N° d'ordre :

REPUBLIQUE ALGERIENNE DEMOCRATIQUE & POPULAIRE

MINISTERE DE L'ENSEIGNEMENT SUPERIEUR & DE LA RECHERCHE SCIENTIFIQUE

UNIVERSITE DJILLALI LIABES
FACULTE DES SCIENCES EXACTES
SIDI BEL ABBÈS

THESE
DE DOCTORAT

Présentée par

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Spécialité : MATHEMATIQUES
Option : Analyse Des Equations Aux Dérivées Partielles

Intitulée

Etude spectrale de certains problèmes d'évolution en présence De feedbacks aux bords

Soutenue le : 28 / Février / 2019.

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Acknowledgements

I would like to express my deep gratitude to **Dr Nour-Eddine Amroun**, my thesis supervisor, for his patience, motivation and enthusiastic encouragement.

I respect and thank **Pr Abbès Benaissa** , for his continuous support through my research. I am extremely thankful to him for providing such a nice support and guidance, although he had busy.

I respect and thank **Pr Ali Hakem, Dr Salima Mimouni** and **Dr Mama Abdelli** having accepted to be part of this jury.

Huge thanks to **Dr Abderrahmane Beniani**, for his guidance, advice and friendship have been invaluable. I am very grateful to him.

Finally, I would like to thank my family: my parents and to my brothers and sisters for supporting me spiritually throughout writing this thesis and my life in general.

Thank you..... N.Taouaf.

Abstract

The present thesis is devoted to the study of Well-Posedness and asymptotic behaviour in time of solution of Lamé system and coupled Lamé system.
This work consists of five chapters, will be devoted to the study of the Well-Posedness and asymptotic behaviour of some evolution equation with linear, and viscoelastic terms. In chapter 1, we recall of some fundamental inequalities. In chapter 2, we consider the Lamé system in 3-dimension bounded domain with distributed delay term. We prove, under some appropriate assumptions, that this system is well-posed and stable. Furthermore, the asymptotic stability is given by using an appropriate Lyapunov functional. In chapter 3, we consider a coupled Lamé system with a viscoelastic damping in the first equation. We prove well-posedness by using Faedo-Galerkin method and establish an exponential decay result by introducing a suitable Lyaponov functional. In chapter 4, we consider a coupled Lamé system with a viscoelastic damping and a strong constant delay in the first equation. We prove well-posedness by using Faedo-Galerkin method and establish an exponential decay result by introducing a suitable Lyaponov functional. In chapter 5, we consider a coupled Lamé system with a viscoelastic term and a strong damping. We prove well posedness by using Faedo-Galerkin method and establish an exponential decay result by introducing a suitable Lyaponov functional.

Key words : Well-Posedness, Coupled system, Exponential decay, Lyapunov method, Galerkin method, Viscoelastic term, Delay term.

subjclass 2000: 35L55, 74D05, 37B25, 35B35, 93D20, 35L80, 35B40, 35L20, 58G16, 35L70.

General introduction

The present thesis is devoted of the study of Well-Posedness , asymptotic behaviour in time of solution to hyperbolic systems.
The problem of stabilization consists in determining the asymptotic behaviour of the energy by $E(t)$, to study its limits in order to determine if this limit is null or not and if this limit is null, to give an estimate of the decay rate of the energy to zero, they are several type of stabilization:

1. Strong stabilization: $E(t) \to 0$, as $t \to \infty$.

2. Uniform stabilization: $E(t) \leqslant C \exp{(-\delta t)}$, $\forall t > 0$, $(c, \delta > 0)$.

3. Polynomial stabilization: $E(t) \leqslant Ct^{-\delta}$, $\forall t > 0$, $(c, \delta > 0)$.

4. Logarithmic stabilization: $E(t) \leqslant C(\ln(t))^{-\delta}$ $\forall t > 0$, $(c, \delta > 0)$.

For wave equation with dissipation of the form $u'' + \Delta_x u + g(u') = 0$, stabilization problems have been investigated by many authors:
When $g : \mathbb{R} \to \mathbb{R}$ is continuous and increasing function such that $g(0) = 0$, global existence of solutions is known for all initial conditions (u_0, u_1) given in $H_0^1(\Omega) \times L^2(\Omega)$. This result is, for instance, a consequence of the general theory of nonlinear semi-groups of contractions generated by a maximal monotone operator (see Brézis [7]).
Moreover, if we impose on g the condition $\forall \lambda \neq 0, g(\lambda) \neq 0$, then strong asymptotic stability of solutions occurs in $H_0^1(\Omega) \times L^2(\Omega)$.

$$i.e\ (u, u') \to (0, 0) \text{ strongly in } H_0^1(\Omega) \times L^2(\Omega)$$

without speed of convergence. These results follows, for instance, from the invariance principle of Lasalle (see for example A. Haraux [11]). If we add the assymption that g has a polynomial growth near zero, we obtain an explicit decay rate of solutions (see M. Nakao

[20]).

This work consists of five chapters:

- In **the chapter 2**: In this chapter, we consider the following Lamé system with a distributed delay term

$$\begin{cases} u''(x,t) - \Delta_e u(x,t) + \int_{\tau_1}^{\tau_2} \mu_2(s) u'(x,t-s)ds + \mu_1 u'(x,t) = 0 & \text{in } \Omega \times \mathbb{R}^+ \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+ \end{cases} \tag{1}$$

  Here Δ denotes the Laplacian operator and Δ_e denotes the elasticity operator, which is the 3×3 matrix-valued differential operator defined by

$$\Delta_e u = \mu \Delta u + (\lambda + \mu) \nabla(\text{div} u), \quad u = (u_1, u_2, u_3)^T$$

  and μ and λ are the Lamé constants which satisfy the conditions

$$\mu > 0, \quad \lambda + \mu \geqslant 0. \tag{2}$$

  Moreover, $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function and $\tau_1 < \tau_2$ are two positive constants.

  In the particular case $\lambda + \mu = 0$, $\Delta_e = \mu \Delta$ gives a vector Laplacian; that is (II.1) describes the vector wave equation.

  The purpose of this chapter is to prove the well-posedness of the problem (II.1). Moreover we show that we can always find initial data in the stable set for which the solution of problem (II.1) decays exponentially, which is based on the construction of a suitable Lyaponov functional.

- In **the chapter 3**: Let us consider the following a coupled Lamé system :

$$\begin{cases} u_{tt}(x,t) + \alpha v - \Delta_e u(x,t) + \int_0^t g(s) \Delta u(t-s) ds - \mu_1 \Delta u_t(x,t) = 0, & \text{in } \Omega \times (0, +\infty), \\ v_{tt}(x,t) + \alpha u - \Delta_e v(x,t) - \mu_2 \Delta v_t(x,t) = 0, & \text{in } \Omega \times (0, +\infty), \\ u(x,t) = v(x,t) = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)) & \text{in } \Omega, \\ (u_t(x,0), v_t(x,0)) = (u_1(x), v_1(x)) & \text{in } \Omega. \end{cases} \tag{3}$$

  The problem of stabilization of coupled systems has also been studied by several authors see [2, 4, 6, 15, 23, 24]and the references therein.Under certain conditions imposed on the subset where the damping term is effective, Komornik [15] proves uniform stabilization of the solutions of a pair of hyperbolic systems coupled in velocities. Alabau and al.[2] studied the indirect internal stabilization of weakly coupled systems where the damping is effective in the whole domain. They prove that the behavior of the first equation is sufficient to stabilize the total system and to have polynomial decay for sufficiently smooth solutions.

In this chapter we prove well-posedness of the problem by using the Faedo-Galerkin method and we prove the exponential decay of the energy when time goes to infinity.

- In **the chapter 4**:

  Let us consider the following problem

$$\begin{cases} u_{tt}(x,t) + \alpha v - \Delta_e u(x,t) + \displaystyle\int_0^t g(s)\Delta u(t-s)ds & \\ \qquad\qquad -\mu_1 \Delta u_t(x,t) - \lambda_1 \Delta_x u_t(x,t-\tau) = 0, & \text{in } \Omega \times (0,+\infty), \\ v_{tt}(x,t) + \alpha u - \Delta_e v(x,t) - \mu_2 \Delta v_t(x,t) = 0, & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = v(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)) & \text{in } \Omega, \\ (u_t(x,0), v_t(x,0)) = (u_1(x), v_1(x)) & \text{in } \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), & \text{in } \Omega \times [0,\tau]. \end{cases} \tag{4}$$

  The main purpose of this work is to prove the well-posedness in Sobolev spaces using Faedo-Galerkin method and to allow a wider class of relaxation functions and improve earlier results in the literature. The basic mechanism behind the decay rates is the relation between the damping and the energy.

- In **the chapter 5**: In this chapter, we consider the following coupled Lamé system :

$$\begin{cases} u_{tt}(x,t) + \alpha v - \Delta_e u(x,t) + \displaystyle\int_0^t g_1(t-s)\Delta u(x,s)ds - \mu_1 \Delta u_t(x,t) = 0, & \text{in } \Omega \times (0,+\infty), \\ v_{tt}(x,t) + \alpha u - \Delta_e v(x,t) + \displaystyle\int_0^t g_2(t-s)\Delta v(x,s)ds - \mu_2 \Delta v_t(x,t) = 0, & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = v(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)) & \text{in } \Omega, \\ (u_t(x,0), v_t(x,0)) = (u_1(x), v_1(x)) & \text{in } \Omega. \end{cases} \tag{5}$$

.

The purpose of this paper is to prove the well-posedness of the problem and exponential decay of the energy when time goes to infinity.

I

Preliminaries

In this chapter, we present some materials which will be used in the following chapters.

1 Function Spaces

We consider the Euclidean space $\mathbb{R}^n$, $n \geqslant 1$ endowed with standard Euclidean topology and for Ω a subset of $\mathbb{R}^n$ we will define various spaces of functions $\Omega \to \mathbb{R}^m$. If endowed by a pointwise addition and multiplication the linear space structure of $\mathbb{R}^m$ is inherited by these spaces. Besides, we will endow them by norms, which makes them normed linear (or, mostly even Banach) spaces. Having two such spaces $U \subset V$, we say that the mapping

$$f : U \to V, u \mapsto u.$$

is a continuous embedding (or, that U is embedded continuously to V) if the linear operator f is continuous (hence bounded). This means that

$$\|u\|_V \leqslant C\|u\|_U$$

for C one can take the norm $\|f\|_{\ell(U,V)}$. If f is compact, we speak about a compact embedding and use the notation $U \subset V$. If U is a dense subset in V , we will speak about a dense embedding; this property obviously depends on the norm of V but not of U. It follows by a general functional-analysis argument that the adjoint mapping

$$f^* : V^* \to U^*.$$

is continuous and injective provided $U \subset V$ continuously and densely, then we can identify V^* as a subset of U^*. Indeed, f^* is injective (because two different linear continuous functionals on V must have also different traces on any dense subset, in particular on U).

1.1 The $L^p(\Omega)$ spaces

Let $1 \leqslant p \leqslant \infty$, and let Ω be an open domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Define the standard Lebesgue space $L^p(\Omega)$, by

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R} : \mathbf{u} \textit{ is measurable and } \int_\Omega |u(x)|^p dx < \infty \right\}.$$

Notation 1.1 *For $p \in \mathbb{R}$ and $1 \leqslant p < \infty$, denote by*

$$\|u\|_p = \left(\int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}}.$$

If $p = \infty$, we have

$$L^\infty(\Omega) = \left\{ u : \Omega \to \mathbb{R} \;\middle|\; \begin{array}{l} \mathbf{u} \textit{ is measurable and there exists a constant } C \\ \textbf{such that, } |u(x)| \leqslant C \textbf{ a.e in } \Omega \end{array} \right\}.$$

with

$$\|u\|_\infty = \inf\{C; |u| \leqslant C \textbf{ a.e On } \Omega\}.$$

Theorem 1.2 *It is well known that $L^p(\Omega)$ supplied with the norm $\|.\|_p$ is a Banach space, for all $1 \leqslant p \leqslant \infty$.*

Remark 1.3 *In particularly, when $p = 2$, $L^2(\Omega)$ equipped with the inner product*

$$\langle f, g \rangle_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx,$$

is a Hilbert space.

Theorem 1.4 *For $1 < p < \infty$, $L^p(\Omega)$ is reflexive space.*

1.2 The $L^p(0, T, V)$ spaces

Definition 1.5 *Let V be a Banach space, denote by $L^p(0, T, V)$ the space of measurable functions*

$$\begin{aligned} u: \ &]0, T[\to V \\ & t \longmapsto u(t) \end{aligned}$$

such that

$$\left(\int_0^T \|u(t)\|_V^p dt\right)^{\frac{1}{p}} = \|u\|_{L^p(0,T,X)} < \infty, \ \textbf{\textit{for}}\ 1 \leqslant p < \infty.$$

If $p = \infty$,

$$\|u\|_{L^\infty(0,T,V)} = \sup_{t \in]0,T[} ess \, \|u(t)\|_V \, .$$

Theorem 1.6 *The space $L^p(0, T, V)$ is complete.*

We denote by $\mathcal{D}'(0, T, V)$ the space of distributions in $]0, T[$ which take its values in V, and let us define

$$\mathcal{D}'(0, T, V) = (\mathcal{D}\,]0, T[\, , V) \, ,$$

where (ϕ, φ) is the space of the linear continuous applications of ϕ to φ. Since $u \in \mathcal{D}'(0, T, V)$, we define the distribution derivation as

$$\frac{\partial u}{\partial t}(\varphi) = -u\left(\frac{d\varphi}{dt}\right), \forall \varphi \in \mathcal{D}\left(]0, T[\right),$$

and since $u \in L^p(0, T, V)$, we have

$$u(\varphi) = \int_0^T u(t)\varphi(t) dt, \forall \varphi \in \mathcal{D}\left(]0, T[\right).$$

We will introduce some basic results on the $L^p(0, T, V)$ space. These results, will be very useful in the other chapters of this thesis.

Lemma 1.7 *Let $u \in L^p(0,T,V)$ and $\frac{\partial u}{\partial t} \in L^p(0,T,V)$, $(1 \leqslant p \leqslant \infty)$, then, the function u is continuous from $[0,T]$ to V .i.e. $u \in C^1(0,T,V)$.*

1.3 Sobolev spaces

Modern theory of differential equations is based on spaces of functions whose derivatives exist in a generalized sense and enjoy a suitable integrability.

Proposition 1.8 *Let Ω be an open domain in $\mathbb{R}^N$, Then the distribution $T \in \mathcal{D}'(\Omega)$ is in $L^p(\Omega)$ if there exists a function $f \in L^p(\Omega)$ such that*

$$\langle T, \varphi \rangle = \int_\Omega f(x)\varphi(x)dx, \ \textbf{\textit{for all}}\ \varphi \in \mathcal{D}(\Omega),$$

where $1 \leqslant p \leqslant \infty$, and it's well-known that f is unique.

Definition 1.9 *Let $m \in \mathbb{N}$ and $p \in [0,\infty]$. The $W^{m,p}(\Omega)$ is the space of all $f \in L^p(\Omega)$, defined as*

$$W^{m,p}(\Omega) = \{f \in L^p(\Omega),\ \textbf{\textit{such that}}\ \partial^\alpha f \in L^p(\Omega)\ \textbf{\textit{for all}}\ \alpha \in \mathbb{N}^m\ \textbf{\textit{such that}}$$
$$|\alpha| = \sum_{j=1}^{n} \alpha_j \leqslant m,\ \textbf{\textit{where,}}\ \partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}..\partial_n^{\alpha_n}\}.$$

Theorem 1.10 *$W^{m,p}(\Omega)$ is a Banach space with their usual norm*

$$\|f\|_{W^{m,p}(\Omega)} = \sum_{|\alpha| \leqslant m} \|\partial^\alpha f\|_{L^p}, 1 \leqslant p < \infty\textbf{\textit{, for all}}\ f \in W^{m,p}(\Omega).$$

Definition 1.11 *Denote by $W_0^{m,p}(\Omega)$ the closure of $D(\Omega)$ in $W^{m,p}(\Omega)$.*

Definition 1.12 *When $p = 2$, we prefer to denote by $W^{m,2}(\Omega) = H^m(\Omega)$ and $W_0^{m,2}(\Omega) = H_0^m(\Omega)$ supplied with the norm*

$$\|f\|_{H^m(\Omega)} = \left(\sum_{|\alpha| \leqslant m} (\|\partial^\alpha f\|_{L^2})^2 \right)^{\frac{1}{2}},$$

which do at $H^m(\Omega)$ a real Hilbert space with their usual scalar product

$$\langle u, v \rangle_{H^m(\Omega)} = \sum_{|\alpha| \leqslant m} \int_\Omega \partial^\alpha u \partial^\alpha v dx$$

Theorem 1.13 *1) $H^m(\Omega)$ supplied with inner product $\langle .,.\rangle_{H^m(\Omega)}$ is a Hilbert space.*

2) If $m \geqslant m'$, $H^m(\Omega) \hookrightarrow H^{m'}(\Omega)$, with continuous imbedding .

Lemma 1.14 *Since $\mathcal{D}(\Omega)$ is dense in $H_0^m(\Omega)$, we identify a dual $H^{-m}(\Omega)$ of $H_0^m(\Omega)$ in a weak subspace on Ω, and we have*

$$\mathcal{D}(\Omega) \hookrightarrow H_0^m(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-m}(\Omega) \hookrightarrow \mathcal{D}'(\Omega),$$

The next results are fundamental in the study of partial differential equations

Theorem 1.15 *Assume that Ω is an open domain in $\mathbb{R}^N$ $(N \geqslant 1)$, with smooth boundary $\partial\Omega$. Then,*

(**i**) *if $1 \leqslant p \leqslant n$, we have $W^{1,p} \subset L^q(\Omega)$, for every $q \in [p, p^*]$, where $p^* = \frac{np}{n-p}$.*

(**ii**) *if $p = n$ we have $W^{1,p} \subset L^q(\Omega)$, for every $q \in [p, \infty)$.*

(**iii**) *if $p > n$ we have $W^{1,p} \subset L^\infty(\Omega) \cap C^{0,\alpha}(\Omega)$, where $\alpha = \frac{p-n}{p}$.*

Theorem 1.16 *If Ω is a bounded, the embedding (ii) and (iii) of Theorem II.18 are compacts. The embedding (i) is compact for all $q \in [p, p^*)$.*

Remark 1.17 *For all $\varphi \in H^2(\Omega)$, $\Delta\varphi \in L^2(\Omega)$ and for $\partial\Omega$ sufficiently smooth, we have*

$$\|\varphi(t)\|_{H^2(\Omega)} \leqslant C \|\Delta\varphi(t)\|_{L^2(\Omega)}.$$

1.4 Weak convergence

Let $(E; \|.\|_E)$ a Banach space and E' its dual space, i.e., the Banach space of all continuous linear forms on E endowed with the norm $\|.\|_{E'}$ defined by

$$\|f\|_{E'} =: \sup_{x \neq 0} \frac{|< f, x >|}{\|x\|}$$

where $< f, x >$ denotes the action of f on x, i.e $< f, x >= f(x)$. In the same way, we can define the dual space of E' that we denote by E''. (The Banach space E'' is also called the bi-dual space of E). An element x of E can be seen as a continuous linear form on E'' by setting $x(f) =:< x, f >$, which means that $E \subset E''$.

