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Je dédie cette thèse :

À mes parents et toute ma famille,

À tous ceux qui me sont chers
et proches.

The beauty of mathematics only shows itself to more patient followers.

Maryam Mirzakhani
(Quated 30-12-2017 in the Toronto Daily Star)

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CHAPITRE 1

Introduction

1.1 Résumé

Dans cette thèse, nous avons étudié les propriétés asymptotiques de la fonction de hasard conditionnelle par l'estimation locale linéaire lorsque la variable aléatoire réponse est réelle et la variable explicative appartient à un espace de dimension infinie.

Nous avons commencé par donnée l'erreur quadratique de la fonction de hasard d'un estimateur local linéaire en donnant l'expression du biais et la variance dans un espace semi-métrique, aussi une étude de simulation sur un échantillon fini, qui montre le performance de l'estimateur local linéaire par rapport à l'estimateur du noyau standard, ce résultat est important pour déterminer le choix du paramètre de lissage.

Dans un second temps, sous des conditions standards, quand la variable explicative prend ses valeurs dans un espace de Hilbert, nous nous intéressons au modèle avec l'indice fonctionnel simple dans le cas où les observations sont indépendantes identiquement distribuées (i.i.d.) lorsque l'indice fonctionnel est connu et nous avons obtenu la convergence ponctuelle et uniforme presque complète avec vitesse. Dans le même modèle, on établit la convergence uniforme presque complète pour le cas dépendent.

1.2 Summary

In this thesis, we studied the asymptotic properties of the conditional hazard function by the local linear estimation when the response random variable is real and the explanatory variable belong to a space of infinite dimension.

We started by giving the quadratic error of the hazard function of a linear local estimator by giving the expression of bias and the variance term in semi-metric space, also a simulation study on a finite sample, which shows the performance of the linear local estimator compared

to the standard kernel estimator.

In a second step, when the explanatory variable takes its values in a Hilbert space, we are interested in the single functional index model where the observations are independent identically distributed (i.i.d.), with the functional index is known and we obtain the pointwise and the uniform consistencies of the constructed estimator with rates. In the same model, we establish the uniform convergence for the dependent case.

1.3 Liste des travaux

Publications

1. Merouan, T., Mechab, B., and Massim, I. (2018). Quadratic error of the conditional hazard function in the local linear estimation for functional data. *Afrika Statistika*, 13(3), 1759-1777.
2. Merouan, T., Mechab, B., and Chouaf, A. (2019). Nonparametric local linear estimator of the conditional hazard function in the functional single-index model. *Journal of Mathematics and Computation*, 30(3), 43-54.
3. Merouan, T., Mechab, B. Uniform convergence of nonparametric conditional hazard function in the single functional modeling for dependent data (Soumis pour publication).

Communications

1. Merouan, T. Estimation of the conditional hazard function in the single functional index. *Onzième Rencontre d'Analyse Mathématique et Applications (RAMA11)*, Sidi Bel Abbès, 21-24 novembre 2019.
2. Merouan, T., Mechab, B., and Massim, I. Quadratic error of the conditional hazard function in the local linear estimation for functional data. *Premières Journées Doctorales de la Faculté des Sciences Exactes (FSE)*, Sidi Bel Abbès, les 14 et 15 décembre 2019.

1.4 Cadre de la statistique fonctionnelle

Nous savons que la statistique classique est de traiter les réalisations de variables aléatoires réelles ou vecteurs aléatoires pour certains phénomènes aléatoires, mais il y a des données apparaissant sous forme continues (courbes, images,...) qui peuvent être considérées comme des fonctions discrétisées (des fonctions observées sur une échelle de discrétisation assez fine), de manière évidente, des données de dimension infinie qui sont appelées données fonctionnelles. Deville (1974) est le premier envisagent à des variables continues/fonctionnelles, Besse et Ramsay (1986) ; Besse (1991), ont approchés l'analyse factorielle ou cas fonctionnel, en particulier l'analyse en composant principales de courbes. Plus tard, en 1997, Ramsay and Silverman ont traité ces données fonctionnelles aussi avec des analyses factorielles, pour des modèles de régression, Besse et Cardot (2003) ont montré que la régression fonctionnelle est bien adaptée et plus efficace qu'une approche vectorielle.
Le secteur de l'analyse de données fonctionnelles a connu un grand intérêt, motivé par des applications différentes dans de nombreux domaines scientifiques. On considère des exemples ci dessous pour de données fonctionnelles comme des courbes aléatoires dépendant du temps.

- Dans la biologie, on peut considérer les données, qui sont des courbes aléatoires au cours du temps, où on dispose un échantillon des courbes de croissance associées pour 54 filles entre 1 et 18 ans, comme indiqué sur la figure 1.1 suivante :

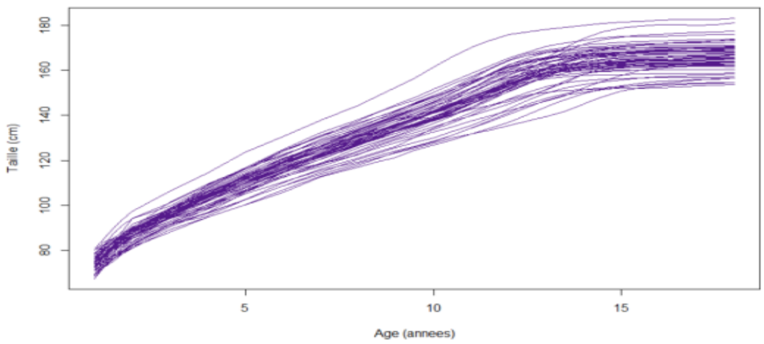

FIGURE 1.1 – Courbes de croissance de 54 filles.

- Dans la climatologie, on prend les courbes des températures moyennes journalières dans 4 stations météorologiques canadiennes (Yarmouth, Québec, Ottawa, London) au cours des années que sont représentées au graphique dans la figure 1.2 :

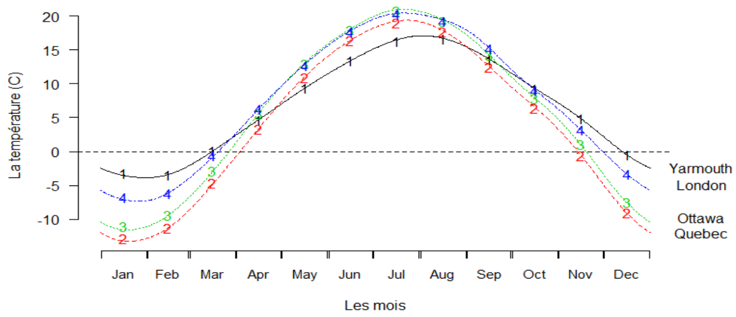

FIGURE 1.2 – Courbes des températures.

– Dans l'économie, on considère la quantité de production du pétrole produit à un certain niveau de la colonne de distillation, les données sont apparaitre dans la figure 1.3 à gauche, et le flux de vapeur dans cette même colonne dans la figure 1.3 à droite.

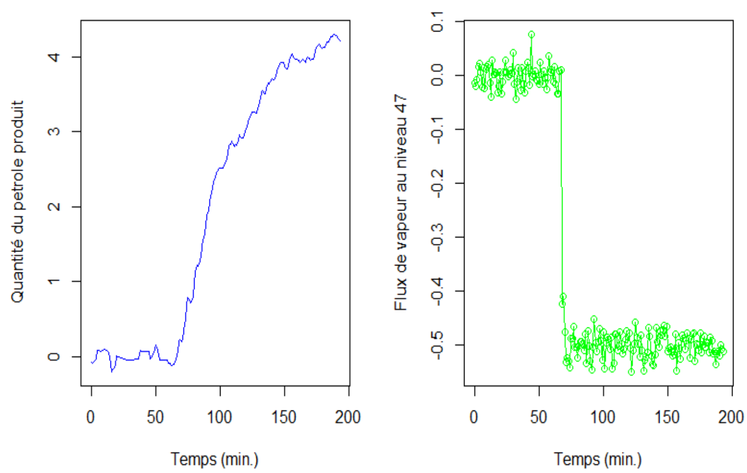

FIGURE 1.3 – Données de production du niveau 47 de la raffinerie en fonction du temps.

On pourra consulter aussi les travaux de Ramsay et Silverman (2002, 2005) et Ferraty et Romain (2011) sur les aspects appliqués pour ces données fonctionnelles, ainsi la contribution de Bosq (2000) qui conduit à un développement de la théorie des processus linéaires en grande dimension.

L'ouvrage de Ferraty et Vieu (2006) pose les bases de l'analyse non paramétrique dans cette thématique par la méthode du noyau (ou méthode de Perzen-Rosenblatt). Cette méthode est un outil classique en statistique et populaire en approche non paramétrique, qui apparaissent principalement pour analyser les données dans les problèmes statistiques en grande dimension. Historiquement, de nombreux travaux ont été consacrés à l'étude d' estimateur à noyau, et largement étudiés dans la littérature, qui connu par Rosenblatt (1956) et Perzan (1962) pour l'estimation non-paramétrique de la densité de probabilité (quand le noyau est défini dans $\mathbb{R}$) et qui permet d'obtenir une densité continue et constitue en ce sens

une généralisation de la méthode de l'histogramme. De plus, Rao (1958) et Tucker (1958) se sont intéressés à l'analyse en composantes principales dans le cas fonctionnel, nous renvoyons aussi à Massy (1965) pour la régression sur composante principale.
Considérons la méthode du noyau dans le cas fonctionnel, Ferraty et Vieu (2000) et Ferraty et *al.* (2002) sont parmi les premiers qui introduisent les modèles non-paramétriques pour les variables fonctionnelles, proposent une approximation du problème de la régression fonctionnelle en utilisant la dimension fractale, ainsi dans l'article de Ferraty et Vieu (2004). Plus encore, la monographie de Ferraty et Vieu (2006) permet de mieux comprendre plusieurs modèles non paramétriques conditionnels (la fonction de régression, la fonction de répartition conditionnelle, la densité conditionnelle et sa dérivé, du mode conditionnel et des quantiles conditionnels) quand la variable explicative prend ces valeurs dans un espace de dimension infinie et quand la variable réponse est scalaire. Ces auteurs ont établi la convergence presque sûre de ces modèles avec la méthode du noyau. Puis, Ferraty et *al.* (2007) ont étudie la convergence en moyenne quadratique, aussi la normalité asymptotique de l'opérateur de régression. En parallèle, Laksaci (2007) a donné l'erreur quadratique de la densité conditionnelle. Par la suite, Ezzahrioui et Ould-Said (2008a, 2008b) ont montré la normalité asymptotique du mode et du quantile conditionnelle dans le cas i.i.d et le cas dépendant.

La méthode du noyau est un cas particulier de la méthode par polynôme locaux. La méthode d'estimation par polynômes locaux est connu par les travaux de Stone (1986), avec l'estimation de la régression dans le cas multivarié. Par la suite, Ioffe et Katkovnik (1987, 1990) ont étudié les propriétés asymptotiques de la régression. Par la méthode locale linéaire, Fan (1992, 1993) ont donné l'erreur quadratique moyenne conditionnelle et l'erreur quadratique moyenne conditionnelle intégrée et aussi résolue le risque minimax. Nous nous référons Fan et Gijbels (1996) pour les applications de cette méthode, qui ont prouvé que cette méthode est bien adapté aux problèmes du biais aux bornes.
La méthode locale linéaire été bien adaptée à l'analyse fonctionnelle, nous renvoyons au travail de Baillo et Grané (2009) qui ont donné la convergence en L^2 dans l'espace de Hilbert et ils ont comparé l'estimateur du noyau avec la méthode de Monte-Carlo pour la régression. Dans le même opérateur, Barrientos et *al.* (2010) ont donné une vaste et facile version de la méthode locale linéaire pour les données fonctionnelles. On cite aussi à Berlinet et *al.* (2011) pour autre version de cette méthode. Par la suite, avec l'estimateur de Barrientos et *al.* (2010), Demongeot et *al.* (2010) ont établi la vitesse de convergence presque complète et uniforme, ainsi que, l'estimateur locale linéaire du mode conditionnel ont été donné comme application. Ce travail a été conduit par ces auteurs en (2011) pour étudier l'estimateur local linéaire de densité et du mode conditionnel dans le cas de mélange, ils ont utilisé ces résultats pour la prévision d'une série temporelle. On considère aussi les résultats de Chouaf et Laksaci (2013) pour les données spatialement dépendantes. Dans Rachdi et al. (2014), ces auteurs ont donné l'erreur quadratique moyenne. Nous citons Naceri et al. (2015) quand les covariables sont des courbes pour l'estimateur local linéaire de régression. En parallèle, l'article de Messaci et *al.* (2015) a traité la distribution conditionnelle, qui ont obtenu la vitesse de convergence presque complète et uniforme de ce modèle et aussi pour le quantile conditionnel.

Les statisticiens se sont été intéressés aussi avec autre méthode, qui est l'approche des k plus proches voisins. Cette approche a été introduit par Burba et *al.* (2008, 2009) ; ces auteurs ont montré la convergence presque complète de la régression. Nous pouvons consulter pour cette méthode les travaux de Lian (2011) ; Attouch et Benchikh (2012) ; kudraszow et Vieu (2013) pour plus de lecture. Attouch et Belabed (2014) ont établi la convergence presque complète et la normalité asymptotique de la fonction de hasard conditionnelle pour k plus proches voisins avec la simulation pour des données réelles. Dans un article récent, Attouch et *al.* (2017) ont construit un estimateur par l'approche locale linéaire et la méthode k plus proches voisins, ils ont établi les propriétés asymptotiques de la fonction de régression pour la variable explicative fonctionnelle et la réponse est scalaire. En 2019, ces auteurs ont développé leurs recherches par donner la convergence uniforme et ponctuelle presque complète sur le nombre de voisins k.

1.5 Modèle non paramétrique fonctionnel existant

1.5.1 Modèle local linéaire

Soit $(X_i, Y_i)_{i\in\mathbb{N}}$ une suite de variables aléatoires de (X, Y), tel que Y est prend sa valeurs dans $\mathbb{R}$ et X à valeurs dans un espace semi-métrique $\mathcal{F}$, et d la semi-métrique sur $\mathcal{F}$. Pour tout $x \in \mathcal{F}$, on note f^x la fonction de densité conditionnelle de Y sachant $X = x$ et F^x la fonction de répartition conditionnelle de Y sachant $X = x$.
Nous pouvons alors définir la fonction de hasard conditionnelle h^x de Y sachant $X = x$, par :

$$h^x(y) = \frac{f^x(y)}{1 - F^x(y)},\ \forall y \in \mathbb{R},$$

avec, $F^x(y) < 1$.
Pour tout $x \in \mathcal{F}$, l'estimateur local linéaire de Barrientos et *al.* (2010) de h^x (resp. F^x, f^x), noté $\widehat{h}^x$ (resp. $\widehat{F}^x$, $\widehat{f}^x$), défini par l'expression :

$$\widehat{h}^x(y) \quad = \quad \frac{\widehat{f}^x(y)}{1 - \widehat{F}^x(y)}$$

où

$$\widehat{F}^x(y) = \frac{\sum\limits_{1\leq i,j\leq n} W_{ij}(x) H({h_H}^{-1}(y - Y_j))}{\sum\limits_{1\leq i,j\leq n} W_{ij}(x)},$$

et

$$\widehat{f}^x(y) = \frac{\sum\limits_{1\leq i,j\leq n} W_{ij}(x) H'({h_H}^{-1}(y - Y_j))}{h_H \sum\limits_{1\leq i,j\leq n} W_{ij}(x)},$$

avec la convention $0/0 := 0$,
et $W_{ij}(x) = \beta(X_i, x)(\beta(X_i, x) - \beta(X_i, x))K(h_K^{-1}\delta(x, X_i))K(h_K^{-1}\delta(x, X_j))$.
Les opérateurs $\beta(.,.)$, $\delta(.,.)$ sont des fonctions bi-fonctionnelles de $\mathcal{F} \times \mathcal{F}$ dans $\mathbb{R}$, et $\forall z \in \mathcal{F},\ \beta(z, z) = 0$ et $d(.,.) = |\delta(.,.)|$.
K est un noyau, H est la fonction de distribution (avec H' est la dérivée de H), et h_K(resp. h_H) est une suite de réels positifs tends vers 0 quand n tend vers l'infinie.

La fonction de hasard ou bien fonction du risque, taux de hasard, taux de survie, c'est un outil très important dans différents domaines scientifiques (en analyse de survie et en sismologie, médecine, fiabilité, etc). Cette fonction est introduit par Watson et Ledadbetter (1964). D'autre part, Hassani et *al.* (1986) ont traité ce modèle en la théorie de la fiabilité. Dans le même contexte et sous des conditions de dépendance Roussas (1989) a donné la convergence ponctuelle et uniforme pour des variables réelles.
Dans la littérature, l'estimation non paramétrique de la fonction de hasard conditionnelle avec la méthode du noyau est très vaste. Ferraty et *al.* (2008) sont parmi les premiers à avoir traité ce modèle où la variable explicative est fonctionnelle, ils ont obtenu la vitesse de convergence presque complète dans le cas i.i.d et dépendants. En parallèles, la normalité asymptotique à été donné par Quintela-del-Rio (2008), y compris la convergence presque complète et l'erreur quadratique pour les variables fonctionnelles dépendants. De plus, Ferraty et *al.* (2010) ont fait la convergence uniforme de cette fonction et autres modèles conditionnels. D'autre part, le cas spatial est donnés dans le travaille de Laksaci et Mechab (2010) pour un estimateur de la fonction de hasard conditionnelle par l'estimation du noyau. Puis, en (2014) dans le même modèle ces auteurs ont donné l'erreur quadratique et la normalité asymptotique avec la simulation et application de données réelles. Un autre estimateur pour ce modèle a été proposé par Attouch et Belabed (2014), ils ont établi la convergence presque complète et la normalité asymptotique par la méthode k plus proche voisin. Récemment, Massim et Mechab (2016) ont construit par l'approche de Barrinetos et *al.* (2010) un estimateur pour le fonction de hasard. Ces auteurs ont établi la convergence presque complète. Avec cet estimateur, Zhou et Lin (2016) ont donné la convergence en moyenne quadratique et la normalité asymptotique de la régression. Récemment, Leulmi (2019) a déterminé la vitesse de convergence ponctuelle et uniforme presque complète de la distribution conditionnelle et du quantile conditionnelle lorsque la variable réponse est dans le cas censurée.

1.5.2 Le modèle dans l'indice fonctionnel simple

Dans cette partie, nous nous intéressons au modèle à indice simple. On considère un couple de variable aléatoire (X, Y) dans lequel X est à valeurs dans un espace de Hilbert $\mathcal{H}$ muni d'un produit scalaire $< .,. >$, et Y est, soit un élément de $\mathbb{R}$, θ est l'indice fonctionnel connu définie dans $\mathcal{H}$, et d_θ la semi métrique qui associé à l'indice fonctionnel, telle que $d_\theta(x_1, x_2) = | < x_1 - x_2, \theta > |$.
On définit la fonction de hasard dans ce type du modèle, et pour tout $x \in \mathcal{H}$, et $\theta \in \mathcal{H}$ on a :

$$h_\theta^x(y) =: h(y| < x, \theta >), \ \forall y \in \mathbb{R},$$

donc

$$h_\theta^x(y) = \frac{f_\theta^x(y)}{1 - F_\theta^x(y)}, \quad \forall y \in \mathbb{R},$$

avec, $F_\theta^x(y) < 1$.
Le modèle est identifiable tel que :

$$h_1(y| < x, \theta_1 >) = h_2(y| < x, \theta_2 >) \Rightarrow h_1 \equiv h_2 \text{ et } \theta_1 = \theta_2,$$

donc un estimateur $\widehat{h}_\theta^x$ de h_θ^x est

$$\widehat{h}_\theta^x(y) = \frac{\widehat{f}_\theta^x(y)}{1 - \widehat{F}_\theta^x(y)}, \ \forall y \in \mathbb{R}$$

On obtient les estimateurs $\widehat{F}_\theta^x$ et $\widehat{f}_\theta^x$, similaire à la méthode de Barrientos et *al.* (2010) comme suit :

$$\widehat{F}_\theta^x(y) = \frac{\sum\limits_{1\leq i,j\leq n} W_{\theta,ij}(x)H({h_H}^{-1}(y-Y_j))}{\sum\limits_{1\leq i,j\leq n} W_{\theta,ij}(x)},$$

et

$$\widehat{f}_\theta^x(y) = \frac{\sum\limits_{1\leq i,j\leq n} W_{\theta,ij}(x)H'({h_H}^{-1}(y-Y_j))}{h_H \sum\limits_{1\leq i,j\leq n} W_{\theta,ij}(x)}$$

où
$W_{\theta,ij}(x) = \beta_\theta(X_i,x)\Big(\beta_\theta(X_i,x)-\beta_\theta(X_j,x)\Big)K(h_K^{-1}d_\theta(x,X_i))K(h_K^{-1}d_\theta(x,X_j))$,
avec $\beta_\theta(X_i,x) =< x - X_i, \theta >$, les fonctions K et H sont des les noyaux (avec H' est la dérivée de H), et h_K(resp. h_H) est une suite de réels positifs.

Le principe des modèles à indice simple (ou modèles à direction révélatrice), est de compromis entre un modèle paramétrique et non paramétrique, en d'autre terme, pour les modèles semi paramétrique. Les premiers résultats pour ce modèle dans le cadre fonctionnel sont introduit par Ferraty et *al.* (2003), lorsque la variable aléatoire explicative dans un espace de Hilbert, pour un estimateur du noyau pour la régression. En 2008, pour cet dernière opérateur, Ait Saidi et *al.* s'intéressent aux l'estimation de l'indice fonctionnel par la méthode de validation croisée. Après deux ans, Bouraine et *al.* (2010) ont utilise cette méthode pour estimé l'indice multi-fonctionnel pour un estimateur de régression. Dans le même context, Attaoui et *al.* (2011) ont établi la convergence presque complète et uniforme de la fonction de densité conditionnel par la méthode du noyau, quand la variable explicative fonctionnelle est liés avec la structure de l'indice fonctionnel simple. Après, Ling et Xu (2012) ont donné la normalité asymptotique de cette fonction mais pour les données fonctionnelles α-mélangeants. Ensuite, Ling et *al.* (2014) ont montré la convergences presque complète et uniforme. De plus, on peut aussi citer le travaille de Attaoui et Ling (2016) pour la fonction de répartition conditionnelle avec des applications et simulation. Plus récemment, Tabti et Ait Saidi (2018) réalisent une étude pour la fonction de hasard avec la méthode du noyau, ils ont précisé la vitesse de convergence presque complète et uniforme de cet estimateur.

1.6 Définitions et inégalités

Définition 1.6.1 (Convergence presque complète)
Nous disons que la suite de variables aléatoires $(X_n)_{n\in\mathbb{N}}$ *converge presque complètement vers une variable aléatoire réelle* X *et nous notons* $X_n \xrightarrow[n\to\infty]{p.co.} X$, *si et seulement si :*

$$\forall\varepsilon > 0, \qquad \sum_{n\in\mathbb{N}} \mathbb{P}(|X_n - X| > \varepsilon) < \infty,$$

Nous disons que la suite de variables aléatoires $(X_n)_{n\in\mathbb{N}}$ *converge presque complètement avec vitesse vers une variable aléatoire réelle* X *d'ordre* (u_n) *et nous notons* $X_n - X = O_{p.co.}(u_n)$, *si et seulement si :*

$$\exists \varepsilon' > 0, \qquad \sum_{n\in\mathbb{N}} \mathbb{P}(|X_n - X| > \varepsilon' u_n) < \infty,$$

Définition 1.6.2 (L'entropie de Kolmogorov)
Soit S un sous-ensemble d'un espace semi-métrique $\mathcal{F}$, et soit $\varepsilon > 0$ donné. Un ensemble fini de points $x_1, x_2, ..., x_N$ dans $\mathcal{F}$ est appelé ε-net pour S, si $S \subset \cup_{k=1}^{N} B(x_k, \varepsilon)$.
La quantité $\psi_S(\varepsilon) = \ln(N_\varepsilon(S))$, où $N_\varepsilon(S)$ désigne le nombre minimum de boules ouvertes en $\mathcal{F}$ de rayon ε qui est nécessaire pour couvrir S, est appelé ε-entropie de Kolmogorov de l'ensemble S.

Définition 1.6.3 (Mélange)
Soit $(X_n)_{n\in\mathbb{N}}$ une suite de variables aléatoires. On appelle coefficient de mélange fort, le nombre réel $\alpha(n)$, si et seulement si :

$$\alpha(n) = \sup_k \sup_{A\in\sigma_1^k(X), B\in\sigma_{n+k}^{\infty}(X)} |\mathbb{P}(AB) - \mathbb{P}(AB)| \xrightarrow{n\to\infty} 0,$$

avec $\sigma_p^q(X)$ est la tribu engendrée par les variables $(X_i,\ p \leq i \leq q)$.

On distingue deux type de mélange fort.

