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Propriétés Topologiques et Géométriques de Certaines Classes d'Équations et d'Inclusions Différentielles Fonctionnelles avec Impulsions

Je dédie ce modeste travail à
mes chers parents
mon mari et mon fils
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et tout ma famille et mes amis.

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Introduction

The dynamics of many evolving processes are subject to abrupt changes, such as shocks, harvesting and natural disasters. These phenomena involve short-term perturbations from continuous and smooth dynamics, whose duration is negligible in comparison with the duration of an entire evolution. In models involving such perturbations, it is natural to assume these perturbations act instantaneously or in the form of impulses. As a consequence, impulsive differential equations have been developed in modeling impulsive problems in physics, population dynamics, ecology, biological systems, biotechnology, industrial robotics, pharmcokinetics, optimal control, and electrical engineering. Important contributions to the study of the mathematical aspects of such equations have been processed in [14, 13, 53, 80].

Meanwhile, differential equations with impulses were considered for the first time in the 1960's by Milman and Myshkis [70, 69]. After a period of active research, mostly in Eastern Europe from 1960-1970, culminating with the monograph by Halanay and Wexler [42].

In many fields of science we can describe various evolutionary process by differential equations with delay and for this reason the study of this type of equations has received great attention during the last years. It is well-known that systems with post effect, with time lag or with delay, are of great theoretical interest and form an important class with regard to their applications. This class of systems can be described by functional differential equations and inclusions, which are also called differential equations and inclusions with deviating argument. Among functional differential equations, one may distinguish some special classes of equations, retarded functional differential equations, advanced functional differential equations and neutral functional equations and inclusions. In particular, retarded functional differential equations and inclusions describe those systems or processes whose rate of change of state is determined by their past and present states. Such equations are frequently encountered as mathematical models of many dynamical processes in mechanics, control theory, physics, chemistry, biology, medicine, economics, atomic energy, information theory, etc. Especially, since the 1960's, many good books, which are in the Russian literature, have been published on delay differential equations; see, for examples, the books of Burton [23], Èl'sgol'ts [34], Èl'sgol'ts and Norkin [35], Gopalsmy [49], Azbelez *et al.* [11], Hale [44], Hale and Lunel [45], Kolmanovskii and Myshkis [63], Kolmanovaskii and Nosov [64], Krasovskii [65], Yoshizawa [85] and the references listed in those books.

Impulsive differential systems and evolution differential systems are used to describe various models of real processes and phenomena studied in physics, chemical technology, population dynamics, biology [71], biotechnology and economics [40]. That is why in recent years they have been the object of investigations. We refer to the monographs of Bainov and Simeonov [14], Benchohra *et al.* [16], Lakshmikantham *et al.* [67], Samoilenko and Perestyuk [81] where numerous properties of their solutions are studied, and a detailed bibliography is given.
Recently, Precup [77] proved the role of matrix convergence and vector metric in the study of semilinear operator systems.
In recent years, many authors studied the existence of solutions for systems of differential equations and impulsive differential equations by using the vector version fixed point theorem; see [18, 75, 78, 73, 74] and in the references therein.
The uniqueness of solution for Cauchy problems does not hold in general. Kneser [62] proved in 1923 that the solution set is a continuum, i.e. closed and connected. For differential inclusions, in 1942, Aronszajn [7] proved that the solution set is in fact compact and acyclic, and he even specified this continuum to be an $R_\delta-$set.

An analogous result was obtained for differential inclusions with *u.s.c.* convex valued nonlinearities by several authors; we quote [4,3,2,6,54,41,46].

The topological and geometric structure of solution set for impulsive differential inclusions on compact intervals were investigated in [30, 57, 55, 56] where contractibility, AR, acyclicity, $R_\delta-$sets properties are obtained. Also, the topological structure of solution set for some Cauchy problems without impulses and posed on non-compact intervals were studied by various techniques in [2, 15, 27, 28].

This dissertation is organised as follows:

In chapter 1, we give some basic concepts about multivalued analysis, and fixed point theory, in the last section we show some recent concept of homologie.
In chapter 2, we give our first main existence of solutions to a Cauchy problem for impulsive ordinary differential equations of first order on an unbounded interval $[0, +\infty)$.

$$\begin{cases} \dot{y}(t) = f(t, y(t)), & a.e. t \in [0, +\infty)/\{t_1, \cdots\} \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), & k = 1, 2, 3, \cdots \\ y(0) = a, & a \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $f : [0, +\infty[\times\mathbb{R}^n \to \mathbb{R}^n$ is Caratheodory function, and $I_k \in C(\mathbb{R}^n, \mathbb{R}^n)$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h), y(t_k^-) = \lim_{h\to 0^-} y(t_k - h)$.

We also investigate the geometric structure of solution set (R_δ, acyclicity) of the problem (1), then as an application, we present an example to illustrate our main results.
In chapter 3, we are mainly concerned with existences results and compactness of solution set of the following first order neutral impulsive functional differential inclusions in a Banach space and Frécht space.

$$\begin{cases} \frac{d}{dt}\left[y(t) - g(t, y_t)\right] & \in & F(t, y_t), \quad a.e.\ t \in J/\{t_1, t_2, \ldots\} \\ y(t_k^+) - y(t_k^-) & = & I_k(y(t_k^-)), \qquad k = 1, 2, \ldots \\ y(t) & = & \phi(t),\ t \in [-r, 0], \end{cases} \tag{2}$$

where $0 < r < \infty$, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} < \ldots$, $J := [0, \infty)$, $F : J \times D \to \mathcal{P}(\mathbb{R}^n)$ is a multifunction, $g : J \times D \to \mathbb{R}^n$, $\lim\limits_{t\to t_k^+} g(t, \psi) = \lim\limits_{t\to t_k^-} g(t, \psi) = g(t_k, \psi)$, $\psi \in D = C([-r, 0], \mathbb{R}^n)$ $\phi \in D$, $I_k \in C(\mathbb{R}^n, \mathbb{R}^n)$ $(k = 1, 2, \ldots)$, and $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$. The notations $y(t_k^+) = \lim\limits_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim\limits_{h\to 0^+} y(t_k - h)$ stand for the right and the left limits of the function y at $t = t_k$, respectively. For any function y defined on $[-r, \infty)$ and any $t \in J$, y_t refers to the element of D such that

$$y_t(\theta) = y(t+\theta), \quad \theta \in [-r, 0].$$

We present three existences of solution for problem (2). Different kinds of growth of the nonlinearity F are considered in case F is *u.s.c.*, *l.s.c.*, Lipschitz or satisfies Nagumo-type condition.

Finally, in chapter 4, we study the existence and solutions set of systems of impulsive differential inclusions with initial conditions.

$$\left\{\begin{array}{rcl} x'(t) & \in & F_1(t, x(t), y(t)), \text{a.e. } t \in [0,1] \\ y'(t) & \in & F_2(t, x(t), y(t)), \text{a.e. } t \in [0,1] \\ x(t_k^+) & = & x(t_k^-) + I_{1,k}(x(t_k), y(t_k)), \ k = 1, \ldots, m \\ y(t_k^+) & = & y(t_k^-) + I_{2,k}(x(t_k), y(t_k)), \ k = 1, \ldots, m \\ x(0) & = & x_0, \\ y(0) & = & y_0, \end{array}\right. \tag{3}$$

where $0 = t_0 < t_1 < \ldots < 1, i = 1, 2, F_i : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ are a multifunction, $I_{1,k}, I_{2,k} \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. The notations $x(t_k^+) = \lim_{h \to 0^+} x(t_k + h)$ and $x(t_k^-) = \lim_{h \to 0^+} x(t_k - h)$ stand for the right and the left limits of the function y at $t = t_k$, respectively.
In this last chapter, we prove some existence results based on a nonlinear alternative of Leray-Schauder type theorem in generalized Banach spaces in the convex case and a multivalued version of Perov's fixed point theorem 4.1.3 for nonconvex case. Finally, we present some topological and geometric structure of the problem (3).

Mots clé: contraction, fixed point, solution set, acyclic, contractible, R_δ, compactness, impulsive differential equation and inclusion, metric and Banach space, Fréchet space, generalized metric space, matrix.
Classification AMS: 34A37, 34A60, 34K30, 34K45.

Chapter 1

Preliminaries

In this chapter, we introduce notations, definitions, lemmas and fixed point theorems which are used throughout this dissertation.

1.1 Notations

Let $J := [a, b]$ be an interval of $\mathbb{R}$. Let $(E, |\cdot|)$ be a real Banach space. $C([a, b], E)$ is the Banach space of all continuous functions from $[a, b]$ into E with the norm

$$\|y\|_\infty = \sup\{|y(t)| : a \leq t \leq b\}.$$

$L^1([a, b], E)$ denotes the Banach space of measurable functions $y : [a, b] \longrightarrow E$ is Lebesgue integrable with the norm

$$\|y\|_{L^1} = \int_0^b |y(t)|dt,$$

denote by $L^1_{loc}([0, \infty), \mathbb{R}_+)$ the space of locally integrable functions

$$L^1_{loc}([0, \infty), \mathbb{R}_+) = \{f : [0, \infty) \to \mathbb{R}_+, \int_0^b f(s)ds < \infty, \forall b \in (0, \infty)\}.$$

$AC^i([a, b], E)$ is the space of i-times differentiable functions $y : (a, b) \to E$, whose i^{th} derivative, $y^{(i)}$, is absolutely continuous.
Let (X, d) be a metric space, the following notations will be used throughout this dissertation.

- $\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$

- $\mathcal{P}_p(X) = \{Y \in P(X) : Y$ has the property "p"$\}$, where p could be: cl=closed, b=bounded, cp=compact, cv=convex, etc.

Thus

- $\mathcal{P}_{cl}(X) = \{Y \in P(X) : Y$ closed$\}$,
- $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y$ bounded$\}$,
- $\mathcal{P}_{cv}(X) = \{Y \in P(X) : Y$ convex$\}$,
- $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y$ compact$\}$,
- $\mathcal{P}_{cv,cp}(X) = \mathcal{P}_{cv}(X) \cap \mathcal{P}_{cp}(X)$, etc.

1.2 Multivalued Analysis

A multivalued map (multimap) F of a set X into a set Y is a correspondence which associates every $x \in X$ to a non empty subset $F(x) \subset Y$, called the value of x. We will write this correspondence as

$$F : X \to \mathcal{P}(Y).$$

1.2.1 Measurable Multivalued Mappings

Throughout this section, we assume that X is a separable metric space and $(\Omega, \mathcal{U}, \mu)$ is a complete $\sigma-$finite measurable space, i.e. a set Ω equipped with σ-algebra $\mathcal{U}$ of subsets and a countably additive measure μ on $\mathcal{U}$.

Definition 1.2.1. *Let Ω be a set. A set $\mathcal{U}$ of subsets of Ω is called a σ-algebra if the following three properties are satisfied:*

(i) $\Omega \in \mathcal{U}$,

(ii) $\forall A \in \mathcal{U} \Rightarrow A^c \in \mathcal{U}$,

(iii) $A_n \in \mathcal{U} \Rightarrow \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{U}$.

A pair $(\Omega, \mathcal{U})$ for which $\mathcal{U}$ is a σ-algebra in Ω is called a measurable space.

Definition 1.2.2. *$(E, \mathcal{O})$ is a topological space, where $\mathcal{O}$ is the set of open sets in E. Then $\sigma(\mathcal{O})$ is called the Borel σ-algebra of the topological space.*

Definition 1.2.3. *A map f from a measure space $(\Omega, \mathcal{U})$ to an other measure space $(\Delta, \mathcal{B})$ is called measurable, if $f^{-1}(B) \in \mathcal{U}$ for all $B \in \mathcal{B}$. The set $f^{-1}(B)$ consists of all points $x \in \Omega$ for which $f(x) \in B$. This pull back set $f^{-1}(B)$ is defined even if f is non-invertible.*

Definition 1.2.4. *A multivalued map $F : \Omega \to \mathcal{P}(X)$ is said:*

a) *measurable if for every closed subset $C \subseteq X$, we have*

$$F_{+}^{-1}(C) = \{\omega \in \Omega : F(\omega) \cap C \neq \emptyset\} \in \mathcal{U},$$

b) *weakly measurable if for every open subset $U \subseteq X$, we have*

$$F_{+}^{-1}(U) = \{\omega \in \Omega : F(\omega) \cap U \neq \emptyset\} \in \mathcal{U},$$

c) *$F(\cdot)$ is said to be K-measurable if for every compact subset $K \subseteq X$, we have*

$$F_{+}^{-1}(K) = \{\omega \in \Omega : F(\omega) \cap K \neq \emptyset\} \in \mathcal{U},$$

d) *graph measurable if*

$$GraF = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \mathcal{U} \otimes B(X),$$

where $B(X)$ is the σ-algebra generated by the family of all open sets from X.

Proposition 1.2.1. *[9, 25] Assume that $\varphi, \psi \colon \Omega \to \mathcal{P}(X)$ are two multivalued mappings. Then the followings hold true*

- *if φ is measurable then φ is also weakly measurable,*
- *if φ has compact values, measurability and weak measurability of φ are equivalent,*
- *if φ is weakly measurable then the graph Γ_{φ} of φ is product measurable,*
- *if φ and ψ are measurable then so is $\varphi \cup \psi$,*
- *if φ and ψ are measurable then so is $\varphi \cap \psi$,*
- *if φ and ψ are measurable then so is $\varphi \times \psi$.*

1.2.2 Upper Semicontinuous Mappings

Let (X, d) and (Y, ρ) be two metric spaces.

Definition 1.2.5. *A multivalued map $F : X \to \mathcal{P}(Y)$ is called upper semicontinuous (u.s.c. for short) provided for every open $U \subset Y$ the set $F^{-1}(U)$ is open in X.*

Now, we give the locally version of u.s.c. multivalued map

Definition 1.2.6. *A multivalued map $F : X \to \mathcal{P}(Y)$ is u.s.c. at the point $x \in X$ if $F(x)$ is a nonempty set, and if for each open set W of Y containing $F(x)$, there exists an open neighborhood U of x such that $F(U) \subset W$.*

Remark 1.2.1. *A multimap is called u.s.c. provided it is upper semicontinuous at every point $x \in X$.*

Proposition 1.2.2. *A multivalued map $F : X \to \mathcal{P}_{cl}(Y)$ is u.s.c. at point $x \in X$ if and only if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x and for any open set $V \subset Y$ such that $F(x) \subset V$, then there exists $n_0 \in \mathbb{N}$ such that*

$$F(x_n) \subset V \textit{ for all } n \geq n_0.$$

Proposition 1.2.3. *A multivalued map $F : X \to \mathcal{P}(Y)$ is u.s.c. if and only if for every closed set $A \subset Y$ the set $F_+^{-1}(A)$ is closed subset of X.*

Definition 1.2.7. *A multivalued map $F : X \to \mathcal{P}(Y)$ is called closed if its graph is a closed subset in $X \times Y$.*

Lemma 1.2.1. *A multivalued map $F : X \to \mathcal{P}(Y)$ is closed if and only if for every sequences $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty, x_n \to x_*, y_n \to y_*$ and $y_n \in F(x_n)$, then $y_* \in F(x_*)$.*

Lemma 1.2.2. *If $F : X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then the graph of F $\mathcal{G}r(F)$ is a closed subset of $X \times Y$.*

Remark 1.2.2. *Notice that for example the map $f : \mathbb{R} \to \mathbb{R}$ defined as follows:*

$$f(x) = \begin{cases} \frac{1}{x}, & \textit{if} \quad x \neq 0 \\ 0, & \textit{if} \quad x = 0, \end{cases}$$

has a closed graph but it is not u.s.c. i.e. continuous.
In general if $f : X \to Y$ is a continuous map from X into Y,then the inverse map $F_f : Y \to \mathcal{P}(X)$ defined by

$$F_f(y) = f^{-1}(y) = \{x \in X : f(x) = y\},$$

has a closed graph but is not necessarily u.s.c.

Lemma 1.2.3. *If $F : X \to \mathcal{P}(Y)$ has a closed graph and locally compact (i.e., for every $x \in X$, there exists an open set U_x such that $\overline{F(U_x)} \in \mathcal{P}_{cp}(Y)$), then $F(.)$ is u.s.c.*

Definition 1.2.8. *F is completely continuous if it is u.s.c. and $F(\mathcal{B})$ is relatively compact for every $\mathcal{B} \in \mathcal{P}_b(X)$. Also, F is compact if $F(X)$ is relatively compact.*

Theorem 1.2.1. *[29] If the multivalued map F is completely continuous with nonempty compact values, then F is u.s.c. if and only if F has a closed graph.*

Definition 1.2.9. A multivalued map $F : X \to \mathcal{P}(Y)$ is quasicompact if its restriction to every compact subset $A \subset X$ is compact.

Theorem 1.2.2. *[59] Let X and Y be metric spaces and $F : X \to \mathcal{P}_{cp}(Y)$ a closed quasicompact multimap. Then F is u.s.c.*

Proposition 1.2.4. *Assume that $G, F : X \to \mathcal{P}(Y)$ are two u.s.c. mappings. Then:*

- *The map $G \cup F : X \to \mathcal{P}(Y)$ is u.s.c.*
- *The map $G \cap F : X \to \mathcal{P}(Y)$ is u.s.c.*

Proposition 1.2.5. *[50] Let X, Y and Z three metric spaces, let $F : X \to \mathcal{P}(Y)$ and $G : X \to \mathcal{P}(Z)$ two u.s.c. mappings. Then the map $F \times G : X \to \mathcal{P}(Y \times Z)$ is u.s.c.*

Proposition 1.2.6. *[50] Let $F : X \to \mathcal{P}_{cp}(Y)$ be an u.s.c. and let A be a compact subset of X. Then $F(A)$ is compact.*

Lemma 1.2.4. *[9]: for a multifunction $F : X \to \mathcal{P}_{cp}(Y)$ u.s.c. we have*

$$\forall x_0 \in X, \lim_{x \to x_0} \sup F(x) = F(x_0).$$

Lemma 1.2.5. *: [9] Let $(K_n)_n \subset K$ such that K is a compact subset of X, and X is a separable Banach space. Then*

$$\overline{co}(\lim_{n \to \infty} \sup K_n) = \cap_{N>0} \overline{co}(\cup_{n \geq N} K_n).$$

Where co is the enveloppe convexe.

1.2.3 Lower Semicontinuous Mappings

Definition 1.2.10. *Let $F : X \to \mathcal{P}(Y)$ a multivalued map. If for every open U of Y the set $F_+^{-1}(U)$ is open in X then F is called lower semicontinuous(l.s.c.).*

Now, we give the locally version of l.s.c.

Definition 1.2.11. *Let $F : X \to \mathcal{P}(Y)$ a multivalued map. F is lower semicontinuous at the point x if for every open set $V \subset Y$ such that $F(x) \cap V \neq \emptyset$ there exists neighborhood U_x of x with the property $F(x') \cap V \neq \emptyset$ for all $x' \in U_x$.*

Remark 1.2.3. *A multivalued map is called lower semicontinuous provided that it is lower semi continuous at every point $x \in X$.*

Proposition 1.2.7. *A map $F : X \to \mathcal{P}(Y)$ is l.s.c. if and only if for every closed $A \subset Y$ the set $F^{-1}(A)$ is a closed subset of X.*

Proposition 1.2.8. *[29] The multimap $F : X \to \mathcal{P}(Y)$ is l.s.c. at the point $x \in X$ if and only if for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ converge to x, then for each $y \in F(x)$ there exists a sequence $\{y_n\}_{n\in\mathbb{N}} \subset Y$, $y_n \in F(x_n)$, such that y_n converge to y.*

Proposition 1.2.9.

- *If $F, G : X \to \mathcal{P}(Y)$ are two l.s.c., then $F \cup G : X \to \mathcal{P}(Y)$ is l.s.c. too.*
- *If $F : X \to \mathcal{P}(Y)$ and $G : Y \to \mathcal{P}(Z)$ are two l.s.c. then the composition $G \circ F : X \to \mathcal{P}(Z)$ is l.s.c. too, provided for every $x \in X$ the set $G(F(x))$ is closed.*

We would like to stress that the intersection of two l.s.c. mappings is not l.s.c.

Example 1.2.1. *Consider two multivalued mappings $F, G : [0, \pi] \to \mathcal{P}(\mathbb{R}^2)$ defined as follows:*

$$F(t) = \{(x, y) \in \mathbb{R}^2 : y \geq 0 \,\text{and}\, x^2 + y^2 \leq 1\}, \text{ for every}\, t \in [0, \pi];$$

$$G(t) = \{(x, y) \in \mathbb{R}^2 : x = \lambda \cos t, y = \lambda \sin t, \lambda \in [-1, 1]\}.$$

Then F is a constant map and hence even continuous, G is l.s.c. map but $F \cap G$ is not l.s.c. (to butter understand consider $t = 0$ or $t = \pi$).

Definition 1.2.12. *A subset $\mathcal{A}$ of $L^1(J, E)$ is decomposable if for all functions $u, v \in \mathcal{A}$ and measurable subset $N \subset J$, the function $u\chi_N + v\chi_{J-N} \in \mathcal{A}$, where χ stands for the characteristic function.*

Definition 1.2.13. *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([a, b], X))$ be a multivalued operator. We say that N has property (BC) if*

1) *N is lower semi-continuous (l.s.c.),*

2) *N has nonempty closed and decomposable values.*

Let $F : [a, b] \times X \to \mathcal{P}(X)$ be a multivalued map with nonempty compact values. Assign to F the multivalued operator

$$\mathcal{F} : C([a, b], X) \to \mathcal{P}(L^1([a, b], X))$$

by letting

$$\mathcal{F}(y) = \{w \in L^1([a, b], X) : w(t) \in F(t, y(t)) \ \ a.e. \ t \in [a, b]\}.$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated with F.

Definition 1.2.14. *Let $F : [a, b] \times X \to \mathcal{P}(X)$ be a multivalued function with nonempty compact values. We say that F is of lower semi-continuous type (l.s.c. type) if its associated Niemytzki operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.*

1.2.4 Hausdorff Continuity

In all this subsection we assume that (X, d) is a metric space

Definition 1.2.15. *Consider the Hausdorff pseudo-metric distance*

$$H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}^+ \cup \{\infty\},$$

defined by

$$H_d(A, B) = \max\{H_d^*(A, B), H_d^*(B, A)\},$$

where

$$H_d^*(A, B) = \sup_{a \in A} d(a, B), H_d^*(B, A) = \sup_{b \in B} d(A, b),$$

and

$$d(A,b)=\inf_{a\in A} d(a,b), d(a,B)=\inf_{b\in B} d(a,b).$$

By convention, $H_d(\emptyset,\emptyset)=0$ and $H_d(A,\emptyset)=\infty$ for every $A\neq\emptyset$, $\inf\emptyset=0$. We can also define the Hausdorff distance in terms of neighborhoods of sets. Let $A\in\mathcal{P}(X)$ and $\varepsilon>0$ we define $\varepsilon-$neighborhood of A by

$$A_\varepsilon=O_\varepsilon(A)=\{x\in X: d(x,A)<\varepsilon\}.$$

We can easily prove that

$$\cap_{\varepsilon>0}O_\varepsilon(A)=\overline{A} \text{ and } O_\varepsilon\left(\cap_{i\in I}A_i\right)=\cup_{i\in I}O_\varepsilon(A_i).$$

Proposition 1.2.10.

$$H_d^*(A,B)=\inf\{\varepsilon>0:\ A\subset B_\varepsilon\}.$$

Lemma 1.2.6. *Let $A,B\in\mathcal{P}_{cl}(X)$, then*

$$H_d(A,B)=\sup\{|d(x,A)-d(x,B)|: x\in X\}.$$

Lemma 1.2.7. *For all A,B,C in $\mathcal{P}(X)\cup\{\emptyset\}$, the following properties are satisfied*

- $H_d(A,B)\geq 0$ *and* $H_d(A,A)=0$.
- $H_d(A,B)=H_d(B,A)$.
- $H_d(A,B)\leq H_d(A,C)+H_d(C,B)$.
- $H_d(A,B)=0$ *if and only if* $\overline{A}=\overline{B}$.

Proposition 1.2.11. *$(\mathcal{P}_{b,cl}(X),H_d)$ is a metric space,and $(\mathcal{P}_{cl}(X),H_d)$ is a generalized metric space.*

Remark 1.2.4. *H_d is a generalized metric(pseudo-metric space)(i.e. H_d satisfies all the conditions distance but in general $H_d(A,B)\nless\infty$ if we take $A=\mathbb{R}$ and $B=\{0\}$ we have $H_d(A,B)=\infty$).*

Theorem 1.2.3. *Let (X,d) be a complete metric space, then $(\mathcal{P}_{cl}(X),H_d)$ is a complete space too.*

Definition 1.2.16. *Given two metric spaces X, Y, a multivalued map $F : X \to \mathcal{P}(Y)$ is said to be H_d-continuous at some point $x_0 \in X$ if*

$$\forall \varepsilon > 0, \exists \delta : \forall x \in X : d(x, x_0) \leq \delta \Rightarrow H_d(F(x), F(x_0)) < \varepsilon.$$

The following facts result immediately from the above definitions.

Proposition 1.2.12. *1. F is $\varepsilon - \delta$ u.s.c. at x_0 if and only if*

$$\forall \varepsilon > 0, \ \exists \delta > 0, \ \forall x \in B(x_0, \delta) : \ H_d^*(F(x), F(x_0)) < \varepsilon.$$

2. F is $\varepsilon - \delta$ l.c.s. at x_0 if and only if

$$\forall \varepsilon > 0, \ \exists \delta > 0, \ \forall x \in B(x_0, \delta) : \ H_d^*(F(x_0), F(x)) < \varepsilon.$$

In other words, $\varepsilon - \delta$ *u.s.c.* at x_0 means that $\lim_{x \to x_0} H_d^*(F(x), F(x_0)) = 0$, $\varepsilon - \delta$ *l.c.s.* at x_0 means that $\lim_{x \to x_0} H_d^*(F(x_0), F(x)) = 0$, and $\varepsilon - \delta$ continuity at x_0 means that $\lim_{x \to x_0} H_d(F(x), F(x_0)) = 0$.

Corollary 1.2.1. *$F : X \to \mathcal{P}(Y)$ is H_d-continuous at $x_0 \in X$ if and only if F is $\varepsilon - \delta$ u.s.c. at x_0 and F is $\varepsilon - \delta$ l.c.s. at x_0.*

Definition 1.2.17. *Let X, Y be metric spaces. A multivalued map $F : X \to \mathcal{P}(Y)$ is said to be*

- *H_d-Lipschitz with constant $k > 0$ if*

$$\forall x_1, x_2 \in X, \ H_d(F(x_1), F(x_2)) \leq k d(x_1, x_2).$$

Proposition 1.2.13.

$$F \, H_d - \textit{Lipschitz with constant } k \ \Rightarrow \ F \, H_d - \textit{continuous.}$$

1.3 Selection Theorems

The following definitions and the result can be found in [50, 41].

Theorem 1.3.1. *{Michael's selection theorem 1956(see [68])}*
Let X be a metric space, E a Banach space and $F \colon X \to \mathcal{P}_{cl,cv}(E)$ a l.s.c. map. Then there exists $f \colon X \to E$, a continuous selection of F ($f \subset F$), i.e. $f(x) \in F(x)$ for every $x \in X$.

Corollary 1.3.1. *Let X be a metric space, Y a separable Banach space, and $F : X \to \mathcal{P}_{cl,cv}(Y)$ a l.s.c. multivalued map. Then F admits a sequence of continuous selections $(f_n)_{n\geq 1}$ such that for all $x \in X$*

$$F(x) = \overline{\{f_n(x)\}_{n\geq 1}}.$$

Definition 1.3.1. *We say that a map $F\colon X \to \mathcal{P}(Y)$ is σ-selectionable, if there exists a decreasing sequence of compact valued u.s.c. maps $F_n\colon X \to \mathcal{P}(Y)$ satisfying:*

1. *F_n has a continuous selection, for all $n \geq 0$,*
2. *$F(x) = \bigcap_n F_n(x)$, for all $x \in X$.*

Definition 1.3.2. *[9,25] Assume that $F : X \to \mathcal{P}(Y)$ is a multi-valued map and $F_n : X \to \mathcal{P}(Y)$, $n = 1, 2, \dots$ is a sequence of multi-valued mappings such that:*

$F_{n+1}(x) \subset F_n(x)$

$F(x) = \cap_{n\geq 0} F_n(x)$, for every $x \in X$ and $n = 0, 1, 2, \dots$.
We say that

F is $\sigma - L-$ selectionable, provided F_n is $L-$ selectionable for every n (i.e for every n there exists a Lipschitz continuous map such that $f_n \subset F_n$)

F is $\sigma - LL-$ selectionable, provided F_n is $LL-$ selectionable for every n (i.e for every n, there exists a locally Lipschitz continuous map such that $f_n \subset F_n$)

F is $\sigma - Ca-$selectionable, provided F_n is $Ca-$ selectionable for every n (i.e for every n, there exists a Carathéodory map such that $f_n \subset F_n$)

F is $\sigma - m-$ selectionable, provided F_n is $m-$ selectionable for every n (i.e for every n, there exists a measurable map such that $f_n \subset F_n$)

F is $\sigma - c-$ selectionable, provided F_n is $c-$ selectionable for every n (i.e for every n, there exists a continuous map such that $f_n \subset F_n$).

F is $\sigma - mLL-$selectionable provided F_n is $mLL-$ selectionable for every n (i.e. for every $n = 0, 1, 2, \dots,$ there exists a measurable-locally Lipchitz map $f_n : [a, b] \times X \to Y$ such that $f_n \subset F_n$).

