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UNIVERSITE DJILLALI LIABES
FACULTE DES SCIENCES EXACTES
SIDI BEL ABBÈS

THESE DE DOCTORAT

Présentée par

AICHI Chahira

Spécialité : MATHEMATIQUES

Option : EQUATIONS AUX DERIVEES PARTIELLES

Intitulée

Stabilisation de quelques problèmes d'évolution dégénérés par des contrôles frontières de type fractionnaire.

Soutenue le **07/07/2019**

Devant le jury composé de :

Président *Soufiane MOKEDDEM* Professeur à l'Université de Sidi Bel Abbes.

Examinateurs *Kacem BELGHABA* Professeur à l'Université d'Oran 1.

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Chahira AICHI ep.BOUDAOUD

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Abbes BENAISSA	Professeur , Université de S.B.A.	Directeur de thèse

Dédicace

Je dédie ce modeste travail à mes parents, mon mari et mes deux fils Mouâdh et Aymen.
A mes frères et mes sœurs .
A tous mes neveux et mes nièces.
A toute ma famille et mes amies .

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Chahira Aïchi

Titre : Stabilisation de quelques problèmes d'évolution dégénérés par des contrôles frontières de type fractionnaire.

Résumé

Dans cette thèse, nous considèrons l'équation des ondes dégénérée avec la présence des termes dissipatifs de type fractionnaires. Sous quelques hypothèses sur les données initiales et aux bords, nous avons concentré notre étude sur l'existence globale et le comportement asymptotique des solutions où nous avons obtenu plusieurs résultats sur la vitesse de décroissance de l'énergie.

D'abords, nous considérons une équation des ondes dégénérée soumis à un contrôle frontière de type fractionnaire. Nous montrons que le problème n'est pas uniformément stable par une méthode spectrale et nous étudions la stabilité polynomiale à l'aide de la théorie des semi-groupes.

Ensuite, nous nous intèressons à l'étude de la stabilisation d'équation des ondes unidimensionnelle faiblement dégénérée $u_{tt} - (x^\gamma u_x)_x = 0$ avec $x \in (0,1)$ et $\gamma \in [0,1)$, contrôlée par un feedback frontière fractionnaire agissant à $x = 0$. On prouve divers type de stabilit: forte, uniforme et polynomiale dans des espaces appropriés. Les résultats sont obtenus par une estimation de la résolvante du générateur associé au semi-groupe. En plus, en utilisant une méthode spectrale, nous établissons l'optimalité.

Enfin, nous nous intéressons à l'étude de la stabilisation d'équation des ondes unidimensionnelles faiblement dégénérée $u_{tt}-(x^\gamma u_x)_x = 0$ avec $x \in (0,1)$ et $\gamma \in [0,1)$, contrôlé par un feedback dynamique frontière de type fractionnaire agissant à $x = 0$. Nous montrons la décroissance polynomiale optimale dans des espaces appropriés. Les résultats sont obtenus par une estimation de la résolvante du générateur associé au semi-groupe et le théorème de Borichev-Tomilov.

Mots Clés:

Équation des ondes dégénérée, Dérivée Fractionnaire, Stabilité polynomiale, la vitesse de décroissance optimale, fonctions de Bessel, C_0 -semi-groupe, Méthode Spectrale.

Title : Stabilization of some degenerate evolution problems with fractional boundary control

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Abstract

In this thesis we consider degenerate wave equation problems with the presence of boundary dissipation of fractional derivative type. Under assumptions on initial data and boundary conditions, we focused our study on the global existence and asymptotic behavior of solutions where we obtained several results on the decay rate.

First, we consider a degenerate wave equation with a boundary control condition of fractional derivative type. We show that the problem is not uniformly stale by a spectrum method and we study the polynomial stability using the semigroup theory of linear operators.

Next, we are concerned with the study of stabilization of one-dimensional weakly degenerate wave equation $u_{tt} - (x^\gamma u_x)_x = 0$ with $x \in (0, 1)$ and $\gamma \in [0, 1)$, controlled by a fractional boundary feedback acting at $x = 0$. Strong, uniform, and nonuniform stabilization are obtained with explicit decay estimates in appropriate spaces. The results are obtained through an estimate on the resolvent of the generator associated with the semigroup. However, using a spectral method, we establish the optimal polynomial decay rate of the energy of the system.

Finally, we are concerned with the study of stabilization of one-dimensional weakly degenerate wave equation $u_{tt} - (x^\gamma u_x)_x = 0$ with $x \in (0, 1)$ and $\gamma \in [0, 1)$, controlled by a dynamic fractional boundary feedback acting at $x = 0$. We prove optimal polynomial decay estimates in appropriate spaces. The results are obtained through an estimate on the resolvent of the generator associated with the semigroup.

Keywords:

Degenerate wave equation, Fractional Derivative, Polynomial stability, Optimal decay rate, Bessel functions, C_0-semigroup, Spectral Method.

Contents

List of symbols

$\mathbb{R}$: The set of real numbers.
$\mathbb{R}_+$: The set of non negative real numbers.
$\mathbb{R}^*$: The set of non zero real numbers.
$\mathbb{C}$: The set of complex numbers.
i: The imaginary unit.
L^p: The Lebesgue space.
H^m: The sobolev space.
C^0: The space of continuous function.
C^1: the space of continuously differentiable functions.
$L(X,Y)$: The space of bounded linear operators from X into Y .
$|.|$: The modulus.
$\|.\|$: The norm.
inf: The infimum.
sup: The supremum.
$\Re$: The real part.
$\Im$: The imaginary part.
∂: The partial derivative.
∂_t: The partial derivative with respect of t.
$\partial_{tt}f$: The second partial derivative of f with respect of t.
$\partial_t^{\alpha,\eta}$: Fractional Derivative.
$D(A)$: Domain of A.
$R(A)$: The range of A.
$ker(A)$: The kernel of A.
A^*: The adjoint operator of A.
$\rho(A)$: The resolvent set of A.
$\sigma(A)$: The spectrum of A.
$\sigma_p(A)$: The punctual spectrum of A.

0.1 Introduction

Control theory is the study of the process of controlling the behavior of an operator system to achieve a certain target. Its application ranges widely from earthquake engineering and seismology to fluid transfer, cooling water and noise reduction in cavities, vehicles, such as pipe systems. Acoustics, aeronautics, hydraulics, are also some of the diverse disciplines where control theory is applied.
Interest for fractional derivation is related to the mechanical modeling of gums and rubbers. In short, all kinds of materials that preserve the memory of previous deformations in particular viscoelastic. Indeed, the fractional derivation is introduced naturally.
The boundary feedback under the consideration in this thesis are of fractional type and are described by the fractional derivatives

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\, ds,\ \eta \geq 0.$$

The order of our derivatives is between 0 and 1. Very little is known in the literature. In addition to being nonlocal, fractional derivatives involve singular and non-integrable kernels ($t^{\alpha}, 0 < \alpha < 1$). This makes the problem more delicate. It has been shown (see [**40**]) that, as ∂_t, the fractional derivative ∂_t^{α} forces the system to become dissipative and the solution to approach the equilibrium state. Therefore, when applied on the boundary, we can consider them as controllers which help to reduce the vibrations.
In the recent years, fractional calculus has been applied successfully in various areas to modify many existing models of physical processes such as heat conduction, diffusion, viscoelasticity, wave propagation, electronics etc. Caputo and Mainardi [**17**] have established the relation between fractional derivative and theory of viscoelasticity. The generalization of the concept of derivative and integral to a non-integer order has been subjected to several approaches and some various alternative definition of fractional derivative appeared in [**29, 30**].

This thesis is divided into 4 Chapter.

CHAPTER 1: PRELIMINARIES

In this Chapter, firstly, we present some well known results on Sobolev spaces and some basic definitions and theorems . Secondly, we recall some results on a C0-semigroup, including some theorems on strong, exponential and polynomial stability of a C0-semigroup. Next, we display a brief historical introduction to fractional derivatives and we define the fractional derivative operator and we present some physical interpretations . After that, we present the Bessel functions and their basic definitions . Finally, we present an appendix that contains almost all the secondary calculations used in this Thesis.

CHAPTER 2: ENERGY DECAY FOR A DEGENERATE WAVE EQUATION UNDER FRACTIONAL DERIVATIVE CONTROLS

In this Chapter, we are concerned with the system

$$(P_1) \qquad \begin{cases} u_{tt}(x,t) - (a(x)u_x(x,t))_x = 0 & \text{in } (0,1)\times(0,+\infty), \\ \begin{cases} u(0,t) = 0 & \text{if } 0 \le \mu_a < 1 \\ (au_x)(0,t) = 0 & \text{if } 1 \le \mu_a < 2 \end{cases} & \text{in } (0,+\infty), \\ \beta u(1,t) + (au_x)(1,t) = -\varrho \partial_t^{\alpha,\eta} u(1,t) & \text{in } (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } (0,1). \end{cases}$$

where $\varrho > 0$ and $\beta \ge 0$.The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α, $(0 < \alpha < 1)$, with respect to the time variable (see **[10]** and **[20]**). It is defined as follows

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\, ds, \; \eta \ge 0.$$

When we show that the problem is not uniformly stale by a spectrum method and we study the polynomial stability using the semigroup theory of linear operators.
Unfortunately we are not able to prove this decay rate by frequency domain method based on multiplier method as the problem (P_1) is degenerate and the control is acting on the degenerate boundary.
For $\eta \neq 0$, by an explicit representation of the resolvent of the generator on the imaginary axis and the use of the theorem by Borichev and Tomilov, we prove an optimal decay rate.

CHAPTER 3: STABILIZATION OF DEGENERATE WAVE EQUATION UNDER FRACTIONAL FEEDBACK ACTING ON THE DEGENERATE BOUNDARY:

In this Chapter, concerned with the boundary stabilization of fractional type for degenerate wave equation of the form:

$$(P_2) \qquad \begin{cases} u_{tt}(x,t) - (x^\gamma u_x(x,t))_x = 0 & \text{in } (0,1)\times(0,+\infty), \\ (x^\gamma u_x)(0,t) = \varrho \partial_t^{\alpha,\eta} u(0,t) & \text{in } (0,+\infty), \\ u(1,t) = 0 & \text{in } (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } (0,1). \end{cases}$$

where $\gamma \in [0,1)$ and $\varrho > 0$. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α, $(0 < \alpha \le 1)$, with respect to the time variable . It is defined as follows

$$\partial_t^{\alpha,\eta} w(t) = \begin{cases} w_t & \text{for } \alpha = 1, \eta \ge 0 \\ \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\, ds, & \text{for } \alpha \neq 1, \eta \ge 0. \end{cases}$$

Where Strong, uniform, and nonuniform stabilization are obtained with explicit decay estimates in appropriate spaces. The results are obtained through an estimate on the resolvent of the generator associated with the semigroup.

CHAPTER 4: STABILIZATION OF DEGENERATE WAVE EQUATION UNDER DYNAMIC FRACTIONAL FEEDBACK ACTING ON THE DEGENERATE BOUNDARY

In this Chapter,we are concerned with the dynamic boundary stabilization of fractional type for degenerate wave equation of the form

$$(P_3) \qquad \begin{cases} u_{tt}(x,t) - (x^\gamma u_x(x,t))_x = 0 & \text{in } (0,1) \times (0,+\infty), \\ -mu_{tt}(0,t) + (x^\gamma u_x)(0,t) = \varrho \partial_t^{\alpha,\eta} u(0,t) & \text{in } (0,+\infty), \\ u(1,t) = 0 & \text{in } (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } (0,1), \end{cases}$$

where $(x,t) \in (0,1) \times (0,+\infty), \gamma \in [0,1), m > 0$ and $\varrho > 0$. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α, $(0 < \alpha \leq 1)$, with respect to the time variable (see [**10**] and [**20**]). It is defined as follows

$$\partial_t^{\alpha,\eta} w(t) = \begin{cases} w_t & \text{for } \alpha = 1, \eta \geq 0 \\ \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\, ds, & \text{for } \alpha \neq 1, \eta \geq 0. \end{cases}$$

Where we discuss and establish the existence, the uniqueness of solution for the degenerate wave equation with a dynamic boundary dissipation of fractional derivative type, and We proved optimal polynomial decay estimates in appropriate spaces. The results are obtained through an estimate on the resolvent of the generator associated with the semigroup.

Chapter 1

PRELIMINARIES

In this chapter, we recall some basic definitions and theorems which will be used in the following chapters.

1.1 Sobolev spaces

We denote by Ω an open domain in $\mathbb{R}^n, n \geq 1$, with a smooth boundary $\Gamma = \partial\Omega$. In general, some regularity of Ω will be assumed. We will suppose that either

$$\Omega \text{ is Lipschitz,}$$

i.e., the boundary Γ is locally the graph of a Lipschitz function, or

$$\Omega \text{ is of class } \mathcal{C}^r, r \geq 1,$$

i.e., the boundary Γ is a manifold of dimension $n \geq 1$ of class $\mathcal{C}^r$. In both cases we assume that Ω is totally on one side of Γ. These definitions mean that locally the domain Ω is below the graph of some function ψ, the boundary Γ is represented by the graph of ψ and its regularity is determined by that of the function ψ. Moreover, it is necessary to note that a domain with a continuous boundary is never on both sides of its boundary at any point of this boundary and that a Lipschitz boundary has almost everywhere a unit normal vector ν.

We will also use the following multi-index notation for partial differential derivatives of a function:

$$\partial_i^k u = \frac{\partial^k u}{\partial x_i^k} \text{ for all } k \in \mathbb{N} \text{ and } i = 1, ..., n,$$
$$D^\alpha u = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\dots\partial_n^{\alpha_n} u = \frac{\partial^{\alpha_1+\dots+\alpha_n} u}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}},$$
$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n, |\alpha| = \alpha_1 + \dots + \alpha_n.$$

We denote by $\mathcal{C}(D)$ (respectively $\mathcal{C}^k(D), k \in \mathbb{N}$ or $k = +\infty$) the space of real continuous functions on D (respectively the space of k times continuously differentiable functions on D), where D plays the role of Ω or its closure $\overline{\Omega}$. The space of real $\mathcal{C}^\infty$ functions on Ω with a compact support in Ω is denoted by $\mathcal{C}_0^\infty(\Omega)$ or $\mathcal{D}(\Omega)$ as in the distributions theory of Schwartz.The distributions space on Ω is denoted by $\mathcal{D}'(\Omega)$, i.e., the space of continuous linear form over $\mathcal{D}(\Omega)$.

For $1 \leq p \leq \infty$, we call $L^p(\Omega)$ the space of measurable functions f on Ω such that

$$\begin{aligned} \|f\|_{L^p(\Omega)} &= \left(\int_\Omega |f(x)|^p dx\right)^{1/p} < +\infty \quad \text{for} \quad p < +\infty \\ \|f\|_{L^\infty(\Omega)} &= \sup_\Omega |f(x)| < +\infty \qquad\qquad \text{for} \quad p = +\infty \end{aligned}$$

The space $L^p(\Omega)$ equipped with the norm $f \longrightarrow \|f\|_{L^p}$ is a Banach space: it is reflexive and separable for $1 < p < \infty$ (its dual is $L^{\frac{p}{p-1}}(\Omega)$), separable but not reflexive for $p = 1$ (its dual is $L^\infty(\Omega)$), and not separable, not reflexive for $p = \infty$ (its dual contains strictly $L^1(\Omega)$). In particular the space $L^2(\Omega)$ is a Hilbert space equipped with the scalar product defined by

$$(f, g)_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx.$$

We denote by $L^p_{loc}(\Omega)$ the space of functions which are L^p on any bounded sub-domain of Ω.

Similar space can be defined on any open set other than Ω, in particular, on the cylinder set $\Omega \times]a, b[$ or on the set $\Gamma \times]a, b[$, where $a, b \in \mathbb{R}$ and $a < b$.

Let U be a Banach space, $1 < p < +\infty$ and $-\infty \leq a < b \leq +\infty$, then $L^p(a, b; U)$ is the space of L^p functions f from (a, b) into U which is a Banach space for the norm

$$\|f\|_{L^p(a,b;U)} = \left(\int_a^b \|f(x)\|_U^p \, dt\right)^{1/p} < +\infty \quad \text{for} \quad p < +\infty$$

and for the norm

$$\|f\|_{L^\infty(a,b;U)} = \sup_{t\in(a,b)} \|f(x)\|_U < +\infty \quad \text{for} \quad p = +\infty$$

Similarly, for a Banach space $U, k \in \mathbb{N}$ and $-\infty < a < b < +\infty$, we denote by $C([a, b]; U)$ (respectively $C^k([a, b]; U)$) the space of continuous functions (respectively the space of k times continuously differentiable functions) f from $[a, b]$ into U, which are Banach spaces, respectively, for the norms

$$\|f\|_{\mathcal{C}(a,b;U)} = \sup_{t\in(a,b)} \|f(x)\|_U, \quad \|f\|_{\mathcal{C}^k(a,b;U)} = \sum_{i=0}^k \left\|\frac{\partial^i f}{\partial t^i}\right\|_{\mathcal{C}(a,b;U)}$$

1.1.1 Definition of Sobolev Spaces

Now, we will introduce the Sobolev spaces: The Sobolev space $W^{k,p}(\Omega)$ is defined to be the subset of L^p such that function f and its weak derivatives up to some order k have a finite L^p norm, for given $p \geq 1$.

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega); D^\alpha f \in L^p(\Omega). \ \forall\alpha; |\alpha| \leq k\} \ ,$$

With this definition, the Sobolev spaces admit a natural norm,

$$f \longrightarrow \|f\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\leq k} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p} , \text{ for } p < +\infty$$

and

$$f \longrightarrow \|f\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha|\leq k} \|D^\alpha f\|_{L^\infty(\Omega)} , \text{ for } p = +\infty$$

Space $W^{k,p}(\Omega)$ equipped with the norm $\| \, . \, \|_{W^{k,p}}$ is a Banach space. Moreover is a reflexive space for $1 < p < \infty$ and a separable space for $1 \leq p < \infty$. Sobolev spaces with $p = 2$ are especially important because of their connection with Fourier series and because they form a Hilbert space. A special notation has arisen to cover this case:

$$W^{k,2}(\Omega) = H^k(\Omega)$$

the H^k inner product is defined in terms of the L^2 inner product:

$$(f, g)_{H^k(\Omega)} = \sum_{|\alpha|\leq k} (D^\alpha f, D^\alpha g)_{L^2(\Omega)} \ .$$

The space $H^m(\Omega)$ and $W^{k,p}(\Omega)$ contain $\mathcal{C}^\infty(\overline{\Omega})$ and $\mathcal{C}^m(\overline{\Omega})$. The closure of $\mathcal{D}(\Omega)$ for the $H^m(\Omega)$ norm (respectively $W^{m,p}(\Omega)$ norm) is denoted by $H^m_0(\Omega)$ (respectively $W^{k,p}_0(\Omega)$).

Now, we introduce a space of functions with values in a space X (a separable Hilbert space).

The space $L^2(a, b; X)$ is a Hilbert space for the inner product

$$(f, g)_{L^2(a,b;X)} = \int_a^b (f(t), g(t))_X \, dt$$

We note that $L^\infty(a, b; X) = (L^1(a, b; X))'$.

Now, we define the Sobolev spaces with values in a Hilbert space X

For $k \in \mathbb{N}$, $p \in [1, \infty]$, we set:

$$W^{k,p}(a, b; X) = \left\{ v \in L^p(a, b; X); \frac{\partial v}{\partial x_i} \in L^p(a, b; X). \ \ \forall i \leq k \right\} ,$$

The Sobolev space $W^{k,p}(a, b; X)$ is a Banach space with the norm

$$\begin{aligned} \|f\|_{W^{k,p}(a,b;X)} &= \left(\sum_{i=0}^k \left\| \frac{\partial f}{\partial x_i} \right\|^p_{L^p(a,b;X)} \right)^{1/p} , \text{ for } p < +\infty \\ \|f\|_{W^{k,\infty}(a,b;X)} &= \sum_{i=0}^k \left\| \frac{\partial f}{\partial x_i} \right\|_{L^\infty(a,b;X)} , \qquad \text{for } p = +\infty \end{aligned}$$

The spaces $W^{k,2}(a, b; X)$ form a Hilbert space and it is noted $H^k(0, T; X)$. The $H^k(0, T; X)$ inner product is defined by:

$$(u, v)_{H^k(a,b;X)} = \sum_{i=0}^k \int_a^b \left(\frac{\partial u}{\partial x^i}, \frac{\partial v}{\partial x^i} \right)_X dt \ .$$

Theorem 1.1.1 *Let $1 \leq p \leq n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$$

where p^* is given by $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}$ (where $p = n, p^* = \infty$). Moreover there exists a constant $C = C(p,n)$ such that

$$\|u\|_{L^{p^*}} \leq C\|\nabla u\|_{L^p(\mathbb{R}^n)} \forall u \in W^{1,p}(\mathbb{R}^n).$$

Corollary 1.1.1 *Let $1 \leq p < n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \quad \forall q \in [p, p^*]$$

with continuous imbedding.

For the case $p = n$, we have

$$W^{1,n}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \quad \forall q \in [n, +\infty[$$

Theorem 1.1.2 *Let $p > n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$$

with continuous imbedding.

Corollary 1.1.2 *Let Ω a bounded domain in $\mathbb{R}^n$ of C^1 class with $\Gamma = \partial\Omega$ and $1 \leq p \leq \infty$. We have*

$$\begin{array}{ll} if & 1 \leq p < \infty,\ then\ W^{1,p}(\Omega) \subset L^{p^*}(\Omega)\ where\ \dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}. \\ if & p = n,\ then\ W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [p, +\infty[. \\ if & p > n,\ then\ W^{1,p}(\Omega) \subset L^\infty(\Omega) \end{array}$$

with continuous imbedding.
Moreover, if $p > n$, we have: $\forall u \in W^{1,p}(\Omega)$,

$$|u(x) - u(y)| \leq C|x-y|^\alpha \|u\|_{W^{1,p}(\Omega)}\ a.e\ x, y \in \Omega$$

with $\alpha = 1 - \dfrac{n}{p} > 0$ and C is a constant which depend on p, n and Ω. In particular $W^{1,p}(\Omega) \subset C(\overline{\Omega})$.

Corollary 1.1.3 *Let Ω a bounded domain in $\mathbb{R}^n$ of C^1 class with $\Gamma = \partial\Omega$ and $1 \leq p \leq \infty$. We have*

$$\begin{array}{ll} if & p < n,\ then\ W^{1,p}(\Omega) \subset L^q(\Omega) \forall q \in [1, p^*[\ where\ \dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}. \\ if & p = n,\ then\ W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [p, +\infty[. \\ if & p > n,\ then\ W^{1,p}(\Omega) \subset C(\overline{\Omega}) \end{array}$$

with compact imbedding.

Remark 1.1.1 *We remark in particular that*

$$W^{1,p}(\Omega) \subset L^q(\Omega)$$

with compact imbedding for $1 \leq p \leq \infty$ *and for* $p \leq q < p^*$.

Corollary 1.1.4

$$\begin{array}{ll} if & \frac{1}{p} - \frac{m}{n} > 0, \; then \; W^{m,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \; where \; \frac{1}{q} = \frac{1}{p} - \frac{m}{n}. \\ if & \frac{1}{p} - \frac{m}{n} = 0, \; then \; W^{m,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \forall q \in [p, +\infty[. \\ if & \frac{1}{p} - \frac{m}{n} < 0, \; then \; W^{m,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n) \end{array}$$

with continuous imbedding.

1.2 Weak convergence

Let $(E; \|.\|_E)$ a Banach space and E' its dual space, i.e., the Banach space of all continuous linear forms on E endowed with the norm $\|.\|'_E$ defined by

$$\|f\|_{E'} =: \sup_{x \neq 0} \frac{|\langle f, x \rangle|}{\|x\|}$$

; where $\langle f, x \rangle$; denotes the action of f on$x$, i.e.$\langle f, x \rangle := f(x)$. In the same way, we can define the dual space of E' that we denote by E''. (The Banach space E'' is also called the bi-dual space of E.) An element x of E can be seen as a continuous linear form on E' by setting $x(f) := \langle x, f \rangle$, which means that $E \subset E''$:

Definition 1.2.1 *The Banach space E is said to be reflexive if* $E = E''$.

Definition 1.2.2 *The Banach space E is said to be separable if there exists a countable subset D of E which is dense in E, i.e.* $\overline{D} = E$.

Theorem 1.2.1 *(Riesz). If* $(H; \langle ., . \rangle)$ *is a Hilbert space,* $\langle ., . \rangle$ *being a scalar product on H, then* $H' = H$ *in the following sense: to each* $f \in H'$ *there corresponds a unique* $x \in H$ *such that* $f = \langle x, . \rangle$ *and* $\|f\|'_H = \|x\|_H$

Remark : From this theorem we deduce that $H'' = H$. This means that a Hilbert space is reflexive.

Proposition 1.2.1 *If E is reflexive and if F is a closed vector subspace of E, then F is reflexive.*

Corollary 1.2.1 *The following two assertions are equivalent: (i) E is reflexive; (ii)* E' *is reflexive.*

1.2.1 Weak and strong convergence

Definition 1.2.3 *(Weak convergence in E). Let $x \in E$ and let $\{x_n\} \subset E$. We say that $\{x_n\}$ weakly converges to x in E, and we write $x_n \rightharpoonup x$ in E, if*

$$\langle f, x_n \rangle \to \langle f, x \rangle$$

for all $f \in E'$.

Definition 1.2.4 *(weak convergence in E'). Let $f \in E'$ and let $\{f_n\} \subset E'$. We say that $\{f_n\}$ weakly converges to f in E', and we write $f_n \rightharpoonup f$ in E', if*

$$\langle f_n, x \rangle \to \langle f, x \rangle$$

for all $x \in E''$.

Definition 1.2.5 *(strong convergence). Let $x \in E$(resp. $f \in E'$) and let $\{x_n\} \subset E$ (resp $\{f_n\} \subset E'$). We say that $\{x_n\}$ (resp. $\{f_n\}$) strongly converges to x (resp. f), and we write $x_n \to x$ in E (resp. $f_n \to f$ in E'), if*

$$\lim_n \|x_n - x\|_E = 0; (\textit{resp. } \lim_n \|f_n - f\|'_E = 0)$$

Proposition 1.2.2 *Let $x \in E$, let $\{x_n\} \subset E$, let $f \in E'$ and let $\{f_n\} \subset E'$.*

i. If $x_n \to x$ in E then $x_n \rightharpoonup x$ in E.

ii. If $x_n \rightharpoonup x$ in E then $\{x_n\}$ is bounded.

iii. If $x_n \rightharpoonup x$ in E then $\liminf_{n\to\infty} \|x_n\|_E \geq \|x\|_E$

iv. If $f_n \to f$ in E' then $f_n \rightharpoonup f$ inE' (and so $f_n \overset{*}{\rightharpoonup} f$ in E').

v. If $f_n \rightharpoonup f$ in E' then $\{f_n\}$ is bounded.

vi. If $f_n \rightharpoonup f$ in E' then then $\liminf_{n\to\infty} \|f_n\|'_E \geq \|f\|'_E$

Proposition 1.2.3 *(finite dimension). If $\dim E < \infty$ then strong, weak and weak star convergence are equivalent.*

1.2.2 Some integral inequalities

We will give here some important integral inequalities. These inequalities play an important role in applied mathematics and also, it is very useful in our next chapters.

Theorem 1.2.2 *(Hölder's inequality). Let $1 \leq p \leq \infty$. Assume that$f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $fg \in L^P(\Omega)$ and*

$$\int_\Omega |fg| dx \leq \|f\|_p \|g\|_q.$$

Lemma 1.2.1 *(Young's inequality). Let* $f \in L^p(\mathbb{R})$ *and* $g \in L^q(\mathbb{R})$ *with* $1 < p < \infty$ *and* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \geq 0$ *then* $f * g \in L^r(\mathbb{R})$ *and*

$$\|f * g\|_{L^r(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})} \|g\|_{L^q(\mathbb{R})}.$$

Lemma 1.2.2 *Let* $1 \leq p \leq r \leq q$, $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$ *and* $1 \leq \alpha \leq 0$. *Then*

$$\|u\|_{L^r} \leq \|f\|^{\alpha}_{L^p} \|g\|^{1-\alpha}_{L^q}.$$

Lemma 1.2.3 *If* $\mu(\Omega) < \infty$, $1 \leq p \leq q \leq \infty$, *then* $L^q \hookrightarrow L^p$ *and*

$$\|u\|_{L^p} \leq \mu(\Omega)^{\frac{1}{p}+\frac{1}{q}} \|u\|_{L^q}.$$

1.2.3 Bounded and Unbounded linear operators

Let $(E, \|.\|_E)$ and $(F, \|.\|_F)$ be two Banach spaces over $\mathbb{C}$, and H will always denote a Hilbert space equipped with the scalar product $< .,. >_H$ and the corresponding norm $\|.\|_H$. A linear operator $T : E \longrightarrow F$ is a transformation which maps linearly E in F, that is

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v), \ \ \forall u, v \in E \text{ and } \alpha, \beta \in \mathbb{C}.$$

Definition 1.2.6 *A linear operator* $T : E \longrightarrow F$ *is said to be bounded if there exists* $C \geq 0$ *such that*

$$\|Tu\|_F \leq C\|u\|_E \ \ \forall u \in E.$$

The set of all bounded linear operators from E into F is denoted by $\mathcal{L}(E, F)$. *Moreover, the set of all bounded linear operators from E into E is denoted by* $\mathcal{L}(E)$.

Definition 1.2.7 *A bounded operator* $T \in \mathcal{L}(E, F)$ *is said to be compact if for each sequence* $(x_n)_{n\in\mathbb{N}} \in E$ *with* $\|x_n\|_E = 1$ *for each* $n \in \mathbb{N}$, *the sequence* $(Tx_n)_{n\in\mathbb{N}}$ *has a subsequence which converges in* F.
The set of all compact operators from E into F is denoted by $\mathcal{K}(E, F)$. *For simplicity one writes* $\mathcal{K}(E) = \mathcal{K}(E, F)$.

Definition 1.2.8 *Let* $T \in \mathcal{L}(E, F)$ *we define*
• *Range of* T *by*

$$\mathcal{R}(T) = \{Tu : u \in E\} \subset F.$$

• *Kernel of* T *by*

$$ker(T) = \{u \in E : Tu = 0\} \subset E.$$

Theorem 1.2.3 (Fredholm alternative)
If $T \in \mathcal{K}(E)$, *then*
• $ker(I - T)$ *is finite dimension, (*I *is the identity operator on* E*) .*
• $\mathcal{R}(I - T)$ *is closed.*
• $ker(I - T) = 0 \Leftrightarrow \mathcal{R}(I - T) = E$.

Definition 1.2.9 *An unbounded linear operator T from E into F is a pair $(T, D(T))$, consisting of a subspace $D(T) \subset E$ (called the domain of T) and a linear transformation.*

$$T : D(T) \subset E \mapsto F.$$

In the case when $E = F$ then we say $(T, D(T))$ is an unbounded linear operator on E. If $D(T) = E$ then $T \in \mathcal{L}(E, F)$.

Definition 1.2.10 *Let $T : D(T) \subset E \mapsto F$ be an unbounded linear operator.*
- *The range of T is defined by*

$$\mathcal{R}(T) = \{Tu : u \in D(T)\} \subset F.$$

- *The Kernel of T is defined by*

$$ker(T) = \{u \in D(T) : Tu = 0\} \subset E.$$

- *The graph of T is defined by*

$$G(T) = \{(u, Tu) : u \in D(T)\} \subset E \times F.$$

Definition 1.2.11 *A map T is said to be closed if $G(T)$ is closed in $E \times F$. The closedness of an unbounded linear operator T can be characterize as following if $u_n \in D(T)$ such that $u_n \to u$ in E and $Tu_n \to v$ in F, then $u \in D(T)$ and $Tu = v$.*

Definition 1.2.12 *Let $T : D(T) \subset E \mapsto F$ be a closed unbounded linear operator.*
- *The resolvent set of T is defined by*

$$\rho(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is bijective from } D(T) \text{ onto } F\}.$$

- *The resolvent of T is defined by*

$$\mathcal{R}(\lambda, T) = \{(\lambda I - T)^{-1} : \lambda \in \rho(T)\}$$

- *The spectrum set of T is the complement of the resolvent set in $\mathbb{C}$, denoted by*

$$\sigma(T) = \mathbb{C}/\rho(T)$$

Definition 1.2.13 *Let $T : D(T) \subset E \mapsto F$ be a closed unbounded linear operator. we can split the spectrum $\sigma(T)$ of T into three disjoint sets, given by*
- *The punctual spectrum of T is define by*

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : ker(\lambda I - T) \neq \{0\}\}$$

in this case λ is called an eigenvalue of T.
- *The continuous spectrum of T is define by*

$$\sigma_c(T) = \{\lambda \in \mathbb{C} \; : \; ker(\lambda I - T) = 0, \; \overline{\mathcal{R}(\lambda I - T)} = F \text{ and } (\lambda I - T)^{-1} \text{ is not bounded } \}.$$

- *The residual spectrum of T is define by*

$$\sigma_r(T) = \{\lambda \in \mathbb{C} \; : \; ker(\lambda I - T) = 0, \text{ and } \mathcal{R}(\lambda I - T) \text{ is not dense in } F\}$$

Definition 1.2.14 *Let $T : D(T) \subset E \longrightarrow F$ be a closed unbounded linear operator and let λ be an eigevalue of A. non-zero element $e \in E$ is called a generalized eigenvector of T associated with the eigenvalue value λ, if there exists $n \in \mathbb{N}^*$ such that*

$$(\lambda I - T)^n e = 0 \quad \text{and} \quad (\lambda I - T)^{n-1} e \neq 0.$$

if $n = 1$, then e is called an eigenvector.

Definition 1.2.15 *Let $T : D(T) \subset E \longrightarrow F$ be a closed unbounded linear operator. We say that T has compact resolvent, if there exist $\lambda_0 \in \rho(T)$ such that $(\lambda_0 I - T)^{-1}$ is compact.*

Theorem 1.2.4 *Let $(T, D(T))$ be a closed unbounded linear operator on H then the space $(D(T), \|.\|_{D(T)})$ where $\|u\|_{D(T)} = \|Tu\|_H + \|u\|_H \quad \forall u \in D(T)$ is Banach space .*

Theorem 1.2.5 *Let $(T, D(T))$ be a closed unbounded linear operator on H then, $\rho(T)$ is an open set of $\mathbb{C}$.*

1.3 Lax-Milgrame Theorem

Let H be a Hilbert space equipped with the inner product $(.,.)_H$ and the induced norm $\|.\|_H$.

Definition 1.3.1 *A bilinear form*

$$a : H \times H \to \mathbb{R}$$

is said to be

- *(i) continuous if there is a constant C such that*

$$|a(u,v)| \leq C\|u\|\|v\|, \;\; \forall u, v \in H$$

- *(ii) coercive if there is a constant $\alpha > 0$ such that*

$$|a(u,u)| \geq \alpha\|u\|^2, \;\; \forall u \in H$$

Theorem 1.3.1 (Lax-Milgrame Theorem) *Assume that $a(.,.)$ is a continuous coercive bilinear form on H. Then, given any $L \in \mathcal{L}(\mathcal{H},\mathbb{C})$, there exists a unique element $u \in H$ such that*

$$a(u,v) = L(v), \;\; \forall v \in H$$

1.4 Semigroups, Existence and uniqueness of solution

The vast majority of the evolution equations can be reduced to the form

$$\begin{cases} U_t(t) = AU(t), \ \ t > 0, \\ U(0) = U_0, \end{cases} \tag{1.1}$$

where $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $S(t)$ over a Hilbert space H. Lets start by basic definitions and theorems.
Let $(X, \|.\|_X)$ be a Banach space, and H be a Hilbert space equipped with the inner product $< .,. >_H$ and the induced norm $\|.\|_H$.

Definition 1.4.1 *Let X be a Banach space and let $I : X \to X$ its identity operator.*

1. *A one parameter family $(S(t))_{t\geq 0}$, of bounded linear operators from X into X is a semigroup of bounded linear operators on X if*

   (i) $S(0) = I$;

   (ii) $S(t+s) = S(t)S(s)$ *for every $s, t \geq 0$.*

2. *A semigroup of bounded linear operators, $(S(t))_{t\geq 0}$, is uniformly continuous if*
$$\lim_{t\to 0} \|S(t) - I\| = 0.$$

3. *A semigroup $(S(t))_{t\geq 0}$ of bounded linear operators on X is a strongly continuous semigroup of bounded linear operators or a C_0-semigroup if*
$$\lim_{t\to 0} S(t)x = x$$

4. *The linear operator $\mathcal{A}$ defined by*
$$\mathcal{A}x = \lim_{t\to 0} \frac{S(t)x - x}{t}, \ \ \forall x \in D(\mathcal{A})$$
where
$$D(\mathcal{A}) = \Big\{x \in X; \ \lim_{t\to 0} \frac{S(t)x - x}{t} \ \ \text{exists}\Big\}$$
is the infinitesimal generator of the semigroup $(S(t))_{t\geq 0}$.