Définition 1.6.4
Soit $(X_n)_{n\in\mathbb{N}}$ une suite de variables aléatoires. On définit que cette suite est arithmétiquement (algébriquement) α-mélange d'ordre $a > 1$, s'il existe $C > 0$, tel que :

$$\alpha(n) \leq Cn^{-a},$$

et géométriquement α-mélange, s'il existe $C > 0$ et $\rho \in]0, 1[$, tel que :

$$\alpha(n) \leq C\rho^n.$$

1.6.1 Inégalité pour le cas indépendant

Corollaire 1.6.1. Soit $(T_n)_{n\in\mathbb{N}}$ une suite de variables aléatoires indépendantes identiquement distribuées, et $\forall m \geq 2,\ \exists C_m > 2,\ \mathbb{E}[|T_1^n|] \leq C_m a^{2(m-1)}$, alors :

$$\forall \varepsilon > 0,\ \mathbb{P}\left[\left|\sum_{i=1}^{n} T_i - \mathbb{E}|T_i|\right| > \varepsilon n\right] \leq 2\exp\left(-\frac{\varepsilon^2 n}{2a^2(1+\varepsilon)}\right) \tag{1}$$

1.6.2 Inégalités pour le cas dépendant

Proposition 1.6.5 (Rio (2000))
Soit $(W_n)_{n\in\mathbb{N}}$ une suite de variables aléatoires réelles centrées et de coefficient de mélange $a > 1$, et $\forall i \in \mathbb{N},\ |W_i| < \infty$. Alors pour tout $r > 1$ et $\varepsilon > 0$ et $C < \infty$:

$$\mathbb{P}\left(\left|\sum_{i=1}^{n} W_i\right| > \varepsilon\right) \leq C\left(\left(1 + \frac{\varepsilon^2}{rS^2}\right)^{-r/2} + nr^{-1}\left(\frac{r}{\varepsilon}\right)^{a+1}\right) \tag{2}$$

avec $S^2 = \sum_{i,j=1}^{n} |Cov(W_i, W_j)|$.

Proposition 1.6.6 (Davydov-Rio's inequality)
Soit $(W_n)_{n\in\mathbb{Z}}$ une famille de variables aléatoires α-mélangeantes. Considérer une variable Z (resp. Z') qui, pour tout $k\in\mathbb{Z}$ est $\sigma_{\infty}^{k}(W_i)$- measurable (resp. $\sigma_{n+k}^{+\infty}(W_i)$-measurable). Si Z et Z' sont bornées, alors :

$$\exists C>0,\ \ Cov(Z,Z')\leq C\alpha(n). \tag{3}$$

Soient p,q,r des nombres réels positifs, tels que $p^{-1}+q^{-1}+r^{-1}=1$ et $\mathbb{E}[Z^p]<\infty$ (resp. $\mathbb{E}[Z'^p]<\infty$), alors :

$$\exists C>0,\ \ Cov(Z,Z')\leq C(\mathbb{E}[Z^p])^{1/p}(\mathbb{E}[Z'^q])^{1/q}\alpha(n)^{1/r}. \tag{4}$$

1.6.3 Résultats obtenus par chapitre

Résultats : l'erreur quadratique de l'estimateur local linéaire de la fonction de hasard conditionnelle

Théorèm 1.6.1. Sous certaines hypothèses dans le chapitre 2, on aura :

$$\mathbb{E}\left[\widehat{h}^x(y)-h^x(y)\right]^2=B_n^2(x,y)+\frac{V_{HK}(x,y)}{nh_H\phi_x(h_K)}+o(h_H^4)+o(h_K^4)+o\left(\frac{1}{nh_H\phi_x(h_K)}\right) \tag{5}$$

où :

$$B_n(x,y)=\frac{(B_{f,H}-h^x(y)B_{F,H})h_H^2+(B_{f,K}-h^x(y)B_{F,K})h_K^2}{1-F^x(y)}$$

avec :

$$\begin{array}{rcl}
B_{f,H}(x,y) &=& \frac{1}{2}\frac{\partial^2 f^x(y)}{\partial y^2}\int t^2H^{(1)}(t)\mathrm{d}t\\
B_{f,K}(x,y) &=& \frac{1}{2}\Psi_{0,1}^{(2)}(0)\left[\frac{\left(K(1)-\int_{-1}^{1}(u^2K(u))^{(1)}\chi_x(u)\mathrm{d}u\right)}{\left(K(1)-\int_{-1}^{1}K^{(1)}(u)\chi_x(u)\mathrm{d}u\right)}\right]\\
B_{F,H}(x,y) &=& \frac{1}{2}\frac{\partial^2 F^x(y)}{\partial y^2}\int t^2H^{(1)}(t)\mathrm{d}t\\
B_{F,K}(x,y) &=& \frac{1}{2}\Psi_{0,0}^{(2)}(0)\left[\frac{\left(K(1)-\int_{-1}^{1}(u^2K(u))^{(1)}\chi_x(u)\mathrm{d}u\right)}{\left(K(1)-\int_{-1}^{1}K^{(1)}(u)\chi_x(u)\mathrm{d}u\right)}\right]
\end{array}$$

et

$$V_{HK}^h(x,y)=\frac{h^x(y)}{(1-F^x(y))}\left[\frac{\left(K^2(1)-\int_{-1}^{1}(K^2(u))^{(1)}\chi_x(u)\mathrm{d}u\right)}{\left(K(1)-\int_{-1}^{1}(K(u))^{(1)}\chi_x(u)\mathrm{d}u\right)^2}\right].$$

La preuve de cet théorème sont détaillées dans le chapitre 2.

Résultats : la convergence presque complète et la convergence uniforme dans le modèle à indice fonctionnel simple

Nous aurons d'après les hypothèses dans le chapitre 3 et quand les variables sont indépendant :

Théorèm 1.6.2.

$$\sup_{y\in S_{\mathbb{R}}}|\widehat{h}_\theta^x(y)-h_\theta^x(y)|=O(h_K^{b_1}+h_H^{b_2})+O_{\text{a.co.}}\left(\sqrt{\frac{\ln n}{nh_H\phi_{\theta,x}(h_K)}}\right). \tag{6}$$

Théorèm 1.6.3.

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}|\widehat{h}^x_\theta(y)-h^x_\theta(y)| = O(h_K^{b_1}+h_H^{b_2}) + O_{\text{a.co.}}\left(\sqrt{\frac{\ln q_n^{S_{\mathcal{H}}}+\ln q_n^{\Theta_{\mathcal{H}}}}{nh_H\phi(h_K)}}\right). \quad (7)$$

Les démonstrations de ces théorèmes sont seront données dans le chapitre 3.

Résultats : la convergence uniforme de hasard conditionnelle dans le modèle à indice fonctionnel simple où les variables sont dépendants

Théorèm 1.6.4. Sous certains conditions dans le chapitre 4, nous aurons :

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}|\widehat{h}^x_\theta(y)-h^x_\theta(y)| = O(h_K^{a_1}+h_H^{a_2}) + O_{\text{a.co.}}\left(\sqrt{\frac{\ln(N^{S_{\mathcal{H}}}N^{\Theta_{\mathcal{H}}})}{nh_H\phi(h_K)}}\right). \quad (8)$$

La preuve de cet théorème est existent dans le chapitre 4.

1.7 Plan de la thèse

Cette thèse organisé comme suit :

Le chapitre 1 est un chapitre introductif présentant le cadre statistique dans lequel s'insère cette thèse, ainsi les résultats obtenus dans ce champs quand les données sont fonctionnelles. Après nous vous rappelons de quelques définitions et inégalités dont nous avons besoin dans ce travail.

Le chapitre 2 est consacré à la convergence en moyenne quadratique de la fonction de hasard conditionnelle par l'estimation locale linéaire quand la variable réponse est réelle et la covariable est fonctionnelle, en précisant le terme explicite du biais et de la variance. Cet résultat asymptotique illustré l'efficacité notre approche par simulation. Ce travail ont été publiés dans le journal international *Afrika Statstika.*

Nous avons étudié dans le chapitre 3 la convergence presque complète et la convergence uniforme de la fonction de hasard. Nous nous intéressons ici à modèle à indice fonctionnel lorsque ce dernier est connu et les variables dont i.i.d. Sous hypothèses, nous établissons la vitesse de convergence de l'estimateur local linéaire. Ces résultats une publication au journal international *Mathematics and Computation.*

Le chapitre 4 à généralise nos résultats aux chapitre précédent, quand les observations sont fonctionnelles et dépendantes. Sous conditions de mélange, nous donnons la convergence uniform comme résultat asymptotique en précisant la vitesse de convergence. Ce travail a fait l'objet d'un article soumis.

Nous terminerons cette thèse par une conclusion générale, aussi dans ce partie, nous présenter les perspectives futures.

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CHAPITRE 2

Quadratic error of the conditional hazard function in the local linear estimation for functional data

Ce chapitre fait l'objet d'une publication accepté dans le Journal of Afrika Statistika.

Quadratic error of the conditional hazard function in the local linear estimation for functional data

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Abstract. In this paper we investigate the asymptotic mean square error and the rates of convergence of the estimator based on the local linear method of the conditional hazard function. Under some general conditions, the expressions of the bias and variance are given. The efficiency of our estimator is evaluated through a simulation study. We proved, theoretically and on the scope of a simulation study, that our proposed estimator has better performance than the estimator based on the standard kernel method.

Keywords : Nonparametric local linear estimation, conditional hazard function, functional variable, mean squared error.

Résumé. Nous étudions dans ce papier, l'estimation non paramétrique de la fonction de hasard conditionnelle basée sur la méthode locale linéaire. Le but est de calculer sous certaines conditions la convergence en moyenne quadratique de notre estimateur, ainsi que les expressions du biais et de la variance de notre estimateur sont données. L'efficacité de notre estimateur est évaluée par une étude de simulation, sur un échantillon fini, qui montre une meilleure performance de l'estimateur introduit par rapport à l'estimateur basée sur la méthode du noyau standard.

Mots-clés : estimation locale linéaire non-parametrique, fonction de hasard conditionnelle, variable fonctionnelle, erreur quadratique moyenne.

2000 Mathematics Subject Classification : 62G05, 62G20.

2.1 Introduction

In recent years, the considerable progress in computing power makes it possible to collect and analyze more and more cumbersome data. These large data sets are available primarily through real time monitoring and computers can efficiently deal with such databases.
Many multivariate statistical techniques, concerning parametric models, have been extended to functional data and a good review on this topic can be found in Ramsay and Silverman (2005) or Bosq (2000). Recently, new studies have been carried out in order to propose non-parametric methods taking into account functional data. For a more comprehensive review

on this subject the reader is referred to Ferraty and Vieu (2006) and to Ferraty and Vieu (2002) for specialized monographs.
However, it is well known that a local polynomial smoothing procedure has many advantages over the kernel method (see, Fan and Yao (2003) and Fan and Gijbels (1996), etc.). In particular, the former method has better properties, in terms of bias estimation. The local linear smoothing in the functional data setting has been considered by many authors. The first results on the regression function were established in Baillo and Grané (2009), Boj et al. (2010), Berlinet et al. (2011) and El methni and Rachdi (2011). Other works have been realized on this subject, for example Barrientos-Marin et al. (2010) developed a smoothing local linear estimation of the regression operator for independent data. Moreover, Demongeot et al. (2010) established the almost complete consistency of local linear estimator of the conditional density when the explanatory variable is functional and the observations are i.i.d. The mean squared error of the last estimator was studied by Rachdi et al. (2014). The asymptotic properties (almost complete convergence and convergence in mean square, with rates) of the local linear estimator of the conditional cumulative distribution were established by Demongeot et al. (2014).
This work deals with the functional nonparametric estimation of the hazard and/or the conditional hazard function. Historically, this function was first introduced by Watson and Leadbetter (1964). Since then, several results have been added by many authors. For example Roussas (1989). States that there is extensive literature on nonparametric estimation of the conditional hazard function using a wide variety of methods. This function is important in a variety of fields such as Medicine, Reliability, Survival Analysis or Seismology, etc.
In nonparametric functional framework, the first result has been obtained by Ferraty et al. (2008), who used an approach based on kernel estimations. The authors introduced a kernel estimator of the conditional hazard function and proved some asymptotic properties (with rates) in various situations including censored and/ or dependent variables. Quintela-Del-Rio (2008) extended the results of Ferraty et al. (2008). They calculated the bias and variance of these estimates, and established their asymptotic normality. Still while using the Kernel method, Rabhi et al. (2013) determined the asymptotic mean square error of the proposed estimator of the conditional hazard function. In the nonfunctional case, a short overview on nonparametric conditional hazard function estimation can be found in Spierdijk (2008). For functional case, Massim and Mechab (2016) have established the almost complete convergence of the estimator of the conditional hazard function based on the local linear approach.
In the light of what precedes on the importance of the hazard function estimation and the availability of a significant number of advanced and detailed asymptotic results based on the kernel approach, we were interested to find analogous result for the estimator introduced in Massim and Mechab (2016), and next to carry out a thorough comparison with available results.
To achieve this work, we address the described estimator in Massim and Mechab (2016). In this paper, we explicitly determine the mean squared error convergence and compare it to the available result and that obtained through a simulation study.

The remainder of our paper is organized as follows. In section 2.2, we present our functional model, give basic notations and describe our assumptions. In Section 2.3. we first state the main theoretical result of the paper about the mean squared convergence in Subsection 2.3.1 and then, in subsection 2.3.2, we present the results and we make a comparison with those obtained through simulation study. The proofs are provided in Section 2.4. We conclude the paper by a conclusion and perspective section 2.5.

2.2 Description of the model, notation and assumptions

2.2.1 Model and estimator

Let us consider a sequence $(X_i, Y_i)_{i\geq 1}$ of independent and identically random pair according to the distribution of the pair (X, Y), all of them defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and taking their values in a space $\mathcal{F} \times \mathbb{R}$, where $(\mathcal{F}, d)$ is a semi-metric space. We suppose that $\mathcal{F} \times \mathbb{R}$ is endowed with the product σ-algebra of the Borel σ-algebras $\mathcal{B}(\mathcal{F})$ and $\mathcal{B}(\mathbb{R})$ on $\mathcal{F}$ and on $\mathbb{R}$ respectively. For a fixed $x \in \mathcal{F}$, we denote by F^x the conditional cumulative distribution function (cdf) of Y given $(X = x)$ and we suppose that F^x is absolutely continuous with respect to the Lebesgue measure with radon-Nikodym derivative f^x, which is the conditional probability density function (pdf) of Y given $(X = x)$. Accordingly, the conditional hazard function (chf) of Y, given $X = x$, is

$$h^x(y) = \frac{f^x(y)}{1 - F^x(y)}, \ y \in \mathbb{R} \text{ and } F^x(y) < 1. \tag{1}$$

Our main objective is to estimate the conditional hazard function $\widehat{h}^x(\cdot)$ for x fixed, in the form.

$$\widehat{h}^x(y) = \frac{\widehat{f}^x(y)}{1 - \widehat{F}^x(y)}, \ y \in \mathbb{R} \text{ and } \widehat{F}^x(y) < 1. \tag{2}$$

By the fast functional local modeling (cf. Fan and Gijbels (1992), the conditional cumulative distribution function $F^x(y)$ is estimated as the argmin value of a in the optimization problem, for each $n \geq 1$, the following equation

$$\widehat{F}^x(y) = arg \min_{(a,b)\in\mathbb{R}^2} \sum_{i=1}^{n} \Big(H(h_H^{-1}(y - Y_i)) - a - b\beta(X_i,\ x)\Big)^2 \ K(h_K^{-1}\delta(x, X_i)) \tag{3}$$

where $\beta(.,.)$ and $\delta(.,.)$ are locating functions defined from $\mathcal{F} \times \mathcal{F}$ into $\mathbb{R}$, such that :

$$\forall \xi \in \mathcal{F}, \quad \beta(\xi, \xi) = 0 \text{ and } d(\cdot, \cdot) = |\delta(\cdot, \cdot)|.$$

K is a kernel appropriately chosen, H is a distribution function and $h_K = h_{K,n}$(respectively, $h_H = h_{H,n}$) is a sequence of positive real numbers which converges to 0 when $n \to \infty$. Clearly, after direct computations, we get

$$\widehat{F}^x(y) = \frac{\sum\limits_{1\leq i,j\leq n} W_{ij}(x) H({h_H}^{-1}(y - Y_j))}{\sum\limits_{1\leq i,j\leq n} W_{ij}(x)}, \qquad \forall y \in \mathbb{R} \tag{4}$$

with $W_{ij}(x) = \beta_i(\beta_i - \beta_j)K(h_K^{-1}\delta(x, X_i))K(h_K^{-1}\delta(x, X_j))$ and $\beta_i = \beta(X_i, x)$.
Further, the estimator $\widehat{f}^x(y)$ of the density function $f^x(y)$ can be deduced from (4), by

$$\widehat{f}^x(y) = \frac{\sum_{1\leq i,j\leq n} W_{ij}(x)H^{(1)}({h_H}^{-1}(y - Y_j))}{h_H \sum_{1\leq i,j\leq n} W_{ij}(x)}, \qquad \forall y \in \mathbb{R} \tag{5}$$

where $H^{(1)}$ denotes the derivative of H. By putting together equations 4 and 5, the final form of our estimator (L.M.M.) is : for $n \geq 1, y \in \mathbb{R}$,

$$\widehat{h}^x(y) = \frac{h_H^{-1} \sum_{1\leq i,j\leq n} W_{ij}(x)H'({h_H}^{-1}(y - Y_j))}{\sum_{1\leq i,j\leq n} W_{ij}(x) - \sum_{1\leq i,j\leq n} W_{ij}(x)H({h_H}^{-1}(y - Y_j))}. \tag{6}$$

(See Massim and Mechab (2016)). To be complete, let us remind the conditional hazard function based on the kernel method (K.M.), given for $n \geq 1, y \in \mathbb{R}$ by

$$\widehat{h}^x(y) = \frac{h_H^{-1} \sum_{i=1}^{n} K_i(x)H'({h_H}^{-1}(y - Y_j))}{\sum_{i=1}^{n} K_i(x) - \sum_{i=1}^{n} K_i(x)H({h_H}^{-1}(y - Y_j))}. \tag{7}$$

(See Quintela-Del-Rio (2008)). Before we treat the asymptotic theory of the estimator (6) and compare it with that of the estimator (7), we need more notations and clear assumptions given below.

2.2.2 Notations and assumptions

Let us introduce a set of hypotheses which will be needed to state our main result. Here and below, x (resp. y) will denote a fixed point in $\mathcal{F}$ (resp. $\mathbb{R}$), $\mathcal{N}_x$ (resp. $\mathcal{N}_y$) a fixed neighborhood of a fixed point x (resp. of y) and $\phi_x(r_1, r_2) = \mathbb{P}(r_2 < \delta(X, x) < r_1)$.

(H1) For any $r > 0$, $\phi_x(r) := \phi_x(-r, r) > 0$. There exists a function $\chi_x(\cdot)$ such that

$$\forall t \in (-1, 1), \lim_{h_K \to 0} \frac{\phi_x(th_K, h_K)}{\phi_x(h_K)} = \chi_x(t).$$

(H2) We denote, for any $l \in \{0, 2\}$ and $j = 0, 1$, the functions

$$\psi_{l,j}(x, y) = \frac{\partial^l F^{x^{(j)}}(y)}{\partial y^l} \text{ and } \Psi_{l,j}(s) = \mathbb{E}[\psi_{l,j}(X, y) - \psi_{l,j}(x, y)|\beta(x, X) = s] \tag{8}$$

where $\Psi_{l,j}^{(1)}(0)$ and $\Psi_{l,j}^{(2)}(0)$ of the function $\Psi_{l,j}(\cdot)$ exist and $g^{(k)}$ denotes the k^{th} order derivative of g.

(H3) The functions $\delta(\cdot, \cdot)$ and $\beta(\cdot, \cdot)$ are such that
$\forall z \in \mathcal{F},\ C_1|\delta(x, z)| \leq |\beta(x, z)| \leq C_2|\delta(x, z)|$, with $C_1 > 0, C_2 > 0$,

$$\sup_{u\in B(x,r)} |\beta(u,x)-\delta(x,u)| = o(r)$$

and

$$h_K \int_{B(x,h_K)} \beta(u,x)\mathrm{d}P_X(u) = o\left(\int_{B(x,h_K)} \beta^2(u,x)\mathrm{d}P_X(u)\right)$$

where $B(x,r)=\{x'\in\mathcal{F} : |\delta(x',x)|\leq r\}$.

(H4) The kernel K is a positive, differentiable function which is supported within $(-1,1)$ satisfies

$$K^2(1)-\int_{-1}^{1}(K^2(u))^{(1)}\chi_x(u)\mathrm{d}u > 0.$$

(H5) The kernel H is a differentiable function which has a bounded first derivative such that

$$\int |t|^2 H^{(1)}(t)\mathrm{d}t < \infty, \int \left(H^{(1)}\right)^2(t)\mathrm{d}t < \infty \text{ and } \int H^{(1)}(t)\mathrm{d}t = 1.$$

(H6) $\exists\alpha<\infty,\ f^x(y)\leq\alpha,\ \forall(x,y)\in\mathcal{F}\times\mathbb{R}$
and $\exists 0<\beta<1,\ F^x(y)\leq 1-\beta,\ \forall(x,y)\in\mathcal{F}\times\mathbb{R}$.

(H7) The bandwidths h_K and h_H satisfy

$$\lim_{n\to\infty} h_K=0, \quad \lim_{n\to\infty} h_H=0, \quad \text{and } \lim_{n\to\infty} n h_H^{(j)}\phi_x(h_K)=\infty, \text{ for } j=0,1.$$

Some Comments on the assumptions : Assumption (H1) is the concentration property of the explanatory variable in small balls. The function $\chi_x(.)$ plays a fundamental role in all asymptotic study, in particular for the variance term. The condition (H2) is used to control the regularity of the functional space of our model and this is needed to evaluate the bias term of the convergence rates. The assumption (H3) is the same assumption as the assumption (H3) in Rachdi et al. (2014), as introduced in Barrientos-Marin et al. (2010). The hypothesis (H4) and (H5) on the kernels K,H and $H^{(1)}$ are standard conditions in the determination of the quadratic error for functional data. The hypotheses (H6) and (H7) are technical conditions and are also similar to those considered in Ferraty et al. (2008).

2.3 Results

In this section we are going to state our theoretical results. In the first subsection the proof of our main Theorem 3.3.1 is demonstrated in terms of Theorems 2.3.2-2.3.3 and Lemmas 2.3.1-2.3.7. The full proofs of all these theoretical results are postponed to Section 2.4. As a result, we will have the time to focus on the simulation study in the second subsection of the current section.

2.3.1 Main results : mean squared convergence

Theorem 2.3.1. Under assumptions (H1)-(H7), we obtain

$$\mathbb{E}\left[\widehat{h}^x(y)-h^x(y)\right]^2 = B_n^2(x,y)+\frac{V_{HK}(x,y)}{nh_H\phi_x(h_K)}+o(h_H^4)+o(h_K^4)+o\left(\frac{1}{nh_H\phi_x(h_K)}\right)$$

where

$$B_n(x,y)=\frac{(B_{f,H}-h^x(y)B_{F,H})h_H^2+(B_{f,K}-h^x(y)B_{F,K})h_K^2}{1-F^x(y)}$$

with

$$\begin{aligned}
B_{f,H}(x,y) &= \tfrac{1}{2}\tfrac{\partial^2 f^x(y)}{\partial y^2}\int t^2 H^{(1)}(t)\mathrm{d}t \\
B_{f,K}(x,y) &= \tfrac{1}{2}\Psi_{0,1}^{(2)}(0)\left[\frac{\left(K(1)-\int_{-1}^{1}(u^2K(u))^{(1)}\chi_x(u)\mathrm{d}u\right)}{\left(K(1)-\int_{-1}^{1}K^{(1)}(u)\chi_x(u)\mathrm{d}u\right)}\right] \\
B_{F,H}(x,y) &= \tfrac{1}{2}\tfrac{\partial^2 F^x(y)}{\partial y^2}\int t^2 H^{(1)}(t)\mathrm{d}t \\
B_{F,K}(x,y) &= \tfrac{1}{2}\Psi_{0,0}^{(2)}(0)\left[\frac{\left(K(1)-\int_{-1}^{1}(u^2K(u))^{(1)}\chi_x(u)\mathrm{d}u\right)}{\left(K(1)-\int_{-1}^{1}K^{(1)}(u)\chi_x(u)\mathrm{d}u\right)}\right]
\end{aligned}$$

and

$$V_{HK}^{h}(x,y) = \frac{h^x(y)}{(1-F^x(y))}\left[\frac{\left(K^2(1)-\int_{-1}^{1}(K^2(u))^{(1)}\chi_x(u)\mathrm{d}u\right)}{\left(K(1)-\int_{-1}^{1}(K(u))^{(1)}\chi_x(u)\mathrm{d}u\right)^2}\right].$$

Comparison remark. It is clear that the bias term is of order of the (L.M.M.) estimator and is given by

$$C_H h_H^2 + C_K h_K^2. \tag{9}$$

We already know form available literature (see for example, Quintela-Del-Rio (2008) and Rabhi et al. (2013)) that bias term of the (K.M.) estimator is

$$C_H h_H^2 + C_K h_K. \tag{10}$$

Further more, both (LMM) and (KM) estimators have equivalent asymptotic variances functions.
From these two remarks, the (LMM) estimator behaves better that the (KM) estimator since $h_K \to 0$ as $n \to +\infty$.
In subsection 2.3.2, we will confirm this important result by simulations.

Herebelow we just show how Theorem 3.3.1 is proved as a subsequent result of Theorems 2.3.2-2.3.3 which are fully proved in Section 2.4.

