N° d'ordre :

REPUBLIQUE ALGERIENNE DEMOCRATIQUE & POPULAIRE

MINISTERE DE L'ENSEIGNEMENT SUPERIEUR & DE LA RECHERCHE SCIENTIFIQUE

UNIVERSITE DJILLALI LIABES
FACULTE DES SCIENCES EXACTES
SIDI BEL ABBES

THESE
DE DOCTORAT

Présentée par :

BENDOUMA BOUHARKET

Spécialité : Mathématiques

Option : Equations différentielles ordinaires

Intitulé

Equations différentielles sur les échelles de temps

Soutenue le : 25 / 07 / 2019.

Devant le jury composé de :

Président : OUAHAB Abdelghani, Pr. U d'Adrar.

Examinateurs : ABBS Said, Pr. U de Saida.

Directeur de thèse: HAMMOUDI Ahmed , Pr. CU d Ain Témouchent.

Co-Directeur de thèse : BENCHOHRA Mouffak , Pr. UDL de SBA.

Année Universitaire: 2018-2019.

People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research

UNIVERSITY OF DJILLALI LIABES
FACULTY OF EXACTES SCIENCES
SIDI BEL ABBES

Phd Thesis

***Presented by*:**

BOUHARKET BENDOUMA

Speciality : ***Mathematics***
Option : ***Ordinary differential equations***

Entitled

Differential Equations on Times Scales

Phd thesis submitted: 25/07/2019.
The jury :

President **: Abdelghani OUAHAB , Pr. Adrar U.**

Examinateurs **: Saïd ABBAS , Pr. Saida U.**

Supervisor **: Ahmed HAMMOUDI , Pr. Ain Témouchent U.**

Co-Supervisor**: Mouffak BENCHOHRA, Pr. SBA U.**

University year : 2018 – 2019

Dédicaces

Ce travail est dédié:

À la mémoire de mon père.

À ma mère, ma femme et mes enfants.

À mes frères, mes soeurs et toute ma famille.

À tous mes amis, mes collègues et mes enseignants.

Remerciements

Avant tout, Je remercie Allah le tout puissant de toute la volonté et le courage qu'il m'a donné pour l'achèvement de ce travail.

Je tiens à remercier vivement le Professeur Ahmed Hammoudi d'avoir accepté de diriger ce travail de thèse et de créer autour de moi un environnement de recherche par ses conseils et ses encouragements. Je remercie également Monsieur le Professeur Mouffak Benchohra pour ses précieux conseils et remarques.

Je tiens aussi à remercier Monsieur Alberto Cabada, professeur à la Faculté de Mathématiques de l'Université de Santiago de Compostela, Espagne, qui m'a accueilli durant deux mois, pendant lesquelles j'ai approfondi mes recherches sur mon sujet de thèse.

Comme je remercie Monsieur le Professeur Abdelghani Ouahab pour avoir accepté la présidence du jury.

Je remercie également le Professeur Abbas Saïd, pour avoir accepté de juger ce travail.

Je n'oublie pas de remercier tous les enseignants d'avoir contribué à notre formation durant l'année de la post-graduation ainsi qu'à toute l'équipe du département de mathématiques de l'université de Sidi Bel Abbès. Je remercie aussi Monsieur Benaissa Cherif Amine pour son soutien.

Finalement, Je tiens à adresser une pensée affectueuse à toute ma famille et à tous mes amis.

Publications

1. **B. Bendouma** and A. Hammoudi, *A nabla conformable fractional calculus on time scales*, Electronic Journal of Mathematical Analysis and Applications. **7** (2019), no. 1, 202-216.

2. **B. Bendouma**, A. Benaissa Cherif and A. Hammoudi, *Systems of first-order nabla dynamic equations on time scales*, Malaya Journal of Mathematik. **6** (2018), no. 4, 757-765.

3. **B. Bendouma** and A. Hammoudi, *Existence results for systems of first-order nabla dynamic inclusions on time scales*, Arab. J. Math. Sci. **25** (2019), no. 1, 106-122.

4. **B. Bendouma**, A. Cabada and A. Hammoudi, *Existence of solutions for conformable fractional problems with nonlinear functional boundary conditions*, Malaya Journal of Mathematik, submitted.

5. **B. Bendouma** and A. Hammoudi, *Nonlinear functional boundary value problems for conformable fractional dynamic equations on time scales*, Mediterr. J. Math. (2019) 16:25 https://doi.org/10.1007/s00009-019-1302-5

6. **B. Bendouma**, A. Cabada and A. Hammoudi, *Existence results for systems of conformable fractional differential equations*, Archivum Mathematicum (BRNO). **55** (2019), 69-83.

Abstract

In this thesis, we present some results of existence of solutions for systems of first order nonlinear nabla dynamic equations and nabla dynamic inclusions on time scales and for systems of conformable fractional differential equations under some boundary conditions. Also, we present existence of solutions for the nonlinear conformable fractional differential equations and for the conformable fractional dynamic equations on time scales, with nonlinear functional boundary value conditions.

These results are obtained by using the notion of solution-tube adapted to these systems. This notion generalizes the definition of lower and upper solution.

Key words and phrases: Conformable fractional derivative, conformable fractional calculus on time scales, systems of nabla dynamic equations and inclusions, conformable fractional dynamic equation, nonlinear boundary conditions, Green function, upper and lower solutions, solution-tube, Schauder's fixed-point theorem, fractional Sobolev's spaces.

AMS (MOS) Subject Classifications: 26A24, 26A33, 26E70, 34A08, 34A12, 34A34, 34A60, 34B15, 34N05, 47E05.

Contents

Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer order. Fractional differential equations play an important role in describing many phenomena and processes in various fields of science such as physics, chemistry, control systems, population dynamics, aerodynamics and electrodynamics, etc. For examples and details, the reader can see the references [10, 39, 71–73, 78, 82, 89]. Many different forms of fractional differential operators like the Grunwald-Letnikow, Riemann-Liouville, Hadamard, Caputo, Riesz, can be found in [21, 23, 38].

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers $\mathbb{R}$ with the subspace topology inherited from the standard topology of $\mathbb{R}$. The theory of time scales was introduced by Stefan Hilger in his PhD thesis [62] in 1988, in order to unify and generalize continuous and discrete analysis. The reader interested on the subject of time scales is referred to [5, 6, 9, 32–34, 40, 59, 63]. In [7, 16, 30, 88, 90], the authors studied fractional calculus on time scales and their important properties.

Recently, a new fractional derivative, called the conformable fractional derivative, was introduced by Khalil et al. [70]. For recent results on conformable fractional derivatives we refer the reader to [1–4, 8, 17, 19, 43, 55, 56, 66, 68–70, 86]. Furthermore, in [8, 17, 19], the authors proved the existence and uniqueness of solutions of initial value problems and boundary value problems for conformable fractional differential equations. In [55], the authors proved existence and uniqueness theorems for sequential linear conformable fractional differential equations. In [69], the authors proved the existence of solutions of boundary value problem involving conformable derivative by the method of upper and lower solutions. We point out that the method of lower and upper solutions has been applied by several authors to obtain the existence of solutions of initial value problems and boundary value problems for fractional differential equations, see [67, 84, 85, 92].

In particular, Benkhettou et al. [31] introduced a conformable fractional calculus on an arbitrary time scale, which provided a natural extension of the conformable fractional calculus. Furthermore, in [76], the author proved mean value theorem for the conformable fractional calculus on arbitrary time Scales. In [87] the authors develop the fractional Sobolev's spaces via conformable fractional calculus on time scales and their important properties. In [58], the authors proved some basic theorems for the conformable fractional Dirac system on time scales.

In this thesis, we present existence of solutions for systems of first order nonlinear nabla dynamic equations and nabla dynamic inclusions on time scales and for systems of conformable fractional differential equations. Also, we present existence of solutions for the nonlinear conformable fractional differential equations and for the conformable fractional dynamic equations on time scales, with nonlinear functional boundary value conditions. Existence results for these problems are obtained by using the method of solution-tube. The purpose of this method is to prove that if a solution $x \in W^{1,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n)$ (*resp.*, $x \in W^{\alpha,1}_{0,b}([0,b], \mathbb{R}^n)$) exists, then it is included in a solution tube, i.e. we can find functions $v \in W^{1,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n)$ and $M \in W^{1,1}_{\nabla}(\mathbb{T}, [0,\infty))$ (*resp.*, $v \in W^{\alpha,1}_{0,b}([0,b], \mathbb{R}^n)$ *and* $M \in W^{\alpha,1}_{0,b}([0,b], [0,\infty))$) such that

$$\|x(t) - v(t)\| \leq M(t) \text{ for every } t \in \mathbb{T} \ (\textit{resp.}, \ t \in [0,b]).$$

We have organized this thesis as follows:

In Chapter 1, we present some definitions and results which are used throughout this thesis.

In Chapter 2, we define and study the nabla conformable fractional derivative and nabla conformable fractional integral on time scales. Many basic properties of the theory are proved.

In Chapter 3, we prove existence of solutions to system of first-order ∇-dynamic equation on time scale:

$$\begin{cases} x^{\nabla}(t) = f(t, x(t)), & \nabla\text{-a.e. } t \in \mathbb{T}_0, \\ x(a) = x(b). \end{cases} \tag{1}$$

Here $\mathbb{T}$ is an arbitrary compact time scale with $a = \min \mathbb{T}$, $b = \max \mathbb{T}$, $\mathbb{T}_0 = \mathbb{T} \backslash \{a\}$ and $f : \mathbb{T}_0 \times \mathbb{R}^n \to \mathbb{R}^n$ is a ∇-Carathéodory function. For this purpose, we use the method of solution-tube and Schauder's fixed-point theorem.

Existence results for system (1) were obtained in [18] with f is a continuous function. In the particular case where $n = 1$, existence results for first-order ∇-dynamic equation on time scales were obtained in [91] for the dynamic initial value problem:

$$x^{\nabla}(t) = f(t, x(t)), \ t \in (0,b]_{\mathbb{T}}, \ \text{ and } \ x(0) = 0,$$

with f is a left-Hilger-continuous function. Their results were established with the method of lower and upper solutions. Existence results were obtained in [44, 47, 53], for systems of Δ-dynamic equations on time scales. In [53], Gilbert introduced the notion of solution-tube to systems of first order Δ-dynamic equations.

In Chapter 4, we establish an existence result for the following system of first-order ∇-dynamic inclusions on time scale:

$$\begin{cases} x^{\nabla}(t) \in F(t, x(\rho(t))), & \nabla\text{-a.e. } t \in \mathbb{T}_0, \\ x \in (\mathfrak{BC}), \end{cases} \tag{2}$$

where $\mathbb{T}$ is an arbitrary compact time scale, with $a = \min \mathbb{T}$, $b = \max \mathbb{T}$, $\mathbb{T}_0 = \mathbb{T} \backslash \{a\}$, $F : \mathbb{T}_0 \times \mathbb{R}^n \to \mathbb{R}^n$ is a multivalued map with compact and convex values, and $(\mathfrak{BC})$

denotes the terminal value or the periodic boundary value conditions:

$$x(b) = x_0, \tag{3}$$

$$x(a) = x(b). \tag{4}$$

In the particular case where $n = 1$, existence results for first order ∇-dynamic inclusion on time scales were obtained in [12] for the general boundary conditions:

$$x^{\nabla}(t) \in F(t, x(t)), \text{ a.e. on } \mathbb{T}_{\kappa}, \quad \text{and} \quad L(x(a), x(b)) = 0,$$

with $F : \mathbb{T}_{\kappa} \times \mathbb{R} \to 2^{\mathbb{R}} \backslash \emptyset$ a multivalued map with compact and convex values and L is a continuous single-valued map. Their results were established with the method of lower and upper solutions. Existence results for systems of first order ∇-dynamic inclusions were obtained in [54] for the initial value problem. Multiplicity results were obtained in [50] for Δ-dynamic inclusions. In [81], the authors proved two variants of the Filippov-Pliss lemma in the case of dynamical inclusions on a time scale. In [49], Frigon and Gilbert introduced the notion of solution-tube to systems of first order Δ-dynamic inclusions (with an initial or a periodic boundary value condition) which generalizes the notions of lower and upper solutions given in [12]. A notion of solution-tube was introduced for first order systems of differential inclusions by B. Mirandette [74]. In order to obtain the existence results for problem (2), we introduce the notion of solution-tube of (2).

In Chapter 5, we present existence of solutions for the nonlinear conformable fractional differential equations, for the conformable fractional dynamic equations on time scales and for systems of conformable fractional differential equations.

This chapter consists of three sections. In Section 5.1, we study the existence of solutions for the nonlinear conformable fractional differential equations with nonlinear functional boundary conditions:

$$x^{(\alpha)}(t) = f(t, x(t)), \qquad \text{for a.e. } t \in I = [0, b], \quad b > 0, \tag{5}$$

where $0 < \alpha \leq 1$, $f : I \times \mathbb{R} \to \mathbb{R}$ is a L^1_{α}-Carathéodory function, and $x^{(\alpha)}(t)$ denotes the conformable fractional derivative of x at t of order α. We consider, depending on the circumstances, nonlinear functional boundary conditions of the type

$$L_1(x, x(b)) = 0 \;\text{ or }\; L_2(x(0), x) = 0,$$

with L_i $(i = 1, 2)$ a continuous function that satisfies suitable monotonicity assumptions. For this purpose, we use the method of upper and lower solutions together with Schauder's fixed point theorem.

In Section 5.2, we are concerned with the existence of solutions for the following conformable fractional dynamic equations:

$$x^{(\alpha)}_{\Delta}(t) = f(t, x^{\sigma}(t)), \qquad \text{for } \Delta\text{-a.e. } t \in I = [a, b]_{\mathbb{T}}, \tag{6}$$

coupled to nonlinear functional boundary conditions:

$$B(x(a), x) = 0, \tag{7}$$

or

$$H(x, x(\sigma(b))) = 0. \tag{8}$$

Here, $\mathbb{T}$ is an arbitrary bounded time scale, $J = [a, \sigma(b)]_{\mathbb{T}}$ with $a, b \in \mathbb{T}$, $0 \leq a < b$ and $f : I \times \mathbb{R} \to \mathbb{R}$ is a $L^1_{\alpha,\Delta}$-Carathéodory function, $x^{(\alpha)}_{\Delta}(t)$ denotes the delta conformable fractional derivative of x at t of order $\alpha \in (0, 1]$, $B : \mathbb{R} \times C(J) \to \mathbb{R}$ and $H : C(J) \times \mathbb{R} \to \mathbb{R}$ are continuous functions. For this purpose, we use the method of upper and lower solutions together with Schauder's fixed point theorem. Existence of solutions were obtained in Section 5.1 for the conformable fractional differential equation (6) with $\mathbb{T} = \mathbb{R}$:

$$x^{(\alpha)}(t) = f(t, x(t)), \qquad \text{for a.e. } t \in [0, b], \quad 0 < \alpha \leq 1,$$

coupled to the nonlinear functional boundary conditions $B(x(0), x) = L_2(x(0), x) = 0$ or $H(x, x(b)) = L_1(x, x(b)) = 0$.

In Section 5.3, we establish existence results for the following system of conformable fractional differential equations:

$$\begin{cases} x^{(\alpha)}(t) = f(t, x(t)), & for\ a.e.\ t \in I = [0, b],\ b > 0, \\ x \in (\mathfrak{B}). \end{cases} \tag{9}$$

Where $0 < \alpha \leq 1$, $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ is a L^1_α-Carathéodory function, $x^{(\alpha)}(t)$ denotes the conformable fractional derivative of x at t of order α, and $(\mathfrak{B})$ denotes the initial value or the periodic boundary value conditions:

$$x(0) = x_0, \tag{10}$$

$$x(0) = x(b). \tag{11}$$

Existence results for problem (9), (10) were obtained in [79], by using the Banach fixed point theorem with f a continuous function. In the particular case where $n = 1$, existence results for problem (9) were obtained in Section 5.1 with nonlinear functional boundary conditions $L_1(x, x(b)) = 0$ or $L_2(x(0), x) = 0$, their results were established, for the scalar case, with the method of lower and upper solutions and cover, as a particular case, the boundary conditions (10) and (11). In [19] the authors solved problem (9),(10) (for $n = 1$), with f a continuous function by the help of the solution-tube method. As we will see, the used definition is equivalent to the existence of a pair of lower and upper solutions of the considered problem.

In order to obtain the existence results for problem (9), we introduce the notion of solution-tube of (9) which generalizes the notions of lower and upper solutions given in Section 5.1. It is inspired by a notion of solution tube for first-order systems of differential equations introduced in [74], (see also [51, 52] and [53] on time scales).

Notations

- $\triangleleft$ *a.e.*: Almost everywhere.
- $\triangleleft$ $\langle .,. \rangle$: The scalar product.
- $\triangleleft$ $\overline{co}A$: The closure of the convex hull of the set A.
- $\triangleleft$ $\|.\|$: The Euclidian norm in $\mathbb{R}^n$.
- $\triangleleft$ $B(x_0, r) = \{x \in \mathbb{R}^n : \|x - x_0\| < r\}$: The open ball of radius r and centre x_0.
- $\triangleleft$ $C(J, E)$: The Banach space of continuous functions from J into E with the norm $\|x\|_\infty = \sup_{t \in J} |x(t)|$, such that $J = [a, b]$ be an interval of $\mathbb{R}$ and $(E, |.|)$ be a real Banach space.
- $\triangleleft$ $L^1([a, b], \mathbb{R}^n)$: The space of Lebesgue-integrable functions $x : [a, b] \to \mathbb{R}^n$, with the norm
$$\|x\|_{L^1} = \int_a^b \|x(s)\| ds.$$
- $\triangleleft$ $\mathbb{T}$: Time scale (is a closed subset of $\mathbb{R}$).
- $\triangleleft$ $\mathbb{T}_0 = \mathbb{T} \backslash \{a\}$, with $a = \min \mathbb{T}$.
- $\triangleleft$ $[a, b]_\mathbb{T} := \{t \in \mathbb{T} : a \leq t \leq b;\ a, b \in \mathbb{T}\}$: The closed interval in $\mathbb{T}$.
- $\triangleleft$ f^Δ (resp., f^∇): The delta (resp., nabla) derivative of f.
- $\triangleleft$ $C_{rd}(\mathbb{T}) := C_{rd}(\mathbb{T}, \mathbb{R})$: The space of all right-dense continuous functions on $\mathbb{T}$.
- $\triangleleft$ $C_{ld}(\mathbb{T}) := C_{ld}(\mathbb{T}, \mathbb{R})$: The space of all left-dense continuous functions on $\mathbb{T}$.
- $\triangleleft$ $f^{(\alpha)}(t)$: The conformable fractional derivative of f of order α at $t \geq 0$.
- $\triangleleft$ $f_\Delta^{(\alpha)}(t)$: The delta conformable fractional derivative of f of order α at $t \in \mathbb{T}^\kappa$.
- $\triangleleft$ $f_\nabla^{(\alpha)}(t)$: The nabla conformable fractional derivative of f of order α at $t \in \mathbb{T}_\kappa$.
- $\triangleleft$ $C^\alpha([a, b]) = C^\alpha([a, b], \mathbb{R}) = \{f : [a, b] \to \mathbb{R},\ is\ conformal\ fractional\ differentiable\ of\ order\ \alpha\ on\ [a, b]\ and\ f^{(\alpha)} \in C([a, b], \mathbb{R})\}$.

- $\triangleleft$ $C_{rd}^{\alpha}([a,b]_{\mathbb{T}})=C_{rd}^{\alpha}([a,b]_{\mathbb{T}},\mathbb{R})=\{f:[a,b]_{\mathbb{T}}\to\mathbb{R}$ *is delta conformal fractional differentiable of order* α *on* $[a,b]_{\mathbb{T}}$ *and* $f_{\Delta}^{(\alpha)}\in C_{rd}([a,b]_{\mathbb{T}},\mathbb{R})\}$.

- $\triangleleft$ $C_{a,b;rd}^{\alpha}([a,b]_{\mathbb{T}},\mathbb{R})=\{f\in C_{rd}^{\alpha}([a,b]_{\mathbb{T}},\mathbb{R}):f(a)=f(b)\}$.

- $\triangleleft$ $L_{\alpha,\Delta}^{1}([a,\ b]_{\mathbb{T}},\mathbb{R})=\{\varphi:\mathbb{T}\longrightarrow\overline{\mathbb{R}}:\int_{[a,\ b]_{\mathbb{T}}}|\varphi(t)|\Delta^{\alpha}t<+\infty\}$ is a Banach space together with the norm

$$\|\varphi\|_{L_{\alpha,\Delta}^{1}([a,\ b]_{\mathbb{T}},\mathbb{R})}:=\int_{[a,\ b]_{\mathbb{T}}}|\varphi(t)|\Delta^{\alpha}t.$$

- $\triangleleft$ $W_{\Delta;a,b}^{\alpha,1}([a,b]_{\mathbb{T}},\mathbb{R})=\Big\{f\in L_{\alpha,\Delta}^{1}([a,\ b]_{\mathbb{T}},\mathbb{R}):f_{\Delta}^{(\alpha)}\in L_{\alpha,\Delta}^{1}([a,\ b]_{\mathbb{T}},\mathbb{R})$ *and there exists* $g:[a,b]_{\mathbb{T}}^{\kappa}\to\overline{\mathbb{R}}$ *such that* $g\in L_{\alpha,\Delta}^{1}([a,b]_{\mathbb{T}},\mathbb{R})$ *and* $\displaystyle\int_{[a,b]_{\mathbb{T}}}f(t)\phi_{\Delta}^{(\alpha)}(t)\Delta^{\alpha}t=-\int_{[a,b]_{\mathbb{T}}}g(t)\phi^{\sigma}(t)\Delta^{\alpha}t,\quad\text{for all }\phi\in C_{a,b;rd}^{\alpha}([a,b]_{\mathbb{T}},\mathbb{R})\Big\}$, *with* $\displaystyle\|\varphi\|_{W_{\Delta;a,b}^{\alpha,1}([a,b]_{\mathbb{T}},\mathbb{R})}:=\int_{[a,b]_{\mathbb{T}}}|\varphi^{\sigma}(t)|\Delta^{\alpha}t+\int_{[a,b]_{\mathbb{T}}}|\varphi_{\Delta}^{(\alpha)}(t)|\Delta^{\alpha}t$, *for* $\varphi\in W_{\Delta;a,b}^{\alpha,1}([a,b]_{\mathbb{T}},\mathbb{R})$.

Chapter 1

Preliminaries

In this chapter, we present some definitions and results which we will use in this Thesis.

1.1 Elements of Functional Analysis

Definition 1.1.1. *[80]. Let E, F be Banach spaces and $T : E \to F$.*

(i) The operator T is said to be bounded if it maps any bounded subset of E into a bounded subset of F.

(ii) The operator T is called compact if $T(E)$ is relatively compact (i.e., $\overline{T(E)}$ is compact).

(iii) The operator T is said to be completely continuous if it is continuous and maps any bounded subset of E into a relatively compact subset of F.

Lemma 1.1.1. *[75]. Let E be a Banach space and $u : [0,1] \to E$ be an absolutely continuous function, then the measure of the set $\{t \in [0,1] : u(t) = 0 \text{ and } u'(t) \neq 0\}$ is zero.*

Theorem 1.1.1. (*Arzela-Ascoli theorem [77]*). *A subset $\mathcal{F}$ of $C([a,b],\mathbb{R}^n)$ is relatively compact (i.e. $\overline{\mathcal{F}}$ is compact) if and only if the following conditions hold:*

1. *$\mathcal{F}$ is uniformly bounded i.e, there exists $M > 0$ such that*

$$\|f(t)\| < M \, for\, each\, t \in [a,b] \text{ and each } f \in \mathcal{F}.$$

2. *$\mathcal{F}$ is equicontinuous i.e, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for each $t_1, t_2 \in [a,b]$, $|t_2 - t_1| \leqslant \delta$ implies $\|f(t_2) - f(t_1)\| \leqslant \varepsilon$, for every $f \in \mathcal{F}$.*

Theorem 1.1.2. (*Schauder's fixed point theorem [57]*). *Let C be a convex (not necessarily closed) subset of a normed linear space E. Then each compact map $N : C \to C$ has at least one fixed point.*

Theorem 1.1.3. (*Dunford-Pettis theorem [46]*). *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $L^1([a,b])$. If there exists a function $g \in L^1([a,b])$ such that $|f_n(t)| \leq |g(t)|$ a.e. $t \in [a,b]$ and for every $n \in \mathbb{N}$, then $\{f_n\}_{n\in\mathbb{N}}$ has a weakly convergent subsequence in $L^1([a,b])$.*

1.2 Multivalued Maps

we recall some definitions and classical results for multivalued maps. The reader is referred to [14, 15, 22, 35, 45, 46, 48, 57, 64, 65] for more details on multivalued maps.

Let X, Y be metric spaces and $G : X \to Y$ a multivalued map. The map G is upper semi-continuous (*u.s.c.*) if $\{x \in X : G(x) \cap C \neq \emptyset\}$ is closed for every closed set $C \subset Y$ and it is compact if $G(X) = \cup_{x \in X} G(x)$ is relatively compact. Let Ω be a measurable space, we say that a multivalued map $G : \Omega \to X$ is measurable (resp. weakly measurable) if $\{t \in \Omega : G(t) \cap C \neq \emptyset\}$ is measurable for every closed (resp. open) set $C \subset X$.

Proposition 1.2.1. *Let $G : \Omega \to X$ be a multivalued map.*

(a) If G is measurable then it is weakly measurable.

(b) If G is weakly measurable and has compact values, then it is measurable.

(c) The map G is weakly measurable if and only if the multivalued map $\overline{G} : \Omega \to X$ defined by $\overline{G}(t) = \overline{G(t)}$ is weakly measurable.

Proposition 1.2.2. *For $n \in \mathbb{N}$, let $G_n : \Omega \to X$ be measurable multivalued maps.*

(a) The map $G : \Omega \to X$ defined by $G(t) = \cup_{n \in \mathbb{N}} G_n(t)$ is measurable.

(b) If X is separable, G_n has closed values, and for each t, at least one $G_{n_t}(t)$ is compact, then $G : \Omega \to X$ defined by $G(t) = \cap_{n \in \mathbb{N}} G_n(t)$ is measurable.

Theorem 1.2.1. (*Kuratowski-Ryll Nardzewski*) *Let X be a separable Banach space and let $G : \Omega \to X$ be a measurable multivalued map. Then G has a measurable selection, i.e. there exists a single-valued measurable map $g : \Omega \to X$ such that $g(t) \in G(t)$ for almost every $t \in \Omega$.*

Theorem 1.2.2. (*Kakutani fixed point theorem*). *Let C be a nonempty convex subset of a normed space X. If $T : C \to C$ is a compact, upper semi-continuous multivalued map with nonempty, compact, convex values. Then T has a fixed point (i.e. there exists $x \in C$ such that $x \in T(x)$).*

1.3 Preliminaries on Time Scales

1.3.1 Definitions and basic properties

Let $\mathbb{T}$ be a time scale, which is a closed subset of $\mathbb{R}$. For $t \in \mathbb{T}$, we define the forward and backward jump operators $\sigma,\ \rho : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \ \ \textit{and} \ \ \rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

respectively. We say that t is right-scattered (resp., left-scattered) if $\sigma(t) > t$ (resp., if $\rho(t) < t$); that t is isolated if it is right-scattered and left-scattered. Also, if $t < \sup \mathbb{T}$

and $t = \sigma(t)$, we say that t is right-dense. If $t > \inf \mathbb{T}$ and $t = \rho(t)$, we say that t is left dense. Points that are right dense and left dense are called dense. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$. If $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^\kappa = \mathbb{T}\backslash\{M\}$, otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. The backward graininess $\nu : \mathbb{T} \to [0, \infty)$ is defined by $\nu(t) := t - \rho(t)$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_\kappa = \mathbb{T}\backslash\{m\}$, otherwise, $\mathbb{T}_\kappa = \mathbb{T}$. For $a, b \in \mathbb{T}$ we define the closed interval $[a, b]_\mathbb{T} := \{t \in \mathbb{T} : a \leq t \leq b\}$.

