) \Rightarrow \overline{\lambda} \in \sigma_p(A^*) \bigcup \sigma_r(A^*)$,

(iii) $\lambda \in \sigma_r(A) \Rightarrow \overline{\lambda} \in \sigma_p(A^*)$,

(iv) $\lambda \in \sigma_c(A) \Leftrightarrow \overline{\lambda} \in \sigma_c(A^*)$.

1.2.3 M-Dissipative Operators on Hilbert spaces

Let H be a Hilbert space equiped with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$.

Definition 1.2.12 *An unbounded linear operator $A : D(A) \subset H \to H$ is said to be dissipative if*

$$\forall x \in D(A), \quad \langle Ax, x \rangle_{\mathcal{H}} \leq 0$$

Remark 1.2.5 *For a complex Hilbert space the previous condition is replaced by*

$$\forall x \in D(A), \quad \Re e \langle Ax, x \rangle_{\mathcal{H}} \leq 0$$

Definition 1.2.13 *An unbounded linear operator $A : D(A) \subset H \to H$, is m-dissipative (or maximal dissipative) if*

1. *A is dissipative,*

2. *$\lambda I - A$ is surjective fo every $\lambda > 0$, i.e., $\forall y \in H$, $\forall \lambda > 0$, $\exists x \in D(A)$ such that*

$$\lambda x - Ax = y$$

Theorem 1.2.3 *Let $A : D(A) \subset H \to H$ be an unbounded linear dissipative operator. The operator A is m-dissipative if and only if $\exists \lambda_0 > 0$ such that $\lambda_0 I - A$ is surjective, i.e., $R(\lambda_0 I - A) = H$.*

Theorem 1.2.4 *If $A : D(A) \subset H \to H$ is an m-dissipative operator, then*

1. *A is closed operator,*

2. *$D(A)$ is dense in H, i.e., $\overline{D(A)} = X$,*

3. *$]0\,, +\infty\,[\subseteq \rho(A)$.*

1.3 Semigroups of Linear Operators in Banach space

In this section we introduce semigroups and their generators.
Let X be a Banach space, and H be a Hilbert space equiped with the inner product $(\cdot,\cdot)_H$ and the iduced norm $\|\cdot\|_H$.

1.3.1 Strongly Continuous Semigroups Generated by Dissipative Operator

We consider the linear Cauchy problem

$$(C) \qquad \begin{cases} u'(t) = Au(t) \\ u(0) = u_0 \end{cases}$$

where A is an unbounded operator on X. By using operator semigroup theory, we establish some results about the existence and uniqueness of solution of (C).

Definition 1.3.1 *A family of bounded linear operators $(S(t))_{t\geq 0}$ on X is a semigroup of bounded linear operators on X if*

1. *$S(0) = I$,*

2. $S(t+s) = S(t)S(s)$ *for every* $s\,,\, t \geq 0$.

Remark 1.3.1 *It follows immediately from the definition that*

$$S(t)S(s) = S(s)S(t)\,,\ \forall t, s \geq 0$$

Definition 1.3.2 *A semigroup* $(S(t))_{t\geq 0}$ *is uniformly continuous if*

$$\lim_{t\to 0^+} \|S(t) - I\|_{\mathcal{L}(\mathcal{X})} = 0$$

Definition 1.3.3 *A semigroup* $(S(t))_{t\geq 0}$ *is a* C_0*-semigroup (or a strongly continuous semigroup) if*

$$\lim_{t\to 0^+} \|S(t)x - x\|_{\mathcal{X}} = 0$$

Theorem 1.3.1 *Let* $(S(t))_{t\geq 0}$ *be a* C_0*-semigroup. Then there exist two constants* $M \geq 1$ *and* $\omega \in \mathbb{R}$ *such that*

$$\|S(t)\|_{\mathcal{L}(\mathcal{X})} \leq Me^{\omega t}\,,\ \ \forall t \geq 0$$

Remark 1.3.2 *If* $\omega = 0$*,i.e.,*

$$\|S(t)\|_{\mathcal{L}(\mathcal{H})} \leq M\,,\ \forall t \geq 0$$

then $(S(t))_{t\geq 0}$ *is called a uniformly bounded* C_0*-semigroup.*

If $\omega = 0$ *and* $M = 1$*,i.e.,*

$$\|S(t)\|_{\mathcal{L}(\mathcal{H})} \leq 1\,,\ \forall t \geq 0$$

then $(S(t))_{t\geq 0}$ *is called a strongly continuous semigroup (or* C_0*-semigroup) of contractions.*

We now define the generator of semigroup.

Definition 1.3.4 *Let* $(S(t))_{t\geq 0}$ *be a* C_0*-semigroup. The infinitesimal generator of the semigroup* $(S(t))_{t\geq 0}$ *is the linear operator* A *defined by*

$$D(A) = \{x \in X\ :\ \lim_{t\to 0^+} \frac{S(t)x - x}{t}\ exists\,in\,X\}$$

and

$$Ax = \lim_{t\to 0^+} \frac{S(t)x - x}{t}, \forall x \in D(A).$$

Remark 1.3.3 *Sometimes we also denote* $S(t)$ *by* e^{At}.

Theorem 1.3.2 *Let $(S(t))_{t\geq 0}$ be a C_0-semigroup and let A be its infinitesimal generator. Then*

$$S(t)x \in D(A)$$

and

$$\frac{d}{dt}S(t)x = AS(t)x = S(t)Ax$$

for $x \in D(A)$ and $t \geq 0$

Remark 1.3.4 *From the above theorem, the solution to the initial value problem (C) admits the following representation*

$$u(t) = S(t)u_0 = e^{At}u_0 \quad \forall t \geq 0$$

The following theorems (Theorem 1.3.3 and Theorem 1.3.4) gives a necessary and sufficient condition for an operator to be the generator of a C_0-semigroup (see Pazy [36]).

Theorem 1.3.3 (Hill-Yosida Theorem in Banach spaces) *An unbounded linear operator $A : D(A) \subset X \to X$ is the infinitesimal generator of a semigroup of contractions if and only if*

1. *$D(A)$ is dense in X, i.e.,$\overline{D(A)} = X$,*

2. *A is a closed operatour,*

3. *The resolvent set $\rho(A)$ of A contains $\mathbb{R}_+$ and for every $\lambda > 0$,*

$$\|R(\lambda, A)\|_{\mathcal{L}(\mathcal{X})} \leq \frac{1}{\lambda}$$

Theorem 1.3.4 (Lumer-Phillips Theorem in Hilbert spaces) *An unbounded linear operator $A : D(A) \subset H \to H$ is the infinitesimal generator of a semigroup of contractions if and only if A is m-dissipative operator.*

The existence and uniqueness of the solution of the initial value problem (C) is justified by the following theorem.

Theorem 1.3.5 (Hill-Yosida Theorem) *Let $A : D(A) \subset H \to H$ be an unbounded linear operator. If A is the infinitesimal generator of $(S(t))_{t\geq 0}$, a C_0-semigroup of contraction, (or A is m-dissipative operator), then*

1. *If $U_0 \in D(A)$, then the initial value problem (C) has a unique strong solution*

$$U \in C^0(\mathbb{R}_+, D(A)) \cap C^1(\mathbb{R}_+, H).$$

2. *If $U_0 \in H$, then the initial value problem (C) has a unique weak solution*

$$U \in C^0(\mathbb{R}_+, H).$$

1.3.2 Stability of Semigroups

The stability theory of semigroups provides powerful tools for the investigation of the convergence to 0 of weak and strong solutions of linear Cauchy problem

$$(C) \qquad \begin{cases} u'(t) = Au(t) \\ u(0) = u_0 \end{cases}$$

where A generates the C_0-semigroup of contraction $(S(t))_{t\geq 0}$ on a Hilbert space H.
In this section, we introduce the notions of stability that will be used throughout this thesis. Let $(S(t))_{t\geq 0}$ be a C_0-semigroup of contractions on a H and let A be its infinitesimal generator.

Definition 1.3.5 (Strong stability) *We say that the semigoup $(S(t))_{t\geq 0}$ is strongly (or asymptotically) stable if for all $x \in H$*

$$\lim_{t\to+\infty} \|e^{tA}x\|_H = 0$$

Definition 1.3.6 (Exponential stability) *We say that the semigoup $(S(t))_{t\geq 0}$ is exponentially (or uniformly) stable if there exist $\alpha, M > 0$ such that*

$$\|S(t)x\|_H \leq Me^{-\alpha t}\,, \ \ \forall t \geq 0, \forall x \in H$$

Definition 1.3.7 (Polynomial stability) *We say that the semigoup $(S(t))_{t\geq 0}$ is polynomially stable if there exist $\beta, C > 0$ such that*

$$\|S(t)x\|_H \leq \frac{C}{t^\beta}\|x\|_H\,, \ \ \forall t \geq 0, \forall x \in H$$

The following theorem (a general criteria of Arendt-Batty) gives a necessary conditions for a strong stability of the C_0-semigroup (see [3]).

Theorem 1.3.6 (Arendt-Batty) *Let A be the generator of a uniformly bounded C_0-semigroup $(S(t))_{t\geq 0}$ on a Hilbert space H. If:*

(i) A does not have eigenvalues on $i\,\mathbb{R}$.

(ii) The intersection of the spectrum $\sigma(A)$ with $i\,\mathbb{R}$ is at most a countable set.

Then the semigroup $(S(t))_{t\geq 0}$ is strongly (or asymptotically) stable, i.e, $\|S(t)z\|_H \to 0$ as $t \to \infty$ for any $z \in H$.

When the C_0-semigroup is asymptotically, we look the type of stability (exponential or polynomial) of the semigroup (see [38], [27] and [7]).

Theorem 1.3.7 (Huang-Pruss) *Let $S(t) = e^{At}$ be a C_0-semigroup of contractions on Hilbert space H. Then $(S(t))_{t\geq 0}$ is exponentially stable if and only if*

$$\rho(A) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R}$$

and

$$\overline{\lim_{|\beta|\to\infty}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$$

This theorem is equivalent to the following theorem:

Theorem 1.3.8 *Let $S(t) = e^{At}$ be a C_0-semigroup of contractions on Hilbert space H. Then $(S(t))_{t\geq 0}$ is exponentially stable if and only if*

$$Sup\,\{\mathrm{Re}\,\lambda,\ \lambda \in \sigma\,(A)\} < 0$$

and

$$\underset{\mathrm{Re}\,\lambda\geq 0}{Sup}\ \|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$$

Theorem 1.3.9 (Borichev-Tomilov) *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup on a Hilbert space H. If*

$$i\mathbb{R} \subset \rho(A) \text{ and } \sup_{|\beta|\geq 1} \frac{1}{\beta^{l'}} \|(i\beta I - A)^{-1}\|_{\mathcal{L}(\mathcal{H})} < M$$

for some l', then there exist c such that

$$\|e^{At}u_0\|^2 \leq \frac{c}{t^{\frac{2}{l'}}} \|u_0\|^2_{D(A)}\,, \quad \forall t > 0, \forall u_0 \in D(A)$$

1.4 Lax-Milgrame Theorem

Let H be a Hilbert space equiped with the inner product $(\cdot,\cdot)_H$ and the iduced norm $\|\cdot\|_H$.

Definition 1.4.1 *A bilinear form*

$$a:\ H \times H \to \mathbb{R}$$

is said to be

(i) continuous if there is a constant C such that

$$|a(u\,,\,v)| \leq C\,\|u\|\,\|v\|, \forall u, v \in H$$

(ii) coercive if there is a constant $\alpha > 0$ such that

$$a(u\,,\,u) \geq \alpha\,\|u\|^2, \forall u \in H$$

Theorem 1.4.1 (Lax-Milgrame Theorem) *Assume that $a(\cdot,\cdot)$ is a continuous coercive bilinear form on H. Then, given any $L \in \mathcal{L}(\mathcal{H},\mathbb{C})$, there exists a unique element $u \in H$ such that*

$$a(u\,,\,v) = L(v), \forall v \in H$$

1.5 Fractional Derivatives: Basic definitions

Fractional derivative, or more precisely derivative of non-integer order, is a generalization of ordinary derivation. The fractional derivatives have been used in various fields of science and engineering, for example in electronics, wave propagation, mechanics, biology, biophysics and viscoelasticity (see [12], [4], [5], [6], [28], [18], [40], [20] and [37]).

In this part, we recall some basic notations and definitions for the fractional derivative (see [21], [33]).

1.5.1 A brief historical introduction to fractional derivatives

In a letter to the French mathematician L'Hospital (1659), Leibniz raised the following question: "Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?" L'Hospital was some what curious about that question and replied by another question to Leibniz: "What if the order will be $\frac{1}{2}$?" Leibnitz in a letter dated September 30, replied: "It will lead to a paradox, from which one day useful consequences will be drawn. Many known mathematicians contributed to this theory over the years. Thus, September 30, 1695 is the exact date of birth of the fractional calculus. Therefore, the fractional calculus it its origin in the works by Leibnitz, L'Hopital (1695), Bernoulli (1697), Euler (1730), and Lagrange (1772). Some years later, Laplace (1812), Fourier (1822), A-bel (1823), Liouville (1832), Riemann (1847), Grünwald (1867), Letnikov (1868), Nekrasov (1888), Hadamard (1892), Heaviside (1892), Hardy (1915), Weyl (1917), Riesz (1922), P. Levy (1923), Davis (1924), Kober (1940), Zygmund (1945), Kuttner (1953), J. L. Lions (1959), and Liverman (1964)...have developed the basic concept of fractional derivatives.

In 1783, Leonhard Euler made his first comments on fractional order derivative. He worked on progressions of numbers and introduced first time the generalization of factorials to *Gamma* function. A little more than fifty year after the death of Leibniz, Lagrange, in 1772, indirectly contributed to the development of exponents law for differential operators of integer order, which can be transferred to arbitrary order under certain conditions. In 1812, Laplace has provided the first detailed definition for fractional derivative. Laplace states that fractional derivative can be defined for functions with representation by an integral, in modern notation it can be written as $\int f(t)t^{-x}dt$. Few years after, Lacroix worked on generalizing the integer order derivative of function $f(t) = t^m$ to fractional order, where m is some natural number. In modern notations, integer order n^{th} derivative derived by *Lacroix* can be given as

$$\frac{d^n f}{dt^n} = \frac{m!}{(m-n)!}t^{m-n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}t^{m-n}, \ m > n$$

where, Γ is the *Gamma function.* Thus, for $n = \frac{1}{2}$ and $m = 1$, one obtains the derivative of order $\frac{1}{2}$ of the function $f(t) = t$

$$\frac{d^{\frac{1}{2}}f(t)}{dt^{\frac{1}{2}}} = \frac{\Gamma(2)}{\Gamma(\frac{3}{2})}t^{\frac{1}{2}} = \frac{2}{\sqrt{\pi}}\sqrt{t}$$

In the period 1900-1970 a modest amount of published work appeared on the subject of the fractional derivative. The year 1974 saw the first international conference on fractional calculus held at the University of New Haven.

In the period 1975 to the present, many papers have been published relating to the application of the fractional derivative to ordinary and partial differential equations.

1.5.2 Some notations and definitions of Fractional derivatives

In this section, we give the definition of the generalized Caputo's fractional derivative and the generalized fractional integral.

Definition 1.5.1 *The Gamma function, denoted by* Γ, *is given by*

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$$

The exponential provides the convergence of this integral in ∞, the convergence at zero obviously occurs for all complex z from the right half of the complex plane ($\Re e(z) > 0$).
The Gamma function is generalization of a factorial in the following form

$$\Gamma(n) = (n-1)!$$

Remark 1.5.1 (Some usefull identities) *We have*

$$\Gamma(z+1) = z\Gamma(z)$$

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}$$

Definition 1.5.2 *The fractional derivative of order* α, $0 < \alpha < 1$, *in sens of Caputo, is defined by*

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{df}{ds}(s)\, ds$$

Definition 1.5.3 *The fractional integral of order* α, $0 < \alpha < 1$, *in sense Riemann-Liouville, is defined by*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds$$

Remark 1.5.2 *From the above definitions, clearly*

$$D^\alpha f = I^{1-\alpha} Df, \quad 0 < \alpha < 1$$

Lemma 1.5.1

$$I^\alpha D^\alpha f(t) = f(t) - f(0), \quad 0 < \alpha < 1$$

Lemma 1.5.2 *If*

$$D^\beta f(0) = 0$$

then

$$D^\alpha D^\beta f = D^{\alpha+\beta} f, \quad 0 < \alpha < 1, \quad 0 < \beta < 1$$

Now, we give the definitions of the generalized Caputo's fractional derivative and the generalized fractional integral. These exponentially modified fractional integro-differential operators were first proposed in [15].

Definition 1.5.4 *The generalized Caputo's fractional derivative is given by*

$$D^{\alpha,\eta} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{df}{ds}(s)\, ds, \qquad 0 < \alpha < 1, \quad \eta \geq 0$$

Remark 1.5.3 *The operators D^{α} and $D^{\alpha,\eta}$ differ just by their kernels.*

Definition 1.5.5 *The generalized fractional integral is given by*

$$I^{\alpha,\eta} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\eta(t-s)} f(s)\, ds, \qquad 0 < \alpha < 1, \quad \eta \geq 0$$

Remark 1.5.4 *We have*

$$D^{\alpha,\eta} f = I^{1-\alpha,\eta} D f, \qquad 0 < \alpha < 1, \quad \eta \geq 0$$

Chapter 2

GLOBAL EXISTENCE AND ENERGY DECAY OF SOLUTIONS TO A BRESSE SYSTEM WITH THREE BOUNDARY DISSIPATIONS OF FRACTIONAL DERIVATIVE TYPE [1]

2.1 Introduction

In this chapiter, we consider a linear Bresse system (in one-dimensionel case) of the type

$$(P) \qquad \begin{cases} \rho_1\varphi_{tt} - Gh(\varphi_x + \psi + l\omega)_x - lEh(\omega_x - l\varphi) = 0 \\ \rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + \psi + l\omega) = 0 \\ \rho_1\omega_{tt} - Eh(\omega_x - l\varphi)_x + lGh(\varphi_x + \psi + l\omega) = 0 \end{cases}$$

[1] A.Benaissa and A.Kasmi, Well-posedeness and energy decay of solutions to a Bresse system with a boundary dissipation of fractional derivative type, *Discrete and Continuous Dynamical Systems B* doi:10.3934/dcdsb.2018168. **Pages:1-35** 2018.

where $(x,t) \in (0,L) \times (0,+\infty)$. This system is subject to the boundary conditions

$$(2.1) \quad \begin{array}{ll} \varphi(0,t) = 0, \quad \psi(0,t) = 0, \quad \omega(0,t) = 0 & \text{in } (0,+\infty) \\ Gh(\varphi_x + \psi + l\omega)(L,t) = -\gamma_1 \partial_t^{\alpha,\eta}\varphi(L,t) & \text{in } (0,+\infty) \\ EI\psi_x(L,t) = -\gamma_2 \partial_t^{\alpha,\eta}\psi(L,t) & \text{in } (0,+\infty) \\ Eh(\omega_x - l\varphi)(L,t) = -\gamma_3 \partial_t^{\alpha,\eta}\omega(L,t) & \text{in } (0,+\infty) \end{array}$$

where $\gamma_i > 0, i = 1,2,3$. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α with respect to the time variable. It is defined as follows

$$(2.2) \quad \partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\, ds, \;\; \eta \geq 0.$$

These exponentially modified fractional integro-differential operators were first proposed in Choi and MacCamy [**15**]. In other words, we investigate three dissipative effects at the boundary. The system is finally completed with initial conditions

$$(2.3) \quad \begin{cases} \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad \psi(x,0) = \psi_0(x), \\ \psi_t(x,0) = \psi_1(x), \quad \omega(x,0) = \omega_0(x), \quad \omega_t(x,0) = \omega_1(x), \quad x \in (0,L) \end{cases}$$

where the initial data $(\varphi_0, \varphi_1, \psi_0, \psi_1, \omega_0, \omega_1)$ belong to a suitable function space. By ω, φ and ψ we are denoting the longitudinal, vertical and shear angle displacements. The coefficients ρ_1, ρ_2, E, G, h, I and l are positive constants characterizing the physical properties of the beam. We note that when $l = 0$, the Bresse model reduces to well-known Timoshenko system.

In our first main result (Theorem 2.3.1), we show that the system (P) is well posed. Our second main result (Theorem 2.4.1) is proving that the dissipation generated by (2.1) can stabilize the system (P).

2.2 Augmented model

This section is concerned with the reformulation of the model (P) into an augmented system. For that, we need the following claims.

Theorem 2.2.1 (see [29]) *Let μ be the function:*

$$(2.4) \quad \mu(\xi) = |\xi|^{(2\alpha-1)/2}, \quad -\infty < \xi < +\infty, \;\; 0 < \alpha < 1.$$

Then the relationship between the 'input' U and the 'output' O of the system

$$(2.5) \quad \partial_t\phi(\xi,t) + \xi^2\phi(\xi,t) + \eta\phi(\xi,t) - U(t)\mu(\xi) = 0, \quad -\infty < \xi < +\infty, \eta \geq 0, t > 0,$$

(2.6)
$$\phi(\xi, 0) = 0,$$

(2.7)
$$O(t) = (\pi)^{-1}\sin(\alpha\pi)\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi, t)\,d\xi$$

is given by

(2.8)
$$O = I^{1-\alpha,\eta}U$$

where

(2.9)
$$[I^{\alpha,\eta}f](t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}e^{-\eta(t-\tau)}f(\tau)\,d\tau$$

Proof. See Appendix 3.

Lemma 2.2.1 *If* $\lambda \in D = \{\lambda \in \mathbb{C} : Re\lambda + \eta > 0\} \cup \{\lambda \in \mathbb{C} : Im\lambda \neq 0\}$ *then*

$$g(\lambda) = \int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\lambda + \eta + \xi^2}\,d\xi = \frac{\pi}{\sin\alpha\pi}(\lambda + \eta)^{\alpha-1}.$$

Proof. Let us set

$$f_\lambda(\xi) = \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2}.$$

We have

$$\left|\frac{\mu^2(\xi)}{\lambda + \eta + \xi^2}\right| \leq \frac{\mu^2(\xi)}{Re\lambda + \eta + \xi^2}.$$

Then the function f_λ is integrable. Moreover

$$\left|\frac{\mu^2(\xi)}{\lambda + \eta + \xi^2}\right| \leq \begin{cases} \dfrac{\mu^2(\xi)}{\eta_0 + \eta + \xi^2} \text{ for all } Re\lambda \geq \eta_0 > -\eta \\ \dfrac{\mu^2(\xi)}{\tilde{\eta}_0 + \xi^2} \text{for all } |Im\lambda| \geq \tilde{\eta}_0 > 0 \end{cases}$$

From Theorem 1.16.1 in [**44**], the function

$$g : D \to \mathbb{C} \text{ is holomorphe.}$$

For a real number $\lambda > -\eta$, we have

$$\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\lambda + \eta + \xi^2}\,d\xi = \int_{-\infty}^{+\infty}\frac{|\xi|^{2\alpha-1}}{\lambda + \eta + \xi^2}\,d\xi = \int_0^{+\infty}\frac{x^{\alpha-1}}{\lambda + \eta + x}\,dx \text{ (with } \xi^2 = x)$$
$$= (\lambda + \eta)^{\alpha-1}\int_1^{+\infty} y^{-1}(y-1)^{\alpha-1}\,dy \text{ (with } y = x/(\lambda + \eta) + 1)$$
$$= (\lambda + \eta)^{\alpha-1}\int_0^1 z^{-\alpha}(1-z)^{\alpha-1}\,dz \text{ (with } z = 1/y)$$
$$= (\lambda + \eta)^{\alpha-1}B(1-\alpha, \alpha) = (\lambda + \eta)^{\alpha-1}\Gamma(1-\alpha)\Gamma(\alpha) = (\lambda + \eta)^{\alpha-1}\frac{\pi}{\sin\pi\alpha}.$$

Both holomorphic functions g and $\lambda \mapsto (\lambda+\eta)^{\alpha-1}\dfrac{\pi}{\sin \pi\alpha}$ coincide on the half line $]-\eta, +\infty[$, hence on D following the principle of isolated zeroes.

We are now in a position to reformulate system (P). Indeed, by using Theorem 2.2.1, system (P) may be recast into the augmented model:

$$
(P') \quad \begin{cases}
\rho_1\varphi_{tt} - Gh(\varphi_x + \psi + l\omega)_x - lEh(\omega_x - l\varphi) = 0 \\
\partial_t\phi_1(\xi, t) + (\xi^2 + \eta)\phi_1(\xi, t) - \varphi_t(L, t)\mu(\xi) = 0 \\
\rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + \psi + l\omega) = 0 \\
\partial_t\phi_2(\xi, t) + (\xi^2 + \eta)\phi_2(\xi, t) - \psi_t(L, t)\mu(\xi) = 0 \\
\rho_1\omega_{tt} - Eh(\omega_x - l\varphi)_x + lGh(\varphi_x + \psi + l\omega) = 0 \\
\partial_t\phi_3(\xi, t) + (\xi^2 + \eta)\phi_3(\xi, t) - \omega_t(L, t)\mu(\xi) = 0 \\
\varphi(0, t) = 0, \quad \psi(0, t) = 0, \quad \omega(0, t) = 0 \\
\varphi(x, 0) = \varphi_0(x), \quad \varphi_t(x, 0) = \varphi_1(x), \quad \psi(x, 0) = \psi_0(x), \\
\psi_t(x, 0) = \psi_1(x), \quad \omega(x, 0) = \omega_0(x), \quad \omega_t(x, 0) = \omega_1(x), \\
\phi_1(\xi, 0) = \phi_{01}(\xi) = 0, \quad \phi_2(\xi, 0) = \phi_{02}(\xi) = 0, \quad \phi_3(\xi, 0) = \phi_{03}(\xi) = 0, \\
Gh(\varphi_x + \psi + l\omega)(L, t) = -\gamma_1(\pi)^{-1}\sin(\alpha\pi)\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_1(\xi, t)\, d\xi \\
EI\psi_x(L, t) = -\gamma_2(\pi)^{-1}\sin(\alpha\pi)\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_2(\xi, t)\, d\xi \\
Eh(\omega_x - l\varphi)(L, t) = -\gamma_3(\pi)^{-1}\sin(\alpha\pi)\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_3(\xi, t)\, d\xi.
\end{cases}
$$

Our main interest is the existence, uniqueness and regularity of the solution of this system. We define the energy associated to the solution of the problem (P') by the following formula:

$$
\begin{aligned}
\mathcal{E}(t) = {} & \frac{\rho_1}{2}\|\varphi_t\|_2^2 + \frac{\rho_2}{2}\|\psi_t\|_2^2 + \frac{\rho_1}{2}\|\omega_t\|_2^2 + \frac{EI}{2}\|\psi_x\|_2^2 + \frac{Gh}{2}\|\varphi_x + \psi + l\omega\|_2^2 + \frac{Eh}{2}\|\omega_x - l\varphi\|_2^2 \\
& +(\pi)^{-1}\sin(\alpha\pi)\sum_{i=1}^{3}\frac{\gamma_i}{2}\int_{-\infty}^{+\infty}(\phi_i(\xi, t))^2\, d\xi.
\end{aligned} \tag{2.10}
$$

Lemma 2.2.2 *Let $(\varphi, \phi_1, \psi, \phi_2, \omega, \phi_3)$ be a solution of the problem (P'). Then, the energy functional defined by (2.10) satisfies*

$$
\mathcal{E}'(t) = -(\pi)^{-1}\sin(\alpha\pi)\sum_{i=1}^{3}\gamma_i\int_{-\infty}^{+\infty}(\xi^2 + \eta)(\phi_i(\xi, t))^2\, d\xi. \tag{2.11}
$$

Proof.