Weak, weak star and strong convergence

Definition 1.18 *(**weak convergence in** E). Let $x \in E$ and let $\{x_n\} \subset E$. We say that $\{x_n\}$ weakly converges to x in E, and we write $x_n \rightharpoonup x$ in E, if*

$$< f, x_n > \to < f, x >$$

for all $x \in E'$.

Definition 1.19 *(**weak convergence in E'**). Let $f \in E'$ and let $\{f_n\} \subset E'$ We say that $\{f_n\}$ weakly converges to f in E', and we write $f_n \rightharpoonup f$ in E', if*

$$< f_n, x > \to < f, x >$$

for all $x \in E''$.

Definition 1.20 *(**weak star convergence**). Let $f \in E'$ and let $\{f_n\} \subset E'$ We say that $\{f_n\}$ weakly star converges to f in E', and we write $f_n \overset{*}{\rightharpoonup} f$ in E' if*

$$< f_n, x > \to < f, x >$$

for all $x \in E$.

Remark 1.21 *As $E \subset E''$ we have $f_n \rightharpoonup f$ in E' imply $f_n \overset{*}{\rightharpoonup} f$ in E'. When E is reflexive, the last definitions are the same, i.e, weak convergence in E' and weak star convergence coincide.*

Definition 1.22 *(**strong convergence**). Let $x \in E$ (resp.$f \in E'$) and let $\{x_n\} \subset E$ (resp.$\{f_n\} \subset E'$). We say that $\{x_n\}$ (resp.$\{f_n\}$) strongly converges to x (resp.f), and we write $x_n \to x$ in E (resp.$f_n \to f$ in E'), if*

$$\lim_{n\to\infty} \|x_n - x\|_E = 0, (resp. \lim_{n\to\infty} \|f_n - f\|_{E'} = 0).$$

Definition 1.23 *(**weak convergence in** $L^p(\Omega)$ **with** $1 \leqslant p < \infty$). Let Ω an open subset of $\mathbb{R}^n$. We say that the sequence $\{f_n\}$ of $L^p(\Omega)$ weakly converges to $f \in L^p(\Omega)$, if*

$$\lim_n \int_\Omega f_n(x)g(x)dx = \int_\Omega f(x)g(x)dx \text{ for all } g \in L^q,\ \frac{1}{p} + \frac{1}{q} = 1.$$

Definition 1.24 ***(weak convergence in*** $W^{1,p}(\Omega)$ ***with*** $1 < p < \infty$ *)We say the* $\{f_n\} \subset W^{1,p}(\Omega)$ *weakly converges to* $f \in W^{1,p}(\Omega)$*, and we write* $f_n \rightharpoonup f$ *in* $W^{1,p}(\Omega)$*, if*

$$f_n \rightharpoonup f \text{ in } L^p(\Omega) \text{ and } \nabla f_n \rightharpoonup \nabla f \text{ in } L^p(\Omega;\mathbb{R}^n)$$

Weak and weak star compactness

In finite dimension, i.e, $\dim E < \infty$, we have Bolzano-Weierstrass's theorem (which is a strong compactness theorem).

Theorem 1.25 ***(Bolzano-Weierstrass)****. If* $\dim E < \infty$ *and if* $\{x_n\} \subset E$ *is bounded, then there exist* $x \in E$ *and a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\{x_{n_k}\}$ *strongly converges to* x*.*

The following two theorems are generalizations, in infinite dimension, of Bolzano- Weierstrass's theorem.

Theorem 1.26 ***(weak star compactness, Banach-Alaoglu-Bourbaki)****. Assume that* E *is separable and consider* $\{f_n\} \subset E'$ *. If* $\{x_n\}$ *is bounded, then there exist* $f \in E'$ *and a subsequence* $\{f_{n_k}\}$ *of* $\{f_n\}$ *such that* $\{f_{n_k}\}$ *weakly star converges to* f *in* E'*.*

Theorem 1.27 ***(weak compactness, Kakutani-Eberlein)****. Assume that* E *is reflexive and consider* $\{x_n\} \subset E$*. If* $\{x_n\}$ *is bounded, then there exist* $x \in E$ *and a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\{x_{n_k}\}$ *weakly converges to* x *in* E*.*

Theorem 1.28 *(weak compactness in* $L^p(\Omega)$*) with* $1 < p < \infty$*. Given* $\{f_n\} \subset L^p(\Omega)$ *, if* $\{f_n\}$ *is bounded, then there exist* $f \in L^p(\Omega)$ *and a subsequence* $\{f_{n_k}\}$ *of* $\{f_n\}$ *such that* $f_n \rightharpoonup f$ *in* $L^p(\Omega)$*.*

Theorem 1.29 ***(weak star compactness in*** $L^\infty(\Omega)$*)*
Given $\{f_n\} \subset L^\infty(\Omega)$*, if* $\{f_n\}$ *is bounded, then there exist* $f \in L^\infty(\Omega)$ *and a subsequence* $\{f_{n_k}\}$ *of* $\{f_n\}$ *such that* $f_n \overset{*}{\rightharpoonup} f$ *in* $L^\infty(\Omega)$*.*

1.5 Aubin -Lions Lemma

The Aubin-Lions Lemma is a result in the theory of Sobolev spaces of Banach space-valued functions. More precisely, it is a compactness criterion that is very useful in the study of nonlinear evolutionary partial differential equations. The result is named after the French mathematicians Thierry Aubin and Jacques-Louis Lions.

Lemma 1.30 *Let X_0, X and X_1 be three Banach spaces with $X_0 \subseteq X \subseteq X_1$. Assume that X_0 is compactly embedded in X and that X is continuously embedded in X_1; assume also that X_0 and X_1 are reflexive spaces. For $1 < p, q < +\infty$, let*

$$W = \{u \in L^p([0,T]; X_0) / \quad \dot{u} \in L^q([0,T]; X_1)\}$$

Then the embedding of W into $L^p([0,T]; X)$ is also compact.

2 Inequalities

Notation 2.1 *Let $1 \leqslant p \leqslant \infty$, we denote by q the conjugate exponent,*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

2.1 Hölder's inequality

Assume that $f \in L^p$ and $g \in L^q$ with $1 \leqslant p \leqslant \infty$. Then $(fg) \in L^1$ and

$$\|fg\| \leqslant \|f\|_p \|g\|_q.$$

Lemma 2.2 ***(Cauchy-Schwarz inequality)*** *Every inner product satisfies the Cauchy-Schwarz inequality*

$$\langle x_1, x_2 \rangle \leqslant \|x_1\| \|x_2\|.$$

The equality sign holds if and only if x_1 and x_2 are dependent.

We will give here some integral inequalities. These inequalities play an important role in applied mathematics and also, it is very useful in our next chapters.

Lemma 2.3 *Let $1 \leqslant p \leqslant r \leqslant q$, $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$, and $0 \leqslant \alpha \leqslant 1$. Then*

$$\|u\|_{L^r} \leqslant \|u\|_{L^p}^{\alpha} \|u\|_{L^q}^{1-\alpha}.$$

Since our study based on some known algebraic inequalities, we want to recall few of them here.

2.2 Young's inequality

For all $a, b \geqslant 0$, the following inequality holds

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q},$$

where, $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 2.4 *For all a, $b \in \mathbb{R}^+$, we have*

$$ab \leqslant \delta a^2 + \frac{b^2}{4\delta},$$

where δ is any positive constant.

2.3 Poincaré's inequality

Let $\Omega \subset \mathbb{R}^n$ is a bounded open subset. Then there exists a constant c, depending on Ω such that:

$$\|f\|_{L^2(\Omega)} \leqslant c\|\nabla f\|_{L^2(\Omega)}, \ \forall f \in H_0^1(\Omega).$$

3 Semigroups, Existence and uniqueness of solution

In this section, we start by introducing some basic concepts concerning the semigroups. The vast majority of the evolution equations can be reduced to the form

$$\begin{cases} U_t = AU, \quad t > 0, \\ \\ U(0) = U_0 \end{cases} \tag{I.1}$$

where A is the infinitesimal generator of a C_0-semigroup $S(t)$ over a Hilbert space H. Lets start by basic definitions and theorems. Let $(X, \|.\|_X)$ be a Banach space, and H be a Hilbert space equipped with the inner product $< .,. >_H$ and the induced norm $\|.\|_H$.

Definition 3.1 *A family $S(t)_{t \geqslant 0}$ of bounded linear operators in X is called a strong continous semigroup (in short, a C_0-semigroup) if*

i) $S(0) = I_d$.

ii) $S(s+t) = S(s)S(t), \ \ \forall t \geqslant 0 \ \forall s \geqslant 0.$

iii) *For each* $u \in H$*,* $S(t)u$ *is continous in* t *on* $[0, +\infty[$.

Sometimes we also denote $S(t)$ *by* e^{At}.

Definition 3.2 *For a semigroup* $S(t)_{t\geqslant 0}$*, we define an linear operator* A *with domain* $D(A)$ *consisting of points* u *such that the limit*

$$Au = \lim_{t \to 0^+} \frac{S(t)u - u}{t} \quad \forall u \in D(A)$$

exists. Then A *is called the infinitesimal generator of the semigroup* $S(t)_{t\geqslant 0}$.

Proposition 3.3 *Let* $S(t)_{t\geqslant 0}$ *be a* C_0*-semigroup in* X*. Then there exist a constant* $M \geqslant 1$ *and* $\omega \geqslant 0$ *such that*

$$\|S(t)\|_{\mathcal{L}(X)} \leqslant Me^{\omega t}. \ \ \forall t \geqslant 0$$

If $\omega = 0$ *then the corresponding semigroup is uniformly bounded. Moreover, if* $M = 1$ *then* $S(t)_{t\geqslant 0}$ *is said to be a* C_0*-semigroup of contractions.*

Definition 3.4 *An unbounded linear operator* $(A, D(A))$ *on* H*, is said to be dissipative if*

$$\Re < Au, u > \leqslant 0, \forall u \in D(A).$$

Definition 3.5 *An unbounded linear operator* $(A, D(A))$ *on* X*, is said to be m-dissipative if*

- A *is a dissipative operator.*
- $\exists \lambda_O$ *such that* $\mathcal{R}(\lambda_0 I - A) = X$

Theorem 3.6 *Let* A *be a m-dissipative operator, then*

- $\mathcal{R}(\lambda_0 I - A) = X, \quad \forall \lambda > 0$
- $]0, \infty[\subseteq \rho(A).$

Theorem 3.7 *(Hille-Yosida)An unbounded linear operator* $(A, D(A))$ *on* X*, is the infinitesimal generator of a* C_0*-semigroup of contractions* $S(t)_{t\geqslant 0}$ *if and only if*

- A *is closed and* $D(\bar{A}) = X$.

- *The resolvent set* $\rho(A)$ *of* A *contains* $\mathbb{R}^+$, *and for all* $\lambda > 0$,

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \leqslant \lambda^{-1}$$

Theorem 3.8 *(Lumer-Phillips) Let* $(A, D(A))$ *be an unbounded linear operator on* X*, with dense domain* $D(A)$ *in* X*.* A *is the infinitesimal generator of a* C_0*-semigroup of contractions if and only if it is a m-dissipative operator.*

Theorem 3.9 *Let* $(A, D(A))$ *be an unbounded linear operator on* X*. If* A *is dissipative with* $\mathcal{R}(I - A) = X$*, and* X *is reflexive then* $D\bar{(}A) = X$*.*

Corrolary 3.10 *Let* $(A, D(A))$ *be an unbounded linear operator on H. A is the infinitesimal generator of a* C_0*-semigroup of contractions if and only if A is a m-dissipative operator.*

Theorem 3.11 *Let* A *be a linear operator with dense domain* $D(A)$ *in a Hilbert space* H*. If* A *is dissipative and* $0 \in \rho(A)$ *then* A *is the infinitesimal generator of a* C_0*-semigroup of contractions on* H*.*

Theorem 3.12 *(Hille-Yosida) Let* $(A, D(A))$ *be an unbounded linear operator on* H*. Assume that* A *is the infinitesimal generator of a* C_0*-semigroup of contractions* $S(t)_{t\geqslant 0}$*.*

1. *For* $U_0 \in D(A)$*, the problem* (I.1) *admits a unique strong solution*

$$U(t) = S(t)U_0 \in C^1([0, \infty[; H) \cap C([0, \infty[; D(A))$$

2. *For* $U_0 \in D(A)$*, the problem* (I.1) *admits a unique weak solution*

$$U(t) \in C^0([0, \infty[; H).$$

II

Well-Posedness And Asymptotic Stability For The Lamé System With Internal Distributed Delay

1 Introduction and statement

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Let us consider the following Lamé system with a distributed delay term:

$$\begin{cases} u''(x,t) - \Delta_e u(x,t) + \int_{\tau_1}^{\tau_2} \mu_2(s) u'(x,t-s)ds + \mu_1 u'(x,t) = 0 & \text{in } \Omega \times \mathbb{R}^+ \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+ \end{cases} \tag{II.1}$$

with initial conditions

$$\begin{cases} u(x) = u_0(x) \quad u'(x,0) = u_1(x), & \text{in } \Omega, \\ u'(x,-t) = f_0(x,-t), & \text{in } \Omega \times (0,\tau_2), \end{cases} \tag{II.2}$$

Where (u_0, u_1, f_0) are given history and initial data . Here Δ denotes the Laplacian operator and Δ_e denotes the elasticity operator, which is the 3×3 matrix-valued differential operator defined by

$$\Delta_e u = \mu\Delta u + (\lambda + \mu)\nabla(\text{div u}), \quad u = (u_1, u_2, u_3)^T$$

and μ and λ are the Lamé constants which satisfy the conditions

$$\mu > 0, \quad \lambda + \mu \geqslant 0. \tag{II.3}$$

Moreover, $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function and $\tau_1 < \tau_2$ are two positive constants.

In the particular case $\lambda+\mu = 0$, $\Delta_e = \mu\Delta$ gives a vector Laplacian; that is (II.1) describes the vector wave equation.

In recent years, the control of partial differential equations with time delay effects has become an active and attractive area of research see ([1, 5, 6, 9, 10, 12]and [17]),and the references therein. Recently, S. A. Messaoudi and al.[17] considered the following problem with a strong damping and a strong distributed delay:

$$\begin{cases} u_{tt} - \Delta_x u(x,t) - \mu_1 \Delta u_t(x,t) - \displaystyle\int_{\tau_1}^{\tau_2} \mu_2(s)\Delta u_t(x, t-s)ds = 0 & \text{in} \quad \Omega \times (0, +\infty), \\ u = 0 & \text{on} \quad \Gamma \times [0, +\infty), \\ u(x,0) = u_0(x) \quad u'(x,0) = u_1(x) & \text{on} \quad \Omega, \\ u_t(x,-t) = f_0(x,-t), & \qquad 0 < t \leqslant \tau_2, \end{cases} \tag{II.4}$$

and under the assumption

$$\mu_1 > \int_{\tau_1}^{\tau_2} \mu_2(s)ds. \tag{II.5}$$

The authors proved that the solution is exponentially stable.

In[3], the authors considered the Bresse system in bounded domain with internal distributed delay

$$\begin{cases} \rho_1\varphi_{tt} - Gh(\varphi_x + lw + \psi)_x - Ehl(w_x - l\varphi) + \mu_1\varphi_t + \mu_2\varphi_t(x, t-\tau_1) = 0, \\ \rho_2\psi_{tt} - El\psi_{xx} - Gh(\varphi_x - lw + \psi) + \displaystyle\int_{\tau_1}^{\tau_2} \mu(s)\psi_t(x, t-s)ds = 0, \\ \rho_1 w_{tt} - Eh(w_x - l\varphi)_x + lGh(\varphi_x + lw + \psi) + \widetilde{\mu_1}w_t + \widetilde{\mu_2}w_t(x, t-\tau_2) = 0, \end{cases} \tag{II.6}$$

where $(x,t) \in]0, L[\times\mathbb{R}_+$, the authors proved, under suitable conditions, that the system is well-posed and its energy converges to zero when time goes to infinity. For Timoshenko-type system with thermoelasticity of second sound,in the presence of a distributed delay

Apalara[1] considered the following system:

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x + \mu\varphi_t + \displaystyle\int_{\tau_1}^{\tau_2} \mu_2(s)\varphi_t(x,t-s)ds = 0 & \text{in} \quad (0,1)\times(0,+\infty), \\ \rho_2\psi_{tt} - b\psi xx + k(\varphi_x + \psi) + \gamma\theta_x = 0 & \text{in} \quad (0,1)\times(0,\infty), \\ \rho_3\theta_t + q_x + \delta\psi_{tx} = 0 & \text{in} \quad (0,1)\times(0,\infty), \\ \tau q_t + Bq + \theta_x = 0, & \text{in} \quad (0,1)\times(0,\infty), \end{cases} \tag{II.7}$$

and proved an exponential decay result under the assumption

$$\mu > \int_{\tau_1}^{\tau_2} \mu_2(s)ds. \tag{II.8}$$

In [4], Beniani and al. considered the following Lamé system with time varing delay term:

$$\begin{cases} u''(x,t) - \Delta_e u(x,t) + \mu_1 g_1(u'(x,t)) + \mu_2 g_2(u'(x,t-\tau(s)) = 0 & \text{in } \Omega\times\mathbb{R}^+ \\ u = 0 & \text{on } \partial\Omega\times\mathbb{R}^+ \end{cases} \tag{II.9}$$

the authors proved, under suitable conditions, that energy converges to zero when time goes to infinity.

The paper is organized as follows: in Section 2, we prove the global existence and uniqueness of solutions of (II.1)-(II.2). In Section 3, we prove the stability results.

2 Well-posedness

In this section, we prove the existence and uniqueness of solutions of (II.1)-(II.2) using semigroup theory.

