République Algérienne Démocratique et Populaire
Ministère de l'Enseignement Supérieur et de la Recherche Scientifique
Université Djillali Liabs - SIDI BEL-ABBES -

THÈSE

Préparée au Département de Mathématiques
de la Faculté des Sciences Exactes

par

Fatima MESRI

Pour obtenir
le grade de **DOCTEUR**

Spécialité : Mathématiques
Option : Equations différentielles fonctionelles

Une étude quantitative et qualitative de certains problèmes d'évolution

Thèse présentée et soutenue publiquement le ,
devant le Jury composé de

Président:

Benchohra Mouffak,	Pr. Univ. Djillali Liabes SBA.

Directeur de thèse:

Arara Amaria,	M.C (A). Univ. Djilali Liabes SBA.

Examinateurs:

Abbas Said,	Pr. Univ. Saïda.
Slimani Boualem Attou,	Pr. Univ. Tlemcen.
Lazreg Jamal-Eddine,	M.C (A). Univ. Djillali Liabes SBA.
Hedia Benaouda,	M.C (A). Univ. Tiaret.

Acknowledgements

In the name of Allah, Most Gracious, Most Merciful.

Praise be to Allah who gave me strength, inspiration and prudence to bring this thesis to a close. Peace be upon His messenger Muhammad and his honorable family.

First and foremost, I would like to express my sincere gratitude to my supervisor *Dr. Amaria Arara* for her continuous support on my Ph.D study and research, for her patience, motivation, enthusiasm, and immense knowledge. her guidance helped me in all the time of research and writing of this thesis.

I would also like to thank *Prof. Mouffak Benchohra*, who accepted to chair this thesis committee. This crucial contribution, made him a backbone of this thesis.
My special thank belongs to *Prof. Said Abbas*, *Prof. Boualem Attou Slimani*, *Dr. Jamal-Eddine Lazreg* and *Dr. Benaouda Hedia* who in the midst of all their activities, accepted to be members of this thesis committee.

Without forgetting my friends and colleagues, to whom I can never be thankful enough for their encouragement, understanding, tolerance and prayers that acted as a giant support for my success.

Last but not least, I dedicate this thesis to my son; without him I would not be here.

Publications

1. S. Abbas, A. Arara, M. Benchohra and **F. Mesri**, Functional random evolution equations in Fréchet spaces, *Advances in the Theory of Nonlinear Analysis and its Applications* , Vol. **2**, No. 3, (2018), 128-137.

2. S. Abbas, A. Arara, M. Benchohra and **F. Mesri**, Evolution equations in Fréchet spaces, *Journal of Mathematical Sciences and Modelling*, Vol. **1**, No. 1, (2018), 33-38.

3. S. Abbas, A. Arara, M. Benchohra and **F. Mesri**, Controllability of Second Order Random Differential Equations in Fréchet Spaces, (submitted).

4. A. Arara, M. Benchohra and **F. Mesri**, Controllability of Semilinear Differential Equations In Fréchet Spaces, (submitted).

5. A. Arara, M. Benchohra and **F. Mesri**, Measure of Noncompactness and Semilinear Differential Equations in Fréchet Spaces, Tbilissi Math. J. (accepted).

Abstract

Functional differential equations occur in a variety of areas of biological, physical, and engineering applications, and such equations have received much attention in recent years. This thesis is devoted to the existence of global mild, integral solutions, random mild solutions and we present the results of controllability of mild solutions and random mild solution for some semilinear first and second order functional differential equations, and other densely and non-densely defined functional differential equations in Fréchet spaces.

The tools used include a generalization of the classical Darbo fixed point theorem for Fréchet spaces associated with the concept of measure of noncompactness.

Key words and phrases :
Functional evolution equation, evolution system, C_0-semigroup, integrated semigroup, densely defined operator, nondensely defned operator, mild solution, integral solutions, random mild solution, cosine and sine families, controllability, Measure of noncompactness, fixed point, Fréchet space.

AMS Subject Classification : 34G20, 34G25, 93B05.

Contents

Introduction

The theory of functional differential equations has emerged as an important branch of nonlinear analysis. During the last decades, existence and uniqueness of mild, strong and classical solutions of functional differential equations has been studied extensively by many authors using the semigroup theory, fixed point argument and measures of noncompactness. We mention, for instance, the books Ahmed [7], Engel and Nagel [32], Kamenski *et al* [44], Pazy [54] and Wu [58]. And other autors used a nonlinear alternative of Leray–Schauder type for contraction operators on Fréchet spaces [38], Baghli and Benchohra [11, 12] provided sufficient conditions for the existence of mild solutions of some classes of evolution equations, while in [14, 15, 21] the authors presented some global existence and stability results for functional evolution equations and inclusions in the space of continuous and bounded functions. In [1], an iterative method is used for the existence of mild solutions of evolution equations and inclusions.

The measure of noncompactness which is one of the fundamental tools in the theory of nonlinear analysis was initiated by the pioneering articles of Kuratowski [43], Darbo [29] and was developed by Bana's and Goebel [16] and many researchers in the literature. The applications of the measure of noncompactness (for the weak case, the measure of weak noncompactenss developed by De Blasi [30]) can be seen in the wide range of applied mathematics: theory of differential equations (see [4, 53] and references therein), difference equations [6].

The nature of a dynamic system in engineering or natural sciences depends on the accuracy of the information we have concerning the parameters that describe that system. If the knowledge about a dynamic system is precise then a deterministic dynamical system arises. Unfortunately in most cases the available data for the description and evaluation of parameters of a dynamic system are inaccurate, imprecise or confusing. In other words, evaluation of parameters of a dynamical system is not without uncertainties. When our knowledge about the parameters of a dynamic system are of statistical nature, that is, the information is probabilistic, the common approach in mathematical modeling of such systems is the use of random differential equations or stochastic differential equations. Random differential equations, as natural extensions of deterministic ones, arise in many applications and have been investigated by many mathematicians. We refer the reader to the monographs [8, 22, 24, 48, 56].

The concept of controllability plays an important role in control theory and engineering because they have close connections to pole assignment, structural decomposition, quadratic optimal control and observer design etc., In recent years, the problem of controllability for various kinds of differential equations and inclusions in Banach and Fréchet spaces have been discussed extensively by several authors, for instance, see papers [20, 35, 39, 50], in particular the controllability of nonlinear systems with and without impulse have studied by several authors, see, for instance [3, 9, 13].

In the following we give an outline of our thesis organization, which consists of six chapters defining the contributed work. The first chapter gives some notations, definitions, lemmas and fixed point theorems which are used throughout the following chapters.

In Chapter 2, we study the existence result of mild solutions for a class of evolution equations of the form:

$$u'(t) = A(t)u(t) + f(t, u(t)); \quad \text{if } t \in \mathbb{R}_+ := [0, \infty), \tag{1}$$

with the initial condition

$$u(0) = u_0 \in E, \tag{2}$$

where $f : \mathbb{R}_+ \times E \to E$ is a given function, $(E, \|\cdot\|)$ is a (real or complex) Banach space, and $\{A(t)\}_{t>0}$ is a family of linear closed (not necessarily bounded) operators from E into E that generate an evolution system of bounded linear operators $\{U(t,s)\}_{(t,s)\in\mathbb{R}_+\times\mathbb{R}_+}$; for $(t,s) \in \Lambda := \{(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+ : 0 \le s \le t < +\infty\}$.

In section one of Chapter 3, we prove the existence of global mild solution in the case where A is densely defined operator generated a C_0-semigroup $(T(t))_{t\ge0}$ on E of the following differential equation of the form:

$$y'(t) = Ay(t) + f(t, y(t)), \quad t \in [0, \infty), \tag{3}$$

with the initial condition

$$y(0) = y_0 \in E \tag{4}$$

where $f : \mathbb{R}_+ \times E \to E$ is given function, $(E, \|\cdot\|)$ is a (real or complex) Banach space. Then in section two, we discuss the existence of integral solution in the case where A is a Hille–Yosida operator nondensely defined on E generated a integrated semigroup $(S(t))_{t\ge0}$ on E of the problem (3)-(4).

In Chapter 4, we study the controllability of mild solutions for functional differential equation

$$y'(t) = Ay(t) + f(t, y(t)) + Bu(t), \quad t \in [0, \infty), \tag{5}$$

with the initial condition

$$y(0) = y_0 \in E \tag{6}$$

where $f : \mathbb{R}_+ \times E \to E$ is given function, $(E, \|\cdot|)$ is a (real or complex) Banach space, and $A : D(A) \to E$ is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\ge0}$ on E.

In Chapter 5, we shall be concerned with the existence of global mild solutions for two classes of second order semi-linear functional equations with random effects. In section 5.1, we will consider the following problem

$$u'(t,w) = A(t)u(t,w) + f(t,u(t,w),w); \quad \text{if } t \in \mathbb{R}_+ := [0,\infty), \ w \in \Omega, \tag{7}$$

with the initial condition

$$u(0,w) = u_0(w) \in E, \ w \in \Omega, \tag{8}$$

where (Ω, F, P) is a complete probability space, $u_0 : \Omega \to E$ is a given function, $f : \mathbb{R}_+ \times E \times \Omega \to E$ is a given function, $(E, \|\cdot\|)$ is a (real or complex) Banach space, and $\{A(t)\}_{t>0}$ is a family of linear closed (not necessarily bounded) operators from E into E that generate an evolution system of bounded linear operators $\{U(t,s)\}_{(t,s)\in\mathbb{R}_+\times\mathbb{R}_+}$; for $(t,s) \in \Lambda := \{(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+ : 0 \leq s \leq t < +\infty\}$.

Later in section 5.2, we discuss the existence of random mild solutions for the following second order evolution problem

$$\begin{cases} u''(t,w) - A(t)u(t,w) = g(t,u(t,w),w); \quad \text{if } t \in \mathbb{R}_+ := [0,\infty), \ w \in \Omega, \\ \\ u(0,w) = \underline{\mathrm{u}}(w), \quad u'(0,w) = \bar{u}(w), \ w \in \Omega, \end{cases} \tag{9}$$

where $E, \{A(t)\}_{t>0}$ are as problem (7)-(8) and $\underline{\mathrm{u}}, \bar{u} : \Omega \to E$ and $g : \mathbb{R}_+ \times E \times \Omega \to E$ are given functions.

In Chapter 6, we study the existence and controllability for second order functional differential equations with random effects of the form:

$$\begin{cases} y''(t,w) = Ay(t,w) + f(t,y(s,w),w) + Bu(t,w), \ \text{a.e. } t \in \mathbb{R}_+ := [0,\infty), \\ y(0,w) = \phi(w), \ y'(0,w) = \varphi(w), \end{cases} \quad w \in \Omega, \tag{10}$$

where $f : \mathbb{R}_+ \times E \times \Omega \to E$ is a given function, $\phi, \varphi : \Omega \to E$ are given measurable functions, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $(C(t))_{t\in\mathbb{R}}$ on E.

Finally we close our thesis with a conclusion and some perspectives

Chapter 1

Preliminaries

In this section we review some fundamental concepts, notations, definitions, fixed point theorems and properties required to establish our main results.

1.1 Notations and Definitions

Let $J = [0, b]$ be an interval of $\mathbb{R}$ and $(E, |\cdot|)$ be a real Banach space. Let $C(J, E)$ be the Banach space of continuous functions from J into E with the norm

$$\|y\|_\infty = \sup \{ \ |y(t)| \ : \ t \in J \ \}.$$

Let $B(E)$ denote the Banach space of bounded linear operators from E into E, with the norm

$$\|N\|_{B(E)} = \sup \{ \ |N(y)| \ : \ |y| = 1 \ \}.$$

A measurable function $y : J \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable.

Let $L^1(J, E)$ denote the Banach space of measurable functions $y : J \to E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^b |y(t)| \ dt.$$

Let $L^\infty(J, E)$ is the Banach space of measurable functions and bounded almost everywhere with

$$\|y\|_{L^\infty} = \operatorname{ess\,sup} |y(t)| = \inf\{c > 0 : \|y(t)\| \le c; \ \text{for a.e.} t \in J\}.$$

1.2 Some Properties in Fréchet Spaces

Let $X := C(\mathbb{R}_+)$ be the Fréchet space of all continuous functions v from $\mathbb{R}_+$ into E, equipped with the family of seminorms

$$\|v\|_n = \sup_{t\in[0,n]} |v(t)|; \ n \in \mathbb{N},$$

and the distance

$$d(u,v)=\sum_{n=1}^{\infty}2^{-n}\frac{\|u-v\|_n}{1+\|u-v\|_n};\ u,v\in X.$$

Let $Y\subset X$, we say that Y is bounded if for every $n\in\mathbb{N}$, there exists $M_n>0$ such that

$$\|y\|_n\leq M_n \quad for\ all\ y\in Y.$$

To X we associate a sequence of Banach spaces $\{(X^n,\|\cdot\|_n)\}$ as follows : For every $n\in\mathbb{N}$, we consider the equivalence relation $\sim_n$defined by : $x\sim_n y$ if and only if $\|x-y\|_n=0$ for all $x,y\in X$. We denote $X^n=(X|_{\sim_n},\|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y\subset X$, we associate a sequence the $\{Y^n\}$ of subsets $Y^n\subset X^n$ as follows : For every $x\in X$, we denote $[x]_n$ the equivalence class of x of subset X^n and we define $Y^n=\{[x]_n : x\in Y\}$. We denote $\overline{Y^n}$, $int_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|$ in X^n. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies :

$$\|x\|_1\leq\|x\|_2\leq\ \|x\|_3\leq ... \quad for\ every\ x\in X.$$

Definition 1.1 *[38] A function $f:X\to X$ is said to be* a contraction *if for each $n\in\mathbb{N}$ there exists $k_n\in[0,1)$ such that :*

$$\|f(x)-f(y)\|_n\leq k_n\ \|x-y\|_n \quad for\ all\ x,y\in X.$$

1.3 Semigroups

1.3.1 C_0-Semigroups

Let E be a Banach space and $B(E)$ be the Banach algebra of linear bounded operators.

Definition 1.2 *One parameter family $\{T(t)|t\geq 0\}\subset B(E)$ satisfying the conditions:*

1. *$T(0)=I$, (I denotes the identity operator in E) ;*
2. *$T(t+s)=T(t)T(s)$, for $t,s\geq 0$;*
3. *the map $t\longrightarrow T(t)(y)$ is strongly continuous, for each $y\in E$, i.e;*

$$\lim_{t\to 0}T(t)y=y \quad \forall\ y\in E.$$

A semigroup of bounded linear operators $T(t)$, is uniformly continuous if

$$\lim_{t\to 0}\|T(t)-I\|=0.$$

Definition 1.3 *Let $T(t)$ be a semigroup of class (C_0) defined on E. The infinitesimal generator A of $T(t)$ is the linear operator defined by*

$$A(y) = \lim_{h\to 0} \frac{T(h)y - y}{h} \quad for\ y \in D(A),$$

where

$$D(A) = \left\{ y \in X \middle| \lim_{h\to 0} \frac{T(h)(y) - y}{h} \quad exists\ in\ E \right\}.$$

Example 1.4 *Let E be the space of continuous functions $\phi : [0,1] \to \mathbb{R}$ endowed with the sup norm. Then the family $(T(t))_{t\geq 0}$ defined by*

$$(T(t)\phi)(x) = \phi(xe^{-t}),\ t \geq 0,\ \phi \in X,\ x \in [0,1],$$

is a $C_0 -$ semigroup on E and its infinitesimal generator A is defined on

$$\begin{aligned} D(A) &= \{\phi \in C([0,1],\mathbb{R}) : \phi'(x)\ exists\ and\ is\ continuous\ for\ x \in [0,1]\} \\ &= C^1([0,1],\mathbb{R}) \end{aligned}$$

by

$$A\phi = \phi'.$$

1.3.2 Analytic semigroups

Definition 1.5 *Let $\Delta = \{z : \varphi_1 < \arg z < \varphi_2, \varphi_1 < 0 < \varphi_2\}$ and for $z \in \Delta$, let $T(z)$ be a bounded linear operator. The family $\{T(z); z \in \Delta\}$ is an analytic semigroup in Δ if*

(i) $z \to T(z)$ *is analytic in* Δ.