Some properties of semigroup and its generator operator $\mathcal{A}$ are given in the following theorems:

Theorem 1.4.1 **(Pazy)** *Let A be the infinitesimal generator of a C_0- semigroup of contractions $(S(t))_{t\geq 0}$. Then, the resolvent $(\lambda I - \mathcal{A})^{-1}$ of $\mathcal{A}$ contains the open right half-plane, i.e., $\rho(\mathcal{A}) \subset \{\lambda : \mathcal{R}(\lambda) > 0\}$ and for such λ we have*

$$\|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(H)} \leq \frac{1}{\mathcal{R}(\lambda)}.$$

***Theorem 1.4.2* (Kato)** *Let $\mathcal{A}$ be a closed operator in a Banach space X such that the resolvent $(I-\mathcal{A})^{-1}$ of $\mathcal{A}$ exists and is compact. Then the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ consists entirely of isolated eigenvalues with finite multiplicities.*

***Theorem 1.4.3* (Pazy)** *Let $(S(t))_{t\geq 0}$ be a C_0-semigroup on a Hilbert space H. Then there exist two constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|S(t)\|_{\mathcal{L}(H)} \leq Me^{\omega t}, \ \ \forall t \geq 0.$$

If $\omega = 0$, the semigroup $(S(t))_{t\geq 0}$ is called uniformly bounded and if moreover $M = 1$, then it is called a C_0-semigroup of contractions. For the existence of solution of problem (1.1), we typically use the following Lumer-Phillips and Hille-Yosida theorems :

***Theorem 1.4.4* (Lumer-Phillips)** *Let $\mathcal{A}$ be a linear operator with dense domain $D(A)$ in a Hilbert space H. If*

(i) *$\mathcal{A}$ is dissipative, i.e., $< \mathcal{R}(< \mathcal{A}x, x >_H) \leq 0, \ \forall x \in D(\mathcal{A})$*
and if

(ii) *there exists a $\lambda_0 > 0$ such that the range $\mathcal{R}(\lambda_0 I - \mathcal{A}) = H$,*
then $\mathcal{A}$ generates a C_0-semigroup of contractions on H.

***Theorem 1.4.5* (Hille-Yosida)** *Let $\mathcal{A}$ be a linear operator on a Banach space X and let $\omega \in \mathbb{R}$, $M \geq 1$ be two constants. Then the following properties are equivalent*

(i) *$\mathcal{A}$ generates a C_0-semigroup $(S(t))_{t\geq 0}$, satisfying*

$$\|S(t)\|_{\mathcal{L}(H)} \leq Me^{\omega t}, \ \ \forall t \geq 0.$$

(ii) *$\mathcal{A}$ is closed, densely defined, and for every $\lambda > \omega$ one has $\lambda \in \rho(\mathcal{A})$ and*

$$\|(\lambda - \omega)^n (\lambda - \mathcal{A})^{-n}\| \leq M, \ \ \forall n \in \mathbb{N}.$$

(iii) *$\mathcal{A}$ is closed, densely defined, and for every $\lambda \in \mathbb{C}$ with $\mathcal{R} > \omega$, one has $\lambda \in \rho(\mathcal{A})$ and*

$$\|(\lambda - \mathcal{A})^{-n}\| \leq \frac{M}{(\mathcal{R}(\lambda) - \omega)^n}, \ \ \forall n \in \mathbb{N}.$$

Consequently, $\mathcal{A}$ is maximal dissipative operator on a Hilbert space H if and only if it generates a C_0-semigroup of contractions $(S(t))_{t\geq 0}$ on H. Thus, the existence of solution is justified by the following corollary which follows from Lumer-Phillips theorem.

Corollary 1.4.1 *Let H be a Hilbert space and let $\mathcal{A}$ be a linear operator defined from $D(\mathcal{A}) \subset H$ into H. If $\mathcal{A}$ is maximal dissipative operator then the initial value problem (1.1) has a unique solution $U(t) = S_A(t)U0$ such that $U \in C([0, +1), H)$, for each initial datum $U_0 \in H$. Moreover, if $U_0 \in D(\mathcal{A})$, then*

$$U \in C([0, +1), D(\mathcal{A})) \cap C^1([0, +1), H).$$

Finally, we also recall the following theorem concerning a perturbations by a bounded linear operators

Theorem 1.4.6 *Let X be a Banach space and let $\mathcal{A}$ be the infinitesimal generator of a C_0-semigroup $(S(t))_{t\geq 0}$ on X, satisfying $\|S_{\mathcal{A}}(t)\|_{\mathcal{L}(H)} \leq Me^{\omega t}$ for all $t \geq 0$. If $\mathcal{B}$ is a bounded linear operator on X , then the operator $\mathcal{A}+\mathcal{B}$ becomes the infinitesimal generator of a C_0-semigroup $(S_{\mathcal{A}+\mathcal{B}}(t))_{t\geq 0}$ on X, satisfying $\|S_{\mathcal{A}+\mathcal{B}}(t)\|_{\mathcal{L}(H)} \leq Me^{(\omega+M\|\mathcal{B}\|)t}$ for all $t \geq 0$.*

1.5 Stability of semigroup

In this section we start by introducing some definition about strong, exponential and polynomial stability of a C_0-semigroup. Then we collect some results about the stability of C_0-semigroup. Let $(X, \|.\|_X)$ be a Banach space, and H be a Hilbert space equipped with the inner product $< .,. >_H$ and the induced norm $\|.\|_H$.

Definition 1.5.1 *Assume that $\mathcal{A}$ is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on X. We say that the C_0-semigroup $(S(t))_{t\geq 0}$ is*

1. *Strongly stable if*
$$\lim_{t\to+\infty} \|S(t)u\|_X = 0, \quad \forall u \in X.$$
2. *Uniformly stable if*
$$\lim_{t\to+\infty} \|S(t)\|_{\mathcal{L}(X)} = 0.$$
3. *Exponentially stable if there exist two positive constants M and ϵ such that*
$$\|S(t)u\|_X \leq Me^{-\epsilon t}\|u\|_X, \quad \forall t > 0, \ \forall u \in X.$$
4. *Polynomially stable if there exist two positive constants C and α such that*
$$\|S(t)u\|_X \leq Ct^{-\alpha}\|u\|_X, \quad \forall t > 0, \ \forall u \in X.$$

Proposition 1.5.1 *Assume that $\mathcal{A}$ is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on X. The following statements are equivalent*

- *$(S(t))_{t\geq 0}$ is uniformly stable.*
- *$(S(t))_{t\geq 0}$ is exponentially stable.*

First, we look for the necessary conditions of strong stability of a C_0-semigroup. The result was obtained by Arendt and Batty.

Theorem 1.5.1 *(Arendt and Batty) Assume that $\mathcal{A}$ is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on a reflexive Banach space X. If*

(i) *$\mathcal{A}$ has no pure imaginary eigenvalues.*

(ii) *$\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable.*

Then $S(t)$ is strongly stable.

Remark 1.5.1 *If the resolvent $(I-T)^{-1}$ of T is compact, then $\sigma(T)=\sigma_p(T)$. Thus, the state of Theorem (...) lessens to $\sigma_p(\mathcal{A})\cap i\mathbb{R}=\emptyset$. Next, when the C_0-semigroup is strongly stable, we look for the necessary and sufficient conditions of exponential stability of a C_0-semigroup. In fact, exponential stability results are obtained using different methods like: multipliers method, frequency domain approach, Riesz basis approach, Fourier analysis or a combination of them .*

Theorem 1.5.2 *(Huang-Pruss)Assume that $\mathcal{A}$ is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on H. $S(t)$ is uniformly stable if and only if*

1. $i\mathbb{R}\subset\rho(\mathcal{A})$.
2. $\sup_{\beta\in\mathbb{R}}\|(i\beta I-\mathcal{A})^{-1}\|_{\mathcal{L}(H)}<+\infty$.

The second one, is a classical method based on the spectrum analysis of the operator $\mathcal{A}$

In the case when the C_0-semigroup is not exponentially stable we look for a polynomial one. In general, polynomial stability results also are obtained using different methods like : multipliers method, frequency domain approach, Riesz basis approach, Fourier analysis or a combination of them .

Theorem 1.5.3 *(Batty , A.Borichev and Y.Tomilov, Z. Liu and B. Rao.)Assume that $\mathcal{A}$ is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on H. If $i\mathbb{R}\subset\rho(\mathcal{A})$, then for a fixed $l>0$ the following conditions are equivalent*

1. $\lim_{|\lambda|\to+\infty}\sup\frac{1}{\lambda^l}\|(\lambda I-\mathcal{A})^{-1}\|_{\mathcal{L}(H)}<+\infty$.
2. $\|S(t)U_0\|_H\leq\frac{C}{t^{l-1}}\|U_0\|_{D(\mathcal{A})}\ \ \forall t>0,\ U_0\in D(\mathcal{A}),$ *for some* $C>0$.

1.6 Fractional Derivative Control

In this part, we introduce a brief reminder of the basic elements of the theory of fractional computation. The concept of fractional computation is a generalization of ordinary derivation and integration to an arbitrary order. Derivatives of non-integer order are now widely applied in many domains, for example in economics, electronics, mechanics, biology, probability and viscoelasticity. A particular interest for fractional derivation is related to the mechanical modeling of gums and rubbers. In short, all kinds of materials that preserve the memory of previous deformations in particular viscoelastic. Indeed, the fractional derivation is introduced naturally. The fractional calculus is an important developing field in both pure and applied mathematics. Many real world problems have been investigated within the fractional derivatives, particularly Caputo fractional derivative is extensively and successfully used in many branches of sciences and engineering.

1.6.1 Some history of fractional calculus

We refer to [**29**].

In a letter dated **September 30th, 1695** L'Hospital wrote to Leibniz asking him about the meaning of $d^n y/dx^n$ if $n = 1/2$, that is "what if n is fractional?". Leibnizs response: An apparent paradox, from which one day useful consequences will be drawn.

In **1819** S. F. Lacroix was the first to mention in some two pages a derivative of arbitrary order.Thus for $y = x^a$, $a \in \mathbb{R}_+$, he showed that

$$\frac{d^{1/2}y}{dx^{1/2}} = \frac{\Gamma(a+1)}{\Gamma(1+1/2)} x^{a-1/2}.$$

In particular he had $(d/dx)^{1/2}x = 2\sqrt{x/\pi}$.

In **1822** J. B. J. Fourier derived an integral representation for f(x),

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} f(\alpha)d\alpha \int_{\mathbb{R}} \cos p(x-\alpha)dp,$$

obtained (formally) the derivative version

$$\frac{d^\nu}{dx^\nu} f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} f(\alpha)d\alpha \int_{\mathbb{R}} p^\nu \cos[p(x-\alpha) + \frac{\nu\pi}{2}]dp$$

where "the number v will be regarded as any quantity whatever, positive or negative".

In **1823** Abel resolved the integral equation arising from the brachistochrone problem, namely

$$\frac{1}{\Gamma(\alpha)} \int_0^x \frac{g(u)}{(x-u)^{1-\alpha}} du = f(x), \qquad 0 < \alpha < 1$$

with the solution

$$g(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{f(u)}{(x-u)^\alpha} du$$

Abel never solved the problem by fractional calculus but, in **1832** Liouville , did solve this integral equation.

Perhaps the first serious attempt to give a logical definition of a fractional derivative is due to Liouville; he published nine papers on the subject between **1832** and **1837**, the last in the field in **1855**. They grew out of Liouville's early work on electromagnetism. There is further work of George Peacock (**1833**), D. F. Gregory (**1841**), Augustus de Morgan (**1842**), P. Kelland (**1846**), William Center (**1848**). Especially basic is Riemann's student paper of **1847**.

After the participation of Riemann and the work of Cayley in **1880** , among the mathematicians spearheading research in the broad area of fractional calculus until **1941** were S.F. Lacroix, J.B.J. Fourier, N.H. Abel, J. Liouville, A. De Morgan, B. Riemann, Hj. Holmgren, K. Griinwald, A.V. Letnikov, N.Ya. Sonine, J. Hadamard, G.H. Hardy, H. Weyl, M. Riesz, H.T. Davis, A. Marchaud, J.E. Littlewood, E.L. Post, E.R. Love, B.Sz.-Nagy, A. Erdelyi and H. Kober.

Fractional calculus has developed especially intensively since **1974** when the first international conference in the field took place.It was organized by Bertram Ross.

Samko et al in their encyclopedic volume state and we cite: "We pay tribute to investigators of recent decades by citing the names of mathematicians who have made a valuable scientific contribution to fractional calculus development from **1941** until the present [**1990**]. These are M.A. Al- Bassam, L.S. Bosanquet, P.L. Butzer, M.M. Dzherbashyan, A. Erdelyi, T.M. Flett, Ch. Fox, S.G. Gindikin, S.L. Kalla, LA. Kipriyanov, H. Kober, P.I. Lizorkin, E.R. Love, A.C. McBride, M. Mikolas, S.M. Nikol'skii, K. Nishimoto, LI. Ogievetskii, R.O. O'Neil, T.J. Osier, S. Owa, B. Ross, M. Saigo, I.N. Sneddon, H.M. Srivastava, A.F. Timan, U. Westphal, A. Zygmund and others". To this list must of course be added the names of the authors of Samko et al and many other mathematicians, particularly those of the younger generation. Books especially devoted to fractional calculus include K.B. Oldham and J. Spanier, S.G. Samko, A.A. Kilbas and O.I. Marichev, V.S. Kiryakova, K.S. Miller and B. Ross, B. Rubin. Books containing a chapter or sections dealing with certain aspects of fractional calculus include H.T. Davis, A. Zygmund, M.M.Dzherbashyan, I.N. Sneddon, P.L. Butzer and R.J. Nessel, P.L. Butzer and W. Trebels, G.O. Okikiolu, S. Fenyo and H.W. Stolle, H.M. Srivastava and H.L. Manocha, R. Gorenfio and S. Vessella.

1.6.2 Various approaches of fractional derivatives

There exists a many mathematical definitions of fractional order integration and derivation. These definitions do not always lead to identical results but are equivalent for a wide large of functions. We introduce the fractional integration operator as well as the two most definitions of fractional derivatives, used, namely that Riemann-Liouville, Caputo and Hadamard.

From the classical fractional calculus, we recall

Definition 1.6.1 *The left Riemann-Liouville fractional integral of order $\alpha > 0$ starting from a has the following form*

$$({}_aI^\alpha f)(n) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt.$$

Definition 1.6.2 *The right Riemann-Liouville fractional integral of order $\alpha > 0$ ending at $b > a$ is defined by*

$$(I_b^\alpha f)(n) = \frac{1}{\Gamma(\alpha)} \int_x^b (x-t)^{\alpha-1} f(t)dt.$$

Definition 1.6.3 *The left Riemann-Liouville fractional derivative of order $\alpha > 0$ starting at a is given below*

$$({}_aD^\alpha f)(x) = (\frac{d}{dx})^n (aI^{n-\alpha} f)(x), \quad n = [\alpha] + 1.$$

Definition 1.6.4 *The right Riemann-Liouville fractional derivative of order $\alpha > 0$ ending at b becomes*

$$(D_b^\alpha f)(x) = (-\frac{d}{dx})^n (I_b^{n-\alpha} f)(x).$$

Definition 1.6.5 *The left Caputo fractional of order $\alpha > 0$ starting from a has the following form*

$$({}_aD^\alpha f)(x) = (aI^{n-\alpha} f^{(n)})(x), \ \ n = [\alpha] + 1.$$

Definition 1.6.6 *The right Caputo fractional derivative of order $\alpha > 0$ ending at b becomes*

$$(D_b^\alpha f)(x) = (I_b^{n-\alpha}(-1)^n f^{(n)})(x).$$

The Hadamard type fractional integrals and derivatives were introduced in [15] as:

Definition 1.6.7 *The left Hadamard fractional integral of order $\alpha > 0$ starting from a has the following form*

$$({}_aI^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (\ln x - \ln t)^{\alpha-1} f(t)dt$$

Definition 1.6.8 *The right Hadamard fractional integral of order $\alpha > 0$ ending at $b > a$ is defined by*

$$(I_b^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\ln t - \ln x)^{\alpha-1} f(t)dt$$

Definition 1.6.9 *The left Hadamard fractional derivative of order $\alpha > 0$ starting at a is given below*

$$({}_aD^\alpha f)(x) = (x\frac{d}{dx})^n (aI^{n-\alpha} f)(x), \ \ n = [\alpha] + 1.$$

Definition 1.6.10 *The right Hadamard fractional derivative of order $\alpha > 0$ ending at b becomes*

$$(D_b^\alpha f)(x) = (-x\frac{d}{dx})^n (I_b^{n-\alpha} f)(x).$$

Definition 1.6.11 *The fractional derivative of order α, $0 < \alpha < 1$, in sense of Caputo, is defined by*

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{df}{ds}(s)ds.$$

Definition 1.6.12 *The fractional integral of order α, $0 < \alpha < 1$, in sense Riemann-Liouville, is defined by*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds.$$

Remark 1.6.1 *From the above definitions, clearly*

$$D^\alpha f = I^{\alpha-1} Df, \ \ 0 < \alpha < 1.$$

Lemma 1.6.1

$$I^\alpha D^\alpha f(t) = f(t) - f(0), \ \ 0 < \alpha < 1.$$

Lemma 1.6.2 *If*

$$D^{\beta} f(0) = 0.$$

then

$$D^{\alpha} D^{\beta} f = D^{\alpha+\beta} f, \;\; 0 < \alpha < 1, \;\; 0 < \beta < 1.$$

Now, we give the definitions of the generalized Caputo's fractional derivative and the generalized fractional integral. These exponentially modified fractional integro-differential operators were first proposed in [].

Definition 1.6.13 *The generalized Caputo's fractional derivative is given by*

$$D^{\alpha,\eta} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{df}{ds}(s)\, ds, \;\; 0 < \alpha < 1, \eta \geq 0.$$

Remark 1.6.2 *The operators D^{α} and $D^{\alpha,\eta}$ differ just by their kernels.*

Definition 1.6.14 *The generalized fractional integral is given by*

$$I^{\alpha,\eta} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\eta(t-s)} f(s)\, ds, \;\; 0 < \alpha < 1, \eta \geq 0.$$

Remark 1.6.3 *We have*

$$D^{\alpha,\eta} f = I^{1-\alpha,\eta} D f, \;\; 0 < \alpha < 1, \eta \geq 0.$$

1.7 Bessel functions

We will discuss a class of functions known as Bessel functions. These are named after the German mathematician and astronomer Friedrich Bessel. Bessel functions occur in many other physical problems, usually in a cylindrical geometry.

Definition 1.7.1 *Bessel's equation can be written in the form*

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2) y = 0, \tag{1.2}$$

with ν real and positive. note that (1.2) has a regular singular point at $x = 0$.

1.7.1 The Gamma Function and Pockhammer Symbol

Definition 1.7.2 *The gamma function is defined by*

$$\Gamma(x) = \int_0^\infty e^{-q} q^{x-1} dq, \quad for\ x > 0. \tag{1.3}$$

Note that the integration is over the dummy variable q and x is treated as constant during the integration

Definition 1.7.3 *The pockhammer symbol is a simple way of writing down long products. It is defined as*

$$(\alpha)_r = \alpha(\alpha+1)(\alpha+2)...(\alpha+r-1)$$

So that, for example, $(\alpha)_1 = \alpha$ and $(\alpha)_2 = \alpha(\alpha+1)$.
Note that $(1)_n = n!$

The relationship between the gamma function and the pockhammer symbol is

$$\Gamma(x)(x)_n = \Gamma(x+n)$$

1.7.2 Series solutions of Bessel's Equation

a.Fundamental solutions of Bessel's equation when $\nu \notin \mathbb{N}$

We can now proceed to consider a Frobenius solution,

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+c}$$

Where we have used the Pockhammer symbol to simplify the expression. So we have

$$y(x) = a_0 x^{\pm\nu} \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{2^{2m}(1\pm\nu)_m m!}$$

With a suitable choice of a_0 we can write this as

$$y(x) = A\frac{x^{\pm\nu}}{2^{\pm\nu}\Gamma(1\pm\nu)} \sum_{m=0}^{\infty} (-1)^m \frac{(\frac{x^2}{4})^m}{(1\pm\nu)_m m!} = AJ_{\pm\nu}(x).$$

These are the Bessel functions of order $\pm\nu$. The general solution of Bessel's equation (1.2), is therefore

$$y(x) = AJ_{+\nu}(x) + BJ_{-\nu}(x),$$

for arbitrary constants A and B, with the first of the two series converges for all values of x and defines the so-called Bessel function of order ν and of the first kind which is denoted by J_ν

$$J_\nu(x) = \frac{x^\nu}{2^\nu\Gamma(1+\nu)}\sum_{m=0}^{\infty}(-1)^m\frac{(\frac{x^2}{4})^m}{(1+\nu)_m m!} = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+\nu+1)}(\frac{x}{2})^{2m+\nu} = \sum_{m=0}^{\infty} c^+_{\nu,m}x^{2m+\nu}\,, x \geq 0. \tag{1.4}$$

The second series converges for all positive values of x and is evidently $J_{-\nu}$

$$J_{-\nu}(x) = \frac{x^{-\nu}}{2^{-\nu}\Gamma(1-\nu)}\sum_{m=0}^{\infty}(-1)^m\frac{(\frac{x^2}{4})^m}{(1-\nu)_m m!} = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m-\nu+1)}(\frac{x}{2})^{2m-\nu} = \sum_{m=0}^{\infty} c^-_{\nu,m}x^{2m-\nu}\,, x > 0. \tag{1.5}$$

b.Fundamental solutions of Bessel's equation when $\nu = n \in \mathbb{N}$

Assume that $\nu = n \in \mathbb{N}$. When looking for solutions of (1.2)of the form of series of ascending powers of x, one sees that J_n and J_{-n} are still solutions of (1.2), where J_n is still by (1.4) and J_{-n} is given by (1.5); when $\nu = n \in \mathbb{N}$, J_{-n} can be written .

$$J_n(x) = \sum_{m\geq n}\frac{(-1)^m}{m!\Gamma(m-n+1)}(\frac{x}{2})^{-n+2m}. \tag{1.6}$$

However now $J_{-n}(x) = (-1)^n J_n(y)$, hence J_n and J_{-n} are linearly dependent. The determination of a fundamental system of solutions in this case requires further investigation. In this purpose, one introduces the Bessel's functions of order ν and of the second kind: among the several definitions of Bessel's functions of second order, we recall here the definition by Weber. The Bessel's functions of order ν and of second kind are denoted by Y_ν and defined by

$$\begin{cases} \forall\nu \notin \mathbb{N}, & Y_\nu(y) := \frac{J_\nu(x)cos(\nu\pi) - J_{-\nu}(x)}{sin(\nu\pi)}, \\ \forall n \in \mathbb{N}, & Y_n(y) := \lim_{\nu\to n} Y_\nu(x), \end{cases}$$

For any $\nu \in \mathbb{R}_+$, the two functions J_ν and Y_ν always are linearly independent. In particular, in the case $\nu = n \in \mathbb{N}$, the pair (J_n, Y_n) forms a fundamental system of solutions of the Bessels equation for functions of order n.

1.7.3 Differential and Recurrence Relations Between Bessel functions

It is often useful to find relationships between Bessel functions with different indices. We will derive two such relationships. We start with (**??**),we multiply by x^ν and differentiate to obtain

$$\frac{d}{dx}\{x^\nu J_\nu(x)\} = x^\nu J_{\nu-1}(x) \tag{1.7}$$

and consequently

$$\frac{d}{dx}\{x^{-\nu} J_\nu(x)\} = -x^{-\nu} J_{\nu+1}(x) \tag{1.8}$$

We can use these relationships to derive recurrence relations between the Bessel functions. We expand the differentials in each expression to give the equations

$$J'_\nu(x) + \frac{\nu}{x} J_\nu(x) = J_{\nu-1}(x), \tag{1.9}$$

where we have divided through by x^ν, and

$$J'_\nu(x) - \frac{\nu}{x} J_\nu(x) = -J_{\nu+1}(x), \tag{1.10}$$

where this time we have multiplied by x^ν. By adding these expressions we find that

$$J'_\nu(x) = \frac{1}{2}\{J_{\nu-1}(x) - J_{\nu+1}(x)\}, \tag{1.11}$$

and by subtracting then

$$\frac{2\nu}{x} J_\nu(x) = J_{\nu-1}(x) + J_{\nu+1}(x), \tag{1.12}$$

which is a pure recurrence relationship. These results can also be used when integrating Bessel functions.

1.7.4 Inhomogeneous Terms in Bessel's Equation

The Inhomogeneous version of Bessel's equation

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = f(x), \tag{1.13}$$

and the solution can be written as

$$y(x) = AJ_\nu(x) + BJ_{-\nu}(x) + \frac{2\nu}{sin\nu\pi}\int_0^x \frac{f(s)}{s}\Big(J_\nu(s)J_{-\nu}(x) - J_\nu(x)Y_{-\nu}(s)\Big)ds. \tag{1.14}$$

1.8 Appendix

Theorem 1.8.1 *Let μ be the function:*

$$\mu(\xi) = |\xi|^{\frac{2\alpha-d}{2}}, \quad \xi \in \mathbb{R}^d \quad \textit{and} \quad 0 < \alpha < 1. \tag{1.15}$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t\omega(\xi,t) + (|\xi|^2 + \eta)\omega(\xi,t) - U(t)\mu(\xi) = 0, \quad \xi \in \mathbb{R}^d, t \in \mathbb{R}^+ \quad \textit{and} \quad \eta \geq 0 \tag{1.16}$$

$$\omega(\xi, 0) = 0, \tag{1.17}$$

$$O(t) = \frac{2\sin(\alpha\pi)\Gamma(\frac{d}{2}+1)}{d\pi^{\frac{d}{2}+1}} \int_{\mathbb{R}^d} \mu(\xi)\omega(\xi,t)d\xi, \tag{1.18}$$

is given by

$$O = I^{1-\alpha,\eta}U = D^{\alpha,\eta}U. \tag{1.19}$$

Proof

Step 1. Take $\eta = 0$, the from equation (1.16) and (1.17),we have

$$\omega(\xi, t) = \int_0^t \mu(\xi) e^{-|\xi|^2(t-\tau)} U(\tau) d\tau \tag{1.20}$$

Then from equations (1.18) and (1.20), we get

$$O(t) = \delta \int_{\mathbb{R}^d} |\xi|^{2\alpha-d} [\int_0^t \mu(\xi) e^{-|\xi|^2(t-\tau)} U(\tau) d\tau] d\xi \tag{1.21}$$

where $\delta = \frac{2\sin(\alpha\pi)\Gamma(\frac{d}{2}+1)}{d\pi^{\frac{d}{2}+1}}$. Next, using the spherical coordinates defined by,

$$\begin{cases} \xi_1 = \rho sin(\phi_1) sin(\phi_2) ... sin(\phi_{d-3}) sin(\phi_{d-2}) sin(\phi_{d-1}), \\ \xi_2 = \rho sin(\phi_1) sin(\phi_2) ... sin(\phi_{d-3}) sin(\phi_{d-2}) cos(\phi_{d-1}), \\ \xi_3 = \rho sin(\phi_1) sin(\phi_2) ... sin(\phi_{d-3}) cos(\phi_{d-2}), \\ \xi_4 = \rho sin(\phi_1) sin(\phi_2) ... cos(\phi_{d-3}), \\ . \\ . \\ . \\ \xi_{d-1} = \rho sin(\phi_1) cos(\phi_2), \\ \xi_d = \rho cos(\phi_1). \end{cases} \tag{1.22}$$

where,$\rho = |\xi| = \sqrt{\sum_{i=1}^d |\xi_i|^2}$, $\phi_j \in [0, \pi]$ if $1 \leq j \leq d-2$ and $\phi_{d-2} \in [0, 2\pi]$. The jacobian J is defined by

$$J = \rho^{d-1} \prod_{j=1}^{d-2} sin^{d-1-j}(\phi_j) \tag{1.23}$$

Since the integrating is a function which depends only on $|\xi| = \rho$, thus we can integrate on all the angles and the calculation reduces that of a simple integral on the positive real axis. Then, from equations (1.21)-(1.23) we get

$$O(t) = \delta \int_0^{+\infty} \rho^{2\alpha-1} \prod_{j=1}^{d-2} (\int_0^\pi \ sin^{d-1-j}(\phi_j) d\phi_j) \int_0^{2\pi} d\phi_{d-1} [\int_0^t e^{-\rho^2(t-\tau)} U(\tau) d\tau] d\rho \tag{1.24}$$

By induction, it easy to see that

$$\prod_{j=1}^{d-2} (\int_0^\pi \ sin^{d-1-j}(\phi_j) d\phi_j) \int_0^{2\pi} d\phi_{d-1} = \frac{d\Pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)} \tag{1.25}$$

Inserting equation (1.25) in equation (1.24), we get

$$O(t) = \frac{sin(\alpha\pi)}{\pi} \int_0^t 2[\int_0^{+\infty} \rho^{2\alpha-1} e^{-\rho^2(t-\tau)} d\rho] U(\tau) d\tau. \tag{1.26}$$

Thus,

$$O(t) = \frac{sin(\alpha\pi)}{\pi} \int_0^t [(t-\tau)^{-\alpha} \Gamma(\alpha)] U(\tau) d\tau. \tag{1.27}$$

Using the fact that $\frac{sin(\alpha\pi)}{\pi} = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}$ in equation, we obtain

$$O(t) = \int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} U(\tau) d\tau. \tag{1.28}$$

It follows that, from equation (1.28) we have

$$O = I^{1-\alpha} U \tag{1.29}$$

Step 2. By simply effecting the following change of function

$$\omega(\xi, t) := e^{-\eta t} \varphi(\xi, t)$$

in equations (1.16) and (1.18), we directly obtain

$$\partial_t \omega(\xi, t) + (|\xi|^2 + \eta)\omega(\xi, t) - U(t)\mu(\xi) = 0, \quad \xi \in \mathbb{R}^N, t \in \mathbb{R}^+ \quad and \quad \eta \geq 0, \tag{1.30}$$

$$\omega(\xi, 0) = 0, \tag{1.31}$$

$$O(t) = \delta e^{-\eta t} \int_{\mathbb{R}^d} \mu(\xi)\omega(\xi, t) d\xi \tag{1.32}$$

Hence, from **Step 1**, (1.30)-(1.32) yield the desired result

$$O(t) = e^{-\eta t} \int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} e^{\eta\tau} U(\tau) d\tau$$

The proof has been completed.

Lemma 1.8.1 *If* $\lambda \in D = \{\lambda \in \mathbb{C} : Re\lambda + \eta > 0\} \cup \{\lambda \in \mathbb{C} : Im\lambda \neq 0\}$ *then*

$$F_1(\lambda) = \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2} \, d\xi = \frac{\pi}{\sin \alpha\pi} (\lambda + \eta)^{\alpha - 1}$$

and

$$F_2(\lambda) = \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{(\lambda + \eta + \xi^2)^2} \, d\xi = (1 - \alpha) \frac{\pi}{\sin \alpha\pi} (\lambda + \eta)^{\alpha - 2}.$$

Proof Let us set

$$f_\lambda(\xi) = \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2}.$$

We have

$$\left| \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2} \right| \leq \frac{\mu^2(\xi)}{Re\lambda + \eta + \xi^2}.$$

Then the function f_λ is integrable. Moreover

$$\left| \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2} \right| \leq \begin{cases} \dfrac{\mu^2(\xi)}{\eta_0 + \eta + \xi^2} \text{ for all } Re\lambda \geq \eta_0 > -\eta \\ \dfrac{\mu^2(\xi)}{\tilde{\eta}_0 + \xi^2} \text{for all } |Im\lambda| \geq \tilde{\eta}_0 > 0 \end{cases}$$

From theorem 1.16.1 in **[44]**, the function

$$f_\lambda : D \to \mathbb{C} \text{ is holomorphe.}$$

For a real number $\lambda > -\eta$, we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\,d\xi = \int_{-\infty}^{+\infty} \frac{|\xi|^{2\alpha-1}}{\lambda+\eta+\xi^2}\,d\xi = \int_0^{+\infty} \frac{x^{\alpha-1}}{\lambda+\eta+x}\,dx \text{ (with } \xi^2 = x)\\
&= (\lambda+\eta)^{\alpha-1}\int_1^{+\infty} y^{-1}(y-1)^{\alpha-1}\,dy \text{ (with } y = x/(\lambda+\eta)+1)\\
&= (\lambda+\eta)^{\alpha-1}\int_0^1 z^{-\alpha}(1-z)^{\alpha-1}\,dz \text{ (with } z = 1/y)\\
&= (\lambda+\eta)^{\alpha-1}B(1-\alpha,\alpha) = (\lambda+\eta)^{\alpha-1}\Gamma(1-\alpha)\Gamma(\alpha) = (\lambda+\eta)^{\alpha-1}\frac{\pi}{\sin\pi\alpha}.
\end{aligned}$$

Both holomorphic functions f_λ and $\lambda \mapsto (\lambda+\eta)^{\alpha-1}\dfrac{\pi}{\sin\pi\alpha}$ coincide on the half line $]-\infty,-\eta[$, hence on D following the principle of isolated zeroes.

Chapter 2

ENERGY DECAY FOR A DEGENERATE WAVE EQUATION UNDER FRACTIONAL DERIVATIVE CONTROLS

2.1 Introduction

In this Chapter, we are concerned with the boundary stabilization of convolution type for degenerate wave equation of the form

$$u_{tt}(x,t) - (a(x)u_x(x,t))_x = 0 \text{ in } (0,1)\times(0,\infty), \tag{2.1}$$

where the coefficient a is a positive function on $]0,1]$ but vanishes at zero. The degeneracy of (2.1) at $x=0$ is measured by the parameter μ_a defined by

$$\mu_a = \sup_{0<x\leq 1} \frac{x|a'(x)|}{a(x)}. \tag{2.2}$$

We distinguish the two following cases:

-*The weakly degenerate case at 0.* When $\mu_a \in [0,1[$, then the problem is called weakly degenerate at 0 and the natural boundary condition associated to (2.1) is the Dirichlet boundary condition $u(0)=0$.

-*The strongly degenerate case at 0.* When $\mu_a > 1$, then the problem is called strongly degenerate at 0 and the natural boundary condition associated to (2.1) is the Neumann boundary condition $(au_x)(0)=0$.

These type of conditions on diffusion coefficient and on the boundary were used before in the context of study of null controlability of degenerate parabolic equation.

Up to now, there are many works concerning the stabilization and controllability of non-degenerate wave equation with different types of damping (see e.g. **[21]**, **[22]**, **[19]**, **[24]** and the references therein). In **[22]**, for $a(x) = a_1 x + a_0$: the authors have established asymptotic stabilization with the following boundary damping

$$\begin{cases} (au_x)(0,t) = 0, \\ (au_x)(1,t) = -ku(1,t) - u_t(1,t), \ k > 0. \end{cases}$$

In **[19]**, the authors considered the following modelization of a flexible torque arm controlled by two feedbacks depending only on the boundary velocities:

$$\begin{cases} u_{tt}(x,t) - (a(x)u_x)_x + \alpha u_t(x,t) + \beta y(x,t) = 0, \ 0 < x < 1, \ t > 0, \\ (a(x)u_x)(0) = k_1 u_t(0,t), t > 0, \\ (a(x)u_x)(1) = -k_2 u_t(1,t), t > 0, \end{cases}$$

where

$$\begin{cases} \alpha \geq 0, \beta > 0, k_1, k_2 \geq 0, k_1 + k_2 \neq 0, \\ a \in W^{1,\infty}(0,1), a(x) \geq a_0 \text{ for all } x \in [0,1]. \end{cases}$$

They proved the exponential decay of the solutions.

On the contrary, when the coefficient $a(x)$ is zero at some points, the equation will be degenerate and few results are known in this case, even though many problems that are relevant for applications are described by hyperbolic equations degenerating at the boundary of the space domain (see **[26]**, **[46]** and **[2]**). In **[26]**, for any $0 < \gamma < 1$, the null controllability of the following degenerate wave equation was considered:

$$(PC) \qquad \begin{cases} u_{tt}(x,t) - (x^\gamma u_x(x,t))_x = 0 & \text{on } (0,1) \times (0,T), \\ u(0,t) = \theta(t), u(1,t) = 0 & \text{on } (0,T), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } (0,1), \end{cases}$$

where $\theta(t)$ is the control variable and it acts on the degenerate boundary. Recently, in **[46]** (see also **[2]**), the authors studied the null controllability problems of one-dimensional degenerate wave equations as in **[26]** but the control acts on the nondegenerate boundary. They proved that any initial value in state space is controllable. Also, an explicit expression for the controllability time is given.

In **[2]**, Alabau has also considered the stabilization of the problem (2.1) together with boundary control of the form

$$u_t(1,t) + u_x(1,t) + \beta u(1,t) = 0, \tag{2.3}$$

where $\beta > 0$. Thanks to the dominant energy approach together with suitable elliptic estimates, she proved that (2.3) stabilizes exponentially the corresponding solution of the degenerate wave equation.