Multiplying the first equation in (P') by φ_t, the third equation by ψ_t, the five equation by ω_t, integrating over $(0, L)$ and using integration by parts, we get

$$
\frac{1}{2}\rho_1\frac{d}{dt}\|\varphi_t\|_2^2 - Gh\int_0^L(\varphi_x + \psi + l\omega)_x\varphi_t dx - lEh\int_0^L(\omega_x - l\varphi)\varphi_t dx = 0
$$

$$\frac{1}{2}\rho_2\frac{d}{dt}\|\psi_t\|_2^2 - EI\int_0^L \psi_{xx}\psi_t\,dx + Gh\int_0^L(\varphi_x+\psi+l\omega)\psi_t dx = 0$$
$$\frac{1}{2}\rho_1\frac{d}{dt}\|\omega_t\|_2^2 - Eh\int_0^L(\omega_x-l\varphi)_x\omega_t dx + lGh\int_0^L(\varphi_x+\psi+l\omega)\omega_t dx = 0.$$

Then

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\rho_1}{2}\|\varphi_t\|_2^2+\frac{\rho_2}{2}\|\psi_t\|_2^2+\frac{\rho_1}{2}\|\omega_t\|_2^2+\frac{EI}{2}\|\psi_x\|_2^2+\frac{Gh}{2}\|\varphi_x+\psi+l\omega\|_2^2+\frac{Eh}{2}\|\omega_x-l\varphi\|_2^2\right)\\&\quad+\zeta_1\varphi_t(L,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi_1(\xi,t)\,d\xi+\zeta_2\psi_t(L,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi_2(\xi,t)\,d\xi\\&\qquad+\zeta_3\omega_t(L,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi_3(\xi,t)\,d\xi=0,\end{aligned} \tag{2.12}$$

where $\zeta_i = (\pi)^{-1}\sin(\alpha\pi)\gamma_i$. Multiplying second, fourth and sixth equations in (P') by $\zeta_i\phi_i$ respectively and integrating over $(-\infty,+\infty)$, to obtain:

$$\begin{aligned}&\tfrac{\zeta_1}{2}\frac{d}{dt}\|\phi_1\|_2^2+\zeta_1\int_{-\infty}^{+\infty}(\xi^2+\eta)(\phi_1(\xi,t))^2\,d\xi-\zeta_1\varphi_t(L,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi_1(\xi,t)\,d\xi=0\\&\tfrac{\zeta_2}{2}\frac{d}{dt}\|\phi_2\|_2^2+\zeta_2\int_{-\infty}^{+\infty}(\xi^2+\eta)(\phi_2(\xi,t))^2\,d\xi-\zeta_2\psi_t(L,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi_2(\xi,t)\,d\xi=0\\&\tfrac{\zeta_3}{2}\frac{d}{dt}\|\phi_3\|_2^2+\zeta_3\int_{-\infty}^{+\infty}(\xi^2+\eta)(\phi_3(\xi,t))^2\,d\xi-\zeta_3\omega_t(L,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi_3(\xi,t)\,d\xi=0.\end{aligned} \tag{2.13}$$

From (2.10), (2.12) and (2.13) we get

$$\mathcal{E}'(t) = -\sum_{i=1}^{3}\zeta_i\int_{-\infty}^{+\infty}(\xi^2+\eta)(\phi_i(\xi,t))^2\,d\xi.$$

This completes the proof of the lemma.

Remark 2.2.1 *From Lemma 2.2.2, we have*

$$\mathcal{E}'(t)\le 0.$$

Then, system (P') is dissipative in the sense that its energy is a nonincreasing function of the time variable t. Our goal is to study the effect of this dissipation on the Bresse system.

2.3 Global existence

In this section we will give well-posedness results for problem (P') using the semigroup techniques. Let us introduce the semigroup representation of the Bresse system (P'). If we set $U = (\varphi,\varphi_t,\phi_1,\psi,\psi_t,\phi_2,\omega,\omega_t,\phi_3)^T$, then $U' = (\varphi_t,\varphi_{tt},\partial_t\phi_1,\psi_t,\psi_{tt},\partial_t\phi_2,\omega_t,\omega_{tt},\partial_t\phi_3)^T$. Putting $u=\varphi_t$, $v=\psi_t$ and $\tilde{\omega}=\omega_t$, the problem (P') can be written as

$$\begin{cases}U'=\mathcal{A}U,\\ U(0)=(\varphi_0,\varphi_1,\phi_{01},\psi_0,\psi_1,\phi_{02},\omega_0,\omega_1,\phi_{03}),\end{cases} \tag{2.14}$$

where the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is defined by

$$\mathcal{A}\begin{pmatrix}\varphi\\ u\\ \phi_1\\ \psi\\ v\\ \phi_2\\ \omega\\ \tilde{\omega}\\ \phi_3\end{pmatrix} = \begin{pmatrix} u\\ \dfrac{Gh}{\rho_1}(\varphi_x+\psi+l\omega)_x+\dfrac{lEh}{\rho_1}(\omega_x-l\varphi)\\ -(\xi^2+\eta)\phi_1+u(L)\mu(\xi)\\ v\\ \dfrac{EI}{\rho_2}\psi_{xx}-\dfrac{Gh}{\rho_2}(\varphi_x+\psi+l\omega)\\ -(\xi^2+\eta)\phi_2+v(L)\mu(\xi)\\ \tilde{\omega}\\ \dfrac{Eh}{\rho_1}(\omega_x-l\varphi)_x-\dfrac{lGh}{\rho_1}(\varphi_x+\psi+l\omega)\\ -(\xi^2+\eta)\phi_3+\tilde{\omega}(L)\mu(\xi)\end{pmatrix} \tag{2.15}$$

with domain

$$D(\mathcal{A}) = \left\{\begin{array}{l} (\varphi,u,\phi_1,\psi,v,\phi_2,\omega,\tilde{\omega},\phi_3)^T \text{ in } \mathcal{H} : \varphi,\psi,\omega \in H^2(0,L)\cap H_L^1(0,L), u,v,\tilde{\omega}\in H_L^1(0,L),\\ -(\xi^2+\eta)\phi_1+u(L)\mu(\xi), -(\xi^2+\eta)\phi_2+v(L)\mu(\xi) \in L^2(-\infty,+\infty),\\ -(\xi^2+\eta)\phi_3+\tilde{\omega}(L)\mu(\xi)\in L^2(-\infty,+\infty)\\ Gh(\varphi_x+\psi+l\omega)(L)+\zeta_1\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi_1(\xi)\,d\xi = 0\\ EI\psi_x(L)+\zeta_2\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi_2(\xi)\,d\xi = 0\\ Eh(\omega_x-l\varphi)(L)+\zeta_3\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi_3(\xi)\,d\xi = 0,\\ |\xi|\phi_1, |\xi|\phi_2, |\xi|\phi_3 \in L^2(-\infty,+\infty)\end{array}\right\}$$

(2.16)

where, the energy space $\mathcal{H}$ is defined as

$$\mathcal{H} = (H_L^1(0,L)\times L^2(0,L)\times L^2(-\infty,+\infty))^3$$

where

$$H_L^1(0,L) = \{\varphi\in H^1(0,L) : \varphi(0)=0\}.$$

For $U = (\varphi,u,\phi_1,\psi,v,\phi_2,\omega,\tilde{\omega},\phi_3)^T, \overline{U} = (\overline{\varphi},\overline{u},\overline{\phi}_1,\overline{\psi},\overline{v},\overline{\phi}_2,\overline{\omega},\overline{\tilde{\omega}},\overline{\phi}_3)^T$, we define the following inner product in $\mathcal{H}$

$$\langle U,\overline{U}\rangle_{\mathcal{H}} = \int_0^L \Big(\rho_1 u\overline{u}+\rho_2 v\overline{v}+\rho_1\tilde{\omega}\overline{\tilde{\omega}}+EI\psi_x\overline{\psi}_x+Gh(\varphi_x+\psi+l\omega)(\overline{\varphi}_x+\overline{\psi}+l\overline{\omega})$$
$$+Eh(\omega_x-l\varphi)(\overline{\omega}_x-l\overline{\varphi})\Big)\,dx+\sum_{i=1}^3\zeta_i\int_{-\infty}^{+\infty}\phi_i\overline{\phi}_i\,d\xi.$$

where $\zeta_i = (\pi)^{-1}\sin(\alpha\pi)\gamma_i$.

Proposition 2.3.1 *The operator $\mathcal{A}$ is the infinitesimal generator of a contraction semigroup $(S(t))_{t\geq 0}$.*

Proof. By Theorem 1.3.4 it sufficient to show that the operator $\mathcal{A}$ is m-dissipative in $\mathcal{H}$. Let $U = (\varphi, u, \phi_1, \psi, v, \phi_2, \omega, \tilde{\omega}, \phi_3)^T \in D(\mathcal{A})$. Using (2.11), (2.14) and the fact that

$$\mathcal{E}(t) = \frac{1}{2}\|U\|^2_{\mathcal{H}}, \tag{2.17}$$

we get

$$\Re e\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\sum_{i=1}^{3} \zeta_i \int_{-\infty}^{+\infty} (\xi^2 + \eta)(\phi_i(\xi))^2 \, d\xi \leq 0. \tag{2.18}$$

Consequently, the operator $\mathcal{A}$ is dissipative.
Now, we will prove that the operator $\lambda I - \mathcal{A}$ is surjective for $\lambda > 0$. For this purpose, let $(f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9)^T \in \mathcal{H}$, we seek $U = (\varphi, u, \phi_1, \psi, v, \phi_2, \omega, \tilde{\omega}, \phi_3)^T \in D(\mathcal{A})$ solution of the following system of equations

$$\begin{cases} \lambda\varphi - u = f_1, \\ \lambda u - \dfrac{Gh}{\rho_1}(\varphi_x + \psi + l\omega)_x - \dfrac{lEh}{\rho_1}(\omega_x - l\varphi) = f_2, \\ \lambda\phi_1 + (\xi^2 + \eta)\phi_1 - u(L)\mu(\xi) = f_3, \\ \lambda\psi - v = f_4, \\ \lambda v - \dfrac{EI}{\rho_2}\psi_{xx} + \dfrac{Gh}{\rho_2}(\varphi_x + \psi + l\omega) = f_5, \\ \lambda\phi_2 + (\xi^2 + \eta)\phi_2 - v(L)\mu(\xi) = f_6, \\ \lambda\omega - \tilde{\omega} = f_7, \\ \lambda\tilde{\omega} - \dfrac{Eh}{\rho_1}(\omega_x - l\varphi)_x + \dfrac{lGh}{\rho_1}(\varphi_x + \psi + l\omega) = f_8, \\ \lambda\phi_3 + (\xi^2 + \eta)\phi_3 - \tilde{\omega}(L)\mu(\xi) = f_9. \end{cases} \tag{2.19}$$

Suppose that we have found φ, ψ and ω. Therefore, the first, the fourth and the seventh equation in (2.19) give

$$\begin{cases} u = \lambda\varphi - f_1, \\ v = \lambda\psi - f_4, \\ \tilde{\omega} = \lambda\omega - f_7. \end{cases} \tag{2.20}$$

It is clear that $u \in H^1_L(0, L), v \in H^1_L(0, L)$ and $\tilde{\omega} \in H^1_L(0, L)$. Furthermore, by (2.19) we can find $\phi_i (i = 1, 2, 3)$ as

$$\begin{cases} \phi_1 &= \dfrac{f_3(\xi) + \mu(\xi)u(L)}{\xi^2 + \eta + \lambda} \\ \phi_2 &= \dfrac{f_6(\xi) + \mu(\xi)v(L)}{\xi^2 + \eta + \lambda} \\ \phi_3 &= \dfrac{f_9(\xi) + \mu(\xi)\tilde{\omega}(L)}{\xi^2 + \eta + \lambda} \end{cases} \tag{2.21}$$

By using (2.19) and (2.20) the functions φ, ψ and ω satisfying the following system

$$
\begin{cases}
\lambda^2\varphi - \dfrac{Gh}{\rho_1}(\varphi_x + \psi + l\omega)_x - \dfrac{lEh}{\rho_1}(\omega_x - l\varphi) = f_2 + \lambda f_1, \\
\lambda^2\psi - \dfrac{EI}{\rho_2}\psi_{xx} + \dfrac{Gh}{\rho_2}(\varphi_x + \psi + l\omega) = f_5 + \lambda f_4, \\
\lambda^2\omega - \dfrac{Eh}{\rho_1}(\omega_x - l\varphi)_x + \dfrac{lGh}{\rho_1}(\varphi_x + \psi + l\omega) = f_8 + \lambda f_7.
\end{cases}
\tag{2.22}
$$

Solving system (2.22) is equivalent to finding $(\varphi, \psi, \omega) \in (H^2(0, L) \cap H^1_L(0, L))^3$ such that

$$
\begin{cases}
\displaystyle\int_0^L (\rho_1\lambda^2\varphi w - Gh(\varphi_x + \psi + l\omega)_x w - lEh(\omega_x - l\varphi)w)\, dx = \int_0^L \rho_1(f_2 + \lambda f_1)w\, dx, \\
\displaystyle\int_0^L (\rho_2\lambda^2\psi\chi - EI\psi_{xx}\chi + Gh(\varphi_x + \psi + l\omega)\chi)\, dx = \int_0^L \rho_2(f_5 + \lambda f_4)\chi\, dx, \\
\displaystyle\int_0^L (\rho_1\lambda^2\omega\zeta - Eh(\omega_x - l\varphi)_x\zeta + lGh(\varphi_x + \psi + l\omega)\zeta)\, dx = \int_0^L \rho_1(f_8 + \lambda f_7)\zeta\, dx
\end{cases}
\tag{2.23}
$$

for all $(w, \chi, \zeta) \in H^1_L(0, L) \times H^1_L(0, L) \times H^1_L(0, L)$. By using (2.23) and (2.21) the functions φ, ψ and ω satisfying the following system

$$
\begin{cases}
\displaystyle\int_0^L (\rho_1\lambda^2\varphi w + Gh(\varphi_x + \psi + l\omega)w_x - lEh(\omega_x - l\varphi)w)\, dx + \tilde{\zeta}_1 u(L)w(L) \\
\qquad \displaystyle= \int_0^L \rho_1(f_2 + \lambda f_1)w\, dx - \zeta_1\int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + \lambda} f_3(\xi)\, d\xi \; w(L), \\
\displaystyle\int_0^L (\rho_2\lambda^2\psi\chi + EI\psi_x\chi_x + Gh(\varphi_x + \psi + l\omega)\chi)\, dx + \tilde{\zeta}_2 v(L)\chi(L) \\
\qquad \displaystyle= \int_0^L \rho_2(f_5 + \lambda f_4)\chi\, dx - \zeta_2\int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + \lambda} f_6(\xi)\, d\xi \; \chi(L), \\
\displaystyle\int_0^L (\rho_1\lambda^2\omega\zeta + Eh(\omega_x - l\varphi)\zeta_x + lGh(\varphi_x + \psi + l\omega)\zeta)\, dx + \tilde{\zeta}_3\tilde{\omega}(L)\zeta(L) \\
\qquad \displaystyle= \int_0^L \rho_1(f_8 + \lambda f_7)\zeta\, dx - \zeta_3\int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + \lambda} f_9(\xi)\, d\xi \; \zeta(L)
\end{cases}
\tag{2.24}
$$

where $\tilde{\zeta}_i = \zeta_i \displaystyle\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2 + \eta + \lambda}\, d\xi$. Using again (2.20), we deduce that

$$
\begin{cases}
u(L) = \lambda\varphi(L) - f_1(L), \\
v(L) = \lambda\psi(L) - f_4(L), \\
\tilde{\omega}(L) = \lambda\omega(L) - f_7(L).
\end{cases}
\tag{2.25}
$$

Inserting (2.25) into (2.24), we get

$$(2.26)\quad \begin{cases} \int_0^L (\rho_1\lambda^2\varphi w + Gh(\varphi_x+\psi+l\omega)_x w_x - lEh(\omega_x - l\varphi)w)\,dx + \lambda\tilde{\zeta}_1\varphi(L)w(L) \\ = \int_0^L \rho_1(f_2+\lambda f_1)w\,dx - \zeta_1\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+\lambda}f_3(\xi)\,d\xi\ w(L) + \tilde{\zeta}_1 f_1(L)w(L), \\ \int_0^L (\rho_2\lambda^2\psi\chi + EI\psi_x\chi_x + Gh(\varphi_x+\psi+l\omega)\chi)\,dx + \lambda\tilde{\zeta}_2\psi(L)\chi(L) \\ = \int_0^L \rho_2(f_5+\lambda f_4)\chi\,dx - \zeta_2\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+\lambda}f_6(\xi)\,d\xi\ \chi(L) + \tilde{\zeta}_2 f_4(L)\chi(L), \\ \int_0^L (\rho_1\lambda^2\omega\zeta + Eh(\omega_x - l\varphi)\zeta_x + lGh(\varphi_x+\psi+l\omega)\zeta)\,dx + \lambda\tilde{\zeta}_3\omega(L)\zeta(L)) \\ = \int_0^L \rho_1(f_8+\lambda f_7)\zeta\,dx - \zeta_3\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+\lambda}f_9(\xi)\,d\xi\ \zeta(L) + \tilde{\zeta}_3 f_7(L)\zeta(L). \end{cases}$$

Consequently, problem (2.26) is equivalent to the problem

$$(2.27)\qquad a((\varphi,\psi,\omega),(w,\chi,\zeta)) = L(w,\chi,\zeta)$$

where the bilinear form $a : [H^1_L(0,L)\times H^1_L(0,L)\times H^1_L(0,L)]^2 \to \mathbb{R}$ and the linear form $L : H^1_L(0,L)\times H^1_L(0,L)\times H^1_L(0,L) \to \mathbb{R}$ are defined by

$$\begin{aligned} a((\varphi,\psi,\omega),(w,\chi,\zeta)) &= \int_0^L (\rho_1\lambda^2\varphi w + Gh(\varphi_x+\psi+l\omega)(w_x+\chi+l\zeta))\,dx \\ &+ \int_0^L (\rho_2\lambda^2\psi\chi + EI\psi_x\chi_x)\,dx + \int_0^L (\rho_1\lambda^2\omega\zeta + Eh(\omega_x - l\varphi)(\zeta_x - lw))\,dx \\ &+ \lambda\tilde{\zeta}_1\varphi(L)w(L) + \lambda\tilde{\zeta}_2\psi(L)\chi(L) + \lambda\tilde{\zeta}_3\omega(L)\zeta(L) \end{aligned}$$

and

$$\begin{aligned} L(w,\chi,\zeta) &= \int_0^L \rho_1(f_2+\lambda f_1)w\,dx - \zeta_1\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+\lambda}f_3(\xi)\,d\xi\ w(L) + \tilde{\zeta}_1 f_1(L)w(L) \\ &+ \int_0^L \rho_2(f_5+\lambda f_4)\chi\,dx - \zeta_2\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+\lambda}f_6(\xi)\,d\xi\ \chi(L) + \tilde{\zeta}_2 f_4(L)\chi(L) \\ &+ \int_0^L \rho_1(f_8+\lambda f_7)\zeta\,dx - \zeta_3\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+\lambda}f_9(\xi)\,d\xi\ \zeta(L) + \tilde{\zeta}_3 f_7(L)\zeta(L). \end{aligned}$$

It is easy to verify that a is continuous and coercive, and L is continuous. So applying the Lax-Milgram theorem, we deduce that for all $(w,\chi,\zeta) \in H^1_L(0,L)\times H^1_L(0,L)\times H^1_L(0,L)$ problem (2.27) admits a unique solution $(\varphi,\psi,\omega) \in H^1_L(0,L)\times H^1_L(0,L)\times H^1_L(0,L)$. Applying the classical elliptic regularity, it follows from (2.26) that $(\varphi,\psi,\omega) \in H^2(0,L)\times H^2(0,L)\times H^2(0,L)$. Therefore, the operator $\lambda I - \mathcal{A}$ is surjective for any $\lambda > 0$. The conclusion of Proposition 2.3.1 follows from Lumer-Phillips Theorem. Then the solution to the linear Cauchy problem (2.14) admits the following representation :

$$U(t) = S(t)U_0 = e^{\mathcal{A}t}U_0 \quad \forall t \geq 0$$

Consequently, using Hille-Yosida theorem, we have the following results.

Theorem 2.3.1 (Existence and uniqueness)

(1) If $U_0 \in D(\mathcal{A})$, then system (2.14) has a unique strong solution

$$U \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

(2) If $U_0 \in \mathcal{H}$, then system (2.14) has a unique weak solution

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

2.4 Asymptotic stability

The problem of stabilization consists in determining the asymptotic behaviour of the energy, to study its limits in order to determine if this limit is null or not and if this limit is null, to give an estimate of the decay rate of the energy to zero.
In this section, we study the stabilization of the bresse system which is given by (P). We prove that the stability (strong stability) of (P) holds.

2.4.1 Strong stability of the system when $\eta \geq 0$

In this part, we use a general criteria of Arendt-Batty in **[3]** to show the strong stability of the C_0-semigroup $e^{t\mathcal{A}}$ associated to the system (P') in the absence of the compactness of the resolvent of $\mathcal{A}$. Our main result is the following theorem:

Theorem 2.4.1 *The C_0-semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ is strongly stable in $\mathcal{H}$, i.e, for all $U_0 \in \mathcal{H}$, the solution of (2.14) satisfies*

$$\lim_{t\to+\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0.$$

The proof will be based on the following lemmas.

Lemma 2.4.1 *If $\eta \geq 0$, we have*

$$\sigma(\mathcal{A}) \cap \{i\lambda,\ \lambda \in \mathbb{R},\ \lambda \neq 0\} = \emptyset.$$

Lemma 2.4.2 *Assume that $\eta \geq 0$. Then, $\lambda = 0$ is not an eigenvalue of $\mathcal{A}$.*

Let us first prove Lemma 2.4.2.

Proof.

From (2.15) we get that $(\varphi, u, \phi_1, \psi, v, \phi_2, \omega, \tilde{\omega}, \phi_3)^T \in N(\mathcal{A}) \subset D(\mathcal{A})$ if and only if

$$\begin{cases} -u = 0, \\ -\dfrac{Gh}{\rho_1}(\varphi_x + \psi + l\omega)_x - \dfrac{lEh}{\rho_1}(\omega_x - l\varphi) = 0, \\ (\xi^2 + \eta)\phi_1 - u(L)\mu(\xi) = 0, \\ -v = 0, \\ -\dfrac{EI}{\rho_2}\psi_{xx} + \dfrac{Gh}{\rho_2}(\varphi_x + \psi + l\omega) = 0, \\ (\xi^2 + \eta)\phi_2 - v(L)\mu(\xi) = 0, \\ -\tilde{\omega} = 0, \\ -\dfrac{Eh}{\rho_1}(\omega_x - l\varphi)_x + \dfrac{lGh}{\rho_1}(\varphi_x + \psi + l\omega) = 0, \\ (\xi^2 + \eta)\phi_3 - \tilde{\omega}(L)\mu(\xi) = 0. \end{cases} \tag{2.28}$$

This implies that $u = v = \tilde{\omega} = 0, \phi_1 = \phi_2 = \phi_3 = 0$ and

$$\begin{cases} -\dfrac{Gh}{\rho_1}(\varphi_x + \psi + l\omega)_x - \dfrac{lEh}{\rho_1}(\omega_x - l\varphi) = 0, \\ -\dfrac{EI}{\rho_2}\psi_{xx} + \dfrac{Gh}{\rho_2}(\varphi_x + \psi + l\omega) = 0, \\ -\dfrac{Eh}{\rho_1}(\omega_x - l\varphi)_x + \dfrac{lGh}{\rho_1}(\varphi_x + \psi + l\omega) = 0, \end{cases} \tag{2.29}$$

with

$$\begin{cases} \varphi(0) = \psi(0) = \omega(0) = 0, \\ \varphi_x(L) + \psi(L) + l\omega(L) = 0, \ \ \psi_x(L) = 0, \ \ \omega_x(L) - l\varphi(L) = 0. \end{cases}$$

This implies that

$$Gh\|\varphi_x + \psi + l\omega\|_2^2 + EI\|\psi_x\|_2^2 + Eh\|\omega_x - l\varphi\|_2^2 = 0. \tag{2.30}$$

(2.30) implies that ψ is a constant function and

$$\varphi_x + \psi + l\omega = 0, \ \ \omega_x - l\varphi = 0.$$

As $\psi(0) = 0$, we deduce that $\psi \equiv 0$. Hence

$$\varphi_x + l\omega = 0, \ \ \omega_x - l\varphi = 0.$$

Then, we have

$$\varphi = c \sin lx, \ \ \omega = -c \cos lx.$$

Hence $\omega(0)=0$ imply that $c=0$. Thus $(\varphi,u,\phi_1,\psi,v,\phi_2,\omega,\tilde{\omega},\phi_3)^T=0$. This concludes the proof of Lemma 2.4.2.