(ii) $T(0) = I$.

(iii) $\lim_{z\to 0} T(z)x = x$ *for every* $x \in E$.

(iv) $T(z_1 + z_2) = T(z_1)T(z_2)$ *for* $z_1, z_2 \in \Delta$.

1.3.3 Integrated semigroups

Definition 1.6 *[10] Let E be a Banach space. An integrated semigroup is a family of operators $(S(t))_{t\geq 0}$ of bounded linear operators $S(t)$ on E with the following properties:*

(i) $S(0) = 0$;

(ii) $t \to S(t)x$ *is continuous on* $[0, +\infty)$ *for each* $x \in E$;

(iii) $S(s)S(t) = \int_0^s (S(t+s) - S(r))dr$, *for all* $t, s \geq 0$.

Definition 1.7 *[10] Let A be the generator of an integrated semigroup $(S(t))_{t\geq 0}$. Then for all $x \in E$ and $t \leq 0$,*

$$\int_0^t S(s)xds \in D(A) \quad \textit{and} \quad S(t)x = A\int_0^t S(s)xds + tx.$$

Definition 1.8 *We say that the linear operator A satisfies the Hille–Yosida condition if there exists $M \leq 0$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ and*

$$\sup\{(\lambda - \omega)^n |(\lambda I - A)^{-n}| : n \in \mathbb{N}, \lambda > \omega\} \leq M.$$

More details of the semigroups and their properties can be found in [54]

1.4 Cosine family

Definition 1.9 *A one parameter family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators in the Banach space E is called a strongly continuous cosine family if and only if*

- *$C(0) = I$ (I is the identity operator);*
- *$C(t)x$ is strongly continuous in t on $\mathbb{R}$ for each fixed $x \in E$;*
- *$C(t+s) + C(t-s) = 2C(t)C(s)$ for all t, $s \in \mathbb{R}$.*

Let $\{C(t) : t \in \mathbb{R}\}$ be a strongly continuous cosine family in E. Define the associated sine family $\{S(t) : t \in \mathbb{R}\}$ by

$$S(t)x = \int_0^t C(s)xds, \quad x \in E, \ t \in \mathbb{R}.$$

The infinitesimal generator $A : E \to E$ of the cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by

$$Ax = \frac{d^2}{dt^2} C(t)x|_{t=0}, \ x \in D(A),$$

where

$$D(A) = \{x \in E : C(\cdot)x \in C^2(\mathbb{R}, E)\}.$$

1.5 Evolution system

In what follows, for the family $\{A(t), \ t \geq 0\}$ of closed densely defined linear unbounded operators on the Banach space E we assume that it satisfies the following assumptions (see [7], p. 158).

(P_1) The domain $D(A(t))$ is independent of t and is dense in E,

(P_2) For $t \geq 0$, the resolvent $R(\lambda, A(t)) = (\lambda I - A(t))^{-1}$ exists for all λ with $Re\lambda \leq 0$, and there is a constant K independent of λ and t such that

$$\|R(t, A(t))\| \leq K(1 + |\lambda|)^{-1}, \text{ for } Re\lambda \leq 0,$$

(P_3) There exist constants $L > 0$ and $0 < \alpha \leq 1$ such that

$$\|(A(t) - A(\theta))A^{-1}(\tau)\| \leq L|t - \tau|^{\alpha}, \text{ for } t, \theta, \tau \in I.$$

Lemma 1.10 *([7], p. 159) Under assumptions $(P_1) - (P_3)$, the Cauchy problem*

$$u'(t) - A(t)u(t) = 0, \ t \in I, \ u(0) = y_0,$$

has a unique evolution system $U(t, s)$, $(t, s) \in \Delta := \{(t, s) \in J \times J : 0 \leq s \leq t \leq T\}$ satisfying the following properties:

1. *$U(t, t) = I$ where I is the identity operator in E,*
2. *$U(t, s)\ U(s, \tau) = U(t, \tau)$ for $0 \leq \tau \leq s \leq t \leq T$,*
3. *$U(t, s) \in B(E)$ the space of bounded linear operators on E, where for every $(t, s) \in \Delta$ and for each $u \in E$, the mapping $(t, s) \to U(t, s)u$ is continuous.*

The existence of solutions to our problem is related to the existence of an evolution operator $U(t, s)$ for the homogeneous problem

$$u''(t) = A(t)u(t); \ t \in \mathbb{R}_+. \tag{1.1}$$

This concept of evolution operator has been developed by Kozak [47].

Definition 1.11 *A family $\mathcal{U}$ of bounded operators $\mathcal{U}(t, s) : E \to E$; $(t, s) \in \{(t, s) : s \leq t\}$, is called an evolution operator of the equation (1.1) if the following conditions hold;*

(P_1) *For any $u \in E$, the map $(t, s) \to \mathcal{U}(t, s)u$ is continuously differentiable and:*

(a) *for any $(t \in \mathbb{R}_+ : \mathcal{U}(t, t) = 0$;*

(b) *for all $(t, s) \in \Delta$ and for any $u \in E$, $\frac{\partial}{\partial t}\mathcal{U}(t, s)u|_{t=s} = u$ and $\frac{\partial}{\partial s}\mathcal{U}(t, s)u|_{t=s} = -u$.*

(P_2) *For all $(t, s) \in \Delta$ if $u \in D(A(t))$, then $\frac{\partial}{\partial s}\mathcal{U}(t, s)u \in D(A(t))$, the map $(t, s) \to \mathcal{U}(t, s)u$ is of class C^2, and*

(a) *$\frac{\partial^2}{\partial t^2}\mathcal{U}(t, s)u = A(t)\mathcal{U}(t, s)u$;*

(b) *$\frac{\partial^2}{\partial s^2}\mathcal{U}(t, s)u = \mathcal{U}(t, s)A(s)u$;*

(c) *$\frac{\partial^2}{\partial t \partial s}\mathcal{U}(t, s)u|_{t=s} = 0$.*

(P_3) *For all* $(t,s) \in \Delta$ *if* $u \in D(A(t))$, *then the map* $(t,s) \to A(t)\frac{\partial}{\partial s}\mathcal{U}(t,s)u$ *is continuous,* $\frac{\partial^3}{\partial t^2 \partial s}\mathcal{U}(t,s)u$ *and* $\frac{\partial^3}{\partial s^2 \partial t}\mathcal{U}(t,s)u$ *exist and*

(a) $\frac{\partial^3}{\partial t^2 \partial s}\mathcal{U}(t,s)u = A(t)\frac{\partial}{\partial s}\mathcal{U}(t,s)u;$

(b) $\frac{\partial^3}{\partial s^2 \partial t}\mathcal{U}(t,s)u = A(t)\frac{\partial}{\partial t}\mathcal{U}(t,s)A(s)u.$

More details on evolution systems and their properties can be found in the books of Ahmed [7] and Pazy [54].

1.6 Measure of Noncompactness

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 1.12 *([16]) Let* E *be a Banach space and* Ω_E *the family of bounded subsets of* E*. The Kuratowski measure of noncompactness is the map* $\alpha : \Omega_E \to [0,\infty)$ *defined by*

$$\alpha(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^{n} B_i \text{ and } diam(B_i) \leq \epsilon\}.$$

Properties 1.13 *The Kuratowski measure of noncompactness satisfies the following properties (for more details see [16]).*

(a_1) $\alpha(B) = 0 \Longleftrightarrow \overline{B}$ *is compact* (B *is relatively compact).*

(b_1) $\alpha(B) = \alpha(\overline{B})$.

(c_1) $A \subset B \Longrightarrow \alpha(A) \leq \alpha(B)$.

(d_1) $\alpha(A+B) \leq \alpha(A) + \alpha(B)$.

(e_1) $\alpha(cB) = |c|\alpha(B); c \in \mathbb{R}$

(f_1) $\alpha(convB) = \alpha(B)$.

Lemma 1.14 *([40]) If* $H \subset C(J,E)$ *is a bounded and equicontinuous set, then*

(ii) $\alpha(H) = \sup_{0 \leq t \leq b} \alpha(H(t))$.

(ii) $\alpha\left(\int_0^b x(s)ds : x \in H\right) \leq \int_0^b \alpha(H(s))ds$ *for* $t \in J$,

where $H(s) = \{x(s) : x \in H\}, \ s \in J$.

Definition 1.15 *Let $\mathcal{M}_{\mathcal{X}}$ be the family of all nonempty and bounded subsets of a Fréchet space $\mathcal{X}$. A family of functions $\{\mu_n\}_n \in \mathbb{N}$ where $\mu_n : \mathcal{M}_X \to [0,\infty)$ is said to be a family of measures of noncompactness in the Fréchet space $\mathcal{X}$ if it satisfies the following conditions for all $B, B_1, B_2 \in \mathcal{M}_{\mathcal{X}}$:*

(a) $\{\mu_n\}_{n\in\mathbb{N}}$ is full, that is: $\mu_n(B) = 0$ for $n \in \mathbb{N}$ if and only if B is precompact,

(b) $\mu_n(B_1) \leq \mu_n(B_2)$ for $B_1 \subset B_2$ and $n \in \mathbb{N}$,

(c) $\mu_n(ConvB) = \mu_n(B)$ for $n \in \mathbb{N}$,

(d) If $\{B_i\}_{i=1,\cdots}$ is a sequence of closed sets from $\mathcal{M}_{\mathcal{X}}$ such that $B_{i+1} \subset B_i$; $i = 1, \cdots$ and if $\lim_{i\to\infty} \mu_n(B_i) = 0$, for each $n \in \mathbb{N}$, then the intersection set $B_\infty := \cap_{i=1}^{\infty} B_i$ is nonempty.

Properties 1.16 *(e) We call the family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$ to be homogeneous if $\mu_n(\lambda B) = |\lambda|\mu_n(B)$; for $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$.*

(f) If the family $\{\mu_n\}_{n\in\mathbb{N}}$ satisfies the condition $\mu_n(B_1 \cup B_2) \leq \mu_n(B_1) + \mu_n(B_2)$, for $n \in \mathbb{N}$, it is called subadditive.

(g) It is sublinear if both conditions (e) and (f) hold.

(h) We say that the family of measures $\{\mu_n\}_{n\in\mathbb{N}}$ has the maximum property if

$$\mu_n(B_1 \cup B_2) = \max\{\mu_n(B_1), \mu_n(B_2)\},$$

(i) The family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$ is said to be regular if and only if the conditions (a), (g) and (h) hold; (full sublinear and has maximum property).

Example 1.17 *For $B \in \mathcal{M}_X$, $x \in B$, $n \in \mathbb{N}$ and $\epsilon > 0$, let us denote by $\omega^n(x,\epsilon)$ for $n \in \mathbb{N}$; the modulus of continuity of the function x on the interval $[0, n]$; that is,*

$$\omega^n(x,\epsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, n], |t - s| \leq \epsilon\}.$$

Further, let us put

$$\omega^n(B,\epsilon) = \sup\{\omega^n(x,\epsilon) : x \in B\},$$

$$\omega_0^n(B) = \lim_{\epsilon\to 0^+} \omega^n(B,\epsilon),$$

$$\bar{\alpha}^n(B) = \sup_{t\in[0,n]} \alpha(B(t)),$$

and

$$\beta_n(B) = \omega_0^n(B) + \bar{\alpha}^n(B).$$

The family of mappings $\{\beta_n\}_{n\in\mathcal{N}}$ where $\beta_n : \mathcal{M}_X \to [0,\infty)$, satisfies the conditions (a)-(d) from Definition 1.15.

Definition 1.18 *A nonempty subset $B \subset \mathcal{X}$ is said to be bounded if*

$$\sup_{v \in \mathcal{X}} \|v\|_n < \infty; \ for \ n \in \mathbb{N}.$$

Lemma 1.19 *[23] If Y is a bounded subset of Fréchet space $\mathcal{X}$, then for each $\epsilon > 0$, there is a sequence $\{y_k\}_{k=1}^{\infty} \subset Y$ such that*

$$\mu_n(Y) \leq 2\mu_n(\{y_k\}_{k=1}^{\infty}) + \epsilon; \ for \ n \in \mathbb{N}.$$

Lemma 1.20 *[51] If $\{u_k\}_{k=1}^{\infty} \subset L^1(I)$ is uniformly integrable, then $\mu_n(\{u_k\}_{k=1}^{\infty})$ is measurable for $n \in \mathbb{N}$, and*

$$\mu_n\left(\left\{\int_0^t u_k(s)ds\right\}_{k=1}^{\infty}\right) \leq 2\int_0^t \mu_n(\{u_k(s)\}_{k=1}^{\infty})ds,$$

for each $t \in [0, n]$.

Definition 1.21 *Let Ω be a nonempty subset of a Fréchet space $\mathcal{X}$, and let $A : \Omega \to \mathcal{X}$ be a continuous operator which transforms bounded subsets of into bounded ones. One says that A satisfies the Darbo condition with constants $(k_n)_{n\in\mathbb{N}}$ with respect to a family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$, if*

$$\mu_n(A(B)) \leq k_n \mu_n(B)$$

for each bounded set $B \subset \Omega$ and $n \in \mathbb{N}$.
If $k_n < 1$; $n \in \mathbb{N}$ then A is called a contraction with respect to $\{\mu_n\}_{n\in\mathbb{N}}$.

1.7 Random operators

Let β_E be the σ-algebra of Borel subsets of E. A mapping $v : \Omega \to E$ is said to be measurable if for any $B \in \beta_E$, one has

$$v^{-1}(B) = \{w \in \Omega : v(w) \in B\} \in \mathcal{A}.$$

To define integrals of sample paths of random process, it is necessary to define a jointly measurable map.

Definition 1.22 *A mapping $T : \Omega \times E \to E$ is called jointly measurable if for any $B \in \beta_E$, one has*

$$T^{-1}(B) = \{(w, v) \in \Omega \times E : T(w, v) \in B\} \subset \mathcal{A} \times \beta_E,$$

where $\mathcal{A} \times \beta_E$ is the direct product of the σ-algebras $\mathcal{A}$ and β_E those defined in Ω and E respectively.

Definition 1.23 *A function $T : \Omega \times E \to E$ is called jointly measurable if $T(\cdot, u)$ is measurable for all $u \in E$ and $T(w, \cdot)$ is continuous for all $w \in \Omega$.*

Definition 1.24 *A function $f : I \times E \times \Omega \to E$ is called random Carathéodory if the following conditions are satisfied:*

(i) The map $(t, w) \to f(t, u, w)$ is jointly measurable for all $u \in E$, and

(ii) The map $u \to f(t, u, w)$ is continuous for all $t \in I$ and $w \in \Omega$.

Let $T : \Omega \times E \to E$ be a mapping. Then T is called a random operator if $T(w, u)$ is measurable in w for all $u \in E$ and it expressed as $T(w)u = T(w, u)$. In this case we also say that $T(w)$ is a random operator on E. A random operator $T(w)$ on E is called continuous (resp. compact, totally bounded and completely continuous) if $T(w, u)$ is continuous (resp. compact, totally bounded and completely continuous) in u for all $w \in \Omega$. The details of completely continuous random operators in Banach spaces and their properties appear in Itoh [42].