In this Chapter, we are concerned with the system

$$(P_1) \qquad \begin{cases} u_{tt}(x,t) - (a(x)u_x(x,t))_x = 0 & \text{in } (0,1) \times (0,+\infty), \\ \begin{cases} u(0,t) = 0 & \text{if } 0 \leq \mu_a < 1 \\ (au_x)(0,t) = 0 & \text{if } 1 \leq \mu_a < 2 \end{cases} & \text{in } (0,+\infty), \\ \beta u(1,t) + (au_x)(1,t) = -\varrho \partial_t^{\alpha,\eta} u(1,t) & \text{in } (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } (0,1). \end{cases}$$

where $\varrho > 0$ and $\beta \geq 0$.The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α, $(0 < \alpha < 1)$, with respect to the time variable (see **[10]** and **[20]**). It is defined as follows

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\, ds, \;\; \eta \geq 0.$$

The degenerate wave equation $(P_1)_1$ can describe the vibration problem of an elastic string. In a neighborhood of an endpoint $x = 0$ of this string, the elastic is sufficiently small or the linear density is large enough. There are a few number of publications concerning the stabilization of distributed systems with fractional damping. In **[39]**, Mbodje studies the energy decay of the wave equation with a boundary fractional derivative control. He used a new approach, when the original model is transformed into an augmented system, and by using energy methods, he proves strong asymptotic stability under the condition $\eta = 0$ and a polynomial type decay rate $E(t) \leq C/t$ if $\eta \neq 0$. Very recently in **[1]**, Benaissa and al. considered the Euler-Bernoulli beam equation with boundary damping of fractional derivative type defined by

$$(PEF) \qquad \begin{cases} u_{tt}(x,t) + u_{xxxx}(x,t) = 0 & \text{in } (0,L) \times (0,+\infty), \\ u(0,t) = u_x(0,t) = 0 & \text{on } (0,+\infty), \\ u_{xx}(L,t) = 0 & \text{on } (0,+\infty), \\ u_{xxx}(L,t) - \gamma \partial_t^{\alpha,\eta} u_t(L,t) = 0 & \text{on } (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } (0,L). \end{cases}$$

They proved, under the condition $\eta = 0$, by a spectral analysis, the non uniform stability. On the other hand, for $\eta > 0$, they also proved that the energy of system (PEF) decay as time goes to infinity as $t^{-1/(1-\alpha)}$.

Fractional calculus so often arise in many physical, chemical, biological, and economical phenomena (see **[4]**, **[5]**, **[6]** and **[38]**). In recent years, the control of PDEs with boundary damping of convolution type has become an active area of research because it improve the performance of the systems.

This Chapter as organized as follows. In section 2, we give preliminaries results and we reformulate the system (P_1) into an augmented system by coupling the degenerate wave equation with a suitable diffusion equation and we show the well-posedness of our problem by semigroup theory. In section 3, uniqueness of strong and weak solutions of the system, when we used Hille-Yosida Theorem. In section 4, we prove lack of exponential stability by spectral analysis for particular case $a(x) = x^\gamma, 0 \leq \gamma < 2$ by using Bessel functions. In section 5, we study asymptotic stability of above model and we establish a polynomial energy decay depending with parameter α for smooth solution. In the last section, we prove an optimal decay rate for the particular case $a(x) = x^\gamma$. The proof heavily relies on multiplier method, Bessel equations and Borichev-Tomilov Theorem.

2.2 Preliminaries results

Let $a \in C([0,1] \cap C^1(]0,1])$ be a function satisfying the following assumptions:

$$\begin{cases} (i) & a(x) > 0 \ \forall x \in]0,1], a(0) = 0, \\ (ii) & \mu_a = \sup\limits_{0<x\leq 1} \dfrac{x|a'(x)|}{a(x)} < 2, \text{ and} \\ (iii) & a \in C^{[\mu_a]}([0,1]), \end{cases} \tag{2.4}$$

where $[\cdot]$ stands for the integer part.

When $\mu_a > 1$, we suppose $\beta > 0$ because if $\beta = 0$ and the feedback law only depends on velocities, we may encounter the situation where the closed-loop system is not well-posed in terms of the semigroups in the Hilbert space.

Examples: 1) Let $\varpi \in (0,2)$ be given. Define

$$a(x) = x^{\varpi} \quad \forall x \in [0,1].$$

satisfies (2.4).
2) Let $\varpi \in [0,2)$ be given and let $\theta \in (0, 1 - \varpi/2)$. The function

$$a(x) = x^{\varpi}(1 + \cos^2(\ln x^{\theta})) \quad \forall x \in [0,1]$$

satisfies (2.4).
3) Let $\varpi \in [0,2)$ be given and let $\theta \in (0,\varpi)$. The function

$$a(x) = x^{\varpi} e^{(\theta-\varpi)x} \quad \forall x \in [0,1]$$

satisfies (2.4).

Now, we introduce, as in **[14]**, **[23]** or **[2]**, the following weighted spaces:

$$H^1_{0,a}(0,1) = \Big\{u \text{ is locally absolutely continuous in } (0,1] : \sqrt{a(x)}u_x \in L^2(0,1)/\ u(0) = 0\Big\} \\ \text{if } \mu_a \in [0,1[,$$

$$H^1_a(0,1) = \Big\{u \text{ is locally absolutely continuous in } (0,1] : \sqrt{a(x)}u_x \in L^2(0,1)\Big\} \text{ if } \mu_a \in [1,2[.$$

It is easy to see that $H^1_a(0,1)$ when $\beta > 0$ is a Hilbert space with the scalar product

$$(u,v)_{H^1_a(0,1)} = \int_0^1 a(x)u'(x)\overline{v'(x)}\,dx + \beta u(1)\overline{v(1)}.$$

Let us also set

$$|u|_{H^1_{0,a}(0,1)} = \Big(\int_0^1 a(x)|u'(x)|^2\,dx\Big)^{1/2} \quad \forall u \in H^1_a(0,1).$$

Actually, $|\cdot|_{H^1_{0,a}(0,1)}$ is an equivalent norm on the closed subspace $H^1_{0,a}(0,1)$ to the norm of $H^1_a(0,1)$ when $\mu_a \in [0,1[$. This fact is a simple consequence of the following version of Poincaré's inequality.

Proposition 2.2.1 *Assume (2.4) with $\mu_a \in [0,1)$. Then there is a positive constant $C_* = C(a)$ such that*

$$\|u\|^2_{L^2(\Omega)} \leq C_* |u|^2_{1,a} \quad \forall u \in H^1_{0,a}(0,1). \tag{2.5}$$

Proof. Let $u \in H^1_{0,a}(0,1)$. For any $x \in]0,1]$ we have that

$$|u(x)| = \left| \int_0^x u'(s)\, ds \right| \leq |u|_{1,a} \left\{ \int_0^1 \frac{1}{a(s)}\, ds \right\}^{1/2}.$$

Therefore

$$\int_0^1 |u(x)|^2\, dx \leq |u|^2_{1,a} \left\{ \int_0^1 \frac{1}{a(s)}\, ds \right\}.$$

Next, we define

$$H^2_a(0,1) = \{u \in H^1_a(0,1) : \; au' \in H^1(0,1)\},$$

where $H^1(0,1)$ denotes the classical Sobolev space.

Now, we state two propositions that will be needed later (see **[14]**, **[23]** and **[2]**).

Proposition 2.2.2 *Assume (2.4). Then the following properties hold.*

(i) *For every $u \in H^1_a(0,1)$*

$$\lim_{x \to 0} x u^2(x) = 0. \tag{2.6}$$

(ii) *For every $u \in H^2_a(0,1)$*

$$\lim_{x \to 0} x a(x) u'(x)^2 = 0. \tag{2.7}$$

(iii) *For every $u \in H^2_a(0,1)$*

$$\lim_{x \to 0} x a(x) u(x) u'(x) = 0. \tag{2.8}$$

Proposition 2.2.3 $H^1_a(0,1) \hookrightarrow L^2(0,1)$ *with compact embedding.*

2.3 Augmented model

This section is concerned with the reformulation of the model (P_1) into an augmented system. For that, we need the following claims.

Theorem 2.3.1 (see [39]) *Let μ be the function:*

$$\mu(\xi) = |\xi|^{(2\alpha-1)/2}, \quad -\infty < \xi < +\infty, \; 0 < \alpha < 1. \tag{2.9}$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t \phi(\xi,t) + \xi^2 \phi(\xi,t) + \eta\phi(\xi,t) - U(t)\mu(\xi) = 0, \quad -\infty < \xi < +\infty, \eta \geq 0, t > 0, \tag{2.10}$$

$$\phi(\xi,0) = 0, \tag{2.11}$$

$$O(t) = (\pi)^{-1}\sin(\alpha\pi)\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi,t)\,d\xi \tag{2.12}$$

is given by

$$O = I^{1-\alpha,\eta}U. \tag{2.13}$$

where

$$[I^{\alpha,\eta}f](t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}e^{-\eta(t-\tau)}f(\tau)\,d\tau$$

Lemma 2.3.1 (see [1]) *If* $\lambda \in D = \{\lambda \in \mathbb{C} : Re\lambda + \eta > 0\} \cup \{\lambda \in \mathbb{C} : Im\lambda \neq 0\}$ *then*

$$F_1(\lambda) = \int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\,d\xi = \frac{\pi}{\sin\alpha\pi}(\lambda+\eta)^{\alpha-1}$$

and

$$F_2(\lambda) = \int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{(\lambda+\eta+\xi^2)^2}\,d\xi = (1-\alpha)\frac{\pi}{\sin\alpha\pi}(\lambda+\eta)^{\alpha-2}.$$

□

We are now in a position to reformulate system (P_1). Indeed, by using Theorem 2.3.1, system (P_1) may be recast into the augmented model:

$$(P_1') \quad \begin{cases} u_{tt}(x,t) - (a(x)u_x(x,t))_x = 0, \\ \phi_t(\xi,t) + (\xi^2+\eta)\phi(\xi,t) - u_t(1,t)\mu(\xi) = 0, \qquad -\infty < \xi < +\infty, \eta \geq 0, t > 0, \\ \begin{cases} u(0,t) = 0 & \text{if } 0 \leq \mu_a < 1 \\ (au_x)(0,t) = 0 & \text{if } 1 \leq \mu_a < 2 \end{cases} \\ \beta u(1,t) + (au_x)(1,t) = -\zeta\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi,t)\,d\xi, \quad \zeta = \varrho(\pi)^{-1}\sin(\alpha\pi), \\ u(x,0) = u_0(x), \\ u_t(x,0) = u_1(x). \end{cases}$$

We define the energy associated to the solution of the problem (P_1') by the following formula:

$$E(t) = \frac{1}{2}\int_0^1 (|u_t|^2 + a(x)|u_x|^2)dx + \frac{\beta}{2}|u(1,t)|^2 + \frac{\zeta}{2}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2\,d\xi. \tag{2.14}$$

Lemma 2.3.2 *Let (u,ϕ) be a regular solution of the problem (P_1'). Then, the energy functional defined by (2.14) satisfies*

$$E'(t) = -\zeta\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,t)|^2\,d\xi \leq 0. \tag{2.15}$$

Remark 2.3.1 *For an initial datum in $D(\mathcal{A})$ (see Theorem 2.4.1 below), we know that (u,ϕ) is of class C^1 in time, thus we can derive the energy $E(t)$.*

Proof of Lemma 2.3.2. Multiplying the first equation in (P_1') by $\overline{u}_t$, integrating over $(0,1)$ and using integration by parts, we get

$$\int_0^1 u_{tt}(x,t)\overline{u}_t dx - \int_0^1 (a(x)u_x(x,t))_x \overline{u}_t dx = 0.$$

Then

$$\frac{d}{dt}\left(\frac{1}{2}\int_0^1 |u_t(x,t)|^2 dx\right) + \frac{1}{2}\frac{d}{dt}\int_0^1 a(x)|u_x(x,t)|^2\,dx - \Re\Big[(a(x)u_x)(x,t)\overline{u}_t\Big]_0^1 = 0.$$

Then

$$\frac{d}{dt}\left(\frac{1}{2}\int_0^1 \left(|u_t(x,t)|^2 + a(x)|u_x(x,t)|^2\right) dx + \frac{\beta}{2}|u(1,t)|^2\right) + \zeta\Re\overline{u}_t(1,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi,t)\,d\xi = 0. \tag{2.16}$$

Multiplying the second equation in (P_1') by $\zeta\overline{\phi}$ and integrating over $(-\infty,+\infty)$, to obtain:

$$\zeta\int_{-\infty}^{+\infty}\phi_t(\xi,t)\overline{\phi}(\xi,t)d\xi + \zeta\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,t)|^2 d\xi - \zeta u_t(1,t)\int_{-\infty}^{+\infty}\mu(\xi)\overline{\phi}(\xi,t)d\xi = 0.$$

Hence

$$\frac{\zeta}{2}\frac{d}{dt}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2 d\xi + \zeta\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,t)|^2 d\xi - \zeta\Re u_t(1,t)\int_{-\infty}^{+\infty}\mu(\xi)\overline{\phi}(\xi,t)d\xi = 0. \tag{2.17}$$

From (2.14), (2.16) and (2.17) we get

$$E'(t) = -\zeta\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,t)|^2\,d\xi \le 0.$$

This completes the proof of the lemma.

□

2.4 Global existence

In this section, we give an existence and uniqueness result for problem (P_1') using the semigroup theory. Introducing the vector function $U = (u, v, \phi)^T$, where $v = u_t$, system (P_1') is equivalent to

$$\begin{cases} U' = \mathcal{A}U, & t > 0,\\ U(0) = (u_0, u_1, \phi_0), \end{cases} \tag{2.18}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} u\\ v\\ \phi\end{pmatrix} = \begin{pmatrix} v\\ (a(x)u_x)_x\\ -(\xi^2+\eta)\phi + v(1)\mu(\xi)\end{pmatrix}. \tag{2.19}$$

We introduce the following Hilbert space (the energy space):

$$\mathcal{H} = H^1_*(0,1) \times L^2(0,1) \times L^2(-\infty, +\infty),$$

where

$$H^1_*(0,1) = \begin{cases} H^1_{0,a}(0,1) & \text{if } \mu_a \in [0,1), \\ H^1_a(0,1) & \text{if } \mu_a \in [1,2). \end{cases}$$

For $U = (u, v, \phi)^T$, $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\phi})^T$ we define the following inner product in $\mathcal{H}$

$$\langle U, \tilde{U} \rangle_{\mathcal{H}} = \int_0^1 a(x) u_x \overline{\tilde{u}}_x dx + \int_0^1 v \overline{\tilde{v}} dx + \zeta \int_{-\infty}^{+\infty} \phi \overline{\tilde{\phi}} \, d\xi + \beta u(1) \overline{\tilde{u}}(1).$$

The domain of $\mathcal{A}$ is then

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (u, v, \phi)^T \text{ in } \mathcal{H} : u \in H^2_a(0,1) \cap H^1_*(0,1), v \in H^1_*(0,1), \\ -(\xi^2 + \eta)\phi + v(1)\mu(\xi) \in L^2(-\infty, +\infty), \\ \beta u(1) + (a u_x)(1) + \zeta \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi) \, d\xi = 0, \\ |\xi|\phi \in L^2(-\infty, +\infty) \end{array} \right\}. \tag{2.20}$$

We have the following existence and uniqueness result.

Theorem 2.4.1 (Existence and uniqueness)

(1) If $U_0 \in D(\mathcal{A})$, then system (2.18) has a unique strong solution

$$U \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

(2) If $U_0 \in \mathcal{H}$, then system (2.18) has a unique weak solution

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

Proof

We use the semigroup approach. In what follows, we prove that $\mathcal{A}$ is monotone. For any $U \in D(\mathcal{A})$ and using (2.18), (2.15) and the fact that

$$E(t) = \frac{1}{2} \|U\|^2_{\mathcal{H}}, \tag{2.21}$$

we have

$$\Re \langle \mathcal{A}U, U \rangle_{\mathcal{H}} = -\zeta \int_{-\infty}^{+\infty} (\xi^2 + \eta) |\phi(\xi)|^2 \, d\xi. \tag{2.22}$$

Hence, $\mathcal{A}$ is monotone. Next, we prove that the operator $\lambda I - \mathcal{A}$ is surjective for $\lambda > 0$. Given $F = (f_1, f_2, f_3)^T \in \mathcal{H}$, we prove that there exists $U \in D(\mathcal{A})$ satisfying

$$(\lambda I - \mathcal{A})U = F. \tag{2.23}$$

Equation (2.23) is equivalent to

$$\begin{cases} \lambda u - v = f_1, \\ \lambda v - (a(x)u_x)_x = f_2, \\ \lambda\phi + (\xi^2+\eta)\phi - v(1)\mu(\xi) = f_3. \end{cases} \tag{2.24}$$

Suppose u is found with the appropriate regularity. Then, $(2.24)_1$ $(2.24)_2$ yield

$$v = \lambda u - f_1 \in H^1_*(0,1) \tag{2.25}$$

and

$$\phi = \frac{f_3(\xi) + \mu(\xi)v(1)}{\xi^2+\eta+\lambda}. \tag{2.26}$$

By using (2.24) and (2.25) it can easily be shown that u satisfies

$$\lambda^2 u - (a(x)u_x)_x = f_2 + \lambda f_1. \tag{2.27}$$

Solving equation (2.27) is equivalent to finding $u \in H^2_a(0,1) \cap H^1_*(0,1)$ such that

$$\int_0^1 (\lambda^2 u\overline{w} - (a(x)u_x)_x\overline{w})\,dx = \int_0^1 (f_2+\lambda f_1)\overline{w}\,dx, \tag{2.28}$$

for all $w \in H^1_*(0,1)$. By using (2.28), the boundary condition $(2.20)_3$ and (2.26) the function u satisfying the following equation

$$\begin{aligned} &\int_0^1 (\lambda^2 u\overline{w} + (a(x)u_x)\overline{w}_x)\,dx + \tilde{\zeta}v(1)\overline{w}(1) + \beta u(1)\overline{w}(1) \\ &\quad = \int_0^1 (f_2+\lambda f_1)\overline{w}\,dx - \zeta\int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_3(\xi)\,d\xi\overline{w}(1) \end{aligned} \tag{2.29}$$

where $\tilde{\zeta} = \zeta \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2+\eta+\lambda}\,d\xi$. Using again (2.25), we deduce that

$$v(1) = \lambda u(1) - f_1(1). \tag{2.30}$$

Inserting (2.30) into (2.29), we get

$$\begin{cases} \int_0^1 (\lambda^2 u\overline{w} + a(x)u_x\overline{w}_x)\,dx + (\lambda\tilde{\zeta}+\beta)u(1)\ \overline{w}(1) \\ = \int_0^1 (f_2+\lambda f_1)\overline{w}\,dx - \zeta\int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_3(\xi)\,d\xi\overline{w}(1) + \tilde{\zeta}f_1(1)\overline{w}(1). \end{cases} \tag{2.31}$$

Problem (2.31) is of the form

$$\mathcal{B}(u,w) = \mathcal{L}(w), \tag{2.32}$$

where $\mathcal{B} : [H^1_*(0,1) \times H^1_*(0,1)] \to \mathbb{C}$ is the bilinear form defined by

$$\mathcal{B}(u,w) = \int_0^1 (\lambda^2 u\overline{w} + a(x)u_x\overline{w}_x)\,dx + (\lambda\tilde{\zeta}+\beta)u(1)\ \overline{w}(1)$$

and $\mathcal{L} : H^1_*(0,1) \to \mathbb{C}$ is the linear functional given by

$$\mathcal{L}(w) = \int_0^1 (f_2 + \lambda f_1)\overline{w}\, dx - \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + \lambda} f_3(\xi)\, d\xi \overline{w}(1) + \tilde{\zeta} f_1(1)\overline{w}(1).$$

It is easy to verify that $\mathcal{B}$ is continuous and coercive, and $\mathcal{L}$ is continuous. Consequently, by the Lax-Milgram Lemma, system (2.32) has a unique solution $u \in H^1_*(0,1)$. By the regularity theory for the linear elliptic equations, it follows that $u \in H^2_a(0,1)$. Therefore, the operator $\lambda I - \mathcal{A}$ is surjective for any $\lambda > 0$. Consequently, using Hille-Yosida theorem, the result of Theorem 2.4.1 follows.

□

2.5 Lack of exponential stability

This section will be devoted to the study of the lack of exponential decay of solutions associated with the system (2.18). In order to state and prove our stability results, we need some lemmas.

Theorem 2.5.1 ([43]) *Let $S(t)$ be a C_0-semigroup of contractions on Hilbert space with generator $\mathcal{A}$. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R}$$

and

$$\overline{\lim_{|\beta|\to\infty}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Theorem 2.5.2 ([11]) *Let $S(t)$ be a bounded C_0-semigroup on a Hilbert space $\mathcal{H}$ with generator $\mathcal{A}$. If*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \text{ and } \overline{\lim_{|\beta|\to\infty}} \frac{1}{\beta^l} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$$

for some l, then there exist c such that

$$\|e^{\mathcal{A}t} U_0\|^2 \leq \frac{c}{t^{\frac{2}{l}}} \|U_0\|^2_{D(\mathcal{A})}.$$

Theorem 2.5.3 ([3]) *Let $\mathcal{A}$ be the generator of a uniformly bounded C_0-semigroup $\{S(t)\}_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. If:*

(i) $\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.

(ii) The intersection of the spectrum $\sigma(\mathcal{A})$ with $i\mathbb{R}$ is at most a countable set,

then the semigroup $\{S(t)\}_{t\geq 0}$ is asymptotically stable, i.e, $\|S(t)z\|_{\mathcal{H}} \to 0$ as $t \to \infty$ for any $z \in \mathcal{H}$.

Our main result is the following.

Theorem 2.5.4 *The semigroup generated by the operator $\mathcal{A}$ is not exponentially stable.*

Proof: We will examine two cases.
•**Case 1** $\eta = 0$: We shall show that $i\lambda = 0$ is not in the resolvent set of the operator $\mathcal{A}$. Indeed, noting that $(\sin x, 0, 0)^T \in \mathcal{H}$, and denoting by $(u, v, \phi)^T$ the image of $(\sin x, 0, 0)^T$ by $\mathcal{A}^{-1}$, we see that $\phi(\xi) = -|\xi|^{\frac{2\alpha-5}{2}} \sin 1$. But, then $\phi \notin L^2(-\infty, +\infty)$, since $\alpha \in]0, 1[$. So $(u, v, \phi)^T \notin D(\mathcal{A})$.
• **Case 2** $\eta \neq 0$: **A)** $a(x) = x^\gamma$**:** We aim to show that an infinite number of eigenvalues of $\mathcal{A}$ approach the imaginary axis which prevents the system (P_1) from being exponentially stable. Indeed we first compute the characteristic equation that gives the eigenvalues of $\mathcal{A}$. Here, we consider only the case $a(x) = x^\gamma, 0 \leq \gamma < 2$ and in particular we treat the case $1 \leq \gamma < 2$. The case $0 \leq \gamma < 1$ is similar. Let λ be an eigenvalue of $\mathcal{A}$ with associated eigenvector $U = (u, v, \phi)^T$. Then $\mathcal{A}U = \lambda U$ is equivalent to

$$\begin{cases} \lambda u - v = 0, \\ \lambda v - (x^\gamma u_x)_x = 0, \\ \lambda\phi + (\xi^2 + \eta)\phi - v(1)\mu(\xi) = 0. \end{cases} \tag{2.33}$$

It is well-known that Bessel functions play an important role in this type of problem. From $(2.33)_1 - (2.33)_2$ for such λ, we find

$$\lambda^2 u - (x^\gamma u_x)_x = 0. \tag{2.34}$$

Using the boundary conditions and $(2.33)_3$, we deduce that

$$\begin{cases} \lambda^2 u - (x^\gamma u_x)_x = 0 \\ (x^\gamma u_x)(0) = 0 \\ u_x(1) + \zeta v(1) \displaystyle\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2 + \lambda + \eta}\, d\xi + \beta u(1) \\ \qquad\qquad = u_x(1) + (\varrho\lambda(\lambda + \eta)^{\alpha-1} + \beta)u(1) = 0. \end{cases} \tag{2.35}$$

Assume that u is a solution of (2.35) associated to eigenvalue $-\lambda^2$, then one easily checks that the function

$$u(x) = x^{\frac{1-\gamma}{2}} \Psi\left(\frac{2}{2-\gamma} i\lambda x^{\frac{2-\gamma}{2}}\right)$$

is a solution of the following boundary problem:

$$\begin{cases} y^2\Psi''(y) + y\Psi'(y) + (y^2 - (\frac{\gamma-1}{2-\gamma})^2)\Psi(y) = 0 \\ (2-\gamma)y^{\frac{1}{2-\gamma}}\Psi'(y) - (\gamma - 1)y^{\frac{\gamma-1}{2-\gamma}}\Psi(y) \to 0 \text{ } as \text{ } y \to 0 \\ \left(\frac{1-\gamma}{2} + \beta + \varrho\lambda(\lambda+\eta)^{\alpha-1}\right)\Psi(\frac{2}{2-\gamma}i\lambda) + i\lambda\Psi'(\frac{2}{2-\gamma}i\lambda) = 0. \end{cases} \tag{2.36}$$

We have

$$u(x) = c_+\Phi_+ + c_-\Phi_-, \tag{2.37}$$

where Φ_+ and Φ_- are defined by

$$\Phi_+(x) := x^{\frac{1-\gamma}{2}} J_{\nu_\gamma}\left(\frac{2}{2-\gamma} i\lambda x^{\frac{2-\gamma}{2}}\right)$$

and

$$\Phi_-(x) := x^{\frac{1-\gamma}{2}} J_{-\nu_\gamma}\left(\frac{2}{2-\gamma} i\lambda x^{\frac{2-\gamma}{2}}\right),$$

where

$$\nu_\gamma = \frac{\gamma-1}{2-\gamma}$$

and J_{ν_γ} and $J_{-\nu_\gamma}$ are Bessel functions of the first kind of order ν_γ and $-\nu_\gamma$. We suppose $\nu_\gamma \notin \mathbb{N}$. So J_{ν_γ} and $J_{-\nu_\gamma}$ are linearly independent and therefore the pair $(J_{\nu_\gamma}, J_{-\nu_\gamma})$ (classical result) forms a fundamental system of solutions $(2.36)_1$.

Using the series expansion of J_{ν_α} and $J_{-\nu_\alpha}$, we deduce that (see **[15]**) $\Phi_+ \in H^1_*(0,1)$, while $\Phi_- \notin H^1_*(0,1)$, so

$$u(x) = c_+\Phi_+(x).$$

Moreover, $x^\gamma \Phi_+'(x) \to 0$ as $x \to 0$, hence the boundary condition in 0 is automatically satisfied.

Our purpose in the sequel is to prove, thanks to Rouché's Theorem, that there is a subsequence of eigenvalues for which their real part tends to 0.

In the sequel, since $\mathcal{A}$ is dissipative, we study the asymptotic behavior of the large eigenvalues λ of $\mathcal{A}$ in the strip $-\alpha_0 \leq \Re(\lambda) \leq 0$, for some $\alpha_0 > 0$ large enough and for such λ, we remark that Φ_+ remains bounded.

Lemma 2.5.1 *There exists $N \in \mathbb{N}$ such that*

$$\{\lambda_k\}_{k\in\mathbf{Z}^*, |k|\geq N} \subset \sigma(\mathcal{A}), \tag{2.38}$$

where

$$\lambda_k = -\frac{2-\gamma}{2} i\left(k + \frac{\nu_\gamma}{2} + \frac{1}{4}\right)\pi + \frac{\tilde{\alpha}}{k^{1-\alpha}} + \frac{\beta}{|k|^{1-\alpha}} + o\left(\frac{1}{k^{1-\alpha}}\right), k \geq N, \tilde{\alpha} \in i\mathbb{R}, \beta \in \mathbb{R}, \beta < 0.$$

$$\lambda_k = \overline{\lambda_{-k}} \text{ if } k \leq -N.$$

Moreover for all $|k| \geq N$, the eigenvalues λ_k are simple.

Proof. The proof is decomposed in three steps:
Step 1.

$$\left(\frac{1-\gamma}{2} + \beta + \varrho\lambda(\lambda+\eta)^{\alpha-1}\right) J_{\nu_\gamma}\left(\frac{2}{2-\gamma} i\lambda\right) + i\lambda J'_{\nu_\gamma}\left(\frac{2}{2-\gamma} i\lambda\right) = 0. \tag{2.39}$$

We known that

$$xJ'_\nu(x) = \nu J(x) - xJ_{\nu+1}(x). \tag{2.40}$$

Then (2.39) is equivalent to

$$
\begin{aligned}
f(\lambda) &= \left(\tfrac{1-\gamma}{2}+\nu_\gamma\tfrac{2-\gamma}{2}+\beta+\varrho\lambda(\lambda+\eta)^{\alpha-1}\right)J_{\nu_\gamma}\left(\tfrac{2}{2-\gamma}i\lambda\right)-i\lambda J_{\nu_\gamma+1}\left(\tfrac{2}{2-\gamma}i\lambda\right)\\
&= -i\lambda\left(J_{\nu_\gamma+1}\left(\tfrac{2}{2-\gamma}i\lambda\right)-\frac{1}{i\lambda}\left(\frac{1-\gamma}{2}+\nu_\gamma\frac{2-\gamma}{2}+\beta+\varrho\lambda(\lambda+\eta)^{\alpha-1}\right)J_{\nu_\gamma}\left(\frac{2}{2-\gamma}i\lambda\right)\right)=0.
\end{aligned}
\tag{2.41}
$$

We set

$$
\tilde{f}(\lambda)=J_{\nu_\gamma+1}\left(\frac{2}{2-\gamma}i\lambda\right)-\frac{1}{i\lambda}\left(\frac{1-\gamma}{2}+\nu_\gamma\frac{2-\gamma}{2}+\beta+\varrho\lambda(\lambda+\eta)^{\alpha-1}\right)J_{\nu_\gamma}\left(\frac{2}{2-\gamma}i\lambda\right).
\tag{2.42}
$$

We will use the following classical asymptotic development (see **[36]** p. 122, (5.11.6)): for all $\delta>0$, the following development holds when $|arg z|\le\pi-\delta$:

$$
J_\nu(z)=\left(\frac{2}{\pi z}\right)^{1/2}\cos\left(z-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)\left(1+O(\frac{1}{|z|^2})\right)-\left(\frac{2}{\pi z}\right)^{1/2}\sin\left(z-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)O\left(\frac{1}{|z|^2}\right).
\tag{2.43}
$$

Then

$$
\tilde{f}(\lambda)=\left(\frac{2}{\pi\tilde{z}}\right)^{1/2}\frac{e^{-iz}}{2i}\tilde{\tilde{f}}(\lambda),
\tag{2.44}
$$

where

$$
\tilde{z}=\frac{2}{2-\gamma}i\lambda,\quad z=\frac{2}{2-\gamma}i\lambda-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4}
$$

and

$$
\begin{aligned}
\tilde{\tilde{f}}(\lambda) &= (e^{2iz}-1)-\frac{\varrho}{\lambda^{1-\alpha}}(e^{2iz}+1)+o\left(\frac{1}{\lambda^{1-\alpha}}\right)\\
&= f_0(\lambda)+\tfrac{f_1(\lambda)}{\lambda^{1-\alpha}}+o\left(\tfrac{1}{\lambda^{1-\alpha}}\right),
\end{aligned}
\tag{2.45}
$$

where

$$
f_0(\lambda)=e^{2iz}-1.
\tag{2.46}
$$

$$
f_1(\lambda)=-\frac{\varrho}{\lambda^{1-\alpha}}(e^{2iz}+1).
\tag{2.47}
$$

Note that f_0 and f_1 remain bounded in the strip $-\alpha_0\le\Re(\lambda)\le 0$.

Step 2. We look at the roots of f_0. From (2.46), f_0 has one family of roots that we denote λ_k^0.

$$
f_0(\lambda)=0\Leftrightarrow e^{2iz}-1=0
$$

Hence

$$
2i\left(\frac{2}{2-\gamma}i\lambda-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4}\right)=2ik\pi,\quad k\in\mathbf{Z},
$$

i.e.,

$$
\lambda_k^0=-\frac{2-\gamma}{2}i\left(k+\frac{\nu_\gamma}{2}+\frac{1}{4}\right)\pi,\quad k\in\mathbf{Z}.
$$

Now with the help of Rouché's Theorem, we will show that the roots of $\tilde{f}$ are close to those of f_0. Let us start with the first family. Changing in (2.41) the unknown λ by $u = 2iz$ then (2.41) becomes

$$\tilde{f}(u) = (e^u - 1) + O\left(\frac{1}{u^{(1-\alpha)}}\right) = f_0(u) + O\left(\frac{1}{u^{(1-\alpha)}}\right)$$

The roots of f_0 are $u_k = -\frac{2-\gamma}{2} i \left(k + \frac{\nu_\gamma}{2} + \frac{1}{4}\right)\pi, k \in \mathbf{Z}$, and setting $u = u_k + re^{it}, t \in [0, 2\pi]$, we can easily check that there exists a constant $C > 0$ independent of k such that $|e^u + 1| \geq Cr$ for r small enough. This allows to apply Rouché's Theorem. Consequently, there exists a subsequence of roots of $\tilde{f}$ which tends to the roots u_k of f_0. Equivalently, it means that there exists $N \in \mathbb{N}$ and a subsequence $\{\lambda_k\}_{|k| \geq N}$ of roots of $f(\lambda)$, such that $\lambda_k = \lambda_k^0 + o(1)$ which tends to the roots $-\frac{2-\gamma}{2} i \left(k + \frac{\nu_\gamma}{2} + \frac{1}{4}\right)\pi$ of f_0. Finally for $|k| \geq N, \lambda_k$ is simple since λ_k^0 is.

Step 3. From Step 2, we can write

$$\lambda_k = -\frac{2-\gamma}{2} i \left(k + \frac{\nu_\gamma}{2} + \frac{1}{4}\right)\pi + \varepsilon_k. \tag{2.48}$$

Using (2.48), we get

$$\begin{aligned} e^{2i\lambda_k} &= e^{-\frac{4}{2-\gamma}\varepsilon_k} \\ &= 1 - \tfrac{4}{2-\gamma}\varepsilon_k + O(\varepsilon_k^2). \end{aligned} \tag{2.49}$$

Substituting (2.49) into (2.45), using that $\tilde{\tilde{f}}(\lambda_k) = 0$, we get:

$$\tilde{f}(\lambda_k) = -\frac{4}{2-\gamma}\varepsilon_k - \frac{2\varrho}{(-\frac{2-\gamma}{2} ik\pi)^{1-\alpha}} + o(\varepsilon_k) + o\left(\frac{1}{k^{1-\alpha}}\right) = 0 \tag{2.50}$$

and hence

$$\begin{aligned} \varepsilon_k &= -\frac{(2-\gamma)\varrho}{2(-\frac{2-\gamma}{2} ik\pi)^{1-\alpha}} + o\left(\frac{1}{k^{1-\alpha}}\right) \\ &= \begin{cases} -\left(\frac{2-\gamma}{2}\right)^{\alpha} \frac{\varrho}{(k\pi)^{1-\alpha}} \left(\cos(1-\alpha)\frac{\pi}{2} - i\sin(1-\alpha)\frac{\pi}{2}\right) + o\left(\frac{1}{k^{1-\alpha}}\right) & \text{for } k \succeq 0, \\ -\left(\frac{2-\gamma}{2}\right)^{\alpha} \frac{\varrho}{(-k\pi)^{1-\alpha}} \left(\cos(1-\alpha)\frac{\pi}{2} + i\sin(1-\alpha)\frac{\pi}{2}\right) + o\left(\frac{1}{k^{1-\alpha}}\right) & \text{for } k \preceq 0. \end{cases} \end{aligned} \tag{2.51}$$

From (2.51) we have in that case $|k|^{1-\alpha}\Re\lambda_k \sim \beta$, with

$$\beta = -\left(\frac{2-\gamma}{2}\right)^{\alpha} \frac{\varrho}{\pi^{1-\alpha}} \cos(1-\alpha)\frac{\pi}{2}.$$

The operator $\mathcal{A}$ has a non exponential decaying branch of eigenvalues. Thus the proof is complete.

Remark 2.5.1 *1) Similarly, we can prove Lack of exponential stability when $\nu_\gamma \in \mathbb{N}$. In this case we define Bessel's functions of order ν_γ of the second kind as following*

$$Y_{\nu_\gamma}(y) = \lim_{\nu \to \nu_\gamma} \frac{J_\nu(y)\cos\nu\pi - J_{-\nu}(y)}{\sin\nu\pi}.$$

Then, J_{ν_γ} and Y_{ν_γ} forms a fundamental system of solutions $(2.36)_1$.
2) Similarly, we can prove Lack of exponential stability when $\gamma \in [0, 1[$.