Now, we prove Lemma 2.4.1. Let us suppose that there is $\lambda\in\mathbb{R},\lambda\neq 0$ and $U\in D(\mathcal{A})$, $U\neq 0$, such that $\mathcal{A}U=i\lambda U$. Our goal is to find a contradiction by proving that $U=0$. Then, we get

$$\left\{\begin{array}{l} i\lambda\varphi-u=0,\\ i\lambda u-\dfrac{Gh}{\rho_1}(\varphi_x+\psi+l\omega)_x-\dfrac{lEh}{\rho_1}(\omega_x-l\varphi)=0,\\ i\lambda\phi_1+(\xi^2+\eta)\phi_1-u(L)\mu(\xi)=0,\\ i\lambda\psi-v=0,\\ i\lambda v-\dfrac{EI}{\rho_2}\psi_{xx}+\dfrac{Gh}{\rho_2}(\varphi_x+\psi+l\omega)=0,\\ i\lambda\phi_2+(\xi^2+\eta)\phi_2-v(L)\mu(\xi)=0,\\ i\lambda\omega-\tilde{\omega}=0,\\ i\lambda\tilde{\omega}-\dfrac{Eh}{\rho_1}(\omega_x-l\varphi)_x+\dfrac{lGh}{\rho_1}(\varphi_x+\psi+l\omega)=0,\\ i\lambda\phi_3+(\xi^2+\eta)\phi_3-\tilde{\omega}(L)\mu(\xi)=0. \end{array}\right. \tag{2.31}$$

Then, from (2.18) we have

$$\phi_i\equiv 0,\ \ i=1,2,3. \tag{2.32}$$

From $(2.31)_3,(2.31)_6$ and $(2.31)_9$, we have

$$u(L)=v(L)=\tilde{\omega}(L)=0. \tag{2.33}$$

Hence, from (2.31) and (P') we obtain

$$\varphi(L)=\psi(L)=\omega(L)=0 \text{ and } \varphi_x(L)=\psi_x(L)=\omega_x(L)=0. \tag{2.34}$$

From (2.31), we have

$$\left\{\begin{array}{l} -\lambda^2\rho_1\varphi-Gh(\varphi_x+\psi+l\omega)_x-lEh(\omega_x-l\varphi)=0,\\ -\lambda^2\rho_2\psi-EI\psi_{xx}+Gh(\varphi_x+\psi+l\omega)=0,\\ -\lambda^2\rho_1\omega-Eh(\omega_x-l\varphi)_x+lGh(\varphi_x+\psi+l\omega)=0. \end{array}\right. \tag{2.35}$$

Consider $X=(\varphi,\psi,\omega,\varphi_x,\psi_x,\omega_x)^T$. Then we can rewrite (2.34) and (2.35) as the initial value problem

$$\begin{array}{l} \dfrac{d}{dx}X=\mathcal{B}X\\ X(L)=0 \end{array} \tag{2.36}$$

where

$$\mathcal{B} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{-\lambda^2\rho_1 + l^2 Eh}{Gh} & 0 & 0 & 0 & -1 & -\frac{(E+G)l}{G} \\ 0 & \frac{-\rho_2\lambda^2 + Gh}{EI} & \frac{Ghl}{EI} & \frac{Gh}{EI} & 0 & 0 \\ 0 & \frac{Gl}{E} & \frac{-\rho_1\lambda^2 + Ghl^2}{Eh} & \frac{(E+G)l}{E} & 0 & 0 \end{pmatrix}$$

By the Picard Theorem for ordinary differential equations the system (2.36) has a unique solution $X = 0$. Therefore $\varphi = 0, \psi = 0, \omega = 0$. It follows from (2.31), that $u = 0, v = 0, \tilde{\omega} = 0$, i.e., $U = 0$.

Lemma 2.4.3 *Assume that $\eta \geq 0$. Then, the operator $i\lambda I - \mathcal{A}$ is surjective for $\lambda \neq 0$.*

Proof.
Let $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9)^T \in \mathcal{H}$, we seek $U = (\varphi, u, \phi_1, \psi, v, \phi_2, \omega, \tilde{\omega}, \phi_3)^T \in D(\mathcal{A})$ solution of the following equation

$$(i\lambda - \mathcal{A})U = F.$$

Equivalently, we have the following system

$$(2.37) \qquad \begin{cases} i\lambda\varphi - u = f_1, \\ i\lambda u - \dfrac{Gh}{\rho_1}(\varphi_x + \psi + l\omega)_x - \dfrac{lEh}{\rho_1}(\omega_x - l\varphi) = f_2, \\ i\lambda\phi_1 + (\xi^2 + \eta)\phi_1 - u(L)\mu(\xi) = f_3, \\ i\lambda\psi - v = f_4, \\ i\lambda v - \dfrac{EI}{\rho_2}\psi_{xx} + \dfrac{Gh}{\rho_2}(\varphi_x + \psi + l\omega) = f_5, \\ i\lambda\phi_2 + (\xi^2 + \eta)\phi_2 - v(L)\mu(\xi) = f_6, \\ i\lambda\omega - \tilde{\omega} = f_7, \\ i\lambda\tilde{\omega} - \dfrac{Eh}{\rho_1}(\omega_x - l\varphi)_x + \dfrac{lGh}{\rho_1}(\varphi_x + \psi + l\omega) = f_8, \\ i\lambda\phi_3 + (\xi^2 + \eta)\phi_3 - \tilde{\omega}(L)\mu(\xi) = f_9 \end{cases}$$

with the following conditions

$$(2.38) \qquad \begin{cases} Gh(\varphi_x + \psi + l\omega)(L) = -\gamma_1(\pi)^{-1}\sin(\alpha\pi)\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_1(\xi, t)\, d\xi \\ EI\psi_x(L) = -\gamma_2(\pi)^{-1}\sin(\alpha\pi)\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_2(\xi, t)\, d\xi \\ Eh(\omega_x - l\varphi)(L) = -\gamma_3(\pi)^{-1}\sin(\alpha\pi)\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_3(\xi, t)\, d\xi. \end{cases}$$

We get

$$(2.39)\qquad \begin{cases} -\lambda^2\varphi - \dfrac{Gh}{\rho_1}(\varphi_x+\psi+l\omega)_x - \dfrac{lEh}{\rho_1}(\omega_x - l\varphi) = f_2 + i\lambda f_1, \\ -\lambda^2\psi - \dfrac{EI}{\rho_2}\psi_{xx} + \dfrac{Gh}{\rho_2}(\varphi_x+\psi+l\omega) = f_5 + i\lambda f_4, \\ -\lambda^2\omega - \dfrac{Eh}{\rho_1}(\omega_x - l\varphi)_x + \dfrac{lGh}{\rho_1}(\varphi_x+\psi+l\omega) = f_8 + i\lambda f_7. \end{cases}$$

Solving system (2.39) is equivalent to finding $(\varphi,\psi,\omega) \in (H^2 \cap H_L^1(0,L))^3$ such that

$$(2.40)\qquad \begin{cases} \displaystyle\int_0^L (-\rho_1\lambda^2\varphi w - Gh(\varphi_x+\psi+l\omega)_x w - lEh(\omega_x - l\varphi)w)\,dx = \int_0^L \rho_1(f_2+i\lambda f_1)w\,dx, \\ \displaystyle\int_0^L (-\rho_2\lambda^2\psi\chi - EI\psi_{xx}\chi + Gh(\varphi_x+\psi+l\omega)\chi)\,dx = \int_0^L \rho_2(f_5+i\lambda f_4)\chi\,dx, \\ \displaystyle\int_0^L (-\rho_1\lambda^2\omega\zeta - Eh(\omega_x - l\varphi)_x\zeta + lGh(\varphi_x+\psi+l\omega)\zeta)\,dx = \int_0^L \rho_1(f_8+i\lambda f_7)\zeta\,dx \end{cases}$$

for all $(w,\chi,\zeta) \in H_L^1(0,L) \times H_L^1(0,L) \times H_L^1(0,L)$. By using (2.40) and (2.21) the functions φ,ψ and ω satisfying the following system

$$(2.41)\qquad \begin{cases} \displaystyle\int_0^L (-\rho_1\lambda^2\varphi w + Gh(\varphi_x+\psi+l\omega)w_x - lEh(\omega_x - l\varphi)w)\,dx + \tilde{\zeta}_1 u(L)w(L) \\ \displaystyle\qquad = \int_0^L \rho_1(f_2+i\lambda f_1)w\,dx - \zeta_1\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+i\lambda}f_3(\xi)\,d\xi\ w(L), \\ \displaystyle\int_0^L (-\rho_2\lambda^2\psi\chi + EI\psi_x\chi_x + Gh(\varphi_x+\psi+l\omega)\chi)\,dx + \tilde{\zeta}_2 v(L)\chi(L) \\ \displaystyle\qquad = \int_0^L \rho_2(f_5+i\lambda f_4)\chi\,dx - \zeta_2\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+i\lambda}f_6(\xi)\,d\xi\ \chi(L), \\ \displaystyle\int_0^L (-\rho_1\lambda^2\omega\zeta + Eh(\omega_x - l\varphi)\zeta_x + lGh(\varphi_x+\psi+l\omega)\zeta)\,dx + \tilde{\zeta}_3\tilde{\omega}(L)w(L) \\ \displaystyle\qquad = \int_0^L \rho_1(f_8+i\lambda f_7)\zeta\,dx - \zeta_3\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+i\lambda}f_9(\xi)\,d\xi\ \zeta(L) \end{cases}$$

where $\tilde{\zeta}_i = \zeta_i \displaystyle\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta+i\lambda}\,d\xi$. Using again (2.20), we deduce that

$$(2.42)\qquad \begin{cases} u(L) = i\lambda\varphi(L) - f_1(L), \\ v(L) = i\lambda\psi(L) - f_4(L), \\ \tilde{\omega}(L) = i\lambda\omega(L) - f_7(L). \end{cases}$$

Inserting (2.42) into (2.41), we get

$$
(2.43)\begin{cases}
\displaystyle\int_0^L(-\rho_1\lambda^2\varphi w + Gh(\varphi_x+\psi+l\omega)w_x - lEh(\omega_x-l\varphi)w)\,dx + i\lambda\tilde{\zeta}_1\varphi(L)w(L)\\
\displaystyle= \int_0^L \rho_1(f_2+i\lambda f_1)w\,dx - \zeta_1\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+i\lambda}f_3(\xi)\,d\xi\ w(L) + \tilde{\zeta}_1 f_1(L)w(L),\\
\displaystyle\int_0^L(-\rho_2\lambda^2\psi\chi + EI\psi_x\chi_x + Gh(\varphi_x+\psi+l\omega)\chi)\,dx + i\lambda\tilde{\zeta}_2\psi(L)\chi(L)\\
\displaystyle= \int_0^L \rho_2(f_5+i\lambda f_4)\chi\,dx - \zeta_2\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+i\lambda}f_6(\xi)\,d\xi\ \chi(L) + \tilde{\zeta}_2 f_4(L)\chi(L),\\
\displaystyle\int_0^L(-\rho_1\lambda^2\omega\zeta + Eh(\omega_x-l\varphi)\zeta_x + lGh(\varphi_x+\psi+l\omega)\zeta)\,dx + i\lambda\tilde{\zeta}_3\omega(L)w(L)\\
\displaystyle= \int_0^L \rho_1(f_8+i\lambda f_7)\zeta\,dx - \zeta_3\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+i\lambda}f_9(\xi)\,d\xi\ \zeta(L) + \tilde{\zeta}_3 f_7(L)\zeta(L).
\end{cases}
$$

We can rewrite (2.43) as

$$
(2.44)\qquad -(L_\lambda U, V)_{H_R^1} + (U,V)_{H_R^1} = l(V).
$$

where

$$
H_R^1(0,L) = H_L^1(0,L)\times H_L^1(0,L)\times H_L^1(0,L),
$$

with the inner product defined by

$$
(U,V)_{H_R^1} = \int_0^L Gh(\varphi_x+\psi+l\omega)(w_x+\chi+l\zeta) + EI\psi_x\chi_x + Eh(\omega_x-l\varphi)(\zeta_x-lw)\,dx
$$

$$
(L_\lambda U,V)_{H_R^1} = \lambda^2\int_0^L(\rho_1\varphi w+\rho_2\psi\chi+\rho_1\omega\zeta)\,dx - i\lambda(\tilde{\zeta}_1\varphi(L)w(L)+\tilde{\zeta}_2\psi(L)\chi(L)+\tilde{\zeta}_3\omega(L)w(L)).
$$

and

$$
\begin{aligned}
l(V) &= \int_0^L \rho_1(f_2+i\lambda f_1)w\,dx - \zeta_1\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+i\lambda}f_3(\xi)\,d\xi\ w(L) + \tilde{\zeta}_1 f_1(L)w(L)\\
&+ \int_0^L \rho_2(f_5+i\lambda f_4)\chi\,dx - \zeta_2\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+i\lambda}f_6(\xi)\,d\xi\ \chi(L) + \tilde{\zeta}_2 f_4(L)\chi(L)\\
&+ \int_0^L \rho_1(f_8+i\lambda f_7)\zeta\,dx - \zeta_3\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+i\lambda}f_9(\xi)\,d\xi\ \zeta(L) + \tilde{\zeta}_3 f_7(L)\zeta(L).
\end{aligned}
$$

Using the compactness embedding from $L^2(0,L)$ into $H^{-1}(0,L)$ and from $H_L^1(0,L)$ into $L^2(0,L)$ we deduce that the operator L_λ is compact from $(L^2(0,L))^3$ into $(L^2(0,L))^3$. Consequently, by Fredholm alternative (see Theorem 1.2.1), proving the existence of U solution of (2.44) reduces to proving that 1 is not an eigenvalue of L_λ. Indeed if 1 is an eigenvalue, then there exists $U \neq 0$, such that

$$
(2.45)\qquad (L_\lambda U,V)_{H_R^1} = (U,V)_{H_R^1}\quad \forall V\in H_R^1.
$$

In particular for $V = U$, it follows that

$$
\begin{aligned}
&\lambda^2\left[\rho_1\|\varphi\|^2_{L^2(0,L)} + \rho_2\|\psi\|^2_{L^2(0,L)} + \rho_1\|\omega\|^2_{L^2(0,L)}\right] - i\lambda(\tilde{\zeta}_1|\varphi(L)|^2 + \tilde{\zeta}^2|\psi(L)|^2 + \tilde{\zeta}_3|\omega(L)|^2)\\
&= Gh\|\varphi_x+\psi+l\omega\|^2_{L^2(0,L)} + EI\|\psi_x\|^2_{L^2(0,L)} + Eh\|w_x - l\varphi\|^2_{L^2(0,L)}.
\end{aligned}
$$

Hence, we have

$$\varphi(L) = \psi(L) = \omega(L) = 0. \tag{2.46}$$

From (2.45), we obtain

$$\varphi_x(L) = \psi_x(L) = \omega_x(L) = 0 \tag{2.47}$$

and

$$\begin{cases} -\lambda^2\varphi - \dfrac{Gh}{\rho_1}(\varphi_x + \psi + l\omega)_x - \dfrac{lEh}{\rho_1}(\omega_x - l\varphi) = 0, \\ -\lambda^2\psi - \dfrac{EI}{\rho_2}\psi_{xx} + \dfrac{Gh}{\rho_2}(\varphi_x + \psi + l\omega) = 0, \\ -\lambda^2\omega - \dfrac{Eh}{\rho_1}(\omega_x - l\varphi)_x + \dfrac{lGh}{\rho_1}(\varphi_x + \psi + l\omega) = 0. \end{cases} \tag{2.48}$$

Consider $X = (\varphi, \psi, \omega, \varphi_x, \psi_x, \omega_x)^T$. Then we can rewrite (2.48), (2.46) and (2.47) as the initial value problem

$$\begin{array}{l} \dfrac{d}{dx}X = \mathcal{B}X \\ X(L) = 0 \end{array} \tag{2.49}$$

where

$$\mathcal{B} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{-\lambda^2\rho_1 + l^2Eh}{Gh} & 0 & 0 & 0 & -1 & -\frac{(E+G)l}{G} \\ 0 & \frac{-\rho_2\lambda^2 + Gh}{EI} & \frac{Ghl}{EI} & \frac{Gh}{EI} & 0 & 0 \\ 0 & \frac{Gl}{E} & \frac{-\rho_1\lambda^2 + Ghl^2}{Eh} & \frac{(E+G)l}{E} & 0 & 0 \end{pmatrix}$$

By the Picard Theorem for ordinary differential equations the system (2.49) has a unique solution $X = 0$. Therefore $\varphi = 0, \psi = 0, \omega = 0$. It follows from (2.31), that $u = 0, v = 0, \tilde{\omega} = 0, \phi_1 = 0, \phi_2 = 0, \phi_3 = 0$. Then $U = 0$, which contradict the hypothesis $U \neq 0$.

Lemma 2.4.4 *If $\eta \neq 0$, we have*

$$0 \in \rho(\mathcal{A}).$$

Proof.

From (2.37)

$$
\begin{cases}
-u = f_1, \\
-\dfrac{Gh}{\rho_1}(\varphi_x+\psi+l\omega)_x - \dfrac{lEh}{\rho_1}(\omega_x - l\varphi) = f_2, \\
(\xi^2+\eta)\phi_1 - u(L)\mu(\xi) = f_3, \\
-v = f_4, \\
-\dfrac{EI}{\rho_2}\psi_{xx} + \dfrac{Gh}{\rho_2}(\varphi_x+\psi+l\omega) = f_5, \\
(\xi^2+\eta)\phi_2 - v(L)\mu(\xi) = f_6, \\
-\tilde{\omega} = f_7, \\
-\dfrac{Eh}{\rho_1}(\omega_x - l\varphi)_x + \dfrac{lGh}{\rho_1}(\varphi_x+\psi+l\omega) = f_8, \\
(\xi^2+\eta)\phi_3 - \tilde{\omega}(L)\mu(\xi) = f_9.
\end{cases} \tag{2.50}
$$

with the following conditions

$$
\begin{cases}
Gh(\varphi_x+\psi+l\omega)(L) = -\gamma_1(\pi)^{-1}\sin(\alpha\pi)\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi_1(\xi,t)\,d\xi \\
EI\psi_x(L) = -\gamma_2(\pi)^{-1}\sin(\alpha\pi)\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi_2(\xi,t)\,d\xi \\
Eh(\omega_x - l\varphi)(L) = -\gamma_3(\pi)^{-1}\sin(\alpha\pi)\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi_3(\xi,t)\,d\xi
\end{cases} \tag{2.51}
$$

Integrating over $(0,L)$ and using integration by parts, we get

$$
\begin{cases}
\displaystyle\int_0^L (Gh(\varphi_x+\psi+l\omega)w_x - lEh(\omega_x-l\varphi)w)\,dx \\
= \displaystyle\int_0^L \rho_1 f_2 w\,dx - \zeta_1\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta}f_3(\xi)\,d\xi\ w(L) + \tilde{\zeta}_1 f_1(L)w(L), \\
\displaystyle\int_0^L (EI\psi_x\chi_x + Gh(\varphi_x+\psi+l\omega)\chi)\,dx \\
= \displaystyle\int_0^L \rho_2 f_5\chi\,dx - \zeta_2\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta}f_6(\xi)\,d\xi\ \chi(L) + \tilde{\zeta}_2 f_4(L)\chi(L), \\
\displaystyle\int_0^L (Eh(\omega_x-l\varphi)\zeta_x + lGh(\varphi_x+\psi+l\omega)\zeta)\,dx \\
= \displaystyle\int_0^L \rho_1 f_8\zeta\,dx - \zeta_3\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta}f_9(\xi)\,d\xi\ \zeta(L) + \tilde{\zeta}_3 f_7(L)\zeta(L).
\end{cases} \tag{2.52}
$$

for all $(w,\chi,\zeta) \in H_L^1(0,L)\times H_L^1(0,L)\times H_L^1(0,L)$. Consequently, problem (2.52) is equivalent to the problem

$$
a_\eta((\varphi,\psi,\omega),(w,\chi,\zeta)) = L_\eta(w,\chi,\zeta) \tag{2.53}
$$

where the bilinear form $a_\eta : [H_L^1(0,L)\times H_L^1(0,L)\times H_L^1(0,L)]^2 \to \mathbb{R}$ and the linear form $L_\eta : H_L^1(0,L)\times H_L^1(0,L)\times H_L^1(0,L) \to \mathbb{R}$ are defined by

$$
a_\eta((\varphi,\psi,\omega),(w,\chi,\zeta)) = \int_0^L Gh(\varphi_x+\psi+l\omega)(w_x+\chi+l\zeta)\,dx
$$
$$
+\int_0^L EI\psi_x\chi_x\,dx + \int_0^L Eh(\omega_x-l\varphi)(\zeta_x-lw)\,dx
$$

and

$$L_\eta(w,\chi,\zeta) = \int_0^L \rho_1 f_2 w\, dx - \zeta_1 \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta} f_3(\xi)\, d\xi\ w(L) + \tilde{\zeta}_1 f_1(L) w(L)$$
$$\int_0^L \rho_2 f_5 \chi\, dx - \zeta_2 \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta} f_6(\xi)\, d\xi\ \chi(L) + \tilde{\zeta}_2 f_4(L)\chi(L)$$
$$+ \int_0^L \rho_1 f_8 \zeta\, dx - \zeta_3 \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta} f_9(\xi)\, d\xi\ \zeta(L) + \tilde{\zeta}_3 f_7(L)\zeta(L).$$

It is easy to verify that a_η is continuous and coercive, and L_η is continuous. So applying the Lax-Milgram theorem, we deduce that for all $(w,\chi,\zeta) \in H_L^1(0,L) \times H_L^1(0,L) \times H_L^1(0,L)$ problem (2.53) admits a unique solution $(\varphi,\psi,\omega) \in H_L^1(0,L) \times H_L^1(0,L) \times H_L^1(0,L)$. Applying the classical elliptic regularity, it follows from (2.52) that $(\varphi,\psi,\omega) \in H^2(0,L) \times H^2(0,L) \times H^2(0,L)$. Therefore, the operator $\mathcal{A}$ is surjective.

Lemma 2.4.5 *If $\eta = 0$, we have*

$$0 \in \sigma(\mathcal{A}).$$

Proof.

We shall show that $i\lambda = 0$ is not in the resolvent set of the operator $\mathcal{A}$. Indeed, noting that $(\sin x, 0,0,0,0,0,0,0,0)^T \in \mathcal{H}$, and denoting by $(\varphi, u, \phi_1, \psi, v, \phi_2, \omega, \tilde{\omega}, \phi_3)^T$ the image of $(\sin x, 0,0,0,0,0,0,0,0)^T$ by $\mathcal{A}^{-1}$, we see that $\phi_1(\xi) = |\xi|^{\frac{2\alpha-5}{2}} \sin L$. But, then $\phi_1 \notin L^2(-\infty,+\infty)$, since $\alpha \in]0,1[$. And so $(\varphi, u, \phi_1, \psi, v, \phi_2, \omega, \tilde{\omega}, \phi_3)^T \notin D(\mathcal{A})$.

Proof of Theorem 2.4.1.

By Lemmas 2.4.1 and 2.4.2, we know that $\mathcal{A}$ does not have eigenvalues on the imaginary axis. Now, from Lemmas 2.4.3, 2.4.4 and 2.4.5 it follows that $\sigma(\mathcal{A}) \cap \{i\,\mathbb{R}\} = \emptyset$ if $\eta > 0$ and $\sigma(\mathcal{A}) \cap \{i\,\mathbb{R}\} = \{0\}$ if $\eta = 0$. This completes the proof of Theorem 2.4.1.

2.4.2 Lack of exponential stability when $\eta \geq 0$

In this part, we show that the semigroup associated with system (P') is not exponentially stable.

Theorem 2.4.2 *The semigroup generated by the operator $\mathcal{A}$ is not exponentially stable.*

We state and prove a lemma that will be needed later. We consider the case when $l \to 0$ i.e, when (P) takes the following form

$$(P_0) \qquad \begin{cases} \rho_1\varphi_{tt} - Gh(\varphi_x+\psi)_x = 0 & \text{in } (0,L)\times(0,\infty) \\ \rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x+\psi) = 0 & \text{in } (0,L)\times(0,\infty) \\ \rho_1\omega_{tt} - Eh(\omega_x)_x = 0 & \text{in } (0,L)\times(0,\infty) \\ \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x) & \text{in } (0,L) \\ \psi(x,0) = \psi_0(x), \psi_t(x,0) = \psi_1(x) & \text{in } (0,L) \\ \omega(x,0) = \omega_0(x), \omega_t(x,0) = \omega_1(x) & \text{in } (0,L) \\ \varphi(0,t) = 0, \quad \psi(0,t) = 0, \omega(0,t) = 0 & \text{in } (0,+\infty) \\ Gh(\varphi_x+\psi)(L,t) = -\gamma_1\partial_t^{\alpha,\eta}\varphi(L,t) & \text{in } (0,+\infty) \\ EI\psi_x(L,t) = -\gamma_2\partial_t^{\alpha,\eta}\psi(L,t) & \text{in } (0,+\infty) \\ Eh\omega_x(L,t) = -\gamma_3\partial_t^{\alpha,\eta}\omega(L,t) & \text{in } (0,+\infty) \end{cases}$$

System (P_0) can be reduced to the Timoshenko system and an independent wave equation:

$$(PT) \qquad \begin{cases} \rho_1\varphi_{tt} - Gh(\varphi_x+\psi)_x = 0 & \text{in } (0,L)\times(0,\infty) \\ \rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x+\psi) = 0 & \text{in } (0,L)\times(0,\infty) \\ \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x) & \text{in } (0,L) \\ \psi(x,0) = \psi_0(x), \psi_t(x,0) = \psi_1(x) & \text{in } (0,L) \\ \varphi(0,t) = 0, \quad \psi(0,t) = 0, & \text{in } (0,+\infty) \\ Gh(\varphi_x+\psi)(L,t) = -\gamma_1\partial_t^{\alpha,\eta}\varphi(L,t) & \text{in } (0,+\infty) \\ EI\psi_x(L,t) = -\gamma_2\partial_t^{\alpha,\eta}\psi(L,t) & \text{in } (0,+\infty) \end{cases}$$

$$(PW) \qquad \begin{cases} \rho_1\omega_{tt} - Eh\omega_{xx} = 0 & \text{in } (0,L)\times(0,\infty) \\ \omega(x,0) = \omega_0(x), \quad \omega_t(x,0) = \omega_1(x) & \text{in } (0,L) \\ \omega(0,t) = 0, & \text{in } (0,+\infty) \\ Eh\omega_x(L,t) = -\gamma_3\partial_t^{\alpha,\eta}\omega(L,t) & \text{in } (0,+\infty). \end{cases}$$

The abstract formulation of (P_0) is:

$$\mathcal{A}_0 \begin{pmatrix} \varphi \\ u \\ \phi_1 \\ \psi \\ v \\ \phi_2 \\ \omega \\ \tilde{\omega} \\ \phi_3 \end{pmatrix} = \begin{pmatrix} u \\ \dfrac{Gh}{\rho_1}(\varphi_x+\psi)_x \\ -(\xi^2+\eta)\phi_1 + u(L)\mu(\xi) \\ v \\ \dfrac{EI}{\rho_2}\psi_{xx} - \dfrac{Gh}{\rho_2}(\varphi_x+\psi) \\ -(\xi^2+\eta)\phi_2 + v(L)\mu(\xi) \\ \tilde{\omega} \\ \dfrac{Eh}{\rho_1}\omega_{xx} \\ -(\xi^2+\eta)\phi_3 + \tilde{\omega}(L)\mu(\xi) \end{pmatrix} \tag{2.54}$$

with domain

$$D(\mathcal{A}_0) = \left\{ \begin{array}{l} (\varphi, u, \phi_1, \psi, v, \phi_2, \omega, \tilde{\omega}, \phi_3)^T \text{ in } \mathcal{H} : \varphi, \psi, \omega \in H^2(0,L) \cap H^1_L(0,L), u, v, \tilde{\omega} \in H^1_L(0,L), \\ -(\xi^2+\eta)\phi_1 + u(L)\mu(\xi), -(\xi^2+\eta)\phi_2 + v(L)\mu(\xi) \in L^2(-\infty,+\infty), \\ -(\xi^2+\eta)\phi_3 + \tilde{\omega}(L)\mu(\xi) \in L^2(-\infty,+\infty) \\ Gh(\varphi_x+\psi)(L) + \zeta_1 \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_1(\xi)\,d\xi = 0 \\ EI\psi_x(L) + \zeta_2 \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_2(\xi)\,d\xi = 0 \\ Eh\omega_x(L) + \zeta_3 \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_3(\xi)\,d\xi = 0, \\ |\xi|\phi_1, |\xi|\phi_2, |\xi|\phi_3 \in L^2(-\infty,+\infty) \end{array} \right\}. \tag{2.55}$$

Proposition 2.4.1 *The semigroup generated by operator $\mathcal{A}_0$ is not exponentially stable.*

Proof. This result is due to the fact that a subsequence of eigenvalues of $\mathcal{A}_0$ is close to the imaginary axis.

