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Intitulée

Sur l'étude asymptotique de quelques modèles non-paramétrique conditionnels en statistique fonctionnelle à direction révélatrice

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Résumé

La problématique traitée dans cette thèse concerne l'estimation non paramétrique de la fonction de répartition conditionnelle et ses dérivées à variable explicative fonctionnelle lorsque les données sont générées à partir d'un modèle de régression à indice fonctionnel simple pour des données complètes et des données censurées. Plus précisément, nous nous intéressons à l'estimation du quantile conditionnel et nous étudions son comportement asymptotique.

Dans un premier temps, nous supposons que les observations sont indépendantes et identiquement distribuées, et la variable explicative est à valeurs dans un espace de Hilbert (dimension infinie). Dans ce contexte, nous construisons un estimateur à double noyau de la fonction de répartition conditionnelle. Ainsi nous établissons la convergence presque-complète et la convergence uniforme de cet estimateur en précisant ses vitesses de convergence. Nous avons déduit un estimateur du quantile conditionnel, pour lequel nous avons établi la vitesse de convergence presque-complète et la convergence uniforme. Puis nous généralisons les résultats obtenus au cas où les observations sont fortement mélangeant.

Dans un second temps, nous traitons notre estimateur sous un modèle de données censurées à droite, et nous étudions son comportement asymptotique. Les résultats obtenus sont illustrés par des exemples sur des données simulées.

Mots clés : Données fonctionnelles, Estimation non paramétrique, Fonction de répartition conditionnelle, Quantile conditionnel, Indice fonctionnel simple, Convergence presque complète, Convergence uniforme, α-mélange, Données censurées.

Abstract

The problematic dealt with in this thesis concerns the nonparametric estimation of the conditional cumulative distribution and its derivatives when the data are generated from a simple functional index regression model for complete and incomplete data. More precisely, we are interested in the conditional quantile function and we study its asymptotic behavior.

First, we assume that the observations are independent and identically distributed, and the explanatory variable takes its values in Hilbert space (infinite dimensional space). In this context, we construct a kernel estimator of the conditional cumulative distribution. Afterward, we establish the pointwise and the uniform almost complete convergence with rates of the estimator, and we deduce similar asymptotic properties of the conditional quantile. Then, we generalize our results to the case where observations are strongly mixed.

Secondly, we treat our estimator under censored data and we study its asymptotic behavior. The established results are illustrated by examples on simulated data.

Keywords : Functional data, Nonparametric estimation, Conditional cumulative distribution, Conditional quantile, Functional single index, Almost complete convergence, Uniform convergence, α-mixing, Censored data.

Liste des travaux

Publications

1. Nadia Kadiri, Souad Mekkaoui, Abbes Rabhi. (2017). *Rate of uniform consistency for nonparametric of the conditional quantile estimate with functional variables in the single functional index model*. International Journal of Mathematics and Statistics. vol. 18(2) 2017, pp. 1-29.
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3. Nadia Kadiri, Abbes Rabhi, and Amina Angelika Bouchentouf. (2018). *Strong uniform consistency rates of conditional quantile estimation in the single functional index model under random censorship*. Depend.Model.6 :197-227.

Communications internationales

1. *Strong uniform consistency rates of conditional quantiles estimation in the single functional index model for dependant functional data under random censorship.*
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2. *Asymptotic normality of conditional density estimation in the single index model under random censorship.*
   International Workshop on Perspectives On High dimensional Data Analysis (HDDA-VIII-2018) Cadi Ayyad University and the Moroccan Association of Probability and Statistics, Marrakesh, Morocco.

Communications nationales

1. *Conditional density estimation in the single functional index model under random censorship.*
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Chapitre Premier

Introduction et Présentation

1.1 Statistique non-paramétrique fonctionnelle

L'analyse des données fonctionnelles occupe désormais une place importante dans la recherche en statistique, et connait un très important développement ces dernières années notamment avec les monographies de Ramsay et Silverman (1997,2002,2005), de Bosq (2000), Bosq et Blanke (2008), de Ferraty et Vieu (2006) et de Horvath et Kokoszka (2012).
Cette branche de statistique vise à étudier des données qui, de part leur structure et le fait qu'elles soient collectées sur des grilles très fines, peuvent être assimilées à des courbes ou à des surfaces, par exemple fonctions du temps ou de l'espace. Le besoin de considérer ce type de données, maintenant couramment rencontré sous le nom de *données fonctionnelles* dans la littérature, est avant tout un besoin pratique. Les situations pouvant fournir de telles données sont multiples et issues de domaines variées comme la météorologie, la chimie quantitative, la biométrie, l'économétrie, ou l'imagerie médicale...
En effet, Compte tenu des capacités actuelles des appareils de mesure et de stockage informatique , nous a permis de traiter de très grands ensembles de données , citons que quelques exemples des données de nature fonctionnelle auxquelles le statisticien peut être confronté ;
les courbes de croissance, de température, des images observées par un satellite, les courbes spectrométriques...

La littérature consacrée à ce type d'observations s'est largement développée, et les chercheurs scientifiques, se sont intéressés au développement d'outils statistiques permettant de traiter ce type d'échantillons. Ce sont Ferraty et Vieu (2000) qui ont permis d'obtenir les premiers travaux, ils ont étudié la convergence presque complète d'un estimateur à noyau de la fonction de régression pour des données indépendantes identiquement distribuées. Le cas dépendant a été traité par Ferraty et al (2002). Dabo-Niang (2002) a obtenu, la convergence presque sûre et la normalité asymptotique d'un estimateur de type histogramme de la densité d'une variable aléatoire dans un espace de dimension infinie. En considérant la propriété de concentration sur des petites boules de la variable explicative fonctionnelle, Dabo-Niang et Rhomari (2004) ont obtenu la convergence en norme L^p de l'estimateur à noyau de la régression non paramétrique. La convergence presque complète pour le cas fortement mélangeant a été étudié par Ferraty et al. (2004). Le premier résultat sur la normalité asymptotique de l'estimateur de la régression non paramétrique a été étudié par Masry (2005) dans le cas où les observations sont alpha-mélangeant.

Les premiers résultats sur les modèles conditionnels ont été obtenus par Ferraty et al (2006) ils ont montré la convergence presque complète de l'estimateur à noyau de quelque modèle conditionnel tel que la fonction de répartition conditionnelle, la densité conditionnelle, les quantiles conditionnelles ainsi que le mode conditionnel. Dans le même contexte, Dabo Niang et al (2004) ont établi la convergence en L^p de l'estimateur à noyau du mode conditionnel à valeur dans un espace vectoriel semi-normé de dimension infinie. Nous renvoyons à Ferraty et Vieu (2006) pour un large éventail d'applications de ces modèles en statistique fonctionnelle. L'estimation de la fonction du hasard conditionnelle a été abordé par Ferraty et al (2008), ils ont établi la convergence presque complète. La normalité asymptotique des estimateurs à noyau du mode conditionnel et des quantiles conditionnels ont été étudié par Ezzahrioui et Ould-saïd (2008a, 2008b,2008c) en traitant le cas i.i.d et cas mixing. L'estimation du quantile conditionnel comme inverse de la fonction de répartition conditionnelle a également été largement étudiée dans des différents types de corrélation par Ferraty, Rabhi et al (2005), Ferraty et al (2006), Ferraty et Vieu (2006a) et Ezzahrioui (2007). Laksaci et Maref (2009) ont aprouvé la version uniforme de la convergence presque complète de l'estimateur du noyau proposé par Ferraty et al. (2006). Lemdani et al . (2009) ont traité les quantiles conditionnels comme un modèle robuste appartenant à la classe des M-estimateurs. Ils ont établi la convergence presque complète et la normalité asymptotique dans le cas i.i.d.

La littérature sur les données fonctionnelles ergodiques est encore limité. Ce problème a été initié par Liab et Louani (2010). Ils ont considéré le problème de l'estimation fonctionnelle pour la fonction de régression non paramétrique dans le cas dépendant moins faible qui est l'état d'ergodicité. Ces derniers ont introduit la même version de l'estimateur que l'estimateur de Nadarya-Watson de type noyau de la fonction de régression classique construite par la méthode du noyau et ils ont établi la convergence uniforme et la vitesse presque complète lorsque les données fonctionnels sont ergodiques stationnaires. En (2011), ils ont étudié la convergence presque sûre et la normalité asymptotique de cet estimateur.

1.2 Modèles conditionnels en statistique non paramétrique fonctionnelle

L'étude statistique du lien entre deux variables aléatoires est un problème trés important. En effet, la régression non-paramétrique est un outil statistique permettant de déterminer la variation de l'espérence mathématique d'une variable dépendante Y en fonction de plusieurs variables explicatives, sans spécifier une forme stricte pour cette relation.

l'estimation non-paramétriqe de la densité conditionnelle qui présente une bonne alternative de régression. Plusieurs auteurs s'y intéresses, citons par exemple, dans le contexte des données fonctionnelles, les premiers travaux ont été obtenus par Ferraty,Laksaci et Vieu (2006) ils ont montré sous des conditions de régularités de la densité conditionnelle, la convergence presque complète de l'estimateur à noyau qu'ils ont introduit pour la densité conditionnelle ainsi que pour le mode conditionnel, et ils ont établi la vitessse de convergence. Une application de leurs résultats aux données issues de l'industrie agro-alimentaire a été présentée. Dans le même contexte Dabo Niang, Ferraty et Vieu (2006) ils ont établi la convergence uniforme presque complète, et ont spécifié les vitesses de convergence en montrant le poids de la mesure des petites boules sur ces derniers. Une application aux données spectrométriques a été présentée. En effet, en considérant des observations α-mélangeantes, Ferraty et al (2005) ont établi la convergence presque-complète d'un estimateur à noyau du mode conditionnel défini par la variable aléatoire maximisant la densité conditionnelle. Ezzahrioui et Ould-Saïd (2008), quant à eux, ont estimé le mode conditionnel par le point qui annule la dérivée de l'estimateur à noyau de la densité conditionnelle, en se concentrant sur la normalité asymptotique de l'estimateur proposé dans le cas i.i.d. Ces auteurs ont généralisé leurs résultats au cas de mélange dans (2010). Laksaci (2007) a précisé ensuite, les termes dominants de l'erreur quadratique de l'estimateur à noyau de la densité conditionnelle. En (2010), Laksaci et al. (2013) ont répondu à la question du choix du paramètre de lissage dans l'estimation de la densité conditionnelle lorsque la variable explicative est fonctionnelle.

Un autre paramètre fonctionnel qui est de grande importance, est le quantile conditionnel. Il donne une alternative majeure dans les prévisions, grâce à son caractère robuste. L'estimation non-paramétrique de la fonction de répartition conditionnelle lorsque la variable explicative est à valeurs dans un espace de dimension infini, a été introduite par Ferraty et al (2006). Ces auteurs ont construit un estimateur à double noyaux pour la fonction de répartition conditionnelle et ils ont précisé la vitesse de convergence presque-complète de cet estimateur lorsque les observations sont indépendantes et identiquement distribuées. Le cas où les observations sont fonctionnelles et α-mélangeantes a été étudié par Ferraty et al (2005), Mahiddine et al (2014) ou encore rabhi et al (2015) et Bouchentouf et al (2014,2016). D'autre part, plusieurs auteurs ont étudié l'estimation de la fonction de répartition conditionnelle comme une étude préliminaire de l'estimation des quantiles conditionnels. Laksaci et al. (2008) par exemple, ont proposé une méthode d'estimation des quantiles conditionnels et ils ont réussi à établir la convergence presque-complète et la normalité asymptotique de leur estimateur quand les observations sont fonctionnelles i.i.d.

La fonction de hasard ou taux de hasard, ou bien taux de défaillance, est un paramètre fonctionnel qui a une grande importance dans de nombreux problèmes pratiques, notamment dans les études de la fiabilité ou de survie. Ferraty et rabhi et vieu (2007) ont établis la convergence presque complètes avec vitesse de convergence dans le cadre des données complètes i.i.d (respectivement dépendantes), ainsi dans le cadre des données censurées i.i.d (respectivement dépendantes).
D'autres auteurs se sont intéressés à l'estimation des modèles conditionnels à partir des observations censurées ou tronquées (voir, par exemple, Lemdani et al (2009) Liang et Una-Álvarez,(2010, 2011) ,Khardani et al. (2010, 2011 et 2012), Ould Saïd et Tatachak (2011) ou Ould Saïd et Djabrane(2011)).

1.3 Données incomplètes

Dans de nombreuses applications statistiques, on est amené à faire intervenir une variable de durée notée T, désignant le temps écoulé jusqu'à la survenus de l'évènement d'intérêt. Ces types des variables sont observés dans divers domaines tels qu'en fiabilités (première panne pour une machine, durée de vie d'un matériel,...), en médecine (décés ou rémission pour un malade...) en économie et assurance (durée de chômage, temps entre deux pannes successives d'un appareil...).
Une spécificité de ces modèles est l'existence d'observations incomplètes, pour lesquelles la variable d'intérêt n'est pas complètement observée pour toutes les données de l'échantillon.

Dans cette thèse, nous étudierons des modèles où la durée est susceptible d'être censurée à droite en faisant alors appel à des techniques adaptées à ce type de contexte pour prendre en compte les observations censurée sans perdre trop d'information sur T.

Nous présentons dans ce qui suit deux cas de données incomplètes ; données censurées et des données tronquées.

1.3.1 Données censurées

La censure est le phénomène le plus couramment rencontré en analyse de survie. Il existe plusieurs catégorie de modèles de censure, mentionnons les deux modèles suivants :

Censure à droite : la durée de vie est dite censurée à droite si l'individu n'atteignent pas l'évènement d'intérêt. Sur toute la durée de l'étude, la variable d'intérêt T n'est pas observée, et on sait seulement qu'elle est supérieures à une certaine valeure de censure connue notée C qui elle a été observée. Au lieu d'observer les variables $T_1, T_2, ..., T_n$, qui nous intéressent, on n'observe T_i que lorsque $T_i < C$, sinon on sait seulement que $T_i > C$.
On en distingue trois types,

- **Censure de type 1 : (fixe)** Soit C une valeur fixée, au lieu d'observer les variables $T_1, ..., T_n$ qui nous intéressent, on observe T_i uniquement lorsque $T_i \leq C$ sinon on sait que $T_i > C$. On utilise la notation suivante $Y_i = T_i \wedge C = \min(T_i, C)$.
- **Censure de type 2 : (attente)** Elle est présente quand on décide d'observer les durées de survie des n patients jusqu'à ce que k d'entre eux soient décédés et d'arrêter l'étude à ce moment là.
  Soient $T_{(i)}$ et $Y_{(i)}$ les statistiques d'ordre des variables T_i et Y_i. La date de censure est donc $Y_{(k)}$ et on observe les variables suivantes
  $Y_{(1)} = T_{(1)}, ..., Y_{(k)} = T_{(k)}, Y_{(k+1)} = T_{(k+1)}, ..., Y_n = T_n$.
- **Censure de type 3 (ou censure aléatoire de type 1)** Soient $C_1, ..., C_n$ des variables aléatoires i.i.d. on observe les variables $Y_i = T_i \wedge C_i$ et $\delta_i = \mathbf{1}_{T_i \leq C_i}$.
  Y_i est la durée réellement observée, $\delta_i = 1$ si l'évènement est observé (d'où $Y_i = T_i$) on observe les "vraies" durées ou les dureés complètes. $\delta = 0$ si l'individu est censuré (d'où $Y_i = C_i$) on observe les durées incomplètes (censurées).

Censure à gauche : une durée de survie est dite censurée à gauche si l'individu a déjà subi l'évènement d'intérêt avant d'être observé. Au lieu d'observer la valeur d'intérêt T, on observe une valeur C avec le fait que T soit inférieur à C. Dans ce cas, on observe le couple (Y, δ) où

$$Y = \max(T, C) \quad et \quad \delta = \mathbf{1}_{\{T \geq C\}}$$

$\delta = 1$ s'il s'agit d'une vraie obsrvation et $\delta = 0$ si c'est une censure à gauche.

Notons que dans un même échantillon peuvent être présenter des données censurées à droite et d'autres censurées à gauche, comme c'est le cas dans la *censure double*, *censure par intervalle* et *censure mixte*.

Dans cette thèse nous nous intéressons uniquement au cas des données censurées à droite de type aléatoire. Celui-ci correspond à le modèle fréquement utilisé dans la pratique. Par exemple, lors d'un essai thérapeutique, elle peut être engendrée par une perte de vue (le patient quitte l'étude en cours), l'arrêt ou le changement d'un traitement alors les patients seront exclus de l'étude, ou bien l'étude se termine alors que certains individus n'ont pas subi l'évènement.

1.3.2 Données tronquées

le phénomène de la troncature est trés différent de la censure, puis ce que dans ce cas on étudie qu'un sous-échantillon ce qu'il engendre une perte d'information.
Dans le cas de la censure, on a connaissance du fait qu'il existe une information, mais on ne connait pas sa valeur précise, simplement le fait qu'elle excède un seuil : dans le cas de la troncature on ne dispose pas de cette information.
On dit qu'il y a une *troncature à droite* (respectivement *à gauche*) lorsque la variable d'intérêt T n'est pas observable quand elle est supérieure (respectivement inférieure) à un seuil C positif fixé ou aléatoire.

1.3.3 Estimateur de Kaplan-Meier

Soit $(\Omega, \mathcal{A}, \mathbb{P})$ un espace probabilisé. Soient $\{T_i, 1 \leq i \leq n\}$ (resp. $\{Y_i, 1 \leq i \leq n\}$)une suite de variable aléatoire (v.a) positives indépendantes et identiquement distribuées (i.i.d) de fonction de répartition $(f.d.r)$ F (resp. H), et $\{C_i, 1 \leq i \leq n\}$ une suite de variable aléatoire de censure, positive i.i.d de $f.d.r$ G.

On note $\tau_G = \sup\{t : G(t) < 1\}$ la borne supérieure de support de H. On a alors la relation, $\tau_H = \tau_F \wedge \tau_G$.

Nous ferons l'hypothèse que la censure est indépendente de l'évènement, c'est-à-dire, que T_i est indépendant de C_i. Cette hypothèse est trés utile d'un point de vue mathématique et indispensable aux modèles classiques d'analyse de survie et rend le modèle identifiable. Par exemple, quand la censure est due à un arrêt du traitement ou quand les malades ne sont plus suivis, l'hypothèse d'indépendance n'est pas vérifiée car la censure apporte une information sur l'état de santé du patient. Dans le cas d'une censure causée par un déménagement ou par la fin d'étude, cette hypothèse est naturelle. Ainsi, nous travaillons sous l'hypothèse générale d'indépendance entre (Y, X) et C.

Une autre hypothèse nécessaire pour obtenir la consistance de l'estimateur K.M (introduit par la suite) et qui est trés courante dans le contexte des données censurées, est de supposer que : $\tau_F \leq \tau_G$. En effet, quand $\tau_F > \tau_G$ toutes les valeurs entre τ_F et τ_G sont des censures et ne sont donc pas observées.

Dans le cas de données complètes, un estimateur naturel de la survie de la variable d'intérêt T est la survie empirique

$$S_n(t) = \bar{F}(t) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{T_i>t}$$

Malheureusement, dans le cas où les données censurées, il est impossible d'utiliser la fonction $S_n(t)$ puisqu'elle fait intervenir des quantités non observées (tous les T_i censurées ne sont pas observés). On fait appel alors à l'estimateur de Kaplan-Meier (E.K.M), qui, lorsqu'il n'y a pas d'ex-aequo (les temps d'évènements "décés" et censure sont distincts) est défini par :

$$\bar{F}_n(t) = 1 - F_n(t) = \begin{cases} \prod_{i=1}^{n}\left(1 - \frac{\delta_{(i)}}{n-i+1}\right)^{\mathbf{1}_{\{Y_{(i)}\le t\}}}, & \text{si } t \le Y_{(n)}; \\ 0, & \text{si } t > Y_{(n)}. \end{cases} \quad (1.1)$$

l'E.K.M pour la survie de la variable de censure est défini de la même façon par :

$$\bar{G}_n(t) = 1 - G_n(t) = \begin{cases} \prod_{i=1}^{n}\left(1 - \frac{1-\delta_{(i)}}{n-i+1}\right)^{\mathbf{1}_{\{Y_{(i)}\le t\}}}, & \text{if } t \le Y_{(n)}; \\ 0, & \text{if } t > Y_{(n)}. \end{cases} \quad (1.2)$$

où $Y_{(1)} < Y_{(2)} < \ldots < Y_{(n)}$ resprésentent les statistiques d'ordre associées à Y_i, et $\delta_{(i)}$ est l'indicatrice correspondante pour chacune des valeurs $Y_{(i)}$.

Remarque 1.1 *Il est à noter, qu'en absence de censure cet estimateur se réduit à la fonction de survie empirique.*

l'E.K.M possède des propriétés analogues à celles de la fonction de répartition empirique. Ces propriétés asymptotiques ont été étudiées par plusieurs auteurs, Breslow & Growley (1974), Peterson (1977), Gill (1980), Anderson (1993),. . . .Pour plus de détails voir le livre de Shorack & Wellner (1986).

1.4 Indice fonctionnelle simple

Le modèle de régression fonctionnelle bien connu avec réponse scalaire postule une relation entre une variable aléatoire réelle Y et une variable aléatoire fonctionnelle X,

$$Y = r(x) + \varepsilon \quad (1.3)$$

où ε est une erreur aléatoire réelle centrée non corrélée au régresseur. Nous pouvons aussi exprimer (1.3) comme suit :

$$r(x) = \mathbb{E}(Y_i|X_i = x) \quad (1.4)$$

Une large classe d'outils flexibles et utiles pour modéliser l'opérateur de régression r est présentée par le modèle à indice fonctionnel simple. Cela consiste à poser une approche semi paramétrique de réduction de dimension sur le modèle en introduisons un paramètre fonctionnel θ. L'idée principale est de chercher la direction de θ sur laquelle la projection de la covariable X capture le plus d'information sur la réponse Y. Alors la fonction de régression sera de la forme

$$r_\theta(x) = \mathbb{E}_\theta(Y_i|X_i = x) = \mathbb{E}(Y_i| < X_i, \theta >=< x, \theta >) \tag{1.5}$$

Le modèle que nous considérons est un modèle à direction révélatrice unique (ou modèle à indice fonctionnel simple). Cette approche présente divers intérêt. Tout d'abord, c'est d'éviter les problèmes dus à la dimensionnalité que l'on peut rencontrer dans l'approche purement non paramétrique (Ferraty et Vieu (2002)). L'estimation non paramétrique de la régression ne sera plus affectée par le fléau de la dimension puisqu'il s'agit d'une fonction dépendante de θ qui est de dimension 1. Enfin, l'estimation du paramètre fonctionnelle θ fournit un outil facilement interprétable.

Pour identifier ce modèle, considérons les mêmes conditions de Ferraty et al (2003). Supposons que r_1 et r_2 sont dérivables et $< \theta, e_1 >= 1$, où e_1 est le premier vecteur propre d'une base orthonormée de $\mathcal{H}$, alors

$$\forall x \in \mathcal{H}, \quad r_1(< \theta_1, x >) = r_2(< \theta_2, x >) \Rightarrow r_1 \equiv r_2 \quad et \quad \theta_1 \equiv \theta_2.$$

L'approche de l'indice simple est bien connue dans le contexte multivarié standard pour son intérêt de ses capacités de prédiction et pour son interprétabilité est attesté par divers œuvres sont apparues au cours des deux dernières décennies Härdle et al. (1993), Härdle and Stoker (1989) et Xia and Härdle (2006). Les extensions au cadre fonctionnelle de telle méthodologie semi-paramétrique fonctionnelle a fait l'objet d'étude approfondie dans la littérature. Le premier travail reliant le modèle à indice simple et le modèle de regression non-paramétrique pour des variables fonctionnelles est faite par Ferraty (2003) dans le cas des observations indépendantes, ils ont établi la convergence presque complète. Leurs résultats ont été étendus aux cas dépendants par Ait Said (2005). Ait Saidi (2008) ont étudiés le cas où l'indice fonctionnel simple est inconnu, ils ont proposé un estimateur de ce paramètre basé sur la technique de validtion croisée. Ces résultats ont été étendus au modèle à indice fonctionnel multiple par Bouraine (2010). Ferraty (2011) ont proposé un nouvel estimateur de ce paramètre basé sur l'idée d'estimation par dérivée fonctionnelle.

En ce qui concerne la densité conditionnelle, Attaoui (2011) ont étudié l'estimation de l'indice fonctionnel simple et ils ont établi la convergence ponctuelle et uniforme presque complète pour des observations indépendantes. Leur résultats ont été étendus au cas de α-mélange par Ling (2014). Tandis que la normalité asymptotiques de cet estimateur a été étudiée par Ling (2012). Attaoui (2014a) ont établi la convergence de la densité conditionnelle pour des donneés dépendantes. Pour les mêmes donneés, Attaoui (2014b) a étudié la convergence uniforme et la normalité asymptotique.

1.5 Outils

Mode de convergence

Définition 1.1 ***(Convergence presque complète)***
Soit $(X_n)_{n\in\mathbb{N}}$ *une suite de variables aléatoires réelles définie sur un espace de probabilité* $(\Omega, \mathcal{A}, \mathbb{P})$*. Nous disons que* X_n *converge presque complètement vers une variable aléatoire réelle* X*, si et seulement si :*

$$\forall\varepsilon > 0, \sum_{n\in\mathbb{N}} \mathbb{P}(|X_n - X| > \varepsilon) < \infty,$$

et la convergence presque complète de $(X_n)_{n\in\mathbb{N}}$ *vers* X *est notée par :*

$$X_n \xrightarrow[n\to\infty]{p.co} X$$

Définition 1.2 ***(Vitesse de convergence presque complète)***
Nous disons que la vitesse de convergence presque complète de la suite de variables aléatoires $(X_n)_{n\in\mathbb{N}}$ *vers* X *est d'ordre un si et seulement si :*

$$\exists\varepsilon_0 > 0, \sum_{n\in\mathbb{N}} \mathbb{P}(|X_n - X| > \varepsilon_0 U_n) < \infty,$$

et nous notons : $X_n - X = \mathcal{O}_{p.co}(U_n)$.

Définition 1.3 ***(Convergence en probabilité)***
Soit (X, X_n)*, une suite de variables aléatoire reélles, définies sur le même espace de probabilité* $(\Omega, \mathcal{F}, \mathbb{P})$*. La suite* X_n *converge en probabilité vers* X *si :*

$$\forall\varepsilon > 0, \lim_{n\to\infty} \mathbb{P}(|X_n - X| \geq \varepsilon) = 0.$$

Remarque 1.2 *La convergence presque complète implique la convergence presque sûre (ainsi que la convergence en probabilité).*

Processus fortement mélangeant

Les conditions de mélange sont souvent considérées dans la littérature pour mesurer la dépendance faible entre les variables, permettant d'obtenir des vitesses de convergence des estimateurs. Il existe différents types de mélange, α-mélange, ϕmélange, ψ-mélange, ρ-mélange, β-mélange. Dans cette thèse, nous nous intéressons essentiellement aux processus α-mélangeant (fortement mélangeant), ce cœfficient est notamment plus faible que les autres cœfficients de mélange qui seront donc forcément α-mélangeantes. Les résultats obtenues dans le cas de l'α-mélange vont donc concerner une large classe de processus. La définition et quelques propriétés sont données ci-aprés.

Définition 1.4 *Soit $\{\Delta_i, i \in \mathbb{Z}\}$ une famille de variables aléatoires dans un même espace probabilisable $(E, \mathcal{A})$. Pour tout couple (i, j) dans $\mathbb{Z} \cup \{-\infty, +\infty\}$, on note σ_i^j la tribu engendrée par $\{\Delta_k, i < k < j\}$. On appelle coefficients de mélange fort, la suite des réels*

$$\alpha(n) = \sup_{\{k \in \mathbb{Z}, A \in \sigma_{-\infty}^{k}, B \in \sigma_{n+k}^{\infty}\}} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|.$$

Définition 1.5 *On dit qu'une famille $\{\Delta_i, i \in \mathbb{Z}\}$ de variables aléatoires dans un même espace probabilisable $(E, \mathcal{A})$ est fortement mélangeante (α-mélangeante), si l'on a*

$$\lim_{n \to \infty} \alpha(n) = 0.$$

Définition 1.6 *On dit qu'une famille $\{\Delta_i, i \in \mathbb{Z}\}$ de variables aléatoires dans un même espace probabilisable $(E, \mathcal{A})$ est algébriquement α-mélangeante, s'il existe deux constantes c, a dans $\mathbb{R}^{*+}$ telles que les coefficients de mélange vérifient*

$$\alpha(n) \leq c n^{-a}.$$

Lemme 1.1 ***(Inégalité de covariance pour variables bornées)*** *Soit $\{\Delta_i, i \in \mathbb{N}\}$ une famille de variables aléatoires à valeur dans $\mathbb{R}$ fortement mélangeantes telle que $\forall i, ||\Delta_i||_\infty < \infty$, alors, pour tout $i \neq j$,*

$$|Cov(\Delta_i, \Delta_j)| \leq 4||\Delta_i||_\infty ||\Delta_j||_\infty \alpha(|i - j|).$$

Quelques inégalités

Pour prouver les résultats asymptotiques des estimateurs proposés dans cette thèse, nous utilisons certaines inégalités proposés ci-après.

Lemme 1.2 ***"Inégalité de type Fuk-Nagaev"****(Bosq et Lecoutre (1987)) Soit $\{\Delta_i, i \in \mathbb{N}\}$ une famille de variables aléatoires à valeur dans $\mathbb{R}$ fortement mélangeantes, de coefficient de mélange algébriquement décroissant. On pose*

$$s_n^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} |Cov(\Delta_i, \Delta_j)|$$

Si $\forall i ||\Delta_i||_\infty < \infty$, alors, pour tout $\varepsilon > 0$ et pour tout $r > 1$, on a :

$$\mathbb{P}\left(\{\sum_{k=1}^{n} \Delta_k\} > 4\varepsilon \right) \leq \left(1 + \frac{\varepsilon^2}{r s_n^2}\right)^{-\frac{r}{2}} + 2ncr^{-1} \left(\frac{2r}{\varepsilon}\right)^{a+1}.$$

Lemme 1.3 ***"Inégalité exponentielle de Bernstein"****(Bosq et Lecoutre (1987)) Soit $(\Delta_n)_{n \in \mathbb{N}}$ une suite des variables aléatoire réelles centrées, indépendantes identiquements distribuées,* *telles qu'il existe deux réelles positifs θ_1 et θ_2 vérifiant : $|\Delta_i| \leq \theta_1$ et $\mathbb{E}(\Delta_i^2) \leq \theta_2$. Alors pour tout $\epsilon \in]0, \frac{\theta_1}{\theta_2}[$, on a :*

$$\mathbb{P}\left(\frac{1}{n}\left|\sum_{i=1}^{n}\Delta_i\right| > \epsilon\right) \leq 2\exp\left(-\frac{n\epsilon^2}{4\theta_2}\right).$$

Les semi-métriques

La principale source de difficulté en estimation non paramétrique pour des données fonctionnelles est le choix de la semi-métrique. Qui permet de réduire la dimension des données et ainsi d'augmenter la vitesse de convergence. La diversité de semi-métrique permet de trouver une topologie qui donne une notion pertinente de proximité entre les courbes, que l'on pourra choisir en fonction de la nature des données. Cependant, il n'existe aucune méthode automatique qui offre un choix optimal de la semi-métrique. Pour plus de discussion nous référons au livre de Ferraty et Vieu (2006).

Nous présentons dans ce qui suit, trois familles de semi-métriques ; Soient $X_1, ..., X_n$ un échantillonn de n courbes indépendantes et identiquement distribuées de la variables aléatoire fonctionnelle $X = \{X(t), t \in [0, 1]\}$.

- **La semi-métrique basé sur la dériveé** Pour deux courbes observées x_1 et x_2 , la semi-métrique dérive est féfinie par :

  $$\forall (x_1, x_2) \in \mathcal{F} \times \mathcal{F}, d_q^{deriv}(x_1, x_2) = (\int (x_1^{(q)}(t) - x_2^{(q)}(t))^2 dt)^{1/2}, q \in \mathbb{N},$$

  où $x^{(q)}$ désigne la dérivée d'ordre q. Cette classe de semi-métriques est adaptée quand on a affaire à des données lisses.
- **La semi-métrique de l'analyse de composantes principales fonctionnelles** Cette semi-métrique est utilisée pour des données irrégulières. En effet, elle définie à partir de la semi-métrique L_2,dont le calcul est basé sur l'analyse des composantes principales. Plus précisément,on cherche les q premier vecteurs propres de l'opérateur de variance covariance empirique, associés au q plus grandes valeurs propres, la métrique est alors définie par :

  $$\forall (x_1, x_2) \in \mathcal{F} \times \mathcal{F}, d_q^{ACP}(x_1, x_2) = \int (\prod_q (x_1^{(q)}(t) - x_2^{(q)}(t)))^2 dt, q \in \mathbb{N},$$

  où $\prod_q$ est le projecteur orthogonal sur l'espace L^2 engendré par les q vecteurs propres associés aux q plus grandes valeurs propres.
- **La semi-métrique de modèle d'indice fonctionnelle** Pour une variable fonctionnelle X dans un espace de Hilbert $\mathcal{H}$ muni de produit scalaires $< \cdot, \cdot >$, on définit la semi-

métrique de modèle d'indice fonctionnel par

$$\forall \theta, x_1, x_2 \in \mathcal{H}, \quad d_\theta(x_1, x_2) = | < x_1 - x_2, \theta > |.$$

Pour sélectionner le paramètre θ, Ait Saidi et al (2008) ont proposé un estimateur basé sur la technique de validation croisée.

Loi du logarithme itéré pour l'estimateur de Kaplan-Meier

Dans le cas de censure à droite, nous notons par $\bar{F_n}$ l'estimateur de Kaplan-Meier, et par F (resp.G) la fonction de répartition de la variable d'intérêt T (resp. de la variable de censure C). Le résultat suivant est une loi du logarithme itéré (notée LIL) pour l'estimateur de kaplan-Meier de la fonction de répartition.

Théorème 1.1 *(Foldes et Rejto 1981). On suppose que F et G sont continues, et que $\tau_F < \tau_G$, alors*

$$\mathbb{P}\left(\sup_{-\infty<t<+\infty} |F_n(t) - F(t)| = \mathcal{O}\left(\sqrt{\frac{\log\log n}{n}}\right)\right) = 1.$$

La condition $\tau_F < \tau_G$ pouvant paraître restrictive, citons le théorème autrement.

Corollaire 1.1 *(Foldes et Rejto 1981). On suppose que F et G sont continues, et on considère le réel T tel que $G(t) < 1$, alors*

$$\mathbb{P}\left(\sup_{-\infty<t<T^*} |F_n(t) - F(t)| = \mathcal{O}\left(\sqrt{\frac{\log\log n}{n}}\right)\right) = 1,$$

où $T^* = min\{T, \tau_F\}$.

Entropie de Kolmogorov

Tous les résultats asymptotiques des statistiques non paramétriques pour les variables fonctionnelles sont étroitement liés aux propriétés de concentration de la mesure de probabilité de la valeur fonctionnelle X.
Soit $\mathcal{S}_\mathcal{F}$ un sous-ensemble fixe de $\mathcal{F}$. Nous considérons l'hypothèse suivante :

$$\forall x \in \mathcal{S}_\mathcal{F}, \quad 0 < C\phi(h) \leq \mathbb{P}(X \in B(x,h)) \leq C'\phi(h) < \infty.$$

On peut dire que la première contribution de la structure topologique de l'espace fonctionnel peut être vue à travers la fonction ϕ contrôlant la concentration de la mesure de probabilité de la variable fonctionnelle sur une petite boule. De plus, pour la cohérence uniforme, où l'outil principal doit couvrir un sous-ensemble $\mathcal{S}_\mathcal{F}$ avec un nombre fini de boules, on introduit un autre concept topologique défini comme suit :

Définition 1.7 ***(Kolmogorov Tikhomirov (1959))*** *Soit $\mathcal{S}_\mathcal{F}$ un sous-ensemble de l'espace métrique $\mathcal{F}$, et soit $\epsilon > 0$, un ensemble fini de points $X_1, X_2, ..., X_N$ dans $\mathcal{F}$ est appelé un ϵ-net pour $\mathcal{S}_\mathcal{F}$ si $\mathcal{S}_\mathcal{F} \subset \bigcup_{k=1}^{N} B(x_k, \epsilon)$. La quantité $\psi_{\mathcal{S}_\mathcal{F}}(\epsilon) = \log(N_\epsilon(\mathcal{S}_\mathcal{F}))$, où $N_\epsilon(\mathcal{S}_\mathcal{F})$ est le nombre minimal de boules ouvertes en $\mathcal{F}$ de rayon ϵ qui est nécessaire pour couvrir $\mathcal{S}_\mathcal{F}$, s'appelle l'entropie de Kolmogorov de $\mathcal{S}_\mathcal{F}$.*

Ce concept a été introduit par Kolmogorov au milieu des années 1950 (voir Kolmogorov et Tikhomirov 1959) et il représente sa mesure de la complexité d'un ensemble dans le sens où une l'entropie élevée signifie que beaucoup d'information est nécessaires pour écrire un élément avec une précision ϵ. Par conséquent le choix de la structure topologique (autrement dit, le choix de la semi métrique), et jouons un rôle professionnel quand on regarde les résultats asymptotiques uniformes sur certains sous-éléments $\mathcal{S}_\mathcal{F}$ de $\mathcal{F}$. Plus précisément, nous avons, par la suite, ce semi métrique peut également augmenter la concentration de la mesure de probabilité de la variable fonctionnelle X pour minimiser ϵ l'entropie de sous ensemble $\mathcal{S}_\mathcal{F}$. Ferraty et al (2006) ont souligné le phénomène de concentration de la mesure de probabilité de la variable fonctionnelle en calculant les probabilités de petite boule dans diverses normes situation. Enfin s'intéresser à ces deux (probabilités d'entropie et de petite boule) ou à l'utilisation de ϵ l'entropie de Kolmogorov dans les problèmes de réduction de la dimensionnalité pour se réfères à Kulbs et Li (1993) ou Odoros et Yannis (1997).

1.6 Plan de la thèse

Cette thèse est organisée en cinq chapitres. Le premier chapitre est un chapitre introductif consacré à la présentation des différents thèmes abordés dans notre thématique de recherche qui est l'estimation non-paramétrique fonctionnelle des modèles conditionnels dans un modèle à indice révélatrice en présence des données censurées. Nous y offrons de nombreuses références bibliographiques. Ainsi nous donnons la définition des données incomplètes essentiellement les données censurées, et nous y présentons la définition de l'estimateur de Kaplan-Meier pour la fonction de survie.

Dans le deuxième chapitre, nous considérons une suite d'observation indépendante et identiquement distribuées, et la variable explicative est à valeurs dans un espace de Hilbert (dimension infinie). Nous construisons un estimateur à double noyau de la fonction de répartition conditionnelle et de ses dérivées. Ainsi nous établissons la convergence presque-complète et la convergence uniforme de cet estimateur en précisant ses vitesses de convergence pour chacun de ces modes de convergence. Nous avons déduit un estimateur du quantile conditionnel, pour lequel nous avons établi la vitesse de convergence presque-complète et la convergence uniforme.

Ensuite, dans le troisième chapitre, nous généralisons les résultats obtenues dans le deuxième chapitre au cas où les observations sont fortement mélangeant (α-mélangeant).

Le quatrième chapitre consiste à étendre les résultats précédents au cas des données censurées à droite. Pour ce faire, nous construisons de nouveaux un estimateur à double noyau de la fonction de répartition conditionnelle dans lesquels sont pris en compte des effets de censure au cours de nos observations. Nous illustrerons nos résultats par des exemples sur des données simulées dont l'objectif est l'étude comparative en cas des données complètes et des données censurées avec différent taux de censure.

Nous concluons ce manuscrit par une conclusion et quelques perspectives de recherche.

1.7 Présentation des résultats

Dans cette section, nous donnons une brève présentation des différents résultats obtenus durant cette thèse.