B) General form of $a(x)$: There exists only few works concerning explicit representation for solutions of Sturm-Liouville equations (see **[34]** and **[35]**). We consider the following eigenvalues problem

$$\begin{cases} \lambda^2 u - (a(x)u_x)_x = 0 \\ (a(x)u_x)(0) = 0 \\ u_x(1) + (\varrho\lambda(\lambda+\eta)^{\alpha-1} + \beta)u(1) = 0. \end{cases} \tag{2.52}$$

Define the functions

$$l(x) = \int_0^x \frac{1}{\sqrt{a(s)}}\, ds, \quad \rho(x) = a(x)^{1/4}$$

and

$$Q(x) = \frac{a(x)}{4}\left[\left(\frac{a'(x)}{a(x)}\right)' + \frac{3}{4}\left(\frac{a'(x)}{a(x)}\right)^2\right].$$

In **[34]**, the authors have derive a Neumann series of Bessel functions (NSBF) representation for solutions of Sturm-Liouville equation with variable coefficients as the following:

Let g be a solution of the equation

$$(a(x)g')' = 0, \quad x \in [0,1].$$

Then the following two families of auxiliary functions are well defined

$$\begin{aligned} \tilde{Y}^{(0)}(x) &\equiv Y^{(0)}(x) \equiv 1, \\ Y^{(n)}(x) &= \begin{cases} n\int_0^x Y^{(n-1)}(x)\frac{1}{g^2(s)a(s)}\, ds, & n \text{ odd }, \\ n\int_0^x Y^{(n-1)}(x) g^2(s)\, ds, & n \text{ even} \end{cases} \\ \tilde{Y}^{(n)}(x) &= \begin{cases} n\int_0^x \tilde{Y}^{(n-1)}(x) g^2(s)\, ds, & n \text{ odd }, \\ n\int_0^x \tilde{Y}^{(n-1)}(x)\frac{1}{g^2(s)a(s)}\, ds, & n \text{ even }. \end{cases} \end{aligned}$$

We define the formal powers associated to equation (2.52)

$$\Phi_k(x) = \begin{cases} g(x)Y^{(k)}(x), & k \text{ odd}, \\ g(x)\tilde{Y}^{(k)}(x), & k \text{ even}, \end{cases} \qquad \Psi_k(x) = \begin{cases} \frac{1}{g(x)}Y^{(k)}(x), & k \text{ even}, \\ \frac{1}{g(x)}\tilde{Y}^{(k)}(x), & k \text{ odd}. \end{cases}$$

Then two linearly independent solutions v_1 and v_2 of equation (2.52) for $\lambda \neq 0$ can be written in the form

$$v_1(x) = \frac{\cos(i\lambda l(x))}{a(x)^{1/4}} + 2\sum_{n=0}^{\infty}(-1)^n \sigma_{2n}(x) j_{2n}(i\lambda l(x)),$$

$$v_2(x) = \frac{\sin(i\lambda l(x))}{a(x)^{1/4}} + 2\sum_{n=0}^{\infty}(-1)^n \sigma_{2n+1}(x) j_{2n+1}(i\lambda l(x)),$$

the coefficients σ_n being defined by the equalities

$$\sigma_n(x) = \frac{2n+1}{2}\left(\sum_{k=0}^{n}\frac{l_{k,n}\Phi_k(x)}{l^k(x)} - \frac{1}{a(x)^{1/4}}\right),$$

where $l_{k,n}$ is the corresponding coefficient of x^k in the Legendre polynomial of order n. Moreover, we obtain

$$\begin{aligned} v_1'(x) &= \frac{1}{\sqrt{a(x)}}\left(\frac{1}{a(x)^{1/4}}(G_1(x)\cos(i\lambda l(x)) - i\lambda\sin(i\lambda l(x))) + 2\sum_{n=0}^{\infty}(-1)^n\mu_{2n}(x)j_{2n}(i\lambda l(x))\right) \\ &\quad -\frac{(a(x)^{1/4})'}{a(x)^{1/4}}v_1(x), \\ v_2'(x) &= \frac{1}{\sqrt{a(x)}}\left(\frac{1}{a(x)^{1/4}}(G_2(x)\sin(i\lambda l(x)) + i\lambda\cos(i\lambda l(x))) + 2\sum_{n=0}^{\infty}(-1)^n\mu_{2n+1}(x)j_{2n+1}(i\lambda l(x))\right) \\ &\quad -\frac{(a(x)^{1/4})'}{a(x)^{1/4}}v_2(x), \end{aligned}$$

where

$$G_1(x) = h + \frac{1}{2}\int_0^{l(x)} Q(s)\,ds, \quad G_2(x) = \frac{1}{2}\int_0^{l(x)} Q(s)\,ds,$$

where

$$h = \lim_{x\to 0}\sqrt{a(x)}\left(\frac{g'(x)}{g(x)} + \frac{\rho'(x)}{\rho(x)}\right),$$

$$j_n(x) = \sqrt{\frac{\pi}{2x}}J_{n+1/2}(x)$$

and

$$\mu_n(x) = \frac{2n+1}{2\rho(x)}\left(\sum_{k=0} n\frac{l_{k,n}}{l^k(x)}\left(k\frac{\Psi_{k-1}(x)}{\rho(x)} + \rho(x)\sqrt{a(x)}\left(\frac{g'(x)}{g(x)} + \frac{(\rho'(x)}{\rho(x)}\right)\Phi_k(x)\right)\right.$$
$$\left. -\frac{n(n+1)}{2l(x)} - G_2(x) - \frac{h}{2}(1+(-1)^n)\right)$$

Now using this explicit representation together with asymptotic behavior of the spherical Bessel function j_n, we can deduce lack of exponential stability of solutions.

Remark 2.5.2 *We mention here the work of Baouendi and Goulaouic* [**7**]. *They studied a degenerate elliptic problem in an open domain of* $\mathbb{R}^n$ *and they gave an estimate of the spectral behavior.*

□

2.6 Asymptotic stability

2.6.1 Strong stability of the system

In this part, we use a general criteria of Theorem 2.5.3 to show the strong stability of the C_0-semigroup $e^{t\mathcal{A}}$ associated to the wave system (P_1) in the absence of the compactness of the resolvent of $\mathcal{A}$. Our main result is the following theorem:

Theorem 2.6.1 *The C_0-semigroup $e^{t\mathcal{A}}$ is strongly stable in $\mathcal{H}$; i.e, for all $U_0 \in \mathcal{H}$, the solution of (2.18) satisfies*

$$\lim_{t\to\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0.$$

For the proof of Theorem 2.6.1, we need the following two lemmas.

Lemma 2.6.1 *$\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.*

Proof
We will argue by contraction. Let us suppose that there $\lambda \in \mathbb{R}$.
•**Case 1**: $\lambda \neq 0$ and $U \neq 0$, such that $\mathcal{A}U = i\lambda U$. Then, we get

$$\begin{cases} i\lambda u - v = 0, \\ i\lambda v - (a(x)u_x)_x = 0, \\ i\lambda\phi + (\xi^2+\eta)\phi - v(1)\mu(\xi) = 0, \end{cases} \tag{2.53}$$

Then, from (2.22) we have

$$\phi \equiv 0. \tag{2.54}$$

From $(2.53)_3$, we have

$$v(1) = 0. \tag{2.55}$$

Hence, from $(2.53)_1$ we obtain

$$u(1) = 0 \text{ and } u_x(1) = 0. \tag{2.56}$$

From $(2.53)_1$ and $(2.53)_2$, we have

$$-\lambda^2 u - (a(x)u_x)_x = 0. \tag{2.57}$$

Hence

$$\begin{cases} \lambda^2 u + (a(x)u_x)_x = 0, \\ u(1) = u_x(1) = 0, \\ \begin{cases} u(0) = 0 & \text{if } \mu_a \in [0,1), \\ (a(x)u_x)(0) = 0 & \text{if } \mu_a \in [1,2). \end{cases} \end{cases} \tag{2.58}$$

Multiplying equation $(2.58)_1$ by $\overline{u}$, using Green formula, Proposition 2.2.2-(iii) and the boundary conditions, we get

$$\lambda^2 \int_0^1 |u|^2\,dx - \int_0^1 a(x)|u_x|^2\,dx = 0. \tag{2.59}$$

Multiplying equation $(2.58)_1$ by $x\overline{u}_x$, we get

$$\lambda^2 \int_0^1 xu\overline{u}_x\,dx + \int_0^1 x\overline{u}_x(a(x)u_x)_x\,dx = 0. \tag{2.60}$$

$U \in D(\mathcal{A})$, then the regularity is sufficiently for applying an integration on the second integral in the left hand side in equation (2.60). Then we obtain

$$\frac{\lambda^2}{2}\int_0^1 x\frac{d}{dx}|u|^2\,dx - \int_0^1 a(x)|u_x|^2\,dx - \frac{1}{2}\int_0^1 xa(x)\frac{d}{dx}|u_x|^2\,dx = 0. \tag{2.61}$$

Using Green formula and the boundary conditions, we get

$$\lambda^2 \int_0^1 |u|^2\,dx + \int_0^1 (a(x) - xa'(x))|u_x|^2\,dx = 0 \tag{2.62}$$

Multiplying equations (2.59) by $-\mu_a/2$, and tacking the sum of this equation and (2.62), we get

$$\frac{2-\mu_a}{2}\lambda^2 \int_0^1 |u|^2\,dx + \int_0^1 \left(a(x) - xa'(x) + \frac{\mu_a}{2}a(x)\right)|u_x|^2\,dx = 0. \tag{2.63}$$

By definition of μ_a, we have

$$(2-\mu_a)a(x) \leq 2(a(x) - xa'(x)) + \mu_a a(x).$$

This, together with (2.63), gives

$$\frac{2-\mu_a}{2}\lambda^2 \int_0^1 |u|^2\,dx + \frac{2-\mu_a}{2}\int_0^1 a(x)|u_x|^2\,dx \leq 0. \tag{2.64}$$

we deduce that

$$u = 0. \tag{2.65}$$

Using equation $(2.53)_1$, we obtain

$$v = 0. \tag{2.66}$$

Consequently, using equations (2.65), (2.66) and (2.54), we obtain $U = 0$, which contradict the hypothesis $U \neq 0$. The proof has been completed.

•**Case 2**: $\lambda = 0$. The system (2.53) becomes

$$\begin{cases} v = 0, \\ (a(x)u_x)_x = 0, \\ (\xi^2 + \eta)\phi - v(1)\mu(\xi) = 0. \end{cases} \tag{2.67}$$

From $(2.67)_1$ and $(2.67)_3$, we have

$$v \equiv 0, \quad \phi \equiv 0. \tag{2.68}$$

Multiplying equation $(2.67)_2$ by $\overline{u}$, using Green formula, Proposition 2.2.2-(iii) and the boundary conditions, we get

$$\int_0^1 a(x)|u_x|^2\,dx + \beta|u(1)|^2 = 0. \tag{2.69}$$

Then

$$(a|u_x|^2)(x) = 0 \quad \forall x \in (0,1). \tag{2.70}$$

Moreover, if $\mu_a \in [1,2)$, then $u(1) = 0$. Hence $(au_x)(1) = 0$ and consequently

$$u_x(1) = 0. \tag{2.71}$$

Moreover, from (2.70), we have

$$u_x(x) = 0 \;\text{ on } (0,1).$$

Hence u is constant in $(0,1)$. As $u(1)=0$, then

$$u \equiv 0$$

Now, if $\mu_a \in [0,1)$, we have $u(0)=0$. Hence $u \equiv 0$. and consequently, we obtain $U=0$, which contradict the hypothesis $U \neq 0$. The proof has been completed.

We have Consequently, $\mathcal{A}$ does not have purely imaginary eigenvalues. So the condition (i) of Theorem 2.5.3 holds. The condition (ii) of Theorem 2.5.3 will be satisfied if we show that $\sigma(\mathcal{A}) \cap \{i\mathbb{R}\}$ is at most a countable set. We have the following lemma.

Lemma 2.6.2 *We have*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \text{ if } \eta \neq 0,$$
$$i\mathbb{R}^* \subset \rho(\mathcal{A}) \text{ if } \eta = 0$$

where $\mathbb{R}^* = \mathbb{R} - \{0\}$.

Proof

•**Case 1**: $\lambda \neq 0$.

We will prove that the operator $i\lambda I - \mathcal{A}$ is surjective for $\lambda \neq 0$. For this purpose, let $F = (f_1, f_2, f_3)^T \in \mathcal{H}$, we seek $X = (u, v, \phi)^T \in D(\mathcal{A})$ solution of the following equation

$$(i\lambda I - \mathcal{A})X = F. \tag{2.72}$$

Equivalently, we have

$$\begin{cases} i\lambda u - v = f_1, \\ i\lambda v - (a(x)u_x)_x = f_2, \\ i\lambda \phi + (\xi^2 + \eta)\phi - v(1)\mu(\xi) = f_3. \end{cases} \tag{2.73}$$

From $(2.73)_1$ and $(2.73)_2$, we have

$$-\lambda^2 u - (a(x)u_x)_x = (f_2 + i\lambda f_1). \tag{2.74}$$

Solving system (2.74) is equivalent to finding $u \in H_a^2 \cap H_*^1(0,1)$ such that

$$\int_0^1 (-\lambda^2 u\overline{w} - (a(x)u_x)_x \overline{w})\, dx = \int_0^1 (f_2 + i\lambda f_1)\overline{w}\, dx \tag{2.75}$$

for all $w \in H_*^1(0,1)$. Then, we get

$$\begin{cases} \displaystyle\int_0^1 (-\lambda^2 u\overline{w} + (a(x)u_x)\overline{w}_x)\, dx + (i\lambda\tilde{\zeta} + \beta)u(1)\ \overline{w}(1) \\ \displaystyle = \int_\Omega (f_2 + i\lambda f_1)\overline{w}\, dx - \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + i\lambda} f_3(\xi)\overline{w}\, d\xi + \tilde{\zeta} f_1(1)\overline{w}(1)\,. \end{cases} \tag{2.76}$$

We can rewrite (2.76) as

$$-(L_\lambda u, w)_{H_*^1} + (u, w)_{H_*^1} = l(w), \tag{2.77}$$

with the inner product defined by

$$(u, w)_{H^1_*} = \int_0^1 a(x) u_x \overline{w}_x \, dx + \beta u(1) \, \overline{w}(1)$$

and

$$(L_\lambda u, w)_{H^1_*} = \int_\Omega \lambda^2 u \overline{w} \, dx - i\lambda\tilde{\zeta} u(1) \, \overline{w}(1).$$

Using the compactness embedding from $L^2(0,1)$ into $H^{-1}_*(0,1)$ and from $H^1_*(0,1)$ into $L^2(0,1)$ we deduce that the operator L_λ is compact from $L^2(0,1)$ into $L^2(0,1)$. Consequently, by Fredholm alternative, proving the existence of u solution of (2.77) reduces to proving that 1 is not an eigenvalue of L_λ. Indeed if 1 is an eigenvalue, then there exists $u \neq 0$, such that

$$(L_\lambda u, w)_{H^1_*} = (u, w)_{H^1_*} \quad \forall w \in H^1_*. \tag{2.78}$$

In particular for $w = u$, it follows that

$$\lambda^2 \|u(x)\|^2_{L^2(0,1)} - i\lambda\tilde{\zeta}|u(1)|^2 = \|\sqrt{a(x)} u_x(x)\|^2_{L^2(0,1)} + \beta|u(1)|^2.$$

Hence, we have

$$u(1) = 0. \tag{2.79}$$

From (2.78), we obtain

$$(au_x)(1) = 0 \tag{2.80}$$

Then

$$\begin{cases} \lambda^2 u + (a(x)u_x)_x = 0 \text{ on } (0,1), \\ \begin{cases} u(0) = 0 & \text{if } \mu_a \in [0,1) \\ (a(x)u_x)(0) = 0 & \text{if } \mu_a \in [1,2) \end{cases} \\ u(1) = 0 \;\; u_x(1) = 0. \end{cases} \tag{2.81}$$

We deduce that $U = 0$.

•**Case 2**: $\lambda = 0$ and $\eta \neq 0$.

The system (2.73) is reduced to the following system

$$\begin{cases} v = -f_1, \\ -(a(x)u_x)_x = f_2, \\ (\xi^2 + \eta)\phi - v(1)\mu(\xi) = f_3. \end{cases} \tag{2.82}$$

Solving system (2.82) is equivalent to finding $u \in H^1_*(0,1)$ such that

$$-\int_0^1 (a(x)u_x)_x \overline{w} \, dx = \int_0^1 f_2 \overline{w} \, dx \tag{2.83}$$

for all $w \in H^1_*(0,1)$. Then, we get

$$\begin{aligned} \int_0^1 a(x) u_x \overline{w}_x \, dx + \beta u(1)\overline{w}(1) \quad &= \quad \int_0^1 f_2 \overline{w} \, dx + \varrho \eta^{\alpha-1} f_1(1)\overline{w}(1) \\ &\quad - \zeta \int_{-\infty}^{\infty} \frac{\mu(\xi) f_3(\xi)}{\xi^2 + \eta} \, d\xi \overline{w}(1). \end{aligned} \tag{2.84}$$

Consequently, problem (2.84) is equivalent to the problem

$$\mathcal{B}(u,w) = \mathcal{L}(w), \tag{2.85}$$

where the bilinear form $\mathcal{B} : H_*^1(0,1) \times H_*^1(0,1) \to \mathbb{C}$ and the linear form $\mathcal{L} : H_*^1(0,1) \to \mathbb{C}$ are defined by

$$\mathcal{B}(u,w) = \int_0^1 (a(x)u_x\overline{w}_x)\,dx + \beta u(1)\overline{w}(1) \tag{2.86}$$

and

$$\mathcal{L}(w) = \int_0^1 f_2\overline{w}\,dx + \varrho\eta^{\alpha-1}f_1(1)\overline{w}(1) - \zeta\int_{-\infty}^{\infty}\frac{\mu(\xi)f_3(\xi)}{\xi^2+\eta}\,d\xi\overline{w}(1).$$

It is easy to verify that $\mathcal{B}$ is continuous and coercive, and $\mathcal{L}$ is continuous. So by applying the Lax-Milgram theorem, we deduce that for all $w \in H_*^1(0,1)$ problem (2.85) admits a unique solution $u \in H_*^1(\Omega)$. Applying the classical elliptic regularity, it follows from (2.84) that $u \in H_a^2(0,1)$. Therefore, the operator $\mathcal{A}$ is surjective.

□

2.6.2 Residual spectrum of $\mathcal{A}$

Lemma 2.6.3 *Let $\mathcal{A}$ be defined by (2.19). Then*

$$\mathcal{A}^*\begin{pmatrix} u \\ v \\ \phi \end{pmatrix} = \begin{pmatrix} -v \\ -(a(x)u_x)_x \\ -(\xi^2+\eta)\phi - v(1)\mu(\xi) \end{pmatrix} \tag{2.87}$$

with domain

$$D(\mathcal{A}^*) = \left\{ \begin{array}{l} (u,v,\phi)^T \ \textit{in}\ \mathcal{H} : u \in H_a^2 \cap H_*^1(0,1), v \in H_*^1(0,1), \\ -(\xi^2+\eta)\phi - v(1)\mu(\xi) \in L^2(-\infty,+\infty), \\ (au_x)(1) + \beta u(1) + \zeta\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)\,d\xi = 0 \\ |\xi|\phi \in L^2(-\infty,+\infty) \end{array} \right\} \tag{2.88}$$

Proof
Let $U = (u,v,\phi)^T$ and $V = (\tilde{u},\tilde{v},\tilde{\phi})^T$. We have
$< \mathcal{A}U, V >_{\mathcal{H}} = < U, \mathcal{A}^*V >_{\mathcal{H}}$.

$$< \mathcal{A}U, V >_{\mathcal{H}} = \int_0^1 a(x)v_x\overline{\tilde{u}_x}\,dx + \int_0^1 (a(x)u_x)_x\overline{\tilde{v}}\,dx + \zeta\int_{-\infty}^{+\infty}[-(\xi^2+\eta)\phi + v(1)\mu(\xi)]\overline{\tilde{\phi}}\,d\xi + \beta v(1)\overline{\tilde{u}}(1)$$
$$= -\int_0^1 (a(x)\overline{\tilde{u}_x})_x v\,dx - \int_0^1 (a(x)u_x)\overline{\tilde{v}_x}\,dx + [(a(x)\overline{\tilde{u}_x})v]_0^1 + [(a(x)u_x)\overline{\tilde{v}}]_0^1 - \zeta\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi\overline{\tilde{\phi}}\,d\xi$$
$$+\zeta v(1)\int_{-\infty}^{+\infty}\mu(\xi)\overline{\tilde{\phi}}\,d\xi + \beta v(1)\overline{\tilde{u}}(1)$$
$$= -\int_0^1 (a(x)\overline{\tilde{u}_x})_x v\,dx - \int_0^1 (a(x)u_x)\overline{\tilde{v}_x}\,dx + (a(x)\overline{\tilde{u}_x})(1)v(1) - \beta u(1)\overline{\tilde{v}(1)} - \zeta\overline{\tilde{v}(1)}\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)\,d\xi$$
$$-\zeta\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi\overline{\tilde{\phi}}\,d\xi + \zeta v(1)\int_{-\infty}^{+\infty}\mu(\xi)\overline{\tilde{\phi}}\,d\xi + \beta v(1)\overline{\tilde{u}}(1).$$

If we set

$$(a(x)\tilde{u}_x)(1) + \beta\tilde{u}_x(1) + \zeta\int_{-\infty}^{+\infty}\mu(\xi)\tilde{\phi}\,d\xi = 0,$$

we get

$$< \mathcal{A}U, V >_{\mathcal{H}} = -\int_0^1 (a(x)\overline{\tilde{u}_x})_x v\,dx - \int_0^1 (a(x)u_x)\overline{\tilde{v}_x}\,dx - \zeta\int_{-\infty}^{+\infty}\overline{[(\xi^2+\eta)\tilde{\phi} + \tilde{v}(1)\mu(\xi)]}\phi\,d\xi - \beta u(1)\overline{\tilde{v}(1)}.$$

Theorem 2.6.2 *$\sigma_r(\mathcal{A}) = \emptyset$, where $\sigma_r(\mathcal{A})$ denotes the set of residual spectrum of $\mathcal{A}$. It is defined as*

$$\sigma_r(\mathcal{A}) = \{\lambda \in \mathbb{C} : ker(\lambda I - \mathcal{A}) = 0 \text{ and } Im(\lambda I - \mathcal{A}) \text{ is not dense in } \mathcal{H}\}.$$

Proof. Since $\lambda \in \sigma_r(\mathcal{A}), \overline{\lambda} \in \sigma_p(\mathcal{A}^*)$ the proof will be accomplished if we can show that $\sigma_p(\mathcal{A}) = \sigma_p(\mathcal{A}^*)$. obviously this is because the eigenvalues of $\mathcal{A}$ are symmetric on the real axis. From (2.87), the eigenvalue problem $\mathcal{A}^*Z = \lambda Z$ for $\lambda \in \mathbb{C}$ and $0 \neq Z = (u, v, \phi) \in D(\mathcal{A}^*)$ we have

$$\begin{cases} \lambda u + v = 0, \\ \lambda v + (a(x)u_x)_x = 0, \\ \lambda\phi + (\xi^2+\eta)\phi + v(1)\mu(\xi) = 0. \end{cases} \tag{2.89}$$

From $(2.89)_1$ and $(2.89)_2$, we find

$$\lambda^2 u - (a(x)u_x)_x = 0. \tag{2.90}$$

As $(au_x)(1) = -\zeta\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)\,d\xi - \beta u(1)$, we deduce from $(2.89)_3$ and $(2.88)_3$ that

$$(au_x)(1) = \zeta v(1)\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\,d\xi - \beta u(1) = -\varrho\lambda(\lambda+\eta)^{\alpha-1}u(1) - \beta u(1) \tag{2.91}$$

with the following conditions

$$\begin{cases} u(0) = 0 & \text{if } \mu_a \in [0,1) \\ (a(x)u_x)(0) = 0 & \text{if } \mu_a \in [1,2) \end{cases} \tag{2.92}$$

Hence $\mathcal{A}^*$ has the same eigenvalues with $\mathcal{A}$. The proof is complete.

Remark 2.6.1 *When $\eta = 0$, then $\lambda = 0$ is in the continuous spectrum.*

□

2.6.3 Polynomial Stability (for $\eta \neq 0$)

Theorem 2.6.3 *The semigroup $S_{\mathcal{A}}(t)_{t\geq 0}$ is polynomially stable and*

$$E(t) = \|S_{\mathcal{A}}(t)U_0\|^2_{\mathcal{H}} \leq \frac{1}{t^{\frac{2}{(1-\alpha)+\frac{\varepsilon}{2}}}}\|U_0\|^2_{D(\mathcal{A})}.$$

Proof
We will need to study the resolvent equation $(i\lambda - \mathcal{A})U = F$, for $\lambda \in \mathbb{R}$, namely

$$\begin{cases} i\lambda u - v = f_1, \\ i\lambda v - (a(x)u_x)_x = f_2, \\ i\lambda\phi + (\xi^2+\eta)\phi - v(1)\mu(\xi) = f_3, \end{cases} \tag{2.93}$$

where $F = (f_1, f_2, f_3)^T \in \mathcal{H}$.
•**Step 1** Taking inner product in $\mathcal{H}$ with U and using (2.22) we get

$$|Re\langle \mathcal{A}U, U\rangle| \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{2.94}$$

This implies that

$$\zeta\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,t)|^2\,d\xi \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} \tag{2.95}$$

and, applying $(2.93)_1$, we obtain

$$\Big||\lambda||u(1)| - |f_1(1)|\Big|^2 \leq |v(1)|^2.$$

We deduce that

$$|\lambda|^2|u(1)|^2 \leq c|f_1(1)|^2 + c|v(1)|^2. \tag{2.96}$$

Moreover, from $(2.93)_4$, we have

$$(au_x)(1) = -\beta u(1) - \zeta\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)\,d\xi.$$

Then

$$\begin{aligned} |(au_x)(1)|^2 &\leq 2\beta^2|u(1)|^2 + 2\zeta^2\left|\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)\,d\xi\right|^2 \\ &\leq 2\beta^2|u(1)|^2 + 2\zeta^2\left(\int_{-\infty}^{+\infty}(\xi^2+\eta)^{-1}|\mu(\xi)|^2\,d\xi\right)\left(\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi)|^2\,d\xi\right) \\ &\leq 2\beta^2|u(1)|^2 + c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \end{aligned} \tag{2.97}$$

From $(2.93)_3$, we obtain

$$v(1)\mu(\xi) = (i\lambda + \xi^2 + \eta)\phi - f_3(\xi). \tag{2.98}$$

By multiplying (2.98) by $(i\lambda + \xi^2+\eta)^{-1}|\xi|^{\frac{1-\varepsilon}{2}}$ (for $\varepsilon > 0$), we get

$$(i\lambda+\xi^2+\eta)^{-1}v(1)\mu(\xi)|\xi|^{\frac{1-\varepsilon}{2}} = |\xi|^{\frac{1-\varepsilon}{2}}\phi - (i\lambda+\xi^2+\eta)^{-1}|\xi|^{\frac{1-\varepsilon}{2}}f_3(\xi). \tag{2.99}$$

Hence, by taking absolute values of both sides of (2.99), integrating over the interval $]-\infty, +\infty[$ with respect to the variable ξ and applying Cauchy-Schwartz inequality, we obtain

$$\mathcal{S}|v(1)| \leq \mathcal{U}\left(\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi|^2\, d\xi\right)^{\frac{1}{2}} + \sqrt{2}\mathcal{V}\left(\int_{-\infty}^{+\infty}|f_3(\xi)|^2\, d\xi\right)^{\frac{1}{2}}, \tag{2.100}$$

where

$$\mathcal{S} = \left|\int_{-\infty}^{+\infty}(i\lambda+\xi^2+\eta)^{-1}|\xi|^{\frac{1-\varepsilon}{2}}\mu(\xi)\, d\xi\right| = \frac{\pi}{\sin(\frac{2(\alpha+1)-\varepsilon}{4})\pi}|i\lambda+\eta|^{\frac{(\alpha-1)}{2}-\frac{\varepsilon}{4}},$$

$$\mathcal{U} = \left(\int_{-\infty}^{+\infty}(\xi^2+\eta)^{-1}|\xi|^{1-\varepsilon}\, d\xi\right)^{\frac{1}{2}},$$

$$\mathcal{V} = \left(\int_{-\infty}^{+\infty}(|\lambda|+\xi^2+\eta)^{-2}|\xi|^{1-\varepsilon}\, d\xi\right)^{\frac{1}{2}} = \left(\frac{\varepsilon}{2}\frac{\pi}{\sin(\frac{2-\varepsilon}{2})\pi}(|\lambda|+\eta)^{-(1+\frac{\varepsilon}{2})}\right)^{1/2}.$$

Thus, by using the inequality $2PQ \leq P^2+Q^2, P \geq 0, Q \geq 0$, again, we get

$$\mathcal{S}^2|v(1)|^2 \leq 2\mathcal{U}^2\left(\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi|^2\, d\xi\right) + 4\mathcal{V}^2\left(\int_{-\infty}^{+\infty}|f_3(\xi)|^2\, d\xi\right). \tag{2.101}$$

We deduce that

$$|v(1)|^2 \leq c|\lambda|^{1-\alpha+\frac{\varepsilon}{2}}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c\|F\|^2_{\mathcal{H}}. \tag{2.102}$$

•**Step 2** Now we use the classical multiplier method.
Let us introduce the following notation

$$\mathcal{I}_u(\alpha) = |\sqrt{a(x)}u_x(\alpha)|^2 + |v(\alpha)|^2,$$

$$\mathcal{E}_u = \int_0^1 \mathcal{I}_u(s)\, ds.$$

Lemma 2.6.4 *We have that*

$$\begin{aligned}&\int_0^1\left[\left((a(x)-xa'(x))+\frac{\mu_a}{2}a(x)\right)|u_x|^2+\left(1-\frac{\mu_a}{2}\right)|v(x)|^2\right]dx\\ &= [x\mathcal{I}_u]_0^1 + \tfrac{\mu_a}{2}[a(x)u_x\overline{u}]_0^1 + R,\end{aligned} \tag{2.103}$$

where R satisfies

$$|R| \leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

for a positive constant C.

Proof
To get (2.103), let us multiply the equation $(2.93)_2$ by $x\overline{u}_x$ Integrating on $(0,1)$ we obtain

$$i\lambda\int_0^1 vx\overline{u}_x\, dx - \int_0^1(a(x)u_x)_x x\overline{u}_x\, dx = \int_0^L f_2x\overline{u}_x\, dx$$

or

$$-\int_0^1 vx(\overline{i\lambda u_x})\, dx - \int_0^1 x(a(x)u_x)_x\overline{u}_x\, dx = \int_0^1 f_2x\overline{u}_x\, dx.$$

Since $i\lambda u_x = v_x + f_{1x}$ taking the real part in the above equality results in

$$-\frac{1}{2}\int_0^1 x\frac{d}{dx}|v|^2\,dx + \frac{1}{2}\int_0^1 xa(x)\frac{d}{dx}|u_x|^2\,dx - [xa(x)|u_x|^2]_0^1 + \int_0^1 a(x)|u_x|^2\,dx$$
$$= Re\int_0^1 vx\overline{f}_{1x}\,dx + Re\int_0^1 f_2 x\overline{u}_x\,dx.$$

Performing an integration by parts we get

$$(2.104)\quad \int_0^1 [|\sqrt{a(x)}u_x|^2 + |v(x)|^2]\,dx - \int_0^1 xa'(x)|u_x(x)|^2\,dx = [x(|\sqrt{a(x)}u_x|^2 + |v(x)|^2)]_0^1 + R_1,$$

where

$$R_1 = 2Re\int_0^1 xf_2\overline{u}_x\,dx + 2Re\int_0^1 xv\overline{f}_{1x}\,dx.$$

Multiplying $(2.93)_2$ by $\overline{u}$ and integrating over $(0,1)$ and using integration by parts we get

$$(2.105)\qquad \int_0^1 a(x)|u_x|^2dx - \int_0^1 |v|^2dx - [a(x)u_x\overline{u}]_0^1 = \int_0^1 v\overline{f_1}\,dx + \int_0^1 f_2\overline{u}dx.$$

Multiplying (2.105) by $\mu_a/2$ and summing with (2.104) we get

$$(2.106)\qquad \begin{aligned}&\int_0^1((a(x) - xa'(x)) + \frac{\mu_a}{2}a(x))|u_x|^2 + (1 - \frac{\mu_a}{2})|v(x)|^2]\,dx\\ &= [x\mathcal{I}_u]_0^1 + \tfrac{\mu_a}{2}[a(x)u_x\overline{u}]_0^1 + R\end{aligned}$$

with:

$$R = R_1 + R_2$$

and

$$R_2 = \frac{\mu_a}{2}\int_0^1 v\overline{f_1}\,dx + \frac{\mu_a}{2}\int_0^1 f_2\overline{u}dx.$$

Moreover

$$\int_0^1 x^2|u_x|^2\,dx \le \int_0^1 x^{\mu_a}|u_x|^2\,dx \le \frac{1}{a(1)}\int_0^1 a(x)|u_x|^2\,dx.$$

Then

$$\left|\int_0^1 xf_2\overline{u}_x\,dx\right| \le C\|f_2\|_{L^2(0,1)}\|xu_x\|_{L^2(0,1)} \le C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}},$$

$$\left|\int_0^1 xv\overline{f}_{1x}\,dx\right| \le C\|v\|_{L^2(0,1)}\|xf_{1x}\|_{L^2(0,1)} \le C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}},$$

$$\left|\int_0^1 v\overline{f_1}\,dx\right| \le C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}$$

and

$$\left|\int_0^1 f_2\overline{u}dx\right| \le C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

Hence, We deduce that

$$(2.107)\qquad |R| \le C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

•**Step 3** We have

$$(a(x)u_x\overline{u})_{x=0} = 0, \ (x|v(x)|^2)_{x=0} = 0, \ (xa(x)|u_x|^2)_{x=0} = 0.$$

By definition of μ_a, we have

$$(2-\mu_a)a \leq 2(a - xa') + \mu_a a.$$

This, together with (2.106), gives

$$\frac{2-\mu_a}{2}\int_0^1 (a(x)|u_x|^2 + |v|^2)\, dx \leq \mathcal{I}_u(1) + \frac{\mu_a}{2}a(1)|u_x(1)\overline{u}(1)| + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{2.108}$$

Then

$$\mathcal{E}_u \leq c|u(1)|^2 + c'|u_x(1)|^2 + c''|v(1)|^2 + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{2.109}$$

As

$$\begin{aligned} |f_1(1)|^2 &\leq 2\int_0^1 |f_1|^2\, dx + 2\int_0^1 |f_1(x) - f_1(1)|^2\, dx \\ &\leq 2\int_0^1 |f_1|^2\, dx + \frac{2}{a(1)(2-\mu_a)}\int_0^1 a(x)|f_{1x}|^2\, dx \\ &\leq C\|F\|^2_{\mathcal{H}} \end{aligned}$$

Moreover, from (2.96) we deduce

$$|u(1)|^2 \leq c\frac{1}{\lambda^2}\|F\|^2 + c\frac{1}{\lambda^2}|v(1)|^2.$$

From (2.97) we deduce

$$|u_x(1)|^2 \leq c\frac{1}{\lambda^2}\|F\|^2 + c\frac{1}{\lambda^2}|v(1)|^2 + c'\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

Since that

$$\int_{-\infty}^{+\infty} |\phi(\xi)|^2\, d\xi \leq C\int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi)|^2\, d\xi \leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

Hence

$$\|U\|^2_{\mathcal{H}} \leq c\frac{1}{\lambda^2}\|F\|^2 + c\frac{1}{\lambda^2}|v(1)|^2 + c|v(1)|^2 + c'\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{2.110}$$

Substitution of inequalities (2.102) into (2.110), we obtain that

$$\|U\|_{\mathcal{H}} \leq c|\lambda|^{1-\alpha+\frac{\varepsilon}{2}}\|F\|_{\mathcal{H}}.$$

The conclusion then follows by applying the Theorem 2.5.2.

$\square$

2.7 Optimality of energy decay when $a(x) = x^\gamma$ and $\eta \neq 0$

By Lemma 2.5.1, the spectrum of $\mathcal{A}$ is at the left of the imaginary axis, but approaches this axis. Hence, the decay of the energy depends on the asymptotic behavior of the real part of these eigenvalues, since Lemma 2.5.1 shows an expected optimal behavior of resolvent like

$$\|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{H}} \equiv |\lambda|^{1-\alpha}.$$

We can expect a decay rate (optimal) of the energy at $t^{-2/(1-\alpha)}$. Unfortunately we were not able to prove this optimal decay rate by frequency domain method based on multiplier method for general function a. In Theorem 2.6.3, we obtain an upper estimate of resolvent like

$$\|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{H}} \leq |\lambda|^{2(1-\alpha)} \text{ as } |\lambda| \to \infty$$

which is less better. In this section, for $a(x) = x^\gamma, 0 \leq \gamma < 2$, by an explicit representation of the resolvent of the generator on the imaginary axis and the use of the theorem by Borichev and Tomilov, we prove an optimal decay rate. We treat only the case $\gamma \in [1, 2)$ and $\nu_\gamma \notin \mathbb{N}$. The cases $\gamma \in [1, 2)$ and $\nu_\gamma \in \mathbb{N}$ and $\gamma \in [0, 1)$ are similar with some modifications.