As in [21], we introduce the variable

$$z(x,\rho,t,s) = u'(x,t-\rho s), \quad (x,\rho,t,s) \in \Omega\times(0,1)\times(0,\infty)\times(\tau_1,\tau_2).$$

Then, it is easy to check that

$$sz_t(x,\rho,t,s) + z_\rho(x,\rho,t,s) = 0, \quad (x,\rho,t,s) \in \Omega\times(0,1)\times(0,\infty)\times(\tau_1,\tau_2). \tag{II.10}$$

Thus, system (II.1) becomes

$$\begin{cases} u''(x,t) - \Delta_e u(x,t) + \mu_1 u'(x,t) + \displaystyle\int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s) ds = 0 & \text{in } \Omega \times \mathbb{R}^+ \\ s z_t(x,\rho,t,s) + z_\rho(x,\rho,t,s) = 0 & \text{in } \Omega \times (0,1) \times (0,\infty) \times (\tau_1,\tau_2) \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+ \\ u(x,0) = u_0(x) \quad u'(x,0) = u_1(x), & \text{in } \Omega \times (0,1) \times (\tau_1,\tau_2), \\ z(x,\rho,0,s) = f_0(x,-\rho s), & \text{in } \Omega \times (0,1) \times (\tau_1,\tau_2), \end{cases} \tag{II.11}$$

Next, we will formulate the system (II.1)-(II.2) in the following abstract linear first-order system:

$$\begin{cases} \mathcal{U}_t(t) = \mathcal{A}\mathcal{U}(t) \quad \forall t > 0, \\ \mathcal{U}(0) = \mathcal{U}_0, \end{cases} \tag{II.12}$$

where $\mathcal{U} = (u, u_t, z)^T$, $\mathcal{U}_0 = (u_0, u_1, f_0)^T \in \mathcal{H}$

$$\mathcal{H} = H_0^1(\Omega)^3 \times (L^2(\Omega))^3 \times L^2((0,1), H)$$

We define the inner product in $\mathcal{H}$,

$$\begin{aligned} \langle V, \bar{V} \rangle_{\mathcal{H}} = &\int_\Omega v\bar{v}\,\mathrm{d}x + \mu \int_\Omega \nabla u \nabla \bar{u}\,\mathrm{d}x + (\lambda+\mu)\int_\Omega \operatorname{div} u.\operatorname{div} \bar{u}\,\mathrm{d}x \\ &+ \int_\Omega \int_{\tau_1}^{\tau_2} s\mu_2(s) \int_0^1 z(x,\rho,t,s)\bar{z}(x,\rho,t,s)\,\mathrm{d}\rho\mathrm{d}s\mathrm{d}x. \end{aligned}$$

The operators $\mathcal{A}$ is linear and given by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ z \end{pmatrix} = \begin{pmatrix} v \\ \Delta_e u(x,t) - \mu_1 v(x,t) - \displaystyle\int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s) ds \\ -\dfrac{1}{s} z_\rho(x,\rho,t,s) \end{pmatrix} \tag{II.13}$$

The domain $D(\mathcal{A})$ of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{ V = (u,v,z)^T \in \mathcal{H}, \mathcal{A}V \in \mathcal{H}, z(,0) = v \right\}.$$

The well-posedness of problem (II.12) is ensured by the following theorem.

Theorem 2.1 *Assume that*

$$\mu_1 > \int_{\tau_1}^{\tau_2} \mu_2(s) ds. \tag{II.14}$$

Then, for any $\mathcal{U}_0 \in \mathcal{H}$, the system (II.12) *has a unique weak solution*

$$\mathcal{U} \in C(\mathbb{R}^+, \mathcal{H}).$$

Moreover, if $\mathcal{U} \in D(\mathcal{A})$*, then the solution of* (II.12) *satisfies (classical solution)*

$$\mathcal{U} \in C^1(\mathbb{R}^+, \mathcal{H}) \cap C(\mathbb{R}^+, D(\mathcal{A})).$$

Proof 2.2 *we prove that $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ is a maximal monotone operator; that is, $\mathcal{A}$ is dissipative and $Id - \mathcal{A}$ is surjective. Indeed, a simple calculation implies that, for any $V = (u, v, z)^T \in D(\mathcal{A})$,*

$$\begin{aligned}
\langle \mathcal{A}V, V\rangle_{\mathcal{H}} &= \mu \int_\Omega \nabla v(x,t)\nabla u(x,t)\,\mathrm{d}x + (\lambda+\mu)\int_\Omega \mathrm{div}\; v(x,t).\mathrm{div}\; u(x,t)\,\mathrm{d}x \\
&\quad + \int_\Omega \Big\{\Delta_e u(x,t) - \mu_1 v(x,t) - \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s)\,\mathrm{d}s\Big\} v(x,t)\,\mathrm{d}x \\
&\quad - \int_\Omega \int_{\tau_1}^{\tau_2} \mu_2(s) \int_0^1 z(x,\rho,t,s) z_\rho(x,\rho,t,s)\,\mathrm{d}\rho\mathrm{d}s\mathrm{d}x \\
&= -\mu_1 \int_\Omega v^2(x,t)\,\mathrm{d}x - \int_\Omega v(x,t)\Big(\int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s)\mathrm{d}s\Big)\mathrm{d}x \\
&\quad - \frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} \mu_2(s) \int_0^1 \frac{\partial}{\partial\rho}|z(x,\rho,t,s)|^2\,\mathrm{d}\rho\mathrm{d}s\mathrm{d}x
\end{aligned} \tag{II.15}$$

Using Young's inequality and taking into account that $z(.,0,.,.) = v$, we get

$$\langle \mathcal{A}V, V\rangle_{\mathcal{H}} = -\Big(\mu_1 - \int_{\tau_1}^{\tau_2} \mu_2(s)\mathrm{d}s\Big)\int_\Omega v^2(x,t)\,\mathrm{d}x \tag{II.16}$$

by virtue of (II.14)*. Therefore, $\mathcal{A}$ is dissipative. On the other hand, we prove that $Id - \mathcal{A}$ is surjective. Indeed, let $F = (f, g, h)^T \in \mathcal{H}$ we show that there exists $V = (u, v, z)^T \in D(\mathcal{A})$ satisfying*

$$(Id - \mathcal{A})V = F \tag{II.17}$$

which is equivalent to

$$\begin{cases}
u - v = f, \\
v - \Delta_e u + \mu_1 v + \displaystyle\int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s) ds = g, \\
s z(x,\rho,t,s) + z_\rho(x,\rho,t,s) = hs,
\end{cases} \tag{II.18}$$

Using the equation in (II.18)*, we obtain*

$$z(x,t,\rho,s) = (u-f)e^{-\rho s} + e^{-\rho s}\int_0^\rho sh(x,\sigma)e^{\sigma s}\mathrm{d}\sigma.$$

Replacing v *by* $u-f$ *in the second equation of* (II.18)*, we get*

$$Ku - \Delta_e u = G \tag{II.19}$$

Where

$$K = 1 + \mu_1 + \int_{\tau_1}^{\tau_2} e^{-s}\mu_2(s)ds > 0 \tag{II.20}$$

and

$$G = g + \Big(1 - \mu_1 - \int_{\tau_1}^{\tau_2} e^{-s}\mu_2(s)ds\Big)f + \int_{\tau_1}^{\tau_2} se^{-s}\mu_2(s)\int_0^1 h(x,\sigma)e^{\sigma s}\mathrm{d}\sigma\mathrm{d}s.$$

So we multiply (II.19) *by a test function* $\varphi \in (H_0^1(\Omega))^3$ *and we integrate by using Green's equality , obtaining the following variational formulation of* (II.19)*:*

$$a(u,\varphi) = L(\varphi) \quad \forall \varphi \in (H_0^1(\Omega))^3, \tag{II.21}$$

where

$$a(u,\varphi) = \int_\Omega (Ku.\varphi + \mu\nabla u.\nabla\varphi + (\lambda+\mu)\mathrm{div}\, u.\mathrm{div}\,\varphi)\,dx \tag{II.22}$$

and

$$L(\varphi) = \int_\Omega G\varphi\,dx \tag{II.23}$$

It is clear that a *is a bilinear and continuous form on* $(H_0^1(\Omega))^3 \times (H_0^1(\Omega))^3$*, and* L *is a linear and continuous form on* $(H_0^1(\Omega))^3$*. On the other hand,* (V.2) *and* (II.20) *imply that there exists a positive constant* a_0 *such that*

$$a(u,u) \geqslant a_0\|u\|_{(H_0^1(\Omega))^3}, \quad \forall v_1 \in (H_0^1(\Omega))^3,$$

which implies that a *is coercive. Therefore, using the Lax-Milgram Theorem, we conclude that* (II.21) *has a unique solution* $u \in (H_0^1(\Omega))^3$ *. By classical regularity arguments, we conclude that the solution* u *of* (II.21) *belongs into* $(H^2(\Omega) \cap H_0^1(\Omega))^3$ *. Consequently, we deduce that* (II.17) *has a unique solution* $V \in D(\mathcal{A})$*. This proves that* $Id - \mathcal{A}$ *is surjective. Finally,* (II.15) *and* (II.17) *mean that* $-\mathcal{A}$ *is maximal monotone operator. Then, using Lummer-Phillips theorem (see [22]), we deduce that* $\mathcal{A}$ *is an infinitesimal generator of a linear* C_0*-semigroup on* $\mathcal{H}$*.*

3 Stability

In this section, we investigate the asymptotic behaviour of the solution of problem (II.12). In fact, using the energy method to produce a suitable Lyapunov functional,

We define the energy associated with the solution of (II.1)-(II.2) by

$$\begin{aligned} E_u(t) = \frac{1}{2}\int_\Omega \Big(\mu|\nabla u|^2 + (\lambda+\mu)|\text{div } u|^2 + |u'|^2\Big)\,dx \\ +\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,t,\rho,s)dsd\rho dx \end{aligned} \tag{II.24}$$

Theorem 3.1 *Assume that* (V.2) *and* (II.14) *hold. Then, for any* $\mathcal{U}_0 \in \mathcal{H}$ *, there exist positive constants* δ_1 *and* δ_2*, such that the solution of* (II.12) *satisfies*

$$E(t) \leqslant \delta_2 e^{-\delta_1 t} \quad \forall t \in \mathbb{R}^+. \tag{II.25}$$

We carry out the proof of Theorem 3.1. Firstly, we will estimate several Lemmas.

Lemma 3.2 *Suppose that* μ_1, μ_2 *satisfy* (II.14)*.Then energy functional satisfies, along the solution* u *of* (II.1)-(II.2),

$$E'(t) \leqslant -\Big(\mu_1 - \int_{\tau_1}^{\tau_2}\mu_2(s)\mathrm{d}s\Big)\int_\Omega u'^2(x,t)\,\mathrm{d}x \leqslant 0 \tag{II.26}$$

Proof 3.3 *A differentiation of* $E(t)$ *gives*

$$\begin{aligned} E'(t) = \int_\Omega (\mu\nabla u\nabla u' + (\lambda+\mu)\text{div } u \text{ div } u' + u'u'')\,dx \\ +\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z'(x,t,\rho,s)z(x,t,\rho,s)dsd\rho dx \end{aligned} \tag{II.27}$$

Using (II.11) *and integrating by parts, we get*

$$\begin{aligned} E'(t) \quad &= -\mu_1\int_\Omega u'^2(x,t)dx - \int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,t,1,s)u'(x,t)dsdx \\ &-\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\frac{\partial}{\partial\rho}\Big(z^2(x,t,\rho,s)\Big)dsd\rho dx \\ &= -\mu_1\int_\Omega u'^2(x,t)dx - \int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,t,1,s)u'(x,t)dsdx \\ &-\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,t,1,s)dsdx + \frac{1}{2}\Big(\int_{\tau_1}^{\tau_2}|\mu_2(s)\,ds\Big)\int_\Omega u'^2(x,t)dx \end{aligned} \tag{II.28}$$

Young's inequality leads to the desired estimate.

Lemma 3.4 *The functional*

$$\phi(t) = \int_\Omega u.u' dx \quad \forall t \in \mathbb{R}^+ \tag{II.29}$$

satisfies, along the solution u of (II.1)-(II.2)

$$\begin{aligned}\phi'(t) \leqslant\ & c\int_\Omega |u'|^2 dx - (\mu - c)\int_\Omega |\nabla u|^2 dx \\ & -(\lambda+\mu)\int_\Omega |\text{div } u|^2 dx + c\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,t,1,s)\, dsdx\end{aligned} \tag{II.30}$$

for a positive constant c.

Proof 3.5 *By differentiating* (II.29) *and using* (II.11)*, yields*

$$\begin{aligned}\phi'(t) =\ & \int_\Omega |u'|^2 dx - \mu\int_\Omega |\nabla u|^2 dx - (\lambda+\mu)\int_\Omega |\text{div } u|^2 dx \\ & -\mu_1 \int_\Omega uu' dx - \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| uz(x,t,1,s)\, dsdx\end{aligned} \tag{II.31}$$

By using Young's inequality, we obtain

$$\begin{aligned}\phi'(t) \leqslant\ & (\frac{\mu_1^2}{2}+1)\int_\Omega |u'|^2 dx - \mu\int_\Omega |\nabla u|^2 dx - (\lambda+\mu)\int_\Omega |\text{div } u|^2 dx \\ & +\frac{1}{2}\int_\Omega u^2(x,t)dx + \frac{1}{2}\Big(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\Big)\int_\Omega u^2(x,t)dx \\ & +\frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,t,1,s)\, dsdx\end{aligned} \tag{II.32}$$

Then,Poincaré's inequality leads to the desired estimate.

Lemma 3.6 *The functional*

$$I(t) = \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho} |\mu_2(s)| z^2(x,t,\rho,s) dsd\rho dx, \quad \forall t \in \mathbb{R}^+ \tag{II.33}$$

satisfy

$$\begin{aligned}I'(t) \leqslant\ & -e^{-\tau_2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,t,1,s)\, dsdx + \Big(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\Big)\int_\Omega u'^2(x,t)\, dx \\ & -e^{-\tau_2}\int_\Omega \int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,t,\rho,s)\, dsd\rho dx.\end{aligned} \tag{II.34}$$

Proof 3.7 *Using* (II.10), *the derivative of I entails*

$$
\begin{aligned}
I'(t) &= 2\int_\Omega \int_{\tau_1}^{\tau_2} se^{-s\rho}|\mu_2(s)|z'(x,t,\rho,s)z(x,t,\rho,s)\,dsd\rho dx \\
&= -\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)|e^{-s\rho}\frac{\partial}{\partial\rho}\Big(z^2(x,t,\rho,s)\Big)dsd\rho dx \\
&= -\int_\Omega \int_{\tau_1}^{\tau_2} se^{-s}|\mu_2(s)|z^2(x,t,1,s)\,dsdx + \left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)\int_\Omega u'^2(x,t)\,dx \\
&- \int_\Omega \int_{\tau_1}^{\tau_2} s|\mu_2(s)|\int_0^1 e^{-s\rho}z^2(x,t,\rho,s)\,d\rho dsdx
\end{aligned}
\tag{II.35}
$$

and the desired estimate follows immediately.

Now, we prove our main stability results (II.25).

Proof of Theorem 3.1 Let

$$
L(t) = NE(t) + \epsilon\phi(t) + I(t), \tag{II.36}
$$

where N and ϵ are positive constants that will be fixed later. Taking the derivative of $L(t)$ with respect to t and making use of (II.26), (II.29) and (II.34), we obtain

$$
\begin{aligned}
L'(t) \leqslant\ & -\left\{\left(\mu_1 - \int_{\tau_1}^{\tau_2}\mu_2(s)\mathrm{d}s\right)N - c\epsilon - \int_{\tau_1}^{\tau_2}\mu_2(s)\mathrm{d}s\right\}\int_\Omega |u'|^2dx \\
& -(\lambda+\mu)\int_\Omega |\mathrm{div}\ u|^2dx - (\mu - c)\epsilon\int_\Omega |\nabla u|^2dx \\
& -(e^{-\tau_2} - c\epsilon)\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,t,1,s)\,dsdx - e^{-\tau_2}\int_\Omega\int_{\tau_1}^{\tau_2}s|\mu_2(s)|z^2(x,t,\rho,s)\,dsd\rho dx
\end{aligned}
\tag{II.37}
$$

At this point, we choose our constants in (II.37), carefully, such that all the coefficients in (II.37) will be negative. It suffices to choose ϵ so small such that

$$
e^{-\tau_2} - c\epsilon > 0
$$

then pick N large enough such that

$$
\left(\mu_1 - \int_{\tau_1}^{\tau_2}\mu_2(s)\mathrm{d}s\right)N - c\epsilon - \int_{\tau_1}^{\tau_2}\mu_2(s)\mathrm{d}s > 0
$$

Consequently, recalling (II.24), we deduce that there exist also $\eta_2 > 0$, such that

$$
\frac{dL(t)}{dt} \leqslant -\eta_2 E(t), \quad \forall t \geqslant 0. \tag{II.38}
$$

On the other hand, it is not hard to see that from (II.36) and for N large enough, there exist two positive constants β_1 and β_2 such that

$$\beta_1\ E(t) \leqslant L(t) \leqslant \beta_2 E(t), \quad \forall t \geqslant 0. \tag{II.39}$$

Combining (II.38) and (II.39), we deduce that there exists $\lambda > 0$ for which the estimate

$$\frac{dL(t)}{dt} \leqslant -\lambda L(t), \quad \forall t \geqslant 0, \tag{II.40}$$

holds. Integrating (II.38) over $(0, t)$ and using (II.38) once again, then (II.25) holds. Then, the proof is complete.

III

Well-posedness and exponential stability for coupled Lamé system with a viscoelastic damping

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Let us consider the following a coupled Lamé system :

$$\begin{cases} u_{tt}(x,t) + \alpha v - \Delta_e u(x,t) + \int_0^t g(s)\Delta u(t-s)ds - \mu_1 \Delta u_t(x,t) = 0, & \text{in } \Omega \times (0,+\infty), \\ v_{tt}(x,t) + \alpha u - \Delta_e v(x,t) - \mu_2 \Delta v_t(x,t) = 0, & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = v(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)) & \text{in } \Omega, \\ (u_t(x,0), v_t(x,0)) = (u_1(x), v_1(x)) & \text{in } \Omega. \end{cases} \tag{III.1}$$

Where μ_1, μ_2 are positive constants and (u_0, u_1, v_0, v_1) are given history and initial data . Here Δ denotes the Laplacian operator and Δ_e denotes the elasticity operator, which is the 3×3 matrix-valued differential operator defined by

$$\Delta_e u = \mu \Delta u + (\lambda + \mu)\nabla(\text{div u}), \quad u = (u_1, u_2, u_3)^T$$

and μ and λ are the Lamé constants which satisfy the conditions

$$\mu > 0, \quad \lambda + \mu \geqslant 0. \tag{III.2}$$

.