Recall that a single-valued map $f : [a,b]\times X \to Y$ is said to be measurable-locally-Lipschitz (mLL) if $f(\cdot, x)$ is measurable for every $x \in X$ and for every $x \in X$, there exists a neighborhood V_x of $x \in X$ and an integrable function $L_x : [a,b] \to [0,\infty)$ such that

$$d(f(t,x_1), f(t,x_2)) \le L_x(t)d(x_1,x_2) \text{ for every } t \in [a,b] \text{ and } x_1, x_2 \in V_x.$$

Lemma 1.3.1. *Let Y be a separable metric space and $F : [a,b] \to \mathcal{P}(Y)$ a measurable multi-valued map with nonempty closed values. Then F has a measurable selection.*

Lemma 1.3.2. *: Let X be a Banach space. Let $F : [0,b] \times X \to \mathcal{P}_{cp,cv}(X)$ is L^1-Caratheodory multifunction with $S_{F,y} \neq \emptyset$, let Γ a continuous linear operator to $L^1([0,b], X)$ in $C([0,b], X)$, then the operator*

$$\begin{array}{rcl} \Gamma \circ S_F : C([0,b],X) & \longrightarrow & \mathcal{P}_{cp,cv}(C([0,b],X)) \\ y & \longrightarrow & (\Gamma \circ S_F)(y) := \Gamma(S_{F,y}) \end{array}$$

has a closed graph in $C([0,b],X) \times C([0,b],X)$, where

$$S_{F,y} = \{v \in L^1([0,b],X) : v(t) \in F(t, y(t));\ t \in [0,b]\}.$$

Theorem 1.3.2. *{Kuratowski-Ryll-Nardzewski's selection theorem 1965}*
Let Y be a separable complete space. Then every measurable $\varphi\colon \Omega \to \mathcal{P}(Y)$ has a measurable selection.

Definition 1.3.3. *A map $\varphi\colon [0,a]\times\mathbb{R}^n \to \mathcal{P}_{cp}(\mathbb{R}^n)$ is called u-Carathéodory (resp. l-Carathéodory; resp. Carathéodory) if it satisfies:*

1. *$t \to \varphi(t,x)$ is measurable for every $x \in \mathbb{R}^n$,*
2. *$x \to \varphi(t,x)$ is u.s.c. (resp. l.s.c.; resp. continuous) for almost all $t \in [0,a]$,*
3. *$\|y\| \le \ell(t)(1 + \|x\|)$, for every $(t,x) \in [0,a] \times \mathbb{R}^n$, $y \in \varphi(t,x)$, where $\ell\colon [0,a] \to [0,+\infty)$ is an integrable function.*

Theorem 1.3.3. *{Theorem of Cellina}*
Let $\varphi\colon [0,a] \times \mathbb{R}^n \to \mathcal{P}_{cp,cv}(\mathbb{R}^n)$ be a multivalued map. If $\varphi(\,\cdot\,,x)$ is u.s.c. for all $x \in \mathbb{R}^n$ and $\varphi(t,\,\cdot\,)$ is l.s.c. for all $t \in [0,a]$, then φ has a Carathéodory selection.

Theorem 1.3.4. *[21](theorem of selection"Bressan-Colombo")*
Let X be a separable metric space, let E be a Banach space. Then for all l.s.c. operator $N : X \to \mathcal{P}_f(L^1(J, E))$ with decomposable closed value has a continuous selection.

For more details on multivalued maps and the proof of the known results cited in this section we refer interested reader to the books of Deimling [29], Gorniewicz [50], Hu and Papageorgiou [47], Smirnov [82] and Tolstonogov [84].

1.4 Homologie

1.4.1 Some definition and property

Let $\mathcal{C}$ be an abelian category. For $\mathcal{C}$ may be the category of abelian groups or sheaves of abelian groups on a topological space.

Definition 1.4.1. *A complex C^* is a family $\{C^n\}_{n\in\mathbb{Z}}$ of objects in $\mathcal{C}$, together with maps $d : C^m \to C^{m+1}$ such that $d \circ d = 0$.*

Definition 1.4.2. *A morphism of complex $\Phi : C^* \to D^*$ is a family of maps $\Phi : C^n \to D^n$ for all $n \in \mathbb{Z}$ such that $\Phi \circ d = d \circ \Phi$.*

The cohomology of a complex C^* is defined as follows

$$H^n(C^*) = \ker(d : C^n \to C^{n+1})/\mathrm{image}(d : C^{n-1} \to C^n).$$

Taking cohomology is an additive functor, i.e. $\Phi : C^* \to D^*$ induces a map $H^n(\Phi) : H^n(C^*) \to H^n(D^*)$ for all $n \in \mathbb{Z}$, with

1. $H^n(id) = id$,
2. $H^n(\Phi \circ \psi) = H^n(\Phi) \circ H^n(\psi)$,
3. $H^n(\Phi + \psi) = H^n(\Phi) + H^n(\psi)$.

Definition 1.4.3. *A morphism $\Phi : C^* \to D^*$ is called a quasi-isomorphism if $H^n(\Phi)$ is an isomorphism for all $n \in \mathbb{Z}$.*

Definition 1.4.4. *A complex C^* is called acyclic if $H^n(C^*) = 0$ for all $n \in \mathbb{Z}$, i.e. $0 \to C^*$ is a quasi-isomorphism.*

If $F \to C^0 \to C^1 \to \cdots$ is an acyclic complex then $C^0 \to C^1 \to \cdots$ is called a resolution of F.

Definition 1.4.5. *A morphism of complex $\Phi : C^* \to D^*$ is called null homotopic if there are maps $K : C^n \to D^{n-1}$ for all $n \in \mathbb{Z}$, such that*

$$\Phi = K \circ d + d \circ K.$$

Two morphism $\Phi : C^* \to D^*$ and $\psi : C^* \to D^*$ are called homotopic if $\Phi - \psi$ is null homotopic.

Proposition 1.4.1. *If Φ and ψ are homotopic then*

$$H^n(\Phi) = H^n(\psi)\, \forall n \in \mathbb{Z}.$$

Definition 1.4.6. *Let $U = (U_i)_{i \in I}$ be an open covering of X. We define the Čech complex by*

$$C^p(U, F) = \amalg_{(i_0, \cdots, i_p) \in I^{p+1}} F(U_{i_0} \cap \cdots \cap U_{i_p}) \text{ for all } p \geq 0.$$

The differential is

$$C^p(U, F) \to C^{p+1}(U, F)$$

$$\delta(a)_{(i_0, \cdots, i_{p+1})} := \sum_{k=0}^{p+1} (-1)^k a_{(i_0, \cdots, i_p)}.$$

We write

$$Z^p(U, F) = \ker(\delta : C^p(U, F) \to C^{p+1}(U, F)),$$

the elements of $Z^p(U, F)$ are called cycles.

Definition 1.4.7. *The cohomology groups*

$$H^i(U, F) := H^i(C^*(U, F))$$

are called the Čech cohomology. They depend on the open covering and the sheaf F. Since F is a sheaf we get

$$H^0(U, F) = H^0(X, F) = \Gamma(X, F).$$

Lemma 1.4.1. *Let $U = (U_i)_{i \in I}$ be an open covering of X with $U_j = X$ for some $j \in I$. Then $H^p(U, F) = 0$ for $p > 0$ and any sheaf F.*

1.5 Topological and Geometric Background

First, we start with some notions from geometric topology. For details, we recommend [58,30,50]. In what follows (X, d) and (Y, d') stand for two metric spaces.

Definition 1.5.1. *(Homotopic)*
Let $f, g : X \longrightarrow Y$ be two continuous functions. We say that f is homotopic to g if there exists is a continuous function $h : X \times [0, 1] \longrightarrow Y$ such that:

(1) $h(x, 0) = f(x); \forall x \in X,$

(2) $h(x, 1) = g(x); \forall x \in X.$

Definition 1.5.2. *(Retract)*
A subset $A \subset X$ is called a retract of X if there exists a continuous mapping $r : X \to A$ such that $r(x) = x,\ \forall\, x \in A$.

Definition 1.5.3. *(Contractible)*
A set $A \subset X$ is called a contractible space provided there exists a continuous homotopy $h : A \times [0, 1] \to A$ and $x_0 \in A$ such that
(a) $h(x, 0) = x$, *for every* $x \in A$,
(b) $h(x, 1) = x_0$, *for every* $x \in A$,
i.e. if the identity map $A \longrightarrow A$ is homotopic to a constant map (A is homotopically equivalent to a one-point space).

Note that any closed convex subset of X is contractible.

The following notion, strictly connected with extendability, was first introduced by K. Borsuk [19].

Definition 1.5.4. *A space X is called an absolute retract (in short $X \in AR$) provided that for every space Y, every closed subset $B \subseteq Y$ and any continuous map $f : B \to X$, there exists a continuous extension $\widetilde{f} : Y \to X$ of f over Y, i.e. $\widetilde{f}(x) = f(x)$ for every $x \in B$. In other words, for every space Y and for any embedding $f : X \longrightarrow Y$, the set $f(X)$ is a retract of Y. If the set $f(X)$ is a retract of U for every open neighborhood U of B in Y we say that $X \in ANR$ and call X to be an absolute neighborhood retract.*

Definition 1.5.5. *($R_\delta-$set)*
A compact nonempty metric space X is called an R_δ-set provided there exists a decreasing sequence of compact nonempty contractible metric spaces

$(X_n)_{n\in\mathbb{N}}$ *such that*

$$X = \bigcap_{n=1}^{\infty} X_n.$$

For compact sets we have

$$Convexsets \subset AR \subset Contractible \subset R_\delta.$$

Definition 1.5.6. *(Acyclic)*
A space A is closed acyclic if
(a) $H_0(A) = \mathbb{Q}$,
(b) $H_n(A) = 0$, for every $n > 0$, where $H_* = \{H_n\}_{n\geq 0}$ is the Čech-homology functor with compact carriers and coefficients in the field of rationals $\mathbb{Q}$. In other words, a space A is acyclic if the map $j : \{p\} \to X,\ j(p) = x_0 \in A$, induces an isomorphism $j_* : H_*(\{p\}) \to H_*(A)$.

Lemma 1.5.1. *[51] Let X be a compact metric space, if X is $R_\delta-$set, then X is an acyclic space.*

Theorem 1.5.1. *[50] (Theorem of Lasota York)*
Let E be a normal space, and X a metric space, and let
$f : X \longrightarrow E$ a continuous map. Then for every $\varepsilon > 0$ there exists a locally Lipschitz function $f_\varepsilon : X \longrightarrow E$ such that

$$\|f(x) - f_\varepsilon(x)\| \leq \varepsilon; \forall x \in X. \tag{1.1}$$

Definition 1.5.7. *A map $f : X \to Y$ is propre, if it is continuous and the inverse image of a compact set is compact.*

Theorem 1.5.2. *:(Theorem of Browder and Gupta)*
[20] Let $(E, \|.\|)$ a Banach spaca, and let $f : X \longrightarrow E$ a propre map, and for every $\varepsilon > 0$ we have a propre map $f_\varepsilon : X \longrightarrow E$, satisfied:

(i) $\|f_\varepsilon(x) - f(x)\| < \varepsilon$ for all $x \in X$.

(ii) For all $u \in E$ such that $\|u\| \leq \varepsilon$, the equation $f_\varepsilon(x) = u$ has a unique solution.
Then the set $S = f^{-1}(0)$ is R_δ.

1.6 Fixed Point Theorems

Fixed point theory plays an important role in our existence results, therefore we state the following fixed point theorems.

1.6.1 Single Mappings Case

Theorem 1.6.1. *(Schauder fixed point theorem)*
Let E be a Banach space, $C \subset E$ be a nonempty compact convex subset of E and $N : C \to C$ be a continuous operator. Then N has at least fixed point in C.

The compactness condition on C is a very strong one and most of the problems in analysis do not have compact setting.

Theorem 1.6.2. *(Granas-Schauder fixed point theorem)*
If $X \in AR$ and $N : X \to X$ is a compact map, then N has a fixed point.

Theorem 1.6.3. *[58, 50](Nonlinear alternative of Leray and Schauder)*
Let X be a Banach space C a convex subset of X, U an open subset in C, and $N : U \to X$ is continuous and compact operator, then

(a) either $\exists u \in \partial U; \exists \lambda \in (0,1)$ such that $u = \lambda N(u)$,

(b) or N has a fixed point in $\overline{U}$.

Definition 1.6.1. *Let* (X, d) and (Y, d_1) *be two metric spaces. Recall that a mapping $f : X \to Y$ is called contractive* (or *Banach contraction) provided there exists $\alpha \in [0, 1)$ such that:*

$$\forall\, x, y \in X,\ d_1(f(x), f(y)) \leq \alpha d(x, y).$$

Theorem 1.6.4. *(Banach fixed point theorem)*
Let (X, d) be a complete metric space and let $f : X \to X$ be a contractive mapping. Then there exists exactly one point $\widetilde{x} \in X$ such that $f(\widetilde{x}) = \widetilde{x}$.

Consider then the situation in which $N : X \to X$ is not necessarily a contraction mapping, but N^n is a contraction for some n.

Example 1.6.1. *Let $N : [0, 2] \to [0, 2]$ be defined by*

$$N(x) = \begin{cases} 0, & \text{if } x \in [0,1], \\ 1, & \text{if } x \in (1,2] \end{cases}$$

Then, $N^2(x) = 0$ for all $x \in [0, 2]$, and so, N^2 is a contraction on $[0, 2]$. Note that N is not continuous and thus not a contraction map.

In [22] Bryant extended Banach fixed point theorem as follows.

Theorem 1.6.5. *Let (X, d) be a complete metric space and let $N : X \to X$ be a mapping such that for some positive integer n, N^n is contraction on X. Then, N has a unique fixed point.*

Next extension of Banach fixed point theorem is due Caccioppoli [24]

Theorem 1.6.6. *Let (X, d) be a complete metric space and let $N : X \to X$ be a mapping such that for each $n \in \mathbb{N}$., there exists a constant $c_n \geq 0$ such that*

$$d(N^n(x), N^n(y)) \leq c_n d(x, y), \text{ for all } x, y \in X,$$

where $\sum_{n=1}^{\infty} c_n < \infty$. Then, N has a unique fixed point.

Theorem 1.6.7. *[36](The Alternative of Frigon and Granas) Let E be a Fréchet space with a family of semi semi−normes $\{\|\cdot\|_n\}_{n\in N^*}$, $N : E \longrightarrow E$ a continuous operator, with*

$$E = \cap_{n\in N^*} E_n, \quad E_n \subset E_{n+1}, \quad \text{and} \quad \|\cdot\|_n \leq \|\cdot\|_{n+1}, \quad n \in \mathbb{N}.$$

assume that for all $n \in \mathbb{N}$, $\exists k_n \in (0, 1)$ such that:

$$\|Ny - Nx\|_n \leq k_n \|y - x\|_n; \quad \forall n \in \mathbb{N}, \text{for all } x, y \in E_n,$$

$\mathcal{O}$ is a closed part in E. Then either,

1) there exists $\lambda \in (0, 1) : y = \lambda Ny, \quad \forall y \in \partial\mathcal{O}$; or

2) there exists a unique $y \in \mathcal{O}$ such that $y = Ny$.

Lemma 1.6.1. *[5](Schauder-Tikhonov fixed point theorem)*
Let E be a locally convex space, C a convex closed subset of E and $N : C \to C$ a continuous, compact map. Then N has at least one fixed point in C.

1.6.2 Multivalued Mappings Case

The question here is to solve the following abstract inclusions:

$$x \in F(x)$$

where $F : X \to \mathcal{P}(X)$ be a given multivalued operator.

Theorem 1.6.8. *(Nonlinear Alternative of Leray-Schauder)*
Let X be a Frechet space and $N : X \to \mathcal{P}_{cl,cv}(X)$ be a completely continuous, u.s.c. multivalued map. Then one of the following conditions holds:

(a) *N has at least one fixed point in X,*

(b) *the set $M := \{x \in X, x \in \lambda N(x), \lambda \in (0,1)\}$ is unbounded*

Lemma 1.6.2. *: We let X be a generalised Banach space. C a convex subset of X, U an open subset in C, and $F : U \to \mathcal{P}_{cp,cv}(X)$ is u.s.c, compact multifunction, then*

(a) *either $\exists u \in \partial U; \exists \lambda \in]0,1[$ such that $u \in \lambda F(u)$,*

(b) *or F has a fixed point in $\overline{U}$.*

Definition 1.6.2. *Let E be a Fréchet space with the topology generated by a family of semi-norms $\|.\|_n$ and corresponding distances $d_n(x,y) = \|x - y\|_n (n \in \mathbb{N})$. A multivalued map $F : E \to \mathcal{P}(E)$ is called an admissible contraction with constant $\{k_n\}_{n\in\mathbb{N}}$ if for each $n \in \mathbb{N}$, there exists $k_n \in (0,1)$ such that*

(a) *$H_{d_n}(F(x), F(y)) \leq k_n \|x - y\|_n$ for all $x, y \in E$, where H_d is the Hausdorff distance,*

(b) *for every $x \in E$ and every $\varepsilon > 0$, there exists $y \in F(x)$ such that*

$$\|x - y\|_n \leq d_n(x, F(x)) + \varepsilon, \ \forall n \in \mathbb{N}.$$

A subset $A \subset E$ is bounded if for every $n \in \mathbb{N}$, there exists $M_n > 0$ such that $\|x\|_n \leq M_n$, for every $x \in A$.

Lemma 1.6.3. *Let E be a Fréchet space, $U \subset E$ an open neighborhood of the origin, and let $N : \overline{U} \to \mathcal{P}(E)$ be a bounded admissible multivalued contraction. Then one of the following statements holds:*

(a) *N has a fixed point,*

(b) *there exists $\lambda \in [0,1)$ and $x \in \partial U$ such that $x \in \lambda N(x)$.*

Chapter 2

Cauchy problem for impulsive ordinary differential equations on unbounded domain

In this chapter, we treate the existence and the uniqueness of solutions to a Cauchy problem for impulsive ordinary differential equations on an unbounded interval $[0, +\infty)$. We also investigate some topological and geometric properties of the solutions set, more precisely consider the following impulsive problem

$$\begin{cases} \dot{y}(t) = f(t, y(t)), & t \in [0, +\infty)/\{t_1, \cdots\} \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), & k = 1, 2, 3, \cdots \\ y(0) = a, & a \in \mathbb{R}^n, \end{cases} \tag{2.1}$$

where $0 = t_0 < t_1 < \dots t_k < t_{k+1}, \dots,\ \lim_{k\to\infty} t_k = \infty,\ f : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is Carathéodory function, and $I_k \in C(\mathbb{R}^n, \mathbb{R}^n)$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$, $y(t_k^-) = \lim_{h\to 0^-} y(t_k - h)$.

2.1 Existence and Uniqueness

Let $J_k = (t_k, t_{k+1}]$, $k \in \mathbb{N}^*$, $J_0 = [0, t_0]$, and let y_k be the restriction of a function y to J_k. In order to define solutions for problem (2.1), consider the spaces:

$$PC = \left\{ \begin{array}{ll} y : [0, +\infty) \longrightarrow \mathbb{R}^n : & y_k \in C(J_k, \mathbb{R}^n), y(t_k^-) \text{ and } y(t_k^+) \\ \text{exist and satisfy} & y(t_k^-) = y(t_k),\ k \in \mathbb{N} \end{array} \right\},$$

and

$$PC_b = \{y \in PC : \|y\|_{PC_b} < \infty\}$$

endowed with the norm

$$\|y\|_{PC_b} = \sup_{t\in[0,\infty)} \|y(t)\|,$$

PC_b is a Banach space.

Definition 2.1.1. *A function $y \in PC \cap \cup_{k=1}^m AC(J_k, \mathbb{R}^n)$ is a solution of the problem (2.1) if and only if*

$$y(t) = a + \int_0^t f(s, y(s))ds + \sum_{0<t_k<t} I_k(y(t_k)), \quad t \in [0, \infty).$$

Our first main result is the existence and uniqueness of the problem (2.1) in PC_b.

Theorem 2.1.1. *[79] Suppose that there exists $p \in L^1([0, +\infty), \mathbb{R}_+)$ such that:*

$\|f(t, y) - f(t, x)\| \leq p(t) \|y - x\|$, *for all $x, y \in \mathbb{R}^n$, and almost all elements $t \in [0, \infty)$,*

and there exist a positives real numbers $c_k > 0; k \in \mathbb{N}$ such that:

$$\|I_k(y) - I_k(x)\| \leq c_k \|y - x\|, \text{ for every } \quad x, y \in \mathbb{R}^n,$$

and

$$\sum_{k=1}^{+\infty} \|I_k(0)\| < +\infty, \quad \int_0^{+\infty} \|f(s, 0)\| ds < \infty.$$

If $\sum_{k=1}^{+\infty} c_k < 1$, then the problem (2.1) has a unique solution on $[0, \infty)$.

Proof. Consider the application

$$\begin{array}{rcl} N : PC_b & \longrightarrow & PC_b \\ y & \longrightarrow & Ny, \end{array}$$

defined by

$$Ny(t) = a + \int_0^t f(s, y(s))ds + \sum_{0<t_k<t} I_k(y(t_k)), \quad t \in [0, \infty).$$

Step 1 N is well defined
Let $y \in PC_b$, we have

$$Ny(t) = a + \int_0^t f(s, y(s))ds + \sum_{0<t_k<t} I_k(y(t_k)), \quad t \in [0, \infty),$$

we will prove that $Ny \in PC_b$,

$$\begin{array}{rcl} \|Ny(t)\| & \leq & \|a\| + \int_0^t \|f(s, y(s))\| \, ds + \sum_{0<t_k<t} \|I_k(y(t_k))\| \\ & \leq & \|a\| + \int_0^{+\infty} \|f(s, y(s)) - f(s, 0)\| \, ds + \int_0^{+\infty} \|f(s, 0)\| \, ds \\ & + & \sum_{k=1}^{+\infty} \|I_k(y(t_k)) - I_k(0)\| + \sum_{k=1}^{+\infty} \|I_k(0)\| \\ & \leq & \|a\| + \int_0^{+\infty} p(s) \|y(s)\| \, ds + \int_0^{+\infty} \|f(s, 0)\| \, ds \\ & + & \sum_{k=1}^{+\infty} c_k \|y(t_k)\| + \sum_{k=1}^{+\infty} \|I_k(0)\| \\ & \leq & \|a\| + \int_0^{+\infty} p(s) \|y\|_{PC_b} \, ds + \int_0^{+\infty} \|f(s, 0)\| \, ds \\ & + & \sum_{k=1}^{+\infty} c_k \|y\|_{PC_b} + \sum_{k=1}^{+\infty} \|I_k(0)\| < \infty. \end{array}$$

Step 2 N is contractive
We let y_1, y_2 in PC_b:

$$\begin{array}{rcl} \|Ny_1(t) - Ny_2(t)\| & \leq & \displaystyle\int_0^t \|f(s, y_1(s)) - f(s, y_2(s))\| \, ds \\ & & + \displaystyle\sum_{0<t_k<t} \|I_k(y_1(t_k)) - I_k(y_2(t_k))\| \end{array}$$

$$
\begin{aligned}
&\leq \int_0^t p(s)\,\|y_1(s)-y_2(s)\|\,ds \\
&+ \sum_{k=1}^{+\infty} c_k\,\|y_1(t_k)-y_2(t_k)\| \\
&\leq \int_0^t \tau\frac{1}{\tau}e^{\tau P(s)}e^{-\tau P(s)}p(s)\,\|y_1(s)-y_2(s)\|\,ds \\
&+ e^{\tau P(t)}e^{-\tau P(t)}\sum_{k=1}^{+\infty} c_k\,\|y_1(t_k)-y_2(t_k)\| \\
&\leq \frac{1}{\tau}\left(\int_0^t \tau p(s)e^{\tau P(s)}ds\right)\|y_1-y_2\|_* + e^{\tau P(t)}\left(\sum_{k=1}^{+\infty} c_k\right)\|y_1-y_2\|_* \\
&\leq \frac{1}{\tau}\left(e^{\tau P(t)}\right)\|y_1-y_2\|_* + e^{\tau P(t)}\left(\sum_{k=1}^{+\infty} c_k\right)\|y_1-y_2\|_*\,,
\end{aligned}
$$

then

$$\|Ny_1-Ny_2\|_* \leq \left(\frac{1}{\tau}+\sum_{k=1}^{+\infty} c_k\right)\|y_1-y_2\|_*\,,$$

where

$$\|x\|_* = \sup_{t\geq 0} e^{-\tau P(t)}\,\|x(t)\|\,, \quad P(t)=\int_0^t p(s)ds,$$

by assumption we have $\sum_{k=1}^{+\infty} c_k < 1$, so there exists ε in $(0,1)$ such that

$$\varepsilon + \sum_{k=1}^{+\infty} c_k < 1,$$

if we take $\tau = \frac{1}{\varepsilon}$, we obtain that N is contractive, hence by the theorem 1.6.4 the problem (2.1) has an unique solution.

$\square$

Next, we present an existence and uniqueness result of the problem (2.1) in the following Fréchet space $PC = \cap_{m\in\mathbb{N}} PC_m$ such that

$$PC_m = PC([0,t_m],\mathbb{R}^n)$$

- $(PC_m, \|\cdot\|_m)$ is a Banach space endowed with the norm $\|\cdot\|_m$ such that

$$\|y\|_m = \sup_{t\in[0,t_m]} \|y(t)\|,$$

and

$$\begin{gathered} PC_1 \subset PC_2 \subset PC_3 \subset \cdots \\ \|\cdot\|_1 \le \|\cdot\|_2 \le \|\cdot\|_3 \le \cdots \end{gathered}$$

- PC is a Fréchet space for the family of semi-norms $\{\|\cdot\|_m\}_{m\in\mathbb{N}}$.

Now, we consider the following hypotheses:

$(\mathcal{A}_1)$ for all $R > 0$; there exists $\ell_R \in L^1_{loc}([0, +\infty), \mathbb{R}_+)$ such that

$$\begin{array}{lcl} \|f(t,y) - f(t,x)\| & \le & \ell_R(t)\,\|y - x\|\,;\ \text{almost all elements } x, y \in \mathbb{R}^n \\ \|y\| \le R, \|x\| \le R & ; & \forall t \in [0, +\infty) \end{array}$$

$(\mathcal{A}_2)$ there exist $c_k > 0; k \in \mathbb{N}$ such that

$$\|I_k(y) - I_k(x)\| \le c_k \|y - x\|\,,\ \text{for all } x, y \in \mathbb{R}^n, \quad \text{and all } t \in [0,\infty),$$

with

$$\sum_{k=1}^{+\infty} c_k < 1,$$

$(\mathcal{A}_3)$ there exist $p \in L^1_{loc}([0,\infty), \mathbb{R}^n)$ and $\psi \in C([0,\infty), (0,\infty))$ such that

$$\|f(t, y(t))\| \le p(t)\psi(\|y(t)\|);\ \text{for all } t \in [0,\infty),\ \text{and all } y \in C([0,\infty), \mathbb{R}^n),$$

with

$$\int_{\|a\|}^{\infty} \frac{du}{\psi(u)} = \infty.$$

Theorem 2.1.2. *[79] Assume $(\mathcal{A}_1) - (\mathcal{A}_3)$ are satisfied. Then the problem (2.1) has an unique solution on $[0, +\infty)$.*

Proof. Consider the operator $N : PC \longrightarrow PC$ defined by

$$(Ny)(t) = a + \int_0^t f(s, y(s))ds + \sum_{0<t_k<t} I_k(y(t_k)); \quad t \in [0, \infty).$$

We let $y \in PC$, $\lambda \in (0,1)$, such that $y = \lambda Ny$, then

$$y(t) = \lambda \left(a + \int_0^t f(s, y(s))ds + \sum_{0<t_k<t} I_k(y(t_k)) \right); \; t \in [0, t_m],$$

we can prove that there exists $M_m > 0$ such that

$$\|y\|_m \leq M_m.$$

Let

$$\mathcal{O} = \{y \in PC : \|y\|_m \leq M_m + 1\}.$$

• $\mathcal{O}$ is closed
• $N : PC_m \longrightarrow PC_m$ is contractive $\forall m \in \mathbb{N}$.
Let $y_1, y_2 \in PC_m$, we have

$$\|Ny_1 - Ny_2\|_{PC_m} \leq \left(\frac{1}{\tau} + \sum_{k=1}^{m} c_k \right) \|y_1 - y_2\|_{PC_m},$$

where

$$\|y\|_{PC_m} = \sup_{t \in [0, t_m]} e^{-\tau P(t)} \|y(t)\|, \quad P(t) = \int_0^t p(s)ds,$$

so, $N : \mathcal{O} \longrightarrow PC$ is contractive. Then by the alternative of Frigon and Granas (theorem 1.6.7), either,

1. there exists $\lambda \in (0,1) : y = \lambda Ny$, for all $y \in \partial\mathcal{O}$; or
2. there exists unique $y \in \mathcal{O}$ such that $y = Ny$.

Assume that $\exists \lambda \in (0,1) : y = \lambda Ny$, for all $y \in \partial\mathcal{O}$,
if $y \in \partial\mathcal{O} \Longrightarrow \|y\|_m = M_m + 1$, we also have

$$\begin{aligned} y = \lambda Ny \quad &\Longrightarrow \quad \|y\|_m \leq \|\lambda Ny\|_m \\ &\Longrightarrow \quad M_m + 1 \leq M_m, \quad \text{contradiction.} \end{aligned}$$

Then there exists unique $y \in \mathcal{O}$ such that $y = Ny$.