1.7.1 Notations

Soient $(X_i, Y_i)_{i=1,\dots,n}$, n couples aléatoires indépendants et de même loi que le couple (X, Y), à valeur dans $\mathcal{H} \times \mathbb{R}$ où $\mathcal{H}$ est un espace de Hilbert muni de la norme induite par sa produit

scalaire $< \cdot, \cdot >$.
Pour tout $x \in \mathcal{H}$ et $y \in \mathbb{R}$, nous définissons la fonction de répartition conditionnelle de Y sachant $< X, \theta >=< x, \theta >$ à travers un indice fonctionnel fixé $\theta \in \mathcal{H}$, notée $F(\theta, \cdot, x)$ par :

$$\forall Y \in \mathbb{R}, \quad F(\theta, y, x) = \mathbb{P}(Y \leq y| < X, \theta >=< x, \theta >).$$

Afin d'assumer l'identifiabilité de ce modèle, nous supposons que F est deux fois différentiable par rapport à (w.r.t) x et θ. tel que $< \theta, e_1 >= 1$ où e_1 est le premier vecteur de la base orthonormée de $\mathcal{H}$. Clairement, sous cette condition, on a pour tout $x \in \mathcal{H}$,

$$F_1(\cdot| < x, \theta_1 >) = F_2(\cdot| < x, \theta_2 >) \Rightarrow F_1 \equiv F_2 \quad et \quad \theta_1 \equiv \theta_2.$$

On définie l'estimateur à noyau $\hat{F}(\theta, t, x)$ de $F(\theta, t, x)$, par

$$\widehat{F}(\theta, y, x) = \sum_{i=1}^{n} W_{ni}(\theta, x) \mathbf{1}_{\{Y_i \leq y\}}, \qquad \forall y \in \mathbb{R}$$

avec $\mathbf{1}_{\{\cdot\}}$ est une fonction indicatrice et $W_{ni}(\theta, x) = \dfrac{K\left(h_n^{-1}(< x - X_i, \theta >)\right)}{\sum_{j=1}^{n} K\left(h_n^{-1}(< x - X_j, \theta >)\right)}$, où K est un noyau, h_n est une suite de réels positifs.

L'estimateur du quantile conditionnel $t_\theta(\gamma)$ d'ordre $\gamma \in]0, 1[$ est défini par

$$t_\theta(\gamma) = \inf\{t \in \mathbb{R} : F(\theta, t, x) \geq \gamma\} = F^{-1}(\theta, \gamma, x).$$

Pour assurer l'existence et l'unicité de $t_\theta(\gamma)$ on suppose que la fonction de répartition conditionnelle $F(\theta, t, x)$ est strictement croissante. On estime le quantile conditionnel $t_\theta(\gamma)$ par $\widehat{t}_\theta(\gamma)$ tel que

$$\widehat{t}_\theta(\gamma) = \hat{F}^{-1}(\theta, \gamma, x).$$

1.7.2 Résultats : cas i.i.d. pour des données complètes

Nous supposons que les observations sont indépendantes identiquement distribuées. Nous établissons la convergence presque complète (p.co) et la convergence uniforme de notre estimateur, en donnant l'expression explicite des termes de vitesse de convergence.

Théorème 1.2 *Sous les hypothèses de concentration fonctionnelle ainsi que les hypothèses de régularités et techniques, nous aurons*

$$\widehat{t}_\theta(\gamma) - t_\theta(\gamma) \underset{n \to \infty}{\longrightarrow} 0, \ p.co.$$

$$\widehat{t}_\theta(\gamma)-t_\theta(\gamma)=\mathcal{O}\left(h_n^{\frac{b_1}{j}}\right)+\mathcal{O}_{p.co.}\left(\left(\frac{\log n}{n\phi_{\theta,x}(h_n)}\right)^{\frac{1}{2j}}\right).$$

Théorème 1.3

$$\sup_{x\in\mathcal{S}_\mathcal{F}}|\widehat{t}_\theta(\gamma)-t_\theta(\gamma)| \underset{n\to\infty}{\longrightarrow} 0,\ p.co.$$

$$\sup_{\theta\in\Theta_\mathcal{F}}\sup_{x\in\mathcal{S}_\mathcal{F}}|\widehat{t}_\theta(\gamma)-t_\theta(\gamma)|=\mathcal{O}\left(\left(h_n^{b_1}\right)^{\frac{1}{j}}\right)+\mathcal{O}_{p.co.}\left(\left(\frac{\log d_n^{\mathcal{S}_\mathcal{F}}d_n^{\Theta_\mathcal{F}}}{nh_n^{2j-1}\phi(h_n)}\right)^{\frac{1}{2j}}\right)$$

Les hypothèses imposées et les preuves de ces résultats ci-dessus seront données dans le chapitre 2.

1.7.3 Résultats : cas α-mélangeant pour des données complètes

Dans cette partie, nous supposons que $(X_i,Y_i)_{i\in\mathbb{N}}$ est une suite α-mélange.
L'estimateur $\hat{F}(\theta,\cdot,x)$ de $F(\theta,\cdot,x)$ est défini par,

$$\widehat{F}(\theta,y,x)=\frac{\sum_{i=1}^n K\left(h_K^{-1}(<x-X_i,\theta>)\right)H\left(h_H^{-1}(y-Y_i)\right)}{\sum_{i=1}^n K\left(h_K^{-1}(<x-X_i,\theta>)\right)},$$

où K est un noyau, H une fonction de répartition et $h_K=h_{K,n}$ (resp. $h_H=h_{H,n}$) sont des suites de réels positifs.

Théorème 1.4

$$\widehat{t}_\theta(\gamma)-t_\theta(\gamma) \underset{n\to\infty}{\longrightarrow} 0,\ p.co.$$

$$\widehat{t}_\theta(\gamma)-t_\theta(\gamma)=\mathcal{O}\left(\left(h_K^{b_1}+h_H^{b_2}\right)^{\frac{1}{j}}\right)+\mathcal{O}_{p.co}\left(\left(\frac{s_n^2\log n}{n^2}\right)^{\frac{1}{2j}}\right).$$

Théorème 1.5

$$\sup_{x\in\mathcal{S}_\mathcal{F}}|\widehat{t}_\theta(\gamma)-t_\theta(\gamma)| \underset{n\to\infty}{\longrightarrow} 0,\ p.co.$$

$$\begin{aligned}\sup_{\theta\in\Theta_\mathcal{F}}\sup_{x\in\mathcal{S}_\mathcal{F}}|\widehat{t}_\theta(\gamma)-t_\theta(\gamma)| &= \mathcal{O}\left(\left(h_K^{b_1}+h_H^{b_2}\right)^{\frac{1}{j}}\right)+\mathcal{O}_{p.co.}\left(\left(\frac{\log d_n^{\mathcal{S}_\mathcal{F}}d_n^{\Theta_\mathcal{F}}}{nh_H^{2j-1}\phi(h_K)}\right)^{\frac{1}{2j}}\right)\\ &+\mathcal{O}_{p.co.}\left(\left(\frac{s_n''^2\log d_n^{\mathcal{S}_\mathcal{F}}d_n^{\Theta_\mathcal{F}}}{n^2}\right)^{\frac{1}{2j}}\right)\end{aligned}$$

les preuves de ces théorèmes seront données dans le chapitre 3.

1.7.4 Résultats : cas α-mélangeant pour des données censurées

Soit (X, T) un couple de variable aléatoire à valeurs dans $\mathcal{H} \times \mathbb{R}$. La suite $(T_i)_{i\geq 1}$ est supposée stationnaire et fortement mélangeante. On note par $(C_i)_{i=1,\ldots,n}$ variables aléatoire de censure indépendantes et identiquement distribués de fonction de distribution continue G, et indépendantes des T_i.
Ainsi, dans le modèle de censure à droite, on construit nos estimateurs par les variables observés $(X_i, Y_i, \delta_i)_{i=1,\ldots,n}$, avec $Y_i = T_i \wedge C_i$ et $\delta_i = \mathbf{1}_{T_i \leq C_i}$, où $\mathbf{1}_A$ désigne la fonction indicatrice de l'ensemble A.

L'estimateur à double noyau de la fonction de répartition conditionnelle adapté à des données censurées, est défini par

$$\hat{F}(\theta, y, x) = \frac{\sum_{i=1}^{n} \frac{\delta_i}{\bar{G}_n(Y_i)} K(h_k^{-1}(< x - X_i, \theta >))H(h_H^{-1}(t - Y_i))}{\sum_{i=1}^{n} K(h_k^{-1}(< x - X_i, \theta >))}$$

avec $\bar{G}_n(Y_i) = 1 - G_n(Y_i)$ où $G_n(Y_i)$ est l'estimateur de Kaplan-Meier de $G(Y_i)$, défini par

$$\bar{G}_n(t) = 1 - G_n(t) = \begin{cases} \prod_{i=1}^{n} \left(1 - \frac{1 - \delta_{(i)}}{n - i + 1}\right)^{\mathbf{1}_{\{Y_{(i)} \leq t\}}}, & \text{if } t \leq Y_{(n)}; \\ 0, & \text{if } t > Y_{(n)}. \end{cases}$$

où $Y_{(1)} < Y_{(2)} < \ldots < Y_{(n)}$ resprésentent les statistiques d'ordre associées à Y_i, et $\delta_{(i)}$ est l'indicatrice correspondante pour chacune des valeurs $Y_{(i)}$.

L'estimateur du quantile conditionnel $\zeta_\theta(\gamma, x)$ est donné par

$$\begin{aligned} \widehat{\zeta}_\theta(\gamma, x) &= \widehat{F}^{-1}(\theta, \gamma, x) \\ &= \inf\{t \in \mathbb{R} : \widehat{F}(\theta, t, x) \geq \gamma\}, \qquad \forall \gamma \in (0, 1). \end{aligned}$$

Théorème 1.6 *Sous la condition de mélange et des hypothèses données dans le chapitre 4, nous aurons*

$$\widehat{\zeta}_\theta(\gamma, x) - \zeta_\theta(\gamma, x) \underset{n \to \infty}{\longrightarrow} 0, \ p.co.$$

$$\widehat{\zeta}_\theta(\gamma, x) - \zeta_\theta(\gamma, x) = \mathcal{O}\left(\left(h_K^{b_1} + h_H^{b_2}\right)^{\frac{1}{l}}\right) + \mathcal{O}_{p.co}\left(\left(\frac{s_n'^2 \log n}{n^2}\right)^{\frac{1}{2l}}\right),$$

Théorème 1.7

$$\sup_{x\in\mathcal{S}_{\mathcal{H}}} |\widehat{\zeta}_\theta(\gamma,x)-\zeta_\theta(\gamma,x)| \underset{n\to\infty}{\longrightarrow} 0, \ p.co.$$

$$\begin{aligned}\sup_{\theta\in\Theta_{\mathcal{F}}}\sup_{x\in\mathcal{S}_{\mathcal{F}}} |\widehat{\zeta}_\theta(\gamma,x)-\zeta_\theta(\gamma,x)| &= \mathcal{O}\left(\left(h_K^{b_1}+h_H^{b_2}\right)^{\frac{1}{l}}\right)+\mathcal{O}_{p.co.}\left(\left(\frac{\log d_n^{\mathcal{S}_{\mathcal{F}}} d_n^{\Theta_{\mathcal{F}}}}{nh_H^{2l-1}\phi(h_K)}\right)^{\frac{1}{2l}}\right)\\ &+\mathcal{O}_{p.co.}\left(\left(\frac{s_n''^2 \log d_n^{\mathcal{S}_{\mathcal{F}}} d_n^{\Theta_{\mathcal{F}}}}{n^2}\right)^{\frac{1}{2l}}\right)\end{aligned}$$

Les démonstrations de ces théorèmes sont détaillées dans le chapitre 4.

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Chapitre Deux

Strong Uniform Consistency Rates of Conditional quantiles estimate with Functional Variables in the Functional Single-Index Model

Ce chapitre a fait l'objet d'une publication internationale dans *Journal of Statistics and Economics.*

Strong Uniform Consistency Rates of Conditional quantiles estimate with Functional Variables in the Functional Single-Index Model

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Abstract. This paper deals with a scalar response conditioned by a functional random variable. The main goal is to estimate nonparametrically the quantiles of such a conditional distribution when the sample is considered as an i.i.d sequence. Firstly, a kernel type estimator for the conditional cumulative distribution function (*cond-cdf*) is introduced. Afterwards, we derive an estimation of the quantiles by inverting this estimated *cond-cdf* and asymptotic properties are stated when the observations are linked with a single-index structure. We establish the pointwise almost complete convergence and the uniform almost complete convergence (with the rate) of the kernel estimate of this model. The functional conditional quantile approach can be used both to forecast and to build confidence prediction bands.

Keywords : Conditional quantile, conditional cumulative distribution, functional random variable, functional single-index process, kernel estimator, nonparametric estimation, small ball probability.

2010 Mathematics Subject Classification : Primary 62G05, Secondary 62G20, 62N02, 62H12.

2.1 Introduction

Estimating quantiles of any distribution is an important part of Statistics. This allows to build confidence ranges and to derive many applications in various fields (chemistry, geophysics, medicine, meteorology,...). On the other hand, Statistics for functional random variables become more and more important. The recent literature in this domain shows the great potential of these new functional statistical methods. The most popular case of functional random variable corresponds to the situation when we observe random curve on different statistical units. Such data are called *Functional Data*. Many multivariate statistical technics, mainly parametric in the functional model terminology, have been extended to functional data and good overviews on this topic can be found in Ramsay and Silverman (1997, 2002) or Bosq (2000). More recently, nonparametric methods taking into account functional variables have been developed with very interesting practical motivations dealing with environmetrics (see Damon and Guillas (2002, Fernández *et al.* (2003), Aneiros *et al.* (2004), chemometrics (see Ferraty and Vieu (2002), meteorological sciences (see Besse *et al.* (2000), Hall and Heckman (2002), speech recognization problem (see Ferraty and Vieu (2003)), radar range profile (see Hall *et al.* (2001), Dabo-Niang *et al.* (2004)), medical data (see Gasser *et al.* (1998)), ... In an other hand, forecasting methods cover a large part of the statistical problems. Because a continuous time series can be viewed as a sequence of dependent functional random variables, the above mentioned functional methodology can be used for time-series forecasting (see for instance Ferraty *et al.* (2002), for a functional forecasting approach of time-series based on conditional expectation estimation). In a functional data setting, the conditioning variable is allowed to take its values in some abstract semi-metric space. In this case, Ferraty *et al.* (2005) define non-parametric estimators of the conditional density and the conditional distribution. They give the rates of convergence (in an almost complete sense) to the corresponding functions, in an a dependence context.

The single-index models are becoming increasingly popular, and have been paid considerable attention recently because of their importance in several areas of science such as econometrics, biostatistics, medicine, financial econometric and so on. we quote for example, Härdle *et al.* (1993), Hristache *et al.* (2001). Delecroix *et al.* (2003) has studied the estimation of the single-index approach of regression function and established some asymptotic properties. The recent literature in this domain shows the great potential of these new functional statistical methods The most popular case of functional random variable corresponds to the situation when we observe random curve on different statistical units. The first work in the fixed functional single-model were obtained by Ferraty *et al.* (2003), where authors have obtained almost complete convergence (with the rate) in the i.i.d. case, of the regression function. Their results were extended to case of the conditional distribution where Kernel type estimators for the conditional cumulative distribution function and the successive derivatives of the conditional density by Bouchentouf *et al.* (2014), where the authors established the pointwise almost complete convergence and the uniform almost complete convergence (with the rate) of the kernel estimate of this model. They are applied to the estimations of the conditional mode. To dependent case by Aït Saidi *et al.* (2005). Aït Saidi *et al.* (2008) studied the case where the functional single-index is unknown. They proposed an estimator of this parameter, based on the the cross-validation procedure.

This paper proposes to put together the three previous statistical aspects in order to derive a method for estimating conditional quantiles in situation when the data are both independent and of functional nature in the functional single-index model are introduced. More precisely, we focus on the nonparametric estimation of the conditional quantiles of a real random variable given a functional random variable of this model. We start by estimating the conditional distribution by means of a kernel estimator and we derive an estimate of the conditional quantiles (see Section 2). From a theoretical point of view, a crucial problem is linked with the so-called *curse of dimensionality*. Indeed, in a nonparametric context, it is known that the rate of convergence decreases with the dimension of the space in which the conditional variable is valued. But here, the conditional variable takes its values in an infinite dimensional space. One way to override this problem is to consider some concentration hypotheses acting on the distribution of the functional variable which allows to obtain asymptotic properties of our kernel estimates and we give their pointwise almost complete convergence (with rate) in Section 3. This approach is used to derive a new method to forecast the functional single-index model. Then, in section 4, we study the uniform almost complete convergence of the conditional quantile estimator given in section 2. We finish our paper by giving technical proofs of Lemmas and Corollary (see Section 5).

2.1.1 On the problematic of single index models

For several years, a increasing interest is worn to models which incorporating of both the parts parametric and nonparametric. Such models type are called semi-parametric model. This consideration is due primarily to problems due to poor specification of some models. Tackle a problem of mis-specification semiparametric way consists in not specify the functional form of some model components. This approach complete those non-parametric models, which can not be useful in small samples, or with a large number of variables. As example, in the classical regression case, the important parameter whose one assumed existence is the regression function of Y knowing the covariate X, denoted $r(x) = \mathbb{E}(Y|X = x)$, $X, Y \in \mathbb{R}^d \times \mathbb{R}$. For this model, the non-parametric method considers only regularity assumptions on the function r. Obviously, this method has some drawbacks. One can cite the problem of curse of dimensionality. This problem appears when the number of regressors d increases, the rate of convergence of the nonparametric estimator r which is supposed k times differentiable is $O(n^{-k/2k+d})$ deteriorate. The second drawback is the lack of means to quantify the effect of each explanatory variable. To alleviate in these drawbacks, an alternative approach is naturally provided by the semi-parametric model which supposes the introduction of a parameter on the regressors, by writing than the regression function is of the form

$$\mathbb{E}_\theta(Y|X) = \mathbb{E}(Y| < X, \theta >= x),$$

The models defined are known in the literature as the single-index models.

These models allow to obtain a compromise between parametric models, generally too restrictive and nonparametric model where the rate of convergence of the estimators deteriorate quickly in the presence of a large number of explanatory variables. In this area, different types of models have been studied in the literature : amongst the most famous, there may be mentioned additive models, partially linear models or single index models. The idea of these models, in the case of estimating the conditional density or regression consists in bring to the covariates a dimension in smaller than dimension of the space variable, thus allowing overcome the problem of curse of dimensionality. For example, for example, in the partially linear model, we decompose the quantity to be estimated, into a linear part and a functional part. This latter quantity does not pose estimation problem since it's expressed as a function of explanatory variables of finite dimension, thus avoiding the problems associated with curse of dimensionality. in order to treat the problem of curse of dimensionality in the case chronologies series, several semi-parametric approaches have been proposed. Without pretend to exhaustively, we quote for example : Xia and An (2002) for the index model. A general presentation of this type of model is given in Ichimura *et al.* (1993) where the convergence and asymptotic normality are obtained. In the case of M-estimators, Delecroix *et al.* (1999) proves the consistency and asymptotic normality of the estimate the index and they study it's effectiveness. The statistical literature on these methods is rich, quote Huber (1985) and Hall (1989) present an estimation method which consists projecting the density and the regression function on a space of dimension one, to bring a non-parametric estimation for dimensional covariates. This amounts exactly to estimate these functions in a single index model. Attaoui *et al.* (2011) have established the pointwise and the uniform almost complete convergence (with the rate) of the kernel estimate of this model. The interest of their study is to show how the estimate of the conditional density can be used to obtain an estimate of the simple functional index if the latter is unknown. More precisely, this parameter can be estimated by pseudo-maximum likelihood method which is based the preliminary estimate of the conditional density. recently Bouchentouf *et al.* (2014) have established the pointwise almost complete convergence and the uniform almost complete convergence (with the rate) of some characteristics of the conditional distribution and the successive derivatives of the conditional density when the observations are linked with a single-index structure and they are applied to the estimations of the conditional mode and conditional quantiles.

The single-index approach is widely applied in econometrics as a reasonable compromise between nonparametric and parametric models. Such kind of modelization is intensively studied in the multivariate case. Without pretend to exhaustivity, we quote for example Härdle *et al.* (1993), Hristache *et al.* (2001). Based on the regression function, Delecroix *et al.* (2003) studied the estimation of the single-index and established some asymptotic properties. The literature is strictly limited in the case where the explanatory variable is functional (that is a curve). The first asymptotic properties in the fixed functional single-model were obtained by Ferraty *et al.* (2003). They established the almost complete convergence, in the i.i.d. case, of the link regression function of this model. Their results were extended to dependent case by Aït Saidi *et al.* (2005). Aït Saidi *et al.* (2008) studied the case where the functional single-index is unknown. They proposed an estimator of this parameter, based on the cross-validation procedure.

2.2 The model and the estimates

2.2.1 The functional nonparametric framework

Consider a random pair (X, Y) with values in $\mathcal{H} \times \mathbb{R}$, where $\mathcal{H}$ is a separable real Hilbert space with the norm $\|\cdot\|$ generated by an inner product $<\cdot,\cdot>$. Let $(X_i, Y_i)_{i=1,\dots,n}$ be the statistical sample of pairs which are identically distributed like (X, Y), but not necessarily independent. Henceforward, X is called functional random variable *f.r.v.* . We consider the semi-metric d_θ, associated to the single-index $\theta \in \mathcal{H}$ defined by $\forall x_1, x_2 \in \mathcal{H}: \ d_\theta(x_1, x_2) := |< x_1 - x_2, \theta >|$. Under such topological structure and for a fixed functional θ, we suppose that the conditional cumulative distribution function (*cond-cdf*) of Y given $< X, \theta >=< x, \theta >$ denoted by $F(\cdot|x)$ exists and is given by :

$$\forall y \in \mathbb{R}, \ F_\theta(y|x) =: F(y| < x, \theta >) = F(\theta, y, x) = \mathbb{P}(Y \leq y| < X, \theta >=< x, \theta >).$$

By considering the same conditions of Ferraty *et al.* (2003) on the regression operator, the identifiability of the model is assumed. More precisely we suppose that f is differentiable with respect to (w.r.t) x and θ such that $< \theta, e_1 >= 1$, where e_1 is the first eigenvector of an orthonormal basis of the space $\mathcal{H}$. Clearly, we have for all $x \in \mathcal{H}$

$$F_1(\cdot| < x, \theta_1 >) = F_2(\cdot| < x, \theta_2 >) \implies F_1 \ \equiv \ F_2 \quad \text{and} \quad \theta_1 \equiv \theta_2.$$

In what follows, we denote by $F(\theta, \cdot, x)$ the conditional distribution of Y given $< \theta, x >$.

Saying that, we are implicitly assuming the existence of a regular version for the conditional distribution of Y given $< \theta, X >$. Now, let t_γ be the γ-order quantile of the distribution of Y given $< \theta, X >=< \theta, x >$. From the *cond-cdf* $F(\theta, \cdot, x)$, it is easy to give the general definition of the γ-order quantile :

$$t_\theta(\gamma) = \inf\{t \in \mathbb{R} : \; F(\theta, t, x) \geq \gamma\}, \qquad \forall \gamma \in (0,1).$$

In order to simplify our framework and to focus on the main interest of our paper (the functional feature of $< \theta, X >$), we assume that $F(\theta, \cdot, x)$ is strictly increasing and continuous in a neighborhood of t_γ. This is insuring that the conditional quantile t_γ is uniquely defined by :

$$t_\theta(\gamma) \;=\; F^{-1}(\theta, \gamma, x). \tag{2.1}$$

In the remaining of the paper, we wish to stay in a free distribution framework. This will lead to assume only smoothness restrictions for the *cond-cdf* $F(\theta, \cdot, x)$ through nonparametric modelling (see Section 2.4 below). The last point of our work is to consider independent data. It will be assumed that $(X_i, Y_i)_{i \in \mathbb{N}}$ is an i.i.d sequence, which is one among the most general independent structures.

In finite dimensional settings, nonparametric modelling is mainly used for estimating functions (such as the *cond-cdf* $F(\theta, \cdot, x)$), in such a way that the words *functional* and *nonparametric* are quite often used equivalently. In our infinite dimensional purpose, things have to be clarified. We use the terminology *functional nonparametric*, where the word *functional* refers to the infinite dimensionality of the data and where the word *nonparametric* refers to the infinite dimensionality of the model. Such *functional nonparametric* statistics is also called *doubly infinite dimensional* (see Ferraty and Vieu, (2003b), for larger discussion). We could also use the terminology *operatorial statistics* since the target object to be estimated (the *cond-cdf* $F(\theta, \cdot, x)$) can be viewed as a nonlinear operator.

2.2.2 The estimators

Following in Bouchentouf *et al.* (2014), the conditional density operator $f(\theta, \cdot, x)$ is defined by using kernel smoothing methods

$$\widehat{f}(\theta, y, x) = \frac{\sum_{i=1}^{n} K\left(h_n^{-1}(< x - X_i, \theta >)\right) H\left(h_n^{-1}(y - Y_i)\right)}{h_n \sum_{i=1}^{n} K\left(h_n^{-1}(< x - X_i, \theta >)\right)},$$

where K and H are kernel functions and h_n is sequence of smoothing parameter. The conditional distribution operator $F(\theta, \cdot, x)$ can be estimated by $\widehat{F}(\theta, \cdot, x)$ defined as follows :

$$\widehat{F}(\theta, y, x) = \sum_{i=1}^{n} W_{ni}(\theta, x) \mathbf{1}_{\{Y_i \leq y\}}, \qquad \forall y \in \mathbb{R} \tag{2.2}$$

with $\mathbf{1}_{\{\cdot\}}$ being the indicator function and where $W_{ni}(\theta, x) = \dfrac{K\left(h_n^{-1}(< x - X_i, \theta >)\right)}{\sum_{j=1}^{n} K\left(h_n^{-1}(< x - X_j, \theta >)\right)}$, K is a kernel function and h_n is a sequence of positive real numbers which goes to zero as n goes to infinity. Note that using similar ideas, Roussas (1969) introduced some related estimate but in the special case when X is real, while Samanta (1989) produced previous asymptotic study.

As a by-product of (4.1) and (4.2), it is easy to derive an estimator $\widehat{t}_\gamma$ of t_γ :

$$\widehat{t}_\theta(\gamma) \;=\; \widehat{F}^{-1}(\theta,\gamma,x). \tag{2.3}$$

As we will see later on, such an estimator is unique as soon as H is an increasing continuous function. Such an approach has been widely used before in situation when the variable X is of finite dimension (see *e.g* Whang and Zhao (1999), Cai (2002), Zhou and Liang (2003) or Gannoun *et al.* (2003)). As far as we know, our paper is the first one about kernel conditional quantile with independent and possibly infinite dimensional variable. According to the finite dimensional knowledges, from a theoretical point of view the main mathematical problem comes from the so-called *curse of dimensionality*. This phenomenon is well-known in the nonparametric statistical community, and it can be summarized by saying that the rate of convergence decreases with the dimension of the space $\mathcal{H}$. Because we consider here an infinite dimensional space $\mathcal{H}$, we have to keep in mind this point. We propose to solve this dimensionality problem by taking into account the distribution of the *f.r.v.* $< X,\theta >$ through its small balls probabilities (see (H0), (H2) and discussion below).

2.2.3 Assumptions on the functional variable

Let N_x be a fixed neighborhood of x and let $B_\theta(x,h)$ be the ball of center x and radius h, namely $B_\theta(x,h_n) = \{f\in\mathcal{H}/0 < |< x-f,\theta >| < h_n\}$. Then, we consider the following hypotheses :

(H1) $\forall h_n>0,\ \mathbb{P}\left(X\in B_\theta(x,h_n)\right)=\phi_{\theta,x}(h_n)>0$,

Note that (H1) can be interpreted as a concentration hypothesis acting on the distribution of the *f.r.v.* X.

2.2.4 The nonparametric model

As usually in nonparametric estimation, we suppose that the *cond-cdf* $F(\theta,\cdot,x)$ verifies some smoothness constraints. Let b_1 and b_2 be two positive numbers ; we assume that $F(\theta,\cdot,x)$ is such that :

(H2) $\forall (x_1,x_2)\in N_x\times N_x,\ \forall (y_1,y_2)\in\mathcal{S}_{\mathbb{R}}^2$,

$$|F(\theta,y_1,x_1)-F(\theta,y_2,x_2)|\le C_{\theta,x}\left(d_\theta(x_1,x_2)^{b_1}+|y_1-y_2|^{b_2}\right),$$

(H3) $F(\theta,\cdot,x)$ is j-times continuously differentiable in some neighborhood of $t_\theta(\gamma)$,

(H4) There exists some integer $j\ge 0$, $\forall (x_1,x_2)\in N_x\times N_x,\ \forall (y_1,y_2)\in\mathcal{S}_{\mathbb{R}}^2$,

$$|f^{(j)}(\theta,y_1,x_1)-f^{(j)}(\theta,y_2,x_2)|\le C_{\theta,x}\left(d_\theta(x_1,x_2)^{b_1}+|y_1-y_2|^{b_2}\right),$$

with N_x is a fixed neighborhood of x and $\mathcal{S}_{\mathbb{R}}$ is a fixed compact subset of $\mathbb{R}$; where, for any positive integer l, $f^{(l)}(\theta,z,x)$ denotes its lth derivative ($i.e.$ $\left.\frac{\partial^l f(\theta,y,x)}{\partial y^l}\right|_{y=z}$).

Later on, (H2) is used to proof the almost complete convergence of $\widehat{t}_\theta(\gamma)$ whereas (H3) and (H4) are needed to establish the rate of convergence.

2.2.5 Comment and Remarks

1. Remarks on the Assumptions : Our conditions are very standard in this context. Assumption (H1) characterizes the probability measure concentration of the functional variable on small balls, it is the basic condition used in the majority of the existing works dealing with this kind of variable. In our work we consider a single index topological structure when the concentration of measure changes a little. More precisely the quantity in (H1) can be written as $\mathbb{P}\left(| < x - X, \theta > | \leq h_n\right) = \phi_{\theta,x}(h_n)$. The regularity of the functional space of our model is controlled by mean of condition (H2) and is needed to evaluate the bias of estimation.
2. Remarks on the single index : It is well known that one of the main advantages of the single index model is its ability to deal with the problem of high dimensional data. A straightforward example is the optimal convergence rate of type $\mathcal{O}\left(n^{-2q/(2q+p)}\right)$ for the estimation of a q-times differentiable regression function, this rate goes to zero dramatically slowly if q is small compared to the dimension p of the explanatory variable ($X \in \mathbb{R}^p$). In this regard, Gaïffas and Lecué (2007) showed that the optimal rate of convergence of regression function in the single index model, is of order $\mathcal{O}\left(n^{-2q/(2q+1)}\right)$ (rather than $\mathcal{O}\left(n^{-2q/(2q+p)}\right)$). The same idea was adopted in the abstract metric spaces by the choice of the semi-metric increasing the probability measure concentration of the explanatory variable in small balls (see Ferraty *et al.* (2006)), Section 13.2). Among these family of semi-metric, one can consider the semi-metric induced by the functional single index estimate.
3. Remarks on the bandwidth : Asymptotically h_n converges to 0 as n tends to infinity. The probability of observing the multiple functional random variable around x (the functional vector which we assess the r regression operator) are interpreted in terms of concentration measurement probability of the multiple small balls of functional random variable X in the direction θ. Because h_n becomes smaller and smaller, these conditional probabilities can be viewed as small-ball conditional probabilities (the notion of ball being defined with the semi-metric $d_\theta(\cdot,\cdot)$). As explained in Ferraty and Vieu (2006b), the notion of small-ball probabilities plays a major role in the asymptotic results.

2.3 Asymptotic study

We start this theoretical section by giving the almost complete convergence (*a.co.*) of the estimated conditional quantile $\widehat{t}_\theta(\gamma)$. After then, we will focus on the rate of convergence. Concerning the notations, as soon as possible, C and C' will denote generic constants.

2.3.1 Pointwise almost complete convergence

Let us start with the statement of an almost complete convergence property [1]. Before to give the result, the following assumptions concerning the kernel estimator $\widehat{F}(\theta,\cdot,x)$ are needed :

1. Recall that a sequence $(T_n)_{n\in\mathbb{N}}$ of random variables is said to converge almost completely to some variable T, if for any $\epsilon > 0$, we have $\sum_n \mathbb{P}(|T_n - T| > \epsilon) < \infty$. This mode of convergence implies both almost sure and in probability convergence (see for instance Bosq and Lecoutre (1987)).

(H5) $\forall (y_1, y_2) \in \mathbb{R}^2,\ \forall j' \leq j,\ |H^{(j)}(y_1) - H^{(j)}(y_2)| \leq C|y_1 - y_2|$ and $\int |t|^{b_2} H(t)dt < \infty$, where, for all $l \in \mathbb{N}^*$, $H^{(l)}(t) = \left.\frac{d^l H(y)}{d y^l}\right|_{y=t}$,

(H6) K is a positive bounded function with support $[-1, 1]$,

(H7) $\lim_{n\to\infty} h_n = 0$ and $\dfrac{\log n}{n\,\phi_{\theta,x}(h_n)} \underset{n\to\infty}{\longrightarrow} 0.$

Remark 2.1 *Note that (H5) insures the existence of $\widehat{t}_\theta(\gamma)$, while (H4) insures its unicity. (H1)-(H4) and (H6) are standard assumptions for the distribution conditional estimation in single functional index model, which have been adopted by Bouchentouf* et al. *(2014) for i.i.d case. (H8) is a technical condition for our results.*

Theorem 2.1 *Under hypotheses (H1)-(H2) and (H5)-(H7), we have :*

$$\widehat{t}_\theta(\gamma) - t_\theta(\gamma) \underset{n \to \infty}{\longrightarrow} 0,\ a.co. \tag{2.4}$$

Proof of Theorem 4.1. this proof is based on previous results concerning the estimation of conditional c.d.f nonlinear operator $F(\theta, y, x)$ by the kernel estimate $\widehat{F}(\theta, y, x)$ (see result (4.11)), whose the proof of the latter follows directly from Lemmas 3.1 and 3.2 below. Because of (H5) and (H6), $\widehat{F}(\theta, ., x)$ is a continuous and strictly increasing function. So, we have :

$$\forall \epsilon > 0,\ \exists \delta(\epsilon) > 0, \forall y, |\widehat{F}(\theta, y, x) - \widehat{F}\left(\theta, t_\theta(\gamma), x\right)| \leq \delta(\epsilon) \Rightarrow |y - t_\theta(\gamma)| \leq \epsilon.$$

This leads us to write

$$\begin{aligned}\forall \epsilon > 0,\ \exists \delta(\epsilon) > 0,\ \mathbb{P}\left(|\widehat{t}_\theta(\gamma) - t_\theta(\gamma)| > \epsilon\right) &\leq \mathbb{P}\left(|\widehat{F}(\theta, \widehat{t}_\theta(\gamma), x) - \widehat{F}(\theta, t_\theta(\gamma), x)| \geq \delta(\epsilon)\right)\\ &= \mathbb{P}\left(|F(\theta, t_\theta(\gamma), x) - \widehat{F}(\theta, t_\theta(\gamma), x)| \geq \delta(\epsilon)\right),\end{aligned}$$

since (4.3) is implying that $\widehat{F}(\theta, \widehat{t}_\theta(\gamma), x) = \gamma = F(\theta, t_\theta(\gamma), x)$. Thus, it is clear that the proof of Theorem 4.1 is achieved as soon as we show the pointwise convergence of $\widehat{F}(\theta, ., x)$ at $t_\theta(\gamma)$:

$$F(\theta, t_\theta(\gamma), x) - \widehat{F}(\theta, t_\theta(\gamma), x) \underset{n \to \infty}{\longrightarrow} 0,\ a.co. \tag{2.5}$$

Consider now, for $i = 1, \ldots, n$, the following notations :

$$K_i(\theta, x) = K(h_n^{-1}(< x - X_i, \theta >)), \qquad \widehat{F}_D(\theta, x) = \frac{1}{n\,\mathbb{E}(K_1(\theta, x))} \sum_{i=1}^n K_i(\theta, x)),$$

$$\widehat{F}_N(\theta, t_\theta(\gamma), x) = \frac{1}{n\,\mathbb{E}(K_1(\theta, x))} \sum_{i=1}^n K_i(\theta, x)\mathbf{1}_{\{Y_i \leq t_\theta(\gamma)\}}.$$

Because some quantities are divided by $\mathbb{E}K_1(\theta, x)$, it is important to remark once for all that (H1) and (H6) imply that

$$0 < C\phi_{\theta,x}(h_n) < \mathbb{E}K_1(\theta, x) < C'\phi_{\theta,x}(h_n).$$

By using the following decomposition

$$\begin{aligned}\widehat{F}(\theta,t_\theta(\gamma),x)-F(\theta,t_\theta(\gamma),x) &= \frac{1}{\widehat{F}_D(\theta,x)}\Big\{\Big(\widehat{F}_N(\theta,t_\theta(\gamma),x)-\mathbb{E}\widehat{F}_N(\theta,t_\theta(\gamma),x)\Big)\\ &\quad -\Big(F(\theta,t_\theta(\gamma),x)-\mathbb{E}\widehat{F}_N(\theta,t_\theta(\gamma),x)\Big)\Big\}\\ &\quad +\frac{F(\theta,t_\theta(\gamma),x)}{\widehat{F}_D(\theta,x)}\Big\{\mathbb{E}\widehat{F}_D(\theta,x)-\widehat{F}_D(\theta,x)\Big\}, \qquad (2.6)\end{aligned}$$

the proof of (4.11) comes from next lemmas :

Lemma 2.1 *Under the conditions of Theorem 4.1 we have*

$$|F(\theta,t_\theta(\gamma),x)-\mathbb{E}\widehat{F}_N(\theta,t_\theta(\gamma),x)| = \mathcal{O}\left(h_n^{b_1}\right). \qquad (2.7)$$

Lemma 2.2 *(See Aït Saidi* et al.*, 2005). Under the assumptions (H1), (H6)-(H7), we have :*

$$|\widehat{F}_D(\theta,x)-\mathbb{E}\widehat{F}_D(\theta,x)| = \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n\phi_{\theta,x}(h_n)}}\right)$$

Lemma 2.3 *Under the assumptions of Theorem 4.1, we have :*

$$|\widehat{F}_N(\theta,t_\theta(\gamma),x)-\mathbb{E}\widehat{F}_N(\theta,t_\theta(\gamma),x)| = \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n\phi_{\theta,x}(h_n)}}\right).$$

Corollary 2.1 *Under the hypotheses of Lemma 3.2, we have*

$$\sum_{n=1}^{\infty}\mathbb{P}\left(|\widehat{F}_D(\theta,x)|\le 1/2\right) < \infty. \qquad (2.8)$$

Our theorem will be proved as soon as these two lemmas are verified.

The Lemmas 3.1, 3.2, are special cases of the Lemmas 2.3.2, 2.3.3 in Ferraty *et al.* (2006) and Lemma 14 in Ferraty *et al.* (2010) with $d(x_1,x_2)=<x_1-x_2,\theta>$ respectively.

2.3.2 Pointwise almost complete rate of convergence

In this section we study the rate of convergence[2] of our conditional quantile estimator $\widehat{t}_\gamma$. Because this kind of result is stronger than the previous one, we have to introduce some additional assumptions. As it is usual in conditional quantiles estimation, the rate of convergence can be linked with the flatness of the *cond-cdf* $F(.|x)$ around the conditional quantile $t_\theta(\gamma)$. This is one reason why we introduced hypotheses (H3) and (H4). But a complementary way to take into account this local shape constraint is to assume that :

(H8) $\exists j>0,\ \forall l,\ 1\le l<j,\ F^{(l)}(\theta,t_\theta(\gamma),x) = 0$ and $\left|F^{(j)}(\theta,t_\theta(\gamma),x)\right|>0.$

2. Recall that a sequence $(T_n)_{n\in\mathbb{N}}$ of random variables is said to be of order of complete convergence u_n, if there exists some $\epsilon>0$ for which $\sum_n \mathbb{P}(|T_n|>\epsilon u_n)<\infty$. This is denoted by $T_n=O_{a.co.}(u_n),.$ (or equivalently by $T_n=O_{a.co.}(u_n)$).