If $f : \mathbb{T} \to \mathbb{R}$, is a function, then we define the function f^ρ (resp., f^σ) by

$$f^\rho(t) = (fo\rho)(t) = f(\rho(t)) \text{ (resp., } f^\sigma(t) = (fo\sigma)(t) = f(\sigma(t))) \text{ for all } t \in \mathbb{T}.$$

Definition 1.3.1. *[33]. The function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$, write $f \in C_{rd}(\mathbb{T}, \mathbb{R})$.*

Definition 1.3.2. *[33](Delta derivative) Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there exists a neighborhood U of t such that*

$$\left|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)\right| \leq \varepsilon |\sigma(t) - s|, \qquad for\ all\ s \in U.$$

We call $f^\Delta(t)$ the delta derivative (Δ-derivative) of f at t and we say that f is delta differentiable on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.
The set of functions $f : \mathbb{T} \to \mathbb{R}$ which are Δ-differentiable and whose Δ-derivative is rd-continuous is denoted by $C^1_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 1.3.3. *[33]. The function $p : \mathbb{T} \to \mathbb{R}$ is μ-regressive if*

$$1 + \mu(t)p(t) \neq 0,\ for\ all\ t \in \mathbb{T}^\kappa.$$

The set of all μ-regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R}_\mu$. We define the set ${\mathcal{R}^+}_\mu = \{p \in \mathcal{R}_\mu : 1 + \mu(t)p(t) > 0\}$ for all $t \in \mathbb{T}$.

Definition 1.3.4. *[33]. If $p \in \mathcal{R}_\mu$, then we define the delta exponential function e_p by:*

$$e_p(t, s) = \exp\Big(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\Big),$$

for $t, s \in \mathbb{T}$, where the μ-cylinder transformation is as in :

$$\xi_h(z) = \begin{cases} \dfrac{1}{h}\log(1 + zh); & if\ h > 0; \\ z; & if\ \ h = 0. \end{cases}$$

where log *is the principal logarithm function.*

Lemma 1.3.1. *[33].*

(1) If $p \in \mathcal{R}_\mu$ and $s, t, t_0 \in \mathbb{T}$, then

$$e_p(t, t) \equiv 1, e_0(t, s) \equiv 1,\ e_p(t, s) = \frac{1}{e_p(s, t)},\ and\ e_p(t, t_0)e_p(t_0, s) = e_p(t, s).$$

(2) If $p \in \mathcal{R}^+_\mu$ and $t_0 \in \mathbb{T}$, then

$$e_p(t, t_0) > 0,\ for\ all\ t \in \mathbb{T}.$$

1.3.2 Nabla calculus on time scales

Next, we introduce the nabla derivative on time scales for vector-valued functions and study some of their important properties.

Definition 1.3.5. *A function $f : \mathbb{T} \to \mathbb{R}^n$ is called ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$, write $f \in C_{ld}(\mathbb{T}, \mathbb{R}^n)$.*

Definition 1.3.6. *[91](Left-Hilger-continuous functions).* A mapping $f : (a,b]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is called left-Hilger-continuous at a point (t,x) if f is continuous at each (t,x) where t is left-dense and the limits

$$\lim_{(s,y)\to(t^+,x)} f(s,y) \quad \text{and} \quad \lim_{y\to x} f(t,y),$$

both exist and are finite at each (t,x) where t is right-dense.

Definition 1.3.7. *[83]. For $f : \mathbb{T} \to \mathbb{R}^n$ and $t \in \mathbb{T}_\kappa$, the ∇-derivative of f at t, denoted by $f^\nabla(t) \in \mathbb{R}^n$, is defined to be the vector (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U_t of t such that*

$$\left\| f^\rho(t) - f(s) - f^\nabla(t)\left(\rho(t) - s\right)\right\| \le \varepsilon \left|\rho(t) - s\right|, \quad for\ all\ s \in U_t.$$

We say that f is ∇-differentiable if $f^\nabla(t)$ exists for every $t \in \mathbb{T}_\kappa$. The function $f^\nabla : \mathbb{T} \to \mathbb{R}^n$ is then called the ∇-derivative of f on $\mathbb{T}_\kappa$. The set of functions $f : \mathbb{T} \to \mathbb{R}^n$ which are ∇-differentiable and whose ∇-derivative is ld-continuous is denoted by $C^1_{ld}(\mathbb{T}, \mathbb{R}^n)$.

The set of functions $f : \mathbb{T} \to \mathbb{R}^n$ which are ∇-differentiable and whose ∇-derivative is ld-continuous is denoted by $C^1_{ld}(\mathbb{T}, \mathbb{R}^n)$.

Theorem 1.3.1. *[18]. Let W be an open set of $\mathbb{R}^n$ and $t \in \mathbb{T}$ be a left-dense point. If $g : \mathbb{T} \to \mathbb{R}^n$ is ∇-differentiable at t and if $f : W \to \mathbb{R}$ is differentiable at $g(t) \in W$, then $f \circ g$ is ∇-differentiable at t and $(f \circ g)^\nabla(t) = \langle f'(g(t)), g^\nabla(t)\rangle$.*

Example 1.3.1. *[18]. Assume $x : \mathbb{T} \to \mathbb{R}^n$ is ∇-differentiable at $t \in \mathbb{T}$. We know that $\|.\| : \mathbb{R}^n \setminus \{0\} \to [0,\infty)$ is differentiable. If $t = \rho(t)$, by the previous theorem, we have*

$$\|x(t)\|^\nabla = \frac{< x(t), x^\nabla(t) >}{\|x(t)\|}.$$

Definition 1.3.8. *[34]. The function $p : \mathbb{T} \to \mathbb{R}$ is ν-regressive if*

$$1 - \nu(t)p(t) \neq 0,\ for\ all\ t \in \mathbb{T}_k.$$

The set of all ν-regressive and ld-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R}_\nu = \mathcal{R}_\nu(\mathbb{T}, \mathbb{R})$. We define the set $\mathcal{R}^+{}_\nu = \{p \in \mathcal{R}_\nu : 1 - \nu(t)p(t) > 0\}$ for all $t \in \mathbb{T}$.

Definition 1.3.9. *Let $f : \mathbb{T} \to \mathbb{R}$. A function $F : \mathbb{T} \to \mathbb{R}$ will be a nabla anti-derivative of f if $F^\nabla(t) = f(t)$, holds for all $t \in \mathbb{T}_\kappa$. We define the Cauchy nabla integral of f by*

$$\int_{t_0}^{t} f(s)\nabla s = F(t) - F(t_0), \quad for\ all\ t_0,\ t \in \mathbb{T}$$

Definition 1.3.10. *[34]. If $p \in \mathcal{R}_\nu$, then we define the nabla exponential function $\hat{e}_p$ by:*

$$\hat{e}_p(t,s) = \exp\Big(\int_s^t \hat{\xi}_{\nu(\tau)}(p(\tau))\nabla\tau\Big),$$

for $t, s \in \mathbb{T}$, where the ν-cylinder transformation is as in :

$$\hat{\xi}_h(z) = \begin{cases} -\dfrac{1}{h}\log(1 - zh); & if \ h > 0; \\ z; & if \ \ h = 0, \end{cases}$$

where log *is the principal logarithm function.*

Theorem 1.3.2. *[34]. If $p \in \mathcal{R}_\nu$, then the nabla exponential function $\hat{e}_p(., t_0) : \mathbb{T} \to \mathbb{R}$ is a solution of the initial value problem*

$$x^\nabla(t) = px(t), \qquad x(t_0) = 1.$$

1.3.3 Lebesgue ∇-measure and Lebesgue ∇-integral

We recall some notions and results related to the theory of ∇-measure and Lebesgue ∇-integration for an arbitrary bounded time scale $\mathbb{T}$ where $a = \min \mathbb{T} < \max \mathbb{T} = b$ introduced in [13, 34, 59].

Definition 1.3.11. *Let $\mathfrak{F}$ denote the family of all right closed and left open intervals of $\mathbb{T}$ of the form*

$$(r, s] = \{t \in \mathbb{T} : r \le t < s\},$$

with $r, s \in \mathbb{T}$ and $r \le s$. The interval $(r, r]$ is understood as the empty set. We define an additive measure $m_1 : \mathfrak{F} \to [0, \infty)$ by $m_1((r, s]) = s - r$. Using m_1, the outer measure $m_1^* : \mathcal{P}(\mathbb{T}) \to \mathbb{R}$, defined for each $E \subset \mathbb{T}$ as:

$$m_1^*(E) = \begin{cases} \inf\left\{\sum_{k=1}^{k=m}(s_k - r_k) : E \subset \bigcup_{k=1}^{k=m}(r_k, s_k] \ \ with \ \ (r_k, s_k] \in \mathfrak{F}\right\} & if \quad a \notin E, \\ +\infty & if \quad a \in E. \end{cases}$$

Definition 1.3.12. *A set $A \subset \mathbb{T}$ is said to be ∇-measurable if, for every set $E \subset \mathbb{T}$*

$$m_1^*(E) = m_1^*(E \cap A) + m_1^*(E \cap (\mathbb{T} \backslash A)).$$

The Lebesgue ∇-measure on $\mathcal{M}(m_1^*) = \{A \subset \mathbb{T} : A \text{ is } \nabla\text{-measurable}\}$, denoted by μ_∇, is the restriction of m_1^* to $\mathcal{M}(m_1^*)$. So, $(\mathbb{T}, \mathcal{M}(m_1^*), \mu_\nabla)$ is a complete measurable space.

Lemma 1.3.2. *[34]. For each t_0 in $\mathbb{T}_0$ the ∇-measure of the single-point set $\{t_0\}$ is given by $\mu_\nabla(\{t_0\}) = t_0 - \rho(t_0)$.*

Lemma 1.3.3. *[34]. 1. If $r, s \in \mathbb{T}$ and $r \le s$, then*

$$\mu_\nabla((r, s]) = s - r, \ \ \mu_\nabla((r, s)) = \rho(s) - r.$$

2. If $r, s \in \mathbb{T}_0$ and $r \le s$, then $\mu_\nabla([r, s)) = \rho(s) - \rho(r)$, $\ \mu_\nabla([r, s]) = s - \rho(r)$.

The following lemma can be proved analogously to Lemma 3.1 in [42].

Lemma 1.3.4. *The set of all left-scattered points of* $\mathbb{T}$ *is at most countable, that is, there are* $J \subseteq \mathbb{N}$ *and* $\{t_j\}_{j\in J} \subset \mathbb{T}$ *such that* $\mathcal{L}_{\mathbb{T}} := \{t \in \mathbb{T}, \rho(t) < t\} = \{t_j\}_{j\in J}$.

The following proposition can be proved analogously to Proposition 3.1 in [42].

Proposition 1.3.1. *Let* $A \subset \mathbb{T}$. *Then* A *is a* ∇*-measurable if and only if,* A *is Lebesgue measurable. In this case the following properties hold for every* ∇*-measurable set* A :
1. If $a \notin A$, *then* $\mu_\nabla(A) = \mu_L(A) + \sum_{j\in J_A} \nu(t_j)$.
2. $\mu_\nabla(A) = \mu_L(A)$ *if and only if* $a \notin A$ *and* A *has no left-scattered point.*

The notions of ∇-measurable and ∇-integrable functions $f : \mathbb{T} \to \mathbb{R}^n$ can be defined similarly to the theory of Lebesgue integral.

Definition 1.3.13. *We say that* $f : \mathbb{T} \longrightarrow \overline{\mathbb{R}} = [-\infty, +\infty]$ *is* ∇*-measurable if for every* $\alpha \in \mathbb{R}$, *the set* $f^{-1}([-\infty, \alpha)) = \{t \in \mathbb{T} : f(t) < \alpha\}$ *is* ∇*-measurable.*

In order to compare the Lebesgue ∇-integral on $\mathbb{T}$ and the Lebesgue integral on $[a, b]$, given a function $f : \mathbb{T} \longrightarrow \mathbb{R}^n$, we need an auxiliary function which extends $\widetilde{f}$ to the interval $[a, b]$ defined as

$$\widetilde{f}(t) := \begin{cases} f(t), & \text{if } t \in \mathbb{T}, \\ f(t_j), & \text{if } t \in (\rho(t_j), t_j)\text{, for all } j \in J. \end{cases} \tag{1.1}$$

Let $E \subset \mathbb{T}$, we define $J_E := \{j \in J : t_j \in E \cap \mathcal{L}_{\mathbb{T}}\}$ and

$$\widetilde{E} := E \cup \bigcup_{j\in J_E} (\rho(t_j), t_j). \tag{1.2}$$

The following theorem can be proved analogously to Theorem 5.1 in [42].

Theorem 1.3.3. *Let* $E \subset \mathbb{T}$ *be a* ∇*-measurable such that* $a \notin E$, *let* $\widetilde{E}$ *be the set defined in* (1.2), *let* $f : \mathbb{T} \longrightarrow \mathbb{R}^n$ *be a* ∇*-measurable function and* $\widetilde{f} : [a, b] \longrightarrow \mathbb{R}^n$ *be the extension of* f *to* $[a, b]$. *Then,* f *is Lebesgue* ∇*-integrable on* E *if and only if* $\widetilde{f}$ *is Lebesgue integrable on* $\widetilde{E}$ *and we have*

$$\int_E f(t)\,\nabla t = \int_{\widetilde{E}} \widetilde{f}(t)\,dt = \int_E f(t)\,dt + \sum_{j\in J_E} \nu(t_j)\, f(t_j). \tag{1.3}$$

1.3.4 Sobolev's spaces on time scales

In this section, we develop the Sobolev's spaces on bounded time scale $\mathbb{T}$ where $a = \min \mathbb{T} < \max \mathbb{T} = b$, $\mathbb{T}_0 = \mathbb{T}\backslash\{a\}$ and their important properties.

Definition 1.3.14. *Let* $p \in [1, +\infty)$, $E \subset \mathbb{T}$ *be a* ∇*-measurable set and* $f : \mathbb{T} \to \mathbb{R}^n$ *be a* ∇*-measurable function. We say that* $f \in L^p_\nabla(E, \mathbb{R}^n)$ *(respectively* $f \in L^p_\nabla(\mathbb{T}, \mathbb{R}^n)$*) provided*

$$\int_E \|f(s)\|^p \nabla s < +\infty \ (\textit{respectively} \int_{\mathbb{T}_0} \|f(s)\|^p \nabla s < +\infty).$$

Proposition 1.3.2. *Assume $f \in L^1_\nabla(E, \mathbb{R}^n)$. Then,*

$$\left\| \int_E f(s)\nabla s \right\| \leq \int_E \|f(s)\| \nabla s.$$

Here is an analog of the Lebesgue dominated convergence theorem.

Theorem 1.3.4. *Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of functions in $L^1_\nabla(\mathbb{T}_0, \mathbb{R}^n)$. If there exists a function $f : \mathbb{T}_0 \to \mathbb{R}^n$ such that $f_k(t) \to f(t)$ ∇-a.e. $t \in \mathbb{T}_0$ and if there exists a function $g \in L^1_\nabla(\mathbb{T}_0)$ such that $\|f_k(t)\| \leq g(t)$ ∇-a.e. $t \in \mathbb{T}_0$ and for every $k \in \mathbb{N}$. Then $f_k \to f$ in $L^1_\nabla(\mathbb{T}_0, \mathbb{R}^n)$.*

The following proposition can be proved analogously to Proposition 3.1 in [20].

Proposition 1.3.3. *Let $p \in [1, +\infty)$, $L^p_\nabla(\mathbb{T}, \mathbb{R}^n)$ is a Banach space equipped with the norm*

$$\|f\|_{L^p_\nabla(\mathbb{T},\mathbb{R}^n)} := \left(\int_{\mathbb{T}_0} \|f(t)\|^p \nabla t \right)^{\frac{1}{p}}.$$

Using Theorem 1.3.3, we obtain the following result.

Theorem 1.3.5. *Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of functions in $L^1_\nabla(\mathbb{T}_0, \mathbb{R}^n)$. If $\{\widetilde{f_k}\}$ converges weakly to γ in $L^1([a,b], \mathbb{R}^n)$, then γ is the extension $\widetilde{f}$ of a function f defined on $\mathbb{T}_0$ in the sense of definition (1.1). Moreover, for every ∇-measurable set $E \subset \mathbb{T}_0$ and every continuous function $g : \mathbb{T} \to \mathbb{R}$, we have*

$$\lim_{k\to\infty} \int_E g(s) f_k(s) \nabla s = \int_E g(s) f(s) \nabla s.$$

Proof. Since $\{\widetilde{f_k}\}$ converges weakly to γ in $L^1([a,b], \mathbb{R}^n)$, we have for every continuous function $g : \mathbb{T} \to \mathbb{R}$,
$\int_A \widetilde{g}(s) \widetilde{f}_k(s) ds \to \int_A \widetilde{g}(s)\gamma(s) ds$ for every measurable set $A \subset [a,b]$.

Thus, for $t_i \in R_\mathbb{T}$,

$$\begin{aligned} \int_{(\rho(t_i),t_i)} \widetilde{g}(s)\widetilde{f}_k(s) ds = \int_{(\rho(t_i),t_i)} g(t_i) f_k(t_i) ds &= g(t_i) f_k(t_i) \nu(t_i) \\ &\to \int_{(\rho(t_i),t_i)} \widetilde{g}(s)\gamma(s) ds. \end{aligned}$$

So, $\{f_k(t_i)\}_{k\in\mathbb{N}}$ converges to some $f(t_i) \in \mathbb{R}^n$. Thus, $\{\widetilde{f_k}\}$ converges strongly to the constant function $f(t_i)$ in $L^1_\nabla((\rho(t_i), t_i), \mathbb{R}^n)$, and we can assume that $\gamma = f(t_i)$ on $(\rho(t_i), t_i]$. The first part of the proposition is proved if we define $f = \gamma \mid_\mathbb{T}$. Finally, by Theorem 1.3.3,

$$\begin{aligned} \int_E g(s) f_k(s) \nabla s &= \int_{\widetilde{E}} \widetilde{g}(s) \widetilde{f}_k(s) ds \\ &\to \int_{\widetilde{E}} \widetilde{g}(s)\gamma(s) ds = \int_{\widetilde{E}} \widetilde{g}(s) f(s) ds = \int_E g(s) f(s) \nabla s. \end{aligned}$$

□

Now we introduce the concept of absolutely continuous function on $\mathbb{T}$.

Definition 1.3.15. *A function $f : \mathbb{T} \to \mathbb{R}^n$ is said to be absolutely continuous on $\mathbb{T}$ if for every $\varepsilon > 0$, there exists a $\eta > 0$ such that if $\{(a_k, b_k], k = 1, ..., m\}$, with $a_k, b_k \in \mathbb{T}$, is a finite pairwise disjoint family of subintervals of $\mathbb{T}$ satisfying*

$$\sum_{k=1}^{k=m}(b_k - a_k) < \eta \quad then \quad \sum_{k=1}^{k=m} \|f(b_k) - f(a_k)\| < \varepsilon.$$

The following theorem can be proved analogously to Theorem 4.1 in [40].

Theorem 1.3.6. *A function $f : \mathbb{T} \to \mathbb{R}^n$ is absolutely continuous on $\mathbb{T}$ if and only if f is ∇-différentiable ∇-almost everywhere on $\mathbb{T}_0$, $f^\nabla \in L^1_\nabla(\mathbb{T}_0, \mathbb{R}^n)$ and*

$$\int_{(t,b]\cap\mathbb{T}} f^\nabla(s)\,\nabla s = f(b) - f(t), \qquad for\ every\ t \in \mathbb{T}.$$

The following two propositions can be proved analogously to Proposition 2.19 and Proposition 2.20 in [53].

Proposition 1.3.4. *Let $f \in L^1_\nabla(\mathbb{T}_0, \mathbb{R}^n)$, then $F : \mathbb{T} \to \mathbb{R}^n$ defined by*

$$F(t) = \int_{(t,b]\cap\mathbb{T}} f(s)\,\nabla s \ satisfies\ F^\nabla(t) = f(t),\ \nabla\text{-a.e. on } \mathbb{T}_0.$$

Proposition 1.3.5. *Let $u : \mathbb{T} \to \mathbb{R}$ be an absolutely continuous function, then the ∇-measure of the set $\{t \in \mathbb{T}_0 \backslash \mathcal{L}_{\mathbb{T}_0} : u(t) = 0$ and $u^\nabla(t) \neq 0\}$ is zero.*

The following theorem can be proved analogously to Theorem 3.2 in [20].

Theorem 1.3.7. *Let $p \in [1, \infty)$, then $\boldsymbol{C}(\mathbb{T}, \mathbb{R}^n)$ is dense in $L^p_\nabla(\mathbb{T}, \mathbb{R}^n)$.*

We now define a notion of Sobolev's space.

Definition 1.3.16. *Let $p \in [1, \infty)$, and $f : \mathbb{T} \to \mathbb{R}^n$. Say that f belongs to $W^{1,p}_\nabla(\mathbb{T}, \mathbb{R}^n)$ if and only if $f \in L^p_\nabla(\mathbb{T}_0, \mathbb{R}^n)$ and there exists $g : \mathbb{T}_k \to \mathbb{R}^n$ such that $g \in L^p_\nabla(\mathbb{T}_0, \mathbb{R}^n)$ and*

$$\int_{\mathbb{T}_0} \left(f.\phi^\nabla\right)(s)\,\nabla s = -\int_{\mathbb{T}_0} (g.\phi^\rho)(s)\,\nabla s, \qquad for\ all\ \phi \in C^1_{0,ld}(\mathbb{T}), \tag{1.4}$$

with

$$C^1_{0,ld}(\mathbb{T}) := \left\{\phi \in C^1_{ld}(\mathbb{T}) : \phi(a) = \phi(b) = 0\right\}.$$

The following theorem can be proved analogously to Theorem 3.4 in [6].

Theorem 1.3.8. *Suppose that $u \in W^{1,1}_\nabla(\mathbb{T}, \mathbb{R}^n)$ and that (1.4) holds for a function $g \in L^1_\nabla(\mathbb{T}, \mathbb{R}^n)$. Then, there exists a unique function $x : \mathbb{T} \longrightarrow \mathbb{R}^n$ absolutely continuous such that ∇-almost everywhere on $\mathbb{T}_0$, one has $x = u$ and $x^\nabla = g$. Moreover, if g is ld-continuous on $\mathbb{T}_0$, then there exists a unique function $x \in C^1_{ld}(\mathbb{T}, \mathbb{R}^n)$ such that $x = u$ ∇-almost everywhere on $\mathbb{T}_0$ and such $x^\nabla = g$ on $\mathbb{T}_0$.*

Theorem 1.3.9. *Let $p \in [1, \infty)$. The set $W^{1,p}_{\nabla}(\mathbb{T}, \mathbb{R}^n)$ is a Banach space together with the norm defined for every $f \in W^{1,p}_{\nabla}(\mathbb{T}, \mathbb{R}^n)$ as*

$$\|f\|_{W^{1,p}_{\nabla}(\mathbb{T},\mathbb{R}^n)} = \|f\|_{L^p_{\nabla}(\mathbb{T},\mathbb{R}^n)} + \left\|f^{\nabla}\right\|_{L^p_{\nabla}(\mathbb{T}_k,\mathbb{R}^n)}.$$

The proof is analogous to that of Theorem 3.5 in [6].

Remark 1.3.1. *If $x \in W^{1,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n)$, then its components $x_i \in W^{1,1}_{\nabla}(\mathbb{T}, \mathbb{R})$. By Theorems 1.3.8 and 1.3.6, x is ∇-differentiable ∇-almost every on $\mathbb{T}$. From Example 1.3.1, we obtain*

$$\|x(t)\|^{\nabla} = \frac{< x(t), x^{\nabla}(t) >}{\|x(t)\|} \quad \nabla\text{-}a.e.\ on\ \{t \in \mathbb{T} :\ t = \rho(t)\}.$$

Next, we define a notion of ∇-Carathéodory functions (resp. multivalued maps) on a compact time scale.

Definition 1.3.17. *A function $f : \mathbb{T}_0 \times \mathbb{R}^n \to \mathbb{R}^n$ is called a ∇-Carathéodory function if the three following conditions hold.*

(i) for every $x \in \mathbb{R}^n$, the function $t \mapsto f(t, x)$ is ∇-measurable;

(ii) the function $x \mapsto f(t, x)$ is continuous ∇-almost every $t \in \mathbb{T}_0$;

(iii) for every $r > 0$, there exists a function $h_r \in L^1_{\nabla}(\mathbb{T}_0, [0, \infty))$ such that $\|f(t, x)\| \leq h_r(t)$ for ∇-almost every $t \in \mathbb{T}_0$ and for all $x \in \mathbb{R}^n$ such that $\|x\| \leq r$.

Definition 1.3.18. *A multivalued map $F : \mathbb{T}_0 \times \mathbb{R}^n \to \mathbb{R}^n$ with compact and convex values is said to be ∇-Carathéodory if the three following conditions hold.*

(i) for every $x \in \mathbb{R}^n$, the function $t \mapsto F(t, x)$ is ∇-measurable;

(ii) the function $x \mapsto F(t, x)$ is upper semi-continuous (u.s.c.) ∇-a.e. $t \in \mathbb{T}_0$;

(iii) for every $q > 0$, there exists a function $h_q \in L^1_{\nabla}(\mathbb{T}_0, [0, \infty))$ such that

$$sup\,\{\|y\| : y \in F(t, x), \|x\| \leq q\} \leq h_q(t), \quad \nabla\text{-a.e. } t \in \mathbb{T}_0.$$

A single-valued mapping $h : \mathbb{T}_0 \times \mathbb{R}^n \to \mathbb{R}^n$ is a ∇-Carathéodory if and only if $F = \{h\}$ is ∇-Carathéodory in the sense of Definition 1.3.18.

1.4 Conformable Fractional Calculus

In this section, we introduce some necessary definitions and properties of the conformable fractional calculus which are used in this thesis and can be found in [1, 66, 70, 79] and in [87](If $\mathbb{T}$ is a real interval $[0, \infty)$) are given:

Definition 1.4.1. *[70]. Given a function $f : [0, \infty) \to \mathbb{R}$ and a real constant $\alpha \in (0, 1]$. The conformable fractional derivative of f of order α is defined by,*

$$f^{(\alpha)}(t) := \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \tag{1.5}$$

for all $t > 0$.

If $f^{(\alpha)}(t)$ exists and is finite, we say that f is α-differentiable at t.