Definition 1.25 *[34] Let $\mathcal{P}(E)$ be the family of all nonempty subsets of E and C be a mapping from Ω into $\mathcal{P}(E)$. A mapping $T : \{(w, y) : w \in \Omega,\ y \in C(w)\} \to E$ is called random operator with stochastic domain C iff C is measurable (i.e., for all closed $A \subset E$, $\{w \in \Omega, C(w) \cap A \neq \emptyset\} \in \mathcal{A}$ is measurable) and for all open $D \subset E$ and all $y \in E$, $\{w \in \Omega : y \in C(w), T(w, y) \in D\} \in \mathcal{A}$. T will be called continuous if every $T(w)$ is continuous. For a random operator T, a mapping $y : \Omega \to E$ is called random (stochastic) fixed point of T iff for $P-$almost all $w \in \Omega$, $y(w) \in C(w)$ and $T(w)y(w) = y(w)$ and for all open $D \subset E$, $\{w \in \Omega : y(w) \in D\} \in \mathcal{A}$(y is measurable).*

Remark 1.26 *If $C(w) \equiv E$, then the definition of random operator with stochastic domain coincides with the definition of random operator.*

Lemma 1.27 *[33] Let $C : \Omega \longrightarrow 2^E$ be measurable with $C(w)$ closed, convex and solid (i.e., int $C(w) \neq \emptyset$) for all $w \in \Omega$. We assume that there exists measurable $y_0 : \Omega \longrightarrow E$ with $y_0 \in$ int $C(w)$ for all $w \in \Omega$. Let T be a continuous random operator with stochastic domain C such that for every $w \in \Omega, \{y \in C(w) : T(w)y = y\} \neq \emptyset$. Then T has a stochastic fixed point.*

Let y be a mapping of $J \times \Omega$ into X. y is said to be a stochastic process if for each $t \in J$, the function $y(t, \cdot)$ is measurable.

1.8 Some Fixed Point Theorems

Theorem 1.28 *(Mönch , [5, 51]).*
Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication

$$V = \overline{conv} N(V) \quad or \quad V = N(V) \cup 0 \Longrightarrow \alpha(V) = 0$$

holds for every subset V of D, then N has a fixed point.

Lemma 1.29 *(Darbo , [40]).*
Let D be a bounded, closed and convex subset of Banach space X. If the operator $N : D \to D$ is a strict set contraction, i.e there is a constant $0 \leq \lambda < 1$ such that $\alpha(N(S)) \leq \lambda\alpha(S)$ for any set $S \subset D$ then N has a fixed point in D.

Theorem 1.30 *[28, 31] Let Ω be a nonempty, bounded, closed, and convex subset of a Fréchet space $\mathcal{X}$ and let $V : \Omega \to \Omega$ be a continuous mapping. Suppose that V is a contraction with respect to a family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$. Then V has at least one fixed point in the set Ω.*

Chapter 2

Evolution of Semilinear Differential Equations

2.1 Introduction

This chapter is concerned with the existence of mild solutions for a class of evolution equations. We discuss the existence of mild solutions for the evolution equation

$$u'(t) = A(t)u(t) + f(t, u(t)); \quad \text{if } t \in \mathbb{R}_+ := [0, \infty), \tag{2.1}$$

with the initial condition

$$u(0) = u_0 \in E, \tag{2.2}$$

where $f : \mathbb{R}_+ \times E \to E$ is a given function, $(E, \|\cdot\|)$ is a (real or complex) Banach space, and $\{A(t)\}_{t>0}$ is a family of linear closed (not necessarily bounded) operators from E into E that generate an evolution system of bounded linear operators $\{U(t,s)\}_{(t,s)\in\mathbb{R}_+\times\mathbb{R}_+}$; for $(t,s) \in \Lambda := \{(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+ : 0 \le s \le t < +\infty\}$.

2.2 Global existence

In this Section, we study the existence of the global mild solutions for our problem. Let us introduce the definition of the mild solution of the problem (2.1)-(2.2).

Definition 2.1 *We say that a continuous function $u(\cdot) : I \to E$ is a mild solution of the problem (2.1)-(2.2), if u satisfies the following integral equation*

$$u(t) = U(t,0)u_0 + \int_0^t U(t,s)\ f(s, u(s))ds, \qquad \textit{for each } t \in \mathbb{R}_+.$$

Let us introduce the following hypotheses.

(H_1) There exists a constant $M \geq 1$ such that

$$\|U(t,s)\|_{B(E)} \leq M; \text{ for every } (t,s) \in \Lambda.$$

(H_2) The function $t \mapsto f(t,u)$ is measurable on $\mathbb{R}_+$ for each $u \in E$, and the function $u \mapsto f(t,u)$ is continuous on E for a.e. $t \in \mathbb{R}_+$.

(H_3) There exists a continuous function $p : \mathbb{R}_+ \to [0,\infty)$ such that

$$\|f(t,u)\| \leq p(t)(1 + \|u\|); \text{ for a.e. } t \in \mathbb{R}_+, \text{ and each } u \in E.$$

(H_4) For each bounded and measurable set $B \subset E$ and for each $t \in \mathbb{R}_+$, we have

$$\mu(f(t,B)) \leq p(t)\mu(B),$$

where μ is a measure of noncompactness on the Banach space E.

For $n \in \mathbb{N}$, let

$$p_n^* = \sup_{t\in[0,n]} p(t),$$

and define on $C(\mathbb{R}_+)$ the family of measure of noncompactness by

$$\mu_n(D) = \sup_{t\in[0,n]} e^{-4Mp_n^*\tau t}\mu(D(t)),$$

where $\tau > 1$, and $D(t) = \{v(t) \in E : v \in D\}; \ t \in [0,n]$.

Theorem 2.2 *Assume that the hypotheses $(H_1) - (H_4)$ are satisfied, and $nMp_n^* < 1$ for each $n \in \mathbb{N}$. Then the problem (2.1)-(2.2) has at least one mild solution.*

Proof. Consider the operator $N : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined by:

$$(Nu)(t) = U(t,0)u_0 + \int_0^t U(t,s)\ f(s,u(s))ds. \tag{2.3}$$

Clearly, the fixed points of the operator N are solution of the problem (2.1)-(2.2).

For any $n \in \mathbb{N}$, let R_n be a positive real number with

$$R_n \geq \frac{M\|u_0\| + np_n^*M}{1 - nMp_n^*},$$

and we consider the ball

$$B_{R_n} := B(0,R_n) = \{w \in C(\mathbb{R}_+) : \|w\|_n \leq R_n\}.$$

For any $n \in \mathbb{N}$, and each $u \in B_{R_n}$ and $t \in [0, n]$ we have

$$\begin{aligned} \|(Nu)(t)\| &\leq \|U(t,0)\|_{B(E)}\|u_0\| + \int_0^t \|U(t,s)\|_{B(E)}\|f(s,u(s))\|ds \\ &\leq M\|u_0\| + M\left(\int_0^t p(s)(1+\|u(s)\|)ds\right) \\ &\leq M\|u_0\| + M(1+\|u\|_n)\int_0^t p(s)ds \\ &\leq M\|u_0\| + nMp_n^*(1+R_n) \\ &\leq R_n. \end{aligned}$$

Thus

$$\|N(u)\|_n \leq R_n. \tag{2.4}$$

This proves that N transforms the ball B_{R_n} into itself. We shall show that the operator $N : B_{R_n} \to B_{R_n}$ satisfies all the assumptions of Theorem 1.30. The proof will be given in several steps.

Step 1. *$N : B_{R_n} \to B_{R_n}$ is continuous.*

Let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence such that $u_k \to u$ in B_{R_n}. Then, for each $t \in [0, n]$, we have

$$\begin{aligned} \|(Nu_k)(t) - (Nu)(t)\| &\leq \int_0^t \|U(t,s)\|_{B(E)}\|f(s,u_k(s)) - f(s,u(s))\|ds \\ &\leq M\int_0^t \|f(s,u_k(s)) - f(s,u(s))\|ds. \end{aligned}$$

Since $u_k \to u$ as $k \to \infty$, the Lebesgue dominated convergence theorem implies that

$$\|N(u_k) - N(u)\|_n \to 0 \quad \text{as } k \to \infty.$$

Step 2. *$N(B_{R_n})$ is bounded.*

Since $N(B_{R_n}) \subset B_{R_n}$ and B_{R_n} is bounded, then $N(B_{R_n})$ is bounded.

Step 3. *For each equicontinuous subset D of B_{R_n}, $\mu_n(N(D)) \leq \frac{1}{\tau}\mu_n(D)$.*

From Lemmas 1.19 and 1.20, for any $D \subset B_{R_n}$ and any $\epsilon > 0$, there exists a sequence

$\{u_k\}_{k=0}^{\infty} \subset D$, such that for all $t \in [0,n]$, we have

$$\begin{aligned}
\mu((ND)(t)) &= \mu\left(\left\{U(t,0)u_0 + \int_0^t U(t,s)\ f(s,u(s))ds;\ u \in D\right\}\right) \\
&\le 2\mu\left(\left\{\int_0^t U(t,s)f(s,u_k(s))ds\right\}_{k=1}^{\infty}\right) + \epsilon \\
&\le 4\int_0^t \mu\left(\|U(t,s)\|_{B(E)}\{f(s,u_k(s))\}_{k=1}^{\infty}\right) ds + \epsilon \\
&\le 4M\int_0^t \mu\left(\{f(s,u_k(s)\}_{k=1}^{\infty}\right) ds + \epsilon \\
&\le 4M\int_0^t p(s)\mu\left(\{u_k(s)\}_{k=1}^{\infty}\right) ds + \epsilon \\
&\le 4Mp_n^*\int_0^t e^{4Mp_n^*\tau s}e^{-4Mp_n^*\tau s}\mu\left(\{u_k(s)\}_{k=1}^{\infty}\right) ds + \epsilon \\
&\le \frac{\left(e^{4Mp_n^*\tau t} - 1\right)}{\tau}\mu_n(D) + \epsilon \\
&\le \frac{e^{4Mp_n^*\tau t}}{\tau}\mu_n(D) + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, then

$$\mu((ND)(t)) \le \frac{e^{4Mp_n^*\tau t}}{\tau}\mu_n(D).$$

Thus

$$\mu_n(N(D)) \le \frac{1}{\tau}\mu_n(D).$$

As a consequence of steps 1 to 3 together with Theorem 1.30, we can conclude that N has at least one fixed point in B_{R_n} which is a mild solution of problem (2.1)-(2.2).

2.3 An Example

As an application of our results, we consider the following functional evolution problem of the form

$$\begin{cases}
\dfrac{\partial z}{\partial t}(t,x) = a(t,x)\dfrac{\partial^2 z}{\partial x^2}(t,x) + Q(t,z(t,x)); & t \in \mathbb{R}_+,\ x \in [0,\pi], \\
z(t,0) = z(t,\pi) = 0; & t \in \mathbb{R}_+, \\
z(0,x) = \Phi(x); & x \in [0,\pi],
\end{cases} \tag{2.5}$$

where $a(t,x) : \mathbb{R}_+ \times [0,\pi] \to \mathbb{R}$ is a continuous function and is uniformly Hölder continuous in t, $Q : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $\Phi : [0,\pi] \to \mathbb{R}$ are continuous functions.

Consider $E = L^2([0,\pi],\mathbb{R})$ and define $A(t)$ by $A(t)w = a(t,x)w''$ with domain

$$D(A) = \{w \in E : w, w' \text{ are absolutely continuous}, w'' \in E,\ w(0) = w(\pi) = 0\}.$$

Then $A(t)$ generates an evolution system $U(t,s)$ (see [37]).

For $x \in [0,\pi]$, we have

$$y(t)(x) = z(t,x); \quad t \in \mathbb{R}_+,$$

$$f(t,u(t),x) = Q(t,z(t,x)); \quad t \in \mathbb{R}_+,$$

and

$$u_0(x) = \Phi(x); \quad x \in [0,\pi].$$

Thus, under the above definitions of f, u_0 and $A(\cdot)$, the system (2.5) can be represented by the functional evolution problem (2.1)-(2.2). Furthermore, more appropriate conditions on Q ensure the hypotheses $(H_1) - (H_4)$. Consequently, Theorem 2.2 implies that the evolution problem (2.5) has at least one mild solution.

Chapter 3

Semilinear Differential Equations

3.1 Introduction

In this chapter, we prove the existence of global mild and integral solutions of the following differential equation for class of semi-linear functional of the form:

$$y'(t) = Ay(t) + f(t, y(t)), \quad t \in [0, \infty), \tag{3.1}$$

with the initial condition

$$y(0) = y_0 \in E \tag{3.2}$$

where $f : \mathbb{R}_+ \times \mathbb{E} \to \mathbb{E}$ is given function, $(E, \|\cdot\|)$ is a (real or complex) Banach space. in section 1, we discuss the existence of mild solution of the problem (3.1)-(3.2), in the case where A is densely defined operator generated a C_0-semigroup $(T(t))_{t\geq 0}$ on E.
Then in section 2, we discuss the existence of integral solution of the problem (3.1)-(3.2), in the case where A is a Hille-Yosida operator, nondensely defined and generates an integrated semigroup $(S(t))_{t\geq 0}$ on E.

3.2 Global existence of mild solutions

In this section, we discuss the existence of global mild solution of the problem (3.1)-(3.2), in the case where A is densely defined operator generated a C_0-semigroup $(T(t))_{t\geq 0}$ on E. Let us introduce the definition of the mild solution of the problem (3.1)-(3.2).

Definition 3.1 *We say that a continuous function $y(.) : I \to E$ is mild solution of the problem* (3.1)-(3.2)*, if y satisfies the following integral equation*

$$y(t) = T(t)y_0 + \int_0^t T(t-s)f(s, y(s))ds, \quad t \in [0, +\infty). \tag{3.3}$$

We will consider the hypotheses (3.1)-(3.2) and we will need to introduce the following one which is assumed hereafter:

(H_1) The operator A is the infinitesimal generator of a C_0-semigroup $T(t), t \in J$ in E and there exists a positive constant $M \geq 1$ such that

$$\{\|T(t)\|_{B(E)} \leq M, \quad t \geq 0\}$$

(H_2) The function $t \to f(t,y)$ is measurable on $\mathbb{R}$ for each $y \in E$, and the function $y \mapsto f(t,y)$ is continuous on E for a.e.$t \in \mathbb{R}_+$.

(H_3) There exists a continuous function $p : \mathbb{R} \to [0,\infty)$ such that

$$\|f(t,y)\| \leq p(t)(1 + \|y\|) \quad \text{for a.e.} t \in \mathbb{R}_+ \quad \text{and each} \quad y \in E.$$

(H_4) For each bounded and measurable set $B \subset E$ and for each $t \in \mathbb{R}_+$, we have

$$\mu(f(t,B)) \leq p(t)\mu(B),$$

where μ is a measure of noncompactness on the Banach space E.

For $n \in \mathbb{N}$, let

$$p_n^* = \sup_{t\in[0,n]} p(t),$$

and define on $C(\mathbb{R}_+)$ the family of noncompactness by

$$\mu_n(D) = \sup_{t\in[0,n]} e^{-4Mp_n^*\tau t}\mu(D(t))$$

where $\tau > 0$ et $D(t) = \{v(t) \in E : v \in D\}; t \in [0,n]$.