Because we focus on the local behavior of $F(\theta,\cdot,x)$ around $t_\theta(\gamma)$ via its derivatives, that leads us to consider the successive derivatives of $\widehat{F}(\theta,\cdot,x)$ and subsequently some assumptions on the successive derivatives of the cumulative kernel H :

(H9) The support of H is compact and $\forall l \geq j$, $H^{(l)}$ exists and is bounded.

(H10) $\forall i \neq i'$, the conditional density of $(Y_i, Y_{i'})$ given $(< X_i, \theta >, < X_{i'}, \theta >)$ is continuous at $(t_\theta(\gamma), t_\theta(\gamma))$.

Theorem 2.2 *Under hypotheses (H1)-(H10), we have*

$$\widehat{t}_\theta(\gamma) - t_\theta(\gamma) = \mathcal{O}\left(h_n^{\frac{b_1}{j}}\right) + \mathcal{O}_{a.co.}\left(\left(\frac{\log n}{n\phi_{\theta,x}(h_n)}\right)^{\frac{1}{2j}}\right). \tag{2.9}$$

Proof of Theorem 4.2. The proof is based on the Taylor expansion of $\widehat{F}(\theta,\cdot,x)$ at $t_\theta(\gamma)$:

$$\begin{aligned}\widehat{F}\left(t_\theta(\gamma),x\right) - \widehat{F}\left(\theta,\widehat{t}_\theta(\gamma),x\right) &= \sum_{l=1}^{j-1}\frac{(t_\theta(\gamma)-\widehat{t}_\theta(\gamma))^l}{l!}\widehat{F}^{(l)}(t_\theta(\gamma),x)\\ &\quad+\frac{(t_\theta(\gamma)-\widehat{t}_\theta(\gamma))^j}{j!}\widehat{F}^{(j)}(t^*_\theta(\gamma)|x),\\ &= \sum_{l=1}^{j-1}\frac{(t_\theta(\gamma)-\widehat{t}_\theta(\gamma))^l}{l!}\left(\widehat{F}^{(l)}(t_\theta(\gamma),x)-F^{(l)}(t_\theta(\gamma),x)\right)\\ &\quad+\frac{(t_\theta(\gamma)-\widehat{t}_\theta(\gamma))^j}{j!}\widehat{F}^{(j)}(t^*_\theta(\gamma)|x),\end{aligned}$$

where, for all $y \in \mathbb{R}$,

$$\widehat{F}^{(j)}(\theta,y,x) = \frac{h_n^{-j}\sum_{i=1}^{n} K(h_n^{-1}(< x - X_i, \theta >))H^{(j-1)}(h_n^{-1}(y-Y_i))}{\sum_{i=1}^{n} K(h_n^{-1}(< x - X_i, \theta >))},$$

and where $\min(t_\theta(\gamma),\widehat{t}_\theta(\gamma)) < t^*_\theta(\gamma) < \max(t_\theta(\gamma),\widehat{t}_\theta(\gamma))$.

Suppose now that we have the following result.

Lemma 2.4 *Suppose that the hypotheses (H1)-(H6), (H8)-(H10) are satisfied and*

$$\lim_{n\to\infty}\frac{\log n}{n\,h_n^{2j-1}\,\phi_{\theta,x}(h_n)} = 0.$$

Then we have :

$$|\widehat{F}^{(j)}(\theta,t_\theta(\gamma),x) - F^{(j)}(\theta,t_\theta(\gamma),x)| = \mathcal{O}\left(h_n^{b_1}\right) + \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n\,h_n^{2j-1}\,\phi_{\theta,z}(h_n)}}\right),$$

Because of Theorem 4.1, Lemma 3.3 and (H8), we have :

$$\widehat{F}^{(j)}(\theta,t^*_\theta,x) \underset{n\to\infty}{\longrightarrow} F^{(j)}(\theta,t_\theta(\gamma),x) > 0, \; a.co.,$$

and we derive

$$(t_\theta(\gamma)-\widehat{t}_\theta(\gamma))^j = \mathcal{O}\left(\widehat{F}(\theta,t_\theta(\gamma),x)-F(\theta,t_\theta(\gamma),x)\right) + \mathcal{O}\left(\sum_{l=1}^{j-1}(t_\theta(\gamma)-\widehat{t}_\theta(\gamma))^l(\widehat{F}^{(l)}(\theta,t_\theta(\gamma),x)-F^{(l)}(\theta,t_\theta(\gamma),x))\right), \ a.co. \quad (2.10)$$

Now, comparing the convergence rates given in Lemmas 3.2 and 3.3, we get

$$(t_\theta(\gamma)-\widehat{t}_\theta(\gamma))^j = \mathcal{O}\left(\widehat{F}(t_\theta(\gamma),x)-F(\theta,t_\theta(\gamma),x)\right), \ a.co.$$

Thus, Lemmas 3.2 and 3.1 allow us to get the claimed result.

The proofs of the the above lemma and corollary are given in the same manner as it was done in (2006), since they are a special case of the Lemmas 2.3.4 and 2.3.5 and Corollary 2.3.1. It suffices to replace $\widehat{f}^{(j-1)}(y|x)$ (resp. $F^{(j-1)}(y|x)$) by $\widehat{F}^{(j)}(\theta,t_\theta(\gamma),x)$ (resp. $F^{(j)}(\theta,t_\theta(\gamma),x)$) and $\widehat{F}_D(x)$, (resp. $F_D(x)$) by $\widehat{F}_D(\theta,x)$ (resp. $F_D(\theta,x)$) with $d(x_1,x_2) =< x_1-x_2,\theta >$.

2.4 Uniform almost complete convergence and rate of convergence

In this section we derive the uniform version of Theorem 4.1 and Theorem 4.2. The study of the uniform consistency is motivated by the fact that the latter is an indispensable tool for studying the asymptotic properties of all estimates of the functional index if is unknown. Noting that, in the multivariate case, the uniform consistency is a standard extension of the pointwise one, however, in our functional case, it requires some additional tools and topological conditions (see Ferraty *et al.* (2010), for more discussion on the uniform convergence in nonparametric functional statistics). Thus, in addition to the conditions introduced previously, we need the following ones. Firstly, consider

$$\mathcal{S}_\mathcal{H} \subset \bigcup_{k=1}^{d_n^{\mathcal{S}_\mathcal{H}}} B_\theta(x_k,r_n) \text{ and } \Theta_\mathcal{H} \subset \bigcup_{m=1}^{d_n^{\Theta_\mathcal{H}}} B_\theta(\theta_m,r_n) \quad (2.11)$$

with x_k (resp. θ_j) $\in \mathcal{H}$ and $r_n, d_n^{\mathcal{S}_\mathcal{H}}, d_n^{\Theta_\mathcal{H}}$ are sequences of positive real numbers which tend to infinity as n goes to infinity and suppose that $d_n^{\mathcal{S}_\mathcal{H}}$, $d_n^{\Theta_\mathcal{H}}$ are the minimal numbers of open balls with radius r_n in $\mathcal{H}$, which are required to cover $\mathcal{S}_\mathcal{H}$ and $\Theta_\mathcal{H}$.

Remark 2.2 *Note that the recovers of the compact subsets imposed in (2.11) are necessary to derive our uniform consistency. The second part of condition (2.11) was explained in Ferraty* et al. *(2006, 2006b), whereas first part of condition (2.11) is a key point to ensure the geometric link between the number $d_n^{\Theta_\mathcal{H}}$ of balls and the sequence of radius r_n. In abstract semi-metric spaces, it is usually assumed that $d_n^{\Theta_\mathcal{H}} r_n$ is bounded, see Ferraty and Vieu (2008) for more discussion.*

2.4.1 Topological considerations

Kolmogorov's entropy

We can say that the first contribution of the topological structure of the functional space can be viewed through the function ϕ controlling the concentration of the measure of probability of the functional variable on a small ball. Moreover, for the uniform consistency, where the main tool is to cover a subset $\mathcal{S}_{\mathcal{H}}$ with a finite number of balls, one introduces an other topological concept defined as follows :

Definition 2.1 *Let $\mathcal{S}$ be a subset of a semi-metric space $\mathcal{F}$, and let $\epsilon > 0$ be given. A finite set of points $x_1, x_2, \ldots, x_n$ in $\mathcal{F}$ is called an ϵ-net for $\mathcal{S}$ if $\mathcal{S} \subset \bigcup_{k=1}^{N} B(x_k, \epsilon)$. The quantity $\log(N_\epsilon(\mathcal{S})$, where $N_\epsilon(\mathcal{S}$ is the minimal number of open balls in $\mathcal{F}$ of radiusϵ which is necessary to cover $\mathcal{S}$, is called Kolmogorov's ϵ-entropy of the set $\mathcal{S}$.*

This concept was introduced by Kolmogorov in the mid-1950s (see, Kolmogorov and Tikhomirov (1959)) and it represents a measure of the complexity of a set, in sense that, high entropy means that much information is needed to describe an element with an accuracy ϵ. Therefore, the choice of the topological structure (with other words, the choice of the semi-metric) will play a crucial role when one is looking at uniform (over some subset $\mathcal{S}_{\mathcal{F}}$ of $\mathcal{F}$) asymptotic results. More precisely, we will see thereafter that a good semi-metric can increase the concentration of the probability measure of the functional variable X as well as minimize the ϵ-entropy of the subset $\mathcal{S}_{\mathcal{F}}$.Inanearliercontribution (see,Ferraty *et al.* (2006)) we highlighted the phenomenon of concentration of the probability measure of the functional variable by computing the small ball probabilities in various standard situations, see Ferraty *et al.* (2010) for more discussion.

2.4.2 Conditional quantile distribution estimation

In this part we propose to study the uniform almost complete convergence of our estimator defined above (4.3) moreover, the following assumptions are also satisfied

(A1) There exists a differentiable function $\phi(\cdot)$ such that $\forall x \in \mathcal{S}_{\mathcal{H}}$ and $\forall \theta \in \Theta_{\mathcal{H}}$,

$$0 < C\phi(h) \leq \phi_{\theta,x}(h) \leq C'\phi(h) < \infty \text{ and } \exists \eta_0 > 0,\ \forall \eta < \eta_0,\ \phi'(\eta) < C,$$

(A2) $\forall (y_1, y_2) \in \mathcal{S}_{\mathbb{R}} \times \mathcal{S}_{\mathbb{R}}, \forall (x_1, x_2) \in \mathcal{S}_{\mathcal{H}} \times \mathcal{S}_{\mathcal{H}}$ and $\forall \theta \in \Theta_{\mathcal{H}}$,

$$|F(\theta, y_1, x_1) - F(\theta, y_2, x_2)| \leq C \left(\|x_1 - x_2\|^{b_1} + |y_1 - y_2|^{b_2} \right),$$

(A3) The kernel K satisfy (H3) and Lipschitz's condition holds

$$|K(x) - K(y)| \leq C\|x - y\|.$$

(A4) For some $\nu \in (0, 1)$, $\lim_{n\to\infty} n^\nu h_n = \infty$, and for $r_n = \mathcal{O}\left(\frac{\log n}{n}\right)$, the sequences $d_n^{\mathcal{S}_{\mathcal{H}}}$ and

$d_n^{\Theta_\mathcal{H}}$ satisfy :

$$\begin{cases} (i)\ \dfrac{(\log n)^2}{n\phi(h_n)} < \log d_n^{\mathcal{S}_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}} < \dfrac{n\phi(h_n)}{\log n}, \\ (ii)\ \sum_{n=1}^{\infty} n^{1/2b_2}(d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}})^{1-\beta} < \infty \text{ for some } \beta > 1, \\ (iii)\ n\phi(h_n) = \mathcal{O}\left((\log n)^2\right). \end{cases}$$

(A5) There exists some integer $j > 0$, $\forall (y_1, y_2) \in \mathcal{S}_\mathbb{R} \times \mathcal{S}_\mathbb{R}, \forall (x_1, x_2) \in \mathcal{S}_\mathcal{H} \times \mathcal{S}_\mathcal{H}$ and $\forall \theta \in \Theta_\mathcal{H}$,

$$|f^{(j)}(\theta, y_1, x_1) - f^{(j)}(\theta, y_2, x_2)| \leq C\left(\|x_1 - x_2\|^{b_1} + |y_1 - y_2|^{b_2}\right), \quad \text{for} \quad j \geq 0$$

(A6) For some $\nu \in (0,1)$, $\lim_{n\to\infty} n^\nu h_n = \infty$, and for $r_n = \mathcal{O}\left(\frac{\log n}{n}\right)$, the sequences $d_n^{\mathcal{S}_\mathcal{H}}$ and $d_n^{\Theta_\mathcal{H}}$ satisfy :

$$\begin{cases} (i)\ \dfrac{(\log n)^2}{nh_n^{2j-1}\phi(h_n)} < \log d_n^{\mathcal{S}_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}} < \dfrac{nh_n^{2j-1}\phi(h_n)}{\log n}, \\ (ii)\ \sum_{n=1}^{\infty} n^{(3\nu+1)/2}(d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}})^{1-\beta} < \infty \text{ for some } \beta > 1, \\ (iii)\ nh_n^{2j-1}\phi(h_n) = \mathcal{O}\left((\log n)^2\right) \end{cases}$$

Remark 2.3 *Hypothesis (A4)-(i) deals with topological considerations by controlling the entropy of $\mathcal{S}_\mathcal{H}$. For a radius not too large, one requires that $\log d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}}$ is not too small and not too large. Moreover, (A4)-(i) implies that $\log d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}} / n\phi(h_n)$ tends to 0 when n tends to $+\infty$. As remarked in Section 2, in some "usual" cases, one has $\log d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}} \sim C \log n$ and (A4)-(i) is satisfied as soon as $(\log n)^2 = \mathcal{O}(n\phi(h_n))$. In a different way, Assumption (A4)-(ii) acts on the Kolmogorov's ϵ-entropy of $\mathcal{S}_\mathcal{H}$. However, if one considers the same particular case as previously, it is easy to see that (A4)-(ii) is verified as soon as $\beta > 2$.*

So, we propose to estimate $t_\theta(\gamma)$ by the estimate $\widehat{t}_\theta(\gamma)$ defined as (4.3) or as

$$\widehat{F}(\theta, \widehat{t}_\theta(\gamma), x) = \gamma. \tag{2.12}$$

To ensure existence and uniqueness of this quantile, we will assume that

(A7) $F(\theta, \cdot, x)$ is strictly increasing.

Note that, because H is a *cdf* satisfying (H6), such a value $\widehat{t}_\theta(\gamma)$ is always exists. It could be the case that it is not unique, but if this happens all the remaining of the paper will concern any among all the values $\widehat{t}_\theta(\gamma)$ satisfying (2.12).

As for the mode estimation problem discussed before, the difficulty occur in estimating the conditional quantile $t_\theta(\gamma)$ is linked with the flatness of the curve of the conditional distribution $F(\theta, \cdot, x)$ around $t_\theta(\gamma)$. More precisely, we will suppose that there exists some integer $j > 0$ such that :

$$(A8) \begin{cases} F^{(l)}(\theta, t_\theta(\gamma), x) = 0, & \text{if}\,;\ 1 \leq l < j \\ \text{and } F^{(j)}(\theta, \cdot, x), & \text{is uniformly continuous on}\,;\ \mathcal{S}_\mathbb{R} \\ \text{such that,} & |F^{(j)}(\theta, t_\theta(\gamma), x)| > C > 0 \end{cases}$$

Theorem 2.3 *Under hypotheses (H5), (A1)-(A4) and (A7), we have*

$$\sup_{x\in\mathcal{S}_\mathcal{F}} |\widehat{t}_\theta(\gamma) - t_\theta(\gamma)| \underset{n\to\infty}{\longrightarrow} 0, \ a.co. \tag{2.13}$$

Proof 2.1 (Proof of Theorem 4.3) *The proof of Theorem 4.3 can be completed by the following lemmas.*

Lemma 2.5 *Under the conditions of Theorem 4.3, we have*

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|F(\theta,t_\theta(\gamma),x)-\mathbb{E}\widehat{F}_N(\theta,t_\theta(\gamma),x)\right|=\mathcal{O}\left(h_n^{b_1}\right). \tag{2.14}$$

Lemma 2.6 *Under the assumptions of Theorem 4.3, we have :*

$$\sup_{\theta\in\Theta_{\mathcal{F}}}\sup_{x\in\mathcal{S}_{\mathcal{F}}}\left|\widehat{F}_D(\theta,x)-\mathbb{E}\widehat{F}_D(\theta,x)\right|=\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{F}}}d_n^{\Theta_{\mathcal{F}}}}{n\phi(h_n)}}\right), \tag{2.15}$$

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\widehat{F}_N(\theta,t_\theta(\gamma),x)-\mathbb{E}\widehat{F}_N(\theta,t_\theta(\gamma),x)\right|=\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{F}}}d_n^{\Theta_{\mathcal{F}}}}{n\phi(h_n)}}\right). \tag{2.16}$$

Corollary 2.2 *Under the assumptions (A1), (A3) and (A4), we have :*

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\inf_{\theta\in\Theta_{\mathcal{H}}}\inf_{x\in\mathcal{S}_{\mathcal{H}}}\widehat{F}_D(\theta,x)<\frac{1}{2}\right)<\infty$$

Theorem 2.4 *Under hypotheses (H9)-(H10), (A1), (A3), (A5)-(A8), we have*

$$\sup_{\theta\in\Theta_{\mathcal{F}}}\sup_{x\in\mathcal{S}_{\mathcal{F}}}|\widehat{t}_\theta(\gamma)-t_\theta(\gamma)|=\mathcal{O}\left(\left(h_n^{b_1}\right)^{\frac{1}{j}}\right)+\mathcal{O}_{a.co.}\left(\left(\frac{\log d_n^{\mathcal{S}_{\mathcal{F}}}d_n^{\Theta_{\mathcal{F}}}}{nh_n^{2j-1}\phi(h_n)}\right)^{\frac{1}{2j}}\right)$$

Corollary 2.3 *Under conditions of Theorem 4.4, we have*

1.

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{y\in\mathcal{S}_{\mathbb{R}}}|\widehat{F}(\theta,t_\theta(\gamma),x)-F(\theta,t_\theta(\gamma),x)|=\mathcal{O}\left(h_n^{b_1}\right)+\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}+\log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}}\right)$$

2.

$$\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{y\in\mathcal{S}_{\mathbb{R}}}|\widehat{F}(\theta,t_\theta(\gamma),x)-F(\theta,t_\theta(\gamma),x)|=\mathcal{O}\left(h_n^{b_1}\right)+\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}}{n\phi(h_n)}}\right)$$

Proof 2.2 (Proof of Theorem 4.4) *Clearly The proofs of these two results namely the Theorem 4.4 and Corollary 4.1 can be deduced from the following intermediate results which are only uniform version of Lemma 3.3.*

Lemma 2.7 *Under conditions of Theorem 4.4 and if*

$$\lim_{n\to\infty} \frac{\log d_n^{\mathcal{S}_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}}}{n\,\phi(h_n)\,h_n^{2j-1}} = 0,$$

then we have :

$$\sup_{\theta\in\Theta_\mathcal{H}}\sup_{x\in\mathcal{S}_\mathcal{H}}\sup_{y\in\mathcal{S}_\mathbb{R}}\left|\widehat{F}^{(j)}(\theta,t_\theta(\gamma),x) - F^{(j)}(\theta,t_\theta(\gamma),x)\right| = \mathcal{O}\left(h_n^{b_1}\right) + \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{F}} + \log d_n^{\Theta_\mathcal{F}}}{n h_n^{2j-1}\phi(h_n)}}\right).$$

2.5 Proofs of technical lemmas

Proof of Lemma 4.6 One has

$$\begin{aligned}\mathbb{E}\widehat{F}_N(\theta,t_\theta(\gamma),x) - F_\theta^x(t_\theta(\gamma)) &= \frac{1}{\mathbb{E}K_1(x,\theta)}\mathbb{E}\left[\sum_{i=1}^n K_i(x,\theta)\mathbf{1}_{\{Y_i\le t_\theta(\gamma)\}}\right] - F_\theta^x(t_\theta(\gamma))\\ &= \frac{1}{\mathbb{E}K_1(x,\theta)}\mathbb{E}\Big(K_1(x,\theta)\Big[\mathbb{E}\Big(\mathbf{1}_{\{Y_1\le t_\theta(\gamma)\}}| < X_1,\theta > \Big)\\ &\qquad -F_\theta^x(t_\theta(\gamma))\Big]\Big). \qquad (2.17)\end{aligned}$$

where $F_\theta^\xi(\cdot) = F(\theta,\cdot,\xi)$.

Moreover, we have

$$\begin{aligned}\mathbb{E}\left(\mathbf{1}_{\{Y_1\le t_\theta(\gamma)\}}| < X_1,\theta >\right) &= \mathbb{E}\left(\mathbf{1}_{\{Y_1\le t_\theta(\gamma)\}}| < X_1,\theta >\right)\\ &= \int_\mathbb{R} H\left(h_n^{-1}(t_\theta(\gamma)-u)\right)F(\theta,u,X_1)du\\ &= \int_\mathbb{R} H(t)F(\theta,t_\theta(\gamma)-h_n t,X_1)dt.\end{aligned}$$

Indeed

$$\mathbb{E}\left[\mathbf{1}_{B_\theta(x,h_n)}(X_1)F_\theta^{X_1}(t_\theta(\gamma))| < X_1,\theta >\right] = \int_\mathbb{R} H(u)F_\theta^{X_1}(t_\theta(\gamma)-uh_n)du.$$

Thus, we have

$$\left|\mathbb{E}\left(\mathbf{1}_{\{Y_1\le t_\theta(\gamma)\}}| < X_1,\theta >\right) - F_\theta^x(t_\theta(\gamma))\right| \le \int_\mathbb{R} H(t)\left|F_\theta^{X_1}(t_\theta(\gamma)-h_n t) - F_\theta^x(t_\theta(\gamma))\right|dt.$$

Finally, the use of (A2) implies that

$$\left|\mathbb{E}\left(\mathbf{1}_{\{Y_1\le t_\theta(\gamma)\}}| < X_1,\theta >\right) - F_\theta^x(t_\theta(\gamma))\right| \le C_\theta\int_\mathbb{R} H(t)\left(h_n^{b_1} + |t|^{b_2}h_n^{b_2}\right)dt. \qquad (2.18)$$

Because this inequality is uniform on $(\theta,y,x)\in\Theta_\mathcal{H}\times\mathcal{S}_\mathcal{H}\times\mathcal{S}_\mathbb{R}$ and because of (H6), (4.19) is a direct consequence of (4.24), (2.18) and of Corollary 2.2.

□

Proof of Lemma 4.7

- Concerning 2.15
  For all $x \in \mathcal{S}_{\mathcal{H}}$ and $\theta \in \Theta_{\mathcal{H}}$, we set

$$k(x) = \arg \min_{k\in\{1\dots r_n\}} \|x - x_k\| \text{ and } j(\theta) =) = \arg \min_{j\in\{1\dots l_n\}} \|\theta - t_j\|.$$

Let us consider the following decomposition

$$\sup_{x\in\mathcal{S}_{\mathcal{H}}} \sup_{\theta\in\Theta_{\mathcal{H}}} \left|\widehat{F}_D(\theta, x) - \mathbb{E}\left(\widehat{F}_D(\theta, x)\right)\right| \leq \Pi_1 + \Pi_2 + \Pi_3 + \Pi_4 + \Pi_5$$

where
$\Pi_1 = \sup_{x\in\mathcal{S}_{\mathcal{H}}} \sup_{\theta\in\Theta_{\mathcal{H}}} \left|\widehat{F}_D(\theta, x) - (\widehat{F}_D(\theta, x_{k(x)})\right|$
$\Pi_2 = \sup_{x\in\mathcal{S}_{\mathcal{H}}} \sup_{\theta\in\Theta_{\mathcal{H}}} \left|\widehat{F}_D(\theta, x_{k(x)}) - \widehat{F}_D(t_{j(\theta)}, x_{k(x)})\right|$
$\Pi_3 = \sup_{x\in\mathcal{S}_{\mathcal{H}}} \sup_{\theta\in\Theta_{\mathcal{H}}} \left|\widehat{F}_D(t_{j(\theta)}, x_{k(x)}) - \mathbb{E}\left(\widehat{F}_D(t_{j(\theta)}, x_{k(x)})\right)\right|$
$\Pi_4 = \sup_{x\in\mathcal{S}_{\mathcal{H}}} \sup_{\theta\in\Theta_{\mathcal{H}}} \left|\mathbb{E}\left(\widehat{F}_D(t_{j(\theta)}, x_{k(x)})\right) - \mathbb{E}\left(\widehat{F}_D(\theta, x_{k(x)})\right)\right|$
$\Pi_5 = \sup_{x\in\mathcal{S}_{\mathcal{H}}} \sup_{\theta\in\Theta_{\mathcal{H}}} \left|\mathbb{E}\left(\widehat{F}_D(\theta, x_{k(x)})\right) - \mathbb{E}\left(\widehat{F}_D(\theta, x)\right)\right|.$
For Π_1 and Π_2, we employe the Hölder continuity condition on K, Cauchy Schwartz's and the Bernstein's inequalities, we get

$$\Pi_1 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}}\right), \quad \Pi_2 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}}\right) \tag{2.19}$$

Then, by using the fact that $\Pi_4 \leq \Pi_1$ and $\Pi_5 \leq \Pi_2$, we get for n tending to infinity

$$\Pi_4 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}}\right), \quad \Pi_5 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}}\right) \tag{2.20}$$

Now, we deal with Π_3, for all $\eta > 0$, we have

$$\mathbb{P}\left(\Pi_3 > \eta\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}}\right)\right) \leq$$

$$d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}} \max_{k\in\{1\dots d_n^{S_{\mathcal{H}}}\}} \max_{j\in\{1\dots d_n^{\Theta_{\mathcal{H}}}\}} \mathbb{P}\left(\Pi_3' > \eta\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}}\right)\right).$$

where $\Pi_3' = \left|\widehat{F}_D(t_{j(\theta)}, x_{k(x)}) - \mathbb{E}\left(\widehat{F}_D(t_{j(\theta)}, x_{k(x)})\right)\right|$
Applying Bernstein's exponential inequality to

$$\frac{1}{\phi(h_n)}\left(K_i\left(t_{j(\theta)}, x_{k(x)}\right) - \mathbb{E}\left(K_i\left(t_{j(\theta)}, x_{k(x)}\right)\right)\right),$$

we get

$$\Pi_3 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}}\right).$$

Lastly the result will be easily deduced from the latter together with (2.19) and (2.20).
- Concerning 2.16

We keep the same notation and we use the compact of $\mathcal{S}_{\mathbb{R}}$, we can write that, for some, $t_1, \ldots, t_{z_n} \in \mathcal{S}_{\mathbb{R}}$, $\mathcal{S}_{\mathbb{R}} \subset \bigcup_{m=1}^{z_n} (\tau_m - l_n, \tau_m + l_n)$ with $l_n = n^{-1/2b_2}$ and $z_n \leq Cn^{-1/2b_2}$. Taking $m(\tau) = \arg \min_{\{1,2,\ldots,z_n\}} |\tau - t_m|$.

Thus, we have the following decomposition :

$$\left|\widehat{F}_N(\theta, t_\theta(\gamma), x) - \mathbb{E}\left(\widehat{F}_N(\theta, t_\theta(\gamma), x)\right)\right| = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5$$

where

$\Gamma_1 = \left|\widehat{F}_N(\theta, t_\theta(\gamma), x) - \widehat{F}_N(\theta, t_\theta(\gamma), x_{k(x)})\right|$

$\Gamma_2 = \left|\widehat{F}_N(\theta, t_\theta(\gamma), x_{k(x)}) - \mathbb{E}\left(\widehat{F}_N(\theta, t_\theta(\gamma), x_{k(x)})\right)\right|$

$\Gamma_3 = 2\left|\widehat{F}_N(t_{j(\theta)}, t_\theta(\gamma), x_{k(x)}) - \widehat{F}_N(t_{j(\theta)}, t_{m(\tau)}, x_{k(x)})\right|$

$\Gamma_4 = 2\left|\mathbb{E}\left(\widehat{F}_N(t_{j(\theta)}, t_\theta(\gamma), x_{k(x)})\right) - \mathbb{E}\left(\widehat{F}_N(t_{j(\theta)}, t_{m(\tau)}, x_{k(x)})\right)\right|$

$\Gamma_5 = \left|\mathbb{E}\left(\widehat{F}_N(\theta, t_\theta(\gamma), x_{k(x)})\right) - \mathbb{E}\left(\widehat{F}_N(\theta, t_\theta(\gamma), x)\right)\right|$.

- Concerning Γ_1 we have

$$\left|\widehat{F}_N(\theta, t_\theta(\gamma), x) - \widehat{F}_N(\theta, t_\theta(\gamma), x_{k(x)})\right|$$

$$\leq$$

$$\frac{1}{n}\sum_{i=1}^{n} |\mathbf{1}_{\{Y_i \leq t_\theta(\gamma)\}}| \left|\frac{1}{\mathbb{E}K_1(\theta, x)} K_i(\theta, x) - \frac{1}{\mathbb{E}K_1(\theta, x_{k(x)})} K_i(\theta, x_{k(x)})\right|,$$

we use the Hölder continuity condition on K, the Cauchy-Schwartz inequality and the Bernstein's inequality. This allows us to get :

$$\begin{aligned}\left|\widehat{F}_N(\theta, t_\theta(\gamma), x) - \widehat{F}_N(\theta, t_\theta(\gamma), x_{k(x)})\right| &\leq \frac{C}{\phi(h_n)} \frac{1}{n}\sum_{i=1}^{n} \left|K_i(\theta, x) - K_i(\theta, x_{k(x)})\right| \\ &\leq \frac{C' r_n}{\phi(h_n)} \end{aligned} \tag{2.21}$$

- Concerning Γ_2, the monotony of the functions $\mathbb{E}\widehat{F}_N(\theta, \cdot, x)$ and $\widehat{F}_N(\theta, \cdot, x)$ permits to write, $\forall m \leq z_n$, $\forall x \in \mathcal{S}_{\mathcal{H}}$, $\forall \theta \in \Theta_{\mathcal{H}}$

$$\begin{aligned}\mathbb{E}\widehat{F}_N(\theta, t_{m(\tau)} - l_n, x_{k(x)}) &\leq \sup_{t_\theta(\gamma) \in (t_{m(\tau)} - l_n, t_{m(\tau)} + l_n)} \mathbb{E}\widehat{F}_N(\theta, t_\theta(\gamma), x) \\ &\leq \mathbb{E}\widehat{F}_N(\theta, t_{m(\tau)} + l_n, x_{k(x)}) \\ \widehat{F}_N(\theta, t_{m(\tau)} - l_n, x_{k(x)}) &\leq \sup_{t_\theta(\gamma) \in (t_{m(\tau)} - l_n, t_{m(\tau)} + l_n)} \widehat{F}_N(\theta, t_\theta(\gamma), x) \\ &\leq \widehat{F}_N(\theta, t_{m(\tau)} + l_n, x_{k(x)}).\end{aligned}$$

Next, we use the Hölder's condition on $F(\theta, \xi, x)$ and we show that, for any $\xi_1,\ \xi_2 \in \mathcal{S}_{\mathbb{R}}$

and for all $x \in \mathcal{S}_{\mathcal{H}},\ \theta \in \Theta_{\mathcal{H}}$

$$\begin{aligned} \left|\mathbb{E}\widehat{F}_N(\theta,\xi_1,x) - \mathbb{E}\widehat{F}_N(\theta,\xi_2,x)\right| &= \frac{1}{\mathbb{E}K_1(x,\theta)}\Big|\mathbb{E}\left(K_1(x,\theta)F_\theta^{X_1}(\xi_1)\right) \\ &\qquad - \mathbb{E}\left(K_1(x,\theta)F_\theta^{X_1}(\xi_2)\right)\Big| \\ &\leq C|\xi_1-\xi_2|^{b_2}. \end{aligned} \tag{2.22}$$

Now, we have, for all $\eta > 0$

$$\mathbb{P}\left(\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{t_\theta(\gamma)\in\mathcal{S}_{\mathbb{R}}}|\Xi_n - \mathbb{E}\Xi_n| > \eta\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}}\right)$$
$$=$$
$$\mathbb{P}\left(\max_{j\in\{1\dots d_n^{\Theta_{\mathcal{H}}}\}}\max_{k\in\{1\dots d_n^{\mathcal{S}_{\mathcal{H}}}\}}\max_{1\leq m\leq z_n}|\Xi_n - \mathbb{E}\Xi_n| > \eta\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}}\right)$$
$$\leq$$
$$z_n d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}}\max_{j\in\{1\dots d_n^{\Theta_{\mathcal{H}}}\}}\max_{k\in\{1\dots d_n^{\mathcal{S}_{\mathcal{H}}}\}}\max_{1\leq m\leq z_n}\mathbb{P}\left(|\Xi_n - \mathbb{E}\Xi_n| > \eta\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}}\right)$$
$$\leq$$
$$2z_n d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}}\exp\left(-C\eta^2\log d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}}\right)$$

where $\Xi_n = \widehat{F}_N(\theta, t_\theta(\gamma), x_{k(x)})$, choosing $z_n = \mathcal{O}\left(l_n^{-1}\right) = \mathcal{O}\left(n^{\frac{1}{2b_2}}\right)$, we get

$$\mathbb{E}\left(|\Xi_n - \mathbb{E}\Xi_n| > \eta\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}}\right) \leq C' z_n\left(d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}}\right)^{1-C\eta^2}$$

putting $C\eta^2 = \beta$ and using (A4), we get

$$\Gamma_2 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_n)}}\right).$$

- Concerning the terms Γ_3 and Γ_4, one can write

$$\begin{aligned} |\Upsilon_N| &\leq C\frac{1}{n\phi(h_n)}\sum_{i=1}^n K_i(t_{j(\theta)}, x_{k(x)})\left|\mathbf{1}_{\{Y_i\leq t_\theta(\gamma)\}} - \mathbf{1}_{\{Y_i\leq t_{m(\tau)}\}}\right| \\ &\leq \frac{Cl_n}{nh_n\phi(h_n)}\sum_{i=1}^n K_i(t_{j(\theta)}, x_{k(x)}). \end{aligned}$$

where $\Upsilon_N = \widehat{F}_N(t_{j(\theta)}, t_\theta(\gamma), x_{k(x)}) - \widehat{F}_N(t_{j(\theta)}, t_{m(\tau)}, x_{k(x)})$.
Once again a standard exponential inequality for a sum of bounded variables allows

us to write

$$\widehat{F}_N(t_{j(\theta)}, t_\theta(\gamma), x_{k(x)}) - \widehat{F}_N(t_{j(\theta)}, t_{m(\tau)}, x_{k(x)}) = \mathcal{O}\left(\frac{l_n}{h_n}\right) + \mathcal{O}_{a.co.}\left(\frac{l_n}{h_n}\sqrt{\frac{\log n}{n\phi_x(h_n)}}\right),$$

Now, the fact that $\lim_{n\to\infty} n^\gamma h_n = \infty$ and $l_n = n^{-1/2b_2}$ imply that :

$$\frac{l_n}{h_n\phi(h_n)} = o\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}}}{n\phi(h_n)}}\right),$$

then

$$\Gamma_3 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}}}{n\phi(h_n)}}\right).$$

Hence, for n large enough, we have

$$\Gamma_3 \le \Gamma_4 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}}}{n\phi(h_n)}}\right).$$

- Concerning Γ_5, we have

$$\mathbb{E}\left(\widehat{F}_N(\theta, t_\theta(\gamma), x_{k(x)})\right) - \mathbb{E}\left(\widehat{F}_N(\theta, t_\theta(\gamma), x)\right) \le \sup_{x\in\mathcal{S}_\mathcal{H}} \left|\widehat{F}_N(\theta, t_\theta(\gamma), x) - \widehat{F}_N(\theta, t_\theta(\gamma), x_{k(x)})\right|,$$

then following similar proof used in the study of Γ_1 and using the same idea as for $\mathbb{E}\left(\widehat{F}_D(\theta, x_{k(x)})\right) - \mathbb{E}\left(\widehat{F}_D(\theta, x)\right)$ we get, for n tending to infinity,

$$\Gamma_5 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}}}{n\phi(h_n)}}\right).$$

□

Proof of Corollary 2.2 It is easy to see that, $\inf_{\theta\in\Theta_\mathcal{H}} \inf_{x\in\mathcal{S}_\mathcal{H}} |\widehat{F}_D(\theta, x)| \le 1/2 \implies \exists\, x \in \mathcal{S}_\mathcal{H},\ \exists\, \theta \in \Theta_\mathcal{H}$, such that

$$1 - \widehat{F}_D(\theta, x) \ge 1/2 \implies \sup_{\theta\in\Theta_\mathcal{H}} \sup_{x\in\mathcal{S}_\mathcal{H}} |1 - \widehat{F}_D(\theta, x)| \ge 1/2.$$

We deduce from first part of Lemma 4.7 the following inequality

$$\mathbb{P}\left(\inf_{\theta\in\Theta_\mathcal{H}} \inf_{x\in\mathcal{S}_\mathcal{H}} |\widehat{F}_D(\theta, x)| \le 1/2\right) \le \mathbb{P}\left(\sup_{\theta\in\Theta_\mathcal{H}} \sup_{x\in\mathcal{S}_\mathcal{H}} |1 - \widehat{F}_D(\theta, x)| \le 1/2\right).$$

Consequently,

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\inf_{\theta\in\Theta_\mathcal{H}} \inf_{x\in\mathcal{S}_\mathcal{H}} \widehat{F}_D(\theta, x) < \frac{1}{2}\right) < \infty$$

□

Proof of Lemma 3.6 We use again the same kind of decomposition as (4.12) :

$$\begin{aligned}\widehat{F}^{(j)}(\theta,t_\theta(\gamma),x)-F^{(j)}(\theta,t_\theta(\gamma),x) &= \frac{1}{\widehat{F}_D(\theta,x)}\Big\{\left(\widehat{F}_N^{(j)}(\theta,t_\theta(\gamma),x)-\mathbb{E}\widehat{F}_N^{(j)}(\theta,t_\theta(\gamma),x)\right)\\ &\quad -\left(F^{(j)}(\theta,t_\theta(\gamma),x)-\mathbb{E}\widehat{F}_N^{(j)}(\theta,t_\theta(\gamma),x)\right)\Big\}\\ &\quad +\frac{F^{(j)}(\theta,t_\theta(\gamma),x)}{\widehat{F}_D(\theta,x)}\left\{\mathbb{E}\widehat{F}_D(\theta,x)-\widehat{F}_D(\theta,x)\right\}. \qquad (2.23)\end{aligned}$$

This proof is very similar to the one of Theorem 4.3. Firstly, we consider the bias term $F^{(j)}(\theta,t_\theta(\gamma),x)-\mathbb{E}\widehat{F}_N^{(j)}(\theta,t_\theta(\gamma),x)$. Using the same arguments as along the proof of Lemma 4.6, replacing $F(\theta,t_\theta(\gamma),x))$ (resp. $\widehat{F}(\theta,t_\theta(\gamma),x))$ with $F^{(j)}(\theta,t_\theta(\gamma),x)$ (resp. $\widehat{F}^{(j)}(\theta,t_\theta(\gamma),x))$ and considering in addition hypotheses (H4), (H5), (H9) and (A5) we get :

$$F^{(j)}(\theta,t_\theta(\gamma),x)-\mathbb{E}\widehat{F}_N^{(j)}(\theta,t_\theta(\gamma),x) = \mathcal{O}\left(h_n^{b_1}\right). \qquad (2.24)$$

Indeed, let $H_i^{(j)}(t_\theta(\gamma))=H^{(j)}\left(h_n^{-1}(t_\theta(\gamma)-Y_i)\right)$, note that

$$\begin{aligned}\Psi_n(\theta,y,x) &= \frac{h_n^{-j}}{\mathbb{E}K_1(x,\theta)}\mathbb{E}\Big(K_1(x,\theta)\Big[\mathbb{E}\left(H_1^{(j)}(t_\theta(\gamma))|<X,\theta>\right)\\ &\quad -h_n^jF^{(j)}(\theta,t_\theta(\gamma),x)\Big]\Big). \qquad (2.25)\end{aligned}$$

where $\Psi_n(\theta,t_\theta(\gamma),x)=\mathbb{E}\widehat{F}_N^{(j)}(\theta,t_\theta(\gamma),x)-F^{(j)}(\theta,t_\theta(\gamma),x)$.