Proof of Theorem 3.3.1. By using the following decomposition

$$\begin{aligned}
\widehat{h}^x(y) - h^x(y) &= \frac{1}{1-\widehat{F}^x(y)}\left[(\widehat{f}^x(y) - f^x(y)) + \frac{f^x(y)}{1-F^x(y)}(\widehat{F}^x(y) - F^x(y))\right] \\
&\leqslant \frac{1}{1-\widehat{F}^x(y)}\left[(\widehat{f}^x(y) - f^x(y)) + \frac{\alpha}{\beta}(\widehat{F}^x(y) - F^x(y))\right] \\
&\leqslant \left[(\widehat{f}^x(y) - f^x(y)) + \frac{\alpha}{\beta}(\widehat{F}^x(y) - F^x(y))\right].
\end{aligned}$$

The proof of Theorem 3.3.1 can be deduced from Theorem 2.3.2, Theorem 2.3.3 and the following result

$$\exists \varepsilon > 0 \text{ such that } \sum_{n\in\mathbb{N}} \mathbb{P}(|1-\widehat{F}^x(y)| < \varepsilon) < \infty. \tag{11}$$

Theorem 2.3.2. Under assumptions (H1)-(H7), we obtain

$$\begin{aligned}\mathbb{E}\left[\widehat{f}^x(y) - f^x(y)\right]^2 &= B^2_{f,H}(x,y)h_H^4 + B^2_{f,K}(x,y)h_K^4 + \frac{V^f_{HK}(x,y)}{nh_H\phi_x(h_K)} \\ &\quad + o(h_H^4) + o(h_K^4) + o\left(\frac{1}{nh_H\phi_x(h_K)}\right)\end{aligned}$$

where $V^f_{HK}(x,y) = f^x(y)\left[\frac{\left(K^2(1) - \int_{-1}^1 (K^2(u))^{(1)}\chi_x(u)\mathrm{d}u\right)}{\left(K(1) - \int_{-1}^1 (K(u))^{(1)}\chi_x(u)\mathrm{d}u\right)^2}\right]\int (H^{(1)}(t))^2\mathrm{d}t.$

We set

$$\widehat{f}^x_N(y) = \frac{1}{n(n-1)h_H\mathbb{E}[W_{12}(x)]}\sum_{1\leq i\neq j\leq n} W_{ij}(x)H^{(1)}(h_H^{-1}(y - Y_j))$$

and

$$\widehat{f}_D(x) = \frac{1}{n(n-1)\mathbb{E}[W_{12}(x)]}\sum_{1\leq i\neq j\leq n} W_{ij}(x)$$

then

$$\widehat{f}^x(y) = \frac{\widehat{f}^x_N(y)}{\widehat{f}_D(x)}.$$

The proof of Theorem 2.3.2 can be deduced from the following intermediates results.

Lemma 2.3.1. Under the hypotheses of Theorem 2.3.2, we get

$$\mathbb{E}\left[\widehat{f}^x_N(y)\right] - f^x(y) = B_{f,H}(x,y)h_H^2 + B_{f,K}(x,y)h_K^2 + o(h_H^2) + o(h_K^2).$$

Lemma 2.3.2. Under the hypotheses of Theorem 2.3.2, we have

$$Var\left[\widehat{f}^x_N(y)\right] = \frac{V^f_{HK}(x,y)}{nh_H\phi_x(h_K)} + o\left(\frac{1}{nh_H\phi_x(h_k)}\right).$$

Lemma 2.3.3. Under the hypotheses of Theorem 2.3.2, we get

$$Cov(\widehat{f}^x_N(y), \widehat{f}_D(x)) = O\left(\frac{1}{n\phi_x(h_K)}\right).$$

Lemma 2.3.4. Under the hypotheses of Theorem 2.3.2, we have

$$Var\left[\widehat{f}_D(x)\right] = O\left(\frac{1}{n\phi_x(h_K)}\right).$$

Theorem 2.3.3. Under assumptions (H1)-(H7), we obtain

$$\begin{aligned}\mathbb{E}\left[\widehat{F}^x(y) - F^x(y)\right]^2 &= B^2_{F,H}(x,y)h_H^4 + B^2_{F,K}(x,y)h_K^4 + \frac{V^F_{HK}(x,y)}{n\phi_x(h_K)} \\ &\quad + o(h_H^4) + o(h_K^4) + o\left(\frac{1}{n\phi_x(h_K)}\right)\end{aligned}$$

where $V_{HK}^{F}(x,y) = F^x(y)(1-F^x(y))\left[\frac{\left(K^2(1)-\int_{-1}^{1}(K^2(u))^{(1)}\chi_x(u)\mathrm{d}u\right)}{\left(K(1)-\int_{-1}^{1}(K(u))^{(1)}\chi_x(u)\mathrm{d}u\right)^2}\right]$.

We note that

$$\widehat{F}^x(y) = \frac{\widehat{F}_N^x(y)}{\widehat{f}_D(x)}$$

where

$$\widehat{F}_N^x(y) = \frac{1}{n(n-1)\mathbb{E}[W_{12}(x)]}\sum_{1\le i\neq j\le n} W_{ij}(x)H(h_H^{-1}(y-Y_j)).$$

The following lemmas will be useful for proof of Theorem 2.3.3.

Lemma 2.3.5. Under the hypotheses of Theorem 2.3.3, we get

$$\mathbb{E}\left[\widehat{F}_N^x(y)\right] - F^x(y) = B_{F,H}(x,y)h_H^2 + B_{F,K}(x,y)h_K^2 + o(h_H^2) + o(h_K^2).$$

Lemma 2.3.6. Under the hypotheses of Theorem 2.3.3, we have

$$Var\left[\widehat{F}_N^x(y)\right] = \frac{V_{HK}^F(x,y)}{n\phi_x(h_K)} + o\left(\frac{1}{n\phi_x(h_k)}\right).$$

Lemma 2.3.7. Under the hypotheses of Theorem 2.3.3, we get

$$Cov(\widehat{F}_N^x(y), \widehat{f}_D^x) = O\left(\frac{1}{n\phi_x(h_K)}\right).$$

2.3.2 Simulation study on the finite samples

We have already justified, as mentioned in the comparison remark given after the statement of Theorem 3.3.1, how the (LMM) estimator given in Formula (6) should behave better that the (KM) estimator given in Formula (7). We are going to illustrate this by a simple simulation experience.

Let us fix a functional regression model,

$$Y_i = m(X_i) + \varepsilon$$

where the random variable ε is normally distributed as $\mathcal{N}(0,1)$ and

$$m(x) = 4\exp\left(\frac{1}{1+\int_0^{\pi}|x(t)|^2dt}\right).$$

The functional variable X is chosen as a real valued function with support $[0,\pi]$, we generate $n=100$ functional data (see Figure 2.1) by :

$$X_i(t) = \cos(W_i(t)), \text{for all } t\in[0,\pi] \text{ et } i=1,...,n$$

where the random variables W_i are i.i.d. and follow the normal distribution $\mathcal{N}(0,1)$. The curves are discretized on the same grid which is composed of 100 equidistant values in $[0,\pi]$.

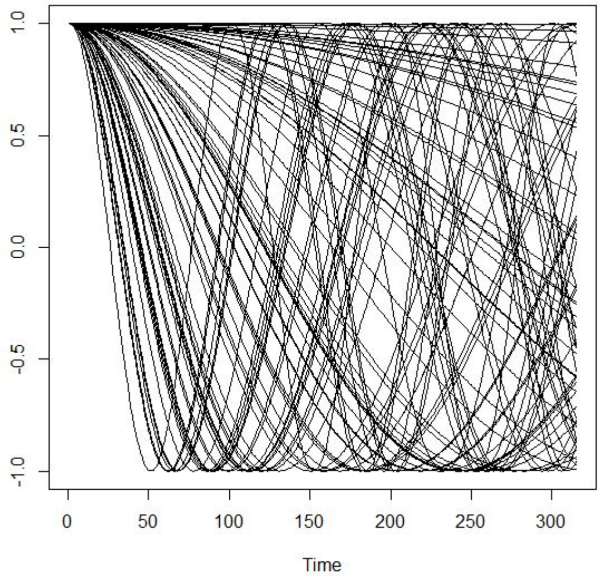

FIGURE 2.1 – Curves X_i

Based on this data, we generated the (LMM) and the (KM) statistics. First, we compare the two obtained graphs, each of both compared with the true conditional hazard function in Figure 2.2.

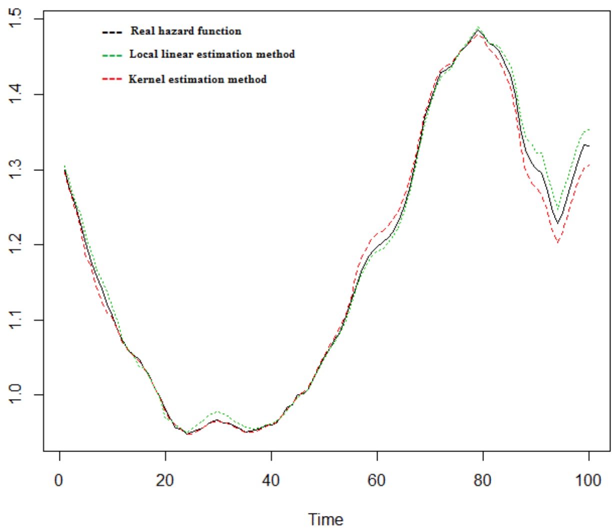

FIGURE 2.2 – Comparison between estimation methods

Next, we compare the performance of both estimators by means of the absolute error (AE) defined

$$AE = |\text{true value} - \text{estimated value}|. \tag{12}$$

Number of sample	AE (L.L.M)$\times 10^3$	AE (K.M)$\times 10^3$
10	11.9	15.1
20	1.6	1.9
40	0.7	1.3
60	6.2	17.4
80	0.6	4.8
100	19.1	25.4

TABLE 2.1 – Comparison of the AE's

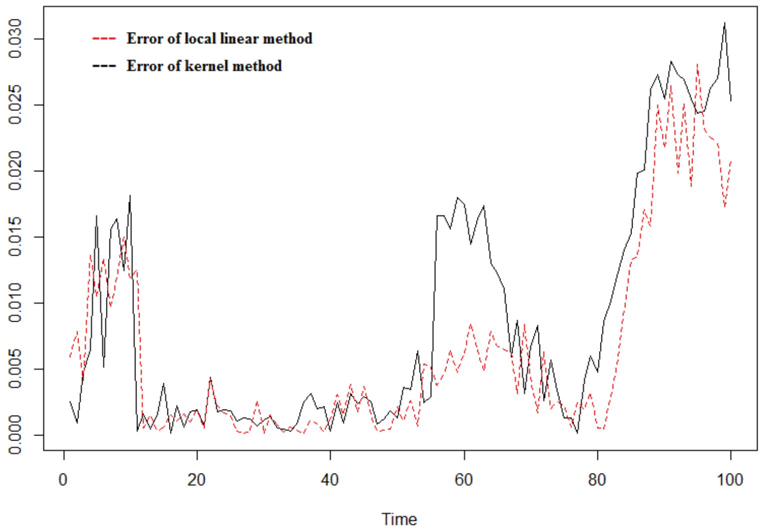

FIGURE 2.3 – The AE-errors of both methods

We report the results of the computations of the AE's in Table 2.1.

Form the graphs and the table, we may draw a number of useful comments.

(a) In Figure 2.1, it can be seen that the (LMM) estimator fits better the chf than the (KM) estimator.

(b) From Table 2.1, we see that the absolute error for the local linear estimation method, in most cases, is smaller than the absolute error in the kernel estimation method.

(c) The full graphs of the AE'S are illustrated in Figure 2.3.

As a general conclusion, we may say the (LMM) estimator performance is better than that of the (KM) with respect to the absolute error and the bias for n large so that h_K is small enough to impact the comparison.

2.4 Proofs

In the proofs below, we will need the following additional notation. C strictly positive generic constant. For all $(i,j) \in \{1,...,n\}^2$, we have

$$K_i = K(h_K^{-1}\delta(X_i,x)),\ W_{ij} = W_{ij}(x),$$

$$H_j = H(h_H^{-1}(y-Y_j)), H_j^{(1)} = H_j^{(1)}(h_H^{-1}(y-Y_j)).$$

The proofs are organized as follows.
Theorem 2.3.2 presents the mean square error of the conditional density estimator. To prove this theorem we need to prove lemmas 2.3.1-2.3.4. Similarly, to prove Theorem 2.3.3 which presents present the mean square error of the conditional distribution estimator, we need to prove lemmas 2.3.5-2.3.7.

Proof of Theorem 2.3.2. We begin by computing the bias and the variance of $\widehat{f}^x(y)$. We have

$$\mathbb{E}\left[\widehat{f}^x(y) - f^x(y)\right]^2 = \left[\mathbb{E}\left[\widehat{f}^x(y)\right] - f^x(y)\right]^2 + Var\left[\widehat{f}^x(y)\right]. \tag{13}$$

By simple calculations, we get

$$\begin{aligned}\widehat{f}^x(y) - f^x(y) &= \left(\widehat{f}_N^x(y) - f^x(y)\right) - \left(\widehat{f}_N^x(y) - \mathbb{E}[\widehat{f}_N^x(y)]\right)\left(\widehat{f}_D(x) - 1\right) \\ &\quad -\mathbb{E}[\widehat{f}_N^x(y)]\left(\widehat{f}_D(x) - 1\right) + \left(\widehat{f}_D(x) - 1\right)^2 \widehat{f}^x(y).\end{aligned}$$

From that fact that $\mathbb{E}[\widehat{f}_D(x)] = 1$, we deduce that :

$$\begin{aligned}\mathbb{E}\left[\widehat{f}^x(y)\right] - f^x(y) &= \left(\mathbb{E}[\widehat{f}_N^x(y)] - f^x(y)\right) - Cov\left(\widehat{f}_N^x(y), \widehat{f}_D(x)\right) \\ &\quad +\mathbb{E}\Big[\left(\widehat{f}_D(x) - \mathbb{E}[\widehat{f}_D(x)]\right)^2 \widehat{f}^x(y)\Big].\end{aligned}$$

Since the kernel $H^{(1)}$ is bounded, we can bound $\widehat{f}^x(y)$ by a constant $C > 0$, where $\widehat{f}^x(y) \leq C/h_H$. Hence

$$\begin{aligned}\mathbb{E}\left[\widehat{f}^x(y)\right] - f^x(y) &= \left(\mathbb{E}[\widehat{f}_N^x(y)] - f^x(y)\right) - Cov\left(\widehat{f}_N^x(y), \widehat{f}_D(x)\right) \\ &\quad +Var\left[\widehat{f}_D(x)\right] O(h_H^{-1}).\end{aligned}$$

Now, by Bosq and Lecoutre (1987), the variance term in (13) is

$$\begin{aligned}Var\left[\widehat{f}^x(y)\right] &= Var\left[\widehat{f}_N^x(y)\right] - 2\mathbb{E}[\widehat{f}_N^x(y)]Cov\left(\widehat{f}_N^x(y), \widehat{f}_D(x)\right) \\ &\quad + \left(\mathbb{E}[\widehat{f}_N^x(y)]\right)^2 Var\left(\widehat{f}_D(x)\right) o\left(\frac{1}{nh_H\phi_x(h_K)}\right). \blacksquare\end{aligned}$$

Proof of Lemma 2.3.1. We have

$$\mathbb{E}[\widehat{f}_N^x(y)] = \frac{1}{h_H \mathbb{E}[W_{12}]}\mathbb{E}\left[W_{12}\mathbb{E}[H_2^{(1)}|X_2]\right]. \tag{14}$$

By using a Taylor's expansion and under assumption $(H5)$, we get

$$\mathbb{E}[H_2^{(1)}|X_2] = f^{X_2}(y) + \frac{h_H^2}{2}\left(\int t^2 H^{(1)}(t)\mathrm{d}t\right)\frac{\partial^2 f^{X_2}(y)}{\partial y^2} + o(h_H^2).$$

The latter can be rewritten as

$$\mathbb{E}[H_2^{(1)}|X_2] = \psi_{0,1}(X_2, y) + \frac{h_H^2}{2}\left(\int t^2 H^{(1)}(t)dt\right)\psi_{2,1}(X_2, y) + o(h_H^2).$$

Thus, from (14), we obtain

$$\mathbb{E}\left[\widehat{f}_N^x(y)\right] = \frac{1}{E[W_{12}]}\left(\mathbb{E}\left[W_{12}\psi_{0,1}(X_2, y)\right] + \frac{h_H^2}{2}\left(\int t^2 H^{(1)}(t)\mathrm{d}t\right)\mathbb{E}\left[W_{12}\psi_{2,1}(X_2, y)\right] + o(h_H^2)\right).$$

Accordingly with to Ferraty et al. (2007), we may shot that for $l \in \{0, 2\}$,

$$\begin{aligned}\mathbb{E}[W_{12}\psi_{l,1}(X_2, y)] &= \psi_{l,1}(x, y)\mathbb{E}[W_{12}] + \mathbb{E}[W_{12}(\psi_{l,1}(X_2, y) - \psi_{l,1}(x, y))] \\ &= \psi_{l,1}(x, y)\mathbb{E}[W_{12}] + \mathbb{E}[W_{12}\mathbb{E}[\psi_{l,1}(X_2, y) - \psi_{l,1}(x, y)|\beta(X_2, x)]] \\ &= \psi_{l,1}(x, y)\mathbb{E}[W_{12}] + \mathbb{E}[W_{12}\Psi_{l,1}(\beta(X_2, x))].\end{aligned}$$

By observing that $\Psi_{l,1}(0) = 0$ and $\mathbb{E}\left[\beta(X_2, x)W_{12}\right] = 0$, we get

$$\mathbb{E}\left[W_{12}\psi_{l,1}(X_2, y)\right] = \psi_{l,1}(x, y)\mathbb{E}[W_{12}] + \frac{1}{2}\Psi_{l,1}^{(2)}(0)\mathbb{E}\left[\beta^2(X_2, x)W_{12}\right] + o(\mathbb{E}\left[\beta^2(X_2, x)W_{12}\right]).$$

So,

$$\begin{aligned}\mathbb{E}\left[\widehat{f}_N^x(y)\right] &= f^x(y) + \frac{h_H^2}{2}\frac{\partial^2 f^x(y)}{\partial y^2}\int t^2 H^{(1)}(t)\mathrm{d}t + o\left(h_H^2\frac{\mathbb{E}\left[\beta^2(X_2, x)W_{12}\right]}{\mathbb{E}[W_{12}]}\right) \\ &\quad + \Psi_{0,1}^{(1)}(0)\frac{\mathbb{E}\left[\beta^2(X_2, x)W_{12}\right]}{2\mathbb{E}[W_{12}]} + o\left(\frac{\mathbb{E}\left[\beta^2(X_2, x)W_{12}\right]}{\mathbb{E}[W_{12}]}\right).\end{aligned}$$

The two quantities $\mathbb{E}\left[\beta(x, X_2)^2 W_{12}\right]$ and $\mathbb{E}[W_{12}]$ are based on the asymptotic evaluation of $\mathbb{E}[K_1^a\beta_1^b]$ (see Rachdi et al. (2014) for more details). To do that, first we treat the case $b = 1$ and $a > 0$. For this case, we use the last part of (H3) and (H4), to get

$$h_K\mathbb{E}[K_1^a\beta_1] = o\left(\int_{B(x,h_K)}\beta^2(u, x)\mathrm{d}P_X(u)\right) = o(h_K^2\phi_x(h_K)).$$

So, we can see that,

$$\mathbb{E}[K_1^a\beta_1] = o(h_K\phi_x(h_K)). \tag{15}$$

On the other hand, for all $b > 1$, and after simplifications of the expressions, we have

$$\mathbb{E}[K_1^a\beta_1^b] = \mathbb{E}[K_1^a\delta^b(x, X)] + o(h_K^b\phi_x(h_K)).$$

Concerning the first term, we write

$$\begin{aligned} h_K^{-b}\mathbb{E}[K_1^a\delta^b] &= \int v^b K^a(v)\mathrm{d}P_X^{h_K^{-1}\delta(x,X)}(v) \\ &= \int_{-1}^1 \left[K^a(1) - \int_v^1 \left((u^b K^a(u))^{(1)}\right)\mathrm{d}u\right]\mathrm{d}P_X^{h_K^{-1}\delta(x,X)}(v) \\ &= \left(K(1)\phi_x(h_K) - \int_{-1}^1 (u^b K^a(u))^{(1)}\phi_x(uh_K,h_K)\mathrm{d}u\right) \\ &= \phi_x(h_K)\left(K(1) - \int_{-1}^1 (u^b K^a(u))^{(1)}\frac{\phi_x(uh_K,h_K)}{\phi_x(h_K)}\mathrm{d}u\right). \end{aligned}$$

Then, under assumptions $(H1)$, we get

$$\mathbb{E}[K_1^a\beta_1^b] = h_K^b\phi_x(h_K)\left(K(1) - \int_{-1}^1 (u^b K^a(u))^{(1)}\chi_x(u)\mathrm{d}u\right) + o(h_K^b\phi_x(h_K)). \tag{16}$$

So,

$$\frac{\mathbb{E}[\beta^2(X_2,x)W_{12}]}{\mathbb{E}[W_{12}]} = h_K^2\left(\frac{K(1) - \int_{-1}^1 (u^2K(u))^{(1)}\chi_x(u)\mathrm{d}u}{K(1) - \int_{-1}^1 (K^{(1)}(u)\chi_x(u)\mathrm{d}u}\right) + o(h_K^2).$$

Hence,

$$\begin{aligned} \mathbb{E}\left[\widehat{f}_N^x(y)\right] &= f^x(y) + \frac{h_H^2}{2}\frac{\partial^2 f^x(y)}{\partial y^2}\int t^2 H^{(1)}(t)\mathrm{d}t + o(h_H^2) \\ &\quad + \frac{h_K^2}{2}\Psi_{0,1}^{(2)}(0)\frac{\left(K(1) - \int_{-1}^1 (u^2K(u))^{(1)}\chi_x(u)\mathrm{d}u\right)}{\left(K(1) - \int_{-1}^1 K^{(1)}(u)\chi_x(u)\mathrm{d}u\right)} + o(h_K^2). \blacksquare \end{aligned}$$

Proof of Lemma 2.3.2. We have

$$Var\left(\widehat{f}_N^x(y)\right) = \frac{1}{(n(n-1)h_H(\mathbb{E}[W_{12}]))^2}Var\left(\sum_{1\le i\ne j\le n} W_{ij}H_j^{(1)}\right)$$

$$\begin{aligned} = \frac{1}{(n(n-1)h_H(\mathbb{E}[W_{12}]))^2}\Big[&n(n-1)\mathbb{E}[W_{12}^2(H_2^{(1)})^2] + n(n-1)\mathbb{E}[W_{12}W_{21}H_2^{(1)}H_1^{(1)}] \\ &+n(n-1)(n-2)\mathbb{E}[W_{12}W_{13}H_2^{(1)}H_3^{(1)}] + n(n-1)(n-2)\mathbb{E}[W_{12}W_{23}H_2^{(1)}H_3^{(1)}] \\ &+n(n-1)(n-2)\mathbb{E}[W_{12}W_{31}H_2^{(1)}H_1^{(1)}] + n(n-1)(n-2)\mathbb{E}[W_{12}W_{32}(H_2^{(1)})^2] \\ &-n(n-1)(4n-6)\mathbb{E}[W_{12}H_2^{(1)}]^2\Big]. \end{aligned} \tag{17}$$

We get, after some direct calculations

$$\begin{cases} \mathbb{E}[W_{12}^2H_2^{(1)}] = O(h_K^4h_H\phi_x^2(h_K)), \quad \mathbb{E}[W_{12}W_{21}H_2^{(1)}H_1^{(1)}] = O(h_K^4h_H^2\phi_x^2(h_K)), \\ \mathbb{E}[W_{12}W_{13}H_2^{(1)}H_3^{(1)}] = \mathbb{E}[W_{12}W_{31}H_2^{(1)}H_1^{(1)}] = \mathbb{E}[W_{12}W_{23}H_2^{(1)}H_3^{(1)}] = O(h_K^4h_H^2\phi_x^3(h_K)), \\ \mathbb{E}[W_{12}W_{32}(H_2^{(1)})^2] = \mathbb{E}^2[\beta_1^2K_1]\mathbb{E}[K_1^2(H_1^{(1)})^2] + o(h_K^4h_H\phi_x^3(h_K)). \end{cases}$$

Clearly, the latter term in the last cases is the leading one, and can be evaluated in (17) by using

$$\frac{(n-2)}{n(n-1)(h_H\mathbb{E}[W_{12}])^2}\mathbb{E}^2[\beta_1^2K_1]\mathbb{E}[K_1^2(H_1^{(1)})^2]$$

So, after the same steps as in the previous Lemma, it suffices to write

$$Var\left(\widehat{f}_N^x(y)\right) = \frac{\mathbb{E}[K_1^2(H_1^{(1)})^2]}{n(h_H\mathbb{E}[K_1])^2} + o\left(\frac{1}{nh_H\phi_x(h_K)}\right). \tag{18}$$

Thus, by the change of variables $t = h_H^{-1}(y-z)$, we get

$$\mathbb{E}[K_1^2(H_1^{(1)})^2] = \mathbb{E}[K_1^2\mathbb{E}((H_1^{(1)})^2|X_1)]$$

and

$$\mathbb{E}((H_1^{(1)})^2|X_1) = h_H\int(H^{(1)})^2(t)f^{X_1}(y-h_Ht)\mathrm{d}t.$$