Theorem 3.2 *Assume that the hypotheses* (H_1)- (H_4) *are satisfied, and*

$$l_n := nMp_n^* < 1; \quad \textit{for each} \quad n \in \mathbb{N}.$$

Then the problem (3.1)-(3.2) *has at least one mild solution.*

Proof. Consider the operator $N : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined by:

$$(Ny)(t) = T(t)y_0 + \int_0^t T(t-s)f(s,y(s))ds, \quad t \in [0,+\infty). \tag{3.4}$$

For any $n \in \mathbb{N}$, let R_n be a positive real number with

$$R_n \geq \frac{M|y_0| + nMp_n^*}{1 - nMp_n^*},$$

and we consider the ball

$$B_{R_n} := B(0,R_n) = \{\omega \in C(R_+) : \|\omega\|_n \leq R_n\}.$$

For any $n \in \mathbb{N}$, and each $y \in B_{R_n}$ and $t \in [0,n]$ we have

$$\begin{aligned}
\|(Ny)(t)\| &\leq \|T(t)\|_{B(E)}|y_0| + \int_0^t \|T(t-s)\|_{B(E)}\|f(s,y(s))\|ds \\
&\leq M|y_0| + M\left(\int_0^t p(s)(1+\|y(s)\|)ds\right) \\
&\leq M|y_0| + M(1+\|y\|_n)\int_0^t p(s)ds \\
&\leq M|y_0| + Mnp_n^*(1+R_n) \\
&\leq R_n.
\end{aligned}$$

Thus

$$\|N(y)\|_n \leq R_n. \tag{3.5}$$

This proves that N transforms the ball B_{R_n} into itself. We shall show that the operator $N : B_{R_n} \to B_{R_n}$ satisfies all the assumptions of Theorem 1.30. The proof will be given several steps.

Step 1. $N : B_{R_n} \to B_{R_n}$ is continuous.

Let $\{y_k\}_{k\in\mathbb{N}}$ be a sequence such that $y_k \to y$ in B_{R_n}. Then, for each $t \in [0,n]$, we have

$$\begin{aligned}
\|(Ny_k)(t) - (Ny)(t)\| &\leq \int_0^t \|T(t-s)\|_{B(E)}\|f(s,y_k(s)) - f(s,y(s))\|ds \\
&\leq M\int_0^t \|f(s,y_k(s)) - f(s,y(s))\|ds.
\end{aligned}$$

Since $y_k \to y$ as $k \to \infty$, the Lebesgue dominated convergence theorem implies that

$$\|N(y_k) - N(y)\|_n \to 0 \quad \text{as} \quad k \to \infty.$$

Step 2. $N(B_{R_n})$ is bounded.

Since $N(B_{R_n}) \subset B_{R_n}$ is bounded, then $N(B_{R_n})$ is bounded.

Step 3. For each equicontinuous subset D of B_{R_n}, $\mu_n((ND) \leq \frac{1}{\tau}\mu_n(D)$.

From lemmas 1.19 and 1.20 , for any $D \subset B_{R_n}$ and any $\epsilon > 0$, there exists a sequence

$\{y_k(s)\}_{k=1}^{\infty} \subset D$, such that for all $t \in [0, n]$, we have

$$\begin{aligned}
\mu(ND)(t)) &= \mu\left(\left\{T(t)y_0 + \int_0^t T(t-s)f(s, y(s))ds; y \in D\right\}\right) \\
&\leq 2\mu(\{\int_0^t T(t-s)f(s, y_k(s))ds\}_{k=1}^{\infty}) + \epsilon \\
&\leq 2\mu\left(\left\{\int_0^t T(t-s)f(s, y_k(s))ds\right\}_{k=1}^{\infty}\right) + \epsilon \\
&\leq 4M\int_0^t \mu(\{f(s, y_k(s))\}_{k=1}^{\infty})ds + \epsilon \\
&\leq 4M\int_0^t p(s)\mu(\{y_k(s)\}_{k=1}^{\infty})ds + \epsilon \\
&\leq 4Mp_n^*\int_0^t \mu(\{y_k(s)\}_{k=1}^{\infty})ds + \epsilon \\
&\leq 4Mp_n^*\int_0^t e^{\frac{16}{n}Mp_n^*s}e^{-\frac{16}{n}Mp_n^*s}\mu(\{y_k(s)\}_{k=1}^{\infty})ds + \epsilon \\
&\leq 4Mp_n^*\mu_n(D)\int_0^t e^{\frac{16}{n}Mp_n^*s}ds + \epsilon \\
&\leq \frac{e^{\frac{16}{n}Mp_n^*\tau t}}{\tau}\mu_n(D) + \epsilon
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, then

$$\mu(ND)(t) \leq \frac{e^{-\frac{16}{n}Mp_n^*\tau t}}{\tau}\mu_n(D)$$

Thus

$$\mu_n(ND) \leq \frac{1}{\tau}\mu_n(D)$$

As a consequence of steps to 1 to 3 together with Theorem 1.30, we can conclude that N has a least one fixed in B_{R_n} which is a mild solution of problem (3.1)-(3.2).

3.3 Existence of global integral solutions

In this section, we discuss the existence of integral solution of the problem (3.1)-(3.2), in the case where A is a Hille-Yosida operator nondensely defined on E generated a integrated semigroup $(S(t))_{t\geq 0}$ on E. Before starting and proving this result, we give the definition of its integral solution.

(P_1) A satisfies the Hille–Yosida condition, namely, there exist $\overline{M} \geq 0$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \varrho(A)$ and

$$|(\lambda I - A)^{-n}| \leq \frac{\overline{M}}{(\lambda - \omega)^n} \quad \text{for} \quad n \in \mathbb{N} \quad \text{and} \quad \lambda > \omega,$$

where $\varrho(A)$ is the resolvent set of A, for more details (see [45]).

Definition 3.3 *We say that* $y : [0, \infty) \to E$ *is an integral solution of (3.1)-(3.2) if*

(i) $y \in C([0, \infty), E)$,

(ii) $\int_0^t y(s)ds \in D(A)$ *for* $t \in J$,

(iii) $y(t) = y_0 + A \int_0^t y(s)ds + \int_0^t f(s, y(s))ds, \quad t \in J$

Moreover, y satisfies the following variation of constants formula:

$$y(t) = S'(t)y_0 + \frac{d}{dt}\int_0^t S(t-s)f(s, y(s))ds, \quad t \in [0, +\infty). \tag{3.6}$$

From the definition it follows that $y(t) \in \overline{D(A)}, \quad t \geq 0$, in particular $y_0 \in \overline{D(A)}$. We will consider the hypotheses (H_2)-(H_4) and we will need to introduce the following one which is assumed hereafter:

(P_2) Let $(S(t))_{t\geq 0}$, be the integrated semigroup generated by A such that

$$\|S'(t)\|_{B(E)} \leq \widetilde{M}, \quad t \geq 0.$$

Theorem 3.4 *Assume that the hypotheses* (P_1)- (P_2) *and* (H_2)-(H_4) *are satisfied, and*

$$l_n := n\widetilde{M}p_n^* < 1; \quad \textit{for each} \quad n \in \mathbb{N}$$

. *Then the problem* (3.1)-(3.2) *has at least one mild solution.*

Proof. Consider the operator $Q : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined by:

$$(Qy)(t) = S'(t)y_0 + \frac{d}{dt}\int_0^t S(t-s)f(s, y(s))ds, \quad t \in [0, +\infty). \tag{3.7}$$

For any $n \in \mathbb{N}$, let R_n be a positive real number with

$$R_n \geq \frac{\widetilde{M}|y_0| + n\widetilde{M}p_n^*}{1 - n\widetilde{M}p_n^*},$$

and we consider the ball

$$B_{R_n} := B(0, R_n) = \{\omega \in C(R_+) : \|\omega\|_n \leq R_n\}.$$

For any $n \in \mathbb{N}$, and each $y \in B_{R_n}$ and $t \in [0, n]$ we have

$$\begin{aligned}
\|(Qy)(t)\| &\leq \|S'(t)\|_{B(E)}|y_0| + \left\|\frac{d}{dt}\int_0^t S(t-s)f(s,y(s))ds\right\| \\
&\leq \widetilde{M}|y_0| + \widetilde{M}\left(\int_0^t p(s)(1+\|y(s)\|)ds\right) \\
&\leq \widetilde{M}|y_0| + \widetilde{M}(1+\|y\|_n)\int_0^t p(s)ds \\
&\leq \widetilde{M}|y_0| + \widetilde{M}np_n^*(1+R_n) \\
&\leq R_n.
\end{aligned}$$

Thus

$$\|Q(y)\|_n \leq R_n. \tag{3.8}$$

This proves that Q transforms the ball B_{R_n} into itself. We shall show that the operator $Q : B_{R_n} \to B_{R_n}$ satisfies all the assumptions of Theorem 1.30. The proof will be given several steps.

Step 1. $Q : B_{R_n} \to B_{R_n}$ is continuous.

Let $\{y_k\}_{k\in\mathbb{N}}$ be a sequence such that $y_k \to y$ in B_{R_n}. Then, for each $t \in [0,n]$, we have

$$\begin{aligned}
\|(Qy_k)(t) - (Qy)(t)\| &\leq \|\frac{d}{dt}\int_0^t S(t-s)(f(s,y_k(s)) - f(s,y(s)))ds\| \\
&\leq \widetilde{M}\int_0^t \|f(s,y_k(s)) - f(s,y(s))\|ds.
\end{aligned}$$

Since $y_k \to y$ as $k \to \infty$, the Lebesgue dominated convergence theorem implies that

$$\|Q(y_k) - Q(y)\|_n \to 0 \quad \text{as} \quad k \to \infty.$$

Step 2. $Q(B_{R_n})$ is bounded.

Since $Q(B_{R_n}) \subset B_{R_n}$ is bounded, then $Q(B_{R_n})$ is bounded.

Step 3. For each equicontinuous subset D of B_{R_n}, $\mu_n(QD) \leq \frac{1}{\tau}\mu_n(D)$.

From lemmas 1.19 and 1.20 , for any $D \subset B_{R_n}$ and any $\epsilon > 0$, there exists a sequence

$\{y_k(s)\}_{k=1}^{\infty} \subset D$, such that for all $t \in [0, n]$, we have

$$\begin{array}{rcl}\mu(QD)(t)) & = & \mu\left(\left\{S'(t)y_0 + \frac{d}{dt}\int_0^t S(t-s)f(s,y(s))ds; y \in D\right\}\right) \\ & \leq & 2\mu\left(\{\frac{d}{dt}\int_0^t S(t-s)f(s,y_k(s))ds\}_{k=1}^{\infty}\right) + \epsilon \\ & \leq & 4\widetilde{M}\int_0^t \mu(\{f(s,y_k(s))\}_{k=1}^{\infty})ds + \epsilon \\ & \leq & 4\widetilde{M}\int_0^t p(s)\mu(\{y_k(s)\}_{k=1}^{\infty})ds + \epsilon \\ & \leq & 4\widetilde{M}p_n^*\int_0^t \mu(\{y_k(s)\}_{k=1}^{\infty})ds + \epsilon \\ & \leq & 4\widetilde{M}p_n^*\int_0^t e^{\widetilde{M}p_n^*s}e^{-\widetilde{M}p_n^*\tau_n s}\mu(\{y_k(s)\}_{k=1}^{\infty})ds + \epsilon \\ & \leq & 4\widetilde{M}p_n^*\mu_n(D)\int_0^t e^{\widetilde{M}p_n^*\tau s}ds + \epsilon \\ & \leq & \frac{e^{\widetilde{M}p_n^*\tau t}}{\tau}\mu_n(D) + \epsilon\end{array}$$

Since $\epsilon > 0$ is arbitrary, then

$$\mu(QD)(t) \leq \frac{e^{\widetilde{M}p_n^*\tau t}}{\tau}\mu_n(D)$$

Thus

$$\mu_n(QD) \leq \frac{1}{\tau}\mu_n(D).$$

As a consequence of steps to 1 to 3 together with Theorem 1.30, we can conclude that Q has a least one fixed in B_{R_n} which is a mild solution of problem (3.1)- (3.2).

3.4 Examples

Example 1. We consider the following problem

$$\begin{cases} \frac{\partial z}{\partial t}(t,x) = \frac{\partial^2 z}{\partial x^2}(t,x) + Q(t,z(t,x)),; & t \in \mathbb{R}_+,\ x \in [0,\pi], \\ z(t,0) = z(t,\pi) = 0; & t \in \mathbb{R}_+. \\ z(0,x) = \Phi(x); & x \in [0,\pi], \end{cases} \tag{3.9}$$

where $Q : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $\Phi : [0,\pi] \to \mathbb{R}$ are continuous functions.

Consider $E = L^2([0, \pi], \mathbb{R})$ and define A by $Aw = w''$ with domain

$$D(A) = \{w \in E : w, w' \text{ are absolutely continuous}, w'' \in E,\ w(0) = w(\pi) = 0\}.$$

Then

$$Aw = \sum_{n=1}^{\infty} n^2 (w, w_n) w_n, \quad w \in D(A),$$

where $(\cdot, \cdot)$ is the inner product in L^2 and $w_n(s) = \sqrt{\frac{2}{\pi}} \sin ns, \quad n = 1, 2, \ldots,$ is orthogonal set of eigenvectors of A. It is well known (see [54]) that A is the infinitesimal generator of an analytic semigroup $T(t), \quad t \geq 0$, in E and is given by the relation

$$T(t)w = \sum_{n=1}^{\infty} \exp(-n^2 t)(w, w_n) w_n, \quad w \in E,$$

and there exists a positive constant M such that

$$\|T(t)\|_{B(E)} \leq M.$$

For $x \in [0, \pi]$, we have

$$\begin{aligned} y(t)(x) &= z(t, x); \quad t \in \mathbb{R}_+, \\ f(t, y(t))(x) &= Q(t, z(t, x)); \quad t \in \mathbb{R}_+, \\ y_0(x) &= \Phi(x). \end{aligned}$$

Then the system (3.9) can be represented by the functional problem (3.1)- (3.2), and conditions $(H_1) - (H_4)$ are satisfied. Consequently, Theorem 3.2 implies that the problem (3.9)has mild solution.

Example 2. We consider the following problem

$$\begin{cases} \dfrac{\partial z}{\partial t}(t, x) = \dfrac{\partial^2 z}{\partial x^2}(t, x) + Q(t, z(t, x)), ; & t \in \mathbb{R}_+,\ x \in [0, \pi], \\ z(t, 0) = z(t, \pi) = 0; & t \in \mathbb{R}_+. \\ z(0, x) = \Phi(x); & x \in [0, \pi], \end{cases} \tag{3.10}$$

where $Q : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $\Phi : [0, \pi] \to \mathbb{R}$ are continuous functions.

Consider $E = C([0, \pi], \mathbb{R})$ and define A by $Aw = w''$ with domain

$$D(A) = \{w \in E : w, w' \text{ are absolutely continuous}, w'' \in E,\ w(0) = w(\pi) = 0\}.$$

It is well known (see [27]) that the operator A Satisfies the Hille-Yosida condition with $(0, +\infty) \subset \varrho(A)$,

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda} \text{ for } \lambda > 0,$$

and

$$\overline{D(A)} = \{w \in E,\ w(0) = w(\pi) = 0\} \neq E.$$

It follows that A generates an integrated semigroup $(S(t))_{t\geq 0}$ and $\|S(t)\| \leq M$ for $t \geq 0$. We can show that problem (3.1)- (3.2) is an abstract formulation of problem (3.10).

For $x \in [0,\pi]$, we have

$$y(t)(x) = z(t,x); \quad t \in \mathbb{R}_+,$$

$$f(t,y(t))(x) = Q(t,z(t,x)); \quad t \in \mathbb{R}_+,$$

$$y_0(x) = \Phi(x).$$

Then the system (3.10) can be represented by the functional problem (3.1)- (3.2), and conditions (P_1)- (P_2) and (H_2)-(H_4) are satisfied. Consequently, Theorem 3.4 implies that the problem (3.10) has an integral solution.

Chapter 4

Controllability of Semilinear Differential Equations

4.1 Introduction

In this chapter, we consider the controllability of mild solutions for semilinear differential equation

$$y'(t) = Ay(t) + f(t, y(t)) + Bu(t), \quad t \in [0, \infty), \tag{4.1}$$

with the initial condition

$$y(0) = y_0 \in E \tag{4.2}$$

where $f : \mathbb{R}_+ \times E \to E$ is given function, $(E, \|\cdot\|)$ is a (real or complex) Banach space, and $A : D(A) \to E$ is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on E. The control function $u(\cdot, w)$ is given in $L^2(\mathbb{R}_+, U)$, a Banach space of admissible control functions with U as a Banach space, and B is a bounded linear operator from U into E.