The problem of stabilization of coupled systems has also been studied by several authors see [2, 4, 6, 15, 19, 23, 24]and the references therein.Under certain conditions imposed on the subset where the damping term is effective, Komornik [15] proves uniform stabilization of the solutions of a pair of hyperbolic systems coupled in velocities. Alabau and al.[2] studied the indirect internal stabilization of weakly coupled systems where the damping is effective in the whole domain. They prove that the behavior of the first equation is sufficient to stabilize the total system and to have polynomial decay for sufficiently smooth solutions. For coupled systems in thermoelasticity, R.Racke [24] considered the following system:

$$\begin{cases} u_{tt}(x,t) - a u_{xx}(x, t-\tau) + b\theta_x(x,t) = 0, & \text{in } (0,L) \times (0,\infty), \\ \theta_t(x,t) - d\theta_{xx}(x,t) + b u_{tx}(x,t) = 0, & \text{in } (0,L) \times (0,\infty), \end{cases}$$

He proved that the internal time delay leads to ill-posedness of the system. However, the system without delay is exponentially stable.

In [18] M.I.Mustafa considered the following system:

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \displaystyle\int_0^t g_1(t-\tau)\Delta u(\tau)d\tau + f_1(u,v) = 0, & \text{in } \Omega \times (0,+\infty), \\ v_{tt}(x,t) - \Delta v(x,t) + \displaystyle\int_0^t g_2(t-\tau)\Delta v(\tau)d\tau + f_2(u,v) = 0, & \text{in } \Omega \times (0,+\infty), \\ u = v = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ (u(.,0) = u_0, u_t(.,0) = u_1, v(.0) = v_0, v_t(.,0) = v_1 & \text{in } \Omega. \end{cases} \tag{III.3}$$

The author proved the well-posedness and, for a wider class of relaxation functions, establish a generalized stability result for this system.

Recently, Beniani and al. [4]considered the following Lamé system with time varying delay term:

$$\begin{cases} u''(x,t) - \Delta_e u(x,t) + \mu_1 g_1(u'(x,t)) + \mu_2 g_2(u'(x,t-\tau(t))) = 0 & \text{in } \Omega \times \mathbb{R}^+ \\ u(x,t) = 0 & \text{on } \partial\Omega \times \mathbb{R}^+ \end{cases} \tag{III.4}$$

and under suitable conditions, they proved general decay of energy.

The paper is organized as follows. The well-posedness of the problem is analyzed in Section 3 using the Faedo-Galerkin method. In Section 4, we prove the exponential decay of the energy when time goes to infinity.

2 Preliminaries and statement of main results

In this section, we present some materials that shall be used for proving our main results. For the relaxation function g, we have the folloing assumptions:

(A1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^1 function satisfying

$$g \in L^1(0,\infty)\, g(0) > 0, \quad 0 < \beta(t) := \mu - \int_0^t g(s)\mathrm{d}s \quad \textit{and} \quad 0 < \beta_0 := \mu - \int_0^\infty g(s)\mathrm{d}s.$$

(A2) There exist a non-increasing differentiable function $\xi(t) : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$g'(t) \leqslant -\xi(t)g(t), \quad \forall t \geqslant 0 \quad \text{and} \quad \int_0^\infty \xi(t)\mathrm{d}t = +\infty.$$

These hypotheses imply that

$$\beta_0 \leqslant \beta(t) \leqslant \mu. \tag{III.5}$$

Let us introduce the following notations:

$$(g * h)(t) := \int_0^t g(t-s)h(s)ds,$$

$$(g \circ h)(t) := \int_0^t g(t-s)|h(t) - h(s)|^2 ds.$$

Lemma 2.1 ([8]) *For any $g, h \in C^1(\mathbb{R})$, the following equation holds*

$$2[g * h]h' = g' \circ h - g(t)|h|^2 - \frac{d}{dt}\Big\{g \circ h - \Big(\int_0^t g(s)ds\Big)|h|^2\Big\}.$$

The existence and uniqueness result is stated as follows:

Theorem 2.2 *Assume that* $(A1)$ *and* $(A2)$ *hold. Then given* $u_0, v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1, v_1 \in (L^2(\Omega))$, *there exists a unique weak solution* u, v *of problem* (III.1) *such that*

$$u, v \in C([0, +\infty[, H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0, +\infty[, L^2(\Omega)).$$

For any regular solution of (III.1), we define the energy as

$$\begin{aligned} E(t) = &\frac{1}{2}\int_\Omega u_t^2(x,t)dx + \frac{\beta(t)}{2}\int_\Omega |\nabla u|^2(x,t)dx + \frac{1}{2}\int_\Omega (g \circ \nabla u)dx + \frac{(\mu+\lambda)}{2}\int_\Omega |\text{div } u|^2 dx \\ &+ \frac{1}{2}\int_\Omega v_t^2(x,t)dx + \frac{\mu}{2}\int_\Omega |\nabla v|^2(x,t)dx + \frac{(\mu+\lambda)}{2}\int_\Omega |\text{div } v|^2 dx + 2\alpha \int_\Omega u(x,t)v(x,t)dx. \end{aligned} \tag{III.6}$$

Our decay result reads as follows:

Theorem 2.3 *Let* (u, v) *be the solution of* (III.1). *Assume that* $(A1)$ *and* $(A2)$ *hold. Then there exist two positives constants* C *and* d, *such that*

$$E(t) \leqslant Ce^{-d\int_0^t \xi(s)ds}, \quad \forall t \geqslant 0. \tag{III.7}$$

3 Well-posedness of the problem

In this section, we will prove the existence and uniqueness of problem (III.1) by using Faedo-Galerkin method.

Proof 3.1 *We divide the proof of Theorem2.2into two steps:the Faedo-Galerkin approximation and the energy estimates.*

Step 1 :Faedo-Galerkin approximation.

We construct approximations of the solution (u, v) *by the Faedo-Galerkin method as follows. For* $n \geqslant 1$, *let* $W_n = span\ \{w_1, ..., w_n\}$ *be a Hilbertian basis of the space* H_0^1 *and the projection of the initial data on the finite dimensional subspace* W_n *is given by*

$$u_0^n = \sum_{i=1}^n a_i w_i, \quad v_0^n = \sum_{i=1}^n b_i w_i, \quad u_1^n = \sum_{i=1}^n c_i w_i, \quad v_1^n = \sum_{i=1}^n d_i w_i$$

where,$(u_0^n, v_0^n, u_1^n, v_1^n) \to (u_0, v_0, u_1, v_1)$ *strongly in* $H^2(\Omega) \cap H_0^1(\Omega)$ *as* $n \to \infty$. *We search the approximate solutions*

$$u^n(x,t)=\sum_{i=1}^{n} f_i^n(t)w_i(x), \quad v^n(x,t)=\sum_{i=1}^{n} h_i^n(t)w_i(x)$$

to the finite dimensional Cauchy problem:

$$\begin{cases} \int_\Omega u_{tt}^n w_i dx+\alpha\int_\Omega v^n w_i dx+\mu\int_\Omega \nabla u^n\nabla w_i dx+(\lambda+\mu)\int_\Omega \text{div } u^n.\text{div } w_i dx \\ -\int_\Omega (g(s)*\nabla u^n)\nabla w_i dx+\mu_1\int_\Omega \nabla u_t^n\nabla w_i dx=0, \\ \int_\Omega v_{tt}^n w_i dx+\alpha\int_\Omega u^n w_i dx+\mu\int_\Omega \nabla v^n\nabla w_i dx \\ +(\lambda+\mu)\int_\Omega \text{div } v^n.\text{div } w_i dx+\mu_2\int_\Omega \nabla v_t^n\nabla w_i dx=0, \\ (u^n(0),v^n(0))=(u_0^n,v_0^n) \quad (u_t^n(0),v_t^n(0))=(u_1^n,v_1^n). \end{cases} \tag{III.8}$$

According to the standard theory of ordinary differential equations, the finite dimensional problem (III.8) *has solution* $f_i^n(t),h_i^n(t)$ *defined on* $[0,t)$*.The a priori estimates that follow imply that in fact* $t_n=T$*.*

Step 2: Energy estimates. Multiplying the first and the second equation of (III.8) *by* $\Big(f_i^n(t)\Big)'$ *and* $\Big(h_i^n(t)\Big)'$ *respectively, we obtain:*

$$\begin{aligned} &\int_\Omega u_{tt}^n u_t^n dx+\alpha\int_\Omega v^n u_t^n dx+\mu\int_\Omega \nabla u^n\nabla u_t^n dx+(\lambda+\mu)\int_\Omega \text{div } u^n.\text{div } u_t^n dx \\ &-\int_\Omega (g(s)*\nabla u^n)\nabla u_t^n dx+\mu_1\int_\Omega |\nabla u_t^n|^2 dx=0. \end{aligned} \tag{III.9}$$

and

$$\begin{aligned} &\int_\Omega v_{tt}^n v_t^n dx+\alpha\int_\Omega u^n v_t^n dx+\mu\int_\Omega \nabla v^n\nabla v_t^n dx+(\lambda+\mu)\int_\Omega \text{div } v^n.\text{div } v_t^n dx \\ &+\mu_2\int_\Omega |\nabla v_t^n|^2 dx=0. \end{aligned} \tag{III.10}$$

Integrating (III.9) *and* (III.10) *over* $(0,t)$*,and using Lemma* (2.1)*, we obtain*

$$\begin{aligned} &\mathcal{E}_n(t)+\mu_1\int_0^t\int_\Omega |\nabla u_t^n|^2 dxds-\frac{1}{2}\int_\Omega (g'\circ\nabla u^n)dx+\frac{1}{2}\int_0^t\int_\Omega g(t)|\nabla u^n|^2 dxds \\ &+\mu_2\int_0^t\int_\Omega |\nabla v_t^n|^2 dxds=\mathcal{E}_n(0) \end{aligned} \tag{III.11}$$

where

$$\begin{aligned}\mathcal{E}_n(t) = & \frac{1}{2}\int_\Omega (u_t^n)^2(x,t)dx + \frac{\beta(t)}{2}\int_\Omega |\nabla u^n|^2(x,t)dx + \frac{1}{2}\int_\Omega (g \circ \nabla u^n)dx \\ & + \frac{(\mu+\lambda)}{2}\int_\Omega |\text{div } u^n|^2 dx + \frac{1}{2}\int_\Omega (v_t^n)^2(x,t)dx + \frac{\mu}{2}\int_\Omega |\nabla v^n|^2(x,t)dx \\ & + \frac{(\mu+\lambda)}{2}\int_\Omega |\text{div } v^n|^2 dx + 2\alpha\int_\Omega u^n(x,t)v^n(x,t)dx.\end{aligned} \tag{III.12}$$

Consequently, fromIII.11, we have the following estimate:

$$\mathcal{E}_n(t) - \frac{1}{2}\int_\Omega (g' \circ \nabla u^n)dx + \frac{1}{2}\int_0^t \int_\Omega g(t)|\nabla u^n|^2 dxds \leqslant \mathcal{E}_n(0). \tag{III.13}$$

Now, since the sequences $\left(u_0^n\right)_{n\in\mathbb{N}}$, $\left(u_1^n\right)_{n\in\mathbb{N}}$, $\left(v_0^n\right)_{n\in\mathbb{N}}$, $\left(v_1^n\right)_{n\in\mathbb{N}}$ *converge and using (A2), in the both cases we can find a positive constant* c *independent of* n *such that*

$$\mathcal{E}_n(t) \leqslant c. \tag{III.14}$$

Therefore, using the fact that $\beta(t) \geqslant \beta(0)$, *the estimate III.14 together with III.13 give us, for all* $n \in \mathbb{N}$,$t_n = T$, *we deduce*

$$\begin{array}{lll}\left(u^n\right)_{n\in\mathbb{N}} & \textit{is bounded in} & L^\infty(0,T;H_0^1(\Omega)), \\ \left(v^n\right)_{n\in\mathbb{N}} & \textit{is bounded in} & L^\infty(0,T;H_0^1(\Omega)), \\ \left(u_t^n\right)_{n\in\mathbb{N}} & \textit{is bounded in} & L^\infty(0,T;L^2(\Omega)), \\ \left(v_t^n\right)_{n\in\mathbb{N}} & \textit{is bounded in} & L^\infty(0,T;L^2(\Omega)).\end{array} \tag{III.15}$$

Consequently, we conclude that

$$\begin{array}{lll}u^n \rightharpoonup u & \textit{weakly star in} & L^\infty(0,T;H_0^1(\Omega)), \\ v^n \rightharpoonup v & \textit{weakly star in} & L^\infty(0,T;H_0^1(\Omega)), \\ u_t^n \rightharpoonup u_t & \textit{weakly star in} & L^\infty(0,T;H_0^1(\Omega)), \\ v_t^n \rightharpoonup v_t & \textit{weakly star in} & L^\infty(0,T;H_0^1(\Omega)).\end{array} \tag{III.16}$$

From III.15, we have $\left(u^n\right)_{n\in\mathbb{N}}$ *and* $\left(v^n\right)_{n\in\mathbb{N}}$ *are bounded in* $L^\infty(0,T;H_0^1(\Omega))$.*Then* $\left(u^n\right)_{n\in\mathbb{N}}$ *and* $\left(v^n\right)_{n\in\mathbb{N}}$ *are bounded in* $L^2(0,T;H_0^1(\Omega))$. *Consequently,* $\left(u^n\right)_{n\in\mathbb{N}}$ *and*

$\left(v^n\right)_{n\in\mathbb{N}}$ *are bounded in* $H^1(0,T;H^1(\Omega))$*. Since the embedding*

$$H^1(0,T;H^1(\Omega)) \hookrightarrow L^2(0,T;L^2(\Omega))$$

is compact,using Aubin-Lion's theorem [16] ,we can extract subsequences $\left(u^k\right)_{k\in\mathbb{N}}$ *of* $\left(u^n\right)_{n\in\mathbb{N}}$ *and* $\left(v^k\right)_{k\in\mathbb{N}}$ *of* $\left(v^n\right)_{n\in\mathbb{N}}$ *such that*

$$u^k \to u \quad \textit{strongly in} \quad L^2(0,T;L^2(\Omega))$$

and

$$v^k \to v \quad \textit{strongly in} \quad L^2(0,T;L^2(\Omega))$$

Therefore,

$$u^k \to u \quad \textit{strongly and a.e} \quad (0,T)\times(\Omega)$$

and

$$v^k \to v \quad \textit{strongly and a.e} \quad (0,T)\times(\Omega)$$

The proof now can be completed arguing as in Theorem 3.1 of [16]

4 Exponential stability

In this section we study the asymptotic behavior of the system (III.1). For the proof of Theorem 2.3 we use the following lemmas.

Lemma 4.1 *Let* (u,v) *be the solution of* (III.1)*, Then we have the inequality*

$$\begin{aligned}\frac{dE(t)}{dt} &\leqslant -\mu_1\int_\Omega |\nabla u_t(x,t)|^2\,dx - \mu_2\int_\Omega |\nabla v_t(x,t)|^2\,dx - \frac{1}{2}g(t)\int_\Omega |\nabla u(x,t)|^2\,dx \\ &+\frac{1}{2}\int_\Omega (g'\circ\nabla u)dx\end{aligned} \tag{III.17}$$

Proof 4.2 *From* (III.6) *we have*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega \Big(u_t^2 + (\lambda+\mu)|\text{div } u|^2 + v_t^2 + \mu|\nabla v|^2 + (\lambda+\mu)|\text{div } v|^2 + 2\alpha vu\Big)\,dx \\ &= -\mu\int_\Omega \nabla u\nabla u_t dx - \mu_1\int_\Omega |\nabla u_t|^2 dx - \mu_2\int_\Omega |\nabla v_t|^2 dx + \int_\Omega\int_0^t g(s)\nabla u(s)\nabla u_t(t)dsdx\end{aligned} \tag{III.18}$$

From Lemma 2.1, the last term in the right-hand side of III.18 can be rewritten as

$$\begin{aligned}&\int_0^t g(s)\int_\Omega \nabla u(s)\nabla u_t(t)dsdx+\frac{1}{2}g(t)\int_\Omega |\nabla u|^2(x,t)dx\\&=\frac{1}{2}\frac{d}{dt}\Big\{\int_0^t g(s)\int_\Omega |\nabla u|^2(x,t)dxds-\int_\Omega (g\circ\nabla u)(t)dx\Big\}+\frac{1}{2}\int_\Omega (g'\circ\nabla u)(t)dx\end{aligned} \tag{III.19}$$

So $\dfrac{dE}{dt}$ *becomes:*

$$\begin{aligned}\frac{dE}{dt}&=-\mu_1\int_\Omega |\nabla u_t|^2dx-\mu_2\int_\Omega |\nabla v_t|^2dx-\frac{1}{2}g(t)\int_\Omega |\nabla u|^2(x,t)dx\\&+\frac{1}{2}\int_\Omega (g'\circ\nabla u)(t)dx\\&\leqslant 0.\end{aligned} \tag{III.20}$$

we show that (III.17) *holds. The proof is complete.*

Now, we define the functional $\mathscr{D}(t)$ as follows

$$\mathscr{D}(t)=\int_\Omega uu_tdx+\int_\Omega vv_tdx+\frac{\mu_1}{2}\int_\Omega |\nabla u|^2dx+\frac{\mu_2}{2}\int_\Omega |\nabla v|^2dx. \tag{III.21}$$

Then, we have the following estimate.

Lemma 4.3 *The functional* $\mathscr{D}(t)$ *satisfies*

$$\begin{aligned}\mathscr{D}'(t)\leqslant C\int_\Omega |\nabla u_t|^2dx+C\int_\Omega |\nabla v_t|^2dx+\Big(\delta+|\alpha|C-\beta(t)\Big)\int_\Omega |\nabla u|^2dx-(\lambda+\mu)\int_\Omega |\text{div }u|^2dx\\+\Big(|\alpha|C-\mu\Big)\int_\Omega |\nabla v|^2dx-(\lambda+\mu)\int_\Omega |\text{div }v|^2dx+\frac{\mu-\beta(t)}{4\delta}\int_\Omega (g\circ\nabla u)(t)dx\end{aligned} \tag{III.22}$$

Proof 4.4 *Taking the derivative of* $\mathscr{D}(t)$ *with respect to* t *and using* (III.1), *we find that:*

$$\begin{aligned}\mathscr{D}'(t)&=\int_\Omega u_t^2dx+\int_\Omega uu_{tt}dx+\int_\Omega v_t^2dx+\int_\Omega vv_{tt}dx+\mu_1\int_\Omega \nabla u_t\nabla udx+\mu_2\int_\Omega \nabla v_t\nabla vdx\\&=\int_\Omega u_t^2dx+\int_\Omega v_t^2dx-\beta(t)\int_\Omega |\nabla u|^2(x,t)dx+\int_\Omega\int_0^t g(s)(\nabla u(s)-\nabla u(t))\nabla u(t)dsdx\\&\quad-\mu\int_\Omega |\nabla v|^2dx-(\lambda+\mu)\int_\Omega |\text{div }u|^2dx-\int_\Omega (\lambda+\mu)|\text{div }v|^2dx-2\alpha\int_\Omega uv\,dx\end{aligned} \tag{III.23}$$

Using the fact that

$$\int_\Omega \int_0^t g(s)|\nabla u(s) - \nabla u(t)|\nabla u(t) ds dx \leqslant \delta \int_\Omega |\nabla u|^2(x,t)dx + \frac{1}{4\delta}\int_\Omega \Big(\int_0^t g(s)|\nabla u(s) - \nabla u(t)| ds \Big)^2 dx$$
$$\leqslant \delta \int_\Omega |\nabla u|^2(x,t)dx + \frac{\mu - \beta(t)}{4\delta} \int_\Omega (g \circ \nabla u)(t)dx. \qquad \text{(III.24)}$$

Inserting the estimate (III.24) *into* (III.23) *and using Young's, Poincaré's inequalities lead to the desired estimate. The proof is complete.*

Proof 4.5 (Proof of Theorem 2.3) *We define the Lyapunov functional*

$$\mathscr{L}(t) = NE(t) + \epsilon \mathscr{D}(t), \qquad \text{(III.25)}$$

where N and ϵ are positive constants that will be fixed later.