Let us consider the resolvent equation

$$\begin{cases} i\lambda u - v = f_1, \\ i\lambda v - (x^\gamma u_x)_x = f_2, \\ i\lambda\phi + (\xi^2 + \eta)\phi - v(1)\mu(\xi) = f_3, \end{cases} \tag{2.111}$$

where $F = (f_1, f_2, f_3)^T \in \mathcal{H}$. From $(2.111)_1$ and $(2.111)_2$, we have

$$\lambda^2 u + (x^\gamma u_x)_x = -(f_2 + i\lambda f_1) \tag{2.112}$$

with

$$\begin{cases} (x^\gamma u_x)_{x=0} = 0 \\ u_x(1) + \beta u(1) + \zeta \int_{-\infty}^{\infty} \mu(\xi)\phi(\xi)\, d\xi = 0. \end{cases} \tag{2.113}$$

The substitution of ϕ given by $(2.111)_3$ into $(2.113)_2$ give us

$$u_x(1) + \beta u(1) + \varrho(i\lambda + \eta)^{\alpha-1} v(1) + \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi) f_3(\xi)}{i\lambda + \xi^2 + \eta}\, d\xi = 0. \tag{2.114}$$

Moreover, from $(2.111)_1$, we have

$$v(1) = i\lambda u(1) - f_1(1).$$

Then, the condition (2.114) become

$$u_x(1) + (\beta + \varrho i\lambda(i\lambda + \eta)^{\alpha-1})u(1) = \varrho(i\lambda + \eta)^{\alpha-1} f_1(1) - \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi) f_3(\xi)}{i\lambda + \xi^2 + \eta}\, d\xi. \tag{2.115}$$

Assume that Φ is a solution of (2.112), then one easily checks that the function Ψ defined by

$$\Phi(x) = x^{\frac{1-\gamma}{2}}\Psi\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right) \tag{2.116}$$

is solution of the following inhomogeneous Bessel equation:

$$\begin{aligned} &y^2\Psi''(y) + y\Psi'(y) + \left(y^2 - \left(\frac{\gamma-1}{2-\gamma}\right)^2\right)\Psi(y) = \\ &-(\tfrac{2}{2-\gamma})^2(\tfrac{2-\gamma}{2}\tfrac{1}{\lambda}y)^{\frac{3-\gamma}{2-\gamma}}\left(f_2\left((\tfrac{2-\gamma}{2}\tfrac{1}{\lambda}y)^{\frac{2}{2-\gamma}}\right) + i\lambda f_1\left((\tfrac{2-\gamma}{2}\tfrac{1}{\lambda}y)^{\frac{2}{2-\gamma}}\right)\right). \end{aligned} \tag{2.117}$$

The solution can be written as

$$\Psi(y) = AJ_{\nu_\gamma}(y) + BJ_{-\nu_\gamma}(y) + \frac{2\nu_\gamma}{\sin\nu_\gamma\pi}\int_0^y \frac{f(s)}{s}\left(J_{\nu_\gamma}(s)J_{-\nu_\gamma}(y) - J_{\nu_\gamma}(y)J_{-\nu_\gamma}(s)\right)ds.$$

Thus,

$$\begin{aligned} &u(x) = Ax^{\frac{1-\gamma}{2}}J_{\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right) + Bx^{\frac{1-\gamma}{2}}J_{-\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right) \\ &-\frac{2\nu_\gamma}{\sin\nu_\gamma\pi}\left(\frac{2}{2-\gamma}\right)x^{\frac{1-\gamma}{2}}\int_0^x s^{\frac{1-\gamma}{2}}(f_2(s)+i\lambda f_1(s))\left(J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda s^{\frac{2-\gamma}{2}}\right)J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right)\right. \\ &\left.-J_{\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right)J_{-\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda s^{\frac{2-\gamma}{2}}\right)\right)ds. \end{aligned}$$

Therefore,

$$\begin{aligned} u(x) &= A\Phi_+(x) + B\Phi_-(x) \\ &\quad -\frac{2\nu_\gamma}{\sin\nu_\gamma\pi}\left(\frac{2}{2-\gamma}\right)\int_0^x (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi_-(x) - \Phi_+(x)\Phi_-(s))\,ds, \end{aligned} \tag{2.118}$$

where Φ_+ and Φ_- are defined by

$$\Phi_+(x) = x^{\frac{1-\gamma}{2}}J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right), \quad \Phi_-(x) = x^{\frac{1-\gamma}{2}}J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right).$$

From where it follows

$$\begin{aligned} u_x(x) &= A\Phi'_+(x) + B\Phi'_-(x) \\ &\quad -\frac{2\nu_\gamma}{\sin\nu_\gamma\pi}\left(\frac{2}{2-\gamma}\right)\int_0^x (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi'_-(x) - \Phi'_+(x)\Phi_-(s))\,ds \end{aligned} \tag{2.119}$$

From (2.115), (2.119) and (2.118), we conclude that

$$\begin{aligned} &A(\Phi'_+(1) + (\beta + \varrho i\lambda(i\lambda+\eta)^{\alpha-1})\Phi_+(1)) = \varrho(i\lambda+\eta)^{\alpha-1}f_1(1) - \zeta\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_3(\xi)}{i\lambda+\xi^2+\eta}\,d\xi \\ &+\frac{2\nu_\gamma}{\sin\nu_\gamma\pi}\left(\frac{2}{2-\gamma}\right)\int_0^1 (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi'_-(1) - \Phi'_+(1)\Phi_-(s))\,ds \\ &+(\beta + \varrho i\lambda(i\lambda+\eta)^{\alpha-1})\frac{2\nu_\gamma}{\sin\nu_\gamma\pi}(\frac{2}{2-\gamma})\int_0^1 (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi_-(1) - \Phi_+(1)\Phi_-(s))\,ds, \end{aligned} \tag{2.120}$$

where

$$\begin{aligned}\Phi'_+(1) &= \frac{1-\gamma}{2} J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda\right) + \lambda J'_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda\right),\\ \Phi'_-(1) &= \frac{1-\gamma}{2} J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda\right) + \lambda J'_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda\right).\end{aligned}$$

Let us set

$$\begin{aligned}D &= \Phi'_+(1) + (\beta + \varrho i\lambda(i\lambda+\eta)^{\alpha-1})\Phi_+(1)\\ &= (\frac{1-\gamma}{2} + \beta + \varrho i\lambda(i\lambda+\eta)^{\alpha-1}) J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda\right) + \lambda J'_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda\right)\\ &= \left(\frac{1-\gamma}{2} + \frac{2-\gamma}{2}\nu_\gamma + \beta + \varrho i\lambda(i\lambda+\eta)^{\alpha-1}\right) J_{\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda\right) - \lambda J_{\nu_\gamma+1}\left(\tfrac{2}{2-\gamma}\lambda\right)\\ &= \frac{i^\alpha\sqrt{2-\gamma}}{\sqrt{\pi}}\lambda^{\alpha-1/2}\cos\left(\frac{2}{2-\gamma}\lambda - \nu_\gamma\frac{\pi}{2} - \frac{\pi}{4}\right) - \frac{i^\gamma\sqrt{2-\gamma}}{\sqrt{\pi}}\lambda^{1/2}\sin\left(\frac{2}{2-\gamma}\lambda - \nu_\gamma\frac{\pi}{2} - \frac{\pi}{4}\right)\\ &\qquad + O\left(\frac{1}{\lambda^{1/2}}\right).\end{aligned}$$

It is clear that

$$|D| \geq c|\lambda|^{\alpha-1/2} \text{ for large } \lambda.$$

The constant A in (2.120) satisfies

$$\begin{aligned}|A||D(\lambda)| &\leq o(1) + \beta\frac{2\nu_\gamma}{\sin\nu_\gamma\pi}(\frac{2}{2-\gamma})\left|\int_0^1 (f_2(s) + i\lambda f_1(s))(\Phi_+(s)\Phi_-(1) - \Phi_+(1)\Phi_-(s))\,ds\right|\\ &\quad + \varrho|\lambda|^\alpha\frac{2\nu_\gamma}{\sin\nu_\gamma\pi}(\frac{2}{2-\gamma})\left|\int_0^1 (f_2(s) + i\lambda f_1(s))(\Phi_+(s)\Phi_-(1) - \Phi_+(1)\Phi_-(s))\,ds\right|\\ &\quad \frac{2\nu_\gamma}{\sin\nu_\gamma\pi}(\frac{2}{2-\gamma})\left|\int_0^1 (f_2(s) + i\lambda f_1(s))(\Phi_+(s)\Phi'_-(1) - \Phi'_+(1)\Phi_-(s))\,ds\right|\\ &\leq o(1) + c\left(\|f_1\|_{H^1_*(0,1)} + \|f_2\|_{L^2(0,1)}\right),\end{aligned} \tag{2.121}$$

where we have used the fact that $f_1 \in H^1_*(0,1)$ and

$$\left|\int_0^1 (f_2(s) + i\lambda f_1(s))(\Phi_+(s)\Phi_-(1) - \Phi_+(1)\Phi_-(s))\,ds\right| \leq \frac{1}{|\lambda|}\left(\|f_1\|_{H^1_*(0,1)} + \|f_2\|_{L^2(0,1)}\right),$$

$$\left|\int_0^1 (f_2(s) + i\lambda f_1(s))(\Phi_+(s)\Phi'_-(1) - \Phi'_+(1)\Phi_-(s))\,ds\right| \leq \left(\|f_1\|_{H^1_*(0,1)} + \|f_2\|_{L^2(0,1)}\right).$$

Then, we conclude that

$$|A| \leq c|\lambda|^{1/2-\alpha}.$$

Then

$$\|u\|_{L^2(0,1)} \leq c|\lambda|^{-\alpha}\left(\|f_1\|_{H^1_*(0,1)} + \|f_2\|_{L^2(0,1)}\right).$$

From (2.119), we deduce that

$$\|x^{\gamma/2}u_x\|_{L^2(0,1)} \leq c|\lambda|^{1-\alpha}\left(\|f_1\|_{H^1_*(0,1)} + \|f_2\|_{L^2(0,1)}\right).$$

From $(2.111)_1$ and (2.118), we get

$$\|v\|_{L^2(0,1)} \leq c|\lambda|^{1-\alpha}\left(\|f_1\|_{H^1(0,1)} + \|f_2\|_{L^2(0,1)}\right).$$

From $(2.111)_3$, we get

$$\begin{aligned}\|\phi\|_{L^2(-\infty,\infty)} &\leq |v(1)|\left\|\frac{\mu(\xi)}{i\lambda+\xi^2+\eta}\right\|_{L^2(-\infty,\infty)} + \left\|\frac{f_3(\xi)}{i\lambda+\xi^2+\eta}\right\|_{L^2(-\infty,\infty)}\\ &\leq c|\lambda|^{-1/2}\left(\|f_1\|_{H^1(0,1)} + \|f_2\|_{L^2(0,1)}\right) + c\frac{1}{|\lambda|}\|f_3\|_{L^2(-\infty,\infty)}.\end{aligned}$$

Thus, we conclude that

$$\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{H}} \leq c|\lambda|^{1-\alpha} \text{ as } |\lambda| \to \infty.$$

The conclusion then follows by applying Theorem 2.5.2.

Chapter 3

STABILIZATION OF DEGENERATE WAVE EQUATION UNDER FRACTIONAL FEEDBACK ACTING ON THE DEGENERATE BOUNDARY

3.1 Introduction

In this Chapter, we are concerned with the boundary stabilization of fractional type for degenerate wave equation of the form

$$(P_2) \qquad \begin{cases} u_{tt}(x,t) - (x^\gamma u_x(x,t))_x = 0 & \text{in } (0,1)\times(0,+\infty),\\ (x^\gamma u_x)(0,t) = \varrho \partial_t^{\alpha,\eta} u(0,t) & \text{in } (0,+\infty),\\ u(1,t) = 0 & \text{in } (0,+\infty),\\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } (0,1), \end{cases}$$

where $\gamma \in [0,1)$ and $\varrho > 0$. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α, $(0 < \alpha \leq 1)$, with respect to the time variable (see **[20]**). It is defined as follows

$$\partial_t^{\alpha,\eta} w(t) = \begin{cases} w_t(t) & \text{for } \alpha = 1, \eta \geq 0,\\ \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\, ds, & \text{for } 0 < \alpha < 1, \eta \geq 0. \end{cases}$$

The degenerate wave equation (P_2) (i.e $\gamma \neq 0$) can describe the vibration problem of an elastic string. In a neighborhood of an endpoint $x = 0$ of this string, the elastic is sufficiently small or the linear density is large enough.

The bibliography of works concerning the stabilization of nondegenerate wave equation with different types of damping is truly long (see e.g. **[21]**, **[22]**, **[19]** and the references therein).

In **[22]**, for $a(x) = a_1 x + a_0$: the authors have established asymptotic stabilization with the following boundary damping

$$\begin{cases} (au_x)(0,t) = 0, \\ (au_x)(1,t) = -ku(1,t) - u_t(1,t), \ k > 0. \end{cases}$$

In **[19]**, the authors considered the following modelization of a flexible torque arm controlled by two feedbacks depending only on the boundary velocities:

$$\begin{cases} u_{tt}(x,t) - (a(x)u_x)_x + \alpha u_t(x,t) + \beta y(x,t) = 0, \ 0 < x < 1, \ t > 0, \\ (a(x)u_x)(0) = k_1 u_t(0,t), t > 0, \\ (a(x)u_x)(1) = -k_2 u_t(1,t), t > 0, \end{cases}$$

where

$$\begin{cases} \alpha \geq 0, \beta > 0, k_1, k_2 \geq 0, k_1 + k_2 \neq 0, \\ a \in W^{1,\infty}(0,1), a(x) \geq a_0 \text{ for all } x \in [0,1]. \end{cases}$$

They proved the exponential decay of the solutions.

On the contrary, when the coefficient $a(x)$ is degenerate very little is known in the literature, even though many problems that are relevant for applications are described by hyperbolic equations degenerating at the boundary of the space domain (see **[26]**, **[46]** and **[2]**). In **[26]**, for any $0 < \gamma < 1$, the null controllability of the following degenerate wave equation was considered:

$$(PC) \qquad \begin{cases} u_{tt}(x,t) - (x^\gamma u_x(x,t))_x = 0 & \text{on } (0,1) \times (0,T), \\ u(0,t) = \theta(t), u(1,t) = 0 & \text{on } (0,T), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } (0,1), \end{cases}$$

where $\theta(t)$ is the control variable and it acts on the degenerate boundary. Recently, in **[46]** (see also **[2]**), the authors studied the null controllability problems of one-dimensional degenerate wave equations as in **[26]** but the control acts on the nondegenerate boundary. They proved that any initial value in state space is controllable. Also, an explicit expression for the controllability time is given.

Very recently, Alabau et al. **[2]** studied the degenerate wave equation of the type

$$(3.1) \qquad u_{tt}(x,t) - (a(x)u_x(x,t))_x = 0 \text{ in } (0,1) \times (0,+\infty),$$

where the coefficient a is a positive function on $]0,1]$ but vanishes at zero. The degeneracy of (3.1) at $x = 0$ is measured by the parameter μ_a defined by

$$(3.2) \qquad \mu_a = \sup_{0<x\leq 1} \frac{x|a'(x)|}{a(x)}$$

and the initial conditions are

$$(3.3) \qquad u(x,0) = u_0(x), u_t(x,0) = u_1(x),$$

followed by the boundary conditions

$$(P_1) \qquad \begin{cases} \begin{cases} u(0,t) = 0 & \text{if } 0 \leq \mu_a < 1 \\ (au_x)(0,t) = 0 & \text{if } 1 \leq \mu_a < 2 \end{cases} & \text{in } (0,+\infty), \\ u_x(1,t) + u_t(1,t) + \beta u(1,t) = 0 & \text{in } (0,+\infty), \end{cases}$$

they obtained exponential stability of the solutions.

Here we want to focus on the following remarks:

- System (3.1), (3.3) and $(P_2)_1$ under study is different from one studied on [**2**]. Indeed, the control is located at $x = 0$.
- The fractional velocity feedbacks considered here provide a weaker damping than the velocity feedbacks (see [**39**]).
- The explicit representation of the resolvent gives us a sharp polynomial decay rate, however in [**2**], stabilization is done under the classical energy method based on multiplier techniques (see [**33**]). Unfortunately, this method does not seem to be applicable in the case of damping acting at $x = 0$.

In this Chapter, we explain the influence of the relation between the degenerate coefficient and the fractional feedback on decay estimates.

This Chapter is organized as follows. In section 2, we give preliminaries results and we reformulate the system (P_2) into an augmented system by coupling the degenerate wave equation with a suitable diffusion equation and we show the well-posedness of our problem by semigroup theory. In section 3, we prove lack of exponential stability by spectral analysis by using Bessel functions. In the last section, we prove an optimal decay rate. The proof heavily relies on Bessel equations and Borichev-Tomilov Theorem.

3.2 Preliminaries results

Now, we introduce, as in [**16**] or [**2**], the following weighted Sobolev spaces:

$$H^1_{0,\gamma}(0,1) = \Big\{u \text{ is locally absolutely continuous in } (0,1] : x^{\gamma/2}u_x \in L^2(0,1)/\ u(1) = 0\Big\}$$

$$H^1_{\gamma}(0,1) = \Big\{u \text{ is locally absolutely continuous in } (0,1] : x^{\gamma/2}u_x \in L^2(0,1)\Big\}.$$

We remark that $H^1_\gamma(0,1)$ is a Hilbert space with the scalar product

$$(u,v)_{H^1_\gamma(0,1)} = \int_0^1 (u\overline{v} + x^\gamma u'(x)\overline{v'(x)})\, dx, \quad \forall u,v \in H^1_\gamma(0,1).$$

Let us also set

$$|u|_{H^1_{0,\gamma}(0,1)} = \left(\int_0^1 x^\gamma |u'(x)|^2\, dx\right)^{1/2} \quad \forall u \in H^1_\gamma(0,1).$$

Actually, $|\cdot|_{H^1_{0,\gamma}(0,1)}$ is an equivalent norm on the closed subspace $H^1_{0,\gamma}(0,1)$ to the norm of $H^1_\gamma(0,1)$. This fact is a simple consequence of the following version of Poincaré's inequality.

Proposition 3.2.1 *There is a positive constant $C_* = C(\gamma)$ such that*

$$\|u\|^2_{L^2(\Omega)} \le C_* |u|^2_{1,\gamma} \quad \forall u \in H^1_{0,\gamma}(0,1). \tag{3.4}$$

Proof. Let $u \in H^1_{0,\gamma}(0,1)$. For any $x \in]0,1]$ we have that

$$|u(x)| = \left|\int_x^1 u'(s)\, ds\right| \leq |u|_{1,\gamma} \left\{\int_0^1 \frac{1}{x^\gamma}\, ds\right\}^{1/2}.$$

Therefore

$$\int_0^1 |u(x)|^2\, dx \leq \frac{1}{1-\gamma}|u|^2_{1,\gamma}.$$

Next, we define

$$H^2_\gamma(0,1) = \{u \in H^1_\gamma(0,1) :\ x^\gamma u' \in H^1(0,1)\},$$

where $H^1(0,1)$ denotes the classical Sobolev space.

Remark 3.2.1 *Notice that if $u \in H^2_\gamma(0,1), \gamma \in [1,2)$,we have $(x^\gamma u_x)_{x=0} \equiv 0$ since $1/x^\gamma$ is not integrable over neighbourhoods of 0. Hence the problem is not well-posed in terms of the semigroups in the Hilbert space.*

3.3 Augmented model

In this section we reformulate (P_2) into an augmented system. For that, we need the following proposition.

Proposition 3.3.1 (see [39]) *Let μ be the function:*

$$\mu(\xi) = |\xi|^{(2\alpha-1)/2}, \quad -\infty < \xi < +\infty,\ 0 < \alpha < 1. \tag{3.5}$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t\phi(\xi,t) + (\xi^2+\eta)\phi(\xi,t) - U(t)\mu(\xi) = 0, \quad -\infty < \xi < +\infty, \eta \geq 0, t > 0, \tag{3.6}$$

$$\phi(\xi,0) = 0, \tag{3.7}$$

$$O(t) = (\pi)^{-1}\sin(\alpha\pi)\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,t)\, d\xi \tag{3.8}$$

is given by

$$O = I^{1-\alpha,\eta}U. \tag{3.9}$$

where

$$[I^{\alpha,\eta} f](t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}e^{-\eta(t-\tau)}f(\tau)\, d\tau$$

Lemma 3.3.1 (see [1]) *If $\lambda \in D_\eta = \mathbb{C}\backslash]-\infty,-\eta]$ then*

$$F(\lambda) = \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\, d\xi = \frac{\pi}{\sin\alpha\pi}(\lambda+\eta)^{\alpha-1}.$$

Using now Proposition 3.3.1 and relation (3.9), system (P_2) may be recast into the following augmented system

$$(P_3') \qquad \begin{cases} u_{tt}(x,t) - (x^\gamma u_x(x,t))_x = 0, \\ \phi_t(\xi,t) + (\xi^2+\eta)\phi(\xi,t) - u_t(0,t)\mu(\xi) = 0, \quad -\infty < \xi < +\infty, \quad t>0, \\ (x^\gamma u_x)(0,t) = \zeta \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,t)\, d\xi, \\ u(1,t) = 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \end{cases}$$

where $\zeta = \varrho(\pi)^{-1}\sin(\alpha\pi)$.

3.4 Well-posedness

In this section, we are interested in showing that system (P_2') is well posed in the sense of semigroups.

We introduce the Hilbert space $\mathcal{H} = H^1_{0,\gamma}(0,1) \times L^2(0,1) \times L^2(-\infty,+\infty)$ with inner product

$$\left\langle \begin{pmatrix} u \\ v \\ \phi \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\phi} \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_0^1 x^\gamma u_x \overline{\tilde{u}}_x dx + \int_0^1 v\overline{\tilde{v}} dx + \zeta \int_{-\infty}^{+\infty} \phi\overline{\tilde{\phi}}\, d\xi.$$

If we put $U = (u, u_t, \phi)^T$ it is clear that (P_2') can be written as

$$U' = \mathcal{A}U, \quad U(0) = U_0, \tag{3.10}$$

where $U_0 = (u_0, u_1, 0)^T$ and $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ \phi \end{pmatrix} = \begin{pmatrix} v \\ (x^\gamma u_x)_x \\ -(\xi^2+\eta)\phi + v(0)\mu(\xi) \end{pmatrix}, \tag{3.11}$$

with domain

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (u,v,\phi) \text{ in } \mathcal{H} : u \in H^2_\gamma(0,1) \cap H^1_{0,\gamma}(0,1), v \in H^1_{0,\gamma}(0,1), \\ -(\xi^2+\eta)\phi + v(0)\mu(\xi) \in L^2(-\infty,+\infty), \\ (x^\gamma u_x)(0) - \zeta \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi = 0, \\ |\xi|\phi \in L^2(-\infty,+\infty) \end{array} \right\}. \tag{3.12}$$

Our main result is giving by the following theorem.

Theorem 3.4.1 *The operator $\mathcal{A}$ defined by (3.11) and (3.12), generates a C_0-semigroup of contractions $e^{t\mathcal{A}}$ in the Hilbert space $\mathcal{H}$.*

Proof.
To prove this result we shall use the Lumer-Phillips' theorem. Since for every $U = (u, v, \phi) \in D(\mathcal{A})$ we have

$$\Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\zeta \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi(\xi)|^2 \, d\xi. \tag{3.13}$$

then the operator $\mathcal{A}$ is dissipative.

Let $\lambda > 0$. we prove that the operator $(\lambda I - \mathcal{A})$ is a surjection. In other words, we shall demonstrate that given any triplet $F = (f_1, f_2, f_3) \in \mathcal{H}$, there is an other triplet $U = (u, v, \phi) \in D(\mathcal{A})$ such that

$$(\lambda I - \mathcal{A})U = F. \tag{3.14}$$

Equation (3.14) is equivalent to

$$\begin{cases} \lambda u - v = f_1, \\ \lambda v - (x^\gamma u_x)_x = f_2, \\ \lambda\phi + (\xi^2 + \eta)\phi - v(0)\mu(\xi) = f_3. \end{cases} \tag{3.15}$$

Suppose u is found with the appropriate regularity. Then, $(3.15)_1$ and $(3.15)_3$ yield

$$v = \lambda u - f_1 \in H^1_{0,\gamma}(0, 1), \tag{3.16}$$

$$\phi = \frac{f_3(\xi) + \mu(\xi)v(0)}{\xi^2 + \eta + \lambda}. \tag{3.17}$$

By using (3.15) and (3.16) it can easily be shown that u satisfies

$$\lambda^2 u - (x^\gamma u_x)_x = f_2 + \lambda f_1. \tag{3.18}$$

Solving equation (3.18) is equivalent to finding $u \in H^2_\gamma(0, 1) \cap H^1_{0,\gamma}(0, 1)$ such that

$$\int_0^1 (\lambda^2 u\overline{w} - (x^\gamma u_x)_x\overline{w}) \, dx = \int_0^1 (f_2 + \lambda f_1)\overline{w} \, dx, \tag{3.19}$$

for all $w \in H^1_{0,\gamma}(0, 1)$. By using (3.19), the boundary condition $(3.12)_3$ and (3.17) the function u satisfying the following equation

$$\begin{aligned} &\int_0^1 (\lambda^2 u\overline{w} + (x^\gamma u_x)\overline{w}_x) \, dx + \tilde{\zeta} v(0)\overline{w}(0) \\ &\quad = \int_0^1 (f_2 + \lambda f_1)\overline{w} \, dx - \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + \lambda} f_3(\xi) \, d\xi \overline{w}(0) \end{aligned} \tag{3.20}$$

where $\tilde{\zeta} = \zeta \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2 + \eta + \lambda} \, d\xi$. Using again (3.16), we deduce that

$$v(0) = \lambda u(0) - f_1(0). \tag{3.21}$$

Inserting (3.21) into (3.20), we get

$$\begin{cases} \int_0^1 (\lambda^2 u\overline{w} + x^\gamma u_x\overline{w}_x) \, dx + \lambda\tilde{\zeta} u(0)\overline{w}(0) \\ = \int_0^1 (f_2 + \lambda f_1)\overline{w} \, dx - \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + \lambda} f_3(\xi) \, d\xi \overline{w}(0) + \tilde{\zeta} f_1(0)\overline{w}(0). \end{cases} \tag{3.22}$$

Problem (3.22) is of the form

$$\mathcal{B}(u,w) = \mathcal{L}(w), \tag{3.23}$$

where $\mathcal{B} : [H^1_{0,\gamma}(0,1) \times H^1_{0,\gamma}(0,1)] \to \mathbb{C}$ is the bilinear form defined by

$$\mathcal{B}(u,w) = \int_0^1 (\lambda^2 u\overline{w} + x^\gamma u_x \overline{w}_x)\, dx + \lambda\tilde{\zeta} u(0)\ \overline{w}(0)$$

and $\mathcal{L} : H^1_{0,\gamma}(0,1) \to \mathbb{C}$ is the linear functional given by

$$\mathcal{L}(w) = \int_0^1 (f_2 + \lambda f_1)\overline{w}\, dx - \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_3(\xi)\, d\xi \overline{w}(0) + \tilde{\zeta} f_1(0)\overline{w}(0).$$

It is easy to verify that $\mathcal{B}$ is continuous and coercive, and $\mathcal{L}$ is continuous. Consequently, by the Lax-Milgram Lemma, system (3.23) has a unique solution $u \in H^1_{0,\gamma}(0,1)$. By the regularity theory for the linear elliptic equations, it follows that $u \in H^2_\gamma(0,1)$. Therefore, the operator $\lambda I - \mathcal{A}$ is surjective for any $\lambda > 0$.

□

As a consequence of Theorem 3.4.1, the system (P_2') is well-posed in the energy space $\mathcal{H}$ and we have the following proposition.

Proposition 3.4.1 *For $(u_0, u_1, 0) \in \mathcal{H}$, the problem (P_2') admits a unique weak solution*

$$(u, u_t, \phi) \in C^0(\mathbb{R}_+, \mathcal{H}).$$

and for $(u_0, u_1, 0) \in D(\mathcal{A})$, the problem (P_2') admits a unique strong solution

$$(u, u_t, \phi) \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

Moreover, from the density $D(\mathcal{A})$ in $\mathcal{H}$ the energy of $(u(t), \phi(t))$ at time $t \geq 0$ by

$$E(t) = \frac{1}{2}\int_0^1 (|u_t|^2 + x^\gamma |u_x|^2) dx + \frac{\zeta}{2}\int_{-\infty}^{+\infty} |\phi(\xi,t)|^2\, d\xi \tag{3.24}$$

decays as follow

$$E'(t) = -\zeta \int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi,t)|^2\, d\xi \leq 0. \tag{3.25}$$

Proof of Proposition 3.4.1. Noting that the regularity of the solution of the problem (P_2') is consequence of the semigroup properties. We have just to prove (3.25).

Multiplying the first equation in (P_2') by $\overline{u}_t$, integrating over $(0,1)$ and using integration by parts, we get

$$\int_0^1 u_{tt}(x,t)\overline{u}_t dx - \int_0^1 (x^\gamma u_x(x,t))_x \overline{u}_t dx = 0.$$

Then

$$\frac{d}{dt}\left(\frac{1}{2}\int_0^1 |u_t(x,t)|^2 dx\right) + \frac{1}{2}\frac{d}{dt}\int_0^1 x^\gamma |u_x(x,t)|^2\, dx - \Re\Big[(x^\gamma u_x)(x,t)\overline{u}_t\Big]_0^1 = 0.$$

Then

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left(|u_t(x,t)|^2 + x^\gamma |u_x(x,t)|^2\right) dx + \zeta \Re \overline{u}_t(0,t) \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,t)\, d\xi = 0. \tag{3.26}$$

Multiplying the second equation in (P_2') by $\zeta\overline{\phi}$ and integrating over $(-\infty,+\infty)$, to obtain:

$$\zeta \int_{-\infty}^{+\infty} \phi_t(\xi,t)\overline{\phi}(\xi,t)d\xi + \zeta \int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi,t)|^2 d\xi - \zeta u_t(0,t)\int_{-\infty}^{+\infty} \mu(\xi)\overline{\phi}(\xi,t)d\xi = 0.$$

Hence

$$\frac{\zeta}{2}\frac{d}{dt}\int_{-\infty}^{+\infty} |\phi(\xi,t)|^2 d\xi + \zeta \int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi,t)|^2 d\xi - \zeta \Re u_t(0,t)\int_{-\infty}^{+\infty} \mu(\xi)\overline{\phi}(\xi,t)d\xi = 0. \tag{3.27}$$

From (3.24), (3.26) and (3.27) we get

$$E'(t) = -\zeta \int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi,t)|^2\, d\xi \le 0.$$

This completes the proof of the lemma.

□

3.5 Lack of exponential stability

This section will be devoted to the study of the lack of exponential decay of solutions associated with the system (3.10). In order to state and prove our stability results, we need some lemmas.

Theorem 3.5.1 ([43]) *Let $S(t)$ be a C_0-semigroup of contractions on Hilbert space $\mathcal{X}$ with generator $\mathcal{A}$. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R}$$

and

$$\overline{\lim_{|\beta|\to\infty}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{X})} < \infty.$$

Our main result is the following.

Theorem 3.5.2 *The semigroup generated by the operator $\mathcal{A}$ is not exponentially stable if $\eta = 0$ or $\alpha \neq 2\nu_\gamma$.*

Proof. We will examine two cases.
•**Case 1** $\eta = 0$: We shall show that $i\lambda = 0$ is not in the resolvent set of the operator $\mathcal{A}$. Indeed, noting that $(\sin(x-1), 0, 0)^T \in \mathcal{H}$, and denoting by $(u, v, \phi)^T$ the image of $(\sin(x-1), 0, 0)^T$ by $\mathcal{A}^{-1}$, we see that $\phi(\xi) = |\xi|^{\frac{2\alpha-5}{2}} \sin 1$. But, then $\phi \notin L^2(-\infty, +\infty)$, since $\alpha \in]0,1[$. So $(u, v, \phi)^T \notin D(\mathcal{A})$.

• **Case 2** $\eta \neq 0$:

We aim to show that an infinite number of eigenvalues of $\mathcal{A}$ approach the imaginary axis which prevents the system (P_2) from being exponentially stable. Indeed we first compute the characteristic equation that gives the eigenvalues of $\mathcal{A}$. Let λ be an eigenvalue of $\mathcal{A}$ with associated eigenvector $U = (u, v, \phi)^T$. Then $\mathcal{A}U = \lambda U$ is equivalent to

$$\begin{cases} \lambda u - v = 0, \\ \lambda v - (x^\gamma u_x)_x = 0, \\ \lambda\phi + (\xi^2 + \eta)\phi - v(0)\mu(\xi) = 0. \end{cases} \tag{3.28}$$

It is well-known that Bessel functions play an important role in this type of problem. From $(3.28)_1 - (3.28)_2$ for such λ, we find

$$\lambda^2 u - (x^\gamma u_x)_x = 0. \tag{3.29}$$

Using the boundary conditions and $(3.28)_3$, we deduce that

$$\begin{cases} \lambda^2 u - (x^\gamma u_x)_x = 0, \\ (x^\gamma u_x)(0) - \zeta v(0) \displaystyle\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2 + \lambda + \eta}\, d\xi \\ \qquad\qquad = (x^\gamma u_x)(0) - \varrho\lambda(\lambda+\eta)^{\alpha-1}u(0) = 0, \\ u(1) = 0. \end{cases} \tag{3.30}$$

Assume that u is a solution of (3.30) associated to eigenvalue $-\lambda^2$, then one easily checks that the function

$$u(x) = x^{\frac{1-\gamma}{2}} \Psi\left(\frac{2}{2-\gamma} i\lambda x^{\frac{2-\gamma}{2}}\right)$$

is a solution of the following problem:

$$y^2\Psi''(y) + y\Psi'(y) + (y^2 - (\frac{\gamma-1}{2-\gamma})^2)\Psi(y) = 0. \tag{3.31}$$

We have

$$u(x) = c_+\Phi_+ + c_-\Phi_-, \tag{3.32}$$

where Φ_+ and Φ_- are defined by

$$\Phi_+(x) := x^{\frac{1-\gamma}{2}} J_{\nu_\gamma}\left(\frac{2}{2-\gamma} i\lambda x^{\frac{2-\gamma}{2}}\right)$$

and

$$\Phi_-(x) := x^{\frac{1-\gamma}{2}} J_{-\nu_\gamma}\left(\frac{2}{2-\gamma} i\lambda x^{\frac{2-\gamma}{2}}\right),$$

where

$$J_\nu(y) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\nu+1)} \left(\frac{y}{2}\right)^{2m+\nu} = \sum_{m=0}^{\infty} c_{\nu,m}^+ y^{2m+\nu}, \tag{3.33}$$

$$J_{-\nu}(y) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m-\nu+1)} \left(\frac{y}{2}\right)^{2m-\nu} = \sum_{m=0}^{\infty} c^-_{\nu,m} y^{2m-\nu} \tag{3.34}$$

$$\nu_\gamma = \frac{1-\gamma}{2-\gamma}$$

and J_{ν_γ} and $J_{-\nu_\gamma}$ are Bessel functions of the first kind of order ν_γ and $-\nu_\gamma$. As $\nu_\gamma \notin \mathbb{N}$, so J_{ν_γ} and $J_{-\nu_\gamma}$ are linearly independent and therefore the pair $(J_{\nu_\gamma}, J_{-\nu_\gamma})$ (classical result) forms a fundamental system of solutions (3.31).

Then, using the series expansion of J_{ν_α} and $J_{-\nu_\alpha}$, one obtains

$$\Phi_+(x) = \sum_{m=0}^{\infty} \tilde{c}^+_{\nu_\gamma,m} x^{1-\gamma+(2-\gamma)m}, \quad \Phi_-(x) = \sum_{m=0}^{\infty} \tilde{c}^-_{\nu_\gamma,m} x^{(2-\gamma)m},$$

with

$$\tilde{c}^+_{\nu_\gamma,m} = c^+_{\nu_\gamma,m} \left(\frac{2}{2-\alpha} i\lambda\right)^{2m+\nu_\gamma}, \quad \tilde{c}^+_{\nu_\gamma,m} = c^-_{\nu_\gamma,m} \left(\frac{2}{2-\alpha} i\lambda\right)^{2m-\nu_\gamma}.$$

Next one easily verifies that $\Phi_+, \Phi_- \in H^1_{0,\gamma}(0,1)$: indeed,

$$\begin{aligned} &\Phi_+(x) \sim_0 \tilde{c}^+_{\nu_\gamma,0} x^{1-\gamma}, \quad x^{\gamma/2}\Phi'_+(x) \sim_0 (1-\gamma)\tilde{c}^+_{\nu_\gamma,0} x^{-\gamma/2}, \\ &\Phi_-(x) \sim_0 \tilde{c}^-_{\nu_\gamma,0}, \quad x^{\gamma/2}\Phi'_-(x) \sim_0 (2-\gamma)\tilde{c}^-_{\nu_\gamma,0} x^{1-\gamma/2}. \end{aligned}$$

where we have used the following relation

$$xJ'_\nu(x) = \nu J(x) - xJ_{\nu+1}(x). \tag{3.35}$$

Hence, given c_+ and c_-, $u(x) = c_+\Phi_+(x) + c_-\Phi_-(x) \in H^1_{0,\gamma}(0,1)$ with the following boundary condition

$$\begin{cases} (x^\gamma u_x)(0) - \varrho\lambda(\lambda+\eta)^{\alpha-1}u(0) = 0, \\ u(1) = 0. \end{cases}$$

Then

$$M(\lambda)C(\lambda) = \begin{pmatrix} (1-\gamma)\tilde{c}^+_{\nu_\gamma,0} & -\varrho\lambda(\lambda+\eta)^{\alpha-1}\tilde{c}^-_{\nu_\gamma,0} \\ J_{\nu_\gamma}\left(\frac{2}{2-\gamma}i\lambda\right) & J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}i\lambda\right) \end{pmatrix} \begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.36}$$

Hence, a non-trivial solution u exists if and only if the determinant of $M(\lambda)$ vanishes. Set $f(\lambda) = det M(\lambda)$ thus the characteristic equation is $f(\lambda) = 0$.

Our purpose in the sequel is to prove, thanks to Rouché's Theorem, that there is a subsequence of eigenvalues for which their real part tends to 0.

In the sequel, since $\mathcal{A}$ is dissipative, we study the asymptotic behavior of the large eigenvalues λ of $\mathcal{A}$ in the strip $-\alpha_0 \le \Re(\lambda) \le 0$, for some $\alpha_0 > 0$ large enough and for such λ, we remark that Φ_+, Φ_- remains bounded.