$\square$

2.2 Solution sets

In this section, we present an existence result, compactness and R_δ solution sets of the problem (2.1).

The following compactness criterion on unbounded domains is a simple extension of a compactness criterion in $PC_b(\mathbb{R}_+, \mathbb{R})$ (see [10, 26]).

Lemma 2.2.1. *Let $C \subset PC_b$. Then C is relatively compact if it satisfies the following conditions:*
(a) C is uniformly bounded in $PC_\ell(\mathbb{R}_+, \mathbb{R}^n)$.
(b) The functions belonging to C are almost equicontinuous on $\mathbb{R}_+$, i.e. equicontinuous on every compact interval of $\mathbb{R}^+$.
(c) The functions from C are equiconvergent, that is, given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $|x(\tau_1) - x(\tau_2)| < \varepsilon$ for any $\tau_1, \tau_2 \geq T(\varepsilon)$ and $x \in C$.

Theorem 2.2.1. *[79] Let $f\colon\ [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function. Assume that the following conditions*

($\mathcal{A}_4$) There exist $c_k, d_k > 0$ such that

$$\sum_{k=1}^{+\infty} c_k < 1 \ \ et \ \ \sum_{k=1}^{\infty} d_k < \infty,$$

with

$$\|I_k(x)\| \leq c_k\|x\| + d_k, \ \ \text{for all} \ \ x \in \mathbb{R}^n, \ \ k \in \mathbb{N}.$$

($\mathcal{A}_5$) There is a continuous non decreasing function $\psi : [0,\infty) \longrightarrow (0,\infty)$ and $p \in L^1([0,\infty), \mathbb{R}_+)$ such that

$$\|f(t,x)\| \leq p(t)\psi(\|x\|), \ \ \text{for almost all elements} \ \ t \in [0,\infty) \ \ \text{and all} \ \ x \in \mathbb{R}^n,$$

with

$$\int_0^{+\infty} m(s)ds < \int_c^{+\infty} \frac{du}{\psi(u)},$$

where

$$m(s) = \frac{p(s)}{1 - \displaystyle\sum_{k=1}^{+\infty} c_k} \ \ \text{and} \ \ c = \frac{\|a\| + \displaystyle\sum_{k=1}^{\infty} d_k}{1 - \displaystyle\sum_{k=1}^{+\infty} c_k}.$$

Then the problem (2.1) has at least one solution. Moreover the solution set is compact, R_δ, acyclic, and the solution operator $S : a \to S(a)$ is u.s.c.

Proof.

- Existence solutions:
  Consider the operator $N\colon\ PC_b([0,\infty),\mathbb{R}^n) \to PC_b([0,\infty),\mathbb{R}^n)$ defined by

$$Ny(t) = a + \int_0^t f(s, y(s))ds + \sum_{0<t_k<t} I_k(y(t_k)),\ t \in [0,\infty).$$

We show that N satisfies all the conditions of theorem 1.6.3 on PC_b.
Step 1 N is well defined
Let $y \in PC_b$ then, we have

$$Ny(t) = a + \int_0^t f(s, y(s))ds + \sum_{0<t_k<t} I_k(y(t_k)),\ t \in [0,\infty), \tag{2.2}$$

then

$$\begin{aligned} \|Ny(t)\| &\leq \|a\| + \int_0^t \|f(s, y(s))\|ds + \sum_{0<t_k<t} \|I_k(y(t_k))\| \\ &\leq \|a\| + \int_0^t p(s)\psi(\|y(s)\|)ds + \sum_{0<t_k<t} (c_k\|y(t_k)\| + d_k), \end{aligned}$$

so

$$\|Ny\|_{PC_b} \leq \|a\| + \psi(\|y\|_{PC_b}) \int_0^\infty p(s)ds + \sum_{k=1}^{\infty}(c_k\|y\|_{PC_b} + d_k) < \infty.$$

Step 2 N is continuous
Let $(y_n)_n$ a sequence in $PC_b([0,\infty),\mathbb{R}^n)$ such that $y_n \longrightarrow y$ as $n \to \infty$, it suffices to prove that $Ny_n \longrightarrow Ny$ as $n \to \infty$.
For all $t \in [0,\infty)$, we have

$$Ny_n(t) = a + \int_0^t f(s, y_n(s))ds + \sum_{0<t_k<t} I_k(y_n(t_k)),$$

then

$$\begin{aligned}\|Ny_n(t) - Ny(t)\| &\leq \int_0^t \|f(s, y_n(s)) - f(s, y(s))\|\, ds \\ &+ \sum_{k=1}^{m} \|I_k(y_n(t_k)) - I_k(y(t_k))\|\,,\end{aligned}$$

we have, $I_k, k = 1, \cdots, m$ are continuous functions, and f is L^1-Carathéodory function. The Lebesgue dominated convergence theorem implies that

$$\begin{aligned}\|Ny_n - Ny\|_{PC_b} &\leq \int_0^{+\infty} \|f(s, y_n(s)) - f(s, y(s))\|\, ds \\ &+ \sum_{k=1}^{+\infty} \|I_k(y_n(t_k)) - I_k(y(t_k))\| \longrightarrow 0 \quad \text{when} \quad n \longrightarrow \infty,\end{aligned}$$

so, N is continuous.
Step 3 N is compact
Let $r > 0$, $B_r := \{y \in PC : \|y\|_{PC_b} \leq r\}$, for to prove that $N(B_r)$ is relatively compact we use the lemma 2.2.1.

- $N(B_r)$ is uniformly bounded in PC_b.
  Let $y \in B_r$, then we have

$$Ny(t) = a + \int_0^t f(s, y(s)ds + \sum_{0<t_k<t} I_k(y(t_k)),\ t \in [0, +\infty), \tag{2.3}$$

then

$$\begin{aligned}\|Ny(t)\| &\leq \|a\| + \int_0^t \|f(s, y(s))\|ds + \sum_{0<t_k<t} \|I_k(y(t_k))\| \\ &\leq \|a\| + \int_0^t p(s)\psi(\|y(s)\|)ds + \sum_{0<t_k<t} (c_k\|y(t_k)\| + d_k),\end{aligned}$$

so

$$|Ny|_{PC_b} \leq \|a\| + \psi(r) \int_0^{\infty} p(s)ds + \sum_{k=1}^{\infty}(c_k r + d_k).$$

- $N(B_r)$ is equicontinuous in PC_b.
For each $\tau_1, \tau_2 \in [0,\infty)$, $\tau_1 < \tau_2$, and $y \in B_r$, we have

$$\begin{aligned}\|Ny(\tau_2) - Ny(\tau_1)\| &\leq \int_{\tau_1}^{\tau_2} \|f(s,y(s))\|ds + \sum_{\tau_1<t_k<\tau_2} \|I_k(y(t_k))\| \\ &\leq \psi(r)\int_{\tau_1}^{\tau_2} p(s)ds \\ &+ \sum_{\tau_1<t_k<\tau_2} (c_k r + d_k) \to 0 \text{ when } \tau_2 \to \tau_1.\end{aligned}$$

Then, we proved the equicontinuity in the case where $\tau_1 \neq t_i$ and $\tau_2 \neq t_i,\ i = 1, \ldots$.

If $\tau_1 = t_i^-$, let $\varepsilon_0 > 0$ such that $\{t_j : j \neq i\} \cap [t_i - \varepsilon_0, t_i + \varepsilon_0] = \emptyset$. Then for all $0 < \varepsilon < \varepsilon_0$, we have

$$\|Ny(t_i) - Ny(t_i - \varepsilon)\| \leq \int_{t_i-\varepsilon}^{t_i} \|f(s,y(s))\|ds \leq \psi(r)\int_{t_i-\varepsilon}^{t_i} p(s)ds.$$

The terms in the right-hand side tend to zero as $\varepsilon \to 0$.
In the same way we have also the equicontinuity if $t_2 = t_i^+$ $(i = 1, \ldots)$.

- $N(B_r)$ is equiconvergent at ∞
We show that for all $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that

$$\|Ny(t) - Ny(\infty)\| \leq \varepsilon \text{ for all } t \geq T_\varepsilon \text{ and all } y \in B_r,$$

let $y \in B_r$, then we have

$$\begin{aligned}\|Ny(t) - Ny(\infty)\| &\leq \int_t^{\infty} \|f(s,y(s))\|ds + \sum_{t\leq t_k<\infty} \|I_k(y(t_k))\| \\ &\leq \psi(r)\int_t^{\infty} p(s)ds + \sum_{t\leq t_k<\infty} (c_k r + d_k),\end{aligned}$$

as $\sum_{k=1}^{\infty} c_k < \infty$, $\sum_{k=1}^{\infty} d_k < \infty$ and $p \in L^1([0,\infty), \mathbb{R}_+)$, so there exist k_0 and $T_\varepsilon > 0$ such that

$$\sum_{k=k_0}^{\infty} (c_k r + d_k) \leq \frac{\varepsilon}{2},$$

and
$$\int_t^\infty p(s) < \frac{\varepsilon}{2\psi(r)}, \forall\, t \geq T_\varepsilon,$$
then
$$\|Ny(t) - Ny(\infty)\| \leq \varepsilon,\ \forall\, t \geq \max(k_0, T_\varepsilon).$$
Then $N(B_r)$ is equiconvergent. Hence by Lemma 2.2.1, the operator N is compact.

Step 4 A priori estimates.
Let $y \in PC_b$ such that $y = \lambda Ny$, et $0 < \lambda < 1$, then
$$y(t) = \lambda \left(a + \int_0^t f(s, y(s))ds + \sum_{0<t_k<t} I_k(y(t_k^-)) \right), \quad \text{for } \ t \in [0, \infty),$$
and
$$\|y(t)\| \leq \|a\| + \int_0^t p(s)\psi(\|y(s)\|)ds + \sum_{0<t_k<t} (c_k\|y(t_k)\| + d_k),$$
let $\alpha(t) = \sup\{\|y(s)\| : s \in [0, t]\}$, we get
$$\alpha(t) \leq \|a\| + \int_0^t p(s)\psi(\alpha(s))ds + \sum_{0<t_k<t} (c_k\alpha(t) + d_k),$$
then
$$\alpha(t) \leq \frac{1}{1 - \sum_{k=1}^\infty c_k} \left(\|a\| + \int_0^t p(s)\psi(\alpha(s))ds + \sum_{k=1}^\infty d_k \right).$$
Thus
$$\|y(t)\| \leq \alpha(t) \leq \beta(t),\ t \in [0, +\infty),$$
where
$$\beta(t) = \frac{1}{1 - \sum_{k=1}^\infty c_k} \left(\|a\| + \int_0^t p(s)\psi(\alpha(s))ds + \sum_{k=1}^\infty d_k \right),$$
hence
$$\beta(0) = \frac{\|a\| + \sum_{k=1}^\infty d_k}{1 - \sum_{k=1}^\infty c_k} \ \text{ and } \ \beta'(t) = \frac{p(t)\psi(\alpha(t))}{1 - \sum_{k=1}^\infty c_k} \leq \frac{p(t)\psi(\beta(t))}{1 - \sum_{k=1}^\infty c_k}.$$

By $(\mathcal{A}_5)$, we have for all $t \in [0,\infty)$

$$\int_{\beta(0)}^{\beta(t)} \frac{ds}{\psi(s)} \leq \frac{1}{1-\sum_{k=1}^{\infty} c_k} \int_0^{\infty} p(s)ds < \int_{\beta(0)}^{\infty} \frac{ds}{\psi(s)}.$$

then

$$\beta(t) \leq \Gamma^{-1}\left(\frac{\|p\|_{L^1}}{1-\sum_{k=1}^{\infty} c_k}\right), \text{ for all } t \in [0,\infty),$$

where $\Gamma(z) = \displaystyle\int_{\beta(0)}^{z} \frac{du}{\psi(u)}$.

Consequently

$$\|y\|_{PC_b} \leq \Gamma^{-1}\left(\frac{\|p\|_{L^1}}{1-\sum_{k=1}^{\infty} c_k}\right) := \widetilde{M},$$

consider the set

$$U := \{y \in PC_b : \|y\|_{PC_b} < \widetilde{M} + 1\}.$$

So, the operator $N : U \to PC_b$ is completely continuous. From theorem 1.6.3, we deduce that N has a fixed point which is a solution of problem (2.1).

- $S(a)$ is compact.

Let

$$S(a) = \{y \in PC_b : y \text{ solution of the problem (2.1) and } y(0) = a\},$$

as in step 3, we can prove that, there exists $\widetilde{M} > 0$ such that, for all $y \in S(a)$, we have

$$\|y\|_{PC_b} \leq \widetilde{M}.$$

Since N is completely continuous, then $N(S(a))$ is relatively compact in PC_b,
let $y \in S(a)$; then $y = N(y)$ so $S(a) \subset \overline{N(S(a))}$,
let $\{y_n : n \in N\} \subset S(a)$ such that $(y_n)_{n\in\mathbb{N}}$ converges to y. Then for all $n \in \mathbb{N}$, we have

$$y_n(t) = a + \int_0^t f(s, y_n(s))ds + \sum_{0<t_k<t} I_k(y_n(t_k)), \quad t \in [0,\infty).$$

Then

$$y_n(t) = (Ny_n)(t), \quad t \in [0, \infty),$$

by the continuity of N, we obtain

$$y_n(t) = N(y_n(t)) \to N(y(t)), \text{ as } n \to \infty,$$

then

$$y(t) = a + \int_0^t f(s, y(s))ds + \sum_{0<t_k<t} I_k(y(t_k)), \ t \in [0, \infty).$$

So, $y \in S(a)$, this implies that $S(a)$ is closed, hence, we deduce that $S(a)$ is compact in PC_b.

- The solution set is R_δ.
It is clear that, $FixN = S(a)$, and by the previous step 4, there exists $\widetilde{M} > 0$ such that for every $y \in S(a)$, we have

$$\|y\|_{PC_b} \le \widetilde{M},$$

let $\widetilde{f} : [0, \infty) \times R^n \to R^n$ be a map defined by

$$\widetilde{f}(t, x) = \begin{cases} f(t, x) & \text{if } \|x\| \le \widetilde{M} \\ f(t, \frac{\widetilde{M}x}{\|x\|}) & \text{if } \|x\| \ge \widetilde{M}, \end{cases}$$

and the function $\widetilde{I}_k : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\widetilde{I}_k(y(t)) = \begin{cases} I_k(x) & \text{if } \|x\| \le \widetilde{M} \\ I_k(\frac{\widetilde{M}x}{\|x\|}) & \text{if } \|x\| \ge \widetilde{M}, \end{cases}$$

f is L^1-Carathéodory, then $\widetilde{f}$ is also L^1-Carathéodory, and there exist $h \in L^1([0, \infty), \mathbb{R}_+)$ such that

$$\left\| \widetilde{f}(t, x) \right\| \le h(t); \quad \text{a.e. } t \in [0, \infty); \quad \text{and} \quad x \in \mathbb{R}^n, \tag{2.4}$$

consider the following modified problem

$$\dot{y}(t) = \widetilde{f}(t, y(t)), \quad t \in [0, \infty)/\{t_1, \cdots, t_m\}$$

$$y(t_k^+) - y(t_k^-) = \widetilde{I}_k(y(t_k^-)), k = 1, 2, \cdots$$
$$y(0) = a.$$

We can easily prove that $FixN = Fix\widetilde{N}$, where $\widetilde{N} : PC_b([0,\infty), \mathbb{R}^n) \longrightarrow PC_b([0,\infty), \mathbb{R}^n)$ defined by

$$\widetilde{N}(y)(t) = a + \int_0^t \widetilde{f}(s, y(s))ds + \sum_{0<t_k<t} \widetilde{I}_k(y(t_k)), \quad t \in [0,\infty). \qquad (2.5)$$

By inequality (2.4) and continuity of I_k, we have

$$\begin{aligned} \left\| \widetilde{N}(y) \right\|_{PC} &\leq \|a\| + \|h\|_{L^1} + \sum_{k=1}^{+\infty} \left(c_k \|y\|_{PC_b} + d_k \right) \\ &\leq \|a\| + \|h\|_{L^1} + \sum_{k=1}^{+\infty} \left(c_k \widetilde{M} + d_k \right) := r, \end{aligned}$$

then $\widetilde{N}$ is uniformly bounded.
We can easily prove that the function $\mathcal{M}$ defined by $\mathcal{M}(y) = y - \widetilde{N}(y)$ is a propre function. Also we have $\widetilde{N}$ is compact, so by the theorem of Lasota Yorke 1.5.1, we can easily prove that the conditions of Theorem 1.5.2 are satisfied, then the set $\mathcal{M}^{-1}(0) = Fix\widetilde{N} = S(a)$ is R_δ, and it is also acyclic and those by the lemma (1.5.1).

- The solution operator S is $u.s.c.$

  1. The graph of S is closed.
     First we show that S has a closed graph. Let G_S be a graph of S defined by

     $$G_S = \{(x, y) \in \mathbb{R}^n \times PC | y \in S(x)\}.$$

     Let $((x_q, y_q))_q$ be a sequence in G_S, and let $(x_q, y_q) \to (x, y)$ when $q \to \infty$.
     As $y_q \in S(x_q)$, then we have

     $$y_q(t) = x_q + \int_0^t f(s, y_q(s))ds + \sum_{0<t_k<t} I_k(y_q(t_k)), \; t \in [0,\infty),$$

let

$$Z(t) = x + \int_0^t f(s, y(s))ds + \sum_{0<t_k<t} I_k(y(t_k)), \quad t \in [0, \infty),$$

let $t \in [0, \infty)$, we have

$$\begin{aligned} \|y_q(t) - Z(t)\| &\leq \|x_n - x\| + \int_0^t \|f(s, y_q(s)) - f(s, y(s))\| \, ds \\ &+ \sum_{0<t_k<t} \|I_k(y_q(t)) - I_k(y(t))\| \\ &\leq \|x_n - x\| + \int_0^{+\infty} \|f(s, y_q(s)) - f(s, y(s))\| \, ds \\ &+ \sum_{k=1}^{+\infty} \|I_k(y_q(t)) - I_k(y(t))\| \,, \end{aligned}$$

by the dominated convergence theorem of Lebesgue, we have

$$\|y_q(t) - Z(t)\| \longrightarrow 0 \ \text{ when } \ q \longrightarrow \infty.$$

Hence, $\lim_{q \longrightarrow 0} y_q = y = Z \in S(x)$.

2. S transforms every bounded set in a relatively compact set
   Let $r > 0, B_r := \{y \in PC_b : \|y\| \leq r\}$.
   (a) $S(B_r)$ is uniformly bounded.
       Let $y \in S(B_r)$, then there exists $x \in B_r$ such that

$$y(t) = x + \int_0^t f(s, y(s))ds + \sum_{0<t_k<t} I_k(y(t_k)), \ t \in [0, \infty).$$

       As in step 4, we can prove that there exists $\widetilde{M} > 0$ such that

$$\|y\|_{PC_b} \leq \widetilde{M}.$$

   (b) $S(B_r)$ is equicontinuous.
       We let $\tau_1, \tau_2 \in [0, +\infty), \tau_1 < \tau_2$, and $y \in B_r$, then we have

$$\|Ny(\tau_2) - Ny(\tau_1)\| \ \leq \ \int_{\tau_1}^{\tau_2} \|f(s, y(s))\| ds + \sum_{\tau_1<t_k<\tau_2} \|I_k(y(t_k))\|$$

$$
\begin{aligned}
&\le \int_{\tau_1}^{\tau_2} p(s)\psi(\|y(s)\|)ds + \sum_{\tau_1<t_k<\tau_2} (c_k\|y(t_k)\| + d_k) \\
&\le \psi(\widetilde{M}) \int_{\tau_1}^{\tau_2} p(s)ds \\
&+ \sum_{\tau_1<t_k<\tau_2} (c_k\widetilde{M} + d_k) \to 0 \text{ when } \tau_1 \to \tau_2.
\end{aligned}
$$

(c) $S(B_r)$ is equiconvergent at ∞.
i.e. for all $\varepsilon > 0$, there exist $T_\varepsilon > 0$ such that $\|y(t)-y(\infty)\| \le \varepsilon$ for all $t \ge T_\varepsilon$ and all $y \in S(B)$.
We take $y \in S(B)$ then there exists $x \in B$, and we have

$$
\begin{aligned}
\|y(t) - y(\infty)\| &\le \int_t^\infty \|f(s, y(s))\|ds + \sum_{t\le t_k<\infty} \|I_k(y(t_k))\| \\
&\le \int_t^\infty p(s)\psi(\|y(s)\|)ds + \sum_{t\le t_k<\infty} (c_k\|y(t_k)\| + d_k) \\
&\le \psi(\widetilde{M}) \int_t^\infty p(s)ds \\
&+ \sum_{t\le t_k<\infty} (c_k\widetilde{M} + d_k) \to 0 \text{ when } t \to \infty.
\end{aligned}
$$

So, the set $\overline{S(B_r)}$ is compact, hence we obtain that the operator S is locally compact, and S has a closed graph, then S is $u.s.c.$

□

2.3 An Example

Consider the problem:

$$
\begin{cases}
y' = \dfrac{1}{100}(1+y)^{\frac{2}{3}}, \quad t \in J = [0, \infty),\ t \neq k; \\
\Delta y(k) = \dfrac{1}{8^k}|y(k)|, \quad k \in \mathbb{N}, \\
y(0) = a \in \mathbb{R}
\end{cases}
\tag{2.6}
$$

$$
f(t, x) = \frac{1}{100}(1+x)^{\frac{2}{3}},
$$

$$I_k(x) = \frac{1}{8^k}|x| \ k \in \mathbb{N}.$$

For every $x \in \mathbb{R}$, we have

$$|f(t,x)| \leq \frac{1}{100}\left(1 + \frac{2}{3}|x|\right),$$

and

$$\int_1^\infty \frac{1}{100(1+\frac{2}{3}u)}du = \infty.$$

Hence the condition $(\mathcal{A}_5)$ holds

Also for all $u, \bar{u}, v, \bar{v} \in \mathbb{R}^+$, we have

$$|I_k(u) - I_k(\bar{u})| \leq \frac{1}{8^k}|u - \bar{u}|, \ k = 1, 2, \cdots,$$

and

$$\sum_{k=1}^\infty \frac{1}{8^k} < 1.$$

Thus $(\mathcal{A}_4)$ holds.

Chapter 3

Impulsive functional differential inclusions on unbounded domain

Consider the problem

$$\begin{cases} \frac{d}{dt}\left[y(t)-g(t,y_t)\right] & \in & F(t,y_t), \quad a.e.\ t\in J/\{t_1,t_2,\ldots\}, \\ y(t_k^+)-y(t_k^-) & = & I_k(y(t_k^-)), \qquad k=1,2,\ldots, \\ y(t) & = & \phi(t),\ t\in[-r,0], \end{cases} \tag{3.1}$$

where $0 < r < \infty$, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} < \ldots$, $J := [0,\infty)$, $F : J \times D \to \mathcal{P}(\mathbb{R}^n)$ is a multifunction, $g : J \times D \to \mathbb{R}^n$ is a single-valued function, $\lim\limits_{t\to t_k^+} g(t,\psi) = \lim\limits_{t\to t_k^-} g(t,\psi) = g(t_k,\psi)$, $\psi \in D = C([-r,0],\mathbb{R}^n)$, $\phi \in D$.
The functions $I_k \in C(\mathbb{R}^n,\mathbb{R}^n)$ characterize the jump of the solutions at impulse points t_k $(k = 1,\ldots,m)$, and $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$. The notations $y(t_k^+) = \lim\limits_{h\to 0^+} y(t_k+h)$, and $y(t_k^-) = \lim\limits_{h\to 0^+} y(t_k-h)$ stand for the right and the left limits of the function y at $t = t_k$, respectively.
For any function y defined on $[-r,\infty)$ and any $t \in J$, y_t refers to the element of D such that

$$y_t(\theta) = y(t+\theta), \quad \theta \in [-r,0].$$

In this chapter, we present some results of existence of solutions as well as the topological structure of solution sets for the problem (3.1).

3.1 Existence results

Let

$$\Omega = \{y : [-r, \infty) \to \mathbb{R}^n : y \in PC(\mathbb{R}^+, \mathbb{R}^n) \cap D\}$$

Ω is a Fréchet space, with the family of semi-norms

$$\|y\|_{\Omega_k} = \sup_{t\in[-r,t_k]} \|y(t)\|,\ k = 1, 2, \cdots$$

Then

$$\|\cdot\|_{\Omega_1} \le \|\cdot\|_{\Omega_2} \le \|\cdot\|_{\Omega_3} \le \cdots \le \|\cdot\|_{\Omega_k} \le \cdots$$

$(\|.\|_{\Omega_k})_{k\in\mathbb{N}}$ is a semi-norms sequence, and $\Omega = \bigcap_{k=1}^{\infty} \Omega_k$, such that

$$\Omega_k = \{y : [-r, +\infty) \to \mathbb{R}^n, y \in PC_k \cap D\},$$

is a Banach space with the norm

$$\|y\|_{\Omega_k} = \sup_{t\in[-r,t_k]} \|y(t)\|.$$

Lemma 3.1.1. *[43] A function $y \in \Omega$ is said to be a solution of problem (3.1) if there exists $f \in L^1(J, \mathbb{R}^n)$ such that $f(t) \in F(t, y_t)$ a.e. on J, and*

$$y(t) = \begin{cases} \phi(t),\ \ if\ \ t \in [-r, 0], \\ \phi(0) + g(t, y_t) - g(0, \phi) - \displaystyle\sum_{0<t_k<t} \Delta_k(g(t_k^-, y_{t_k^-})) \\ + \displaystyle\int_0^t f(s)ds + \sum_{0<t_k<t} I_k(y(t_k^-)),\ \ if\ \ t \in [0, \infty), \end{cases} \tag{3.2}$$

where $\Delta_k(g(t_k^-, y_{t_k^-})) = g(t_k^+, y_{t_k^+}) - g(t_k, y_{t_k})$.

3.1.1 The upper semi-continuous case

In this subsection, we present a global existence result and prove the compactness of solution set for the problem (3.1) by using a nonlinear alternative for multivalued maps combined with a compactness argument. The nonlinearity is u.s.c. with respect to the second variable and satisfies a Nagumo growth condition. We will consider the following assumptions.

$(\mathcal{H}_1)$ There exist $c_k, d_k > 0$ such that

$$\|I_k(x)\| \leq c_k\|x\| + d_k, \text{ for every } x \in \mathbb{R}^n, k = 1, 2, \ldots$$

With

$$\sum_{k=1}^{\infty} c_k < 1 \text{ and } \sum_{k=1}^{\infty} d_k < \infty.$$

The carathéodory multivalued map $F : J \times \mathbb{R}^n \longrightarrow \mathcal{P}_{cp,cv}(\mathbb{R}^n)$ satisfies:

$(\mathcal{H}_2)$ For every bounded set $B \in \Omega$, the set $\{t : t \mapsto g(t, y_t), \ y \in B\}$ is equicontinuous and equiconvergent in Ω, g is continuous and there exists a constant $\ell_1 \in [0, 1)$ and $\ell_2 > 0$ such that

$$\|g(t, x)\| \leq \ell_1\|x\|_D + \ell_2 \text{ for all } x \in D.$$

$(\mathcal{H}_3)$ There exists a continuous non decreasing function $\psi : [0, \infty) \to (0, \infty)$ and $p \in L^1(J, \mathbb{R}^+)$ suth that

$$\|F(t, x)\|_{\mathcal{P}} \leq p(t)\psi(\|x\|_D) \ for \ a.e. \ t \in J \ and \ each \ x \in D,$$

with

$$\int_0^{\infty} m(s)ds < \int_c^{\infty} \frac{du}{\psi(u)}$$

where

$$m(s) = \frac{p(s)}{1 - \ell_1 - \sum_{k=1}^{\infty} c_k} \text{ and } c = \frac{2\|\phi\|_D + 2\ell_2 + \sum_{k=1}^{\infty} d_k}{1 - \ell_1 - \sum_{k=1}^{\infty} c_k}.$$

Theorem 3.1.1. *Assume that the conditions* $(\mathcal{H}_1) - (\mathcal{H}_3)$ *hold. Then the problem (3.1) has at least one solution. Moreover, the solution set*

$$S_F(\phi) = \{y \in \Omega : y \text{ is solution of } (3.1)\}, \phi \in D,$$

is compact and the multivalued map $S_F : \phi \longrightarrow S_F(\phi)$ *is u.s.c.*

In the next proof we will use the nonlinear alternative of Leray-Schauder for multi-valued maps (see theorem1.6.8, [50]).

Proof.

Step 1. Existence of solutions.
Consider the operator $N : \Omega \to \mathcal{P}(\Omega)$ defined for $y \in \Omega$ by

$$N(y) = \left\{ h \in \Omega : \ h(t) = \left\{ \begin{array}{ll} \phi(0) + g(t, y_t) - g(0, \phi) & \\ + \int_0^t f(s)ds & \\ + \sum_{0<t_k<t} I_k(y(t_k^-)), \text{ a.e. } t \in J & \\ & \\ \phi(t), & t \in [-r, 0] \end{array} \right. \right\} \tag{3.3}$$

where $f \in S_{F,y} = \{v \in L^1(J, \mathbb{R}^n) : v(t) \in F(t, y_t), \text{ a.e.} t \in J\}$, the set $S_{F,y}$ is nonempty if and only if the mapping $t \to \inf\{\|v\| : v \in F(t, y(t))\}$ belong to L^1. It is further bounded if and only if the mapping $t \to \|F(t, y_t)\|_{\mathcal{P}} = \sup\{\|v\| : v \in F(t, y_t)\}$ belong to L^1, this particularly holds true when F satisfies $(\mathcal{H}_3)$. Moreover, fixed points of the operator N are mild solutions of problem (3.1). We shall show that N satisfies the assumptions of theorem1.6.8. First notice that since $S_{F,y}$ is convex (because F has convex values), then N takes convex values.