If f is α-differentiable in some interval $(0, a)$, $a > 0$, and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then the conformable fractional derivative of f of order α at $t = 0$ is defined as

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t).$$

Example 1.4.1. *Conformable fractional derivatives of certain functions as follow:*

1. $(t^p)^{(\alpha)} = p\, t^{p-\alpha}$, *for all* $p \in \mathbb{R}$.
2. $(\lambda)^{(\alpha)} = 0$, *for all* $\lambda \in \mathbb{R}$.
3. $(e^{pt})^{(\alpha)} = p\, t^{1-\alpha} e^{pt}$, *and* $(e^{\frac{p}{\alpha} t^{\alpha}})^{(\alpha)} = p\, e^{\frac{p}{\alpha} t^{\alpha}}$, *for all* $p \in \mathbb{R}$.

Definition 1.4.2. *[87]. Assume $f : [0, \infty) \to \mathbb{R}^n$, $f(t) := (f_1(t), f_2(t), ..., f_n(t))$ and let $\alpha \in (0, 1]$ and $t \geq 0$. Then one defines $f^{(\alpha)}(t) = (f_1^{(\alpha)}(t), f_2^{(\alpha)}(t), \cdots, f_n^{(\alpha)}(t))$ (provided it exists). One calls $f^{(\alpha)}(t)$ the conformable fractional derivative of f of order α at $t > 0$. Function f is conformal fractional differentiable of order α provided $f^{(\alpha)}(t)$ exists for all $t > 0$, in such a case, we say that f is α-differentiable at t. We define the conformable fractional derivative at 0 as $f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t)$, provided it exists.*

Definition 1.4.3. *[87]. Let $\alpha \in (m,\ m + 1]$, $m \in \mathbb{N}$, and $f : [0, \infty) \to \mathbb{R}^n$, where $f^{(m)}(t)$ exists at $t > 0$. We define the conformable fractional derivative of f of order α as*

$$f^{(\alpha)}(t) := (f^{(m)})^{(\alpha - m)}(t).$$

Theorem 1.4.1. *[87]. If a function $f : [0, \infty) \to \mathbb{R}^n$ is α-differentiable at $t > 0$, $\alpha \in (0, 1]$, then f is continuous at t.*

Remark 1.4.1. *(i) The Riemann-liouville derivative D_a^{α} does not satisfy $D_a^{\alpha}(1) = 0$, if f is not a natural number. ($D_a^{\alpha}(1) = 0$ for the Caupto derivative).*

(ii) All fractional derivatives do not satisfy the Known product rule:

$$D_a^{\alpha}(fg) = f D_a^{\alpha}(g) + g D_a^{\alpha}(f).$$

(iii) All fractional derivatives do not satisfy the known quotient rule:

$$D_a^{\alpha}(f/g) = \frac{g D_a^{\alpha}(f) - f D_a^{\alpha}(g)}{g^2}.$$

(iv) All fractional derivatives do not satisfy the chain rule:

$$D_a^{\alpha}(fog) = f^{\alpha}(g)g^{\alpha}.$$

(v) All fractional derivatives don't satisfy: $D^{\alpha}D^{\beta}f = D^{\alpha+\beta}f$ *in general.*

(vi) The Caputo definition assumes that the function f *is differentiable.*

Theorem 1.4.2. *[87]. Let* $\alpha \in (0,1]$ *and assume* $f, g : [0,\infty) \to \mathbb{R}^n$ *are* α*-differentiable at* $t > 0$*. Then, by denoting* $(fg)(t) = (f_1(t)\, g_1(t), \cdots, f_n(t)\, g_n(t))$*, we have the following properties:*

(i) $(af + bg)^{(\alpha)} = af^{(\alpha)} + bg^{(\alpha)}$, *for all* $a, b \in \mathbb{R}$*;*

(ii) $(fg)^{(\alpha)} = fg^{(\alpha)} + gf^{(\alpha)}$*;*

(iii) $(f/g)^{(\alpha)} = \dfrac{gf^{(\alpha)} - fg^{(\alpha)}}{g^2}$.

(iv) If, in addition, f *is differentiable at a point* $t > 0$*, then*

$$f^{(\alpha)}(t) = t^{1-\alpha}f'(t).$$

Remark 1.4.2. *It is not difficult to verify the following assertions:*

(i) The function $x : t \mapsto e^{\frac{p}{\alpha}t^{\alpha}}$, $p \in \mathbb{R}$, *is the unique solution to the conformable fractional differential equation*

$$x^{(\alpha)}(t) = p\,x(t),\ t \in [0,\infty),\ x(0) = 1.$$

(ii) If f *is differentiable at* t*, then* f *is* α*-differentiable at* t*.*

We introduce the following spaces: we assume $I = [0,b],\ b > 0$.

$$\begin{aligned}
C^{\alpha}(I,\mathbb{R}^n) &= \{f : I \to \mathbb{R}^n, \text{ is } \alpha\text{-differentiable on } I \text{ and } f^{(\alpha)} \in C(I,\mathbb{R}^n)\}.\\
C_0^{\alpha}(I,\mathbb{R}^n) &= \{f \in C^{\alpha}(I,\mathbb{R}^n) : f(0) = f(b) = 0\}.\\
C_{0,b}^{\alpha}(I,\mathbb{R}^n) &= \{f \in C^{\alpha}(I,\mathbb{R}^n) : f(0) = f(b)\}.
\end{aligned}$$

Definition 1.4.4. *[70]. Let* $\alpha \in (0,1]$ *and* $f : [0,\infty) \to \mathbb{R}$*. The conformable fractional integral of* f *of order* α *from* 0 *to* t*, denoted by* $I_{\alpha}(f)(t)$*, is defined by*

$$I_{\alpha}(f)(t) := I_1(t^{\alpha-1}f)(t) = \int_0^t f(s)d_{\alpha}s := \int_0^t f(s)s^{\alpha-1}ds.$$

The considered integral is the usual improper Riemann one.

Definition 1.4.5. *[87]. Let $f : [0, \infty) \to \mathbb{R}^n$ and $\alpha \in (0, 1]$. The conformable fractional integral of f of order α from 0 to t, denoted by $I_\alpha(f)(t)$, is defined by*

$$I_\alpha(f)(t) = \int_0^t f(s) d_\alpha s = \Big(I_\alpha(f_1)(t), I_\alpha(f_2)(t), \cdots, I_\alpha(f_n)(t)\Big),$$

where $I_\alpha(f_i)(t)$ is the conformable fractional integral of f_i of order α from 0 to t, for $i = 1, ..., n$.

Lemma 1.4.1. *[70, 79]. Let $0 < \alpha \leq 1$ and $f : [0, \infty) \to \mathbb{R}^n$ be a continuous function in the domain of I_α. Then for all $t \geq 0$ we have*

$$(I_\alpha(f))^{(\alpha)}(t) = f(t).$$

Corollary 1.4.1. *[1, 87]. Let $f : [0, b) \to \mathbb{R}^n$ be such that $I_\alpha(f^\alpha)(t)$ exists for $0 < t < b$. Then, f is differentiable on $(0, b)$.*

Lemma 1.4.2. *[1, 87]. Let $f : (0, b) \to \mathbb{R}^n$ be differentiable and $0 < \alpha \leq 1$. Then, for all $t > 0$ we have*

$$I_\alpha(f^\alpha)(t) = f(t) - f(0). \tag{1.6}$$

The next result is an adaptation of Lemma 2 in [79]

Proposition 1.4.1. *Let $0 < \alpha \leq 1$, and W be an open set of $\mathbb{R}^n$. If $g : I \to \mathbb{R}^n$ is α-differentiable at $t > 0$ and $f : W \to \mathbb{R}^m$ is differentiable at $g(t) \in W$. Then $f \circ g$ is α-differentiable at t and*

$$(f \circ g)^{(\alpha)}(t) = f'(g(t)) \left(g^{(\alpha)}(t)\right)^T.$$

Here v^T denotes the transpose vector of v.

Example 1.4.2. *Let $\alpha \in (0, 1]$, and $x : [0, \infty) \to \mathbb{R}^n$ α-differentiable at t. It is not difficult to verify that the Eucliden norm $\| \cdot \| : \mathbb{R}^n \setminus \{0\} \to [0, \infty)$ defined as*

$$\|x(t)\| =< x(t), x(t) >^{1/2},$$

with $< \cdot, \cdot >$ the usual scalar product in $\mathbb{R}^n$, is differentiable.

By the previous Proposition, we have

$$\|x(t)\|^{(\alpha)} = \frac{< x(t), x^{(\alpha)}(t) >}{\|x(t)\|}.$$

Next, we develop the fractional Sobolev's spaces via conformable fractional calculus and their important properties. The basic definitions and relations based on [87] (If $\mathbb{T}$ is a real interval $[0, \infty)$) are given:

Definition 1.4.6. *Let $B \subset I$. B is called null set if the measure of B is zero. We say that a property P holds almost everywhere (a.e.) on B, or for almost all (a.a.) $t \in B$ if there is a null set $E_0 \subset B$ such that P holds for all $t \in B \setminus E_0$.*

Definition 1.4.7. *Let A be a Lebesgue measurable subset of I. We say that function $f : I \to \mathbb{R}$, is a function α-integrable on A if and only if $t^{\alpha-1}f(t)$ is Lebesgue integrable on A. In such a case, we denote*

$$\int_A f(t)\, d_\alpha t = \int_A t^{\alpha-1}\, f(t)\, dt.$$

Definition 1.4.8. *[87]. Let $E \subset \mathbb{R}$ be a measurable set, and let $\varphi : E \to \mathbb{R}$ be a measurable function. We say that φ belongs to $L^1_\alpha(E, \mathbb{R})$ is the following property is fulfilled*

$$\int_E |\varphi(s)|\, d_\alpha s = \int_E |\varphi(s)|\, s^{\alpha-1} ds < +\infty.$$

We say that a measurable function $f : E \to \mathbb{R}^n$ is in the set $L^1_\alpha(E, \mathbb{R}^n)$ provided

$$\int_E \|f(s)\|\, d_\alpha s = \int_E \|f(s)\|\, s^{\alpha-1} ds < +\infty.$$

$$i.e.\quad f_i \in L^1_\alpha(E, \mathbb{R}), \quad for\ each\ of\ its\ components\ f_i : E \to \mathbb{R}, i = 1, ..., n.$$

Theorem 1.4.3. *[87]. The set $L^1_\alpha(I, \mathbb{R}^n)$ is a Banach space together with the norm defined for $\varphi \in L^1_\alpha(I, \mathbb{R}^n)$ as*

$$\|\varphi\|_{L^1_\alpha(I,\mathbb{R}^n)} := \int_I \|\varphi(t)\| d_\alpha t.$$

Remark 1.4.3. *It is not difficult to verify the following assertions for all $\alpha \in (0, 1]$:*

(i) $L^1_\alpha(I, \mathbb{R}^n) \subset L^1(I, \mathbb{R}^n)$.

(ii) *For $t \in I$, $t > 0$ and $\varphi : I \to \mathbb{R}^n$, it is satisfied that $\varphi^{(\alpha)} \in L^1_\alpha(I, \mathbb{R}^n)$ if and only if $\varphi' \in L^1(I, \mathbb{R}^n)$.*

Theorem 1.4.4. *[87]. Let $f \in L^1_\alpha(I, \mathbb{R}^n)$. Then, a necessary and sufficient condition for the validity of the equality:*

$$\int_I f(t) h^{(\alpha)}(t) d_\alpha t = 0 \quad for\ every\ h\ \in C^\alpha_{0,b}(I, \mathbb{R}^n),$$

is the existence of a constant $C \in \mathbb{R}^n$ such that $f \equiv C$ a.e. on I.

Definition 1.4.9. *A function $f : I \to \mathbb{R}^n$ is said to be absolutely continuous on I (i.e., $f \in AC(I, \mathbb{R}^n)$) if for every $\varepsilon > 0$, there exists $\eta > 0$ such that if $\{[a_k, b_k[\}_{k=1}^m$, is a finite pairwise disjoint family of subintervals of I satisfying*

$$\sum_{k=1}^{k=m} (b_k - a_k) < \eta, \ then\ \sum_{k=1}^{k=m} \|f(b_k) - f((a_k))\| < \varepsilon.$$

Theorem 1.4.5. *[87]. Assume function $f : I \to \mathbb{R}^n$ is absolutely continuous on I, then f is conformable fractional differentiable of order α a.e. on I and the following equality is valid:*

$$f(t) = f(0) + \int_{[0,t]} f^{(\alpha)}(s) d_\alpha s, \quad for\ all\ t \in I.$$

Definition 1.4.10. *Let $\alpha \in (0,1]$ and $f : I \to \mathbb{R}^n$. One says that $f \in W^{\alpha,1}_{0,b}(I, \mathbb{R}^n)$ if and only if $f \in L^1_\alpha(I, \mathbb{R}^n)$ and there exists $g : I \to \mathbb{R}^n$ such that $g \in L^1_\alpha(I, \mathbb{R}^n)$ and*

$$\int_I f(t)\phi^{(\alpha)}(t) d_\alpha t = -\int_I g(t)\phi(t) d_\alpha t, \qquad for\ all\ \phi \in C^\alpha_{0,b}(I, \mathbb{R}^n). \tag{1.7}$$

- We denote

$$V^{\alpha,1}_{0,b}(I, \mathbb{R}^n) = \{f \in AC(I, \mathbb{R}^n) : f^{(\alpha)} \in L^1_\alpha(I, \mathbb{R}^n), f(0) = f(b)\}.$$

Remark 1.4.4. *We have $V^{\alpha,1}_{0,b}(I, \mathbb{R}^n) \subset W^{\alpha,1}_{0,b}(I, \mathbb{R}^n)$.*

Theorem 1.4.6. *[87]. Assume that $f \in W^{\alpha,1}_{0,b}(I, \mathbb{R}^n)$ and that (1.7) holds for some $g \in L^1_\alpha(I, \mathbb{R}^n)$. Then, there exists a unique function $x \in V^{\alpha,p}_{a,b}([a,b], \mathbb{R}^n)$ such that*

$$x = f,\ x^{(\alpha)} = g \quad a.e.\ on\ I.$$

Theorem 1.4.7. *[87]. The set $W^{\alpha,1}_{0,b}(I, \mathbb{R}^n)$ is a Banach space together with the norm defined as*

$$\|\varphi\|_{W^{\alpha,1}_{0,b}(I,\mathbb{R}^n)} := \int_I \|\varphi(t)\| d_\alpha t + \int_I \|\varphi^{(\alpha)}(t)\| d_\alpha t,$$

for every $\varphi \in W^{\alpha,1}_{0,b}(I, \mathbb{R}^n)$.

Proposition 1.4.2. *Let $x \in W^{\alpha,1}_{0,b}(I, \mathbb{R}^n)$. Then $\|x\| \in W^{\alpha,1}_{0,b}(I, \mathbb{R})$ and*

$$\|x(t)\|^{(\alpha)} = \frac{< x(t), x^\alpha(t) >}{\|x(t)\|}, \ a.e.\ on\ \{t \in I : \|x(t)\| > 0\}.$$

Proof. If $x \in W^{\alpha,1}_{0,b}(I, \mathbb{R}^n)$. By Theorems 1.5.9 and 1.4.5, x is α-differentiable *a.e. on I*. From Example 1.4.2, we obtain

$$\|x(t)\|^{(\alpha)} = \frac{< x(t), x^\alpha(t) >}{\|x(t)\|}, \quad a.e.\ on\ \{t \in I : \|x(t)\| > 0\}.$$

□

We now define a notion of L^1_α-Carathéodory function.

Definition 1.4.11. *A function $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ is called a L^1_α-Carathéodory function if the three following conditions hold.*

(i) for every $x \in \mathbb{R}^n$, the function $t \mapsto f(t,x)$ is Lebesgue measurable;

(ii) the function $x \mapsto f(t,x)$ is continuous almost every $t \in I$;

(iii) for every $r > 0$, there exists a function $h_r \in L^1_\alpha(I, [0,\infty))$ such that $\|f(t,x)\| \leq h_r(t)$ for almost every $t \in I$ and for all $x \in \mathbb{R}^n$ such that $\|x\| \leq r$.

1.5 Conformable Fractional Calculus on Time Scales

We begin by introducing the notion of delta conformable fractional derivative of order $\alpha \in]0,1]$ for function defined on arbitrary time scale $\mathbb{T}$.

Definition 1.5.1. *[31]. Let $f : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}^\kappa$, and $\alpha \in]0,1]$. For $t > 0$, we define $f_\Delta^{(\alpha)}(t)$ to be the number (provided it exists) with the property that, given any $\epsilon > 0$, there is a δ-neighborhood $\mathcal{V}_t \subset \mathbb{T}$ (i.e., $\mathcal{V}_t :=]t-\delta, t+\delta[\cap \mathbb{T}$) of t, $\delta > 0$, such that*

$$\left| [f(\sigma(t)) - f(s)]\, t^{1-\alpha} - f_\Delta^{(\alpha)}(t)\, [\sigma(t) - s] \right| \leq \epsilon\, |\sigma(t) - s| \ \ for\ all\ s \in \mathcal{V}_t.$$

We call $f_\Delta^{(\alpha)}(t)$ the delta conformable fractional derivative of f of order α at t, and we define the delta conformable fractional derivative at 0 as $f_\Delta^{(\alpha)}(0) = \lim\limits_{t \to 0^+} f_\Delta^{(\alpha)}(t)$. The function f is delta conformal fractional differentiable of order α on $\mathbb{T}^\kappa$ provided $f_\Delta^{(\alpha)}(t)$ exists for all t in $\mathbb{T}^\kappa$.

Remark 1.5.1. *(i) If $\alpha = 1$, we have $f_\Delta^{(\alpha)} = f^\Delta$.*

(ii) If $\alpha = 0$, we denote $f_\Delta^{(\alpha)} = f$.

(iii) If $\mathbb{T} = \mathbb{R}$, then $f_\Delta^{(\alpha)} = f^{(\alpha)}$ is the conformable fractional derivative of f of order α (see Definition 1.4.1).

We introduce the following spaces:

$$C_{rd}^{\alpha}([a,b]_{\mathbb{T}}, \mathbb{R}) = \{f\ is\ delta\ conformal\ fractional\ differentiable\ of\ order\ \alpha\ on\ [a,b]_{\mathbb{T}} \\ and\ f_\Delta^{(\alpha)} \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})\}.$$

$$C_{0;rd}^{\alpha}([a,b]_{\mathbb{T}}, \mathbb{R}) = \{f \in C_{rd}^{\alpha}([a,b]_{\mathbb{T}}, \mathbb{R}) : f(a) = f(b) = 0\}.$$

$$C_{a,b;rd}^{\alpha}([a,b]_{\mathbb{T}}, \mathbb{R}) = \{f \in C_{rd}^{\alpha}([a,b]_{\mathbb{T}}, \mathbb{R}) : f(a) = f(b)\}.$$

Definition 1.5.2. *[31]. Let $\mathbb{T}$ be a time scale, $\alpha \in (n,\ n+1]$, $n \in \mathbb{N}$, and let f be n times delta differentiable at $t \in \mathbb{T}^{\kappa^n}$. We define the delta conformable fractional derivative of f of order α as*

$$f_\Delta^{(\alpha)}(t) := \left(f^{\Delta^n}\right)_\Delta^{(\alpha-n)}(t).$$

Theorem 1.5.1. *[31]. Let $\alpha \in (n,\ n+1]$, $n \in \mathbb{N}$. The following relation holds:*

$$f_\Delta^{(\alpha)}(t) = t^{1+n-\alpha} f^{\Delta^{1+n}}(t). \tag{1.8}$$

Theorem 1.5.2. *[31]. Let $\alpha \in]0,1]$ and $\mathbb{T}$ be a time scale. Assume $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^\kappa$. The following properties hold.*

(i) If f is delta conformal fractional differentiable of order α at $t > 0$, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is delta conformable fractional differentiable of order α at t with

$$f_{\Delta}^{(\alpha)}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} t^{1-\alpha} = t^{1-\alpha} f^{\Delta}(t).$$

(iii) If t is right-dense, then f is delta conformable fractional differentiable of order α at t if and only if the limit $\lim\limits_{s \to t} \frac{f(t) - f(s)}{(t-s)} t^{1-\alpha}$ exists as a finite number. In this case,

$$f_{\Delta}^{(\alpha)}(t) = t^{1-\alpha} f'(t).$$

(iv) If f is delta conformable fractional differentiable of order α at t, then

$$f(\sigma(t)) = f(t) + (\mu(t)) t^{\alpha-1} f_{\Delta}^{(\alpha)}(t).$$

Theorem 1.5.3. *[31]. Assume $f, g : \mathbb{T} \to \mathbb{R}$ are delta conformable fractional differentiable of order α. Then,*

(i) the sum $f+g$ is delta conformable fractional differentiable with $(f+g)_{\Delta}^{(\alpha)} = f_{\Delta}^{(\alpha)} + g_{\Delta}^{(\alpha)}$;

(ii) for any $\lambda \in \mathbb{R}$, λf is delta conformable fractional differentiable with $(\lambda f)_{\Delta}^{(\alpha)} = \lambda f_{\Delta}^{(\alpha)}$;

(iii) if f and g are continuous, then the product fg is delta conformable fractional differentiable with $(fg)_{\Delta}^{(\alpha)} = f_{\Delta}^{(\alpha)} g + (f \circ \sigma) g_{\Delta}^{(\alpha)} = f_{\Delta}^{(\alpha)} (g \circ \sigma) + f g_{\Delta}^{(\alpha)}$;

(iv) if f is continuous, then $1/f$ is delta conformable fractional differentiable with

$$\left(\frac{1}{f}\right)_{\Delta}^{(\alpha)} = -\frac{f_{\Delta}^{(\alpha)}}{f(f \circ \sigma)},$$

valid at all points $t \in \mathbb{T}^{\kappa}$ for which $f(t) f(\sigma(t)) \neq 0$;

(v) if f and g are continuous, then f/g is delta conformable fractional differentiable with

$$\left(\frac{f}{g}\right)_{\Delta}^{(\alpha)} = \frac{f_{\Delta}^{(\alpha)} g - f g_{\Delta}^{(\alpha)}}{g(g \circ \sigma)},$$

valid at all points $t \in \mathbb{T}^{\kappa}$ for which $g(t) g(\sigma(t)) \neq 0$.

Example 1.5.1. *Let $\alpha \in (0,1]$. Functions $f, g, h : \mathbb{T} \to \mathbb{R} : f(t) = t$, $p \in \mathbb{R}$, $g(t) \equiv \lambda$, $\lambda \in \mathbb{R}$, and $h(t) = e_p(t,a)$, $p \in \mathcal{R}_{\mu}$, are delta conformable fractional derivatives of order α with: $f_{\Delta}^{(\alpha)}(t) = t^{1-\alpha}$; $g_{\Delta}^{(\alpha)}(t) = 0$ and $h_{\Delta}^{(\alpha)}(t) = t^{1-\alpha} p\ e_p(t,a)$.*

Now we introduce the delta α-conformable fractional integral (or delta α-fractional integral) on time scales.

Definition 1.5.3. *[31]. Let $f : \mathbb{T} \to \mathbb{R}$ be a regulated function. Then the delta α-fractional integral of f, $0 < \alpha \leq 1$, is defined by $\int f(t)\Delta^\alpha t := \int f(t)t^{\alpha-1}\Delta t$.*

Definition 1.5.4. *[31]. Suppose $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Denote the indefinite delta α-fractional integral of f of order α, $\alpha \in (0,\ 1]$, as follows: $F(t) = \int f(t)\Delta^\alpha t$. Then, for all $a, b \in \mathbb{T}$, we define the Cauchy delta α-fractional integral by $\int_a^b f(t)\Delta^\alpha t = F(b) - F(a)$.*

Theorem 1.5.4. *[31]. Let $\alpha \in (0,\ 1]$. Then, for any rd-continuous function $f : \mathbb{T} \to \mathbb{R}$, there exist a function $F : \mathbb{T} \to \mathbb{R}$ such that $F_\Delta^{(\alpha)}(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$. Function F is said to be an delta α-antiderivative of f.*

The notions of Δ-measurable and Δ-integrable functions $f : \mathbb{T} \to \mathbb{R}$ are defined the same as those in [59].

Definition 1.5.5. *[6]. Let $B \subset \mathbb{T}$. B is called Δ-null set if the Δ-measure of B is zero. We say that a property P holds Δ-almost everywhere (Δ-a.e.) on B, or for Δ-almost all (Δ-a.a.) $t \in B$ if there is a Δ-null set $E_0 \subset B$ such that P holds for all $t \in B \setminus E_0$.*

Definition 1.5.6. *Assume $f : \mathbb{T} \to \mathbb{R}$, is a function. Let A is a Δ-measurable subset of $\mathbb{T}$. f is delta α-integrable on A if and only if $t^{\alpha-1}f(t)$ is integrable on A, and $\int_A f(t)\Delta^\alpha t = \int_A t^{\alpha-1}f(t)\Delta t$.*

Theorem 1.5.5. *[31]. Let $\alpha \in (0, 1]$, $a, b, c \in \mathbb{T}$, $\lambda \in \mathbb{R}$, and f, g be two rd-continuous functions. Then,*

(i) $$\int_a^b [\lambda f(t) + g(t)]\Delta^\alpha t = \lambda \int_a^b f(t)\Delta^\alpha t + \int_a^b g(t)\Delta^\alpha t;$$

(ii) $$\int_a^b f(t)\Delta^\alpha t = -\int_b^a f(t)\Delta^\alpha t;$$

(iii) $$\int_a^b f(t)\Delta^\alpha t = \int_a^c f(t)\Delta^\alpha t + \int_c^b f(t)\Delta^\alpha t;$$

(iv) $$\int_a^a f(t)\Delta^\alpha t = 0;$$

(v) if there exist $g : \mathbb{T} \to \mathbb{R}$ with $|f(t)| \leq g(t)$ for all $t \in [a,\ b]_\mathbb{T}$, then $\left|\int_a^b f(t)\Delta^\alpha t\right| \leq \int_a^b g(t)\Delta^\alpha t$;

(vi) if $f(t) > 0$ for all $t \in [a,\ b]_\mathbb{T}$, then $\int_a^b f(t)\Delta^\alpha t \geq 0$.

Theorem 1.5.6. *[31]. If $f : \mathbb{T}^\kappa \to \mathbb{R}$ is a rd-continuous function and $t \in \mathbb{T}^\kappa$, then*

$$\int_t^{\sigma(t)} f(s)\Delta^\alpha s = f(t)\mu(t)t^{\alpha-1}.$$

Theorem 1.5.7. *[31]. Let $f : \mathbb{T} \to \mathbb{R}$ be a function. If $f_\Delta^{(\alpha)}(t) \geq 0$ for all $t \in [a,b]_\mathbb{T}$, then f is an increasing function on $[a,b]_\mathbb{T}$.*

Now we introduce the concept of absolutely continuous function.