Let $\mathcal{H}_1$, be the subspaces of $\mathcal{H}$ defined by

$$\mathcal{H}_1 = \{U \in \mathcal{H} \backslash U = (0,0,0,0,0,0,\omega,\tilde{\omega},\phi_3)\}$$

and

$$\mathcal{A}_1 = \mathcal{A}_0|_{\mathcal{H}_1}.$$

Observe that the generator $\mathcal{A}_0$ becomes the operator $\mathcal{A}_1$ defined by

$$D(\mathcal{A}_1) = \left\{ \begin{array}{l} (0,0,0,0,0,0,\omega,\tilde{\omega},\phi_3)^T \text{ in } \mathcal{H}_1 : \omega \in H^2(0,L) \cap H^1_L(0,L), \\ \tilde{\omega} \in H^1_L(0,L), -(\xi^2+\eta)\phi_3 + \tilde{\omega}(L)\mu(\xi) \in L^2(-\infty,+\infty), \\ Eh\omega_x(L) + \zeta_3 \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_3(\xi)\,d\xi = 0 \\ |\xi|\phi_3 \in L^2(-\infty,+\infty) \end{array} \right\} \tag{2.56}$$

and

$$
(2.57) \qquad \mathcal{A}_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \omega \\ \tilde{\omega} \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \tilde{\omega} \\ \dfrac{Eh}{\rho_1}\omega_{xx} \\ -(\xi^2+\eta)\phi_3 + \tilde{\omega}(L)\mu(\xi) \end{pmatrix}
$$

for all $U = (0,0,0,0,0,0,\omega,\tilde{\omega},\phi_3)^T \in D(\mathcal{A}_1)$.

We first compute the characteristic equation that gives the eigenvalues of $\mathcal{A}_1$. Let λ be an eigenvalue of $\mathcal{A}_1$ with associated eigenvector $U = (0,0,0,0,0,0,\omega,\tilde{\omega},\phi_3)^T$. Then $\mathcal{A}_1 U = \lambda U$ is equivalent to

$$
(2.58) \qquad \begin{cases} \lambda\omega - \tilde{\omega} = 0, \\ \lambda\tilde{\omega} - \dfrac{Eh}{\rho_1}\omega_{xx} = 0, \\ \lambda\phi_3 + (\xi^2+\eta)\phi_3 - \tilde{\omega}(L)\mu(\xi) = 0. \end{cases}
$$

From $(2.58)_1$, we have

$$
(2.59) \qquad \tilde{\omega} = \lambda\omega.
$$

Inserting (2.59) in $(2.58)_2$, we get

$$
(2.60) \qquad \lambda^2\omega - \frac{Eh}{\rho_1}\omega_{xx} = 0
$$

with the following conditions (according to the Lemma 2.2.1)

$$
(2.61) \qquad \begin{cases} \omega(0) = 0, \\ Eh\omega_x(L) = -\gamma_3\lambda(\lambda+\eta)^{\alpha-1}\omega(L). \end{cases}
$$

The matrix of the system determining is not singular. Set $X = (\omega, \omega_x)^T$

$$
(2.62) \qquad \frac{d}{dx}X = \tilde{\mathcal{B}}X,
$$

where

$$
\tilde{\mathcal{B}} = \begin{pmatrix} 0 & 1 \\ \dfrac{\rho_1}{Eh}\lambda^2 & 0 \end{pmatrix}.
$$

The characteristic polynomial of $\tilde{\mathcal{B}}$ is

$$
s^2 - \frac{\rho_1}{Eh}\lambda^2 = 0.
$$

We find the roots

$$t_1(\lambda) = \sqrt{\frac{\rho_1}{Eh}}\lambda, \quad t_2(\lambda) = -\sqrt{\frac{\rho_1}{Eh}}\lambda.$$

Here and below, for simplicity we denote $t_i(\lambda)$ by t_i. The solution ω is given by

$$\omega(x) = \sum_{i=1}^{2} c_i e^{t_i x} \tag{2.63}$$

Thus the boundary conditions may be written as the following system:

$$\tilde{M}(\lambda)C(\lambda) = \begin{pmatrix} 1 & 1 \\ h(t_1)e^{t_1 L} & h(t_2)e^{t_2 L} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.64}$$

where we have set

$$h(r) = Kr + \gamma_3\lambda(\lambda+\eta)^{\alpha-1} \text{ (we set } Eh = K).$$

Hence a non-trivial solution ω exists if and only if the determinant of $\tilde{M}(\lambda)$ vanishes. Set $f(\lambda) = det\tilde{M}(\lambda)$, thus the characteristic equation is $f(\lambda) = 0$.

Our purpose in the sequel is to prove, thanks to Rouché's Theorem (see [**43**]), that there is a subsequence of eigenvalues for which their real part tends to 0.

In the sequel, since $\mathcal{A}_1$ is dissipative, we study the asymptotic behavior of the large eigenvalues λ of $\mathcal{A}_1$ in the strip $-\alpha_0 \leq \Re e(\lambda) \leq 0$, for some $\alpha_0 > 0$ large enough and for such λ, we remark that $e^{t_i}, i = 1, 2$ remains bounded.

Lemma 2.4.6 *There exists $N \in \mathbb{N}$ such that*

$$\{\lambda_k\}_{k \in Z^*, |k| \geq N} \subset \sigma(\mathcal{A}_1), \tag{2.65}$$

where

$$\lambda_k = i\frac{1}{rL}\left(k + \frac{1}{2}\right)\pi + \frac{\tilde{\alpha}}{k^{1-\alpha}} + \frac{\beta}{|k|^{1-\alpha}} + o\left(\frac{1}{k^{1-\alpha}}\right), r = \sqrt{\frac{\rho_1}{K}}, k \geq N, \tilde{\alpha} \in i\mathbb{R}, \beta \in \mathbb{R}, \beta < 0.$$

$$\lambda_k = \overline{\lambda_{-k}} \text{ if } k \leq -N.$$

Moreover for all $|k| \geq N$, the eigenvalues λ_k are simple.

Proof.

The proof is decomposed in three steps:

Step 1.

$$
\begin{aligned}
f(\lambda) &= e^{Lt_2}h(t_2) - e^{Lt_1}h(t_1) \\
&= -e^{-\sqrt{\frac{\rho_1}{K}}\lambda L} h(\sqrt{\tfrac{\rho_1}{K}}\lambda)\left(e^{2\sqrt{\frac{\rho_1}{K}}\lambda L} + \frac{K(\lambda+\eta)^{1-\alpha}-\gamma_3}{K(\lambda+\eta)^{1-\alpha}+\gamma_3}\right) \\
&= -e^{-\sqrt{\frac{\rho_1}{K}}\lambda L} h(\lambda)\left(e^{2\sqrt{\frac{\rho_1}{K}}\lambda L} + 1 - \frac{2\gamma_3}{\gamma_3+K(\lambda+\eta)^{1-\alpha}}\right)
\end{aligned} \tag{2.66}
$$

We set

$$
\begin{aligned}
\tilde{f}(\lambda) &= e^{2\sqrt{\frac{\rho_1}{K}}\lambda L} + 1 - \frac{2\gamma_3}{\gamma_3+K(\lambda+\eta)^{1-\alpha}} \\
&= f_0(\lambda) + \tfrac{f_1(\lambda)}{\lambda^{1-\alpha}} + \tfrac{f_2(\lambda)}{\lambda^{2-2\alpha}} + o\left(\tfrac{1}{\lambda^{2-2\alpha}}\right)
\end{aligned} \tag{2.67}
$$

$$
f_0(\lambda) = e^{2\sqrt{\frac{\rho_1}{K}}\lambda L} + 1 \tag{2.68}
$$

$$
f_1(\lambda) = -2\gamma_3/K \tag{2.69}
$$

$$
f_2(\lambda) = 2\gamma_3^2/K^2 \tag{2.70}
$$

Note that f_0, f_1 and f_2 remain bounded in the strip $-\alpha_0 \leq \Re e(\lambda) \leq 0$.

Step 2. We look at the roots of f_0. From (2.68), f_0 has one familie of roots that we denote λ_k^0.

$$
f_0(\lambda) = 0 \Leftrightarrow e^{2\sqrt{\frac{\rho_1}{K}}\lambda L} = -1.
$$

Hence

$$
2r\lambda L = i(2k+1)\pi, \quad k \in Z, r = \sqrt{\frac{\rho_1}{K}}
$$

i.e.,

$$
\lambda_k^0 = \frac{i(2k+1)\pi}{2rL}, \quad k \in Z.
$$

Now with the help of Rouché's Theorem, we will show that the roots of $\tilde{f}$ are close to those of f_0. Let us start with the first family. Changing in (2.67) the unknown λ by $u = 2\sqrt{\frac{\rho_1}{K}}\lambda L$ then (2.67) becomes

$$
\tilde{f}(u) = (e^u+1) + O\left(\frac{1}{u^{(1-\alpha)}}\right) = f_0(u) + O\left(\frac{1}{u^{(1-\alpha)}}\right)
$$

The roots of f_0 are $u_k = \frac{i\left(k+\frac{1}{2}\right)}{rL}\pi, k \in Z$, and setting $u = u_k + re^{it}, t \in [0, 2\pi]$, we can easily check that there exists a constant $C > 0$ independent of k such that $|e^u + 1| \geq Cr$ for r small enough. This allows to apply Rouché's Theorem. Consequently, there exists a subsequence of roots of $\tilde{f}$ which tends to the roots u_k of f_0. Equivalently, it means that there exists $N \in \mathbb{N}$ and a subsequence $\{\lambda_k\}_{|k|\geq N}$ of roots of $f(\lambda)$, such that $\lambda_k = \lambda_k^0 + o(1)$ which tends to the roots $\frac{i\left(k+\frac{1}{2}\right)}{rL}\pi$ of f_0. Finally for $|k| \geq N, \lambda_k$ is simple since λ_k^0 .

Step 3. From Step 2, we can write

$$\lambda_k = i\frac{1}{rL}\left(k+\frac{1}{2}\right)\pi + \varepsilon_k. \tag{2.71}$$

Using (2.71), we get

$$e^{2r\lambda_k L} = -1 - 2rL\varepsilon_k - 2rL^2\varepsilon_k^2 + o(\varepsilon_k^2). \tag{2.72}$$

Substituting (2.72) into (2.67), using that $\tilde{f}(\lambda_k) = 0$, we get:

$$\tilde{f}(\lambda_k) = -2rL\varepsilon_k - \frac{2\gamma_3}{K(\lambda_k^0)^{1-\alpha}} + o(\varepsilon_k) = 0 \tag{2.73}$$

and hence

$$\varepsilon_k = -\frac{\gamma_3}{Kr^\alpha L^\alpha((k+\frac{1}{2})i\pi)^{1-\alpha}} + o\left(\frac{1}{k^{1-\alpha}}\right) \tag{2.74}$$

Using the fact that

$$\frac{1}{i^{1-\alpha}} = cos(1-\alpha)\frac{\pi}{2} - i\sin(1-\alpha)\frac{\pi}{2} \tag{2.75}$$

we get

$$\varepsilon_k = \begin{cases} -\dfrac{\gamma_3}{Kr^\alpha L^\alpha((k+\frac{1}{2})\pi)^{1-\alpha}}\left(\cos(1-\alpha)\dfrac{\pi}{2} - i\sin(1-\alpha)\dfrac{\pi}{2}\right) + o\left(\dfrac{1}{k^{1-\alpha}}\right) & \text{for } k \succeq 0 \\ -\dfrac{\gamma_3}{Kr^\alpha L^\alpha(-(k+\frac{1}{2})\pi)^{1-\alpha}}\left(\cos(1-\alpha)\dfrac{\pi}{2} + i\sin(1-\alpha)\dfrac{\pi}{2}\right) + o\left(\dfrac{1}{k^{1-\alpha}}\right) & \text{for } k \preceq 0 \end{cases} \tag{2.76}$$

From (2.76) we have in that case $|k|^{1-\alpha}\Re e\lambda_k \sim \beta$, with

$$\beta = -\frac{\gamma_3}{Kr^\alpha L^\alpha \pi^{1-\alpha}}\cos(1-\alpha)\frac{\pi}{2}.$$

The operator $\mathcal{A}_1$ has a non exponential decaying branche of eigenvalues. Thus the proof of Proposition 2.4.1 is complete.

Now, we return to the proof of Theorem 2.4.2.

Proof

We will examine two cases.

Case 1 $\eta = 0$: This case follows directly from the Lemma 2.4.5.

Case 2 $\eta > 0$: We aim to show that an infinite number of eigenvalues of $\mathcal{A}$ approach the imaginary axis which prevents the Bresse system (P) from being exponentially stable. Indeed We first compute the characteristic equation that gives the eigenvalues of $\mathcal{A}$.

Let λ be an eigenvalue of $\mathcal{A}$ with associated eigenvector $U = (\varphi, u, \phi_1, \psi, v, \phi_2, \omega, \tilde{\omega}, \phi_3)^T$. We set $\rho_1 = \rho_2 = 1, Gh = 1, EI = a, Eh = k$. To solve $\mathcal{A}U = \lambda U$ is enough to solve

$$(2.77) \qquad \begin{cases} -\lambda^2\varphi + (\varphi_x + \psi + l\omega)_x + lk(\omega_x - l\varphi) = 0, \\ -\lambda^2\psi + a\psi_{xx} - (\varphi_x + \psi + l\omega) = 0, \\ -\lambda^2\omega + k(\omega_x - l\varphi)_x - l(\varphi_x + \psi + l\omega) = 0, \\ \lambda\phi_1 + (\xi^2 + \eta)\phi_1 - u(L)\mu(\xi) = 0, \\ \lambda\phi_2 + (\xi^2 + \eta)\phi_2 - v(L)\mu(\xi) = 0, \\ \lambda\phi_3 + (\xi^2 + \eta)\phi_3 - \tilde{\omega}(L)\mu(\xi) = 0. \end{cases}$$

From the boundary conditions, we get

$$(2.78) \qquad \begin{cases} (\varphi_x + \psi + l\omega)(L) = -\gamma_1\lambda(\lambda + \eta)^{\alpha-1}\varphi(L) \\ a\psi_x(L) = -\gamma_2\lambda(\lambda + \eta)^{\alpha-1}\psi(L) \\ k(\omega_x - l\varphi)(L) = -\gamma_3\lambda(\lambda + \eta)^{\alpha-1}\omega(L) \\ \varphi(0) = \psi(0) = \omega(0) = 0 \end{cases}$$

We set

$$\tilde{\varphi} = (\varphi_x + \psi + l\omega), \quad \tilde{\psi} = \psi_x, \quad \tilde{\omega} = (\omega_x - l\varphi).$$

Hence from (2.77), we have

$$(2.79) \qquad \begin{cases} -\lambda^2\varphi + \tilde{\varphi}_x + lk\tilde{\omega} = 0, \\ -\lambda^2\psi + a\tilde{\psi}_x - \tilde{\varphi} = 0, \\ -\lambda^2\omega + k\tilde{\omega}_x - l\tilde{\varphi} = 0. \end{cases}$$

Then

$$(2.80) \quad \begin{cases} -(\lambda^2 + l^2 + 1)\tilde{\varphi} + \tilde{\varphi}_{xx} + a\tilde{\psi}_x + 2lk\tilde{\omega}_x = 0 \quad ((2.79)_1 x + (2.79)_2 + l(2.79)_3), \\ -\lambda^2\tilde{\psi} + a\tilde{\psi}_{xx} - \tilde{\varphi}_x = 0, \\ -(\lambda^2 + l^2 k)\tilde{\omega} + k\tilde{\omega}_{xx} - 2l\tilde{\varphi}_x = 0. \end{cases}$$

From $(2.80)_3$, after derivation, we have

$$(2.81) \qquad -(\lambda^2 + l^2 k)\tilde{\omega}_x + k\tilde{\omega}_{xxx} - 2l\tilde{\varphi}_{xx} = 0.$$

From $(2.80)_1$, after derivation, we have

$$(2.82) \qquad \begin{aligned} \tilde{\omega}_x &= \frac{1}{2lk}((\lambda^2 + l^2 + 1)\tilde{\varphi} - \tilde{\varphi}_{xx} - a\tilde{\psi}_x), \\ \tilde{\omega}_{xxx} &= \frac{1}{2lk}((\lambda^2 + l^2 + 1)\tilde{\varphi}_{xx} - \tilde{\varphi}_{xxxx} - a\tilde{\psi}_{xxx}). \end{aligned}$$

Inserting $(2.82)_1, (2.82)_2$ into (2.81) , we find

$$(2.83) \qquad \begin{aligned} &-\frac{(\lambda^2 + l^2 k)}{2lk}\left((\lambda^2 + l^2 + 1)\tilde{\varphi} - \tilde{\varphi}_{xx} - a\tilde{\psi}_x\right) \\ &\qquad + \frac{1}{2l}\left((\lambda^2 + l^2 + 1)\tilde{\varphi}_{xx} - \tilde{\varphi}_{xxxx} - a\tilde{\psi}_{xxx}\right) - 2l\tilde{\varphi}_{xx} = 0 \end{aligned}$$

From $(2.79)_2$, after derivation, we have

$$(2.84)\qquad \begin{cases} \tilde{\varphi} &= -\lambda^2\psi + a\tilde{\psi}_x \\ \tilde{\varphi}_{xx} &= -\lambda^2\psi_{xx} + a\tilde{\psi}_{xxx} \\ \tilde{\varphi}_{xxxx} &= -\lambda^2\psi_{xxxx} + a\tilde{\psi}_{xxxxx}. \end{cases}$$

Inserting $(2.84)_1, (2.84)_2, (2.84)_3$ into (2.83), we find

$$(2.85)\qquad \begin{aligned} &\psi_{xxxxxx} + \left(-\left(\tfrac{1}{k}+\tfrac{1}{a}+1\right)\lambda^2 + 2l^2\right)\psi_{xxxx} \\ &\ +\left(\left(\tfrac{1}{a}+\tfrac{1}{ak}+\tfrac{1}{k}\right)\lambda^4 + \left(\tfrac{1}{a}+(-\tfrac{2}{a}+\tfrac{1}{k}+1)l^2\right)\lambda^2 + l^4\right)\psi_{xx} \\ &-\left(\tfrac{1}{ak}\lambda^6 + \left(\tfrac{1}{ak}l^2+\tfrac{1}{a}l^2+\tfrac{1}{ak}\right)\lambda^4 + \left(\tfrac{1}{a}l^4+\tfrac{1}{a}l^2\right)\lambda^2\right)\psi = 0, \end{aligned}$$

whose general solutions depends on the roots of the polynomial

$$(2.86)\qquad \begin{aligned} &p(s) = s^6 + \left(-\left(\tfrac{1}{k}+\tfrac{1}{a}+1\right)\lambda^2 + 2l^2\right)s^4 \\ &\ +\left(\left(\tfrac{1}{a}+\tfrac{1}{ak}+\tfrac{1}{k}\right)\lambda^4 + \left(\tfrac{1}{a}+(-\tfrac{2}{a}+\tfrac{1}{k}+1)l^2\right)\lambda^2 + l^4\right)s^2 \\ &-\left(\tfrac{1}{ak}\lambda^6 + \left(\tfrac{1}{ak}l^2+\tfrac{1}{a}l^2+\tfrac{1}{ak}\right)\lambda^4 + \left(\tfrac{1}{a}l^4+\tfrac{1}{a}l^2\right)\lambda^2\right). \end{aligned}$$

If we put $s^2 = S$, we can write (2.86) as

$$(2.87)\qquad \begin{aligned} &\tilde{p}(S) = S^3 + \left(-\left(\tfrac{1}{k}+\tfrac{1}{a}+1\right)\lambda^2 + 2l^2\right)S^2 \\ &\ +\left(\left(\tfrac{1}{a}+\tfrac{1}{ak}+\tfrac{1}{k}\right)\lambda^4 + \left(\tfrac{1}{a}+(-\tfrac{2}{a}+\tfrac{1}{k}+1)l^2\right)\lambda^2 + l^4\right)S \\ &-\left(\tfrac{1}{ak}\lambda^6 + \left(\tfrac{1}{ak}l^2+\tfrac{1}{a}l^2+\tfrac{1}{ak}\right)\lambda^4 + \left(\tfrac{1}{a}l^4+\tfrac{1}{a}l^2\right)\lambda^2\right). \end{aligned}$$

The polynomial $\tilde{p}(S)$, given in (2.87), has one real root and two complex conjugate roots.

The general solution of (2.85) must be of the form

$$(2.88)\qquad \psi(x) = \sum_{i=1}^{6} c_i e^{t_i x},$$

where $t_i(\lambda)$ $(i = 1, \ldots, 6)$ are the roots of (2.86) such that $t_2(\lambda) = -t_1(\lambda), t_4(\lambda) = -t_3(\lambda), t_6(\lambda) = -t_5(\lambda)$. $t_3(\lambda)^2$ and $t_5(\lambda)^2$ are complex conjugate. Then, we write (2.78) uniquely in function of ψ. From $(2.79)_1$ we have

$$\varphi = \frac{1}{\lambda^2}(\tilde{\varphi}_x + lk\tilde{\omega}).$$

From $(2.80)_3$ we have

$$\tilde{\omega} = -\frac{2l}{\lambda^2 + l^2k}\tilde{\varphi}_x + \frac{k}{\lambda^2 + l^2k}\tilde{\omega}_{xx}.$$

From $(2.80)_1$, after derivation, we get

$$\tilde{\omega}_{xx} = \frac{\lambda^2 + l^2 + 1}{2lk}\tilde{\varphi}_x - \frac{1}{2lk}\tilde{\varphi}_{xxx} - \frac{a}{2lk}\tilde{\psi}_{xx}.$$

Thus we find

$$\begin{aligned}
\tilde{\omega} &= -\frac{2l}{\lambda^2+l^2k}\tilde{\varphi}_x + \frac{\lambda^2+l^2+1}{2l(\lambda^2+l^2k)}\tilde{\varphi}_x - \frac{1}{2l(\lambda^2+l^2k)}\tilde{\varphi}_{xxx} - \frac{a}{2l(\lambda^2+l^2k)}\tilde{\psi}_{xx} \\
&= \frac{\lambda^2-3l^2+1}{2l(\lambda^2+l^2k)}\tilde{\varphi}_x - \frac{1}{2l(\lambda^2+l^2k)}\tilde{\varphi}_{xxx} - \frac{a}{2l(\lambda^2+l^2k)}\tilde{\psi}_{xx} \\
&= -\lambda^2\frac{\lambda^2-3l^2+1}{2l(\lambda^2+l^2k)}\psi_x + \frac{(a+1)\lambda^2-3al^2}{2l(\lambda^2+l^2k)}\psi_{xxx} - \frac{a}{2l(\lambda^2+l^2k)}\psi_{xxxxx}.
\end{aligned}$$

We deduce that

$$\varphi = -\frac{(k+2)\lambda^2+k(1-l^2)}{2(\lambda^2+kl^2)}\psi_x + \frac{((k+2)a+k)\lambda^2-akl^2}{2\lambda^2(\lambda^2+kl^2)}\psi_{xxx} - \frac{ka}{2\lambda^2(\lambda^2+kl^2)}\psi_{xxxxx}.$$

From $(2.79)_3$ and $(2.80)_1$ we have

$$\omega = \frac{1}{\lambda^2}(k\tilde{\omega}_x - l\tilde{\varphi})$$

$$\tilde{\omega}_x = \frac{\lambda^2+l^2+1}{2lk}\tilde{\varphi} - \frac{1}{2lk}\tilde{\varphi}_{xx} - \frac{a}{2lk}\tilde{\psi}_x. \tag{2.89}$$

Then from $(2.79)_2$ and $(2.80)_2$ and (2.89), we get

$$\omega = -\frac{\lambda^2-l^2+1}{2l}\psi + \frac{(a+1)\lambda^2-al^2}{2l\lambda^2}\psi_{xx} - \frac{a}{2l\lambda^2}\psi_{xxxx}.$$