Lemma 3.5.1 *There exists $N \in \mathbb{N}$ such that*

$$\{\lambda_k\}_{k\in\mathbf{Z}^*,|k|\ge N} \subset \sigma(\mathcal{A}), \tag{3.37}$$

where

- *If* $\gamma = 0$ *and* $\alpha = 1$

$$\lambda_k = \begin{Bmatrix} \ln\sqrt{\dfrac{\varrho-1}{\varrho+1}} + ik\pi & \text{if } \rho > 1 \\ \ln\sqrt{\dfrac{1-\varrho}{\varrho+1}} + i\left(k+\dfrac{1}{2}\right)\pi & \text{if } \rho < 1 \end{Bmatrix}, \quad k \in \mathbf{Z}.$$

- *If* $0 < \gamma < 1$ *and* $\alpha = 1$

$$\lambda_k = -\frac{2-\gamma}{2} i\left(k + \frac{\nu_\gamma}{2} + \frac{3}{4}\right)\pi + \frac{\beta}{|k|^{1-2\nu_\gamma}} + o\left(\frac{1}{k^{1-2\nu_\gamma}}\right), k \geq N, \tilde{\alpha} \in i\mathbb{R}, \beta \in \mathbb{R}, \beta < 0.$$

$$\lambda_k = \overline{\lambda_{-k}} \text{ if } k \leq -N,$$

where

$$\beta = -\frac{1-\gamma}{\varrho}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\frac{\sin\nu_\gamma\pi}{\pi^{1-2\nu_\gamma}}.$$

- *If* $\alpha = 2\nu_\gamma$

$$\lambda_k = -i\frac{2-\gamma}{4}\left(2k\pi + \theta - \frac{\pi}{2}\right) - \frac{2-\gamma}{4}\ln\frac{1+\tilde{A}}{\sqrt{1+\tilde{A}^2+2\tilde{A}\cos 2\nu_\gamma\pi}} + O\left(\frac{1}{k}\right), \quad k \in \mathbf{Z},$$

$$\lambda_k = \overline{\lambda_{-k}} \text{ if } k \leq -N,$$

where

$$\tilde{A} = \frac{1-\gamma}{\varrho}\left(\frac{2}{2-\gamma}\right)^{2\nu_\gamma}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}$$

and θ *is such that*

$$\begin{cases} \cos\theta = \dfrac{(1+\tilde{A})\cos\nu_\gamma\pi}{\sqrt{1+\tilde{A}^2+2\tilde{A}\cos 2\nu_\gamma\pi}}, \\ \sin\theta = \dfrac{(1-\tilde{A})\sin\nu_\gamma\pi}{\sqrt{1+\tilde{A}^2+2\tilde{A}\cos 2\nu_\gamma\pi}}. \end{cases}$$

- *If* $\alpha > 2\nu_\gamma$

$$\lambda_k = -\frac{2-\gamma}{2} i\left(k + \frac{\nu_\gamma}{2} + \frac{3}{4}\right)\pi + \frac{\tilde{\alpha}}{k^{\alpha-2\nu_\gamma}} + \frac{\beta}{|k|^{\alpha-2\nu_\gamma}} + o\left(\frac{1}{k^{\alpha-2\nu_\gamma}}\right), k \geq N, \tilde{\alpha} \in i\mathbb{R}, \beta \in \mathbb{R}, \beta < 0,$$

$$\lambda_k = \overline{\lambda_{-k}} \text{ if } k \leq -N,$$

where

$$\beta = -\frac{1-\gamma}{\varrho}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\left(\frac{2-\gamma}{2}\right)^{1-\alpha}\frac{\cos(1-\alpha)\frac{\pi}{2}\sin\nu_\gamma\pi}{\pi^{\alpha-2\nu_\gamma}}.$$

- *If* $\alpha < 2\nu_\gamma$

$$\lambda_k = -\frac{2-\gamma}{2} i\left(k - \frac{\nu_\gamma}{2} + \frac{3}{4}\right)\pi + \frac{\tilde{\alpha}}{k^{2\nu_\gamma-\alpha}} + \frac{\beta}{|k|^{2\nu_\gamma-\alpha}} + o\left(\frac{1}{k^{2\nu_\gamma-\alpha}}\right), k \geq N, \tilde{\alpha} \in i\mathbb{R}, \beta \in \mathbb{R}, \beta < 0,$$

$$\lambda_k = \overline{\lambda_{-k}} \text{ if } k \leq -N,$$

where

$$\beta = -\frac{\varrho}{1-\gamma} \frac{c^-_{\nu_\gamma,0}}{c^+_{\nu_\gamma,0}} \left(\frac{2-\gamma}{2}\right)^{1+\alpha} \frac{\cos(1-\alpha)\frac{\pi}{2}\sin\nu_\gamma\pi}{\pi^{\alpha-2\nu_\gamma}}.$$

Moreover for all $|k| \geq N$, the eigenvalues λ_k are simple.

Proof.
• $\gamma = 0$ and $\alpha = 1$.
System (3.29)-(3.30) becomes

$$\begin{cases} \lambda^2 u - u_{xx} = 0, \\ u_x(0) = \varrho u(0), \\ u(1) = 0. \end{cases}$$

The solution u is given by

$$u = c_1 e^{\lambda x} + c_2 e^{-\lambda x}.$$

Thus, the boundary conditions give

$$e^{2\lambda} = \frac{\varrho - 1}{\varrho + 1}.$$

If $\varrho > 1$ and if we set $\lambda = x + iy$, then

$$e^{2x} = \frac{\varrho-1}{\varrho+1} \text{ and } e^{2iy} = 1.$$

Hence

$$x = \frac{1}{2}\ln\frac{\varrho-1}{\varrho+1} \text{ and } y = k\pi, \quad k \in \mathbf{Z}.$$

Then

$$\lambda = \frac{1}{2}\ln\frac{\varrho-1}{\varrho+1} + ik\pi, \quad k \in \mathbf{Z}.$$

Now if $\varrho < 1$, we have

$$e^{2x} = \frac{1-\varrho}{\varrho+1} \text{ and } e^{2iy} = -1.$$

Hence

$$x = \frac{1}{2}\ln\frac{1-\varrho}{\varrho+1} \text{ and } y = (k+\frac{1}{2})\pi, \quad k \in \mathbf{Z}.$$

Then

$$\lambda = \frac{1}{2}\ln\frac{1-\varrho}{\varrho+1} + i(k+\frac{1}{2})\pi, \quad k \in \mathbf{Z}.$$

• $0 < \gamma < 1$ and $\alpha = 1$.
Step 1. From (3.36), we have

$$f(\lambda) = (1-\gamma)\tilde{c}^+_{\nu_\gamma,0} J_{-\nu_\gamma}\left(\frac{2}{2-\gamma} i\lambda\right) + \varrho\lambda\tilde{c}^-_{\nu_\gamma,0} J_{\nu_\gamma}\left(\frac{2}{2-\gamma} i\lambda\right) = 0$$

We will use the following classical asymptotic development (see **[36]** p. 122, (5.11.6)): for all $\delta > 0$, the following development holds when $|argz| \leq \pi - \delta$:

$$J_\nu(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos\left(z - \nu\frac{\pi}{2} - \frac{\pi}{4}\right)\left(1 + O(\frac{1}{|z|^2})\right) - \left(\frac{2}{\pi z}\right)^{1/2} \sin\left(z - \nu\frac{\pi}{2} - \frac{\pi}{4}\right) O\left(\frac{1}{|z|^2}\right). \tag{3.38}$$

Then

$$f(\lambda) = \left(\frac{2}{\pi \tilde{z}}\right)^{1/2} \varrho \lambda^{1-\nu_\gamma} c^-_{\nu_\gamma,0} \left(\frac{2}{2-\gamma} i\right)^{-\nu_\gamma} \frac{e^{-i(\tilde{z} - \nu_\gamma \frac{\pi}{2} - \frac{\pi}{4})}}{2} \tilde{f}(\lambda), \tag{3.39}$$

where

$$\tilde{z} = \frac{2}{2-\gamma} i\lambda$$

and

$$\begin{aligned} \tilde{f}(\lambda) &= (e^{2i(\tilde{z} - \nu_\gamma \frac{\pi}{2} - \frac{\pi}{4})} + 1) + \frac{1-\gamma}{\varrho} \left(\frac{2}{2-\gamma} i\right)^{2\nu_\gamma} \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}} \frac{e^{2i(\tilde{z} - \frac{\pi}{4})} + e^{-i\nu_\gamma \pi}}{\lambda^{1-2\nu_\gamma}} + O\left(\frac{1}{\lambda^2}\right) \\ &= f_0(\lambda) + \frac{f_1(\lambda)}{\lambda^{1-2\nu_\gamma}} + O\left(\frac{1}{\lambda^2}\right), \end{aligned} \tag{3.40}$$

where

$$f_0(\lambda) = e^{2i(\tilde{z} - \nu_\gamma \frac{\pi}{2} - \frac{\pi}{4})} + 1. \tag{3.41}$$

$$f_1(\lambda) = \frac{1-\gamma}{\varrho} \left(\frac{2}{2-\gamma} i\right)^{2\nu_\gamma} \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}} (e^{2i(\tilde{z} - \frac{\pi}{4})} + e^{-i\nu_\gamma \pi}). \tag{3.42}$$

Note that f_0 and f_1 remain bounded in the strip $-\alpha_0 \leq \Re(\lambda) \leq 0$.
Step 2. We look at the roots of f_0. From (3.41), f_0 has one family of roots that we denote λ_k^0.

$$f_0(\lambda) = 0 \Leftrightarrow e^{2i(\tilde{z} - \nu_\gamma \frac{\pi}{2} - \frac{\pi}{4})} + 1 = 0$$

Hence

$$2i\left(\frac{2}{2-\gamma} i\lambda - \nu_\gamma \frac{\pi}{2} - \frac{\pi}{4}\right) = i(2k+1)\pi, \quad k \in \mathbf{Z},$$

i.e.,

$$\lambda_k^0 = -\frac{2-\gamma}{2} i \left(k + \frac{\nu_\gamma}{2} + \frac{3}{4}\right)\pi, \quad k \in \mathbf{Z}.$$

Now with the help of Rouché's Theorem, we will show that the roots of $\tilde{f}$ are close to those of f_0. Let us start with the first family. Changing in (3.40) the unknown λ by $u = 2iz$ then (3.40) becomes

$$\tilde{f}(u) = (e^u + 1) + O\left(\frac{1}{u^{(1-2\nu_\gamma)}}\right) = f_0(u) + O\left(\frac{1}{u^{(1-2\nu_\gamma)}}\right)$$

The roots of f_0 are $u_k = -\dfrac{2-\gamma}{2} i \left(k + \dfrac{\nu_\gamma}{2} + \dfrac{3}{4}\right)\pi, k \in \mathbf{Z}$, and setting $u = u_k + re^{it}, t \in [0, 2\pi]$, we can easily check that there exists a constant $C > 0$ independent of k such that $|e^u + 1| \geq Cr$ for r small enough. This allows to apply Rouché's Theorem. Consequently, there exists a subsequence of roots of $\tilde{f}$ which tends to the roots u_k of f_0. Equivalently, it means that there exists $N \in \mathbb{N}$

and a subsequence $\{\lambda_k\}_{|k|\geq N}$ of roots of $f(\lambda)$, such that $\lambda_k = \lambda_k^0 + o(1)$ which tends to the roots $-\frac{2-\gamma}{2} i \left(k + \frac{\nu_\gamma}{2} + \frac{3}{4}\right)\pi$ of f_0. Finally for $|k| \geq N, \lambda_k$ is simple since λ_k^0 is.

Step 3. From Step 2, we can write

$$\lambda_k = -\frac{2-\gamma}{2} i \left(k + \frac{\nu_\gamma}{2} + \frac{3}{4}\right)\pi + \varepsilon_k. \tag{3.43}$$

Using (3.43), we get

$$\begin{aligned} e^{2i\lambda_k} &= -e^{-\frac{4}{2-\gamma}\varepsilon_k} \\ &= -1 + \tfrac{4}{2-\gamma}\varepsilon_k + O(\varepsilon_k^2). \end{aligned} \tag{3.44}$$

Substituting (3.44) into (3.41), using that $\tilde{f}(\lambda_k) = 0$, we get:

$$(3.45)\tilde{f}(\lambda_k) = \frac{4}{2-\gamma}\varepsilon_k - \frac{1-\gamma}{\varrho}\left(\frac{2}{2-\gamma} i\right)^{2\nu_\gamma} \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}} \frac{2i\sin\nu_\gamma\pi}{(-\frac{2-\gamma}{2} ik\pi)^{1-2\nu_\gamma}} + o(\varepsilon_k) + o\left(\frac{1}{k^{1-2\nu_\gamma}}\right) = 0$$

and hence

$$\varepsilon_k = -\frac{1-\gamma}{\varrho} \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}} \frac{\sin\nu_\gamma\pi}{(k\pi)^{1-2\nu_\gamma}} + o\left(\frac{1}{k^{1-2\nu_\gamma}}\right) \tag{3.46}$$

From (3.46) we have in that case $|k|^{1-\alpha}\Re\lambda_k \sim \beta$, with

$$\beta = -\frac{1-\gamma}{\varrho} \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}} \frac{\sin\nu_\gamma\pi}{(\pi)^{1-2\nu_\gamma}}.$$

- $\alpha = 2\nu_\gamma$.

  From (3.36), we have

$$f(\lambda) = (1-\gamma)\tilde{c}^+_{\nu_\gamma,0} J_{-\nu_\gamma}\left(\frac{2}{2-\gamma} i\lambda\right) + \varrho\lambda(\lambda+\eta)^{\alpha-1}\tilde{c}^-_{\nu_\gamma,0} J_{\nu_\gamma}\left(\frac{2}{2-\gamma} i\lambda\right) = 0$$

Then

$$f(\lambda) = \left(\frac{2}{\pi\tilde{z}}\right)^{1/2} \varrho\lambda^{1-\nu_\gamma}(\lambda+\eta)^{\alpha-1} c^-_{\nu_\gamma,0}\left(\frac{2}{2-\gamma} i\right)^{-\nu_\gamma} \frac{e^{-i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}}{2}\tilde{f}(\lambda), \tag{3.47}$$

where

$$\begin{aligned} \tilde{f}(\lambda) &= (e^{2i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}+1) + \frac{1-\gamma}{\varrho}\left(\frac{2}{2-\gamma} i\right)^{2\nu_\gamma} \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}(e^{2i(\tilde{z}-\frac{\pi}{4})} + e^{-i\nu_\gamma\pi}) + O\left(\frac{1}{\lambda}\right) \\ &= f_0(\lambda) + O\left(\tfrac{1}{\lambda}\right). \end{aligned} \tag{3.48}$$

We look at the roots of f_0. From (3.41), f_0 has one family of roots that we denote λ_k^0.

$$f_0(\lambda) = 0 \Leftrightarrow e^{2i\tilde{z}} = -i\frac{1+\tilde{A}}{e^{-i\nu_\gamma\pi} + \tilde{A}e^{-i\nu_\gamma\pi}},$$

where

$$\tilde{A} = \frac{1-\gamma}{\varrho}\left(\frac{2}{2-\gamma}\right)^{2\nu_\gamma} \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}.$$

If we set $\lambda = x + iy$. Then

$$\begin{cases} e^{-\frac{4}{2-\gamma}x} = \dfrac{1+\tilde{A}}{\sqrt{1+\tilde{A}^2+2\tilde{A}\cos 2\nu_\gamma\pi}}, \\ -\frac{4}{2-\gamma}y = 2k\pi - \dfrac{\pi}{2} + \theta, \quad k \in \mathbf{Z}, \end{cases}$$

where θ is such that

$$\begin{cases} \cos\theta = \dfrac{(1+\tilde{A})\cos\nu_\gamma\pi}{\sqrt{1+\tilde{A}^2+2\tilde{A}\cos 2\nu_\gamma\pi}}, \\ \sin\theta = \dfrac{(1-\tilde{A})\sin\nu_\gamma\pi}{\sqrt{1+\tilde{A}^2+2\tilde{A}\cos 2\nu_\gamma\pi}}. \end{cases}$$

Hence

$$\begin{cases} x = -\frac{2-\gamma}{4}\ln\dfrac{1+\tilde{A}}{\sqrt{1+\tilde{A}^2+2\tilde{A}\cos 2\nu_\gamma\pi}}, \\ y = -\frac{2-\gamma}{4}(2k\pi - \dfrac{\pi}{2} + \theta), \quad k \in \mathbf{Z}. \end{cases}$$

Now with the help of Rouché's Theorem, we conclude.

• $\alpha > 2\nu_\gamma$.

Step 1. From (3.36), we have

$$f(\lambda) = \left(\frac{2}{\pi\tilde{z}}\right)^{1/2} \varrho\lambda^{1-\nu_\gamma}(\lambda+\eta)^{\alpha-1}c^-_{\nu_\gamma,0}\left(\frac{2}{2-\gamma}i\right)^{-\nu_\gamma}\frac{e^{-i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}}{2}\tilde{f}(\lambda), \tag{3.49}$$

$$\begin{aligned} \tilde{f}(\lambda) &= (e^{2i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}+1) + \frac{1-\gamma}{\varrho}\left(\frac{2}{2-\gamma}i\right)^{2\nu_\gamma}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\frac{e^{2i(\tilde{z}-\frac{\pi}{4})}+e^{-i\nu_\gamma\pi}}{\lambda^{\alpha-2\nu_\gamma}} + O\left(\frac{1}{\lambda^2}\right) \\ &= f_0(\lambda) + \frac{f_1(\lambda)}{\lambda^{\alpha-2\nu_\gamma}} + O\left(\frac{1}{\lambda^2}\right), \end{aligned} \tag{3.50}$$

where

$$f_0(\lambda) = e^{2i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})} + 1. \tag{3.51}$$

$$f_1(\lambda) = \frac{1-\gamma}{\varrho}\left(\frac{2}{2-\gamma}i\right)^{2\nu_\gamma}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}(e^{2i(\tilde{z}-\frac{\pi}{4})}+e^{-i\nu_\gamma\pi}). \tag{3.52}$$

We look at the roots of f_0. From (3.51), f_0 has one family of roots that we denote λ^0_k.

$$f_0(\lambda) = 0 \Leftrightarrow e^{2i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})} + 1 = 0.$$

Hence

$$2i\left(\frac{2}{2-\gamma}i\lambda - \nu_\gamma\frac{\pi}{2} - \frac{\pi}{4}\right) = i(2k+1)\pi, \quad k \in \mathbf{Z},$$

i.e.,

$$\lambda_k^0 = -\frac{2-\gamma}{2} i \left(k + \frac{\nu_\gamma}{2} + \frac{3}{4}\right)\pi, \quad k \in \mathbf{Z}.$$

Step 2. From Step 1, we can write

$$\lambda_k = -\frac{2-\gamma}{2} i \left(k + \frac{\nu_\gamma}{2} + \frac{3}{4}\right)\pi + \varepsilon_k. \tag{3.53}$$

Using (3.53), we get

$$\begin{array}{rcl} e^{2i\lambda_k} & = & -e^{-\frac{4}{2-\gamma}\varepsilon_k} \\ & = & -1 + \frac{4}{2-\gamma}\varepsilon_k + O(\varepsilon_k^2). \end{array} \tag{3.54}$$

Substituting (3.54) into (3.50), using that $\tilde{f}(\lambda_k) = 0$, we get:

$$\tilde{f}(\lambda_k) = \frac{4}{2-\gamma}\varepsilon_k - \frac{1-\gamma}{\varrho}\left(\frac{2}{2-\gamma}i\right)^{2\nu_\gamma} \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}} \frac{2i\sin\nu_\gamma\pi}{(-\frac{2-\gamma}{2}ik\pi)^{\alpha-2\nu_\gamma}} + o(\varepsilon_k) + o\left(\frac{1}{k^{\alpha-2\nu_\gamma}}\right) = 0 \tag{3.55}$$

and hence

$$\begin{array}{rcl} \varepsilon_k & = & -\frac{1-\gamma}{\varrho}\left(\frac{2-\gamma}{2}\right)^{1-\alpha} \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}} \frac{\sin\nu_\gamma\pi}{(k\pi)^{\alpha-2\nu_\gamma}}(-i)^{1-\alpha} + o\left(\frac{1}{k^{\alpha-2\nu_\gamma}}\right) \\ & = & \begin{cases} -\frac{1-\gamma}{\varrho}\left(\frac{2-\gamma}{2}\right)^{1-\alpha} \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}} \frac{\sin\nu_\gamma\pi}{(k\pi)^{\alpha-2\nu_\gamma}}\left(\cos(1-\alpha)\frac{\pi}{2} - i\sin(1-\alpha)\frac{\pi}{2}\right) + o\left(\frac{1}{k^{\alpha-2\nu_\gamma}}\right) \\ \text{for } k \succeq 0, \\ -\frac{1-\gamma}{\varrho}\left(\frac{2-\gamma}{2}\right)^{1-\alpha} \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}} \frac{\sin\nu_\gamma\pi}{(-k\pi)^{\alpha-2\nu_\gamma}}\left(\cos(1-\alpha)\frac{\pi}{2} + i\sin(1-\alpha)\frac{\pi}{2}\right) + o\left(\frac{1}{k^{\alpha-2\nu_\gamma}}\right) \\ \text{for } k \preceq 0. \end{cases} \end{array} \tag{3.56}$$

From (3.56) we have in that case $|k|^{\alpha-2\nu_\gamma}\Re\lambda_k \sim \beta$, with

$$\beta = -\frac{1-\gamma}{\varrho}\left(\frac{2-\gamma}{2}\right)^{1-\alpha} \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}} \frac{1}{\pi^{\alpha-2\nu_\gamma}} \sin\nu_\gamma\pi\cos(1-\alpha)\frac{\pi}{2}.$$

• $\alpha < 2\nu_\gamma$.
step 1.

$$f(\lambda) = \left(\frac{2}{\pi\tilde{z}}\right)^{1/2}(1-\gamma)\lambda^{\nu_\gamma} c^+_{\nu_\gamma,0}\left(\frac{2}{2-\gamma}i\right)^{\nu_\gamma} \frac{e^{-i(\tilde{z}+\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}}{2}\tilde{f}(\lambda), \tag{3.57}$$

$$\begin{array}{rcl} \tilde{f}(\lambda) & = & (e^{2i(\tilde{z}+\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}+1) + \frac{\varrho}{1-\gamma}\left(\frac{2}{2-\gamma}i\right)^{-2\nu_\gamma} \frac{c^-_{\nu_\gamma,0}}{c^+_{\nu_\gamma,0}} \frac{e^{2i(\tilde{z}-\frac{\pi}{4})}+e^{i\nu_\gamma\pi}}{\lambda^{2\nu_\gamma-\alpha}} + O\left(\frac{1}{\lambda^2}\right) \\ & = & f_0(\lambda) + \frac{f_1(\lambda)}{\lambda^{\alpha-2\nu_\gamma}} + O\left(\frac{1}{\lambda^2}\right), \end{array} \tag{3.58}$$

where

$$f_0(\lambda) = e^{2i(\tilde{z}+\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})} + 1. \tag{3.59}$$

$$f_1(\lambda) = \frac{\varrho}{1-\gamma}\left(\frac{2}{2-\gamma}i\right)^{-2\nu_\gamma} \frac{c^-_{\nu_\gamma,0}}{c^+_{\nu_\gamma,0}}(e^{2i(\tilde{z}-\frac{\pi}{4})} + e^{i\nu_\gamma\pi}). \tag{3.60}$$

We look at the roots of f_0. From (3.59), f_0 has one family of roots that we denote λ_k^0.

$$2i\left(\frac{2}{2-\gamma}i\lambda+\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4}\right)=i(2k+1)\pi,\quad k\in\mathbf{Z},$$

i.e.,

$$\lambda_k^0=-\frac{2-\gamma}{2}i\left(k-\frac{\nu_\gamma}{2}+\frac{3}{4}\right)\pi,\quad k\in\mathbf{Z}.$$

Step 2. From Step 1, we can write

$$\lambda_k=-\frac{2-\gamma}{2}i\left(k-\frac{\nu_\gamma}{2}+\frac{3}{4}\right)\pi+\varepsilon_k. \tag{3.61}$$

Using (3.61), we get

$$\begin{aligned} e^{2i\lambda_k} &= -e^{-\frac{4}{2-\gamma}\varepsilon_k}\\ &= -1+\tfrac{4}{2-\gamma}\varepsilon_k+O(\varepsilon_k^2). \end{aligned} \tag{3.62}$$

Substituting (3.62) into (3.58), using that $\tilde{f}(\lambda_k)=0$, we get:

$$\tilde{f}(\lambda_k)=\frac{4}{2-\gamma}\varepsilon_k+\frac{\varrho}{1-\gamma}\left(\frac{2}{2-\gamma}i\right)^{-2\nu_\gamma}\frac{c^-_{\nu_\gamma,0}}{c^+_{\nu_\gamma,0}}\frac{2i\sin\nu_\gamma\pi}{(-\frac{2-\gamma}{2}ik\pi)^{2\nu_\gamma-\alpha}}+o(\varepsilon_k)+o\left(\frac{1}{k^{\alpha-2\nu_\gamma}}\right)=0 \tag{3.63}$$

and hence

$$\begin{aligned}\varepsilon_k &= -\frac{\varrho}{1-\gamma}\left(\frac{2-\gamma}{2}\right)^{1+\alpha}\frac{c^-_{\nu_\gamma,0}}{c^+_{\nu_\gamma,0}}\frac{\sin\nu_\gamma\pi}{(k\pi)^{2\nu_\gamma-\alpha}}(-i)^{\alpha-1}+o\left(\frac{1}{k^{2\nu_\gamma-\alpha}}\right)\\ &= \begin{cases} -\dfrac{\varrho}{1-\gamma}\left(\dfrac{2-\gamma}{2}\right)^{1+\alpha}\dfrac{c^-_{\nu_\gamma,0}}{c^+_{\nu_\gamma,0}}\dfrac{\sin\nu_\gamma\pi}{(k\pi)^{2\nu_\gamma-\alpha}}\left(\cos(1-\alpha)\dfrac{\pi}{2}+i\sin(1-\alpha)\dfrac{\pi}{2}\right)+o\left(\dfrac{1}{k^{2\nu_\gamma-\alpha}}\right)\\ \text{for } k\succeq 0,\\ -\dfrac{\varrho}{1-\gamma}\left(\dfrac{2-\gamma}{2}\right)^{1+\alpha}\dfrac{c^-_{\nu_\gamma,0}}{c^+_{\nu_\gamma,0}}\dfrac{\sin\nu_\gamma\pi}{(-k\pi)^{2\nu_\gamma-\alpha}}\left(\cos(1-\alpha)\dfrac{\pi}{2}-i\sin(1-\alpha)\dfrac{\pi}{2}\right)+o\left(\dfrac{1}{k^{2\nu_\gamma-\alpha}}\right)\\ \text{for } k\preceq 0.\end{cases}\end{aligned} \tag{3.64}$$

From (3.64) we have in that case $|k|^{2\nu_\gamma-\alpha}\Re\lambda_k\sim\beta$, with

$$\beta=-\frac{\varrho}{1-\gamma}\left(\frac{2-\gamma}{2}\right)^{1+\alpha}\frac{c^-_{\nu_\gamma,0}}{c^+_{\nu_\gamma,0}}\frac{1}{\pi^{2\nu_\gamma-\alpha}}\sin\nu_\gamma\pi\cos(1-\alpha)\frac{\pi}{2}.$$

The operator $\mathcal{A}$ has a non exponential decaying branch of eigenvalues for $\alpha\neq 2\nu_\gamma$. Thus the proof is complete.

Remark 3.5.1 *From Lemma 3.5.1, the operator $\mathcal{A}$ does not have eigenvalues on imaginary axis $i\mathbb{R}$.*

3.6 Optimality of energy decay when $\eta \neq 0$

By Lemma 3.5.1, the spectrum of $\mathcal{A}$ is at the left of the imaginary axis, but approaches this axis for $\alpha \neq 2\nu_\gamma$. Hence, the decay of the energy depends on the asymptotic behavior of the real part of these eigenvalues.

Unfortunately we were not able to prove this decay rate by frequency domain method based on multiplier method as the problem (P_2) is degenerate and the control is acting on the degenerate boundary.

To state and prove our stability results, we need some results from semigroup theory.

Theorem 3.6.1 ([11]) *Let $S(t)$ be a bounded C_0-semigroup on a Hilbert space $\mathcal{X}$ with generator $\mathcal{A}$. If*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \text{ and } \overline{\lim_{|\beta|\to\infty}} \frac{1}{\beta^l} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{X})} < \infty$$

for some l, then there exist c such that

$$\|e^{\mathcal{A}t}U_0\|^2 \leq \frac{c}{t^{\frac{2}{l}}} \|U_0\|^2_{D(\mathcal{A})}.$$

Theorem 3.6.2 ([3]) *Let $\mathcal{A}$ be the generator of a uniformly bounded C_0-semigroup $\{S(t)\}_{t\geq 0}$ on a Hilbert space $\mathcal{X}$. If:*

(i) $\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.

(ii) The intersection of the spectrum $\sigma(\mathcal{A})$ with $i\mathbb{R}$ is at most a countable set,

then the semigroup $\{S(t)\}_{t\geq 0}$ is asymptotically stable, i.e, $\|S(t)z\|_{\mathcal{X}} \to 0$ as $t \to \infty$ for any $z \in \mathcal{X}$.

In this section, by an explicit representation of the resolvent of the generator on the imaginary axis and the use of the Theorem by Borichev and Tomilov, we prove an optimal decay rate. Our main result is the following.

Theorem 3.6.3 *The C_0-semigroup $e^{t\mathcal{A}}$ is strongly stable in $\mathcal{H}$; i.e, for all $U_0 \in \mathcal{H}$, the solution of (3.10) satisfies*

$$\lim_{t\to\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0.$$

If $\eta \neq 0$, then the global solution of the problem (P_2) has the following energy decay property

$$E(t) = \|S_{\mathcal{A}}(t)U_0\|^2_{\mathcal{H}} \leq \begin{cases} \dfrac{c}{t^{\frac{2}{\alpha-2\nu_\gamma}}} \|U_0\|^2_{D(\mathcal{A})} & \text{if } \alpha > 2\nu_\gamma, \\ \dfrac{c}{t^{\frac{2}{2\nu_\gamma-\alpha}}} \|U_0\|^2_{D(\mathcal{A})} & \text{if } \alpha < 2\nu_\gamma, \\ ce^{-\omega t} \|U_0\|^2_{D(\mathcal{A})} & \text{if } \alpha = 2\nu_\gamma. \end{cases}$$

Moreover, the rate of energy decay is optimal for any initial data in $D(\mathcal{A})$.

Proof.
Let us consider the resolvent equation

$$\begin{cases} i\lambda u - v = f_1, \\ i\lambda v - (x^\gamma u_x)_x = f_2, \\ i\lambda \phi + (\xi^2+\eta)\phi - v(0)\mu(\xi) = f_3, \end{cases} \tag{3.65}$$

where $F = (f_1, f_2, f_3)^T \in \mathcal{H}$. From $(3.65)_1$ and $(3.65)_2$, we have

$$\lambda^2 u + (x^\gamma u_x)_x = -(f_2 + i\lambda f_1) \tag{3.66}$$

with

$$\begin{cases} (x^\gamma u_x)_{x=0} = \zeta \int_{-\infty}^{\infty} \mu(\xi)\phi(\xi)\, d\xi \\ u(1) = 0. \end{cases} \tag{3.67}$$

The substitution of ϕ given by $(3.65)_3$ into $(3.67)_1$ give us

$$(x^\gamma u_x)_{x=0} = \varrho(i\lambda+\eta)^{\alpha-1} v(0) + \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi) f_3(\xi)}{i\lambda + \xi^2 + \eta}\, d\xi. \tag{3.68}$$

Moreover, from $(3.65)_1$, we have

$$v(0) = i\lambda u(0) - f_1(0).$$

Then, the condition (3.68) become

$$(x^\gamma u_x)_{x=0} - \varrho i\lambda (i\lambda+\eta)^{\alpha-1} u(0) = -\varrho(i\lambda+\eta)^{\alpha-1} f_1(0) + \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi) f_3(\xi)}{i\lambda + \xi^2 + \eta}\, d\xi. \tag{3.69}$$

Assume that Φ is a solution of (3.66), then one easily checks that the function Ψ defined by

$$\Phi(x) = x^{\frac{1-\gamma}{2}} \Psi\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right) \tag{3.70}$$

is solution of the following inhomogeneous Bessel equation:

$$\begin{aligned} &y^2\Psi''(y) + y\Psi'(y) + \left(y^2 - \left(\frac{\gamma-1}{2-\gamma}\right)^2\right)\Psi(y) = \\ &-(\tfrac{2}{2-\gamma})^2 (\tfrac{2-\gamma}{2}\tfrac{1}{\lambda}y)^{\frac{3-\gamma}{2-\gamma}} \left(f_2\left((\tfrac{2-\gamma}{2}\tfrac{1}{\lambda}y)^{\frac{2}{2-\gamma}}\right) + i\lambda f_1\left((\tfrac{2-\gamma}{2}\tfrac{1}{\lambda}y)^{\frac{2}{2-\gamma}}\right)\right). \end{aligned} \tag{3.71}$$

The solution can be written as

$$\Psi(y) = AJ_{\nu_\gamma}(y) + BJ_{-\nu_\gamma}(y) + \frac{2\nu_\gamma}{\sin \nu_\gamma \pi} \int_0^y \frac{f(s)}{s} \left(J_{\nu_\gamma}(s)J_{-\nu_\gamma}(y) - J_{\nu_\gamma}(y)J_{-\nu_\gamma}(s)\right) ds.$$

Thus,

$$\begin{aligned} &u(x) = Ax^{\frac{1-\gamma}{2}} J_{\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right) + Bx^{\frac{1-\gamma}{2}} J_{-\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right) \\ &-\frac{2\nu_\gamma}{\sin\nu_\gamma\pi}\left(\frac{2}{2-\gamma}\right) x^{\frac{1-\gamma}{2}} \int_0^x s^{\frac{1-\gamma}{2}} (f_2(s) + i\lambda f_1(s)) \left(J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda s^{\frac{2-\gamma}{2}}\right) J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right)\right. \\ &\left. -J_{\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right) J_{-\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda s^{\frac{2-\gamma}{2}}\right)\right) ds. \end{aligned}$$

Therefore,

$$\begin{aligned} u(x) &= A\Phi_+(x) + B\Phi_-(x) \\ &\quad -\frac{2\nu_\gamma}{\sin\nu_\gamma\pi}\left(\frac{2}{2-\gamma}\right)\int_0^x (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi_-(x)-\Phi_+(x)\Phi_-(s))\,ds, \end{aligned} \tag{3.72}$$

where Φ_+ and Φ_- are defined by

$$\Phi_+(x) = x^{\frac{1-\gamma}{2}} J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right), \quad \Phi_-(x) = x^{\frac{1-\gamma}{2}} J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right). \tag{3.73}$$

From where it follows

$$\begin{aligned} u_x(x) &= A\Phi'_+(x) + B\Phi'_-(x) \\ &\quad -\frac{2\nu_\gamma}{\sin\nu_\gamma\pi}\left(\frac{2}{2-\gamma}\right)\int_0^x (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi'_-(x)-\Phi'_+(x)\Phi_-(s))\,ds \end{aligned} \tag{3.74}$$

From (3.69), (3.74) and (3.72), we conclude that

$$(1-\gamma)\tilde{c}^+_{\nu_\gamma,0}A - \varrho i\lambda(i\lambda+\eta)^{\alpha-1}\tilde{c}^-_{\nu_\gamma,0}B = -\varrho(i\lambda+\eta)^{\alpha-1}f_1(0) + \zeta\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_3(\xi)}{i\lambda+\xi^2+\eta}\,d\xi \tag{3.75}$$

$$A\Phi_+(1) + B\Phi_-(1) = \frac{2\nu_\gamma}{\sin\nu_\gamma\pi}\left(\frac{2}{2-\gamma}\right)\int_0^1 (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi_-(1)-\Phi_+(1)\Phi_-(s))\,ds \tag{3.76}$$

where

$$\tilde{c}^+_{\nu_\gamma,m} = c^+_{\nu_\gamma,m}\left(\frac{2}{2-\alpha}\lambda\right)^{2m+\nu_\gamma}, \quad \tilde{c}^+_{\nu_\gamma,m} = c^-_{\nu_\gamma,m}\left(\frac{2}{2-\alpha}\lambda\right)^{2m-\nu_\gamma}$$

and

$$\Phi_+(1) = J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda\right), \qquad \Phi_-(1) = J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda\right).$$

Using (3.75) and (3.76), a linear system in A and B is obtained

$$\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} C \\ \tilde{C} \end{pmatrix}, \tag{3.77}$$

where

$$\begin{aligned} r_{11} &= (1-\gamma)\tilde{c}^+_{\nu_\gamma,0}, \\ r_{12} &= -\varrho i\lambda(i\lambda+\eta)^{\alpha-1}\tilde{c}^-_{\nu_\gamma,0}, \\ r_{21} &= J_{\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda\right), \\ r_{22} &= J_{-\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda\right), \end{aligned}$$

$$\begin{aligned} C &= -\varrho(i\lambda+\eta)^{\alpha-1}f_1(0) + \zeta\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_3(\xi)}{i\lambda+\xi^2+\eta}\,d\xi, \\ \tilde{C} &= \frac{2\nu_\gamma}{\sin\nu_\gamma\pi}\left(\frac{2}{2-\gamma}\right)\int_0^1 (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi_-(1)-\Phi_+(1)\Phi_-(s))\,ds. \end{aligned}$$

Let the determinant of the linear system given in (3.77) be denoted by D. Then

$$
\begin{aligned}
D &= (1-\gamma)\tilde{c}^{+}_{\nu_\gamma,0}J_{-\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda\right) + \varrho i\lambda(i\lambda+\eta)^{\alpha-1}\tilde{c}^{-}_{\nu_\gamma,0}J_{\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda\right)\\
&= (1-\gamma)c^{+}_{\nu_\gamma,0}\left(\tfrac{2}{2-\gamma}\right)^{\nu_\gamma}\lambda^{\nu_\gamma}\left[\left(\frac{2-\gamma}{\pi\lambda}\right)^{1/2}\cos\left(\tfrac{2}{2-\gamma}\lambda+\nu_\gamma\tfrac{\pi}{2}-\tfrac{\pi}{4}\right)+O\left(\frac{1}{\lambda^{5/2}}\right)\right]\\
&\quad+\varrho i\lambda(i\lambda+\eta)^{\alpha-1}c^{-}_{\nu_\gamma,0}\left(\tfrac{2}{2-\gamma}\right)^{-\nu_\gamma}\lambda^{-\nu_\gamma}\left[\left(\frac{2-\gamma}{\pi\lambda}\right)^{1/2}\cos\left(\tfrac{2}{2-\gamma}\lambda-\nu_\gamma\tfrac{\pi}{2}-\tfrac{\pi}{4}\right)+O\left(\frac{1}{\lambda^{5/2}}\right)\right]\\
&= (1-\gamma)c^{+}_{\nu_\gamma,0}\left(\tfrac{2}{2-\gamma}\right)^{\nu_\gamma}\left(\frac{2-\gamma}{\pi}\right)^{1/2}\lambda^{\nu_\gamma-\frac{1}{2}}\cos\left(\tfrac{2}{2-\gamma}\lambda+\nu_\gamma\tfrac{\pi}{2}-\tfrac{\pi}{4}\right)\\
&\quad+\varrho i^{\alpha}c^{-}_{\nu_\gamma,0}\left(\tfrac{2}{2-\gamma}\right)^{-\nu_\gamma}\left(\frac{2-\gamma}{\pi}\right)^{1/2}\lambda^{\alpha-\nu_\gamma-\frac{1}{2}}\cos\left(\tfrac{2}{2-\gamma}\lambda-\nu_\gamma\tfrac{\pi}{2}-\tfrac{\pi}{4}\right)+O\left(\frac{1}{\lambda^{3/2+\nu_\gamma-\alpha}}\right).
\end{aligned}
$$

As $D \neq 0$ for all $\lambda \neq 0$, then A and B are uniquely determined by (3.77). Hence the operator $(i\lambda I - \mathcal{A})$ is surjective for all $\lambda \neq 0$. Moreover for $\lambda = 0$ and $\eta \neq 0$, using Lax-Milgram Theorem, we can deduce that the operator $\mathcal{A}$ is surjective. Taking account of Remark 3.5.1 and from Theorem 3.6.2 The C_0-semigroup $e^{t\mathcal{A}}$ is strongly stable in $\mathcal{H}$.