Moreover,

$$\begin{aligned}\mathbb{E}\left(H_1^{(j)}(t_\theta(\gamma))|<X,\theta>\right) &= \int_{\mathbb{R}}H^{(j)}\left(h_n^{-1}(t_\theta(\gamma)-z)\right)f(\theta,z,X_1)dz,\\ &= -\sum_{l=1}^{j-1}h_n^l\Big[H^{(j-l)}\left(h_n^{-1}(t_\theta(\gamma)-z)\right)\\ &\qquad F^{(l)}(\theta,z,X_1)\Big]_{-\infty}^{+\infty}\\ &\quad + h_n^{j-1}\int_{\mathbb{R}}H^{(1)}\left(h_n^{-1}(t_\theta(\gamma)-z)\right)F^{(j)}(\theta,z,X_1)dz. \qquad (2.26)\end{aligned}$$

Condition (H8) allows us to cancel the first term in the right side of (2.26) and we can write :

$$\left|\mathbb{E}\left(H_1^{(j)}(t_\theta(\gamma))|<X,\theta>\right)-h_n^jF^{(j-1)}(\theta,t_\theta(\gamma),x)\right|$$
$$\leq$$
$$h_n^j\int_{\mathbb{R}}H^{(1)}(t)\left|F^{(j)}(\theta,t_\theta(\gamma)-h_nt,X_1)-F^{(j)}(\theta,t_\theta(\gamma),x)\right|dt.$$

Finally, (A5) allows to write

$$\left|\mathbb{E}\left(H_1^{(j)}(t_\theta(\gamma))|<X,\theta>\right)-h_n^jF^{(j)}(\theta,t_\theta(\gamma),x)\right|$$
$$\leq$$

$$C_{\theta,x}h_n^j\int_{\mathbb{R}} H(t)\left(h_n^{b_1}+(|t|h_n)^{b_2}\right)dt.$$

because this last inequality is uniform on $(\theta,t_\theta(\gamma),x)\in\Theta_{\mathcal{H}}\times\mathcal{S}_{\mathcal{H}}\times\mathcal{S}_{\mathbb{R}}$, the use of now to finish the proof it is sufficient to use (H5).

Now, we focus on the term $\widehat{F}_N^{(j)}(\theta,t_\theta(\gamma),x)-\mathbb{E}\widehat{F}_N^{(j)}(\theta,t_\theta(\gamma),x)$. To get the asymptotic behavior of this quantity, we follow the framework of the proof of Lemma 4.7. To do that, once again replace $F(\theta,t_\theta(\gamma),x))$ (resp. $\widehat{F}(\theta,t_\theta(\gamma),x))$ with $F^{(j)}(\theta,t_\theta(\gamma),x)$ (resp. $\widehat{F}^{(j)}(\theta,t_\theta(\gamma),x)$). Note that (H9), (H10) and (A6) allow to show that

$$\mathbb{E}\left(H^{(j)}(h_n^{-1}(t_\theta(\gamma)-Y_i))H^{(j)}(h_n^{-1}(t_\theta(\gamma)-Y_{i'}))|(X_i,X_{i'})\right)=\mathcal{O}(h_n^2),$$

while (A1) and (A5) imply that

$$\mathbb{E}\left(H^{(j)}(h_n^{-1}(t_\theta(\gamma)-Y_i))|<X_i,\theta>\right)=\mathcal{O}(h_n).$$

Consider the following decomposition

$$\left|\widehat{F}_N^{(j)}(\theta,t_\theta(\gamma),x)-\mathbb{E}\left(\widehat{F}_N^{(j)}(\theta,t_\theta(\gamma),x)\right)\right|=\Delta_1+\Delta_2+\Delta_3+\Delta_4+\Delta_5.$$

where

$$\Delta_1=\left|\widehat{F}_N^{(j)}(\theta,t_\theta(\gamma),x)-\widehat{F}_N^{(j)}(\theta,t_\theta(\gamma),x_{k(x)})\right|$$
$$\Delta_2=\left|\widehat{F}_N^{(j)}(\theta,t_\theta(\gamma),x_{k(x)})-\mathbb{E}\left(\widehat{F}_N^{(j)}(\theta,t_\theta(\gamma),x_{k(x)})\right)\right|$$
$$\Delta_3=2\left|\widehat{F}_N^{(j)}(t_{j(\theta)},t_\theta(\gamma),x_{k(x)})-\widehat{F}_N^{(j)}(t_{j(\theta)},t_{m(\tau)},x_{k(x)})\right|$$
$$\Delta_4=2\left|\mathbb{E}\left(\widehat{F}_N^{(j)}(t_{j(\theta)},t_\theta(\gamma),x_{k(x)})\right)-\mathbb{E}\left(\widehat{F}_N^{(j)}(t_{j(\theta)},t_{m(\tau)},x_{k(x)})\right)\right|$$
$$\Delta_5=\left|\mathbb{E}\left(\widehat{F}_N^{(j)}(\theta,t_\theta(\gamma),x_{k(x)})\right)-\mathbb{E}\left(\widehat{F}_N^{(j)}(\theta,t_\theta(\gamma),x)\right)\right|.$$

- concerning Δ_1, Δ_2 and Δ_5 the same arguments as those invoked in (2.21) are used; of which by substituting by $H^{(j)}$ and we apply the Lipschitz condition (H65) and (H9). This allows us to get :

  $$\left|\widehat{F}_N^{(j)}(\theta,y,x)-\widehat{F}_N^{(j)}(\theta,y,x_{k(x)})\right|\le\frac{Cr_n}{h_n^j\phi(h_n)},$$

  note that (A6) and so, for n large enough, we have

  $$\Delta_1=\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}}}{nh_n^{2j-1}\phi(h_n)}}\right).$$

  Then using the fact that $\Delta_5\le\Delta_1$, we obtain

  $$\Delta_5\le\Delta_1=\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}}}{nh_n^{2j-1}\phi(h_n)}}\right).\tag{2.27}$$

  For Δ_2, we follow the same idea given for Γ_2, we get for a choice to $z_n\le Cn^{-\frac{3}{2}\nu-\frac{1}{2}}$,

  $$\Delta_2=\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}}}{nh_n^{2j-1}\phi(h_n)}}\right).$$

Concerning Δ_3 and Δ_4, using Lipschitz's condition on the kernel $H^{(j)}$,

$$\left|\widehat{F}_N^{(j)}(t_{j(\theta)}, y, x_{k(x)}) - \widehat{F}_N^{(j)}(t_{j(\theta)}, y_{m(y)}, x_{k(x)})\right| \leq \frac{l_n}{h_n^{j+1}\phi(h_n)},$$

using the fact that $\lim_{n\to\infty} n^{\nu} h_n = \infty$ and choosing $l_n = n^{-\frac{3}{2}\nu-\frac{1}{2}}$ implies

$$\frac{l_n}{h_n^{j+1}\phi(h_n)} = o\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{nh_n^{2j-1}\phi(h_n)}}\right).$$

So, for n large enough, we have

$$\Delta_3 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}}{nh_n^{2j-1}\phi(h_n)}}\right),$$

and as $\Delta_4 \leq \Delta_3$, we obtain

$$\Delta_4 \leq \Delta_3 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}}{nh_n^{2j-1}\phi(h_n)}}\right). \tag{2.28}$$

Finally, the result can be easily deduced from (2.27) and (2.28).

□

2.6 Concluding Remarks

Single-index models and varying coefficient models are powerful tools for dimension reduction and semi-parametric modeling because they can effectively avoid "curse of dimensionality". Both models largely relax some restrictive assumptions on linear and nonparametric models. Specially, the single-index model is usually treated as the first step of the famous projection pursuit regression. The single-index model has also been extensively used in the projection pursuit regression,

For example, firstly the motivation of this model also comes from an analysis of environmental data, consisting of daily measurement of pollutants and other environment factors. Of interest is to examine the association between the levels of pollutants, two environment factors (temperature and relative humidity) and the total number of daily hospital admissions for respiratory problems.

Secondly, single-index models cannot reflect the additivity of covariates, while the performance of varying coefficient models can be poor if the varying coefficient contains moderate or high dimensional covariates.

In standard multivariate regression, the single-index model is a good compromise between nonparametric and parametric regression models when one wishes to regress a response variable on several real-valued explanatory ones. It assumes the existence of a latent one-dimensional explanatory variable which allows us to explain the response through a nonparametric regression model.

In addition, the latent explanatory variable is supposed to be a linear combination of the explanatory variables. The vector of the linear combination is called "single index". Such models are useful tools for interpreting some situations. This is particularly the case in econometrics where single-index models and various extensions have been intensively studied.

Single-index models, or projection pursuit regression, have proven to be an efficient way of coping with the high-dimensional problem in nonparametric regressions (see, e.g., Hall (1989) ; Ichimura (1993)). The idea is restricting the general multivariate regression function to a special form.

For the functional single-index models, the literature is closely limited, and only a few theoretical results have been obtained until now.

For the past two decades, the single-index model, a special case of projection pursuit regression, has proven to be an efficient way of coping with the high dimensional problem in nonparametric regression. Here we deal with single index modeling when the explanatory variable is functional. More precisely, we consider the problem of estimating the conditional density of a real variable Y given a functional variable X when the explanation of Y given X is done through its projection on one functional direction.

The single-index approach is widely applied in econometrics as a reasonable compromise between nonparametric and parametric models. Such kind of modelization is intensively studied in the multivariate case. Without pretend to exhaustivity.

In practice, this study has great importance, because, it permit us to construct a prediction method based on the conditional mode estimator. Moreover, in the case where the functional single index is unknown, our estimate can be used to estimate this parameter via the pseudo-maximum likelihood estimation method. Noting that the estimation of the functional single-index has great interest on the semi-metric choice in nonparametric functional data analysis but it has been not attacked in this paper.

In this article, we examine conditional quantile estimation in the single functional index model for independent functional data. The asymptotic properties such as pointwise almost complete consistency and the uniform almost complete convergence of the kernel estimator with rate are presented under some mild conditions. Although α-mixing is reasonably weak among various weak dependence process and has many practical applications such as in time series prediction, we also address other dependence settings such as long memory dependence functional data (see Benhenni *et al.* 2008). In this case, the asymptotic properties of the estimation of successive derivatives of the conditional density function, conditional hazard function, conditional distribution function and conditional quantile in the single functional index model have been investigated in our other works.

The goal of this work is to contribute to the functional data literature by studying some classes of semi-parametric models. Note that such models are important in the statistical and econometric modelization, due to its flexibility for dimension reduction, it provides the best new ways to investigate problems in substantive economics (see Horowitz 2009). As a particular case, the single-index model has proven useful in providing an optimal approach to compromise between nonparametric and parametric models.

Single-index models when the explanatory variable is an element of a finite-dimensional space have been studied extensively in both statistical and econometric literatures, we quote, for instance, Härdle *et al.* (1993), Horowitz (1996), Hristache *et al.* (2001,2001b) and Delecroix *et al.* (2003).

For the functional single-index models, the literature is closely limited, and only a few theoretical results have been obtained until now. The first asymptotic properties in the fixed functional single-model were obtained by Ferraty *et al.* (2003). They established the almost complete convergence, in the i.i.d. case, of the link regression function of this model. Their results were extended to the dependent case by Aït Saidi *et al.* (2005). Where the functional single-index is unknown, Aït Saidi *et al.* (2008) proposed an estimator of this parameter, based on the cross-validation procedure. Newly, Ferraty and Park (2011) proposed a new estimator of the single index based on the idea of functional derivative estimation.

2.6.1 About the Functional Single Index Estimate

Among the interest of our study is to show how the conditional density estimate can be used to derive an estimate of the functional single index if the latter is unknown. The leave-out-one-curve cross-validation procedure was adapted by Aït Saidi *et al.* (2008) to estimate the single index. Newly, Hall and Müller (2005) proposed a method for estimating functional derivatives and Ferraty and Park (2011) adopted this technique to estimate the parameter θ. Alternatively, this parameter can be estimated via the pseudo-maximum likelihood method which is based on the preliminary estimation of the conditional density of Y given X by

$$\widehat{\theta} = \arg\max_{\theta \in \Theta_{\mathcal{F}}} \widehat{L}(\theta)$$

where

$$\widehat{L}(\theta) = \sum_{k=1}^{n} \log \widehat{f}(\theta, Y_k, X_k).$$

Note that, this method has been studied by Delecroix *et al.* (2003) in the real case where they showed that this technique has minimal variance among all estimators. The asymptotic optimality of this procedure in functional statistic, is an important prospect of the present work.

As an application, this approach can be used for answering the semi-metric choice question. Indeed, it is well known that, in nonparametric functional statistic, the projection-type semi-metric is very important for increasing the concentration property. The functional index model is a particular case of this family of semi-metric, because it is based on the projection on one functional direction. So, the estimation procedures of this direction permit us to compute adaptive semi-metrics in the general context of nonparametric functional data analysis. Finally, the theoretical justification and practice should be established.

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Chapitre Trois

Rate of uniform consistency for nonparametric of the conditional quantile estimate with functional variables in the single functional index model

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Rate of uniform consistency for nonparametric of the conditional quantile estimate with functional variables in the single functional index model

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Abstract. *The main objective of this paper is to estimate non-parametrically the quantiles of a conditional distribution when the sample is considered as an α-mixing sequence. First of all, a kernel type estimator for the conditional cumulative distribution function (*cond-cdf*) is introduced. Afterwards, we give an estimation of the quantiles by inverting this estimated* cond-cdf*, the asymptotic properties are stated when the observations are linked with a single-index structure. The pointwise almost complete convergence and the uniform almost complete convergence (with rate) of the kernel estimate of this model are established. This approach can be applied in time series analysis. For that, the whole observed time series has to be split into a set of functional data, and the functional conditional quantile approach can be employed both in foreseeing and building confidence prediction bands.*

Keywords : Conditional quantile, conditional cumulative distribution, derivatives of conditional cumulative distribution, functional random variable, kernel estimator, nonparametric estimation, strong mixing processes.

2000 Mathematics Subject Classification : 62G05, 62G99, 62M10.

3.1 Introduction

Estimating quantiles of any distribution is a substantial part of Statistics, it guaranties to build confidence ranges deriving many applications in numerous fields, chemistry, geophysics, medicine, meteorology,.... Furthermore, Statistics for functional random variables become progressively important, the latest literature in this domain presents the great potential of these functional statistical methods. The most famous case of functional random variable corresponds to the situation when we observe random curve on different statistical units. Such data are called *Functional Data*. Numerous multivariate statistical technics, mainly parametric in the functional model terminology, have been extended to functional data and good analysis on this area can be found in Ramsay and Silverman (1997 and 2002) or Bosq (2000). Lately, nonparametric methods considering functional variables have been grown with very interesting practical motivations dealing with environmetrics, (see Damon and Guillas (2002), Fernández *et al.* (2005), Aneiros *et al.* (2004)), chemometrics (see Ferraty and Vieu (2002)), meteorological sciences (see Besse *et al.* (2000), Hall and Heckman (hall2002)), speech recognization problem (see Ferraty and Vieu (2003)), radar range profile (see Hall *et al.* (2001), Dabo-Niang *et al.* (2004)), medical data (see Gasser *et al.* (1998)), ... Moreover, forecasting techniques cover a big part of the statistical problems. Because a continuous time series can be seen as a sequence of dependent functional random variables, the above mentioned functional methodology can be used for time-series forecasting (see for instance Ferraty *et al.*, (2002), for a functional forecasting approach of time-series based on conditional expectation estimation). This article suggests to bring together the three former statistical aspects in order to derive a method for estimating conditional quantiles in situation when the data are both dependent and of functional nature. In particular, we focus on the nonparametric estimation of the conditional quantiles of a real random variable given a functional random variable under mixing assumption. We start by estimating the conditional distribution by means of a kernel estimator and we derive an estimate of the conditional quantiles (see Section 2). From a theoretical point of view, a crucial problem is linked with the so-called *curse of dimensionality*. Actually, in a nonparametric context, it is well known that the rate of convergence decreases with the dimension of the space in which the conditional variable is valued. But here, the conditional variable takes its values in an infinite dimensional space. One way to override this problem is to consider some concentration hypotheses acting on the distribution of the functional variable which allows to obtain asymptotic properties of our kernel estimates (see Section 3). This approach is used to derive a new method to forecast time series.

3.1.1 On the problematic of single index models

For several years, a increasing interest is worn to models which incorporating of both the parts parametric and nonparametric. Such models type are called semi-parametric model. This consideration is due primarily to problems due to poor specification of some models. Tackle a problem of mis-specification semiparametric way consists in not specify the functional form of some model components. This approach complete those non-parametric models, which can not be useful in small samples, or with a large number of variables. As example, in the classical regression case, the important parameter whose one assumed existence is the regression function of Y knowing the covariate X, denoted $r(x) = \mathbb{E}(Y|X = x)$, $X, Y \in \mathbb{R}^d \times \mathbb{R}$. For this model, the non-parametric method considers only regularity assumptions on the function r. Obviously, this method has some drawbacks. One can cite the problem of curse of dimensionality. This problem appears when the number of regressors d increases, the rate of convergence of the nonparametric estimator r which is supposed k times differentiable is $O(n^{-k/2k+d})$ deteriorate. The second drawback is the lack of means to quantify the effect of each explanatory variable. To alleviate in these drawbacks, an alternative approach is naturally provided by the semi-parametric model which supposes the introduction of a parameter on the regressors, by writing than the regression function is of the form

$$\mathbb{E}_\theta(Y|X) = \mathbb{E}(Y| < X, \theta >= x),$$

The models defined are known in the literature as the single-index models.

These models allow to obtain a compromise between parametric models, generally too restrictive and nonparametric model where the rate of convergence of the estimators deteriorate quickly in the presence of a large number of explanatory variables. In this area, different types of models have been studied in the literature : amongst the most famous, there may be mentioned additive models, partially linear models or single index models. The idea of these models, in the case of estimating the conditional density or regression consists in bring to the covariates a dimension in smaller than dimension of the space variable, thus allowing overcome the problem of curse of dimensionality. For example, for example, in the partially linear model, we decompose the quantity to be estimated, into a linear part and a functional part. This latter quantity does not pose estimation problem since it's expressed as a function of explanatory variables of finite dimension, thus avoiding the problems associated with curse of dimensionality. in order to treat the problem of curse of dimensionality in the case chronologies series, several semi-parametric approaches have been proposed. Without pretend to exhaustively, we quote for example : Xia and An (2002) for the index model. A general presentation of this type of model is given in Ichimura *et al.* (1993) where the convergence and asymptotic normality are obtained. In the case of M-estimators, Delecroix *et al.* (1999) proves the consistency and asymptotic normality of the estimate the index and they study it's effectiveness. The statistical literature on these methods is rich, quote Huber (1985) and Hall (1989) present an estimation method which consists projecting the density and the regression function on a space of dimension one, to bring a non-parametric estimation for dimensional covariates. This amounts exactly to estimate these functions in a single index model. Attaoui *et al.* (2011) have established the pointwise and the uniform almost complete convergence (with the rate) of the kernel estimate of this model. The interest of their study is to show how the estimate of the conditional density can be used to obtain an estimate of the simple functional index if the latter is unknown. More precisely, this parameter can be estimated by pseudo-maximum likelihood method which is based the preliminary estimate of the conditional density. recently Bouchentouf *et al.* (2014) have established the pointwise almost complete convergence and the uniform almost complete convergence (with the rate) of some characteristics of the conditional distribution and the successive derivatives of the conditional density when the observations are linked with a single-index structure and they are applied to the estimations of the conditional mode and conditional quantiles.

The single-index approach is widely applied in econometrics as a reasonable compromise between nonparametric and parametric models. Such kind of modelization is intensively studied in the multivariate case. Without pretend to exhaustivity, we quote for example Härdle *et al.* (1993), Hristache *et al.* (2001). Based on the regression function, Delecroix *et al.* (2003) studied the estimation of the single-index and established some asymptotic properties. The literature is strictly limited in the case where the explanatory variable is functional (that is a curve). The first asymptotic properties in the fixed functional single-model were obtained by Ferraty *et al.* (2003). They established the almost complete convergence, in the i.i.d. case, of the link regression function of this model. Their results were extended to dependent case by Aït Saidi *et al.* (2005). Aït Saidi *et al.* (2008) studied the case where the functional single-index is unknown. They proposed an estimator of this parameter, based on the cross-validation procedure.

3.2 The model and the estimates

3.2.1 The functional nonparametric framework

Consider a random pair (X, Y) with values in $\mathcal{H} \times \mathbb{R}$, where $\mathcal{H}$ is a separable real Hilbert space with the norm $\| \cdot \|$ generated by an inner product $< \cdot, \cdot >$. Let $(X_i, Y_i)_{i=1,\dots,n}$ be the statistical sample of pairs which are identically distributed like (X, Y), but not necessarily independent. Henceforward, X is called functional random variable *f.r.v.* . We consider the semi-metric d_θ, associated to the single-index $\theta \in \mathcal{H}$ defined by $\forall x_1, x_2 \in \mathcal{H}: \; d_\theta(x_1, x_2) := |< x_1 - x_2, \theta >|$. Under such topological structure and for a fixed functional θ, we suppose that the conditional cumulative distribution function (*cond-cdf*) of Y given $X = x$ denoted by $F(\cdot|x)$ exists and is given by :

$$\forall y \in \mathbb{R}, \; F_\theta(y|x) =: F(y| < x, \theta >) = F(\theta, y, x) = \mathbb{P}(Y \leq y| < X, \theta >=< x, \theta >).$$

By considering the same conditions of Ferraty et al. (2003) on the regression operator, the identifiability of the model is assumed. More precisely we suppose that f is differentiable with respect to (w.r.t) x and θ such that $< \theta, e_1 >= 1$, where e_1 is the first eigenvector of an orthonormal basis of the space $\mathcal{H}$. Clearly, we have for all $x \in \mathcal{H}$

$$F_1(\cdot| < x, \theta_1 >) = F_2(\cdot| < x, \theta_2 >) \implies F_1 \;\equiv\; F_2 \quad \text{and} \quad \theta_1 = \theta_2.$$

In what follows, we denote by $F(\theta, \cdot, x)$ the conditional distribution of Y given $< \theta, x >$.

Saying that, we are implicitly assuming the existence of a regular version for the conditional distribution of Y given $< \theta, x >$. Now, let t_γ be the γ-order quantile of the distribution of Y given $< \theta, X >=< \theta, x >$. From the *cond-cdf* $F(\theta, \cdot, x)$, the general definition of the γ-order quantile is given as :

$$t_\theta(\gamma) = \inf\{t \in \mathbb{R} : \; F(\theta, t, x) \geq \gamma\}, \qquad \forall \gamma \in (0, 1).$$

In order to simplify our framework and to focus on the main interest of our paper (the functional feature of $< \theta, X >$), we assume that $F(\theta, ., x)$ is strictly increasing and continuous in a neighborhood of t_γ. This is insuring that the conditional quantile t_γ is uniquely defined by :

$$t_\theta(\gamma) \;=\; F^{-1}(\theta, \gamma, x). \tag{3.1}$$

Next, in all what follows, we assume only smoothness restrictions for the *cond-cdf* $F(\theta, ., x)$ through nonparametric modelling (Section 2.4). We suppose also that $(X_i, Y_i)_{i \in \mathbb{N}}$ is an α-mixing sequence, which is one among the most general mixing structures. The α-mixing condition together with the functional approach allow to deal with continuous time processes (see Section 4 for instance).

Recall that a process $(X_i, Y_i)_{i \geq 1}$ is called α-mixing or strongly mixing (see Lin and Lu (1996)) for more details and examples), if

$$\sup_{A \in \mathcal{F}_1^k} \sup_{B \in \mathcal{F}_{n+k}^\infty} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| = \alpha(n) \to 0 \;\text{ as }\; n \to \infty,$$

where $\mathcal{F}_i^j$ is the σ-field generated by $X_i, \dots, X_j$.

In our infinite dimensional purpose, we use the terminology *functional nonparametric*, where the word *functional* referees to the infinite dimensionality of the data and where the word *nonparametric* referees to the infinite dimensionality of the model. Such *functional nonparametric* statistics is also called *doubly infinite dimensional* (see Ferraty and Vieu (2003b), for more details). We also use the terminology *operatorial statistics* since the target object to be estimated (the *cond-cdf* $F(\theta, ., x)$) can be viewed as a nonlinear operator.

3.2.2 The estimators

The kernel estimator $\widehat{F}(\theta, ., x)$ of $F(\theta, ., x)$ is presented as follows :

$$\widehat{F}(\theta, y, x) = \frac{\sum_{i=1}^{n} K\left(h_K^{-1}(< x - X_i, \theta >)\right) H\left(h_H^{-1}(y - Y_i)\right)}{\sum_{i=1}^{n} K\left(h_K^{-1}(< x - X_i, \theta >)\right)}, \tag{3.2}$$

where K is a kernel function, H a cumulative distribution function and $h_K = h_{K,n}$ (resp. $h_H = h_{H,n}$) a sequence of positive real numbers. Note that using similar ideas, Roussas (1969) introduced some related estimates but in the special case when X is real, while Samanta (1989) produced previous asymptotic study.

As a by-product of ((4.1)) and ((4.2)), it is easy to derive an estimator $\widehat{t}_\gamma$ of t_γ :

$$\widehat{t}_\theta(\gamma) \;=\; \widehat{F}^{-1}(\theta, \gamma, x). \tag{3.3}$$

Such an estimator is unique as soon as H is an increasing continuous function. Such an approach has been largely used in the case where the variable X is of finite dimension (see *e.g* Whang and Zhao, (1999), Cai (2002), Zhou and Liang (2003) or Gannoun *et al.* (2003)).

3.2.3 Assumptions on the functional variable

Let N_x be a fixed neighborhood of x and let $B(x, h_K)$ be the ball of center x and radius h_K, namely $B_\theta(x, h_K) = \{z \in \mathcal{H}/0 < | < x - z, \theta > | < h_K\}$. Then, let's consider the following hypotheses :

(H0) $\mathbb{P}(X \in B_\theta(x, h_k)) = \phi_{\theta,x}(h_k) > 0$, $\phi_{\theta,x}(h_k) \to 0$ as $h_K \to 0$

(H1) $(X_i, Y_i)_{i\in\mathbb{N}}$ is an α-mixing sequence whose the coefficients of mixture verify :

$$\exists a > (5 + \sqrt{17})/2,\ \exists c > 0 :\ \forall n \in \mathbb{N},\ \alpha(n) \le cn^{-a}.$$

(H2) $0 < \sup_{i\neq j} \mathbb{P}\left((X_i, X_j) \in B_\theta(x, h_k) \times B_\theta(x, h_k)\right) \;=\; \mathcal{O}\left(\frac{(\phi_{\theta,x}(h_K))^{(a+1)/a}}{n^{1/a}}\right).$

• (H0) can be interpreted as a concentration hypothesis acting on the distribution of the *f.r.v.* X, while (H2) concerns the behavior of the joint distribution of the pairs (X_i, X_j). Indeed, this hypothesis is equivalent to assume that, for n large enough

$$\frac{\sup_{i\neq j} \mathbb{P}\left((X_i, X_j) \in B_\theta(x, h_K) \times B_\theta(x, h_K)\right)}{\mathbb{P}\left(X \in B_\theta(x, h_K)\right)} \le C\left(\frac{\phi_{\theta,x}(h_K)}{n}\right)^{1/a}.$$

This is one way to control the local asymptotic ratio between the joint distribution and its margin. Remark that the upper bound increases with a. In other words, the stronger the dependence, more restrictive is (H2). The hypothesis (H1) specifies the asymptotic behavior of the α-mixing coefficients.

3.2.4 The nonparametric model

As usually in nonparametric estimation, we suppose that the *cond-cdf* $F(\theta, ., x)$ verifies some smoothness constraints. Let b_1 and b_2 be two positive numbers ; such that :

(H3) $\forall (x_1, x_2) \in N_x \times N_x, \forall (y_1, y_2) \in \mathcal{S}_{\mathbb{R}}^2, |F(\theta, y_1, x_1) - F(\theta, y_2, x_2)| \leq C_{\theta,x} \left(\|x_1 - x_2\|^{b_1} + |y_1 - y_2|^{b_2} \right),$

(H4) $F(\theta, ., x)$ is j-times continuously differentiable in some neighborhood of $t_\theta(\gamma)$,

(H5) $\forall (x_1, x_2) \in N_x \times N_x$, $\forall (y_1, y_2) \in \mathcal{S}_{\mathbb{R}}^2$,

$$|F^{(j)}(\theta, y_1, x_1) - F^{(j)}(\theta, y_2, x_2)| \leq C_{\theta,x} \left(\|x_1 - x_2\|^{b_1} + |y_1 - y_2|^{b_2} \right),$$

where, for any positive integer l, $F^{(l)}(\theta, z, x)$ denotes its lth derivative ($i.e.$ $\left.\frac{\partial^l F(\theta, y, x)}{\partial y^l}\right|_{y=z}$).

Let's note that (H3) is used for the proof of the the almost complete convergence of $\widehat{t}_\theta(\gamma)$ whereas (H4) and (H5) are needed to establish the rate of convergence.

3.2.5 Comment and Remarks

1. Remarks on the Assumptions : Our conditions are very standard in this context. Assumption (H0) characterizes the probability measure concentration of the functional variable on small balls, it is the basic condition used in the majority of the existing works dealing with this kind of variable. In our work we consider a single index topological structure when the concentration of measure changes a little. More precisely the quantity in (H0) can be written as $\mathbb{P}\left(| < x - X, \theta > | \leq h_K\right) = \phi_{\theta,x}(h_k)$. The regularity of the functional space of our model is controlled by mean of condition (H3) and is needed to evaluate the bias of estimation. In order to establish the almost complete convergence rate of our model under the ?-mixing hypothesis, we need to the assumption (H2), that described the asymptotic behavior of the joint distribution of the couple (X_i, X_j). Nothing that, the asymptotic expression imposed in (H2) was obtained as n go to infinity by the following equivalent :

   The assumption (H2) can be seen differently, based on the idea of maximum concentration between the quantities $\mathbb{P}\left(X_i \in B_\theta(x, h_K)\right)$ and $\mathbb{P}\left(X_j \in B_\theta(x, h_K)\right)$ (see Ferraty et al. (2005)). The rest assumptions are the technical conditions imposed for brevity of proofs.

2. Remarks on the single index : It is well known that one of the main advantages of the single index model is its ability to deal with the problem of high dimensional data. A straightforward example is the optimal convergence rate of type $\mathcal{O}\left(n^{-2q/(2q+p)}\right)$ for the estimation of a q-times differentiable regression function, this rate goes to zero dramatically slowly if q is small compared to the dimension p of the explanatory variable ($X \in \mathbb{R}^p$). In this regard, Gaïffas and Lecué (2007) showed that the optimal rate of convergence of regression function in the single index model, is of order $\mathcal{O}\left(n^{-2q/(2q+1)}\right)$ (rather than $\mathcal{O}\left(n^{-2q/(2q+p)}\right)$). The same idea was adopted in the abstract metric spaces by the choice of the semi-metric increasing the probability measure concentration of the explanatory variable in small balls (see Ferraty *et al.* (2006)), Section 13.2). Among these

family of semi-metric, one can consider the semi-metric induced by the functional single index estimate.

3.3 Asymptotic study

This part of paper is devoted, to the theoretical analysis, we start it by giving the almost complete convergence ($a.co.$) of the estimated conditional quantile $\widehat{t}_\theta(\gamma)$. After that, we will focus on the rate of convergence. Concerning the notations, as soon as possible, C and C' will denote generic constants. Moreover, from now on, h_H (resp. h_K) is a sequence which tends to 0 with n.

3.3.1 Pointwise almost complete convergence

Let's begin with the statement of an almost complete convergence property [1]. To this end, we need some assumptions concerning the kernel estimator $\widehat{F}(\theta, ., x)$:

(H6) The restriction of H to the set $\left\{u \in \mathbb{R},\ H(u) \in (0,1)\right\}$ is a strictly increasing function,

(H7) $\forall (y_1, y_2) \in \mathbb{R}^2,\ |H(y_1) - H(y_2)| \leq C|y_1 - y_2|$ and $\int |t|^{b_2} H^{(1)}(t)dt < \infty$,
where, for all $l \in \mathbb{N}^*$, $H^{(l)}(t) = \left.\frac{d^l H(y)}{dy^l}\right|_{y=t}$,

(H8) K is a positive bounded function with support $[-1, 1]$ such that $\forall u \in (0,1) \quad 0 < K(u)$,

(H9) $\dfrac{\log n}{n\, \phi_{\theta,x}(h_K)} \underset{n\to\infty}{\rightarrow} 0.$

(H10) (X_i, Y_i) for $i = 1, \ldots, n$ are strongly mixing with arithmetic coefficient of order $a > 1$, and $\exists \beta > 2$ such that
(i) $s_{n,l}^{-(a+1)} = o\left(n^{-\beta}\right)$ for $l = 0, 1, 2\,;$
(ii) $s_{n,k}^{-(a+1)} = o\left(n^{-\beta}\right)$ for $k = 3, 4, 5, 6, 7\,;$

Remark 3.1 • *(H7) insures the existence of $\widehat{t}_\theta(\gamma)$, while (H6) insures its unicity.*
• *(H0)-(H5) and (H8) are standard assumptions for the distribution conditional estimation in single functional index model, which have been adopted by Bouchentouf et al. (2014) for i.i.d case.*
• *(H9) is a technical condition for our results.* • *(H10) is similar to that appeared in Ferraty and Vieu (2006), it shows the influence of covariance structure on the convergence rate. Here, $s_{n,l}$ and $s_{n,k}$ will be defined bellow.*

Theorem 3.1 *Put $s_n = \max\{s_{n,0}; s_{n,1}\}$, and suppose that either (H10)-(i) is satisfied together with hypotheses (H0)-(H3) and (H6)-(H9), thus we have :*

$$\widehat{t}_\theta(\gamma) - t_\theta(\gamma) \underset{n\,\to\,\infty}{\longrightarrow} 0,\ a.co. \tag{3.4}$$

Proof 3.1 (Proof of Theorem 4.1) *The proof is based on the pointwise convergence of $\widehat{F}(\theta, ., x)$ at $t_\theta(\gamma)$:*

$$F(\theta, t_\theta(\gamma), x) - \widehat{F}(\theta, t_\theta(\gamma), x) \underset{n\,\to\,\infty}{\longrightarrow} 0,\ a.co. \tag{3.5}$$

1. Recall that a sequence $(T_n)_{n\in\mathbb{N}}$ of random variables is said to converge a.co. to some variable T, if for any $\epsilon > 0$, we have $\sum_{n\geq 1} \mathbb{P}(|T_n - T| > \epsilon) < \infty$. Note this type of convergence implies both the almost-sure convergence and the convergence in probability (see for instance Bosq and Lecoutre, (1987)).

Where the proof of the latter follows directly from Lemmas 3.1 and 3.2 which will be given below.

First of all, let's note that because of (H6) and (H7), $\widehat{F}(\theta,.,x)$ is a continuous and strictly increasing function. So, we have :

$$\forall\epsilon>0,\ \exists\delta(\epsilon)>0,\forall y,|\widehat{F}(\theta,y,x)-\widehat{F}\left(\theta,t_\theta(\gamma),x\right)|\leq\delta(\epsilon)\Rightarrow|y-t_\theta(\gamma)|\leq\epsilon.$$

This leads us to write

$$\begin{aligned}\forall\epsilon>0,\ \exists\delta(\epsilon)>0,\ \mathbb{P}\left(|\widehat{t}_\theta(\gamma)-t_\theta(\gamma)|>\epsilon\right) &\leq \mathbb{P}\left(|\widehat{F}(\theta,\widehat{t}_\theta(\gamma),x)-\widehat{F}(\theta,t_\theta(\gamma),x)|\geq\delta(\epsilon)\right)\\ &= \mathbb{P}\left(|F(\theta,t_\theta(\gamma),x)-\widehat{F}(\theta,t_\theta(\gamma),x)|\geq\delta(\epsilon)\right),\end{aligned}$$

since (4.3) is implying that $\widehat{F}(\theta,\widehat{t}_\theta(\gamma),x)=\gamma=F(\theta,t_\theta(\gamma),x)$.

Consider now, for $i=1,\ldots,n$, the following notations :

$$K_i(\theta,x)=K(h_K^{-1}(<x-X_i,\theta>)),\qquad H_i(t_\theta(\gamma))=H\left(h_H^{-1}\left(t_\theta(\gamma)-Y_i\right)\right),$$

$$\widehat{F}_N(\theta,t_\theta(\gamma),x)=\frac{1}{n\,\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^{n}K_i(\theta,x)H_i(t_\theta(\gamma))\ \text{and}\ \widehat{F}_D(\theta,x)=\frac{1}{n\,\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^{n}K_i(\theta,x)).$$

By using the following decomposition

$$\begin{aligned}\widehat{F}(\theta,t_\theta(\gamma),x)-F(\theta,t_\theta(\gamma),x) &= \frac{1}{\widehat{F}_D(\theta,x)}\left\{\left(\widehat{F}_N(\theta,t_\theta(\gamma),x)-\mathbb{E}\widehat{F}_N(\theta,t_\theta(\gamma),x)\right)\right\}\\ &-\frac{1}{\widehat{F}_D(\theta,x)}\left\{\left(F(\theta,t_\theta(\gamma),x)-\mathbb{E}\widehat{F}_N(\theta,t_\theta(\gamma),x)\right)\right\}\\ &+\frac{F(\theta,t_\theta(\gamma),x)}{\widehat{F}_D(\theta,x)}\left\{\mathbb{E}\widehat{F}_D(\theta,x)-\widehat{F}_D(\theta,x)\right\},\end{aligned}\tag{3.6}$$

In what follows, let's denote

$$s_{n,0}^2=\sum_{i=1}^{n}\sum_{j=1}^{n}|Cov\left(\Delta_i(x,\theta),\Delta_j(x,\theta)\right)|$$

$$s_{n,1}^2=\sum_{i=1}^{n}\sum_{j=1}^{n}|Cov\left(\Delta_i(x,\theta)H_i\left(t_\theta(\gamma)\right),\Delta_j(x,\theta)H_j\left(t_\theta(\gamma)\right)\right)|$$

where $\Delta_i(x,\theta)=\dfrac{K(h_K^{-1}(<x-X_i,\theta>))}{\mathbb{E}K_1(\theta,x)}$. Let now present the following lemmas

Lemma 3.1 *Under the conditions of Theorem (H0)-(H3) and (H6)-(H9), we have*

$$|F(\theta,t_\theta(\gamma),x)-\mathbb{E}\widehat{F}_N(\theta,t_\theta(\gamma),x)|=\mathcal{O}\left(h_K^{b_1}\right)+\mathcal{O}\left(h_H^{b_2}\right).\tag{3.7}$$

Lemma 3.2 *Under the assumptions of Theorem 4.1, we have :*

i) $\widehat{F}_D(\theta,x)-\mathbb{E}\widehat{F}_D(\theta,x)=\mathcal{O}_{a.co.}\left(\dfrac{\sqrt{s_{n,0}^2\log n}}{n}\right)$,

ii) $\widehat{F}_N(\theta,t_\theta(\gamma),x)-\mathbb{E}\widehat{F}_N(\theta,t_\theta(\gamma),x)=\mathcal{O}_{a.co.}\left(\dfrac{\sqrt{s_{n,1}^2\log n}}{n}\right)$.

The proof of these two lemmas will be done in the same manner as it was given in in Ferraty et al. (2005), (since they are a special case of the Lemmas 3.2 and 3.3), the reader may also refer to Ferraty and Vieu (2006). It suffices to replace $\widehat{F}(t_\gamma|x)$ (resp. $F(t_\gamma|x)$) by $\widehat{F}(\theta, t_\theta(\gamma), x)$ (resp. $F(\theta, t_\theta(\gamma), x)$), and $\widehat{F}_D(x)$, (resp. $F_D(x)$) by $\widehat{F}_D(\theta, x)$ (resp. $F_D(\theta, x)$) with $d(x_1, x_2) =< x_1 - x_2, \theta >$.