Definition 1.5.7. *[87]. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be absolutely continuous on $[a,\ b]_\mathbb{T}$ (i.e., $f \in AC([a,\ b]_\mathbb{T}, \mathbb{R})$) if for every $\varepsilon > 0$, there exists a $\eta > 0$ such that if $\{[a_k, b_k]_\mathbb{T}\}_{k=1}^n$, is a finite pairwise disjoint family of subintervals of $[a,\ b]_\mathbb{T}$ satisfying*

$$\sum_{k=1}^{k=n}(b_k - a_k) < \eta \text{ then } \sum_{k=1}^{k=n} |f(\rho(b_k)) - f(a_k)| < \varepsilon.$$

Theorem 1.5.8. *[87]. Assume function $f : [a,b]_\mathbb{T} \to \mathbb{R}$ is absolutely continuous on $[a,b]_\mathbb{T}$, then f is delta conformable fractional differentiable of order α Δ-a.e. on $[a,b]_\mathbb{T}$ and the following equality is valid:*

$$f(t) = f(a) + \int_{[a,t]_\mathbb{T}} f_\Delta^{(\alpha)}(s)\Delta^\alpha s \quad \text{for all } t \in [a,\ b]_\mathbb{T}.$$

Next, we develop the fractional Sobolev's spaces via conformable fractional calculus on time scales and their important properties. The basic definitions and relations based on [87] are given:

Definition 1.5.8. *[87]. Let $E \subset \mathbb{T}$ be a Δ-measurable set and let $\varphi : \mathbb{T} \longrightarrow \overline{\mathbb{R}}$ be a Δ-measurable function. Say that φ belongs to $L^1_{\alpha,\Delta}(E, \mathbb{R})$ provided that either*

$$\int_E |\varphi(s)|\Delta^\alpha s < +\infty.$$

Proposition 1.5.1. *[87]. The set $L^1_{\alpha,\Delta}([a,\ b]_\mathbb{T}, \mathbb{R})$ is a Banach space together with the norm defined for $\varphi \in L^1_{\alpha,\Delta}([a,\ b]_\mathbb{T}, \mathbb{R})$ as*

$$\|\varphi\|_{L^1_{\alpha,\Delta}([a,\ b]_\mathbb{T},\mathbb{R})} := \int_{[a,\ b]_\mathbb{T}} |\varphi(t)|\Delta^\alpha t.$$

Definition 1.5.9. *[87]. Let $f : [a,b]_\mathbb{T} \to \overline{\mathbb{R}}$. One says that $f \in W^{\alpha,1}_{\Delta;a,b}([a,b]_\mathbb{T}, \mathbb{R})$ if and only if $f \in L^1_{\alpha,\Delta}([a,\ b]_\mathbb{T}, \mathbb{R})$, $f_\Delta^{(\alpha)} \in L^1_{\alpha,\Delta}([a,\ b]_\mathbb{T}, \mathbb{R})$ and there exists $g : [a,b]_\mathbb{T}^\kappa \to \overline{\mathbb{R}}$ such that $g \in L^1_{\alpha,\Delta}([a,b]_\mathbb{T}, \mathbb{R})$ and*

$$\int_{[a,b]_\mathbb{T}} f(t)\phi_\Delta^{(\alpha)}(t)\Delta^\alpha t = -\int_{[a,b]_\mathbb{T}} g(t)\phi^\sigma(t)\Delta^\alpha t, \quad \text{for all } \phi \in C^\alpha_{a,b;rd}([a,b]_\mathbb{T}, \mathbb{R}). \tag{1.9}$$

We denote

$$V^{\alpha,1}_{\Delta;a,b}([a,\ b]_\mathbb{T}, \mathbb{R}) = \{u \in AC([a,\ b]_\mathbb{T}; \mathbb{R}) : u_\Delta^{(\alpha)} \in L^1_{\alpha,\Delta}([a,\ b]_\mathbb{T}, \mathbb{R}), u(a) = u(b)\}.$$

Remark 1.5.2. *It is not difficult to verify the following assertions for all $\alpha \in (0,1]$:*

(i) $L^1_{\alpha,\Delta}([a,b]_{\mathbb{T}}) \subset L^1_{\Delta}([a,b]_{\mathbb{T}})$.

(ii) For $t \in [a,b]_{\mathbb{T}}$, $t > 0$ and $\varphi : [a,\ b]_{\mathbb{T}} \to \mathbb{R}$, it is satisfied that $\varphi^{(\alpha)}_{\Delta} \in L^1_{\alpha,\Delta}([a,\ b]_{\mathbb{T}})$ if and only if $\varphi^{\Delta} \in L^1_{\Delta}([a,\ b]_{\mathbb{T}})$.

(iii) $V^{\alpha,1}_{\Delta;a,b}([a,b]_{\mathbb{T}},\mathbb{R}) \subset W^{\alpha,1}_{\Delta;a,b}([a,b]_{\mathbb{T}},\mathbb{R})$.

Theorem 1.5.9. *[87]. Assume that $f \in W^{\alpha,1}_{\Delta;a,b}([a,b]_{\mathbb{T}},\mathbb{R})$ and that equality (1.9) holds for $g \in L^1_{\alpha,\Delta}([a,b]_{\mathbb{T}},\mathbb{R})$. Then, there exists a unique function $x \in V^{\alpha,1}_{\Delta;a,b}([a,b]_{\mathbb{T}},\mathbb{R})$ such that*

$$x = f,\ x^{(\alpha)}_{\Delta} = g \quad \Delta\text{-a.e. on } [a,b]_{\mathbb{T}}.$$

Theorem 1.5.10. *[87]. The set $W^{\alpha,1}_{\Delta;a,b}([a,b]_{\mathbb{T}},\mathbb{R})$ is a Banach space together with the norm defined as*

$$\|\varphi\|_{W^{\alpha,1}_{\Delta;a,b}([a,b]_{\mathbb{T}},\mathbb{R})} := \int_{[a,b]_{\mathbb{T}}} |\varphi^{\sigma}(t)|\Delta^{\alpha}t + \int_{[a,b]_{\mathbb{T}}} |\varphi^{(\alpha)}_{\Delta}(t)|\Delta^{\alpha}t,$$

for every $\varphi \in W^{\alpha,1}_{\Delta;a,b}([a,b]_{\mathbb{T}},\mathbb{R})$.

We now define a notion of $L^1_{\alpha,\Delta}$-Carathéodory function.

Definition 1.5.10. *A function $f : [a,b]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is called a $L^1_{\alpha,\Delta}$-Carathéodory function if the three following conditions hold.*

1. *for every $x \in \mathbb{R}$, the function $t \mapsto f(t,x)$ is Δ-measurable;*
2. *the function $x \mapsto f(t,x)$ is continuous Δ-almost every $t \in [a,\ b]_{\mathbb{T}}$;*
3. *for every $r > 0$, there exists a function $h_r \in L^1_{\alpha,\Delta}([a,\ b]_{\mathbb{T}},[0,\infty))$ such that $\|f(t,x)\| \leq h_r(t)$ for Δ-almost every $t \in [a,\ b]_{\mathbb{T}}$ and for all $x \in \mathbb{R}$ such that $\|x\| \leq r$.*

Chapter 2

A nabla conformable fractional calculus on time scales

2.1 Introduction

In 2014, Khalil et al. [70] defined a new fractional derivative which is called the conformable fractional derivative (see Definition 1.4.1). In particular, Benkhettou et al. [31] extended this definition to an arbitrary time scale, which is a natural extension of the conformable fractional calculus (see Definition 1.5.1), then developed later in [76, 87].

Motivated by results in [31, 76, 87], in this chapter, we introduce definitions of nabla conformable fractional derivative and integral on time scales and study their important properties.

The original results of this chapter are published in [26].

2.2 Nabla Conformable Fractional Derivative

We begin by introducing the notion of nabla conformable fractional derivative of order $\alpha \in]0,1]$ for function defined on arbitrary time scale $\mathbb{T}$.

Definition 2.2.1. *Let $f : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}_\kappa$, and $\alpha \in]0,1]$. For $t > 0$, we define $f_\nabla^{(\alpha)}(t)$ to be the number (provided it exists) with the property that, given any $\epsilon > 0$, there is a δ-neighborhood $\mathcal{V}_t \subset \mathbb{T}$ (i.e., $\mathcal{V}_t =]t-\delta, t+\delta[\cap \mathbb{T}$) of t, $\delta > 0$, such that*

$$\left|\left(f(\rho(t)) - f(s)\right) t^{1-\alpha} - f_\nabla^{(\alpha)}(t)\left(\rho(t) - s\right)\right| \leq \epsilon \left|\rho(t) - s\right|,$$

for all $s \in \mathcal{V}_t$. We call $f_\nabla^{(\alpha)}(t)$ the nabla conformable fractional derivative of f of order α at t, and we define the nabla conformable fractional derivative at 0 as $f_\nabla^{(\alpha)}(0) = \lim\limits_{t \to 0^+} f_\nabla^{(\alpha)}(t)$. The function f is nabla conformal fractional differentiable of order α on $\mathbb{T}_\kappa$ provided $f_\nabla^{(\alpha)}(t)$ exists for all t in $\mathbb{T}_\kappa$.
Note that If $\alpha = 1$, and f is nabla conformable fractional derivative of order α, then

$f_{\nabla}^{(\alpha)}(t) = f^{\nabla}(t)$.
We denote:

(i) $C^{\alpha}([a,b]_{\mathbb{T}}, \mathbb{R}) = \Big\{ f : [a,b]_{\mathbb{T}} \to \mathbb{R},\ f$ *is nabla conformal fractional differentiable of order* α *on* $[a,b]_{\mathbb{T}}$ *and* $f_{\nabla}^{(\alpha)} \in C([a,b]_{\mathbb{T}}, \mathbb{R}) \Big\}$.

(ii) $C_{ld}^{\alpha}([a,b]_{\mathbb{T}}, \mathbb{R}) = \Big\{ f : [a,b]_{\mathbb{T}} \to \mathbb{R},\ f$ *is nabla conformal fractional differentiable of order* α *on* $[a,b]_{\mathbb{T}}$ *and* $f_{\nabla}^{(\alpha)} \in C_{ld}([a,b]_{\mathbb{T}}, \mathbb{R}) \Big\}$.

Some useful properties of the nabla conformable fractional derivative of f of order α are given in the following theorem.

Theorem 2.2.1. *Let* $\alpha \in]0,1]$ *and* $\mathbb{T}$ *be a time scale. Assume* $f : \mathbb{T} \to \mathbb{R}$ *and let* $t \in \mathbb{T}_{\kappa}$. *The following properties hold.*

(i) If f *is nabla conformal fractional differentiable of order* α *at* $t > 0$, *then* f *is continuous at* t.

(ii) If f *is continuous at* t *and* t *is left-scattered, then* f *is nabla conformable fractional differentiable of order* α *at* t *with*

$$f_{\nabla}^{(\alpha)}(t) = \frac{f(t) - f(\rho(t))}{\nu(t)} t^{1-\alpha}. \tag{2.1}$$

(iii) If t *is left-dense, then* f *is nabla conformable fractional differentiable of order* α *at* t *if, and only if, the limit* $\lim_{s \to t} \dfrac{f(t) - f(s)}{(t-s)} t^{1-\alpha}$ *exists as a finite number. In this case,*

$$f_{\nabla}^{(\alpha)}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t-s} t^{1-\alpha}. \tag{2.2}$$

(iv) If f *is nabla conformable fractional differentiable of order* α *at* t, *then*

$$f(\rho(t)) = f(t) - (\nu(t)) t^{\alpha-1} f_{\nabla}^{(\alpha)}(t).$$

Proof. (i) Assume that f is nabla conformable fractional differentiable at t. Then, there exists a neighborhood $\mathcal{V}_t$ of t such that

$$\left| \left(f(\rho(t)) - f(s)\right) t^{1-\alpha} - f_{\nabla}^{(\alpha)}(t) \left(\rho(t) - s\right) \right| \leq \epsilon \left|\rho(t) - s\right|$$

for $s \in \mathcal{V}_t$. Therefore,

$$\begin{aligned} |f\,(t) - f\,(s)| \leq &\left| \left(f(\rho(t) - f(s)\right) - f_{\nabla}^{(\alpha)}(t) \left(\rho(t) - s\right) t^{\alpha-1} \right| + \left|\left(f(\rho(t)) - f(t)\right)\right| \\ &+ \left| f_{\nabla}^{(\alpha)}(t) \right| \left|\left(\rho(t) - s\right)\right| \left|t^{\alpha} - 1\right|, \end{aligned}$$

for all $s \in \mathcal{V}_t \cap]t-\epsilon, t+\epsilon[$ and, since t is a left-dense point,

$$\begin{aligned}|f(t) - f(s)| &\leq \left|(f^\rho(t) - f(s)) - f^{(\alpha)}(t)\,(\rho(t) - s)^\alpha\right| + \left|f_\nabla^{(\alpha)}(t)\,(t-s)^\alpha\right| \\ &\leq \epsilon\delta + \left|t^{\alpha-1}\right| \left|f_\nabla^{(\alpha)}(t)\right| \delta.\end{aligned}$$

Since $\delta \to 0$ when $s \to t$, and $t > 0$, it follows the continuity of f at t.
(ii) Assume that f is continuous at t and t is left-scattered. By continuity,

$$\lim_{s\to t} \frac{f(\rho(t)) - f(s)}{\rho(t) - s} t^{1-\alpha} = \frac{f(\rho(t)) - f(t)}{\rho(t) - t} t^{1-\alpha} = \frac{f(t) - f(\rho(t))}{\nu(t)} t^{1-\alpha}.$$

Hence, given $\epsilon > 0$ and $\alpha \in]0,1]$, there is a neighborhood $\mathcal{V}_t$ of t such that

$$\left|\frac{f(\rho(t)) - f(s)}{\rho(t) - s} t^{1-\alpha} - \frac{f(t) - f(\rho(t))}{\nu(t)} t^{1-\alpha}\right| \leq \epsilon$$

for all $s \in \mathcal{V}_t$. It follows that

$$\left|[f(\rho(t)) - f(s)]\, t^{1-\alpha} - \frac{f(t) - f(\rho(t))}{\nu(t)} t^{1-\alpha}(\rho(t) - s)\right| \leq \epsilon|\rho(t) - s|$$

for all $s \in \mathcal{V}_t$. The desired equality (2.1) follows from Definition 2.2.1.
(iii) Assume that f is nabla conformable fractional differentiable of order α at t and t is left-dense. Let $\epsilon > 0$ be given. Since f is nabla conformable fractional differentiable of order α at t, there is a neighborhood $\mathcal{V}_t$ of t such that

$$\left|[f(\rho(t)) - f(s)]t^{1-\alpha} - f_\nabla^{(\alpha)}(t)(\rho(t) - s)\right| \leq \epsilon|\rho(t) - s|$$

for all $s \in \mathcal{V}_t$. Because $\rho(t) = t$,

$$\left|\frac{f(t) - f(s)}{t - s} t^{1-\alpha} - f_\nabla^{(\alpha)}(t)\right| \leq \epsilon$$

for all $s \in \mathcal{V}_t$, $s \neq t$. Therefore, we get the desired result (2.2). Now, assume that the limit on the right-hand side of (2.2) exists and is equal to L, and t is left-dense. Then, there exists $\mathcal{V}_t$ such that $|(f(t) - f(s))t^{1-\alpha} - L(t-s)| \leq \epsilon|t-s|$ for all $s \in \mathcal{V}_t$. Because t is left-dense,

$$\left|(f(\rho(t)) - f(s))t^{1-\alpha} - L(\rho(t) - s)\right| \leq \epsilon|\rho(t) - s|,$$

which lead us to the conclusion that f is nabla conformable fractional differentiable of order α at t and $T_{\nabla,\alpha}(f)(t) = L$.
(iv) If t is left-dense, i.e., $\rho(t) = t$, then $\nu(t) = 0$ and $f(\rho(t)) = f(t) = f(t) - \nu(t)f_\nabla^{(\alpha)}(t)t^{1-\alpha}$. On the other hand, if t is left-scattered, i.e., $\rho(t) < t$, then by (iii)

$$f(\rho(t)) = f(t) - \nu(t)t^{\alpha-1} \cdot \frac{f(t) - f(\rho(t))}{\nu(t)} t^{1-\alpha} = f(t) - (\nu(t))t^{\alpha-1} f_\nabla^{(\alpha)}(t).$$

The proof is complete. $\square$

Example 2.2.1. *(i) If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = c$ for all $t \in \mathbb{T}$, $c \in \mathbb{R}$, then*

$$f_{\nabla}^{(\alpha)}(t) = (c)_{\nabla}^{(\alpha)} = 0.$$

(ii) If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = t$ for all $t \in \mathbb{T}$, then

$$f_{\nabla}^{(\alpha)}(t) = (t)_{\nabla}^{(\alpha)} = \begin{cases} t^{1-\alpha} & \text{if } \alpha \neq 1, \\ 1 & \text{if } \alpha = 1. \end{cases}$$

(iii) Let $p \in \mathcal{R}_{\nu}$, fix $t_0 \in \mathbb{T}$ and $f(t) = \hat{e}_p(t, t_0)$ for $t \in \mathbb{T}$, the nabla exponential function given in Definition 1.3.10, then

$$f_{\nabla}^{(\alpha)}(t) = t^{1-\alpha} p \hat{e}_p(t, t_0).$$

Example 2.2.2. *(i) Function $f : \mathbb{R} \to \mathbb{R}$ is nabla conformable fractional differentiable of order α at point $t \in \mathbb{R}$ if, and only if, the limit $\lim\limits_{s \to t} \dfrac{f(t) - f(s)}{t - s} t^{1-\alpha}$ exists as a finite number. In this case,*

$$f_{\nabla}^{(\alpha)}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} t^{1-\alpha}. \tag{2.3}$$

If $\alpha = 1$, then $f_{\nabla}^{(\alpha)} = f^{\nabla}(t) = f'(t)$.
The identity (2.3) corresponds to the conformable fractional derivative given in Definition 1.4.1.

(ii) Let $h > 0$. If $f : h\mathbb{Z} \to \mathbb{R}$, then f is nabla conformable fractional differentiable of order α at $t \in h\mathbb{Z}$ with

$$f_{\nabla}^{(\alpha)}(t) = \frac{f(t) - f(t-h)}{h} t^{1-\alpha}.$$

If $\alpha = 1$ and $h = 1$, then $f_{\nabla}^{(\alpha)} = \nabla f(t) = f(t) - f(t-1)$, where ∇ is the backward difference operator.

Next, we would like to be able to find the derivatives of sums, products, and quotients of nabla conformable fractional differentiable functions. This is possible according to the following theorem.

Theorem 2.2.2. *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are nabla conformable fractional differentiable of order α. Then,*

(i) the sum $f + g : \mathbb{T} \to \mathbb{R}$ is nabla conformable fractional differentiable with

$$(f+g)_{\nabla}^{(\alpha)} = f_{\nabla}^{(\alpha)} + g_{\nabla}^{(\alpha)};$$

(ii) for any $\lambda \in \mathbb{R}$, $\lambda f : \mathbb{T} \to \mathbb{R}$ is nabla conformable fractional differentiable with

$$(\lambda f)_{\nabla}^{(\alpha)} = \lambda f_{\nabla}^{(\alpha)};$$

(iii) *if f and g are continuous, then the product $fg : \mathbb{T} \to \mathbb{R}$ is nabla conformable fractional differentiable with*

$$(fg)_{\nabla}^{(\alpha)} = f_{\nabla}^{(\alpha)} g + f^{\rho} g_{\nabla}^{(\alpha)} = f_{\nabla}^{(\alpha)} g^{\rho} + f g_{\nabla}^{(\alpha)};$$

(iv) *if f is continuous, then $1/f$ is nabla conformable fractional differentiable with*

$$\left(\frac{1}{f}\right)_{\nabla}^{(\alpha)} = -\frac{f_{\nabla}^{(\alpha)}}{f f^{\rho}},$$

valid at all points $t \in \mathbb{T}_{\kappa}$ for which $f(t) f^{\rho}(t) \neq 0$;

(v) *if f and g are continuous, then f/g is nabla conformable fractional differentiable with*

$$\left(\frac{f}{g}\right)_{\nabla}^{(\alpha)} = \frac{f_{\nabla}^{(\alpha)} g - f g_{\nabla}^{(\alpha)}}{g g^{\rho}},$$

valid at all points $t \in \mathbb{T}_{\kappa}$ for which $g(t) g^{\rho}(t) \neq 0$.

Proof. Let us consider that $\alpha \in]0, 1]$, and let us assume that f and g are nabla conformable fractional differentiable at $t \in \mathbb{T}_{\kappa}$.
(i) Let $\epsilon > 0$. Then there exist neighborhoods $\mathcal{V}_t$ and $\mathcal{U}_t$ of t for which

$$\left|[f(\rho(t)) - f(s)] t^{1-\alpha} - f_{\nabla}^{(\alpha)}(t)\,(\rho(t) - s)\right| \leq \frac{\epsilon}{2} |\rho(t) - s| \quad \text{for all } s \in \mathcal{V}_t$$

and

$$\left|[g(\rho(t)) - g(s)] t^{1-\alpha} - g_{\nabla}^{(\alpha)}(t)(\rho(t) - s)\right| \leq \frac{\epsilon}{2} |\rho(t) - s| \quad \text{for all } s \in \mathcal{U}_t.$$

Let $\mathcal{W}_t = \mathcal{V}_t \cap \mathcal{U}_t$. Then

$$\left|[(f+g)(\rho(t)) - (f+g)(s)] t^{1-\alpha} - \left[f_{\nabla}^{(\alpha)}(t) + g_{\nabla}^{(\alpha)}(t)\right](\rho(t) - s)\right| \leq \epsilon |\rho(t) - s|$$

for all $s \in \mathcal{W}_t$. Thus, $f + g$ is nabla conformable differentiable at t and

$$(f+g)_{\nabla}^{(\alpha)}(t) = f_{\nabla}^{(\alpha)}(t) + g_{\nabla}^{(\alpha)}(t).$$

(ii) Let $\epsilon > 0$. Then $\left|[f(\rho(t)) - f(s)] t^{1-\alpha} - f_{\nabla}^{(\alpha)}(t)(\rho(t) - s)\right| \leq \epsilon |\rho(t) - s|$ for all s in a neighborhood $\mathcal{V}_t$ of t. It follows that

$$\left|[(\lambda f)(\rho(t)) - (\lambda f)(s)] t^{1-\alpha} - \lambda f_{\nabla}^{(\alpha)}(t)(\rho(t) - s)\right| \leq \epsilon |\lambda|\, |\rho(t) - s| \text{ for all } s \in \mathcal{V}_t.$$

Therefore, λf is nabla conformable fractional differentiable at t and $(\lambda f)_{\nabla}^{(\alpha)} = \lambda f_{\nabla}^{(\alpha)}$ holds at t.
(iii) If t is left-scattered, then

$$\begin{aligned}(fg)_{\nabla}^{(\alpha)}(t) &= \left[\frac{f(t) - f(\rho(t))}{\nu(t)} t^{1-\alpha}\right] g(\rho(t)) + \left[\frac{g(t) - g(\rho(t))}{\nu(t)} t^{1-\alpha}\right] f(t) \\ &= f_{\nabla}^{(\alpha)}(t) g(\rho(t)) + f(t) g_{\nabla}^{(\alpha)}(t).\end{aligned}$$

If t is left-dense, then

$$(fg)_{\nabla}^{(\alpha)}(t) = \underset{s\to t}{lim}\left[\frac{f(t)-f(s)}{t-s}t^{1-\alpha}\right] g\,(t) + \underset{s\to t}{lim}\left[\frac{g(t)-g(s)}{t-s}t^{1-\alpha}\right] f\,(s)$$
$$= f_{\nabla}^{(\alpha)}(t)g(t) + g_{\nabla}^{(\alpha)}(t)f(t) = f_{\nabla}^{(\alpha)}(t)g(\rho(t)) + g_{\nabla}^{(\alpha)}(t)f(t).$$

The other product rule formula follows by interchanging the role of functions f and g.
(iv) From Example 2.2.1 (i), we know that $\left(f\cdot\frac{1}{f}\right)_{\nabla}^{(\alpha)}(t) = (1)_{\nabla}^{(\alpha)} = 0$. Therefore, by (iii)

$$\left(\frac{1}{f}\right)_{\nabla}^{(\alpha)}(t)f(\rho(t)) + f_{\nabla}^{(\alpha)}(t)\frac{1}{f(t)} = 0.$$

Since we are assuming $f(\rho(t)) \neq 0$, $\left(\frac{1}{f}\right)_{\nabla}^{(\alpha)}(t) = -\dfrac{f_{\nabla}^{(\alpha)}(t)}{f(t)f(\rho(t))}$.
(v) We use *(ii)* and *(iv)* to obtain

$$\left(\frac{f}{g}\right) f_{\nabla}^{(\alpha)}(t) = \left(f\cdot\frac{1}{g}\right)_{\nabla}^{(\alpha)}(t) = f(t)\left(\frac{1}{g}\right)_{\nabla}^{(\alpha)}(t) + f_{\nabla}^{(\alpha)}(t)\frac{1}{g(\rho(t))}$$
$$= \frac{f_{\nabla}^{(\alpha)}(t)g(t) - f(t)g_{\nabla}^{(\alpha)}(t)}{g(t)g(\rho(t))}.$$

The proof is complete. □

Theorem 2.2.3. *Let c be a constant, $m \in \mathbb{N}$, $\alpha \in]0,1]$ and $f(t) = (t-c)^m$. Then*

$$f_{\nabla}^{(\alpha)}(t) = t^{1-\alpha}\sum_{i=0}^{m-1}(t-c)^{m-1-i}\,(\rho(t)-c)^i. \tag{2.4}$$

If $c = 0$, then $f_{\nabla}^{(\alpha)}(t) = (t^m)_{\nabla}^{(\alpha)} = t^{1-\alpha}\sum_{i=0}^{m-1}(t)^{m-1-i}(\rho(t))^i$.