Thus the boundary conditions (2.78) may be written as the following system:

$$\psi(0)=0 \Longrightarrow \sum_{i=1}^{6} c_i = 0,$$
$$\varphi(0)=0 \Longrightarrow \sum_{i=1}^{6}(\alpha_1 t_i + \alpha_3 t_i^3 + \alpha_5 t_i^5)c_i = 0,$$
$$\omega(0)=0 \Longrightarrow \sum_{i=1}^{6}(\alpha_0 + \alpha_2 t_i^2 + \alpha_4 t_i^4)c_i = 0,$$

$$a\psi_x(L) = -\gamma_2\lambda(\lambda+\eta)^{\alpha-1}\psi(L) \Longrightarrow \sum_{i=1}^{6}(at_i + \gamma_2\lambda(\lambda+\eta)^{\alpha-1}))e^{t_iL}c_i = 0,$$

$$\tilde{\varphi}(L) = -\gamma_1\lambda(\lambda+\eta)^{\alpha-1}\varphi(L) \Longrightarrow \sum_{i=1}^{6}(-\lambda^2 + at_i^2 + \gamma_1\lambda(\lambda+\eta)^{\alpha-1}(\alpha_1 t_i + \alpha_3 t_i^3 + \alpha_5 t_i^5))e^{t_iL}c_i = 0,$$

$$k\tilde{\omega}(L) = -\gamma_3\lambda(\lambda+\eta)^{\alpha-1}\omega(L) \Longrightarrow \sum_{i=1}^{6}(k(\delta_1 t_i + \delta_3 t_i^3 + \delta_5 t_i^5) + \gamma_3\lambda(\lambda+\eta)^{\alpha-1}(\alpha_0 + \alpha_2 t_i^2 + \alpha_4 t_i^4))e^{t_iL}c_i = 0.$$

Thus the boundary conditions may be written as the following system:

$$\mathcal{M}_l C(\lambda) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where $C(\lambda) = (c_1, c_2, c_3, c_4, c_5, c_6)^T$ and

$$\mathcal{M}_l = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ h_1(t_1) & h_1(t_2) & h_1(t_3) & h_1(t_4) & h_1(t_5) & h_1(t_6) \\ h_2(t_1) & h_2(t_2) & h_2(t_3) & h_2(t_4) & h_2(t_5) & h_2(t_6) \\ h_3(t_1)e^{t_1 L} & h_3(t_2)e^{t_2 L} & h_3(t_3)e^{t_3 L} & h_3(t_4)e^{t_4 L} & h_3(t_5)e^{t_5 L} & h_3(t_6)e^{t_6 L} \\ h_4(t_1)e^{t_1 L} & h_4(t_2)e^{t_2 L} & h_4(t_3)e^{t_3 L} & h_4(t_4)e^{t_4 L} & h_4(t_5)e^{t_5 L} & h_4(t_6)e^{t_6 L} \\ h_5(t_1)e^{t_1 L} & h_5(t_2)e^{t_2 L} & h_5(t_3)e^{t_3 L} & h_5(t_4)e^{t_4 L} & h_5(t_5)e^{t_5 L} & h_5(t_6)e^{t_6 L} \end{pmatrix},$$

where

$$\begin{aligned} h_1(r) &= \alpha_1 r + \alpha_3 r^3 + \alpha_5 r^5, \\ h_2(r) &= \alpha_0 + \alpha_2 r^2 + \alpha_4 r^4, \\ h_3(r) &= ar + \gamma_2 \lambda(\lambda+\eta)^{\alpha-1}, \\ h_4(r) &= -\lambda^2 + ar^2 + \gamma_1 \lambda(\lambda+\eta)^{\alpha-1}(\alpha_1 r + \alpha_3 r^3 + \alpha_5 r^5), \\ h_5(r) &= k(\delta_1 r + \delta_3 r^3 + \delta_5 r^5), +\gamma_3 \lambda(\lambda+\eta)^{\alpha-1}(\alpha_0 + \alpha_2 r^2 + \alpha_4 r^4), \end{aligned}$$

where

$$\alpha_1 = -\frac{(k+2)\lambda^2 + k(1-l^2)}{2(\lambda^2+kl^2)}, \alpha_3 = \frac{((k+2)a+k)\lambda^2 - akl^2}{2\lambda^2(\lambda^2+kl^2)}, \alpha_5 = -\frac{ka}{2\lambda^2(\lambda^2+kl^2)},$$
$$\alpha_0 = -(\lambda^2 - l^2 + 1), \alpha_2 = \frac{(a+1)\lambda^2 - al^2}{\lambda^2}, \alpha_4 = -\frac{a}{\lambda^2},$$
$$\delta_1 = -\lambda^2 \frac{\lambda^2 - 3l^2 + 1}{(\lambda^2 + l^2 k)}, \delta_3 = \frac{(a+1)\lambda^2 - 3al^2}{(\lambda^2 + l^2 k)}, \delta_5 = -\frac{a}{(\lambda^2 + l^2 k)}.$$

Hence a non-trivial solution ψ exists if and only if the determinant of $\mathcal{M}_l$ vanishes. Set $f_l(\lambda) = det\mathcal{M}_l$, thus the characteristic equation is $f_l(\lambda) = 0$.

We remark that $f_l(\lambda) = det\mathcal{M}_l$ is a smooth function with the parameter l. In the expansion of $t_i(\lambda)(i = 1, \ldots, 6)$ and $h_i(t_j)(i = 1, \ldots 5, j = 1, \ldots, 6)$, the parameter l appears only in lower terms. Hence, in the development of $det\mathcal{M}_l$ in power series following λ, we obtain same development as $det\mathcal{M}_0$ modulo lower terms depending on l. Hence $\mathcal{A}_l$ (if we note $\mathcal{A}$ by $\mathcal{A}_l$) and $\mathcal{A}_0$ have same branches of eigenvalues modulo lower terms depending on l.

From Proposition 2.4.1 and the fact that $f_0(\lambda) = det\mathcal{M}_0 = 0$ give the eigenvalues of Bresse system when $l = 0$ we conclude our result. The proof is completed.

Remark 2.4.1 *We can also show the lack of exponential stability by proving that the second condition in Theorem 1.3.7 does not hold. In particular, it can be shown that there is a sequence $(\lambda_n)_n \subset \mathbb{R}$ diverging to ∞, and a bounded sequence $(F_n)_n \subset \mathcal{H}$ such that*

$$\|(i\lambda_n I - \mathcal{A})^{-1}F_n\| \to \infty \text{ for all } n \text{ large enough .}$$

We give an idea of the proof in the Appendix 1.

2.4.3 Polynomial stability when $\eta > 0$

When the semigoup is not exponentialy stable we study the polynomial stability of the semigroup.

Theorem 2.4.3 *Suppose that $\eta > 0$. Then the semigroup $(S(t))_{t\geq 0}$ is polynomially stable and*

$$\|S_{\mathcal{A}}(t)U_0\|_{\mathcal{H}} \leq \frac{1}{t^{\frac{1}{2(1-\alpha)}}}\|U_0\|_{D(\mathcal{A})}$$

Proof.
By Lemma 2.4.3 and Lemma 2.4.4 we know that $i\,\mathbb{R} \subset \rho(A)$ for $\eta \neq 0$. We will need to study the resolvent equation $(i\lambda - \mathcal{A})U = F$, for $\lambda \in \mathbb{R}$, namely

$$(2.90)\qquad \begin{cases} i\lambda\varphi - u = f_1, \\ i\lambda\rho_1 u - Gh(\varphi_x + \psi + l\omega)_x - lEh(\omega_x - l\varphi) = \rho_1 f_2, \\ i\lambda\phi_1 + (\xi^2 + \eta)\phi_1 - u(L)\mu(\xi) = f_3, \\ i\lambda\psi - v = f_4, \\ i\lambda\rho_2 v - EI\psi_{xx} + Gh(\varphi_x + \psi + l\omega) = \rho_2 f_5, \\ i\lambda\phi_2 + (\xi^2 + \eta)\phi_2 - v(L)\mu(\xi) = f_6, \\ i\lambda\omega - \tilde{\omega} = f_7, \\ i\lambda\rho_1\tilde{\omega} - Eh(\omega_x - l\varphi)_x + lGh(\varphi_x + \psi + l\omega) = \rho_1 f_8, \\ i\lambda\phi_3 + (\xi^2 + \eta)\phi_3 - \tilde{\omega}(L)\mu(\xi) = f_9, \end{cases}$$

where $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9)^T$. Taking inner product in $\mathcal{H}$ with U and using (2.18) we get

$$|Re\langle \mathcal{A}U, U\rangle| = |Re\langle (i\lambda - \mathcal{A})U, U\rangle| \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{2.91}$$

This implies that

$$\sum_{i=1}^{3} \zeta_i \int_{-\infty}^{+\infty} (\xi^2 + \eta)(\phi_i(\xi, t))^2 \, d\xi \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{2.92}$$

and, applying $(2.90)_{1,4,7}$, we obtain

$$\begin{array}{rcl} ||\lambda||\varphi(L)| - |f_1(L)||^2 & \leq & |u(L)|^2 \\ ||\lambda||\psi(L)| - |f_4(L)||^2 & \leq & |v(L)|^2 \\ ||\lambda||\omega(L)| - |f_7(L)||^2 & \leq & |\tilde{\omega}(L)|^2. \end{array}$$

We deduce that

$$\begin{array}{rcl} |\lambda|^2|\varphi(L)|^2 & \leq & C|f_1(L)|^2 + C|u(L)|^2 \\ |\lambda|^2|\psi(L)|^2 & \leq & C|f_4(L)|^2 + C|v(L)|^2 \\ |\lambda|^2|\omega(L)|^2 & \leq & C|f_1(L)|^2 + C|\tilde{\omega}(L)|^2. \end{array}$$

Moreover, since

$$\begin{aligned} & Gh|(\varphi_x + \psi + l\omega)(L)|^2 + EI|\psi_x(L)|^2 + Eh|(\omega_x - l\varphi)(L)|^2 \leq \sum_{i=1}^{3} \zeta_i^2 \left| \int_{-\infty}^{+\infty} \mu(\xi)\phi_i(\xi)\, d\xi \right|^2 \\ & \leq \left(\int_{-\infty}^{+\infty} (\xi^2 + \eta)^{-1} |\mu(\xi)|^2 \, d\xi \right) \sum_{i=1}^{3} \zeta_i^2 \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi_i(\xi)|^2 \, d\xi \\ & \leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \end{aligned}$$

From $(2.90)_{3,6,9}$, we obtain

$$\begin{cases} u(L)\mu(\xi) = (i\lambda + \xi^2 + \eta)\phi_1 - f_3(\xi) \\ v(L)\mu(\xi) = (i\lambda + \xi^2 + \eta)\phi_2 - f_6(\xi) \\ \omega(L)\mu(\xi) = (i\lambda + \xi^2 + \eta)\phi_3 - f_9\xi). \end{cases} \tag{2.93}$$

By multiplying $(2.93)_1$ by $(i\lambda + \xi^2 + \eta)^{-1}\mu(\xi)$, we get

$$(i\lambda + \xi^2 + \eta)^{-1} u(L)\mu^2(\xi) = \mu(\xi)\phi_1 - ((i\lambda + \xi^2 + \eta)^{-1}\mu(\xi) f_3(\xi). \tag{2.94}$$

Hence, by taking absolute values of both sides of (2.94), integrating over the interval $]-\infty, +\infty[$ with respect to the variable ξ and applying Cauchy-Schwartz inequality, we obtain

$$\mathcal{S}|u(L)| \leq \mathcal{U} \left(\int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi_1|^2 \, d\xi \right)^{\frac{1}{2}} + \mathcal{V} \left(\int_{-\infty}^{+\infty} |f_3(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} \tag{2.95}$$

where

$$\mathcal{S} = \int_{-\infty}^{+\infty} (|\lambda| + \xi^2 + \eta)^{-1} |\mu(\xi)|^2 \, d\xi$$

$$\mathcal{U} = \left(\int_{-\infty}^{+\infty} (\xi^2+\eta)^{-1}|\mu(\xi)|^2\, d\xi\right)^{\frac{1}{2}}$$

$$\mathcal{V} = \left(\int_{-\infty}^{+\infty} (|\lambda|+\xi^2+\eta)^{-2}|\mu(\xi)|^2\, d\xi\right)^{\frac{1}{2}}.$$

Thus, by using again the inequality $2PQ \leq P^2+Q^2, P\geq 0, Q\geq 0$, we get

$$\mathcal{S}^2|u(L)|^2 \leq 2\mathcal{U}^2\left(\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi_1|^2\, d\xi\right) + 2\mathcal{V}^2\left(\int_{-\infty}^{+\infty}|f_3(\xi)|^2\, d\xi\right). \tag{2.96}$$

We deduce that

$$|u(L)|^2+|v(L)|^2+|\tilde{\omega}(L)|^2 \leq C|\lambda|^{2-2\alpha}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|F\|^2_{\mathcal{H}}. \tag{2.97}$$

It follows that

$$|\varphi_x(L)|^2+|\psi_x(L)|^2+|\omega_x(L)|^2 \leq C\left(\frac{1}{|\lambda|^{2\alpha}}+1\right)\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\frac{1}{|\lambda|^2}\|F\|^2_{\mathcal{H}}. \tag{2.98}$$

Let us introduce the following notation

$$\begin{aligned}
\mathcal{I}_\varphi(\alpha) &= \rho_1|u(\alpha)|^2 + Gh|\varphi_x(\alpha)|^2,\\
\mathcal{I}_\psi(\alpha) &= \rho_2|v(\alpha)|^2 + EI|\psi_x(\alpha)|^2,\\
\mathcal{I}_\omega(\alpha) &= \rho_1|\tilde{\omega}(\alpha)|^2 + Eh|\omega_x(\alpha)|^2,\\
\mathcal{I}(\alpha) &= \mathcal{I}_\varphi(\alpha)+\mathcal{I}_\psi(\alpha)+\mathcal{I}_\omega(\alpha),
\end{aligned}$$

$$\mathcal{E}_\varphi(L) = \int_0^L \mathcal{I}_\varphi(s)\, ds, \quad \mathcal{E}_\psi(L) = \int_0^L \mathcal{I}_\psi(s)\, ds, \quad \mathcal{E}_\omega(L) = \int_0^L \mathcal{I}_\omega(s)\, ds.$$

Lemma 2.4.7 *Let $q \in H^1(0,L)$. We have that*

$$\begin{aligned}
\mathcal{E}_\varphi(L) = [q\mathcal{I}_\varphi]_0^L - Ehl^2[q|\varphi|^2]_0^L + 2Gh\ Re\int_0^L q\psi_x\overline{\varphi}_x\, dx + Ehl^2\int_0^L q'(x)|\varphi|^2\, dx\\
+2(G+E)hlRe\int_0^L q\omega_x\overline{\varphi}_x\, dx + R_1
\end{aligned} \tag{2.99}$$

$$\begin{aligned}
\mathcal{E}_\psi(L) = [q\mathcal{I}_\psi]_0^L - Gh[q|\psi|^2]_0^L - 2Gh\ Re\int_0^L q\varphi_x\overline{\psi}_x\, dx + Gh\int_0^L q'(x)|\psi|^2\, dx\\
-2GhlRe\int_0^L q\omega\overline{\psi}_x\, dx + R_2
\end{aligned} \tag{2.100}$$

and

$$\begin{aligned}
\mathcal{E}_\omega(L) = [q\mathcal{I}_\omega]_0^L - Ghl^2[q|\omega|^2]_0^L - 2Ghl\ Re\int_0^L q\psi\overline{\omega}_x\, dx + Ghl^2\int_0^L q'(x)|\omega|^2\, dx\\
-2(G+E)hlRe\int_0^L q\varphi_x\overline{\omega}_x\, dx + R_3
\end{aligned} \tag{2.101}$$

where R_i satisfies

$$|R_i| \leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}, \quad i=1,2,3.$$

for a positive constant C.

Proof.

To get (2.99), let us multiply the equation $(2.90)_2$ by $q\overline{\varphi}_x$ Integrating on $(0,L)$ we obtain

$$i\lambda\rho_1\int_0^L uq\overline{\varphi}_x\,dx - Gh\int_0^L(\varphi_x+\psi+l\omega)_x q\overline{\varphi}_x\,dx - lEh\int_0^L(\omega_x-l\varphi)q\overline{\varphi}_x\,dx = \rho_1\int_0^L f_2 q\overline{\varphi}_x\,dx$$

or

$$\begin{aligned}&-\rho_1\int_0^L uq(\overline{i\lambda\varphi_x})\,dx - Gh\int_0^L q\varphi_{xx}\overline{\varphi}_x\,dx - Gh\int_0^L q\psi_x\overline{\varphi}_x\,dx\\ &-(G+E)lh\int_0^L q\omega_x\overline{\varphi}_x\,dx + l^2Eh\int_0^L q\varphi\overline{\varphi}_x\,dx = \rho_1\int_0^L f_2 q\overline{\varphi}_x\,dx.\end{aligned}$$

Since $i\lambda\varphi_x = u_x + f_{1x}$ taking the real part in the above equality results in

$$\begin{aligned}&-\frac{\rho_1}{2}\int_0^L q\frac{d}{dx}|u|^2\,dx - \frac{Gh}{2}\int_0^L q\frac{d}{dx}|\varphi_x|^2\,dx = \rho_1 Re\int_0^L f_2 q\overline{\varphi}_x\,dx + \rho_1 Re\int_0^L uq\overline{f}_{1x}\,dx\\ &+Gh\ Re\int_0^L q\psi_x\overline{\varphi}_x\,dx + +(G+E)lh\ Re\int_0^L q\omega_x\overline{\varphi}_x\,dx - \frac{l^2Eh}{2}\int_0^L q\frac{d}{dx}|\varphi|^2\,dx.\end{aligned}$$

Performing an integration by parts we get

$$\begin{aligned}&\int_0^L q'(s)[\rho_1|u(s)|^2 + Gh|\varphi_x(s)|^2]\,ds\\ &= [q\mathcal{I}_\varphi]_0^L - l^2Eh[q|\varphi|^2]_0^L + 2Gh\ Re\int_0^L q\psi_x\overline{\varphi}_x\,dx\\ &+l^2Eh\int_0^L q'(s)|\varphi(s)|^2\,dx + 2(G+E)lh\ Re\int_0^L q\omega_x\overline{\varphi}_x\,dx + R_1\end{aligned}$$

where

$$R_1 = 2\rho_1 Re\int_0^L f_2 q\overline{\varphi}_x\,dx + 2\rho_1 Re\int_0^L uq\overline{f}_{1x}\,dx.$$

Similarly, multiplying equation $(2.90)_5$ by $q\overline{\varphi}_x$, integrating on $(0,L)$ and taking the real part we obtain

$$i\lambda\rho_2\int_0^L vq\overline{\psi}_x\,dx - EI\int_0^L \psi_{xx}q\overline{\psi}_x\,dx + Gh\int_0^L(\varphi_x+\psi+l\omega)q\overline{\psi}_x\,dx = \rho_2\int_0^L f_5 q\overline{\psi}_x\,dx$$

or

$$\begin{aligned}&-\rho_2\int_0^L vq(\overline{i\lambda\psi_x})\,dx - EI\int_0^L q\psi_{xx}\overline{\psi}_x\,dx + Gh\int_0^L q\varphi_x\overline{\psi}_x\,dx\\ &+Gh\int_0^L q\psi\overline{\psi}_x\,dx + Ghl\int_0^L q\omega\overline{\psi}_x\,dx = \rho_2\int_0^L f_5 q\overline{\psi}_x\,dx.\end{aligned}$$

Since $i\lambda\psi_x = v_x + f_{4x}$ taking the real part in the above equality results in

$$\begin{aligned}&-\frac{\rho_2}{2}\int_0^L q\frac{d}{dx}|v|^2\,dx - \frac{EI}{2}\int_0^L q\frac{d}{dx}|\psi_x|^2\,dx = \rho_2 Re\int_0^L f_5 q\overline{\psi}_x\,dx\\ &+\rho_2\ Re\int_0^L qv\overline{f}_{4x}\,dx - Gh\ Re\int_0^L q\varphi_x\overline{\psi}_x\,dx - Ghl\ Re\int_0^L q\omega\overline{\psi}_x\,dx - \frac{Gh}{2}\int_0^L q\frac{d}{dx}|\psi|^2\,dx.\end{aligned}$$

Performing an integration by parts we get

$$\begin{aligned}&\int_0^L q'(s)[\rho_2|v(s)|^2 + EI|\psi_x(s)|^2]\,ds\\ &= [q\mathcal{I}_\psi]_0^L - Gh[q|\psi|^2]_0^L - 2Gh\ Re\int_0^L q\varphi_x\overline{\psi}_x\,dx\\ &-2Ghl\ Re\int_0^L q\omega\overline{\psi}_x\,dx + Gh\int_0^L q'|\psi|^2\,dx + R_2\end{aligned}$$

where

$$R_2 = 2\rho_2 Re \int_0^L f_5 q\overline{\psi}_x \, dx + 2\rho_2 \; Re \int_0^L qv\overline{f}_{4x} \, dx.$$

Finally, multiplying equation $(2.90)_8$ by $q\overline{\omega}_x$, integrating on $(0, L)$ and taking the real part, after some algebric manipulations we obtain (2.101) for

$$i\lambda\rho_1 \int_0^L \tilde{\omega} q\overline{\omega}_x \, dx - Eh \int_0^L (\omega_x - l\varphi)_x q\overline{\omega}_x \, dx + lGh \int_0^L (\varphi_x + \psi + l\omega) q\overline{\omega}_x \, dx = \rho_1 \int_0^L f_8 q\overline{\omega}_x \, dx$$

or

$$\begin{aligned} & -\rho_1 \int_0^L \tilde{\omega} q(\overline{i\lambda\omega_x}) \, dx - Eh \int_0^L q\omega_{xx}\overline{\omega}_x \, dx + lGh \int_0^L q\psi\overline{\omega}_x \, dx \\ & +(G+E)lh \int_0^L q\varphi_x\overline{\omega}_x \, dx + l^2 Gh \int_0^L q\omega\overline{\omega}_x \, dx = \rho_1 \int_0^L f_8 q\overline{\omega}_x \, dx. \end{aligned}$$

Since $i\lambda\omega_x = \tilde{\omega}_x + f_{7x}$ taking the real part in the above equality results in

$$\begin{aligned} & -\frac{\rho_1}{2} \int_0^L q \frac{d}{dx} |\tilde{\omega}|^2 \, dx - \frac{Eh}{2} \int_0^L q \frac{d}{dx} |\omega_x|^2 \, dx = \rho_1 Re \int_0^L f_8 q\overline{\omega}_x \, dx + \rho_1 Re \int_0^L \overline{f}_{7x} q\tilde{\omega} \, dx \\ & -lGh \; Re \int_0^L q\psi\overline{\omega}_x \, dx - (G+E)lh \; Re \int_0^L q\varphi_x\overline{\omega}_x \, dx - \frac{l^2 Gh}{2} \int_0^L q \frac{d}{dx} |\omega|^2 \, dx. \end{aligned}$$

Performing an integration by parts we get

$$\begin{aligned} & \int_0^L q'(s)[\rho_1 |\tilde{\omega}(s)|^2 + Eh|\omega_x(s)|^2] \, ds \\ & = [q\mathcal{I}_\omega]_0^L - l^2 Gh[q|\omega|^2]_0^L - 2lGh \; Re \int_0^L q\psi\overline{\omega}_x \, dx - 2(G+E)lh \; Re \int_0^L q\varphi_x\overline{\omega}_x \, dx \\ & + l^2 Gh \int_0^L q'|\omega|^2 \, dx + R_3 \end{aligned}$$

where

$$R_3 = 2\rho_1 Re \int_0^L f_8 q\overline{\omega}_x \, dx + 2\rho_1 Re \int_0^L \overline{f}_{7x} q\tilde{\omega} \, dx.$$

If we take $q(x) = x$ in Lemma 2.4.7 and if we add (2.99)-(2.101) we arrive at

$$\begin{aligned} & \mathcal{E}_\varphi(L) + \mathcal{E}_\psi(L) + \mathcal{E}_\omega(L) \\ & = L\mathcal{I}_\varphi(L) - Ehl^2 L|\varphi(L)|^2 + Ehl^2 \int_0^L |\varphi|^2 \, dx \\ & + L\mathcal{I}_\psi(L) - GhL|\psi(L)|^2 + Gh \int_0^L |\psi|^2 \, dx + L\mathcal{I}_\omega(L) - Ghl^2 L|\omega(L)|^2 \\ & + Ghl^2 \int_0^L |\omega|^2 \, dx + R_1 + R_2 + R_3 \\ & -2Ghl Re \int_0^L x\omega\overline{\psi}_x \, dx - 2Ghl \; Re \int_0^L x\psi\overline{\omega}_x \, dx. \end{aligned}$$

Since

$$\begin{aligned} & -2Ghl Re \int_0^L x\omega\overline{\psi}_x \, dx - 2Ghl \; Re \int_0^L x\psi\overline{\omega}_x \, dx \\ & = -2GhlL Re\, \omega(L)\overline{\psi}(L) + 2Ghl \; Re \int_0^L \psi\overline{\omega} \, dx. \end{aligned}$$

using Lemma 2.4.7 and the Young inequality we get

$$
\begin{aligned}
&\mathcal{E}_\varphi(L) + \mathcal{E}_\psi(L) + \mathcal{E}_\omega(L) \\
&\le L\mathcal{I}_\varphi(L) + Ehl^2 \int_0^L |\varphi|^2\,dx \\
&+L\mathcal{I}_\psi(L) + GhL|\psi(L)|^2 + 2Gh\int_0^L |\psi|^2\,dx + L\mathcal{I}_\omega(L) + Ghl^2L|\omega(L)|^2 \\
&+2Ghl^2\int_0^L |\omega|^2\,dx + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}
\end{aligned}
$$

for a positive constant C. It results by (2.97) and (2.98) that we can find a positive constant C such that

$$
\begin{aligned}
&\mathcal{E}_\varphi(L) + \mathcal{E}_\psi(L) + \mathcal{E}_\omega(L) \le Ehl^2\int_0^L |\varphi|^2\,dx + +2Gh\int_0^L |\psi|^2\,dx + 2Ghl^2\int_0^L |\omega|^2\,dx \\
&+C|\lambda|^{2\alpha-2}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|F\|^2_{\mathcal{H}}
\end{aligned}
$$

for $\lambda \neq 0$. Since that $\varphi = \frac{u+f_1}{i\lambda}, \psi = \frac{v+f_4}{i\lambda}$ and $\omega = \frac{\tilde{\omega}+f_7}{i\lambda}$ we obtain

$$
\begin{aligned}
&\mathcal{E}_\varphi(L) + \mathcal{E}_\psi(L) + \mathcal{E}_\omega(L) \le C|\lambda|^{2\alpha-2}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|F\|^2_{\mathcal{H}} \\
&+\frac{C}{|\lambda|^2}\|U\|^2_{\mathcal{H}} + \frac{C}{|\lambda|^2}\|F\|^2_{\mathcal{H}} + \frac{C}{|\lambda|^2}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.
\end{aligned}
$$

Since that

$$\int_{-\infty}^{+\infty} (\phi_i(\xi))^2\,d\xi \le C\int_{-\infty}^{+\infty} (\xi^2+\eta)(\phi_i(\xi))^2\,d\xi$$

for $\lambda \neq 0$. If $|\lambda| > 1$ we get

$$\|U\|^2_{\mathcal{H}} \le C|\lambda|^{4(1-\alpha)}\|F\|^2_{\mathcal{H}}.$$

It follows that

$$\frac{1}{|\lambda|^{2-2\alpha}}\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \le C, \quad \forall\lambda \in \mathbb{R}, |\lambda| > 1.$$

for a positive constant C. Thus, by Theorem 1.3.9 we obtain the polynomial stability for $\eta > 0$.