3.3.2 Pointwise almost complete rate of convergence

In this section we study the rate of convergence[2] of our conditional quantile estimator $\widehat{t}_\gamma$. Because this kind of result is stronger than the previous one, we have to introduce some additional assumptions. As it is usual in conditional quantiles estimation, the rate of convergence can be linked with the flatness of the *cond-cdf* $F(.|x)$ around the conditional quantile $t_\theta(\gamma)$. This is one reason why we introduced hypotheses (H4) and (H5). But a complementary way to take into account this local shape constraint is to suppose that :

(H11) $\exists j > 0,\ \forall l,\ 1 \le l < j,\ F^{(l)}(\theta, t_\theta(\gamma), x) = 0$ and $\left|F^{(j)}(\theta, t_\theta(\gamma), x)\right| > 0$.

Because we focus on the local behavior of $F(\theta, ., x)$ around $t_\theta(\gamma)$ via its derivatives, that leads us to consider the successive derivatives of $\widehat{F}(\theta, ., x)$ and subsequently some assumptions on the successive derivatives of the cumulative kernel H :

(H12) The support of $H^{(1)}$ is compact and $\forall l \ge j$, $H^{(l)}$ exists and is bounded.

(H13) $\forall i \ne i'$, the conditional density of $(Y_i, Y_{i'})$ given $(< X_i, \theta >, < X_{i'}, \theta >)$ is continuous at $(t_\theta(\gamma), t_\theta(\gamma))$.

Theorem 3.2 *Put $s_n = \max\{s_{n,0}; s_{n,1}\}$, and assume that either (H10)-(i) is satisfied together with hypotheses (H0)-(H9) and (H11)-(H13), we have*

$$\widehat{t}_\theta(\gamma) - t_\theta(\gamma) = \mathcal{O}\left(\left(h_K^{b_1} + h_H^{b_2}\right)^{\frac{1}{j}}\right) + \mathcal{O}_{a.co}\left(\left(\frac{s_n^2 \log n}{n^2}\right)^{\frac{1}{2j}}\right). \tag{3.8}$$

Proof 3.2 (Proof of Theorem 4.2) *The proof is based on the Taylor expansion of $\widehat{F}(\theta, ., x)$ at $t_\theta(\gamma)$ and on the use of (H10) :*

$$\begin{aligned} \widehat{F}\left(t_\theta(\gamma), x\right) - \widehat{F}\left(\theta, \widehat{t}_\theta(\gamma), x\right) &= \sum_{l=1}^{j-1} \frac{(t_\theta(\gamma) - \widehat{t}_\theta(\gamma))^l}{l!} \widehat{F}^{(l)}(t_\theta(\gamma), x) + \frac{(t_\theta(\gamma) - \widehat{t}_\theta(\gamma))^j}{j!} \widehat{F}^{(j)}(t_\theta^*|x), \\ &= \sum_{l=1}^{j-1} \frac{(t_\theta(\gamma) - \widehat{t}_\theta(\gamma))^l}{l!} \left(\widehat{F}^{(l)}(t_\theta(\gamma), x) - F^{(l)}(t_\theta(\gamma), x)\right) \\ &\quad + \frac{(t_\theta(\gamma) - \widehat{t}_\theta(\gamma))^j}{j!} \widehat{F}^{(j)}(t_\theta^*|x), \end{aligned}$$

where, for all $y \in \mathbb{R}$,

$$\widehat{F}^{(j)}(\theta, y, x) = \frac{h_H^{-j} \sum_{i=1}^{n} K(h_K^{-1}(< x - X_i, \theta >)) H^{(j)}(h_H^{-1}(y - Y_i))}{\sum_{i=1}^{n} K(h_K^{-1}(< x - X_i, \theta >))},$$

2. Recall that a sequence $(T_n)_{n\in\mathbb{N}}$ of random variables is said to be of order of complete convergence u_n, if there exists some $\epsilon > 0$ for which $\sum_{ngeq1} \mathbb{P}(|T_n| > \epsilon u_n) < \infty$. This is denoted by $T_n = \mathcal{O}_{a.co.}(u_n),.$ (or equivalently by $T_n = \mathcal{O}_{a.co.}(u_n)$)

and where $\min(t_\theta(\gamma), \widehat{t}_\theta(\gamma)) < t^*_\theta < \max(t_\theta(\gamma), \widehat{t}_\theta(\gamma))$. *Suppose now that we have the following result.*

Lemma 3.3 *Put* $s^*_n = \max\{s_{n,0}; s_{n,2}\}$*, and assume that either (H10)-(ii) is satisfied together with under conditions (H0)-(H8) and (H11)-(H12) and if*

$$\lim_{n\to\infty} \frac{\log n}{n\, h_H^{2j-1}\, \phi_{\theta,x}(h_K)} = 0,$$

then we have :

$$\widehat{F}^{(j)}(\theta, t_\theta(\gamma), x) - F^{(j)}(\theta, t_\theta(\gamma), x)| = \mathcal{O}\left(h_K^{b_1} + h_H^{b_2}\right) + \mathcal{O}_{a.co.}\left(\frac{\sqrt{s^{*\,2}_n \log n}}{n}\right),$$

where $s^2_{n,2} = \sum_{i=1}^{n}\sum_{j=1}^{n}\left|Cov\left(h_H^{-l}\Delta_i(x,\theta)H_i^{(l)}\left(t_\theta(\gamma)\right), h_H^{-l}\Delta_j(x,\theta)H_j^{(l)}\left(t_\theta(\gamma)\right)\right)\right|$

Because of Theorem 4.1, Lemma 3.3 and (H10), we have :

$$\widehat{F}^{(j)}(\theta, t^*_\theta, x) \underset{n\,\to\infty}{\longrightarrow} F^{(j)}(\theta, t_\theta(\gamma), x) > 0,\ a.co.,$$

then we derive

$$\begin{aligned}(t_\theta(\gamma) - \widehat{t}_\theta(\gamma))^j &= \mathcal{O}\left(\widehat{F}(\theta, t_\theta(\gamma), x) - F(\theta, t_\theta(\gamma), x)\right)\\ &+\mathcal{O}\left(\sum_{l=1}^{j-1}(t_\theta(\gamma) - \widehat{t}_\theta(\gamma))^l(\widehat{F}^{(l)}(\theta, t_\theta(\gamma), x) - F^{(l)}(\theta, t_\theta(\gamma), x))\right),\ a.co. \qquad (3.9)\end{aligned}$$

Now, comparing the convergence rates given in Lemmas 3.2 and 3.3, we get

$$(t_\theta(\gamma) - \widehat{t}_\theta(\gamma))^j = \mathcal{O}\left(\widehat{F}(t_\theta(\gamma), x) - F(\theta, t_\theta(\gamma), x)\right),\ a.co.$$

Thus, Lemmas 3.1 and 3.2 allow us to get the claimed result. The proof of Lemma 3.3 will be given in the same manner as it was done in Ferraty et al (2005) (they are a special case of the Lemmas 3.5). It suffices to replace $\widehat{F}^{(j)}(t_\gamma|x)$ *(resp.* $F^{(j)}(t_\gamma|x)$*) by* $\widehat{F}^{(j)}(\theta, t_\theta(\gamma), x)$ *(resp.* $F^{(j)}(\theta, t_\theta(\gamma), x)$*), with* $d(x_1, x_2) = < x_1 - x_2, \theta >$ *. The proof of these latter will be given briefly in the appendix.*

3.4 Uniform almost complete convergence and rate of convergence

In this section we derive the uniform version of Theorem 4.1 and Theorem 4.2. The study of the uniform consistency is a crucial tool for studying the asymptotic properties of all estimates of the functional index if is unknown. In the multivariate case, the uniform consistency is a standard extension of the pointwise one, nevertheless, in the studied case, it requires some additional tools and topological conditions (see Ferraty *et al.* (2010)). Consequently, coupled with the conditions introduced antecedently, we need the following ones. Firstly, consider

$$\mathcal{S}_{\mathcal{H}} \subset \bigcup_{k=1}^{d_n^{\mathcal{S}_{\mathcal{H}}}} B_\theta(x_k, r_n) \text{ and } \Theta_{\mathcal{H}} \subset \bigcup_{m=1}^{d_n^{\Theta_{\mathcal{H}}}} B_\theta(\theta_m, r_n) \tag{3.10}$$

with x_k (resp. θ_j) $\in \mathcal{H}$ and $r_n, d_n^{\mathcal{S}_{\mathcal{H}}}, d_n^{\Theta_{\mathcal{H}}}$ are sequences of positive real numbers which tend to infinity as n goes to infinity and suppose that $d_n^{\mathcal{S}_{\mathcal{H}}}$, $d_n^{\Theta_{\mathcal{H}}}$ are the minimal numbers of open balls with radius r_n in $\mathcal{H}$, which are required to cover $\mathcal{S}_{\mathcal{H}}$ and $\Theta_{\mathcal{H}}$.

3.4.1 Conditional quantile distribution estimation

In this subpart we propose to study the uniform almost complete convergence of our estimator (4.3), to this end, we need to state the following assumptions

(A1) There exists a differentiable function $\phi(\cdot)$ such that $\forall x \in \mathcal{S}_{\mathcal{H}}$ and $\forall \theta \in \Theta_{\mathcal{H}}$,

$$0 < C\phi(h) \leq \phi_{\theta,x}(h) \leq C'\phi(h) < \infty \text{ and } \exists \eta_0 > 0,\ \forall \eta < \eta_0,\ \phi'(\eta) < C,$$

(A2) $\forall (y_1, y_2) \in \mathcal{S}_{\mathbb{R}} \times \mathcal{S}_{\mathbb{R}}, \forall (x_1, x_2) \in \mathcal{S}_{\mathcal{H}} \times \mathcal{S}_{\mathcal{H}}$ and $\forall \theta \in \Theta_{\mathcal{H}}$,

$$|F(\theta, y_1, x_1) - F(\theta, y_2, x_2)| \leq C \left(\|x_1 - x_2\|^{b_1} + |y_1 - y_2|^{b_2} \right),$$

(A3) The kernel K satisfy (H3) and Lipschitz's condition holds

$$|K(x) - K(y)| \leq C\|x - y\|,$$

(A4) For some $\nu \in (0, 1)$, $\lim_{n\to\infty} n^\nu h_H = \infty$, and for $r_n = \mathcal{O}\left(\frac{\log n}{n}\right)$, the sequences $d_n^{\mathcal{S}_{\mathcal{H}}}$ and $d_n^{\Theta_{\mathcal{H}}}$ satisfy :

$$\begin{cases} (i)\ \dfrac{(\log n)^2}{n\phi(h_K)} < \log d_n^{\mathcal{S}_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}} < \dfrac{n\phi(h_K)}{\log n}, \\ (ii)\ \displaystyle\sum_{n=1}^{\infty} n^{1/2b_2} (d_n^{\mathcal{S}_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}})^{1-\beta} < \infty \text{ for some } \beta > 1, \\ (iii)\ n\phi(h_K) = \mathcal{O}\left((\log n)^2\right). \end{cases}$$

(A5) $\forall (y_1, y_2) \in \mathcal{S}_{\mathbb{R}} \times \mathcal{S}_{\mathbb{R}}, \forall (x_1, x_2) \in \mathcal{S}_{\mathcal{H}} \times \mathcal{S}_{\mathcal{H}}$ and $\forall \theta \in \Theta_{\mathcal{H}}$,

$$|F^{(j)}(\theta, y_1, x_1) - F^{(j)}(\theta, y_2, x_2)| \leq C \left(\|x_1 - x_2\|^{b_1} + |y_1 - y_2|^{b_2} \right),$$

(A6) For some $\nu \in (0, 1)$, $\lim_{n\to\infty} n^\nu h_H = \infty$, and for $r_n = \mathcal{O}\left(\frac{\log n}{n}\right)$, the sequences $d_n^{\mathcal{S}_{\mathcal{H}}}$ and $d_n^{\Theta_{\mathcal{H}}}$ satisfy :

$$\begin{cases} (i)\ \dfrac{(\log n)^2}{nh_H^{2j-1}\phi(h_K)} < \log d_n^{\mathcal{S}_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}} < \dfrac{nh_H^{2j-1}\phi(h_K)}{\log n}, \\ (ii)\ nh_H^{2j-1}\phi(h_K) = \mathcal{O}\left((\log n)^2\right) \end{cases}$$

And let

$$s_{n,3}^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} |Cov\left(\Lambda_i, \Lambda_j\right)|, \quad s_{n,4}^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} |Cov\left(\Omega_i, \Omega_j\right)|$$

$$s_{n,5}^2=\sum_{i=1}^{n}\sum_{j=1}^{n}\left|Cov\left(\Delta_i\left(x_{k(x)},\theta_{m(\theta)}\right),\Delta_j\left(x_{k(x)},\theta_{m(\theta)}\right)\right)\right|,\quad s_{n,6}^2=\sum_{i=1}^{n}\sum_{j=1}^{n}\left|Cov\left(\Gamma_i,\Gamma_j\right)\right|$$

$$s_{n,7}^2=\sum_{i=1}^{n}\sum_{j=1}^{n}\left|Cov\left(\Gamma_i^{(l)},\Gamma_j^{(l)}\right)\right|$$

where

$$\Lambda_i(x,\theta)=\frac{1}{h_K\phi(h_K)}\mathbf{1}_{B_\theta(x,h)\cup B_\theta(x_{k(x)},h)}(X_i),$$

$$\Omega_i(x,\theta)=\frac{1}{h_K\phi(h_K)}\mathbf{1}_{B_\theta(x_{k(x)},h)\cup B_{\theta_{m(\theta)}}(x_{k(x)},h)}(X_i),$$

$$\Delta_i\left(x_{k(x)},\theta_{m(\theta)}\right)=\frac{K\left(h_K^{-1}<x_{k(x)}-X_i,\theta_{m(\theta)}>\right)}{\mathbb{E}K\left(h_K^{-1}<x_{k(x)}-X_i,\theta_{m(\theta)}>\right)},$$

$$\begin{aligned}\Gamma_i &= \frac{K\left(h_K^{-1}<x_{k(x)}-X_i,\theta_{m(\theta)}>\right)}{\mathbb{E}K\left(h_K^{-1}<x_{k(x)}-X_i,\theta_{m(\theta)}>\right)}H\left(h_H^{-1}(t_y-Y_i)\right)\\ &- \mathbb{E}\left(\frac{K\left(h_K^{-1}<x_{k(x)}-X_i,\theta_{m(\theta)}>\right)}{\mathbb{E}K\left(h_K^{-1}<x_{k(x)}-X_i,\theta_{m(\theta)}>\right)}H\left(h_H^{-1}(t_y-Y_i)\right)\right)\end{aligned}$$

and

$$\begin{aligned}\Gamma_i^{(l)} &= \frac{1}{h_H^l}\frac{K\left(h_K^{-1}<x_{k(x)}-X_i,\theta_{m(\theta)}>\right)}{\mathbb{E}K\left(h_K^{-1}<x_{k(x)}-X_i,\theta_{m(\theta)}>\right)}H^{(l)}\left(h_H^{-1}(t_y-Y_i)\right)\\ &- \frac{1}{h_H^l}\mathbb{E}\left(\frac{K\left(h_K^{-1}<x_{k(x)}-X_i,\theta_{m(\theta)}>\right)}{\mathbb{E}K\left(h_K^{-1}<x_{k(x)}-X_i,\theta_{m(\theta)}>\right)}H^{(l)}\left(h_H^{-1}(t_y-Y_i)\right)\right)\end{aligned}$$

Theorem 3.3 *Put $s_n'=\max\{s_{n,3};s_{n,4};s_{n,5};s_{n,6}\}$, and assume that either (H10)-(ii) is satisfied together with under hypotheses (H0)-(H3) and (H6)-(H9), (A1) and (A3)-(A4), we have*

$$\sup_{x\in\mathcal{S}_\mathcal{F}}\left|\widehat{t}_\theta(\gamma)-t_\theta(\gamma)\right| \underset{n\to\infty}{\longrightarrow} 0,\ a.co. \tag{3.11}$$

Proof 3.3 (Proof of Theorem 4.3) *The proof of the theorem can be completed by using the following results.*

Lemma 3.4 *Under the conditions (H0)-(H3) and (H6)-(H9), we have*

$$\sup_{\theta\in\Theta_\mathcal{H}}\sup_{x\in\mathcal{S}_\mathcal{H}}\sup_{t\in\mathcal{S}_\mathbb{R}}\left|F(\theta,t_\theta(\gamma),x)-\mathbb{E}\widehat{F}_N(\theta,t_\theta(\gamma),x)\right|=\mathcal{O}\left(h_K^{b_1}\right)+O\left(h_H^{b_2}\right). \tag{3.12}$$

Lemma 3.5 *Under the assumptions of Theorem 4.3, we have :*

1. $\sup_{\theta\in\Theta_\mathcal{F}}\sup_{x\in\mathcal{S}_\mathcal{F}}\left|\widehat{F}_D(\theta,x)-\mathbb{E}\widehat{F}_D(\theta,x)\right|=\mathcal{O}_{a.co.}\left(\frac{\sqrt{\max\{s_{n,3}^2;s_{n,4}^2;s_{n,5}^2\}\log d_n^{\mathcal{S}_\mathcal{F}}d_n^{\Theta_\mathcal{F}}}}{n}\right),$

2.

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\widehat{F}_N(\theta,t_\theta(\gamma),x)-\mathbb{E}\widehat{F}_N(\theta,t_\theta(\gamma),x) = \mathcal{O}_{a.co.}\left(\frac{\sqrt{\max\{s^2_{n,3};s^2_{n,4};s^2_{n,6}\}\log d_n^{\mathcal{S}_{\mathcal{F}}}d_n^{\Theta_{\mathcal{F}}}}}{n}\right) + \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{F}}}d_n^{\Theta_{\mathcal{F}}}}{n\phi(h_K)}}\right).$$

Theorem 3.4 *Under hypotheses (H0)-(H3), (H6)-(H10) and (A1)-(A4), we have*

$$\sup_{\theta\in\Theta_{\mathcal{F}}}\sup_{x\in\mathcal{S}_{\mathcal{F}}}|\widehat{t}_\theta(\gamma)-t_\theta(\gamma)| = \mathcal{O}\left(\left(h_K^{b_1}+h_H^{b_2}\right)^{\frac{1}{j}}\right)+\mathcal{O}_{a.co.}\left(\left(\frac{\log d_n^{\mathcal{S}_{\mathcal{F}}}d_n^{\Theta_{\mathcal{F}}}}{nh_H^{2j-1}\phi(h_K)}\right)^{\frac{1}{2j}}\right) +\mathcal{O}_{a.co.}\left(\left(\frac{s_n''^2\log d_n^{\mathcal{S}_{\mathcal{F}}}d_n^{\Theta_{\mathcal{F}}}}{n^2}\right)^{\frac{1}{2j}}\right)$$

where $s_n''=\max\{s_{n,3};s_{n,4};s_{n,5};s_{n,7}\}$

Remark 3.2 *These results extends Theorem 3 or Theorem 4 given in Bouchentouf* et al. *(2014) to the mixing case. The effect of covariance structure for dependence case on the convergence rate is reflected in the last term. Specially, if the functional single-index is fixed, it is easy to prove the following corollary that are similar the one given in Bouchentouf* et al. *(2014).*

Corollary 3.1 *Under the conditions of Theorem 4.4, we have*

1.

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{y\in\mathcal{S}_{\mathbb{R}}}|\widehat{F}(\theta,y,x)-F(\theta,y,x)| = \mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right)+\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}+\log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right) + \mathcal{O}_{a.co.}\left(\frac{\sqrt{s_n'^{*2}\log d_n^{\mathcal{S}_{\mathcal{F}}}d_n^{\Theta_{\mathcal{F}}}}}{n}\right)$$

2.

$$\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{y\in\mathcal{S}_{\mathbb{R}}}|\widehat{F}(\theta,y,x)-F(\theta,y,x)| = \mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right)+\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}}{n\phi(h_K)}}\right)+\mathcal{O}_{a.co.}\left(\frac{\sqrt{s_n'^{*2}\log d_n^{\mathcal{S}_{\mathcal{F}}}}}{n}\right)$$

where $s_n'^{*}=\max\{s_{n,3};s_{n,5};s_{n,6}\}$

Proof 3.4 (Proof of Theorem 4.4) *Obviously, the proofs of these two results, namely Theorem 4.4 and Corollary 4.1 can be deduced from the following intermediate results which are only uniform version of Lemma 3.3.*

Lemma 3.6 *Put* $s_n^*=\max\{s_{n,0};s_{n,2}\}$*, and assume that either (H10)-(ii) is satisfied together with under conditions (H0)-(H8) and (H11)-(H12) and if*

$$\lim_{n\to\infty}\frac{\log n}{n\,h_H^{2j-1}\,\phi_{\theta,x}(h_K)}=0,$$

then we have :

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{y\in\mathcal{S}_{\mathbb{R}}}\left|\widehat{F}^{(j)}(\theta,t_\theta(\gamma),x)-F^{(j)}(\theta,t_\theta(\gamma),x)\right| = \mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right)+\mathcal{O}_{a.co.}\left(\frac{\sqrt{s_n^{*2}\log d_n^{\mathcal{S}_{\mathcal{F}}}d_n^{\Theta_{\mathcal{F}}}}}{n}\right),$$

where $s_{n,2}^2=\sum_{i=1}^n\sum_{j=1}^n\left|Cov\left(h_H^{-l}\Delta_i(x,\theta)H_i^{(l)}\left(t_\theta(\gamma)\right),h_H^{-l}\Delta_j(x,\theta)H_j^{(l)}\left(t_\theta(\gamma)\right)\right)\right|$

3.5 Proofs of technical lemmas

In order to highlight the main contribution of our paper (i.e. α-mixing and functional variables) some details are voluntarily omitted.

Proof 3.5 (Proof of Lemma 3.1) *The asymptotic behavior of bias term is standard, in the sense that it is not affected by the dependence structure of the data. We have*

$$\begin{aligned}\mathbb{E}\widehat{F}_N(\theta,t_\theta(\gamma),x)-F_\theta^x(t_\theta(\gamma)) &= \frac{1}{\mathbb{E}K_1(x,\theta)}\mathbb{E}\Big(K_1(x,\theta)\Big[\mathbb{E}\Big(H_1(t_\theta(\gamma))|<X_1,\theta>\Big)\\ &\qquad -F_\theta^x(t_\theta(\gamma))\Big]\Big).\end{aligned}\tag{3.13}$$

and by noting that

$$\mathbb{E}\left(H_1(t_\theta(\gamma))|<X_1,\theta>\right) = \int_{\mathbb{R}}H^{(1)}(t)F(\theta,t_\theta(\gamma)-h_Ht,X_1)dt,$$

we can write, because of (H3) and (H7) :

$$|\mathbb{E}\left(H_1(t_\theta(\gamma))|<X_1,\theta>\right)-F_\theta^x(t_\theta(\gamma))| \le C_{x,\theta}\int_{\mathbb{R}}H^{(1)}(t)\left(h_K^{b_1}+|t|^{b_2}h_H^{b_2}\right)dt.$$

Combining this last result with (4.24) allows us to achieve the proof.

Proof 3.6 (Proof of Lemma 3.2) *Following the ideas used in regression (see Ferraty and Vieu (2004)), the key fact consists in using a pseudo-exponential inequality taking considering the α-mixing structure. We start by writing*

1. *Concerning (i), in fact, it can be found that.*

$$\begin{aligned}\widehat{F}_D(\theta,x)-\mathbb{E}\widehat{F}_D(\theta,x) &= \frac{1}{n\,\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^n K_i(\theta,x)-\frac{1}{n\,\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^n\mathbb{E}K_i(\theta,x)\\ &= \frac{1}{n\,\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^n K_i(\theta,x)-\mathbb{E}K_i(\theta,x)\\ &= \frac{1}{n}\sum_{i=1}^n\Delta_i(\theta,x)-\mathbb{E}\Delta_i(\theta,x)=\frac{1}{n}\sum_{i=1}^n\Psi_i(\theta,x)\end{aligned}$$

where $\Psi_i(\theta,x)=K_i(\theta,x)-\mathbb{E}K_i(\theta,x)$ *has zero mean and satisfies*

$$|\Psi_i(\theta,x)|\le C_{x,\theta}/\phi_{\theta,x}(h_K),$$

then it allows us to use directly a dependent version of the Fuk-Nagaev's exponential inequality (see Corollary A.12-ii of Ferraty and Vieu (2006)) and obtain

$$\widehat{F}_D(\theta,x)-\mathbb{E}\widehat{F}_D(\theta,x)=\mathcal{O}_{a.co.}\left(\frac{\sqrt{s_{n,0}^2\log n}}{n}\right).$$

2. *Concerning (ii), it performs along the same steps and by invoking the same arguments, just changing the variable $\Psi_i(\theta,x)$ into the following ones :*

$$\Xi_i\left(\theta,t_\theta(\gamma),x\right)=H_i\left(t_\theta(\gamma)\right)\,\Delta_i(\theta,x)-\mathbb{E}\,H_i\left(t_\theta(\gamma)\right)\,\Delta_i(\theta,x).$$

Because H is a cumulative kernel, we have $H_i\left(t_\theta(\gamma)\right)\le 1$. By using systematically this fact to bound the variables H_i, all the calculus made previously with the variables $\Psi_i(\theta,x)$ remain valid with the variables $\Xi_i\left(\theta,t_\theta(\gamma),x\right)$.

Thus $\Psi_i(\theta,x)=K_i(\theta,x)-\mathbb{E}K_i(\theta,x)$ has zero mean and satisfies

$$|\Psi_i(\theta,x)|\le C_{x,\theta}/\phi_{\theta,x}(h_K),$$

the Fuk-Nagaev's inequality (Rio (2000)) allows one to get

$$\widehat{F}_N(\theta,t_\theta(\gamma),x)-\mathbb{E}\widehat{F}_N(\theta,t_\theta(\gamma),x)=\mathcal{O}_{a.co.}\left(\frac{\sqrt{s_{n,1}^2\log n}}{n}\right).$$

Consequently, the proof of Lemma 3.2 is achieved.

Proof 3.7 (Proof of Lemma 3.3) *We use again the same kind of decomposition as 4.12 :*

$$\begin{aligned}\widehat{F}^{(j)}\left(\theta,t_\theta(\gamma),x\right)-F^{(j)}\left(\theta,t_\theta(\gamma),x\right) &= \frac{1}{\widehat{F}_D(\theta,x)}\Big\{\Big(\widehat{F}_N^{(j)}\left(\theta,t_\theta(\gamma),x\right)-\mathbb{E}\widehat{F}_N^{(j)}\left(\theta,t_\theta(\gamma),x\right)\Big)\\ &\quad-\Big(F^{(j)}\left(\theta,t_\theta(\gamma),x\right)-\mathbb{E}\widehat{F}_N^{(j)}\left(\theta,t_\theta(\gamma),x\right)\Big)\Big\}\\ &\quad+\frac{F^{(j)}\left(\theta,t_\theta(\gamma),x\right)}{\widehat{F}_D(\theta,x)}\Big\{\mathbb{E}\widehat{F}_D(\theta,x)-\widehat{F}_D(\theta,x)\Big\}.\end{aligned}\tag{3.14}$$

This proof is very similar to the one of Theorem 4.1.

First of all, let's consider the bias term $F^{(j)}\left(\theta,t_\theta(\gamma),x\right)-\mathbb{E}\widehat{F}_N^{(j)}\left(\theta,t_\theta(\gamma),x\right)$. Using the same arguments in the proof of Lemma 3.1, replacing $F\left(\theta,t_\theta(\gamma),x\right)$ (resp. $\widehat{F}\left(\theta,t_\theta(\gamma),x\right)$) with $F^{(j)}\left(\theta,t_\theta(\gamma),x\right)$ (resp. $\widehat{F}_N^{(j)}\left(\theta,t_\theta(\gamma),x\right)$) and considering hypotheses (H5), (H7) and (H12) we get :

$$F^{(j)}\left(\theta,t_\theta(\gamma),x\right)-\mathbb{E}\widehat{F}_N^{(j)}\left(\theta,t_\theta(\gamma),x\right)=\mathcal{O}\left(h_K^{b_1}+h_K^{b_2}\right).\tag{3.15}$$

Now, we focus on the term $\widehat{F}_N^{(j)}\left(\theta,t_\theta(\gamma),x\right)-\mathbb{E}\widehat{F}_N^{(j)}\left(\theta,t_\theta(\gamma),x\right)$. To get the asymptotic behaviour of this quantity, we comeback to the proof of Lemma 3.2, and we replace $F\left(\theta,t_\theta(\gamma),x\right)$ (resp. $\widehat{F}\left(\theta,t_\theta(\gamma),x\right)$) with $F^{(j)}\left(\theta,t_\theta(\gamma),x\right)$ (resp. $\widehat{F}_N^{(j)}\left(\theta,t_\theta(\gamma),x\right)$).

Note that (H11) and (H13) permit to show that

$$\mathbb{E}\left(H^{(j)}\left(h_H^{-1}(t_\theta(\gamma)-Y_i)\right)H^{(j)}\left(h_H^{-1}(t_\gamma-Y_{i'})\right)|(X_i,X_{i'})\right)=\mathcal{O}(h_H^2),$$

while (H1) and (H5) imply that

$$\mathbb{E}\left(H^{(j)}\left(h_H^{-1}(t_\theta(\gamma)-Y_i)\right)|X_i\right)=\mathcal{O}(h_H).$$

Consequently, we have by using successively (H8), (H0), (H2) and (H10)-(i)

$$Cov\left(\Xi_i^*\left(\theta,t_\theta(\gamma),x\right),\Xi_{i'}^*\left(\theta,t_\theta(\gamma),x\right)\right)=\mathcal{O}\left(h_H^2\left(\frac{\phi_{\theta,x}(h_K)}{n}\right)^{1/a}\phi_{\theta,x}(h_K)\right),$$

where

$$\Xi_i^*\left(\theta,t_\theta(\gamma),x\right)=H^{(j)}\left(h_H^{-1}\left(t_\theta(\gamma)-Y_i\right)\right)K_i(\theta,x)-\mathbb{E}\left(H^{(j)}\left(h_H^{-1}\left(t_\theta(\gamma)-Y_i\right)\right)K_i(x)\right).$$

has zero mean and satisfies

$$\left|\Xi_i^*\left(\theta,t_\theta(\gamma),x\right)\right|\leq Ch_H^{-j},$$

because $H^{(j)}$ is bounded. Indeed, it can be found that

$$\widehat{F}_N^{(j)}\left(\theta,t_\theta(\gamma),x\right)-\mathbb{E}\widehat{F}_N^{(j)}\left(\theta,t_\theta(\gamma),x\right)=\frac{h_H^{-j}}{n\ \mathbb{E}K_1(\theta,x)}\sum_{i=1}^n\Xi_i^*\left(\theta,t_\theta(\gamma),x\right),$$

then it allows us to use directly similar arguments of Lemma 3.2, we obtain

$$\widehat{F}_N^{(j)}\left(\theta,t_\theta(\gamma),x\right)-\mathbb{E}\widehat{F}_N^{(j)}\left(\theta,t_\theta(\gamma),x\right)=\mathcal{O}_{a.co.}\left(\frac{\sqrt{s_n^{*2}\log n}}{n}\right),$$

which leads directly to the result of Lemma 3.3

Proof 3.8 (Proof of Lemma 4.6) *It is omitted as it very similar to that of Lemma 4.6 in Bouchentouf et al. (2014).*

Proof 3.9 (Proof of Lemma 4.7) *The proof can be completed following the same steps as of Lemmas 4.4 and 4.7 in Bouchentouf et al. (2014).*

- *Concerning* 1
  In fact, for $\forall x\in\mathcal{S_H}$ and $\forall\theta\in\Theta_\mathcal{H}$, we have the decomposition as follows. For all $x\in\mathcal{S_H}$ and $\theta\in\Theta_\mathcal{H}$, we set
  Let us consider the following decomposition

$$\begin{aligned}\sup_{x\in\mathcal{S_H}}\sup_{\theta\in\Theta_\mathcal{H}}\left|\widehat{F}_D(\theta,x)-1\right| &= \sup_{x\in\mathcal{S_H}}\sup_{\theta\in\Theta_\mathcal{H}}\left|\widehat{F}_D(\theta,x)-\mathbb{E}\left(\widehat{F}_D(\theta,x)\right)\right|\\
&\leq \sup_{x\in\mathcal{S_H}}\sup_{\theta\in\Theta_\mathcal{H}}\left|\widehat{F}_D(\theta,x)-(\widehat{F}_D(\theta,x_{k(x)})\right|\\
&\quad+\sup_{x\in\mathcal{S_H}}\sup_{\theta\in\Theta_\mathcal{H}}\left|\widehat{F}_D(\theta,x_{k(x)})-\widehat{F}_D(t_{m(\theta)},x_{k(x)})\right|\\
&\quad+\sup_{x\in\mathcal{S_H}}\sup_{\theta\in\Theta_\mathcal{H}}\left|\widehat{F}_D(t_{m(\theta)},x_{k(x)})-\mathbb{E}\left(\widehat{F}_D(t_{m(\theta)},x_{k(x)})\right)\right|\\
&\quad+\sup_{x\in\mathcal{S_H}}\sup_{\theta\in\Theta_\mathcal{H}}\left|\mathbb{E}\left(\widehat{F}_D(t_{m(\theta)},x_{k(x)})\right)-\mathbb{E}\left(\widehat{F}_D(\theta,x_{k(x)})\right)\right|\\
&\quad+\sup_{x\in\mathcal{S_H}}\sup_{\theta\in\Theta_\mathcal{H}}\left|\mathbb{E}\left(\widehat{F}_D(\theta,x_{k(x)})\right)-\mathbb{E}\left(\widehat{F}_D(\theta,x)\right)\right|\end{aligned}$$

$$= F_1 + F_2 + F_3 + F_4 + F_5 + \tag{3.16}$$

where $k(x) = \arg \min_{k \in \{1 \dots r_n\}} \|x - x_k\|$ *and* $m(\theta) = \arg \min_{m \in \{1 \dots l_n\}} \|\theta - t_m\|$.
In order to complete the proof of Lemma 4.6, we only need to give the convergence rate of five terms in (3.16) respectively.
Firstly, we treat F_1. *Let* $\lambda = \lambda_0 \sqrt{\frac{\log d_n^{\mathcal{S_F}} d_n^{\Theta_{\mathcal{F}}} s_{n,5}^2}{n^2}}$ *for all* $\lambda_0 > 0$, *we have that*

$$\begin{aligned}
\mathbb{P}\left(F_3 > \lambda)\right) &= \mathbb{P}\left(\max_{k \in \{1 \dots d_n^{\mathcal{S_H}}\}} \max_{m \in \{1 \dots d_n^{\Theta_{\mathcal{H}}}\}} F_3' > \lambda \right) \\
&\leq d_n^{\mathcal{S_H}} d_n^{\Theta_{\mathcal{H}}} \max_{k \in \{1 \dots d_n^{\mathcal{S_H}}\}} \max_{m \in \{1 \dots d_n^{\Theta_{\mathcal{H}}}\}} \mathbb{P}\left(F_3' > \lambda\right).
\end{aligned} \tag{3.17}$$

where $F_3' = \left| \widehat{F}_D(t_{m(\theta)}, x_{k(x)}) - \mathbb{E}\left(\widehat{F}_D(t_{m(\theta)}, x_{k(x)}) \right) \right|$.
By using the Fuk-Nagaev's inequality (Proposition A.11(ii), see Ferraty and Vieu (2006)) with taking $r = (\log n)^2$ *and* $q = a + 1$, *one will obtain that*

$$\mathbb{P}\left(F_3' > \lambda\right) \leq C_1 A_1 + C_2 A_2 \tag{3.18}$$

where

$$\begin{aligned}
A_1 &= \left(1 + \frac{\lambda_0^2 (\log d_n^{\mathcal{S_F}} d_n^{\Theta_{\mathcal{F}}})}{(\log n)^2}\right)^{-\frac{(\log n)^2}{2}} \\
A_2 &= \frac{n (\log n)^{2a} \lambda_0^{-(a+1)}}{(\log d_n^{\mathcal{S_F}} d_n^{\Theta_{\mathcal{F}}})^{(a+1)/2}} \frac{1}{s_{n,5}^{a+1}}.
\end{aligned}$$

By hypotheses (A4)-(i) and (iii), we get $\frac{\log d_n^{\mathcal{S_H}} + \log d_n^{\Theta_{\mathcal{H}}}}{(\log n)^2} \longrightarrow 0$ *as* $n \to 0$, *which leads to*

$$A_1 \leq d_n^{\mathcal{S_H}} \; d_n^{\Theta_{\mathcal{H}}}, \tag{3.19}$$

for some $\beta > 1$ *and* $\lambda_0 > 0$ *such that* $\lambda_0^2 = 2\beta$.
On the other hand,

$$A_2 \leq C n (\log n)^{2a} (\log d_n^{\mathcal{S_H}} \; d_n^{\Theta_{\mathcal{H}}})^{-(a+1)/2} n^{-\eta} \leq C' \frac{1}{n^{\eta - \tau - 1}}, \tag{3.20}$$

where $\tau > 0$ *such that* $\eta > \eta - \tau > 2$. *Meanwhile, by the selection of* β *and* η, *we can find that*

$$(d_n^{\mathcal{S_H}} \; d_n^{\Theta_{\mathcal{H}}})^{\beta} = \mathcal{O}(n^{\eta - \tau - 1}). \tag{3.21}$$

Combining the equations (3.17)-(3.21) with hypothesis (A4)-(iii), we have

$$F_3 = \mathcal{O}_{a.co.} \left(\sqrt{\frac{s_{n,5}^2 \log d_n^{\mathcal{S_F}} d_n^{\Theta_{\mathcal{F}}}}{n^2}} \right) \tag{3.22}$$

Next, let us treat F_1 and F_2, respectively. By Assumption (A1), it follows

$$\begin{aligned}\sup_{x\in\mathcal{S}_\mathcal{H}}\sup_{\theta\in\Theta_\mathcal{H}}\frac{1}{n}\left|\sum_{i=1}^{n}\left(\Delta_i(x,\theta)-\Delta_i\left(x_{k(x)},\theta\right)\right)\right| &\leq \frac{C}{\phi(h_K)}\sup_{x\in\mathcal{S}_\mathcal{H}}\sup_{\theta\in\Theta_\mathcal{H}}\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{B_\theta(x,h)\cup B_\theta(x_{k(x)},h)}(X_i)\\ &= C\sup_{x\in\mathcal{S}_\mathcal{H}}\sup_{\theta\in\Theta_\mathcal{H}}\frac{1}{n}\sum_{i=1}^{n}\Lambda_i(x,\theta)\end{aligned}$$

and

$$\begin{aligned}&\sup_{x\in\mathcal{S}_\mathcal{H}}\sup_{\theta\in\Theta_\mathcal{H}}\frac{1}{n}\left|\sum_{i=1}^{n}\left(\Delta_i\left(x_{k(x)},\theta\right)-\Delta_i\left(x_{k(x)},\theta_{m(\theta)}\right)\right)\right|\\ &\qquad\leq \frac{C}{\phi(h_K)}\sup_{x\in\mathcal{S}_\mathcal{H}}\sup_{\theta\in\Theta_\mathcal{H}}\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{B_\theta(x_{k(x)},h)\cup B_\theta(x_{k(x)},h)}(X_i)\\ &\qquad= C\sup_{x\in\mathcal{S}_\mathcal{H}}\sup_{\theta\in\Theta_\mathcal{H}}\frac{1}{n}\sum_{i=1}^{n}\Omega_i(x,\theta).\end{aligned}$$

Therefore, similar to the argument for (3.22), we can get

$$F_1=\mathcal{O}_{a.co.}\left(\sqrt{\frac{s_{n,3}^2\log d_n^{\mathcal{S}_\mathcal{F}}d_n^{\Theta_\mathcal{F}}}{n^2}}\right)\tag{3.23}$$

and

$$F_2=\mathcal{O}_{a.co.}\left(\sqrt{\frac{s_{n,4}^2\log d_n^{\mathcal{S}_\mathcal{F}}d_n^{\Theta_\mathcal{F}}}{n^2}}\right)\tag{3.24}$$

Thus, by using the same arguments as that in Bouchentouf et al. (2014), it leads $F_5\leq F_1$ and $F_4\leq F_1$, respectively. then, as $n\to\infty$,

$$F_4=\mathcal{O}_{a.co.}\left(\sqrt{\frac{s_{n,4}^2\log d_n^{\mathcal{S}_\mathcal{F}}d_n^{\Theta_\mathcal{F}}}{n^2}}\right)\quad\text{and}\quad F_5=\mathcal{O}_{a.co.}\left(\sqrt{\frac{s_{n,3}^2\log d_n^{\mathcal{S}_\mathcal{F}}d_n^{\Theta_\mathcal{F}}}{n^2}}\right).\tag{3.25}$$

Finally, the first part of Lemma 4.7 can be easily deduced from (3.22)-(3.25).