Then, by Taylor's expansion of order 1 of $f^{X_1}(\cdot)$ we obtain

$$f^{X_1}(y-h_Ht) = f^{X_1}(y) + O(h_H) = f^{X_1}(y) + o(1).$$

Now, it follows from (18) that :

$$\mathbb{E}[K_1^2(H_1^{(1)})^2] = h_H\int(H^{(1)})^2(t)\mathrm{d}t\mathbb{E}\Big[K_1^2f^X(y)\Big] + o(h_H\mathbb{E}[K_1^2]).$$

Again, by the same steps in proof of Lemma 2.3.1, we get

$$\mathbb{E}\Big[K_1^2f^{X_1}(y)\Big] = f^x(y)\mathbb{E}[K_1^2] + o(\mathbb{E}[K_1^2])$$

which implies :

$$\mathbb{E}[K_1^2(H_1^{(1)})^2] = h_Hf^x(y)\mathbb{E}[K_1^2]\int(H^{(1)})^2(t)\mathrm{d}t + o(h_H\mathbb{E}[K_1^2]). \tag{19}$$

Consequently, we obtain from (16), (18) and (19), that

$$Var\left(\widehat{f}_N^x(y)\right) = \frac{f^x(y)}{nh_H\phi_x(h_K)}\left(\int H^{(1)}(t)^2\mathrm{d}t\right)\left[\frac{\left(K^2(1)-\int_{-1}^1(K^2(u))^{(1)}\chi_x(u)\mathrm{d}u\right)}{\left(K(1)-\int_{-1}^1(K(u))^{(1)}\chi_x(u)\mathrm{d}u\right)^2}\right]$$

$$+o\left(\frac{1}{nh_H\phi_x(h_K)}\right).\blacksquare$$

Proof of Lemma 2.3.3. By simple computations, we have

$$\begin{aligned}
Cov\left(\widehat{f}_N^x(y),\widehat{f}_D(x)\right) &= \frac{1}{(n(n-1)h_H(\mathbb{E}[W_{12}]))^2}Cov\Big(\sum_{1\leq i\neq j\leq n}W_{ij}H_j^{(1)},\sum_{1\leq i'\neq j'\leq n}W_{i'j'}\Big)\\
&= \frac{1}{(n(n-1)h_H(\mathbb{E}[W_{12}]))^2}\Big[n(n-1)\mathbb{E}[W_{12}^2H_1^{(1)}] + n(n-1)\mathbb{E}[W_{12}W_{21}H_2^{(1)}]\\
&\quad +n(n-1)(n-2)\mathbb{E}[W_{12}W_{13}H_2^{(1)}] + n(n-1)(n-2)\mathbb{E}[W_{12}W_{23}H_2^{(1)}]\\
&\quad +n(n-1)(n-2)\mathbb{E}[W_{12}W_{31}H_2^{(1)}] + n(n-1)(n-2)\mathbb{E}[W_{12}W_{32}H_2^{(1)}]\\
&\quad -n(n-1)(4n-6)(\mathbb{E}[W_{12}H_2^{(1)}]\mathbb{E}[W_{12}]\Big].
\end{aligned}$$

By direct manipulations, we get

$$\begin{cases} \mathbb{E}[W_{12}^2 H_2^{(1)}] = \mathbb{E}[W_{12}W_{21}H_2^{(1)}] = O(h_K^4 h_H \phi_x^2(h_K)), \\ \mathbb{E}[W_{12}W_{13}H_2^{(1)}] = \mathbb{E}[W_{12}W_{31}H_2^{(1)}] = O(h_K^4 h_H \phi_x^3(h_K)), \\ \mathbb{E}[W_{12}W_{23}H_2^{(1)}] = \mathbb{E}[W_{12}W_{32}H_2^{(1)}] = O(h_K^4 h_H \phi_x^3(h_K)). \end{cases}$$

Since $\mathbb{E}[W_{12}] = O(h_K^2 \phi_x^2(h_K))$, we obtain

$$Cov\left(\widehat{f}_N^x(y), \widehat{f}_D(x)\right) = O\left(\frac{1}{n\phi_x(h_K)}\right). \blacksquare$$

Proof of Lemma 2.3.4. The demonstration of this result follows the lines of the proof of the previous lemma, step by step, by replacing $H^{(1)}$ by 1. Thus,

$$\begin{aligned} Var(\widehat{f}_D^x) &= \frac{1}{(n(n-1)\mathbb{E}[W_{12}])^2} Var\left(\sum_{1\le i\neq j\le n} W_{ij}\right) \\ &= \frac{1}{(n(n-1)\mathbb{E}[W_{12}])^2}\Big(n(n-1)\mathbb{E}[W_{12}^2] + n(n-1)\mathbb{E}[W_{12}W_{21}] \\ &\quad + n(n-1)(n-2)\mathbb{E}[W_{12}W_{13}] + n(n-1)(n-2)\mathbb{E}[W_{12}W_{23}] \\ &\quad + n(n-1)(n-2)\mathbb{E}[W_{12}W_{31}] + n(n-1)(n-2)\mathbb{E}[W_{12}W_{32}] \\ &\quad - n(n-1)(4n-6)(\mathbb{E}[W_{12}])^2\Big). \end{aligned}$$

Still by straightforward manipulations, we get

$$\begin{cases} \mathbb{E}[W_{12}^2] = \mathbb{E}[W_{12}W_{21}] = O(h_K^4 \phi_x^2(h_K)), \\ \mathbb{E}[W_{12}W_{13}] = \mathbb{E}[W_{12}W_{31}] = O(h_K^4 \phi_x^3(h_K)), \\ \mathbb{E}[W_{12}W_{23}] = \mathbb{E}[W_{12}W_{32}] = O(h_K^4 \phi_x^3(h_K)). \end{cases}$$

So, we have

$$Var\left(\widehat{f}_D^x\right) = O\left(\frac{1}{n\phi_x(h_K)}\right). \blacksquare$$

Proof of Theorem 2.3.3. The proof of this theorem is based on the same techniques as in the proof of Theorem 2.3.2, where

$$\mathbb{E}\left[\widehat{F}^x(y) - F^x(y)\right]^2 = \left[\mathbb{E}\left[\widehat{F}^x(y)\right] - F^x(y)\right]^2 + Var\left[\widehat{F}^x(y)\right]$$

and to simplify the bias and the variance of the second term in the right equality, we use the results of Ferraty et al. (2007), to obtain

$$\mathbb{E}\left[\widehat{F}^x(y)\right] - F^x(y) = \left(\mathbb{E}[\widehat{F}_N^x(y)] - F^x(y)\right) + \frac{\mathbb{E}[\widehat{F}_N^x(y)(\widehat{f}_D^x - \mathbb{E}[\widehat{f}_D^x])]}{(\mathbb{E}[\widehat{f}_D^x])^2} + \frac{\mathbb{E}[\widehat{F}^x(y)(\widehat{f}_D^x - \mathbb{E}[\widehat{f}_D^x])^2]}{(\mathbb{E}[\widehat{f}_D^x])^2}$$

and

$$\begin{aligned} Var\left[\widehat{F}^x(y)\right] &= Var\left(\widehat{F}_N^x(y)\right) - 4\left(\mathbb{E}[\widehat{F}_N^x(y)]\right) Cov\left(\widehat{F}_N^x(y), \widehat{f}_D(x)\right) \\ &\quad + 3\left(\mathbb{E}[\widehat{F}_N^x(y)]\right)^2 Var\left(\widehat{f}_D(x)\right) + o\left(\frac{1}{n\phi(h_K)}\right). \blacksquare \end{aligned}$$

Proof of Lemma 2.3.5. Concerning the quantities $\mathbb{E}[\widehat{F}_N^x(y)]$, we use an integration by part to arrive at

$$\mathbb{E}[\widehat{F}_N^x(y)] = \frac{1}{\mathbb{E}[W_{12}]}\mathbb{E}[W_{12}\mathbb{E}[H_2|X_2]] \text{ with } \mathbb{E}[H_2|X_2] = \int H_2^{(1)}(t)F^{X_2}(y-h_H t)\mathrm{d}t.$$

Then, the same steps used in studying $\mathbb{E}[\widehat{f}_N^x(y)]$ can be reused to prove that

$$\begin{aligned}\mathbb{E}\left[\widehat{F}_N^x(y)\right] &= F^x(y) + \frac{h_H^2}{2}\frac{\partial^2 F^x(y)}{\partial y^2}\int t^2 H_2^{(1)}(t)\mathrm{d}t + o(h_H^2)\\ &\quad + \frac{h_K^2}{2}\Psi_{0,0}^{(2)}(0)\frac{\left(K(1)-\int_{-1}^1 (u^2K(u))^{(1)}\chi_x(u)\mathrm{d}u\right)}{\left(K(1)-\int_{-1}^1 K^{(1)}(u)\chi_x(u)\mathrm{d}u\right)} + o(h_K^2).\blacksquare\end{aligned}$$

Proof of Lemma 2.3.6. It clear that

$$\begin{aligned} Var[\widehat{F}_N^x(y)] &= \frac{1}{(n(n-1)h_H(\mathbb{E}[W_{12}]))^2}\Big[n(n-1)\mathbb{E}[W_{12}^2(H_2)^2] + n(n-1)\mathbb{E}[W_{12}W_{21}H_2H_1]\\ &+n(n-1)(n-2)\mathbb{E}[W_{12}W_{13}H_2H_3] + n(n-1)(n-2)\mathbb{E}[W_{12}W_{23}H_2H_3]\\ &+n(n-1)(n-2)\mathbb{E}[W_{12}W_{31}H_2H_1] + n(n-1)(n-2)\mathbb{E}[W_{12}W_{32}(H_2)^2]\\ &-n(n-1)(4n-6)\mathbb{E}[W_{12}H_2]^2\Big]. \end{aligned} \tag{20}$$

For these terms, we use the same steps used in Lemma 2.3.1 and we get

$$\begin{cases} \mathbb{E}[W_{12}^2H_2^2] = O(h_K^4\phi_x^2(h_K)), \mathbb{E}[W_{12}W_{21}H_1H_2] = O(h_K^4\phi_x^2(h_K)),\\ \mathbb{E}[W_{12}W_{13}H_2H_3] = (F^x(y))^2\mathbb{E}[\beta_1^4K_1^2]\mathbb{E}^2[K_1] + o(h_K^4\phi_x^3(h_K)),\\ \mathbb{E}[W_{12}W_{23}H_2H_3] = (F^x(y))^2\mathbb{E}[\beta_1^2K_1]\mathbb{E}[\beta_1^2K_1^2]\mathbb{E}[K_1] + o(h_K^4\phi_x^3(h_K)),\\ \mathbb{E}[W_{12}W_{31}H_2H_1] = (F^x(y))^2\mathbb{E}[\beta_1^2K_1]\mathbb{E}[\beta_1^2K_1^2]\mathbb{E}[K_1] + o(h_K^4\phi_x^3(h_K)),\\ \mathbb{E}[W_{12}W_{32}H_2^2] = F^x(y)\mathbb{E}^2[\beta_1^2K_1]\mathbb{E}[K_1^2] + o(h_K^4\phi_x^3(h_K)),\\ \mathbb{E}[W_{12}H_1] = O(h_K^2\phi_x^2(h_K)). \end{cases} \tag{21}$$

Hence, it follows from (20) and (21) :

$$Var[\widehat{F}_N^x(y)] = \frac{F^x(y)(1-F^x(y))}{\mathbb{E}[K_1^2]}(\mathbb{E}[K_1])^2 + o\left(\frac{1}{n\phi_x(h_K)}\right).$$

Finally,

$$Var[\widehat{F}_N^x(y)] = \frac{F^x(y)(1-F^x(y))}{n\phi_x(h_K)}\left[\frac{\left(K^2(1)-\int_{-1}^1 (K^2(u))^{(1)}\chi_x(u)\mathrm{d}u\right)}{\left(K(1)-\int_{-1}^1 (K(u))^{(1)}\chi_x(u)\mathrm{d}u\right)^2}\right] + o\left(\frac{1}{n\phi_x(h_K)}\right).\blacksquare$$

Proof of Lemma 2.3.7. Both assertions of this lemma are direct consequences of Lemma 2.3.3.■

2.5 Conclusion and perspectives

We presented in this paper the leading term of the mean square error of the estimator of the conditional hazard by the local linear approach. In terms of mean squared error our estimator performs competitively in comparison to existing estimators for the conditional hazard function. Our theoretical and practical studies confirm the superiority of the linear local approach over the classical kernel approach. From a theoretical point of view, there are interesting prospects. It would be very important in the next future to study the asymptotic normality of our estimator to make statistical tests. The kNN method is an alternative smoothing approach that offers an adaptive estimator. The very important feature of this method is that it allows the construction of a neighbourhood adapted to the local structure of the data. So, It would be also of interest to study the asymptotic properties of the kNN estimator of the conditional hazard function. This will be considered in future works.

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CHAPITRE 3

Nonparametric local linear estimator of the conditional hazard function in the functional single-index model

Ce chapitre fait l'objet d'une publication accepté dans le Journal of Mathematics and Computation.

Nonparametric local linear estimator of the conditional hazard function in the functional single-index model

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Abstract. In this paper, we present a nonparametric estimate of the conditional hazard function by the local linear approach of a scalar response variable given a Hilbertian random variable via the single-index structure. We establish the pointwise and the uniform consistencies (with rates) of the constructed estimator under some general conditions.

Keywords : nonparametric local linear estimation, Conditional hazard function, Conditional single-index, Functional variable.

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3.1 Introduction

Thanks to technological progress, measurement techniques one of the major topics which received growing interest in the field of statistical analysis is to focus on the context of studying the high/infinite dimension data, due to its effect on modern statistics and enormous scope of applications, such as biology, chemometrics, econometrics, medicine, etc. This latter has been popularized in the book of Ramsay and Silverman (2005) for parametric models and in Ferraty and Vieu (2006) for nonparametric conditional models (regression, cumulative distribution, density, etc).

The nonparametric estimation of conditional hazard function is important to a large degree in statistics. For example, we find their use in survival analysis and in seismology, medicine, reliability, etc. Concerning the first dealing with this function, when the regressors are functional, we refer to the work of Ferraty et al. (2008). They proved the almost complete convergence (with rate) of the kernel estimation. Quintela-Del-Rio (2008) has studied the asymptotic normality of the same estimator in α-mixing data case. On the other hand, the local linear regression model has been considered by others authors, Baillo and Grané (2009) proposed a local linear estimator of the regression operator when the explanatory variable takes values in a Hilbert space. We can cite also the work of Barrientos-Marin et al. (2010), they introduce the functional locally modeled regression method which is a direct extension of the functional nonparametric regression model. Demongeot et al. (2013) used the same approach to estimate the conditional density. Recently, Massim and Mechab (2016) established the almost complete convergence of conditional hazard function in the same set of estimation.

The single-index model is one of the most important methods for reducing the dimensionality,

where we find their application in econometrics, which its a tradeoff between nonparametric and parametric models. In nonparametric functional statistics, this model was introduced by Ferraty et al. (2003) of the kernel regression estimation. Attaoui et al. (2011) specified the pointwise and the uniform almost complete convergence (with the rate) for the conditional density of this model. After that, this method has been used to estimate the conditional hazard function in the quasi-associated data see Tabti and Ait Saïdi (2018).
The aim of this paper is to study the local linear kernel estimator of the conditional hazard function in the single functional index structure when the data are independent and identically distributed. We prove the pointwise almost complete convergence (a.co.) and the uniform almost complete convergence with rate of the constructed estimator.
The paper is organized as follows : in section 2, we present our model and estimator. In section 3, we introduce our assumptions and give the main results. Finally, in Section 4, we give The proofs of our results.

3.2 Single functional index and estimator

Consider the sample $(X_i, Y_i)_{1\leq i\leq n}$ of n independent pairs identically distributed as a couple of random variables (X, Y), taking its values in $\mathcal{H}\times\mathbb{R}$, where $\mathcal{H}$ is a separable real Hilbert space with the norm $||\cdot||$ generated by an inner product $<\cdot,\cdot>$. We consider the semi metric d_θ, associated to the single index $\theta\in\mathcal{H}$ defined by $\forall x_1, x_2\in\mathcal{H} : d_\theta(x_1, x_2) := | < x_1 - x_2, \theta > |$. In our context, we assume that the conditional hazard function of Y given X has a single-index structure θ in $\mathcal{H}$, where this latter denoted by $h^x_\theta(\cdot)$ and is given by

$$\forall y\in\mathbb{R},\quad h^x_\theta(y) := h(y| < x, \theta >).$$

Distinctly, the identifiability of the model is assured such that for all $x\in\mathcal{H}$, we get,

$$h_1(y| <\cdot, \theta_1 >) = h_2(y| <\cdot, \theta_2 >) \Rightarrow h_1\equiv h_2 \text{ and } \theta_1 = \theta_2.$$

The expression of the conditional hazard function is defined as follows by

$$h^x_\theta(y) = \frac{f^x_\theta(y)}{1 - F^x_\theta(y)},\quad F^x_\theta(y) < 1,\ \forall y\in\mathbb{R}.$$

In this paper, we define the local linear estimator $\widehat{h}^x_\theta(y)$ of $h^x_\theta(y)$ by

$$\widehat{h}^x_\theta(y) = \frac{\widehat{f}^x_\theta(y)}{1 - \widehat{F}^x_\theta(y)}$$

where, $\forall y\in\mathbb{R}$

$$\widehat{F}^x_\theta(y) = \frac{\sum\limits_{1\leq i,j\leq n} W_{\theta,ij}(x) H({h_H}^{-1}(y - Y_j))}{\sum\limits_{1\leq i,j\leq n} W_{\theta,ij}(x)},\quad \widehat{f}^x_\theta(y) = \frac{\sum\limits_{1\leq i,j\leq n} W_{\theta,ij}(x) H'({h_H}^{-1}(y - Y_j))}{h_H \sum\limits_{1\leq i,j\leq n} W_{\theta,ij}(x)}$$

with

$$W_{\theta,ij}(x) = \beta_\theta(X_i, x)\Big(\beta_\theta(X_i, x) - \beta_\theta(X_j, x)\Big) K(h_K^{-1} d_\theta(x, X_i)) K(h_K^{-1} d_\theta(x, X_j))$$

and $\beta_\theta(X_i, x) =< x - X_i, \theta >$ is a known bi-functional operator from $\mathcal{H}^2$ into $\mathbb{R}$, such that $\forall x'\in\mathcal{H}$, $\forall\theta\in\mathcal{H}$, $< x' - x', \theta >= 0$, with the function K is a kernel, H is a distribution function (respectively, H' is the derivative of H) and $h_K = h_{K,n}$(respectively, $h_H = h_{H,n}$) is a sequence of positive real numbers.

3.3 Assumptions and main results

In the following of this paper, we will denote by C, C', C'' and $C_{\theta,x}$ some strictly positive constants and by $K_{\theta,i}(x) := K(h_K^{-1}d_\theta(x, X_i)),\ \forall x \in \mathcal{H}, i = 1, ..., n$ and $H_j(y) := H(h_H{}^{-1}(y - Y_j)), \forall y \in \mathbb{R}, j = 1, ..., n$. Moreover, we denote x a fixed point in $\mathcal{H}$, $\mathcal{N}_x$ is a fixed neighborhood of x and $S_\mathbb{R}$ is a fixed compact of $\mathbb{R}$.

3.3.1 Pointwise almost complete convergence

In order to establish the almost complete convergence of our estimator, the following assumptions will be considered.

(H1) $\mathbb{P}(| < X - x, \theta > | < h_K) =: \phi_{\theta,x}(h_K) > 0.$

(H2) The Operators F_θ^x and f_θ^x satisfy the Hölder condition :

- (i) $\forall (x, y) \in \mathcal{H} \times S_\mathbb{R},\ \forall \theta \in \Theta, \exists 0 < \tau < 1,\ F_\theta^x(y) \leq 1 - \tau,\ \exists b_1, b_2 > 0,\ \forall (y, y') \in S_\mathbb{R}^2, \forall (x, x') \in \mathcal{N}_x \times \mathcal{N}_x,\ \forall \theta \in \Theta,$

$$|F_\theta^x(y) - F_\theta^{x'}(y')| \leq C_{\theta,x}(||x - x'||^{b_1} + |y - y'|^{b_2}).$$

- (ii) $\forall (x, y) \in \mathcal{H} \times S_\mathbb{R},\ \forall \theta \in \Theta, \exists \alpha < \infty,\ f_\theta^x(y) \leq \alpha,\ \exists b_1, b_2 > 0,\ \forall (y, y') \in S_\mathbb{R}^2, \forall (x, x') \in \mathcal{N}_x \times \mathcal{N}_x,\ \forall \theta \in \Theta,$

$$|f_\theta^x(y) - f_\theta^{x'}(y')| \leq C_{\theta,x}(||x - x'||^{b_1} + |y - y'|^{b_2}).$$

(H3) The bi-functional function $\beta_\theta(\cdot\,,\cdot)$ satisfies :

$$\forall x' \in \mathcal{H},\ C d_\theta(x', x) \leq |\beta_\theta(x, x')| \leq C' d_\theta(x', x).$$

(H4)
- (i) The kernel $K_{\theta,i}(\cdot)$ is a positive, differentiable function and supported within $(-1, 1)$.
- (ii) H is a Lipschitzian and differentiable function which has a bounded first derivative, such that :

$$\int |t|^{b_2} H'(t)\mathrm{d}t < \infty \text{ and } \int H'^2(t)\mathrm{d}t < \infty.$$

(H5) The bandwidth h_K satisfies :

- (i) $\exists \eta_0 \in \mathbb{N},\ \forall \eta > \eta_0,\quad -\frac{1}{\phi_{\theta,x}(h_K)} \int_{-1}^{1} \phi_{\theta,x}(t h_K, h_K) \frac{\mathrm{d}}{\mathrm{d}t}(t^2 K(t))\mathrm{d}t > C'' > 0$
  and $\quad h_K \int_{B(x,h_K)} \beta_\theta(u, x)\mathrm{d}P(u) = o\Big(\int_{B(x,h_K)} \beta_\theta^2(u, x)\mathrm{d}P(u)\Big)$
  where $B(x, h) = \{z \in \mathcal{H}|\ d_\theta(z, x) \leq h\}$ denotes the closed ball centered at x and of radius h, $\mathrm{d}P(u)$ is the probability measure of X.
- (ii) $\lim\limits_{n\to\infty} \frac{\ln n}{n\phi_{\theta,x}(h_K)} = 0.$

(H6) For some $\lambda > 0$ the bandwidth h_H satisfies

$$\lim_{n\to\infty} n^\lambda h_H = \infty, \text{ and } \lim_{n\to\infty} \frac{\ln n}{n h_H \phi_{\theta,x}(h_K)} = 0.$$

Comments on assumptions : Assumptions (H1) and (H3) are the same used in Ferraty et al. (2003). Assumption (H2) is a regularity condition where is needed to evaluate the bias term of our asymptotic results. Assumptions (H4)- (H6) are technical conditions and are also similar to those in Barrientos-Marin et al. (2010).