Remark 2.4.2 *For the case $\eta = 0$, we have prove only strong asymptotic stability. The decay rate is polynomial but we did not obtain any exponent depending on parameter α. As $\lambda = 0$ is a spectral value, the method based on multiplier technic and Bourichev-Tomilov method do not work.*

2.5 Appendix

2.5.1 Appendix 1

We will show the lack of exponential stability by frequency domain method.

We show the existence of a sequence $(\lambda_\mu) \subset \mathbb{R}$ with $\lim_{\mu\to\infty} |\lambda_\mu| = \infty$ and $(U_\mu) \subset D(\mathcal{A})$, $(F_\mu) \subset \mathcal{H}$ such that $(i\lambda_\mu I - \mathcal{A})U_\mu = F_\mu$ is bounded in $\mathcal{H}$ and $\lim_{\mu\to\infty} \|U_\mu\|_{\mathcal{H}} = \infty$. Let $F = F_\mu = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9)^T$ with $U_\mu = (\varphi_\mu, u_\mu, \phi_{1\mu}, \psi_\mu, v_\mu, \phi_{2\mu}, \omega_\mu, \tilde{\omega}_\mu, \phi_{3\mu})^T$.

Equivalently, we have

$$\begin{cases} \lambda^2\varphi + \tilde{\varphi}_x + lk\tilde{\omega} = -(f_2 + i\lambda f_1), \\ \lambda^2\psi + a\tilde{\psi}_x - \tilde{\varphi} = -(f_5 + i\lambda f_4), \\ \lambda^2\omega + k\tilde{\omega}_x - l\tilde{\varphi} = -(f_8 + i\lambda f_7). \end{cases} \tag{2.102}$$

Then

$$\begin{cases} (\lambda^2 - l^2 - 1)\tilde{\varphi} + \tilde{\varphi}_{xx} + a\tilde{\psi}_x + 2lk\tilde{\omega}_x = -[(f_{2x} + i\lambda f_{1x}) + (f_5 + i\lambda f_4) + l(f_8 + i\lambda f_7)], \\ \lambda^2\tilde{\psi} + a\tilde{\psi}_{xx} - \tilde{\varphi}_x = -(f_{5x} + i\lambda f_{4x}), \\ (\lambda^2 - l^2 k)\tilde{\omega} + k\tilde{\omega}_{xx} - 2l\tilde{\varphi}_x = -(f_{8x} + i\lambda f_{7x}) + l(f_2 + i\lambda f_1). \end{cases} \tag{2.103}$$

From $(2.103)_3$, after derivation, we have

$$(\lambda^2 - l^2 k)\tilde{\omega}_x + k\tilde{\omega}_{xxx} - 2l\tilde{\varphi}_{xx} = -L_0 \tag{2.104}$$

where

$$L_0 = (f_{8xx} + i\lambda f_{7xx}) + l(f_{2x} + i\lambda f_{1x}).$$

From $(2.103)_1$, after derivation, we have

$$\begin{aligned} \tilde{\omega}_x &= -\frac{1}{2lk}((\lambda^2 - l^2 - 1)\tilde{\varphi} + \tilde{\varphi}_{xx} + a\tilde{\psi}_x + L_1) \\ \tilde{\omega}_{xxx} &= -\frac{1}{2lk}((\lambda^2 - l^2 - 1)\tilde{\varphi}_{xx} + \tilde{\varphi}_{xxxx} + a\tilde{\psi}_{xxx} + L_{1xx}) \end{aligned}$$

where

$$L_1 = [(f_{2x} + i\lambda f_{1x}) + (f_5 + i\lambda f_4) + l(f_8 + i\lambda f_7)].$$

Now, we replace in (2.104), we get

$$\begin{aligned} &-\frac{(\lambda^2 - l^2 k)}{2lk}\Big((\lambda^2 - l^2 - 1)\tilde{\varphi} + \tilde{\varphi}_{xx} + a\tilde{\psi}_x + L_1\Big) \\ &\qquad -\frac{1}{2l}\Big((\lambda^2 - l^2 - 1)\tilde{\varphi}_{xx} + \tilde{\varphi}_{xxxx} + a\tilde{\psi}_{xxx} + L_{1xx}\Big) - 2l\tilde{\varphi}_{xx} = -L_0 \end{aligned}$$

Then

$$\begin{aligned} &\frac{(\lambda^2 - l^2 k)(\lambda^2 - l^2 - 1)}{2lk}\tilde{\varphi} + \frac{((k+1)\lambda^2 + k(2l^2 - 1))}{2lk}\tilde{\varphi}_{xx} \\ &+\frac{1}{2l}\tilde{\varphi}_{xxxx} + \frac{(\lambda^2 - l^2 k)a}{2lk}\tilde{\psi}_x + \frac{a}{2l}\tilde{\psi}_{xxx} \\ &= -\frac{(\lambda^2 - l^2 k)}{2lk}L_1 - \frac{1}{2l}L_{1xx} + L_0. \end{aligned}$$

From $(2.103)_2$, after derivation, we have

$$\begin{array}{rcl}\tilde{\varphi} & = & \lambda^2\psi + a\tilde{\psi}_x + (f_5 + i\lambda f_4)\\ \tilde{\varphi}_{xx} & = & \lambda^2\psi_{xx} + a\tilde{\psi}_{xxx} + (f_{5xx} + i\lambda f_{4xx})\\ \tilde{\varphi}_{xxxx} & = & \lambda^2\psi_{xxxx} + a\tilde{\psi}_{xxxxx} + (f_{5xxxx} + i\lambda f_{4xxxx}).\end{array}$$

Then, we deduce that

$$\frac{(\lambda^2 - l^2k)(\lambda^2 - l^2 - 1)}{2lk}(\lambda^2\psi + a\tilde{\psi}_x + L_2) + \frac{(k+1)\lambda^2 + k(2l^2-1)}{2lk}(\lambda^2\psi_{xx} + a\tilde{\psi}_{xxx} + L_{2xx})$$
$$+\frac{1}{2l}(\lambda^2\psi_{xxxx} + a\tilde{\psi}_{xxxxx} + L_{2xxxx}) + \frac{(\lambda^2 - l^2k)a}{2lk}\tilde{\psi}_x + \frac{a}{2l}\tilde{\psi}_{xxx}$$
$$= -\frac{(\lambda^2 - l^2k)}{2lk}L_1 - \frac{1}{2l}L_{1xx} + L_0.$$

Hence

$$\psi_{xxxxxx} + \left(\left(\tfrac{1}{k} + \tfrac{1}{a} + 1\right)\lambda^2 + 2l^2\right)\psi_{xxxx}$$
$$+\left(\left(\tfrac{1}{a} + \tfrac{1}{ak} + \tfrac{1}{k}\right)\lambda^4 + \left(-\tfrac{1}{a} + (\tfrac{2}{a} - \tfrac{1}{k} - 1)l^2\right)\lambda^2 + l^4\right)\psi_{xx}$$
$$+\left(\tfrac{1}{ak}\lambda^6 - \left(\tfrac{1}{ak}l^2 + \tfrac{1}{a}l^2 + \tfrac{1}{ak}\right)\lambda^4 + \left(\tfrac{1}{a}l^4 + \tfrac{1}{a}l^2\right)\lambda^2\right)\psi$$
$$= \left[-\frac{(\lambda^2 - l^2k)}{ak}L_1 - \frac{1}{a}L_{1xx} + L_0 - \frac{(\lambda^2 - l^2k)(\lambda^2 - l^2 - 1)}{ak}L_2 - \frac{(k+1)\lambda^2 + k(2l^2-1)}{ak}L_{2xx} - \frac{1}{a}L_{2xxxx}\right] \tag{2.105}$$

The general solution of the homogeneous differential equation is of the form

$$c_1e^{r_1x} + c_2e^{-r_1x} + c_3\cos r_2xe^{r_2x} + c_4\sin r_2xe^{r_2x} + c_5\cos r_2xe^{-r_2x} + c_6\sin r_2xe^{-r_2x}$$

Using the variation of constants method with boundary conditions and choosing $f_2 = f_3 = f_4 = f_5 = f_6 = f_7 = f_8 = f_9 = 0$ and $f_1 \in H_L(0, L)$, we find an explicit solution. Thus, we calculate $\|\psi_x\|_2$ and choosing $\lambda = \lambda_n = \frac{\pi}{L}(n + \frac{1}{2})$ (if $k = a = 1$), we can prove

$$\|\psi_x\|_2 \geq C\lambda_n^{1-\alpha}\|F\|_{\mathcal{H}}.$$

This implies that there exists some constant $\tilde{M} > 0$ independent of λ such that

$$\|(i\lambda_n - \mathcal{A})^{-1}\|_{\mathcal{H}} \geq \tilde{M}\lambda_n^{(1-\alpha)}.$$

The proof is completed.

2.5.2 Appendix 2

Let $\mathcal{A}$ be defined by (2.15). In order to investigate the residual spectrum of $\mathcal{A}$, we need the adjoint operator $\mathcal{A}^*$.

Lemma 2.5.1 *The adjoint operator $\mathcal{A}^*$ is defined by*

$$\mathcal{A}^* \begin{pmatrix} \varphi \\ u \\ \phi_1 \\ \psi \\ v \\ \phi_2 \\ \omega \\ \varpi \\ \phi_3 \end{pmatrix} = \begin{pmatrix} -u \\ -\dfrac{Gh}{\rho_1}(\varphi_x + \psi + l\omega)_x - \dfrac{lEh}{\rho_1}(\omega_x - l\varphi) \\ -(\xi^2+\eta)\phi_1 - u(L)\mu(\xi) \\ -v \\ -\dfrac{EI}{\rho_2}\psi_{xx} + \dfrac{Gh}{\rho_2}(\varphi_x + \psi + l\omega) \\ -(\xi^2+\eta)\phi_2 - v(L)\mu(\xi) \\ -\varpi \\ -\dfrac{Eh}{\rho_1}(\omega_x - l\varphi)_x + \dfrac{lGh}{\rho_1}(\varphi_x + \psi + l\omega) \\ -(\xi^2+\eta)\phi_3 - \varpi(L)\mu(\xi) \end{pmatrix} \tag{2.106}$$

with domain

$$D(\mathcal{A}^*) = \left\{ \begin{array}{l} (\varphi, u, \phi_1, \psi, v, \phi_2, \omega, \varpi, \phi_3)^T \ \textit{in}\ \mathcal{H} : \varphi, \psi, \omega \in H^2(0,L) \cap H^1_L(0,L), u, v, \varpi \in H^1_L(0,L), \\ -(\xi^2+\eta)\phi_1 - u(L)\mu(\xi), -(\xi^2+\eta)\phi_2 - v(L)\mu(\xi) \in L^2(-\infty,+\infty), \\ -(\xi^2+\eta)\phi_3 - \varpi(L)\mu(\xi) \in L^2(-\infty,+\infty) \\ Gh(\varphi_x + \psi + l\omega)(L) + \zeta_1 \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_1(\xi)\, d\xi = 0 \\ EI\psi_x(L) + \zeta_2 \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_2(\xi)\, d\xi = 0 \\ Eh(\omega_x - l\varphi)(L) + \zeta_3 \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_3(\xi)\, d\xi = 0, \\ |\xi|\phi_1, |\xi|\phi_2, |\xi|\phi_3 \in L^2(-\infty,+\infty) \end{array} \right\} \tag{2.107}$$

Proof.
Let $U = (\varphi, u, \phi_1, \psi, v, \phi_2, \omega, \varpi, \phi_3)^T$ and $V = (\tilde{\varphi}, \tilde{u}, \tilde{\phi_1}, \tilde{\psi}, \tilde{v}, \tilde{\phi_2}, \tilde{\omega}, \tilde{\varpi}, \phi_3)^T$. We have

$< \mathcal{A}U, V >_{\mathcal{H}} = < U, \mathcal{A}^* V >_{\mathcal{H}}$.

$$\begin{aligned}
< \mathcal{A}U, V >_{\mathcal{H}} &= Gh\int_0^L \tilde{u}(\varphi_x+\psi+l\omega)_x\,dx + lEh\int_0^L \tilde{u}(\omega_x - l\varphi)\,dx + EI\int_0^L \tilde{v}\psi_{xx}\,dx \\
&-Gh\int_0^L \tilde{v}(\varphi_x+\psi+l\omega)\,dx + Eh\int_0^L \tilde{\varpi}(\omega_x-l\varphi)_x\,dx - lGh\int_0^L \tilde{\varpi}(\varphi_x+\psi+l\omega)\,dx \\
&+Gh\int_0^L (\tilde{\varphi}_x+\tilde{\psi}+l\tilde{\omega})(u_x+v+l\varpi)\,dx + Eh\int_0^L(\tilde{\omega}_x - l\tilde{\varphi})(\varpi_x - lu)\,dx \\
&+EI\int_0^L \tilde{\psi}_x v_x\,dx + \zeta_1\int_{-\infty}^{+\infty}[-(\xi^2+\eta)\phi_1 + u(L)\mu(\xi)]\tilde{\phi}_1\,d\xi \\
&+\zeta_2\int_{-\infty}^{+\infty}[-(\xi^2+\eta)\phi_2 + v(L)\mu(\xi)]\tilde{\phi}_2\,d\xi + \zeta_3\int_{-\infty}^{+\infty}[-(\xi^2+\eta)\phi_3 + \varpi(L)\mu(\xi)]\tilde{\phi}_3\,d\xi \\
&= -Gh\int_0^L(\tilde{u}_x+\tilde{v}+\tilde{\varpi})(\varphi_x+\psi+l\omega)\,dx - Eh\int_0^L(\tilde{\varpi}_x - l\tilde{u})(\omega_x - l\varphi)\,dxdx \\
&-EI\int_0^L \tilde{v}_x\psi_x\,dx + Gh(\tilde{\varphi}_x+\tilde{\psi}+l\tilde{\omega})(L)u(L) + \zeta_1 u(L)\int_{-\infty}^{+\infty}\mu(\xi)\tilde{\phi}_1\,d\xi \\
&+EI\tilde{\psi}_x(L)v(L) + \zeta_2 u(L)\int_{-\infty}^{+\infty}\mu(\xi)\tilde{\phi}_2\,d\xi \\
&+Eh(\tilde{\omega}_x - l\tilde{\varphi})(L)\varpi(L) + \zeta_3\varpi(L)\int_{-\infty}^{+\infty}\mu(\xi)\tilde{\phi}_3\,d\xi \\
&-\int_0^L u[Gh(\tilde{\varphi}_x+\tilde{\psi}+l\tilde{\omega})_x + lEh(\tilde{\omega}_x - l\tilde{\varphi})]\,dx \\
&-\int_0^L v[EI\tilde{\psi}_{xx} - Gh(\tilde{\varphi}_x+\tilde{\psi}+l\tilde{\omega})]\,dx \\
&-\int_0^L \varpi[Eh(\tilde{\omega}_x - l\tilde{\varphi})_x - lGh(\tilde{\varphi}_x+\tilde{\psi}+l\tilde{\omega})]\,dx
\end{aligned}$$

If we set

$$\begin{aligned}
&Gh(\tilde{\varphi}_x+\tilde{\psi}+l\tilde{\omega})(L)u(L) + \zeta_1 u(L)\int_{-\infty}^{+\infty}\mu(\xi)\tilde{\phi}_1\,d\xi = 0, \\
&EI\tilde{\psi}_x(L)v(L) + \zeta_2 u(L)\int_{-\infty}^{+\infty}\mu(\xi)\tilde{\phi}_2\,d\xi = 0, \\
&Eh(\tilde{\omega}_x - l\tilde{\varphi})(L)\varpi(L) + \zeta_3\varpi(L)\int_{-\infty}^{+\infty}\mu(\xi)\tilde{\phi}_3\,d\xi = 0,
\end{aligned}$$

we get

$$\begin{aligned}
< \mathcal{A}U, V >_{\mathcal{H}} &= -\int_0^L u[Gh(\tilde{\varphi}_x+\tilde{\psi}+l\tilde{\omega})_x + lEh(\tilde{\omega}_x - l\tilde{\varphi})]\,dx \\
&-\int_0^L v[EI\tilde{\psi}_{xx} - Gh(\tilde{\varphi}_x+\tilde{\psi}+l\tilde{\omega})]\,dx \\
&-\int_0^L \varpi[Eh(\tilde{\omega}_x - l\tilde{\varphi})_x - lGh(\tilde{\varphi}_x+\tilde{\psi}+l\tilde{\omega})]\,dx \\
&-Gh\int_0^L(\tilde{u}_x+\tilde{v}+\tilde{\varpi})(\varphi_x+\psi+l\omega)\,dx - EI\int_0^L \tilde{v}_x\psi_x\,dx \\
&-Eh\int_0^L(\tilde{\varpi}_x - l\tilde{u})(\omega_x - l\varphi)\,dxdx
\end{aligned}$$

Theorem 2.5.1 *$\sigma_r(\mathcal{A}) = \emptyset$, where $\sigma_r(\mathcal{A})$ denotes the set of residual spectrum of $\mathcal{A}$.*

Proof.

Since $\lambda \in \sigma_r(\mathcal{A})$ implies $\overline{\lambda} \in \sigma_p(\mathcal{A}^*)$ the proof will be accomplished if we can show that $\sigma_p(\mathcal{A}) = \sigma_p(\mathcal{A}^*)$. This is because obviously the eigenvalues of A are symmetric on the real axis. From (2.106), the eigenvalue problem $\mathcal{A}^*Z = \lambda Z$ for $\lambda \in \mathbb{C}$ and $0 \neq Z = (\varphi, u, \phi_1, \psi, v, \phi_2, \omega, \varpi, \phi_3) \in D(\mathcal{A}^*)$ we have

$$\text{(2.108)} \qquad \begin{cases} \lambda\varphi + u = 0, \\ \lambda u + \dfrac{Gh}{\rho_1}(\varphi_x + \psi + l\omega)_x + \dfrac{lEh}{\rho_1}(\omega_x - l\varphi) = 0, \\ \lambda\phi_1 + (\xi^2 + \eta)\phi_1 + u(L)\mu(\xi) = 0, \\ \lambda\psi + v = 0, \\ \lambda v + \dfrac{EI}{\rho_2}\psi_{xx} - \dfrac{Gh}{\rho_2}(\varphi_x + \psi + l\omega) = 0, \\ \lambda\phi_2 + (\xi^2 + \eta)\phi_2 + v(L)\mu(\xi) = 0, \\ \lambda\omega + \varpi = 0, \\ \lambda\varpi + \dfrac{Eh}{\rho_1}(\omega_x - l\varphi)_x - \dfrac{lGh}{\rho_1}(\varphi_x + \psi + l\omega) = 0, \\ \lambda\phi_3 + (\xi^2 + \eta)\phi_3 + \varpi(L)\mu(\xi) = 0. \end{cases}$$

From $(2.108)_1$ and $(2.108)_2$, $(2.108)_4$ and $(2.108)_5$, $(2.108)_8$ and $(2.108)_9$, we get

$$\text{(2.109)} \qquad \begin{cases} -\lambda^2\varphi + \dfrac{Gh}{\rho_1}(\varphi_x + \psi + l\omega)_x + \dfrac{lEh}{\rho_1}(\omega_x - l\varphi) = 0, \\ -\lambda^2\psi + \dfrac{EI}{\rho_2}\psi_{xx} - \dfrac{Gh}{\rho_2}(\varphi_x + \psi + l\omega) = 0, \\ -\lambda^2\omega + \dfrac{Eh}{\rho_1}(\omega_x - l\varphi)_x - \dfrac{lGh}{\rho_1}(\varphi_x + \psi + l\omega) = 0, \end{cases}$$

From $(2.108)_3, (2.108)_6, (2.108)_9$ and the boundary conditions, we get

$$\text{(2.110)} \qquad \begin{cases} Gh(\varphi_x + \psi + l\omega)(L) = -\gamma_1\lambda(\lambda + \eta)^{\alpha-1}\varphi(L) \\ EI\psi_x(L) = -\gamma_2\lambda(\lambda + \eta)^{\alpha-1}\psi(L) \\ Eh(\omega_x - l\varphi)(L) = -\gamma_3\lambda(\lambda + \eta)^{\alpha-1}\omega(L) \\ \varphi(0) = \psi(0) = \omega(0) = 0 \end{cases}$$

System (2.109)-(2.110) is exactly the eigenvalue problem of $\mathcal{A}$. Hence $\mathcal{A}^*$ has the same eigenvalues of $\mathcal{A}$. The proof is complete.

2.5.3 Appendix 3

Proof of Theorem 2.2.1

The proof is decomposed in two steps:

Step 1. $\eta = 0$:

From (2.5) and (2.6), we have

$$\phi\,(\xi\,,\,t) = \int_0^t \mu\,(\xi)\, e^{-\xi^2 (t-s)}\, U\,(s)\; ds \tag{2.111}$$

Hence, by using (2.7), we get

$$O\,(t) = (\pi)^{-1} \sin\,(\alpha\,\pi) \int_0^t \left[2 \int_0^{+\infty} |\xi|^{2\,\alpha-1} e^{-\xi^2 (t-s)}\, d\xi \right] U\,(s)\; ds \tag{2.112}$$

Thus

$$O\,(t) = (\pi)^{-1} \sin\,(\alpha\,\pi) \int_0^t \left[(t-s)^{-\alpha} \Gamma\,(\alpha) \right] U\,(s)\; ds \tag{2.113}$$

Using the fact that $\Gamma(1-\alpha)\Gamma(\alpha) = \dfrac{\pi}{\sin \pi\alpha}$, we obtain

$$O(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} U(s)\, ds \tag{2.114}$$

It follows that

$$O = I^{1-\alpha} U \tag{2.115}$$

Step 2. $\eta \geq 0$:

By simply effecting the following change of function

$$\phi\,(\xi\,,\,t) = e^{-\eta\, t}\, \omega\,(\xi\,,\,t) \tag{2.116}$$

in equations (2.5), (2.6) and (2.7), we directly obtain

$$\partial_t \omega(\xi,t) + \xi^2 \omega(\xi,t) - [e^{\eta t} U(t)] \mu(\xi) = 0, \quad -\infty < \xi < +\infty, \eta \geq 0, t > 0, \tag{2.117}$$

$$\omega(\xi, 0) = 0, \tag{2.118}$$

$$O(t) = (\pi)^{-1} \sin(\alpha\pi) e^{-\eta t} \int_{-\infty}^{+\infty} \mu(\xi) \omega(\xi,t)\, d\xi \tag{2.119}$$

Hence, from Step 1, (2.117), (2.118) and (2.119) yield the desired result

$$O\,(t) = e^{-\eta\, t} \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma\,(1-\alpha)}\, e^{\eta\, s}\, U\,(s)\; ds \tag{2.120}$$

The proof is completed.

Chapter 3

GLOBAL EXISTENCE AND ENERGY DECAY OF SOLUTIONS TO A BRESSE SYSTEM WITH TWO BOUNDARY DISSIPATIONS OF FRACTIONAL DERIVATIVE TYPE

3.1 Introduction

In this chapiter, we consider a linear Bresse system (in one-dimensionel case) of the type

$$(P) \qquad \begin{cases} \rho_1\varphi_{tt} - Gh(\varphi_x + \psi + l\omega)_x - lEh(\omega_x - l\varphi) = 0 \\ \rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + \psi + l\omega) = 0 \\ \rho_1\omega_{tt} - Eh(\omega_x - l\varphi)_x + lGh(\varphi_x + \psi + l\omega) = 0 \end{cases}$$

where $(x, t) \in (0, L) \times (0, +\infty)$. This system is subject to the boundary conditions

$$\begin{array}{ll} \varphi(0,t), \quad \psi(0,t) = 0, \quad \omega(0,t) = 0 & \text{in } (0,+\infty) \\ Gh(\varphi_x + \psi + l\omega)(L,t) = 0 & \text{in } (0,+\infty) \\ EI\psi_x(L,t) = -\gamma_2\partial_t^{\alpha,\eta}\psi(L,t) & \text{in } (0,+\infty) \\ Eh(\omega_x - l\varphi)(L,t) = -\gamma_3\partial_t^{\alpha,\eta}\omega(L,t) & \text{in } (0,+\infty) \end{array}$$

where $\gamma_1 = 0, \gamma_2 > 0, \gamma_3 > 0$. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α with respect to the time variable. It is defined as follows

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\, ds, \ \eta \geq 0. \tag{3.1}$$

We consider the following initial conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad \psi(x,0) = \psi_0(x), \\ \psi_t(x,0) = \psi_1(x), \quad \omega(x,0) = \omega_0(x), \quad \omega_t(x,0) = \omega_1(x), \quad x \in (0,L) \end{cases}$$

where the initial data $(\varphi_0, \varphi_1, \psi_0, \psi_1, \omega_0, \omega_1)$ belong to a suitable function space. By ω, φ and ψ we are denoting the longitudinal, vertical and shear angle displacements.
First, we show that the system (P) is well posed (Theorem 3.3.1). Second, we prove that the dissipation generated by (3.1) can stabilize the system (P) (Theorem 3.4.1).