4.2 Controllability Result

In this section, we study the controllability for the differential system (4.1)-(4.2).

Definition 4.1 *We say that a continuous function $y(\cdot) : I \to E$ is mild solution of the problem* (4.1)-(4.2)*, if y satisfies the following integral equation*

$$y(t) = T(t)y_0 + \int_0^t T(t-s)f(s, y(s))ds + \int_0^t T(t-s)Bu(s)ds, \quad t \in [0, +\infty). \tag{4.3}$$

Definition 4.2 *The problem* (4.1)-(4.2) *is said to be controllable if for every $y_0, y_1 \in E$ where y_0 is initial condition and y_1 is final state, there is some control $u \in L^2([0, n], E)$ such that the mild solution $y(\cdot)$ of* (4.1)-(4.2) *satisfies the terminal condition $y(n) = y_1$.*

We will consider the hypotheses (4.1)-(4.2) and we will need to introduce the following one which is assumed hereafter:

(H_1) The operator A is the infinitesimal generator of a C_0-semigroup $T(t), t \geq 0$ in E and there exists a positive constant $M \geq 1$ such that

$$M = \sup\{\|T(t)\|_{B(E)}, t \geq 0\}$$

(H_2) The function $t \to f(t, y)$ is measurable on $\mathbb{R}$ for each $y \in E$, and the function $y \mapsto f(t, y)$ is continuous on E for a.e.$t \in \mathbb{R}_+$.

(H_3) There exists a continuous function $p : \mathbb{R}_+ \to [0, \infty)$ such that

$$\|f(t, y)\| \leq p(t)(1 + \|y\|) \quad \text{for a.e.} t \in \mathbb{R}_+ \quad \text{and each} \quad y \in E.$$

(H_4) For each bounded and measurable set $B \subset E$ and for each $t \in \mathbb{R}_+$, we have

$$\mu(f(t, B)) \leq p(t)\mu(B),$$

where μ is a measure of noncompactness on the Banach space E.

(H_5) For each $n \in \mathbb{N}$, the linear operator $W : L^2([0, n], E) \to E$ is defined by

$$Wu = \int_0^n T(n - s)Bu(s)ds,$$

has a bounded pseudo inverse operator W^{-1} which takes values in $L^2([0, n], E)/\ker W$ and there exist positive constant K such that:

$$\|BW^{-1}\| \leq K$$

Remark 4.3 *For the construction of W see [57].*

For each $t \in [0, n], \quad n \in \mathbb{N}$

$$u_y(t) = W^{-1}\left(y_1 - T(n)y_0 - \int_0^n T(n - s)f(s, y(s))ds\right)(t). \tag{4.4}$$

For $n \in \mathbb{N}$, let

$$p_n^* = \sup_{t \in [0,n]} p(t),$$

and define on $C(\mathbb{R}_+)$ the family of noncompactness by

$$\mu_n(D) = \sup_{t \in [0,n]} e^{-4Mp_n^*(1+nMK)\tau t}\mu(D(t))$$

where $\tau > 1$ and $D(t) = \{v(t) \in E : v \in D\}; t \in [0, n]$.

Theorem 4.4 *Assume that the hypotheses (H_1)- (H_5) are satisfied, and*

$$l_n := nMp_n^* + n^2M^2Kp_n^* < 1; \quad \textit{for each} \quad n \in \mathbb{N}.$$

Then the problem (4.1)-(4.2) is controllable.

Proof. Consider the operator $N : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined by:

$$(Ny)(t) = T(t)y_0 + \int_0^t T(t-s)f(s,y(s))ds + \int_0^t T(t-s)Bu_y(s)ds, \quad t \in [0,+\infty). \tag{4.5}$$

Using assumption (H_5), for arbitrary function $y(.)$, we define the control

$$u_y(t) = W^{-1}\left(y_1 - T(n)y_0 - \int_0^n T(n-s)f(s,y(s))ds\right)(t), \tag{4.6}$$

we obtain

$$\begin{aligned}(Ny)(t) &= T(t)y_0 + \int_0^t T(t-s)f(s,y(s))ds \\ &+ \int_0^t T(t-s)BW^{-1}(y_1 - T(n)y_0 - \int_0^n T(n-\tau)f(\tau,y(\tau))d\tau)ds, \quad t \in [0,+\infty).\end{aligned} \tag{4.7}$$

Noting that, for $t \in [0,n]$ we have

$$\begin{aligned}\|u_y(t)\| &\leq \|W^{-1}\| \left[|y_1| + \|T(n)\|_{B(E)}|y_0| + \int_0^n \|T(n-s)\|_{B(E)}\|f(s,y(s))\|\right] ds \\ &\leq M_2\left[|y_1| + M|y_0| + M\int_0^n p(s)(1+\|y(s)\|)\right] \\ &\leq M_2\left[|y_1| + M|y_0| + Mnp_n^*(1+\|y\|_n)\right].\end{aligned}$$

For any $n \in \mathbb{N}$, let R_n be a positive real number with

$$R_n \geq \frac{M|y_0| + nMK|y_1| + nM^2K|y_0| + nMp_n^* + n^2M^2Kp_n^*}{1 - nMp_n^* - n^2M^2Kp_n^*},$$

and we consider the ball

$$B_{R_n} := B(0,R_n) = \{v \in C(\mathbb{R}_+) : \|v\|_n \leq R_n\}.$$

For any $n \in \mathbb{N}$, and each $y \in B_{R_n}$ and $t \in [0, n]$ we have

$$\begin{aligned}
\|(Ny)(t)\| &\leq \|T(t)\|_{B(E)}|y_0| + \int_0^t \|T(t-s)\|_{B(E)}\|f(s, y(s))\|ds \\
&+ MK\left[\int_0^t |y_1| + \|T(n)\|_{B(E)}|y_0|ds\right. \\
&+ \left.\int_0^t \int_0^n \|T(n-\tau)\|_{B(E)}\|f(\tau, y(\tau))\|d\tau ds\right] \\
&\leq M|y_0| + M\left(\int_0^t p(s)(1 + \|y(s)\|)ds\right) \\
&+ MKn|y_1| + M^2Kn|y_0| + M^2Kn\left(\int_0^t p(\tau)(1 + \|y(\tau)\|)d\tau\right) \\
&\leq M|y_0| + M(1 + \|y\|_n)\int_0^t p(s)ds \\
&+ MKn|y_1| + M^2Kn|y_0| + M^2Kn(1 + \|y\|_n)\int_0^t p(s)ds \\
&\leq M|y_0| + Mnp_n^*(1 + R_n) \\
&+ MKn|y_1| + M^2Kn|y_0| + M^2Kn^2p_n^*(1 + R_n) \\
&\leq R_n.
\end{aligned}$$

Thus

$$\|N(y)\|_n \leq R_n. \tag{4.8}$$

This proves that N transforms the ball B_{R_n} into itself. We shall show that the operator $N : B_{R_n} \to B_{R_n}$ satisfies all the assumptions of Theorem 1.30. The proof will be given several steps.

Step 1. $N : B_{R_n} \to B_{R_n}$ is continuous.
Let $\{y_k\}_{k\in\mathbb{N}}$ be a sequence such that $y_k \to y$ in B_{R_n}. Then, for each $t \in [0, n]$, we have

$$
\begin{aligned}
\|(Ny_k)(t)-(Ny)(t)\| &\le \int_0^t \|T(t-s)\|_{B(E)}\|f(s,y_k(s))-f(s,y(s))\|ds \\
&+ \int_0^t \|T(t-s)\|_{B(E)}\|B(u_{y_k}(s)-u_y(s))\|ds \\
&\le M\int_0^t \|f(s,y_k(s))-f(s,y(s))\|ds \\
&+ MK\int_0^t\int_0^n \|T(n-\tau)\|_{B(E)}\|f(\tau,y_k(\tau))-f(\tau,y(\tau))\|d\tau ds \\
&\le M\int_0^t \|f(s,y_k(s))-f(s,y(s))\|ds \\
&+ M^2K\int_0^t\int_0^n \|f(\tau,y_k(\tau))-f(\tau,y(\tau))\|d\tau ds \\
&\le M\int_0^t \|f(s,y_k(s))-f(s,y(s))\|ds \\
&+ nM^2K\int_0^n \|f(\tau,y_k(\tau))-f(\tau,y(\tau))\|d\tau.
\end{aligned}
$$

Since $y_k \to y$ as $k \to \infty$, the Lebesgue dominated convergence theorem implies that

$$\|N(y_k)-N(y)\|_n \to 0 \quad \text{as} \quad k \to \infty.$$

Step 2. $N(B_{R_n})$ is bounded.
Since $N(B_{R_n}) \subset B_{R_n}$ is bounded, then $N(B_{R_n})$ is bounded.

Step 3. For each subset D of B_{R_n}, $\mu_n(ND) \le \frac{1}{\tau}\mu_n(D)$.
From lemmas 1.19 and 1.20 , for any $D \subset B_{R_n}$ and any $\epsilon > 0$, there exists a sequence

$\{y_k(s)\}_{k=1}^{\infty} \subset D$, such that for all $t \in [0,n]$, we have

$$\begin{aligned}
\mu((ND)(t)) &= \mu\left(\left\{T(t)y_0 + \int_0^t T(t-s)f(s,y(s))ds + \int_0^t T(t-s)Bu_y(s)ds; y \in D\right\}\right) \\
&\leq 2\mu(\{\int_0^t T(t-s)f(s,y_k(s))ds + \int_0^t T(t-s)BW^{-1} \\
&\times \left(y_1 - T(n)y_0 - \int_0^n T(n-\tau)f(\tau,y_k(\tau))d\tau\right) ds\}_{k=1}^{\infty}) + \epsilon \\
&\leq 2\mu\left(\left\{\int_0^t T(t-s)f(s,y_k(s))ds\right\}_{k=1}^{\infty}\right) \\
&+ 2\mu\left(\left\{\int_0^t T(t-s)BW^{-1}\left(y_1 - T(n)y_0 - \int_0^n T(n-\tau)f(\tau,y_k(\tau))d\tau\right) ds\right\}_{k=1}^{\infty}\right) \\
&+ \epsilon \\
&\leq 4M\int_0^t \mu(\{f(s,y_k(s))\}_{k=1}^{\infty})ds \\
&+ 4MK\int_0^t \mu(y_1 - T(n)y_0 + \int_0^n T(n-\tau)\{f(\tau,y_k(\tau))\}_{k=1}^{\infty})d\tau)ds) + \epsilon \\
&\leq 4M\int_0^t p(s)\mu(\{y_k(s)\}_{k=1}^{\infty})ds \\
&+ 4MK\int_0^t \mu(y_1 - T(n)y_0) + 4MK\int_0^t\int_0^n \mu(T(n-\tau)\{f(\tau,y_k(\tau))\}_{k=1}^{\infty})d\tau)ds) + \epsilon \\
&\leq 4Mp_n^*\int_0^t \mu(\{y_k(s)\}_{k=1}^{\infty})ds + 4nM^2K\int_0^n \mu\{f(\tau,y_k(\tau))\}_{k=1}^{\infty})d\tau) + \epsilon \\
&\leq 4Mp_n^*\int_0^t \mu(\{y_k(s)\}_{k=1}^{\infty})ds + 4nM^2Kp_n^*\int_0^t \mu(\{y_k(s)\}_{k=1}^{\infty})ds + \epsilon \\
&\leq 4Mp_n^*\int_0^t e^{4Mp_n^*(1+nMK)\tau s}e^{-4Mp_n^*(1+nMK)\tau s}\mu(\{y_k(s)\}_{k=1}^{\infty})ds \\
&+ 4nM^2Kp_n^*\int_0^t e^{4Mp_n^*(1+nMK)\tau s}e^{-4Mp_n^*(1+nMK\tau s}\mu(\{y_k(s)\}_{k=1}^{\infty})ds + \epsilon \\
&\leq 4Mp_n^*(1+nMK)\int_0^t e^{4Mp_n^*(1+nMK)\tau s}e^{-4Mp_n^*(1+nMK)\tau s}\mu(\{y_k(s)\}_{k=1}^{\infty})ds + \epsilon \\
&\leq 4Mp_n^*(1+nMK)\mu_n(D)\int_0^t e^{4Mp_n^*(1+nMK)\tau s}ds + \epsilon \\
&\leq e^{4Mp_n^*(1+nMK)\tau t}\mu_n(D) + \epsilon
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, then

$$\mu((ND)(t)) \leq \frac{e^{4Mp_n^*(1+nMK)\tau t}}{\tau}\mu_n(D).$$

Thus

$$\mu_n(ND) \leq \frac{1}{\tau}\mu_n(D).$$

As a consequence of steps to 1 to 3 together with Theorem 1.30, we can conclude that N has a least one fixed in B_{R_n} which is a mild solution of problem (4.1)-(4.2).

4.3 An Example

As an application of our results, we consider the following functional evolution problem of the form

$$\begin{cases} \frac{\partial z}{\partial t}(t,x) = \frac{\partial^2 z}{\partial x^2}(t,x) + Q(t, z(t,x)) + Bu(t),; & t \in \mathbb{R}_+,\ x \in [0,\pi], \\ z(t,0) = z(t,\pi) = 0; & t \in \mathbb{R}_+. \\ z(0,x) = \Phi(x); & x \in [0,\pi], \end{cases} \tag{4.9}$$

where $Q : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $\Phi : [0,\pi] \to \mathbb{R}$ are continuous functions.

Consider $E = L^2([0,\pi],\mathbb{R})$ and define A by $Aw = w''$ with domain

$$D(A) = \{w \in E : w, w' \text{ are absolutely continuous}, w'' \in E,\ w(0) = w(\pi) = 0\}.$$

Then

$$Aw = \sum_{n=1}^{\infty} n^2(w, w_n)w_n, \quad w \in D(A),$$

where $(\cdot,\cdot)$ is the inner product in L^2 and $w_n(s) = \sqrt{\frac{2}{\pi}}\sin ns, \quad n = 1, 2, \ldots,$ is orthogonal set of eigenvectors of A. It is well known (see [54]) that A is the infinitesimal generator of an analytic semigroup $T(t), \quad t \geq 0$, in E and is given by the relation

$$T(t)w = \sum_{n=1}^{\infty} \exp(-n^2 t)(w, w_n)w_n, \quad w \in E.$$

Since the analytic semigroup $T(t)$ is compact, there exists a positive constant M such that

$$\|T(t)\|_{B(E)} \leq M.$$

Assume that $B : U \to Y, \quad U \subset [0,\infty)$, is a bounded linear operator and, for every $n > 0$, the operator

$$Wy = \int_0^n T(n-s)By(s)ds$$

has the bounded pseudo inverse W^{-1}, which takes values in $L^2([0,\infty), U)\backslash \ker W$.

For $x \in [0, \pi]$, set

$$y(t)(x) = z(t, x); \quad t \in \mathbb{R}_+,$$

$$f(t, y(t))(x) = Q(t, z(t, x)); \quad t \in \mathbb{R}_+,$$

and

$$y_0(x) = \Phi(x).$$

Then the system (4.9) can be represented by the functional problem (4.1)- (4.2), and conditions $(H_1) - (H_5)$ are satisfied. Consequently, Theorem 4.4 implies that the problem (4.9) has at least one mild solution.