Taking the derivative of (III.25) *with respect to t and making use of* (III.17), (III.22), *we obtain*

$$\begin{aligned}\frac{d}{dt}\mathscr{L}(t) \leqslant & -\Big\{ N\mu_1 - \epsilon C \Big\} \int_\Omega |\nabla u_t(x,t)|^2 dx - \Big\{ N\mu_2 - \epsilon C \Big\} \int_\Omega |\nabla v_t(x,t)|^2 dx \\ & - \Big(\beta(t) - \delta - |\alpha| C \Big) \epsilon \int_\Omega |\nabla u|^2 dx - \Big(\mu - |\alpha| C \Big) \epsilon \int_\Omega |\nabla v|^2 dx \\ & - (\lambda + \mu)\epsilon \int_\Omega |\text{div } u|^2 dx \ - (\lambda+\mu)\epsilon \int_\Omega |\text{div } v|^2 dx \\ & + \frac{N}{2}\int_\Omega (g' \circ \nabla u)(t) dx + \frac{(\mu - \beta(t))\epsilon}{4\delta} \int_\Omega (g \circ \nabla u)(t) dx \\ & - \frac{N}{2} g(t) \int_\Omega |\nabla u|^2(x,t) dx. \end{aligned} \qquad \text{(III.26)}$$

At this point, we choose our constants in (III.26)*, carefully, such that all the coefficients in* (III.26) *will be negative. It suffices to choose ϵ so small and N large enough such that*

$$N\mu_1 - \epsilon C > 0,$$

and

$$N\mu_2 - \epsilon C > 0,$$

Further, we choose α small enough such that

$$\beta(t) - \delta - |\alpha| C > 0,$$

and

$$\mu - |\alpha|C > 0.$$

Consequently, from the above, we deduce that there exist there exists two positive constants η_1 *and* η_2*such that* (III.26) *becomes*

$$\frac{d\mathscr{L}(t)}{dt} \leqslant -\eta_1 E(t) + \eta_2 \int_\Omega (g \circ \nabla u) dx \tag{III.27}$$

By multiplying (III.27) *by* $\xi(t)$*, we arrive at*

$$\xi(t)\mathscr{L}'(t) \leqslant -\eta_1 \xi(t) E(t) + \eta_2 \xi(t) \int_\Omega (g \circ \nabla u) dx \tag{III.28}$$

Recalling (A2) and using (III.17)*,we get*

$$\begin{aligned} \xi(t)\mathscr{L}'(t) &\leqslant -\eta_1 \xi(t) E(t) - \eta_2 \int_\Omega (g' \circ \nabla u) dx \\ &\leqslant -\eta_1 \xi(t) E(t) - 2\eta_2 E'(t) \end{aligned} \tag{III.29}$$

That is

$$\Big(\xi(t)\mathscr{L}(t) + 2\eta_2 E(t)\Big)' - \xi'(t)\mathscr{L} \leqslant -\eta_1 \xi(t) E(t)$$

Using the fact that $\xi'(t) \leqslant 0, \quad \forall t \geqslant 0$ *and letting*

$$\mathscr{F}(t) = \xi(t)\mathscr{L}(t) + 2\eta_2 E(t) \sim E(t) \tag{III.30}$$

we obtain

$$\mathscr{F}'(t) \leqslant -\eta_1 \xi(t) E(t) \leqslant -\eta_3 \xi(t) \mathscr{F}(t) \tag{III.31}$$

A simple integration of (III.31) *over* $(0,t)$ *leads to*

$$\mathscr{F}(t) \leqslant \mathscr{F}(0) e^{-\eta_3 \int_0^t \xi(s) ds}, \quad \forall t \geqslant 0 \tag{III.32}$$

A combination of (III.30) *and* (III.32) *leads to* (III.7)*. Then, the proof is complete.*

IV

well-posedness and exponential stability for coupled Lamé system with a viscoelastic term and a strong delay

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Let us consider the following a coupled Lamé system :

$$\begin{cases} u_{tt}(x,t) + \alpha v - \Delta_e u(x,t) + \int_0^t g(s)\Delta u(t-s)ds & \\ \qquad\qquad -\mu_1 \Delta u_t(x,t) - \lambda_1 \Delta_x u_t(x,t-\tau) = 0, & \text{in } \Omega \times (0,+\infty), \\ v_{tt}(x,t) + \alpha u - \Delta_e v(x,t) - \mu_2 \Delta v_t(x,t) = 0, & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = v(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)) & \text{in } \Omega, \\ (u_t(x,0), v_t(x,0)) = (u_1(x), v_1(x)) & \text{in } \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), & \text{in } \Omega \times [0,\tau]. \end{cases} \tag{IV.1}$$

Where μ_1, μ_2 are positive constants and (u_0, u_1, v_0, v_1) are given history and initial data . Here Δ denotes the Laplacian operator and Δ_e denotes the elasticity operator, which is the 3×3 matrix-valued differential operator defined by

$$\Delta_e u = \mu \Delta u + (\lambda + \mu) \nabla (\text{div u}), \quad u = (u_1, u_2, u_3)^T$$

and μ and λ are the Lamé constants which satisfy the conditions

$$\mu > 0, \quad \lambda + \mu \geqslant 0. \tag{IV.2}$$

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The problem of stabilization of coupled systems has also been studied by several authors see [2, 4, 6, 13, 15, 23, 24]and the references therein.Under certain conditions imposed on the subset where the damping term is effective, Komornik [15] proves uniform stabilization of the solutions of a pair of hyperbolic systems coupled in velocities. Alabau and al.[2] studied the indirect internal stabilization of weakly coupled systems where the damping is effective in the whole domain. They prove that the behavior of the first equation is sufficient to stabilize the total system and to have polynomial decay for sufficiently smooth solutions. For coupled systems in thermoelasticity, R.Racke [24] considered the following system:

$$\begin{cases} u_{tt}(x,t) - a u_{xx}(x, t-\tau) + b\theta_x(x,t) = 0, & \text{in } (0, L) \times (0, \infty), \\ \theta_t(x,t) - d\theta_{xx}(x,t) + b u_{tx}(x,t) = 0, & \text{in } (0, L) \times (0, \infty), \end{cases}$$

He proved that the internal time delay leads to ill-posedness of the system. However, the system without delay is exponentially stable.

In [25] the authors examined a transmission problem with a viscoelastic term and a delay:

$$\begin{cases} u_{tt}(x,t) - a u_{xx}(x,t) + \displaystyle\int_0^t g(t-s) u_{xx}(x,s) \mathrm{d}s & \\ +\mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau) = 0, & (x,t) \in \Omega \times (0, +\infty), \\ v_{tt}(x,t) - b v_{xx}(x,t) = 0, & (x,t) \in (L_1, L_2) \times (0, +\infty), \end{cases}$$

under appropriate hypotheses on the relaxation function and the relationship between the weight of the damping and the weight of the delay, they proved the well-posedness result and exponential decay of the energy.

In [18] M.I.Mustafa considered the following system:

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \displaystyle\int_0^t g_1(t-\tau)\Delta u(\tau)d\tau + f_1(u,v) = 0, & \text{in } \Omega \times (0,+\infty), \\ v_{tt}(x,t) - \Delta v(x,t) + \displaystyle\int_0^t g_2(t-\tau)\Delta v(\tau)d\tau + f_2(u,v) = 0, & \text{in } \Omega \times (0,+\infty), \\ u = v = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ (u(.,0) = u_0, u_t(.,0) = u_1, v(.0) = v_0, v_t(.,0) = v_1 & \text{in } \Omega. \end{cases}$$

The author proved the well-posedness and, for a wider class of relaxation functions, establish a generalized stability result for this system.

Recently, Beniani and al. [4]considered the following Lamé system with time varying delay term:

$$\begin{cases} u''(x,t) - \Delta_e u(x,t) + \mu_1 g_1(u'(x,t)) + \mu_2 g_2(u'(x,t-\tau(t))) = 0 & \text{in } \Omega \times \mathbb{R}^+ \\ u(x,t) = 0 & \text{on } \partial\Omega \times \mathbb{R}^+ \end{cases}$$

and under suitable conditions, they proved general decay of energy.

The paper is organized as follows. The well-posedness of the problem is analyzed in Section 3 using the Faedo-Galerkin method. In Section 4, we prove the exponential decay of the energy when time goes to infinity.

2 Preliminaries and statement of main results

In this section, we present some materials that shall be used for proving our main results. For the relaxation function g, we have the folloing assumptions:

(A1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^1 function satisfying

$$g \in L^1(0,\infty)\, g(0) > 0, \quad 0 < \beta(t) := \mu - \int_0^t g(s)\mathrm{d}s \quad \textit{and} \quad 0 < \beta_0 := \mu - \int_0^\infty g(s)\mathrm{d}s.$$

(A2) There exist a non-increasing differentiable function $\xi(t) : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$g'(t) \leqslant -\xi(t)g(t), \quad \forall t \geqslant 0 \quad \text{and} \quad \int_0^\infty \xi(t)\mathrm{d}t = +\infty.$$

These hypotheses imply that

$$\beta_0 \leqslant \beta(t) \leqslant \mu. \tag{IV.3}$$

Let us introduce the following notations:

$$(g * h)(t) := \int_0^t g(t-s)h(s)ds,$$

$$(g \circ h)(t) := \int_0^t g(t-s)|h(t)-h(s)|^2 ds.$$

Lemma 2.1 ([8]) *For any $g, h \in C^1(\mathbb{R})$, the following equation holds*

$$2[g * h]h' = g' \circ h - g(t)|h|^2 - \frac{d}{dt}\Big\{g \circ h - \Big(\int_0^t g(s)ds\Big)|h|^2\Big\}.$$

As in [21], we introduce the following new variable:

$$z(x,\rho,t) = u_t(x, t-\tau\rho) \quad \text{in } \Omega \times (0,1) \times (0,+\infty). \tag{IV.4}$$

Then, we obtain

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad \text{in } \Omega \times (0,1) \times (0,+\infty). \tag{IV.5}$$

Therefore, problem (IV.1) is equivalent to

$$\begin{cases} u_{tt}(x,t) + \alpha v - \Delta_e u(x,t) + \displaystyle\int_0^t g(s)\Delta u(t-s)ds & \\ \qquad\qquad -\mu_1 \Delta u_t(x,t) - \lambda_1 \Delta_x z(x,1,t) = 0, & \text{in } \Omega \times (0,+\infty), \\ v_{tt}(x,t) + \alpha u - \Delta_e v(x,t) - \mu_2 \Delta v_t(x,t) = 0, & \text{in } \Omega \times (0,+\infty), \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & \text{in } \Omega \times (0,1) \times (0,+\infty), \\ u(x,t) = v(x,t) = 0, & \text{on } \partial\Omega \times (0,+\infty), \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)), & \text{in } \Omega, \\ (u_t(x,0), v_t(x,0)) = (u_1(x), v_1(x)), & \text{in } \Omega, \\ z(x,1,t) = f_0(x, t-\tau), & \text{in } \Omega \times [0,\tau]. \end{cases} \tag{IV.6}$$

The existence and uniqueness result is stated as follows:

Theorem 2.2 *Assume that $|\lambda_1| \leqslant \mu_1$, (A1) and (A2) hold.*

Then given $u_0, v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$,$u_1, v_1 \in (L^2(\Omega))^2$, there exists a unique weak solution u, v, z of problem (IV.6) *such that*

$$u, v \in C([0,+\infty[, H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,+\infty[, (L^2(\Omega))^2),$$

$$z \in C([0,+\infty[;L^2((0,1),\Omega)).$$

For any regular solution of (IV.1), we define the energy as

$$\begin{aligned}E(t) &= \frac{1}{2}\int_\Omega u_t^2(x,t)dx + \frac{\beta(t)}{2}\int_\Omega |\nabla u|^2(x,t)dx + \frac{1}{2}\int_\Omega (g\circ\nabla u)dx + \frac{(\mu+\lambda)}{2}\int_\Omega |\mathrm{div}\ u|^2dx \\ &+ \frac{1}{2}\int_\Omega v_t^2(x,t)dx + \frac{\mu}{2}\int_\Omega |\nabla v|^2(x,t)dx + \frac{(\mu+\lambda)}{2}\int_\Omega |\mathrm{div}\ v|^2dx + 2\alpha\int_\Omega u(x,t)v(x,t)dx \\ &+ \frac{\tau\lambda_1}{2}\int_\Omega\int_0^1 |\nabla z(x,\rho,t)|^2\,d\rho\,dx.\end{aligned} \tag{IV.7}$$

Our decay result reads as follows:

Theorem 2.3 *Let (u,v,z) be the solution of* (IV.6). *Assume that $|\lambda_1| \leqslant \mu_1$, (A1) and (A2) hold. Then there exist two positive constants C and d, such that*

$$E(t) \leqslant Ce^{-d\int_0^t \xi(s)ds}, \quad \forall t \geqslant 0. \tag{IV.8}$$

3 Well-posedness of the problem

In this section, we will prove the existence and uniqueness of problem (IV.1) by using Faedo-Galerkin method.

Proof 3.1 *We divide the proof of Theorem2.2into two steps:the Faedo-Galerkin approximation and the energy estimates.*

Step 1 :Faedo-Galerkin approximation.

We construct approximations of the solution (u,v,z) by the Faedo-Galerkin method as follows. For $n \geqslant 1$, let W_n = span $\{w_1, ..., w_n\}$ be a Hilbertian basis of the space H_0^1. Now, we we define for $1 \leqslant i \leqslant n$ the sequence $\varphi_i(x,\rho)$ as follows:

$$\varphi_i(x,0) = w_i(x)$$

Then we may extend $\varphi_i(x,\rho)$ over $L^2((0,1),\Omega)$ and denote V_n = span $\{\varphi_1, ..., \varphi_n\}$. We choose sequences $(u_0^n), (u_1^n), (v_0^n), (v_1^n)$ in W_n and (z_0^n) in V_n such that

$(u_0^n, v_0^n, u_1^n, v_1^n) \to (u_0, v_0, u_1, v_1)$ strongly in $H^2(\Omega) \cap H_0^1(\Omega)$ and $z_0^n \to f_0$ strongly in $L^2((0,1),\Omega)$ as $n \to \infty$.

We search the approximate solutions

$$u^n(x,t)=\sum_{i=1}^{n}f_i^n(t)w_i(x),\quad v^n(x,t)=\sum_{i=1}^{n}h_i^n(t)w_i(x)\quad and\quad z^n(x,\rho,t)=\sum_{i=1}^{n}k_i^n(t)\varphi_i(x,\rho)$$

to the finite dimensional Cauchy problem:

$$\begin{cases}\int_\Omega u_{tt}^n w_i dx+\alpha\int_\Omega v^n w_i dx+\mu\int_\Omega \nabla u^n\nabla w_i dx+(\lambda+\mu)\int_\Omega \text{div } u^n.\text{div } w_i dx\\ -\int_\Omega (g(s)*\nabla u^n)\nabla w_i dx+\mu_1\int_\Omega \nabla u_t^n\nabla w_i dx+\lambda_1\int_\Omega \nabla z^n(x,1,t)\nabla w_i dx=0,\\ \int_\Omega v_{tt}^n w_i dx+\alpha\int_\Omega u^n w_i dx+\mu\int_\Omega \nabla v^n\nabla w_i dx\\ +(\lambda+\mu)\int_\Omega \text{div } v^n.\text{div } w_i dx+\mu_2\int_\Omega \nabla v_t^n\nabla w_i dx=0,\\ (u^n(0),v^n(0))=(u_0^n,v_0^n)\quad (u_t^n(0),v_t^n(0))=(u_1^n,v_1^n),\end{cases}\tag{IV.9}$$

and

$$\begin{cases}\int_\Omega (\tau z_t^n(x,\rho,t)+z_\rho^n(x,\rho,t))\varphi_i dx=0,\\ z^n(x,\rho,0)=z_0^n.\end{cases}\tag{IV.10}$$

According to the standard theory of ordinary differential equations , the finite dimensional problem (IV.9)-(IV.10) *has solution* $f_i^n(t),h_i^n(t),k_i^n(t)$ *defined on* $[0,t)$*.The a priori estimates that follow imply that in fact* $t_n=T$*.*

Step 2: Energy estimates. Multiplying the first and the second equation of (IV.9) *by* $\left(f_i^n(t)\right)'$ *and* $\left(h_i^n(t)\right)'$ *respectively, we obtain:*

$$\begin{aligned}&\int_\Omega u_{tt}^n u_t^n dx+\alpha\int_\Omega v^n u_t^n dx+\mu\int_\Omega \nabla u^n\nabla u_t^n dx+(\lambda+\mu)\int_\Omega \text{div } u^n.\text{div } u_t^n dx\\ &+\lambda_1\int_\Omega \nabla z^n(x,1,t)\nabla u_t^n dx-\int_\Omega (g(s)*\nabla u^n)\nabla u_t^n dx+\mu_1\int_\Omega |\nabla u_t^n|^2 dx=0.\end{aligned}\tag{IV.11}$$

and

$$\begin{aligned}&\int_\Omega v_{tt}^n v_t^n dx+\alpha\int_\Omega u^n v_t^n dx+\mu\int_\Omega \nabla v^n\nabla v_t^n dx+(\lambda+\mu)\int_\Omega \text{div } v^n.\text{div } v_t^n dx\\ &+\mu_2\int_\Omega |\nabla v_t^n|^2 dx=0.\end{aligned}\tag{IV.12}$$