3.2 Augmented model

We are in position to reformulte system (P) into the augmented model. Using Theorem 2.2.1 the system (P) is equivalently to the augmented model defined by

$$(P') \qquad \begin{cases} \rho_1 \varphi_{tt} - Gh(\varphi_x + \psi + l\omega)_x - lEh(\omega_x - l\varphi) = 0 \\ \rho_2 \psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + \psi + l\omega) = 0 \\ \partial_t \phi_2(\xi,t) + (\xi^2 + \eta)\phi_2(\xi,t) - \psi_t(L,t)\mu(\xi) = 0 \\ \rho_1 \omega_{tt} - Eh(\omega_x - l\varphi)_x + lGh(\varphi_x + \psi + l\omega) = 0 \\ \partial_t \phi_3(\xi,t) + (\xi^2 + \eta)\phi_3(\xi,t) - \omega_t(L,t)\mu(\xi) = 0 \\ \varphi(0,t) = 0, \quad \psi(0,t) = 0, \quad \omega(0,t) = 0 \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad \psi(x,0) = \psi_0(x), \\ \psi_t(x,0) = \psi_1(x), \quad \omega(x,0) = \omega_0(x), \quad \omega_t(x,0) = \omega_1(x), \\ \phi_2(\xi,0) = \phi_{02}(\xi) = 0, \quad \phi_3(\xi,0) = \phi_{03}(\xi) = 0, \\ Gh(\varphi_x + \psi + l\omega)(L,t) = 0 \\ EI\psi_x(L,t) = -\gamma_2(\pi)^{-1}\sin(\alpha\pi) \int_{-\infty}^{+\infty} \mu(\xi)\phi_2(\xi,t)\, d\xi \\ Eh(\omega_x - l\varphi)(L,t) = -\gamma_3(\pi)^{-1}\sin(\alpha\pi) \int_{-\infty}^{+\infty} \mu(\xi)\phi_3(\xi,t)\, d\xi. \end{cases}$$

We define the energy associated to the solution of the problem (P') by the following formula:

$$\begin{aligned} \mathcal{E}(t) = {} & \frac{\rho_1}{2}\|\varphi_t\|_2^2 + \frac{\rho_2}{2}\|\psi_t\|_2^2 + \frac{\rho_1}{2}\|\omega_t\|_2^2 + \frac{EI}{2}\|\psi_x\|_2^2 + \frac{Gh}{2}\|\varphi_x + \psi + l\omega\|_2^2 + \frac{Eh}{2}\|\omega_x - l\varphi\|_2^2 \\ & + (\pi)^{-1}\sin(\alpha\pi) \sum_{i=2}^{3} \frac{\gamma_i}{2} \int_{-\infty}^{+\infty} (\phi_i(\xi,t))^2\, d\xi. \end{aligned} \tag{3.2}$$

Lemma 3.2.1 *Let $(\varphi, \psi, \phi_2, \omega, \phi_3)$ be a solution of the problem (P'). Then, the energy functional defined by (3.2) satisfies*

$$\mathcal{E}'(t) = -(\pi)^{-1}\sin(\alpha\pi)\sum_{i=2}^{3}\gamma_i\int_{-\infty}^{+\infty}(\xi^2+\eta)(\phi_i(\xi,t))^2\,d\xi \le 0. \tag{3.3}$$

Proof.

Multiplying the first equation in (P') by φ_t, the second equation by ψ_t, the fourth equation by ω_t, integrating over $(0, L)$ and using integration by parts, we get

$$\frac{1}{2}\rho_1\frac{d}{dt}\|\varphi_t\|_2^2 - Gh\int_0^L(\varphi_x+\psi+l\omega)_x\varphi_t dx - lEh\int_0^L(\omega_x - l\varphi)\varphi_t dx = 0$$
$$\frac{1}{2}\rho_2\frac{d}{dt}\|\psi_t\|_2^2 - EI\int_0^L\psi_{xx}\psi_t\,dx + Gh\int_0^L(\varphi_x+\psi+l\omega)\psi_t dx = 0$$
$$\frac{1}{2}\rho_1\frac{d}{dt}\|\omega_t\|_2^2 - Eh\int_0^L(\omega_x - l\varphi)_x\omega_t dx + lGh\int_0^L(\varphi_x+\psi+l\omega)\omega_t dx = 0.$$

Then

$$\frac{d}{dt}\left(\frac{\rho_1}{2}\|\varphi_t\|_2^2 + \frac{\rho_2}{2}\|\psi_t\|_2^2 + \frac{\rho_1}{2}\|\omega_t\|_2^2 + \frac{EI}{2}\|\psi_x\|_2^2 + \frac{Gh}{2}\|\varphi_x+\psi+l\omega\|_2^2 + \frac{Eh}{2}\|\omega_x - l\varphi\|_2^2\right)$$
$$+\zeta_2\psi_t(L,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi_2(\xi,t)\,d\xi + \zeta_3\omega_t(L,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi_3(\xi,t)\,d\xi = 0, \tag{3.4}$$

where $\zeta_i = (\pi)^{-1}\sin(\alpha\pi)\gamma_i, i = 2, 3$. Multiplying equation $(P')_2$ by $\zeta_2\phi_2$, equation $(P')_5$ by $\zeta_3\phi_3$ and integrating over $(-\infty, +\infty)$, to obtain:

$$\begin{aligned}&\tfrac{\zeta_2}{2}\frac{d}{dt}\|\phi_2\|_2^2 + \zeta_2\int_{-\infty}^{+\infty}(\xi^2+\eta)(\phi_2(\xi,t))^2\,d\xi - \zeta_2\psi_t(L,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi_2(\xi,t)\,d\xi = 0\\ &\tfrac{\zeta_3}{2}\frac{d}{dt}\|\phi_3\|_2^2 + \zeta_3\int_{-\infty}^{+\infty}(\xi^2+\eta)(\phi_3(\xi,t))^2\,d\xi - \zeta_3\omega_t(L,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi_3(\xi,t)\,d\xi = 0.\end{aligned} \tag{3.5}$$

From (3.2), (3.4) and (3.5) we get

$$\mathcal{E}'(t) = -\sum_{i=2}^{3}\zeta_i\int_{-\infty}^{+\infty}(\xi^2+\eta)(\phi_i(\xi,t))^2\,d\xi \le 0.$$

This completes the proof of the lemma.

3.3 Global existence

The aim of this section is to prove the existence and uniqueness of solutions for the problem (P'). As in chapter 2, (P') can be formulated as a first order system of the form

$$\begin{cases} U' = \mathcal{A}U, \\ U(0) = (\varphi_0, \varphi_1, \psi_0, \psi_1, \phi_{02}, \omega_0, \omega_1, \phi_{03}), \end{cases} \tag{3.6}$$

where the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is defined by

$$\mathcal{A}\begin{pmatrix} \varphi \\ u \\ \psi \\ v \\ \phi_2 \\ \omega \\ \tilde{\omega} \\ \phi_3 \end{pmatrix} = \begin{pmatrix} u \\ \dfrac{Gh}{\rho_1}(\varphi_x+\psi+l\omega)_x + \dfrac{lEh}{\rho_1}(\omega_x - l\varphi) \\ v \\ \dfrac{EI}{\rho_2}\psi_{xx} - \dfrac{Gh}{\rho_2}(\varphi_x+\psi+l\omega) \\ -(\xi^2+\eta)\phi_2 + v(L)\mu(\xi) \\ \tilde{\omega} \\ \dfrac{Eh}{\rho_1}(\omega_x - l\varphi)_x - \dfrac{lGh}{\rho_1}(\varphi_x+\psi+l\omega) \\ -(\xi^2+\eta)\phi_3 + \tilde{\omega}(L)\mu(\xi) \end{pmatrix} \tag{3.7}$$

with domain

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (\varphi, u, \psi, v, \phi_2, \omega, \tilde{\omega}, \phi_3)^T \text{ in } \mathcal{H} : \varphi, \psi, \omega \in H^2(0,L) \cap H^1_L(0,L), u, v, \tilde{\omega} \in H^1_L(0,L), \\ -(\xi^2+\eta)\phi_2 + v(L)\mu(\xi), -(\xi^2+\eta)\phi_3 + \tilde{\omega}(L)\mu(\xi) \in L^2(-\infty,+\infty), \\ Gh(\varphi_x+\psi+l\omega)(L) = 0 \\ EI\psi_x(L) + \zeta_2 \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_2(\xi)\, d\xi = 0 \\ Eh(\omega_x - l\varphi)(L) + \zeta_3 \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_3(\xi)\, d\xi = 0, \\ |\xi|\phi_2, |\xi|\phi_3 \in L^2(-\infty,+\infty) \end{array} \right\} \tag{3.8}$$

where, the energy space $\mathcal{H}$ is defined as

$$\mathcal{H} = (H^1_L(0,L) \times L^2(0,L)) \times (H^1_L(0,L) \times L^2(0,L) \times L^2(-\infty,+\infty))^2$$

where $H^1_L(0,L)$ is given by

$$H^1_L(0,L) = \{\varphi \in H^1(0,L) : \varphi(0) = 0\}.$$

For $U = (\varphi, u, \psi, v, \phi_2, \omega, \tilde{\omega}, \phi_3)^T, \overline{U} = (\overline{\varphi}, \overline{u}, \overline{\psi}, \overline{v}, \overline{\phi}_2, \overline{\omega}, \overline{\tilde{\omega}}, \overline{\phi}_3)^T$, we define the following inner product in $\mathcal{H}$

$$\begin{aligned} \langle U, \overline{U}\rangle_{\mathcal{H}} = \int_0^L &\Big(\rho_1 u\overline{u} + \rho_2 v\overline{v} + \rho_1\tilde{\omega}\overline{\tilde{\omega}} + EI\psi_x\overline{\psi}_x + Gh(\varphi_x+\psi+l\omega)(\overline{\varphi}_x + \overline{\psi} + l\overline{\omega}) \\ &+ Eh(\omega_x - l\varphi)(\overline{\omega}_x - l\overline{\varphi})\Big)\, dx + \sum_{i=2}^{3}\zeta_i \int_{-\infty}^{+\infty} \phi_i\overline{\phi}_i\, d\xi. \end{aligned}$$

Proposition 3.3.1 *The operator $\mathcal{A}$ is m-dissipative in $\mathcal{H}$.*

Proof. First, it is straightforward to see that for any $U \in D(\mathcal{A})$

$$\Re e\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = \mathcal{E}'(t) = -\sum_{i=2}^{3}\zeta_i \int_{-\infty}^{+\infty} (\xi^2+\eta)(\phi_i(\xi))^2\, d\xi \leq 0. \tag{3.9}$$

So $\mathcal{A}$ is dissipative. Now, let $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)^T \in \mathcal{H}$ and look for $U \in D(\mathcal{A})$ satisfying

$$(\lambda I - \mathcal{A})U = F \quad for \quad \lambda > 0.$$

Equivalently, we have the following system

$$(3.10) \qquad \begin{cases} \lambda\varphi - u = f_1, \\ \lambda u - \dfrac{Gh}{\rho_1}(\varphi_x + \psi + l\omega)_x - \dfrac{lEh}{\rho_1}(\omega_x - l\varphi) = f_2, \\ \lambda\psi - v = f_3, \\ \lambda v - \dfrac{EI}{\rho_2}\psi_{xx} + \dfrac{Gh}{\rho_2}(\varphi_x + \psi + l\omega) = f_4, \\ \lambda\phi_2 + (\xi^2 + \eta)\phi_2 - v(L)\mu(\xi) = f_5, \\ \lambda\omega - \tilde{\omega} = f_6, \\ \lambda\tilde{\omega} - \dfrac{Eh}{\rho_1}(\omega_x - l\varphi)_x + \dfrac{lGh}{\rho_1}(\varphi_x + \psi + l\omega) = f_7, \\ \lambda\phi_3 + (\xi^2 + \eta)\phi_3 - \tilde{\omega}(L)\mu(\xi) = f_8. \end{cases}$$

Substituting $u, v, \tilde{\omega}$ from $(3.10)_1$, $(3.10)_3$ and $(3.10)_6$ into $(3.10)_2$, $(3.10)_4$ and $(3.10)_7$, we get

$$(3.11) \qquad \begin{cases} \lambda^2\varphi - \dfrac{Gh}{\rho_1}(\varphi_x + \psi + l\omega)_x - \dfrac{lEh}{\rho_1}(\omega_x - l\varphi) = f_2 + \lambda f_1, \\ \lambda^2\psi - \dfrac{EI}{\rho_2}\psi_{xx} + \dfrac{Gh}{\rho_2}(\varphi_x + \psi + l\omega) = f_4 + \lambda f_3, \\ \lambda^2\omega - \dfrac{Eh}{\rho_1}(\omega_x - l\varphi)_x + \dfrac{lGh}{\rho_1}(\varphi_x + \psi + l\omega) = f_7 + \lambda f_6. \end{cases}$$

Solving system (3.11) is equivalent to finding $(\varphi, \psi, \omega) \in (H^2(0, L) \cap H^1_L(0, L))^3$ such that

$$(3.12) \quad \begin{cases} \displaystyle\int_0^L (\rho_1\lambda^2\varphi w - Gh(\varphi_x + \psi + l\omega)_x w - lEh(\omega_x - l\varphi)w)\, dx = \int_0^L \rho_1(f_2 + \lambda f_1)w\, dx, \\ \displaystyle\int_0^L (\rho_2\lambda^2\psi\chi - EI\psi_{xx}\chi + Gh(\varphi_x + \psi + l\omega)\chi)\, dx = \int_0^L \rho_2(f_4 + \lambda f_3)\chi\, dx, \\ \displaystyle\int_0^L (\rho_1\lambda^2\omega\zeta - Eh(\omega_x - l\varphi)_x\zeta + lGh(\varphi_x + \psi + l\omega)\zeta)\, dx = \int_0^L \rho_1(f_7 + \lambda f_6)\zeta\, dx \end{cases}$$

$(w, \chi, \zeta) \in H^1_L(0, L) \times H^1_L(0, L) \times H^1_L(0, L)$. Integrating by parts $(3.12)_1$, $(3.12)_2$ and $(3.12)_3$, we get

$$(3.13) \quad \begin{cases} \displaystyle\int_0^L (\rho_1\lambda^2\varphi w + Gh(\varphi_x + \psi + l\omega)w_x - lEh(\omega_x - l\varphi)w)\, dx = \int_0^L \rho_1(f_2 + \lambda f_1)w\, dx, \\ \displaystyle\int_0^L (\rho_2\lambda^2\psi\chi + EI\psi_x\chi_x + Gh(\varphi_x + \psi + l\omega)\chi)\, dx - EI\psi_x(L)\chi(L) \\ \displaystyle\qquad\qquad = \int_0^L \rho_2(f_4 + \lambda f_3)\chi\, dx, \\ \displaystyle\int_0^L (\rho_1\lambda^2\omega\zeta + Eh(\omega_x - l\varphi)\zeta_x + lGh(\varphi_x + \psi + l\omega)\zeta)\, dx - Eh(\omega_x - l\varphi)(L)\zeta(L) \\ \displaystyle\qquad\qquad = \int_0^L \rho_1(f_7 + \lambda f_6)\zeta\, dx \end{cases}$$

Substituting ϕ_2, ϕ_3 from $(3.10)_5$ and $(3.10)_8$ into boundary conditions and replacing in (3.13), we get

$$(3.14)\quad \begin{cases} \int_0^L (\rho_1\lambda^2\varphi w + Gh(\varphi_x+\psi+l\omega)w_x - lEh(\omega_x - l\varphi)w)\,dx = \int_0^L \rho_1(f_2+\lambda f_1)w\,dx \\ \int_0^L (\rho_2\lambda^2\psi\chi + EI\psi_x\chi_x + Gh(\varphi_x+\psi+l\omega)\chi)\,dx + \tilde{\zeta}_2 v(L)\chi(L) \\ \qquad = \int_0^L \rho_2(f_4+\lambda f_3)\chi\,dx - \zeta_2\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+\lambda}f_5(\xi)\,d\xi\ \chi(L) \\ \int_0^L (\rho_1\lambda^2\omega\zeta + Eh(\omega_x - l\varphi)\zeta_x + lGh(\varphi_x+\psi+l\omega)\zeta)\,dx + \tilde{\zeta}_3\tilde{\omega}(L)\zeta(L) \\ \qquad = \int_0^L \rho_1(f_7+\lambda f_6)\zeta\,dx - \zeta_3\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+\lambda}f_8(\xi)\,d\xi\ \zeta(L) \end{cases}$$

where $\tilde{\zeta}_i = \zeta_i\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta+\lambda}\,d\xi$.
Substituting $v(L)$ and $\tilde{\omega}(L)$ from $(3.10)_3$ and $(3.10)_6$ into (3.14), we get

$$(3.15)\quad \begin{cases} \int_0^L (\rho_1\lambda^2\varphi w + Gh(\varphi_x+\psi+l\omega)w_x - lEh(\omega_x - l\varphi)w)\,dx = \int_0^L \rho_1(f_2+\lambda f_1)w\,dx \\ \int_0^L (\rho_2\lambda^2\psi\chi + EI\psi_x\chi_x + Gh(\varphi_x+\psi+l\omega)\chi)\,dx + \lambda\tilde{\zeta}_2\psi(L)\chi(L) \\ \qquad = \int_0^L \rho_2(f_4+\lambda f_3)\chi\,dx - \zeta_2\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+\lambda}f_5(\xi)\,d\xi\ \chi(L) + \tilde{\zeta}_2 f_3(L)\chi(L), \\ \int_0^L (\rho_1\lambda^2\omega\zeta + Eh(\omega_x - l\varphi)\zeta_x + lGh(\varphi_x+\psi+l\omega)\zeta)\,dx + \lambda\tilde{\zeta}_3\omega(L)\zeta(L) \\ \qquad = \int_0^L \rho_1(f_7+\lambda f_6)\zeta\,dx - \zeta_3\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+\lambda}f_8(\xi)\,d\xi\ \zeta(L) + \tilde{\zeta}_3 f_6(L)\zeta(L). \end{cases}$$

We define the bilinear form a over $[H_L^1(0,L)\times H_L^1(0,L)\times H_L^1(0,L)]^2$ by

$$\begin{aligned} a((\varphi,\psi,\omega),(w,\chi,\zeta)) &= \int_0^L (\rho_1\lambda^2\varphi w + Gh(\varphi_x+\psi+l\omega)(w_x+\chi+l\zeta))\,dx \\ &+ \int_0^L (\rho_2\lambda^2\psi\chi + EI\psi_x\chi_x)\,dx + \int_0^L (\rho_1\lambda^2\omega\zeta + Eh(\omega_x - l\varphi)(\zeta_x - lw)\,dx \\ &+ \lambda\tilde{\zeta}_2\psi(L)\chi(L) + \lambda\tilde{\zeta}_3\omega(L)\zeta(L) \end{aligned}$$

and the linear form

$$\begin{aligned} L(w,\chi,\zeta) &= \int_0^L \rho_1(f_2+\lambda f_1)w\,dx + \int_0^L \rho_2(f_4+\lambda f_3)\chi\,dx + \int_0^L \rho_1(f_7+\lambda f_6)\zeta\,dx \\ &-\zeta_2\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+\lambda}f_5(\xi)\,d\xi\ \chi(L) + \tilde{\zeta}_2 f_3(L)\chi(L) - \zeta_3\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+\lambda}f_8(\xi)\,d\xi\ \zeta(L) + \tilde{\zeta}_3 f_6(L)\zeta(L). \end{aligned}$$

It is easy to verify that a is continuous and coercive, and L is continuous. So, by the Lax-Milgram theorem, we conclude the existence of a unique$(\varphi,\psi,\omega)\in H_L^1(0,L)\times H_L^1(0,L)\times H_L^1(0,L)$ satisfying

$$a((\varphi,\psi,\omega),(w,\chi,\zeta)) = L(w,\chi,\zeta) \tag{3.16}$$

Applying the classical elliptic regularity, it follows from (3.15) that $(\varphi, \psi, \omega) \in H^2(0, L) \times H^2(0, L) \times H^2(0, L)$. Therefore, the operator $\lambda I - \mathcal{A}$ is surjective for any $\lambda > 0$. This completes the proof of the Proposition 3.3.1. Hence from Lumer-Phillips theorem, we conclude that $\mathcal{A}$ generates a C_0-semigroup $(S(t))_{t\geq 0}$. Then the solution to the linear Cauchy problem (3.6) admits the following representation :

$$U(t) = S(t)U_0 = e^{At}U_0 \quad \forall t \geq 0$$

Consequently, using Hille-Yosida theorem, we have the following results.

Theorem 3.3.1 (Existence and uniqueness)

(1) If $U_0 \in D(\mathcal{A})$, then system (3.6) has a unique strong solution

$$U \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

(2) If $U_0 \in \mathcal{H}$, then system (3.6) has a unique weak solution

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

3.4 Strong stability

In this section, we study the strong stability of the system (P') in the sense that its energy converges to zero when t goes to infinity for all initial data in $\mathcal{H}$. Our result is the following theorem:

Theorem 3.4.1 *The C_0-semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ is strongly stable in $\mathcal{H}$, i.e, for all $U_0 \in \mathcal{H}$, the solution of (3.6) satisfies*

$$\lim_{t\to+\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0.$$

First, we need to prove the following lemmas.

Lemma 3.4.1 *$\mathcal{A}$ does not have eigenvalues on the imaginary axis, i.e, for all $\lambda \in \mathbb{R}$, we have*

$$N(i\lambda I - \mathcal{A}) = \{0\}.$$

Proof. We distinguish two cases:

Case 1 $\lambda = 0$: The idea of the proof is similar to the proof of Lemma 2.4.2.

Case 2 $\lambda \neq 0$: Let $U \in D(\mathcal{A})$, $U \neq 0$, such that $\mathcal{A}U = i\lambda U$.

Equivalently, we have the following system

$$
\begin{cases}
i\lambda\varphi - u = 0, \\
i\lambda u - \dfrac{Gh}{\rho_1}(\varphi_x + \psi + l\omega)_x - \dfrac{lEh}{\rho_1}(\omega_x - l\varphi) = 0, \\
i\lambda\psi - v = 0, \\
i\lambda v - \dfrac{EI}{\rho_2}\psi_{xx} + \dfrac{Gh}{\rho_2}(\varphi_x + \psi + l\omega) = 0, \\
i\lambda\phi_2 + (\xi^2 + \eta)\phi_2 - v(L)\mu(\xi) = 0, \\
i\lambda\omega - \tilde{\omega} = 0, \\
i\lambda\tilde{\omega} - \dfrac{Eh}{\rho_1}(\omega_x - l\varphi)_x + \dfrac{lGh}{\rho_1}(\varphi_x + \psi + l\omega) = 0, \\
i\lambda\phi_3 + (\xi^2 + \eta)\phi_3 - \tilde{\omega}(L)\mu(\xi) = 0.
\end{cases} \tag{3.17}
$$

Then, from (3.9) we have

$$\phi_2 = \phi_3 \equiv 0. \tag{3.18}$$

Hence, from $(3.17)_5$ and $(3.17)_8$, we obtain

$$v(L) = \tilde{\omega}(L) = 0. \tag{3.19}$$

It follows, from (3.17) and (P') , that

$$\psi(L) = \omega(L) = \psi_x(L) = \varphi_x(L) = 0 \text{ and } \omega_x(L) = l\varphi(L). \tag{3.20}$$

From $(3.17)_1, (3.17)_2, (3.17)_3, (3.17)_4, (3.17)_6$ and $(3.17)_7$, we have

$$
\begin{cases}
-\lambda^2\varphi - \dfrac{Gh}{\rho_1}(\varphi_x + \psi + l\omega)_x - \dfrac{lEh}{\rho_1}(\omega_x - l\varphi) = 0, \\
-\lambda^2\psi - \dfrac{EI}{\rho_2}\psi_{xx} + \dfrac{Gh}{\rho_2}(\varphi_x + \psi + l\omega) = 0, \\
-\lambda^2\omega - \dfrac{Eh}{\rho_1}(\omega_x - l\varphi)_x + \dfrac{lGh}{\rho_1}(\varphi_x + \psi + l\omega) = 0.
\end{cases} \tag{3.21}
$$

Consider $X = (\varphi, \psi, \omega, \varphi_x, \psi_x, \omega_x)^T$. Then we can rewrite the system (3.21) as

$$\frac{d}{dx}X = \mathcal{B}X \tag{3.22}$$

where

$$
\mathcal{B} = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\frac{-\lambda^2\rho_1 + l^2Eh}{Gh} & 0 & 0 & 0 & -1 & -\frac{(E+G)l}{G} \\
0 & \frac{-\rho_2\lambda^2 + Gh}{EI} & \frac{Ghl}{EI} & \frac{Gh}{EI} & 0 & 0 \\
0 & \frac{Gl}{E} & \frac{-\rho_1\lambda^2 + Ghl^2}{Eh} & \frac{(E+G)l}{E} & 0 & 0
\end{pmatrix}
$$

We observe that $X = 0$ satisfy the conditions (3.20). Thus, by the Cauchy-Lipschitz theorem for ordinary differential equations, the (unique) solution of (3.22) with the conditions (3.20) is $X = 0$. Therefore $\varphi = 0, \psi = 0, \omega = 0$. It follows from (3.17), that $u = 0, v = 0, \tilde{\omega} = 0$, i.e., $U = 0$.

Lemma 3.4.2 *If $\eta \geq 0$, for all $\lambda \in \mathbb{R}^*$, we have*

$$R(i\lambda I - \mathcal{A}) = H.$$

Proof. Let $\lambda \in \mathbb{R}^*$ and $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)^T \in \mathcal{H}$, then we look for $U = (\varphi, u, \psi, v, \phi_2, \omega, \tilde{\omega}, \phi_3)^T \in D(\mathcal{A})$ solution of the following equation

$$(i\lambda - \mathcal{A})U = F.$$

Equivalently, we have the following system

$$\begin{cases} i\lambda\varphi - u = f_1, \\ i\lambda u - \dfrac{Gh}{\rho_1}(\varphi_x + \psi + l\omega)_x - \dfrac{lEh}{\rho_1}(\omega_x - l\varphi) = f_2, \\ i\lambda\psi - v = f_3, \\ i\lambda v - \dfrac{EI}{\rho_2}\psi_{xx} + \dfrac{Gh}{\rho_2}(\varphi_x + \psi + l\omega) = f_4, \\ i\lambda\phi_2 + (\xi^2 + \eta)\phi_2 - v(L)\mu(\xi) = f_5, \\ i\lambda\omega - \tilde{\omega} = f_6, \\ i\lambda\tilde{\omega} - \dfrac{Eh}{\rho_1}(\omega_x - l\varphi)_x + \dfrac{lGh}{\rho_1}(\varphi_x + \psi + l\omega) = f_7, \\ i\lambda\phi_3 + (\xi^2 + \eta)\phi_3 - \tilde{\omega}(L)\mu(\xi) = f_8 \end{cases} \tag{3.23}$$

with the following conditions

$$\begin{cases} Gh(\varphi_x + \psi + l\omega)(L) = 0 \\ EI\psi_x(L) = -\gamma_2(\pi)^{-1}\sin(\alpha\pi)\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_2(\xi, t)\, d\xi \\ Eh(\omega_x - l\varphi)(L) = -\gamma_3(\pi)^{-1}\sin(\alpha\pi)\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_3(\xi, t)\, d\xi. \end{cases} \tag{3.24}$$

By eliminating u, v and $\tilde{\omega}$ from the system (3.23) we get the following system

$$\begin{cases} -\lambda^2\varphi - \dfrac{Gh}{\rho_1}(\varphi_x + \psi + l\omega)_x - \dfrac{lEh}{\rho_1}(\omega_x - l\varphi) = f_2 + i\lambda f_1, \\ -\lambda^2\psi - \dfrac{EI}{\rho_2}\psi_{xx} + \dfrac{Gh}{\rho_2}(\varphi_x + \psi + l\omega) = f_4 + i\lambda f_3, \\ -\lambda^2\omega - \dfrac{Eh}{\rho_1}(\omega_x - l\varphi)_x + \dfrac{lGh}{\rho_1}(\varphi_x + \psi + l\omega) = f_7 + i\lambda f_6. \end{cases} \tag{3.25}$$

Solving system (3.25) is equivalent to finding $(\varphi,\psi,\omega)\in(H^2\cap H^1_L(0,L))^3$ such that

$$(3.26)\quad \begin{cases} \displaystyle\int_0^L(-\rho_1\lambda^2\varphi w-Gh(\varphi_x+\psi+l\omega)_x w-lEh(\omega_x-l\varphi)w)\,dx=\int_0^L\rho_1(f_2+i\lambda f_1)w\,dx,\\ \displaystyle\int_0^L(-\rho_2\lambda^2\psi\chi-EI\psi_{xx}\chi+Gh(\varphi_x+\psi+l\omega)\chi)\,dx=\int_0^L\rho_2(f_4+i\lambda f_3)\chi\,dx,\\ \displaystyle\int_0^L(-\rho_1\lambda^2\omega\zeta-Eh(\omega_x-l\varphi)_x\zeta+lGh(\varphi_x+\psi+l\omega)\zeta)\,dx=\int_0^L\rho_1(f_7+i\lambda f_6)\zeta\,dx\end{cases}$$

for all $(w,\chi,\zeta)\in H^1_L(0,L)\times H^1_L(0,L)\times H^1_L(0,L)$.