- **Claim 1**. $N(PC_b \cap D) \subset PC_\ell \cap D$. Indeed, let $y \in PC_b \cap D$ and $h \in N(y)$ then there exists $f \in S_{F,y}$ such that

$$h(t) = \left\{ \begin{array}{ll} \phi(0) + g(t, y_t) - g(0, \phi) + \int_0^t f(s)ds & \\ + \displaystyle\sum_{0<t_k<t} I_k(y(t_k^-)), & \text{a.e. } t \in J \\ & \\ \phi(t), & t \in [-r, 0] \end{array} \right.$$

by $(\mathcal{H}_1) - (\mathcal{H}_3)$, we get

$$\begin{aligned} \|h(t)\| \ \leq \ & \|\phi\| + \|g(t, y_t)\| + \|g(0, \phi)\| \\ & + \int_0^t \|f(s)\|ds + \sum_{k=1}^{\infty} \|I_k(y(t_k))\|, \end{aligned}$$

hence

$$\begin{aligned} \|h\|_{PC_b} \ \leq \ & \|\phi\| + \ell_1\|y\|_{PC_b} + \ell_2 + \|g(0, \phi)\| \\ & + \psi(\|y\|_{PC_b}) \int_0^{\infty} p(s)ds + \|y\|_{PC_b} + \sum_{k=1}^{\infty} d_k. \end{aligned}$$

This shows that N transforms bounded sets in $PC_b \cap D$ into bounded sets of $PC_\ell \cap D$.

- **Claim 2**. N transforms bounded sets in $PC_b \cap D$ into almost equicontinuous sets of $PC_\ell \cap D$. Let $q > 0$, $B_q := \{y \in PC_b : \|y\|_\infty \leq q\}$, $B_q \cap D$ be a bounded set in $PC_b \cap D$, $\tau_1, \tau_2 \in J, \tau_1 < \tau_2$, and $y \in B_q \cap D$. For each $h \in N(y)$, we have

$$\begin{aligned}\|h(\tau_2) - h(\tau_1)\| &\leq \|g(\tau_1, y_{\tau_1}) - g(\tau_2, y_{\tau_2})\| + \int_{\tau_1}^{\tau_2} \|f(s)\| ds \\ &\quad + \sum_{\tau_1 < t_k < \tau_2} \|I_k(y(t_k))\| \\ &\leq \|g(\tau_1, y_{\tau_1}) - g(\tau_2, y_{\tau_2})\| + \psi(q) \int_{\tau_1}^{\tau_2} p(s) ds \\ &\quad + \sum_{\tau_1 < t_k < \tau_2} (c_k q + d_k).\end{aligned}$$

Since $\sum_{k=1}^{\infty} c_k < \infty$, $\sum_{k=1}^{\infty} d_k < \infty$ and $p \in L^1(J, \mathbb{R}^+)$, and $(\mathcal{H}_2)$, the right-hand term tends to zero as $|\tau_2 - \tau_1| \to 0$, proving equicontinuity for the case where $t \neq t_i, i = 1, 2, \cdots$ To prove equicontinuity at $t = t_i$ for some $i \in \mathbb{N}^*$, we fix $\varepsilon_0 > 0$ such that $\{t_j : j \neq i\} \cap [t_i - \varepsilon_0, t_i + \varepsilon_0] = \emptyset$. Then for each $0 < \varepsilon < \varepsilon_0$, we have the estimates

$$\begin{aligned}\|h(t_i) - h(t_i - \varepsilon)\| &\leq \|g(t_i, y_{t_i}) - g(t_i - \varepsilon, y_{t_i - \varepsilon})\| + \int_{t_i - \varepsilon}^{t_i} \|f(s)\| ds \\ &\leq \|g(t_i, y_{t_i}) - g(t_i - \varepsilon, y_{t_i - \varepsilon})\| + \psi(q) \int_{t_i - \varepsilon}^{t_i} p(s) ds.\end{aligned}$$

Since $p \in L^1(J, \mathbb{R}^+)$, and $(\mathcal{H}_2)$, the right-hand term tends to 0 as $\varepsilon \to 0$. The equicontinuity at $t_i^+ (i = 1, \cdots)$ is proved in the same way.

- **Claim 3**. Let $\overline{B}(0, q)$ be the closed ball centered at the origin with radius $q > 0$. We show that the set $N(\overline{B}(0, q) \cap D)$ is equiconvergent at ∞, i.e. for every $\varepsilon > 0$, there exists $T(\varepsilon) > 0$ such that $\|h(t) - h(\infty)\| \leq \varepsilon$ for every $t \geq T$ and each $h \in N(\overline{B}(0, q) \cap D)$. If $h \in N(y)$ for some $y \in \overline{B}(0, q) \cap D$, then there exists $f \in S_{F,y}$ such that h satisfies (3.11). Then

$$\|h(t) - h(\infty)\| \leq \|g(t, y_t) - g(\infty, y_\infty)\| + \int_t^\infty \|f(s)\| ds$$

$$
\begin{aligned}
& + \sum_{t<t_k<\infty} \|I_k(y(t_k))\| \\
\leq \;& \|g(t,y_t) - g(\infty, y_\infty)\| + \psi(q)\int_t^\infty p(s)ds \\
& + \sum_{t<t_k<\infty} (c_k q + d_k).
\end{aligned}
$$

Since $\sum_{k=1}^{\infty} c_k < \infty$, $\sum_{k=1}^{\infty} d_k < \infty$ and $p \in L^1(J, \mathbb{R}^+)$, and $(\mathcal{H}_2)$, then there exists k_0 and $T_1(\varepsilon) > 0$ and $T_2(\varepsilon) > 0$ such that

$$\sum_{k=k_0}^{\infty} (c_k q + d_k) \leq \frac{\varepsilon}{3},$$

and

$$\int_t^\infty p(s)ds < \frac{\varepsilon}{3\psi(q)}, \; \forall t \geq T_1(\varepsilon)$$

$$\|g(t,y_t) - g(\infty, y_\infty)\| \leq \frac{\varepsilon}{3}, \; \forall t \geq T_2(\varepsilon),$$

hence

$$\|h(t) - h(\infty)\| \leq \varepsilon, \; \forall t \geq \max(k_0, T_1(\varepsilon), T_2(\varepsilon)).$$

Then $N(\overline{B}(0,q) \cap D)$ is equiconvergent. With Lemma 2.2.1 and Claims $1-3$, we conclude that N is completely continuous.

- **Claim 4**. N is *u.s.c.*
  To this end, it is sufficient to show that N has a closed graph. Let $h_n \in N(y_n)$ be such that $h_n \to h$ and $y_n \to y$, as $n \to +\infty$. Then there exists $M > 0$ such that $\|y_n\|_{PC_b} \leq M$. We shall prove that $h \in N(y)$. $h_n \in N(y_n)$ means that there exists $f_n \in S_{F,y_n}$ such that, for a.e. $t \in J$, we have

$$h_n(t) = \phi(0) + g(t, (y_n)_t) - g(0,\phi) + \int_0^t f_n(s)ds + \sum_{0<t_k<t} I_k(y_n(t_k)).$$

$(\mathcal{H}_3)$ implies that $f_n(t) \in p(t)\psi(M)B(0,1)$. Then $(f_n)_{n\in\mathbb{N}}$ is integrably bounded in $L^1(J, \mathbb{R}^n)$. Since F has compact values, we deduce that $(f_n)_{n\in\mathbb{N}}$, there exists a subsequence, still denoted $(f_n)_{n\in\mathbb{N}}$, which converges weakly to some limit $f \in L^1(J, \mathbb{R}^n)$. Moreover for a.e. $t \in J$,

$y_n(t)$ converges to $y(t)$ and the functions g and I_k are continuous for $k = 1, \cdots$ Then we have

$$h(t) = \phi(0) + g(t, y_t) - g(0, \phi) + \int_0^t f(s)ds + \sum_{0<t_k<t} I_k(y(t_k)).$$

It remains to prove that $f \in F(t, y(t))$, a.e. $t \in J$. By Mazur's lemma there exists $\alpha_i^n \geq 0, i = 1, \cdots, k(n)$ such that $\sum_{i=1}^{k(n)} \alpha_i^n = 1$ and the sequence of convex combinations $v_n(.) = \sum_{i=1}^{k(n)} \alpha_i^n f_i(.)$ converges strongly to f in L^1. Hence

$$\begin{array}{rcl} f(t) & \in & \bigcap_{n\geq 1} \overline{\{f_k(t) : k \geq n\}}, \ a.e. t \in J \\ & \subset & \bigcap_{n\geq 1} \overline{co}\{f_k(t), k \geq n\} \\ & \subset & \bigcap_{n\geq 1} \overline{co}\left\{\bigcup_{k\geq n} F(t, y_k(t))\right\} \end{array}$$

However, the fact that the multivalued $x \multimap F(., x)$ is $u.s.c.$ and has compact convex values, we obtain

$$f(t) \in \overline{co} F(t, y(t)) = F(t, y(t)).$$

Therefore $h \in N(y)$, proving that N has a closed graph. Then, N is $u.s.c.$

- **Claim 5**. A priori bounds on solutions. Let $y \in PC \cap D$ be such that $y \in \lambda N(y)$ and $\lambda \in (0, 1)$. Then there exists $f \in S_{F,y}$ suth that

$$y(t) = \lambda \begin{cases} \phi(0) + g(t, y_t) - g(0, \phi) \\ + \int_0^t f(s)ds \\ + \sum_{0<t_k<t} I_k(y(t_k^-)), \ \text{a.e. } t \in J \\ \\ \phi(t), & t \in [-r, 0] \end{cases} \tag{3.4}$$

Arguing as in Claim 1, we get the estimates

$$\begin{array}{rcl} \|y(t)\|_{PC_k} & \leq & \|\phi(0)\| + \|g(t, y_t) - g(0, \phi)\| + \displaystyle\int_0^t p(s)\psi(\|y(s)\|)ds + \\ & & \displaystyle\sum_{0<t_k<t} (c_k\|y(t_k)\| + d_k), \ a.e. \, t \in J. \end{array}$$

Letting $\alpha(t) = \sup\{\|y(s)\| : s \in [-r, t]\}$ and using the non decreasing character of ψ, we obtain that

$$\begin{aligned}
\alpha(t) &\leq \|\phi(0)\| + \|g(t, y_t) - g(0, \phi)\| + \int_0^t p(s)\psi(\alpha(s))ds \\
&\quad + \sum_{0<t_k<t} (c_k\alpha(t) + d_k) \\
&\leq \|\phi\|_D + \ell_1\alpha(t) + \ell_2 + \ell_1\|\phi\|_D + \ell_2 + \int_0^t p(s)\psi(\alpha(s))ds \\
&\quad + \alpha(t) \sum_{0<t_k<t} c_k + \sum_{0<t_k<t} d_k.
\end{aligned}$$

Hence

$$\alpha(t) \leq \frac{1}{1 - \ell_1 - \sum_{k=1}^{\infty} c_k} \left(2\|\phi\|_D + 2\ell_2 + \int_0^t p(s)\psi(\alpha(s))ds + \sum_{k=1}^{\infty} d_k \right).$$

Denoting the right-hand side by $\beta(t)$, we have

$$\|y(t)\| \leq \alpha(t) \leq \beta(t),\ t \in [-r, \infty),$$

as well as

$$\beta(0) = \frac{2\|\phi\|_D + 2\ell_2 + \sum_{k=1}^{\infty} d_k}{1 - \ell_1 - \sum_{k=1}^{\infty} c_k},$$

and

$$\grave{\beta}(t) = \frac{p(t)\psi(\alpha(t))}{1 - \ell_1 - \sum_{k=1}^{\infty} c_k} \leq \frac{p(t)\psi(\beta(t))}{1 - \ell_1 - \sum_{k=1}^{\infty} c_k}$$

From $(\mathcal{H}_3)$, this implies that for $t \in J$

$$\Gamma(\beta(t)) = \int_{\beta(0)}^{\beta(t)} \frac{ds}{\psi(s)} \leq \frac{1}{1 - \ell_1 - \sum_{k=1}^{\infty} c_k} \int_0^{\infty} p(s)ds.$$

Thus

$$\beta(t) \leq \Gamma^{-1} \left(\frac{\|p\|_{L^1}}{1 - \ell_1 - \sum_{k=1}^{\infty} c_k} \right),\ \forall t \in J,$$

where $\Gamma(z) = \int_{\beta(0)}^{z} \frac{du}{\psi(u)}$. As a consequence,

$$\|y\|_{\Omega_k} \leq \|y\|_{PC_b} \leq \Gamma^{-1} \left(\frac{\|p\|_{L^1}}{1 - \ell_1 - \sum_{k=1}^{\infty} c_k} \right) := \widetilde{M}.$$

So, the set

$$M = \{y \in \Omega : y \in \lambda N(y), \lambda \in (0,1)\} \tag{3.5}$$

is bounded, as a consequence of the multivalued version of the nonlinear alternative of Leray–Schauder (theorem1.6.8), N has a fixed point y in U which is a solution of problem (3.1).

Step 2. Compactness of the solution set. For each $\phi \in D$, let

$$S_F(\phi) = \{y \in \Omega : y \text{ is a solution of problem (3.1)}\}.$$

From Step 1, there exists $\widetilde{M}$ such that for every $y \in S_F(\phi), \|y\|_{\Omega_k} \leq \widetilde{M}$. Since N is completely continuous, $N(S_F(\phi))$ is relatively compact in Ω. Let $y \in S_F(\phi)$; then $y \in N(y)$ and hence $S_F(\phi) \subset N(S_F(\phi))$. It remains to prove that $S_F(\phi)$ is a closed subset in Ω. Let $\{y_n : n \in \mathbb{N}\} \subset S_F(\phi)$ be such that the sequence $(y_n)_{n\in\mathbb{N}}$ converges to y. For every $n \in \mathbb{N}$, there exists f_n such that $f_n(t) \in F(t, y_t), a.e.t \in J$, and

$$y_n(t) = \phi(0) + g(t, y_t) - g(0, \phi) + \int_0^t f_n(s)ds + \sum_{0<t_k<t} I_k(y_n(t_k)). \tag{3.6}$$

Arguing as in Claim 4, we can prove that there exists f such that $f(t) \in F(t, y_t)$ and

$$y(t) = \phi(0) + g(t, y_t) - g(0, \phi) + \int_0^t f(s)ds + \sum_{0<t_k<t} I_k(y(t_k)). \tag{3.7}$$

Therefore $y \in S_F(\phi)$ which yields that $S_F(\phi)$ is closed, and hence compact in Ω.

Step 3. $S_F(.)$ is $u.s.c.$ For this, we prove that the graph of S_F

$$\Gamma_{S_F} := \{(\phi, y) : y \in S_F(\phi)\}$$

is closed. Let $(\phi_n, y_n) \in \Gamma_{S_F}$ be such that $(\phi_n, y_n) \to (\phi, y)$ as $n \to \infty$. Since $y_n \in S_F(\phi_n)$, there exists $f_n \in L^1(J, \mathbb{R}^n)$ such that

$$y_n(t) = \phi_n(0) + g(t, y_t) - g(0, \phi) + \int_0^t f_n(s)ds + \sum_{0<t_k<t} I_k(y_n(t_k)), \ t \in J. \tag{3.8}$$

Arguing as in Claim 4, we can prove that there exists $f \in S_{F,y}$ such that y satisfies (3.10). Thus, $y \in S_F(\phi)$. Now, we show that S_F maps bounded sets into relatively compact sets of Ω. Let B be a bounded sequence, there exists a subsequence of $\phi_n \subset B$ and $f_n \in S_{F,y_n}$, $n \in \mathbb{N}$ such that (3.11) is satisfied. Since ϕ_n is a bounded sequence, there exists a subsequence of ϕ_n converging to ϕ. As in Claims 2 and 3, we can show that $y_n : n \in \mathbb{N}$ is equicontinuous on every compact of J and is equiconvergent at ∞. As a consequence of Lemma 2.2.1 , we conclude that there exists a subsequence of y_n converging to y in Ω. By an argument similar to Claim 4, we can prove that y satisfies (3.10) for some $f \in S_{F,y}$. Thus $y \in \overline{S_F(B)}$. This implies that $S_F(.)$ is u.s.c., ending the proof of theorem 3.1.1.

□

Remark 3.1.1. *If we consider the function g defined by*

$$\begin{aligned} g : J \times D &\to \mathbb{R}^n \\ (t, \psi) &\to g(t, \psi) = \frac{1}{t+1}\psi(-r). \end{aligned}$$

The assumption $(\mathcal{H}_2)$ is satisfied.

3.1.2 The Lipschitz case

In this subsection, we prove the existence of solutions under Hausdorff-Lipschitz conditions.

Now present our second existence result for problem (3.1). Here and hereafter $J_k = [0, t_k]/\{t_j, 0 < j < k\}$.

Theorem 3.1.2. *Suppose the multivalued map $F : J \times \mathbb{R}^n \to \mathcal{P}_{cp}(\mathbb{R}^n)$ is such that $t \multimap F(t, .)$ is measurable and*

$(\mathcal{H}_4)$ for each $k = 1, 2, \cdots$, there exist $l_k \in L^1([0, t_k], \mathbb{R}^+)$ such that

$$H_{d_k}(F(t, x), F(t, y)) \leq l_k(t)\|x - y\|, \text{ for all } x, y \in \mathbb{R}^n \text{ and a.e. } t \in J_k$$

and

$$F(t, 0) \subset l_k(t)\overline{B}(0, 1), \text{ for a.e. } t \in J_k.$$

($\mathcal{H}_5$) $\sum_{k=1}^{\infty}\|I_k(0)\|<\infty$ *and there exist constants* $c_k \neq 0$ *such that*

$\sum_{k=1}^{\infty} c_k < 1$ *and*

$$\|I_k(x)-I_k(y)\| \leq c_k\|x-y\|, \text{ for each } x,y\in\mathbb{R}^n.$$

($\mathcal{H}_6$) There exists constant $m\in(0,1)$ *such that*

$$\|g(t,y)-g(t,x)\| \leq m\|y-x\|_D, \text{ for all } t\in J \text{and all } x,y\in D.$$

The problem (3.1) has at least one mild solution.

Remark 3.1.2. *(a) Note that* $(\mathcal{H}_5)$ *implies* $(\mathcal{H}_1)$ *with* $d_k=\|I_k(0)\|$.

(b) $(\mathcal{H}_4)$ *implies that the nonlinearity F has at most linear growth:*

$$\|F(t,x)\|_{\mathcal{P}} \leq l_k(t)(1+\|x\|),\ l_k\in L^1(J_k,\mathbb{R}^+), a.e.t\in J_k, x\in D,$$

and thus $(\mathcal{H}_3)$ *is satisfied locally. However, F is not Carathéodory and may take nonconvex values.*

Proof of theorem 3.1.2. We begin by defining a family of semi-norms on Ω, thus rendering Ω a Fréchet space. Let τ be a sufficiently large real parameter, say

$$m+\frac{1}{\tau}+\sum_{k=1}^{\infty} c_k<1.$$

For each $n\in\mathbb{N}$, define in Ω the family of weighted semi-norms

$$\|y\|_n=\sup\{e^{-\tau L_n(t)}\|y(t)\| : 0\leq t\leq t_n\},$$

where

$$L_n(t)=\int_0^t l_n(s)ds.$$

Thus $\Omega=\cap_{n\geq 1}\Omega_n$ where $\Omega_n=\{y:[-r,+\infty)\to\mathbb{R}^n, y\in PC_n\cap D\}$, where $PC_k=PC([0,t_k],\mathbb{R}^n)$. Then Ω is a Fréchet space with the family of semi-norms $\{|.|_n\}$. In order to transform problem (3.1) into a fixed point problem, define the operator $N:\Omega\to\mathcal{P}(\Omega)$ by (3.3). Since the fixed points of the operator N are solutions of problem (3.1), we first show that $N:\overline{U}\to\mathcal{P}_{cl}(\Omega)$ is an admissible multivalued contraction, where $U\subset\Omega$ is some open subset

to be defined later on.
Step1. We proof that there exists $\gamma < 1$ such that

$$H_{d_n}(N(y), N(\overline{y})) \leq \gamma \|y - \overline{y}\|_n, \text{ for each } y, \overline{y} \in \Omega_n.$$

Let $y, \overline{y} \in \Omega_n$ and $h \in N(y)$. Then there exists $f \in S_{F,y}$ such that

$$h(t) = \phi(0) + g(t, y_t) - g(0, \phi) + \int_0^t f(s)ds + \sum_{0<t_k<t} I_k(y(t_k)),\ a.e.t \in J_n.$$

$(\mathcal{H}_4)$ implies that

$$H_{d_n}(F(t, y(t)), F(t, \overline{y}(t))) \leq l_n(t)\|y(t) - \overline{y}(t)\|,\ a.e.t \in J_n.$$

Hence, there is some $w_0 \in F(t, \overline{y}(t))$ such that

$$\|f(t) - w_0\| \leq l_n(t)\|y(t) - \overline{y}(t)\|,\ t \in J_n.$$

Consider the multivalued map $U_n : J_n \to \mathcal{P}(\mathbb{R}^n)$ defined by

$$U_n(t) = \{w \in F(t, \overline{y}(t)) : \|f(t) - w\| \leq l_n(t)\|y(t) - \overline{y}(t)\|,\ a.e.t \in J_n\}.$$

Then U_n is a nonempty set because it contains w_0 and Theorem III.4.1 in [25] tells us that U_n is measurable. Moreover, the multivalued intersection operator $V_n(.) = U_n(.) \cap F(., \overline{y}(.))$ is also measurable. Therefore, by theorem 1.3.2, there exists a function $t \to \overline{f}_n(t)$ which is a measurable selection for V_n, that is $\overline{f}_n(t) \in F(t, \overline{y}(t))$ and

$$\|f(t) - \overline{f}_n(t)\| \leq l_n(t)\|y(t) - \overline{y}(t)\|,\ a.e.t \in J_n.$$

Define $\overline{h}$ by

$$\overline{h}(t) = \phi(0) + g(t, \overline{y}_t) - g(0, \phi) + \int_0^t \overline{f}(s)ds + \sum_{0<t_k<t} I_k(\overline{y}(t_k)),\ a.e.t \in J_n.$$

Then, for a.e. $t \in J_n$, we have

$$\begin{aligned}\|h(t) - \overline{h}(t)\| \;\leq\; & \|g(t, y_t) - g(t, \overline{y}_t)\| + \int_0^t \|f(s) - \overline{f}(s)\|ds + \\ & \sum_{0<t_k<t} \|I_k(y(t_k)) - I_k(\overline{y}(t_k))\|\end{aligned}$$

$$\begin{aligned}
&\leq m\|y_t-\overline{y}_t\|+\int_0^t l_n(s)\|y(s)-\overline{y}(s)\|ds+\\
&\quad\sum_{0<t_k<t} c_k\|y(t_k)-\overline{y}(t_k)\|\\
&\leq m\|y_t-\overline{y}_t\|+\int_0^t l_n(s)e^{\tau L_n(s)}e^{-\tau L_n(s)}\|y(s)-\overline{y}(s)\|ds+\\
&\quad\sum_{0<t_k<t} c_k e^{\tau L_n(t)}e^{-\tau L_n(t)}\|y(t_k)-\overline{y}(t_k)\|\\
&\leq m\|y_t-\overline{y}_t\|+\int_0^t l_n(s)e^{\tau L_n(s)}ds\|y-\overline{y}\|_n+\\
&\quad\sum_{0<t_k<t} c_k e^{\tau L_n(t)}\|y-\overline{y}\|_n\\
&\leq m e^{\tau L_n(t)}\|y-\overline{y}\|_n+\int_0^t \frac{1}{\tau}(e^{\tau L_n(s)})' ds\|y-\overline{y}\|_n+\\
&\quad\sum_{k=1}^{n} c_k e^{\tau L_n(t)}\|y-\overline{y}\|_n\\
&\leq e^{\tau L_n(t)}\left(m+\frac{1}{\tau}+\sum_{k=1}^{n} c_k\right)\|y-\overline{y}\|_n.
\end{aligned}$$

It follows that

$$e^{-\tau L_n(t)}\|h(t)-\overline{h}(t)\|\leq\left(m+\frac{1}{\tau}+\sum_{k=1}^{n} c_k\right)\|y-\overline{y}\|_n.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, we finally arrive at

$$H_{d_n}(N(y),N(\overline{y}))\leq\left(m+\frac{1}{\tau}+\sum_{k=1}^{n} c_k\right)\|y-\overline{y}\|_n.$$

Moreover, since F has compact valued, we can prove that N has compact values too. Let $x\in\overline{U}$ and $\varepsilon>0$. If $x\notin N(x)$, then $d_n(x,N(x))\neq 0$. Since $N(x)$ is compact, then there exists $y\in N(x)$ such that $d_n(x,N(x))=\|x-y\|_n$ and we have

$$\|x-y\|_n\leq d_n(x,N(x))+\varepsilon.$$

In the case where $x \in N(x)$, we may take $y = x$. Therefore N is an admissible contractive operator.
Step2. A priori estimates. Given $t \in J_n$, let $y \in \lambda N(y)$ for some $\lambda \in (0,1]$. Then there exists $f \in S_{F,y}$ such that (3.4) is satisfied. Then we have

$$\begin{aligned}
\|y(t)\| &\leq \|\phi(0)\| + \|g(t,y_t) - g(0,\phi)\| + \int_0^t \|f(s)\|ds + \sum_{0<t_k<t} \|I_k(y(t_k))\|. \\
&\leq \|\phi(0)\| + m\|y_t - \phi\| + \int_0^t l_n(s)(1+\|y(s)\|)ds + \\
&\quad \sum_{k=1}^n c_k \|y(t_k)\| + \sum_{k=1}^n \|I_k(0)\|.
\end{aligned}$$

Consider the function μ defined on J_n by

$$\mu(t) = \sup\{\|y(s)\| : 0 \leq s \leq t\}.$$

By the previous inequality, we have for $t \in J_n$

$$\mu(t) \leq \frac{1}{1 - m - \sum_{k=1}^n c_k}\left((1+m)\|\phi\| + \sum_{k=1}^n \|I_k(0)\| + \int_0^t l_n(s)(1+\mu(s))ds\right).$$

Let us take the right-hand side of the above inequality as $\beta(t)$. Then we have

$$\beta(0) = \frac{(1+m)\|\phi\| + \sum_{k=1}^n \|I_k(0)\|}{1 - m - \sum_{k=1}^n c_k}$$

$$\mu(t) \leq \beta(t),\ t \in J_n,$$

and

$$\beta'(t) = \frac{l_n(t)(1+\mu(t))}{1 - m - \sum_{k=1}^n c_k} \leq \frac{l_n(t)(1+\beta(t))}{1 - m - \sum_{k=1}^n c_k},\ t \in J_n.$$

Integrating over $t \in J_n$ yields

$$\int_{\beta(0)}^{\beta(t)} \frac{ds}{1+s} \leq \frac{1}{1 - m - \sum_{k=1}^n c_k}\int_0^t l_n(s)ds =: M_n.$$

Hence $\beta(t) \leq K_n := (1+\beta(0))e^{M_n}$ and as a consequence

$$\|y(t)\| \leq \mu(t) \leq \beta(t) \leq K_n,\ t \in J_n.$$

Therefore
$$\|y\|_n \leq K_n,\ n \in \mathbb{N}^*.$$
Let
$$U = \{y \in \Omega : \|y\|_n < K_n + 1,\ \text{ for all } n \in \mathbb{N}\}.$$
Clearly, U is a open subset of Ω and there is no $y \in \partial U$ such that $y \in \lambda N(y)$ and $\lambda \in (0,1)$, by lemma 1.6.3 and Steps 1, 2, N has at least one fixed point y solution to problem (3.1).

3.1.3 The lower semi-continuous case

Our third existence result for problem (3.1) deals with the case where the nonlinearity is lower semi-continuous with respect to the second argument and does not necessarily have convex values. In the proof, we will make use of the nonlinear alternative of Leray–Schauder type (theorem 1.6.8) combined with a selection theorem for lower semi-continuous multivalued maps with decomposable values. Consider a Banach space E and $I = [a,b]$ an interval of the real line.

Theorem 3.1.3. *Suppose that:*

($\widetilde{\mathcal{H}_1}$) *There exist $c_k, d_k > 0$ such that*
$$\|I_k(x)\| \leq c_k\|x\| + d_k,\ \textit{for every } x \in \mathbb{R}^n, k = 1, 2, \cdots,$$
with
$$\sum_{k=1}^{\infty} c_k < \infty \ \textit{and} \ \sum_{k=1}^{\infty} d_k < \infty.$$

($\mathcal{H}_2$) *For every bounded set $B \in \Omega$, the set $\{t : t \mapsto g(t, y_t),\ y \in B\}$ is equicontinuous and equiconvergent in Ω, g is continuous and there exists a constant $\ell_1 \in [0,1)$ and $\ell_2 > 0$ such that*
$$\|g(t,x)\| \leq \ell_1\|x\|_D + \ell_2 \textit{ for all } x \in D.$$

($\widetilde{\mathcal{H}_3}$) *There exists $p \in L^1_{loc}([0,\infty), \mathbb{R}^n)$ and a continuous nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ such that*
$$\|F(t,x)\|_{\mathcal{P}} \leq p(t)\psi(\|x\|) \textit{ for a.e. } t \in J \textit{ and each } x \in D,$$
with
$$\int_0^{\infty} \frac{du}{\psi(u)} = \infty.$$

($\mathcal{H}_7$) $F : [0,\infty) \times D \to \mathcal{P}(\mathbb{R}^n)$ is a nonempty compact valued multimap such that

(a) the mapping $(t, y) \multimap F(t, y)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable,

(b) the mapping $y \multimap F(t, y)$ is lower semi-continuous for a.e. $t \in [0,\infty)$.