Now, it is easy to prove that

$$
|D| \geq \begin{cases} c|\lambda|^{\nu_\gamma-1/2} \text{ for large } \lambda \text{ if } \alpha \geq 2\nu_\gamma,\\ c|\lambda|^{\alpha-\nu_\gamma-1/2} \text{ for large } \lambda \text{ if } \alpha \leq 2\nu_\gamma.\end{cases}
$$

Now

$$
A = \frac{1}{D}(Cr_{22} - \tilde{C}r_{12})
$$

$$
B = \frac{1}{D}(-Cr_{21} + \tilde{C}r_{11})
$$

Considering only the dominant terms of λ, the following is obtained:

$$
\begin{aligned}
|D||A| &\leq c_1|\lambda|^{\alpha-\frac{3}{2}} + c_2|\lambda|^{\alpha-\nu_\gamma-1} \leq c_3|\lambda|^{\alpha-\nu_\gamma-1},\\
|D||B| &\leq c_1|\lambda|^{\alpha-\frac{3}{2}} + c_2|\lambda|^{\nu_\gamma-1} \leq \begin{cases} c_3|\lambda|^{\alpha-\nu_\gamma-1} \text{ if } \alpha > 2\nu_\gamma,\\ \tilde{c}_3|\lambda|^{\nu_\gamma-1} \text{ if } \alpha < 2\nu_\gamma,\end{cases}
\end{aligned}
$$

where we have used the fact that $f_1 \in H^1_{0,\gamma}(0,1)$ and

$$
\left|\int_0^1 (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi_-(1)-\Phi_+(1)\Phi_-(s))\,ds\right| \leq \frac{1}{|\lambda|}\left(\|f_1\|_{H^1_{0,\gamma}(0,1)} + \|f_2\|_{L^2(0,1)}\right),
$$

$$
\left|\int_0^1 (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi'_-(1)-\Phi'_+(1)\Phi_-(s))\,ds\right| \leq \left(\|f_1\|_{H^1_{0,\gamma}(0,1)} + \|f_2\|_{L^2(0,1)}\right).
$$

Then, we conclude that

$$
|A| \leq \begin{cases} c|\lambda|^{\alpha-2\nu_\gamma-\frac{1}{2}} \text{ if } \text{ if } \alpha > 2\nu_\gamma,\\ c|\lambda|^{-1/2} \text{ if } \text{ if } \alpha < 2\nu_\gamma,\end{cases}
$$

$$
|B| \leq \begin{cases} c|\lambda|^{\alpha-2\nu_\gamma-\frac{1}{2}} \text{ if } \text{ if } \alpha > 2\nu_\gamma,\\ c|\lambda|^{2\nu_\gamma-\alpha-\frac{1}{2}} \text{ if } \text{ if } \alpha < 2\nu_\gamma.\end{cases}
$$

Then

$$\|u\|_{L^2(0,1)} \leq \begin{cases} c|\lambda|^{\alpha-2\nu_\gamma-1}\left(\|f_1\|_{H^1_{0,\gamma}(0,1)}+\|f_2\|_{L^2(0,1)}\right) & \text{if } \alpha > 2\nu_\gamma, \\ c|\lambda|^{2\nu_\gamma-\alpha-1}\left(\|f_1\|_{H^1_{0,\gamma}(0,1)}+\|f_2\|_{L^2(0,1)}\right) & \text{if if } \alpha < 2\nu_\gamma. \end{cases}$$

Using $(3.65)_1$ and (3.72), we get

$$\|v\|_{L^2(0,1)} \leq \begin{cases} c|\lambda|^{\alpha-2\nu_\gamma}\left(\|f_1\|_{H^1_{0,\gamma}(0,1)}+\|f_2\|_{L^2(0,1)}\right) & \text{if } \alpha > 2\nu_\gamma, \\ c|\lambda|^{2\nu_\gamma-\alpha}\left(\|f_1\|_{H^1_{0,\gamma}(0,1)}+\|f_2\|_{L^2(0,1)}\right) & \text{if if } \alpha < 2\nu_\gamma. \end{cases}$$

From (3.73) and (3.35), we have

$$\begin{cases} x^{\gamma/2}\Phi'_+(x) &=& (\frac{1-\gamma}{2}+\frac{2\nu_\gamma}{2-\gamma})x^{-1/2}J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda\right)-\lambda x^{\frac{1-\gamma}{2}}J_{1+\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda\right), \\ x^{\gamma/2}\Phi'_-(x) &=& (\frac{1-\gamma}{2}-\frac{2\nu_\gamma}{2-\gamma})x^{-1/2}J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda\right)-\lambda x^{\frac{1-\gamma}{2}}J_{1-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda\right). \end{cases}$$

Then from (3.74), we can get

$$\|x^{\gamma/2}u_x\|_{L^2(0,1)} \leq \begin{cases} c|\lambda|^{\alpha-2\nu_\gamma}\left(\|f_1\|_{H^1_{0,\gamma}(0,1)}+\|f_2\|_{L^2(0,1)}\right) & \text{if } \alpha > 2\nu_\gamma, \\ c|\lambda|^{2\nu_\gamma-\alpha}\left(\|f_1\|_{H^1_{0,\gamma}(0,1)}+\|f_2\|_{L^2(0,1)}\right) & \text{if } \alpha < 2\nu_\gamma. \end{cases}$$

Moreover from $(3.65)_3$, we have

$$\|\phi\|^2_{L^2(-\infty,\infty)} \leq \int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi)|^2\,d\xi \leq c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

Thus, we conclude that

$$\|(i\lambda I-\mathcal{A})^{-1}\|_{\mathcal{H}} \leq \begin{cases} c|\lambda|^{\alpha-2\nu_\gamma} & \text{as } |\lambda|\to\infty & \text{if } \alpha > 2\nu_\gamma \\ c|\lambda|^{2\nu_\gamma-\alpha} & \text{as } |\lambda|\to\infty & \text{if } \alpha < 2\nu_\gamma \\ c & \text{as } |\lambda|\to\infty & \text{if } \alpha = 2\nu_\gamma \end{cases}$$

The conclusion then follows by applying Theorem 3.6.1 for $\alpha \neq 2\nu_\gamma$ and Theorem 3.5.1 for $\alpha = 2\nu_\gamma$.

Besides, we prove that the decay rate is optimal. Indeed, the decay rate is consistent with the asymptotic expansion of eigenvalues.

□

3.7 Future works

1) It seems to be interesting to develop some multipliers method to treat the following problem (also in the case $a(x) = x^\gamma$)

$$(P) \qquad \begin{cases} u_{tt}(x,t)-(a(x)u_x(x,t))_x = 0 & \text{in } (0,1)\times(0,+\infty), \\ (a(x)u_x)(0,t) = \varrho\partial_t^{\alpha,\eta}u(0,t) & \text{in } (0,+\infty), \\ u(1,t) = 0 & \text{in } (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } (0,1). \end{cases}$$

Here a is weakly degenerate at $x = 0$ in the sense that

$$\int_0^1 \frac{1}{a(s)}\, ds < +\infty.$$

Moreover, an explicit representation need to develop some tools similar to Bessel equations. This is an interesting problem.

2) More general problem is the following

$$\begin{cases} u_{tt}(x,t) - M(\|\sqrt{a(x)}u_x\|^2_{L^2(0,L)})(a(x)u_x)_x(x,t) = 0 & \text{in } (0,1)\times(0,+\infty) \\ M(\|\sqrt{a(x)}u_x\|^2_{L^2(0,1)})(a(x)u_x)(0,t) = \varrho\partial_t^{\alpha,\eta}u(0,t) & \text{in } (0,+\infty) \\ u(1,t) = 0 & \text{in } (0,+\infty) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } (0,1). \end{cases}$$

The problem of global existence and energy decay is open. It is clear that the energy decay rate depends on the order of degeneracy of M, a and the parameter α.

Chapter 4

STABILIZATION OF DEGENERATE WAVE EQUATION UNDER DYNAMIC FRACTIONAL FEEDBACK ACTING ON THE DEGENERATE BOUNDARY

4.1 Introduction

In this chapter, we are concerned with the dynamic boundary stabilization of fractional type for degenerate wave equation of the form

$$(P_3) \qquad \begin{cases} u_{tt}(x,t) - (x^\gamma u_x(x,t))_x = 0 & \text{in } (0,1) \times (0,+\infty), \\ -mu_{tt}(0,t) + (x^\gamma u_x)(0,t) = \varrho \partial_t^{\alpha,\eta} u(0,t) & \text{in } (0,+\infty), \\ u(1,t) = 0 & \text{in } (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } (0,1), \end{cases}$$

where $(x,t) \in (0,1) \times (0,+\infty), \gamma \in [0,1), m > 0$ and $\varrho > 0$. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α, $(0 < \alpha \leq 1)$, with respect to the time variable (see **[20]**). It is defined as follows

$$\partial_t^{\alpha,\eta} w(t) = \begin{cases} w_t(t) & \text{for } \alpha = 1, \eta \geq 0, \\ \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\, ds, & \text{for } 0 < \alpha < 1, \eta \geq 0. \end{cases}$$

The problem (P_3) describes the motion of a pinched vibration cable with tip mass $m > 0$ (see **[41]** and **[27]**), where, for simplicity, the wave speed is chosen to be unity and a subscript letter denotes a partial differential with respect to the corresponding variable.

The bibliography of works concerning the stabilization of nondegenerate wave equation with different types of damping is truly long (see e.g. **[21]**, **[22]**, **[19]** and the references therein).

In **[22]**, for $a(x) = a_1 x + a_0$: the authors have established asymptotic stabilization with the following boundary damping

$$\begin{cases} (au_x)(0,t) = 0, \\ (au_x)(1,t) = -ku(1,t) - u_t(1,t), \ k > 0. \end{cases}$$

In **[19]**, the authors considered the following modelization of a flexible torque arm controlled by two feedbacks depending only on the boundary velocities:

$$\begin{cases} u_{tt}(x,t) - (a(x)u_x)_x + \alpha u_t(x,t) + \beta y(x,t) = 0, \ 0 < x < 1, \ t > 0, \\ (a(x)u_x)(0) = k_1 u_t(0,t), t > 0, \\ (a(x)u_x)(1) = -k_2 u_t(1,t), t > 0, \end{cases}$$

where

$$\begin{cases} \alpha \geq 0, \beta > 0, k_1, k_2 \geq 0, k_1 + k_2 \neq 0, \\ a \in W^{1,\infty}(0,1), a(x) \geq a_0 \text{ for all } x \in [0,1]. \end{cases}$$

They proved the exponential decay of the solutions.

Let us mention here that the case $\alpha = 1$ in (CF) corresponds to a boundary damping and it has been extensively studied by many authors (see, for instance, **[37]**, **[25]**, and references therein). It has been proved, in particular that solutions exist globally with an optimal decay rate that is $E(t) \sim c/t$ by using Riesz basis property of the generalized eigenvector of the system.

Recently in **[9]**, Benaissa and Benkhedda considered the stabilization for the following wave equation with dynamic boundary control of fractional derivative type (CF):

$$(PF) \qquad \begin{cases} y_{tt}(x,t) - y_{xx}(x,t) = 0 & \text{in } (0,L) \times (0,+\infty) \\ y(0,t) = 0 & \text{in } (0,+\infty) \\ my_{tt}(L,t) + y_x(L,t) = -\gamma \partial_t^{\alpha,\eta} y(L,t) & \text{in } (0,+\infty) \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x) & \text{in } (0,L) \end{cases}$$

They proved that the decay of the energy is not exponential but polynomial that is $E(t) \leq C1/t^{(2-\alpha)}$.

Very recently in **[18]**, Benaissa and Benkhedda considered the stabilization for the following wave equation with a general dynamic boundary control of diffusive type (CF):

$$(P) \qquad \begin{cases} y_{tt}(x,t) - y_{xx}(x,t) = 0 & \text{in } (0,L) \times (0,+\infty) \\ y(0,t) = 0 & \text{in } (0,+\infty) \\ my_{tt}(L,t) + y_x(L,t) = -\zeta \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,t)\, d\xi & \text{in } (0,+\infty) \\ \partial_t \phi(\xi,t) + (\xi^2 + \eta)\phi(\xi,t) - y_t(L,t)\mu(\xi) = 0 & \text{in } (-\infty,\infty) \times (0,+\infty) \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x) & \text{in } (0,L) \\ \phi(\xi,0) = \phi_0 & \text{in } (-\infty,\infty) \end{cases}$$

They proved that the decay of the energy is not exponential. Moreover, they obtained a precise and optimal energy decay estimate for a general control of diffusive type, from which the usual control of fractional derivative type is a special case.

On the contrary, when the coefficient $a(x)$ is degenerate very little is known in the literature, even though many problems that are relevant for applications are described by hyperbolic equations degenerating at the boundary of the space domain (see **[8]** and **[2]**). In **[2]**, Alabau et al. **[2]** studied the degenerate wave equation of the type

$$u_{tt}(x,t) - (a(x)u_x(x,t))_x = 0 \text{ in } (0,1) \times (0,+\infty), \tag{4.1}$$

where the coefficient a is a positive function on $]0,1]$ but vanishes at zero. The degeneracy of (4.1) at $x = 0$ is measured by the parameter μ_a defined by

$$\mu_a = \sup_{0<x\leq 1} \frac{x|a'(x)|}{a(x)} \tag{4.2}$$

and the initial conditions are

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \tag{4.3}$$

followed by the boundary conditions

$$(P1) \qquad \begin{cases} \begin{cases} u(0,t) = 0 & \text{if } 0 \leq \mu_a < 1 \\ (au_x)(0,t) = 0 & \text{if } 1 \leq \mu_a < 2 \end{cases} & \text{in } (0,+\infty), \\ u_x(1,t) + u_t(1,t) + \beta u(1,t) = 0 & \text{in } (0,+\infty), \end{cases}$$

they obtained exponential stability of the solutions.

Very recently in **[8]**, Benaissa and Aichi considered the stabilization of the problem (4.1)-(P1) but with a feed back of fractional time derivative type instead of the feedback of the usual time derivative type. They proved an optimal polynomial decay rate. It is proved that the presence of a degenerate coefficient has no effect on the stabilization results in **[2]**) and **[8]**.

Here we want to focus on the following remarks:

- System (4.1), (4.3) and $(P1)$ under study is different from one studied on **[2]**. Indeed, the control is located at $x = 0$.
- The fractional velocity feedbacks considered here provide a weaker damping than the velocity feedbacks (see **[39]**).
- The explicit representation of the resolvent gives us a sharp polynomial decay rate, however in **[2]**, stabilization is done under the classical energy method based on multiplier techniques (see **[33]**). Unfortunately, this method does not seem to be applicable in the case of damping acting at $x = 0$.

In this work, we explain the influence of the relation between the degenerate coefficient and the fractional feedback on decay estimates.

This chapter is organized as follows. In section 2, we give preliminaries results and we reformulate the system (P_3) into an augmented system by coupling the degenerate wave equation with a suitable diffusion equation and we show the well-posedness of our problem by semigroup theory. In section 3, we prove lack of exponential stability by spectral analysis by using Bessel functions. In the last section, we prove an optimal decay rate. The proof heavily relies on Bessel equations and Borichev-Tomilov Theorem.

4.2 Preliminaries results

Now, we introduce, as in **[15]** or **[2]**, the following weighted Sobolev spaces:

$$H^1_{0,\gamma}(0,1) = \Big\{u \text{ is locally absolutely continuous in } (0,1] : x^{\gamma/2}u_x \in L^2(0,1)/\ u(1)=0\Big\}$$

$$H^1_{\gamma}(0,1) = \Big\{u \text{ is locally absolutely continuous in } (0,1] : x^{\gamma/2}u_x \in L^2(0,1)\Big\}.$$

We remark that $H^1_\gamma(0,1)$ is a Hilbert space with the scalar product

$$(u,v)_{H^1_\gamma(0,1)} = \int_0^1 (u\overline{v} + x^\gamma u'(x)\overline{v'(x)})\,dx, \quad \forall u,v \in H^1_\gamma(0,1).$$

Let us also set

$$|u|_{H^1_{0,\gamma}(0,1)} = \left(\int_0^1 x^\gamma |u'(x)|^2\,dx\right)^{1/2} \quad \forall u \in H^1_\gamma(0,1).$$

Actually, $|\cdot|_{H^1_{0,\gamma}(0,1)}$ is an equivalent norm on the closed subspace $H^1_{0,\gamma}(0,1)$ to the norm of $H^1_\gamma(0,1)$. This fact is a simple consequence of the following version of Poincaré's inequality.

Proposition 4.2.1 *There is a positive constant $C_* = C(\gamma)$ such that*

$$\|u\|^2_{L^2(\Omega)} \leq C_*|u|^2_{1,\gamma} \quad \forall u \in H^1_{0,\gamma}(0,1). \tag{4.4}$$

Proof. Let $u \in H^1_{0,\gamma}(0,1)$. For any $x \in]0,1]$ we have that

$$|u(x)| = \left|\int_x^1 u'(s)\,ds\right| \leq |u|_{1,\gamma}\left\{\int_0^1 \frac{1}{x^\gamma}\,dx\right\}^{1/2}.$$

Therefore

$$\int_0^1 |u(x)|^2\,dx \leq \frac{1}{1-\gamma}|u|^2_{1,\gamma}.$$

Next, we define

$$H^2_\gamma(0,1) = \{u \in H^1_\gamma(0,1) :\ x^\gamma u' \in H^1(0,1)\},$$

where $H^1(0,1)$ denotes the classical Sobolev space.

Remark 4.2.1 *Notice that if $u \in H^2_\gamma(0,1), \gamma \in [1,2)$,we have $(x^\gamma u_x)_{x=0} \equiv 0$ since $1/x^\gamma$ is not integrable over neighborhoods of 0. Hence the problem is not well-posed in terms of the semigroups in the Hilbert space.*

4.2.1 Augmented model

In this section we reformulate (P_3) into an augmented system. For that, we need the following proposition.

Proposition 4.2.2 (see [39]) *Let μ be the function:*

$$\mu(\xi) = |\xi|^{(2\alpha-1)/2}, \quad -\infty < \xi < +\infty, \ 0 < \alpha < 1. \tag{4.5}$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t \phi(\xi, t) + (\xi^2 + \eta)\phi(\xi, t) - U(t)\mu(\xi) = 0, \quad -\infty < \xi < +\infty, \eta \geq 0, t > 0, \tag{4.6}$$

$$\phi(\xi, 0) = 0, \tag{4.7}$$

$$O(t) = (\pi)^{-1} \sin(\alpha\pi) \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi, t)\, d\xi \tag{4.8}$$

is given by

$$O = I^{1-\alpha,\eta} U. \tag{4.9}$$

where

$$[I^{\alpha,\eta} f](t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{-\eta(t-\tau)} f(\tau)\, d\tau$$

Lemma 4.2.1 (see [1]) *If $\lambda \in D_\eta = \mathbb{C} \backslash]-\infty, -\eta]$ then*

$$F(\lambda) = \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2}\, d\xi = \frac{\pi}{\sin \alpha\pi} (\lambda + \eta)^{\alpha - 1}.$$

Using now Proposition 4.2.2 and relation (4.9), system (P_3) may be recast into the following augmented system

$$(P') \qquad \begin{cases} u_{tt}(x,t) - (x^\gamma u_x(x,t))_x = 0, \\ \phi_t(\xi, t) + (\xi^2 + \eta)\phi(\xi, t) - u_t(0,t)\mu(\xi) = 0, \qquad -\infty < \xi < +\infty, \quad t > 0, \\ -m u_{tt}(0,t) + (x^\gamma u_x)(0,t) = \zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,t)\, d\xi, \\ u(1,t) = 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \end{cases}$$

where $\zeta = \varrho(\pi)^{-1} \sin(\alpha\pi)$.

4.3 Well-posedness

In this section, we give an existence and uniqueness result for problem (P_3) using the semigroup theory. To define the semigroup associated with (P_3), we consider the right-end boundary condition

$$u_t(0,t) = \theta(t), t > 0,$$

where v solve the equation

$$-m\theta_t(t) + (x^\gamma u_x)(0,t) - \zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,t)\, d\xi = 0. \tag{4.10}$$

with the initial condition

$$\theta(0) = u_1(0) = \theta_0. \tag{4.11}$$

Let us denote $U := (u, u_t, \phi, \theta)^T$, then U satisfies the following Cauchy problem:

$$\partial_t U = \mathcal{A}U, \qquad U(0) = U_0, \tag{4.12}$$

where $U_0 = (u_0, u_1, 0, \theta_0)^T$ and $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is given by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ \phi \\ \theta \end{pmatrix} = \begin{pmatrix} v \\ (x^\gamma u_x)_x \\ -(\xi^2+\eta)\phi + v(0)\mu(\xi) \\ \dfrac{1}{m}(x^\gamma u_x)(0) - \dfrac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\,d\xi \end{pmatrix}, \tag{4.13}$$

We introduce the Hilbert space $\mathcal{H} = H^1_{0,\gamma}(0,1) \times L^2(0,1) \times L^2(-\infty,+\infty) \times \mathbb{C}$ with inner product

$$\left\langle \begin{pmatrix} u \\ v \\ \phi \\ \theta \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\phi} \\ \tilde{\theta} \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_0^1 x^\gamma u_x \overline{\tilde{u}}_x dx + \int_0^1 v\overline{\tilde{v}} dx + \zeta \int_{-\infty}^{+\infty} \phi\overline{\tilde{\phi}}\, d\xi + m\theta\overline{\tilde{\theta}}.$$

The domain of the operator $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (u,v,\phi) \text{ in } \mathcal{H} : u \in H^2_\gamma(0,1) \cap H^1_{0,\gamma}(0,1), v \in H^1_{0,\gamma}(0,1), \theta \in \mathbb{C} \\ -(\xi^2+\eta)\phi + v(0)\mu(\xi) \in L^2(-\infty,+\infty), v(0) = \theta \\ |\xi|\phi \in L^2(-\infty,+\infty) \end{array} \right\}. \tag{4.14}$$

Our main result is giving by the following theorem.

Theorem 4.3.1 *The operator $\mathcal{A}$ defined by (4.13) and (4.14), generates a C_0-semigroup of contractions $e^{t\mathcal{A}}$ in the Hilbert space $\mathcal{H}$.*

Proof.
To prove this result we shall use the Lumer-Phillips' theorem. Since for every $U = (u,v,\phi) \in D(\mathcal{A})$ we have

$$\Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\zeta \int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi)|^2\, d\xi. \tag{4.15}$$

then the operator $\mathcal{A}$ is dissipative.

Let $\lambda > 0$. we prove that the operator $(\lambda I - \mathcal{A})$ is a surjection. In other words, we shall demonstrate that given any triplet $F = (f_1, f_2, f_3) \in \mathcal{H}$, there is an other triplet $U = (u,v,\phi) \in D(\mathcal{A})$ such that

$$(\lambda I - \mathcal{A})U = F. \tag{4.16}$$

Equation (4.16) is equivalent to

$$\begin{cases} \lambda u - v = f_1, \\ \lambda v - (x^\gamma u_x)_x = f_2, \\ \lambda\phi + (\xi^2+\eta)\phi - v(0)\mu(\xi) = f_3. \\ \lambda\theta - \frac{1}{m}(x^\gamma u_x)(0) + \frac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\,d\xi = f_4. \end{cases} \tag{4.17}$$

Suppose u is found with the appropriate regularity. Then, $(4.17)_1$ and $(4.17)_3$ yield

$$v = \lambda u - f_1 \in H^1_{0,\gamma}(0,1), \tag{4.18}$$

Using equations $(4.17)_3, (4.17)_1$ and the fact that $\eta \geq 0$, we get

$$\phi(\xi) = \frac{f_3(\xi)}{\xi^2+\eta+\lambda} + \frac{\lambda u(0)\mu(\xi)}{\xi^2+\eta+\lambda} - \frac{f_1(0)\mu(\xi)}{\xi^2+\eta+\lambda}. \tag{4.19}$$

By using (4.17) and (4.18) it can easily be shown that u satisfies

$$\lambda^2 u - (x^\gamma u_x)_x = f_2 + \lambda f_1. \tag{4.20}$$

Solving equation (4.20) is equivalent to finding $u \in H^2_\gamma(0,1) \cap H^1_{0,\gamma}(0,1)$ such that

$$\int_0^1 (\lambda^2 u\overline{w} - (x^\gamma u_x)_x \overline{w})\, dx = \int_0^1 (f_2 + \lambda f_1)\overline{w}\, dx, \tag{4.21}$$

for all $w \in H^1_{0,\gamma}(0,1)$. By using (4.21), the boundary condition $(4.14)_3$ and (4.19) the function u satisfying the following equation

$$\begin{aligned}&\int_0^1 (\lambda^2 u\overline{w} + (x^\gamma u_x)\overline{w}_x)\, dx + (\tilde{\zeta} + m\lambda)v(0)\overline{w}(0)\\ &\quad = \int_0^1 (f_2+\lambda f_1)\overline{w}\, dx - \zeta\int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_3(\xi)\, d\xi\overline{w}(0) - mf_4\overline{w}(0)\end{aligned} \tag{4.22}$$

where $\tilde{\zeta} = \zeta \displaystyle\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2+\eta+\lambda}\, d\xi$. Using again (4.18), we deduce that

$$v(0) = \lambda u(0) - f_1(0). \tag{4.23}$$

Inserting (4.23) into (4.22), we get

$$\left\{\begin{aligned}&\int_0^1 (\lambda^2 u\overline{w} + x^\gamma u_x\overline{w}_x)\, dx + \lambda(m\lambda + \tilde{\zeta})u(0)\overline{w}(0)\\ &= \int_0^1 (f_2+\lambda f_1)\overline{w}\, dx - \zeta\int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_3(\xi)\, d\xi\overline{w}(0) + (m\lambda+\tilde{\zeta})f_1(0)\overline{w}(0) - mf_4\overline{w}(0).\end{aligned}\right. \tag{4.24}$$

Problem (4.24) is of the form

$$\mathcal{B}(u,w) = \mathcal{L}(w), \tag{4.25}$$

where $\mathcal{B} : [H^1_{0,\gamma}(0,1) \times H^1_{0,\gamma}(0,1)] \to \mathbb{C}$ is the bilinear form defined by

$$\mathcal{B}(u,w) = \int_0^1 (\lambda^2 u\overline{w} + x^\gamma u_x\overline{w}_x)\, dx + \lambda(m\lambda+\tilde{\zeta})u(0)\,\overline{w}(0)$$

and $\mathcal{L} : H^1_{0,\gamma}(0,1) \to \mathbb{C}$ is the linear functional given by

$$\mathcal{L}(w) = \int_0^1 (f_2+\lambda f_1)\overline{w}\, dx - \zeta\int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_3(\xi)\, d\xi\overline{w}(0) + (m\lambda+\tilde{\zeta})f_1(0)\overline{w}(0) - mf_4\overline{w}(0).$$

It is easy to verify that $\mathcal{B}$ is continuous and coercive, and $\mathcal{L}$ is continuous. Consequently, by the Lax-Milgram Lemma, system (4.25) has a unique solution $u \in H^1_{0,\gamma}(0,1)$. By the regularity theory for the linear elliptic equations, it follows that $u \in H^2_\gamma(0,1)$. Therefore, the operator $\lambda I - \mathcal{A}$ is surjective for any $\lambda > 0$.

$\square$

As a consequence of Theorem 4.3.1, the system (P_3') is well-posed in the energy space $\mathcal{H}$ and we have the following proposition.

Proposition 4.3.1 *For $(u_0, u_1, 0, \theta_0) \in \mathcal{H}$, the problem (P_3') admits a unique weak solution*

$$(u, u_t, \phi, \theta) \in C^0(\mathbb{R}_+, \mathcal{H}).$$

and for $(u_0, u_1, 0, \phi_0) \in D(\mathcal{A})$, the problem (P_3') admits a unique strong solution

$$(u, u_t, \phi, \theta) \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

Moreover, from the density $D(\mathcal{A})$ in $\mathcal{H}$ the energy of $(u(t), \phi(t))$ at time $t \geq 0$ by

$$E(t) = \frac{1}{2}\int_0^1 (|u_t|^2 + x^\gamma |u_x|^2)dx + \frac{m}{2}|u_t(L,t)|^2 + \frac{\zeta}{2}\int_{-\infty}^{+\infty} |\phi(\xi,t)|^2\, d\xi \tag{4.26}$$

decays as follow

$$E'(t) = -\zeta \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi(\xi,t)|^2\, d\xi \leq 0. \tag{4.27}$$

Proof of Proposition 4.3.1. Noting that the regularity of the solution of the problem (P_3') is consequence of the semigroup properties. We have just to prove (4.27).

Multiplying the first equation in (P_3') by $\overline{u}_t$, integrating over $(0,1)$ and using integration by parts, we get

$$\int_0^1 u_{tt}(x,t)\overline{u}_t dx - \int_0^1 (x^\gamma u_x(x,t))_x \overline{u}_t dx = 0.$$

Then

$$\frac{d}{dt}\left(\frac{1}{2}\int_0^1 |u_t(x,t)|^2 dx\right) + \frac{1}{2}\frac{d}{dt}\int_0^1 x^\gamma |u_x(x,t)|^2\, dx - \Re\Big[(x^\gamma u_x)(x,t)\overline{u}_t\Big]_0^1 = 0.$$

Then

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left(|u_t(x,t)|^2 + x^\gamma |u_x(x,t)|^2\right) dx + \frac{m}{2}|u_t(0,t)|^2 + \zeta \Re \overline{u}_t(0,t)\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,t)\, d\xi = 0. \tag{4.28}$$

Multiplying the second equation in (P_3') by $\zeta\overline{\phi}$ and integrating over $(-\infty, +\infty)$, to obtain:

$$\zeta\int_{-\infty}^{+\infty} \phi_t(\xi,t)\overline{\phi}(\xi,t)d\xi + \zeta\int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi,t)|^2 d\xi - \zeta u_t(0,t)\int_{-\infty}^{+\infty} \mu(\xi)\overline{\phi}(\xi,t)d\xi = 0.$$

Hence

$$\frac{\zeta}{2}\frac{d}{dt}\int_{-\infty}^{+\infty} |\phi(\xi,t)|^2 d\xi + \zeta\int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi,t)|^2 d\xi - \zeta\Re u_t(0,t)\int_{-\infty}^{+\infty} \mu(\xi)\overline{\phi}(\xi,t)d\xi = 0. \tag{4.29}$$

From (4.26), (4.28) and (4.29) we get

$$E'(t) = -\zeta \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi(\xi, t)|^2 \, d\xi \leq 0.$$

This completes the proof of the lemma.

□

4.4 Lack of exponential stability

This section will be devoted to the study of the lack of exponential decay of solutions associated with the system (4.12). In order to state and prove our stability results, we need some lemmas.

Theorem 4.4.1 ([43]) *Let $S(t)$ be a C_0-semigroup of contractions on Hilbert space $\mathcal{X}$ with generator $\mathcal{A}$. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R}$$

and

$$\overline{\lim_{|\beta| \to \infty}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{X})} < \infty.$$

Our main result is the following.

Theorem 4.4.2 *The semigroup generated by the operator $\mathcal{A}$ is not exponentially stable.*

Proof. We will examine two cases.
•**Case 1** $\eta = 0$: We shall show that $i\lambda = 0$ is not in the resolvent set of the operator $\mathcal{A}$. Indeed, noting that $(\sin(x-1), 0, 0, 0)^T \in \mathcal{H}$, and denoting by $(u, v, \phi, \theta)^T$ the image of $(\sin(x-1), 0, 0, 0)^T$ by $\mathcal{A}^{-1}$, we see that $\phi(\xi) = |\xi|^{\frac{2\alpha-5}{2}} \sin 1$. But, then $\phi \notin L^2(-\infty, +\infty)$, since $\alpha \in]0, 1[$. So $(u, v, \phi, \theta)^T \notin D(\mathcal{A})$.
• **Case 2** $\eta \neq 0$:

We aim to show that an infinite number of eigenvalues of $\mathcal{A}$ approach the imaginary axis which prevents the system (P) from being exponentially stable. Indeed we first compute the characteristic equation that gives the eigenvalues of $\mathcal{A}$. Let λ be an eigenvalue of $\mathcal{A}$ with associated eigenvector $U = (u, v, \phi, \theta)^T$. Then $\mathcal{A}U = \lambda U$ is equivalent to

$$\begin{cases} \lambda u - v = 0, \\ \lambda v - (x^\gamma u_x)_x = 0, \\ \lambda \phi + (\xi^2 + \eta)\phi - v(0)\mu(\xi) = 0. \\ \lambda \theta - \frac{1}{m}(x^\gamma u_x)(0) + \frac{\zeta}{m} \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi) \, d\xi = 0 \end{cases} \tag{4.30}$$

It is well-known that Bessel functions play an important role in this type of problem. From $(4.30)_1 - (4.30)_2$ for such λ, we find

$$\lambda^2 u - (x^\gamma u_x)_x = 0. \tag{4.31}$$

Using the boundary conditions and $(4.30)_3$, we deduce that

$$\begin{cases} \lambda^2 u - (x^\gamma u_x)_x = 0, \\ (x^\gamma u_x)(0) - (m\lambda^2 + \varrho\lambda(\lambda+\eta)^{\alpha-1})u(0) = 0, \\ u(1) = 0. \end{cases} \tag{4.32}$$

Assume that u is a solution of (4.57) associated to eigenvalue $-\lambda^2$, then one easily checks that the function

$$u(x) = x^{\frac{1-\gamma}{2}} \Psi\left(\frac{2}{2-\gamma} i\lambda x^{\frac{2-\gamma}{2}}\right)$$

is a solution of the following problem:

$$y^2\Psi''(y) + y\Psi'(y) + (y^2 - (\frac{\gamma-1}{2-\gamma})^2)\Psi(y) = 0. \tag{4.33}$$

We have

$$u(x) = c_+\Phi_+ + c_-\Phi_-, \tag{4.34}$$

where Φ_+ and Φ_- are defined by

$$\Phi_+(x) := x^{\frac{1-\gamma}{2}} J_{\nu_\gamma}\left(\frac{2}{2-\gamma} i\lambda x^{\frac{2-\gamma}{2}}\right)$$

and

$$\Phi_-(x) := x^{\frac{1-\gamma}{2}} J_{-\nu_\gamma}\left(\frac{2}{2-\gamma} i\lambda x^{\frac{2-\gamma}{2}}\right),$$

where

$$J_\nu(y) = \sum_{m=0}^\infty \frac{(-1)^m}{m!\Gamma(m+\nu+1)}\left(\frac{y}{2}\right)^{2m+\nu} = \sum_{m=0}^\infty c^+_{\nu,m} y^{2m+\nu}, \tag{4.35}$$

$$J_{-\nu}(y) = \sum_{m=0}^\infty \frac{(-1)^m}{m!\Gamma(m-\nu+1)}\left(\frac{y}{2}\right)^{2m-\nu} = \sum_{m=0}^\infty c^-_{\nu,m} y^{2m-\nu} \tag{4.36}$$

$$\nu_\gamma = \frac{1-\gamma}{2-\gamma}$$

and J_{ν_γ} and $J_{-\nu_\gamma}$ are Bessel functions of the first kind of order ν_γ and $-\nu_\gamma$. As $\nu_\gamma \notin \mathbb{N}$, so J_{ν_γ} and $J_{-\nu_\gamma}$ are linearly independent and therefore the pair $(J_{\nu_\gamma}, J_{-\nu_\gamma})$ (classical result) forms a fundamental system of solutions (4.33).