- *Concerning 2 The proof follows the same steps as that in Ferraty et al. (2005). It is also adopted by Bouchentouf et al. (2014). In fact, by the compact property of $\mathcal{S}_\mathbb{R}\subset\mathbb{R}$, we have $\mathcal{S}_\mathbb{R}\subset\bigcup_{\xi=1}^{z_n}(t_\xi-l_n,t_\xi+l_n)$ and l_n, z_n can be selected such as $z_n=\mathcal{O}\left(l_n^{-1}\right)=\mathcal{O}\left(n^{\frac{1}{2b_2}}\right)$. By taking $\xi(t_\theta(\gamma))=\arg\min_{\{1,2,\ldots,z_n\}}|t_\theta(\gamma)-t_\xi|$, then similar to the decomposition given in Bouchentouf et al. (2014), it leads*

$$\sup_{\theta\in\Theta_\mathcal{H}}\sup_{x\in\mathcal{S}_\mathcal{H}}\sup_{t\in\mathcal{S}_\mathbb{R}}\left|\widehat{F}_N(\theta,t_\theta(\gamma),x)-\mathbb{E}\left(\widehat{F}_N(\theta,t_\theta(\gamma),x)\right)\right|=\Psi_1+\Psi_2+\Psi_3+\Psi_4+\Psi_5+\Psi_6+\Psi_7$$

where

$$\Psi_1=\sup_{\theta\in\Theta_\mathcal{H}}\sup_{x\in\mathcal{S}_\mathcal{H}}\sup_{t\in\mathcal{S}_\mathbb{R}}\left|\widehat{F}_N(\theta,t_\theta(\gamma),x)-\widehat{F}_N(\theta,t_\theta(\gamma),x_{k(x)})\right|$$

$$\Psi_2=\sup_{\theta\in\Theta_\mathcal{H}}\sup_{x\in\mathcal{S}_\mathcal{H}}\sup_{t\in\mathcal{S}_\mathbb{R}}\left|\widehat{F}_N(\theta,t_\theta(\gamma),x_{k(x)})-\widehat{F}_N(t_{m(\theta)},t_\theta(\gamma),x_{k(x)})\right|$$

$$\Psi_3=\sup_{\theta\in\Theta_\mathcal{H}}\sup_{x\in\mathcal{S}_\mathcal{H}}\sup_{t\in\mathcal{S}_\mathbb{R}}\left|\widehat{F}_N(t_{m(\theta)},t_\theta(\gamma),x_{k(x)})-\widehat{F}_N(t_{m(\theta)},t_{\xi(y)},x_{k(x)})\right|$$

$$\Psi_4 = \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in\mathcal{S}_{\mathcal{H}}} \sup_{t\in\mathcal{S}_{\mathbb{R}}} \left|\widehat{F}_N(t_{m(\theta)}, t_{\xi(y)}, x_{k(x)}) - \mathbb{E}\left(\widehat{F}_N(t_{m(\theta)}, t_{\xi(y)}, x_{k(x)})\right)\right|$$
$$\Psi_5 = \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in\mathcal{S}_{\mathcal{H}}} \sup_{t\in\mathcal{S}_{\mathbb{R}}} \left|\mathbb{E}\left(\widehat{F}_N(t_{m(\theta)}, t_{\xi(y)}, x_{k(x)})\right) - \mathbb{E}\left(\widehat{F}_N(t_{m(\theta)}, t_\theta(\gamma), x_{k(x)})\right)\right|$$
$$\Psi_6 = \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in\mathcal{S}_{\mathcal{H}}} \sup_{t_\theta(\gamma)\in\mathcal{S}_{\mathbb{R}}} \left|\mathbb{E}\left(\widehat{F}_N(t_{m(\theta)}, t_\theta(\gamma), x_{k(x)})\right) - \mathbb{E}\left(\widehat{F}_N(\theta, t_\theta(\gamma), x_{k(x)})\right)\right|$$
$$\Psi_7 = \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in\mathcal{S}_{\mathcal{H}}} \sup_{t_\theta(\gamma)\in\mathcal{S}_{\mathbb{R}}} \left|\mathbb{E}\left(\widehat{F}_N(\theta, t_\theta(\gamma), x_{k(x)})\right) - \mathbb{E}\left(\widehat{F}_N(\theta, t_\theta(\gamma), x)\right)\right|$$

Since
$$\Psi_1 \le \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in\mathcal{S}_{\mathcal{H}}} \frac{1}{n}\left|\sum_{i=1}^n \left(\Delta_i(x,\theta) - \Delta_i(\theta, x_{k(x)})\right)\right| = F_1$$

and
$$\Psi_2 \le \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in\mathcal{S}_{\mathcal{H}}} \frac{1}{n}\left|\sum_{i=1}^n \left(\Delta_i(\theta, x_{k(x)}) - \Delta_i(t_{m(\theta)}, x_{k(x)})\right)\right| = F_2$$

then using the fact that $\Gamma_1 \le F_1$ and $\Psi_2 \le F_2$ and using equations (3.23) and (3.24), we get
$$\Psi_1 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{s_{n,3}^2 \log d_n^{\mathcal{S}_{\mathcal{F}}} d_n^{\Theta_{\mathcal{F}}}}{n^2}}\right)$$

and
$$\Psi_2 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{s_{n,4}^2 \log d_n^{\mathcal{S}_{\mathcal{F}}} d_n^{\Theta_{\mathcal{F}}}}{n^2}}\right)$$

On the other hand, since $\Psi_7 \le \Psi_1$ and $\Psi_6 \le \Psi_2$, we get
$$\Psi_7 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{s_{n,3}^2 \log d_n^{\mathcal{S}_{\mathcal{F}}} d_n^{\Theta_{\mathcal{F}}}}{n^2}}\right)$$

and
$$\Psi_6 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{s_{n,4}^2 \log d_n^{\mathcal{S}_{\mathcal{F}}} d_n^{\Theta_{\mathcal{F}}}}{n^2}}\right)$$

respectively.

- *Concerning Ψ_3 and Ψ_5; by conditions (H4) and (H5), boundness of K and selection of l_n, using the same arguments of Lemma 4.7 in Bouchentouf el al. (2014), we get*

$$\begin{aligned}\left|\widehat{F}_N(t_{m(\theta)}, t_\theta(\gamma), x_{k(x)}) - \widehat{F}_N(t_{m(\theta)}, t_{\xi(y)}, x_{k(x)})\right| &\le \frac{C}{n}\left|\sum_{i=1}^n \Delta_i\left(x_{k(x)}, \theta_{m(\theta)}\right)\right|\left|\frac{t_\theta(\gamma) - t_{\xi(y)}}{h_H}\right| \\ &\le \mathcal{O}\left(\frac{l_n}{h_H}\right) = \mathcal{O}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right)\end{aligned}$$

as $n \to \infty$, where $\Delta_i\left(x_{k(x)}, \theta_{m(\theta)}\right) = \frac{K\left(h_K^{-1} < x_{k(x)} - X_i, \theta_{m(\theta)} >\right)}{\mathbb{E}K\left(h_K^{-1} < x_{k(x)} - X_i, \theta_{m(\theta)} >\right)}$, therefore, it follows
$$\Psi_5 \le \Psi_3 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right)$$

Concerning Ψ_4*, let us consider* $\varepsilon = \varepsilon_0\sqrt{\frac{s_{n,6}^2 \log d_n^{\mathcal{S}_\mathcal{F}} d_n^{\Theta_\mathcal{F}}}{n^2}}$*. Since*

$$\begin{aligned}\mathbb{P}\left(\Psi_4 > \varepsilon_0\sqrt{\frac{s_{n,6}^2 \log d_n^{\mathcal{S}_\mathcal{F}} d_n^{\Theta_\mathcal{F}}}{n^2}}\right) &= \mathbb{P}\left(\max_{m\in\{1\dots d_n^{\Theta_\mathcal{H}}\}}\max_{k\in\{1\dots d_n^{\mathcal{S}_\mathcal{H}}\}}\max_{\xi(y)\in\{1,2,\dots,z_n\}}|\Upsilon_n - \mathbb{E}\Upsilon_n| > \varepsilon\right)\\ &\leq z_n d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}}\mathbb{P}\left(|\Upsilon_n - \mathbb{E}\Upsilon_n| > \varepsilon\right)\end{aligned}$$

where $\Upsilon_n = \widehat{F}_N(t_{m(\theta)}, t_{\xi(y)}, x_{k(x)})$*, the application of Fuk-Nagaev's inequality (Proposition A.11-(ii), see Ferraty and Vieu (2006)) with* $r = (\log n)^2 > 1$ *and* $q = a + 1$*, we get that*

$$\begin{aligned}\mathbb{P}\left(\left|\sum_{i=1}^{n}(\Gamma_i - \mathbb{E}\Gamma_i)\right| > \varepsilon\right) &\leq C\left(1 + \frac{\varepsilon_0^2(\log d_n^{\mathcal{S}_\mathcal{F}} d_n^{\Theta_\mathcal{F}})}{(\log n)^2}\right)^{-(\log n)^2/2}\\ &\quad + \frac{n(\log n)^{2a}\varepsilon_0^{-(a+1)}}{(\log d_n^{\mathcal{S}_\mathcal{F}} d_n^{\Theta_\mathcal{F}})^{(a+1)/2}}\frac{1}{s_{n,6}^{a+1}}\\ &= C_1B_1 + C_2B_2.\end{aligned}$$

Similarly to (3.22), it yields

$$\Psi_4 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{s_{n,6}^2 \log d_n^{\mathcal{S}_\mathcal{F}}}{n^2}}\right).$$

Finally, the proof of Lemma 4.7 is achieved.

Proof 3.10 (Proof of Lemma 3.6) *The proof is an immediate consequence of the second part of Lemma 4.7, it suffices to replace the conditional cumulative distribution function by its successive derivatives.*

3.6 Concluding Remarks

Single-index models and varying coefficient models are powerful tools for dimension reduction and semiparametric modeling because they can effectively avoid "curse of dimensionality". Both models largely relax some restrictive assumptions on linear and nonparametric models. Specially, the single-index model is usually treated as the first step of the famous projection pursuit regression. The single-index model has also been extensively used in the projection pursuit regression,

For example, firsly the motivation of this model also comes from an analysis of environmental data, consisting of daily measurement of pollutants and other environment factors. Of interest is to examine the association between the levels of pollutants, two environment factors (temperature and relative humidity) and the total number of daily hospital admissions for respiratory problems.

Secondly, single-index models cannot reflect the additivity of covariates, while the performance of varying coefficient models can be poor if the varying coefficient contains moderate or high dimensional covariates.

In standard multivariate regression, the single-index model is a good compromise between nonparametric and parametric regression models when one wishes to regress a response variable on several real-valued explanatory ones. It assumes the existence of a latent one-dimensional explanatory variable which allows us to explain the response through a nonparametric regression model.

In addition, the latent explanatory variable is supposed to be a linear combination of the explanatory variables. The vector of the linear combination is called "single index". Such models are useful tools for interpreting some situations. This is particularly the case in econometrics where single-index models and various extensions have been intensively studied.

Single-index models, or projection pursuit regression, have proven to be an efficient way of coping with the highdimensional problem in nonparametric regressions (see, e.g., Hall (1989) ; Ichimura (1993)). The idea is restricting the general multivariate regression function to a special form.

For the functional single-index models, the literature is closely limited, and only a few theoretical results have been obtained until now.

For the past two decades, the single-index model, a special case of projection pursuit regression, has proven to be an efficient way of coping with the high dimensional problem in nonparametric regression. Here we deal with single index modeling when the explanatory variable is functional. More precisely, we consider the problem of estimating the conditional density of a real variable Y given a functional variable X when the explanation of Y given X is done through its projection on one functional direction.

The single-index approach is widely applied in econometrics as a reasonable compromise between nonparametric and parametric models. Such kind of modelization is intensively studied in the multivariate case. Without pretend to exhaustivity.

In practice, this study has great importance, because, it permit us to construct a prediction method based on the conditional mode estimator. Moreover, in the case where the functional single index is unknown, our estimate can be used to estimate this parameter via the pseudo-maximum likelihood estimation method. Noting that the estimation of the functional single-index has great interest on the semi-metric choice in nonparametric functional data analysis but it has been not attacked in this paper.

In this article, we examine conditional quantile estimation in the single functional index model for α-mixing functional data. The asymptotic properties such as pointwise almost complete consistency and the uniform almost complete convergence of the kernel estimator with rate are presented under some mild conditions. Although α-mixing is reasonably weak among various weak dependence process and has many practical applications such as in time series prediction, we also address other dependence settings such as long memory dependence functional data (see Benhenni *et al.* (2008)). In this case, the asymptotic properties of the estimation of successive derivatives of the conditional density function, conditional hazard function, conditional distribution function and conditional quantile in the single functional index model have been investigated in our other works.

The goal of this work is to contribute to the functional data literature by studying some classes of semiparametric models. Note that such models are important in the statistical and econometric modelization, due to its flexibility for dimension reduction, it provides the best new ways to investigate problems in substantive economics (see Horowitz (2009)). As a particular case, the single-index model has proven useful in providing an optimal approach to compromise between nonparametric and parametric models.

Single-index models when the explanatory variable is an element of a finite-dimensional space have been studied extensively in both statistical and econometric literatures, we quote, for instance, Härdle *et al.* (1993), Horowitz (1996), Hristache *et al.* (2001a, b) and Delecroix *et al.* (2003).

For the functional single-index models, the literature is closely limited, and only a few theoretical results have been obtained until now. The first asymptotic properties in the fixed functional single-model were obtained by Ferraty et al. (2003). They established the almost complete convergence, in the i.i.d. case, of the link regression function of this model. Their results were extended to the dependent case by Aït Saidi *et al.* (2005). Where the functional single-index is unknown, Aït Saidi *et al.* (2008) proposed an estimator of this parameter, based on the cross-validation procedure. Newly, Ferraty and Park (2011) proposed a new estimator of the single index based on the idea of functional derivative estimation.

3.6.1 About the Functional Single Index Estimate

Among the interest of our study is to show how the conditional density estimate can be used to derive an estimate of the functional single index if the latter is unknown. The leave-out-one-curve cross-validation procedure was adapted by Aït Saidi *et al.* (2008) to estimate the single index. Newly, Hall and Müller (2009) proposed a method for estimating functional derivatives and Ferraty and Park (2011) adopted this technique to estimate the parameter θ. Alternatively, this parameter can be estimated via the pseudo-maximum likelihood method which is based on the preliminary estimation of the conditional density of Y given X by

$$\widehat{\theta} = \arg\max_{\theta \in \Theta_{\mathcal{F}}} \widehat{L}(\theta)$$

where

$$\widehat{L}(\theta) = \sum_{k=1}^{n} \log \widehat{f}(\theta, Y_k, X_k).$$

Note that, this method has been studied by Delecroix *et al.* (2003) in the real case where they showed that this technique has minimal variance among all estimators. The asymptotic optimality of this procedure in functional statistic, is an important prospect of the present work.

As an application, this approach can be used for answering the semimetric choice question. Indeed, it is well known that, in nonparametric functional statistic, the projection-type semi-metric is very important for increasing the concentration property. The functional index model is a particular case of this family of semi-metric, because it is based on the projection on one functional direction. So, the estimation procedures of this direction permit us to compute adaptive semi-metrics in the general context of nonparametric functional data analysis. Finally, the theoretical justification and practice should be established.
For example, the motivation of this model also comes from an analysis of environmental data, consisting of daily measurement of pollutants and other environment factors. Of interest is to examine the association between the levels of pollutants, two environment factors (temperature and relative humidity) and the total number of daily hospital admissions for respiratory problems.

The estimators for single-index function and function coefficients are sensitive enough to capture the true shapes.

Wong et al. [11] and Li and Zhang [12] proposed the varying-coefficient single-index model (VCSIM) to analysis the real data based on local smoothing method and penalized spline (P-spline) method, respectively.

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Chapitre Quatre

Strong uniform consistency rates of conditional quantile estimation in the single functional index model under random censorship

Ce chapitre a fait l'objet d'une publication internationale dans *Dependence Modeling.*

Strong uniform consistency rates of conditional quantile estimation in the single functional index model under random censorship

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Abstract.The main objective of this paper is to estimate non-parametrically the quantiles of a conditional distribution in the censorship model when the sample is considered as an α-mixing sequence. First of all, a kernel type estimator for the conditional cumulative distribution function (*cond-cdf*) is introduced. Afterwards, we give an estimation of the quantiles by inverting this estimated *cond-cdf*, the asymptotic properties are stated when the observations are linked with a single-index structure. The pointwise almost complete convergence and the uniform almost complete convergence (with rate) of the kernel estimate of this model are established. This approach can be applied in time series analysis.

Keywords :Conditional quantile ; conditional cumulative distribution ; derivatives of conditional cumulative distribution ; functional data ; kernel estimator ; nonparametric estimation ; probabilities of small balls ; strong mixing processes.

2010 Mathematics Subject Classification : Primary 62G05, Secondary 62G99, Thirdly 62M10.

4.1 Introduction

Estimating quantiles of any distribution guaranties to build confidence ranges deriving many applications in numerous fields, chemistry, geophysics, medicine, meteorology,.... Furthermore, Statistics for functional random variables become progressively important. Numerous multivariate statistical technics, mainly parametric in the functional model terminology, have been extended to functional data and good analysis on this area can be found in Ramsay and Silverman (1997) and (2002) or Bosq (2000). Lately, nonparametric methods considering functional variables have been grown with very interesting practical motivations dealing with environmetrics, (see Damon and Guillas (2002), Fernández *et al.* (2005), Aneiros *et al.* (2004)), chemometrics (see Ferraty and Vieu (2002)), meteorological sciences (see Besse *et al.* (2000), Hall and Heckman (2002)), speech recognization problem (see Ferraty and Vieu (2003a)), radar range profile (see Hall *et al.* (2001), Dabo-Niang *et al.* (2004)), medical data (see Gasser *et al.* (1998)), ...

On the other hand, the single index model is a powerful tool to incorporate the multivariate regressors into a regression model, thus ensuring some flexibility, while avoiding the "curse of dimensionality".

For functional regressors, the single functional index model is a very simple approach for reducing dimensionality by assuming that a functional explanatory variable acts on a scalar response only through its production on one functional direction. This model was first introduced by Ferraty *et al.* (2003) for regression problems, and then has received a lot of attention.

For example, Aït Saidi *et al.* (2005) studied the single functional index model for functional time series data ; Aït Saidi *et al.* (2008) proposed to estimate the unknown functional index via the cross-validation technique. Ferraty *et al.* (2011) proposed a new estimator of this parameter based on the idea of functional derivative estimation.

Chen *et al.* (2011) introduced a new technique for estimating the link function nonparametrically, and they proposed the multi-index modeling using an adaptive linear projections approach.

Attaoui *et al.* (2011) dealt with the single index via its conditional density and established the pointwise and the uniform almost complete convergence of the kernel density estimator.

Their results were extended to the α-mixing case by Ling *et al.* (2014).

Recently Attaoui (2014a) established the a.co. of the kernel conditional density and the mode estimators for α-mixing data. For the same data, Attaoui (2014b) investigated the strong uniform convergence rate sand the asymptotic normality of the conditional density estimator in the single index model, and gave an approach to estimate this functional parameter.

In a recent survey of semiparametric FDA, we refer to Goia and Vieu (2014),(2015) this authors gave a methodology to approximate the unknown regression operator semiparametrically through a single index approach.

Our purpose in this work is to investigate nonparametrically the estimation of the conditional distribution of a scalar response variable Y, given a functional Hilbertian explanatory variable X, when the observations are linked with a single index relationship.

In the censoring case, instead of observing the lifetime T, we observe the censored lifetime of items under study. That is, assuming that $(T_i)_{i\geq 1}$ is a stationary sequence of lifetimes which satisfy some kind of dependency and $(C_i)_{i\geq 1}$ is a sequence of i.i.d censoring rv with common unknown continuous *df* G, where $Y_i = \min\{T_i, C_i\}$ and $\delta_i = \mathbf{1}_{T_i \leq C_i}$.

To ensure the identifiability of the model, we suppose that $(T_i)_i$ are independent of $(C_i)_i$. Let $\{(Y_i, \delta_i, X_i)\}$ be a sequence of strictly stationary random vectors where $X_{i\geq 1}$ is valued in infinite dimensional semi-metric vector space, and Y_i is real valued. To follow the convention in biomedical studies and as indicated before, we assume that $(C_i)_{i\geq 1}$ and $\{(X_i, T_i)_{i\geq 1}\}$ are independent ; this condition is plausible whenever the censoring is independent of the patients modality. Furthermore this condition permits to get an unbiased Kernel estimator. Along this paper we assume that $\tau_G < \infty$ where $\tau_G := \sup\{t : G(t) < 1\}$ and let τ be a positive real number such that $\tau < \tau_G$.

4.2 The model and the estimates

4.2.1 The functional nonparametric framework

Consider a random pair (X, T) where T is valued in $\mathbb{R}$ and X is valued in some infinite dimensional Hilbertian space $\mathcal{H}$ with scalar product $< \cdot, \cdot >$. Let $(X_i, T_i)_{i=1,\dots,n}$ be the statistical sample of pairs which are identically distributed like (X, T), but not necessarily independent. Henceforward, X is called functional random variable *f.r.v.* . Let x be fixed in $\mathcal{H}$ and let $F(\theta, t, x)$ be the conditional cumulative distribution function (*cond-cdf*) of T given $< \theta, X >=< \theta, x >$, specifically :

$$\forall t \in \mathbb{R}, \ F(\theta, t, x) = \mathbb{P}(T \leq t| < X, \theta >=< x, \theta >).$$

Saying that, we are implicitly assuming the existence of a regular version for the conditional distribution of T given $< \theta, X >$. Now, let $\zeta_\theta(\gamma, x)$ be the γ-order quantile of the distribution of T given $< \theta, X >=< \theta, x >$. From the *cond-cdf* $F(\theta, \cdot, x)$, the general definition of the γ-order quantile is given as :

$$\zeta_\theta(\gamma, x) = \inf\{t \in \mathbb{R} : \ F(\theta, t, x) \geq \gamma\}, \qquad \forall \gamma \in (0, 1).$$

In order to simplify our framework and to focus on the main interest of our paper (the functional feature of $< \theta, X >$), we assume that $F(\theta, \cdot, x)$ is strictly increasing and continuous in a neighborhood of $\zeta_\theta(\gamma, x)$. This is insuring that the conditional quantile $\zeta_\theta(\gamma, x)$ is uniquely defined by :

$$\zeta_\theta(\gamma, x) \ = \ F^{-1}(\theta, \gamma, x). \tag{4.1}$$

Next, in all what follows, we assume only smoothness restrictions for the *cond-cdf* $F(\theta, \cdot, x)$ through nonparametric modelling (Section 2.4). We suppose also that $(X_i, T_i)_{i\in\mathbb{N}}$ is an α-mixing sequence, which is one among the most general mixing structures. The α-mixing condition together with the functional approach allow to deal with continuous time processes (see Section 4 for instance).

In our infinite dimensional purpose, we use the terminology *functional nonparametric*, where the word *functional* referees to the infinite dimensionality of the data and where the word *nonparametric* referees to the infinite dimensionality of the model. Such *functional nonparametric* statistics is also called *doubly infinite dimensional* (see Ferraty and Vieu (2003b), for more details). We also use the terminology *operational statistics* since the target object to be estimated (the *cond-cdf* $F(\theta, \cdot, x)$) can be viewed as a nonlinear operator.

4.2.2 The estimators

The kernel estimator $F_n(\theta, \cdot, x)$ of $F(\theta, \cdot, x)$ is presented as follows :

$$F_n(\theta, t, x) = \frac{\sum_{i=1}^{n} K\left(h_K^{-1}(< x - X_i, \theta >)\right) H\left(h_H^{-1}(t - T_i)\right)}{\sum_{i=1}^{n} K\left(h_K^{-1}(< x - X_i, \theta >)\right)}, \tag{4.2}$$

where K is a kernel function, H a cumulative distribution function and $h_K = h_{K,n}$ (resp. $h_H = h_{H,n}$) a sequence of positive real numbers. Note that using similar ideas, Roussas (1969) introduced some related estimates but in the special case when X is real, while Samanta (1989) produced previous asymptotic study.

As a by-product of (4.1) and (4.2), it is easy to derive an estimator $\zeta_{\theta,n}(\gamma, x)$ of $\zeta_\theta(\gamma, x)$:

$$\zeta_{\theta,n}(\gamma, x) \ = \ F_n^{-1}(\theta, \gamma, x). \tag{4.3}$$

Such an estimator is unique as soon as H is an increasing continuous function. Such an approach has been largely used in the case where the variable X is of finite dimension (see *e.g* Whang and Zhao, (1999), Cai (2002), Zhou and Liang (2003) or Gannoun *et al.* (2003)).

In practice, in particularly medical applications, we can be in the presence of censored variables. This problem is usually modeled by considering a positive C variable called censorship, and the observed random variables are not couples (T_i, X_i), but rather (Y_i, δ_i, X_i) where $Y_i = \min(T_i,\ C_i)$ and $\delta_i = \mathbf{1}_{T_i \le C_i}$. In the following we will use the notations F_1^X and f_1^X to describe the conditional distribution function and the conditional density C knowing the covariate X.

Such censorship models have been amply studied in the literature for real or multi-dimensional random variables, and in nonparametric frameworks the kernel techniques are particularly used (see Tanner and Wong (1983), Padgett (1988), Lecoutre and Ould-Saïd (1995) and Van Keilegom and Veraverbeke (2001), for a necessarily non-exhaustive sample of literature in this area).

The objective of this paragraph is to adapt these ideas under functional random variable X, and build a kernel type estimator of the conditional distribution $F(\theta, \cdot, X)$ adapted for censored samples. Thus we can reformulate the expression (4.2) as follows :

$$\widetilde{F}(\theta, t, x) = \frac{\sum_{i=1}^{n} \frac{\delta_i}{\bar{G}(Y_i)} K\left(h_K^{-1}(< x - X_i, \theta >)\right) H\left(h_H^{-1}(t - Y_i)\right)}{\sum_{i=1}^{n} K\left(h_K^{-1}(< x - X_i, \theta >)\right)}, . \tag{4.4}$$

In practice $\bar{G}(\cdot) = 1 - G(\cdot)$ is unknown, hence it is impossible to use the estimator (4.4). Then, we replace $\bar{G}(\cdot)$ by its Kaplan and Meier (1958) estimate $\bar{G}_n(\cdot)$ given by

$$\bar{G}_n(t) = 1 - G_n(t) = \begin{cases} \prod_{i=1}^{n} \left(1 - \frac{1 - \delta_{(i)}}{n - i + 1}\right)^{\mathbf{1}_{\{Y_{(i)} \le t\}}}, & \text{if } t < Y_{(n)}; \\ 0, & \text{if } t \ge Y_{(n)}. \end{cases} \tag{4.5}$$

where $Y_{(1)} < Y_{(2)} < \ldots < Y_{(n)}$ are the order statistics of Y_i and $\delta_{(i)}$ is the non-censoring indicator corresponding to $Y_{(i)}$.

Therefore feasible estimator of the conditional distribution function $F(\theta, \cdot, x)$ is given by

$$\widehat{F}(\theta, t, x) = \frac{\sum_{i=1}^{n} \frac{\delta_i}{\bar{G}_n(Y_i)} K\left(h_K^{-1}(< x - X_i, \theta >)\right) H\left(h_H^{-1}(t - Y_i)\right)}{\sum_{i=1}^{n} K\left(h_K^{-1}(< x - X_i, \theta >)\right)}. \tag{4.6}$$

Then a natural estimator of $\zeta_\theta(\gamma, x)$ is given by

$$\widehat{\zeta}_\theta(\gamma, x) = \widehat{F}^{-1}(\theta, \gamma, x) \tag{4.7}$$
$$= \inf\{t \in \mathbb{R} : \ \widehat{F}(\theta, t, x) \geq \gamma\}, \tag{4.8}$$

which satisfies

$$\widehat{F}(\theta, \widehat{\zeta}_\theta(\gamma, x) \ , x) = \gamma \tag{4.9}$$

4.2.3 Assumptions on the functional variable

Let N_x be a fixed neighborhood of x and let $B_\theta(x, h)$ be the ball of center x and radius h, namely $B_\theta(x, h) = \{f \in \mathcal{H} : \ 0 < | < x - f, \theta > | < h\}$ and $\mathcal{S}_\mathbb{R}$ is a fixed compact of $\mathbb{R}^+$. Assume that, $(C_i)_{i \geq 1}$ are independent and let's consider the following hypotheses :

(H0) $\forall h > 0, \ \mathbb{P}\left(X \in B_\theta(x, h)\right) = \phi_{\theta,x}(h) > 0,$

(H1) $(X_i, Y_i)_{i \in \mathbb{N}}$ is an α-mixing sequence whose the coefficients of mixture verify :

$$\exists a > 0, \ \exists c > 0 : \ \forall n \in \mathbb{N}, \ \alpha(n) \leq cn^{-a}.$$

(H2) $0 < \sup_{i \neq j} \mathbb{P}\left((X_i, X_j) \in B_\theta(x, h) \times B_\theta(x, h)\right) \ = \ \mathcal{O}\left(\frac{(\phi_{\theta,x}(h_K))^{(a+1)/a}}{n^{1/a}}\right).$

• (H0) can be interpreted as a concentration hypothesis acting on the distribution of the *f.r.v.* X, while (H2) concerns the behavior of the joint distribution of the pairs (X_i, X_j). Indeed, this hypothesis is equivalent to assume that, for n large enough for some positive constant C

$$\sup_{i \neq j} \frac{\mathbb{P}\left((X_i, X_j) \in B_\theta(x, h) \times B_\theta(x, h)\right)}{\mathbb{P}\left(X \in B_\theta(x, h)\right)} \leq C \left(\frac{\phi_{\theta,x}(h_K)}{n}\right)^{1/a}.$$

This is one way to control the local asymptotic ratio between the joint distribution and its margin. Remark that the upper bound increases with a. In other words, more the dependence is strong, more restrictive is (H2). The hypothesis (H1) specifies the asymptotic behavior of the α-mixing coefficients.

4.2.4 The nonparametric model

As usually in nonparametric estimation, we suppose that the *cond-cdf* $F(\theta, \cdot, x)$ verifies some smoothness constraints. Let b_1 and b_2 be two positive numbers ; such that :

(H3) $\forall (x_1, x_2) \in N_x \times N_x, \ \forall (t_1, t_2) \in \mathcal{S}_\mathbb{R}^2,$

(i) $|F(\theta, t_1, x_1) - F(\theta, t_2, x_2)| \leq C_{\theta,x}\left(\|x_1 - x_2\|^{b_1} + |t_1 - t_2|^{b_2}\right),$

(ii) $\int_\mathbb{R} t f(\theta, t, x) dt < \infty$ for all $\theta, x \in \mathcal{H}$.

(H4) $F(\theta, \cdot, x)$ is l-times continuously differentiable in some neighborhood of $\zeta_\theta(\gamma, x)$,

(H5) $\forall (x_1, x_2) \in N_x \times N_x, \ \forall (t_1, t_2) \in \mathcal{S}_\mathbb{R}^2,$

$$|F^{(l)}(\theta, t_1, x_1) - F^{(l)}(\theta, t_2, x_2)| \leq C_{\theta,x}\left(\|x_1 - x_2\|^{b_1} + |t_1 - t_2|^{b_2}\right),$$

where, for any positive integer l, $F^{(l)}(\theta, z, x)$ denotes its lth derivative ($i.e.$ $\left.\frac{\partial^l F(\theta, t, x)}{\partial t^l}\right|_{t=z}$).

Let's note that (H3) is used for the prove of the the almost complete convergence of $\widehat{\zeta}_\theta(\gamma, x)$ whereas (H4) and (H5) are needed to establish the rate of convergence.

4.3 Asymptotic study

This part of paper is devoted, to the theoretical analysis, we start it by giving the almost complete convergence $(a.co.)$ of the estimated conditional quantile $\widehat{\zeta}_\theta(\gamma, x)$. After that, we will focus on the rate of convergence. Concerning the notations, as soon as possible, C and C' will denote generic constants. Moreover, from now on, h_H (resp. h_K) is a sequence which tends to 0 with n.

4.3.1 Pointwise almost complete convergence

Let's begin with the statement of an almost complete convergence property [1]. To this end, we need some assumptions concerning the kernel estimator $\widehat{F}(\theta, \cdot, x)$:

(H6) (i) $\forall (t_1, t_2) \in \mathbb{R}^2,\ |H(t_1) - H(t_2)| \leq C|t_1 - t_2|$ and $\int |t|^{b_2} H^{(1)}(t)dt < \infty$,

where, for all $l \in \mathbb{N}^*$, $H^{(l)}(t) = \left.\frac{d^l H(z)}{dz^l}\right|_{z=t}$ and $\lim_{n\to\infty} n^{\varsigma} h_H = \infty$, for some $\varsigma > 0$

(ii) The support of $H^{(1)}$ is compact and $\forall l \geq j$, $H^{(l)}$ exists and is bounded.

(H7) The restriction of H to the set $\{u \in \mathbb{R},\ H(u) \in (0,1)\}$ is a strictly increasing function.

(H8) K is a positive bounded function with support $[0, 1]$ such that $\forall u \in (0, 1) \quad 0 < K(u)$,

(H9) (i) $\dfrac{\log n}{n\,\phi_{\theta,x}(h_K)} \underset{n\to\infty}{\to} 0.$

(ii) $\exists C > 0,\ \phi_{\theta,x}(h_K) \geq C/n^{2/a+1}$ and $\left(\frac{\phi_{\theta,x}(h_K)}{n}\right)^{1/a} + \phi_{\theta,x}(h_K) = o\left(\frac{1}{n^{a/a+1}}\right)$

(H10) (X_i, Y_i) for $i = 1, \ldots, n$ are strongly mixing with arithmetic coefficient of order $a > 1$, and $\exists \beta > 2$ such that

(i) $s_{n,l}^{-(a+1)} = o\left(n^{-\beta}\right)$ for $l = 1, 2, 3$;

(ii) $s_{n,k}^{-(a+1)} = o\left(n^{-\beta}\right)$ for $k = 4, 5, 6, 7, 8$;

Remark 4.1 • *(H7) insures the existence of $\widehat{\zeta}_\theta(\gamma, x)$, while (H6) insures its unicity.*

• *(H0)-(H5) and (H8) are standard assumptions for the distribution conditional estimation in single functional index model, which have been adopted by Bouchentouf et al. (2014) for i.i.d case.*

• *(H9) is a technical condition for our results.*

• *(H10) is similar to that appeared in Ferraty and Vieu (2006), it shows the influence of covariance structure on the convergence rate. Here, $s_{n,l}$ and $s_{n,k}$ will be defined below.*

First observe that (4.6) can be rewritten as :

$$\widehat{F}(\theta, t, x) = \frac{\widehat{F}_N(\theta, t, x)}{\widehat{F}_D(\theta, x)} \tag{4.10}$$

1. Recall that a sequence $(S_n)_{n\in\mathbb{N}}$ of random variables is said to converge almost completely to some variable S, if for any $\epsilon > 0$, we have $\sum_n \mathbb{P}(|S_n - S| > \epsilon) < \infty$. This mode of convergence implies both almost sure and in probability convergence (see for instance Bosq and Lecoutre, (1987)).

Proposition 4.1 *Under conditions (H0)-(H4), (H7)-(H9), and assume that (H6)-(i) and (H10)-(i) are satisfied, then we have*

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\widehat{F}(\theta,t,x)-F(\theta,t,x)\right| \;=\; \mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right)+\mathcal{O}_{a.co.}\left(\frac{\sqrt{s_n^2\log n}}{n}\right) \tag{4.11}$$

where $s_n^2=\max\{s_{n,1}^2;s_{n,2}^2\}$

Proof 4.1 *Consider now, for* $i=1,\ldots,n$, *in what follows, let's denote :*

$$K_i(\theta,x)=K(h_K^{-1}(<x-X_i,\theta>)),\;\; H_i(t)=H\left(h_H^{-1}\left(t-Y_i\right)\right),\;\; \bar{G}_i=\bar{G}(Y_i),$$

$$\begin{aligned}
\widehat{F}_N(\theta,t,x) &= \frac{1}{n\,\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^{n}\frac{\delta_i}{\bar{G}_n(Y_i)}K_i(\theta,x)H_i(t)\\
\widetilde{F}_N(\theta,t,x) &= \frac{1}{n\,\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^{n}\frac{\delta_i}{\bar{G}(Y_i)}K_i(\theta,x)H_i(t)\\
\widehat{F}_D(\theta,x) &= \frac{1}{n\,\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^{n}K_i(\theta,x)\\
\Delta_i(x,\theta) &= \frac{K(h_K^{-1}(<x-X_i,\theta>))}{\mathbb{E}K_1(\theta,x)}
\end{aligned}$$

and

$$s_{n,1}^2=\sum_{i=1}^{n}\sum_{j=1}^{n}\left|Cov\left(\Delta_i(x,\theta),\Delta_j(x,\theta)\right)\right|,$$

$$s_{n,2}^2=\sum_{i=1}^{n}\sum_{j=1}^{n}\left|Cov\left(\frac{\delta_i}{\bar{G}(Y_i)}H_i(t)\Delta_i(x,\theta),\frac{\delta_j}{\bar{G}(Y_j)}H_j(t)\Delta_j(x,\theta)\right)\right|.$$

The proof is based on the following decomposition, valid for any $t\in\mathcal{S}_{\mathbb{R}}$ *:*

$$\begin{aligned}
\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\widehat{F}(\theta,t,x)-F(\theta,t,x)\right| \;\leq\; & \frac{1}{\widehat{F}_D(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left\{\left|\widehat{F}_N(\theta,t,x)-\widetilde{F}_N(\theta,t,x)\right|\right\}\\
&+\frac{1}{\widehat{F}_D(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left\{\left|\widetilde{F}_N(\theta,t,x)-\mathbb{E}\widetilde{F}_N(\theta,t,x)\right|\right\}\\
&+\frac{1}{\widehat{F}_D(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left\{\left|\mathbb{E}\widetilde{F}_N(\theta,t,x)-F(\theta,t,x)\right|\right\}\\
&+\frac{F(\theta,t,x)}{\widehat{F}_D(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|1-\widehat{F}_D(\theta,x)\right|,
\end{aligned} \tag{4.12}$$

Finally, the proof of this proposition is a direct consequence of the following intermediate results.

Lemma 4.1 *Assume that either (H6)-(i) is satisfied together with under conditions (H7)-(H8) and if*

$$\left(\frac{\log\log n}{n}\right)^{1/2}=o\left(\phi_{\theta,x}(h_K)\right),$$

we have

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left\{\left|\widehat{F}_N^{(l)}(\theta,t,x)-\widetilde{F}_N^{(l)}(\theta,t,x)\right|\right\} = \mathcal{O}_{a.s.}\left(\frac{\log\log n}{n}\right), \quad \textit{for } l\geq 0$$

The following lemma shows the asymptotic bias term of $\widetilde{F}_N(\theta,t,x)$ *and* $\widehat{F}_D(\theta,x)$ *as* n *tends to infinity.*

Lemma 4.2 *Under hypotheses (H0), (H4) and (H6)-(i), we have as* $n\to\infty$

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\mathbb{E}\left[\widetilde{F}_N(\theta,t,x)\right]-F(\theta,t,x)\right| = \mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right). \tag{4.13}$$

The following result deals with the variance term of the right-hand side of (4.12) which is expressed by : $\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left\{\left|\widetilde{F}_N(\theta,t,x)-\mathbb{E}\widetilde{F}_N(\theta,t,x)\right|\right\}$. *For* $\widehat{F}_D(\theta,x)-\mathbb{E}\left[\widehat{F}_D(\theta,x)\right]$ *the same arguments will be used with a slight difference.*

Lemma 4.3 *Under the assumptions of Proposition 4.1 and if*

$$\left(\frac{\log\log n}{n}\right)^{1/2} = o\left(\phi_{\theta,x}(h_K)\right),$$

we have

(i) $\widehat{F}_D(\theta,x)-\mathbb{E}\widehat{F}_D(\theta,x) = \mathcal{O}_{a.co.}\left(\frac{\sqrt{s_{n,1}^2\log n}}{n}\right),$

(ii) $\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left\{\left|\widetilde{F}_N(\theta,t,x)-\mathbb{E}\widetilde{F}_N(\theta,t,x)\right|\right\} = \mathcal{O}_{a.co.}\left(\frac{\sqrt{s_{n,2}^2\log n}}{n}\right)$

We conclude the proof of the Proposition 4.1 by making use the inequality (4.12), in conjunction with lemmas Lemma 4.1, Lemma 4.2 and Lemma 4.3.