Then problem (3.1) has at least one solution.

For the proof, we need some auxiliary lemmas.

Lemma 3.1.2. *[37]. Let $F : I \times E \to \mathcal{P}_{cp}(E)$ be a locally integrably bounded multivalued map satisfying ($\mathcal{H}_7$). Then F is of lower semi-continuous type.*

The following result is known as the Gronwall-Bihari Theorem.

Lemma 3.1.3. *[17] Let $u, g : I \to \mathbb{R}$ be positive real continuous functions. Assume there exists $c > 0$ and a continuous nondecreasing function $h : \mathbb{R} \to (0,+\infty)$ such that*

$$u(t) \leq c + \int_a^t g(s)h(u(s))ds, \; \forall t \in I.$$

Then

$$u(t) \leq H^{-1}\left(\int_a^t g(s)ds\right), \; \forall t \in I$$

provided

$$\int_c^{+\infty} \frac{dy}{h(y)} > \int_a^b g(s)ds.$$

Here H^{-1} refers to the inverse of the function $H(u) = \displaystyle\int_c^u \frac{dy}{h(y)}$ for $u \geq c$.

Proof of Theorem 3.1.3. Let $F : J_m \times D \to \mathcal{P}(\mathbb{R}^n)$. ($\widetilde{\mathcal{H}_3}$) and ($\mathcal{H}_7$) imply, by Lemma 3.1.2, that F is of lower semi-continuous type. From Lemma 1.3.4, there is a continuous selection $f_m : PC(J_m, \mathbb{R}^n) \to L^1(J_m, \mathbb{R}^n)$ such that $f_m(y) \in \mathcal{F}_m(y)$ for every $y \in PC(J_m, \mathbb{R}^n)$ where $\mathcal{F}_m$ is the Nemyts'kii operator associated with F on J_m

$$\mathcal{F}_m(y) = \{v \in L^1(J_m, \mathbb{R}^n) : v(t) \in F(t, y_t), a.e. t \in J_m\}.$$

Let $f : PC \to L^1_{loc}([0,\infty),\mathbb{R}^n)$ be defined by

$$f(y)(t) = f_m(y)(t),\ a.e.t \in J_m.$$

Then $\Omega = \cap_{m\geq 1}\Omega_m$ is a Fréchet space with family of semi-norms $\{\|.\|_m\}$ where

$$\|y\|_m = \sup\{\|y(t)\| : t \in J_m\}.$$

Consider the problem

$$\begin{cases} \dfrac{d}{dt}\left[y(t) - g(t,y_t)\right] = f(y)(t) & a.e.t \in J \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), & k = 1,2,\ldots, \\ y(t) = \phi(t), & t \in [-r,0], \end{cases} \tag{3.9}$$

and the operator $L : \Omega \to \Omega$ defined by

$$L(y)(t) = \phi(0) + g(t,y_t) - g(0,\phi) + \int_0^t f(s)ds + \sum_{0<t_k<t} I_k(y(t_k)),\ a.e.t \in J.$$

Clearly, the fixed points of the operator L are mild solutions of problem (3.1).

Step 1. A priori estimates. Let y be a possible solution of problem (3.1). For $t \in [0,t_1]$, we have

$$y(t) = \phi(0) + g(t,y_t) - g(0,\phi) + \int_0^t f(s)ds.$$

Then

$$\|y(t)\| \leq \|\phi(0)\| + \ell_1\|y_t\| + \ell_2 + \|g(0,\phi)\| + \int_0^t p(s)\psi(\|y(s)\|)ds.$$

Consider the function μ defined by

$$\mu(t) = \max\left\{\sup_{s\in[0,t]}\|y(s)\|,\ \sup_{s\in[-r,t]}\|y_s\|\right\},$$

then

$$\mu(t) \leq \frac{\|\phi(0)\| + \ell_2 + \|g(0,\phi)\|}{1-\ell_1} + \int_0^t \frac{p(s)}{1-\ell_1}\psi(\mu(s))ds.$$

By lemma 3.1.3 and $(\widetilde{\mathcal{H}_3})$, we have

$$\mu(t) \leq \Gamma_1^{-1}\left(\int_0^t \frac{p(s)}{1-\ell_1}ds\right), \quad t \in [0,t_1],$$

where $\Gamma_1(z) = \displaystyle\int_c^z \frac{du}{\psi(u)}$, and $c = \dfrac{\|\phi(0)\| + \ell_2 + \|g(0,\phi)\|}{1-\ell_1}$.
For $t \in (t_1,t_2]$, we have

$$y(t) = \phi(0) + g(t,y_t) - g(0,\phi) + I_1(y(t_1)) + \int_0^t f(s)ds.$$

Then

$$\|y(t)\| \leq \|\phi(0)\| + \ell_1\|y_t\| + \ell_2 + \|g(0,\phi)\| + K_1 + \int_0^t p(s)\psi(\|y(s)\|)ds.$$

$$\mu(t) \leq \frac{\|\phi(0)\| + \ell_2 + \|g(0,\phi)\| + K_1}{1-\ell_1} + \int_0^t \frac{p(s)}{1-\ell_1}\psi(\mu(s))ds,$$

where

$$K_1 = \sup\{\|I_1(z)\| : z \in \overline{B}(0,M_0)\} \text{ and } M_0 = \Gamma_1^{-1}\left(\int_0^{t_1} \frac{p(s)}{1-\ell_1}ds\right).$$

By lemma 3.1.3, we again have

$$\mu(t) \leq \Gamma_2^{-1}\left(\int_0^t \frac{p(s)}{1-\ell_1}ds\right), \quad t \in (t_1,t_2],$$

where $\Gamma_2(z) = \displaystyle\int_{c+\frac{K_1}{1-\ell_1}}^z \frac{du}{\psi(u)}$.
We continue this process until we obtain, for every $t \in (t_{m-1},t_m]$, the estimate

$$\|y(t)\| \leq \mu(t) \leq \Gamma_m^{-1}\left(\int_0^t \frac{p(s)}{1-\ell_1}ds\right), \quad t \in (t_{m-1},t_m],$$

where

$$\Gamma_m(z) = \int_{c+\frac{K_{m-1}}{1-\ell_1}}^z \frac{du}{\psi(u)},$$

$$\begin{aligned}
K_{m-1} &= \sup\{\|I_{m-1}(z)\| : z \in \overline{B}(0, M_{m-2})\} \\
M_{m-2} &= \Gamma_{m-1}^{-1}\left(\int_0^{t_{m-1}} \frac{p(s)}{1-\ell_1}ds\right).
\end{aligned}$$

Let

$$C = \left\{ y \in \Omega : \|y(t)\| \leq \Gamma_m^{-1}\left(\int_0^t \frac{p(s)}{1-\ell_1}ds\right),\ t \in (t_{m-1}, t_m], m = 1, 2, \cdots \right\}$$

It is clear that C is a convex closed and bounded subset in Ω.
Step 2. $L(C) \subset C$. Given $y \in C$ we have for $t \in [0, t_1]$

$$\begin{aligned}
\|L(y)(t)\| &\leq \|\phi\| + \ell_1\|y(t)\| + \ell_2 + \|g(0,\phi)\| + \int_0^t p(s)\psi(\|y(s)\|)ds \\
&\leq \|\phi\| + \ell_1\left(\Gamma_1^{-1}\left(\int_0^t \frac{p(s)}{1-\ell_1}ds\right)\right) + \ell_2 + \|g(0,\phi)\| \\
&+ \int_0^t p(s)\psi\left(\Gamma_1^{-1}\left(\int_0^s \frac{p(r)}{1-\ell_1}dr\right)\right)ds \\
&= \|\phi\| + \ell_1\left(\Gamma_1^{-1}\left(\int_0^t \frac{p(s)}{1-\ell_1}ds\right)\right) + \ell_2 + \|g(0,\phi)\| \\
&+ \int_0^t p(s)(\Gamma_1^{-1})'\left(\int_0^s \frac{p(r)}{1-\ell_1}dr\right)ds \\
&= \|\phi\| + \ell_1\left(\Gamma_1^{-1}\left(\int_0^t \frac{p(s)}{1-\ell_1}ds\right)\right) + \ell_2 + \|g(0,\phi)\| \\
&+ (1-\ell_1)\int_0^t \left(\Gamma_1^{-1}\left(\int_0^s \frac{p(r)}{1-\ell_1}dr\right)\right)' ds
\end{aligned}$$

We have used the fact that $\Gamma_1^{-1}(0) = \|\phi\| + \ell_2 + \|g(0,\phi)\|$ and

$$\psi(z) = \frac{1}{(\Gamma_1)'(z)} = (\Gamma_1^{-1})'(\Gamma_1)(z).$$

Lemma 3.1.3 implies that

$$\|L(y)(t)\| \leq \Gamma_1^{-1}\left(\int_0^t \frac{p(s)}{1-\ell_1}ds\right),\ a.e. t \in [0, t_1]. \tag{3.10}$$

Also for $t \in (t_1, t_2]$, we have

$$\|L(y)(t)\| \leq \|\phi\| + \ell_1\|y(t)\| + \ell_2 + \|g(0,\phi)\| + \|I_1(y(t_1))\| + \int_0^t p(s)\psi(\|y(s)\|)ds$$

$$\leq \quad \|\phi\| + \ell_1\|y(t)\| + \ell_2 + \|g(0,\phi)\| + K_1 + \int_0^t p(s)\psi(\|y(s)\|)ds$$

Arguing as above, we obtain

$$\|L(y)(t)\| \leq \Gamma_2^{-1}\left(\int_0^t \frac{p(s)}{1-\ell_1}ds\right), \ a.e.t \in (t_1, t_2]. \tag{3.11}$$

We continue this process until we arrive at the estimate

$$\|L(y)(t)\| \leq \Gamma_m^{-1}\left(\int_0^t \frac{p(s)}{1-\ell_1}ds\right), \ a.e.t \in (t_{m-1}, t_m], \tag{3.12}$$

proving that $L(C) \subset C$; this implies that $L(C)$ is a bounded set in the Fréchet space Ω. As in claims 2 and 3, step 1 of the proof of theorem 3.1.1, we can prove that for every $m \in \mathbb{N}$, the operator $L : \Omega_m \to \Omega_m$ is completely continuous; hence $L : \Omega \to \Omega$ is continuous and $L(C)$ is relatively compact. By lemma 1.6.1, we conclude that L has at least one fixed point, a solution of problem (3.9), and hence a solution of problem (3.1).

Chapter 4

Existences and solutions sets of systems of impulsive differential inclusions

In this chapter, we treat the existence of solutions and even a few properties of the set of solutions and the solutions operator for a system of differential inclusions with impulse effects. For the Cauchy problem, under various assumptions on the nonlinear term, we present several existence results. We appeal to the fixed point theorems in vector metric space. Finally, we prove some precise geometric properties about the structure of the solution set such as AR, R_δ, contractibility and acyclicity, corresponding to Aronszajn-Browder-Gupta type results.
consider the following problem:

$$\left\{\begin{array}{rcl} x'(t) & \in & F_1(t,x(t),y(t)), \text{a.e. } t \in [0,1] \\ y'(t) & \in & F_2(t,x(t),y(t)), \text{a.e. } t \in [0,1] \\ x(t_k^+) & = & x(t_k^-) + I_{1,k}(x(t_k),y(t_k)),\ k=1,\dots,m \\ y(t_k^+) & = & y(t_k^-) + I_{2,k}(x(t_k),y(t_k)),\ k=1,\dots,m \\ x(0) & = & x_0, \\ y(0) & = & y_0, \end{array}\right. \tag{4.1}$$

where $0 = t_0 < t_1 < \ldots < 1, i = 1,2, F_i : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ are a multifunction, $I_{1,k}, I_{2,k} \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. The notations $x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$ and $x(t_k^-) = \lim_{h\to 0^+} x(t_k - h)$ stand for the right and the left limits of the function x at $t = t_k$, respectively.

4.1 Notations and Definitions

Definition 4.1.1. *Let X be a nonempty set. By a vector-valued metric on X we mean a map $d : X \times X \to \mathbb{R}^n$ with the following properties:*

(i) $d(u, v) \geq 0$ *for all* $u, v \in X$; *if* $d(u, v) = 0$ *then* $u = v$;

(ii) $d(u, v) = d(v, u)$ *for all* $u, v \in X$;

(iii) $d(u, v) \leq d(u, w) + d(w, v)$ *for all* $u, v, w \in X$;

We call the pair (X, d) a generalized metric space. For $r = (r_1, \cdots, r_n) \in \mathbb{R}^n_+$, we denote by

$$B(x_0, r) = \{x \in X : d(x_0, x) < r\}$$

the open ball centered in x_0 with radius r and

$$\overline{B(x_0, r)} = \{x \in X : d(x_0, x) \leq r\}$$

the closed ball centered in x_0 with radius r.
We mention that for generalized metric space, the notation of open subset, closed set, convergence, Cauchy sequence and completeness are similar to those in usual metric spaces. If, $x, y \in \mathbb{R}^n, x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for all $i = 1, \cdots, n$. Also $\|x\| = (|x_1|, \cdots, |x_n|)$ and $\max(x, y) = (\max(x_1, y_1), \cdots, \max(x_n, y_n))$. If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c$ for each $i = 1, \cdots, n$.

Definition 4.1.2. *A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc (i.e. $|\lambda| < 1$, for every $\lambda \in C$ with $det(M - \lambda I) = 0$, where I denote the identity matrix of $\mathcal{M}_{n\times n}(\mathbb{R})$).*

Theorem 4.1.1. *Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$. The following assertions are equivalent:*

(i) M *is convergent towards zero;*

(ii) $M^k \to 0$ *as* $k \to \infty$;

(iii) *The matrix* $(I - M)$ *is nonsingular and*

$$(I - M)^{-1} = I + M + M^2 + \cdots + M^k + \cdots ;$$

(iv) The matrix $(I - M)$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.

Definition 4.1.3. *We say that a non-singular matrix $A = (a_{ij})_{1 \le i,j \le n} \in \mathcal{M}_{n \times n}(\mathbb{R})$ has the absolute value property if*

$$A^{-1} |A| \le I,$$

where

$$|A| = (|a_{ij}|)_{1 \le i,j \le n} \in \mathcal{M}_{n \times n}(\mathbb{R}).$$

Definition 4.1.4. *Let (X, d) be a generalized metric space. An operator $N : X \to X$ is said to be contractive if there exists a convergent to zero matrix M such that*

$$d(N(x), N(y)) \le M d(x, y), \forall x, y \in X.$$

Theorem 4.1.2. *Let (X, d) be a complete generalized metric space and $N : X \to X$ a contractive operator with Lipschitz matrix M. Then N has a unique fixed point x_* and for each $x_0 \in X$ we have*

$$d(N^k(x_0), x_*) \le M^k (I - M)^{-1} d(x_0, n(x_0)), \forall k \in \mathbb{N}.$$

Let (X, d) be a metric space, we will denote by H_{d_*} the Hausdorff pseudo-metric distance on $\mathcal{P}(X)$, defined as

$$H_{d_*} : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}, H_{d_*}(A, B) = \max\{\sup_{a \in A} d_*(a, B), \sup_{b \in B} d_*(A, b)\}.$$

where $d_*(A, b) = \inf_{a \in A} d_*(a, b)$ and $d_*(a, B) = \inf_{b \in B} d_*(a, b)$. Then $(\mathcal{P}_{b,cl}(X), H_{d_*})$ is a metric space and $(\mathcal{P}_{cl}(X), H_{d_*})$ is a generalized metric space. In particular, H_{d_*} satisfies the triangle inequality.
Let (X, d) be a generalized metric space with

$$d(x, y) := \begin{pmatrix} d_1(x, y) \\ \vdots \\ d_n(x, y) \end{pmatrix}.$$

Notice that d is a generalized metric space on X if and only if $d_i, i = 1, \cdots, n$ are metrics on X. Consider the generalized Hausdorff pseudo-metric distance

$$H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}^n_+ \cup \{\infty\}$$

defined by

$$H_d(A,B) := \begin{pmatrix} H_{d_1}(A,B) \\ \vdots \\ H_{d_n}(A,B) \end{pmatrix}.$$

Definition 4.1.5. *Let (X,d) be a generalized metric space. A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is said to be contractive if there exists a matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ such that*

$$M^k \to 0 ask \to \infty$$

and

$$H_d(N(u), N(v)) \leq Md(u,v), \forall u,v \in X.$$

Theorem 4.1.3. *Let (X,d) be a generalized complete metric space, and let $N : X \to \mathcal{P}_{cl}(X)$ be a multivalued map. Assume that there exist $A, B, C \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ such that*

$$H_d(N(x), N(y)) \leq Ad(x,y) + Bd(y, N(x)) + Cd(x, N(x)) \tag{4.2}$$

where $A+C$ converge to zero. Then there exist $x \in X$ such that $x \in N(x)$.

Definition 4.1.6. *Let E be a vector space on $K = \mathbb{R}$ or C. By a vector-valued norm on E we mean a map $\|.\| : E \to \mathbb{R}^n$ with the following properties:*

(i) $\|x\| \geq 0$ *for all $x \in E$; if $\|x\| = 0$ then $x = (0, \cdots, 0)$;*

(ii) $\|\lambda x\| = |\lambda| \, \|x\|$ *for all $x \in E$ and $\lambda \in K$;*

(iii) $\|x + y\| \leq \|x\| + \|y\|$ *for all $x, y \in E$.*

The pair $(E, \|.\|)$ is called a generalized normed space. If the generalized metric generated by $\|.\|$ (i.e. $d(x,y) = \|x - y\|$) is complete then the space $(E, \|.\|)$ is called a generalized Banach space.

Lemma 4.1.1. *: Let $F : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp}(J, \mathbb{R})$ a multivalued map integrally bounded, such that*

(a) $(t,x,y) \to F_i(t,x,y)$ *is $\mathcal{L} \otimes \mathcal{B}$ measurable for $i = 1, 2$.*

(b) $(x,y) \to F_i(t,x,y)$ *is l.s.c. a.e. $t \in J$.*

Then F has a l.s.c. type.

Definition 4.1.7. *We say that a multi-valued map $\phi : [0,a] \times E \to \mathcal{P}(E)$ with closed values is upper-Scorza-Dragoni if, given $\delta > 0$, there exists a closed subset $A_\delta \subset [0,a]$ such that the measure $\mu([0,a] \setminus A_\delta) \leq \delta$ and the restriction ϕ_δ of ϕ to $A_\delta \times E$ is u.s.c.*

Theorem 4.1.4. *[50] Let E, E_1 be two separable Banach spaces and let $F : [a,b] \times E \to \mathcal{P}_{cp,cv}(E_1)$ be an upper-Scorza-Dragoni map. Then F is $\sigma-$Ca-selectionable, the maps $F_n : [a,b] \times E \to E_1,\ n \in \mathbb{N}$ are almost upper semi-continuous and we have*

$$F_n(t,x) \subset \overline{conv}\left(\bigcup_{x\in E} F_n(t,x)\right).$$

Moreover, if F is integrally bounded, then F is $\sigma - mLL$-selectionable.

4.2 Existence Results

Consider the space $PC([0,1],\mathbb{R})$, endowed with the norm

$$\|y\|_{PC} = \sup\{\|y(t)\| : t \in J\},\ J = [0,1].$$

PC is a Banach space.

4.2.1 Convexe case

Theorem 4.2.1. *Assume that there exists a continuous nondecreasing map $\psi : [0,+\infty[\longrightarrow (0,+\infty)$, and $p \in L^1(J,\mathbb{R}_+)$ such that*

$$\|F_i(t,u,v)\| \leq p(t)\psi(|u|+|v|) \quad a.e. \quad t \in J\ , i \in \{1,2\} \text{ and } \ (u,v) \in \mathbb{R}^2,$$

assume also $F_1, F_2 : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$ are Carathéodory.
Then the problem (4.1) has at least one solution.

Proof. Consider the operator $N : PC \times PC \to \mathcal{P}(PC \times PC)$ defined by

$$N(x,y) = \left\{ (h_1,h_2) \in PC \times PC : \begin{pmatrix} h_1(t) & h_2(t) \end{pmatrix} = \left\{ \begin{array}{ll} x_0 + \int_0^t f_1(s)ds & \\ + \sum_{0<t_k<t} I_{1,k}(x(t_k),y(t_k)), & t \in J \\ & \\ y_0 + \int_0^t f_2(s)ds & \\ + \sum_{0<t_k<t} I_{2,k}(x(t_k),y(t_k)), & t \in J \end{array} \right. \right\}$$

where $f_i \in S_{F_i} = \{f \in L^1(J,\mathbb{R}) : f(t) \in F_i(t, x(t), y(t)), a.e. t \in J\}$. Clearly, fixed points of the operator N are solutions of problem (4.1).
We are going to prove that N is u.s.c, compact, and N has convex compact values. The proof is given by the following steps.

Step 1: $N(x, y)$ is convexe for all $(x, y) \in PC \times PC$.
Let $(h_1, h_2), (h_3, h_4) \in N(x, y)$ so there exist $f_1, f_3 \in S_{F_1(.,x(.),y(.))}$, and $f_2, f_4 \in S_{F_2(.,x(.),y(.))}$ such that for all $t \in J$ we have

$$\begin{aligned} h_1(t) &= x_0 + \int_0^t f_1(s)ds + \sum_{0<t_k<t} I_{1,k}(x(t_k), y(t_k)) \\ h_2(t) &= y_0 + \int_0^t f_2(s)ds + \sum_{0<t_k<t} I_{2,k}(x(t_k), y(t_k)), \end{aligned}$$

and

$$\begin{aligned} h_3(t) &= x_0 + \int_0^t f_3(s)ds + \sum_{0<t_k<t} I_{1,k}(x(t_k), y(t_k)) \\ h_4(t) &= y_0 + \int_0^t f_4(s)ds + \sum_{0<t_k<t} I_{2,k}(x(t_k), y(t_k)). \end{aligned}$$

Let $l \in [0, 1]$ for each $t \in J$, we have

$$\begin{aligned} \left(l \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + (1-l) \begin{pmatrix} h_3 \\ h_4 \end{pmatrix}\right)(t) &= \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} \int_0^t (lf_1 + (1-l)f_3)(s)ds \\ \int_0^t (lf_2 + (1-l)f_4)(s)ds \end{pmatrix} \\ &+ \begin{pmatrix} \sum_{0<t_k<t} I_{1,k}(x(t_k), y(t_k)) \\ \sum_{0<t_k<t} I_{2,k}(x(t_k), y(t_k)) \end{pmatrix}, \end{aligned}$$

as S_{F_1} and S_{F_2} are convexe (because F_1, F_2 have a convexe value) then

$$l \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + (1-l) \begin{pmatrix} h_3 \\ h_4 \end{pmatrix} \in N(x, y).$$

Step 2: N transforms every bounded set to a bounded set in $PC \times PC$.
It suffices to show that there exists $\ell = \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} > 0$,

$$\text{for all } (x, y) \in \mathcal{B}_q = \{(x, y) \in PC \times PC : \|(x, y)\| \le q\},$$

$$\text{and all } (h,g) \in N(x,y), \quad \text{we have } \|(h,g)\| \le \ell, q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} > 0,$$

if $(h,g) \in N(x,y)$, then there exists $f_1 \in S_{F_1(.,x(.),y(.))}$, and $f_2 \in S_{F_2(.,x(.),y(.))}$ such that for all $t \in J$ we have

$$h(t) = x_0 + \int_0^t f_1(s)ds + \sum_{0<t_k<t} I_{1,k}(x(t_k), y(t_k)),$$

and

$$g(t) = y_0 + \int_0^t f_2(s)ds + \sum_{0<t_k<t} I_{2,k}(x(t_k), y(t_k))$$

$$\|(h,g)\|_{PC\times PC} = \begin{pmatrix} \|h\|_{PC} \\ \|g\|_{PC} \end{pmatrix},$$

for all $t \in J$, we have

$$\begin{aligned}
\|h(t)\| &\le \|x_0\| + \int_0^t \|f_1(s)\|ds + \sum_{0<t_k<t} \|I_{1,k}(x(t_k), y(t_k))\| \\
&\le \|x_0\| + \int_0^1 \|F_1(s, x(s), y(s))\|ds + \sum_{k=1}^{m} \sup_{(x,y)\in\mathcal{B}_q} \|I_{1,k}(x,y)\| \\
&\le \|x_0\| + \psi(q_1+q_2)\|p\|_{L^1} + \sum_{k=1}^{m} \sup_{(x,y)\in\mathcal{B}_q} \|I_{1,k}(x,y)\| := \tilde{\ell},
\end{aligned}$$

and

$$\begin{aligned}
\|g(t)\| &\le \|y_0\| + \int_0^t \|f_2(s)\|ds + \sum_{0<t_k<t} \|I_{2,k}(x(t_k), y(t_k))\| \\
&\le \|y_0\| + \int_0^b \|F_2(s, x(s), y(s))\|ds + \sum_{k=1}^{m} \sup_{(x,y)\in\mathcal{B}_q} \|I_{2,k}(x,y)\| \\
&\le \|y_0\| + \psi(q_1+q_2)\|p\|_{L^1} + \sum_{k=1}^{m} \sup_{(x,y)\in\mathcal{B}_q} \|I_{2,k}(x,y)\| := \tilde{\tilde{\ell}},
\end{aligned}$$

then,

$$\begin{pmatrix} \|h\|_{PC} \\ \\ \|g\|_{PC} \end{pmatrix} \le \begin{pmatrix} \tilde{\ell} \\ \\ \tilde{\tilde{\ell}} \end{pmatrix} = \ell.$$

Step 3: N transforms every bounded set to a equicontinuous set to $PC\times PC$
We let $\tau_1,\tau_2 \in J, \tau_1 < \tau_2$ and $\mathcal{B}_q = \{(x,y) \in PC \times PC : \|(x,y)\| \leq q\}$, $q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} > 0$; for all $(x,y) \in \mathcal{B}_q$ and $(h,g) \in N(x,y)$, there exist $f_1 \in S_{F_1(.,x(.),y(.))}$, and $f_2 \in S_{F_2(.,x(.),y(.))}$ such that for all $t \in J$ we have

$$\begin{aligned} h(t) &= x_0 + \int_0^t f_1(s)ds + \sum_{0<t_k<t} I_{1,k}(x(t_k),y(t_k)) \\ g(t) &= y_0 + \int_0^t f_2(s)ds + \sum_{0<t_k<t} I_{2,k}(x(t_k),y(t_k)). \end{aligned}$$

Then

$$\begin{aligned} \|h(\tau_2)-h(\tau_1)\| &\leq \int_{\tau_1}^{\tau_2} \|f_1(s)\|ds + \sum_{\tau_1\leq t_k<\tau_2} \|I_{1,k}(x(t_k),y(t_k))\| \\ &\leq \psi(q_1+q_2)\int_{\tau_1}^{\tau_2} p(s)ds + \sum_{\tau_1\leq t_k<\tau_2} \sup_{(x,y)\in\mathcal{B}_q} \|I_{1,k}(x,y)\| \longrightarrow 0 \text{ when } \tau_2\to\tau_1, \end{aligned}$$

and

$$\begin{aligned} \|g(\tau_2)-g(\tau_1)\| &\leq \int_{\tau_1}^{\tau_2} \|f_2(s)\|ds + \sum_{\tau_1\leq t_k<\tau_2} \|I_{2,k}(x(t_k),y(t_k))\| \\ &\leq \psi(q_1+q_2)\int_{\tau_1}^{\tau_2} p(s)ds + \sum_{\tau_1\leq t_k<\tau_2} \sup_{(x,y)\in\mathcal{B}_q} \|I_{2,k}(x,y)\| \longrightarrow 0 \text{ when } \tau_2\to\tau_1. \end{aligned}$$

So by step 2 and 3, which is obtained, N is compact.
Step 4: The graph of N is closed.
Let $(x_n,y_n) \to (x_*,y_*)$, $(h_n,g_n) \in N(x_n,y_n)$, and $h_n \to h_*$ and $g_n \to g_*$. It suffices to show that there exists $f_1 \in S_{F_1(.,x_*(.),y(_*.))}$, and $f_2 \in S_{F_2(.,x_*(.),y_*(.))}$ such that for all $t \in J$ we have

$$h_*(t) = x_0 + \int_0^t f_1(s)ds + \sum_{0<t_k<t} I_{1,k}(x_*(t_k),y_*(t_k)),$$

and

$$g_*(t) = y_0 + \int_0^t f_2(s)ds + \sum_{0<t_k<t} I_{2,k}(x_*(t_k),y_*(t_k)).$$

$(h_n, g_n) \in N(x_n, y_n)$, so there exist $f_{1,n} \in S_{F_1(.,x_n(.),y_n(.))}$, and $f_{2,n} \in S_{F_2(.,x_n(.),y_n(.))}$ such that for all $t \in J$ we have

$$h_n(t) = x_0 + \int_0^t f_{1,n}(s)ds + \sum_{0<t_k<t} I_{1,k}(x_n(t_k), y_n(t_k)),$$

and

$$g_n(t) = y_0 + \int_0^t f_{2,n}(s)ds + \sum_{0<t_k<t} I_{2,k}(x_n(t_k), y_n(t_k)).$$

Since $I_{i,k}, k = 1, \cdots, m; i = 1, 2$ are continuous,

$$\left\| \left(h_n(t) - x_0 - \sum_{0<t_k<t} I_{1,k}(x_n(t_k), y_n(t_k)) \right) - \left(h_*(t) - x_0 - \sum_{0<t_k<t} I_{1,k}(x_*(t_k), y_*(t_k)) \right) \right\|_{PC} \to 0,$$

and

$$\left\| \left(g_n(t) - y_0 - \sum_{0<t_k<t} I_{2,k}(x_n(t_k), y_n(t_k)) \right) - \left(g_*(t) - y_0 - \sum_{0<t_k<t} I_{2,k}(x_*(t_k), y_*(t_k)) \right) \right\|_{PC} \to 0,$$

when $n \to \infty$.