Proof. We prove the first formula by induction. If $m = 1$, then $f(t) = t - c$ and $f_{\nabla}^{(\alpha)}(t) = t^{1-\alpha}$ holds from Example 2.2.1 and Theorem 2.2.2 *(i)*. Now assume that

$$f_{\nabla}^{(\alpha)}(t) = t^{1-\alpha}\sum_{i=0}^{m-1}(t-c)^{m-1-i}(\rho(t)-c)^i$$

holds for $f(t) = (t-c)^m$ and let $F(t) = (t-c)^{m+1} = (t-c)f(t)$. We use Theorem 2.2.2 *(iii)* to obtain

$$(F(t))^{(\alpha)} = (t-c)f_{\nabla}^{(\alpha)}f(\rho(t)) + f_{\nabla}^{(\alpha)}(t)(t-c) = t^{1-\alpha}\sum_{i=0}^{m}(t-c)^{m-p}(\rho(t)-c)^i.$$

Hence, by mathematical induction, (2.4) holds. If $c = 0$, then we have

$$f_{\nabla}^{(\alpha)}(t) = t^{1-\alpha}\sum_{i=0}^{m-1}(t)^{m-1-i}(\rho(t))^i.$$

Note that if t is left-dense, then $f_{\nabla}^{(\alpha)}(t) = mt^{m-\alpha}$. $\square$

Theorem 2.2.4 (Chain rule)**.** *Let $\alpha \in]0,1]$. Assume $g : \mathbb{T} \to \mathbb{R}$ is continuous and nabla conformable fractional differentiable of order α at $t \in \mathbb{T}_{\kappa}$, and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then there exists c in the real interval $[\rho(t), t]$ with*

$$(f \circ g)_{\nabla}^{(\alpha)}(t) = f'(g(c))\, g_{\nabla}^{(\alpha)}(t). \tag{2.5}$$

Proof. Let $t \in \mathbb{T}_{\kappa}$. First we consider t to be left-scattered. In this case,

$$(f \circ g)_{\nabla}^{(\alpha)}(t) = \frac{f(g(t)) - f(g(\rho(t)))}{\nu(t)} t^{1-\alpha}.$$

If $g(\rho(t)) = g(t)$, then we get $(f \circ g)_{\nabla}^{(\alpha)}(t) = 0$ and $g_{\nabla}^{(\alpha)}(t) = 0$. Therefore, (2.5) holds for any c in the real interval $[\rho(t), t]$. Now assume that $g(\rho(t)) \neq g(t)$. By the mean value theorem we have

$$(f \circ g)_{\nabla}^{(\alpha)}(t) = \frac{f(g(\rho(t))) - f(g(t))}{g(\rho(t)) - g(t)} \cdot \frac{g(t) - g(\rho(t))}{\nu(t)} t^{1-\alpha} = f'(\xi) g_{\nabla}^{(\alpha)}(t),$$

where ξ between $g(\rho(t))$ and $g(t)$. Since $g : \mathbb{T} \to \mathbb{R}$ is continuous, there is a $c \in [\rho(t), t]$ such that $g(c) = \xi$, which gives the desired result. Now let us consider the case when t is left-dense. In this case

$$(f \circ g)_{\nabla}^{(\alpha)}(t) = \lim_{s \to t} \frac{f(g(t)) - f(g(s))}{g(t) - g(s)} \cdot \frac{g(t) - g(s)}{t - s} t^{1-\alpha}.$$

By the mean value theorem, there exist ξ_s between $g(\rho(t))$ and $g(t)$ such that

$$(f \circ g)_{\nabla}^{(\alpha)}(t) = \lim_{s \to t} \left\{ f'(\xi_s) \cdot \frac{g(t) - g(s)}{t - s} t^{1-\alpha} \right\}.$$

By the continuity of g, we get that $\lim_{s \to t} \xi_s = g(t)$. Then $(f \circ g)_{\nabla}^{(\alpha)}(t) = f'(g(t)) \cdot g_{\nabla}^{(\alpha)}(t)$. Since t is left-dense, we conclude that $c = t = \rho(t)$, which gives the desired result. $\square$

We define the nabla conformable fractional derivative $(.)_{\nabla}^{(\alpha)}$ for $\alpha \in (m,\ m+1]$, where m is some natural number.

Definition 2.2.2. *Let $\mathbb{T}$ be a time scale, $\alpha \in (m,\ m+1]$, $m \in \mathbb{N}$, and let f be m times nabla differentiable at $t \in \mathbb{T}_{\kappa^m}$. We define the nabla conformable fractional derivative of f of order α as $f_{\nabla}^{(\alpha)}(t) = \left(f^{\nabla^m}\right)_{\nabla}^{(\alpha-m)}(t)$.*

Theorem 2.2.5. *Let $\alpha \in (m,\ m+1]$, $m \in \mathbb{N}$. The following relation holds:*

$$f_{\nabla}^{(\alpha)}(t) = t^{1+m-\alpha} f^{\nabla^{1+m}}(t). \tag{2.6}$$

Proof. Let f be a function m times nabla-differentiable. For $\alpha \in (m, m+1]$, there exist $\beta \in (0,1]$ such that $\alpha = m + \beta$. Using Definition 2.2.2, $f_{\nabla}^{(\alpha)} = \left(f^{\nabla^m}\right)_{\nabla}^{(\beta)}$. From the definition of (higher-order) nabla derivative and Theorem 2.2.1 (*ii*) and (*iii*), it follows that $f_{\nabla}^{(\alpha)}(t) = t^{1-\beta}\left(f^{\nabla^m}\right)^{\nabla}(t)$. $\square$

Remark 2.2.1. *In(2.6), when $m = 0$, we have $f_{\nabla}^{(\alpha)}(t) := t^{1-\alpha} f^{\nabla}(t),\ \alpha \in (0,1]$.*

Next, we introduce the nabla conformable fractional derivative on time scales for vector-valued functions and study some of their important properties.

Definition 2.2.3. *Assume $f : \mathbb{T} \to \mathbb{R}^n$ is a function, $f(t) = (f_1(t), f_2(t), ..., f_n(t))$ and let $t \in \mathbb{T}_{\kappa}$. Then one defines*

$$f_{\nabla}^{(\alpha)}(t) = \left((f_1)_{\nabla}^{(\alpha)}(t), (f_2)_{\nabla}^{(\alpha)}(t), ..., (f_n)_{\nabla}^{(\alpha)}(t)\right)$$

provided it exists. One calls $f_{\nabla}^{(\alpha)}(t)$ the nabla conformable fractional derivative of f of order α at $t > 0$. The function f is nabla conformal fractional differentiable of order α on $\mathbb{T}_{\kappa}$ provided $f_{\nabla}^{(\alpha)}(t)$ exists for all t in $\mathbb{T}_{\kappa}$. The function $f_{\nabla}^{(\alpha)} : \mathbb{T}_{\kappa} \to \mathbb{R}^n$ is then called the nabla conformable fractional derivative of f of order α, and we define the nabla conformable fractional derivative at 0 as $f_{\nabla}^{(\alpha)}(0) = \lim\limits_{t \to 0^+} f_{\nabla}^{(\alpha)}(t)$.

Definition 2.2.4. *Let $\mathbb{T}$ be a time scale, $\alpha \in (m,\ m+1]$, $m \in \mathbb{N}$, and let $f : \mathbb{T} \to \mathbb{R}^n$ be m times nabla differentiable at $t \in \mathbb{T}_{\kappa^m}$. We define the nabla conformable fractional derivative of f of order α as $f_{\nabla}^{(\alpha)}(t) := \left(f^{\nabla^m}\right)_{\nabla}^{(\alpha-m)}(t)$.*

Combining Definition 2.2.3 and Theorems 2.2.1, 2.2.2 we have the following theorems.

Theorem 2.2.6. *Let $\alpha \in]0,1]$. Assume $f : \mathbb{T} \to \mathbb{R}^n$ and let $t \in \mathbb{T}_{\kappa}$. The following properties hold:*

(i) If f is nabla conformal fractional differentiable of order α at $t > 0$, then f is continuous at t.

(ii) If f is continuous at t and t is left-scattered, then f is nabla conformable fractional differentiable of order α at t with

$$f_{\nabla}^{(\alpha)}(t) = \frac{f(t) - f(\rho(t))}{\nu(t)} t^{1-\alpha}. \tag{2.7}$$

(iii) If t is left-dense, then f is nabla conformable fractional differentiable of order α at t if and only if the limit $\lim\limits_{s \to t} \dfrac{f(t) - f(s)}{(t-s)} t^{1-\alpha}$ exists as a finite number. In this case,

$$f_{\nabla}^{(\alpha)}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{(t-s)} t^{1-\alpha}. \tag{2.8}$$

(iv) If f is nabla conformable fractional differentiable of order α at t, then

$$f(\rho(t)) = f(t) - (\nu(t))t^{\alpha-1} f_{\nabla}^{(\alpha)}(t).$$

Theorem 2.2.7. *Assume $f, g : \mathbb{T} \to \mathbb{R}^n$ are nabla conformable fractional differentiable of order α. Then,*

(i) the sum $f + g : \mathbb{T} \to \mathbb{R}^n$ is nabla conformable fractional differentiable with

$$(f+g)_{\nabla}^{(\alpha)} = f_{\nabla}^{(\alpha)} + g_{\nabla}^{(\alpha)};$$

(ii) for any $\lambda \in \mathbb{R}$, $\lambda f : \mathbb{T} \to \mathbb{R}^n$ is nabla conformable fractional differentiable with $(\lambda f)_{\nabla}^{(\alpha)} = \lambda f_{\nabla}^{(\alpha)}$;

(iii) if f and g are continuous, then the product $fg : \mathbb{T} \to \mathbb{R}^n$ is nabla conformable fractional differentiable with

$$(fg)_{\nabla}^{(\alpha)} = f_{\nabla}^{(\alpha)} g + (f \circ \rho) g_{\nabla}^{(\alpha)} = f_{\nabla}^{(\alpha)} (g \circ \rho) + f g_{\nabla}^{(\alpha)}.$$

2.3 Nabla Conformable Fractional Integral

Now we introduce the nabla conformable fractional integral (or nabla α-fractional integral) on time scales.

Definition 2.3.1. *Let $f : \mathbb{T} \to \mathbb{R}$ be a regulated function. Then the nabla α-fractional integral of f, $0 < \alpha \leq 1$, is defined by $\int f(t)\nabla_{\alpha} t = \int f(t) t^{\alpha-1} \nabla t$.*
Note that If $\alpha = 1$, then $\int f(t)\nabla_{\alpha} t = \int f(t)\nabla t$ is the indefinite nabla integral. If $\mathbb{T} = \mathbb{R}$, then $\int f(t)\nabla_{\alpha} t = \int t^{\alpha-1} f(t) dt$ is the conformable fractional integral given in Definition 1.4.4.

Definition 2.3.2. *Suppose $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Denote the indefinite nabla α-fractional integral of f of order α, $\alpha \in (0,\ 1]$, as follows: $F_{\nabla,\alpha}(t) = \int f(t)\nabla_{\alpha} t$. Then, for all $a, b \in \mathbb{T}$, we define the Cauchy nabla α-fractional integral by:*

$$\int_a^b f(t)\nabla_{\alpha} t = F_{\nabla,\alpha}(b) - F_{\nabla,\alpha}(a).$$

Definition 2.3.3. *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function. Let A is a ∇-measurable subset of $\mathbb{T}$. f is nabla α-integrable on A if and only if $t^{\alpha-1} f(t)$ is integrable on A, and $\int_A f(t)\nabla_{\alpha} t = \int_A t^{\alpha-1} f(t)\nabla t$*

Theorem 2.3.1. *Let $\alpha \in (0,\ 1]$. Then, for any ld-continuous function $f : \mathbb{T} \to \mathbb{R}$, there exist a function $F_{\nabla,\alpha} : \mathbb{T} \to \mathbb{R}$ such that $(F_{\nabla,\alpha})_{\nabla}^{(\alpha)}(t) = f(t)$ for all $t \in \mathbb{T}_{\kappa}$. Function $F_{\nabla,\alpha}$ is said to be an nabla α-antiderivative of f.*

Proof. The case $\alpha = 1$ is proved in [33]. Let $\alpha \in (0,\ 1)$. Suppose f is *ld*-continuous. By Theorem 1.16 of [34], f is regulated. Then, $F_{\nabla,\alpha}(t) = \int f(t)\nabla_\alpha t$ is nabla conformable fractional differentiable on $\mathbb{T}_\kappa$. Using (2.6) and Definition 2.3.1, we obtain that

$$\left(F_{\nabla,\alpha}\right)_{\nabla}^{(\alpha)}(t) = t^{1-\alpha}\left(F_{\nabla,\alpha}(t)\right)^{\nabla} = f(t),\ t \in \mathbb{T}_\kappa.$$

□

Theorem 2.3.2. *Let $\alpha \in (0,\ 1]$, $a, b, c \in \mathbb{T}$, $\lambda, \gamma \in \mathbb{R}$, and f, g be two ld-continuous functions. Then,*

(i) $\displaystyle\int_a^b [\lambda f(t) + \gamma g(t)]\nabla_\alpha t = \lambda \int_a^b f(t)\nabla_\alpha t + \gamma \int_a^b g(t)\nabla_\alpha t;$

(ii) $\displaystyle\int_a^b f(t)\nabla_\alpha t = -\int_b^a f(t)\nabla_\alpha t;$

(iii) $\displaystyle\int_a^b f(t)\nabla_\alpha t = \int_a^c f(t)\nabla_\alpha t + \int_c^b f(t)\nabla_\alpha t;$

(iv) $\displaystyle\int_a^a f(t)\nabla_\alpha t = 0;$

(v) if there exist $g : \mathbb{T} \to \mathbb{R}$ with $|f(t)| \le g(t)$ for all $t \in [a,\ b]$, then

$$\left|\int_a^b f(t)\nabla_\alpha t\right| \le \int_a^b g(t)\nabla_\alpha t;$$

(vi) if $f(t) > 0$ for all $t \in [a,\ b]_{\mathbb{T}}$, then $\displaystyle\int_a^b f(t)\nabla_\alpha t \ge 0.$

Proof. The relations follow from Definitions 2.3.1 and 2.3.2, analogous properties of the nabla-integral, and the properties of Section 2.2 for the nabla conformable fractional derivative on time scales. □

Theorem 2.3.3. *If $f : \mathbb{T}_\kappa \to \mathbb{R}$ is a ld-continuous function and $t \in \mathbb{T}_\kappa$, then*

$$\int_{\rho(t)}^t f(s)\nabla_\alpha s = \nu(t)f(t)t^{\alpha-1}.$$

Proof. Let f be a *ld*-continuous function on $\mathbb{T}_\kappa$. Then f is a regulated function. By Definition 2.3.2 and Theorem 2.3.1, there exist an antiderivative $F_{\nabla,\alpha}$ of f satisfying

$$\int_{\rho(t)}^t f(s)\nabla_\alpha s = F_{\nabla,\alpha}(t) - F_{\nabla,\alpha}(\rho(t)) = \left(F_{\nabla,\alpha}\right)_{\nabla}^{(\alpha)}(t)\nu(t)t^{1-\alpha} = \nu(t)f(t)t^{1-\alpha}.$$

This concludes the proof. □

Theorem 2.3.4. *Let $a, b \in \mathbb{T}$, $\alpha \in (0,\ 1]$ and $f : \mathbb{T} \to \mathbb{R}$ be ld-continuous function. Then we have the following.*

(i) If $\mathbb{T} = \mathbb{R}$, then $\int_a^b f(t)\nabla_\alpha t = \int_a^b f(t)t^{\alpha-1}dt$ where the integral on the right is the conformable fractional integral (see Definition 1.4.4). If $\alpha = 1$, then it reduces to the usual Riemann integral.

(ii) If $[a, b]_\mathbb{T}$ consists of only isolated points, then

$$\int_a^b f(t)\nabla_\alpha t = \begin{cases} \sum_{t\in(a,b]_\mathbb{T}} \nu(t)t^{\alpha-1}f(t) & if\ a < b \\ 0 & if\ \ a = b \\ -\sum_{t\in(b,a]_\mathbb{T}} \nu(t)t^{\alpha-1}f(t) & if\ a > b \end{cases}$$

(iii) If $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, where $h > 0$, then

$$\int_a^b f(t)\nabla_\alpha t = \begin{cases} \sum_{k=\frac{a}{h}+1}^{\frac{b}{h}} h(kh)^{\alpha-1}f(kh) & if\ a < b \\ 0 & if\ \ a = b \\ -\sum_{k=\frac{b}{h}+1}^{\frac{a}{h}} h(kh)^{\alpha-1}f(kh) & if\ a > b \end{cases}$$

(iv) If $\mathbb{T} = \mathbb{Z}$ then

$$\int_a^b f(t)\nabla_\alpha t = \begin{cases} \sum_{t=a+1}^{b} t^{\alpha-1}f(t) & if\ a < b \\ 0 & if\ \ a = b \\ -\sum_{t=b+1}^{a} t^{\alpha-1}f(t) & if\ a > b \end{cases}$$

Proof. Part (i). It follows from Example 2.2.2 (i).
Part (ii). First, note that $[a, b]_\mathbb{T}$ contains only finitely many points since each point in $[a, b]_\mathbb{T}$ is isolated. Assume that $a < b$ and let $[a, b] = \{t_0, t_1, ..., t_n\}$, where

$$a = t_0 < t_1 < t_2 < ... < t_n = b$$

By virtue of Theorem 2.3.2 (iii),

$$\int_a^b f(t)\nabla_\alpha t = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f(t)\nabla_\alpha t = \sum_{i=0}^{n-1} \int_{\rho(t_{i+1})}^{\sigma(t_{i+1})} f(t)\nabla_\alpha t = \sum_{i=0}^{n-1} \nu(t_{i+1})f(t_{i+1})t_{i+1}^{\alpha-1}.$$

Consequently,

$$\int_a^b f(t)\nabla_\alpha t = \sum_{t\in(a,b]_\mathbb{T}} \nu(t)t^{\alpha-1}f(t).$$

If $a > b$, then the result follows from what we just proved and Theorem 2.3.2 (ii). If $a = b$, then the result follows from Theorem 2.3.2 (vi). Part (iii) and (iv) are special cases of Part (ii). The proof is complete. □

Example 2.3.1. *(i) If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = ct^{1-\alpha}$ for all $t \in \mathbb{T}$, $c \in \mathbb{R}$, then*

$$\int_a^b f(t)\nabla_\alpha t = c(b-a)$$

(ii) If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(t) = t$ for all $t \in \mathbb{R}$, then

$$\int_a^b f(t)\nabla_\alpha t = \int_a^b t^\alpha dt = \frac{1}{\alpha+1}(b^{\alpha+1} - a^{\alpha+1})$$

(iii) If $f : \frac{1}{2}\mathbb{N} \to \mathbb{R}$ is defined by $f(t) = 2^t$ and $\alpha = \frac{1}{2}$, then

$$\begin{aligned}\int_1^3 2^t \nabla_{\frac{1}{2}} t &= \frac{1}{2} \sum_{t \in (1,3]_{\frac{1}{2}\mathbb{N}}} \sqrt{\frac{1}{t}}\, 2^t = \frac{1}{2}\Big(\sqrt{\frac{2}{3}}\, 2^{\frac{3}{2}} + \sqrt{\frac{1}{2}}\, 2^2 + \sqrt{\frac{2}{5}}\, 2^{\frac{5}{2}} + \sqrt{\frac{1}{3}}\, 2^3\Big) \\ &= \sqrt{2} + \frac{6}{\sqrt{3}} + \frac{4}{\sqrt{5}}.\end{aligned}$$

Lemma 2.3.1. *Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$ with $a < b$. If $f_\nabla^{(\alpha)}(t) \geq 0$ for all $t \in [a,b]_\mathbb{T}$, then f is an increasing function on $[a,b]_\mathbb{T}$.*

Proof. Assume $f_\nabla^{(\alpha)}$ exist on $[a,b]_\mathbb{T}$ and $f_\nabla^{(\alpha)}(t) \geq 0$ for all $t \in [a,b]_\mathbb{T}$. Then, by (i) of Theorem 2.2.1, $f_\nabla^{(\alpha)}$ is continuous on $[a,b]_\mathbb{T}$ and, therefore, by Theorem 2.3.2 (vi),

$$\int_s^t f_\nabla^{(\alpha)}(\xi)\nabla_\alpha \xi \geq 0 \quad \textit{for } s,t \textit{ such that } a \leq s \leq t \leq b.$$

From Definition 2.3.2, $f(t) = f(s) + \int_s^t f_\nabla^{(\alpha)}(\xi)\nabla_\alpha \xi \geq f(s)$. ∎

Theorem 2.3.5. *Let $f : \mathbb{T} \to \mathbb{R}$ be a continuous function on $[a,b]_\mathbb{T}$ that is nabla conformal fractional differentiable of order α on $(a,b]_\mathbb{T}$ and satisfies $f(a) = f(b)$. Then there exist $\xi, \eta \in [a,b]_{\kappa,\mathbb{T}}$ such that*

$$f_\nabla^{(\alpha)}(\xi) \leq 0 \leq f_\nabla^{(\alpha)}(\eta).$$

Proof. Since the function f is continuous on the compact set $[a,b]_\mathbb{T}$, f assumes its minimum m and its maximum M Therefore there exist $\xi, \eta \in [a,b]_\mathbb{T}$ such that $m = f(\xi)$ and $M = f(\eta)$. Since $f(a) = f(b)$, we may assume that $\xi, \eta \in [a,b]_{\kappa,\mathbb{T}}$. By Lemma 2.3.1, we have

$$f_\nabla^{(\alpha)}(\xi) \leq 0 \leq f_\nabla^{(\alpha)}(\eta).$$

∎

Theorem 2.3.6. (*Mean value theorem*). *Let $0 < a < b$ and f be a continuous function on $[a,b]_{\mathbb{T}}$ which is nabla conformal fractional differentiable of order α on $[a,b]_{\kappa,\mathbb{T}}$. Then there exist $\xi, \eta \in [a,b]_{\kappa,\mathbb{T}}$ such that*

$$\xi^{\alpha-1} f_{\nabla}^{(\alpha)}(\xi) \leq \frac{(f)(b)-(f)(a)}{b-a} \leq \eta^{\alpha-1} f_{\nabla}^{(\alpha)}(\eta).$$

Proof. It follows from Theorem 2.2.5 that

$$(t)_{\nabla}^{(\alpha)} = \begin{cases} t^{1-\alpha} & \text{if } 0<\alpha<1, \\ 1 & \text{if } \alpha = 1. \end{cases} \tag{2.9}$$

Let $h(t) = f(t) - f(b) - \dfrac{(f(b)-f(a))}{(b-a)}(t-b)$. Then, the function h is continuous function on $[a,b]_{\mathbb{T}}$ which is nabla conformal fractional differentiable of order α on $[a,b)_{\mathbb{T}}$ and $h(a) = h(b)$. Combining Theorem 2.2.2 and (2.9), we have

$$h_{\nabla}^{(\alpha)}(t) = \begin{cases} f_{\nabla}^{(\alpha)}(t) - \frac{(f(b)-f(a))}{(b-a)} & \text{if } \alpha = 1, \\ f_{\nabla}^{(\alpha)}(t) - \frac{(f(b)-f(a))}{(b-a)} t^{1-\alpha} & \text{if } 0<\alpha<1. \end{cases} \tag{2.10}$$

Applying Theorem 2.3.5 to h, there exist $\xi, \eta \in (a,b]_{\mathbb{T}}$ such that $h_{\nabla}^{(\alpha)}(\xi) \leq 0 \leq h_{\nabla}^{(\alpha)}(\eta)$. That is

$$\xi^{\alpha-1} f_{\nabla}^{(\alpha)}(\xi) \leq \frac{(f)(b)-(f)(a)}{b-a} \leq \eta^{\alpha-1} f_{\nabla}^{(\alpha)}(\eta).$$

The proof is complete. □

In the next theorems we give a relationship between the nabla conformable fractional differentiable and the delta conformable fractional differentiable given in Definition 1.5.1.

Theorem 2.3.7. *Assume $f : \mathbb{T} \to \mathbb{R}$ is delta conformable fractional differentiable (Definition 1.5.1) on $\mathbb{T}^{\kappa}$ and if $f_{\Delta}^{(\alpha)}$ is continuous on $\mathbb{T}^{\kappa}$, then f is nabla conformable fractional differentiable on $\mathbb{T}_{\kappa}$ and*

$$f_{\nabla}^{(\alpha)}(t) = f_{\Delta}^{(\alpha)}(\rho(t)) \quad for\ all\ t \in \mathbb{T}_{\kappa}.$$

Proof. Fix $t \in \mathbb{T}_k$. First we consider the case where t is left-scattered. Since f is delta conformable fractional differentiable, it will be continuous function. Therefore, f will be nabla conformable fractional differentiable at t and

$$f_{\nabla}^{(\alpha)}(t) = \frac{f(\rho(t)) - f(t)}{\rho(t) - t} t^{1-\alpha}.$$

On the other hand, since $\rho(t)$ will be right-scattered, we have

$$f_{\Delta}^{(\alpha)}(\rho(t)) = \frac{f(\sigma(\rho(t))) - f(\rho(t))}{\sigma(\rho(t)) - \rho(t)} t^{1-\alpha} = \frac{f(t) - f(\rho(t))}{t - \rho(t)} t^{1-\alpha}.$$

Therefore $f_{\nabla}^{(\alpha)}(t) = f_{\Delta}^{(\alpha)}(\rho(t))$ which is the desired result.
Let now t be left-dense and right-dense, simultaneously. In this case from the existence of $f_{\Delta}^{(\alpha)}(t)$ it follows that the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{(t-s)} t^{1-\alpha} \tag{2.11}$$

exists as a finite number and is equal to $f_{\Delta}^{(\alpha)}(t)$. On the other hand since t is left-dense, from the existence of the limit 2.11 it follows that $f_{\nabla}^{(\alpha)}(t)$ exists and is equal to this limit. Therefore $f_{\nabla}^{(\alpha)}(t) = f_{\Delta}^{(\alpha)}(f)(t)$.
Finally, let t be left-dense and right-scattered. Applying mean value Theorem 15 of [87] to f, we can write

$$\xi^{\alpha-1} f_{\Delta}^{(\alpha)}(\xi) \leq \frac{(f)(t) - (s)(a)}{t-s} \leq \eta^{\alpha-1} f_{\Delta}^{(\alpha)}(\eta), \tag{2.12}$$

where ξ, η are between s and t. Since $\xi \to t,\ \eta \to t\ as\ s \to t$ and since, by the condition, $f_{\Delta}^{(\alpha)}$ is continuous, it follows from 2.12 that

$$\lim_{s \to t} \frac{f(t) - f(s)}{(t-s)} = t^{\alpha-1} f_{\Delta}^{(\alpha)}. \tag{2.13}$$

On the other hand since t is left-dense, the left-hand side of (2.13) is equal to $t^{\alpha-1} f_{\nabla}^{(\alpha)}(t)$. So, $f_{\nabla}^{(\alpha)}(t) = f_{\Delta}^{(\alpha)}$. The theorem is proved. □

The following theorem can be proved in a similar way using an analogous mean value Theorem 2.3.6.

Theorem 2.3.8. *Assume $f : \mathbb{T} \to \mathbb{R}$ is nabla conformable fractional differentiable on $\mathbb{T}_\kappa$ and if $f_{\nabla}^{(\alpha)}$ is continuous on $\mathbb{T}_\kappa$, then f is delta conformable fractional differentiable $(Definition$ 1.5.1) on $\mathbb{T}^\kappa$ and*

$$f_{\Delta}^{(\alpha)}(t) = f_{\nabla}^{(\alpha)}(\sigma(t)) \quad for\ all\ t \in \mathbb{T}^\kappa.$$

Similar to the Definition 1.5.7, we give the following definition of absolutely continuous function.

Definition 2.3.4. *A function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is said to be absolutely continuous on $[a,b]_{\mathbb{T}}$ (i.e., $f \in AC([a,b]_{\mathbb{T}}, \mathbb{R})$) if for every $\varepsilon > 0$, there exists a $\eta > 0$ such that if $\{(a_k, b_k]_{\mathbb{T}}\}_{k=1}^{m}$, is a finite pairwise disjoint family of subintervals of $[a,b]_{\mathbb{T}}$ satisfying*

$$\sum_{k=1}^{k=m} (b_k - a_k) < \eta \ then\ \sum_{k=1}^{k=m} |f(b_k) - f(\sigma(a_k))| < \varepsilon.$$

The following analogue for nabla differentiable of Theorem 4.1 in [40] can be proved in a similar way.