Chapter 5

Random Evolution Equations

5.1 Introduction

In this Chapter, we provide sufficient conditions for the existence of global mild solutions for two classes of second order semi-linear functional equations with random effects. First, we will consider the following problem

$$u'(t,w) = A(t)u(t,w) + f(t,u(t,w),w); \quad \text{if } t \in \mathbb{R}_+ := [0,\infty), \; w \in \Omega, \tag{5.1}$$

with the initial condition

$$u(0,w) = u_0(w) \in E, \; w \in \Omega, \tag{5.2}$$

where $u_0 : \Omega \to E$ is a given function, $f : \mathbb{R}_+ \times E \times \Omega \to E$ is a given function, $(E, \|\cdot\|)$ is a (real or complex) Banach space, and $\{A(t)\}_{t>0}$ is a family of linear closed (not necessarily bounded) operators from E into E that generate an evolution system of bounded linear operators $\{U(t,s)\}_{(t,s)\in\mathbb{R}_+\times\mathbb{R}_+}$; for $(t,s) \in \Lambda := \{(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+ : 0 \le s \le t < +\infty\}$.

Later, we discuss the existence of random mild solutions for the following second order evolution problem

$$\begin{cases} u''(t,w) - A(t)u(t,w) = g(t,u(t,w),w); \quad \text{if } t \in \mathbb{R}_+ := [0,\infty), \; w \in \Omega, \\ \\ u(0,w) = \underline{\mathrm{u}}(w), \quad u'(0,w) = \bar{u}(w), \; w \in \Omega, \end{cases} \tag{5.3}$$

where $E, \{A(t)\}_{t>0}$ are as problem (5.1)-(5.2) and $\underline{\mathrm{u}}, \bar{u} : \Omega \to E$ and $g : \mathbb{R}_+ \times E \times \Omega \to E$ are given functions.

5.2 First Order Random Evolution Equations

In this section, we present the main results for the global existence of random mild solutions for the problem (5.1)-(5.2).

Definition 5.1 *We say that a continuous function $u(\cdot, w) : \mathbb{R}_+ \times \Omega \to E$ is a mild solution of the problem (5.1)-(5.2), if u satisfies the following integral equation*

$$u(t, w) = U(t, 0)u_0(w) + \int_0^t U(t, s)\ f(s, u(s, w), w)ds;\ for\ each\ t \in \mathbb{R}_+,\ and\ w \in \Omega.$$

Let us introduce the following hypotheses.

(H_1) There exists a constant $M \geq 1$ such that

$$\|U(t, s)\|_{B(E)} \leq M;\ \text{for every}\ (t, s) \in \Lambda.$$

(H_2) The function f is random Carathéodory on $\mathbb{R}_+ \times E \times \Omega$.

(H_3) There exists a continuous function $p : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ such that for any $w \in \Omega$, we have

$$\|f(t, u, w)\| \leq p(t, w)(1 + \|u\|);\ \text{for a.e.}\ t \in \mathbb{R}_+,\ \text{and each}\ u \in E.$$

(H_4) For each bounded set $B \subset E$ and for any $w \in \Omega$, we have

$$\mu(f(t, B, w)) \leq p(t, w)\mu(B);\ \text{for a.e.}\ t \in \mathbb{R}_+,$$

where μ is a measure of noncompactness on the Banach space E.

Set

$$p_n^*(w) = \operatorname*{ess\,sup}_{t \in [0,n]} p(t, w);\ for\ n \in \mathbb{N}.$$

Define on X the family of measure of noncompactness by

$$\mu_n(D) = \sup_{t \in [0,n]} e^{-4Mp_n^*(w)\tau t}\mu(D(t)),$$

where $\tau > 1$, and $D(t) = \{v(t) \in E : v \in D\};\ t \in [0, n]$.

Theorem 5.2 *Assume that the hypotheses $(H_1) - (H_4)$ are satisfied, and $nMp_n^*(w) < 1$ for each $n \in \mathbb{N}$, and $w \in \Omega$. Then the problem (5.1)-(5.2) has at least one random mild solution.*

Proof. Consider the operator $N : \Omega \times X \to X$ defined by:

$$(N(w)u)(t) = U(t, 0)u_0(w) + \int_0^t U(t, s)\ f(s, u(s, w), w)ds. \tag{5.4}$$

The function f is continuous on $\mathbb{R}_+$, then $N(w)$ defines a mapping $N : \Omega \times X \to X$. Thus u is a random solution for the problem (5.1)-(5.2) if and only if $u = (N(w))u$. We shall show that the operator $N(w)$ satisfies all conditions of Theorem 1.30. The proof will be given in several steps.

Step 1. *$N(w)$ is a random operator with stochastic domain on X.*
Since $f(t,u,w)$ is random Carathéodory, the map $w \to f(t,u,w)$ is measurable in view of Definition 1.23. Therefore, the map

$$w \mapsto U(t,0)u_0(w) + \int_0^t U(t,s) f(s,u(s,w),w)ds,$$

is measurable. As a result, N is a random operator on $\Omega \times X$ into X.

Let $W : \Omega \to \mathcal{P}(X)$ be the ball

$$W(w) := B(0, R_n(w)) = \{v \in X : \|v\|_n \le R_n(w)\};\ w \in \Omega,\ n \in \mathbb{N},$$

where $R_n : \Omega \to (0,\infty)$ is a function such that

$$R_n(w) \ge \frac{M\|u_0(w)\| + nMp_n^*(w)}{1 - nMp_n^*(w)}.$$

Since $W(w)$ bounded, closed, convex and solid for all $w \in \Omega$, then W is measurable by Lemma 17 of [34]. Let $w \in \Omega$ be fixed, then from (H_3), for any $u \in W(w)$, and each $t \in [0,n]$ we have

$$\begin{aligned}
\|(N(w)u)(t)\|_E &\le \|U(t,0)u_0(w) + \int_0^t U(t,s)\ f(s,u(s,w),w)ds\|_E \\
&\le M\|u_0(w)\| + M\left(\int_0^t p(s,w)(1 + \|u(s,w)\|)ds\right) \\
&\le M\|u_0(w)\| + nMp_n^*(w) + nMp_n^*(w)R_n \\
&\le R_n(w).
\end{aligned}$$

Therefore, N is a random operator with stochastic domain W and $N(w) : W(w) \to W(w)$. Furthermore, $N(w)$ maps bounded sets into bounded sets in X.

Step 2. *$N(w) : W(w) \to W(w)$ is continuous.*
Let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence such that $u_k \to u$ in $W(w)$. Then, for each $t \in [0,n]$ and $w \in \Omega$, we have

$$\|(N(w)u_k)(t) - (N(w)u)(t)\|$$

$$\begin{aligned}
&\le \int_0^t \|U(t,s)\|_{B(E)} \|f(s,u_k(s.w),w) - f(s,u(s,w),w)\|ds \\
&\le M \int_0^t \|f(s,u_k(s,w),w) - f(s,u(s,w),w)\|ds.
\end{aligned}$$

Since $u_k \to u$ as $k \to \infty$, the Lebesgue dominated convergence theorem implies that

$$\|N(w)(u_k) - N(w)(u)\|_n \to 0 \quad \text{as } k \to \infty.$$

As a consequence of Steps 1 and 2, we can conclude that $N(w) : W(w) \to W(w)$ is a continuous random operator with stochastic domain W, and $N(w)(W(w))$ is bounded.

Step 3. *For each equicontinuous subset D of $W(w)$, $\mu_n(N(w)(D)) \leq \frac{1}{\tau}\mu_n(D)$.* From Lemmas 1.19 and 1.20, for any $D \subset B_{R_n}$ and any $\epsilon > 0$, there exists a sequence $\{u_k\}_{k=0}^{\infty} \subset D$, such that for all $t \in [0, n]$ and $w \in \Omega$, we have

$$\begin{aligned}
\mu((N(w)D)(t)) &= \mu\left(\left\{U(t,0)u_0 + \int_0^t U(t,s)\ f(s,u(s,w),w)ds;\ u \in D\right\}\right) \\
&\leq 2\mu\left(\left\{\int_0^t U(t,s)f(s,u_k(s,w),w)ds\right\}_{k=1}^{\infty}\right) + \epsilon \\
&\leq 4\int_0^t \mu\left(\|U(t,s)\|_{B(E)}\{f(s,u_k(s,w),w)\}_{k=1}^{\infty}\right) ds + \epsilon \\
&\leq 4M\int_0^t \mu\left(\{f(s,u_k(s,w),w)\}_{k=1}^{\infty}\right) ds + \epsilon \\
&\leq 4M\int_0^t p_n(s)\mu\left(\{u_k(s,w)\}_{k=1}^{\infty}\right) ds + \epsilon \\
&\leq 4Mp_n^*(w)\int_0^t e^{4Mp_n^*(w)\tau s}e^{-4Mp_n^*(w)\tau s}\mu\left(\{u_k(s,w)\}_{k=1}^{\infty}\right) ds + \epsilon \\
&\leq \frac{\left(e^{4Mp_n^*\tau t} - 1\right)}{\tau}\mu_n(D) + \epsilon \\
&\leq \frac{e^{4Mp_n^*(w)\tau t}}{\tau}\mu_n(D) + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, then

$$\mu((N(w)D)(t)) \leq \frac{e^{4Mp_n^*(w)\tau t}}{\tau}\mu_n(D).$$

Thus

$$\mu_n(N(w)(D)) \leq \frac{1}{\tau}\mu_n(D).$$

As a consequence of steps 1 to 3 together with Theorem 1.30, we can conclude that N has at least one fixed point in $W(w)$ which is a random mild solution of problem (5.1)-(5.2).

5.3 Second Order Random Evolution Equations

In this section, we present the main results for the global existence of random mild solutions for problem (5.3). In what follows, let $\{A(t),\ t \geq 0\}$ be a family of closed linear operators on the Banach space E with domain $D(A(t))$ that is dense in E and independent of t. The existence of solutions to our problem is related to the existence of an evolution operator $U(t, s)$ for the homogeneous problem

$$u''(t) = A(t)u(t);\ t \in \mathbb{R}_+. \tag{5.5}$$

This concept of evolution operator has been developed by Kozak [47].

Let $X := C(\mathbb{R}_+)$ be the Fréchet space of all continuous functions from $\mathbb{R}_+$ into E. Let us introduce the definition of the mild solution of the problem (5.3).

Definition 5.3 *We say that a function $u \in X$ is a random mild solution of the problem (5.3) if u satisfies the following integral equation*

$$u(t) = -\frac{\partial}{\partial s}U(t,0)\underline{u}(w) + U(t,0)\bar{u}(w) + \int_0^t U(t,s)\ g(s,u(s,w),w)ds;$$

$t \in \mathbb{R}_+,\ w \in \Omega.$

Let us introduce the following hypotheses.

(H_1') There exist constants $M_1,\ M_2 > 0$ such that for every $(t,s) \in \Lambda$, we have

$$\left\|\frac{\partial}{\partial s}U(t,s)\right\|_{B(E)} \leq M_1 \quad \text{and} \quad \|U(t,s)\|_{B(E)} \leq M_2.$$

(H_2') The function g is random Carathéodory on $\mathbb{R}_+ \times E \times \Omega$.

(H_3') There exists a continuous function $q : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ such that for any $w \in \Omega$, we have

$$\|g(t,u,w)\| \leq q(t,w)(1 + \|u\|);\ \text{for a.e. } t \in \mathbb{R}_+,\ \text{and each } u \in E.$$

(H_4') For each bounded and measurable set $B \subset E$ and for any $w \in \Omega$, we have

$$\mu(g(t,B,w)) \leq q(t,w)\mu(B);\ \text{for a.e. } t \in \mathbb{R}_+,$$

Set

$$q_n^*(w) = \operatorname*{ess\,sup}_{t\in[0,n]} q(t,w);\ for\ n \in \mathbb{N}.$$

Now we present an existence of random mild solution for problem (5.3).

Theorem 5.4 *Assume that the hypotheses $(H_1') - (H_4')$ are satisfied. If $nM_2q_n^*(w) < 1$ for each $n \in \mathbb{N}$, and $w \in \Omega$, then the problem (5.3) has at least one random mild solution.*

Proof. Consider the operator $L(w) : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined by:

$$(L(w)u)(t) = -\frac{\partial}{\partial s}U(t,0)\underline{\mathrm{u}}(w) + U(t,0)\bar{u}(w) + \int_0^t U(t,s)\ g(s,u(s,w),w)ds. \tag{5.6}$$

Clearly, the fixed points of the operator $L(w)$ are solution of the problem (5.3). We shall show that the operator L satisfies all conditions of Theorem 1.30. The proof will be given in several steps.

Step 1. *$L(w)$ is a random operator with stochastic domain on X.*
Since $f(t,u,w)$ is random Carathéodory, the map $w \to g(t,u,w)$ is measurable in view of Definition 1.23. Therefore, the map

$$w \mapsto -\frac{\partial}{\partial s}U(t,0)\underline{u}(w) + U(t,0)\bar{u}(w) + \int_0^t U(t,s)\ g(s,u(s,w),w)ds,$$

is measurable. As a result, L is a random operator on $\Omega \times X$ into X.

For any $n \in \mathbb{N}$, let $R_n : \Omega \to (0,\infty)$ be a positive function such that

$$R_n(w) \geq \frac{M_1\|\underline{u}(w)\| + M_2\|\bar{u}(w)\| + nq_n^* M_2}{1 - nM_2 q_n^*(w)},$$

and we consider the ball

$$V(w) := B(0, R_n(w)) = \{\nu \in C(\mathbb{R}_+) : \|\nu\|_n \leq R_n(w)\}.$$

For any $n \in \mathbb{N}$, and $\omega \in \Omega$.
Since $V(w)$ bounded, closed, convex and solid for all $w \in \Omega$, then V is measurable by Lemma 17 of [34]. Let $w \in \Omega$ be fixed, then from (H_3'), for any $u \in V(w)$, and each $t \in [0,n]$ we have and each $u \in B_{R_n}$ and $t \in [0,n]$ we have

$$\begin{aligned}
\|(L(w)u)(t)\| &\leq \|\frac{\partial}{\partial s}U(t,0)\underline{u}(w)\| + \|U(t,0)\bar{u}(w)\| + \|\int_0^t U(t,s)\ g(s,u(s,w),w)ds\| \\
&\leq M_1\|\underline{u}(w)\| + M_2\|\bar{u}(w)\| + \int_0^t \|U(t,s)\|_{B(E)}\|f(s,u(s))\|ds \\
&\leq M_1\|\underline{u}(w)\| + M_2\|\bar{u}(w)\| + M_2\left(\int_0^t q(s,w)(1+\|u(s,w)\|)ds\right) \\
&\leq M_1\|\underline{u}(w)\| + M_2\|\bar{u}(w)\| + M_2 n q_n^*(w)(1+R_n(w)) \\
&\leq R_n(w).
\end{aligned}$$

Therefore, L is a random operator with stochastic domain V and $L(w) : V(w) \to V(w)$. Furthermore, $L(w)$ maps bounded sets into bounded sets in X.

Step 2. *$L(w) : V(w) \to V(w)$ is continuous.*
Let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence such that $u_k \to u$ in $V(w)$. Then, for each $t \in [0,n]$ and $w \in \Omega$, we have

$$\|(L(w)u_k)(t) - (L(w)u)(t)\|$$

$$\begin{aligned}
&\leq \int_0^t \|U(t,s)\|_{B(E)}\|g(s,u_k(s.w),w) - g(s,u(s,w),w)\|ds \\
&\leq M_2\int_0^t \|g(s,u_k(s,w),w) - g(s,u(s,w),w)\|ds.
\end{aligned}$$

Since $u_k \to u$ as $k \to \infty$, the Lebesgue dominated convergence theorem implies that

$$\|L(w)(u_k) - L(w)(u)\|_n \to 0 \quad \text{as } k \to \infty.$$

As a consequence of Steps 1 and 2, we can conclude that $L(w) : V(w) \to V(w)$ is a continuous random operator with stochastic domain W, and $L(w)(V(w))$ is bounded.