Let Γ a continuous linear operator, defined as

$$\begin{array}{rcl} \Gamma : L^1(J, \mathbb{R}) & \longrightarrow & PC(J, \mathbb{R}) \\ r & \longrightarrow & \Gamma(r), \end{array}$$

such that

$$\Gamma(r)(t) = \int_0^t r(s)ds; \ \ \forall t \in [0, 1].$$

By lemma 1.3.2, the operator $\Gamma \circ S_F$ has a closed graph, moreover we have

$$\left(h_n(t) - x_0 - \sum_{0<t_k<t} I_{1,k}(x_n(t_k), y_n(t_k)) \right) \in \Gamma(S_{F_1(.,x_n(.),y_n(.))}),$$

and

$$\left(g_n(t) - y_0 - \sum_{0<t_k<t} I_{2,k}(x_n(t_k), y_n(t_k)) \right) \in \Gamma(S_{F_2(.,x_n(.),y_n(.))}).$$

So

$$\left(h_*(t) - x_0 - \sum_{0<t_k<t} I_{1,k}(x_*(t_k), y_*(t_k))\right) = \int_0^t f_1(s)ds,$$

$$\left(g_*(t) - y_0 - \sum_{0<t_k<t} I_{2,k}(x_*(t_k), y_*(t_k))\right) = \int_0^t f_2(s)ds,$$

then $f_1 \in S_{F_1(.,x_*(.),y_*(.))}$ and $f_2 \in S_{F_2(.,x_*(.),y_*(.))}$.

Step 5: A priori estimation
Let $(x, y) \in PC(J, \mathbb{R}) \times PC(J, \mathbb{R})$ such that $(x, y) \in \lambda N(x, y)$, and $0 < \lambda < 1$, so there exists $f_1 \in S_{F_1(.,x(.),y(.))}$, and $f_2 \in S_{F_2(.,x(.),y(.))}$.
For all $t \in [0, t_1]$, we have

$$x(t) = \lambda x_0 + \lambda \int_0^t f_1(s, x(s), y(s))ds,$$

and

$$y(t) = \lambda y_0 + \lambda \int_0^t f_2(s, x(s), y(s))ds,$$

then

$$\|x(t)\| \leq \|x_0\| + \int_0^t p(s)\psi(\|x(s)\| + \|y(s))\|)ds, \quad t \in [0, t_1],$$

and

$$\|y(t)\| \leq \|y_0\| + \int_0^t p(s)\psi(\|x(s)\| + \|y(s)\|)ds, \quad t \in [0, t_1].$$

Consider the function ϑ_1, $\mathcal{W}_1$ such that

$$\vartheta_1(t) = \|x_0\| + \int_0^t p(s)\psi(\|x(s)\| + \|y(s)\|)ds, \quad t \in [0, t_1],$$

and

$$\mathcal{W}_1(t) = \|y_0\| + \int_0^t p(s)\psi(\|x(s)\| + \|y(s)\|)ds, \quad t \in [0, t_1].$$

So we have

$$(\vartheta_1(0), \mathcal{W}_1(0)) = (\|x_0\|, \|y_0\|); \quad \|x(t)\| \leq \vartheta_1(t), \|y(t)\| \leq \mathcal{W}_1(t) \quad t \in [0, t_1],$$

and

$$\dot{\mathcal{W}}_1(t) = \dot{\vartheta}_1(t) = p(t)\psi(\|x(t)\| + \|y(t)\|), \quad t \in [0, t_1].$$

As ψ is nondecreasing map, we have

$$\dot{\vartheta}_1(t) \leq p(t)\psi(\vartheta_1(t)); \quad \dot{\mathcal{W}}_1(t) \leq p(t)\psi(\mathcal{W}_1(t)), \quad t \in [0, t_1].$$

What implies that for every $t \in [0, t_1]$

$$\int_{\vartheta_1(0)}^{\vartheta_1(t)} \frac{du}{\psi(u)} \leq \int_0^{t_1} p(s)ds; \quad \int_{\mathcal{W}_1(0)}^{\mathcal{W}_1(t)} \frac{du}{\psi(u)} \leq \int_0^{t_1} p(s)ds.$$

The map $\Gamma_0^1(z) = \displaystyle\int_{\vartheta_1(0)}^{z} \frac{du}{\psi(u)}$ and the map $\Gamma_0^2(z) = \displaystyle\int_{\mathcal{W}_1(0)}^{z} \frac{du}{\psi(u)}$ are continuous and nondecreasing, then $(\Gamma_0^1)^{-1}$ and $(\Gamma_0^2)^{-1}$ exist and are nondecreasing, and we have

$$\vartheta_1(t) \leq (\Gamma_0^1)^{-1}\left(\int_0^{t_1} p(s)ds\right) := M_0; \quad \mathcal{W}_1(t) \leq (\Gamma_0^2)^{-1}\left(\int_0^{t_1} p(s)ds\right) := \ell_0.$$

As for every $t \in [0, t_1]$, $\|x(t)\| \leq \vartheta_1(t)$, and $\|y(t)\| \leq \mathcal{W}_1(t)$, then

$$\sup_{t\in[0,t_1]} \|y(t)\| \leq \ell_0; \quad \sup_{t\in[0,t_1]} \|x(t)\| \leq M_0.$$

Now, for $t \in (t_1, t_2]$, we have

$$\begin{aligned} \left\|x(t_1^+)\right\| &\leq \|I_{1,1}(x(t_1), y(t_1))\| + \|x(t_1)\| \\ &\leq \sup_{(\alpha,\beta)\in\overline{B}(0,M_0)\times\overline{B}(0,\ell_0)} \|I_{1,1}(\alpha, \beta)\| + M_0 := N_1, \end{aligned}$$

and

$$\begin{aligned} \left\|y(t_1^+)\right\| &\leq \|I_{2,1}(x(t_1), y(t_1))\| + \|y(t_1)\| \\ &\leq \sup_{(\alpha,\beta)\in\overline{B}(0,M_0)\times\overline{B}(0,\ell_0)} \|I_{2,1}(\alpha, \beta)\| + \ell_0 := D_1, \end{aligned}$$

$$x(t) = \lambda\left(x(t_1) + I_{1,1}(x(t_1), y(t_1))\right) + \lambda \int_{t_1}^{t} f_1(s, x(s), y(s))ds,$$

and

$$y(t) = \lambda\left(y(t_1) + I_{2,1}(x(t_1), y(t_1))\right) + \lambda \int_{t_1}^{t} f_2(s, x(s), y(s))ds,$$

so

$$\|x(t)\| \leq N_1 + \int_{t_1}^{t} p(s)\psi(\|x(s)\| + \|y(s)\|)ds, \quad t \in (t_1, t_2],$$

$$\|y(t)\| \leq D_1 + \int_{t_1}^{t} p(s)\psi(\|x(s)\| + \|y(s)\|)ds, \quad t \in (t_1, t_2],$$

consider the map ϑ_2 and the map $\mathcal{W}_2$ such that

$$\vartheta_2(t) = N_1 + \int_{t_1}^{t} p(s)\psi(\|x(s)\| + \|y(s)\|)ds, \quad t \in (t_1, t_2],$$

and

$$\mathcal{W}_2(t) = D_1 + \int_{t_1}^{t} p(s)\psi(\|x(s)\| + \|y(s)\|)ds, \quad t \in (t_1, t_2].$$

Then

$$\vartheta_2(t_1^+) = N_1, \quad \|x(t)\| \leq \vartheta_2(t), \quad t \in (t_1, t_2],$$
$$\mathcal{W}_2(t_1^+) = D_1, \quad \|y(t)\| \leq \mathcal{W}_2(t), \quad t \in (t_1, t_2],$$

and

$$\dot{\vartheta}_2(t) = p(t)\psi(\|x(t)\| + \|y(t)\|), \;\; \dot{\mathcal{W}}_2(t) = p(t)\psi(\|x(t)\| + \|y(t)\|), \;\; t \in (t_1, t_2],$$

as ψ is nondecreasing, then

$$\dot{\vartheta}_2(t) \leq p(t)\psi(\vartheta_2(t)), \;\; \dot{\mathcal{W}}_2(t) \leq p(t)\psi(\mathcal{W}_2(t)), t \in (t_1, t_2].$$

What implies that for every $t \in (t_1, t_2]$, we have

$$\int_{\vartheta_2(t_1^+)}^{\vartheta_2(t)} \frac{du}{\psi(u)} \leq \int_{t_1}^{t_2} p(s)ds, \quad \int_{\mathcal{W}_2(t_1^+)}^{\mathcal{W}_2(t)} \frac{du}{\psi(u)} \leq \int_{t_1}^{t_2} p(s)ds.$$

If we consider the map $\Gamma_1^1(z) = \int_{\vartheta_2(t_1^+)}^{z} \frac{du}{\psi(u)}$, and the map $\Gamma_1^2(z) = \int_{\mathcal{W}_2(t_1^+)}^{z} \frac{du}{\psi(u)}$, we get

$$\vartheta_2(t) \leq (\Gamma_1^1)^{-1}\left(\int_{t_1}^{t_2} p(s)ds\right) := M_1,$$

and

$$\mathcal{W}_2(t) \leq (\Gamma_1^2)^{-1}\left(\int_{t_1}^{t_2} p(s)ds\right) := \ell_1,$$

for all $t \in (t_1, t_2]$, $\|x(t)\| \leq \vartheta_2(t)$, $\|y(t)\| \leq \mathcal{W}_2(t)$, then

$$\sup_{t\in(t_1,t_2]} \|x(t)\| \leq M_1, \quad \sup_{t\in(t_1,t_2]} \|y(t)\| \leq \ell_1.$$

We continue the process until the we reach the interval $(t_m, b]$, then we obtain that there exists M_m and ℓ_m such that

$$\sup_{t\in(t_m,b]} \|x(t)\| \leq (\Gamma^1_m)^{-1}\left(\int_{t_m}^b p(s)ds\right) := M_m,$$

and

$$\sup_{t\in(t_m,b]} \|y(t)\| \leq (\Gamma^2_m)^{-1}\left(\int_{t_m}^b p(s)ds\right) := \ell_m.$$

As we choose x and y arbitrarily, then for all solution of the problem (4.1), we have

$$\|(x,y)\|_{PC\times PC} \leq \max\{\begin{pmatrix} M_k \\ \ell_k \end{pmatrix} : k = 0, 1, \cdots, m\} := b^*.$$

Consider the set

$$U = \{(x,y) \in PC \times PC : \|(x,y)\|_{PC\times PC} < b^* + 1\}.$$

So we have $N : \overline{U} \times \overline{U} \to \mathcal{P}_{cv}(PC \times PC)$ is compact and u.s.c. and by the definition of U it doses not exist a $(x,y) \in \partial U \times \partial U$ such that $(x,y) \in \lambda N(x,y)$, for all $\lambda \in (0,1)$. Then by the lemma 1.6.2, the problem (4.1) has at least one solution.

□

4.2.2 Non Convexe case

Assume the following conditions:

($\mathcal{C}_1$) $F_i : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$; $t \to F_i(t,u,v)$ are measurable for each $u, v \in \mathbb{R}, i = 1, 2$.

($\mathcal{C}_2$) There exist functions $l_i \in L^1(J, \mathbb{R}^+), i = 1, \cdots, 4$ such that

$$H_d(F_1(t,u,v), F_1(t,\overline{u},\overline{v})) \leq l_1(t)\|u-\overline{u}\|+l_2(t)\|v-\overline{v}\|, t \in J; \forall u, \overline{u}, v, \overline{v} \in \mathbb{R}$$

$$H_d(F_2(t,u,v), F_2(t,\overline{u},\overline{v})) \leq l_3(t)\|u-\overline{u}\|+l_4(t)\|v-\overline{v}\|, t \in J; \forall u, \overline{u}, v, \overline{v} \in \mathbb{R},$$

$(\mathcal{C}_3)$ There exist a constants $a_i, b_i \geq 0, i = 1, 2$ such that

$$\|I_{1,k}(u,v) - I_{1,k}(\overline{u} - \overline{v}\| \leq a_1\|u - \overline{u}\| + a_2\|v - \overline{v}\|, \forall u, \overline{u}, v, \overline{v} \in \mathbb{R}, k : 1, \cdots, m,$$

and

$$\|I_{2,k}(u,v) - I_{2,k}(\overline{u} - \overline{v}\| \leq b_1\|u - \overline{u}\| + b_2\|v - \overline{v}\|, \forall u, \overline{u}, v, \overline{v} \in \mathbb{R}, k : 1, \cdots, m, .$$

Theorem 4.2.2. *Assume that $(\mathcal{C}_1) - (\mathcal{C}_3)$ are satisfied and the matrix*

$$M = \begin{pmatrix} \|l_1\|_{L^1} + a_1 & \|l_2\|_{L^1} + a_2 \\ \|l_3\|_{L^1} + b_1 & \|l_4\|_{L^1} + b_2 \end{pmatrix}$$

converge to zero, then the problem (4.1) has at least one solution.

Proof. Consider the operator $N : PC \times PC \to \mathcal{P}(PC \times PC)$ defined by

$$N(x,y) = \left\{ (h_1, h_2) \in PC \times PC : \begin{pmatrix} h_1(t) & h_2(t) \end{pmatrix} = \left\{ \begin{array}{ll} x_0 + \int_0^t f_1(s)ds + & \\ \sum_{0<t_k<t} I_{1,k}(x(t_k), y(t_k)), & t \in J \\ & \\ y_0 + \int_0^t f_2(s)ds + & \\ \sum_{0<t_k<t} I_{2,k}(x(t_k), y(t_k)), & t \in J \end{array} \right. \right\}$$

where $f_i \in S_{F_i} = \{f \in L^1(J, \mathbb{R}) : f(t) \in F_i(t, x(t), y(t)), a.e. t \in J\}$. Clearly, fixed points of the operator N are solutions of problem (4.1).

Let

$$N_i(x,y) = \left\{ h \in PC : h(t) = x_i + \int_0^t f_i(s)ds + \sum_{0<t_k<t} I_{i,k}(x(t_k), y(t_k)), t \in J \right\},$$

where $x_1 = x_0, x_2 = y_0, f_i \in S_{F_i} = \{f \in L^1(J, \mathbb{R}) : f(t) \in F_i(t, x(t), y(t)), a.e. t \in J\}$. We show that N satisfies the assumptions of Theorem 4.1.3.
Let $(x, y), (\overline{x}, \overline{y}) \in PC \times PC$ and $(h_1, h_2) \in N(x, y)$. Then there exist $f_i \in S_{F_i}, i = 1, 2$ such that

$$(h_1(t), h_2(t)) = \left\{ \begin{array}{l} x_0 + \int_0^t f_1(s)ds + \sum_{0<t_k<t} I_{1,k}(x(t_k), y(t_k)), \ t \in J \\ y_0 + \int_0^t f_2(s)ds + \sum_{0<t_k<t} I_{2,k}(x(t_k), y(t_k)), \ t \in J \end{array} \right.$$

$(\mathcal{C}_2)$ implies that

$$H_d(F_1(t, x(t), y(t)), F_1(t, \overline{x}(t), \overline{y}(t)) \leq l_1(t)\,|x(t) - \overline{x}(t)| + l_2(t)\,|y(t) - \overline{y}(t)|,$$

and

$$H_d(F_2(t,x(t),y(t)),F_2(t,\overline{x}(t),\overline{y}(t)))\leq l_3(t)\left|x(t)-\overline{x}(t)\right|+l_4(t)\left|y(t)-\overline{y}(t)\right|.$$

Hence, there are some $(\omega,\overline{\omega})\in F_1(t,\overline{x}(t),\overline{y}(t))\times F_2(t,\overline{x}(t),\overline{y}(t))$ such that

$$|f_1(t)-\omega|\leq l_1(t)\left|x(t)-\overline{x}(t)\right|+l_2(t)\left|y(t)-\overline{y}(t)\right|,$$

and

$$|f_2(t)-\overline{\omega}|\leq l_3(t)\left|x(t)-\overline{x}(t)\right|+l_4(t)\left|y(t)-\overline{y}(t)\right|.$$

Consider the multi-valued maps $U_i:J\to\mathcal{P}(\mathbb{R}), i=1,2$ defined by

$$U_1(t)=\left\{\begin{array}{ccc}\omega\in F_1(t,\overline{x}(t),\overline{y}(t)) & : & \\ |f_1(t)-\omega|\leq l_1(t)\left|x(t)-\overline{x}(t)\right| & + & l_2(t)\left|y(t)-\overline{y}(t)\right|, a.e. t\in J\end{array}\right\}$$

and

$$U_2(t)=\left\{\begin{array}{ccc}\overline{\omega}\in F_2(t,\overline{x}(t),\overline{y}(t)) & : & \\ |f_2(t)-\overline{\omega}|\leq l_3(t)\left|x(t)-\overline{x}(t)\right| & + & l_4(t)\left|y(t)-\overline{y}(t)\right|, a.e. t\in J\end{array}\right\}$$

Then $U_i(t)$ are a nonempty set and Theorem $III.4.1$ in [25] tells us that U_i are measurable. Moreover, the multi-valued intersection operator $V_i(.)=U_i(.)\cap F_i(.,\overline{x}(.),\overline{y}(.))$ are measurable. Therefore, by Lemma 1.3.1, there exists a function $t\to\overline{f}_i(t)$, which are measurable selection for V_i, that is $\overline{f}_i(t)\in F_i(t,\overline{x}(t),\overline{y}(t))$ and

$$\left|f_1(t)-\overline{f}_1(t)\right|\leq l_1(t)\left|x(t)-\overline{x}(t)\right|+l_2(t)\left|y(t)-\overline{y}(t)\right|, a.e. t\in J,$$

and

$$\left|f_2(t)-\overline{f}_2(t)\right|\leq l_3(t)\left|x(t)-\overline{x}(t)\right|+l_4(t)\left|y(t)-\overline{y}(t)\right|, a.e. t\in J.$$

Define $\overline{h}_1,\overline{h}_2$ by

$$\overline{h}_1(t)=x_0+\int_0^t\overline{f}_1(s)ds+\sum_{0<t_k<t}I_{1,k}(\overline{x}(t_k),\overline{y}(t_k)), t\in J,$$

and

$$\overline{h}_2(t)=y_0+\int_0^t\overline{f}_2(s)ds+\sum_{0<t_k<t}I_{2,k}(\overline{x}(t_k),\overline{y}(t_k)), t\in J.$$

Then we have, for $t \in J$,

$$\left|h_1(t) - \overline{h}_1(t)\right| \leq (\|l_1\|_{L^1} + a_1)\,|x - \overline{x}|_{PC} + (\|l_2\|_{L^1} + a_2)\,\|y - \overline{y}\|_{PC}\,.$$

Thus

$$\left\|h_1 - \overline{h}_1\right\|_{PC} \leq (\|l_1\|_{L^1} + a_1)\,|x - \overline{x}|_{PC} + (\|l_2\|_{L^1} + a_2)\,\|y - \overline{y}\|_{PC}\,.$$

By an analogous relation, we finally arrive at the estimate

$$\begin{array}{rcl} H_d(N_1(x,y), N_1(\overline{x},\overline{y})) & \leq & (\|l_1\|_{L^1} + a_1)\|x - \overline{x}\|_{PC} \\ & & +(\|l_2\|_{L^1} + a_2)\|y - \overline{y}\|_{PC}. \end{array}$$

Similarly we have

$$\begin{array}{rcl} H_d(N_2(x,y), N_2(\overline{x},\overline{y})) & \leq & (\|l_3\|_{L^1} + b_1)\|x - \overline{x}\|_{PC} \\ & & +(\|l_4\|_{L^1} + b_2)\|y - \overline{y}\|_{PC}. \end{array}$$

Therefore

$$H_d(N(x,y), N(\overline{x},\overline{y})) \leq M \begin{pmatrix} \|x - \overline{x}\|_{PC}, & \|y - \overline{y}\|_{PC} \end{pmatrix},$$

for each (x,y) and $(\overline{x},\overline{y})$ in $PC \times PC$. Hence, by Theorem 4.1.3, the operator N has at least one fixed point which is solution of (4.1).

$\square$

Theorem 4.2.3. *Assume that exist a continuous nondecreasing maps $\psi_i : [0,+\infty) \longrightarrow (0,+\infty)$, and $p_i \in L^1(J,\mathbb{R}_+)$ such that*

$$\|F_i(t,u,v)\| \leq p_i(t)\psi_i(\|u\| + \|v\|) \quad a.e. \quad t \in J\ , i \in \{1,2\} \ \text{and} \ (u,v) \in \mathbb{R}^2,$$

assume also $F_1, F_2 : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$ are Carathéodory, and

(a) $(t,x,y) \to F_i(t,x,y)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable for $i = 1,2$.

(b) $(x,y) \to F_i(t,x,y)$ is l.s.c. a.e. $t \in J$.

Then the problem (4.1) has at least one solution.

Proof. Since F_i are l.s.c. type, so by theorem 1.3.4 there exists continuous functions $f_i : PC \to L^1(J,\mathbb{R})$, $i = 1,2$ such that $f_i(x,y) \in S_{F_i(\cdot,x,y)}$, for all

$(x, y) \in PC(J, \mathbb{R}) \times PC(J, \mathbb{R})$.
Consider the following impulsive system

$$\left\{\begin{array}{rcl} x'(t) & = & f_1(t, x, y),\ a.e.\ t \in J \\ y'(t) & = & f_2(t, x, y), a.e.t \in J \\ x(t_k^+) - x(t_k^-) & = & I_{1,k}(x(t_k), y(t_k)),\ k = 1, 2, \ldots, m \\ y(t_k^+) - y(t_k^-) & = & I_{2,k}(x(t_k), y(t_k)),\ k = 1, 2, \ldots, m \\ x(0) = x_0, & & y(0) = y_0. \end{array}\right. \tag{4.3}$$

It is clear if (x, y) is a solution of the problem (4.3) then (x, y) is also a solution of the problem (4.1).
The operator $N_* : PC \times PC \to \mathcal{P}(PC \times PC)$ defined by

$$N_*(x, y) = \left\{ (h_1, h_2) \in PC \times PC : \begin{pmatrix} h_1(t), & h_2(t) \end{pmatrix} = \left\{ \begin{array}{l} x_0 + \int_0^t f_1(s)ds + \\ \sum_{0<t_k<t} I_{1,k}(x(t_k), y(t_k)), \quad t \in J \\ \\ y_0 + \int_0^t f_2(s)ds + \\ \sum_{0<t_k<t} I_{2,k}(x(t_k), y(t_k)), \quad t \in J \end{array} \right. \right\}$$

By the proof of theorem 4.2.1 the problem (4.1), has at least one solution.

□

4.3 Structure of solutions sets

Consider the first-order impulsive single-value problem

$$\left\{\begin{array}{rcl} x'(t) & = & f_1(t, x(t), y(t)),\ a.e.t \in [0, 1] \\ y'(t) & = & f_2(t, x(t), y(t)), a.e.t \in [0, 1] \\ x(t_k^+) - x(t_k^-) & = & I_{1,k}(x(t_k), y(t_k)),\ k = 1, \cdots, m \\ y(t_k^+) - y(t_k^-) & = & I_{2,k}(x(t_k), y(t_k)), k = 1, \cdots, m \\ x(0) = x_0, & & y(0) = y_0, \end{array}\right. \tag{4.4}$$

where $f_1, f_2 \in L^1(J \times \mathbb{R}^2, \mathbb{R})$ are a given functions, $0 = t_o < t_1 < \cdots < t_m < t_{m+1} = 1$.
Then (x, y) is a solution of (4.4) if and only if (x, y) is a solution of the impulsive integral system

$$\left\{\begin{array}{rcl} x(t) & = & x_0 + \int_0^t f_1(s, x(s), y(s))ds + \sum_{0<t_k<t} I_{1,k}(x(t_k), y(t_k)), a.e.t \in J \\ y(t) & = & y_0 + \int_0^t f_2(s, x(s), y(s))ds + \sum_{0<t_k<t} I_{2,k}(x(t_k), y(t_k)), a.e.t \in J. \end{array}\right. \tag{4.5}$$

Denote by $S(f_{1,2}, (x_0, y_0))$ the set of all solutions of problem (4.4).

Theorem 4.3.1. *Suppose that there exists a function* $\ell \in L^1([0,1], \mathbb{R}^+)$ *such that:*

$|f_i(t, x_1, y_1) - f_i(t, x_2, y_2)| < \ell(t)|x_1 - x_2| + |y_1 - y_2)|; \forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2, \forall i = 1, 2.$

Then the problem (4.4) *has a unique solution.*

Proof. 1.The existence:
We consider the problem (4.4) on $[0, t_1]$

$$\begin{array}{c} x'(t) = f_1(t, x(t), y(t)), y'(t) = f_2(t, x(t), y(t)), \quad a.e. t \in [0, t_1], \\ x(0) = x_0, y(0) = y_0, \end{array} \tag{4.6}$$

we consider the operator N_1 defined by

$$\begin{array}{ccc} N_1 : C([0, t_1]; \mathbb{R}) \times C([0, t_1]; \mathbb{R}) & \longrightarrow & C([0, t_1]; \mathbb{R}) \times C([0, t_1]; \mathbb{R}) \\ (x, y) & \longrightarrow & N_1(x, y) \end{array}$$

$$N_1(x, y)(t) = \left(x_0 + \int_0^t f_1(s, x(s), y(s))ds, \; y_0 + \int_0^t f_2(s, x(s), y(s))ds \right).$$

Let's $(x_1, y_1), (x_2, y_2) \in C([0, t_1]; \mathbb{R}) \times C([0, t_1]; \mathbb{R})$, and $t \in [0, t_1]$

$$\|N_1(x_1, y_1)(t) - N_1(x_2, y_2)(t)\| \;=\; \|(\alpha, \beta)\| = \begin{pmatrix} \|\alpha\| \\ \|\beta\| \end{pmatrix},$$

where

$$\alpha \;=\; \int_0^t (f_1(s, x_1(s), y_1(s)) - f_1(s, x_2(s), y_2(s))))ds,$$

and

$$\beta \;=\; \int_0^t (f_2(s, x_1(s), y_1(s) - f_2(s, x_2(s), y_2(s))))ds$$

then

$$\|\alpha\| \;\leq\; \int_0^t \ell_1(s) \, \|(x_1(s), y_1(s)) - (x_2(s), y_2(s))\| \, ds$$

$$\begin{aligned}
&\leq \frac{1}{\tau}\int_0^t \tau\ell(s)e^{\tau L(s)}ds \left\| \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix} \right\|_{BC} \\
&\leq \frac{1}{\tau}e^{\tau L(t)} \left\| \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix} \right\|_{BC} \\
&\leq e^{\tau L(t)} \left(\frac{1}{\tau}\|x_1 - x_2\| + \frac{1}{\tau}\|y_1 - y_2\| \right),
\end{aligned}$$

similarly

$$\|\beta\| \leq e^{\tau L(t)} \left(\frac{1}{\tau}\|x_1 - x_2\| + \frac{1}{\tau}\|y_1 - y_2\| \right),$$

where

$$L(t) = \int_0^t \ell(s)ds.$$

So

$$e^{-\tau L(t)} \|N_1(x_1, y_1)(t) - N_1(x_2, y_2)(t)\| \leq \begin{pmatrix} \frac{1}{\tau} & \frac{1}{\tau} \\ & \\ \frac{1}{\tau} & \frac{1}{\tau} \end{pmatrix} \begin{pmatrix} \|x_1 - x_2\| \\ \\ \|y_1 - y_2\| \end{pmatrix}.$$

Then

$$\|N_1(x_1, y_1) - N_1(x_2, y_2)\|_{BC} \leq \frac{1}{\tau} \begin{pmatrix} 1 & 1 \\ & \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \|x_1 - x_2\| \\ \\ \|y_1 - y_2\| \end{pmatrix},$$

where $\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_{BC} = \sup_{t\in[0,t_1]} e^{-\tau L(t)} \left\| \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \right\|$.
Let

$$B = \frac{1}{\tau} \begin{pmatrix} 1 & 1 \\ & \\ 1 & 1 \end{pmatrix}$$

and

$$A = \begin{pmatrix} 1 & 1 \\ & \\ 1 & 1 \end{pmatrix}$$

we have $\|A\| = 1$ so $\|A^n\| = 1$ and $\|B^n\| = \|(\frac{1}{\tau})^n A^n\| \leq \left|(\frac{1}{\tau})^n\right| \|A^n\| \leq \left|(\frac{1}{\tau})^n\right|$
For $\tau \in (1, +\infty)$; N_1 is contractive, then there exists unique

$$(x^0, y^0) \in C([0, t_1]; \mathbb{R}) \times C([0, t_1]; \mathbb{R}) : N_1(x^0, y^0) = (x^0, y^0).$$

(x^0, y^0) is the solution of (4.6).