Theorem 3.3.1. Under assumptions (H1)-(H6), we have

$$\sup_{y\in S_{\mathbb{R}}}|\widehat{h}^x_\theta(y)-h^x_\theta(y)| = O(h_K^{b_1}+h_H^{b_2}) + O_{\text{a.co.}}\left(\sqrt{\frac{\ln n}{nh_H\phi_{\theta,x}(h_K)}}\right).$$

Proof of Theorem 3.3.1. The proof is focused on the decomposition :

$$\widehat{h}^x_\theta(y)-h^x_\theta(y) = \frac{1}{1-\widehat{F}^x_\theta(y)}\Big(\widehat{f}^x_\theta(y)-f^x_\theta(y)\Big) + \frac{h^x_\theta(y)}{1-\widehat{F}^x_\theta(y)}\Big(\widehat{F}^x_\theta(y)-F^x_\theta(y)\Big). \tag{1}$$

From this decomposition, the result of the Theorem 3.3.1 is a direct consequence of the Lemmas 3.3.1-3.3.3 and Corollary 4.3.1, for which the proofs are given in the Appendix.
We write for $p=0,1$,

$$\widehat{F}^{x(p)}_\theta(y)-F^{x(p)}_\theta(y) =$$

$$\frac{1}{\widehat{g}^x_{\theta,D}}\Big\{\Big(\widehat{F}^{x(p)}_{\theta,N}(y)-\mathbb{E}[\widehat{F}^{x(p)}_{\theta,N}(y)]\Big)-\Big(F^{x(p)}_\theta(y)-\mathbb{E}[\widehat{F}^{x(p)}_{\theta,N}(y)]\Big)\Big\}+\frac{F^{x(p)}_\theta(y)}{\widehat{g}^x_{\theta,D}}(1-\widehat{g}^x_{\theta,D})$$

where

$$\widehat{F}^{x(p)}_{\theta,N}(y) = \frac{1}{n(n-1)h_H^{(p)}\mathbb{E}[W_{\theta,12}(x)]}\sum_{1\le i\ne j\le n} W_{\theta,ij}(x)H^{(p)}(h_H^{-1}(y-Y_j))$$

and

$$\widehat{g}^x_{\theta,D} = \frac{1}{n(n-1)\mathbb{E}[W_{\theta,12}(x)]}\sum_{1\le i\ne j\le n} W_{\theta,ij}(x).$$

Lemma 3.3.1. Under assumptions (H1),(H2) and (H4), we obtain :

$$\sup_{y\in S_{\mathbb{R}}}|F^x_\theta(y)-\mathbb{E}[\widehat{F}^x_{\theta,N}(y)]| = O(h_K^{b_1})+O(h_H^{b_2})$$

and

$$\sup_{y\in S_{\mathbb{R}}}|f^x_\theta(y)-\mathbb{E}[\widehat{f}^x_{\theta,N}(y)]| = O(h_K^{b_1})+O(h_H^{b_2}).$$

Lemma 3.3.2. Under assumptions (H1)-(H6), we get

$$\sup_{y\in S_{\mathbb{R}}}|\widehat{F}^x_{\theta,N}(y)-\mathbb{E}[\widehat{F}^x_{\theta,N}(y)]| = O_{\text{a.co.}}\left(\sqrt{\frac{\ln n}{n\phi_{\theta,x}(h_K)}}\right)$$

and

$$\sup_{y\in S_{\mathbb{R}}}|\widehat{f}^x_{\theta,N}(y)-\mathbb{E}[\widehat{f}^x_{\theta,N}(y)]| = O_{\text{a.co.}}\left(\sqrt{\frac{\ln n}{nh_H\phi_{\theta,x}(h_K)}}\right).$$

Lemma 3.3.3. (cf. Barrientos-Marin et al. (2010)) Under assumptions (H1), (H3), (H4)(i) and (H5), we obtain

$$1-\widehat{g}^x_{\theta,D} = O_{\text{a.co.}}\left(\sqrt{\frac{\ln n}{n\phi_{\theta,x}(h_K)}}\right) \text{ and } \exists\varepsilon>0, \sum_{i=1}^{\infty}\mathbb{P}(\widehat{g}^x_{\theta,D}<\varepsilon)<\infty.$$

Corollary 3.3.1. Under the conditions of Theorem 3.3.1, we get

$$\exists\mu>0,\ \sum_{n=1}^{\infty}\mathbb{P}\Big(\inf_{y\in S_{\mathbb{R}}}|1-\widehat{F}^x_\theta(y)|<\mu\Big)<\infty.$$

3.3.2 Uniform almost complete convergence

While the functional index is unknown, the study of the uniform consistency is an indispensable tool for studying the asymptotic properties of the estimator (cf. Attaoui et al. (2011)), so, we need some additional tools and topological conditions for our estimator on some subset $S_{\mathcal{H}}$ and $\Theta_{\mathcal{H}}$. We consider that

$$S_{\mathcal{H}} \subset \bigcup_{k=1}^{q_n^{S_{\mathcal{H}}}} B(x_k, v_n) \text{ and } \Theta_{\mathcal{H}} \subset \bigcup_{k'=1}^{q_n^{\Theta_{\mathcal{H}}}} B(t_{k'}, v_n)$$

and we put, $k(x) = \arg \min_{k\in\{1,\dots,q_n^{S_{\mathcal{H}}}\}} ||x - x_k||$, $k'(\theta) = \arg \min_{k'\in\{1,\dots,q_n^{\Theta_{\mathcal{H}}}\}} ||\theta - t_{k'}||$ with x_k (resp, $t_{k'}$)$\in \mathcal{H}$ and $v_n, q_n^{S_{\mathcal{H}}}, q_n^{\Theta_{\mathcal{H}}}$ are sequences of positive real numbers which tend to infinity as n goes to infinity.

For uniform almost complete convergence, we need the following assumptions :

(U1) The assumptions (H1) holds and there exists a differentiable function $\phi(\cdot)$ such that $\forall x \in S_{\mathcal{H}}$ and $\forall \theta \in \Theta_{\mathcal{H}}$

$$0 < C\phi(h_K) \leq \phi_{\theta,x}(h_K) \leq C\phi^{'}(h_K) < \infty \quad \text{and } \exists \eta_0 > 0,\ \forall \eta < \eta_0, \phi'(\eta) < 0,$$

where ϕ' is the first derivative function of ϕ and $\phi(0) = 0$.

(U2) The Operators F_θ^x and f_θ^x satisfy the Hölder condition :

(i) $\forall (x, y) \in \mathcal{H} \times S_{\mathbb{R}}, \forall \theta \in \Theta_{\mathcal{H}}, \exists 0 < \tau < 1,\ F_\theta^x(y) \leq 1 - \tau,\ \exists b_1, b_2 > 0,\ \forall (y, y') \in S_{\mathbb{R}}^2, \forall (x, x') \in \mathcal{N}_x \times \mathcal{N}_x$,

$$|F_\theta^x(y) - F_\theta^{x'}(y')| \leq C(||x - x'||^{b_1} + |y - y'|^{b_2}).$$

(ii) $\forall (x, y) \in \mathcal{H} \times S_{\mathbb{R}}, \forall \theta \in \Theta_{\mathcal{H}}, \exists \alpha < \infty,\ f_\theta^x(y) \leq \alpha$,

$$|f_\theta^x(y) - f_\theta^{x'}(y')| \leq C(||x - x'||^{b_1} + |y - y'|^{b_2}).$$

(U3) The function $\beta_\theta(\cdot,\cdot)$ satisfies (H3) and the following Lipschitz's condition :

$$\forall (x_1, x_2) \in \mathcal{H} \times \mathcal{H}, \forall \theta \in \Theta_{\mathcal{H}}, \forall x' \in \mathcal{H},\ |\beta_\theta(x_1, x') - \beta_\theta(x_2, x')| \leq C d_\theta(x_1, x_2).$$

(U4) The kernel $K_{\theta,i}(\cdot)$ is a Lipchitzian function and satisfies (H4)(i).

(U5) For $v_n = O\left(\frac{\ln n}{n}\right)$ the sequences $q_n^{S_{\mathcal{H}}}$ and $q_n^{\Theta_{\mathcal{H}}}$ satisfy :

$$\frac{(\ln n)^2}{n\phi(h_K)} < \ln q_n^{S_{\mathcal{H}}} + \ln q_n^{\Theta_{\mathcal{H}}} < \frac{n\phi(h_K)}{\ln n},$$

$$\frac{(\ln n)^2}{n h_H \phi(h_K)} < \ln q_n^{S_{\mathcal{H}}} + \ln q_n^{\Theta_{\mathcal{H}}} < \frac{n h_H \phi(h_K)}{\ln n},$$

and for some $\lambda > 0$:

$$\lim_{n\to+\infty} n^\lambda h_H = \infty,\ \sum_{n=1}^{\infty} n^{(\lambda+1)/2} (q_n^{S_{\mathcal{H}}} q_n^{\Theta_{\mathcal{H}}})^{1-m} < \infty, \text{ for some } m > 1.$$

Comments on assumptions : Note that assumptions (U1) and (U2) are successive, the uniform versions of (H1) and (H2), whereas assumption (U3) and (U4) are respectively, Lipschitz's condition on $\beta_{\theta,i}(\cdot,\cdot)$ and $K_{\theta,i}(\cdot)$. Assumptions (U5) is a technical condition, allows to establish the asymptotic results.

Theorem 3.3.2. Under assumptions (U1)-(U5), we have

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}|\widehat{h}^x_\theta(y)-h^x_\theta(y)| = O(h_K^{b_1}+h_H^{b_2})+O_{\text{a.co.}}\left(\sqrt{\frac{\ln q_n^{S_{\mathcal{H}}}+\ln q_n^{\Theta_{\mathcal{H}}}}{nh_H\phi(h_K)}}\right).$$

Proof of Theorem 3.3.2.
The proof can be deduced from decomposition (1) and Lemmas 3.3.4-3.3.6, also, Corollary 3.3.2, which is given as follow :

Lemma 3.3.4. Under assumptions (U1),(U2) and (H4), we obtain

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}|F^x_\theta(y)-\mathbb{E}[\widehat{F}^x_{\theta,N}(y)]| = O(h_K^{b_1})+O(h_H^{b_2})$$

and

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}|f^x_\theta(y)-\mathbb{E}[\widehat{f}^x_{\theta,N}(y)]| = O(h_K^{b_1})+O(h_H^{b_2}).$$

Lemma 3.3.5. Under assumptions (U1), (U3), (U4)(i) and (U5), we obtain

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}|1-\widehat{g}^x_{\theta,D}| = O_{\text{a.co.}}\left(\sqrt{\frac{\ln q_n^{S_{\mathcal{H}}}+\ln q_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right) \text{ and } \sum_{i=1}^{\infty}\mathbb{P}(\inf_{\theta\in\Theta_{\mathcal{H}}}\inf_{x\in S_{\mathcal{H}}}\widehat{g}^x_{\theta,D}<1/2)<\infty.$$

Lemma 3.3.6. Under assumptions (U1)-(U5), we obtain

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}|\widehat{F}^x_{\theta,N}(y)-\mathbb{E}[\widehat{F}^x_{\theta,N}(y)]| = O_{\text{a.co.}}\left(\sqrt{\frac{\ln q_n^{S_{\mathcal{H}}}+\ln q_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right)$$

and

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}|\widehat{f}^x_{\theta,N}(y)-\mathbb{E}[\widehat{f}^x_{\theta,N}(y)]| = O_{\text{a.co.}}\left(\sqrt{\frac{\ln q_n^{S_{\mathcal{H}}}+\ln q_n^{\Theta_{\mathcal{H}}}}{nh_H\phi(h_K)}}\right).$$

Corollary 3.3.2. Under the conditions of Theorem 3.3.2, we obtain

$$\exists\varepsilon>0,\ \sum_{n=1}^{\infty}\mathbb{P}\Big(\inf_{\theta\in\Theta_{\mathcal{H}}}\inf_{x\in S_{\mathcal{H}}}\inf_{y\in S_{\mathbb{R}}}|1-\widehat{F}^x_\theta(y)|<\varepsilon\Big)<\infty.$$

3.4 Concluding remarks

This article provides theoretical results for estimate nonparametrically conditional hazard function in the single-index model using a local linear approach. Specifically, the rate of almost and uniform complete convergence is presented under certain conditions in i.i.d. case, when we have employed similar ideas in the context of functional analysis. In addition, one can see that the rate of our method is a good option than the Kernel method. Finally, for future research, this work is an encouragement to obtain the dependent case, also, to study the case where the functional single-index is unknown, such that, the estimation of this parameter is important in the choice of the semi-metric in this topic.

3.5 Appendix

We based on the results of the Lemmas 3.2-3.4 in Demongeot et al. (2013), we obtain the Lemmas 3.3.1-3.3.3 in very easy manner, so, we will focus on the Lemmas 3.3.4-3.3.6, which the proofs are also close to the Lemmas 4.4 and 4.5 in Deomongeot et al. (2013).

Proof of Lemma 3.3.4. We have that the sample $(X_i, Y_j)_{i,j=1,\dots,n}$ is identically distributed, thus for $p = 0, 1$. We get

$$\mathbb{E}[\widehat{F}^{x(p)}_{\theta,N}(y)] = \frac{1}{\mathbb{E}[W_{12}(x)]}\mathbb{E}\Big[W_{12}(x)\mathbb{E}\Big(\frac{H_1^{(p)}(y)}{h_H^p}\Big| < \theta, X_1 > \Big)\Big],\ \forall y \in S_{\mathbb{R}},\ \forall \theta \in \Theta$$

and by a classical computed, when we integrate by parts

$$\mathbb{E}\Big(\frac{H_1^{(p)}(y)}{h_H^p}\Big| < \theta, X_1 > \Big) = \int H^{(p)}(t)F_\theta^{X^{(p)}}(y - h_H t)\mathrm{d}t$$

and we get

$$\mathbb{E}[H_1^{(p)}(y)| < \theta, X_1 >] - F_\theta^{x^{(p)}}(y) = \int H^{(p)}(t)\Big(F_\theta^{X^{(p)}}(y - h_H t) - F_\theta^{x^{(p)}}(y)\Big)\mathrm{d}t.$$

So, by the Hölder property (U2) and (H4), we get

$$\begin{aligned} |\mathbb{E}[H_1^{(p)}(y)| < \theta, X_1 >] - F_\theta^{x^{(p)}}(y)| &\leq C_{\theta,x}\int H^{(p)}(t)\Big(h_K^{b_1} + |t|^{b_2}h_H^{b_2}\Big)\mathrm{d}t \\ &= O(h_K^{b_1}) + O(h_H^{b_2}). \end{aligned}$$

Proof of Lemma 3.3.5. We have by adoption from the proof of Lemma 4.4 in Barrientos-Marin et al. (2010)

$$\begin{aligned} \widehat{g}^x_{\theta,D} = \underbrace{\frac{n^2h_K^2\phi^2_{\theta,x}(h_K)}{n(n-1)\mathbb{E}[W_{\theta,12}(x)]}}_{S_1}\Big[\underbrace{\Big(\frac{1}{n}\sum_{j=1}^n\frac{K_{\theta,j}(x)}{\phi_{\theta,x}(h_K)}\Big)}_{S^x_{\theta,2}}\underbrace{\Big(\frac{1}{n}\sum_{i=1}^n\frac{K_{\theta,i}(x)\beta^2_{\theta,i}(x)}{h_K^2\phi_{\theta,x}(h_K)}\Big)}_{S^x_{\theta,4}} \\ - \underbrace{\Big(\frac{1}{n}\sum_{j=1}^n\frac{K_{\theta,j}(x)\beta_{\theta,j}(x)}{h_K\phi_{\theta,x}(h_K)}\Big)}_{S^x_{\theta,3}}\underbrace{\Big(\frac{1}{n}\sum_{i=1}^n\frac{K_{\theta,i}(x)\beta_{\theta,i}(x)}{h_K\phi_{\theta,x}(h_K)}\Big)}_{S^x_{\theta,3}}\Big]. \end{aligned}$$

We notice that $\mathbb{E}[\widehat{g}^x_{\theta,D}] = 1$.
Then,

$$\widehat{g}^x_{\theta,D} - \mathbb{E}[\widehat{g}^x_{\theta,D}] = S_1\Big((S^x_{\theta,2}S^x_{\theta,4} - \mathbb{E}[S^x_{\theta,2}S^x_{\theta,4}]) - ((S^x_{\theta,3})^2 - \mathbb{E}[(S^x_{\theta,3})^2])\Big).$$

We put

$$\begin{aligned} S^x_{\theta,2}S^x_{\theta,4} - \mathbb{E}[S^x_{\theta,2}S^x_{\theta,4}] &= (S^x_{\theta,2} - \mathbb{E}[S^x_{\theta,2}])(S^x_{\theta,4} - \mathbb{E}[S^x_{\theta,4}]) + (S^x_{\theta,4} - \mathbb{E}[S^x_{\theta,4}])\mathbb{E}[S^x_{\theta,2}] \\ &\quad + (S^x_{\theta,2} - \mathbb{E}[S^x_{\theta,2}])\mathbb{E}[S^x_{\theta,4}] + \mathbb{E}[S^x_{\theta,2}]\mathbb{E}[S^x_{\theta,4}] - \mathbb{E}[S^x_{\theta,2}S^x_{\theta,4}], \\ (S^x_{\theta,3})^2 - \mathbb{E}[(S^x_{\theta,3})^2] &= (S^x_{\theta,3} - \mathbb{E}[S^x_{\theta,3}])^2 + 2(S^x_{\theta,3} - \mathbb{E}[S^x_{\theta,3}])\mathbb{E}[S^x_{\theta,3}] - \mathrm{Var}[S^x_{\theta,3}] \end{aligned}$$

and under hypothesis (U1),(U3) and (U4), we have the following results

$$S_1 = O(1) \text{ and } \mathbb{E}[S^x_{\theta,z}] = O(1) \text{ for z} = 2, 3, 4. \qquad (2)$$

$$\Big|\mathbb{E}[S^x_{\theta,2}]\mathbb{E}[S^x_{\theta,4}] - \mathbb{E}[S^x_{\theta,2}S^x_{\theta,4}] + \mathrm{Var}[S^x_{\theta,3}]\Big| = o_{a.co.}\left(\sqrt{\frac{\ln q_n^{S_\mathcal{H}} + \ln q_n^{\Theta_\mathcal{H}}}{n\phi(h_K)}}\right) \tag{3}$$

$$\sum_{n=1}^{\infty}\mathbb{P}\left\{|S^x_{\theta,z} - \mathbb{E}[S^x_{\theta,z}]| > \eta\sqrt{\frac{\ln q_n^{S_\mathcal{H}} + \ln q_n^{\Theta_\mathcal{H}}}{n\phi(h_K)}}\right\} < \infty, \text{ for z} = 2,3,4, \tag{4}$$

obviously, equations (11) and (12) are consequences of Lemma 4.4 and Lemma A.1 in Barrientos-Marin et al. (2010). About equation (13), corresponding to Demongeot et al. (2013) , we obtain

$$\begin{aligned}
|S^x_{\theta,z} - \mathbb{E}[S^x_{\theta,z}]| &\leq \underbrace{\sup_{\theta\in\Theta_\mathcal{H}}\sup_{x\in S_\mathcal{H}}|S^x_{\theta,z} - S^{x_{k(x)}}_{\theta,z}|}_{R_1^z} + \underbrace{\sup_{\theta\in\Theta_\mathcal{H}}\sup_{x\in S_\mathcal{H}}|S^{x_{k(x)}}_{\theta,z} - S^{x_{k(x)}}_{t_{k'(\theta)},z}|}_{R_2^z} \\
&+ \underbrace{\sup_{\theta\in\Theta_\mathcal{H}}\sup_{x\in S_\mathcal{H}}|S^{x_{k(x)}}_{t_{k'(\theta)},z} - \mathbb{E}[S^{x_{k(x)}}_{t_{k'(\theta)},z}]|}_{R_3^z} + \underbrace{\sup_{\theta\in\Theta_\mathcal{H}}\sup_{x\in S_\mathcal{H}}|\mathbb{E}[S^{x_{k(x)}}_{t_{k'(\theta)},z}] - \mathbb{E}[S^{x_{k(x)}}_{\theta,z}]|}_{R_4^z} \\
&+ \underbrace{\sup_{\theta\in\Theta_\mathcal{H}}\sup_{x\in S_\mathcal{H}}|\mathbb{E}[S^{x_{k(x)}}_{\theta,z}] - \mathbb{E}[S^x_{\theta,z}]|}_{R_5^z}.
\end{aligned}$$

- Concerning R_1^z and R_5^z, we write for $z = 2,3,4$ that :

$$\begin{aligned}
R_1^z &\leq \frac{1}{nh_K^{z-2}\phi_{\theta,x}(h_K)}\sup_{\theta\in\Theta_\mathcal{H}}\sup_{x\in S_\mathcal{H}}\sum_{i=1}^n|K_{\theta,i}(x)\beta^{k-2}_{\theta,i}(x)\mathbb{I}_{B(x,h_K)}(X_i) - K_{\theta,i}(x_{k(x)})\beta^{k-2}_{\theta,i}(x_{k(x)})\mathbb{I}_{B(x_{k(x)},h_K)}(X_i)| \\
&\leq \frac{C}{nh_K^{z-2}\phi_{\theta,x}(h_K)}\sup_{\theta\in\Theta_\mathcal{H}}\sup_{x\in S_\mathcal{H}}\sum_{i=1}^n|K_{\theta,i}(x)\mathbb{I}_{B(x,h_K)}(X_i)|\beta^{z-2}_{\theta,i}(x) - \beta^{z-2}_{\theta,i}(x_{k(x)})\mathbb{I}_{B(x_{k(x)},h_K)}(X_i)| \\
&+ \frac{1}{nh_K^{z-2}\phi_{\theta,x}(h_K)}\sup_{\theta\in\Theta_\mathcal{H}}\sup_{x\in S_\mathcal{H}}\sum_{i=1}^n\beta^{z-2}_{\theta,i}(x_{k(x)})\mathbb{I}_{B(x_{k(x)},h_K)}(X_i)|K_{\theta,i}(x)\mathbb{I}_{B(x,h_K)}(X_i) - K_{\theta,i}(x_{k(x)})|
\end{aligned}$$

we can write under assumption (U4) on $K_{\theta,i}(\cdot)$, that

$$\mathbb{I}_{B(x_{k(x)},h_K)}(X_i)|K_{\theta,i}(x)\mathbb{I}_{B(x,h_K)}(X_i) - K_{\theta,i}(x_{k(x)})| \leq Cv_n\mathbb{I}_{B(x,h_K)\cap B(x_{k(x)},h_K)}(X_i) + C\mathbb{I}_{\overline{B(x,h_K)}\cap B(x_{k(x)},h_K)}(X_i).$$

The same way, under assumption (U3) on $\beta_{\theta,i}(\cdot,\cdot)$, we obtain :

$$\mathbb{I}_{B(x,h_K)}(X_i)|\beta_{\theta,i}(x) - \beta_{\theta,i}(x_{k(x)})\mathbb{I}_{B(x_{k(x)},h_K)}(X_i)| \leq$$
$$v_n\mathbb{I}_{B(x,h_K)\cap B(x_{k(x)},h_K)}(X_i) + h_K^2\mathbb{I}_{B(x,h_K)\cap\overline{B(x_{k(x)},h_K)}}(X_i)$$

and for $z = 2$,

$$\mathbb{I}_{B(x,h_K)}(X_i)|\beta^2_{\theta,i}(x) - \beta^2_{\theta,i}(x_{k(x)})\mathbb{I}_{B(x_{k(x)},h_K)}(X_i)| \leq$$
$$v_nh_K\mathbb{I}_{B(x,h_K)\cap B(x_{k(x)},h_K)}(X_i) + h_K^2\mathbb{I}_{B(x,h_K)\cap\overline{B(x_{k(x)},h_K)}}(X_i).$$

For $z = 3,4$

$$\mathbb{I}_{B(x,h_K)}(X_i)|\beta^{z-2}_{\theta,i}(x) - \beta^{z-2}_{\theta,i}(x_{k(x)})\mathbb{I}_{B(x_{k(x)},h_K)}(X_i)| \leq$$
$$v_nh_K^{z-3}\mathbb{I}_{B(x,h_K)\cap B(x_{k(x)},h_K)}(X_i) + h_K^{z-2}\mathbb{I}_{B(x,h_K)\cap\overline{B(x_{k(x)},h_K)}}(X_i).$$

So, after simplifying the previous result and under (U1), we obtain

$$R_1^z \leq \frac{Cv_n}{nh_K\phi(h_K)} \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sum_{i=1}^{n} \mathbb{I}_{B(x,h_K)\cup B(x_{k(x)},h_K)}(X_i).$$

Now, we apply a standard inequality for sums of bounded random variables (see. Ferraty and Vieu (2006) Corollary A.9), where

$$|\Lambda_i| \leq \frac{Cv_n}{h_K\phi(h_K)},\ \mathbb{E}(|\Lambda_i|) \leq \frac{Cv_n}{h_K},\ \mathbb{E}(\Lambda_i^2) \leq \frac{Cv_n^2}{h_K^2\phi(h_K)}$$

for

$$\Lambda = \frac{Cv_n}{nh_K\phi(h_K)} \mathbb{I}_{B(x,h_K)\cup B(x_{k(x)},h_K)}(X_i)$$

and the assumption (U5) permit us to get

$$R_1^z = O_{\text{a.co.}}\left(\sqrt{\frac{\ln q_n^{S_{\mathcal{H}}} + \ln q_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right).$$

Finally

$$R_5^z \leq R_1^z = O_{\text{a.co.}}\left(\sqrt{\frac{\ln q_n^{S_{\mathcal{H}}} + \ln q_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right).$$

- For the terms R_2^z and R_4^z, we using the same ideas those used in R_1^z and R_5^z for $z = 2, 3, 4$ such that :

$$R_4^z \leq R_2^z = O_{\text{a.co.}}\left(\sqrt{\frac{\ln q_n^{S_{\mathcal{H}}} + \ln q_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right).$$

- Finally, For the terms R_3^z. For all $\eta > 0$, we have that

$$\mathbb{P}\left(R_3^z > \eta\sqrt{\frac{\ln q_n^{S_{\mathcal{H}}}+\ln q_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right) \leq$$

$$q_n^{\Theta_{\mathcal{H}}} q_n^{S_{\mathcal{H}}} \max_{1\leq t_{k'(\theta)}\leq q_n^{\Theta_{\mathcal{H}}}} \max_{1\leq x_{k(x)}\leq q_n^{S_{\mathcal{H}}}} \mathbb{P}\left(|S_{\theta,z}^{x_{k(x)}} - \mathbb{E}[S_{\theta,z}^{x_{k(x)}}] > \eta\sqrt{\frac{\ln q_n^{S_{\mathcal{H}}} + \ln q_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right).$$

By applying Lemma A.1 in Barrinetos-Marin et al. (2010). We have that

$$\mathbb{E}|\Upsilon_{\theta,i}^{z^M}| = O\Big(\phi(h_K)^{-M+1}\Big).$$

Therefore, by using a Bernstein-type inequality in Ferraty and Vieu (2006) Corollary A.8 for :

$$\Upsilon_{\theta,i}^z = \frac{1}{h_K^{z-2}\phi(h_K)}\Big(K_{\theta,i}(x_{k(x)})\beta_{\theta,i}^{z-2}(x_{k(x)}) - \mathbb{E}[K_{\theta,i}(x_{k(x)})\beta_{\theta,i}^{z-2}(x_{k(x)})]\Big),\ \text{for } z = 2, 3, 4.$$

We obtain

$$\mathbb{P}\left(\frac{1}{n}\Big|\sum_{n=1}^{n}\Upsilon_{\theta,i}^z\Big| > \eta\sqrt{\frac{\ln q_n^{S_{\mathcal{H}}} + \ln q_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right) \leq 2\exp\Big(-C\eta^2\ln(q_n^{S_{\mathcal{H}}} q_n^{\Theta_{\mathcal{H}}})\Big).$$

Then, we take $m = C\eta^2$:

$$q_n^{\Theta_{\mathcal{H}}} q_n^{S_{\mathcal{H}}} \max_{1\leq t_{k'(\theta)}\leq q_n^{\Theta_{\mathcal{H}}}} \max_{1\leq x_{k(x)}\leq q_n^{S_{\mathcal{H}}}} \mathbb{P}\left(\frac{1}{n}\Big|\sum_{n=1}^{n}\Upsilon_{\theta,i}^{z}\Big| > \eta\sqrt{\frac{\ln q_n^{S_{\mathcal{H}}} + \ln q_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right) \leq C'(q_n^{S_{\mathcal{H}}} q_n^{\Theta_{\mathcal{H}}})^{1-m}$$

finally, since $\sum\limits_{n=1}^{\infty}(q_n^{S_{\mathcal{H}}} q_n^{\Theta_{\mathcal{H}}})^{1-m} < \infty$, we get

$$R_3^z = O_{\text{a.co.}}\left(\sqrt{\frac{\ln q_n^{S_{\mathcal{H}}} + \ln q_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right).$$

Now, we prove $\sum\limits_{i=1}^{\infty}\mathbb{P}(\inf\limits_{\theta\in\Theta_{\mathcal{H}}}\inf\limits_{x\in S_{\mathcal{H}}}\widehat{g}_{\theta,D}^{x} < 1/2) < \infty$,
we remark that

$$\inf_{\theta\in\Theta_{\mathcal{H}}}\inf_{x\in S_{\mathcal{H}}}\widehat{g}_{\theta,D}^{x} < 1/2 \Longrightarrow \exists\theta\in\Theta_{\mathcal{H}},\ \exists x\in S_{\mathcal{H}},\ 1-g_{\theta,D}^{x} > 1/2,$$

so,

$$\inf_{\theta\in\Theta_{\mathcal{H}}}\inf_{x\in S_{\mathcal{H}}}\widehat{g}_{\theta,D}^{x} < 1/2 \Longrightarrow \sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}|1-\widehat{g}_{\theta,D}^{x}| > 1/2,$$

and,

$$\mathbb{P}(\inf_{\theta\in\Theta_{\mathcal{H}}}\inf_{x\in S_{\mathcal{H}}}\widehat{g}_{\theta,D}^{x} < 1/2) \Longrightarrow \mathbb{P}(\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\widehat{|}1-g_{\theta,D}^{x}| > 1/2).$$

Then, the proof is a consequence from the above results, and from this latter, we finished the proof of the theorem .