Multiplying the first equation of (IV.10) *by* $\lambda_1 k_i^n(t)$ *and integrating over* $(0,1)\times(0,t)$*,* *we get*

$$\frac{\lambda_1\tau}{2}\int_\Omega\int_0^1(z^n(x,\rho,t))^2d\rho dx+\lambda_1\int_0^t\int_\Omega\int_0^1 z^n_\rho z^n(x,\rho,s)d\rho dxds=\frac{\lambda_1\tau}{2}\int_\Omega\int_0^1(z^n(x,\rho,0))^2d\rho dx, \tag{IV.13}$$

we remark that

$$\lambda_1\int_0^t\int_\Omega\int_0^1 z^n_\rho z^n(x,\rho,s)d\rho dxds=\frac{\lambda_1}{2}\int_\Omega\int_0^t\Big((z^n(x,1,s))^2-(z^n(x,0,s))^2\Big)dsdx, \tag{IV.14}$$

Integrating (IV.11) *and* (IV.12) *over* $(0,t)$, *taking into account* (IV.13),(IV.14) *up and using Lemma* (2.1), *we obtain*

$$\begin{aligned}&\mathcal{E}_n(t)+(\mu_1-\frac{\lambda_1}{2})\int_0^t\int_\Omega|\nabla u^n_t|^2dxds+\lambda_1\int_0^t\int_\Omega\nabla z^n(x,1,s)\nabla u^n_tdxds+\frac{\lambda_1}{2}\int_\Omega\int_0^t((z^n(x,1,s))^2dsdx\\&-\frac{1}{2}\int_\Omega(g'\circ\nabla u^n)dx+\frac{1}{2}\int_0^t\int_\Omega g(t)|\nabla u^n|^2dxds+\mu_2\int_0^t\int_\Omega|\nabla v^n_t|^2dxds\\&=\mathcal{E}_n(0),\end{aligned} \tag{IV.15}$$

where

$$\begin{aligned}\mathcal{E}_n(t)=&\frac{1}{2}\int_\Omega(u^n_t)^2(x,t)dx+\frac{\beta(t)}{2}\int_\Omega|\nabla u^n|^2(x,t)dx+\frac{1}{2}\int_\Omega(g\circ\nabla u^n)dx+\frac{(\mu+\lambda)}{2}\int_\Omega|\text{div }u^n|^2dx\\&+\frac{1}{2}\int_\Omega(v^n_t)^2(x,t)dx+\frac{\mu}{2}\int_\Omega|\nabla v^n|^2(x,t)dx+\frac{(\mu+\lambda)}{2}\int_\Omega|\text{div }v^n|^2dx+2\alpha\int_\Omega u^n(x,t)v^n(x,t)dx\\&+\frac{\lambda_1\tau}{2}\int_\Omega\int_0^1(z^n(x,\rho,t))^2d\rho dx.\end{aligned} \tag{IV.16}$$

Young's inequality gives us that

$$\begin{aligned}&\mathcal{E}_n(t)+(\mu_1-\lambda_1)\int_0^t\int_\Omega|\nabla u^n_t|^2dxds-\frac{1}{2}\int_\Omega(g'\circ\nabla u^n)dx\\&+\frac{1}{2}\int_0^t\int_\Omega g(t)|\nabla u^n|^2dxds+\mu_2\int_0^t\int_\Omega|\nabla v^n_t|^2dxds\\&\leqslant\mathcal{E}_n(0).\end{aligned} \tag{IV.17}$$

Consequently, using that $|\lambda_1|\leqslant\mu_1$, *we have the following estimate:*

$$\mathcal{E}_n(t) - \frac{1}{2}\int_\Omega (g' \circ \nabla u^n)dx + \frac{1}{2}\int_0^t \int_\Omega g(t)|\nabla u^n|^2 dxds \leqslant \mathcal{E}_n(0). \tag{IV.18}$$

Now, since the sequences $\left(u_0^n\right)_{n\in\mathbb{N}}$, $\left(u_1^n\right)_{n\in\mathbb{N}}$, $\left(v_0^n\right)_{n\in\mathbb{N}}$, $\left(v_1^n\right)_{n\in\mathbb{N}}$, $\left(z_0^n\right)_{n\in\mathbb{N}}$ *converge and using (A2), in the both cases we can find a positive constant c independent of n such that*

$$\mathcal{E}_n(t) \leqslant c. \tag{IV.19}$$

Therefore, using the fact that $\beta(t) \geqslant \beta(0)$, *the estimate (IV.16) together with (IV.13) give us, for all* $n \in \mathbb{N}, t_n = T$, *we deduce*

$$\begin{array}{lll} \left(u^n\right)_{n\in\mathbb{N}} & \text{is bounded in} & L^\infty(0,T;H_0^1(\Omega)), \\ \left(v^n\right)_{n\in\mathbb{N}} & \text{is bounded in} & L^\infty(0,T;H_0^1(\Omega)), \\ \left(u_t^n\right)_{n\in\mathbb{N}} & \text{is bounded in} & L^\infty(0,T;H_0^1(\Omega)), \\ \left(v_t^n\right)_{n\in\mathbb{N}} & \text{is bounded in} & L^\infty(0,T;H_0^1(\Omega)), \\ \left(z^n\right)_{n\in\mathbb{N}} & \text{is bounded in} & L^\infty(0,T;L^2((0,1),\Omega)). \end{array} \tag{IV.20}$$

Consequently, we conclude that

$$\begin{array}{lll} u^n \rightharpoonup u & \text{weakly star in} & L^\infty(0,T;H_0^1(\Omega)), \\ v^n \rightharpoonup v & \text{weakly star in} & L^\infty(0,T;H_0^1(\Omega)), \\ u_t^n \rightharpoonup u_t & \text{weakly star in} & L^\infty(0,T;H_0^1(\Omega)), \\ v_t^n \rightharpoonup v_t & \text{weakly star in} & L^\infty(0,T;H_0^1(\Omega)), \\ z^n \rightharpoonup z & \text{weakly star in} & L^\infty(0,T;L^2((0,1),\Omega)). \end{array} \tag{IV.21}$$

From IV.18, we have $\left(u^n\right)_{n\in\mathbb{N}}$, $\left(v^n\right)_{n\in\mathbb{N}}$ *are bounded in* $L^\infty(0,T;H_0^1(\Omega))$ *and* $\left(z^n\right)_{n\in\mathbb{N}}$ *is bounded in* $L^\infty(0,T;L^2((0,1),\Omega))$. *Then* $\left(u^n\right)_{n\in\mathbb{N}}$, $\left(v^n\right)_{n\in\mathbb{N}}$ *are bounded in* $L^2(0,T;H_0^1(\Omega))$, *and* $\left(z^n\right)_{n\in\mathbb{N}}$ *is bounded in* $L^2(0,T;L^2((0,1),\Omega))$. *Consequently,* $\left(u^n\right)_{n\in\mathbb{N}}$, $\left(v^n\right)_{n\in\mathbb{N}}$ *are bounded in* $H^1(0,T;H^1(\Omega))$ *and* $\left(z^n\right)_{n\in\mathbb{N}}$ *is bounded in* $H^1(0,T;L^2((0,1),\Omega))$. *Since the embedding*

$$H^1(0,T;H^1(\Omega)) \hookrightarrow L^2(0,T;L^2(\Omega))$$

is compact,using Aubin-Lion's theorem [16] ,we can extract subsequences $\left(u^k\right)_{k\in\mathbb{N}}$ *of* $\left(u^n\right)_{n\in\mathbb{N}}$, $\left(v^k\right)_{k\in\mathbb{N}}$ *of* $\left(v^n\right)_{n\in\mathbb{N}}$ *and* $\left(z^k\right)_{k\in\mathbb{N}}$ *of* $\left(z^n\right)_{n\in\mathbb{N}}$ *such that*

$$u^k \to u \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)),$$

$$v^k \to v \quad \textit{strongly in} \quad L^2(0,T;L^2(\Omega))$$

and

$$z^k \to z \quad \textit{strongly in} \quad L^2(0,T;L^2((0,1),\Omega))$$

Therefore,

$$u^k \to u \quad \textit{strongly and a.e} \quad (0,T)\times(\Omega),$$

$$v^k \to v \quad \textit{strongly and a.e} \quad (0,T)\times(\Omega)$$

and

$$z^k \to z \quad \textit{strongly and a.e} \quad (0,T)\times(0,1)\times(\Omega)$$

The proof now can be completed arguing as in Theorem 3.1 of [16]

4 Exponential stability

In this section we study the asymptotic behavior of the system (IV.1). For the proof of Theorem 2.3 we use the following lemmas.

Lemma 4.1 *Let* (u,v) *be the solution of* (IV.1)*, Then we have the inequality*

$$\begin{aligned}&\frac{dE(t)}{dt} \leqslant -(\mu_1-\lambda_1)\int_\Omega |\nabla u_t(x,t)|^2\,dx - \mu_2\int_\Omega |\nabla v_t(x,t)|^2\,dx - \frac{1}{2}g(t)\int_\Omega |\nabla u(x,t)|^2\,dx\\&+\frac{1}{2}\int_\Omega (g'\circ\nabla u)dx\end{aligned} \tag{IV.22}$$

Proof 4.2 *From* (IV.7) *we have*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega \Big(u_t^2+(\lambda+\mu)|\text{div }u|^2+v_t^2+\mu|\nabla v|^2+(\lambda+\mu)|\text{div }v|^2+2\alpha vu\Big)\,dx\\&= -\mu\int_\Omega \nabla u\nabla u_t dx - \mu_1\int_\Omega |\nabla u_t|^2 dx - \lambda_1\int_\Omega \nabla z(x,1,t)\nabla u_t dx - \mu_2\int_\Omega |\nabla v_t|^2 dx\\&+\int_\Omega\int_0^t g(s)\nabla u(s)\nabla u_t(t)ds dx\end{aligned} \tag{IV.23}$$

From Lemma 2.1, the last term in the right-hand side of V.21 can be rewritten as

$$\begin{aligned}&\int_0^t g(s)\int_\Omega \nabla u(s)\nabla u_t(t)dsdx+\frac{1}{2}g(t)\int_\Omega|\nabla u|^2(x,t)dx\\&=\frac{1}{2}\frac{d}{dt}\Big\{\int_0^t g(s)\int_\Omega|\nabla u|^2(x,t)dxds-\int_\Omega(g\circ\nabla u)(t)dx\Big\}+\frac{1}{2}\int_\Omega(g'\circ\nabla u)(t)dx.\end{aligned}\tag{IV.24}$$

Using the fact that

$$\begin{aligned}\frac{d}{dt}\frac{\lambda_1\tau}{2}\int_\Omega\int_0^1|\nabla z(x,\rho,t)|^2\,d\rho\,dx&=\lambda_1\tau\int_\Omega\int_0^1\nabla z(x,\rho,t)\nabla z_t(x,\rho,t)\,d\rho\,dx\\&=-\lambda_1\int_\Omega\int_0^1\nabla z_\rho(x,\rho,t)\nabla z(x,\rho,t)d\rho\,dx\\&=-\frac{\lambda_1}{2}\int_\Omega\int_0^1\frac{d}{d\rho}|\nabla z(x,\rho,t)|^2\,d\rho\,dx\\&=-\frac{\lambda_1}{2}\int_\Omega(|\nabla z(x,1,t)|^2-|\nabla z(x,0,t)|^2)\,dx\,,\end{aligned}\tag{IV.25}$$

So $\dfrac{dE}{dt}$ *becomes:*

$$\begin{aligned}\frac{dE}{dt}\;&=-(\mu_1-\frac{\lambda_1}{2})\int_\Omega|\nabla u_t|^2dx-\mu_2\int_\Omega|\nabla v_t|^2dx-\frac{1}{2}g(t)\int_\Omega|\nabla u|^2(x,t)dx\\&\quad-\lambda_1\int_\Omega\nabla z(x,1,t)\nabla u_tdx+\frac{1}{2}\int_\Omega(g'\circ\nabla u)(t)dx-\frac{\lambda_1}{2}\int_\Omega|\nabla z(x,1,t)|^2\,dx.\end{aligned}\tag{IV.26}$$

Applying Young's inequality, we show that (IV.22) *holds. The proof is complete.*

Now, we define the functional $\mathscr{D}(t)$ as follows

$$\mathscr{D}(t)=\int_\Omega uu_tdx+\int_\Omega vv_tdx+\frac{\mu_1}{2}\int_\Omega|\nabla u|^2dx+\frac{\mu_2}{2}\int_\Omega|\nabla v|^2dx.\tag{IV.27}$$

Then, we have the following estimate.

Lemma 4.3 *The functional* $\mathscr{D}(t)$ *satisfies*

$$\begin{aligned}\mathscr{D}'(t)\leqslant C\int_\Omega|\nabla u_t|^2dx+C\int_\Omega|\nabla v_t|^2dx+\Big(\delta+|\alpha|C+\frac{1}{2}-\beta(t)\Big)\int_\Omega|\nabla u|^2dx-(\lambda+\mu)\int_\Omega|\text{div }u|^2dx\\+\Big(|\alpha|C-\mu\Big)\int_\Omega|\nabla v|^2dx-(\lambda+\mu)\int_\Omega|\text{div }v|^2dx+\frac{\mu-\beta(t)}{4\delta}\int_\Omega(g\circ\nabla u)(t)dx\\+\frac{\lambda_1^2}{2}\int_\Omega|\nabla z(x,1,t)|^2dx\end{aligned}\tag{IV.28}$$

Proof 4.4 *Taking the derivative of* $\mathscr{D}(t)$ *with respect to* t *and using* (IV.6), *we find that:*

$$\begin{aligned}
\mathscr{D}'(t) &= \int_\Omega u_t^2 dx + \int_\Omega u u_{tt} dx + \int_\Omega v_t^2 dx + \int_\Omega v v_{tt} dx + \mu_1 \int_\Omega \nabla u_t \nabla u dx + \mu_2 \int_\Omega \nabla v_t \nabla v dx \\
&= \int_\Omega u_t^2 dx + \int_\Omega v_t^2 dx - \beta(t) \int_\Omega |\nabla u|^2(x,t) dx + \int_\Omega \int_0^t g(s)(\nabla u(s) - \nabla u(t)) \nabla u(t) ds dx \\
&\quad -\mu \int_\Omega |\nabla v|^2 dx - (\lambda+\mu) \int_\Omega |\text{div } u|^2 dx - \int_\Omega (\lambda+\mu) |\text{div } v|^2 dx - 2\alpha \int_\Omega uv\, dx \\
&\quad -\lambda_1 \int_\Omega \nabla z(x,1,t) \nabla u dx \qquad \text{(IV.29)}
\end{aligned}$$

Using the fact that

$$\begin{aligned}
\int_\Omega \int_0^t g(s) |\nabla u(s) - \nabla u(t)| \nabla u(t) ds dx &\leqslant \delta \int_\Omega |\nabla u|^2(x,t) dx + \frac{1}{4\delta} \int_\Omega \Big(\int_0^t g(s) |\nabla u(s) - \nabla u(t)| ds \Big)^2 dx \\
&\leqslant \delta \int_\Omega |\nabla u|^2(x,t) dx + \frac{\mu - \beta(t)}{4\delta} \int_\Omega (g \circ \nabla u)(t) dx. \qquad \text{(IV.30)}
\end{aligned}$$

Inserting the estimate (IV.30) *into* (IV.29) *and using Young's, Poincaré's inequalities lead to the desired estimate. The proof is complete.*

We define the functionals

$$I(t) = \tau \int_\Omega \int_0^1 e^{-\tau\rho} |\nabla z(x,\rho,t)|^2 \mathrm{d}\rho \mathrm{d}x,$$

and state the following lemma.

Lemma 4.5 *Let* (u, v, z) *be the solution of* (IV.6). *Then*

$$\frac{dI(t)}{dt} \leqslant -e^{-\tau} \Big(\int_\Omega |\nabla z(x,1,t)|^2 \mathrm{d}x + \tau \int_\Omega \int_0^1 |\nabla z(x,\rho,t)|^2 \mathrm{d}\rho \mathrm{d}x \Big) + \int_\Omega |\nabla u_t(x,t)|^2 \mathrm{d}x. \qquad \text{(IV.31)}$$

Proof 4.6

$$\begin{aligned}
\frac{d}{dt} I(t) &= 2\tau \int_0^1 \int_\Omega e^{-\tau\rho} \nabla z_t(x,\rho,t) \nabla z(x,\rho,t) \mathrm{d}\rho \mathrm{d}x \\
&= -2 \int_0^1 \int_\Omega e^{-\tau\rho} \nabla z_\rho(x,\rho,t) \nabla z(x,\rho,t) \mathrm{d}\rho \mathrm{d}x \\
&= - \int_0^1 \int_\Omega e^{-\tau\rho} \frac{\partial}{\partial\rho} \Big(|\nabla z(x,\rho,t|^2) \Big) \mathrm{d}\rho \mathrm{d}x \\
&= -\tau \int_0^1 \int_\Omega e^{-\tau\rho} |\nabla z(x,\rho,t)|^2 \mathrm{d}\rho \mathrm{d}x + \int_\Omega |\nabla u_t(x,t)|^2 \mathrm{d}x - e^{-\tau} \int_\Omega |\nabla z(x,1,t)|^2 \mathrm{d}x \\
&\leqslant -e^{-\tau} \Big(\tau \int_0^1 \int_\Omega |\nabla z(x,\rho,t)|^2 \mathrm{d}\rho \mathrm{d}x + \int_\Omega |\nabla z(x,1,t)|^2 \mathrm{d}x \Big) + \int_\Omega |\nabla u_t(x,t)|^2 \mathrm{d}x.
\end{aligned}$$

Proof 4.7 (Proof of Theorem 2.3) *We define the Lyapunov functional*

$$\mathscr{L}(t) = NE(t) + \epsilon\mathscr{D}(t) + I(t), \tag{IV.32}$$

where N and ϵ are positive constants that will be fixed later.