Then, using the series expansion of J_{ν_α} and $J_{-\nu_\alpha}$, one obtains

$$\Phi_+(x) = \sum_{m=0}^\infty \tilde{c}^+_{\nu_\gamma,m} x^{1-\gamma+(2-\gamma)m}, \quad \Phi_-(x) = \sum_{m=0}^\infty \tilde{c}^-_{\nu_\gamma,m} x^{(2-\gamma)m},$$

with

$$\tilde{c}^+_{\nu_\gamma,m} = c^+_{\nu_\gamma,m}\left(\frac{2}{2-\alpha} i\lambda\right)^{2m+\nu_\gamma}, \quad \tilde{c}^+_{\nu_\gamma,m} = c^-_{\nu_\gamma,m}\left(\frac{2}{2-\alpha} i\lambda\right)^{2m-\nu_\gamma}.$$

Next one easily verifies that $\Phi_+, \Phi_- \in H^1_{0,\gamma}(0,1)$: indeed,

$$\begin{array}{ll}\Phi_+(x) \sim_0 \tilde{c}^+_{\nu_\gamma,0} x^{1-\gamma}, & x^{\gamma/2}\Phi'_+(x) \sim_0 (1-\gamma)\tilde{c}^+_{\nu_\gamma,0} x^{-\gamma/2},\\ \Phi_-(x) \sim_0 \tilde{c}^-_{\nu_\gamma,0}, & x^{\gamma/2}\Phi'_-(x) \sim_0 (2-\gamma)\tilde{c}^-_{\nu_\gamma,0} x^{1-\gamma/2}.\end{array}$$

where we have used the following relation

$$xJ'_\nu(x) = \nu J(x) - xJ_{\nu+1}(x). \tag{4.37}$$

Hence, given c_+ and c_-, $u(x) = c_+\Phi_+(x) + c_-\Phi_-(x) \in H^1_{0,\gamma}(0,1)$ with the following boundary condition

$$\begin{cases} (x^\gamma u_x)(0) - (m\lambda^2 + \varrho\lambda(\lambda+\eta)^{\alpha-1})u(0) = 0,\\ u(1) = 0.\end{cases}$$

Then

$$M(\lambda)C(\lambda) = \begin{pmatrix} (1-\gamma)\tilde{c}^+_{\nu_\gamma,0} & -(m\lambda^2 + \varrho\lambda(\lambda+\eta)^{\alpha-1})\tilde{c}^-_{\nu_\gamma,0} \\ J_{\nu_\gamma}\left(\frac{2}{2-\gamma}i\lambda\right) & J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}i\lambda\right)\end{pmatrix}\begin{pmatrix} c_+ \\ c_-\end{pmatrix} = \begin{pmatrix} 0 \\ 0\end{pmatrix}. \tag{4.38}$$

Hence, a non-trivial solution u exists if and only if the determinant of $M(\lambda)$ vanishes. Set $f(\lambda) = detM(\lambda)$ thus the characteristic equation is $f(\lambda) = 0$.

Our purpose in the sequel is to prove, thanks to Rouché's Theorem, that there is a subsequence of eigenvalues for which their real part tends to 0.

In the sequel, since $\mathcal{A}$ is dissipative, we study the asymptotic behavior of the large eigenvalues λ of $\mathcal{A}$ in the strip $-\alpha_0 \le \Re(\lambda) \le 0$, for some $\alpha_0 > 0$ large enough and for such λ, we remark that Φ_+, Φ_- remains bounded.

Lemma 4.4.1 *There exists $N \in \mathbb{N}$ such that*

$$\{\lambda_k\}_{k\in\mathbf{Z}^*, |k|\ge N} \subset \sigma(\mathcal{A}), \tag{4.39}$$

where

$$\begin{aligned}\lambda_k \;=\; & -\frac{2-\gamma}{2} i \left(k + \frac{\nu_\gamma}{2} + \frac{3}{4}\right)\pi - i\frac{1-\gamma}{m}\left(\frac{2}{2-\gamma}\right)\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\frac{\sin\nu_\gamma\pi}{(k\pi)^{2-2\nu_\gamma}}\\ & -(\frac{1-\gamma}{m})^2(\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}})^2\frac{8}{(2-\gamma)^3}\frac{\sin\nu_\gamma\cos\nu_\gamma}{(\pi k)^{4-4\nu_\gamma}} i\\ & -i(\frac{2}{2-\gamma})^{3-\alpha}\frac{\varrho(1-\gamma)}{m^2}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\frac{\sin\nu_\gamma\pi\sin(1-\alpha)\frac{\pi}{2}}{\pi^{4-\alpha-2\nu_\gamma}}\frac{1}{k^{4-\alpha-2\nu_\gamma}}\\ & -(\frac{2}{2-\gamma})^{3-\alpha}\frac{\varrho(1-\gamma)}{m^2}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\frac{\sin\nu_\gamma\pi\cos(1-\alpha)\frac{\pi}{2}}{\pi^{4-\alpha-2\nu_\gamma}}\frac{1}{k^{4-\alpha-2\nu_\gamma}} + O\left(\frac{1}{k^\omega}\right)\end{aligned}$$

$$\lambda_k = \overline{\lambda_{-k}} \text{ if } k \le -N,$$

where $\omega = \max\{4-\alpha-2\nu_\gamma, 4-4\nu_\gamma\}$. Moreover for all $|k| \ge N$, the eigenvalues λ_k are simple.

Proof.
From (4.38), we have

$$f(\lambda) = (1-\gamma)\tilde{c}^+_{\nu_\gamma,0} J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}i\lambda\right) + (m\lambda^2 + \varrho\lambda(\lambda+\eta)^{\alpha-1}))\tilde{c}^-_{\nu_\gamma,0} J_{\nu_\gamma}\left(\frac{2}{2-\gamma}i\lambda\right) = 0$$

We will use the following classical asymptotic development (see **[36]** p. 122, (5.11.6)): for all $\delta > 0$, the following development holds when $|argz| \leq \pi - \delta$:

$$J_\nu(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos\left(z - \nu\frac{\pi}{2} - \frac{\pi}{4}\right)\left(1 + O(\frac{1}{|z|^2})\right) - \left(\frac{2}{\pi z}\right)^{1/2} \sin\left(z - \nu\frac{\pi}{2} - \frac{\pi}{4}\right) O\left(\frac{1}{|z|^2}\right). \tag{4.40}$$

step 1.

$$f(\lambda) = m\left(\frac{2}{\pi\tilde{z}}\right)^{1/2} \lambda^{2-\nu_\gamma} c^-_{\nu_\gamma,0}\left(\frac{2}{2-\gamma}i\right)^{-\nu_\gamma} \frac{e^{-i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}}{2}\tilde{f}(\lambda), \tag{4.41}$$

$$\begin{aligned} \tilde{f}(\lambda) &= (e^{2i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}+1) + \frac{1-\gamma}{m}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\left(\frac{2}{2-\gamma}i\right)^{2\nu_\gamma} \frac{e^{2i(\tilde{z}-\frac{\pi}{4})}+e^{-i\nu_\gamma\pi}}{\lambda^{2-2\nu_\gamma}} + \frac{\varrho}{m}\frac{e^{2i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}+1}{\lambda^{2-\alpha}} + O\left(\frac{1}{\lambda^2}\right) \\ &= f_0(\lambda) + \frac{f_1(\lambda)}{\lambda^{2-2\nu_\gamma}} + \frac{f_2(\lambda)}{\lambda^{2-\alpha}} + O\left(\frac{1}{\lambda^2}\right), \end{aligned} \tag{4.42}$$

where

$$f_0(\lambda) = e^{2i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})} + 1. \tag{4.43}$$

$$f_1(\lambda) = \frac{1-\gamma}{m}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\left(\frac{2}{2-\gamma}i\right)^{2\nu_\gamma}(e^{2i(\tilde{z}-\frac{\pi}{4})}+e^{-i\nu_\gamma\pi}) \tag{4.44}$$

$$f_2(\lambda) = \frac{\varrho}{m}(e^{2i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})}+1) \tag{4.45}$$

Note that f_0 and f_1 remain bounded in the strip $-\alpha_0 \leq \Re(\lambda) \leq 0$.
Step 2. We look at the roots of f_0. From (4.43), f_0 has one family of roots that we denote λ^0_k.

$$f_0(\lambda) = 0 \Leftrightarrow e^{2i(\tilde{z}-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})} + 1 = 0$$

Hence

$$2i\left(\frac{2}{2-\gamma}i\lambda - \nu_\gamma\frac{\pi}{2} - \frac{\pi}{4}\right) = i(2k+1)\pi, \quad k \in \mathbf{Z},$$

i.e.,

$$\lambda^0_k = -\frac{2-\gamma}{2}i\left(k + \frac{\nu_\gamma}{2} + \frac{3}{4}\right)\pi, \quad k \in \mathbf{Z}.$$

Now with the help of Rouché's Theorem, we will show that the roots of $\tilde{f}$ are close to those of f_0. Let us start with the first family. Changing in (4.42) the unknown λ by $u = 2iz$ then (4.42) becomes

$$\tilde{f}(u) = (e^u + 1) + O\left(\frac{1}{u^{(1-2\nu_\gamma)}}\right) = f_0(u) + O\left(\frac{1}{u^{(1-2\nu_\gamma)}}\right)$$

The roots of f_0 are $u_k = -\dfrac{2-\gamma}{2}i\left(k + \dfrac{\nu_\gamma}{2} + \dfrac{3}{4}\right)\pi, k \in \mathbf{Z}$, and setting $u = u_k + re^{it}, t \in [0, 2\pi]$, we can easily check that there exists a constant $C > 0$ independent of k such that $|e^u + 1| \geq Cr$ for r small enough. This allows to apply Rouché's Theorem. Consequently, there exists a subsequence of roots of $\tilde{f}$ which tends to the roots u_k of f_0. Equivalently, it means that there exists $N \in \mathbb{N}$ and a subsequence $\{\lambda_k\}_{|k|\geq N}$ of roots of $f(\lambda)$, such that $\lambda_k = \lambda^0_k + o(1)$ which tends to the roots $-\dfrac{2-\gamma}{2}i\left(k + \dfrac{\nu_\gamma}{2} + \dfrac{3}{4}\right)\pi$ of f_0. Finally for $|k| \geq N, \lambda_k$ is simple since λ^0_k is.

Step 3. From Step 2, we can write

$$\lambda_k = -\frac{2-\gamma}{2} i \left(k + \frac{\nu_\gamma}{2} + \frac{3}{4}\right)\pi + \varepsilon_k. \tag{4.46}$$

Using (4.46), we get

$$\begin{aligned} e^{2i((\frac{2}{2-\gamma} i\lambda_k) - \nu_\gamma \frac{\pi}{2} - \frac{\pi}{4})} &= -e^{-\frac{4}{2-\gamma}\varepsilon_k} \\ &= -1 + \tfrac{4}{2-\gamma}\varepsilon_k + O(\varepsilon_k^2). \end{aligned} \tag{4.47}$$

Substituting (4.47) into (4.42), using that $\tilde{f}(\lambda_k) = 0$, we get:

$$\tilde{f}(\lambda_k) = \frac{4}{2-\gamma}\varepsilon_k + \frac{1-\gamma}{m} \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}} \left(\frac{2}{2-\gamma}\right)^2 \frac{2i \sin \nu_\gamma \pi}{(k\pi)^{2-2\nu_\gamma}} + o(\varepsilon_k) + o\left(\frac{1}{k^{\alpha-2\nu_\gamma}}\right) = 0 \tag{4.48}$$

and hence

$$\varepsilon_k = -\frac{1-\gamma}{m} \left(\frac{2}{2-\gamma}\right) \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}} i \frac{\sin \nu_\gamma \pi}{(k\pi)^{2-2\nu_\gamma}}$$

From Step 3, we can write

$$\lambda_k = -\frac{2-\gamma}{2} i \left(k + \frac{\nu_\gamma}{2} + \frac{3}{4}\right)\pi - i\frac{1-\gamma}{m} \left(\frac{2}{2-\gamma}\right) \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}} \frac{\sin \nu_\gamma \pi}{(k\pi)^{2-2\nu_\gamma}} + \varepsilon_k. \tag{4.49}$$

Using (4.46), we get

$$\begin{aligned} e^{2i((\frac{2}{2-\gamma} i\lambda_k) - \nu_\gamma \frac{\pi}{2} - \frac{\pi}{4})} &= -e^{-\frac{4}{2-\gamma}\varepsilon_k + \frac{4c}{2-\gamma}\frac{1}{k^{2-2\nu_\gamma}}} \\ &= -1 + \tfrac{4}{2-\gamma}\varepsilon_k - \tfrac{4c}{2-\gamma}\tfrac{1}{k^{2-2\nu_\gamma}} + O(\varepsilon_k^2). \end{aligned} \tag{4.50}$$

where

$$c = \frac{1-\gamma}{m} \left(\frac{2}{2-\gamma}\right) \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}} \frac{\sin \nu_\gamma \pi}{\pi^{2-2\nu_\gamma}} i$$

Substituting (4.50) into (4.42), using that $\tilde{f}(\lambda_k) = 0$, we get:

$$\tilde{f}(\lambda_k) = \frac{4}{2-\gamma}\varepsilon_k - i\frac{1-\gamma}{m} \left(\frac{8}{(2-\gamma)^2}\right) \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}} \frac{(2-2\nu_\gamma)(\frac{\nu_\gamma}{2} + \frac{3}{4})}{\pi^{2-2\nu_\gamma} k^{3-2\nu_\gamma}} \sin \nu_\gamma \pi \tag{4.51}$$

and hence

$$\varepsilon_k = i\frac{1-\gamma}{m} \left(\frac{2}{2-\gamma}\right) \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}} \frac{(2-2\nu_\gamma)(\frac{\nu_\gamma}{2} + \frac{3}{4})}{\pi^{2-2\nu_\gamma} k^{3-2\nu_\gamma}} \sin \nu_\gamma \pi$$

From Step 3, we can write

$$\begin{aligned} \lambda_k &= -\frac{2-\gamma}{2} i \left(k + \frac{\nu_\gamma}{2} + \frac{3}{4}\right)\pi - i\frac{1-\gamma}{m} \left(\frac{2}{2-\gamma}\right) \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}} \frac{\sin \nu_\gamma \pi}{(k\pi)^{2-2\nu_\gamma}} \\ &\quad + i\frac{1-\gamma}{m} \left(\frac{2}{2-\gamma}\right) \frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}} \frac{(2-2\nu_\gamma)(\frac{\nu_\gamma}{2} + \frac{3}{4})}{\pi^{2-2\nu_\gamma} k^{3-2\nu_\gamma}} \sin \nu_\gamma \pi + \varepsilon_k. \end{aligned} \tag{4.52}$$

Using (4.46), we get

$$\begin{array}{rcl} e^{2i((\frac{2}{2-\gamma}i\lambda_k)-\nu_\gamma\frac{\pi}{2}-\frac{\pi}{4})} & = & -e^{-\frac{4}{2-\gamma}\varepsilon_k+\frac{4c}{2-\gamma}\frac{1}{k^{2-2\nu_\gamma}}-\frac{4\tilde{c}}{2-\gamma}\frac{1}{k^{3-2\nu_\gamma}}} \\ & = & -1+\frac{4}{2-\gamma}\varepsilon_k-\frac{4c}{2-\gamma}\frac{1}{k^{2-2\nu_\gamma}}+\frac{4\tilde{c}}{2-\gamma}\frac{1}{k^{3-2\nu_\gamma}}-\frac{1}{2}(\frac{4c}{2-\gamma})^2\frac{1}{k^{4-4\nu_\gamma}}+O(\varepsilon_k^2). \end{array} \tag{4.53}$$

where

$$\begin{array}{rcl} c & = & \dfrac{1-\gamma}{m}\left(\dfrac{2}{2-\gamma}\right)\dfrac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\dfrac{\sin\nu_\gamma\pi}{\pi^{2-2\nu_\gamma}}i \\ \tilde{c} & = & i\dfrac{1-\gamma}{m}\left(\dfrac{2}{2-\gamma}\right)\dfrac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\dfrac{(2-2\nu_\gamma)(\frac{\nu_\gamma}{2}+\frac{3}{4})}{\pi^{2-2\nu_\gamma}k^{3-2\nu_\gamma}}\sin\nu_\gamma\pi \end{array}$$

Substituting (4.53) into (4.42), using that $\tilde{f}(\lambda_k)=0$, we get:

$$\begin{array}{rcl} \tilde{f}(\lambda_k) & = & \frac{4}{2-\gamma}\varepsilon_k-\frac{4c}{2-\gamma}\frac{1}{k^{2-2\nu_\gamma}}+\frac{4\tilde{c}}{2-\gamma}\frac{1}{k^{3-2\nu_\gamma}} \\ & & -\frac{1}{2}(\frac{4c}{2-\gamma})^2\frac{1}{k^{4-4\nu_\gamma}}-2i\dfrac{\tilde{\tilde{c}}}{\delta^{2-2\nu_\gamma}}\dfrac{\sin\nu_\gamma\pi}{k^{2-2\nu_\gamma}} \\ & & +2i(2-2\nu_\gamma)(\frac{\nu_\gamma}{2}+\frac{3}{4})\dfrac{\tilde{\tilde{c}}}{\delta^{2-2\nu_\gamma}}\dfrac{\sin\nu_\gamma\pi}{k^{3-2\nu_\gamma}}-\dfrac{\tilde{\tilde{c}}c}{\delta^{2-2\nu_\gamma}}\dfrac{4}{2-\gamma}\dfrac{e^{i\nu_\gamma\pi}}{k^{4-4\nu_\gamma}}-\dfrac{\varrho}{m}\dfrac{4}{2-\gamma}\dfrac{c}{\delta^{2-\alpha}}\dfrac{1}{k^{4-\alpha-2\nu_\gamma}} \\ & = & \frac{4}{2-\gamma}\varepsilon_k+(\dfrac{1-\gamma}{m})^2(\dfrac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}})^2\dfrac{32}{(2-\gamma)^4}\dfrac{\sin\nu_\gamma\cos\nu_\gamma}{(\pi k)^{4-4\nu_\gamma}}i-\dfrac{\varrho}{m}\dfrac{4}{2-\gamma}\dfrac{c}{\delta^{2-\alpha}}\dfrac{1}{k^{4-\alpha-2\nu_\gamma}}=0 \end{array} \tag{4.54}$$

where

$$\delta=-\frac{2-\gamma}{2}i\pi,\quad \tilde{\tilde{c}}=\frac{1-\gamma}{m}\frac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}(\frac{2}{2-\gamma}i)^{2\nu_\gamma}.$$

and hence

$$\begin{array}{rcl} \varepsilon_k & = & -(\dfrac{1-\gamma}{m})^2(\dfrac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}})^2\dfrac{8}{(2-\gamma)^3}\dfrac{\sin\nu_\gamma\cos\nu_\gamma}{(\pi k)^{4-4\nu_\gamma}}i \\ & & -i(\dfrac{2}{2-\gamma})^{3-\alpha}\dfrac{\varrho(1-\gamma)}{m^2}\dfrac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\dfrac{\sin\nu_\gamma\pi\sin(1-\alpha)\frac{\pi}{2}}{\pi^{4-\alpha-2\nu_\gamma}}\dfrac{1}{k^{4-\alpha-2\nu_\gamma}} \\ & & -(\dfrac{2}{2-\gamma})^{3-\alpha}\dfrac{\varrho(1-\gamma)}{m^2}\dfrac{c^+_{\nu_\gamma,0}}{c^-_{\nu_\gamma,0}}\dfrac{\sin\nu_\gamma\pi\cos(1-\alpha)\frac{\pi}{2}}{\pi^{4-\alpha-2\nu_\gamma}}\dfrac{1}{k^{4-\alpha-2\nu_\gamma}} \end{array}$$

The operator $\mathcal{A}$ has a non exponential decaying branch of eigenvalues. Thus the proof is complete.

Remark 4.4.1 *From Lemma 4.4.1, the operator $\mathcal{A}$ does not have eigenvalues on imaginary axis $i\mathbb{R}$.*

4.4.1 Polynomial Stability (for $\eta \neq 0$)

To state and prove our stability results, we need some results from semigroup theory.

Theorem 4.4.3 ([11]) *Let $S(t)$ be a bounded C_0-semigroup on a Hilbert space $\mathcal{X}$ with generator $\mathcal{A}$. If*

$$i\mathbb{R}\subset\rho(\mathcal{A}) \textit{ and } \overline{\lim_{|\beta|\to\infty}}\frac{1}{\beta^l}\|(i\beta I-\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{X})}<\infty$$

for some l, then there exist c such that

$$\|e^{\mathcal{A}t}U_0\|^2 \leq \frac{c}{t^{\frac{2}{l}}}\|U_0\|^2_{D(\mathcal{A})}.$$

Our main result is the following.

Theorem 4.4.4 *The semigroup $S_{\mathcal{A}}(t)_{t\geq 0}$ is polynomially stable and*

$$E(t) = \|S_{\mathcal{A}}(t)U_0\|^2_{\mathcal{H}} \leq \frac{1}{t^{\frac{2}{(4-\alpha-2\nu_\gamma)}}}\|U_0\|^2_{D(\mathcal{A})}.$$

Proof
We will need to study the resolvent equation $(i\lambda - \mathcal{A})U = F$, for $\lambda \in \mathbb{R}$, namely

$$\begin{cases} i\lambda u - v = f_1, \\ i\lambda v - (x^\gamma u_x)_x = f_2, \\ i\lambda\phi + (\xi^2+\eta)\phi - v(0)\mu(\xi) = f_3. \\ i\lambda\theta - \frac{1}{m}(x^\gamma u_x)(0) + \frac{\zeta}{m}\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)\,d\xi = f_4. \end{cases} \tag{4.55}$$

where $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$. From $(4.55)_1$ and $(4.55)_2$, we have

$$\lambda^2 u + (x^\gamma u_x)_x = -(f_2 + i\lambda f_1) \tag{4.56}$$

with

$$\begin{cases} \lambda^2 u - (x^\gamma u_x)_x = 0, \\ -(x^\gamma u_x)(0) + (-m\lambda^2 + i\varrho\lambda(i\lambda+\eta)^{\alpha-1})u(0) \\ \qquad\qquad = mf_4 - \zeta\displaystyle\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_3(\xi)}{i\lambda+\eta+\xi^2}\,d\xi + (mi\lambda + \varrho(i\lambda+\eta)^{\alpha-1})f_1(0), \\ u(1) = 0. \end{cases} \tag{4.57}$$

Assume that Φ is a solution of (4.56), then one easily checks that the function Ψ defined by

$$\Phi(x) = x^{\frac{1-\gamma}{2}}\Psi\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right) \tag{4.58}$$

is solution of the following inhomogeneous Bessel equation:

$$\begin{aligned} &y^2\Psi''(y) + y\Psi'(y) + \left(y^2 - \left(\frac{\gamma-1}{2-\gamma}\right)^2\right)\Psi(y) = \\ &-(\tfrac{2}{2-\gamma})^2(\tfrac{2-\gamma}{2}\tfrac{1}{\lambda}y)^{\frac{3-\gamma}{2-\gamma}}\left(f_2\left((\tfrac{2-\gamma}{2}\tfrac{1}{\lambda}y)^{\frac{2}{2-\gamma}}\right) + i\lambda f_1\left((\tfrac{2-\gamma}{2}\tfrac{1}{\lambda}y)^{\frac{2}{2-\gamma}}\right)\right). \end{aligned} \tag{4.59}$$

The solution can be written as

$$\Psi(y) = AJ_{\nu_\gamma}(y) + BJ_{-\nu_\gamma}(y) + \frac{2\nu_\gamma}{\sin\nu_\gamma\pi}\int_0^y\frac{f(s)}{s}\left(J_{\nu_\gamma}(s)J_{-\nu_\gamma}(y) - J_{\nu_\gamma}(y)J_{-\nu_\gamma}(s)\right)ds.$$

Thus,

$$
\begin{aligned}
&u(x) = Ax^{\frac{1-\gamma}{2}} J_{\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right) + Bx^{\frac{1-\gamma}{2}} J_{-\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right)\\
&-\frac{2\nu_\gamma}{\sin\nu_\gamma\pi}\left(\frac{2}{2-\gamma}\right) x^{\frac{1-\gamma}{2}} \int_0^x s^{\frac{1-\gamma}{2}}(f_2(s)+i\lambda f_1(s))\left(J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda s^{\frac{2-\gamma}{2}}\right) J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right)\right.\\
&\left.-J_{\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right) J_{-\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda s^{\frac{2-\gamma}{2}}\right)\right) ds.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
u(x) &= A\Phi_+(x) + B\Phi_-(x)\\
&\quad -\frac{2\nu_\gamma}{\sin\nu_\gamma\pi}\left(\frac{2}{2-\gamma}\right)\int_0^x (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi_-(x)-\Phi_+(x)\Phi_-(s))\,ds,
\end{aligned}
\tag{4.60}
$$

where Φ_+ and Φ_- are defined by

$$
\Phi_+(x) = x^{\frac{1-\gamma}{2}} J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right), \quad \Phi_-(x) = x^{\frac{1-\gamma}{2}} J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right).
\tag{4.61}
$$

From where it follows

$$
\begin{aligned}
u_x(x) &= A\Phi'_+(x) + B\Phi'_-(x)\\
&\quad -\frac{2\nu_\gamma}{\sin\nu_\gamma\pi}\left(\frac{2}{2-\gamma}\right)\int_0^x (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi'_-(x)-\Phi'_+(x)\Phi_-(s))\,ds
\end{aligned}
\tag{4.62}
$$

From $(4.57)_2$, (4.62) and (4.60), we conclude that

$$
\begin{aligned}
&(1-\gamma)\tilde{c}^+_{\nu_\gamma,0}A - (-m\lambda^2 + \varrho i\lambda(i\lambda+\eta)^{\alpha-1})\tilde{c}^-_{\nu_\gamma,0}B\\
&\qquad = -mf_4 + \zeta\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_3(\xi)}{i\lambda+\eta+\xi^2}\,d\xi - (mi\lambda + \varrho(i\lambda+\eta)^{\alpha-1})f_1(0)
\end{aligned}
\tag{4.63}
$$

$$
A\Phi_+(1) + B\Phi_-(1) = \frac{2\nu_\gamma}{\sin\nu_\gamma\pi}(\frac{2}{2-\gamma})\int_0^1 (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi_-(1)-\Phi_+(1)\Phi_-(s))\,ds
\tag{4.64}
$$

where

$$
\tilde{c}^+_{\nu_\gamma,m} = c^+_{\nu_\gamma,m}\left(\frac{2}{2-\gamma}\lambda\right)^{2m+\nu_\gamma}, \quad \tilde{c}^+_{\nu_\gamma,m} = c^-_{\nu_\gamma,m}\left(\frac{2}{2-\gamma}\lambda\right)^{2m-\nu_\gamma}
$$

and

$$
\Phi_+(1) = J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda\right), \qquad \Phi_-(1) = J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda\right).
$$

Using (4.63) and (4.64), a linear system in A and B is obtained

$$
\begin{pmatrix} r_{11} & r_{12}\\ r_{21} & r_{22}\end{pmatrix}\begin{pmatrix} A\\ B\end{pmatrix} = \begin{pmatrix} C\\ \tilde{C}\end{pmatrix},
\tag{4.65}
$$

where

$$
\begin{aligned}
r_{11} &= (1-\gamma)\tilde{c}^+_{\nu_\gamma,0},\\
r_{12} &= (m\lambda^2 - \varrho i\lambda(i\lambda+\eta)^{\alpha-1})\tilde{c}^-_{\nu_\gamma,0},\\
r_{21} &= J_{\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda\right),\\
r_{22} &= J_{-\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda\right),
\end{aligned}
$$

$$\begin{aligned}
C &= -mf_4 + \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi) f_3(\xi)}{i\lambda + \eta + \xi^2}\, d\xi - (mi\lambda + \varrho(i\lambda+\eta)^{\alpha-1}) f_1(0),\\
\tilde{C} &= \frac{2\nu_\gamma}{\sin \nu_\gamma \pi}\Big(\frac{2}{2-\gamma}\Big) \int_0^1 (f_2(s) + i\lambda f_1(s))(\Phi_+(s)\Phi_-(1) - \Phi_+(1)\Phi_-(s))\, ds.
\end{aligned}$$

Let the determinant of the linear system given in (4.65) be denoted by D. Then

$$\begin{aligned}
D &= (1-\gamma)\tilde{c}^+_{\nu_\gamma,0} J_{-\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda\right) - (m\lambda^2 - \varrho i\lambda(i\lambda+\eta)^{\alpha-1})\tilde{c}^-_{\nu_\gamma,0} J_{\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda\right)\\
&= (1-\gamma) c^+_{\nu_\gamma,0}\left(\tfrac{2}{2-\gamma}\right)^{\nu_\gamma} \lambda^{\nu_\gamma}\left[\left(\frac{2-\gamma}{\pi\lambda}\right)^{1/2} \cos\left(\tfrac{2}{2-\gamma}\lambda + \nu_\gamma\tfrac{\pi}{2} - \tfrac{\pi}{4}\right) + O(\frac{1}{\lambda^{5/2}})\right]\\
&\quad -(m\lambda^2 - \varrho i\lambda(i\lambda+\eta)^{\alpha-1}) c^-_{\nu_\gamma,0}\left(\tfrac{2}{2-\gamma}\right)^{-\nu_\gamma}\lambda^{-\nu_\gamma}\left[\left(\frac{2-\gamma}{\pi\lambda}\right)^{1/2}\cos\left(\tfrac{2}{2-\gamma}\lambda - \nu_\gamma\tfrac{\pi}{2} - \tfrac{\pi}{4}\right) + O(\frac{1}{\lambda^{5/2}})\right]\\
&= (1-\gamma) c^+_{\nu_\gamma,0}\left(\tfrac{2}{2-\gamma}\right)^{\nu_\gamma}\left(\frac{2-\gamma}{\pi}\right)^{1/2} \lambda^{\nu_\gamma - \frac{1}{2}} \cos\left(\tfrac{2}{2-\gamma}\lambda + \nu_\gamma\tfrac{\pi}{2} - \tfrac{\pi}{4}\right)\\
&\quad + \varrho i^\alpha c^-_{\nu_\gamma,0}\left(\tfrac{2}{2-\gamma}\right)^{-\nu_\gamma}\left(\frac{2-\gamma}{\pi}\right)^{1/2}\lambda^{\alpha-\nu_\gamma-\frac{1}{2}}\cos\left(\tfrac{2}{2-\gamma}\lambda - \nu_\gamma\tfrac{\pi}{2} - \tfrac{\pi}{4}\right) + O(\frac{1}{\lambda^{3/2+\nu_\gamma-\alpha}}).
\end{aligned}$$

As $D \neq 0$ for all $\lambda \neq 0$, then A and B are uniqueley determined by (4.65). Hence the operator $(i\lambda I - \mathcal{A})$ is surjective for all $\lambda \neq 0$. Moreover for $\lambda = 0$ and $\eta \neq 0$, using Lax-Milgram Theorem, we can deduce that the operator $\mathcal{A}$ is surjective. Taking account of Remark 4.4.1 and from Theorem 2.5.3 The C_0-semigroup $e^{t\mathcal{A}}$ is strongly stable in $\mathcal{H}$.

Now, it is easy to prove that

$$|D| \geq c|\lambda|^{-5/2+\nu_\gamma+\alpha} \text{ for large } \lambda,$$

Now

$$A = \frac{1}{D}(Cr_{22} - \tilde{C}r_{12})$$

$$B = \frac{1}{D}(-Cr_{21} + \tilde{C}r_{11})$$

Considering only the dominant terms of λ, the following is obtained:

$$\begin{aligned}
|D||A| &\leq c_1|\lambda|^{\frac{1}{2}} + c_2|\lambda|^{1-\nu_\gamma} \leq c_3|\lambda|^{1-\nu_\gamma},\\
|D||B| &\leq c_1|\lambda|^{\frac{1}{2}} + c_2|\lambda|^{\nu_\gamma-1} \leq c|\lambda|^{\frac{1}{2}},
\end{aligned}$$

where we have used the fact that $f_1 \in H^1_{0,\gamma}(0,1)$ and

$$\left|\int_0^1 (f_2(s) + i\lambda f_1(s))(\Phi_+(s)\Phi_-(1) - \Phi_+(1)\Phi_-(s))\, ds\right| \leq \frac{1}{|\lambda|}\left(\|f_1\|_{H^1_{0,\gamma}(0,1)} + \|f_2\|_{L^2(0,1)}\right),$$

$$\left|\int_0^1 (f_2(s) + i\lambda f_1(s))(\Phi_+(s)\Phi'_-(1) - \Phi'_+(1)\Phi_-(s))\, ds\right| \leq \left(\|f_1\|_{H^1_{0,\gamma}(0,1)} + \|f_2\|_{L^2(0,1)}\right).$$

Then, we conclude that

$$|A| \leq c|\lambda|^{\frac{7}{2}-\alpha-2\nu_\gamma}$$

$$|B| \leq c|\lambda|^{3-\alpha-\nu_\gamma},$$

Then
$$\|u\|_{L^2(0,1)} \leq c|\lambda|^{3-\alpha-2\nu_\gamma}\left(\|f_1\|_{H^1_{0,\gamma}(0,1)} + \|f_2\|_{L^2(0,1)}\right),$$
Using $(4.55)_1$ and (4.60), we get
$$\|v\|_{L^2(0,1)} \leq c|\lambda|^{4-\alpha-2\nu_\gamma}\left(\|f_1\|_{H^1_{0,\gamma}(0,1)} + \|f_2\|_{L^2(0,1)}\right)$$
From (4.61) and (4.37), we have
$$\begin{cases} x^{\gamma/2}\Phi'_+(x) &= (\frac{1-\gamma}{2} + \frac{2-\gamma}{2}\nu_\gamma)x^{-1/2}J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right) - \lambda x^{\frac{1-\gamma}{2}}J_{1+\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right), \\ x^{\gamma/2}\Phi'_-(x) &= (\frac{1-\gamma}{2} - \frac{2-\gamma}{2}\nu_\gamma)x^{-1/2}J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right) - \lambda x^{\frac{1-\gamma}{2}}J_{1-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right). \end{cases}$$
Then from (4.62), we can get
$$\|x^{\gamma/2}u_x\|_{L^2(0,1)} \leq c|\lambda|^{4-\alpha-2\nu_\gamma}\left(\|f_1\|_{H^1_{0,\gamma}(0,1)} + \|f_2\|_{L^2(0,1)}\right)$$
Moreover from $(4.55)_3$, we have
$$\|\phi\|^2_{L^2(-\infty,\infty)} \leq \int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi)|^2\,d\xi \leq c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$
Thus, we conclude that
$$\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{H}} \leq c|\lambda|^{4-\alpha-2\nu_\gamma} \text{ as } |\lambda| \to \infty$$
The conclusion then follows by applying Theorem 4.4.3.

Besides, we prove that the decay rate is optimal. Indeed, the decay rate is consistent with the asymptotic expansion of eigenvalues.

□

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الملخص (بالعربية) :

في هذه الأطروحة اقترحنا بعض المسائل الرياضية لمعادلات و جمل معادلات تطورية منحلة بوجود آليات للتبديد ذات أشكال كسرية من زوايا مختلفة. ندرس خاصة معادلات الموجة. تحت بعض الفرضيات على الشروط الابتدائية و الشروط الحدية، ركزنا دراستنا على وجود الحلول ودراسة السلوك المقارب للحلول الموجودة عند اللانهاية الزمنية أين توصلنا لإيجاد عدة نتائج حول طريقة تناقص الطاقة.

الكلمات المفتاحية :
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Résumé (en Français) :

Dans cette thèse, nous avons considéré quelques problèmes d'évolution hyperbolique dégénérés avec la présence des termes dissipatifs de type fractionnaires. En particulier on considère l'équation des ondes dégénérée. Sous quelques hypothèses sur les données initiales et aux bords, nous avons concentré notre étude sur l'existence globale et le comportement asymptotique des solutions où nous avons obtenu plusieurs résultats sur la vitesse de décroissance de l'énergie.

Les mots clés : équation des ondes dégénérée, dérivée fractionnaire, stabilité polynomiale, vitesse de décroissance optimale, fonctions de Bessel, C_0-semi-groupe, Méthode spectrale.

Abstract (en Anglais) :

In this thesis we considered some degenerate evolution problems with the presence of boundary dissipation of fractional derivative type. In particular, we consider degenerate wave equation. Under assumptions on initial data and boundary conditions, we focused our study on the global existence and asymptotic behavior of solutions where we obtained several results on the decay rate.

Keywords: degenerate wave equation, fractional derivative, polynomial stability, optimal decay rate, Bessel functions, C_0-semigroup, spectral method.