• We consider the problem (4.4) on $(t_1, t_2]$

$$\begin{array}{ll} x'(t) = f_1(t, x(t), y(t)), y'(t) = f_2(t, x(t), y(t)), & a.e. t \in J_1 = (t_1, t_2] \\ x(t_1^+) = x^0(t_1) + I_{1,k}(x^0(t_1), y^0(t_1)), & y(t_1^+) = y^0(t_1) + I_{2,k}(x^0(t_1), y^0(t_1)) \end{array} \tag{4.7}$$

We consider the space $C_* = \{(x, y) \in C(J_1, \mathbb{R}) \times C(J_1, \mathbb{R}) \ /(x(t_1^+), y(t_1^+)) \text{ exist}\}$, $(C_*; \|.\|_{J_1})$ is a Banach space.

Let

$$\begin{array}{rcl} N_2 : C_* & \longrightarrow & C_* \\ (x, y) & \longrightarrow & N_2(x, y) \end{array}$$

$$N_2(x, y)(t) = \left(x^0(t_1) + I_{1,k}(x^0(t_1), y^0(t_1)) + \int_{t_1}^t f_1(s, x(s), y(s))ds \right. ,$$

$$\left. y^0(t_1) + I_{2,k}(x^0(t_1), y^0(t_1)) + \int_{t_1}^t f_2(s, x(s), y(s))ds \right).$$

Let $(x_1, y_1), (x_2, y_2) \in C_* \times C_*$, and $t \in (t_1, t_2]$

$$\|N_2(x_1, y_1)(t) - N_2(x_2, y_2)(t)\| = \|(\alpha, \beta)\| = \begin{pmatrix} \|\alpha\| \\ \\ \|\beta\| \end{pmatrix},$$

where

$$\begin{aligned} \|\alpha\| &\le \int_{t_1}^t \ell(s) \, \|(x_1(s), y_1(s)) - (x_2(s), y_2(s))\| \, ds \\ &\le \frac{1}{\tau} \int_{t_1}^t \tau \ell(s) e^{\tau L(s)} ds \left\| \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix} \right\|_{BC}, \\ &\le e^{\tau L(t)} \left(\frac{1}{\tau} \|x_1 - x_2\| + \frac{1}{\tau} \|y_1 - y_2\| \right). \end{aligned}$$

Similarly

$$\|\beta\| \le e^{\tau L(t)} \left(\frac{1}{\tau} \|x_1 - x_2\| + \frac{1}{\tau} \|y_1 - y_2\| \right),$$

such that

$$L(t) = \int_{t_1}^t \ell(s) ds,$$

so

$$e^{-\tau L(t)} \|N_2(x_1,y_1)(t) - N_2(x_2,y_2)(t)\| \leq \frac{1}{\tau}\begin{pmatrix} 1 & 1 \\ & \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \|x_1 - x_2\| \\ \\ \|y_1 - y_2\| \end{pmatrix}.$$

Then

$$\|N_2(x_1,y_1) - N_2(x_2,y_2)\|_{BC} \leq \frac{1}{\tau}\begin{pmatrix} 1 & 1 \\ & \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \|x_1 - x_2\| \\ \\ \|y_1 - y_2\| \end{pmatrix}.$$

Then for $\tau \in (1, +\infty)$; N_2 is contractive, so there exists unique

$$(x^1, y^1) \in C(]t_1, t_2]; \mathbb{R}) : \; N_2(x^1, y^1) = (x^1, y^1),$$

we have

$$\begin{aligned} (x^1,y^1)(t_1^+) &= N_2(x^1,y^1)(t_1^+) = (x^0(t_1) + I_{1,k}(x^0(t_1), y^0(t_1)) \\ &+ \lim_{t \longrightarrow t_1^+} \textstyle\int_{t_1}^t f_1(s, x(s), y(s))ds, \end{aligned}$$

$$y^0(t_1) + I_{2,k}(x^0(t_1), , y^0(t_1)) + \lim_{t \longrightarrow t_1^+} \int_{t_1}^t f_2(s, x(s), y(s))ds).$$

Then (x^1, y^1) is the solution of the problem (4.7). As a consequence, the solution of the problem (4.4) is given by

$$(x^*, y^*)(t) = \begin{cases} (x^0, y^0)(t), & t \in [0, t_1] \\ (x^1, y^1)(t), & t \in (t_1, t_2] \\ \vdots & \vdots \\ (x^m, y^m)(t), & t \in (t_m, b]. \end{cases}$$

2.The uniqueness:
Lets $(x^*, y^*), (x^{**}, y^{**})$ are two solutions of the problem (4.4); we are going to show that:

$$(x^*, y^*)(t) = (x^{**}, y^{**})(t), \; \text{for all} \; t \in [0, 1],$$

if $t \in J_0 = [0, t_1]$ then $(x^*, y^*)(t) = (x^{**}, y^{**})(t)$, for all $t \in [0, t_1]$
if $t \in J_i = (t_i, t_{i+1}]$ then $(x^*, y^*)(t) = (x^{**}, y^{**})(t)$, for each $t \in (t_i, t_{i+1}]$.
Now it is enough to show that
$(x^*, y^*)(t_k^+) = (x^{**}, y^{**})(t_k^+)$, $k \in \{1, 2, \cdots, m\}$
we have:
$(x^*, y^*)(t_i^+) - (x^*, y^*)(t_i^-) = (I_{1,i}(x^*(t_i), y^*(t_i)), I_{2,i}(x^*(t_i), y^*(t_i)))$
implies that

$$\begin{aligned}(x^*, y^*)(t_i^+) &= (x^*, y^*)(t_i^-) + (I_{1,i}(x^*(t_i), y^*(t_i)), I_{2,i}(x^*(t_i), y^*(t_i)))\\ &= (x^{**}, y^{**})(t_i) + (I_{1,i}(x^{**}(t_i), y^{**}(t_i)), I_{2,i}(x^{**}(t_i), y^{**}(t_i)))\\ &= (x^{**}, y^{**})(t_i^+).\end{aligned}$$

□

Theorem 4.3.2. *Suppose there exists a continuous nondecreasing function $\psi : [0, \infty) \longrightarrow (0, \infty)$, and $p \in L^1(J, \mathbb{R}_+)$ such that*

$$\|f_i(t, x, y)\| \leq p(t)\psi(\|x\| + \|y\|); \text{ for all } t \in J, \text{ and all } x, y \in \mathbb{R},$$

with

$$\int_0^1 p(s)ds < \int_{\|(x_0, y_0)\|}^{\infty} \frac{du}{\psi(u)}.$$

Then the problem (4.4) *has at least one solution.*

Proof. For the proof we use "The non linear alternative of Leray and Schauder".
Consider the operator

$$N : PC(J, \mathbb{R}) \times PC(J, \mathbb{R}) \longrightarrow PC(J, \mathbb{R}) \times PC(J, \mathbb{R})$$

defined by

$$\begin{aligned}N(x, y)(t) &= (x_0 + \int_0^t f_1(s, x(s), y(s))ds + \sum_{0<t_k<t} I_{1,k}(x(t_k), y(t_k)),\\ &\quad y_0 + \int_0^t f_2(s, x(s), y(s))ds + \sum_{0<t_k<t} I_{2,k}(x(t_k), y(t_k))).\end{aligned}$$

The fixed point of N is solution of the problem (4.4). It is enough to prove that N is completely continuous, the proof is given by the following steps.

Step 1 N is continuous:
Let $(x_n, y_n)_n$ be a sequence in $PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$ such that $(x_n, y_n) \longrightarrow (x, y)$, it is enough to prove that $N(x_n, y_n) \longrightarrow N(x, y)$. For all $t \in J$ we have:

$$\begin{aligned} N(x_n, y_n)(t) &= (x_0 + \int_0^t f_1(s, x_n(s), y_n(s))ds + \sum_{0<t_k<t} I_{1,k}(x_n(t_k), y_n(t_k)), \\ & \quad\; y_0 + \int_0^t f_2(s, x_n(s), y_n(s))ds + \sum_{0<t_k<t} I_{2,k}(x_n(t_k), y_n(t_k))). \end{aligned}$$

Then

$$\begin{aligned} \|N(x_n, y_n)(t) - N(x, y)(t)\| &= \|\,(\alpha, \beta)\,\| \\ &= \begin{pmatrix} \|\alpha\| \\ \|\beta\| \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} \|\alpha\| &= \| \int_0^t (f_1(s, x_n(s), y_n(s)) - f_1(s, x(s), y(s)))ds \\ &+ \sum_{0<t_k<t} (I_{1,k}(x_n(t_k), y_n(t_k)) - I_{1,k}(x(t_k), y(t_k)))\| \\ &\leq \int_0^t \|f_1(s, x_n(s), y_n(s)) - f_1(s, x(s), y(s))\|ds \\ &+ \sum_{0<t_k<t} \|I_{1,k}(x_n(t_k), y_n(t_k)) - I_{1,k}(x(t_k), y(t_k))\|, \end{aligned}$$

as $I_k, k = 1, \cdots, m$ are continuous functions, and f_1, f_2 are L^1-Caratheodory functions, and by the dominated convergence theorem of Lebesgue we have

$$\begin{aligned} \|\alpha\| &\leq \int_0^b \|f_1(s, x_n(s), y_n(s)) - f_1(s, x(s), y(s))\|ds \\ &+ \sum_{0<t_k<m} \|I_{1,k}(x_n(t_k), y_n(t_k)) - I_{1,k}(x(t_k), y(t_k))\| \to 0 \text{ when } n \to \infty, \end{aligned}$$

similarly

$$\|\beta\| \leq \int_0^b \|f_2(s, x_n(s), y_n(s)) - f_2(s, x(s), y(s))\|ds$$

$$+ \sum_{0<t_k<m} \|I_{2,k}(x_n(t_k), y_n(t_k)) - I_{2,k}(x(t_k), y(t_k))\| \to 0 \,\text{when}\, n \to \infty,$$

so

$$\|N(x_n, y_n) - N(x, y)\| \;\; \leq \;\; \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then N is continuous.

Step 2 N transforms every bounded set to a bounded set in $PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$:

It suffices to show that, for all $q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} > 0$, there exists $\ell = \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} > 0$, for each (x, y) in $\mathcal{B}_q = \{(x, y) \in PC \times PC : \|(x, y)\| \leq q\}$, we have $\|N(x, y)\| \leq \ell$.

Let $(x, y) \in \mathcal{B}_q$, we have

$$\begin{aligned} \|N(x,y)\| \;\; &\leq \;\; (\|x_0\| + \int_0^b \|f_1(s, x(s), y(s))\| \, ds + \sum_{k=1}^m \|I_{1,k}(x(t_k), y(t_k))\| \\ &\quad , \;\; \|y_0\| + \int_0^b \|f_2(s, x(s), y(s))\| \, ds + \sum_{k=1}^m \|I_{2,k}(x(t_k), y(t_k))\|) \\ &= \;\; (\alpha, \beta), \end{aligned}$$

where

$$\begin{aligned} \|\alpha\| \;\; &\leq \;\; \|x_0\| + \int_0^b p(t)\psi(\|x\|_{PC} + \|y\|_{PC})dt + \sum_{k=1}^m \|I_{1,k}(x(t_k), y(t_k))\| \\ &\leq \;\; \|x_0\| + \int_0^b p(t)\psi(\|x\|_{PC} + \|y\|_{PC})dt + \sum_{k=1}^m \sup_{(x,y)\in\overline{B_q}} \|I_{1,k}(x, y)\| := \ell_1, \end{aligned}$$

similarly

$$\|\beta\| \leq \|y_0\| + \int_0^b p(t)\psi(\|x\|_{PC} + \|y\|_{PC})dt + \sum_{k=1}^m \sup_{(x,y)\in\overline{B_q}} \|I_{2,k}(x, y)\| := \ell_2.$$

Step 3 N transforms every bounded set to a equicontinuous set to $PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$

We let $\tau_1, \tau_2 \in J, \tau_1 < \tau_2$ and $\mathcal{B}_q = \{(x, y) \in PC \times PC : \|(x, y)\| \leq q\}$, $q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} > 0$; let $(x, y) \in \mathcal{B}_q$, then:

1. If $\tau_1 \neq t_k$ (or $\tau_2 \neq t_k$), $\forall k \in \{1, 2, \cdots, m\}$, we have

$$\begin{aligned}\|N(x,y)(\tau_2) - N(x,y)(\tau_1)\| &\leq (\int_{\tau_1}^{\tau_2} p(s)\psi(q_1+q_2)ds + \sum_{\tau_1 \leq t_k < \tau_2} \sup_{(x,y)\in\overline{B_q}} \|I_{1,k}(x,y)\| \\ &, \quad \int_{\tau_1}^{\tau_2} p(s)\psi(q_1+q_2)ds + \sum_{\tau_1 \leq t_k < \tau_2} \sup_{(x,y)\in\overline{B_q}} \|I_{2,k}(x,y)\|) \\ &\longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{when} \quad \tau_1 \longrightarrow \tau_2\end{aligned}$$

2. If $\tau_1 = t_i^-$, we consider $\delta_1 > 0$ such that $\{t_k, k \neq i\} \cap [t_i - \delta_1, t_i + \delta_1] = \emptyset$, so for $0 < h < \delta_1$ we have

$$\begin{aligned}\|N(x,y)(t_i) - N(x,y)(t_i - h)\| &\leq (\int_{t_i-h}^{t_i} p(s)\psi(q_1+q_2)ds, \int_{t_i-h}^{t_i} p(s)\psi(q_1+q_2)ds) \\ &\longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{when} \quad h \longrightarrow 0\end{aligned}$$

3. If $\tau_2 = t_i^+$, we consider $\delta_2 > 0$ such that $\{t_k, k \neq i\} \cap [t_i - \delta_2, t_i + \delta_2] = \emptyset$, so for $0 < h < \delta_2$ we have

$$\begin{aligned}\|N(x,y)(t_i + h) - N(x,y)(t_i)\| &\leq (\int_{t_i}^{t_i+h} p(s)\psi(q_1+q_2)ds, \int_{t_i}^{t_i+h} p(s)\psi(q_1+q_2)ds) \\ &\longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{when} \quad h \longrightarrow 0.\end{aligned}$$

So by steps 1, 2 and 3, which is obtained, and by Arzela-Ascoli's theorem, N is completely continuous.

Step 4 A priori estimation:
Let $(x, y) \in PC(J, \mathbb{R}) \times PC(J, \mathbb{R})$ such that $(x, y) = \lambda N(x, y)$, and $0 < \lambda < 1$. Then for all $t \in [0, t_1]$ we have

$$x(t) = \lambda x_0 + \lambda \int_0^t f_1(s, x(s), y(s))ds,$$

and

$$y(t) = \lambda y_0 + \lambda \int_0^t f_2(s, x(s), y(s))ds,$$

so

$$\begin{aligned}\|(x,y)(t)\| &\leq (\|x_0\| + \int_0^t p(s)\psi(\|x(s)\| + \|y(s)\|)ds \\ &, \quad \|y_0\| + \int_0^t p(s)\psi(\|x(s)\| + \|y(s)\|)ds).\end{aligned}$$

Consider the map $\vartheta = (\vartheta_1, \vartheta_2)$ such that

$$\vartheta_1(t) = \|x_0\| + \int_0^t p(s)\psi(\|x(s)\| + \|y(s)\|)ds,$$

and

$$\vartheta_2(t) = \|y_0\| + \int_0^t p(s)\psi(\|x(s)\| + \|y(s)\|)ds,$$

then we have

$$\vartheta(0) = (\|x_0\|, \|y_0\|), \quad \|(x,y)(t)\| \leq \vartheta(t),$$

and

$$\dot{\vartheta}_i(t) = p(t)\psi(\|x(s)\| + \|y(t)\|), \forall i = 1, 2.$$

As ψ is nondecreasing map, we have

$$\dot{\vartheta}_i(t) \leq p(t)\psi(\vartheta_i(t)), \forall i = 1, 2.$$

What implies that for every $t \in [0, t_1]$, we have

$$\int_{\vartheta_i(0)}^{\vartheta_i(t)} \frac{du}{\psi(u)} \leq \int_0^{t_1} p(s)ds, \forall i = 1, 2.$$

The map $\Gamma_{i,0}(z) = \displaystyle\int_{\vartheta_i(0)}^{z} \frac{du}{\psi(u)}, i = 1, 2$ is continuous and nondecreasing, then $\Gamma_{i,0}^{-1}$ exist and they are nondecreasing, we have

$$\vartheta_i(t) \leq \Gamma_{i,0}^{-1}\left(\int_0^{t_1} p(s)ds\right) := M_{i,0}, i = 1, 2.$$

As for all $t \in [0, t_1]$, $\|(x,y)(t)\| \leq \vartheta(t)$, then

$$\sup_{t\in[0,t_1]} \|(x,y)(t)\| \leq \begin{pmatrix} M_{1,0} \\ M_{2,0} \end{pmatrix}.$$

Now, for $t \in (t_1, t_2]$, we have

$$\begin{aligned}\left\|x(t_1^+)\right\| &\leq \|I_{1,1}(x(t_1), y(t_1))\| + \|x(t_1)\| \\ &\leq \sup_{(x,y)\in \overline{B}_q} \|I_{1,1}(x,y)\| + M_{1,0} := N_1,\end{aligned}$$

and

$$\begin{aligned}\left\|y(t_1^+)\right\| &\leq \|I_{2,1}(x(t_1), y(t_1))\| + \|y(t_1)\| \\ &\leq \sup_{(x,y)\in \overline{B}_q} \|I_{2,1}(x,y)\| + M_{2,0} := N_2,\end{aligned}$$

where

$$q = \begin{pmatrix} M_{1,0} \\ M_{2,0} \end{pmatrix}$$

$$y(t) = \lambda\left(x(t_1) + I_{1,1}(x(t_1), y(t_1))\right) + \lambda \int_{t_1}^{t} f_1(s, x(s), y(s))ds,$$

and

$$y(t) = \lambda\left(y(t_1) + I_{2,1}(x(t_1), y(t_1))\right) + \lambda \int_{t_1}^{t} f_2(s, x(s), y(s))ds.$$

Then

$$\|x(t)\| \leq N_1 + \int_{t_1}^{t} p(s)\psi(\|x(s)\| + \|y(s)\|)ds,$$

and

$$\|y(t)\| \leq N_2 + \int_{t_1}^{t} p(s)\psi(\|x(s)\| + \|y(s)\|)ds.$$

Consider the map $W = (W_1, W_2)$ such that

$$\begin{aligned}W_1(t) &= N_1 + \int_{t_1}^{t} p(s)\psi(\|x(s)\| + \|y(s)\|)ds \\ W_2(t) &= N_2 + \int_{t_1}^{t} p(s)\psi(\|x(s)\| + \|y(s)\|)ds,\end{aligned}$$

so we have

$$W(t_1^+) = (N_1, N_2), \quad \|(x,y)(t)\| \leq W(t),$$

and

$$\dot{W}_i(t) = p(t)\psi(\|x(s)\| + \|y(t)\|), \forall i = 1, 2,$$

ψ is nondecreasing, then

$$\dot{W}_i(t) \leq p(t)\psi(W_i(t)), \forall i = 1, 2.$$

What implies is that for every $t \in (t_1, t_2]$, we have

$$\int_{W_i(t_1^+)}^{W_i(t)} \frac{du}{\psi(u)} \leq \int_{t_1}^{t_2} p(s)ds; i = 1, 2.$$

If we consider the map $\Gamma_{i,1}(z) = \displaystyle\int_{W_i(t_1^+)}^{z} \frac{du}{\psi(u)}, i = 1, 2,$ we get

$$W_i(t) \leq \Gamma_{i,1}^{-1}\left(\int_{t_1}^{t_2} p(s)ds\right) := M_{i,1}; i = 1, 2.$$

For all $t \in (t_1, t_2]$, $\|(x, y)(t)\| = \begin{pmatrix} \|x(t)\| \\ \|y(t)\| \end{pmatrix} \leq \begin{pmatrix} W_1(t) \\ W_2(t) \end{pmatrix}$, then

$$\sup_{t \in (t_1, t_2]} \|(x, y)(t)\| \leq \begin{pmatrix} M_{1,1} \\ M_{2,1} \end{pmatrix},$$

we continue the process until we reach the interval $(t_m, 1]$, $(x, y)|_{(t_m,1]}$ is the solution of the problem $(x, y) = \lambda N(x, y)$ in $(t_m, 1]$, for $0 < \lambda < 1$. We get there exist $M_{i,m}, i = 1, 2$ such that

$$\sup_{t \in (t_m, b]} \|(x, y)(t)\| \leq \Gamma_{i,m}^{-1}\left(\int_{t_m}^{b} p(s)ds\right) := M_{i,m}.$$

As we choose (x, y) arbitrarily, then for all solution of the problem (4.4) we have

$$\|(x, y)\| \leq \begin{pmatrix} \max_{k=0,1,\cdots,m}(M_{1,k}) \\ \max_{k=0,1,\cdots,m}(M_{2,k}) \end{pmatrix} := \begin{pmatrix} b_1^* \\ b_2^* \end{pmatrix}.$$

Consider the set

$$U = \{(x, y) \in PC \times PC : \|(x, y)\|_{PC} < \begin{pmatrix} b_1^* + 1 \\ b_2^* + 1 \end{pmatrix}\}.$$

So we get $N : \overline{U} \longrightarrow PC \times PC$ is completely continuous, and by the definition of U it doses not exist a $(x, y) \in \partial U$ such that $(x, y) = \lambda N(x, y)$ for all $\lambda \in (0, 1)$.
Then by the theorem 1.6.3, N has a fixed point $(x, y) \in \overline{U}$ which is solution of the problem (4.4).

□

Theorem 4.3.3. *Suppose that we have the same conditions of the theorem 4.3.2, then the set of all solutions of the problem* (4.4) *is non empty, compact,* R_δ, *acyclic. Moreover the solution operator* S *is u.s.c., where*

$$\begin{array}{rcl} S:\mathbb{R}\times\mathbb{R} & \longrightarrow & \mathcal{P}_{cp}(PC\times PC)\\ (x_0,y_0) & \longrightarrow & S(x_0,y_0)\end{array}$$

$$\begin{array}{rcl} S(x_0,y_0) & = & \{(x,y)\in PC\times PC:(x,y) \ \textit{solution of the problem (4.4) with}\\ & & (x(0),y(0))=(x_0,y_0)\}.\end{array}$$

Proof. • The solution set is compact:
Let $(a,b)\in\mathbb{R}\times\mathbb{R}$,

$$\begin{array}{rcl} S(a,b) & = & \{(x,y)\in PC\times PC:(x,y) \ \text{solution of the problem (4.4) with}\\ & & (x(0),y(0))=(a,b)\}.\end{array}$$

1. S(a,b) is a closed set
   Let $(x_q,y_q)_q$ be a sequence in $S(a,b)$, such that

   $$\lim_{q\longrightarrow\infty}(x_q,y_q)=(x,y).$$

   Let

   $$\begin{array}{rcl} Z_1(t) & = & a+\displaystyle\int_0^t f_1(s,x(s),y(s))ds+\sum_{0<t_k<t}I_{1,k}(x(t_k),y(t_k))\\ Z_2(t) & = & b+\displaystyle\int_0^t f_2(s,x(s),y(s))ds+\sum_{0<t_k<t}I_{2,k}(x(t_k),y(t_k)),\end{array}$$

   let $t\in[0,1]$, we have

   $$\begin{array}{rcl} \|x_q(t)-Z_1(t)\| & \leq & \displaystyle\int_0^t\|f_1(s,x_q(s),y_q(s))-f_1(s,x(s),y(s))\|\,ds\\ & + & \displaystyle\sum_{0<t_k<t}\|I_{1,k}(x_q(t_k),y_q(t))-I_{1,k}(x(t_k),y(t_k))\|\\ & \leq & \displaystyle\int_0^1\|f-1(s,x_q(s),y_q(s))-f_1(s,x(s),y(s))\|\,ds\end{array}$$

$$+ \sum_{k=1}^{m} \|I_{1,k}(x_q(t_k), y_q(t)) - I_{1,k}(x(t_k), y(t_k))\|,$$

by the dominated convergence theorem of Lebesgue we have

$$\|x_q(t) - Z_1(t)\| \longrightarrow 0 \ \text{ when } \ q \longrightarrow \infty,$$

similarly

$$\|y_q(t) - Z_2(t)\| \longrightarrow 0 \ \text{ when } \ q \longrightarrow \infty.$$

So, $\lim_{q\longrightarrow\infty}(x_q, y_q) = (x, y) = (Z_1, Z_2) \in S(a, b)$.

2. S(a,b) is uniformly bounded:
   Let $(x, y) \in S(a, b)$, then (x, y) is solution of the problem (4.4), so $\exists b^* > 0$ such that
   $$\|(x, y)\| \leq b^*.$$
3. S(a,b) is equicontinuous:
   Let $r_1, r_2 \in [0, 1], r_1 < r_2$ and $(x, y) \in S(a, b)$:

$$\begin{aligned}\|(x,y)(r_1) - (x,y)(r_2)\| \ &\leq \ (\int_{r_1}^{r_2} \|f_1(s, x(s), y(s))\| \, ds + \sum_{r_1<t_k<r_2} \|I_{1,k}(x(t), y(t))\| \\ &\quad , \ \int_{r_1}^{r_2} \|f_2(s, x(s), y(s))\| \, ds + \sum_{r_1<t_k<r_2} \|I_{2,k}(x(t), y(t))\|)\end{aligned}$$

$$\begin{aligned}\int_{r_1}^{r_2} \|f_1(s, x(s), y(s))\| \, ds \ &+ \ \sum_{r_1<t_k<r_2} \|I_{1,k}(x(t), y(t))\| \leq \int_{r_1}^{r_2} p(s)\psi(\|x(s)\| + \|y(s)\|)ds \\ &+ \ \sum_{r_1<t_k<r_2} \sup_{(x,y)\in\overline{B}_{b^*}} \|I_{1,k}(x, y)\| \\ &\leq \ \int_{r_1}^{r_2} p(s)\psi(b_1^* + b_2^*)ds + \sum_{r_1<t_k<r_2} \sup_{(x,y)\in\overline{B}_{b^*}} \|I_{1,k}(x, y)\| \\ &\longrightarrow 0 \ \text{ when } \ r_1 \longrightarrow r_2.\end{aligned}$$

Then, $S(a, b)$ is compact.
• The solution set $S(a, b)$ is R_δ
Let $N : PC \times PC \longrightarrow PC \times PC$ defined by

$$N(x, y)(t) \ = \ (a + \int_0^t f_1(s, x(s), y(s))ds + \sum_{0<t_k<t} I_{1,k}(x(t_k), y(t_k))$$

$$, \quad b+\int_0^t f_2(s,x(s),y(s))ds+\sum_{0<t_k<t} I_{2,k}(x(t_k),y(t_k))) \quad t\in[0,1].$$

then, $FixN = S(a,b)$, by the step 4 of the proof of the theorem 4.3.2, there exists $b^* > 0$ such that

$$\|(x,y)\| \leq b^*; \forall (x,y) \in S(a,b),$$

for $i = 1,2$ we defined

$$\widetilde{f}_i(t,y(t)) = \begin{cases} f_i(t,x(t),y(t)) & \text{if } \|(x,y)(t)\| \leq b^*, \\ f_i(t, \dfrac{b_1^* x(t)}{\|x(t)\|}, \dfrac{b_2^* y(t)}{\|y(t)\|}) & \text{if } \|(x,y)(t)\| \geq b^*, \end{cases}$$

and

$$\widetilde{I}_{i,k}(x(t),y(t)) = \begin{cases} I_{i,k}(x(t),y(t)) & \text{if } \|(x,y)(t)\| \leq b^*, \\ I_{i,k}(\dfrac{b_1^* x(t)}{\|x(t)\|}, \dfrac{b_2^* y(t)}{\|y(t)\|}) & \text{if } \|(x,y)(t)\| \geq b^*, \end{cases}$$

as f_i are L^1-Carathéodory, then $\widetilde{f^i}$ are also L^1-Carathéodory, and there exists $h \in L^1(J,\mathbb{R}_+)$ such that

$$\left\|\widetilde{f}_i(t,x,y)\right\| \leq h(t); \forall i=1,2, \quad \text{a.e.} \quad t\in J; \quad \text{and} \quad (x,y)\in\mathbb{R}\times\mathbb{R}. \tag{4.8}$$

Consider the problem

$$\begin{cases} \dot{x}(t) &= \widetilde{f}_1(t,x(t),y(t)), \ t\in[0,1] \\ \dot{y}(t) &= \widetilde{f}_2(t,x(t),y(t)), \ t\in[0,1] \\ x(t_k^+)-x(t_k^-) &= \widetilde{I}_{1,k}(x(t_k),y(t_k^-)), \ k=1,2,\cdots,m \\ y(t_k^+)-y(t_k^-) &= \widetilde{I}_{2,k}(x(t_k),y(t_k^-)), \ k=1,2,\cdots,m, \\ x(0) &= a, \quad y(0)=b. \end{cases} \tag{4.9}$$

We can easily prove that $FixN = Fix\widetilde{N}$, where $\widetilde{N} : PC\times PC \longrightarrow PC\times PC$ defined by

$$\begin{aligned} \widetilde{N}(x,y)(t) &= (a+\int_0^t \widetilde{f}_i(s,x(s),y(s))ds+\sum_{0<t_k<t}\widetilde{I}_{1,k}(x(t_k),y(t_k)) \\ &\quad , \quad b+\int_0^t \widetilde{f}_2(s,x(s),y(s))ds+\sum_{0<t_k<t}\widetilde{I}_{2,k}(x(t_k),y(t_k))). \end{aligned}$$

By the inequalities (4.8), and the continuity of $I_{i,k}, i = 1, 2$, we have

$$\begin{aligned}\left\|\widetilde{N}(x,y)\right\| &\leq (\|a\| + \|h\|_{L^1} + \sum_{k=1}^{m} \sup_{(x,y)\in\overline{B}_b^*} \|I_{1,k}(x,y)\| \\ &, \quad \|b\| + \|h\|_{L^1} + \sum_{k=1}^{m} \sup_{(x,y)\in\overline{B}_b^*} \|I_{2,k}(x,y)\|) := (r_1, r_2) = r,\end{aligned}$$

then $\widetilde{N}$ is uniformly bounded.
We can easily prove that the function $\mathcal{M}$ defined by $\mathcal{M}(x,y) = (x,y) - \widetilde{N}(x,y)$ is a propre function, and as $\widetilde{N}$ is compact and by Lasota Yorke's theorem (theorem 1.5.1), We can prove easily that the conditions of theorem 1.5.2 are verified, then the set $\mathcal{M}^{-1}(0) = Fix\widetilde{N} = S(a,b)$ is R_δ-set, and it is also acyclic by the lemma 1.5.1.