Lemma 2.3.2. *Assume function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ is absolutely continuous on $[a, b]_{\mathbb{T}}$, if and only if f is nabla differentiable ∇-a.e. on $[a, b]_{\mathbb{T}}$ and*

$$f(t) = f(a) + \int_{[a,t]_{\mathbb{T}}} f^{\nabla}(s)\nabla s, \quad \textit{for all } t \in [a, b]_{\mathbb{T}}.$$

The following analogue for nabla conformable fractional differentiable of Theorem 18 in [87] can be proved in a similar way.

Theorem 2.3.9. *Assume function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ is absolutely continuous on $[a, b]_{\mathbb{T}}$, then f is nabla conformable fractional differentiable of order α ∇-a.e. on $[a, b]_{\mathbb{T}}$ and the following equality is valid:*

$$f(t) = f(a) + \int_{[a,t]_{\mathbb{T}}} f_{\nabla}^{(\alpha)}(s)\nabla_{\alpha} s \quad \textit{for all } t \in [a, b]_{\mathbb{T}}.$$

Next, we introduce the nabla conformable fractional integral (or nabla α-fractional integral) on time scales for vector-valued functions.

Definition 2.3.5. *Assume $f : \mathbb{T} \to \mathbb{R}^n$, is a function and $f(t) = (f_1(t), f_2(t), ..., f_n(t))$. Let A be a ∇-measurable subset of $\mathbb{T}$. Then f is nabla α-integrable on A if and only if $f_i (i = 1, 2, ..., n)$ are nabla α-integrable on A, and*

$$\int_A f(t)\nabla_{\alpha} t = \Big(\int_A f_1(t)\nabla_{\alpha} t, \int_A f_2(t)\nabla_{\alpha} t, ..., \int_A f_n(t)\nabla_{\alpha} t \Big).$$

Combining Definition 2.3.5 and Theorem 2.3.2, we have the following theorem.

Theorem 2.3.10. *Let $\alpha \in (0,\ 1]$, $a, b, c \in \mathbb{T}$, $\lambda, \gamma \in \mathbb{R}$, and $f, g : \mathbb{T} \to \mathbb{R}^n$ be two ld-continuous functions. Then,*

(i) $\displaystyle\int_a^b [\lambda f(t) + \gamma g(t)]\nabla_{\alpha} t = \lambda \int_a^b f(t)\nabla_{\alpha} t + \gamma \int_a^b g(t)\nabla_{\alpha} t;$

(ii) $\displaystyle\int_a^b f(t)\nabla_{\alpha} t = -\int_b^a f(t)\nabla_{\alpha} t;$

(iii) $\displaystyle\int_a^b f(t)\nabla_{\alpha} t = \int_a^c f(t)\nabla_{\alpha} t + \int_c^b f(t)\nabla_{\alpha} t;$

(iv) $\displaystyle\int_a^a f(t)\nabla_{\alpha} t = 0;$

(v) if there exist $g : \mathbb{T} \to \mathbb{R}$ with $\|f(t)\| \leq g(t)$ for all $t \in [a,\ b]$, then

$$\Big\| \int_a^b f(t)\nabla_{\alpha} t \Big\| \leq \int_a^b g(t)\nabla_{\alpha} t.$$

Similar to the Definition 37 in [87], we give the following definition of absolutely continuous function.

Definition 2.3.6. *A function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}^n$, $f(t) = (f_1(t), f_2(t), ..., f_n(t))$. We say f absolutely continuous on $[a, b]_{\mathbb{T}}$ (i.e., $f \in AC([a, b]_{\mathbb{T}}, \mathbb{R}^n)$), if for every $\varepsilon > 0$, there exists a $\eta > 0$ such that if $\{(a_k, b_k]_{\mathbb{T}}\}_{k=1}^{m}$, is a finite pairwise disjoint family of subintervals of $[a, b]_{\mathbb{T}}$ satisfying*

$$\sum_{k=1}^{k=m}(b_k - a_k) < \eta \text{ then } \sum_{k=1}^{k=m}\|f(b_k) - f(\sigma(a_k))\| < \varepsilon.$$

Combining Definitions 2.2.3, 2.3.2 and Theorem 2.3.9, we have the following theorem.

Theorem 2.3.11. *Assume function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}^n$ is absolutely continuous on $[a, b]_{\mathbb{T}}$, then f is nabla conformable fractional differentiable of order α ∇-a.e. on $[a, b]_{\mathbb{T}}$ and the following equality is valid:*

$$f(t) = f(a) + \int_{[a,t]_{\mathbb{T}}} f_{\nabla}^{(\alpha)}(s)\nabla_{\alpha} s \quad for\ all\ t \in [a, b]_{\mathbb{T}}.$$

Chapter 3

Systems of first-order nabla dynamic equations on time scales

3.1 Introduction

In this chapter, we prove existence of solutions to system of first-order ∇-dynamic equations on time scale:

$$\begin{cases} x^{\nabla}(t) = f(t, x(t)), & \nabla\text{-a.e. } t \in \mathbb{T}_0, \\ x(a) = x(b), \end{cases} \tag{3.1}$$

where $\mathbb{T}$ is an arbitrary compact time scale, with $a = \min \mathbb{T}$, $b = \max \mathbb{T}$, $\mathbb{T}_0 = \mathbb{T} \backslash \{a\}$ and $f : \mathbb{T}_0 \times \mathbb{R}^n \to \mathbb{R}^n$ is a ∇-Carathéodory function. For this purpose, we use the method of solution-tube and Schauder's fixed-point theorem.

Existence results for system (3.1) were obtained in [18] with f is a continuous function. In the particular case where $n = 1$, existence results for first-order ∇-dynamic equation on time scales were obtained in [91] for the dynamic initial value problem:

$$x^{\nabla}(t) = f(t, x(t)),\ t \in (0, b]_{\mathbb{T}}, \quad \text{and} \quad x(0) = 0,$$

with f is a left-Hilger-continuous function, their results were established with the method of lower and upper solutions. Existence results were obtained in [44, 47, 53], for systems of Δ-dynamic equations on time scales. In [53] Gilbert introduced the notion of solution-tube to systems of first order Δ-dynamic equations which generalizes the notions of lower and upper solutions.

The original results of this chapter are published in [27].

3.2 Existence Theorem

In this section, we establish an existence result for the problem (3.1). A solution of this problem will be a function $x \in W^{1,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n)$ for which (3.1) is satisfied. We introduce the notion of solution tube for the problem (3.1).

Definition 3.2.1. *Let* $(v, M) \in W^{1,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n) \times W^{1,1}_{\nabla}(\mathbb{T}, [0, \infty))$. *We say that* (v, M) *is a solution tube of (3.1) if*

1. $\langle x - v(t),\ f(t, x) - v^{\nabla}(t)\rangle \leq M(t)M^{\nabla}(t)$ ∇*-a.e.* $t \in \mathbb{T}_0$ *and for every* $x \in \mathbb{R}^n$ *such that* $\|x - v(t)\| = M(t)$,
2. $v^{\nabla}(t) = f(t, v(t))$ ∇*-a.e.* $t \in \mathbb{T}_0$ *such that* $M(t) = 0$,
3. $\|v(b) - v(a)\| \leq M(a) - M(b)$.

If $\mathbb{T}$ is a real interval $[a, b]$, our definition of solution tube is equivalent to the notion of solution tube introduced in [74] for first order systems of ordinary differential equations.

We denote

$$T(v, M) = \{x \in W^{1,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n) : \|x(t) - v(t)\| \leq M(t) \text{ for every } t \in \mathbb{T}\}.$$

We consider the following problem.

$$\begin{cases} x^{\nabla}(t) + x(t) = f(t, \overline{x}(t)) + \overline{x}(t), & \nabla\text{-.a.e. } t \in \mathbb{T}_0, \\ x(a) = x(b). \end{cases} \tag{3.2}$$

where

$$\overline{x}(t) = \begin{cases} \frac{M(t)}{\|x - v(t)\|}(x - v(t)) + v(t), & \text{if } \|x - v(t)\| > M(t), \\ x(t), & \textit{otherwise.} \end{cases} \tag{3.3}$$

Lemma 3.2.1. *For every* $g \in L^1_{\nabla}(\mathbb{T}_0, \mathbb{R}^n)$, *the problem*

$$\begin{cases} x^{\nabla}(t) + x(t) = g(t), & \nabla\text{-a.e. } t \in \mathbb{T}_0, \\ x(a) = x(b), \end{cases} \tag{3.4}$$

has a unique solution $x \in W^{1,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n)$ *given by:*

$$x(t) = \hat{e}_{-1}(t, b)\left(\frac{\hat{e}_{-1}(a, b)}{\hat{e}_{-1}(a, b) - 1}\int_{(a,b]\cap\mathbb{T}} \frac{g(s)}{\hat{e}_{-1}(\rho(s), b)}\nabla s - \int_{(t,b]\cap\mathbb{T}} \frac{g(s)}{\hat{e}_{-1}(\rho(s), b)}\nabla s\right).$$

Proof. Let x be a solution to (3.4). By Theorem 3.3 in [34], consider

$$\left[\frac{x(t)}{\hat{e}_{-1}(t, b)}\right]^{\nabla} = \frac{x^{\nabla}(t)\hat{e}_{-1}(t, b) + \hat{e}_{-1}(t, b)x(t)}{\hat{e}_{-1}(t, b)\hat{e}_{-1}(\rho(t), b)} = \frac{g(t)}{\hat{e}_{-1}(\rho(t), b)},$$

and hence integrating the above on $(t, b] \cap \mathbb{T}$ obtain

$$x(t) = \hat{e}_{-1}(t, b)\left(x(b) - \int_{(t,b]\cap\mathbb{T}} \frac{g(s)}{\hat{e}_{-1}(\rho(s), b)}\nabla s\right). \tag{3.5}$$

If follows from the boundary condition in (3.4) and (3.5) that

$$x(a) = x(b) = \frac{\hat{e}_{-1}(a,b)}{\hat{e}_{-1}(a,b)-1}\int_{(a,b]\cap\mathbb{T}} \frac{g(s)}{\hat{e}_{-1}(\rho(s),b)}\nabla s. \tag{3.6}$$

So by substituting (3.6) into (3.5), the result follows.

$\square$

The following lemma is similar to Lemma 2.11 in [18].

Lemma 3.2.2. *Let $r \in W^{1,1}_{\nabla}(\mathbb{T})$ such that $r^{\nabla}(t) < 0$ ∇-a.e. $t \in \{t \in \mathbb{T}_0 : r(t) > 0\}$. If $r(a) \leq r(b)$, then $r(t) \leq 0$, for every $t \in \mathbb{T}$.*

Let us define the operator $T_2 : C(\mathbb{T}, \mathbb{R}^n) \to C(\mathbb{T}, \mathbb{R}^n)$ by:

$$\begin{aligned} T_2(x)(t) =& \hat{e}_{-1}(t,b)\left(\frac{\hat{e}_{-1}(a,b)}{\hat{e}_{-1}(a,b)-1}\int_{(a,b]\cap\mathbb{T}} \frac{(f(s,\overline{x}(s))+\overline{x}(s))}{\hat{e}_{-1}(\rho(s),b)}\nabla s \right. \\ & \left. - \int_{(t,b]\cap\mathbb{T}} \frac{(f(s,\overline{x}(s))+\overline{x}(s))}{\hat{e}_{-1}(\rho(s),b)}\nabla s\right). \end{aligned}$$

Proposition 3.2.1. *If $(v, M) \in W^{1,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n) \times W^{1,1}_{\nabla}(\mathbb{T}, [0,\infty))$ is a solution tube of (3.1) then the operator $T_2 : C(\mathbb{T}, \mathbb{R}^n) \to C(\mathbb{T}, \mathbb{R}^n)$ is compact.*

Proof. We first observe that from Definitions 1.3.17 and 3.2.1, there exists a function $h \in L^1_{\nabla}(\mathbb{T}_0, [0,\infty))$ such that $\|f(t,\overline{x}(t)) + \overline{x}(t)\| \leq h(t), \nabla$-a.e. $t \in \mathbb{T}_0$ for every $x \in C(\mathbb{T}, \mathbb{R}^n)$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of $C(\mathbb{T}, \mathbb{R}^n)$ converging to $x \in C(\mathbb{T}, \mathbb{R}^n)$. By Proposition 1.3.2,

$$\begin{aligned} &\|T_2(x_n(t)) - T_2(x(t))\| \\ &\quad \leq \frac{K(C+1)}{m}\int_{(a,b]\cap\mathbb{T}} \Big\| (f(s,\overline{x}_n(s)) + \overline{x}_n(s)) - (f(s,\overline{x}(s)) + \overline{x}(s)) \Big\| \nabla s, \end{aligned}$$

where $K := \max_{t\in\mathbb{T}} |\hat{e}_{-1}(t,b)|$, $C = \left|\dfrac{\hat{e}_{-1}(a,b)}{\hat{e}_{-1}(a,b)-1}\right|$ and $m := \min_{t\in\mathbb{T}} |\hat{e}_{-1}(t,b)|$. Then, we must show that the sequence $\{g_n\}_{n\in\mathbb{N}}$ defined by $g_n(s) = f(s, \overline{x_n}(s)) + \overline{x_n}(s)$ converges to the function $g(s) \in L^1_{\nabla}(\mathbb{T}_0, \mathbb{R}^n)$ where $g(s) = f(s, \overline{x}(s)) + \overline{x}(s)$. We can easily check that $\overline{x_n}(t) \to \overline{x}(t)$ for every $t \in \mathbb{T}_0$ and, then, by *(ii)* of Definition 1.3.17, $g_n(s) \to g(s)$ ∇-a.e. $s \in \mathbb{T}_0$. Using also the fact that $\|g_n(s)\| \leq h(s), \nabla$-a.e. $s \in \mathbb{T}_0$, we deduce that $g_n(s) \to g(s)$ in $L^1_{\nabla}(\mathbb{T}_0, \mathbb{R}^n)$ by Theorem 1.3.4. This prove the continuity of T_2. For the second part of the proof, we have to show that the set $T_2(C(\mathbb{T}, \mathbb{R}^n))$ is relatively compact. Let $y = T_2(x) \in T_2(C(\mathbb{T}, \mathbb{R}^n))$. Therefore,

$$\|T_2(x(t))\| \quad \leq \quad \frac{K(C+1)}{m}\Big(\|h(s)\|_{L^1_{\nabla}(\mathbb{T}_0,\mathbb{R}^n)}\Big).$$

So, $T_2(C(\mathbb{T}, \mathbb{R}^n))$ is uniformly bounded. This set is also equicontinuous since for every $t_1 < t_2 \in \mathbb{T}$,

$$\begin{aligned} &\|T_2(x)(t_2) - T_2(x)(t_1)\| \\ &\qquad \leq \left|\hat{e}_{-1}(b,t_2) - \hat{e}_{-1}(b,t_1)\right|\frac{(C+1)}{m}\int_{(a,b]\cap\mathbb{T}} h(s)\nabla s + K\int_{(t_1,t_2]\cap\mathbb{T}} h(s)\nabla s. \end{aligned}$$

By an analogous version of the Arzelà-Ascoli theorem adapted to our context, $T_2(C(\mathbb{T}, \mathbb{R}^n))$ is relatively compact in $C(\mathbb{T}, \mathbb{R}^n)$. Hence, T_2 is compact. □

Here is the main existence theorem for problem (3.1).

Theorem 3.2.1. *If $(v, M) \in W_{\nabla}^{1,1}(\mathbb{T}, \mathbb{R}^n) \times W_{\nabla}^{1,1}(\mathbb{T}, [0, \infty))$ is a solution tube of (3.1) then the problem (3.1) has a solution $x \in W_{\nabla}^{1,1}(\mathbb{T}, \mathbb{R}^n) \cap T(v, M)$.*

Proof. By Proposition 3.2.1, T_2 is compact. It has a fixed point by the Schauder fixed-point theorem. Lemma 3.2.1 implies that this fixed point is a solution for the problem (3.2). Then, it suffices to show that for every solution x of (3.2), $x \in T(v, M)$. Consider the set $A = \{t \in \mathbb{T}_0 : \|x(t) - v(t)\| > M(t)\}$. By Remark 1.3.1, ∇-a.e. on the set $\tilde{A} = \{t \in A : t = \rho(t)\}$, we have

$$(\|x(t) - v(t)\| - M(t))^{\nabla} = \frac{\langle x(t) - v(t), x^{\nabla}(t) - v^{\nabla}(t)\rangle}{\|x(t) - v(t)\|} - M^{\nabla}(t) \tag{3.7}$$

If $t \in A$ is left scattered, then $\nu(t) = t - \rho(t) > 0$ and

$$\begin{aligned}
&(\|x(t) - v(t)\| - M(t))^{\nabla} \\
&\qquad = \frac{\|x(t) - v(t)\|^2 - \|x(\rho(t)) - v(\rho(t))\|\|x(t) - v(t)\|}{\nu(t)\|x(t) - v(t)\|} - M^{\nabla}(t) \\
&\qquad \leq \frac{\langle x(t) - v(t),\ (x(t) - v(t)) - (x(\rho(t)) - v(\rho(t)))\rangle}{\nu(t)\|x(t) - v(t)\|} - M^{\nabla}(t) \\
&\qquad = \frac{\langle x(t) - v(t),\ x^{\nabla}(t) - v^{\nabla}(t)\rangle}{\|x(t) - v(t)\|} - M^{\nabla}(t).
\end{aligned}$$

Since (v, M) is a solution tube of (3.1), we have ∇-a.e. on $\{t \in A : M(t) > 0\}$,

$$\begin{aligned}
&(\|x(t) - v(t)\| - M(t))^{\nabla} \\
&\qquad \leq \frac{\langle x(t) - v(t),\ f(t, \overline{x}(t)) + \overline{x}(t) - x(t) - v^{\nabla}(t)\rangle}{\|x(t) - v(t)\|} - M^{\nabla}(t) \\
&\qquad = \frac{\langle \overline{x}(t) - v(t),\ f(t, \overline{x}(t)) - v^{\nabla}(t)\rangle}{M(t)} \\
&\qquad\quad + \frac{\langle x(t) - v(t),\ \dfrac{(M(t) - \|x(t) - v(t)\|)}{\|x(t) - v(t)\|}(x(t) - v(t))\rangle}{\|x(t) - v(t)\|} - M^{\nabla}(t) \\
&\qquad = \frac{M(t)M^{\nabla}(t)}{M(t)} - (\|x(t) - v(t)\| - M(t)) - M^{\nabla}(t) < 0.
\end{aligned}$$

On the other hand, we have ∇-a.e. on $\{t \in A : M(t) = 0\}$, that

$$
\begin{aligned}
&(\|x(t)-v(t)\|-M(t))^{\nabla} \\
&\quad \leq \frac{\langle x(t)-v(t),\ f(t,\overline{x}(t))+\bar{x}(t)-x(t)-v^{\nabla}(t)\rangle}{\|x(t)-v(t)\|}-M^{\nabla}(t) \\
&\quad = \frac{\langle x(t)-v(t),\ f(t,v(t))-v^{\nabla}(t)\rangle}{\|x(t)-v(t)\|}-\|x(t)-v(t)\|-M^{\nabla}(t)<0.
\end{aligned}
$$

If we set $r(t)=\|x(t)-v(t)\|-M(t)$, then $r^{\nabla}(t)<0$ ∇-a.e. $t\in\{t\in\mathbb{T}_0 : r(t)>0\}$. Moreover, since (v,M) is a solution tube of (3.1) and $x(a)=x(b)$, then $r(a)-r(b)\leq \|v(a)-v(b)\|-(M(a)-M(b))\leq 0$. Lemma 3.2.2 implies that $A=\emptyset$. So, $x\in T(v,M)$ and the theorem is proved. □

Example 3.2.1. *The following is a modified version, considering a periodic condition, of Example 4.1 in [53]:*

$$
\begin{cases} x^{\nabla}(t)=a_1\|x(t)\|^2x(t)-a_2x(t)+a_3\varphi(t), & t\in\mathbb{T}_0, \\ x(a)=x(b). \end{cases} \tag{3.8}
$$

where $a_1,a_2,a_3\in\mathbb{R}_+$ such that $a_2\geq a_1+a_3$ and $\varphi:\mathbb{T}_0\to\mathbb{R}^n$ is a continuous function satisfying $\|\varphi(t)\|=1$ for every $t\in\mathbb{T}_k$. It is easy to check that $v=0$ and $M=1$, is a tube solution. By Theorem 3.2.1, problem (3.8) has a solution $x\in W^{1,1}_{\nabla}(\mathbb{T},\mathbb{R}^n)$ such that $\|x(t)\|\leq 1$ for every $t\in\mathbb{T}$.

Remark 3.2.1. *Definition 3.2.1 generalizes the notions of lower and upper solutions α and β introduced in [91] in the particular case where the problem (3.1) is considered with $n=1$, and the periodic boundary condition replaced by $x(0)=0$ and f is left-Hilger continuous on $(0,b]_{\mathbb{T}}\times\mathbb{R}$. We recall these definitions.*
Consider the problem:

$$
\begin{cases} x^{\nabla}(t)=f(t,x(t)), & \textit{for all } t\in(0,\ b]_{\mathbb{T}}, \\ x(0)=x(b). \end{cases} \tag{3.9}
$$

Definition 3.2.2. Let α, β be nabla differentiable functions on $(0,b]_{\mathbb{T}}$. We call α a lower solution to (3.9) on $[0,b]_{\mathbb{T}}$ if

(i) $\alpha^{\nabla}(t)\leq f(t,\alpha(t))$, for all $t\in(0,b]_{\mathbb{T}}$;

(ii) $\alpha(0)=\alpha(b)$.

Similarly, we call β an upper solution to (3.9) on $[0,b]_{\mathbb{T}}$ if

(i) $\beta^{\nabla}(t)\geq f(t,\beta(t))$, *for all* $t\in(0,b]_{\mathbb{T}}$*;*

(ii) $\beta(0)=\beta(b)$.

Remark If $\alpha,\ \beta \in \mathbb{R}$ are, respectively, lower and upper solutions of (3.9) such that $\alpha(t) \leq \beta(t)$ for every $t \in (0,b]_{\mathbb{T}}$, then $v = (\alpha+\beta)/2$ and $M = (\beta-\alpha)/2$ is a solution tube for this problem. Conversely, if (v, M) is a solution tube of (3.9) with v and M of class C^1,$v(0) = v(b)$, and $M(0) = M(b)$, then $\alpha = v - M$ and $\beta = v + M$ are, respectively, lower and upper solutions of (3.9).

Example 3.2.2. Consider the problem:

$$\begin{cases} x^{\nabla}(t) = -x^3(t) - t, & \text{for all } t \in (0,1]_{\mathbb{T}}; \\ x(0) = x(1). \end{cases} \tag{3.10}$$

Verify that with $v = 0$ and $M = 1$, (v, M) is a solution-tube of (3.10). By Theorem 3.2.1, the problem (3.10) has a solution x such that $|x(t)| \leq 1$ for every $t \in (0,1]_{\mathbb{T}}$. Observe that $\alpha = v - M$ and $\beta = v + M$ are, respectively, lower and upper solutions of (3.10) and $-1 \leq x(t) \leq 1$ for every $t \in [0,1]_{\mathbb{T}}$.

Chapter 4

Systems of first-order nabla dynamic inclusions on time scales

4.1 Introduction

In this chapter, we establish an existence result for the following system of first-order ∇-dynamic inclusions on time scale:

$$\begin{cases} x^{\nabla}(t) \in F(t, x(\rho(t))), & \nabla\text{-a.e. } t \in \mathbb{T}_0, \\ x \in (\mathfrak{BC}), \end{cases} \tag{4.1}$$

where $\mathbb{T}$ to be an arbitrary compact time scale, with $a = \min \mathbb{T}$, $b = \max \mathbb{T}$, $\mathbb{T}_0 = \mathbb{T} \backslash \{a\}$, $F : \mathbb{T}_0 \times \mathbb{R}^n \to \mathbb{R}^n$ is a multivalued map with compact and convex values, and $(\mathfrak{BC})$ denotes the terminal value or the periodic boundary value conditions:

$$x(b) = x_0, \tag{4.2}$$

$$x(a) = x(b). \tag{4.3}$$

In the particular case where $n = 1$, existence results for first order ∇-dynamic inclusion on time scales were obtained in [12] for the general boundary conditions:

$$x^{\nabla}(t) \in F(t, x(t)), \text{ a.e. } on\ \mathbb{T}_{\kappa}, \quad \text{and} \quad L(x(a), x(b)) = 0,$$

with $F : \mathbb{T}_{\kappa} \times \mathbb{R} \to 2^{\mathbb{R}} \backslash \emptyset$ a multivalued map with compact and convex values and L is a continuous single-valued map. Their results were established with the method of lower and upper solutions. Existence results for systems of first order ∇-dynamic inclusions were obtained in [54] for the initial value problem. In [49] Frigon and Gilbert introduced the notion of solution-tube to systems of first order Δ-dynamic inclusions (with an initial or a periodic boundary value condition) which generalizes the notions of lower and upper solutions given in [12]. In order to obtain the existence results for problem (4.1), we introduce the notion of solution-tube of (4.1).

The original results of this chapter are published in [28].

4.2 Existence Theorem

In this section, we are concerned with the existence of solutions for the problem (4.1). A solution of this problem will be a function $x \in W^{1,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n)$ for which (4.1) is satisfied, we introduce the notion of solution-tube of this problem.

Definition 4.2.1. *Let $(v, M) \in W^{1,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n) \times W^{1,1}_{\nabla}(\mathbb{T}, [0, \infty))$. We say that (v, M) is a solution tube of (4.1) if*

1. *∇-a.e. $t \in \mathbb{T}_0$ and for every $x \in \mathbb{R}^n$ such that $\|x - v(\rho(t))\| = M(\rho(t))$, there exists $\delta > 0$ such that, for every $u \in \mathbb{R}^n$ such that $\|u - x\| < \delta$, and $\|u - v(\rho(t))\| \geq M(\rho(t))$, there exists $y \in F(t, u)$ such that*
$$\langle u - v(\rho(t)), y - v^{\nabla}(t)\rangle \geq M^{\nabla}(t)\|u - v(\rho(t))\|.$$
2. *$v^{\nabla}(t) \in F(t, v(\rho(t)))$ ∇-a.e. $t \in \mathbb{T}_0$ such that $M(\rho(t)) = 0$,*
3. *$M(t) = 0$, for every $t \in \mathbb{T}_0$ such that $M(\rho(t)) = 0$,*
4. *- If $(\mathfrak{BC})$ denotes (4.2), then $\|x_0 - v(b)\| \leq M(b)$.*
   - If $(\mathfrak{BC})$ denotes (4.3), then $\|v(a) - v(b)\| \leq M(b) - M(a)$.

We denote
$$T(v, M) = \{x \in W^{1,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n) : \|x(t) - v(t)\| \leq M(t) \ for\ every\ t \in \mathbb{T}\}.$$

We need the following auxiliary lemmas.