Step 3. *For each equicontinuous subset D of $V(w)$, $\mu_n(L(w)(D)) \leq \frac{1}{\tau}\mu_n(D)$.*
From Lemmas 1.19 and 1.20, for any $D \subset B_{R_n}$ and any $\epsilon > 0$, there exists a sequence $\{u_k\}_{k=0}^{\infty} \subset D$, such that for all $t \in [0, n]$ and $w \in \Omega$, we have

$$\begin{aligned}
\mu((N(w)D)(t)) &= \mu\Bigg(\Bigg\{-\frac{\partial}{\partial s}U(t,0)\underline{u}(w) \\
&\qquad +U(t,0)\bar{u}(w) + \int_0^t U(t,s)\ g(s,u(s,w),w)ds;\ u \in D\Bigg\}\Bigg) \\
&\leq 2\mu\left(\left\{\int_0^t U(t,s)g(s,u_k(s,w),w)ds\right\}_{k=1}^{\infty}\right) + \epsilon \\
&\leq 4\int_0^t \mu\left(\|U(t,s)\|_{B(E)}\{g(s,u_k(s,w),w)\}_{k=1}^{\infty}\right) ds + \epsilon \\
&\leq 4M_2\int_0^t \mu\left(\{g(s,u_k(s,w),w)\}_{k=1}^{\infty}\right) ds + \epsilon \\
&\leq 4M_2\int_0^t q_n(s)\mu\left(\{u_k(s,w)\}_{k=1}^{\infty}\right) ds + \epsilon \\
&\leq 4M_2 q_n^*(w)\int_0^t e^{4M_2q_n^*(w)\tau s}e^{-4M_2q_n^*(w)\tau s}\mu\left(\{u_k(s,w)\}_{k=1}^{\infty}\right) ds + \epsilon \\
&\leq \frac{\left(e^{4M_2q_n^*\tau t} - 1\right)}{\tau}\mu_n(D) + \epsilon \\
&\leq \frac{e^{4M_2q_n^*(w)\tau t}}{\tau}\mu_n(D) + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, then

$$\mu((L(w)D)(t)) \leq \frac{e^{4M_2q_n^*(w)\tau t}}{\tau}\mu_n(D).$$

Thus

$$\mu_n(L(w)(D)) \leq \frac{1}{\tau}\mu_n(D).$$

As a consequence of steps 1 to 3 together with Theorem 1.30, we can conclude that L has at least one fixed point in $V(w)$ which is a random mild solution of problem (5.3).

5.4 An Example

Let be equipped with the usual σ-algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$. Given a measurable function $u : \Omega \to L^2([0,\pi],\mathbb{R})$, we consider the following functional random evolution problem of the form

$$\begin{cases} \frac{\partial z}{\partial t}(t,x,w) = a(t,x,w)\frac{\partial^2 z}{\partial x^2}(t,x) \\ +Q(t,z(t,x,w)); & t \in \mathbb{R}_+,\ x \in [0,\pi],\ w \in \Omega, \\ z(t,0,w) = z(t,\pi,w) = 0; & t \in \mathbb{R}_+,\ w \in \Omega, \\ z(0,x,w) = \Phi(x,w); & x \in [0,\pi],\ w \in \Omega, \end{cases} \tag{5.7}$$

where $a(t,x,w) : \mathbb{R}_+ \times [0,\pi] \times \Omega \to \mathbb{R}$ is a continuous function and is uniformly Hölder continuous in t, $Q : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $\Phi : [0,\pi] \times \Omega \to \mathbb{R}$ are continuous functions.

Consider $E = L^2([0,\pi],\mathbb{R})$ and define $A(t)$ by $A(t)y = a(t,x,w)y''$ with domain

$$D(A) = \{y \in E : y, y' \text{ are absolutely continuous}, y'' \in E,\ y(0) = y(\pi) = 0\}.$$

Then $A(t)$ generates an evolution system $U(t,s)$ (see [37]).

For $x \in [0,\pi]$, we have

$$u(t,w)(x) = z(t,x,w); \quad t \in \mathbb{R}_+,\ w \in \Omega,$$

$$f(t,u(t,w),x,w) = Q(t,z(t,x,w)); \quad t \in \mathbb{R}_+,\ w \in \Omega,$$

and

$$u_0(x,w) = \Phi(x,w); \quad x \in [0,\pi],\ w \in \Omega.$$

Thus, under the above definitions of f, u_0 and $A(\cdot)$, the system (5.7) can be represented by the functional evolution problem (5.1)-(5.2). Furthermore, more appropriate conditions on Q ensure the hypotheses $(H_1) - (H_5)$. Consequently, Theorem 5.2 implies that the evolution problem (5.7) has at least one random mild solution.

Chapter 6

Controllability of Random Differential Equations

6.1 Introduction

In this Chapter, we study the controllability of the following functional differential equation with random effect (random parameters) of the form:

$$\begin{cases} y''(t,w) = Ay(t,w) + f(t,y(s,w),w) + Bu(t,w), \text{ a.e. } t \in \mathbb{R}_+ := [0,\infty), \\ y(0,w) = \phi(w), \ y'(0,w) = \varphi(w), \end{cases} \quad w \in \Omega, \tag{6.1}$$

where (Ω, F, P) is a complete probability space, $f : \mathbb{R}_+ \times E \times \Omega \to E$ is a given function, $\phi, \varphi : \Omega \to E$ are given measurable functions, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $(C(t))_{t\in \mathbb{R}}$ on E, and $(E, |\cdot|)$ is a real Banach space. The control function $u(\cdot, w)$ is given in $L^2(\mathbb{R}_+, U)$, a Banach space of admissible control functions with U as a Banach space, and B is a bounded linear operator from U into E.

6.2 Existence and Controllability Results

In this section, we study the controllability for the differential system with delay and random effect (6.1).

Definition 6.1 *The problem (6.1) is said to be controllable on $\mathbb{R}_+$, if for every initial states $\phi(w)$, $\varphi(w)$, and final state $y^1(w)$, and for each $n \in \mathbb{N}$, there is a control $u(\cdot, w)$ in $L^2(\mathbb{R}_+, U)$, such that the solution $y(t, w)$ of (6.1) satisfies $y(n, w) = y^1(w)$.*

Now we give our main existence result for problem (6.1).

Consider the Fréchet space $X = \mathcal{C}(\mathbb{R}_+)$ equipped with the family of seminorms

$$\|y\|_n = \sup_{t\in[0,n]} |y(t)|; \ n \in \mathbb{N},$$

and the distance

$$d(u,v)=\sum_{n=1}^{\infty}2^{-n}\frac{\|u-v\|_n}{1+\|u-v\|_n};\ u,v\in X.$$

Definition 6.2 *A stochastic process $y:\mathbb{R}_+\times\Omega\to E$ is said to be a random mild solution of problem (6.1) if $y(0,w)=\phi(w),\ y^{'}(0,w)=\varphi(w)$, and $y(\cdot,w)$ is continuous and satisfies the following integral equation:*

$$\begin{aligned} y(t,w) &= C(t)\phi(w)+S(t)\varphi(w)+\int_0^t S(t-s)f(s,y(s,w),w)ds \\ &\quad +\int_0^t S(t-s)Bu(t,w)ds \end{aligned}$$

Let

$$M=\sup\left\{\|C(t)\|_{B(E)}:t\geq 0\right\}\ \text{and}\ M^{'}=\sup\left\{\|S(t)\|_{B(E)}:t\geq 0\right\}.$$

We will need to introduce the following hypotheses:

(H_1) The function f is random Carathéeodory on $\mathbb{R}_+\times E\times\Omega$.

(H_2) There exists a continuous function $p:\mathbb{R}_+\times\Omega\to\mathbb{R}_+$ such that for any $w\in\Omega$, we have

$$\|f(t,u,w)\|\leq p(t,w)(1+\|u\|);\ \text{for a.e. } t\in\mathbb{R}_+,\ \text{and each } u\in E.$$

(H_3) For each bounded and measurable set $B\subset E$ and for any $w\in\Omega$, we have

$$\mu(f(t,B,w))\leq p(t,w)\mu(B);\ \text{for a.e. } t\in\mathbb{R}_+,$$

where μ is a measure of noncompactness on the Banach space E.

(H_4) For any $n\in\mathbb{N}^*$, the linear operator $W:L^2(\mathbb{R}_+,U)\to E$ defined by:

$$W(u)=\int_0^n S(n-s)Bu(s,w)ds$$

has a pseudo inverse operator W^{-1} which takes values in $L^2([0,n],U)/ker(W)$ and there exists a positive constant K such that $\|BW^{-1}\|\leq K$.

Set

$$p_n^*(w)=\operatorname*{ess\,sup}_{t\in[0,n]}p(t,w);\ \text{for } n\in\mathbb{N}.$$

and define on X the family of measure of noncompactness by

$$\mu_n(B)=\sup_{t\in[0,n]}e^{-4Mp_n^*(w)(1+nMK)\tau t}\mu(B(t)),$$

where $\tau>1$, and $B(t)=\{v(t)\in E:v\in B\};\ t\in[0,n]$.

Theorem 6.3 *Assume that hypotheses $(H_1)-(H_4)$ are satisfied, and*

$$nM'p_n^*(w)(1+nM'K)<1,$$

for each $n\in\mathbb{N}$, and $w\in\Omega$. Then the problem (6.1) is controllable.

Proof. Using (H_4) we define the control

$$u(t,w)=W^{-1}\left(y^1(w)-C(n)\phi(0,w)-S(n)\varphi(w)-\int_0^n S(n-s)f(s,y(s,w),w)ds\right),$$

for $n\in\mathbb{N}$. We shall show that using the control function $u(t,w)$, the operator $N:\Omega\times X\longrightarrow X$ defined by:

$$(N(w)y)(t)=C(t)\phi(w)+S(t)\varphi(w)+\int_0^t S(t-s)\ f(s,y(s,w),w)ds+\int_0^t S(t-s)BW^{-1}$$

$$\times\left(y^1(w)-C(n)\phi(w)-S(n)\varphi(w)-\int_0^n S(n-\tau)f(\tau,y(\tau,w),w)d\tau\right)ds, \tag{6.2}$$

has a fixed point $y(t,w)$ which is a random mild solution of the problem (6.1). This implies that the problem (6.1) is controllable. We shall show that the operator N defined in (6.2) satisfies all conditions of Theorem 1.30. The proof will be given in several steps.

Step 1. *$N(w)$ is a random operator with stochastic domain on X.*
We need to prove that for any $y\in X$, $N(\cdot)(y):\Omega\longrightarrow X$ is a random variable. Then we prove that $N(\cdot)(y):\Omega\longrightarrow X$ is measurable. As the mapping $f(t,y,\cdot)$, $t\in\mathbb{R}_+$, $y\in X$ is measurable by assumption (H_1) and (H_4). Let $D:\Omega\longrightarrow 2^X$ be defined by:

$$D(w)=\{y\in X:\|y\|_n\le R_n(w)\},$$

where $R_n:\Omega\longrightarrow(0,\infty)$ is a random function such that

$$R_n(w)\ge\frac{(M|\phi(w)|+M'|\varphi(w)|+nM'p_n^*(w))(1+nM'K)+nM'K\,|y^1(w)|}{1-nM'p_n^*(w)(1+nM'K)}.$$

Since $D(w)$ is bounded, closed, convex and solid for all $w\in\Omega$, then D is measurable by Lemma 17 (see [34]). Let $w\in\Omega$ be fixed, then for any $y\in D(w)$ and from (H_2) and (H_3), we have

$$\begin{aligned}
|(N(w)y)(t)| &\leq M|\phi(w)| + M^{'}|\varphi(w)| + M^{'}\int_0^t |f(s,y(s,w),w)|ds \\
&\quad +M^{'}K\int_0^t \left|y^1(w)\right| + M|\phi(w)| + M^{'}|\varphi(w)|ds \\
&\quad +M^{'}K\int_0^t\int_0^n \|S(\tau-s)\|\,|f(\tau,y(\tau,w),w)|\ d\tau ds \\
&\leq M|\phi(w)| + M^{'}|\varphi(w)| + M^{'}\ \int_0^n p(s,w)(1+|y(s,w)|)\ ds \\
&\quad +nM^{'}K\left|y^1(w)\right| + nMM^{'}K|\phi(w)| + nM^{'^2}K|\varphi(w)| \\
&\quad +nM^{'^2}K\int_0^n p(\tau,w)(1+|y(\tau,w)|)\ d\tau \\
&\leq M(1+nM^{'}K)|\phi(w)| + nM^{'}K\left|y^1(w)\right| + M^{'}(1+nM^{'}K)|\varphi(w)| \\
&\quad +M^{'}\left(1+nM^{'}K\right)\ \int_0^n p(s,w)(1+|y(s,w)|)ds \\
&\leq (M|\phi(w)| + M^{'}|\varphi(w)| + nM^{'}p_n^*(w))(1+nM^{'}K) + nM^{'}K\left|y^1(w)\right| \\
&\quad +nM^{'}p_n^*(w)(1+nM^{'}K)R_n(w) \\
&\leq R_n(w).
\end{aligned}$$

Thus

$$\|(N(w)y)\|_X \leq R_n(w).$$

This implies that N is a random operator with stochastic domain D and $N(w) : D(w) \longrightarrow D(w)$ for each $w \in \Omega$.

Step 2. $N(w) : D(w) \to D(w)$ *is continuous.*

Let $\{y_k\}_{k\in\mathbb{N}}$ be a sequence such that $y_k \to y$ in $D(w)$. Then, for each $t \in [0,n]$ and $w \in \Omega$, we have

$$\begin{aligned}
&|(N(w)y^n)(t) - (N(w)y)(t)| \\
\leq\ & M^{'}\int_0^t\ |f(s,y^n(s,w) - f(s,y(s,w)|\ ds \\
&+KM^{'}\int_0^t\ \int_0^n \|S(n-\tau)\|\,|f(\tau,y^n(\tau,w),w) - f(\tau,y(\tau,w),w)|d\tau ds \\
\leq\ & M^{'}\int_0^t\ |f(s,y^n(s,w),w) - f(s,y(s,w),w)|\ ds \\
&+bM^{'^2}K\int_0^n |f(\tau,y^n(\tau,w),w) - f(\tau,y(\tau,w),w)|d\tau \\
\leq\ & M^{'}\left(1+nM^{'}K\right)\int_0^n |f(\tau,y^n(\tau,w),w) - f(\tau,y(\tau,w),w)|d\tau.
\end{aligned}$$

Since $f(s,\cdot,w)$ is continuous, and $y_k \to y$ as $k\to\infty$, the Lebesgue dominated convergence theorem implies that

$$\|N(w)(y_k) - N(w)(y)\|_n \to 0 \quad \text{as } k\to\infty.$$

As a consequence of Steps 1 and 2, we can conclude that $N(w) : D(w) \to D(w)$ is a continuous random operator with stochastic domain W, and $N(w)(D(w))$ is bounded.