Proof of Lemma 3.3.6. Consider the following decomposition, for $p = 0, 1$:

$$\widehat{F}_{\theta,N}^{x(p)}(y) = S_1\Big[T_{\theta,1}^{x}(y)T_{\theta,2}^{x} - T_{\theta,3}^{x}(y)T_{\theta,4}^{x}\Big]$$

where, S_1 is the same term in Lemma 3.3.5 and

$$T_{\theta,1}^{x}(y) = \frac{1}{n}\sum_{j=1}^{n}\frac{K_{\theta,j}(x)H_j^{(p)}(y)}{h_H^{(p)}\phi_{\theta,x}(h_K)},\ T_{\theta,2}^{x} = \frac{1}{n}\sum_{i=1}^{n}\frac{K_{\theta,i}(x)\beta_{\theta,i}^{2}(x)}{h_K^{2}\phi_{\theta,x}(h_K)},$$
$$T_{\theta,3}^{x}(y) = \frac{1}{n}\sum_{j=1}^{n}\frac{K_{\theta,j}(x)\beta_{\theta,j}(x)H_j^{(p)}(y)}{h_H^{(p)}h_K\phi_{\theta,x}(h_K)},\ T_{\theta,4}^{x} = \frac{1}{n}\sum_{i=1}^{n}\frac{K_{\theta,i}(x)\beta_{\theta,i}(x)}{h_K\phi_{\theta,x}(h_K)}.$$

Therefor,

$$\widehat{F}_{\theta,N}^{x(p)}(y) - \mathbb{E}[\widehat{F}_{\theta,N}^{x(p)}(y)] = S_1\Big(\Big(T_{\theta,1}^{x}(y)T_{\theta,2}^{x} - \mathbb{E}[T_{\theta,1}^{x}(y)T_{\theta,2}^{x}]\Big) - \Big(T_{\theta,3}^{x}(y)T_{\theta,4}^{x} - \mathbb{E}[T_{\theta,3}^{x}(y)T_{\theta,4}^{x}]\Big)\Big).$$

We have from Lemma 3.3.5

$$T_{\theta,2}^{x} = S_{\theta,4}^{x},\ T_{\theta,4}^{x} = S_{\theta,3}^{x}.$$

So, in the same strides of the Lemma 3.3.5, the following assertions it remains :

$$\mathbb{E}[T_{\theta,z}^{x}(y)] = O(1),\ \text{for z} = 1, 3. \tag{5}$$

$$\sum_{n=1}^{\infty}\mathbb{P}\left\{|T_{\theta,z}^{x}(y) - \mathbb{E}[T_{\theta,z}^{x}(y)]| > \eta\sqrt{\frac{\ln q_n^{S_{\mathcal{H}}} + \ln q_n^{\Theta_{\mathcal{H}}}}{nh_H^{(p)}\phi(h_K)}}\right\} < \infty,\ \text{for z} = 1, 3 \tag{6}$$

Using again Lemma A.1 in Barrientos-Marin et al. (2010) and the boundless of H, we get the equation (14). Then, for equation (6), the compactness of $S_{\mathbb{R}}$ allows us to write that,

$$S_{\mathbb{R}} \subset \bigcup_{k''=1}^{s_n} (y_{k''} - l_n, y_{k''} + l_n) \text{ with } l_n = n^{\frac{(-\lambda-1)}{2}} \text{ and } s_n = O(l_n^{-1}).$$

By taking $k''(y) = \arg \min_{k'' \in \{1,\dots,s_n\}} |y - y_{k''}|$,
$|T^x_{\theta,z}(y) - \mathbb{E}[T^x_{\theta,z}(y)]| \leq E^z_{\theta,1} + E^z_{\theta,2} + E^z_{\theta,3} + E^z_{\theta,4} + E^z_{\theta,5} + E^z_{\theta,6} + E^z_{\theta,7}$
where

$$\begin{aligned}
E^z_{\theta,1} &= \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sup_{y\in S_{\mathbb{R}}} |T^x_{\theta,z}(y) - T^{x_{k(x)}}_{\theta,z}(y)| \\
E^z_{\theta,2} &= \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sup_{y\in S_{\mathbb{R}}} |T^{x_{k(x)}}_{\theta,z}(y) - T^{x_{k(x)}}_{t_{k'(\theta)},z}(y)| \\
E^z_{\theta,3} &= \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sup_{y\in S_{\mathbb{R}}} |T^{x_{k(x)}}_{t_{k'(\theta)},z}(y) - T^{x_{k(x)}}_{t_{k'(\theta)},z}(y_{k''(y)})| \\
E^z_{\theta,4} &= \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sup_{y\in S_{\mathbb{R}}} |T^{x_{k(x)}}_{t_{k'(\theta)},z}(y_{k''(y)}) - \mathbb{E}[T^{x_{k(x)}}_{t_{k'(\theta)},z}(y_{k''(y)})]| \\
E^z_{\theta,5} &= \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sup_{y\in S_{\mathbb{R}}} |\mathbb{E}[T^{x_{k(x)}}_{t_{k'(\theta)},z}(y_{k''(y)})] - \mathbb{E}[T^{x_{k(x)}}_{t_{k'(\theta)},z}(y)]| \\
E^z_{\theta,6} &= \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sup_{y\in S_{\mathbb{R}}} |\mathbb{E}[T^{x_{k(x)}}_{t_{k'(\theta)},z}(y)] - \mathbb{E}[T^{x_{k(x)}}_{\theta,z}(y)]| \\
E^z_{\theta,7} &= \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sup_{y\in S_{\mathbb{R}}} |\mathbb{E}[T^{x_{k(x)}}_{\theta,z}(y)] - \mathbb{E}[T^x_{\theta,z}(y)]|.
\end{aligned}$$

- For $E^z_{\theta,1}$ and $E^z_{\theta,7}$, the boundless of H permit us to obtain that for all $z = 1, 3$

$$E^z_{\theta,1} = R^z_{\theta,1} \text{ and } E^z_{\theta,7} = R^z_{\theta,5}$$

since, $E^z_{\theta,7} \leq E^z_{\theta,1}$, we get

$$E^z_{\theta,7} \leq E^z_{\theta,1} = O_{\text{a.co.}}\left(\sqrt{\frac{\ln q_n^{S_{\mathcal{H}}} + \ln q_n^{\Theta_{\mathcal{H}}}}{n h_H^{(p)} \phi(h_K)}}\right).$$

- Concerning the term $E^z_{\theta,2}$ and $E^z_{\theta,6}$ for all $z = 1, 3$, by using the same ideas as for $E^z_{\theta,1}$ and $E^z_{\theta,7}$, we get

$$E^z_{\theta,6} \leq E^z_{\theta,2} = O_{\text{a.co.}}\left(\sqrt{\frac{\ln q_n^{S_{\mathcal{H}}} + \ln q_n^{\Theta_{\mathcal{H}}}}{n h_H^{(p)} \phi(h_K)}}\right).$$

- Now, the term $E^z_{\theta,3}$ and $E^z_{\theta,5}$, by using Lipschitz's conditions on H,

$$|T^{x_{k(x)}}_{t_{k'(\theta)},z}(y) - T^{x_{k(x)}}_{t_{k'(\theta)},z}(y_{k''(y)})| \leq$$

$$\frac{C}{n h_K^{z-2} h_H^{(p)} \phi_{\theta,x}(h_K)} \sum_{I=1}^{n} K_{\theta,I}(x_{k(x)}) \quad \beta^{z-2}_{\theta,I}(x_{k(x)}) \mathbb{I}_{B(x_{k(x)},h_K)}(X_I) |H_I^{(p)}(y) - H_I^{(p)}(y_{k''(y)})|$$

$$\leq \frac{l_n}{h_H h_H^{(p)}} S^{x_{k(x)}}_{\theta,z}$$

where $S^{x_{k(x)}}_{\theta,z}$, for all $z = 2, 3$ are treated in the proof of Lemma 3.3.5 and by taking $\lim_{n\to\infty} n^{\lambda} h_H = \infty$ and $l_n = n^{-\frac{\lambda}{2}-\frac{1}{2}}$, we obtain

$$E^z_{\theta,5} \leq E^z_{\theta,3} = O_{\text{a.co.}}\left(\sqrt{\frac{\ln q_n^{S_{\mathcal{H}}} + \ln q_n^{\Theta_{\mathcal{H}}}}{n h_H^{(p)} \phi(h_K)}}\right).$$

- Finally, for the term $E^z_{\theta,4}$, for all $\eta > 0$, we obtain by the classical Bernstein inequality (see, Ferraty and Vieu (2006)) that

$$\mathbb{P}\left(E^z_{\theta,4} > \eta\sqrt{\frac{\ln q_n^{S_\mathcal{H}} + \ln q_n^{\Theta_\mathcal{H}}}{nh_H^{(p)}\phi(h_K)}}\right) \leq s_n q_n^{S_\mathcal{H}} q_n^{\Theta_\mathcal{H}} \max_{1\leq t_{k'(\theta)}\leq q_n^{\Theta_\mathcal{H}}} \max_{1\leq x_{k(x)}\leq q_n^{S_\mathcal{H}}} \max_{1\leq y_{k''(y)}\leq s_n}$$

$$\mathbb{P}\left(|T^{x_{k(x)}}_{t_{k'(\theta)},z}(y_{k''(y)}) - \mathbb{E}[T^{x_{k(x)}}_{t_{k'(\theta)},z}(y_{k''(y)})]| > \eta\sqrt{\frac{\ln q_n^{S_\mathcal{H}} + \ln q_n^{\Theta_\mathcal{H}}}{n\phi(h_K)}}\right) < \infty$$

where, we take $s_n = O(l_n^{-1}) = n^{(\lambda+1)/2}$ and $\sum_{i=1}^{\infty} n^{(\lambda+1)/2}(q_n^{S_\mathcal{H}} q_n^{\Theta_\mathcal{H}})^{1-m} < \infty$, for $m > 0$.
Finally, we get

$$E^z_{\theta,4} = O_{\text{a.co.}}\left(\sqrt{\frac{\ln q_n^{S_\mathcal{H}} + \ln q_n^{\Theta_\mathcal{H}}}{nh_H^{(p)}\phi(h_K)}}\right).$$

Proof of of Corollary 4.3.1 and Corollary 3.3.2. The Corollary 4.3.1 (resp. Corollary 3.3.2) is a consequence of Lemmas 3.3.1-3.3.3 (resp. Lemmas 3.3.4-3.3.6).

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CHAPITRE 4

Uniform convergence of nonparametric conditional hazard function in the single functional modeling for dependent data

Ce travail fait l'objet d'un article soumis.

Uniform convergence of nonparametric conditional hazard function in the single functional modeling for dependent data

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Abstract. We study the nonparametric local linear estimation of the conditional hazard function of a scalar response variable given a functional explanatory variable, when the functional data are α-mixing dependency and we give the uniform almost complete convergence with rates of this function.

Keywords : nonparametric local linear estimation, conditional distribution, conditional single-index, α-mixing dependency.

2000 Mathematics Subject Classification : Primary : 62G05, Secondary : 62G20, 62G07

4.1 Introduction

The contribution of this work is to study the conditional hazard in the single functional index model, for its excellence in many characteristics and due to the flexibility of the model in dimension reduction and used in econometrics fields as accord between nonparametric and parametric models. The single-index models have been considered in the multivaraite case by Hardle et al. (1993), Hristache et al. (2001) and Delecroix et al. (2003). Then, by nonparametric kernel estimation, Ferraty et al. (2003) started to deal with the single functioanl index, they obtained the almost complete convergence in the independent and identically distributed (i.i.d) case for regression function. After that, for the same estimator Attaoui et al. (2011) established the pointwise and uniform almost complete convergence of the conditional density, moreover, the depnedent case were achiveted by Ait Saidi et al. ((2005), (2008)), where they proposed an estimator for thr single functional index more.
In our study, we estimate the conditional hazard in the single index model for dependent data of a real variable Y given a functional variable X in the local linear method (see, Barrientos et al. (2010)). We point out that the single functional index in this method is intimately limited until now. Concerning this model of conditional density in the kernel method for dependent observation, one we cite the work of Ling and Xu (2012), Attaoui (2014) and Ling et al. (2014). Particularly, in the quasi-associated, Hadjila and Ait Saidi (2018) studied the pointwise almost complete convergence and the uniform almost complete convergence(with the rate) of the kernel estimate of the hazard function of a real random variable conditioned by a functional predictor, also, gave a simulation to illustrate their methodology. Our work count on the study of the conditional hazard function of a scalar response variable Y given a Hilbertian random variable in functioanl single-index model for dependence case in the local linear method, such that under certain conditions we prove its uniform almost complete convergence.
In this paper, we will see the model and the estimator in the local linear estimation in this section 1. Then, we give in section 2 assumptions and results. Finally, we finished by the proofs of our results in section 3.

4.2 Model

Let $\{Y_i, X_i\}_{i\in\mathbb{N}}$ be a random processes identically distributed as (Y, X) where Y_i's are valued in $\mathbb{R}$ and X_i takes values in seperable Hilbert space $\mathcal{H}$ with the norm $||.||$ generated by an inner product $< .,. >$. We assume that the regular version of the conditional probability of Y given X exists and bounded. Moreover, we suppose that the conditional hazard function of Y given X has a known single-index θ in $\mathcal{H}$ and we denote the conditional density by $f_\theta^x(y)$ respect to Lebesgue's measure over $\mathbb{R}$. So, denote the conditional hazard function of Y given X by

$$h_\theta^x(y) = \frac{f_\theta^x(y)}{1 - F_\theta^x(y)}, \quad \forall y \in \mathbb{R}$$

where, $F_\theta^x(y) < 1$.
As usually, in the single-index model the identifiability is assured such that, $\forall x \in \mathcal{H}$, we have

$$h_1(y| <\cdot\,, \theta_1>) = h_2(y| <\cdot\,, \theta_2>) \Rightarrow h_1 \equiv h_2 \text{ and } \theta_1 = \theta_2.$$

The local linear estimator (see, Demongeot al. (2010)) of $F_\theta^x(y)$ and $f_\theta^x(y)$ was defined as follows

$$\widehat{F}_\theta^x(y) = \frac{\sum\limits_{1\le i,j\le n} W_{\theta,ij}(x) H({h_H}^{-1}(y - Y_j))}{\sum\limits_{1\le i,j\le n} W_{\theta,ij}(x)}$$

and

$$\widehat{f}_\theta^x(y) = \frac{\sum\limits_{1\le i,j\le n} W_{\theta,ij}(x) H'({h_H}^{-1}(y - Y_j))}{h_H \sum\limits_{1\le i,j\le n} W_{\theta,ij}(x)},$$

with

$$W_{\theta,ij}(x) = \beta_\theta(X_i, x)\Big(\beta_\theta(X_i, x) - \beta_\theta(X_j, x)\Big)K(h_K^{-1} d_\theta(x, X_i))K(h_K^{-1} d_\theta(x, X_j))$$

with
$\beta_\theta(X_i, x) =< x - X_i, \theta >$ is a known bi-functional operator from $\mathcal{H}^2$ into $\mathbb{R}$, such that $\forall x_1, x_2 \in \mathcal{H}$, $\forall \theta \in \mathcal{H}$, d_θ is a semi-metric associated to the single index $\theta \in \mathcal{H}$ defined by $d_\theta(x_1, x_2) := | < x_1 - x_2, \theta > |$, with the kernel K. H is a distribution function (respectively, H' is the derivative of H), and $h_K = h_{K,n}$(respectively, $h_H = h_{H,n}$) is a sequence of positive real numbers.
Finally, the local linear estimator of the hazard function is given by

$$\widehat{h}_\theta^x(y) = \frac{\widehat{f}_\theta^x(y)}{1 - \widehat{F}_\theta^x(y)}.$$

Now, we define the definition of $\alpha-$mixing sequence. The sequence is said to be $\alpha-$mixing (strong mixing), if the mixing coefficient $\alpha(n) \xrightarrow{n\to\infty} 0$ such that

$$\alpha(n) = \sup_k \sup_{A\in\sigma_1^k(X), B\in\sigma_{n+k}^\infty(X)} \{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|, k \in \mathbb{N}^*\}$$

and σ_j^k denote the σ-algebra generated by the random variales $\{(Y_i, X_i), j \le i \le k\}$.

4.3 Assumptions and Results

4.3.1 Uniform almost complete convergence

In this paper, we will study the uniform almost complete convergence denote by C, C' and C'', also, $C_{\theta,x}$, some strictly positive constants, and $\forall x \in \mathcal{H}$, and $i, j = 1, ., n$,

$$K_{\theta,i}(x) := K(h_K^{-1} d_\theta(x, X_i)) \text{ and }, \ \forall y \in \mathbb{R}, \ H_j(y) := H({h_H}^{-1}(y - Y_j)).$$

On the other hand, we denotes x a fixed point in $\mathcal{H}$, $\mathcal{N}_x$ is a fixed neighborhood of x and $S_\mathbb{R}$ is a fixed compact of $\mathbb{R}$. We consider the following cover of the compacts $S_\mathcal{H}$ and $\Theta_\mathcal{H}$:

$$S_\mathcal{H} \in \bigcup_{j=1}^{N^{S_\mathcal{H}}} B(x_j, r_n) \text{ and } \Theta_\mathcal{H} \in \bigcup_{j''=1}^{N^{\Theta_\mathcal{H}}} B(t_{j''}, r_n)$$

and $\forall x \in \mathcal{H}, \forall \theta \in \Theta_\mathcal{H}$ we set

$$j(x) = \arg \min_{j \in \{1,\ldots,N^{S_\mathcal{H}}\}} ||x - x_j|| \text{ and } j''(\theta) = \arg \min_{j'' \in \{1,\ldots,N^{\Theta_\mathcal{H}}\}} ||\theta - t''_j||$$

where, $x_j, t_{j''} \in \mathcal{H}$ and r_n is a sequence of positif numbers.
Suppose that $N^{S_\mathcal{H}}, N^{\Theta_\mathcal{H}}$, are the minimal numbers of open balls (see, Kolmogorov et Tikhomirov (1959)) with radius r_n in $\mathcal{H}$, which are required to cover $S_\mathcal{H}$ and $\Theta_\mathcal{H}$.

For our context, we need assumptions for our estimate.

$$\forall h_K > 0, \ \mathbb{P}(| < X - x, \theta > | < h_K) =: \phi_{\theta,x}(h_K) > 0.$$

(U1) There exists a differentiable function $\phi(\cdot)$ such that $\forall x \in S_\mathcal{H}$, and $\forall \theta \in \Theta_\mathcal{H}$

$$0 < C\phi(h_K) \leq \phi_{\theta,x}(h_K) \leq C\phi'(h_K) < \infty \quad \text{and } \exists \eta_0 > 0, \ \forall \eta < \eta_0, \phi'(\eta) < 0,$$

where ϕ' is the first derivative function of ϕ, and $\phi(0) = 0$.

(U2) The function F_θ^x and f_θ^x satisfy :

$$\begin{cases} \exists a_1, a_2 > 0, \ \forall (y, y') \in S_\mathbb{R}^2, \forall (x, x') \in \mathcal{N}_x \times \mathcal{N}_x, \ \forall \theta \in \Theta, \\ (i) \ |f_\theta^x(y) - f_\theta^{x'}(y')| \leq C'(||x - x'||^{a_1} + |y - y'|^{a_2}), \\ (ii) \ |F_\theta^x(y) - F_\theta^{x'}(y')| \leq C''(||x - x'||^{a_1} + |y - y'|^{a_2}). \end{cases}$$

(U3) The pairs $(X_i, Y_j)_{i,j \in \mathbb{N}}$ satisfies :
  (i) $\exists a > 0, \exists c > 0 : \forall n \in \mathbb{N}, \alpha(n) \leq cn^{-a}$.
  (ii) $\exists 0 < d \leq 1, 0 \leq C\phi(h_K)^{1+d} \leq \varphi_{\theta,x}(h_K) \leq C'\phi(h_K)^{1+d}$.

  where $\varphi_{\theta,x}(h_K) := \sup_{i \neq j} \mathbb{P}\Big((X_i, X_j) \in B(x, h_K) \times B(x, h_K)\Big)$.

(U4) The bi-functional function $\beta_\theta(\cdot, \cdot)$ is Lipschitsian continuos function and satisfying :

$$\forall x' \in S_\mathcal{H}, \ Cd_\theta(x', x) \leq |\beta_\theta(x, x')| \leq C'd_\theta(x', x).$$

(U5) (i) The kernel K is a positive, Lipschitzian and differentiable function, supported within $(-1, 1)$.

(ii) The kernel H is a positive, bounded and Lipschitzian continuous function, such that :

$$\int |t|^{a_2} H(t)\mathrm{d}t < \infty \text{ and } \int H^2(t)\mathrm{d}t < \infty.$$

(U6) The bandwidth h_K satisfies : $\exists n_0 \in \mathbb{N},\ \forall \eta > n_0,\quad -\frac{1}{\phi_{\theta,x}(h_K)} \int_{-1}^{1} \phi_{\theta,x}(th_K, h_K) \frac{\mathrm{d}}{\mathrm{d}t}(t^2 K(t))\mathrm{d}t > C'' > 0$ and $\quad h_K \int_{B(x,h_K)} \beta_\theta(u,x)\mathrm{d}P(u) = o\Big(\int_{B(x,h_K)} \beta_\theta^2(u,x)\mathrm{d}P(u)\Big)$
where $B(x,h) = \{z \in \mathcal{H} |\ d_\theta(z,x) \le h\}$ and $\mathrm{d}P(u)$ is the probability measure of X.

(U7) For some $\lambda > 0$ the bandwidth h_H satisfies

$$\lim_{n\to\infty} n^\lambda h_H = \infty, \text{ and } \lim_{n\to\infty} \frac{\ln n}{n h_H^{(j)} \phi(h_K)} = 0, \text{ for } j = 0, 1.$$

(U8) $\exists 0 < t < 1,\ C n^{\frac{3-a}{a+1}+\eta_0} \le h_H^{(j)} \phi(h_K) \le C n^{-t}$
where $\eta_0 > \frac{\lambda+1}{a+1}$, for $j = 0, 1$.

Comments on assumption

As usually in functional statistics and in the independent case, the conditions (U1) and (U4) are standard hypotheses (see, Ferraty et al. (2003)). (U2) is about regularity and boundary conditions. Hypotheses (U5) and (U7) are a technical conditions (see, Barrientos et al. (2010)). Particulary, for the dependence frame, (U3) indicate that the observations are α-mixing dependency. Likewise, we find the condition (U6) in Barrientos et al. (2010) and we put (U8) that needed for our asymptotic results.