Lemma 4.2.1. *Let $g \in L^1_{\nabla}(\mathbb{T}_0, \mathbb{R}^n)$. The function $x : \mathbb{T} \to \mathbb{R}^n$ defined by*
$$x(t) = \hat{e}_{-1}(b, t)\Big(x_0 - \int_{(t,b]\cap\mathbb{T}} g(s)\hat{e}_{-1}(s, b)\nabla s\Big) \tag{4.4}$$
is a unique solution of the problem
$$\begin{cases} x^{\nabla}(t) - x(\rho(t)) = g(t), & \nabla\text{-a.e. } t \in \mathbb{T}_0 \\ x(b) = x_0. \end{cases} \tag{4.5}$$

Proof. We check (4.5) for each pair $(x_i,\ g_i)$, $i \in \{1, 2, ..., n\}$, by direct calculation. From Theorem 3.3 in [34] and Proposition 1.3.4 we have that

$$\begin{aligned} x^{\nabla}(t) - x(\rho(t)) &= x_0 \left(\hat{e}_{-1}(b,t)\right)^{\nabla} - \left(\hat{e}_{-1}(b,t)\right)^{\nabla} \int_{(\rho(t),b]\cap\mathbb{T}} g(s)\hat{e}_{-1}(s,b)\nabla s \\ &\quad -\hat{e}_{-1}(b,t)\left(\int_{(t,b]\cap\mathbb{T}} g(s)\hat{e}_{-1}(s,b)\nabla s\right)^{\nabla} - \hat{e}_{-1}(b,\rho(t))x_0 \\ &\quad +\hat{e}_{-1}(b,\rho(t)) \int_{(\rho(t),b]\cap\mathbb{T}} g(s)\hat{e}_{-1}(s,b)\nabla s \\ &= g(t), \end{aligned}$$

for all $t \in \mathbb{T}$. It is easy to verify that $x(b) = x_0$. $\square$

Lemma 4.2.2. *Let $g \in L^1_\nabla(\mathbb{T}_0, \mathbb{R}^n)$. The function $x : \mathbb{T} \to \mathbb{R}^n$ defined by*

$$x(t) = \frac{1}{\hat{e}_{-1}(t,b)} \left(\frac{1}{1 - \hat{e}_{-1}(a,b)} \int_{(a,b]\cap\mathbb{T}} g(s)\hat{e}_{-1}(s,b)\nabla s - \int_{(t,b]\cap\mathbb{T}} g(s)\hat{e}_{-1}(s,b)\nabla s \right)$$

is a unique solution of the problem

$$\begin{cases} x^\nabla(t) - x(\rho(t)) = g(t), & \nabla\text{-}a.e.\ t \in \mathbb{T}_0, \\ x(a) = x(b). \end{cases} \tag{4.6}$$

Proof. The result follows in a similar way to the proof of *Lemma* 4.2.1. $\square$

The following lemma can be proved analogously to Lemma 2.24 in [53].

Lemma 4.2.3. *Let $r \in W^{1,1}_\nabla(\mathbb{T})$ such that $r^\nabla(t) > 0$ ∇-a.e. $t \in \{t \in \mathbb{T}_0 : r(\rho(t)) > 0\}$. If one of the following conditions holds,*

(i) $r(b) \leq 0$*;*

(ii) $r(b) \leq r(a)$*;*

then $r(t) \leq 0$, for every $t \in \mathbb{T}$.

We assume the following hypothesis

(H_1) $F : \mathbb{T}_0 \times \mathbb{R}^n \to \mathbb{R}^n$ is a ∇-Carathéodory multivalued map with compact and convex values.

(H_2) There exists $(v, M) \in W^{1,1}_\nabla(\mathbb{T}, \mathbb{R}^n) \times W^{1,1}_\nabla(\mathbb{T}, [0,\infty))$ a solution tube of (4.1).

To prove our existence theorem, we consider the following modified problem:

$$\begin{cases} x^\nabla(t) - x(\rho(t)) \in F_u(t, x(\rho(t))) - \overline{x}(\rho(t)), & \nabla\text{-a.e. } t \in \mathbb{T}_0, \\ x \in (\mathfrak{BC}), \end{cases} \tag{4.7}$$

where $F_u : \mathbb{T}_0 \times \mathbb{R}^n \to \mathbb{R}^n$ is defined by :

$$F_u(t,x) = F(t, \overline{x}(\rho(t))) \cap G(t,x); \tag{4.8}$$

with

$$G(t,x) = \begin{cases} v^\nabla(t) & \text{if M}(\rho(\text{t})) = 0, \\ \mathbb{R}^n & \text{if } \|x(\rho(t)) - v(\rho(t))\| \leq M(\rho(t)), \\ & \text{ and M}(\rho(\text{t})) > 0, \\ \big\{ z \in \mathbb{R}^n : \langle x - v(\rho(t)), z - v^\nabla(t) \rangle & \\ \qquad\qquad \geq M^\nabla(t)\|x - v(\rho(t))\| \big\}, & \text{otherwise,} \end{cases}$$

and

$$\overline{x}(t) = \begin{cases} \frac{M(t)}{\|x-v(t)\|}(x - v(t)) + v(t), & \text{if } \|x - v(t)\| > M(t), \\ x(t), & \textit{otherwise}. \end{cases}$$

Remark 4.2.1. *For every (t, x) such that $\|x - v(\rho(t))\| > M(\rho(t))$,*

$$G(t, x) = G(t, \overline{x}_\theta(\rho(t))) \ \textit{for all} \ \theta \in [0, 1[, \tag{4.9}$$

where

$$\overline{x}_\theta(\rho(t)) = \theta\overline{x}(\rho(t)) + (1 - \theta)x.$$

Indeed, for $\theta \in [0, 1[$,

$$\overline{x}_\theta(\rho(t)) - v(\rho(t)) = \left(1 - \theta + \frac{\theta M(\rho(t))}{\|x - v(\rho(t))\|}\right)(x - v(\rho(t))).$$

Thus,

$$\begin{aligned} G(t, x) &= \left\{z \in \mathbb{R}^n : \langle x - v(\rho(t)), z - v^\nabla(t)\rangle \geq M^\nabla(t)\|x - v(\rho(t))\|\right\} \\ &= \left\{z \in \mathbb{R}^n : \langle \overline{x}_\theta(\rho(t)) - v(\rho(t)), z - v^\nabla(t)\rangle \geq M^\nabla(t)\|\overline{x}_\theta(\rho(t)) - v(\rho(t))\|\right\}. \end{aligned}$$

So, for $\theta \in [0, 1[$, $G(t, x) = G(t, \overline{x}_\theta(\rho(t)))$ since $\|\overline{x}_\theta(\rho(t)) - v(\rho(t))\| > M(\rho(t))$.

Similar to the Propositions 3.3 and 3.4 in [49], we give the following propositions.

Proposition 4.2.1. *The multivalued map $G : \mathbb{T}_0 \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following properties:*

(i) $G(t, x)$ has nonempty, closed, convex values for all $x \in \mathbb{R}^n$, and for ∇-almost every $t \in \mathbb{T}_0$;

(ii) $x \mapsto G(t, x)$ has closed graph for ∇-almost every $t \in \mathbb{T}_0$;

(iii) $t \mapsto G(t, x)$ is ∇-measurable for every $x \in \mathbb{R}^n$.

Proof. (i) It is obvious that G has nonempty, closed, convex values.
(ii) To show that

$$A_t = \left\{(x, y) \in \mathbb{R}^{2n} : y \in G(t, x)\right\}$$

is closed for ∇-a.e. $t \in \mathbb{T}_0$, we just have to check the case where $t \in \mathbb{T}_0$ is such that $M(\rho(t)) \neq 0$. Let $\{(x_k, y_k)\}$ be in A_t such that $x_k \to x$ and $y_k \to y$. If $\|x - v(\rho(t))\| \leq M(\rho(t))$ then $y \in G(t, x) = \mathbb{R}^n$. So, $(x, y) \in A_t$. Otherwise, $\|x - v(\rho(t))\| > M(\rho(t))$ and for k sufficiently large $\|x_k - v(\rho(t))\| > M(\rho(t))$ and

$$\langle x_k - v(\rho(t)), y_k - v^\nabla(t)\rangle \geq M^\nabla(t)\|x_k - v(\rho(t))\|.$$

Therefore,

$$\langle x - v(\rho(t)), y - v^\nabla(t)\rangle \geq M^\nabla(t)\|x - v(\rho(t))\|, \ \textit{and hence} \ (x, y) \in A_t.$$

(iii) Let C be a nonempty, closed subset of $\mathbb{R}^n$, and fix $x \in \mathbb{R}^n$. Let $\{y_m : m \in \mathbb{N}\}$ be a countable, dense subset of C. Observe that

$$\mathcal{B}_x = \big\{t \in \mathbb{T}_0 : G(t,x) \cap C \neq \emptyset\big\} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup (\mathcal{B}_3 \cap \mathcal{B}_4),$$

where

$$\begin{aligned}
\mathcal{B}_1 &= \Big\{t \in \mathbb{T}_0 : v^\nabla(t) \in C\Big\} \cap \Big\{t \in \mathbb{T}_0 : M(\rho(t)) = 0\Big\},\\
\mathcal{B}_2 &= \Big\{t \in \mathbb{T}_0 : \|x - v(\rho(t))\| - M(\rho(t)) \leq 0\Big\} \cap \Big\{t \in \mathbb{T}_0 : M(\rho(t)) > 0\Big\},\\
\mathcal{B}_3 &= \Big\{t \in \mathbb{T}_0 : \|x - v(\rho(t))\| - M(\rho(t)) > 0\Big\} \cap \Big\{t \in \mathbb{T}_0 : M(\rho(t)) > 0\Big\},\\
\mathcal{B}_4 &= \bigcap_{k\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \Big\{t \in \mathbb{T}_0 : \big\langle x - v(\rho(t)), y_m - v^\nabla(t)\big\rangle \geq M^\nabla(t)\|x - v(\rho(t))\| - \frac{1}{k}\Big\}.
\end{aligned}$$

The ∇-measurability of the maps $t \mapsto v(\rho(t))$, $t \mapsto M(\rho(t))$, $t \mapsto v^\nabla(t)$, and $t \mapsto M^\nabla(t)$ imply that $\mathcal{B}_x$ is ∇-measurable, and so is $t \mapsto G(t,x)$. $\square$

We now define the multivalued map $\mathcal{H} : C(\mathbb{T},\mathbb{R}^n) \to L^1_\nabla(\mathbb{T}_0,\mathbb{R}^n)$ by

$$\mathcal{H}(x) = \Big\{w \in L^1_\nabla(\mathbb{T}_0,\mathbb{R}^n) : w(t) \in F_u(t, x(\rho(t))) \ \ \nabla\text{-a.e. } t \in \mathbb{T}_0\Big\}.$$

Proposition 4.2.2. *Assume (H_1) and (H_2). Then, $\mathcal{H}$ has nonempty, convex values, and there exists $h \in L^1_\nabla(\mathbb{T}_0,[0,\infty))$ such that*

$$\|w(t)\| \leq h(t) \quad \nabla\text{-a.e. on } \mathbb{T}_0 \text{ for all } w \in \mathcal{H}(x) \text{ and all } x \in C(\mathbb{T},\mathbb{R}^n). \tag{4.10}$$

Proof. First of all, we want to show that $\mathcal{H}$ has nonempty values. Let $x \in C(\mathbb{T},\mathbb{R}^n)$. There exists a sequence of simple functions $\{x_m\}_{m\in\mathbb{N}}$ such that

$$\|x_m(\rho(t)) - v(\rho(t))\| > M(\rho(t)) \qquad \nabla\text{-a.e. on } \Big\{t : \|x(\rho(t)) - v(\rho(t))\| > M(\rho(t))\Big\},$$

and such that $x_m \to \bar{x}$ in $C(\mathbb{T},\mathbb{R}^n)$. Since the multivalued maps $t \mapsto F(t,y)$ and $t \mapsto G(t,y)$ are ∇-measurable for every $y \in \mathbb{R}^n$, the maps $t \mapsto F(t, x_m(\rho(t)))$ and $t \mapsto G(t, x_m(\rho(t)))$ are also ∇-measurable for every $m \in \mathbb{N}$. Proposition 1.2.2 implies that, for every $m \in \mathbb{N}$,

$$t \mapsto F(t, x_m(\rho(t))) \cap G(t, x_m(\rho(t)))$$

is ∇-measurable, and for every $k \in \mathbb{N}$,

$$t \mapsto \bigcup_{m\geq k} \Big(F(t, x_m(\rho(t))) \cap G(t, x_m(\rho(t)))\Big)$$

is ∇-measurable. Again, Propositions 1.2.2 and 1.2.1 imply that

$$t \mapsto \bigcap_{k\in\mathbb{N}} \overline{\bigcup_{m\geq k} \Big(F(t, x_m(\rho(t))) \cap G(t, x_m(\rho(t)))\Big)}$$

is ∇-measurable. Definition 4.2.1 (1) guarantees that this map has nonempty values ∇-almost everywhere on $\{t : M(\rho(t)) \neq 0\}$. Indeed, ∇-almost everywhere on

$$\{t : M(\rho(t)) \neq 0 \textit{ and } \|\bar{x}(\rho(t)) - v(\rho(t))\| < M(\rho(t))\},$$

for $m \geq k$ sufficiently large, $\|x_m(\rho(t)) - v(\rho(t))\| < M(\rho(t))$ and

$$F(t, x_m(\rho(t))) \cap G(t, x_m(\rho(t))) = F(t, x_m(\rho(t))) \cap \mathbb{R}^n \neq \emptyset.$$

On the other hand, for ∇-almost every

$$t \in \{t : \|\bar{x}(\rho(t)) - v(\rho(t))\| = M(\rho(t)) > 0\},$$

if there exists $m \geq k$ such that $\|x_m(\rho(t)) - v(\rho(t))\| \leq M(\rho(t))$, then as before, $F(t, x_m(\rho(t))) \cap G(t, x_m(\rho(t))) \neq \emptyset$. Otherwise, there exists a $\delta > 0$ given by Definition 4.2.1 (1) and $m \geq k$ sufficiently large such that

$$\|x_m(\rho(t)) - \bar{x}(\rho(t))\| < \delta, \quad \|x_m(\rho(t)) - v(\rho(t))\| > M(\rho(t)),$$

and there exists $z \in F(t, x_m(\rho(t)))$ such that

$$\langle x_m(\rho(t)) - v(\rho(t)), z - v^{\nabla}(t)\rangle \geq \|x_m(\rho(t)) - v(\rho(t))\| M^{\nabla}(t),$$

i.e. $z \in F(t, x_m(\rho(t))) \cap G(t, x_m(\rho(t)))$.
Thus, the multivalued map $\Phi : \mathbb{T}_0 \to L^1_{\nabla}(\mathbb{T}_0, \mathbb{R}^n)$ defined by

$$\Phi(t) = \begin{cases} \bigcap_{k\in\mathbb{N}} \overline{\bigcup_{m\geq k} \Big(F(t, x_m(\rho(t))) \cap G(t, x_m(\rho(t)))\Big)}, & \text{if } t \in \{t : M(\rho(t)) \neq 0\}, \\ v^{\nabla}(t), & \text{if } t \in \{t : M(\rho(t)) = 0\}, \end{cases}$$

is ∇-measurable and has nonempty and compact values. Finally, Theorem 1.2.1 guarantees the existence of a ∇-measurable selection w of Φ.
We must show that $w \in \mathcal{H}(x)$. Since $w(t) \in \Phi(t)$ ∇-a.e., we have,

$$w(t) \in \overline{\bigcup_{m\geq k} \Big(F(t, x_m(\rho(t))) \cap G(t, x_m(\rho(t)))\Big)} \quad \nabla\text{-a.e. in}\{t : M(\rho(t)) \neq 0\},$$

for every $k \in \mathbb{N}$. So, for ∇-almost every $t \in \{t : M(\rho(t)) \neq 0\}$, there exists a subsequence

$$u_{m_l}(t) \in F(t, x_{m_l}(\rho(t))) \cap G(t, x_{m_l}(\rho(t)))$$

such that $u_{m_l}(t) \to w(t)$. If $\|x(\rho(t)) - v(\rho(t))\| \leq M(\rho(t))$, since $y \mapsto F(t, y)$ and $y \mapsto G(t, y)$ have closed graph and since $x_{m_l}(\rho(t)) \to \bar{x}(\rho(t)) = x(\rho(t))$, we deduce that

$$w(t) \in F(t, \overline{x}(\rho(t))) \cap G(t, x(\rho(t))) = F_u(t, x(\rho(t))).$$

On the other hand, if $\|x(\rho(t)) - v(\rho(t))\| > M(\rho(t))$, since $x_{m_l}(\rho(t)) \to \overline{x}(\rho(t))$, there exists a sequence $\{y_{m_l}\}$ such that $y_{m_l} \to x(\rho(t))$ and

$$x_{m_l}(\rho(t)) = \theta_{m_l}\overline{x}_{m_l}(\rho(t)) + (1 - \theta_{m_l})y_{m_l} = \overline{(y_{m_l})}_{\theta_{m_l}}(\rho(t)) \quad \textit{for some } \theta_{m_l} \in [0, 1[.$$

By (4.9),

$$u_{m_l}(t) \in F(t, x_{m_l}(\rho(t))) \cap G(t, x_{m_l}(\rho(t))) = F(t, x_{m_l}(\rho(t))) \cap G(t, y_{m_l}).$$

Again, since $y \in F(t, y)$ and $y \in G(t, y)$ have closed graph and since $x_{m_l}(\rho(t)) \to \bar{x}(\rho(t))$ and $y_{m_l} \to x(\rho(t))$, we can deduce that

$$w(t) \in F(t, \overline{x}(\rho(t))) \cap G(t, x(\rho(t))) = F_u(t, x(\rho(t))).$$

Moreover, Definition 4.2.1 (2) implies that ∇-a.e. on $\{t : M(\rho(t)) = 0\}$,

$$w(t) = v^{\nabla}(t) \in F(t, \overline{x}(\rho(t))) \cap G(t, x(\rho(t))) = F_u(t, x(\rho(t))).$$

Hence, we can conclude that $w \in \mathcal{H}(x)$ since by hypothesis (H_1), $w \in L^1_{\nabla}(\mathbb{T}_0, \mathbb{R}^n)$.
The convexity of $\mathcal{H}(x)$ follows from convexity of the values of F and G.
Finally, hypothesis (H_1) guarantees the existence of $h := h_q \in L^1_{\nabla}(\mathbb{T}_0, [0, \infty))$ with $q = max\{\|v(t)\| + M(t) : t \in \mathbb{T}\}$, such that for every $x \in C(\mathbb{T}, \mathbb{R}^n)$ and every $w \in \mathcal{H}(x)$,

$$\|w(t)\| \leq h(t) \quad \nabla\text{-a.e. } t \in \mathbb{T}_0.$$

□

Let us define the multivalued operators $\mathcal{N}_I, \mathcal{N}_p : C(\mathbb{T}, \mathbb{R}^n) \to C(\mathbb{T}, \mathbb{R}^n)$ by

$$\mathcal{N}_I(x)(t) = \Big\{u : u(t) = \hat{e}_{-1}(b,t)\Big(x_0 - \int_{(t,b]\cap\mathbb{T}} \hat{e}_{-1}(s,b)\Big(w(s) - \bar{x}(\rho(s))\Big)\nabla s\Big), \\ \textit{where } w \in \mathcal{H}(x)\Big\}.$$

and

$$\mathcal{N}_p(x)(t) = \Big\{u \in C(\mathbb{T}, \mathbb{R}^n) : \\ u(t) = \frac{1}{\hat{e}_{-1}(t,b)}\Big[\frac{1}{1 - \hat{e}_{-1}(a,b)} \int_{(a,b]\cap\mathbb{T}} \Big(w(s) - \overline{x}(\rho(s))\Big)\hat{e}_{-1}(s,b)\nabla s \\ - \int_{(t,b]\cap\mathbb{T}} \Big(w(s) - \overline{x}(\rho(s))\Big)\hat{e}_{-1}(s,b)\nabla s\Big],\ \textit{where } w \in \mathcal{H}(x)\Big\}.$$

Clearly, from Lemma 4.2.1 (resp. Lemma 4.2.2), the solutions of problem (4.7), (4.2) (resp. problem (4.7), (4.3)) coincide with the fixed points of operator $\mathcal{N}_I$ (resp. $\mathcal{N}_p$).

Proposition 4.2.3. *Assume (H_1) and (H_2). The operator $\mathcal{N}_I$ is compact, u.s.c., with nonempty, convex and compact values.*

Proof. The previous proposition insures that $\mathcal{N}_I$ has nonempty, convex values, and guarantees the existence of $h \in L^1_{\nabla}(\mathbb{T}_0, [0, \infty))$ satisfying (4.10).

Set $K := \max\{|\hat{e}_{-1}(t,s)|, s,t \in \mathbb{T}\}$ and $q := \max\{\|v(t)\| + M(t) : t \in \mathbb{T}\}$. To show that $\mathcal{N}_I(C(\mathbb{T},\mathbb{R}^n))$ is bounded, we just have to remark that for every $u \in \mathcal{N}_I(C(\mathbb{T},\mathbb{R}^n))$,

$$\begin{aligned}\|u(t)\| &\leq K\Big(\|x_0\| + \int_{(a,b]\cap\mathbb{T}} K\,\|w(s) - \bar{x}(\rho(s))\|\,\nabla s\Big)\\ &\leq K\Big(\|x_0\| + \int_{(a,b]\cap\mathbb{T}} K\,(h(s)+q)\,\nabla s\Big) \ \textit{for all}\ t \in \mathbb{T}.\end{aligned}$$

On the other hand, for every $t_2 > t_1 \in \mathbb{T}$,

$$\begin{aligned}&\|u(t_2) - u(t_1)\|\\ &\leq |\hat{e}_{-1}(b,t_2) - \hat{e}_{-1}(b,t_1)| \left(\|x_0\| + \int_{(t_2,b]\cap\mathbb{T}} |\hat{e}_{-1}(s,b)|\|w(s) - \bar{x}(\rho(s))\|\nabla s\right)\\ &\quad + \int_{(t_1,t_2]\cap\mathbb{T}} |\hat{e}_{-1}(s,t_1)|\,\|w(s) - \overline{x}(\rho(s))\|\,\nabla s\\ &\leq |\hat{e}_{-1}(b,t_2) - \hat{e}_{-1}(b,t_1)| \left(\|x_0\| + \int_{(a,b]\cap\mathbb{T}} K(q+h(s))\nabla s\right)\\ &\quad + K^2 \int_{(t_1,t_2]\cap\mathbb{T}} (q+h(s))\nabla s.\end{aligned}$$

Thus, $\mathcal{N}_I(C(\mathbb{T},\mathbb{R}^n))$ is equicontinuous since

$$t \to \hat{e}_{-1}(b,t) \quad \textit{and} \quad t \to \int_{(t,b]\cap\mathbb{T}} (q+h(s))\nabla s$$

are continuous on $\mathbb{T}$. By an analogous version of the Arzelà-Ascoli theorem adapted to our context, we conclude that $\mathcal{N}_I(C(\mathbb{T},\mathbb{R}^n))$ is relatively compact in $C(\mathbb{T},\mathbb{R}^n)$.

We now prove that $\mathcal{N}_I$ has closed graph.

Let $\{x_m\}$ and $\{u_m\}$ be convergent sequences in $C(\mathbb{T},\mathbb{R}^n)$ such that $x_m \to x$, $u_m \to u$ and $u_m \in \mathcal{N}_I(x_m)$. Let $w_m \in \mathcal{H}(x_m)$ be such that

$$u_m(t) = \hat{e}_{-1}(b,t)\Big(x_0 - \int_{(t,b]\cap\mathbb{T}} \hat{e}_{-1}(s,b)\Big(w_m(s) - \overline{x}_m(\rho(s))\Big)\nabla s\Big).$$

Let h be the function given in (4.10). Considering the extensions $\widetilde{w}_m$ and $\widetilde{h}$ in $L^1([a,b])$, we have

$$\|\widetilde{w}_m(t)\| \leq \widetilde{h}(t) \ \textit{for almost every}\ t \in [a,b].$$

By Dunford-Pettis theorem, there exists $g \in L^1([a,b],\mathbb{R}^n)$ and a subsequence still denoted $\{\widetilde{w}_m\}$ such that $\widetilde{w}_m \to g$ in $L^1([a,b],\mathbb{R}^n)$. Since a closed convex set is weakly closed, there exist $\widetilde{z}_m \in co\{\widetilde{w}_m, \widetilde{w}_{m+1}, ...\}$ such that $\widetilde{z}_m \to g$ in $L^1([a,b],\mathbb{R}^n)$.

Thus, there exists a subsequence again noted $\{\widetilde{z}_m\}$ such that, $\widetilde{z}_m(t) \to g(t)$ for almost every $t \in [a,b]$. Therefore, for almost every $t \in [a,b]$,

$$\widetilde{z}_m(t) \in co\Big\{\bigcup_{l\geq m} \widetilde{w}_l(t)\Big\} \subset co\Big\{\bigcup_{l\geq m} \widetilde{F}(t,\overline{x}_l(\rho(t))) \cap \widetilde{G}(t,x_l(\rho(t)))\Big\}.$$

where the multivalued maps $\widetilde{F}$ and $\widetilde{G}$ are respectively extensions of the multivalued maps F and G in the sense of (1.1). Taking the limit, we get

$$
\begin{aligned}
g(t) \in \bigcap_{m\in\mathbb{N}} \overline{co}\Big\{ \bigcup_{l\geq m} \widetilde{F}(t, \overline{x}_l(\rho(t))) \cap \widetilde{G}(t, x_l(\rho(t)))\Big\} \\
\subset \widetilde{F}(t, \overline{x}(\rho(t))) \cap \widetilde{G}(t, x(\rho(t))) = \widetilde{F}_u(t, x(\rho(t))),
\end{aligned}
$$

since $x_m \to x$ in $C(\mathbb{T}, \mathbb{R}^n)$ and since $y \to \widetilde{F}(t, y)$ and $y \to \widetilde{G}(t, y)$ have closed graph and closed, convex values. By Theorem 1.3.5, there exists a function $w : \mathbb{T}_0 \to \mathbb{R}^n$ such that $g = \widetilde{w}$. So,

$$w(t) \in \widetilde{F}_u(t, x(\rho(t))) = F_u(t, x(\rho(t))) \quad \nabla\text{-a.e. } t \in \mathbb{T}_0.$$

Thus, $w \in \mathcal{H}(x)$.
Finally, since $\widetilde{w}_m \to \widetilde{w}$ in $L^1([a, b], \mathbb{R}^n)$ and $x_m \to x$ in $C(\mathbb{T}, \mathbb{R}^n)$, again by Theorem 1.3.5, we deduce that for every $t \in \mathbb{T}$,

$$\int_{(t,b]\cap\mathbb{T}} \hat{e}_{-1}(s, b)\Big(w_m(s) - \overline{x}_m(\rho\,(s))\Big)\nabla s \longrightarrow \int_{(t,b]\cap\mathbb{T}} \hat{e}_{-1}(s, b)\Big(w(s) - \overline{x}(\rho\,(s))\Big)\nabla s.$$

Moreover, since $u_m \to u$ in $C(\mathbb{T}, \mathbb{R}^n)$, we get that for every $t \in \mathbb{T}$,

$$u(t) = \hat{e}_{-1}(b, t)\Big(x_0 - \int_{(t,b]\cap\mathbb{T}} \hat{e}_{-1}(s, b)\Big(w(s) - \overline{x}(\rho\,(s))\Big)\nabla s\Big).$$

Thus, $u \in \mathcal{N}_I(x)$ and hence, $\mathcal{N}_I$ has closed graph. Since $\mathcal{N}_I$ is compact and has closed graph, N_I has compact values.
We now prove that $\mathcal{N}_I$ is upper semi-continuous.
Let $B \subset C(\mathbb{T}, \mathbb{R}^n)$ be a closed set and $\mathcal{A} = \{x \in C(\mathbb{T}, \mathbb{R}^n) : \mathcal{N}_I(x) \cap B \neq \emptyset\}$. Let $\{x_m\}$ be a sequence in $\mathcal{A}$ converging to x in $C(\mathbb{T}, \mathbb{R}^n)$. There exists $u_m \in \mathcal{N}_I(x_m) \cap B$. The compacity of $\mathcal{N}_I$ guarantees the existence of a subsequence still denoted $\{u_m\}$ converging to u in $C(\mathbb{T}, \mathbb{R}^n)$. Since B is closed and $\mathcal{N}_I$ has closed graph, we deduce that $u \in \mathcal{N}_I(x) \cap B$. Thus $x \in \mathcal{A}$. □

The following result can be proved as the previous one.