Integrating by parts $(3.26)_1$, $(3.26)_2$ and $(3.26)_3$ and substituting ϕ_2, ϕ_3 from $(3.23)_5$ and $(3.23)_8$ into boundary conditions, we get

$$(3.27)\quad \begin{cases} \displaystyle\int_0^L(-\rho_1\lambda^2\varphi w+Gh(\varphi_x+\psi+l\omega)w_x-lEh(\omega_x-l\varphi)w)\,dx=\int_0^L\rho_1(f_2+i\lambda f_1)w\,dx,\\ \displaystyle\int_0^L(-\rho_2\lambda^2\psi\chi+EI\psi_x\chi_x+Gh(\varphi_x+\psi+l\omega)\chi)\,dx+\tilde{\zeta}_2 v(L)\chi(L)\\ \displaystyle\qquad=\int_0^L\rho_2(f_4+i\lambda f_3)\chi\,dx-\zeta_2\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+i\lambda}f_5(\xi)\,d\xi\;\chi(L),\\ \displaystyle\int_0^L(-\rho_1\lambda^2\omega\zeta+Eh(\omega_x-l\varphi)\zeta_x+lGh(\varphi_x+\psi+l\omega)\zeta)\,dx+\tilde{\zeta}_3\tilde{\omega}(L)w(L)\\ \displaystyle\qquad=\int_0^L\rho_1(f_7+i\lambda f_6)\zeta\,dx-\zeta_3\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+i\lambda}f_8(\xi)\,d\xi\;\zeta(L)\end{cases}$$

where $\tilde{\zeta}_i=\zeta_i\displaystyle\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta+i\lambda}\,d\xi$.

Substituting $v(L)$ and $\tilde{\omega}(L)$ from $(3.23)_3$ and $(3.23)_6$ into (3.27), we get

$$(3.28)\quad \begin{cases} \displaystyle\int_0^L(-\rho_1\lambda^2\varphi w+Gh(\varphi_x+\psi+l\omega)w_x-lEh(\omega_x-l\varphi)w)\,dx=\int_0^L\rho_1(f_2+i\lambda f_1)w\,dx,\\ \displaystyle\int_0^L(-\rho_2\lambda^2\psi\chi+EI\psi_x\chi_x+Gh(\varphi_x+\psi+l\omega)\chi)\,dx+i\lambda\tilde{\zeta}_2\psi(L)\chi(L)\\ \displaystyle\qquad=\int_0^L\rho_2(f_4+i\lambda f_3)\chi\,dx-\zeta_2\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+i\lambda}f_5(\xi)\,d\xi\;\chi(L)+\tilde{\zeta}_2 f_3(L)\chi(L),\\ \displaystyle\int_0^L(-\rho_1\lambda^2\omega\zeta+Eh(\omega_x-l\varphi)\zeta_x+lGh(\varphi_x+\psi+l\omega)\zeta)\,dx+i\lambda\tilde{\zeta}_3\omega(L)w(L)\\ \displaystyle\qquad=\int_0^L\rho_1(f_7+i\lambda f_6)\zeta\,dx-\zeta_3\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+i\lambda}f_8(\xi)\,d\xi\;\zeta(L)+\tilde{\zeta}_3 f_6(L)\zeta(L).\end{cases}$$

Adding equations $(3.28)_1$, $(3.28)_2$ and $(3.28)_3$, we obtain

$$(3.29)\qquad -(L_\lambda U,V)_{H^1_R}+(U,V)_{H^1_R}=l(V).$$

where

$$H^1_R(0,L)=H^1_L(0,L)\times H^1_L(0,L)\times H^1_L(0,L),$$

$$(U,V)_{H^1_R}=\int_0^L Gh(\varphi_x+\psi+l\omega)(w_x+\chi+l\zeta)+EI\psi_x\chi_x+Eh(\omega_x-l\varphi)(\zeta_x-lw)\,dx$$

and

$$(L_\lambda U, V)_{H^1_R} = \lambda^2 \int_0^L (\rho_1 \varphi w + \rho_2 \psi \chi + \rho_1 \omega \zeta)\, dx - i\lambda(\tilde{\zeta}_2 \psi(L)\chi(L) + \tilde{\zeta}_3 \omega(L) w(L)).$$

As in chapter 2, the operator L_λ is compact. Consequently, by Fredholm alternative (see Theorem 1.2.1), proving the existence of U solution of (3.29) reduces to proving that 1 is not an eigenvalue of L_λ. Indeed if 1 is an eigenvalue, then there exists $U \neq 0$, such that

$$(L_\lambda U, V)_{H^1_R} = (U, V)_{H^1_R} \quad \forall V \in H^1_R. \tag{3.30}$$

In particular for $V = U$, it follows that

$$\begin{aligned} &\lambda^2 \left[\rho_1 \|\varphi\|^2_{L^2(0,L)} + \rho_2 \|\psi\|^2_{L^2(0,L)} + \rho_1 \|\omega\|^2_{L^2(0,L)}\right] - i\lambda(\tilde{\zeta}^2 |\psi(L)|^2 + \tilde{\zeta}_3 |\omega(L)|^2) \\ &= Gh\|\varphi_x + \psi + l\omega\|^2_{L^2(0,L)} + EI\|\psi_x\|^2_{L^2(0,L)} + Eh\|w_x - l\varphi\|^2_{L^2(0,L)}. \end{aligned}$$

Hence, we have

$$\psi(L) = \omega(L) = 0. \tag{3.31}$$

From (3.30), we obtain

$$\psi_x(L) = \varphi_x(L) = 0 \text{ and } \omega_x(L) = l\varphi(L). \tag{3.32}$$

and

$$\begin{cases} -\lambda^2 \varphi - \dfrac{Gh}{\rho_1}(\varphi_x + \psi + l\omega)_x - \dfrac{lEh}{\rho_1}(\omega_x - l\varphi) = 0, \\ -\lambda^2 \psi - \dfrac{EI}{\rho_2}\psi_{xx} + \dfrac{Gh}{\rho_2}(\varphi_x + \psi + l\omega) = 0, \\ -\lambda^2 \omega - \dfrac{Eh}{\rho_1}(\omega_x - l\varphi)_x + \dfrac{lGh}{\rho_1}(\varphi_x + \psi + l\omega) = 0. \end{cases} \tag{3.33}$$

Consider $X = (\varphi, \psi, \omega, \varphi_x, \psi_x, \omega_x)^T$. Then we can rewrite the system (3.33) as

$$\frac{d}{dx} X = \mathcal{B} X \tag{3.34}$$

where

$$\mathcal{B} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{-\lambda^2 \rho_1 + l^2 Eh}{Gh} & 0 & 0 & 0 & -1 & -\frac{(E+G)l}{G} \\ 0 & \frac{-\rho_2 \lambda^2 + Gh}{EI} & \frac{Ghl}{EI} & \frac{Gh}{EI} & 0 & 0 \\ 0 & \frac{Gl}{E} & \frac{-\rho_1 \lambda^2 + Ghl^2}{Eh} & \frac{(E+G)l}{E} & 0 & 0 \end{pmatrix}$$

Then the same as proof of Lemma 3.4.1 , we know that the (unique) solution of (3.34) with the conditions (3.31) and (3.32) is $X = 0$. Therefore $\varphi = 0, \psi = 0, \omega = 0$. It follows from (3.17), that $u = 0, v = 0, \tilde{\omega} = 0, \phi_2 = 0, \phi_3 = 0$, i.e, $U = 0$. The proof is thus complete.

Lemma 3.4.3 *If $\eta \neq 0$, we have system*

$$0 \in \rho(\mathcal{A}).$$

Proof.

If $\lambda = 0$ and $\eta \neq 0$, the system (3.23) is reduced to the following

$$\left\{\begin{array}{l} -u = f_1, \\ -\dfrac{Gh}{\rho_1}(\varphi_x + \psi + l\omega)_x - \dfrac{lEh}{\rho_1}(\omega_x - l\varphi) = f_2, \\ -v = f_3, \\ -\dfrac{EI}{\rho_2}\psi_{xx} + \dfrac{Gh}{\rho_2}(\varphi_x + \psi + l\omega) = f_4, \\ (\xi^2 + \eta)\phi_2 - v(L)\mu(\xi) = f_5, \\ -\tilde{\omega} = f_6, \\ -\dfrac{Eh}{\rho_1}(\omega_x - l\varphi)_x + \dfrac{lGh}{\rho_1}(\varphi_x + \psi + l\omega) = f_7, \\ (\xi^2 + \eta)\phi_3 - \tilde{\omega}(L)\mu(\xi) = f_8. \end{array}\right. \tag{3.35}$$

with the following conditions

$$\left\{\begin{array}{l} Gh(\varphi_x + \psi + l\omega)(L) = 0 \\ EI\psi_x(L) = -\gamma_2(\pi)^{-1}\sin(\alpha\pi)\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_2(\xi, t)\, d\xi \\ Eh(\omega_x - l\varphi)(L, t) = -\gamma_3(\pi)^{-1}\sin(\alpha\pi)\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_3(\xi, t)\, d\xi \end{array}\right. \tag{3.36}$$

Integrating by parts $(3.35)_2$, $(3.35)_4$ and $(3.35)_7$ we get

$$\left\{\begin{array}{l} \displaystyle\int_0^L (Gh(\varphi_x + \psi + l\omega)w_x - lEh(\omega_x - l\varphi)w)\, dx = \int_0^L \rho_1 f_2 w \\ \displaystyle\int_0^L (EI\psi_x\chi_x + Gh(\varphi_x + \psi + l\omega)\chi)\, dx \\ \displaystyle= \int_0^L \rho_2 f_4 \chi\, dx - \zeta_2 \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta} f_5(\xi)\, d\xi\ \chi(L) + \tilde{\zeta}_2 f_3(L)\chi(L), \\ \displaystyle\int_0^L (Eh(\omega_x - l\varphi)\zeta_x + lGh(\varphi_x + \psi + l\omega)\zeta)\, dx \\ \displaystyle= \int_0^L \rho_1 f_7 \zeta\, dx - \zeta_3 \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta} f_8(\xi)\, d\xi\ \zeta(L) + \tilde{\zeta}_3 f_6(L)\zeta(L). \end{array}\right. \tag{3.37}$$

for all $(w, \chi, \zeta) \in H^1_L(0, L) \times H^1_L(0, L) \times H^1_L(0, L)$.
Adding equations $(3.37)_1$, $(3.37)_2$ and $(3.37)_3$, we obtain

$$a_\eta((\varphi, \psi, \omega), (w, \chi, \zeta)) = L_\eta(w, \chi, \zeta) \tag{3.38}$$

where the bilinear form $a_\eta : [H_0^1(0,L) \times H_L^1(0,L) \times H_L^1(0,L)]^2 \to \mathbb{R}$ and the linear form $L_\eta : H_0^1(0,L) \times H_L^1(0,L) \times H_L^1(0,L) \to \mathbb{R}$ are defined by

$$a_\eta((\varphi,\psi,\omega),(w,\chi,\zeta)) = \int_0^L Gh(\varphi_x+\psi+l\omega)(w_x+\chi+l\zeta)\,dx + \int_0^L EI\psi_x\chi_x\,dx + \int_0^L Eh(\omega_x-l\varphi)(\zeta_x-lw)\,dx$$

and

$$L_\eta(w,\chi,\zeta) = \int_0^L \rho_1 f_2 w\,dx + \int_0^L \rho_2 f_4 \chi\,dx - \zeta_2\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta}f_5(\xi)\,d\xi\ \chi(L) + \tilde{\zeta}_2 f_3(L)\chi(L) + \int_0^L \rho_1 f_7\zeta\,dx - \zeta_3\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta}f_8(\xi)\,d\xi\ \zeta(L) + \tilde{\zeta}_3 f_6(L)\zeta(L).$$

By the Lax-Milgram Theorem there exists a unique $(\varphi,\psi,\omega) \in H_L^1(0,L)\times H_L^1(0,L)\times H_L^1(0,L)$ such that

$$a_\eta((\varphi,\psi,\omega),(w,\chi,\zeta)) = L_\eta(w,\chi,\zeta), \quad \forall (w,\chi,\zeta) \in H_L^1(0,L)\times H_L^1(0,L)\times H_L^1(0,L).$$

Using the classical elliptic regularity, it follows from (3.37) that $(\varphi,\psi,\omega) \in H^2(0,L)\times H^2(0,L)\times H^2(0,L)$. Therefore, the operator $\mathcal{A}$ is surjective and consequently $0 \in \rho(\mathcal{A})$.

Lemma 3.4.4 *If $\eta = 0$, we have*

$$0 \in \sigma(\mathcal{A})$$

i.e. the operator $-\mathcal{A}$ is not invertible.

Proof.
Let $F \in \mathcal{H}$ such that

$$F = (0,0,\sin x,0,0,0,0,0)^T.$$

Assume that there exists $(\varphi,u,\psi,v,\phi_2,\omega,\tilde{\omega},\phi_3)^T \in D(\mathcal{A})$ such that

$$-\mathcal{A}U = F.$$

It follows, from (3.23), that

$$\phi_2(\xi) = -|\xi|^{\frac{2\alpha-5}{2}}\sin L.$$

But, then $\phi_2 \notin L^2(-\infty,+\infty)$, since $\alpha \in]0,1[$. So $(\varphi,u,\psi,v,\phi_2,\omega,\tilde{\omega},\phi_3)^T \notin D(\mathcal{A})$ and consequently, the operator $-\mathcal{A}$ is not invertible.

Proof of Theorem 3.4.1.
From Lemma 3.4.1, we directly deduce that $\mathcal{A}$ does not have eigenvalues on the imaginary axis. Now, using Lemmas 3.4.2, 3.4.3 and 3.4.4, we conclude that $\sigma(\mathcal{A}) \cap \{i\,\mathbb{R}\} = \emptyset$ if $\eta > 0$ and $\sigma(\mathcal{A}) \cap \{i\,\mathbb{R}\} = \{0\}$ if $\eta = 0$. This completes the proof of Theorem 3.4.1.

Conclusion and future work

In this thesis we have studied the global existence, uniqueness, continuous dependence (well-posendess) and decay properties of solutions for the Bresse system with three or two control on the boundary conditions given by fractional derivative type. The main result of this work is polynomial stability of the solution for the Bresse system with three control. We have considered two case: $\eta = 0$ and $\eta > 0$.

For the case $\eta = 0$, we have prove only strong asymptotic stability. The decay rate is polynomial but we did not obtain any exponent depending on parameter α. As $\lambda = 0$ is a spectral value, the result obtained by Bourichev and Tomilov do not work.

For the case $\eta > 0$, we have succeed to prouve decay rate depending on parameter α using the semigroup theory of linear operators and a result obtained by Borichev and Tomilov. This tool is flexible and can be adapted to the multi-dimensional case and other complex systems. But, in general do not give optimal decay rate. Hence, the optimality of the polynomial decay rate of Bresse system with a boundary dissipation of fractional derivative type remain an open problem.

General Notation

H^1, H^2	The sobolev spaces,
H^{-1}	The dual space of H_0^1,
C^0	The space of continuous function,
C^1	The space of continuously differentiable functions,
$L(X, Y)$	The space of bounded linear operators from X into Y,
X'	The dual space of X,
B_X	The unit ball in X,
$\lvert . \rvert$	The modulus,
$\lVert . \rVert$	The norm,
$\langle . , . \rangle$	The scalar product,
sup	The supreme,
inf	The infimum, ll
∂_t	The partial derivative with respect of t,
∂_{tt}	The second partial derivative of f with respect of t,
$\partial_t^{\alpha,\eta}$	The generalized fractional derivative,
$I^{\alpha,\eta}$	The generalized fractional integral,
$\mathbb{R}$	The set of real numbers,
$\mathbb{C}$	The set of complex numbers,
Z	The set of integer numbers,
$\mathbb{N}$	The set of natural numbers,
$\Re e$	The real part,
$\Im m$	The imaginary part,
$\overline{\lambda}$	The conjugate of a complex number λ,
$D(A)$	The domain of A,

$R(A)$	The range of A,
$N(A)$	The kernel of A,
C^T	The Transpose of a matrix C,
A^*	The adjoint operator of A,
$\rho(A)$	The resolvent set of A,
$\sigma(A)$	The spectrum of A,
$\sigma_p(A)$	The ponctuel spectrum of A,
$\sigma_r(A)$	The residual spectrum of A,
$\sigma_c(A)$	The continuous spectrum of A,
$R(.\,,A)$	The resolvent of A,
o	The little o: $f(x) = o(g(x))$ for $x \to \infty$ if $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 0$,
O	The big O: $f(x) = O(g(x))$ for $x \to \infty$ if $\lvert f(x)\rvert \leq C\lvert g(x)\rvert$ for all $x \geq x_0$,
$\sim$	The asymptotically equivalent: $f \sim g$ for $x \to \infty$ if $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 1$,
Γ	The Gamma function: $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$,
B	The Beta function: $B(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1}dt = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$,
	$\overline{\lim\limits_{\lvert\beta\rvert\to\infty}} \lVert(i\beta I - A)^{-1}\rVert = \inf\limits_{\lvert\beta\rvert}(\sup\limits_{k\geq\lvert\beta\rvert} \lVert(ikI - A)^{-1}\rVert)$.

Bibliography

[1] F. Alabau-Boussouira, J.M. Rivera & D. Almeida-Júnior. Stability to weak dissipative Bresse system, *Journal of Mathematical Analysis and Applications.* **374** (2011)-(2), 481-498.

[2] M. S. Alves, Octavio Vera, Jaime Munoz Rivera & Amelie Rambaudm. Exponential stability to the bresse system with boundary dissipation conditions, arXiv150601657A (2015).

[3] W. Arendt & C. J. K. Batty. Tauberian theorems and stability of one-parameter semigroups, *Trans. Amer. Math. Soc.,* **306** (1988)-(2), 837-852.

[4] R. L. Bagley & P. J. Torvik. A theoretical basis for the application of fractional calculus to viscoelasticity, *J. Rheology.* **27** (1983), 201210

[5] R. L. Bagley & P. J. Torvik. A different approach to the analysis of viscoelastically damped structures, *AIAA J.* **21** (1983), 741-748.

[6] R. L. Bagley & P. J. Torvik. On the appearance of the fractional derivative in the behavior of real material, *J. Appl. Mech.* **51** (1983), 294-298.

[7] A. Borichev & Y. Tomilov. Optimal polynomial decay of functions and operator semigroups, *Math. Ann.* **347** (2010)-2, 455-478.

[8] J. A. C. Bresse. Cours de Mécanique Appliquée, Mallet Bachelier, Paris, 1859.

[9] H. Brézis. Analyse fonctionelle. Théorie et applications.. Collection Mathématiques Appliques pour la Matrise. Masson, Paris, 1983.

[10] H. Brézis. Functional analysis, Sobolev spaces and partial diferential equations. Universitext. Springer, New York, 2011.

[11] H. Brézis. Operateurs Maximaux Monotones et semi-groupes de contractions dans les espaces de Hilbert, Notas de Matemàtica (50), Universidade Federal do Rio de Janeiro and University of Rochester, North-Holland, Amsterdam, (1973).

[12] M. Caputo. Vibrations of an infinite plate with a frequency independent, *Q. J. Acoust. Soc. Am.,* **60** (1976), 634-639.

[13] M. M. Cavalcanti. V. D. Cavalcanti & I. Lasiecka. Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping-source interaction, *J. Diff. Equa.,* **236** (2007), 407-459.

[14] W. Charles, J. A. Soriano, F.A.F. Nascimento & J. H. Rodrigues. Decay rates for Bresse System with arbitrary nonlinear localized damping,.*J. Diff. Equa.,* **255** (2013), 2267-2290.

[15] J. U. Choi & R. C. Maccamy. Fractional order Volterra equations with applications to elasticity,.*J. Math. Anal. Appl.,* **139** (1989), 448-464.

[16] K. J. Engel & R. Nagel. One-parameter Semigroups for Linear Evolution Equation-s,*Graduate Texts in Mathematics,* **194**, Berlin (2000).

[17] L.H. Fatori & J.M. Rivera. Rates of decay to weak thermoelastic Bresse system, *IMA J. Appl. Math.* **75** (2010)-(6), 881-904.

[18] W.G. Glockle & T.F. Nonnenmacher. A fractional calculus approach to self-similar protein dynamics, *Biophysical Joumal,* **68** (1995)-1, 46-53.

[19] A. Haraux. Two remarks on dissipative hyperbolic problems, *Research Notes in Mathematics,* **122.** Pitman: Boston, MA, 1985; 161-179.

[20] R. Hilfer. Applications of Fractional Calculus in Physics . *World Scientific Publ.Co* 2000.

[21] A.A. Kilbas, Hari M. Srivastava & Juan J. Trujillo. Theory and Applications of Fractional Differential Equations. *North-Holland Mathematics Studies,* **204.** Elsevier Science B.V., Amsterdam, 2006.

[22] J. U. Kim & Y. Renardy. Boundary control of the Timoshenko beam, *SIAM J. Control Optim.,* **25** (1987), 1417-1429.

[23] V. Komornik, Exact Controllability and Stabilization. The Multiplier Method, Masson-John Wiley, Paris, 1994.

[24] J.E. Lagnese, G. Leugering & J.P.G. Schmidt. Modelling Analysis and Control of Dynamic Elastic Multi-Link Structures. *Systems Control Found. Appl.,* 1994.

[25] M. Lima, A. Soufyane & D. Almeida Júnior. Asymptotic behavior to Bresse system with past history, *Quarterly of Applied Mathematics.* **73** (2015)-(1), 23-53.

[26] Z. Liu & B. Rao. Energy decay rate of the thermoelastic Bresse system, *Z. Angew. Math. Phys.* **60** (2009), 54-69.

[27] Z. Liu & S. Zheng. Semigroups Associated with Dissipative Systems, *Chapman-Hall/CRC,* 1999.

[28] F. Mainardi & E. Bonetti. The applications of real order derivatives in linear viscoelasticity, *Rheol. Acta* **26** (1988), 64-67.

[29] B. Mbodje. Wave energy decay under fractional derivative controls, *IMA Journal of Mathematical Control and Information.,* **23** (2006), 237-257.

[30] B. Mbodje & G. Montseny. Boundary fractional derivative control of the wave equation, *IEEE Transactions on Automatic Control.,* **40** (1995), 368-382.

[31] S. A. Messaoudi & M. I. Mustapha. On the internal and boundary stabilization of Timoshenko beams, *Nonlinear Differ. Equ. Appl.,* **15** (2008), 655-671.

[32] S. A. Messaoudi & M. I. Mustapha. On the stabilization of the Timochenko system by a weak nonlinear dissipation, *Math. Meth. Appl. Sci.,* **32** (2009), 454-469.

[33] K. Miller & B. Ross. An Introduction to the Fractional Calculus and Fractional Differential Equations. *John Wiley and Sons, Inc., New York,* 1993.

[34] N. Najda & A. Wehbe. Weakly locally thermal stabilization of Bresse systems, *Electronic Journ. od Diff.Equ.* **182** (2014), 1-19.

[35] J. H. Park & J. R. Kang. Energy decay of solutions for Timoshenko beam with a weak non-linear dissipation, *IMA J. Appl. Math.* **76** (2011), 340-350.

[36] A. Pazy. Semigroups of Linear Operators and Applications to Partial Differential Equations, *Springer-Verlag, Berlin, Heidelberg, New-York,* 1983.

[37] I. Podlubny. Fractional differential equations, *Mathematics in Science and Engineering,* **198** (1999), Academic Press.

[38] J. Pruss. On the spectrum of C0-semigroups, *Transactions of the American Mathematical Society,* **284** (1984)-2, 847-857.

[39] C.A. Raposo, J. Ferreira, J. Santos & N. N. O. Castro. Exponential stability for the Timoshenko system with two weak dampings, *Appl. Math. Lett.* **18** (2005)-5, 535-541.

[40] Y.A. Rossikhin & M. V. Shitikova. Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids , *Appl. Mech. Rev.* **50** (1997), 15-67.

[41] J.A. Soriano, W. Charles & R. Schulz. Asymptotic stability for Bresse systems, *J. Math. Anal. Appl.* **412** (2014)-1, 369-380.

[42] J.A. Soriano, J.M. Rivera & L.H. Fatori. Bresse system with indefinite damping, *J. Math. Anal. Appl.* **387** (2012)-1, 284-290.

[43] P. Tauvel. Analyse complexe, Dunod, Paris, 2006.

[44] C. Wagschal. Fonctions holomorphes - Equations différentielles : Exercices corrigés, Herman, Paris, 2003.

[45] A. Wehbe & W. Youssef. Exponentialy and polynomial stability of an elastic Bresse system with two Locally Distributed Feedbacks, *J. Math. Phy.* **51** 103523 (2010).