The proof of these latter will be collected in Section 5.

Theorem 4.1 *Put* $s_n=\max\{s_{n,1};s_{n,2}\}$, *and under the hypotheses of Proposition 4.1, thus we have :*

$$\widehat{\zeta}_\theta(\gamma,x)-\zeta_\theta(\gamma,x) \underset{n\to\infty}{\longrightarrow} 0, \ \ a.co. \tag{4.14}$$

Proof 4.2 (Proof of Theorem 4.1) *The proof is based on the pointwise convergence of* $\widehat{F}(\theta,\cdot,x)$.

First of all, let's note that because of (H6)-(i) and (H7), $\widehat{F}(\theta,\cdot,x)$ *is a continuous and strictly increasing function. So, we have :*

$$\forall\epsilon>0,\ \exists\delta(\epsilon)>0,\forall y,|\widehat{F}(\theta,t,x)-\widehat{F}\left(\theta,\zeta_\theta(\gamma,x),x\right)|\leq\delta(\epsilon)\Rightarrow|t-\zeta_\theta(\gamma,x)|\leq\epsilon.$$

This leads us to write

$$\begin{aligned}\forall\epsilon>0,\ \exists\delta(\epsilon)>0,\ \mathbb{P}\left(|\widehat{\zeta}_\theta(\gamma,x)-\zeta_\theta(\gamma,x)|>\epsilon\right) &\leq \mathbb{P}\left(|\widehat{F}(\theta,\widehat{\zeta}_\theta(\gamma,x),x)-\widehat{F}(\theta,\zeta_\theta(\gamma,x),x)|\geq\delta(\epsilon)\right)\\ &= \mathbb{P}\left(|F(\theta,\zeta_\theta(\gamma,x),x)-\widehat{F}(\theta,\zeta_\theta(\gamma,x),x)|\geq\delta(\epsilon)\right),\end{aligned}$$

since (4.1) and (4.9) and is implying that $\widehat{F}(\theta, \widehat{\zeta}_\theta(\gamma, x), x) = \gamma = F(\theta, \zeta_\theta(\gamma, x), x)$.

Moreover, we have,

$$\begin{aligned}|F(\theta, \widehat{\zeta}_\theta(\gamma, x), x) - F(\theta, \zeta_\theta(\gamma, x), x)| &= |F(\theta, \widehat{\zeta}_\theta(\gamma, x), x) - \widehat{F}(\theta, \widehat{\zeta}_\theta(\gamma, x), x)| \\ &\leq \sup_{t \in \mathcal{S}_\mathbb{R}} |\widehat{F}(\theta, t, x) - F(\theta, t, x)|. \end{aligned} \tag{4.15}$$

The consistency of $\widehat{\zeta}_\theta(\gamma, x)$ *follows then immediately from Proposition 4.1 and the following inequality*

$$\sum_{n\geq 1} \mathbb{P}\left(|\zeta_{\theta,n}(\gamma, x) - \zeta_\theta(\gamma, x)| \geq \epsilon\right) \leq \sum_{n\geq 1} \mathbb{P}\left(\sup_{t\in\mathcal{S}_\mathbb{R}} \left|\widehat{F}(\theta, t, x) - F(\theta, t, x)\right| \geq \delta(\epsilon)\right).$$

4.3.2 Pointwise almost complete rate of convergence

In this section we study the rate of convergence of our conditional quantile estimator $\widehat{\zeta}_\theta(\gamma, x)$. Because this kind of result is stronger than the previous one, we have to introduce some additional assumptions. As it is usual in conditional quantiles estimation, the rate of convergence can be linked with the flatness of the *cond-cdf* $F(\theta, \cdot, x)$ around the conditional quantile $\zeta_\theta(\gamma, x)$. This is one reason why we introduced hypotheses (H4) and (H5). But a complementary way to take into account this local shape constraint is to suppose that :

(H11) $\exists l > 0,\ \forall m,\ 1 \leq m < l, \widehat{F}^{(m)}(\theta, \zeta_\theta(\gamma, x), x) = 0$ and $0 < \left|F^{(l)}(\theta, \zeta_\theta(\gamma, x), x)\right| < \infty$.

Because we focus on the local behavior of $F(\theta, \cdot, x)$ around $\zeta_\theta(\gamma, x)$ via its derivatives, that leads us to consider the successive derivatives of $\widehat{F}(\theta, \cdot, x)$ and subsequently some assumptions on the successive derivatives of the cumulative kernel H :

(H12) $\forall i \neq j$, the conditional density of (Y_i, Y_j) given $(< X_i, \theta >, < X_j, \theta >)$ is continuous at $(\zeta_\theta(\gamma, x), \zeta_\theta(\gamma, x))$.

Proposition 4.2 *Suppose that the hypotheses (H0)-(H2), (H4)-(H9) are satisfied and if*

$$\lim_{n\to\infty} \frac{\log n}{n\, h_H^{2l-1}\, \phi_{\theta,x}(h_K)} = 0.$$

Then we have

$$\left|\widehat{F}^{(l)}(\theta, t, x) - F^{(l)}(\theta, t, x)\right| = \mathcal{O}\left(h_K^{b_1} + h_H^{b_2}\right) + \mathcal{O}_{a.co.}\left(\frac{\sqrt{s_n'^2 \log n}}{n}\right).$$

Theorem 4.2 *Put* $s_n'^2 = \max\{s_{n,1}^2; s_{n,3}^2\}$, *and assume that either (H10)-(i) is satisfied together with hypotheses (H0)-(H9) and (H11)-(H12), we have*

$$\widehat{\zeta}_\theta(\gamma, x) - \zeta_\theta(\gamma, x) = \mathcal{O}\left(\left(h_K^{b_1} + h_H^{b_2}\right)^{\frac{1}{l}}\right) + \mathcal{O}_{a.co}\left(\left(\frac{s_n'^2 \log n}{n^2}\right)^{\frac{1}{2l}}\right), \tag{4.16}$$

where

$$s_{n,3}^2 = \sum_{i=1}^n \sum_{j=1}^n \left|Cov\left(h_H^{-l} \frac{\delta_i}{\bar{G}(Y_i)} H_i^{(l)}(t)\Delta_i(x,\theta), h_H^{-l} \frac{\delta_j}{\bar{G}(Y_j)} H_j^{(l)}(t)\Delta_j(x,\theta)\right)\right|$$

Proof 4.3 (Proof of Theorem 4.2) *The proof is based on the Taylor expansion of* $\widehat{F}(\theta,\cdot,x)$ *at* $\zeta_\theta(\gamma,x)$ *and on the use of (H11) :*

$$\begin{aligned}
\widehat{F}\left(\theta,\zeta_\theta(\gamma,x),x\right)-\widehat{F}\left(\theta,\widehat{\zeta}_\theta(\gamma,x),x\right) &= \sum_{m=1}^{l-1}\frac{(\zeta_\theta(\gamma,x)-\widehat{\zeta}_\theta(\gamma,x))^{m-1}}{m!}\widehat{F}^{(m)}(\theta,\zeta_\theta(\gamma,x),x)\\
&\quad+\frac{(\zeta_\theta(\gamma,x)-\widehat{\zeta}_\theta(\gamma,x))^{l}}{l!}\widehat{F}^{(l)}(\theta,\zeta^*_\theta,x),\\
&= \sum_{m=1}^{l-1}\frac{(\zeta_\theta(\gamma,x)-\widehat{\zeta}_\theta(\gamma,x))^{m-1}}{m!}\Big(\widehat{F}^{(m)}(\theta,\zeta_\theta(\gamma,x),x)-\\
&\qquad F^{(m)}(\theta,\zeta_\theta(\gamma,x),x)\Big)\\
&\quad+\frac{(\zeta_\theta(\gamma,x)-\widehat{\zeta}_\theta(\gamma,x))^{l}}{l!}\widehat{F}^{(l)}(\theta,\zeta^*_\theta,x),
\end{aligned}$$

where, for all $t\in\mathbb{R}$*,*

$$\widehat{F}^{(l)}(\theta,t,x)=\frac{h_H^{-l}\sum_{i=1}^{n}\frac{\delta_i}{\bar{G}_n(Y_i)}K(h_K^{-1}(<x-X_i,\theta>))H^{(l)}(h_H^{-1}(t-Y_i))}{\sum_{i=1}^{n}K(h_K^{-1}(<x-X_i,\theta>))},$$

$$\widetilde{F}^{(l)}(\theta,t,x)=\frac{h_H^{-l}\sum_{i=1}^{n}\frac{\delta_i}{\bar{G}(Y_i)}K(h_K^{-1}(<x-X_i,\theta>))H^{(l)}(h_H^{-1}(t-Y_i))}{\sum_{i=1}^{n}K(h_K^{-1}(<x-X_i,\theta>))},$$

and where $\min(\zeta_\theta(\gamma,x),\widehat{\zeta}_\theta(\gamma,x))<\zeta^*_\theta<\max(\zeta_\theta(\gamma,x),\widehat{\zeta}_\theta(\gamma,x))$*. Suppose now that we have the following result.*

Because of Theorem 4.1, Proposition 4.2 and (H11), we have :

$$\widehat{F}^{(l)}(\theta,\zeta^*_\theta,x)\underset{n\to\infty}{\longrightarrow}F^{(l)}(\theta,\zeta_\theta(\gamma),x)\neq 0,\ a.co.,$$

then we derive

$$\begin{aligned}
\left|\zeta_\theta(\gamma,x)-\widehat{\zeta}_\theta(\gamma,x)\right|^l &= \mathcal{O}\left(\widehat{F}(\theta,\zeta_\theta(\gamma,x),x)-F(\theta,\zeta_\theta(\gamma,x),x)\right)\\
&+\mathcal{O}\left(\sum_{m=1}^{l-1}(\zeta_\theta(\gamma,x)-\widehat{\zeta}_\theta(\gamma,x))^m(\widehat{F}^{(m)}(\theta,\zeta_\theta(\gamma,x),x)-F^{(m)}(\theta,\zeta_\theta(\gamma,x),x))\right),\ a.co.
\end{aligned}$$

Now, comparing the convergence rates given in Proposition 4.1 and 4.2, we get

$$\left|\zeta_\theta(\gamma,x)-\widehat{\zeta}_\theta(\gamma,x)\right|^l=\mathcal{O}\left(\widehat{F}(\theta,t,x)-F(\theta,t,x)\right),\ a.co.$$

Thus, first part of Lemma 4.3 with lemmas Lemma 4.4 and Lemma 4.5 allow us to get the claimed result.

Lemma 4.4 *Under the hypotheses (H0) and (H5)-(H8) we have*

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}} |F^{(l)}(\theta,t,x) - \mathbb{E}(\widetilde{F}_N^{(l)}(\theta,t,x)| = \mathcal{O}\left(h_K^{b_1} + h_H^{b_2}\right),$$

Lemma 4.5 *Under the assumptions of Proposition 4.2, we have*

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}} |\widetilde{F}_N^{(l)}(\theta,t,x) - \mathbb{E}(\widetilde{F}_N^{(l)}(\theta,t,x)| = \mathcal{O}_{a.co.}\left(\sqrt{\frac{s_{n,3}^2 \log n}{n^2}}\right),$$

4.4 Uniform almost complete convergence and rate of convergence

In this section we derive the uniform version of Theorem 4.1 and Theorem 4.2. The study of the uniform consistency is a crucial tool for studying the asymptotic properties of all estimates of the functional index if is unknown. In the multivariate case, the uniform consistency is a standard extension of the pointwise one, nevertheless, in the studied case, it requires some additional tools and topological conditions (see Ferraty *et al.* (2010)). Consequently, coupled with the conditions introduced antecedently, we need the following ones. Firstly, consider

$$\mathcal{S}_{\mathcal{H}} \subset \bigcup_{k=1}^{d_n^{\mathcal{S}_{\mathcal{H}}}} B_\theta(x_k, r_n) \text{ and } \Theta_{\mathcal{H}} \subset \bigcup_{q=1}^{d_n^{\Theta_{\mathcal{H}}}} B_\theta(\theta_q, r_n) \tag{4.17}$$

with x_k (resp. θ_q) $\in \mathcal{H}$ and $r_n, d_n^{\mathcal{S}_{\mathcal{H}}}, d_n^{\Theta_{\mathcal{H}}}$ are sequences of positive real numbers which tend to infinity as n goes to infinity and suppose that $d_n^{\mathcal{S}_{\mathcal{H}}}$, $d_n^{\Theta_{\mathcal{H}}}$ are the minimal numbers of open balls with radius r_n in $\mathcal{H}$, which are required to cover $\mathcal{S}_{\mathcal{H}}$ and $\Theta_{\mathcal{H}}$.

4.4.1 Conditional quantile distribution estimation

In this subpart we propose to study the uniform almost complete convergence of our estimator (4.3), to this end, we need to state the following assumptions

(A1) There exists a differentiable function $\phi(\cdot)$ such that $\forall x \in \mathcal{S}_{\mathcal{H}}$ and $\forall \theta \in \Theta_{\mathcal{H}}$,

$$0 < C\phi(h) \leq \phi_{\theta,x}(h) \leq C'\phi(h) < \infty \text{ and } \exists \eta_0 > 0,\ \forall \eta < \eta_0,\ \phi'(\eta) < C,$$

(A2) The kernel K satisfy (H3) and Lipschitz's condition holds

$$|K(x) - K(y)| \leq C\|x - y\|,$$

(A3) $\forall (t_1, t_2) \in \mathcal{S}_{\mathbb{R}} \times \mathcal{S}_{\mathbb{R}}, \forall (x_1, x_2) \in \mathcal{S}_{\mathcal{H}} \times \mathcal{S}_{\mathcal{H}}$ and $\forall \theta \in \Theta_{\mathcal{H}}$,

$$|F(\theta, t_1, x_1) - F(\theta, t_2, x_2)| \leq C\left(\|x_1 - x_2\|^{b_1} + |t_1 - t_2|^{b_2}\right),$$

(A4) For some $\nu \in (0,1)$, $\lim_{n\to\infty} n^\nu h_H = \infty$, and for $r_n = \mathcal{O}\left(\frac{\log n}{n}\right)$, the sequences $d_n^{\mathcal{S}_{\mathcal{H}}}$ and

$d_n^{\Theta_{\mathcal{H}}}$ satisfy :

$$\begin{cases} (i) \ \dfrac{(\log n)^2}{n\phi(h_K)} < \log d_n^{\mathcal{S}_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}} < \dfrac{n\phi(h_K)}{\log n}, \\ (ii) \ \sum_{n=1}^{\infty} n^{1/2b_2} (d_n^{\mathcal{S}_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}})^{1-\beta} < \infty \text{ for some } \beta > 1, \\ (iii) \ n\phi(h_K) = \mathcal{O}\left((\log n)^2\right). \end{cases}$$

(A5) $\forall (t_1, t_2) \in \mathcal{S}_{\mathbb{R}} \times \mathcal{S}_{\mathbb{R}}, \forall (x_1, x_2) \in \mathcal{S}_{\mathcal{H}} \times \mathcal{S}_{\mathcal{H}}$ and $\forall \theta \in \Theta_{\mathcal{H}}$,

$$|F^{(l)}(\theta, t_1, x_1) - F^{(l)}(\theta, t_2, x_2)| \leq C \left(\|x_1 - x_2\|^{b_1} + |t_1 - t_2|^{b_2} \right),$$

(A6) For some $\nu \in (0, 1)$, $\lim_{n\to\infty} n^{\nu} h_H = \infty$, and for $r_n = \mathcal{O}\left(\frac{\log n}{n}\right)$, the sequences $d_n^{\mathcal{S}_{\mathcal{H}}}$ and $d_n^{\Theta_{\mathcal{H}}}$ satisfy :

$$\begin{cases} (i) \ \dfrac{(\log n)^2}{n h_H^{2l-1} \phi(h_K)} < \log d_n^{\mathcal{S}_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}} < \dfrac{n h_H^{2l-1} \phi(h_K)}{\log n}, \\ (ii) \ n h_H^{2l-1} \phi(h_K) = \mathcal{O}\left((\log n)^2\right) \end{cases}$$

And let

$$s_{n,4}^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} |Cov(\Lambda_i, \Lambda_j)|, \quad s_{n,5}^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} |Cov(\Omega_i, \Omega_j)|$$

$$s_{n,6}^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \left| Cov\left(\Delta_i \left(x_{k(x)}, \theta_{q(\theta)} \right), \Delta_j \left(x_{k(x)}, \theta_{q(\theta)} \right) \right) \right|, \quad s_{n,7}^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} |Cov(\Gamma_i, \Gamma_j)|$$

$$s_{n,8}^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \left| Cov\left(\Gamma_i^{(l)}, \Gamma_j^{(l)} \right) \right|$$

where

$$\Lambda_i(x, \theta) = \frac{1}{h_K \phi(h_K)} \mathbf{1}_{B_\theta(x,h) \cup B_\theta(x_{k(x)},h)}(X_i),$$

$$\Omega_i(x, \theta) = \frac{1}{h_K \phi(h_K)} \mathbf{1}_{B_\theta(x_{k(x)},h) \cup B_{\theta_{q(\theta)}}(x_{k(x)},h)}(X_i),$$

$$\Delta_i \left(x_{k(x)}, \theta_{q(\theta)} \right) = \frac{K\left(h_K^{-1} < x_{k(x)} - X_i, \theta_{q(\theta)} > \right)}{\mathbb{E}K\left(h_K^{-1} < x_{k(x)} - X_i, \theta_{q(\theta)} > \right)},$$

$$\begin{aligned} \Gamma_i &= \frac{K\left(h_K^{-1} < x_{k(x)} - X_i, \theta_{q(\theta)} > \right)}{\mathbb{E}K\left(h_K^{-1} < x_{k(x)} - X_i, \theta_{q(\theta)} > \right)} \frac{\delta_i}{\bar{G}(Y_i)} H\left(h_H^{-1}(v_{k_t} - Y_i) \right) \\ &- \mathbb{E}\left(\frac{K\left(h_K^{-1} < x_{k(x)} - X_i, \theta_{q(\theta)} > \right)}{\mathbb{E}K\left(h_K^{-1} < x_{k(x)} - X_i, \theta_{q(\theta)} > \right)} \frac{\delta_i}{\bar{G}(Y_i)} H\left(h_H^{-1}(v_{k_t} - Y_i) \right) \right) \end{aligned}$$

and

$$\begin{aligned}\Gamma_i^{(l)} &= \frac{1}{h_H^l}\frac{K\left(h_K^{-1}<x_{k(x)}-X_i,\theta_{q(\theta)}>\right)}{\mathbb{E}K\left(h_K^{-1}<x_{k(x)}-X_i,\theta_{q(\theta)}>\right)}H^{(l)}\left(h_H^{-1}(v_{k_t}-Y_i)\right)\\ &- \frac{1}{h_H^l}\mathbb{E}\left(\frac{K\left(h_K^{-1}<x_{k(x)}-X_i,\theta_{q(\theta)}>\right)}{\mathbb{E}K\left(h_K^{-1}<x_{k(x)}-X_i,\theta_{q(\theta)}>\right)}H^{(l)}\left(h_H^{-1}(v_{k_t}-Y_i)\right)\right)\end{aligned}$$

Theorem 4.3 *Put $s_n^* = \max\{s_{n,4}; s_{n,5}; s_{n,6}; s_{n,7}\}$, and assume that either (H10)-(ii) is satisfied together with under hypotheses (H0)-(H3) and (H6)-(H9), (A1)-(A4), we have*

$$\sup_{x\in\mathcal{S}_\mathcal{H}} |\widehat{\zeta}_\theta(\gamma,x)-\zeta_\theta(\gamma,x)| \underset{n\to\infty}{\longrightarrow} 0, \ a.co. \tag{4.18}$$

Proof 4.4 (Proof of Theorem 4.3) *The proof of the theorem can be completed by using the following results.*

Lemma 4.6 *Under the conditions (H0)-(H3) and (H6)-(H9), we have*

$$\sup_{\theta\in\Theta_\mathcal{H}}\sup_{x\in\mathcal{S}_\mathcal{H}}\sup_{t\in\mathcal{S}_\mathbb{R}}\left|F(\theta,t,x)-\mathbb{E}\widehat{F}_N(\theta,t,x)\right| = \mathcal{O}\left(h_K^{b_1}\right)+O\left(h_H^{b_2}\right). \tag{4.19}$$

Lemma 4.7 *Under the assumptions of Theorem 4.3, we have :*

1. $\sup_{\theta\in\Theta_\mathcal{F}}\sup_{x\in\mathcal{S}_\mathcal{F}}\left|\widehat{F}_D(\theta,x)-\mathbb{E}\widehat{F}_D(\theta,x)\right| = \mathcal{O}_{a.co.}\left(\frac{\sqrt{\max\{s_{n,4}^2;s_{n,5}^2;s_{n,6}^2\}\log d_n^{\mathcal{S}_\mathcal{F}}d_n^{\Theta_\mathcal{F}}}}{n}\right),$
2.

$$\begin{aligned}\sup_{\theta\in\Theta_\mathcal{H}}\sup_{x\in\mathcal{S}_\mathcal{H}}\sup_{t\in\mathcal{S}_\mathbb{R}}\widehat{F}_N(\theta,t,x)-\mathbb{E}\widehat{F}_N(\theta,t,x) &= \mathcal{O}_{a.co.}\left(\frac{\sqrt{\max\{s_{n,4}^2;s_{n,5}^2;s_{n,7}^2\}\log d_n^{\mathcal{S}_\mathcal{F}}d_n^{\Theta_\mathcal{F}}}}{n}\right)\\ &+ \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{F}}d_n^{\Theta_\mathcal{F}}}{n\phi(h_K)}}\right).\end{aligned}$$

Theorem 4.4 *Under hypotheses (H0)-(H3), (H6)-(H10), (A1)-(A2) and (A5)-(A6), we have*

$$\begin{aligned}\sup_{\theta\in\Theta_\mathcal{F}}\sup_{x\in\mathcal{S}_\mathcal{F}}|\widehat{\zeta}_\theta(\gamma,x)-\zeta_\theta(\gamma,x)| &= \mathcal{O}\left(\left(h_K^{b_1}+h_H^{b_2}\right)^{\frac{1}{l}}\right)+\mathcal{O}_{a.co.}\left(\left(\frac{\log d_n^{\mathcal{S}_\mathcal{F}}d_n^{\Theta_\mathcal{F}}}{nh_H^{2l-1}\phi(h_K)}\right)^{\frac{1}{2l}}\right)\\ &+\mathcal{O}_{a.co.}\left(\left(\frac{s_n^{\prime\prime 2}\log d_n^{\mathcal{S}_\mathcal{F}}d_n^{\Theta_\mathcal{F}}}{n^2}\right)^{\frac{1}{2l}}\right)\end{aligned}$$

where $s_n'' = \max\{s_{n,4}; s_{n,5}; s_{n,6}; s_{n,8}\}$

Proof 4.5 *Obviously, the proof of these results, can be deduced from the decomposition (4.25) and the following intermediate results which are only uniform version of Proposition 4.2.*

Lemma 4.8 *Under the conditions (H0)-(H3) and (H6)-(H9), we have*

$$\sup_{\theta\in\Theta_\mathcal{H}}\sup_{x\in\mathcal{S}_\mathcal{H}}\sup_{t\in\mathcal{S}_\mathbb{R}}\left|F^{(l)}(\theta,t,x)-\mathbb{E}\widehat{F}_N^{(l)}(\theta,t,x)\right| = \mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right). \tag{4.20}$$

Lemma 4.9 *Under the assumptions of Theorem 4.3, we have :*

$$\sup_{\theta\in\Theta_{\mathcal{F}}}\sup_{x\in\mathcal{S}_{\mathcal{F}}}\left|\widehat{F}_D(\theta,x)-\mathbb{E}\widehat{F}_D(\theta,x)\right| = \mathcal{O}_{a.co.}\left(\frac{\sqrt{\max\{s_{n,3}^2;s_{n,4}^2;s_{n,5}^2\}\log d_n^{\mathcal{S}_{\mathcal{F}}}d_n^{\Theta_{\mathcal{F}}}}}{n}\right).$$

Lemma 4.10 *Under the assumptions of Theorem 4.3, we have :*

$$\begin{aligned}\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\widehat{F}_N^{(l)}(\theta,t,x)-\mathbb{E}\widehat{F}_N^{(l)}(\theta,t,x) &= \mathcal{O}_{a.co.}\left(\frac{\sqrt{s_n''^2\log d_n^{\mathcal{S}_{\mathcal{F}}}d_n^{\Theta_{\mathcal{F}}}}}{n}\right)\\ &+ \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{F}}}d_n^{\Theta_{\mathcal{F}}}}{nh_H^{2l-1}\phi(h_K)}}\right).\end{aligned}$$

Remark 4.2

1. *These results extends Theorem 3 or Theorem 4 given in Bouchentouf* et al. *(2014) to the mixing case. The effect of covariance structure for dependence case on the convergence rate is reflected in the last term. Specially, if the functional single-index is fixed, it is easy to prove the following corollary that are similar the one given in Bouchentouf* et al. *(2014).*
2. *Theorems 4.3 and 4.4 are obtained when the functional parameter θ is known. However, if the functional parameter θ is unknown, it would be very important for practical using of the model and would be more interesting.*

Corollary 4.1 *Under the conditions of Theorem 4.3 for $l=0$, we have*

$$\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{t\in\mathcal{S}_{\mathbb{R}}}|\widehat{F}(\theta,t,x)-F(\theta,t,x)| = \mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right)+\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}}{n\phi(h_K)}}\right)+\mathcal{O}_{a.co.}\left(\frac{\sqrt{s_n'^{*2}\log d_n^{\mathcal{S}_{\mathcal{F}}}}}{n}\right)$$

where $s_n'^{*}=\max\{s_{n,4};s_{n,6};s_{n,7}\}$

4.5 Simulation study

In this section we carry out a numerical simulation to evaluate the performance of the proposed method for finite samples data. For this aim, we compare our model CFSIM (4.6) (censored functional single index model) with CNPFDA (4.21) (censored nonparametric functional data analysis) with censored data in mixing case.

$$\widehat{F}(t,x)=\frac{\sum_{i=1}^{n}\frac{\delta_i}{\bar{G}_n(Y_i)}K\left(h_K^{-1}d(x,X_i)\right)H\left(h_H^{-1}(t-Y_i)\right)}{\sum_{i=1}^{n}K\left(h_K^{-1}d(x,X_i)\right)}. \tag{4.21}$$

Example 4.1 *Consider the curves generated in the following way :*

$$X_i(t)=1-sin(W_i(t-\frac{\pi}{3})),\quad i=1,\ldots,200;\quad t\in[0,\frac{\pi}{3}] \tag{4.22}$$

where $W_i=\frac{1}{3}W_{i-1}+\epsilon$ and ϵ standard normal distribution ($\epsilon\sim\mathcal{N}(0,1)$) and ($W_0\sim\mathcal{N}(0,1)$). We carry out the simulation with a 200 sample of the curves $X(t)$ (see Figure 4.1). We simulates

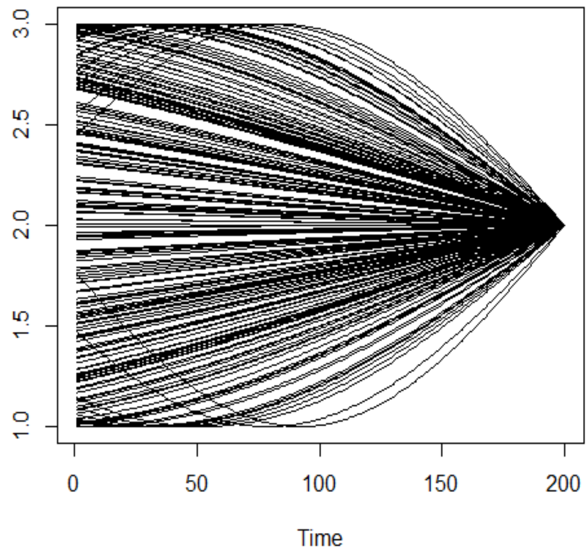

FIGURE 4.1 – The curves $X_{i=1,\ldots,200}(t), \quad t \in [0, \frac{pi}{3}]$.

the single functional index model as follows.

First, we choose the functional parameter θ*. So for a training sample* $\mathcal{L} = \{1, \ldots, 150\}$ *, the best approximation of* θ *is to estimate the eigenfunctions of the covariance operator* $\mathbb{E}[(X' - \mathbb{E}(X')) < X', \cdot >_{\mathcal{H}}]$ *by its empirical covariance* $\frac{1}{\mathcal{L}} \sum_{i \in \mathcal{L}} (X_i' - \mathbb{E}(X'))^t (X_i' - \mathbb{E}(X'))$ *(2014b). (Fig4.2) show the descretization of all the eigenfunctions* $\theta_i(t)$*.*

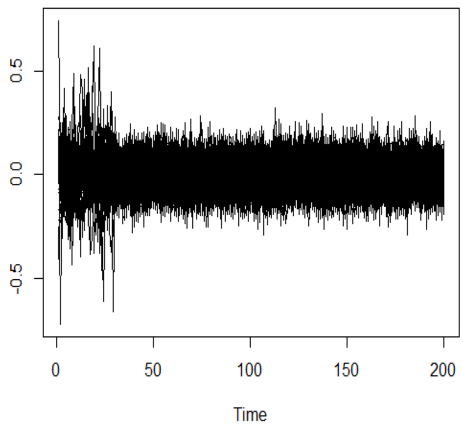

FIGURE 4.2 – The curves $\theta_{i=1,\ldots,150}(t), \quad t \in [0, \frac{pi}{3}]$.

Taking θ^* *the first eigenfunction corresponding to the first higher eigenvalue, and compute the inner product* $< \theta^*, X_1 >, \ldots, < \theta^*, X_{200} >$*, Then simulate the response variables* $T_i = r(<$

$\theta^*, X_i >) + \epsilon$, where $r(<\theta^*, X_i>) = \exp(10(<\theta^*, X_i> -0.05))$ and ε generate independently from a centred gaussian of variance equal to 0.075.

We simulate n i.i.d rv $C_i, i = 1, \ldots, n$ with the exponential distribution $\mathcal{E}(1, 5)$.

Noting that the computation of thos estimators are based on the observed data $(X_i, Y_i, \delta_i)_{i=1,\ldots,n}$, where $Y_i = min(T_i, C_i)$ and $\delta_i = \mathbf{1}_{\{T_i \leq C_i\}}$. On the other hand, take into account of the smoothness of the curves $X_i(t)$, we choose for the CNPFDA model the semi-metric in $\mathcal{H}$:

$$d(x_i, x_j) = \sqrt{\int_0^{\pi/3} \left(x_i'(t) - x_j'(t)\right)^2 dt}, \qquad x_i, x_j \in \mathcal{H}$$

Then, we choose the quadratic kernel defined by :

$$K(u) = \frac{3}{2}(1 - u^2)\mathbf{1}_{(0,1)}(u)$$

and the distribution function $H(.)$

$$H(t) = \int_{-\infty}^{t} \frac{3}{4}(1 - t^2)\mathbf{1}_{(-1,1)}(t)dt$$

For the bandwidths $h_H \sim h_K =: h$ is automatically selected by the procedure of the cross-validation method on the K-nearest neighbours (2006).

In our simulation, sample sizes are $n = 200$, we take it into two parts, one is a learning sample of 150 observations and the others 50 observations are a test sample.

Then using the learning sample to compute the estimator of $\widehat{Y}_i = \widehat{\zeta}_{\theta^*}(\gamma, x)$ and $\widehat{Y}_{ni} = \widehat{\zeta}(\gamma, x)$ for j={151,...,200}.

Fig 4.3 show the results by plotting the true values versus the predicted values for the MSE in complet data for both estimators.

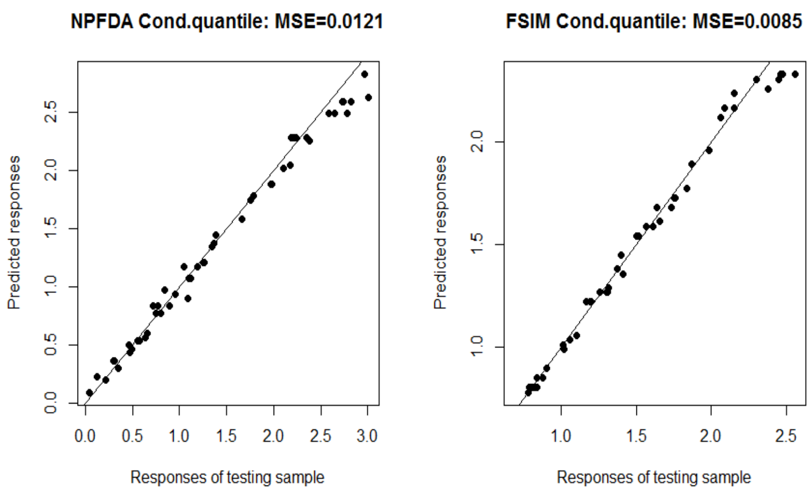

FIGURE 4.3 – Prediction via the conditional quantile for complete data

Finally, to show how the different censored rates (CRs) effects the prediction results, we

n	CR%	CNPFDA	CFSIM
	4%	0.0784	0.0471
200	20%	0.2694	0.1915
	45%	0.7030	0.4844
	4%	0.0653	0.0237
300	20%	0.2235	0.1219
	45%	0.6172	0.3014

TABLE 4.1 – MSE comparaison for CFSIM and CNPFDA

present some CRs and their corresponding MSE, which are defined as :

$$CFSIM.MSE = \frac{1}{50}\sum_{i=151}^{200}(Y_i - \widehat{Y}_i)^2, \; CNPFDA.MSE = \frac{1}{50}\sum_{i=151}^{200}(Y_i - \widehat{Y}_{ni})^2$$

respectively.

For each explicit value of CRs and n, we carried 100 *independent replications of the experiment, then we computed the average of mean squared error. These quantities are presented in Table (4.1).*

Example 4.2 *We consider a diffusion process on the interval* $[0,1]$ *:*

$$X_i(t) = cos(\pi b_i t) + a_i t^2, \quad i = 1, \dots, 200; \quad t \in [0,1] \tag{4.23}$$

where a_i *are uniformly distributed on* $[0,1]$ $(a \sim \mathcal{U}(0,1))$ *and* $b_i = \frac{1}{3}b_{i-1} + \xi_i$, $\xi_i \sim \mathcal{N}(0,1)$ *are independent from* a_i *and* b_i*, which is generated by are standard normal distribution* ($b_0 \sim \mathcal{N}(0,1)$)*. we carry out the simulation with a* 200 *sample of the curves X(t) (see Figure 4.4).*

We simulates the single functional index model as follows.

First, we choose the functional parameter θ*. So for a training sample* $\mathcal{L} = \{1, \dots, 150\}$ *, the best approximation of* θ *is to estimate the eigenfunctions of the covariance operator* $\mathbb{E}[(X' - \mathbb{E}(X')) < X', \cdot >_{\mathcal{H}}]$ *by its emprical covariance* $\frac{1}{\mathcal{L}}\sum_{i\in\mathcal{L}}(X_i' - \mathbb{E}(X'))^t(X_i' - \mathbb{E}(X'))$ *(2014b). (Fig4.5)shows the descretization of all the eingenfunctions* $\theta_i(t)$ *respectively.*

Taking θ^* *the first eigenfunction corresponding to the first higher eigenvalue, and compute the inner product* $< \theta^*, X_1 >, \dots, < \theta^*, X_{200} >$*,Then simulate the response variables* $T_i = r(< \theta^*, X_i >) + \epsilon$*, where* $r(< \theta^*, X_i >) = \exp(10(< \theta^*, X_i > -0.05))$ *and* ε *generate independently from a centred gaussian of variance equal to 0.075.*

We simulate n i.i.d rv $C_i, i = 1, \dots, n$ *with the exponential distribution* $\mathcal{E}(1,5)$*.*

Noting that the computation of those estimators are based on the observed data $(X_i, Y_i, \delta_i)_{i=1,\dots,n}$*, where* $Y_i = min(T_i, C_i)$ *and* $\delta_i = \mathbf{1}_{\{T_i \le C_i\}}$*. On the other hand, take into account of the smoothness of the curves* $X_i(t)$*, we choose the semi-metric in* $\mathcal{H}$ *:*

$$d(x_i, x_j) = \sqrt{\int_0^1 \left(x_i'(t) - x_j'(t)\right)^2 dt}, \qquad x_i, x_j \in \mathcal{H}$$

Then, we choose the quadratic kernels defined by :

$$K(u) = \frac{3}{2}(1 - u^2)\mathbf{1}_{(0,1)}(u)$$

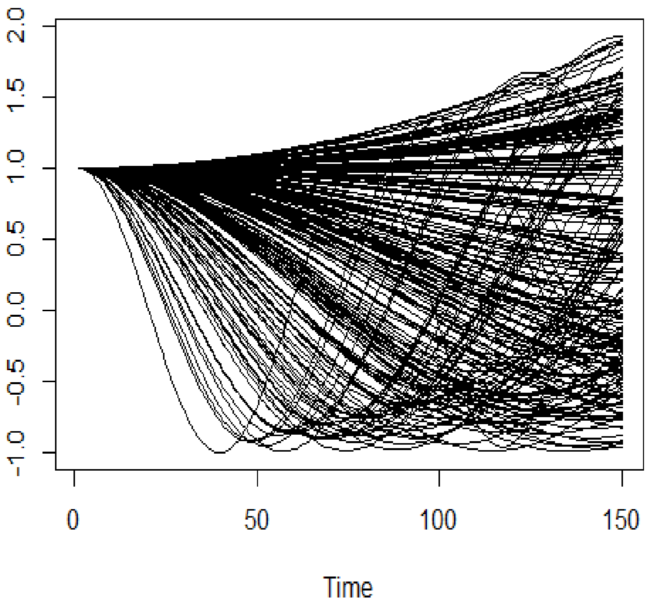

FIGURE 4.4 – The curves $X_{i=1,...,200}(t), \quad t \in [0,1]$.

and the distribution function $H(\cdot)$

$$H(t) = \int_{-\infty}^{t} \frac{3}{4}(1-t^2)\mathbf{1}_{(-1,1)}(t)dt$$

For the bandwidths $h_H \sim h_K =: h$ *is automatically selected by the procedure of the cross-validation method on the K-nearest neighbours (2006).*

In our simulation, sample sizes are n=200, we take it into two parts, one is a learning sample of 150 observations and the others 50 observations are a test sample.

Then using the learning sample to compute the estimator of $\widehat{Y}_i = \widehat{\zeta}_{\theta^*}(\gamma, x)$ *and* $\widehat{Y}_{ni} = \widehat{\zeta}(\gamma, x)$ *for j={151,...,200}.*

Finally we show the results by plotting the true values versus the predicted values for the MSE under censored data for both estimators with different censored rate (CR) 4.6 and 4.21 which are defined as :

$$FSIM.MSE = \frac{1}{50}\sum_{i=151}^{200}(Y_i - \widehat{Y}_i)^2$$

$$NPFDA.MSE = \frac{1}{50}\sum_{i=151}^{200}(Y_i - \widehat{Y}_{ni})^2$$

respectively.