Theorem 4.3.1. Under assumptions $(U1) - (U8)$, we have :

$$\sup_{\theta\in\Theta_\mathcal{H}} \sup_{x\in S_\mathcal{H}} \sup_{y\in S_\mathbb{R}} |\widehat{h}_\theta^x(y) - h_\theta^x(y)| = O(h_K^{a_1} + h_H^{a_2}) + O_{\text{a.co.}}\left(\sqrt{\frac{\ln(N^{S_\mathcal{H}} N^{\Theta_\mathcal{H}})}{n h_H \phi(h_K)}}\right).$$

Proof of Theorem 4.3.1. The proof is based on the decomposition in Theorem 3.1 of Merouan *et al.* (2019) which we remind it and the Lemmas below.

$$\widehat{h}_\theta^x(y) - h_\theta^x(y) = \frac{1}{1-\widehat{F}_\theta^x(y)}\Big(\widehat{f}_\theta^x(y) - f_\theta^x(y)\Big) + \frac{h_\theta^x(y)}{1-\widehat{F}_\theta^x(y)}\Big(\widehat{F}_\theta^x(y) - F_\theta^x(y)\Big). \quad (1)$$

Where for $p = 0, 1$, we have :
$\widehat{F}_\theta^{x(p)}(y) - F_\theta^{x(p)}(y) = \frac{1}{\widehat{g}_{\theta,D}^x}\Big\{\Big(\widehat{F}_{\theta,N}^{x(p)}(y) - \mathbb{E}[\widehat{F}_{\theta,N}^{x(p)}(y)]\Big) - \Big(F_\theta^{x(p)}(y) - \mathbb{E}[\widehat{F}_{\theta,N}^{x(p)}(y)]\Big)\Big\} + \frac{F_\theta^{x(p)}(y)}{\widehat{g}_{\theta,D}^x}(1 - \widehat{g}_{\theta,D}^x)$

and

$$\widehat{F}_{\theta,N}^{x(p)}(y) = \frac{1}{n(n-1)h_H^{(p)}\mathbb{E}[W_{\theta,12}(x)]} \sum_{1\le i\ne j\le n} W_{\theta,ij}(x) H^{(p)}(h_H^{-1}(y - Y_j))$$

$$\widehat{g}_{\theta,D}^x = \frac{1}{n(n-1)\mathbb{E}[W_{\theta,12}(x)]} \sum_{1\le i\ne j\le n} W_{\theta,ij}(x).$$

Lemma 4.3.1. Under assumptions $(U1), (U2)$ and $(U5)$, we obtain :

$$\sup_{\theta\in\Theta_\mathcal{H}} \sup_{x\in S_\mathcal{H}} \sup_{y\in S_\mathbb{R}} |f_\theta^x(y) - \mathbb{E}[\widehat{f}_{\theta,N}^x(y)]| = O(h_K^{a_1}) + O(h_H^{a_2}).$$

and

$$\sup_{\theta\in\Theta_\mathcal{H}} \sup_{x\in S_\mathcal{H}} \sup_{y\in S_\mathbb{R}} |F_\theta^x(y) - \mathbb{E}[\widehat{F}_{\theta,N}^x(y)]| = O(h_K^{a_1}) + O(h_H^{a_2}).$$

Lemma 4.3.2. Under assumptions $(U1)$ and $(U3)-(U8)$, we get :

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}|1-\widehat{g}^x_{\theta,D}| = O_{\text{a.co.}}\left(\sqrt{\frac{\ln(N^{S_{\mathcal{H}}}N^{\Theta_{\mathcal{H}}})}{n\phi(h_K)}}\right) \text{ and } \sum_{i=1}^{\infty}\mathbb{P}(\inf_{\theta\in\Theta_{\mathcal{H}}}\inf_{x\in S_{\mathcal{H}}}\widehat{g}^x_{\theta,D}<1/2)<\infty.$$

Lemma 4.3.3. Under assumptions $(U1),(U2)-(U8)$, we obtain :

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}|\widehat{f}^x_{\theta,N}(y)-\mathbb{E}[\widehat{f}^x_{\theta,N}(y)]| = O_{\text{a.co.}}\left(\sqrt{\frac{\ln(N^{S_{\mathcal{H}}}N^{\Theta_{\mathcal{H}}})}{nh_H\phi(h_K)}}\right).$$

and

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}|\widehat{F}^x_{\theta,N}(y)-\mathbb{E}[\widehat{F}^x_{\theta,N}(y)]| = O_{\text{a.co.}}\left(\sqrt{\frac{\ln(N^{S_{\mathcal{H}}}N^{\Theta_{\mathcal{H}}})}{n\phi(h_K)}}\right).$$

Corollary 4.3.1. Under the conditions of Theorem 3.3.1, we get :

$$\exists\mu>0,\ \sum_{n=1}^{\infty}\mathbb{P}\Big(\inf_{\theta\in\Theta_{\mathcal{H}}}\inf_{x\in S_{\mathcal{H}}}\inf_{y\in S_{\mathbb{R}}}|1-\widehat{F}^x_{\theta}(y)|<\mu\Big)<\infty.$$

4.4 Conclusion

In this paper, we establish the estimation of the conditional hazard function in the single functional index model for α-mixing functional data. Under some conditions, we present the uniform almost complete convergence of the local linear estimator with rate. However, the normality of this model in the same method have been invistgated in our other work.

4.5 Appendix

We use the procedure of Masry (1986) for proofing the Lemma 4.5.1 and Lemma 4.5.2 that we need for proved our lemmas.

Lemma 4.5.1. Under assumptions $(U1)-(U8)$, we have

$$S^2_{\theta,n}=O(n\phi(h_K)),$$

where, for $k=0,1,2$, we define $S^2_{\theta,n}$ as follows

$$S^2_{\theta,n}=\sum_{i,j=1}^{n}|\text{Cov}(\Lambda^{ik}_{\theta}(x),\Lambda^{jk}_{\theta}(x))| \text{ and } \Lambda^{ik}_{\theta}(x)=\frac{1}{h_K^k}\left(K_{\theta,i}(x)\beta^k_{\theta,i}(x)-\mathbb{E}[K_{\theta,i}(x)\beta^k_{\theta,i}(x)]\right).$$

Proof of Lemma 4.5.1
For determine $S^2_{\theta,n}$, we have for $k=0,1,2$ that

$$S^2_{\theta,n}=\sum_{i\neq j=1}^{n}\text{Cov}(\Lambda^{ik}_{\theta}(x),\Lambda^{jk}_{\theta}(x))+n\text{Var}(\Lambda^{1k}_{\theta}(x)).$$

First, we calculate the quantity $\sum_{i\neq j=1}^{n}\text{Cov}(\Lambda^{ik}_{\theta}(x),\Lambda^{jk}_{\theta}(x))$. So, collecting this latter over the sets S'_1 and S'_2, where, for the sequence v_n, we define the sets as follow

$$S'_1=\{(i,j),1\le i-j\le v_n\} \text{ and } S'_2=\{(i,j),v_n+1\le i-j\le n-1\}.$$

Then, the sum over S_1' and under $(U1), (U3_{ii})$, and $(U5_{ii})$, permit us to write

$$\begin{aligned}\sum_{(i,j)\in S_1'} \mathrm{Cov}(\Lambda_\theta^{i^k}(x), \Lambda_\theta^{j^k}(x)) &\leq Cnv_n\Big(\mathbb{P}(X_i, X_j) \in B(x, h_K) \times B(x, h_K) \\ &\qquad + \mathbb{P}(X_i \in B(x, h_K))\mathbb{P}(X_j \in B(x, h_K))\Big) \\ &\leq Cnv_n\Big(\phi(h_K)^{1+d} + \phi(h_K)^2\Big) \\ &\leq Cnv_n\phi(h_K)^{1+d}. \qquad (2)\end{aligned}$$

Second, regarding the sum over the set S_2', we use Davydov-Rio's inequality, for all $i \neq j$

$$\Big|\mathrm{Cov}\Big(\Lambda_\theta^{i^k}(x), \Lambda_\theta^{j^k}(x)\Big)\Big| \leq C\alpha(|i-j|),$$

and under $(U3_i)$

$$\Big|\sum_{(i,j)\in S_2'} \mathrm{Cov}(\Lambda_\theta^{i^k}(x), \Lambda_\theta^{j^k}(x))\Big| \leq C\frac{nv_n^{-a+1}}{a-1}, \qquad (3)$$

then, under $(U8)$ and by choosing $v_n = (\phi(h_K))^{-1/a}$ in (2), we get

$$\sum_{(i,j)\in S_1'} \mathrm{Cov}(\Lambda_\theta^{i^k}(x), \Lambda_\theta^{j^k}(x)) \leq Cn\phi(h_K)n^{-t(\frac{ad-1}{a})},$$

thus,

$$\sum_{(i,j)\in S_1'} \mathrm{Cov}(\Lambda_\theta^{i^k}(x), \Lambda_\theta^{j^k}(x)) = O(n\phi(h_K)). \qquad (4)$$

Also, for (3) we use the same v_n, for $a > 2$ and for $k = 0, 1, 2$, we obtain

$$\sum_{(i,j)\in S_2'} \mathrm{Cov}\Big(\Lambda_\theta^{i^k}(x), \Lambda_\theta^{j^k}(x)\Big) = O(n\phi^{(a-1)/a}(h_K)). \qquad (5)$$

About the variance term, and under $(U1)$, we have

$$\begin{aligned}\mathrm{Var}(\Lambda_\theta^{1^k}(x)) &\leq C\Big(\phi(h_K) + \phi^2(h_K)\Big) \\ &\leq C\phi(h_K). \qquad (6)\end{aligned}$$

Finally, from (4), (5), and (6) we get

$$S_{\theta,n}^2 = O(n\phi(h_K)).$$

Lemma 4.5.2. Under assumptions $(U1) - (U8)$, we have

$$\tilde{S}_{\theta,n}^2 = O(nh_H\phi(h_K)),$$

where, for $k = 0, 1$, we define $\tilde{S}_{\theta,n}^2$ as follows

$$\tilde{S}_{\theta,n}^2 = \sum_{i,j=1}^{n} |\mathrm{Cov}(\Upsilon_{\theta,i}^k(x), \Upsilon_{\theta,j}^k(x))| \text{ and } \Upsilon_{\theta,i}^k = \frac{1}{h_K^k}[K_{\theta,i}(x)H_i(y)\beta_{\theta,i}^k(x) - \mathbb{E}[K_{\theta,i}(x)H_i(y)\beta_{\theta,i}^k(x)].$$

Proof of Lemma 4.5.2
In the same manner of Lemma 4.5.1 where $v'_n \to \infty$, we calculate the summation of $\sum_{i\neq j}^{n} |\mathrm{Cov}(\Upsilon_i^k, \Upsilon_j^k)|$ over $\tilde{S}_1$ and $\tilde{S}_2$

$$\tilde{S}_1 = \{(i,j) \text{ such that } 0 < |i-j| < v'_n\} \text{ and } \tilde{S}_2 = \{(i,j) \text{ such that } v'_n + 1 < |i-j| < n-1\},$$

So, under $(U1)$ and $(U3)-(U6)$ and some calculate, we get

$$\sum_{(i,j)\in\tilde{S}_1}^{n} \mathrm{Cov}(\Upsilon_i^k, \Upsilon_j^k) \leq Cnh_H^2\phi(h_K), \tag{7}$$

and for $\tilde{S}_2$ we use again Davydov-Rio's inequality for bounded mixing processes,

$$\sum_{(i,j)\in\tilde{S}_2}^{n} \mathrm{Cov}(\Upsilon_i^k, \Upsilon_j^k) \leq Cnv_n'^{1-a}, \tag{8}$$

then, taking $v'_n = \left(\frac{1}{h_H^2\phi(h_K)}\right)^{1/a}$ in (7) and (8), we obtain

$$\sum_{i\neq j}^{n} |\mathrm{Cov}(\Upsilon_i^k, \Upsilon_j^k)| = O(n(h_H^2\phi(h_K))^{(a-1)/a}), \tag{9}$$

now, for the variance term, we use the same concepts in Lemma 4.5.2 to get

$$\mathrm{Var}(\Upsilon_1^k) \leq h_H\phi(h_K). \tag{10}$$

Finally, from (9) and (10), we obtain

$$\tilde{S}_{\theta,n}^2 = O(nh_H\phi(h_K)).$$

Proof of Lemma 4.3.1
By the Lemma 4.3 in Barrientos *et al.* (2010) in the regression function, the proofs of Lemma 1.1 can be completed, due to the bias term is not constrained by the dependence shape.

Proof of Lemma 4.3.2
We have by adoption from the proof of Lemma 4.4 in Barrientos *et al.* (2010)

$$\widehat{g}_{\theta,D}^x = \underbrace{\frac{n^2h_K^2\phi_{\theta,x}^2(h_K)}{n(n-1)\mathbb{E}[W_{\theta,12}(x)]}}_{S_1}\Big[\underbrace{\Big(\frac{1}{n}\sum_{j=1}^{n}\frac{K_{\theta,j}(x)}{\phi_{\theta,x}(h_K)}\Big)}_{S_{\theta,2}^x}\underbrace{\Big(\frac{1}{n}\sum_{i=1}^{n}\frac{K_{\theta,i}(x)\beta_{\theta,i}^2(x)}{h_K^2\phi_{\theta,x}(h_K)}\Big)}_{S_{\theta,4}^x}$$
$$-\underbrace{\Big(\frac{1}{n}\sum_{j=1}^{n}\frac{K_{\theta,j}(x)\beta_{\theta,j}(x)}{h_K\phi_{\theta,x}(h_K)}\Big)}_{S_{\theta,3}^x}\underbrace{\Big(\frac{1}{n}\sum_{i=1}^{n}\frac{K_{\theta,i}(x)\beta_{\theta,i}(x)}{h_K\phi_{\theta,x}(h_K)}\Big)}_{S_{\theta,3}^x}\Big].$$

We have that $\mathbb{E}[\widehat{g}_{\theta,D}^x] = 1$.
So,

$$\widehat{g}_{\theta,D}^x - \mathbb{E}[\widehat{g}_{\theta,D}^x] = S_1\Big((S_{\theta,2}^xS_{\theta,4}^x - \mathbb{E}[S_{\theta,2}^xS_{\theta,4}^x]) - ((S_{\theta,3}^x)^2 - \mathbb{E}[(S_{\theta,3}^x)^2])\Big),$$

and we put

$$S^x_{\theta,2}S^x_{\theta,4} - \mathbb{E}[S^x_{\theta,2}S^x_{\theta,4}] = (S^x_{\theta,2} - \mathbb{E}[S^x_{\theta,2}])(S^x_{\theta,4} - \mathbb{E}[S^x_{\theta,4}]) + (S^x_{\theta,4} - \mathbb{E}[S^x_{\theta,4}])\mathbb{E}[S^x_{\theta,2}] + (S^x_{\theta,2} - \mathbb{E}[S^x_{\theta,2}])\mathbb{E}[S^x_{\theta,4}] - \mathrm{Cov}(S^x_{\theta,2}S^x_{\theta,4}),$$

$$(S^x_{\theta,3})^2 - \mathbb{E}[(S^x_{\theta,3})^2] = (S^x_{\theta,3} - \mathbb{E}[S^x_{\theta,3}])^2 + 2(S^x_{\theta,3} - \mathbb{E}[S^x_{\theta,3}])\mathbb{E}[S^x_{\theta,3}] - \mathrm{Var}[S^x_{\theta,3}].$$

thus, these decompositions allows us to proved the following equations

$$S_1 = O(1) \text{ and } \mathbb{E}[S^x_{\theta,z}] = O(1) \text{ for z} = 2,3,4. \tag{11}$$

$$\mathrm{Var}[S^x_{\theta,3}] = \mathrm{o}\left(\sqrt{\frac{\ln(N^{S_\mathcal{H}}N^{\Theta_\mathcal{H}})}{n\phi(h_K)}}\right). \tag{12}$$

$$\mathrm{Cov}(S^x_{\theta,2}, S^x_{\theta,4}) = \mathrm{o}\left(\sqrt{\frac{\ln(N^{S_\mathcal{H}}N^{\Theta_\mathcal{H}})}{n\phi(h_K)}}\right). \tag{13}$$

$$\sum_{n=1}^{\infty}\mathbb{P}\left\{|S^x_{\theta,z} - \mathbb{E}[S^x_{\theta,z}]| > \eta\sqrt{\frac{\ln(N^{S_\mathcal{H}}N^{\Theta_\mathcal{H}})}{n\phi(h_K)}}\right\} < \infty, \text{ for z} = 2,3,4. \tag{14}$$

We have that (11) is proved in Barrientos et al. (2010). So, it requires to prove (12), (13), and (14). Concerning (14), we consider the following decomposition

$$\begin{aligned}\sup_{\theta\in\Theta_\mathcal{H}}\sup_{x\in S_\mathcal{H}}|S^x_{\theta,z} - \mathbb{E}[S^x_{\theta,z}]| \le &\sup_{\theta\in\Theta_\mathcal{H}}\sup_{x\in S_\mathcal{H}}|S^x_{\theta,z} - S^{x_{j(x)}}_{\theta,z}|\}_{\mathrm{E}_1}\\ &+ \sup_{\theta\in\Theta_\mathcal{H}}\sup_{x\in S_\mathcal{H}}|S^{x_{j(x)}}_{\theta,z} - S^{x_{j(x)}}_{t_{j''(\theta)},z}|\}_{\mathrm{E}_2}\\ &+ \sup_{\theta\in\Theta_\mathcal{H}}\sup_{x\in S_\mathcal{H}}|S^{x_{j(x)}}_{t_{j''(\theta)},z} - \mathbb{E}[S^{x_{j(x)}}_{t_{j''(\theta)},z}]|\}_{\mathrm{E}_3}\\ &+ \sup_{\theta\in\Theta_\mathcal{H}}\sup_{x\in S_\mathcal{H}}|\mathbb{E}[S^{x_{j(x)}}_{t_{j''(\theta)},z}] - \mathbb{E}[S^{x_{j(x)}}_{\theta,z}]|\}_{\mathrm{E}_4}\\ &+ \sup_{\theta\in\Theta_\mathcal{H}}\sup_{x\in S_\mathcal{H}}|\mathbb{E}[S^{x_{j(x)}}_{\theta,z}] - \mathbb{E}[S^x_{\theta,z}]|\}_{\mathrm{E}_5}\end{aligned}$$

- **Analyse E_3** Applying the Fuck-Nagaev exponential inequality (Proposition A.11(ii), Ferraty and Vieu (2006), for all $\varepsilon > 0, r > 0$, we have

$$\begin{aligned}\mathbb{P}(\mathrm{E}_3 > \varepsilon) &= \mathbb{P}\left(\max_{j''\in\{1,\dots,N^{\Theta_\mathcal{H}}\}}\max_{j'\in\{1,\dots,N^{S_\mathcal{H}}\}}|S^{x_{j(x)}}_{t_{j''(\theta)},z} - \mathbb{E}[S^{x_{j(x)}}_{t_{j''(\theta)},z}]| > \varepsilon\right)\\ &\le N^{\Theta_\mathcal{H}}N^{S_\mathcal{H}}\max_{j''\in\{1,\dots,N^{\Theta_\mathcal{H}}\}}\max_{j'\in\{1,\dots,N^{S_\mathcal{H}}\}}\mathbb{P}\left(|S^{x_{j(x)}}_{t_{j''(\theta)},z} - \mathbb{E}[S^{x_{j(x)}}_{t_{j''(\theta)},z}]| > \varepsilon\right)\\ &\le N^{\Theta_\mathcal{H}}N^{S_\mathcal{H}}\max_{j''\in\{1,\dots,N^{\Theta_\mathcal{H}}\}}\max_{j'\in\{1,\dots,N^{S_\mathcal{H}}\}}\mathbb{P}\left(\Big|\sum_{i=1}^{n}\Lambda^i_{t_{j''(\theta)},z}(x_{j(x)})\Big| > n\phi(h_K)\varepsilon\right)\end{aligned} \tag{15}$$

where, for $k = 0,1,2,$ and for $z = 2,3,4$,
$\Lambda^{ik}_{t_{j''(\theta)},z}(x_{j(x)}) = \frac{1}{h_K^k}\left(K_{t_{j''(\theta)},i}(x_{j(x)})\beta^k_{t_{j''(\theta)},i}(x_{j(x)}) - \mathbb{E}[K_{t_{j''(\theta)},i}(x_{j(x)})\beta^k_{t_{j''(\theta)},i}(x_{j(x)})]\right)$,
and since $\mathbb{E}[|\Lambda^i_{t_{j''(\theta)},z}(x_{j(x)})|^p] \le C\phi(h_K)$, for $p > 2$, $0 < C < \infty$, we get

$$\mathbb{P}(\mathrm{E}_3 > \varepsilon) \le N^{\Theta_\mathcal{H}}N^{S_\mathcal{H}}\Big(1 + \frac{\varepsilon^2n^2\phi(h_K)^2}{rS_n^2}\Big)^{-r/2} + N^{\Theta_\mathcal{H}}N^{S_\mathcal{H}}nr^{-1}\Big(\frac{r}{\varepsilon n\phi(h_K)}\Big)^{a+1}$$

we put

$$A_1 = N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}} \Big(1 + \frac{\varepsilon^2 n^2 \phi(h_K)^2}{r S_n^2}\Big)^{-r/2}, \; A_2 = N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}} n r^{-1} \Big(\frac{r}{\varepsilon n \phi(h_K)}\Big)^{a+1}.$$

Now, taking $r = C(\ln n)^2$, $\varepsilon = \frac{\iota\sqrt{S_n^2 \ln N^{S_{\mathcal{H}}} N^{\Theta_{\mathcal{H}}}}}{n\phi(h_K)}$ in A_1 and by Lemma 4.5.2, we get

$$A_1 \leq N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}} \Big(1 + \frac{\iota^2 \ln n}{(\ln n)^2}\Big)^{-(\ln n)^2/2} = N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}} \exp\Big(-\iota^2 \frac{\ln n}{2}\Big) = C N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}} n^{-\iota^2/2},$$

therefore, for favorable ι

$$A_1 \leq C n^{-\iota^2/2},$$

about A_2, the same r and ε in pervious calculation and Lemma 4.5.2, we obtain

$$A_2 \leq C N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}} n^{1-(a+1)/2} \phi(h_K)^{-(a+1)/4} (\ln n)^{(3a-1)/2}$$

then, by $(U8)$, there exist $\nu > 0$ such that

$$A_2 \leq C n^{-1-\nu},$$

So,

$$\mathbb{P}\left(\mathrm{E}_3 > \iota\sqrt{\frac{\ln N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}}}{n\phi(h_K)}}\right) < \infty.$$

- **Analyse $\mathbf{E}_1$ and $\mathbf{E}_5$** We can write the term E_1 by

$$\begin{aligned}
\mathrm{E}_1 &\leq \frac{C}{n h_K^{k-2} \phi(h_K)} \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sum_{i=1}^{n} |K_{\theta,i}(x)\beta_{\theta,i}^{k-2}(x) 1_{B(x,h)}(X_i) - K_{\theta,i}(x_{j(x)})\beta_{\theta,i}^{k-2}(x_{j(x)}) 1_{B(x_{j(x)},h)}(X_i)| \\
&\leq \frac{C}{n h_K^{k-2} \phi(h_K)} \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sum_{i=1}^{n} K_{\theta,i}(x) 1_{B(x,h)(X_i)} |\beta_{\theta,i}^{k-2}(x) - \beta_{\theta,i}^{k-2}(x_{j(x)}) 1_{B(x_{j(x)},h)}(X_i)| \\
&\quad + \frac{C}{n h_K^{k-2} \phi(h_K)} \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sum_{i=1}^{n} \beta_{\theta,i}^{k-2}(x_{j(x)}) 1_{B(x_{j(x)},h)(X_i)} |K_{\theta,i}(x) 1_{B(x,h)(X_i)} - K_{\theta,i}(x_{j(x)})| \\
&:= \mathrm{E}_{1,1} + \mathrm{E}_{1,2}.
\end{aligned}$$

Concerning the term $\mathrm{E}_{1,1}$

$$1_{B(x,h)}(X_i)|\beta_{\theta,i}(x) - \beta_{\theta,i}(x_{j(x)}) 1_{B(x_{j(x)},h)}(X_i)| \leq C r_n 1_{B(x,h)\cap B(x_{j(x)},h)}(X_i) + C h_K 1_{B(x,h)\cap \overline{B(x_{j(x)},h)}}(X_i)$$

and

$$1_{B(x,h)}(X_i)|\beta_{\theta,i}^2(x) - \beta_{\theta,i}^2(x_{j(x)}) 1_{B(x_{j(x)},h)}(X_i)| \leq C r_n h_K 1_{B(x,h)\cap B(x_{j(x)},h)}(X_i) + C h_K^2 1_{B(x,h)\cap \overline{B(x_{j(x)},h)}}(X_i)$$

for $k = 3, 4$

$$1_{B(x,h)}(X_i)|\beta_{\theta,i}^{k-2}(x) - \beta_{\theta,i}^{k-2}(x_{j(x)}) 1_{B(x_{j(x)},h)}(X_i)| \leq C r_n h_K^{k-2} 1_{B(x,h)\cap B(x_{j(x)},h)}(X_i) + C h_K^{k-2} 1_{B(x,h)\cap \overline{B(x_{j(x)},h)}}(X_i)$$