Taking the derivative of (IV.32) *with respect to t and making use of* (IV.22), (IV.28) *and* (IV.31), *we obtain*

$$\begin{aligned}\frac{d}{dt}\mathscr{L}(t) \leqslant &-\Big\{N(\mu_1-\lambda_1)-\epsilon C-1\Big\}\int_\Omega |\nabla u_t(x,t)|^2dx - \Big\{N\mu_2-\epsilon C\Big\}\int_\Omega |\nabla v_t(x,t)|^2dx\\ &-\Big(\beta(t)-\delta-|\alpha|C-\frac{1}{2}\Big)\epsilon\int_\Omega|\nabla u|^2dx - \Big(\mu-|\alpha|C\Big)\epsilon\int_\Omega|\nabla v|^2dx-(\lambda+\mu)\epsilon\int_\Omega|\mathrm{div}\ u|^2dx\\ &-(\lambda+\mu)\epsilon\int_\Omega|\mathrm{div}\ v|^2dx - \Big(e^{-\tau}-\frac{\lambda_1^2}{2}\epsilon\Big)\int_\Omega|\nabla z(x,1,t)|^2\mathrm{d}x - \tau e^{-\tau}\int_\Omega\int_0^1|\nabla z(x,\rho,t)|^2\mathrm{d}\rho\mathrm{d}x\\ &+\frac{N}{2}\int_\Omega(g'\circ\nabla u)(t)dx + \frac{(\mu-\beta(t))\epsilon}{4\delta}\int_\Omega(g\circ\nabla u)(t)dx\\ &-\frac{N}{2}g(t)\int_\Omega|\nabla u|^2(x,t)dx.\end{aligned} \tag{IV.33}$$

At this point, we choose our constants in (IV.33), *carefully, such that all the coefficients in* (IV.33) *will be negative. It suffices to choose ϵ so small and N large enough such that*

$$N(\mu_1-\lambda_1)-\epsilon C-1>0,$$

$$N\mu_2-\epsilon C>0,$$

and

$$e^{-\tau}-\frac{\lambda_1^2}{2}\epsilon>0.$$

Further, we choose α small enough such that

$$\beta(t)-\delta-|\alpha|C-\frac{1}{2}>0,$$

and

$$\mu-|\alpha|C>0.$$

Consequently, from the above, we deduce that there exist there exists two positive constants η_1 and $\eta_2$such that (IV.33) *becomes*

$$\frac{d\mathscr{L}(t)}{dt} \leqslant -\eta_1 E(t) + \eta_2\int_\Omega(g\circ\nabla u)dx \tag{IV.34}$$

By multiplying (IV.34) *by* $\xi(t)$, *we arrive at*

$$\xi(t)\mathscr{L}'(t) \leqslant -\eta_1\xi(t)E(t) + \eta_2\xi(t)\int_\Omega (g\circ\nabla u)dx \tag{IV.35}$$

Recalling (A2) and using (IV.35),*we get*

$$\begin{aligned}\xi(t)\mathscr{L}'(t) &\leqslant -\eta_1\xi(t)E(t) - \eta_2\int_\Omega (g'\circ\nabla u)dx\\ &\leqslant -\eta_1\xi(t)E(t) - 2\eta_2 E'(t)\end{aligned}$$

That is

$$\Big(\xi(t)\mathscr{L}(t) + 2\eta_2 E(t)\Big)' - \xi'(t)\mathscr{L} \leqslant -\eta_1\xi(t)E(t)$$

Using the fact that $\xi'(t) \leqslant 0, \quad \forall t \geqslant 0$ *and letting*

$$\mathscr{F}(t) = \xi(t)\mathscr{L}(t) + 2\eta_2 E(t) \sim E(t) \tag{IV.36}$$

we obtain

$$\mathscr{F}'(t) \leqslant -\eta_1\xi(t)E(t) \leqslant -\eta_3\xi(t)\mathscr{F}(t) \tag{IV.37}$$

A simple integration of (IV.37) *over* $(0,t)$ *leads to*

$$\mathscr{F}(t) \leqslant \mathscr{F}(0)e^{-\eta_3\int_0^t \xi(s)ds}, \quad \forall t \geqslant 0 \tag{IV.38}$$

A combination of (IV.36) *and* (IV.38) *leads to* (IV.8). *Then, the proof is complete.*

V

well-posedness and exponential stability for coupled Lamé system with a viscoelastic term and strong damping

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Let us consider the following a coupled Lamé system :

$$\begin{cases} u_{tt}(x,t)+\alpha v-\Delta_e u(x,t)+\int_0^t g_1(t-s)\Delta u(x,s)ds-\mu_1\Delta u_t(x,t)=0, & \text{in } \Omega\times(0,+\infty),\\ v_{tt}(x,t)+\alpha u-\Delta_e v(x,t)+\int_0^t g_2(t-s)\Delta v(x,s)ds-\mu_2\Delta v_t(x,t)=0, & \text{in } \Omega\times(0,+\infty),\\ u(x,t)=v(x,t)=0 & \text{on } \partial\Omega\times(0,+\infty),\\ (u(x,0),v(x,0))=(u_0(x),v_0(x)) & \text{in } \Omega,\\ (u_t(x,0),v_t(x,0))=(u_1(x),v_1(x)) & \text{in } \Omega. \end{cases} \tag{V.1}$$

Where μ_1,μ_2 are positive constants and (u_0,u_1,v_0,v_1) are given history and initial data

. Here Δ denotes the Laplacian operator and Δ_e denotes the elasticity operator, which is the 3×3 matrix-valued differential operator defined by

$$\Delta_e u = \mu \Delta u + (\lambda + \mu)\nabla(\text{div u}), \quad u = (u_1, u_2, u_3)^T$$

and μ and λ are the Lamé constants which satisfy the conditions

$$\mu > 0, \quad \lambda + \mu \geqslant 0. \tag{V.2}$$

.

The problem of stabilization of coupled systems has also been studied by several authors see [2, 4, 6, 15, 23, 24]and the references therein.Under certain conditions imposed on the subset where the damping term is effective, Komornik [15] proves uniform stabilization of the solutions of a pair of hyperbolic systems coupled in velocities. Alabau and al.[2] studied the indirect internal stabilization of weakly coupled systems where the damping is effective in the whole domain. They prove that the behavior of the first equation is sufficient to stabilize the total system and to have polynomial decay for sufficiently smooth solutions. For coupled systems in thermoelasticity, R.Racke [24] considered the following system:

$$\begin{cases} u_{tt}(x,t) - a u_{xx}(x, t-\tau) + b\theta_x(x,t) = 0, & \text{in } (0,L) \times (0,\infty), \\ \theta_t(x,t) - d\theta_{xx}(x,t) + b u_{tx}(x,t) = 0, & \text{in } (0,L) \times (0,\infty), \end{cases}$$

He proved that the internal time delay leads to ill-posedness of the system. However, the system without delay is exponentially stable.

In[4],beniani and al. considered the following Lamé system with time varying delay term:

$$\begin{cases} u''(x,t) - \Delta_e u(x,t) + \mu_1 g_1(u'(x,t)) + \mu_2 g_2(u'(x,t-\tau(t))) = 0 & \text{in } \Omega \times \mathbb{R}^+ \\ u(x,t) = 0 & \text{on } \partial\Omega \times \mathbb{R}^+ \end{cases} \tag{V.3}$$

and under suitable conditions, they proved general decay of energy.

In [17], authors considered the following problem:

$$\begin{cases} u_{tt} - \Delta_x u(x,t) - \mu_1 \Delta u_t(x,t) - \displaystyle\int_{\tau_1}^{\tau_2} \mu_2(s) \Delta u_t(x, t-s)\,ds = 0 & \text{in} \quad \Omega \times (0, +\infty), \\ u = 0 & \text{on} \quad \Gamma \times [0, +\infty), \\ u(x,0) = u_0(x) \quad u'(x,0) = u_1(x) & \text{on} \quad \Omega, \\ u_t(x,-t) = f_0(x,-t), & \qquad 0 < t \leq \tau_2, \end{cases} \tag{V.4}$$

and under the assumption

$$\mu_1 > \int_{\tau_1}^{\tau_2} |\mu_2| ds \tag{V.5}$$

they proved that the solution is exponentially stable.

Recently, Bouzettouta and al. [3] considered the Bresse system in bounded domain with internal distributed delay:

$$\begin{cases} \rho_1\varphi_{tt} - Gh(\varphi_x + lw + \psi)_x - Ehl(w_x - l\varphi) + \mu_1\varphi_t + \mu_2\varphi_t(x, t-\tau_1) = 0 & , \\ \rho_2\psi_{tt} - El\psi_{xx} - Gh(\varphi_x - lw + \psi) + \int_{\tau_1}^{\tau_2} \mu(s)\psi_t(x, t-s)ds = 0 & , \\ \rho_1 w_{tt} - Eh(w_x - l\varphi)_x + lGh(\varphi_x + lw + \psi) + \widetilde{\mu_1} w_t + \widetilde{\mu_2} w_t(x, t-\tau_2) = 0. & \end{cases} \tag{V.6}$$

where $(x,t) \in]0, L[\times\mathbb{R}_+$, the authors proved, under suitable conditions, that the system is well-posed and its energy converges to zero when time goes to infinity.

The paper is organized as follows.In Section 2, we give some materials needed for our work and state our main results. The well-posedness of the problem is analyzed in Section 3, by using Faedo-Galerkin method. In Section 4, we prove the exponential decay of the energy when time goes to infinity.

2 Preliminaries and statement of main results

In this section, we present some materials that shall be used for proving our main results. For the relaxation functions g_1, g_2, we have the folloing assumptions:

(A1) $g_i : \mathbb{R}_+ \to \mathbb{R}_+$ $(for\, i = 1, 2)$ are C^1 functions satisfying

$$g_i(0) > 0, \quad 0 < \beta_i(t) := \mu - \int_0^t g_i(s)\mathrm{d}s \quad and \quad 0 < \beta_i^0 := \mu - \int_0^\infty g_i(s)\mathrm{d}s.$$

(A2) There exist non-increasing differentiable functions $\xi_1(t), \xi_2(t) : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$g_i'(t) \le -\xi_i(t) g_i(t), \quad \forall t \geqslant 0 \quad \text{and} \quad \int_0^\infty \xi_i(t)\mathrm{d}t = +\infty, \quad for \quad i = 1, 2.$$

These hypotheses imply that

$$\beta_i^0 \le \beta_i(t) \le \mu, \quad for \quad i = 1, 2. \tag{V.7}$$

Let us introduce the following notations:

$$(g * h)(t) := \int_0^t g(t-s)h(s)ds,$$

$$(g \circ h)(t) := \int_0^t g(t-s)|h(t) - h(s)|^2 ds,$$

Lemma 2.1 ([8]) *For any $g, h \in C^1(\mathbb{R})$, the following equation holds*

$$2[g * h]h' = g' \circ h - g(t)|h|^2 - \frac{d}{dt}\Big\{g \circ h - \Big(\int_0^t g(s)ds\Big)|h|^2\Big\}.$$

The existence and uniqueness result is stated as follows:

Theorem 2.2 *Assume that $(A1)$ and $(A2)$ hold. Then given $u_0, v_0 \in (H^2(\Omega) \cap H_0^1(\Omega))^2$, $u_1, v_1 \in (L^2(\Omega))^2$, there exists a unique weak solution u, v of problem* (V.1) *such that*

$$u, v \in C([0, +\infty[, H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0, +\infty[, (L^2(\Omega))^2).$$

For any regular solution of (V.1), we define the energy as

$$\begin{aligned} E(t) = & \frac{1}{2}\int_\Omega u_t^2(x,t)dx + \frac{\beta_1(t)}{2}\int_\Omega |\nabla u|^2(x,t)dx + \frac{1}{2}\int_\Omega (g_1 \circ \nabla u)dx + \frac{(\mu+\lambda)}{2}\int_\Omega |\text{div } u|^2 dx \\ & + \frac{1}{2}\int_\Omega v_t^2(x,t)dx + \frac{\beta_2(t)}{2}\int_\Omega |\nabla v|^2(x,t)dx + \frac{(\mu+\lambda)}{2}\int_\Omega |\text{div } v|^2 dx + \frac{1}{2}\int_\Omega (g_2 \circ \nabla v)dx \\ & + 2\alpha \int_\Omega u(x,t)v(x,t)dx. \end{aligned} \tag{V.8}$$

Our decay result reads as follows:

Theorem 2.3 *Let (u, v) be the solution of* (V.1). *Assume that $(A1)$ and $(A2)$ hold. Then there exist two positive constants C and d, such that*

$$E(t) \leqslant C e^{-d\int_0^t \xi(s)ds}, \quad \forall t \geqslant 0. \tag{V.9}$$

3 Well-posedness

In this section, we will prove the existence and uniqueness of problem (V.1) by using Faedo-Galerkin method.

Proof 3.1 *We divide the proof of Theorem2.2 into two steps: the Faedo-Galerkin approximation and the energy estimates.*

Step 1 :Faedo-Galerkin approximation.

We construct approximations of the solution (u, v) by the Faedo-Galerkin method as follows. For $n \geqslant 1$, let $W_n = span \{w_1, ..., w_n\}$ be a Hilbertian basis of the space $H^2(\Omega) \cap H_0^1(\Omega)$ and the projection of the initial data on the finite dimensional subspace W_n is given by

$$u_0^n = \sum_{i=1}^{n} a_i w_i, \quad v_0^n = \sum_{i=1}^{n} b_i w_i, \quad u_1^n = \sum_{i=1}^{n} c_i w_i, \quad v_1^n = \sum_{i=1}^{n} d_i w_i$$

where,$(u_0^n, v_0^n, u_1^n, v_1^n) \rightarrow (u_0, v_0, u_1, v_1)$ strongly in $H^2(\Omega) \cap H_0^1(\Omega)$ as $n \rightarrow \infty$. We search the approximate solutions

$$u^n(x,t) = \sum_{i=1}^{n} f_i^n(t) w_i(x), \quad v^n(x,t) = \sum_{i=1}^{n} h_i^n(t) w_i(x)$$

to the finite dimensional Cauchy problem:

$$\begin{cases} \int_\Omega u_{tt}^n w_i dx + \alpha \int_\Omega v^n w_i dx + \mu \int_\Omega \nabla u^n \nabla w_i dx + (\lambda + \mu) \int_\Omega \text{div } u^n . \text{div } w_i dx \\ - \int_\Omega (g_1(s) * \nabla u^n) \nabla w_i dx + \mu_1 \int_\Omega \nabla u_t^n \nabla w_i dx = 0, \\ \int_\Omega v_{tt}^n w_i dx + \alpha \int_\Omega u^n w_i dx + \mu \int_\Omega \nabla v^n \nabla w_i dx + (\lambda + \mu) \int_\Omega \text{div } v^n . \text{div } w_i dx \\ - \int_\Omega (g_2(s) * \nabla v^n) \nabla w_i dx + \mu_2 \int_\Omega \nabla v_t^n \nabla w_i dx = 0, \\ (u^n(0), v^n(0)) = (u_0^n, v_0^n) \quad (u_t^n(0), v_t^n(0)) = (u_1^n, v_1^n). \end{cases} \tag{V.10}$$

According to the standard theory of ordinary differential equations , the finite dimensional problem (V.10) *has solution $f_i^n(t), h_i^n(t)$ defined on $[0, t)$.The a priori estimates that follow imply that in fact $t_n = T$.*

Step 2: Energy estimates. Multiplying the first and the second equation of (V.10) *by $\left(f_i^n(t)\right)'$ and $\left(h_i^n(t)\right)'$ respectively, we obtain:*

$$\begin{aligned} &\int_\Omega u_{tt}^n u_t^n dx + \alpha \int_\Omega v^n u_t^n dx + \mu \int_\Omega \nabla u^n \nabla u_t^n dx + (\lambda + \mu) \int_\Omega \text{div } u^n . \text{div } u_t^n dx \\ &- \int_\Omega (g_1(s) * \nabla u^n) \nabla u_t^n dx + \mu_1 \int_\Omega |\nabla u_t^n|^2 dx = 0. \end{aligned} \tag{V.11}$$

and

$$\begin{aligned}&\int_\Omega v_{tt}^n v_t^n dx + \alpha \int_\Omega u^n v_t^n dx + \mu \int_\Omega \nabla v^n \nabla v_t^n dx + (\lambda+\mu)\int_\Omega \operatorname{div} v^n . \operatorname{div} v_t^n dx \\ &- \int_\Omega (g_2(s) * \nabla v^n)\nabla v_t^n dx + \mu_2 \int_\Omega |\nabla v_t^n|^2 dx = 0.\end{aligned} \tag{V.12}$$

Integrating (V.11) *and* (V.12) *over* $(0,t)$,*and using Lemma* (2.1), *we obtain*

$$\begin{aligned}&\mathcal{E}_n(t) + \mu_1 \int_0^t \int_\Omega |\nabla u_t^n|^2 dxds - \frac{1}{2}\int_\Omega (g_1' \circ \nabla u^n)dx + \frac{1}{2}\int_0^t \int_\Omega g_1(t)|\nabla u^n|^2 dxds \\ &+ \mu_2 \int_0^t \int_\Omega |\nabla v_t^n|^2 dxds - \frac{1}{2}\int_\Omega (g_2' \circ \nabla v^n)dx + \frac{1}{2}\int_0^t \int_\Omega g_2(t)|\nabla v^n|^2 dxds = \mathcal{E}_n(0)\end{aligned} \tag{V.13}$$

where

$$\begin{aligned}\mathcal{E}_n(t) = &\frac{1}{2}\int_\Omega (u_t^n)^2(x,t)dx + \frac{\beta_1(t)}{2}\int_\Omega |\nabla u^n|^2(x,t)dx + \frac{1}{2}\int_\Omega (g_1 \circ \nabla u^n)dx + \frac{(\mu+\lambda)}{2}\int_\Omega |\operatorname{div} u^n|^2 dx \\ &+ \frac{1}{2}\int_\Omega (v_t^n)^2(x,t)dx + \frac{\beta_1(t)}{2}\int_\Omega |\nabla v^n|^2(x,t)dx + \frac{(\mu+\lambda)}{2}\int_\Omega |\operatorname{div} v^n|^2 dx \\ &+ \frac{1}{2}\int_\Omega (g_2 \circ \nabla v^n)dx + 2\alpha \int_\Omega u^n(x,t)v^n(x,t)dx.\end{aligned} \tag{V.14}$$

Consequently, fromV.13, we have the following estimate:

$$\begin{aligned}\mathcal{E}_n(t) &- \frac{1}{2}\int_\Omega (g_1' \circ \nabla u^n)dx + \frac{1}{2}\int_0^t \int_\Omega g_1(t)|\nabla u^n|^2 dxds \\ &- \frac{1}{2}\int_\Omega (g_2' \circ \nabla v^n)dx + \frac{1}{2}\int_0^t \int_\Omega g_2(t)|\nabla v^n|^2 dxds \\ &\leqslant \mathcal{E}_n(0).\end{aligned} \tag{V.15}$$

Now, since the sequences $\left(u_0^n\right)_{n\in\mathbb{N}}$, $\left(u_1^n\right)_{n\in\mathbb{N}}$, $\left(v_0^n\right)_{n\in\mathbb{N}}$, $\left(v_1^n\right)_{n\in\mathbb{N}}$ *converge and using (A2),in the both cases we can find a positive constant* c *independent of* n *such that*

$$\mathcal{E}_n(t) \leqslant c. \tag{V.16}$$

Therefore, using the fact that $\beta_i(t) \geqslant \beta_i^0$, *the estimate V.16 together with V.14 give us,* *for all* $n \in \mathbb{N}, t_n = T$*, we deduce*

$$\begin{array}{lll} \left(u^n\right)_{n\in\mathbb{N}} & \textit{is bounded in} & L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)), \\ \left(v^n\right)_{n\in\mathbb{N}} & \textit{is bounded in} & L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)), \\ \left(u_t^n\right)_{n\in\mathbb{N}} & \textit{is bounded in} & L^\infty(0,T;L^2(\Omega)), \\ \left(v_t^n\right)_{n\in\mathbb{N}} & \textit{is bounded in} & L^\infty(0,T;L^2(\Omega)). \end{array} \tag{V.17}$$

Consequently, we conclude that

$$\begin{array}{lll} u^n \rightharpoonup u & \textit{weakly star in} & L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)), \\ v^n \rightharpoonup v & \textit{weakly star in} & L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)), \\ u_t^n \rightharpoonup u_t & \textit{weakly star in} & L^\infty(0,T;L^2(\Omega)), \\ v_t^n \rightharpoonup v_t & \textit{weakly star in} & L^\infty(0,T;L^2(\Omega)). \end{array} \tag{V.18}$$

From V.17, we have $\left(u^n\right)_{n\in\mathbb{N}}$ *and* $\left(v^n\right)_{n\in\mathbb{N}}$ *are bounded in* $L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega))$*. Then* $\left(u^n\right)_{n\in\mathbb{N}}$ *and* $\left(v^n\right)_{n\in\mathbb{N}}$ *are bounded in* $L^2(0,T;H_0^1(\Omega))$*. Consequently,* $\left(u^n\right)_{n\in\mathbb{N}}$ *and* $\left(v^n\right)_{n\in\mathbb{N}}$ *are bounded in* $H^1(0,T;H^2(\Omega)\cap H_0^1(\Omega))$*. Using AubinŰLion's theorem [16], we can extract subsequences* $\left(u^k\right)_{k\in\mathbb{N}}$ *of* $\left(u^n\right)_{n\in\mathbb{N}}$ *and* $\left(v^k\right)_{k\in\mathbb{N}}$ *of* $\left(v^n\right)_{n\in\mathbb{N}}$ *such that*

$$u^k \to u \quad \textit{strongly in} \quad L^2(0,T;L^2(\Omega))$$

and

$$v^k \to v \quad \textit{strongly in} \quad L^2(0,T;L^2(\Omega))$$

Therefore,

$$u^k \to u \quad \textit{strongly and a.e} \quad (0,T)\times(\Omega)$$

and

$$v^k \to v \quad \textit{strongly and a.e} \quad (0,T)\times(\Omega)$$

The proof now can be completed arguing as in Theorem 2.2 of [16]

Uniqueness.