Step 3. *For each bounded subset B of $D(w)$, $\mu_n(N(w)(B)) \leq \frac{1}{\tau}\mu_n(B)$.*

From Lemmas 1.19 and 1.20, for any $B\subset D(w)$ and any $\epsilon > 0$, there exists a sequence $\{y_k\}_{k=0}^{\infty}\subset B$, such that for all $t\in[0,n]$ and $w\in\Omega$, we have

$$\begin{aligned}
\mu((N(w)B)(t)) &= \mu(\{M|\phi(w)| + M^{'}|\varphi(w)| + M^{'}\int_0^t |f(s,y(s,w),w)|ds \\
&\quad +M^{'}K\int_0^t \left|y^1(w)\right| + M|\phi(w)| + M^{'}|\varphi(w)|ds \\
&\quad +M^{'}K\int_0^t\int_0^n \|S(\tau-s)\|\,|f(\tau,y(\tau,w),w)|\;d\tau ds;\; y\in D\}) \\
&\leq 2\mu\left(\left\{M^{'}\int_0^t |f(s,y_k(s,w),w)|ds\right.\right. \\
&\quad \left.\left. +M^{'}K\int_0^t\int_0^n \|S(\tau-s)\|\,|f(\tau,y_k(\tau,w),w)|\;d\tau ds\right\}_{k=1}^{\infty}\right) + \epsilon \\
&\leq 4M^{'}\int_0^t \mu\left(\{f(s,y_k(s,w),w)\}_{k=1}^{\infty}\right)ds \\
&+ 4M^{'}K\int_0^t\int_0^n \mu\left(\left\{\|S(\tau-s)\|_{B(E)}f(\tau,y_k(\tau,w),w)\right\}_{k=1}^{\infty}\right)d\tau ds + \epsilon \\
&\leq 4M^{'}\int_0^t p_n(s)\mu\left(\{y_k(s,w)\}_{k=1}^{\infty}\right)ds \\
&+ 4nM^{'2}K\int_0^n p_n(\tau)\mu\left(\{y_k(\tau,w)\}_{k=1}^{\infty}\right)ds + \epsilon \\
&\leq 4M^{'}p_n^*(w)(1+nM^{'}K)\int_0^t e^{4M^{'}p_n^*(w)(1+nM^{'}K)\tau s}e^{-4M^{'}p_n^*(w)(1+nM^{'}K)\tau s} \\
&\times \mu\left(\{y_k(s,w)\}_{k=1}^{\infty}\right)ds + \epsilon \\
&\leq \frac{1}{\tau}\left(e^{4M^{'}p_n^*(1+nM^{'}K)\tau t} - 1\right)\mu_n(D) + \epsilon \\
&\leq \frac{1}{\tau}e^{4M^{'}p_n^*(w)(1+nM^{'}K)\tau t}\mu_n(D) + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, then

$$\mu((N(w)B)(t)) \leq \frac{e^{4M^{'}p_n^*(w)(1+nM^{'}K)\tau t}}{\tau}\mu_n(B).$$

Thus

$$\mu_n(N(w)(B)) \leq \frac{1}{\tau}\mu_n(B).$$

As a consequence of steps 1 to 3 together with Theorem 1.30, we can conclude that N has at least one fixed point in $D(w)$ which is a random mild solution of problem (6.1).

6.3 An Example

Consider the partial differential equation

$$\frac{\partial^2 z}{\partial t^2}(t,x,w) = \frac{\partial^2 z}{\partial x^2}(t,x,w)$$

$$+f(t,z(t,x,w),w) + Bu\,(t,w) \;\; x \in [0,\pi];\;\; t \in \mathbb{R}_+, \tag{6.3}$$

$$z(t,0,w) = z(t,\pi,w) = 0;\;\; t \in \mathbb{R}_+,\; w \in \Omega, \tag{6.4}$$

$$z(0,x,w) = \phi(x,w), \frac{\partial z}{\partial t}(0,x,w) = v(x,w);\;\; t \in \mathbb{R}_+,\;\; w \in \Omega, \tag{6.5}$$

where $f : \mathbb{R}_+ \times \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$ is a given function. Let $E = L^2[0,\pi]$, (Ω, F, P) is a complete probability space and define the operator $A : E \longrightarrow E$ by $Av = v''$ with domain

$$D(A) = \{v \in E; v, v' \text{ are absolutely continuous, } v'' \in E,\; v(0) = v(\pi) = 0\}.$$

It is well known that A is the infinitesimal generator of a strongly continuous cosine function $(C(t))_{t\in\mathbb{R}}$ on E, respectively. Moreover, A has discrete spectrum, the eigenvalues are $-n^2, n \in \mathbb{N}$ with corresponding normalized eigenvectors

$$z_n(\tau) := \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin(n\tau),$$

and the following properties hold:

(a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of E,

(b) If $y \in E$, then $Ay = -\sum_{n=1}^{\infty} n^2 \langle y, z_n \rangle z_n$,

(c) For $y \in E$,

$$C(t)y = \sum_{n=1}^{\infty} cos(nt) \langle y, z_n \rangle z_n,$$

and the associated sine family is

$$S(t)y = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} \langle y, z_n \rangle z_n,$$

and that

$$\|C(t)\| = \|S(t)\| \leq 1, \text{ for all } t \geq 0.$$

(d) If we denote the group of translations on E defined by

$$\Phi(t)y(\zeta,w)=\widetilde{y}(\zeta+t,w),$$

where $\widetilde{y}$ is the extension of y with period 2π, then

$$C(t)=\frac{1}{2}(\Phi(t)+\Phi(-t));A=D,$$

where D is the infinitesimal generator of the group on

$$X=\{y(\cdot,w)\in H^1(0,\pi):y(0,w)=y(\pi,w)=0\}.$$

Assume that B is a bounded linear operator from U into E and the linear operator $W:L^2(\mathbb{R}_+,U)\to E$ defined by:

$$W(u)=\int_0^n S(n-s)Bu(s,w)ds;\ n\in\mathbb{N},$$

has a pseudo inverse operator W^{-1} which takes values in $L^2(\mathbb{R}_+,U)/kerW$. Then the problem (6.1) is an abstract formulation of the problem (6.3)-(6.5). If conditions $(H_1)-(H_4)$ are satisfied, then Theorem 6.3 implies that the problem (6.3)-(6.5) is controllable.

Conclusion

In this thesis we have considered the problem of existence and controllability of mild and integral solutions for some classes of evolution equation and existence of solutions for random semilinear equations have been considered on the real half line $[0,\infty)$. The main results are based on a fixed point theorem for contraction operators in Fréchet space, the semigroup theory, the cosine families operators and measure of noncompactness.
Our hope is to extend the application of this method to the functional evolution equations and inclusions with delay in Fréchet space. Also, we plan to study the S-asymptotically ω-positive periodic solutions for semilinear neutral function evolution equations and inclusions.

Bibliography

[1] S. Abbas, W. Albarakati and M. Benchohra, Successive approximations for functional evolution equations and inclusions, *J. Nonlinear Funct. Anal.*, Vol. 2017 (2017), Article ID 39, pp. 1-13.

[2] S. Abbas, M. Benchohra, *Advanced Functional Evolution Equations and Inclusions.* Developments in Mathematics, 39. Springer, Cham, 2015.

[3] R.P. Agarwal, S. Baghli, M. Benchohra, Controllability for semilinear functional and neutral functional evolution equations with infinite delay in Fréchet spaces, *Appl. Math. Optim.*, **60** (2) (2009), 253-274.

[4] R. P. Agarwal, M. Benchohra, D. Seba, On the applications of Measure of noncompactness to the existence of solutions for fractional difference equations, *Results Math.*, **55** (2009), 221-230.

[5] R. P. Agarwal, M. Meechan and D. O'Regan, *Fixed Point Theory and Applications*, Cambridge University Press, Cambridge, 2001.

[6] R. P. Agarwal, D. O'Regan, Difference equations in abstract spaces, *J. Aust. Math. Soc.* (Series A), **64** (1998), 277-284.

[7] N. U. Ahmed, *Semigroup Theory with Applications to Systems and Control*, Harlow John Wiley & Sons, Inc., New York, 1991.

[8] E. Alaidarous, A. Benaissa, M. Benchohra, J.J. Nieto, Second order functional differential equations with delay and random effects, *Nonlinear Anal. Forum* **21** (2) (2016), 125-143.

[9] D. Aoued, S. Baghli-Bendimerad, Controllability of mild solutions for evolution equations with infinite state-dependent delay, *Eur. J. Pure Appl. Math.* **9** (4) (2016), 383-401.

[10] W. Arendt, C. Batty, M., Hieber, and F. Neubrander, *Vector-Valued Laplace Transforms and Cauchy Problems*, Monographs in Mathematics, **96** Birkhauser, Basel, 2001.

[11] S. Baghli and M. Benchohra, Global uniqueness results for partial functional and neutral functional evolution equations with infinite delay, *Differential Integral Equations*, **23** (2010), 31–50.

[12] S. Baghli and M. Benchohra, Multivalued evolution equations with infinite delay in Fréchet spaces, *Electron. J. Qual. Theo. Differ. Equ.* 2008, No. 33, 24 pp.

[13] S. Baghli, M. Benchohra and K. Ezzinbi, Controllability results for semilinear functional and neutral functional evolution equations with infinite delay, *Surv. Math. Appl.* **4** (2009), 15-39.

[14] A. Baliki and M. Benchohra, Global existence and asymptotic behaviour for functional evolution equations, *J. Appl. Anal. Comput.* **4** (2) (2014), 129–138.

[15] A. Baliki and M. Benchohra, Global existence and stability for neutral functional evolution equations, *Rev. Roumaine Math. Pures Appl.* **LX** (1) (2015), 71-82.

[16] J. Banaѐs and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Marcel Dekker, New York, 1980.

[17] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff International Publishing, Leiden, 1976.

[18] K. Balachandran and J. P. Dauer, Controllability of nonlinear systems in Banach spaces: A survey. Dedicated to Professor Wolfram Stadler, *J. Optim. Theory Appl.* **115** (2002), 7-28.

[19] K. Balachandran, J. H. Kim; Remarks on the paper Controllability of second order differentialinclusion in Banach spaces *J. Math. Anal. Appl.* **324** (2006), 46-749.

[20] M. Benchohra, L. Górniewicz, S. K. Ntouyas, A. Ouahab: Controllability results for nondensely defined semilinear functional differential equations. *Z. Anal. Anwend.* **25** (2006), 311-325.

[21] M. Benchohra and I. Medjadj, Global existence results for second order neutral functional differential equation with state-dependent delay. *Comment. Math. Univ. Carolin.* **57** (2016), 169-183.

[22] A. T. Bharucha-Reid, *Random Integral Equations*, Academic Press, New York, 1972.

[23] D. Bothe, Multivalued perturbation of m-accretive differential inclusions, *Isr. J. Math.* **108** (1998), 109-138.

[24] T. A. Burton, T. Furumochi, A note on stability by Schauders theorem, *Funkcial. Ekvac.* **44** (2001), 73-82.

[25] Y. K. Chang, J. J. Nieto, W. S. Li; Controllability of semilinear differential systems with nonlocal initial conditions in Banach spaces, *J. Optim. Theory Appl.* **142** (2009), 267-273.

[26] E. N. Chukwu, S. M. Lenhart; Controllability questions for nonlinear systems in abstract spaces, *J. Optim. Theory Appl.* **68** (1991), 437-462.

[27] G. DA Prato, E. Sinestrari, Differential operators with non-dense domains, *Ann. Scuola. Norm. Sup. Pisa Sci.*, **14** (1987), 285-344.

[28] S. Dudek, Fixed point theorems in Fréchet Algebras and Fréchet spaces and applications to nonlinear integral equations, *Appl. Anal. Discrete Math.* **11** (2017), 340-357.

[29] G. Darbo, Punti uniti in transformazioni a condominio non compatto, *Rend. Semin. Math.* Univ. Padova, **24** (1955), 84-92.

[30] F. S. DeBlasi, On a property of unit sphere in a Banach space,*Bull. Math. Soc. Sci. Math.* Roumanie (N.S.), **21** (1977), 259-262.

[31] S. Dudek and L. Olszowy, Continuous dependence of the solutions of nonlinear integral quadratic Volterra equation on the parameter, *J. Funct. Spaces*, V. 2015, Article ID 471235, 9 pages.

[32] K. J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations,* Graduate Texts in Mathematics, **194**, Springer-Verlag, New York, 2000.

[33] H. W. Engel, A general stochastic fixed-point theorem for continuos random operators on stochastic domains. *Anal. Appl* **66** (1978), 220-231.

[34] H. W. Engl, A general stochastic fixed-point theorem for continuous random operators on stochastic domains, *J. Math. Anal. Appl.* **66** (1978), 220-231.

[35] K. Ezzinbi, S. Lalaoui Rhali, Existence and controllability for nondensly defined partial neutral functional differential inclusions, *Appl Math*, **60**(3) (2015), 321-340.

[36] H. O. Fattorini, *Second Order Linear Differential Equations in Banach Spaces*, North-Holland, Mathematical Studies, **108**, North-Holland, Amsterdam, 1985.

[37] A. Freidman, *Partial Differential Equations*, Holt, Rinehat and Winston, New York, 1969.

[38] M. Frigon and A. Granas, Résultats de type Leray-Schauder pour des contractions sur des espaces de Fréchet, *Ann. Sci. Math. Québec.* **22** (2) (1998), 161-168.

[39] X. Fu, Controllability of neutral functional differential systems in abstract spaces, *Appl. Math. Comput.*, **141** (2003), 281-296.

[40] D.J. Guo, V. Lakshmikantham, X. Liu, *Nonlinear Integral Equations in Abstract Spaces,* Kluwer Academic Publishers, Dordrecht, 1972.

[41] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functtional Differential Equations*, Appl. Math. Sci., **99**, Springer-Verlag, New York, (1993).

[42] S. Itoh, Random fixed point theorems with applications to random differential equations in Banach spaces, *J. Math. Anal. Appl.,* **67** (1979), 261-273.

[43] K. Kuratowski, Sur les espaces complets, *Fund. Math.*, **15** (1930), 301-309.

[44] M. Kamenskii, V. Obukhovskii and P. Zecca, *Condensing Multivalued maps and Semilinear Differential Inclusions in Banach Spaces*, Walter de Gruyter & Co., Berlin, 2001.

[45] H. Kellermann, M. Hieber, Integrated semigroup, *J. Funct. Anal.* **84** (1989) 160-180.

[46] J. Klamka, *Controllability of Dynamical Systems*, Kluwer Academic, Dordrecht, 1993.

[47] M. Kozak, A fundamental solution of a second-order differential equation in a Banach space, *Univ. Iagel. Acta Math.*, **32** (1995), 275-289.

[48] G.S. Ladde and V. Lakshmikantham, *Random Differential Inequalities*, Academic Press, New York, 1980.

[49] C. Lizama, Regularized solutions for abstract Volterra equations, *J. Math. Anal. Appl.* **243** (2000), 278-292.

[50] N. I. Mahmudov, Approximate controllability of evolution systems with nonlocal conditions, *Nonlinear Anal*, **68** (2008) 536-546.

[51] H. Mönch, Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces, *Nonlinear Anal.*, **4** (5) (1980), 985-999.

[52] K. Naito, On controllability for a nonlinear Volterra equation,Anal.: *Nonlinear Theory Methods Appl.* **18** (1992), 99-108.

[53] D. O'Regan, Measures of noncompactness, Darbo maps and differential equations in abstract spaces,*Acta Math.* Hungar. **69** (3) (1995), 233-261.

[54] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[55] J. Prüss, *Evolutionary Integral Equations and Applications* Monographs Math. **87**, Bikhäuser Verlag, 1993.

[56] C.P. Tsokos and W.J. Padgett, *Random Integral Equations with Applications to Life Sciences and Engineering*, Academic Press, New York, 1974.

[57] Quinn, M.D., Carmichael, N.: An approach to nonlinear control problem using fixed point methods, degree theory and pseudo-inverses. Numer. *Funct. Anal. Optim.* **7** (1984), 197-219.

[58] J. Wu, *Theory and Applications of Partial Functional-Differential Equations.* Applied Mathematics Sciences, vol. **119** Springer, New-York, 1996.

[59] Z. Yan, Controllability of semilinear integrodifferential systems with nonlocal conditions, *International J. Comp. Appl. Math.,* **3** (2007), 221-236.

[60] X. Zhang, Exact controllability of semilinear evolution systems and its application, *J. Optim.Theory Appl.* **107** (2000), 415-432.