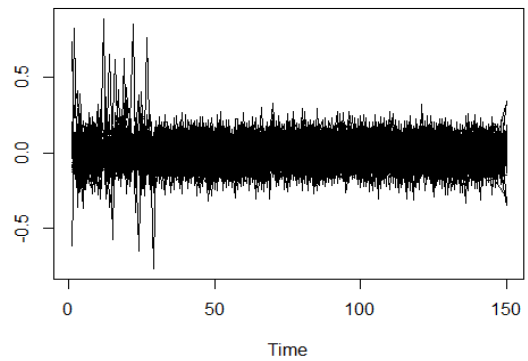

FIGURE 4.5 – The curves $\theta_{i=1,\ldots,150}(t), \quad t \in [0,1]$.

4.6 Remarks on the single index

It is well known that one of the main advantages of the single index model is its ability to deal with the problem of high dimensional data. A straightforward example is the optimal convergence rate of type $\mathcal{O}(n^{-2k/(2k+p)})$ for the estimation of a k-times differentiable regression function, this rate goes to zero dramatically slowly if k is small compared to the dimension p of the explanatory variable ($X \in \mathbb{R}^p$). In this regard, Gaïffas and Lecué (2007) showed that the optimal rate of convergence of regression function in the single index model, is of order $\mathcal{O}(n^{-2k/(2k+p)})$ (rather than $\mathcal{O}(n^{-2k/(2k+p)})$, thereby answering a conjecture of Stone (1982).

The same idea was adopted in the abstract metric spaces by the choice of the semi-metric increasing the probability measure concentration of the explanatory variable in small balls (see Ferraty and Vieu (2006)), Section 13.2). Among these family of semi-metric, one can consider the semi-metric induced by the functional single index estimate.

In the classical nonparametric multivariate regression context the first problem appearing is the so-called curse of dimensionality caused by the sparsity of data in high dimensional spaces.

One possibility to circumvent this problem is to impose additional assumptions on the regression function, in order to reduce the effect of the dimensionality of the regressor variables.

4.6.1 About the Functional Single Index Estimate

Besides using the pseudo-maximum likelihood method proposed by Attaouti *et al.* (2011) to estimate θ, the cross-validation method presented by Aït-Saïdi *et al.* (2008)) for the estimation of θ in the single functional index regression model would be the first natural candidate method for choosing θ in this case.

The second candidate way to estimate θ may be wild bootstrap method, which was shown by Ferraty *et al.* (2010) to estimate the functional nonparametric regression operator. Thus, the theoretical justification and practice have been established.

Among the interest of our study is to show how the conditional density estimate can be used to derive an estimate of the functional single index if the latter is unknown. The leave-out-one-curve cross-validation procedure was adapted by Aït Saidi *et al.* (2008) to estimate the single index. Newly, Hall and Müller (2009) proposed a method for estimating functional derivatives and Ferraty *et al.* (2011) adopted this technique to estimate the parameter θ.

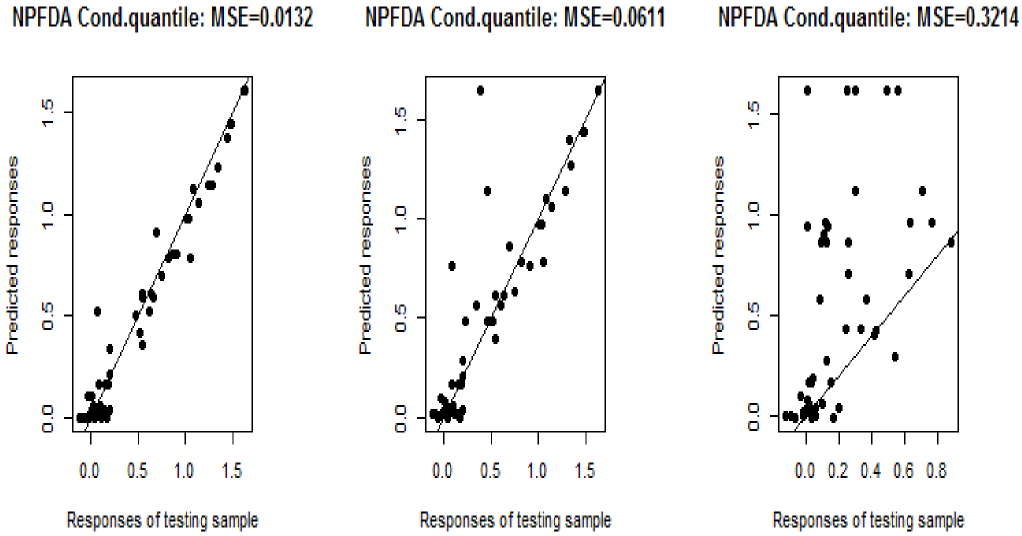

FIGURE 4.6 – CR ∼ 3.5%, MSE=0.0132 on the left, CR ∼ 18%, MSE=0.0611 on the center, and CR ∼ 48%, MSE=0.3214 on the right

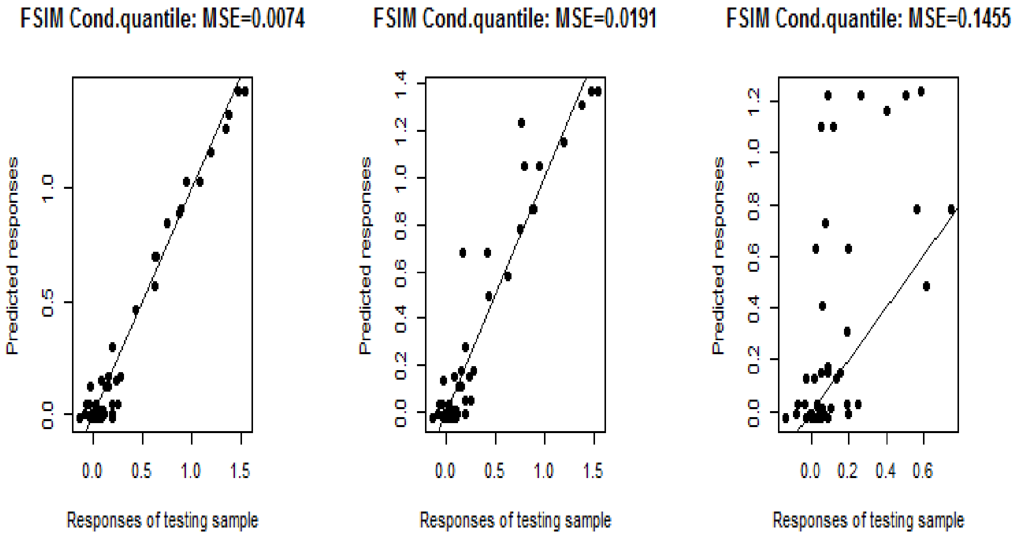

FIGURE 4.7 – CR ∼ 3.5%, MSE=0.0074 on the left, CR ∼ 18%, MSE=0.0191 on the center, and CR ∼ 48%, MSE=0.1455 on the right

Alternatively, this parameter can be estimated via the pseudo-maximum likelihood method which is based on the preliminary estimation of the conditional density of Y given X by

$$\widehat{\theta} = \arg\max_{\theta \in \Theta_{\mathcal{F}}} \widehat{L}(\theta)$$

where

$$\widehat{L}(\theta) = \sum_{k=1}^{n} \log \widehat{f}(\theta, Y_k, X_k).$$

Note that, this method has been studied by Delecroix *et al.* (2003) in the real case where they showed that this technique has minimal variance among all estimators. The asymptotic optimality of this procedure in functional statistic, is an important prospect of the present work.

As an application, this approach can be used for answering the semi-metric choice question. Indeed, it is well known that, in nonparametric functional statistic, the projection-type semi-metric is very important for increasing the concentration property. The functional index model is a particular case of this family of semi-metric, because it is based on the projection on one functional direction. So, the estimation procedures of this direction permit us to compute adaptive semi-metrics in the general context of nonparametric functional data analysis. Finally, the theoretical justification and practice should be established.

4.6.2 Concluding Remarks

Single-index models and varying coefficient models are powerful tools for dimension reduction and semiparametric modeling because they can effectively avoid "curse of dimensionality". Both models largely relax some restrictive assumptions on linear and nonparametric models. Specially, the single-index model is usually treated as the first step of the famous projection pursuit regression. The single-index model has also been extensively used in the projection pursuit regression,

For example, firsly the motivation of this model also comes from an analysis of environmental data, consisting of daily measurement of pollutants and other environment factors. Of interest is to examine the association between the levels of pollutants, two environment factors (temperature and relative humidity) and the total number of daily hospital admissions for respiratory problems.

Secondly, single-index models cannot reflect the additivity of covariates, while the performance of varying coefficient models can be poor if the varying coefficient contains moderate or high dimensional covariates.

In standard multivariate regression, the single-index model is a good compromise between nonparametric and parametric regression models when one wishes to regress a response variable on several real-valued explanatory ones. It assumes the existence of a latent one-dimensional explanatory variable which allows us to explain the response through a nonparametric regression model.

In addition, the latent explanatory variable is supposed to be a linear combination of the explanatory variables. The vector of the linear combination is called "single index". Such models are useful tools for interpreting some situations. This is particularly the case in econometrics where single-index models and various extensions have been intensively studied.

Single-index models, or projection pursuit regression, have proven to be an efficient way of coping with the high dimensional problem in nonparametric regressions. The idea is restricting the general multivariate regression function to a special form.

For the functional single-index models, the literature is closely limited, and only a few theoretical results have been obtained until now.

For the past two decades, the single-index model, a special case of projection pursuit regression, has proven to be an efficient way of coping with the high dimensional problem in nonparametric regression. Here we deal with single index modeling when the explanatory variable is functional. More precisely, we consider the problem of estimating the conditional density of a real variable Y given a functional variable X when the explanation of Y given X

is done through its projection on one functional direction.

The single-index approach is widely applied in econometrics as a reasonable compromise between nonparametric and parametric models. Such kind of modelization is intensively studied in the multivariate case. Without pretend to exhaustivity.

In practice, this study has great importance, because, it permit us to construct a prediction method based on the conditional mode estimator. Moreover, in the case where the functional single index is unknown, our estimate can be used to estimate this parameter via the pseudo-maximum likelihood estimation method. Noting that the estimation of the functional single-index has great interest on the semi-metric choice in nonparametric functional data analysis but it has been not attacked in this paper.

4.7 Proofs of technical lemmas

In order to highlight the main contribution of our paper (i.e. α-mixing and functional variables) some details are voluntarily omitted.

Proof 4.6 (Proof of Lemma 4.1) *Let*

$$\begin{aligned}|\widehat{F}_N^{(l)}(\theta,t,x)-\widetilde{F}_N^{(l)}(\theta,t,x)| &\leq \frac{h_H^{-l}}{n\mathbb{E}K_1(\theta,x)}\sum_{i=1}^n\left|\frac{\delta_i}{\bar{G}_n(Y_i)}K_i(\theta,x)H_i^{(l)}(t)-\frac{\delta_i}{\bar{G}(Y_i)}K_i(\theta,x)H_i^{(l)}(t)\right| \\ &\leq \frac{h_H^{-l}}{n\mathbb{E}K_1(\theta,x)}\sum_{i=1}^n|\delta_iK_i(\theta,x)H_i(t)|\left|\frac{1}{\bar{G}_n(Y_i)}-\frac{1}{\bar{G}(Y_i)}\right| \\ &\leq \frac{h_H^{-l}}{\phi_{\theta,x}(h_K)}\frac{C}{\bar{G}_n(\tau_F)\bar{G}(\tau_F)}\sup_{t\in\mathcal{S}_\mathbb{R}}|\bar{G}_n(t)-\bar{G}(t)|\frac{1}{n}\sum_{i=1}^n\delta_i\end{aligned}$$

In conjunction with the SLLN and the LIL on the censoring law (see Deheuvels and Einmahl (2000)), hypotheses (H2), (H7) and $\left(\frac{\log\log n}{n}\right)^{1/2}=o\left(\phi_{\theta,x}(h_K)\right)$ *complete the proof.*

Proof 4.7 (Proof of Lemma 4.2) *The asymptotic behavior of bias term is standard, in the sense that it is not affected by the dependence structure of the data. We have*

$$\begin{aligned}\mathbb{E}\widetilde{F}_N(\theta,t,x)-F(\theta,t,x) &= \frac{1}{\mathbb{E}K_i(x,\theta)}\mathbb{E}\left(\frac{\delta_i}{\bar{G}(Y_i)}K_i(x,\theta)H_i(t)\right)-F(\theta,t,x) \\ &= \frac{1}{\mathbb{E}K_1(x,\theta)}\mathbb{E}\left(\frac{\delta_i}{\bar{G}(Y_i)}K_i(x,\theta)\left[\mathbb{E}\left(H_i(t)|<X_1,\theta>\right)\right]\right)-F(\theta,t,x)\end{aligned} \quad (4.24)$$

integrating by parts and using the fact that H is a cdf *and the use a double conditioning with* *respect to T_1, we can easily get*

$$\begin{aligned}
I &= \mathbb{E}\left(\frac{\delta_i}{\bar{G}(Y_i)}H_i(t)| < X_1,\theta >\right) \\
&= \mathbb{E}\left(\mathbb{E}\left[\frac{\mathbf{1}_{T_1\leq C_1}}{\bar{G}(T_1)}H\left(\frac{t-T_1}{h_H}\right)| < X_1,\theta >, T_1\right]\right) \\
&= \mathbb{E}\left(\frac{1}{\bar{G}(T_1)}H\left(\frac{t-T_1}{h_H}\right)\mathbb{E}\left[\mathbf{1}_{T_1\leq C_1}|T_1\right]| < X_1,\theta >\right) \\
&= \mathbb{E}\left[H\left(\frac{t-T_1}{h_H}\right)| < X_1,\theta >\right] \\
&= \int_{\mathbb{R}} H\left(\frac{t-u}{h_H}\right) f(\theta,u,X_1)du, \\
&= \int_{\mathbb{R}} H\left(\frac{t-u}{h_H}\right) dF(\theta,u,X_1), \\
&= \int_{\mathbb{R}} H^{(1)}\left(\frac{t-u}{h_H}\right) F(\theta,u,X_1)du, \\
&= \int_{\mathbb{R}} H^{(1)}(v)F(\theta,t-vh_H,X_1)dv, \\
&= \int_{\mathbb{R}} H^{(1)}(v)\left(F(\theta,t-vh_H,X_1)-F(\theta,t,x)\right)dv + F(\theta,t,x)\int_{\mathbb{R}} H^{(1)}(v)dv,
\end{aligned}$$

we can write, because of (H3) and (H6)-(i) :

$$\begin{aligned}
I &\leq C_{x,\theta}\int_{\mathbb{R}} H^{(1)}(v)\left(h_K^{b_1}+|v|^{b_2}h_H^{b_2}\right)dv + F(\theta,t,x) \\
&= \mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right)+F(\theta,t,x).
\end{aligned}$$

Combining this last result with (4.24) allows us to achieve the proof.

Proof 4.8 (Proof of Lemma 4.3) *Using the compactness of $\mathcal{S}_{\mathbb{R}}$, we can write that $\mathcal{S}_{\mathbb{R}} \subset \bigcup_{k=1}^{\tau_n}(z_k-l_n, z_k+l_n)$ with l_n and τ_n can be chosen such that $l_n = C\tau_n^{-1} \sim Cn^{-\varsigma-1/2}$. Taking $k_t = \arg\min_{\{z_1,\ldots,z_{\tau_n}\}} |t-z_k|$.*

Thus, we have the following decomposition :

$$\begin{aligned}
\frac{1}{\widehat{F}_D(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\widetilde{F}_N(\theta,t,x)-\mathbb{E}\widetilde{F}_N(\theta,t,x)\right| &\leq \frac{1}{\widehat{F}_D(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\widetilde{F}_N(\theta,t,x)-\widetilde{F}_N(\theta,t_k,x)\right| \\
&\quad+\frac{1}{\widehat{F}_D(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\widetilde{F}_N(\theta,t_k,x)-\mathbb{E}\widetilde{F}_N(\theta,t_k,x)\right| \\
&\quad+\frac{1}{\widehat{F}_D(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\mathbb{E}\widetilde{F}_N(\theta,t_k,x)-\mathbb{E}\widetilde{F}_N(\theta,t,x)\right| \\
&\leq B_1+B_2+B_3
\end{aligned}$$

On the one hand, as the first and the third terms can be treated in the same manner, we *deal only with first term. Making use of (H6)-(i) we get*

$$\begin{aligned}
\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\widetilde{F}_N(\theta,t,x)-\widetilde{F}_N(\theta,t_k,x)\right| &\leq \frac{1}{n\mathbb{E}K_1(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\sum_{i=1}^{n}\left|\frac{\delta_i}{\bar{G}(Y_i)}H_i(t)-\frac{\delta_i}{\bar{G}(Y_i)}H_i(t_k)\right||K_i(\theta,x)| \\
&\leq \frac{1}{n\mathbb{E}K_1(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\sum_{i=1}^{n}\left|\frac{\delta_i}{\bar{G}(Y_i)}H_i(t)-\frac{\delta_i}{\bar{G}_n(Y_i)}H_i(t_k)\right||K_i(\theta,x)| \\
&\leq \frac{C}{n\mathbb{E}K_1(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\frac{|t-t_k|}{h_H}\times\left(\sum_{i=1}^{n}K_i(\theta,x)\left(\frac{1}{\bar{G}(Y_i)}-\frac{1}{\bar{G}_n(Y_i)}\right)\right) \\
&\leq \frac{Cl_n}{h_H\overline{G}_n(a_F)\bar{G}(a_F)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}|G_n(t)-G(t)|\widehat{F}_D(\theta,x)
\end{aligned}$$

Using $l_n=n^{-\varsigma-1/2}$ we obtain

$$A_1 \leq \frac{Cn^{-\varsigma-1/2}}{h_H\bar{G}_n(a_F)\bar{G}(a_F)}\left(\frac{\log n\log n}{n}\right)^{1/2}$$

and note that, because of (H6)-(i), we have

$$\frac{l_n}{h_H}=o\left(\sqrt{\frac{\log n}{n\phi_{\theta,x}(h_K)}}\right).$$

Thus, for n large enough, we have

$$B_1 = \mathcal{O}_{a.co}\left(\sqrt{\frac{\log n}{n\phi_{\theta,x}(h_K)}}\right).$$

Following similar arguments, we can write

$$B_3 \leq B_1.$$

Concerning B_2, let us consider $\varepsilon=\epsilon_0\sqrt{\frac{{s_{n,2}}^2\log n}{n^2}}$. Since for all $\epsilon_0>0$, we have that

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\mathbb{E}\widehat{F}_N(\theta,t_k,x)-\mathbb{E}\widehat{F}_N(\theta,t,x)\right|>\varepsilon)\right) &\leq \mathbb{P}\left(\max_{k\in\{1\ldots\tau_n\}}\left|\mathbb{E}\widehat{F}_N(\theta,t_k,x)-\mathbb{E}\widehat{F}_N(\theta,t,x)\right|>\varepsilon\right) \\
&\leq \tau_n\mathbb{P}\left(\left|\mathbb{E}\widehat{F}_N(\theta,t_k,x)-\mathbb{E}\widehat{F}_N(\theta,t,x)\right|>\varepsilon\right).
\end{aligned}$$

The application of Fuk-Nagaev's inequality (see Proposition A.11-ii of Ferraty and Vieu (2006)) with $r=(\log n)^2$ and $q=a+1$, we get that

$$\begin{aligned}
\mathbb{P}\left(\left|\mathbb{E}\widehat{F}_N(\theta,t_k,x)-\mathbb{E}\widehat{F}_N(\theta,t,x)\right|>\varepsilon\right) &\leq \left(1+\frac{\epsilon_0^2}{(\log n)^2}\right)^{-(\log n)^2/2}+n(\log n)^{-2}\left(\frac{\sqrt{\log n}}{\epsilon_0 s_{n,2}}\right)^{a+1} \\
&\leq C_{\theta,x}F_1+C'_{\theta,x}F_2.
\end{aligned}$$

Finally, the use that $s_{n,2}^2=\mathcal{O}(n\phi_{\theta,x}(h_K))$, allows to get directly that there exist some $\eta>0$

such that

$$F_1 + F_2 \ \leq \ Cn^{-1-\eta}.$$

Finally, we arrive at

$$\sup_{t\in\mathcal{S}_\mathbb{R}} \left|\widetilde{F}_N(\theta,t,x) - \mathbb{E}\widetilde{F}_N(\theta,t,x)\right| = \mathcal{O}_{a.co.}\left(\frac{\sqrt{s_{n,2}^2 \log n}}{n}\right)$$

Proof 4.9 (Proof of Proposition 4.2) *It is omitted as it very similar to that of Proposition 4.1, we use again the same kind of decomposition as (4.12) :*

$$\begin{aligned}\sup_{t\in\mathcal{S}_\mathbb{R}} \left|\widehat{F}^{(l)}(\theta,t,x) - F^{(l)}(\theta,t,x)\right| \ &= \ \frac{1}{\widehat{F}_D(\theta,x)} \sup_{t\in\mathcal{S}_\mathbb{R}} \left\{\left|\widehat{F}_N^{(l)}(\theta,t,x) - \widetilde{F}_N^{(l)}(\theta,t,x)\right|\right\} \\ &+ \frac{1}{\widehat{F}_D(\theta,x)} \sup_{t\in\mathcal{S}_\mathbb{R}} \left\{\left|\widetilde{F}_N^{(l)}(\theta,t,x) - \mathbb{E}\widetilde{F}_N^{(l)}(\theta,t,x)\right|\right\} \\ &+ \frac{1}{\widehat{F}_D(\theta,x)} \sup_{t\in\mathcal{S}_\mathbb{R}} \left\{\left|\mathbb{E}\widetilde{F}_N^{(l)}(\theta,t,x) - F^{(l)}(\theta,t,x)\right|\right\} \\ &+ \frac{F^{(l)}(\theta,t,x)}{\widehat{F}_D(\theta,x)} \sup_{t\in\mathcal{S}_\mathbb{R}} \left|1 - \widehat{F}_D(\theta,x)\right|. \qquad (4.25)\end{aligned}$$

This proof is very similar to the one of Proposition 4.1. Indeed the proof of the Proposition 4.2 by making use the decomposition (4.25), in conjunction of first part of Lemma 4.3 with lemmas Lemma 4.1, Lemma 4.4 and Lemma 4.5.

Proof 4.10 (Proof of Lemma 4.4) *Using the same arguments in the proof of Lemma 4.2, replacing $F(\theta,t,x)$ (resp. $\widetilde{F}(\theta,t,x)$) with $F^{(l)}(\theta,t,x)$ (resp. $\widetilde{F}_N^{(l)}(\theta,t,x)$) and considering hypotheses (H5) and (H6) we get :*

$$F^{(l)}(\theta,t,x) - \mathbb{E}\widehat{F}_N^{(l)}(\theta,t,x) \ = \ \mathcal{O}\left(h_K^{b_1} + h_K^{b_2}\right). \qquad (4.26)$$

Proof 4.11 (Proof of Lemma 4.5) *To get the asymptotic behaviour of the quantity $\widetilde{F}_N^{(l)}(\theta,t,x) - \mathbb{E}\widetilde{F}_N^{(l)}(\theta,t,x)$, we comeback to the proof of Lemma 4.3, and we replace $F(\theta,t,x)$ (resp. $\widetilde{F}(\theta,t,x)$) with $F^{(l)}(\theta,t,x)$ (resp. $\widetilde{F}_N^{(l)}(\theta,t,x)$).*

Note that (H6)-(ii) and (H12) permit to show that

$$\mathbb{E}\left(\frac{\delta_i}{\bar{G}(Y_i)} H^{(l)}\left(h_H^{-1}(t-Y_i)\right) \frac{\delta_j}{\bar{G}(Y_j)} H^{(l)}\left(h_H^{-1}(t-Y_j)\right) |(X_i,X_j)\right) = \mathcal{O}(h_H^2),$$

while (H1) and (H5) imply that

$$\mathbb{E}\left(\frac{\delta_i}{\bar{G}(Y_i)} H^{(l)}\left(h_H^{-1}(t_\theta(\gamma)-Y_i)\right) |X_i\right) = \mathcal{O}(h_H).$$

Consequently, we have by using successively (H0), (H2), (H8) and (H10)-(i)

$$Cov\left(\Xi_i(\theta,t,x), \Xi_j(\theta,t,x)\right) = \mathcal{O}\left(h_H^2 \left(\frac{\phi_{\theta,x}(h_K)}{n}\right)^{1/a}\right),$$

where

$$\Xi_i = h_H^{-l} \frac{\delta_i}{\bar{G}(Y_i)} H_i^{(l)}(t)\Delta_i(x,\theta) - \mathbb{E} h_H^{-l} \frac{\delta_i}{\bar{G}(Y_i)} H_i^{(l)}(t)\Delta_i(x,\theta)$$

has zero mean and satisfies $|\Xi_i(\theta,t,x)| \leq C h_H^{-l} \phi_{\theta,x}^{-1}(h_K)$. *Now, we show that because* $H^{(l)}$ *is bounded. Indeed, it can be found that*

$$\widetilde{F}_N^{(l)}(\theta,t,x) - \mathbb{E}\widetilde{F}_N^{(l)}(\theta,t,x) = \frac{1}{n}\sum_{i=1}^{n} \Xi_i(\theta,t,x),$$

then it allows us to use directly similar arguments of second part by Lemma 4.3, we obtain

$$\widetilde{F}_N^{(l)}(\theta,t,x) - \mathbb{E}\widetilde{F}_N^{(l)}(\theta,t,x) = \mathcal{O}_{a.co.}\left(\frac{\sqrt{{s_{n,2}}^2 \log n}}{n}\right),$$

which leads directly to the result of Lemma 4.5

Proof 4.12 (Proof of Lemma 4.8) *The proof can be completed following the same steps as Lemma 4.4.*

Proof 4.13 (Proof of Lemma 4.9) *Similar to the proof of Lemma 3.4 in Ling* et al. *(2014), it can be completed easily. Here we omit its proof.*

Proof 4.14 (Proof of Lemma 4.10) *For all* $x \in \mathcal{S}_{\mathcal{H}}$ *and* $\forall \theta \in \Theta_{\mathcal{H}}$, *we set* $k(x) = \arg\min_{k\in\{1\ldots d_n^{\mathcal{S}_{\mathcal{H}}}\}} \|x - x_k\|$ *and* $q(\theta) = \arg\min_{m\in\{1\ldots d_n^{\Theta_{\mathcal{H}}}\}} \|\theta - \theta_q\|$ *and by the compact property of* $\mathcal{S}_{\mathbb{R}} \subset \mathbb{R}$, *we have* $\mathcal{S}_{\mathbb{R}} \subset \bigcup_{k=1}^{\tau_n} (v_k - l_n, v_k + l_n)$ *with* l_n *and* τ_n *can be selected such as* $l_n = \mathcal{O}\left(\tau_n^{-1}\right) = \mathcal{O}\left(n^{-(3\varsigma+1)/2}\right)$. *Taking* $k_t = \arg\min_{\{v_1,\ldots,v_{\tau_n}\}} |t - v_k|$.

Let us consider the following decomposition

$$\begin{aligned}
\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\widetilde{F}_N^{(l)}(\theta,t,x) - \mathbb{E}\left(\widetilde{F}_N^{(l)}(\theta,t,x)\right)\right| \leq\ & \sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left\{\left|\widetilde{F}_N^{(l)}(\theta,t,x) - \widetilde{F}_N^{(l)}(\theta,t,x_{k(x)})\right|\right.\\
& + \left|\widetilde{F}_N^{(l)}(\theta,t,x) - \widetilde{F}_N^{(l)}(\theta,t,x_{k(x)})\right|\\
& + \left|\widetilde{F}_N^{(l)}(\theta_{q(\theta)},t,x_{k(x)}) - \widetilde{F}_N^{(l)}(\theta_{q(\theta)},v_{k_t},x_{k(x)})\right|\\
& + \left|\widetilde{F}_N^{(l)}(\theta_{q(\theta)},v_{k_t},x_{k(x)}) - \mathbb{E}\left(\widetilde{F}_N^{(l)}(\theta_{q(\theta)},v_{k_t},x_{k(x)})\right)\right|\\
& + \left|\mathbb{E}\left(\widetilde{F}_N^{(l)}(\theta_{q(\theta)},v_{k_t},x_{k(x)})\right) - \mathbb{E}\left(\widetilde{F}_N^{(l)}(\theta_{q(\theta)},t,x_{k(x)})\right)\right|\\
& + \left|\mathbb{E}\left(\widetilde{F}_N^{(l)}(\theta_{q(\theta)},t,x_{k(x)})\right) - \mathbb{E}\left(\widetilde{F}_N^{(l)}(\theta,t,x_{k(x)})\right)\right|\\
& \left. + \left|\mathbb{E}\left(\widetilde{F}_N^{(l)}(\theta,t,x_{k(x)})\right) - \mathbb{E}\left(\widetilde{F}_N^{(l)}(\theta,t,x)\right)\right|\right\}\\
\leq\ & \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 + \Psi_5 + \Psi_6 + \Psi_7.
\end{aligned}$$

- *Concerning Ψ_3 and Ψ_5; by conditions (H6)-(ii) and (A6), boundness of K, we obtain*

$$\begin{aligned}
\left|\widetilde{F}_N^{(l)}(\theta_{q(\theta)}, t, x_{k(x)}) - \widetilde{F}_N^{(l)}(\theta_{q(\theta)}, \upsilon_{k_t}, x_{k(x)})\right| &\leq \frac{1}{nh_H^l \mathbb{E}K_1(\theta, x)} \sup_{t\in\mathcal{S}_\mathbb{R}} \sum_{i=1}^n \left|\frac{\delta_i}{\bar{G}(Y_i)} K_i(\theta_{q(\theta)}, x_{k(x)})\right| \\
&\quad \left|H^{(l)}\left(\frac{t-Y_i}{h_H}\right) H^{(l)}\left(\frac{\upsilon_{k_t}-Y_i}{h_H}\right)\right| \\
&\leq \sup_{t\in\mathcal{S}_\mathbb{R}} C\frac{|t-\upsilon_{k_t}|}{h_H^{l+1}} \left(\frac{1}{n\mathbb{E}(K_1(\theta_{q(\theta)}, x_{k(x)}))}\right. \\
&\quad \left.\sum_{i=1}^n \left|K_i(\theta_{q(\theta)}, x_{k(x)})\frac{1}{\bar{G}(Y_i)}\right|\right) \\
&\leq \frac{Cl_n}{\phi(h_K)h_H^{l+1}} = \mathcal{O}\left(\frac{l_n}{h_H^{l+1}\phi(h_K)}\right).
\end{aligned}$$

Now, the fact that $\lim_{n\to\infty} n^\nu h_H = \infty$, and choosing $l_n = n^{-(3\nu+1)/2}$ and using the second part of (A6), imply that

$$\frac{l_n}{h_H^{l+1}\phi(h_K)} = o\left(\sqrt{\frac{logn}{nh_H^{2l-1}\phi(h_K)}}\right)$$

as $n\to\infty$, therefore, it follows

$$\Psi_5 \leq \Psi_3 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}}}{nh_H^{2l-1}\phi(h_K)}}\right).$$

Concerning Ψ_4, let us consider $\varepsilon = \varepsilon_0\sqrt{\frac{s_{n,8}^2 \log d_n^{\mathcal{S}_\mathcal{F}} d_n^{\Theta_\mathcal{F}}}{n^2}}$. Since

$$\begin{aligned}
\mathbb{P}\left(\Psi_4 > \varepsilon_0\sqrt{\frac{s_{n,6}^2 \log d_n^{\mathcal{S}_\mathcal{F}} d_n^{\Theta_\mathcal{F}}}{n^2}}\right) &= \mathbb{P}\left(\max_{q\in\{1\ldots d_n^{\Theta_\mathcal{H}}\}} \max_{k\in\{1\ldots d_n^{\mathcal{S}_\mathcal{H}}\}} \max_{k_t\in\{1,2,\ldots,\tau_n\}} \left|\Gamma_i^{(l)} - \mathbb{E}\Gamma_i^{(l)}\right| > \varepsilon\right) \\
&\leq \tau_n d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}} \mathbb{P}\left(\left|\Gamma_i^{(l)} - \mathbb{E}\Gamma_i^{(l)}\right| > \varepsilon\right) \qquad (4.27)
\end{aligned}$$

the application of Fuk-Nagaev's inequality (Proposition A.11-(ii), see Ferraty and Vieu (2006)) with $r = (\log n)^2 > 1$ and $q = a+1$, we get fu

$$\begin{aligned}
\mathbb{P}\left(\left|\sum_{i=1}^n (\Gamma_i^{(l)} - \mathbb{E}\Gamma_i^{(l)})\right| > \varepsilon\right) &\leq C\left(1 + \frac{\varepsilon_0^2(\log d_n^{\mathcal{S}_\mathcal{F}} d_n^{\Theta_\mathcal{F}})}{(\log n)^2}\right)^{-(\log n)^2/2} \\
&+ \frac{n(\log n)^{2a}\varepsilon_0^{-(a+1)}}{(\log d_n^{\mathcal{S}_\mathcal{F}} d_n^{\Theta_\mathcal{F}})^{(a+1)/2}} \frac{1}{s_{n,8}^{a+1}} \\
&\leq C(J_1 + J_2). \qquad (4.28)
\end{aligned}$$

By assumptions (A6), it has $\frac{\log d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}}}{(\log n)^2} \longrightarrow \infty$ as $n\to\infty$, which leads to

$$J_1 \leq \left(d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}}\right)^{-\lambda} \qquad (4.29)$$

for some $\lambda > 1$ and $\varepsilon_0 > 0$ such that $\varepsilon_0^2 = 2\lambda$.

On the other hand,

$$J_2 \leq Cn\,(\log n)^{2a}\left(d_n^{\mathcal{S}_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}\right)^{-\frac{a+1}{2}} n^{-\beta} \leq C' n^{1+\eta-\beta} \tag{4.30}$$

where $\eta > 0$ *such that* $\beta > \beta - \eta > 2$. *Meanwhile, by the selection of* λ *and* β, *we can find that*

$$\left(d_n^{\mathcal{S}_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}\right)^{\lambda} = \mathcal{O}\left(n^{\beta-\eta-1}\right) \tag{4.31}$$

Combining (4.27)-(4.31) with assumption (A6)-(ii)

$$\Psi_4 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{s_{n,8}^2 \log d_n^{\mathcal{S}_{\mathcal{F}}}}{n^2}}\right). \tag{4.32}$$

- *Concerning* Ψ_1 *and* Ψ_2

$$\begin{aligned}
\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{t\in\mathcal{S}_{\mathbb{R}}} |\widetilde{F}_N^{(l)}(\theta,t,x) - \widetilde{F}_N^{(l)}(\theta,t,x_{k(x)})| &\leq \frac{1}{nh_H^l \mathbb{E}K_1(\theta,x)} \sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{t\in\mathcal{S}_{\mathbb{R}}} \sum_{i=1}^{n} \left|\frac{\delta_i}{\bar{G}(Y_i)}\right| \left|H_i^{(l)(t)}\right| \\
&\qquad |(K_i(\theta,x) - K_i(\theta,x_k))| \\
&\leq \frac{1}{nh_H^l \phi(h_K)} \sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}} \sum_{i=1}^{n} \left|\Delta_i(x,\theta) - \Delta_i\left(x_{k(x)},\theta\right)\right| \\
&\leq \frac{1}{h_H^l \phi(h_K)} \sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}} \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{B_\theta(x,h)\cup B_\theta(x_{k(x)},h)}(X_i) \\
&\leq \frac{C}{h_H^l} \sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}} \frac{1}{n}\sum_{i=1}^{n} \Lambda_i(x,\theta)
\end{aligned}$$

Therefore, similar to the arguments for (4.32), we can get that

$$\Psi_1 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{s_{n,4}^2 \log d_n^{\mathcal{S}_{\mathcal{F}}}}{n^2}}\right).$$

$$\begin{aligned}
\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{t\in\mathcal{S}_{\mathbb{R}}} |\widetilde{F}_N^{(l)}(\theta,t,x) - \widetilde{F}_N^{(l)}(\theta_{q(\theta)},t,x_{k(x)})| &\leq \frac{h_H^{-l}}{n\mathbb{E}K_1(\theta,x)} \sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{t\in\mathcal{S}_{\mathbb{R}}} \sum_{i=1}^{n} \left|\frac{\delta_i}{\bar{G}(Y_i)}\right| \left|H_i^{(l)(t)}\right| \\
&\qquad \left|(K_i(\theta,x_k) - K_i(\theta_{q(\theta)},x_k))\right| \\
&\leq \frac{Ch_H^{-l}}{n\phi(h_K)} \sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}} \sum_{i=1}^{n} \left|\Delta_i(\theta,x_k) - \Delta_i\left(\theta_{q(\theta)},x_{k(x)}\right)\right| \\
&\leq \frac{Ch_H^{-l}}{\phi(h_K)} \sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}} \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{B_\theta(x_k,h)\cup B_{q(\theta)}(x_{k(x)},h)}(X_i) \\
&\leq \frac{C}{h_H^l} \sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}} \frac{1}{n}\sum_{i=1}^{n} \Omega_i(x,\theta)
\end{aligned}$$

Similar to the deduce of (4.32), it yields

$$\Psi_2 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{s_{n,5}^2 \log d_n^{\mathcal{S}_\mathcal{F}}}{n^2}}\right).$$

On the other hand, since $\Psi_7 \leq \Psi_1$ and $\Psi_6 \leq \Psi_2$, it also leads to

$$\Psi_6 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{s_{n,4}^2 \log d_n^{\mathcal{S}_\mathcal{F}}}{n^2}}\right).$$

and

$$\Psi_7 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{s_{n,5}^2 \log d_n^{\mathcal{S}_\mathcal{F}}}{n^2}}\right).$$

Then the proof of Lemma 4.10 can be completed.

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Chapitre Cinq

Conclusion et Perspectives

Conclusion

Dans notre travail, nous avons illustré une méthodologie dans le cadre de la modélisation du quantile conditionnelle par une approche à indice fonctionnel simple, tout en tenant compte des éventuels changements structurels.

Dans la première partie, nous avons établis la convergence ponctuelle presque complète et la convergence uniforme presque complète de l'estimateur de la fonction de répartition conditionnelle sous un modèle d'indice fonctionnel simple, en précisant ses vitesses de convergences. Deux cas ont été étudiés, le premier lorsque les variables sont indépendantes et identiquement distribuées tandis que dans le deuxième cas nous avons traité le cas d'α-mélange.

Dans la deuxième partie de cette thèse, nous avons présenté un modèle à indice fonctionnel simple qui est adapté à des données censurées. Ensuite, nous avons prouvé sa convergence uniforme presque complète avec un taux de convergence. Nous avons illustré les résultats obtenues par des modèles de simulation, où nous avons montré la performance de notre modèle.

En conclusion, le modèle à indice fonctionnel simple représente une alternative valable au purement non paramétrique. Par rapport à cela, certains avantages peuvent être mis en évidence, le problème de la dimensionnalité est évité et la tâche de choisir une bonne semi norme est ignorée.

Perspectives

Pour conclure les travaux de cette thèse, nous exposons dans ce qui suit, quelques développements futurs possibles en vue d'améliorer et d'étendre nos résultats.

- La première perspective à traiter à cours terme est la question liée à la normalité asymptotique de nos estimateurs dans les deux cas i.i.d et α-mélangeant. Notons que cette propriété asymptotique permettra d'optimiser les intervalles de confiance et de faire des tests statistiques.

- Etendre les résultats dans le cas ergodique, spatial.

- D'autres problèmes méritent d'être soulevés dans le modèle à indice fonctionnelle simple, tels que l'estimation de la régression relative, la régression robuste, et l'estimation par la méthode des polynômes locaux.

- Une question qui nous semble naturelle est l'estimation du modèle à indice fonctionnelle simple dans le cas des données tronquées, les données aberrantes.

- Le choix du paramètre de lissage qui est un paramètre trés important en estimation non paramétrique car il intervient dans toutes les propriétés asymptotiques que nous avons étudier, ainsi il permet d'améliorer la qualité de l'estimateur.

D'autres problèmes peuvent être abordés notamment celui où les variables réponses sont fonctionnelles dans le modèle à indice fonctionnelle simple.

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