So,

$$\mathrm{E}_{1,1} \leq \frac{C r_n}{n h_K \phi(h_K)} \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sum_{i=1}^{n} K_{\theta,i}(x) 1_{B(x,h)\cap B(x_{j(x)},h)}(X_i) + \frac{C}{n\phi(h)} \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sum_{i=1}^{n} K_{\theta,i}(x) 1_{B(x,h)\cap \overline{B(x_{j(x)},h)}}(X_i)$$

Concerning the term $\mathrm{E}_{1,2}$

$$
\begin{aligned}
&1_{B(x_{j(x)},h)(X_i)}|K_{\theta,i}(x)1_{B(x,h)}(X_i) - K_{\theta,i}(x_{j(x)})1_{B(x,h)\cup\overline{B(x,h)}}(X_i)| \\
&\leq 1_{B(x_{j(x)},h)\cap B(x,h)}(X_i)|K_{\theta,i}(x) - K_{\theta,i}(x_{j(x)})| + K_{\theta,i}(x_{j(x)})1_{B(x_{j(x)},h)\cap\overline{B(x,h)}}(X_i)
\end{aligned}
$$

and by

$$
\begin{aligned}
&|\beta_{\theta,i}^{k-2}(x_{j(x)})|1_{B(x_{j(x)},h)}(X_i)1_{B(x_{j(x)},h)(X_i)}|K_{\theta,i}(x)1_{B(x,h)}(X_i) - K_{\theta,i}(x_{j(x)})| \\
&\leq Ch_K^{k-2}\frac{r_n}{h_K}1_{B(x_{j(x)},h)\cap B(x,h)}(X_i) + K_{\theta,i}(x_{j(x)})1_{B(x_{j(x)},h)\cap\overline{B(x,h)}}(X_i)
\end{aligned}
$$

we get

$$
\begin{aligned}
\mathrm{E}_{1,2} \leq{}& \frac{Cr_n}{nh_K\phi(h_K)}\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sum_{i=1}^{n}1_{B(x,h)\cap B(x_{j(x)},h)}(X_i) \\
&+\frac{C}{n\phi(h_K)}\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sum_{i=1}^{n}K_{\theta,i}(x_{j(x)})1_{B(x_{j(x)},h_K)\cap\overline{B(x,h_K)}}(X_i)
\end{aligned}
$$

finally, by $\mathrm{E}_{1,1}$ and $\mathrm{E}_{1,2}$, we get

$$
\mathrm{E}_1 \leq \frac{Cr_n}{nh_K\phi(h_K)}\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sum_{i=1}^{n}1_{B(x_{j(x)},h_K)\cup B(x,h_K)}(X_i),
$$

we put

$$
T_i = \frac{Cr_n}{h_K}\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}1_{B(x_{j(x)},h_K)\cup B(x,h_K)}(X_i)
$$

and under hypotheses $(U1),(U3_{ii}),(U8)$, and Lemma 4.5.1, we have that

$$
S_n^2 = \sum_{i,j=1}^{n}|\mathrm{Cov}(T_i,T_j)| = O(n\phi(h_K)).
$$

So, from E_3, we obtain $\mathrm{E}_1 = \mathrm{O}_{a.co.}\left(\sqrt{\frac{\ln(N^{\Theta_{\mathcal{H}}}N^{S_{\mathcal{H}}})}{n\phi(h_K)}}\right)$, and we deduce

$$
\mathrm{E}_5 \leq \mathrm{E}_1 = \mathrm{O}_{a.co.}\left(\sqrt{\frac{\ln(N^{\Theta_{\mathcal{H}}}N^{S_{\mathcal{H}}})}{n\phi(h_K)}}\right).
$$

• **Analyse $\mathbf{E}_2$ and $\mathbf{E}_4$** We using the same ideas those used in E_1 and E_5, we get

$$
\mathrm{E}_4 \leq \mathrm{E}_2 = \mathrm{O}_{a.co.}\left(\sqrt{\frac{\ln(N^{\Theta_{\mathcal{H}}}N^{S_{\mathcal{H}}})}{n\phi(h_K)}}\right).
$$

Now, for (12) and (13), we will use results that obtained in Lemma 4.5.2, we obtain

$$
\mathrm{Var}[S_{\theta,3}^x] = \mathrm{o}\left(\frac{1}{n\phi(h_K)}\right) = \mathrm{o}\left(\sqrt{\frac{\ln(N^{\Theta_{\mathcal{H}}}N^{S_{\mathcal{H}}})}{n\phi(h_K)}}\right),
$$

and

$$
\mathrm{Cov}(S_{\theta,2}^x, S_{\theta,4}^x) = \mathrm{O}\left(\frac{1}{n\phi(h_K)}\right) = \mathrm{o}\left(\sqrt{\frac{\ln(N^{\Theta_{\mathcal{H}}}N^{S_{\mathcal{H}}})}{n\phi(h_K)}}\right).
$$

Finally, from this later results, we deduce that

$$\sum_{i=1}^{\infty}\mathbb{P}(\inf_{\theta\in\Theta_{\mathcal{H}}}\inf_{x\in S_{\mathcal{H}}}\widehat{g}^{x}_{\theta,D}<1/2)<\infty,$$

because we have $\inf_{\theta\in\Theta_{\mathcal{H}}}\inf_{x\in S_{\mathcal{H}}}\widehat{g}^{x}_{\theta,D}<1/2$ and this latter, involved that

$$1-\widehat{g}^{x}_{\theta,D}<1/2\Rightarrow\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}|1-\widehat{g}^{x}_{\theta,D}>|<1/2,$$

so, after the step of insertion the probability on both sides, we finished the proof of lemma.

Proof of Lemma 4.3.3
Firstly, regerding to $\mathbb{E}[\widehat{f}^{x}_{\theta,N}(y)]$, so, by the fact that $S_{\mathbb{R}}$ is a compact set, we can write that $S_{\mathbb{R}}\in\bigcup_{J=1}^{N^{\mathbb{R}}}(y_J-l_n,y_J+l_n)$ and putting $l_n=n^{-\lambda-1/2}$and $z_y=\arg\min_{J\in\{1,\ldots,N^{\mathbb{R}}\}}|y-y_J|$, we obtain

$$\begin{aligned}\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}|f^{x}_{\theta,N}(y)-\mathbb{E}[f^{x}_{\theta,N}(y)]|\le&\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}|f^{x}_{\theta,N}(y)-f^{x_{j(x)}}_{\theta,N}(y)|\}_{\text{E'}_1}\\&+\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}|f^{x_{j(x)}}_{\theta,N}(y)-f^{x_{j(x)}}_{t_{j''(\theta)},N}(y)|\}_{\text{E'}_2}\\&+\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}|f^{x_{j(x)}}_{t_{j''(\theta)},N}(y)-f^{x_{j(x)}}_{t_{j''(\theta)},N}(z_y)|\}_{\text{E'}_3}\\&+\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}|f^{x_{j(x)}}_{t_{j''(\theta)},N}(z_y)-\mathbb{E}[f^{x_{j(x)}}_{t_{j''(\theta)},N}(z_y)]|\}_{\text{E'}_4}\\&+\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}|\mathbb{E}[f^{x_{j(x)}}_{t_{j''(\theta)},N}(z_y)]|-\mathbb{E}[f^{x_{j(x)}}_{t_{j''(\theta)},N}(y)]|\}_{\text{E'}_5}\\&+\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}|\mathbb{E}[f^{x_{j(x)}}_{t_{j''(\theta)},N}(y)]|-\mathbb{E}[f^{x_{j(x)}}_{\theta,N}(y)]|\}_{\text{E'}_6}\\&+\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}|\mathbb{E}[f^{x_{j(x)}}_{\theta,N}(y)]|-\mathbb{E}[f^{x}_{\theta,N}(y)]|\}_{\text{E'}_7}\end{aligned}$$

- **Analyse E'$_1$ and E'$_7$** The same steps that using in E_1 and E_5, we get

$$\text{E'}_7\le\text{E'}_1=\text{O}_{a.co.}\left(\sqrt{\frac{\ln(N^{\Theta_{\mathcal{H}}}N^{S_{\mathcal{H}}})}{n\phi(h_K)}}\right).$$

- **Analyse E'$_4$** we have for all $\eta>0$

$$\begin{aligned}&\mathbb{P}\left(\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}|f^{x_{j(x)}}_{t_{j''(\theta)},N}(z_y)-\mathbb{E}[f^{x_{j(x)}}_{t_{j''(\theta)},N}(z_y)]|>\eta\sqrt{\frac{\ln(N^{\Theta_{\mathcal{H}}}N^{S_{\mathcal{H}}})}{nh_H\phi(h_K)}}\right)\\&=\mathbb{P}\left(\max_{j''\in\{1,\ldots,N^{\Theta_{\mathcal{H}}}\}}\max_{j'\in\{1,\ldots,N^{S_{\mathcal{H}}}\}}\max_{z_y\in\{1,\ldots,N^{\mathbb{R}}\}}|f^{x_{j(x)}}_{t_{j''(\theta)},N}(z_y)-\mathbb{E}[f^{x_{j(x)}}_{t_{j''(\theta)},N}(z_y)]>\eta\sqrt{\frac{\ln(N^{\Theta_{\mathcal{H}}}N^{S_{\mathcal{H}}})}{nh_H\phi(h_K)}}\right)\\&\le N^{\Theta_{\mathcal{H}}}N^{S_{\mathcal{H}}}N^{\mathbb{R}}\max_{j''\in\{1,\ldots,N^{\Theta_{\mathcal{H}}}\}}\max_{j'\in\{1,\ldots,N^{S_{\mathcal{H}}}\}}\max_{z_y\in\{1,\ldots,N^{\mathbb{R}}\}}\\&\qquad\mathbb{P}\left(|f^{x_{j(x)}}_{t_{j''(\theta)},N}(z_y)-\mathbb{E}[f^{x_{j(x)}}_{t_{j''(\theta)},N}(z_y)]|>\eta\sqrt{\frac{\ln(N^{\Theta_{\mathcal{H}}}N^{S_{\mathcal{H}}})}{nh_H\phi(h_K)}}\right),\end{aligned}$$

then, by adaption of the proof of Lemma 4.3.2, we have

$$f^{x}_{\theta,N}(y)=S_1[T_{\theta,2}(y)S^{x}_{\theta,4}-T_{\theta,5}(y)S^{x}_{\theta,3}]$$

where $S_1, S^x_{\theta,3}$ and $S^x_{\theta,4}$ are the same terms in Lemma 4.3.2, such that

$$T_{\theta,2}(y) = \frac{1}{n}\sum_{j=1}^{n}\frac{K_{\theta,j}(x)H_j(y)}{h_H\phi_{\theta,x}(h_K)},\ T_{\theta,5}(y) = \frac{1}{n}\sum_{j=1}^{n}\frac{K_{\theta,j}(x)\beta_j(x)H_j(y)}{h_Hh_K\phi_{\theta,x}(h_K)}.$$

Then, we have the decomposition

$$f^x_{\theta,N}(y) - \mathbb{E}[f^x_{\theta,N}(y)] = S_1\Bigg(\Big(T_{\theta,2}(y)S^x_{\theta,4} - \mathbb{E}[T_{\theta,2}(y)S^x_{\theta,4}]\Big) - \Big(T_{\theta,5}(y)S^x_{\theta,3} - \mathbb{E}[T_{\theta,5}(y)S^x_{\theta,3}]\Big)\Bigg),$$

which implies that

$$\begin{aligned}T_{\theta,2}(y)S^x_{\theta,4} - \mathbb{E}[T_{\theta,2}(y)S^x_{\theta,4}] &= (T^x_{\theta,2}(y) - \mathbb{E}[T^x_{\theta,2}(y)])(S^x_{\theta,4} - \mathbb{E}[S^x_{\theta,4}]) + (S^x_{\theta,4} - \mathbb{E}[S^x_{\theta,4}])\mathbb{E}[T^x_{\theta,2}(y)]\\ &+(T^x_{\theta,2}(y) - \mathbb{E}[T^x_{\theta,2}(y)])\mathbb{E}[S^x_{\theta,4}] - \mathrm{Cov}(T^x_{\theta,2}(y), S^x_{\theta,4}).\end{aligned}$$

$$\begin{aligned}T_{\theta,5}(y)S^x_{\theta,3} - \mathbb{E}[T_{\theta,5}(z_y)S^x_{\theta,3}] &= (T^x_{\theta,5}(y) - \mathbb{E}[T^x_{\theta,5}(y)])(S^x_{\theta,3} - \mathbb{E}[S^x_{\theta,3}]) + (S^x_{\theta,3} - \mathbb{E}[S^x_{\theta,3}])\mathbb{E}[T^x_{\theta,5}(y)]\\ &+(T^x_{\theta,5}(y) - \mathbb{E}[T^x_{\theta,5}(y)])\mathbb{E}[S^x_{\theta,3}] - \mathrm{Cov}(T^x_{\theta,5}(y), S^x_{\theta,3}).\end{aligned}$$

So, our results are from the following assertions

$$S_1 = O(1) \text{ and } \mathbb{E}[S^x_{\theta,s}] = O(1),\ \text{ for } s = 3, 4 \tag{16}$$

$$\mathbb{E}[T_{\theta,s}(y)] = O(1),\ \text{ for } s = 2, 5 \tag{17}$$

$$\mathrm{Cov}(T_{\theta,2}(y), S^x_{\theta,4}) = \mathrm{o}\left(\sqrt{\frac{\ln N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}}}{n\phi(h_K)}}\right), \tag{18}$$

$$\mathrm{Cov}(T_{\theta,5}(y), S^x_{\theta,3}) = \mathrm{o}\left(\sqrt{\frac{\ln (N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}})}{n\phi(h_K)}}\right), \tag{19}$$

$$\sum_{n=1}^{\infty}\mathbb{P}\left\{\Big|T_{\theta,s}(y) - \mathbb{E}[T_{\theta,s}(y)]\Big| > \eta\sqrt{\frac{\ln (N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}})}{nh_H\phi(h_K)}}\right\} < \infty,\ \text{ for } s = 2, 5. \tag{20}$$

Again, by the lemma in Barrientos et al. (2010), we obtain (16) and (17).
In other hand for (20), by the same manner in Lemma 4.3.2 for $\varepsilon, r > 1$, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\mathbb{P}\left\{\Big|T_{\theta,s}(y) - \mathbb{E}[T_{\theta,s}(y)]\Big| > \varepsilon\right\} &\le N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}} N^{\mathbb{R}}\mathbb{P}\left\{\Big|\Upsilon_{\theta,i}(y) - \mathbb{E}[\Upsilon_{\theta,i}(y)]\Big| > nh_H\phi(h_K)\varepsilon\right\}\\ &\le N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}} N^{\mathbb{R}}(A'_1 + A'_2),\end{aligned} \tag{21}$$

where $A'_1 = N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}} N^{\mathbb{R}}\Big(1 + \frac{n^2h_H^2\phi(h_K)^2\varepsilon^2}{r\tilde{S}_n^2}\Big)^{-r/2}$ and $A'_2 = N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}} N^{\mathbb{R}}nr^{-1}\Big(\frac{r}{nh_H\phi(h_K)\varepsilon}\Big)^{a+1}$.
The choice of $\varepsilon = C\eta\frac{\sqrt{\tilde{S}_n^2 \ln n}}{nh_H\phi(h_K)}$ and $r = C(\ln n)^2$, and from Lemma 4.5.2 we have that $\tilde{S}_n^2 = O(nh_H\phi(h_K))$, and by taking $N^{\mathbb{R}} = \frac{1}{l_n} = n^{\lambda+1/2}$, for $\nu > 0$ we obtain

$$CN^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}} N^{\mathbb{R}}(A'_1 + A'_2) \le Cn^{-1-\nu},$$

finally

$$\sum_{n=1}^{\infty}\mathbb{P}\left\{\Big|T_{\theta,s(y)} - \mathbb{E}[T_{\theta,s}(y)]\Big| > \eta\sqrt{\frac{\ln (N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}})}{nh_H\phi(h_K)}}\right\} < \infty.$$

About the covariance terms, by the same arguments used in (4.5) for (18) and (19) to get

$$\mathrm{Cov}(T_{\theta,2}(y), S_{\theta,4}) = o\left(\sqrt{\frac{(\ln N^{\Theta_\mathcal{H}} N^{S_\mathcal{H}})}{n\phi(h_K)}}\right),$$

$$\mathrm{Cov}(T_{\theta,5}(y), S_{\theta,3}) = o\left(\sqrt{\frac{\ln(N^{\Theta_\mathcal{H}} N^{S_\mathcal{H}})}{n\phi(h_K)}}\right).$$

- **Analyse E'$_2$ and E'$_6$** By using the same ideas of E'$_1$ and E'$_7$, we get

$$\text{E'}_6 \leq \text{E'}_2 = O_{a.co.}\left(\sqrt{\frac{\ln(N^{\Theta_\mathcal{H}} N^{S_\mathcal{H}})}{nh_H\phi(h_K)}}\right).$$

- **Analyse E'$_3$ and E'$_5$** The Lipschitz's condition on H permit us to write

$$\begin{aligned}
|f_{\theta,N}(y) - f^x_{\theta,N}(z_y)| &\leq \frac{C}{nh_H\phi_{\theta,x}(h_K)} \sup_{\theta\in\Theta_\mathcal{H}} \sup_{x\in S_\mathcal{H}} \sup_{y\in S_\mathbb{R}} \sum_{i\neq j=1}^{n} W_{\theta,ij}(x)|H^{'}(y) - H^{'}(z_y)| \\
&\leq C\frac{l_n}{h_H^2} \sup_{\theta\in\Theta_\mathcal{H}} \sup_{x\in S_\mathcal{H}} S^x_{\theta,s} \\
&\leq C\frac{l_n}{h_H^2}.
\end{aligned}$$

and we have that $S^x_{\theta,s}$ is proved in Lemma 4.3.2, and taking the same l_n in pervious calculs , we get

$$\text{E'}_5 \leq \text{E'}_3 = o_{a.co.}\left(\sqrt{\frac{\ln(N^{\Theta_\mathcal{H}} N^{S_\mathcal{H}})}{nh_H\phi(h_K)}}\right).$$

Secondly, for $\mathbb{E}[\widehat{F}^x_{\theta,N}(y)]$ The same steps that proved in pervious calculation.

Proof of Corollary 4.3.1
We have that

$$\begin{aligned}
&\inf_{\theta\in\Theta_\mathcal{H}} \inf_{x\in S_\mathcal{H}} \inf_{y\in S_\mathbb{R}} |1 - \widehat{F}^x(y)| \leq (1 - \sup_{\theta\in\Theta_\mathcal{H}} \sup_{x\in S_\mathcal{H}} \sup_{y\in S_\mathbb{R}} \hat{F}^x(y))/2 \\
\Longrightarrow \quad &\sup_{\theta\in\Theta_\mathcal{H}} \sup_{x\in S_\mathcal{H}} \sup_{y\in S_\mathbb{R}} |\widehat{F}^x(y) - F^x(y)| \geq (1 - \sup_{\theta\in\Theta_\mathcal{H}} \sup_{x\in S_\mathcal{H}} \sup_{y\in S_\mathbb{R}} \hat{F}^x(y))/2
\end{aligned}$$

then, by the probability on this terms, we get

$$\begin{aligned}
&\mathbb{P}\left\{\inf_{\theta\in\Theta_\mathcal{H}} \inf_{x\in S_\mathcal{H}} \inf_{y\in S_\mathbb{R}} |1 - \widehat{F}^x(y) \leq (1 - \sup_{\theta\in\Theta_\mathcal{H}} \sup_{x\in S_\mathcal{H}} \sup_{y\in S_\mathbb{R}} \hat{F}^x(y))/2\right\} \\
&\leq \mathbb{P}\left\{\sup_{\theta\in\Theta_\mathcal{H}} \sup_{x\in S_\mathcal{H}} \sup_{y\in S_\mathbb{R}} |\widehat{F}^x(y) - F^x(y)| \geq (1 - \sup_{\theta\in\Theta_\mathcal{H}} \sup_{x\in S_\mathcal{H}} \sup_{y\in S_\mathbb{R}} \hat{F}^x(y))/2\right\} < \infty.
\end{aligned}$$

So, taking $\mu = (1 - \sup_{\theta\in\Theta_\mathcal{H}} \sup_{x\in S_\mathcal{H}} \sup_{y\in S_\mathbb{R}} \hat{F}^x(y))/2$ and under the Lemmas 4.3.2 and Lemma 4.3.3, the proved of colloralry is finished.

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Conclusion et perspectives

Conclusion

Nous avons introduit dans cette thèse l'estimation non paramétrique de la fonction de hasard conditionnelle par la méthode locale linéaire quand la variable explicative est fonctionnelle et la réponse est réelle dans le cas indépendants et identiquement distribuées et dans le cas dépendant, notre étude bibliographique a été dirigée dans la première partie sur l'analyse des statistiques fonctionnelles. Puis d'après des hypothèses, et à partir de fonction de répartition et de densité conditionnelle, nous avons construit notre estimateur de la fonction de hasard conditionnelle, nous avons pu voir une partie pour étudier l'erreur quadratique de cet estimateur, tandis que le biais et la variance sont données. Nous illustrons ces résultats par simulation sur une échantillon fini, nous avons comparu l'estimateur qu'introduit dans cette thèse par un estimateur basée sur la méthode du noyau standard.

Dans la troisième partie, sous des conditions, nous avons présenté le modèle de l'indice simple, quand l'indice est fonctionnelle et connu dans le cas i.i.d., et nous avons établi la vitesse de convergence presque complète et la convergence uniforme. Enfin, sous des conditions sur la dépendance, ce qui est le cas α-mélangeant, nous avons obtenu la convergence uniforme de notre estimateur local linéaire qui définit sous le modèle de l'indice simple. Nos résultats confirment que la méthode locale linéaire est efficace et préférable par rapport la méthode du noyau.

Perspectives

Nous terminons ce travail par quelques perspectives dans cette thématique de recherche.

- Étudier la normalité asymptotique similaires à ceux du chapitre 3 dans le cas dépendant et indépendant avec des applications, aussi les tests pour la fonction de hasard conditionnelle dans cette méthode.
- Établir la convergence presque complète et uniforme de ce modèle avec d'autre type de données (incomplète, censurées,...).
- Estimation de l'indice fonctionnel de la fonction de hasard conditionnelle.
- Le choix de la fenêtre pour la fonction de hasard conditionnelle.

- Utiliser un autre type de dépendance entre les observations, est l'association, ainsi dans le cas ergodique.
- L'estimation récursive de la fonction de hasard conditionnelle dans le cas i.i.d. et spatial.
- L'estimation de la fonction de hasard conditionnelle pour variables réponse et explicative fonctionnelles.

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الملخص

في هذه الأطروحة ، ندرس الخصائص التقريبية لوظيفة الصدفة الشرطية بواسطة التقدير المحلي الخطي عندما يكون المتغير العشوائي الاستجوابي حقيقي و المتغير العشوائي التوضيحي ينتمي إلى فضاء ذو بعد غير محدود .

نبدأ بإعطاء الخطأ التربيعي لوظيفة الصدفة لمقدر محلي خطي من خلال إعطاء تعبير عن التحيز والتباين في الفضاء الشبه المتري ، وكذلك دراسة محاكاة لعينة محدودة ، التي تُظهر فعالية المقدر الخطي المحلي مقارنةً بتقدير النواة القياسي ، هذه النتيجة مهمة في تحديد اختيار معامل التنعيم.

في الخطوة الثانية ، عندما يأخذ المتغير التوضيحي قيمه في فضاء هيلبرت، نهتم بنموذج المؤشر الوظيفي البسيط في حالة استقلال الملاحظات بالتوزيع المتماثل عندما يكون المؤشر الوظيفي غير معروف و نحصل على التقارب النقطي و الشبه كامل القانوني مع السرعة. في نفس النموذج، نحدد التقارب الشبه الكامل القانوني في حالة الارتباط.

Résumé

Dans cette thèse, nous avons étudié les propriétés asymptotiques de la fonction de hasard conditionnelle par l'estimation locale linéaire lorsque la variable aléatoire réponse est réelle et la variable explicative appartient à un espace de dimension infinie.

Nous avons commencé par donnée l'erreur quadratique de la fonction de hasard d'un estimateur local linéaire en donnant l'expression du biais et la variance dans un espace semi-métrique, aussi une étude de simulation sur un échantillon fini, qui montre la performance de l'estimateur local linéaire par rapport à l'estimateur du noyau standard, ce résultat est important pour déterminer le choix du paramètre de lissage.

Dans un second temps, sous des conditions standards, quand la variable explicative prend ses valeurs dans un espace de Hilbert, nous nous intéressons au modèle avec l'indice fonctionnel simple dans le cas où les observations sont indépendantes identiquement distribuées (i.i.d.) lorsque l'indice fonctionnel est connu et nous avons obtenu la convergence ponctuelle et uniforme presque complète avec vitesse. Dans le même modèle, on établit la convergence uniforme presque complète pour le cas dépendent.

Summary

In this thesis, we studied the asymptotic properties of the conditional hazard function by the local linear estimation when response random variable is real and the explanatory variable belong to a space of infinite dimension.

We started by giving the quadratic error of the hazard function of a linear local estimator by giving the expression of bias and the variance term in semi-metric space, also a simulation study on a finite sample, which shows the performance of the linear local estimator compared to the standard kernel estimator.

In a second step, when the explanatory variable takes its values in a Hilbert space, we are interested in the single functional index model where the observations are independent identically distributed (i.i.d.) with the functional index is known and we obtain the pointwise and the uniform consistencies of the constructed estimator with rates. In the same model, we establish the uniform convergence for the dependent case.