Let (u_1, v_1) and (u_2, v_2) be two solutions of problem (V.1) Then $(u, v) = (u_1 - u_2, v_1 - v_2)$ satisfies

$$\begin{cases} u_{tt}(x,t) + \alpha v - \Delta_e u(x,t) + \int_0^t g_1(t-s)\Delta u(x,s)ds - \mu_1 \Delta u_t(x,t) = 0, & \text{in } \Omega \times (0,+\infty), \\ v_{tt}(x,t) + \alpha u - \Delta_e v(x,t) + \int_0^t g_2(t-s)\Delta v(x,s)ds - \mu_2 \Delta v_t(x,t) = 0, & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = v(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ (u(x,0), v(x,0)) = (0,0) & \text{in } \Omega, \\ (u_t(x,0), v_t(x,0)) = (0,0) & \text{in } \Omega. \end{cases} \tag{V.19}$$

Following Lemma 4.1, the energy function associated to the problem (V.19) satisfies $E'(t) \leqslant 0$. Then $E(t) = E(0) = 0$. As $u(x,t) = v(x,t) = 0$ on $\partial\Omega \times (0,+\infty)$, we deduce that $u = v = 0$. The proof is complete.

4 Exponential stability

In this section we study the asymptotic behavior of the system (V.1). For the proof of Theorem 2.3 we use the following lemmas.

Lemma 4.1 *Let (u, v) be the solution of* (V.1)*, Then we have the inequality*

$$\begin{aligned} \frac{dE(t)}{dt} &\leqslant -\mu_1 \int_\Omega |\nabla u_t(x,t)|^2\, dx - \mu_2 \int_\Omega |\nabla v_t(x,t)|^2\, dx - \frac{1}{2} g_1(t) \int_\Omega |\nabla u(x,t)|^2\, dx \\ &+ \frac{1}{2}\int_\Omega (g_1' \circ \nabla u) dx - \frac{1}{2} g_2(t) \int_\Omega |\nabla v(x,t)|^2\, dx + \frac{1}{2}\int_\Omega (g_2' \circ \nabla v) dx \end{aligned} \tag{V.20}$$

Proof 4.2 *From* (V.8) *we have*

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega \Big(u_t^2 + (\lambda+\mu)|\text{div } u|^2 + v_t^2 + (\lambda+\mu)|\text{div } v|^2 + 2\alpha v u\Big)\, dx \\ &= -\mu \int_\Omega \nabla u \nabla u_t dx - \mu \int_\Omega \nabla v \nabla v_t dx - \mu_1 \int_\Omega |\nabla u_t|^2 dx - \mu_2 \int_\Omega |\nabla v_t|^2 dx \\ &+ \int_\Omega \int_0^t g_1(s) \nabla u(s) \nabla u_t(t) ds dx + \int_\Omega \int_0^t g_2(s) \nabla v(s) \nabla v_t(t) ds dx. \end{aligned} \tag{V.21}$$

From Lemma 2.1,the last terms in the right-hand side of V.21 can be rewritten as

$$\begin{aligned}&\int_0^t g_1(s)\int_\Omega \nabla u(s)\nabla u_t(t)dsdx+\frac{1}{2}g_1(t)\int_\Omega|\nabla u|^2(x,t)dx\\&=\frac{1}{2}\frac{d}{dt}\Big\{\int_0^t g_1(s)\int_\Omega|\nabla u|^2(x,t)dxds-\int_\Omega(g_1\circ\nabla u)(t)dx\Big\}+\frac{1}{2}\int_\Omega(g_1'\circ\nabla u)(t)dx\end{aligned}\tag{V.22}$$

and

$$\begin{aligned}&\int_0^t g_2(s)\int_\Omega \nabla v(s)\nabla v_t(t)dsdx+\frac{1}{2}g_2(t)\int_\Omega|\nabla v|^2(x,t)dx\\&=\frac{1}{2}\frac{d}{dt}\Big\{\int_0^t g_2(s)\int_\Omega|\nabla v|^2(x,t)dxds-\int_\Omega(g_2\circ\nabla v)(t)dx\Big\}+\frac{1}{2}\int_\Omega(g_2'\circ\nabla v)(t)dx\end{aligned}\tag{V.23}$$

So $\dfrac{dE}{dt}$ *becomes:*

$$\begin{aligned}\frac{dE}{dt}&=-\mu_1\int_\Omega|\nabla u_t|^2dx-\mu_2\int_\Omega|\nabla v_t|^2dx-\frac{1}{2}g_1(t)\int_\Omega|\nabla u|^2(x,t)dx\\&+\frac{1}{2}\int_\Omega(g_1'\circ\nabla u)(t)dx-\frac{1}{2}g_2(t)\int_\Omega|\nabla v|^2(x,t)dx+\frac{1}{2}\int_\Omega(g_2'\circ\nabla v)(t)dx\\&\leqslant 0.\end{aligned}\tag{V.24}$$

we show that (V.20) *holds. The proof is complete.*

Now, we define the functional $\mathscr{D}(t)$ as follows

$$\mathscr{D}(t)=\int_\Omega uu_tdx+\int_\Omega vv_tdx+\frac{\mu_1}{2}\int_\Omega|\nabla u|^2dx+\frac{\mu_2}{2}\int_\Omega|\nabla v|^2dx.\tag{V.25}$$

Then, we have the following estimate.

Lemma 4.3 *The functional* $\mathscr{D}(t)$ *satisfies*

$$\begin{aligned}\mathscr{D}'(t)\leqslant C&\int_\Omega|\nabla u_t|^2dx+C\int_\Omega|\nabla v_t|^2dx+\Big(\delta+|\alpha|C-\beta_1(t)\Big)\int_\Omega|\nabla u|^2dx\\&-(\lambda+\mu)\int_\Omega|\text{div }u|^2dx+\Big(\delta+|\alpha|C-\beta_2(t)\Big)\int_\Omega|\nabla v|^2dx-(\lambda+\mu)\int_\Omega|\text{div }v|^2dx\\&+\frac{\mu-\beta_1(t)}{4\delta}\int_\Omega(g_1\circ\nabla u)(t)dx+\frac{\mu-\beta_2(t)}{4\delta}\int_\Omega(g_2\circ\nabla v)(t)dx.\end{aligned}\tag{V.26}$$

Proof 4.4 *Taking the derivative of* $\mathscr{D}(t)$ *with respect to* t *and using* (V.1), *we find that:*

$$\begin{aligned}
\mathscr{D}'(t) &= \int_\Omega u_t^2 dx + \int_\Omega u u_{tt} dx + \int_\Omega v_t^2 dx + \int_\Omega v v_{tt} dx + \mu_1 \int_\Omega \nabla u_t \nabla u dx + \mu_2 \int_\Omega \nabla v_t \nabla v dx \\
&= \int_\Omega u_t^2 dx + \int_\Omega v_t^2 dx - \beta_1(t) \int_\Omega |\nabla u|^2(x,t) dx + \int_\Omega \int_0^t g_1(s)(\nabla u(s) - \nabla u(t)) \nabla u(t) ds dx \\
&\quad - \beta_1(t) \int_\Omega |\nabla v|^2 dx + \int_\Omega \int_0^t g_2(s)(\nabla v(s) - \nabla v(t)) \nabla v(t) ds dx - (\lambda+\mu) \int_\Omega |\text{div } u|^2 dx \\
&\quad - (\lambda+\mu) \int_\Omega |\text{div } v|^2 dx - 2\alpha \int_\Omega uv \, dx. \qquad (V.27)
\end{aligned}$$

Using the fact that

$$\begin{aligned}
\int_\Omega \int_0^t g_1(s)|\nabla u(s) - \nabla u(t)| \nabla u(t) ds dx &\leqslant \delta \int_\Omega |\nabla u|^2(x,t) dx + \frac{1}{4\delta} \int_\Omega \Big(\int_0^t g_1(s) |\nabla u(s) - \nabla u(t)| ds \Big)^2 dx \\
&\leqslant \delta \int_\Omega |\nabla u|^2(x,t) dx + \frac{\mu - \beta_1(t)}{4\delta} \int_\Omega (g_1 \circ \nabla u)(t) dx. \qquad (V.28)
\end{aligned}$$

By the same, we have

$$\begin{aligned}
\int_\Omega \int_0^t g_2(s)|\nabla v(s) - \nabla v(t)| \nabla v(t) ds dx &\leqslant \delta \int_\Omega |\nabla v|^2(x,t) dx + \frac{1}{4\delta} \int_\Omega \Big(\int_0^t g_2(s) |\nabla v(s) - \nabla v(t)| ds \Big)^2 dx \\
&\leqslant \delta \int_\Omega |\nabla v|^2(x,t) dx + \frac{\mu - \beta_2(t)}{4\delta} \int_\Omega (g_2 \circ \nabla v)(t) dx. \qquad (V.29)
\end{aligned}$$

Inserting the estimates (V.28), (V.28) *into* (V.27) *and using Young's, Poincaré's inequalities lead to the desired estimate. The proof is complete.*

Proof 4.5 (Proof of Theorem 2.3) *We define the Lyapunov functional*

$$\mathscr{L}(t) = NE(t) + \epsilon \mathscr{D}(t), \qquad (V.30)$$

where N *and* ϵ *are positive constants that will be fixed later.*

Taking the derivative of (V.30) *with respect to* t *and making use of* (V.20), (V.26), *we* *obtain*

$$\begin{aligned}
\frac{d}{dt}\mathscr{L}(t) \leqslant & -\Big\{N\mu_1 - \epsilon C\Big\}\int_\Omega |\nabla u_t(x,t)|^2 dx - \Big\{N\mu_2 - \epsilon C\Big\}\int_\Omega |\nabla v_t(x,t)|^2 dx \\
& - \Big(\beta_1(t) - \delta - |\alpha| C\Big)\epsilon \int_\Omega |\nabla u|^2 dx - \Big(\beta_2(t) - \delta - |\alpha| C\Big)\epsilon \int_\Omega |\nabla v|^2 dx \\
& - (\lambda+\mu)\int_\Omega |\text{div } u|^2 dx \; - (\lambda+\mu)\int_\Omega |\text{div } v|^2 dx \\
& + \frac{N}{2}\int_\Omega (g_1' \circ \nabla u)(t) dx + \frac{N}{2}\int_\Omega (g_2' \circ \nabla v)(t) dx \\
& + \frac{(\mu - \beta_1(t))\epsilon}{4\delta}\int_\Omega (g_1 \circ \nabla u)(t) dx + \frac{(\mu - \beta_2(t))\epsilon}{4\delta}\int_\Omega (g_2 \circ \nabla v)(t) dx \\
& - \frac{N}{2} g_1(t)\int_\Omega |\nabla u|^2(x,t) dx - \frac{N}{2} g_2(t)\int_\Omega |\nabla v|^2(x,t) dx.
\end{aligned} \tag{V.31}$$

At this point, we choose our constants in (V.31), *carefully, such that all the coefficients in* (V.31) *will be negative. It suffices to choose* ϵ *so small and* N *large enough such that*

$$N\mu_1 - \epsilon C > 0,$$

and

$$N\mu_2 - \epsilon C > 0,$$

Further, we choose α *small enough such that*

$$\beta_1(t) - \delta - |\alpha| C > 0,$$

and

$$\beta_2(t) - \delta - |\alpha| C > 0.$$

Consequently, from the above, we deduce that there exist there exists two positive constants η_1, η_2 *and* η_3*such that* (V.31) *becomes*

$$\frac{d\mathscr{L}(t)}{dt} \leqslant -\eta_1 E(t) + \eta_2 \int_\Omega (g_1 \circ \nabla u) dx + \eta_3 \int_\Omega (g_2 \circ \nabla v) dx. \tag{V.32}$$

Therefore, if $\xi(t) = min \xi_1(t), \xi_2(t), \quad \forall t \geqslant 0$, *then using (A2) and* (V.20), *we get*

$$
\begin{aligned}
\xi(t)\mathscr{L}'(t) &\leqslant -\eta_1\xi(t)E(t) + \eta_2\xi(t)\int_\Omega (g_1 \circ \nabla u)dx + \eta_3\xi(t)\int_\Omega (g_2 \circ \nabla v)dx \\
&\leqslant -\eta_1\xi(t)E(t) + \eta_2\xi_1(t)\int_\Omega (g_1 \circ \nabla u)dx + \eta_3\xi_2(t)\int_\Omega (g_2 \circ \nabla v)dx \\
&\leqslant -\eta_1\xi(t)E(t) + \eta_2\int_\Omega\int_0^t \xi_1(t-s)g_1(t-s)|\nabla u(t) - \nabla u(s)|^2 dsdx \\
&+ \eta_3\int_\Omega\int_0^t \xi_2(t-s)g_2(t-s)|\nabla v(t) - \nabla v(s)|^2 dsdx \\
&\leqslant -\eta_1\xi(t)E(t) - \eta_2\int_\Omega\int_0^t g_1'(t-s)|\nabla u(t) - \nabla u(s)|^2 dsdx \\
&- \eta_3\int_\Omega\int_0^t g_2'(t-s)|\nabla v(t) - \nabla v(s)|^2 dsdx \\
&\leqslant -\eta_1\xi(t)E(t) - cE'(t), \quad \forall t \geqslant 0.
\end{aligned} \tag{V.33}
$$

Which gives

$$
\Big(\xi(t)\mathscr{L}(t) + cE(t)\Big)' - \xi'(t)\mathscr{L}(t) \leqslant -\eta_1\xi(t)E(t).
$$

Using the fact that $\xi'(t) \leqslant 0, \quad \forall t \geqslant 0$ *and letting*

$$
\mathscr{F}(t) = \xi(t)\mathscr{L}(t) + cE(t) \sim E(t) \tag{V.34}
$$

we obtain

$$
\mathscr{F}'(t) \leqslant -\eta_1\xi(t)E(t) \leqslant -\eta_3\xi(t)\mathscr{F}(t). \tag{V.35}
$$

A simple integration of (V.35) *over* $(0,t)$ *leads to*

$$
\mathscr{F}(t) \leqslant \mathscr{F}(0)e^{-\eta_3\int_0^t \xi(s)ds} \qquad \forall t \geqslant 0. \tag{V.36}
$$

A combination of (V.34) *and* (V.36) *leads to* (V.9)*. Then, the proof is complete.*

Publications

The following results were published or submitted:

1. N. Taouaf, N. Amroun, A. Benaissa and A. Beniani, *Well-Posedness And Asymptotic Stability For The Lamé System With Internal Distributed Delay,* Mathematica Moravica, Vol. 22, No. 1 (2018), 31-41.

2. A. Beniani, N. Taouaf and A. Benaissa, *Well-Posedness And Exponential Stability For Coupled Lamé System With A Viscoelastic Term And Strong Damping,* Computers and Mathematics with Applications 75.12 (2018), 4397-4404.

3. N. Taouaf, N. Amroun, A. Benaissa and A. Beniani, *Well-Posedness And Exponential Stability For Coupled Lamé System With A Viscoelastic Damping,* Filomat, Vol 32, No 10 (2018).

4. N. Taouaf, N. Amroun, A. Benaissa and A. Beniani, *Well-Posedness And Exponential Stability For Coupled Lamé System With A Viscoelastic Term And A Strong Delay,* .

5. N. Taouaf, N. Amroun, A. Benaissa and A. Beniani, *Energy Decay Of Solution For Nonlinear Transmission Problems With A Delay Term,* .

6. N. Taouaf, N. Amroun, A. Benaissa and A. Beniani, *well-posedness and exponential stability of solutions for laminated viscoelastic timoshenko beams,* .

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Abstract

The present thesis is devoted to the study of Well-Posedness and asymptotic behaviour in time of solution of Lamé system and coupled Lamé system. This work consists of five chapters, will be devoted to the study of the Well-Posedness and asymptotic behaviour of some evolution equation with linear, and viscoelastic terms. We recall of some fundamental inequalities.

Résumé

La présente thèse est consacrée à l'étude de la position bien posée et du comportement asymptotique dans le temps de résolution du système de Lamé et du système de Lamé couplé. Ce travail, composé de cinq chapitres, sera consacré à du comportement asymptotique d'une équation d'évolution avec des termes linéaires et viscoélastiques. Nous rappelons quelques inégalités fondamentales.

ملخص

تُخصّص الرسالة الحالية لدراسة الموقف الجيد والسلوك التقاربي في وقت حل نظام Lamé ونظام Lamé المقترن .سيتم تكريس هذا العمل ، الذي يتكون من خمسة فصول ، للسلوك التقاربي لمعادلة تطور مع المصطلحات الخطية والمرونة .نتذكر بعض أوجه عدم المساواة الأساسية.