- The solution operator is u.s.c.:

1 S has a closed graph:
The graph of S is the set

$$G_S = \{((a,b);(x,y)) \in (\mathbb{R}\times\mathbb{R}) \times (PC \times PC) | (x,y) \in S(a,b)\},$$

let $((a_q, b_q);(x_q, y_q))_q$ a sequence in G_S, and let $((a_q, b_q);(x_q, y_q))_q \to ((a,b);(x,y))$ when $q \to \infty$.
As $(x_q, y_q) \in S(a_q, b_q)$, then we have

$$x_q(t) = a_q + \int_0^t f_1(s, x_q(s), y_q(s))ds + \sum_{0<t_k<t} I_{1,k}(x_q(s), y_q(t_k)),$$

and

$$y_q(t) = b_q + \int_0^t f_2(s, x_q(s), y_q(s))ds + \sum_{0<t_k<t} I_{2,k}(x_q(s), y_q(t_k)),$$

let

$$\begin{aligned}Z(t) &= (Z_1(t), Z_2(t)) = (a + \int_0^t f_1(s, x(s), y(s))ds + \sum_{0<t_k<t} I_{1,k}(x(s), y(t_k)) \\ &, \quad b + \int_0^t f_2(s, x(s), y(s))ds + \sum_{0<t_k<t} I_{2,k}(x(s), y(t_k))),\end{aligned}$$

let $t \in [0,1]$, we have

$$\begin{aligned}\|(x_q,y_q)(t)-Z(t)\| \;\leq\; & (\|a_q-a\|+\int_0^b \|f_1(s,x_q(s),y_q(s))-f_1(s,x(s),y(s))\|\,ds \\ & +\sum_{k=1}^{m}\|I_{1,k}(x_q(t),y_q(t))-I_{1,k}(x(t),y(t))\| \\ , \;\; & \|b_q-b\|+\int_0^b \|f_2(s,x_q(s),y_q(s))-f_2(s,x(s),y(s))\|\,ds \\ & +\sum_{k=1}^{m}\|I_{2,k}(x_q(t),y_q(t))-I_{2,k}(x(t),y(t))\|),\end{aligned}$$

by the dominated convergence theorem of Lebesgue we have

$$\|(x_q,y_q)(t)-Z(t)\| \longrightarrow 0 \;\text{ when }\; q\longrightarrow\infty.$$

Then,

$$(x,y)(t)=Z(t)$$

what implies that $(x,y)\in S(a,b)$.

2 S transforms every bounded set to a relatively compact set
Let $r=\begin{pmatrix} r_1 \\ r_2\end{pmatrix}>0, \overline{B}_r := \{(x,y)\in PC\times PC : \|(x,y)\|\leq r\}$.
(a) $S(\overline{B}_r)$ is unifomly bounded:
Let $(x,y)\in S(\overline{B}_r)$, then there exists $(a,b)\in\overline{B}_r$ such that

$$x(t)=a+\int_0^t f_1(s,x(s),y(s))ds+\sum_{0<t_k<t} I_{1,k}(x(t_k),y(t_k)),$$

and

$$y(t)=b+\int_0^t f_2(s,x(s),y(s))ds+\sum_{0<t_k<t} I_{2,k}(x(t_k),y(t_k)),$$

by the same method detailed in step 4 of the proof of the theorem 4.3.2 we find that there exists $b^*>0$ such that

$$\|(x,y)\|\leq b^*.$$

(b) $S(\overline{B}_r)$ is equicontinuous set:
Let $\tau_1, \tau_2 \in [0,1], \tau_1 < \tau_2$, and $(x,y) \in S(\overline{B}_r)$, then

$$\begin{aligned}
\|(x,y)(\tau_2) - (x,y)(\tau_1)\| \;\leq\; & (\int_{\tau_1}^{\tau_2} \|f_1(s,x(s),y(s))\|ds + \sum_{\tau_1<t_k<\tau_2} \|I_{1,k}(x(t_k),y(t_k))\| \\
, & \int_{\tau_1}^{\tau_2} \|f_2(s,x(s),y(s))\|ds + \sum_{\tau_1<t_k<\tau_2} \|I_{2,k}(x(t_k),y(t_k))\|) \\
\leq & (\int_{\tau_1}^{\tau_2} p(s)\psi(\|x(s)\| + \|y(s)\|)ds + \sum_{\tau_1<t_k<\tau_2} \|I_{1,k}(x(t_k),y(t_k))\| \\
, & \int_{\tau_1}^{\tau_2} p(s)\psi(\|x(s)\| + \|y(s)\|)ds + \sum_{\tau_1<t_k<\tau_2} \|I_{2,k}(x(t_k),y(t_k))\|) \\
\leq & (\int_{\tau_1}^{\tau_2} p(s)\psi(b_1^* + b_2^*)ds + \sum_{\tau_1<t_k<\tau_2} \sup_{(x,y)\in\overline{B_{b^*}}} \|I_{1,k}(x,y))\| \\
, & \int_{\tau_1}^{\tau_2} p(s)\psi(b_1^* + b_2^*)ds + \sum_{\tau_1<t_k<\tau_2} \sup_{(x,y)\in\overline{B_{b^*}}} \|I_{2,k}(x,y)\|) \\
& \to 0 \text{ when } \tau_1 \to \tau_2.
\end{aligned}$$

Then the set $\overline{S(\overline{B}_r)}$ is compact, the operator S is locally compact and has a closed graph, then S is u.s.c.

□

Theorem 4.3.4. *Suppose that we have the conditions of the theorem 4.2.1, where $F_1, F_2 : [0,b] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$ are Carathédory, u.s.c. and mLL-sectionnable. Then the set of all solutions of the problem* (4.1) *is contractible.*
Proof. Let $f_i \in S_{F_i}$ be a measurable locally lipchitzienne selection of F_i, $i = 1, 2$.
Let us consider the problem

$$\left\{ \begin{array}{rcl} x'(t) &=& f_1(t,x(t),y(t)), \;\; a.e.t \in [0,1] \\ y'(t) &=& f_2(t,x(t),y(t)), \;\; a.e.t \in [0,1] \\ x(t_k^+) - x(t_k^-) &=& I_{1,k}(x(t_k),y(t_k)), k = 1,\cdots,m \\ y(t_k^+) - y(t_k^-) &=& I_{2,k}(x(t_k),y(t_k)), k = 1,\cdots,m \\ x(0) &=& x_0, \;\; y(0) = y_0, \end{array} \right. \tag{4.10}$$

by the theorem 4.3.1 the problem (4.10) has a unique solution.
Consider the homotopic function $h : S(x_0,y_0) \times [0,1] \longrightarrow S(x_0,y_0)$ defined

by

$$h((x,y),\alpha)(t) = \begin{cases} (x,y)(t), & \text{if } 0 \le t \le \alpha, \\ (x^*,y^*)(t), & \text{if } \alpha < t \le 1, \end{cases}$$

where (x^*, y^*) is the solution of problem (4.10), and $S(x_0, y_0)$ is the set of all solutions of problem (4.1). In particular

$$h((x,y),\alpha) = \begin{cases} (x,y), & \text{if } \alpha = 1, \\ (x^*,y^*), & \text{if } \alpha = 0. \end{cases}$$

Thus to prove that $S(x_0, y_0)$ is contractible it is enough to show that the homotopic h is continuous.
Let $((x_n, y_n), \alpha_n) \in S(x_0, y_0) \times [0, 1]$ such that $((x_n, y_n), \alpha_n) \to ((x, y), \alpha)$, when $n \to \infty$. we have

$$h((x_n,y_n),\alpha_n)(t) = \begin{cases} (x_n,y_n)(t), & \text{if } 0 \le t \le \alpha_n, \\ (x^*,y^*)(t), & \text{if } \alpha_n < t \le 1. \end{cases}$$

(a) If $\lim\limits_{n\to\infty} \alpha_n = 0$, then

$$h((x,y),0)(t) = (x^*,y^*)(t), \quad \text{for all } \ t \in [0,1].$$

Thus

$$\|h((x_n,y_n),\alpha_n) - h((x,y),\alpha)\|_\infty \le \|(x_n,y_n) - (x^*,y^*)\|_{[0,\alpha_n]} \to 0 \ \text{ when } \ n \to \infty.$$

(b) If $\lim\limits_{n\to\infty} \alpha_n = 1$, then

$$h((x,y),1)(t) = (x,y)(t), \quad \text{for all } \ t \in [0,1].$$

Thus

$$\|h((x_n,y_n),\alpha_n) - h((x,y),\alpha)\|_\infty \le \|(x_n,y_n) - (x,y)\|_{[0,\alpha_n]} \to 0 \ \text{ when } \ n \to \infty.$$

(c) If $0 < \lim\limits_{n\to\infty} \alpha_n = \alpha < 1$, then we distinguish following both cases
(1) If $t \in [0, \alpha]$, we have $(x_n, y_n) \in S(x_0, y_0)$ thus there exists $(v_{1n}, v_{2n}) \in S_{F_1} \times S_{F_2}$ such that for all $t \in [0, \alpha_n]$

$$x_n(t) = x_0 + \int_0^t v_{1n}(s)ds + \sum_{0<t_k<t} I_{1,k}(x_n(t_k), y_n(t_k)),$$

$$y_n(t) = y_0 + \int_0^t v_{2n}(s)ds + \sum_{0<t_k<t} I_{2,k}(x_n(t_k), y_n(t_k)),$$

by the step 5, in the proof of theorem 4.2.1 we have

$$\|(x_n, y_n)\|_{PC\times PC} \leq b^* = \begin{pmatrix} b_1^* \\ \\ b_2^* \end{pmatrix},$$

and by hypothesis, we have

$$\|(v_{1n}, v_{2n})(t)\| \leq p(t)\psi(b_1^* + b_2^*) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ for all } \; n \in \mathbb{N},$$

which implies

$$(v_{1n}, v_{2n})(t) \in p(t)\psi(b_1^* + b_2^*)\overline{B}(0, 1),$$

as $p(t)\psi(b_1^*+b_2^*)\overline{B}(0,1)$ is compact, thus it exist a sub-sequence $(v_{1n_m}, v_{2n_m})(.)$ which converges towards $(v_1, v_2)(.)$. We have $F_i(t, .)$ are u.s.c. then

$$\begin{array}{l} \forall\varepsilon > 0, \exists n_0 \geq 0; \forall n \geq n_0 : \\ v_{1n}(t) \in F_1(t, x_n(t), y_n(t)) \subset F_1(t, x(t), y(t)) + \varepsilon B(0, 1), \quad \text{a.e.} \quad t \in [0, \alpha], \\ v_{2n}(t) \in F_2(t, x_n(t), y_n(t)) \subset F_2(t, x(t), y(t)) + \varepsilon B(0, 1), \quad \text{a.e.} \quad t \in [0, \alpha]. \end{array}$$

And by the lemma 1.2.4 And the lemma 1.2.5, and as F has compact convex values, we obtain that

$$v_1(t) \in F_1(t, x(t), y(t)), \quad \text{a.e.} \quad t \in [0, \alpha].$$

$$v_2(t) \in F_2(t, x(t), y(t)), \quad \text{a.e.} \quad t \in [0, \alpha].$$

And by the dominated convergence theorem of Lebesgue, we find that

$$v_i \in L^1([0, 1], \mathbb{R}) \Longrightarrow v_i \in S_{F_i}, i = 1, 2.$$

Hence, for every $t \in [0, 1]$

$$x(t) = x_0 + \int_0^t v_1(s)ds + \sum_{0<t_k<t} I_{1,k}(x(t_k), y(t_k)),$$

and

$$y(t) = y_0 + \int_0^t v_2(s)ds + \sum_{0<t_k<t} I_{2,k}(x(t_k), y(t_k)).$$

(2) If $t \in (\alpha_n, 1]$, then

$$h((x_n, y_n), \alpha_n)(t) = h((x, y), \alpha)(t) = (x^*, y^*)(t).$$

Thus

$$\|h((x_n, y_n), \alpha_n) - h((x, y), \alpha)\| \to 0, \quad \text{when} \quad n \to \infty.$$

Hence, h is continuous, so the set $S(x_0, y_0)$ is contractible.

□

Theorem 4.3.5. *Suppose we have the conditions of the theorem 4.2.1, with $F_1, F_2 : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R} \times \mathbb{R})$ are Carathéodory, u.s.c and $\sigma-Ca-$selectionnable. Then the set of all solutions of the problem* (4.1) *is $R_\delta-$contractible and acyclic.*

Proof. Lets $f_i \in S_{F_i}$ a Carathéodory selection of F_i, $i = 1, 2$. Consider the homotopic multifunction $\Pi : S(x_0, y_0) \times [0, 1] \to \mathcal{P}(S(x_0, y_0))$ defined by

$$\Pi((x, y), \alpha) = \begin{cases} S(x_0, y_0)(t), & \text{if } 0 \leq t \leq \alpha \\ S(f, \alpha, (x, y)), & \text{if } \alpha < t \leq 1, \end{cases}$$

where

- $S(x_0, y_0)$ is the set of all solutions of problem (4.1),
- $S(f, \alpha, (x, y))$ is the set of all solutions of next problem

$$\left\{ \begin{array}{rcl} z_1'(t) & = & f_1(t, z_1(t), z_2(t)), \ a.e. t \in [\alpha, 1] \\ z_2'(t) & = & f_2(t, z_1(t), z_2(t)), \ a.e. t \in [\alpha, 1] \\ z_1(t_k^+) - z_1(t_k^-) & = & I_{1,k}(z_1(t_k), z_2(t_k)), \ k = 1, \cdots, m \\ z_2(t_k^+) - z_2(t_k^-) & = & I_{2,k}(z_1(t_k), z_2(t_k)), \ k = 1, \cdots, m \\ z_1(\alpha) & = & x(\alpha), \quad z_2(\alpha) = y(\alpha). \end{array} \right. \tag{4.11}$$

By the definition of Π, for all $(x, y) \in S(x_0, y_0)$, $(x, y) \in \Pi((x, y), 1)$ and $\Pi((x, y), 0) = S(f, 0, (x, y))$ which is $R_\delta-$set by the theorem 4.3.3.
It remains to show that Π is u.s.c. and $\Pi((x, y), \alpha)$ is $R_\delta-$set for all $((x, y), \alpha) \in S(x_0, y_0) \times [0, 1]$. The proof is given by the following steps.
Step 1: Π is locally compact.

(a) The multifunction $\widetilde{S} : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(PC([0, 1], \mathbb{R}) \times PC([0, 1], \mathbb{R}))$ defined by

$$\widetilde{S}(\widetilde{t}, (\widetilde{x}, \widetilde{y})) = S(f, \widetilde{t}, (\widetilde{x}, \widetilde{y})),$$

is u.s.c. where $S(f,\widetilde{t},(\widetilde{x},\widetilde{y}))$ is the set of all solutions of the problem

$$\left\{\begin{array}{rcl} z_1'(t) &=& f_1(t,z_1(t),z_2(t)),\ a.e.t\in[\widetilde{t},1]\\ z_2'(t) &=& f_2(t,z_1(t),z_2(t)),\ a.e.t\in[\widetilde{t},1]\\ z_1(t_k^+)-z_1(t_k^-) &=& I_{1,k}(z_1(t_k),z_2(t_k)),\ k=1,\cdots,m\\ z_2(t_k^+)-z_2(t_k^-) &=& I_{2,k}(z_1(t_k),z_2(t_k)),\ k=1,\cdots,m\\ z_1(\widetilde{t}) &=& \widetilde{x},\quad z_2(\widetilde{t})=\widetilde{y}. \end{array}\right. \tag{4.12}$$

Assume the opposite, ie. $\widetilde{S}$ is not u.s.c. Then for a point $(\widetilde{t},(\widetilde{x},\widetilde{y}))$ there exists an open neighborhood U of $\widetilde{S}(\widetilde{t},(\widetilde{x},\widetilde{y}))$ in $PC([0,1],\mathbb{R})\times PC([0,1],\mathbb{R})$, such that for any open neighborhood V of $(\widetilde{t},(\widetilde{x},\widetilde{y}))$ in $[0,1]\times\mathbb{R}\times\mathbb{R}$, there exists $(\widetilde{t_1},(\widetilde{x_1},\widetilde{y_1}))\in V$ such that $\widetilde{S}(\widetilde{t_1},(\widetilde{x_1},\widetilde{y_1}))\not\subset U$.
Let $V_n=\left\{(t,(x,y))\in[0,1]\times\mathbb{R}\times\mathbb{R} : d((t,(x,y)),(\widetilde{t},(\widetilde{x},\widetilde{y})))<\begin{pmatrix}1/n\\1/n\\1/n\end{pmatrix}\right\}$,
where d is the generalized metric of the space $[0,1]\times(\mathbb{R}\times\mathbb{R})$. Then for each $n\in\mathbb{N}$ we take $(t_n,(x_n,y_n))\in V_n$ and $(x_n,y_n)\in\widetilde{S}(t_n,(x_n,y_n))$ such that $(x_n,y_n)\notin U$. We define the functions

$G_{\widetilde{t},(\widetilde{x},\widetilde{y})}, F_{\widetilde{t},(\widetilde{x},\widetilde{y})} : PC([0,1],\mathbb{R})\times PC([0,1],\mathbb{R})\to PC([0,1],\mathbb{R})\times PC([0,1],\mathbb{R})$

by

$$F_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x,y)(t)=(\widetilde{x}+\int_{\widetilde{t}}^t f_1(s,(x(s),y(s)))ds+\sum_{\widetilde{t}<t_k<t} I_{1k}(x(t_k),y(t_k)),$$

$$\widetilde{y}+\int_{\widetilde{t}}^t f_2(s,(x(s),y(s)))ds+\sum_{\widetilde{t}<t_k<t} I_{2k}(x(t_k),y(t_k))); \ t\in[\widetilde{t},1],$$

$$G_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x,y)=(x,y)-F_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x,y),\quad \text{for}\quad t\in[0,1],$$

and

$$(x,y)\in PC([0,1],\mathbb{R})\times PC([0,1],\mathbb{R}).$$

Then for $(x,y)\in C([0,1],\mathbb{R})\times C([0,1],\mathbb{R}), t,\widetilde{t}\in[0,1]$, and $(\widetilde{x},\widetilde{y})\in\mathbb{R}\times\mathbb{R}$, we have

$$F_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x,y)(t)=(\widetilde{x},\widetilde{y})-F_{0,(\widetilde{x},\widetilde{y})}(x,y)(\widetilde{t})+F_{0,(\widetilde{x},\widetilde{y})}(x,y)(t),$$

consequently

$$G_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x,y)(t)=-(\widetilde{x},\widetilde{y})+F_{0,(\widetilde{x},\widetilde{y})}(x,y)(t)+G_{0,(\widetilde{x},\widetilde{y})}(x,y)(t),$$

then we obtain

$$\widetilde{S}(\widetilde{t},(\widetilde{x},\widetilde{y})) = G^{-1}_{\widetilde{t},(\widetilde{x},\widetilde{y})}(0), \quad \text{for all} \quad (\widetilde{t},(\widetilde{x},\widetilde{y})) \in [0,1]\times\mathbb{R}\times\mathbb{R},$$

as $F_{\widetilde{t},(\widetilde{x},\widetilde{y})}$ is compact (See proof of theorem 4.3.3), then $G_{\widetilde{t},(\widetilde{x},\widetilde{y})}$ is propre, and as $(x_n, y_n) \in \widetilde{S}(t_n,(x_n,y_n))$, then

$$x_n(t) = x_n(t_n) + \int_{t_n}^{t} f_1(s, x_n(s), y_n(s))ds + \sum_{t_n<t_k<t} I_{1,k}(x_n(t_k), y_n(t_k)),$$

and

$$y_n(t) = y_n(t_n) + \int_{t_n}^{t} f_2(s, x_n(s), y_n(s))ds + \sum_{t_n<t_k<t} I_{2,k}(x_n(t_k), y_n(t_k)),$$

which gives

$$0 = G_{t_n,(x_n,y_n)}(x_n,y_n)(t) = -(x_n,y_n)(t_n)+F_{0,(x_n,y_n)}(x_n,y_n)(t_n)+G_{0,(x_n,y_n)}(x_n,y_n)(t),$$

and

$$G_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x_n,y_n)(t) = -(\widetilde{x},\widetilde{y}) + F_{0,(\widetilde{x},\widetilde{y})}(x_n,y_n)(\widetilde{t}) + G_{0,(\widetilde{x},\widetilde{y})}(x_n,y_n)(t),$$

then

$$\begin{aligned}
\|G_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x_n,y_n)(t) &- G_{t_n,(x_n,y_n)}(x_n,y_n)(t)\| = \|G_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x_n,y_n)(t)\| \\
&= \|-(\widetilde{x},\widetilde{y}) + (x_n,y_n)(t_n) \\
&+ F_{0,(\widetilde{x},\widetilde{y})}(x_n,y_n)(\widetilde{t}) - F_{0,(x_n,y_n)}(x_n,y_n)(t_n)\| \\
&= \|\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\| \\
&= \begin{pmatrix}\|\alpha\|\\ \|\beta\|\end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
\alpha &= -\widetilde{x} + x_n(t_n) + \left(\widetilde{x} + \int_0^{\widetilde{t}} f_1(s,x_n(s),y_n(s))ds + \sum_{0<t_k<\widetilde{t}} I_{1,k}(x_n(t_k),y_n(t_k))\right) \\
&- \left(x_n(t_n) + \int_0^{t_n} f_1(s,x_n(s),y_n(s))ds + \sum_{0<t_k<t_n} I_{1,k}(x_n(t_k),y_n(t_k))\right),
\end{aligned}$$

then

$$\begin{aligned}\|\alpha\| &\leq \int_{t_n}^{\widetilde{t}} \|f_1(s,x_n(s),y_n(s))\|ds + \sum_{t_n<t_k<\widetilde{t}} \|I_{1,k}(x_n(t_k),y_n(t_k))\| \\ &\leq \int_{t_n}^{\widetilde{t}} p(s)\psi(b_1^*+b_2^*)ds + \sum_{t_n<t_k<\widetilde{t}} \|I_{1,k}(x_n(t_k),y_n(t_k))\|,\end{aligned}$$

similarly

$$\begin{aligned}\beta &= -\widetilde{y} + y_n(t_n) + \left(\widetilde{y} + \int_0^{\widetilde{t}} f_2(s,x_n(s),y_n(s))ds + \sum_{0<t_k<\widetilde{t}} I_{2,k}(x_n(t_k),y_n(t_k))\right) \\ &- \left(y_n(t_n) + \int_0^{t_n} f_2(s,x_n(s),y_n(s))ds + \sum_{0<t_k<t_n} I_{2,k}(x_n(t_k),y_n(t_k))\right) \\ \|\beta\| &\leq \int_{t_n}^{\widetilde{t}} p(s)\psi(b_1^*+b_2^*)ds + \sum_{t_n<t_k<\widetilde{t}} \|I_{2,k}(x_n(t_k),y_n(t_k))\|\end{aligned}$$

$\lim_{n\to\infty}(x_n,y_n) = (\widetilde{x},\widetilde{y})$ and $\lim_{n\to\infty} t_n = \widetilde{t}$ implies that $\lim_{n\to\infty} G_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x_n,y_n) = 0$. Then the set $A = \overline{\{G_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x_n,y_n)\}}$ is compact thus $G^{-1}_{\widetilde{t},(\widetilde{x},\widetilde{y})}(A)$ is also compact. It is clear that $\{(x_n,y_n)\} \subset A$. As $\lim_{n\to\infty}(x_n,y_n) = (\widetilde{x},\widetilde{y})$, then $(\widetilde{x},\widetilde{y}) \in \widetilde{S}(\widetilde{t},(\widetilde{x},\widetilde{y})) \subset U$, thus we find a contradiction with the hypothesis $(x_n,y_n) \notin U$ for every n.

(b) Π is locally compact.
For $r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} > 0$, consider the set

$$B \times I = \{((x,y),\alpha) \in S(x_0,y_0) \times [0,1] : \|(x,y)\| \leq r\},$$

and let $\{u_n\} \in \Pi(B\times I)$, then it exist $((x_n,y_n),\alpha_n) \in B \times I$ such that

$$u_n(t) = \begin{cases} (x_n,y_n), & \text{if } 0 \leq t \leq \alpha_n, \\ v_n(t), & \text{if } \alpha_n < t \leq 1,\ v_n \in S(f,\alpha_n,(x_n,y_n)). \end{cases}$$

As $S(x_0,y_0)$ is compact then there exists a subsequence of $(x_n,\alpha_n)_n$ which converges towards $((x,y),\alpha)$. $\widetilde{S}$ is u.s.c. implies that for all $\varepsilon > 0$

there exists $n_0(\varepsilon)$ such that $v_n(t) \in \widetilde{S}(t,(x,y)) = S(f,\alpha,(x,y))$, for all $n \geq n_0(\varepsilon)$, and by the compactness of $S(f,\alpha,(x,y))$, it is concluded that there exists a subsequence of $\{v_n\}$ which converges towards $v \in S(f,\alpha,(x,y))$. Hence Π is locally compact.

Step 2: Π has closed graph.
Let $((x_n,y_n),\alpha_n) \to ((x_*,y_*),\alpha), h_n \in \Pi(x_n,\alpha_n)$ and $h_n \to h_*$ when $n \to +\infty$. We are going to prove that $h_* \in \Pi((x_*,y_*),\alpha)$.
$h_n \in \Pi((x_n,y_n),\alpha_n)$, then it exists $z_n \in S(f,\alpha_n,(x_n,y_n))$ such that for all $t \in J$

$$h_n(t) = \begin{cases} (x_n,y_n), & \text{if } 0 \leq t \leq \alpha_n, \\ z_n(t), & \text{if } \alpha_n < t \leq 1, \end{cases}$$

and it is enough to prove that it exists $z_* \in S(f,\alpha,(x_*,y_*))$ such that for all $t \in J$

$$h_*(t) = \begin{cases} (x_*,y_*), & \text{if } 0 \leq t \leq \alpha, \\ z_*(t), & \text{if } \alpha < t \leq 1. \end{cases}$$

it is clear that $(\alpha_n,(x_n,y_n)) \to (\alpha,(x_*,y_*))$ when $n \to \infty$ and it can easily be proved that there exists a subsequence of $\{z_n\}$ which converges towards z_*. So we can handle the cases $\alpha = 0$ and $\alpha = 1$ as we did in the proof of the theorem 4.3.4, and we obtain finally that $z_* \in S(f,\alpha,(x_*,y_*))$.
Step 3: $\Pi((x,y),\alpha)$ is R_δ-set for all $((x,y),\alpha) \in S(x_0,y_0) \times [0,1]$.
As F is $\sigma - Ca-$selectionnable, Then there is a decreasing sequence of multi-functions $F_k : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R} \times \mathbb{R})$, $k \in \mathbb{N}$ which admits Carathéodory selections and

$$F_{k+1}(t,u) \subset F_k(t,u) \quad \text{for all} \quad t \in [0,1], u \in \mathbb{R} \times \mathbb{R},$$

and

$$F(t,u) = \cap_{k=0}^{\infty} F_k(t,u),\ u \in \mathbb{R} \times \mathbb{R}.$$

then

$$\Pi((x,y),\alpha) = \cap_{k=0}^{\infty} S(F_k,(x,y)).$$

By the theorem 4.3.3, the sets $\Pi((x,y),\alpha)$, and $S(F_k,(x,y))$ are compact. Furthermore by the theorem 4.3.4 the set $S(F_k,(x,y))$ is contractible. Then $\Pi((x,y),\alpha)$ is $R_\delta-$set.

□

Lemma 4.3.1. *Suppose that the multifunction $F_i : [0,b] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R}), i = 1,2$, are Carathéodory and u.s.c. to the type of Scorza-Dragoni, then the set of all solutions of the problem (4.1) is $R_\delta-$contractible.*

Proof. By the theorem 4.1.4 we have that F_i is $\sigma-$Ca-selectionnable, then we have the same conditions of the last theorem.

$\square$

Conclusion and Perspective

In this dissertation, we have presented some results to the theory of existence of solutions of some classes of impulsive differential equations and impulsive differential inclusions and system of impulsive differential inclusions, we have proved also some property topological geometrical of solutions set as: compactness,R_δ, contractible, acyclic.

It would be interesting, for a future research, to look for system of impulsive differential equations with delay, system of impulsive differential equations non instantaneous, and system of impulsive differential equations depends on the state.

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