Proposition 4.2.4. *Assume (H_1) and (H_2). The operator $\mathcal{N}_p$ is compact, u.s.c., with nonempty, convex and compact values.*

Now, we can obtain our main theorem.

Theorem 4.2.1. *Assume (H_1) and (H_2). The problem (4.1) has a solution $x \in W^{1,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n) \cap T(v, M)$.*

Proof. By Proposition 4.2.3 (resp. Proposition 4.2.4), the operator $\mathcal{N}_I$ (resp. $\mathcal{N}_p$) is compact and upper semi-continuous with nonempty, convex, and compact values. It has a fixed point by the Kakutani fixed point theorem. If $(\mathfrak{BC})$ denotes (4.2) (resp. (4.3)), Lemma 4.2.1 (resp. Lemma 4.2.2), implies that this fixed point of $\mathcal{N}_I$ (resp. $\mathcal{N}_p$) is a solution of Problem (4.7), (4.2) (resp. problem (4.7), (4.3). Then, it suffices to show that for every solution x of (4.7), $x \in \mathbf{T}(v, M)$.

Consider the set $A = \{t \in \mathbb{T}_0 : \|x(\rho(t)) - v(\rho(t))\| > M(\rho(t))\}$. By Remark 1.3.1, ∇-a.e. on the set $\{t \in A : t = \rho(t)\}$, we have

$$(\|x(t) - v(t)\| - M(t))^\nabla = \frac{< x(\rho(t)) - v(\rho(t)), x^\nabla(t) - v^\nabla(t) >}{\|x(\rho(t)) - v(\rho(t))\|} - M^\nabla(t). \tag{4.11}$$

If $t \in A$ is left scattered, then $\nu(t) = t - \rho(t) > 0$ and

$$\begin{aligned}
&(\|x(t) - v(t)\| - M(t))^\nabla \\
&\quad = \frac{\|x(\rho(t)) - v(\rho(t))\|\|x(t) - v(t)\| - \|x(\rho(t)) - v(\rho(t))\|^2}{\nu(t)\|x(\rho(t)) - v(\rho(t))\|} - M^\nabla(t) \\
&\quad \geq \frac{\langle x(\rho(t)) - v(\rho(t)),\ (x(t) - v(t)) - (x(\rho(t)) - v(\rho(t)))\rangle}{\nu(t)\|x(\rho(t)) - v(\rho(t))\|} - M^\nabla(t) \\
&\quad = \frac{\langle x(\rho(t)) - v(\rho(t)),\ x^\nabla(t) - v^\nabla(t)\rangle}{\|x(\rho(t)) - v(\rho(t))\|} - M^\nabla(t).
\end{aligned} \tag{4.12}$$

Let us denote $y(t) = \Big(x^\nabla(t) - x(\rho(t)) + \overline{x}(\rho(t))\Big) \in F_u(t, x(\rho(t)))$ ∇-a.e. on $\mathbb{T}_0$. Since (v, M) is a solution-tube of (4.1) and from (4.8), (4.11), (4.12) and Remark 4.2.1, we deduce that ∇-a.e. on $\{t \in A : M(\rho(t)) > 0\}$,

$$\begin{aligned}
&(\|x(t) - v(t)\| - M(t))^\nabla \\
&\quad \geq \frac{\langle x(\rho(t)) - v(\rho(t)), y(t) - \Big(\overline{x}(\rho(t)) + x(\rho(t))\Big) - v^\nabla(t)\rangle}{\|x(\rho(t)) - v(\rho(t))\|} - M^\nabla(t) \\
&\quad = \frac{\langle \overline{x}(\rho(t)) - v(\rho(t)), y(t) - v^\nabla(t)\rangle}{M(\rho(t))} \\
&\qquad - \Big(M(\rho(t)) - \|x(\rho(t)) - v(\rho(t))\|\Big) - M^\nabla(t) \\
&\quad > \frac{M(\rho(t))M^\nabla(t)}{M(\rho(t))} - M^\nabla(t) = 0.
\end{aligned}$$

On the other hand, if $M(\rho(t)) = 0$, then $F_u(t, x(\rho(t))) = \{v^\nabla(t)\}$ and ∇-a.e. on $\{t \in A : M(\rho(t)) = 0\}$, we have

$$\begin{aligned}
&(\|x(t) - v(t)\| - M(t))^\nabla \\
&\quad \geq \frac{\langle x(\rho(t)) - v(\rho(t)), y(t) - \Big(\overline{x}(\rho(t)) + x(\rho(t))\Big) - v^\nabla(t)\rangle}{\|x(\rho(t)) - v(\rho(t))\|} - M^\nabla(t) \\
&\quad = \frac{\langle x(\rho(t)) - v(\rho(t)), v^\nabla(t) - v^\nabla(t)\rangle}{\|x(\rho(t)) - v(\rho(t))\|} + \|x(\rho(t)) - v(\rho(t))\| - M^\nabla(t) \\
&\quad > -M^\nabla(t) = 0.
\end{aligned}$$

This last equality follows from Definition 4.2.1 (3) and Proposition 1.3.5.

If we set $r(t) = \|x(t)-v(t)\|-M(t)$, then $r^{\nabla}(t) > 0$ ∇-a.e. on $A = \{t \in \mathbb{T}_0 : r(\rho(t)) > 0\}$. Moreover, since (v, M) is a solution tube of (4.1) and x satisfies (4.2) (resp. (4.3)), then $r(b) < 0$ (resp. $r(b) - r(a) \leq \|v(a) - v(b)\| - (M(b) - M(a)) \leq 0$), Lemma 4.2.3 implies that $A = \emptyset$. So, $x \in T(v, M)$ and the theorem is proved. $\square$

Chapter 5

Existence of solutions for conformable fractional differential equations and dynamic equations and for systems of conformable fractional differential equations

In this chapter, we present existence of solutions for the nonlinear conformable fractional differential equations, for the nonlinear conformable fractional dynamic equations on time scales with nonlinear functional boundary value conditions and for systems of nonlinear conformable fractional differential equations with periodic boundary value or initial value conditions.

5.1 Existence of solutions for conformable fractional problems with nonlinear functional boundary conditions

The results of this section are original and are submitted for publication [24].

5.1.1 Introduction

In this section, we study the existence of solutions for the nonlinear conformable fractional differential equations with nonlinear functional boundary conditions:

$$x^{(\alpha)}(t) = f(t, x(t)), \qquad \text{for a.e. } t \in I = [0, b], \quad b > 0, \tag{5.1}$$

where $0 < \alpha \leq 1$, $f : I \times \mathbb{R} \to \mathbb{R}$ is a L^1_α-Carathéodory function, and $x^{(\alpha)}(t)$ denotes the conformable fractional derivative of x at t of order α. We consider, depending on the circumstances, nonlinear functional boundary conditions of the type

$$L_1(x, x(b)) = 0 \quad \text{or} \quad L_2(x(0), x) = 0,$$

with L_i $(i = 1,2)$ a continuous function that satisfies suitable monotonicity assumptions. For this purpose, we use the method of upper and lower solutions together with Schauder's fixed point theorem. Existence results for the conformable fractional differential equations with linear and nonlinear functional boundary conditions are obtained with new comparison results and definitions of upper and lower solutions. For first order ordinary differential equations with nonlinear boundary conditions, we refer the reader to the papers [36, 60].

5.1.2 Green's Functions and Comparison Results

In this subsection, we study the expression of the solutions of a linear conformable fractional differential equation of order $\alpha \in (0,1]$ coupled to two-point linear conditions. This study is mainly devoted to obtain the expression of the fractional Green's function related to the considered problem. Once we have such expression, we derive comparison results for the considered problems.

To be concise, we look for $x \in W^{\alpha,1}_{0,b}(I)$, the solution of the following linear problem:

$$x^{(\alpha)}(t) + p(t)\,x(t) = g(t), \quad \text{a.e. } t \in I, \qquad a_0\,x(0) - b_0\,x(b) = \lambda_0, \tag{5.2}$$

with $p,\ g \in L^1_\alpha(I)$, and $a_0, b_0, \lambda_0 \in \mathbb{R}$.

Theorem 5.1.1. *If $a_0 \neq b_0\,e^{-\int_0^b p(r)\,d_\alpha r}$, then problem* (5.2) *has a unique solution $x \in W^{\alpha,1}_{0,b}(I)$, and it is given by the following expression:*

$$x(t) := \int_0^b G(t,s)g(s)d_\alpha s + \frac{\lambda_0\,e^{-\int_0^t p(r)\,d_\alpha r}}{a_0 - b_0\,e^{-\int_0^b p(r)\,d_\alpha r}}, \tag{5.3}$$

where

$$G(t,s) = \frac{e^{-\int_s^t p(r)\,d_\alpha r}}{a_0 - b_0\,e^{-\int_0^b p(r)\,d_\alpha r}} \begin{cases} a_0, & 0 \le s \le t \le b, \\ b_0\,e^{-\int_0^b p(r)\,d_\alpha r}, & 0 \le t < s \le b. \end{cases} \tag{5.4}$$

Proof. Let x be a solution of problem (5.2). Since $x \in W^{\alpha,1}_{0,b}(I)$, from Remark 1.4.3, we have that x is differentiable a.e. on I. Thus, Theorem 1.4.2 (iv), ensures that, it is a solution of the following singular differential equation:

$$t^{1-\alpha}\,x'(t) + p(t)\,x(t) = g(t), \quad \text{a.e. } t \in I, \qquad a_0\,x(0) - b_0\,x(b) = \lambda_0,$$

or, which is the same,

$$x'(t) + t^{\alpha-1}\,p(t)\,x(t) = t^{\alpha-1}\,g(t), \quad \text{a.e. } t \in I, \qquad a_0\,x(0) - b_0\,x(b) = \lambda_0. \tag{5.5}$$

Now, by using that $p,\ g \in L^1_\alpha(I)$, we have that, for a.e. $t \in I$,

$$\frac{d}{dt}\left(x(t)\,e^{\int_0^t p(r)\,d_\alpha r}\right) = e^{\int_0^t p(r)\,d_\alpha r}\left(x'(t) + t^{\alpha-1}\,p(t)\,x(t)\right) = e^{\int_0^t p(r)\,d_\alpha r} t^{\alpha-1}\,g(t).$$

Thus, by direct integration, we have that

$$x(t) = e^{-\int_0^t p(r)\, d_\alpha r} x(0) + \int_0^t e^{-\int_s^t p(r)\, d_\alpha r} g(s) d_\alpha s \quad \text{for all } t \in I. \tag{5.6}$$

If follows from (5.6) and the boundary condition in (5.2) that

$$x(0) = \frac{b_0}{a_0 - b_0\, e^{-\int_0^b p(r)\, d_\alpha r}} \int_0^b e^{-\int_s^b p(r)\, d_\alpha r} g(s) d_\alpha s + \frac{\lambda_0}{a_0 - b_0\, e^{-\int_0^b p(r)\, d_\alpha r}}. \tag{5.7}$$

Now, by substituting (5.7) into (5.6), we arrive to

$$\begin{aligned} x(t) = & \frac{b_0\, e^{-\int_0^t p(r)\, d_\alpha r}}{a_0 - b_0\, e^{-\int_0^b p(r)\, d_\alpha r}} \int_0^t e^{-\int_s^b p(r)\, d_\alpha r} g(s) d_\alpha s + \int_0^t e^{-\int_s^t p(r)\, d_\alpha r} g(s) d_\alpha s \\ & + \frac{b_0\, e^{-\int_0^t p(r)\, d_\alpha r}}{a_0 - b_0\, e^{-\int_0^b p(r)\, d_\alpha r}} \int_t^b e^{-\int_s^b p(r)\, d_\alpha r} g(s) d_\alpha s + \frac{\lambda_0\, e^{-\int_0^t p(r)\, d_\alpha r}}{a_0 - b_0\, e^{-\int_0^b p(r)\, d_\alpha r}} \\ = & \frac{e^{-\int_s^t p(r)\, d_\alpha r}}{a_0 - b_0\, e^{-\int_0^b p(r)\, d_\alpha r}} \left(a_0 \int_0^t g(s) d_\alpha s + b_0 \int_t^b e^{-\int_0^b p(r)\, d_\alpha r}\, g(s) d_\alpha s \right) \\ & + \frac{\lambda_0\, e^{-\int_0^t p(r)\, d_\alpha r}}{a_0 - b_0\, e^{-\int_0^b p(r)\, d_\alpha r}} = \int_0^b G(t,s) g(s) d_\alpha s + \frac{\lambda_0\, e^{-\int_0^t p(r)\, d_\alpha r}}{a_0 - b_0\, e^{-\int_0^b p(r)\, d_\alpha r}}. \end{aligned}$$

□

As a direct consequence, we deduce the following result:

Lemma 5.1.1. *The fractional Green's function G, related to the linear problem* (5.2), *and given by the expression* (5.4), *satisfies the following properties for every $p \in L^1_\alpha(I)$:*

(i) *$G > 0$ on $I \times I$ if and only if*

$$\frac{a_0}{a_0 - b_0\, e^{-\int_0^b p(r)\, d_\alpha r}} > 0 \quad \text{and} \quad \frac{b_0}{a_0 - b_0\, e^{-\int_0^b p(r)\, d_\alpha r}} > 0. \tag{5.8}$$

(ii) *$G < 0$ on $I \times I$ if and only if*

$$\frac{a_0}{a_0 - b_0\, e^{-\int_0^b p(r)\, d_\alpha r}} < 0 \quad \text{and} \quad \frac{b_0}{a_0 - b_0\, e^{-\int_0^b p(r)\, d_\alpha r}} < 0. \tag{5.9}$$

As a direct consequence of previous result, we deduce the following expressions for the particular cases of the initial, terminal and periodic problems.

Corollary 5.1.1. *The initial problem*

$$\begin{cases} x^{(\alpha)}(t) + p(t)\, x(t) = g(t), & for\ a.e.\ t \in I, \\ x(0) = x_0, \end{cases} \tag{5.10}$$

with $p,\ g \in L^1_\alpha(I)$, has a unique solution $x \in W^{\alpha,1}_{0,b}(I)$, and it is given by the following expression

$$x(t) := \int_0^b G_I(t,s)g(s)d_\alpha s + x_0\, e^{-\int_0^t p(r)\, d_\alpha r}, \tag{5.11}$$

where

$$G_I(t,s) = e^{-\int_s^t p(r)\, d_\alpha r} \begin{cases} 1, & 0 \le s \le t \le b, \\ 0, & 0 \le t < s \le b. \end{cases} \tag{5.12}$$

Corollary 5.1.2. *The terminal problem*

$$\begin{cases} x^{(\alpha)}(t) + p(t)\, x(t) = g(t), & \text{for a.e. } t \in I, \\ x(b) = x_0, \end{cases} \tag{5.13}$$

with $p,\ g \in L^1_\alpha(I)$, has a unique solution $x \in W^{\alpha,1}_{0,b}(I)$, and it is given by the following expression

$$x(t) := \int_0^b G_T(t,s)g(s)d_\alpha s + x_0\, e^{-\int_t^b p(r)\, d_\alpha r}, \tag{5.14}$$

where

$$G_T(t,s) = -e^{-\int_s^t p(r)\, d_\alpha r} \begin{cases} 0, & 0 \le s \le t \le b, \\ 1, & 0 \le t < s \le b. \end{cases} \tag{5.15}$$

From expressions (5.12) and (5.15), it is obvious that $G_I \ge 0$ and $G_T \le 0$ on $I \times I$. Thus, as a direct consequence of expressions (5.11) and (5.14), we deduce the following comparison result:

Lemma 5.1.2. *Let $x \in W^{\alpha,1}_{0,b}(I)$, then the following comparison principles hold for every $p \in L^1_\alpha(I)$:*

(i) If $x^{(\alpha)}(t) + p(t)\, x(t) \ge 0$ a.e. $t \in I$ and $x(0) \ge 0$ then $x \ge 0$ on I.

(ii) If $x^{(\alpha)}(t) + p(t)\, x(t) \ge 0$ a.e. $t \in I$ and $x(b) \le 0$ then $x \le 0$ on I.

Concerning the non homogeneous periodic problem, which follows directly by the choice of $a_0 = b_0 = 1$, as a corollary of Theorem 5.1.1, we deduce the following result.

Corollary 5.1.3. *The non homogeneous periodic problem*

$$\begin{cases} x^{(\alpha)}(t) + p(t)\, x(t) = g(t), & \text{for a.e. } t \in I, \\ x(0) - x(b) = \lambda_0, \end{cases} \tag{5.16}$$

with $p,\ g \in L^1_\alpha(I)$, has a unique solution $x \in W^{\alpha,1}_{0,b}(I)$, and it is given by the following expression

$$x(t) := \int_0^b G_P(t,s) g(s) d_\alpha s + \lambda_0 \frac{e^{-\int_0^t p(r)\, d_\alpha r}}{1 - e^{-\int_0^b p(r)\, d_\alpha r}}, \tag{5.17}$$

where

$$G_P(t,s) = \frac{e^{-\int_s^t p(r)\, d_\alpha r}}{1 - e^{-\int_0^b p(r)\, d_\alpha r}} \begin{cases} 1, & 0 \le s \le t \le b, \\ e^{-\int_0^b p(r)\, d_\alpha r}, & 0 \le t < s \le b. \end{cases} \tag{5.18}$$

As a consequence, it is immediate to verify, from expression (5.18), that the periodic problem has a unique solution if and only if

$$\int_0^b p(r)\, d_\alpha r \neq 0.$$

Moreover the fractional Green's function G_P has the same sign of the previous integral, i.e.,

Corollary 5.1.4. *Let $p \in L^1_\alpha(I)$, then the following properties hold:*

(i) $G_P > 0$ on $I \times I$ if and only if $\int_0^b p(r)\, d_\alpha r > 0$.

(ii) $G_P < 0$ on $I \times I$ if and only if $\int_0^b p(r)\, d_\alpha r < 0$.

As a direct consequence of previous result and equality (5.17), denoting $y \succ 0$ on I as $y \geq 0$ and $y \not\equiv 0$ on I, we deduce the following comparison result.

Corollary 5.1.5. *Let $x \in W^{\alpha,1}_{0,b}(I)$ be such that*

$$x^{(\alpha)}(t) + p(t)\, x(t) \succ 0 \text{ on } I; \text{ and } x(0) \geq x(b).$$

Then the following comparison principles are fulfilled:

(i) If $\int_0^b p(r)\, d_\alpha r > 0$ then $x > 0$ on I.

(ii) If $\int_0^b p(r)\, d_\alpha r < 0$ then $x < 0$ on I.

5.1.3 Nonlinear Functional Boundary Conditions

In this subsection, we prove the existence of solutions of the nonlinear conformable fractional differential equation (5.1) coupled to nonlinear functional boundary conditions. In particular, we will consider the two following kind of functional boundary conditions:

$$L_1(x, x(b)) = 0 \tag{5.19}$$

and

$$L_2(x(0), x) = 0. \tag{5.20}$$

Here $L_1 : C(I) \times \mathbb{R} \to \mathbb{R}$ and $L_2 : \mathbb{R} \times C(I) \to \mathbb{R}$ are continuous functions that satisfy suitable monotonicity assumptions.

The used tool will be the well known method of upper and lower solutions. A solution of these problems will be a function $x \in W^{\alpha,1}_{0,b}(I)$ that satisfies equation (5.1) a.e. on I coupled to the corresponding boundary conditions (either (5.19) or (5.20) in each case).

First, we consider the problem (5.1), (5.19). To this end, we introduce the following definition of lower and upper solution related to such problem.

Definition 5.1.1. *Let $\gamma \in W^{\alpha,1}_{0,b}(I)$. We say that γ is a lower solution of the boundary value problem* (5.1), (5.19) *if*

(i) $\gamma^{(\alpha)}(t) \geq f(t, \gamma(t)), \quad a.e.\ t \in I;$

(ii) $L_1(\gamma, \gamma(b)) \geq 0.$

Let $\delta \in W^{\alpha,1}_{0,b}(I)$. We say that δ is an upper solution of the boundary value problem (5.1), (5.19) *if*

(i) $\delta^{(\alpha)}(t) \leq f(t, \delta(t)), \quad a.e.\ t \in I;$

(ii) $L_1(\delta, \delta(b)) \leq 0.$

In order to obtain existence and location results for the considered nonlinear problems, we define the sector

$$[\gamma, \delta] = \{x \in C(I) : \gamma(t) \leq x(t) \leq \delta(t), \text{ for all } t \in I\}.$$

Now we give the main result on the existence of solutions for the nonlinear problem (5.1), (5.19). The proof is on the basis on the one given in [36, Theorem 3.1] for two-point nonlinear boundary conditions.

Theorem 5.1.2. *If there exist γ and δ in $W^{\alpha,1}_{0,b}(I)$, $\gamma \leq \delta$ in I, a pair of well ordered lower and upper solutions respectively for problem* (5.1), (5.19), *with L_1 a continuous function in $[\gamma, \delta] \times [\gamma(b), \delta(b)]$ and nondecreasing in the first variable on $[\gamma, \delta]$, then problem* (5.1), (5.19) *has at least one solution $x \in [\gamma, \delta]$.*

Proof.

We consider the following modified problem:

$$\begin{cases} x^{(\alpha)}(t) = f(t, \tau(t, x(t))), \quad for\ a.e.\ t \in I, \\ x(b) = \tau(b, x(b) + L_1(\tau(\cdot, x(\cdot)), \tau(b, x(b)))), \end{cases} \tag{5.21}$$

where τ is the truncated function, defined for any $x \in C(I)$, as follows:

$$\tau(t, x(t)) = \max\Big\{\gamma(t),\ \min\{x(t), \delta(t)\}\Big\}, \quad \text{for all } t \in I.$$

By the definition of function τ, it is obvious that $\gamma(b) \leq x(b) \leq \delta(b)$.

Suppose now that $x(0) < \gamma(0)$. From the continuity of both functions we know that there exists $t_0 \in (0, b]$ such that $\gamma(t_0) = x(t_0)$ with $\gamma > x$ on $[0, t_0)$. In this case, due to the linearity of the conformable α-derivative and the definition of the truncated function τ, we have that

$$(\gamma - x)^{(\alpha)}(t) \geq 0, \quad a.e.\ t \in [0, t_0], \quad (\gamma - x)(t_0) = 0.$$

So, Lemma 5.1.2 (ii) implies that $x \geq \gamma$ on $[0,\ t_0]$, and we arrive to a contradiction.

Analogously, we can prove that $x(0) \leq \delta(0)$.

If there exists $c \in (0, b)$ with $x(c) < \gamma(c)$, then there exists a subinterval $(t_1, t_2) \subset (0, b)$, such that $(\gamma - x)(t_1) = (\gamma - x)(t_2) = 0$, with $\gamma > x$ on (t_1, t_2).

But, arguing as before, we deduce that

$$(\gamma - x)^{(\alpha)}(t) \geq 0, \text{ a.e. } t \in [t_1, t_2].$$

Now, using Lemma 5.1.2, (ii) again, we deduce that $\gamma \leq x$ on $[t_1, t_2]$ and we attain a contradiction.

A similar argument is valid to show that $x \leq \delta$ on I.

Therefore, every solution x of problem (5.21) belongs to the sector $[\gamma, \delta]$. Let's see now that it satisfies the functional boundary condition (5.19).

Clearly, if $x(b) + L_1(\tau(\cdot, x(\cdot)), \tau(b, x(b))) < \gamma(b)$, we obtain that $x(b) = \gamma(b)$ and, in consequence, $\gamma(b) > x(b) + L_1(\tau(\cdot, x(\cdot)), \gamma(b))$.

The nondecreasing character of L_1 with respect to the first variable on the sector $[\gamma, \delta]$, and the definition of function τ, allow us to arrive at the following contradiction

$$\gamma(b) > x(b) + L_1(\gamma, \gamma(b)) \geq x(b) = \gamma(b).$$

Analogously, we can verify that $x(b) + L_1(\tau(\cdot, x(\cdot)), \tau(b, x(b))) \leq \delta(b)$, and, as consequence, every solution x of the truncated problem (5.21) is a solution of (5.1), (5.19).

Now, to finalize the proof, we must ensure that the truncated problem (5.21) has a solution.

To this end, let us define the operator $\mathcal{F} : C(I) \to C(I)$ as follows:

$$(\mathcal{F}\, x)(t) = -\int_t^b \Big(f(s, \tau(s, x(s)))\Big) d_\alpha s + \tau(b, x(b) + L_1(\tau(\cdot, x(\cdot)), \tau(b, x(b)))).$$

First, notice that the solutions of problem (5.21) coincide with the fixed points of the operator $\mathcal{F}$. This property holds from equation (5.14) and the expression of the fractional Green's function G_T, related to the terminal problem (5.13), with $p \equiv 0$, which is given in (5.15).

In order to ensure that operator $\mathcal{F}$ has a fixed point, we will prove that it is compact. We first observe that, from Definition 1.3.17 of a L^1_α-Carathéodory function and the definition of τ, function $f(\cdot, \tau(\cdot, x(\cdot)))$ is Lebesgue measurable on I for any continuous function x [11, Theorem 1.1], and there exists $h \in L^1_\alpha(I, [0, \infty))$ such that

$$|f(t, \tau(t, x(t)))| \leq h(t), \quad \text{for } a.e.\ t \in I \text{ and all } x \in C(I).$$

The continuity of operator $\mathcal{F}$ follows from the continuous dependence with respect to x of function f, the definition of τ and the Lebesgue's dominated convergence theorem.

To see that $\mathcal{F}(C(I))$ is a relatively compact set on $C(I)$, consider $x \in C(I)$. Therefore

$$|\mathcal{F}(x)(t)| \leq \|h\|_{L^1_\alpha(I)} + \max\{|\gamma(b)|, |\delta(b)|\}, \quad \text{for all } t \in I,$$

and, as a consequence, $\mathcal{F}(C(I))$ is uniformly bounded on $C(I)$.

This set is also equicontinuous since for every $t_1 < t_2 \in I$,

$$|\mathcal{F}(x)(t_2) - \mathcal{F}(x)(t_1)| \leq \int_{t_1}^{t_2} |h(s)| d_\alpha s.$$

By Arzelà-Ascoli theorem, we conclude that the set $\mathcal{F}(C(I))$ is relatively compact in $C(I)$. Hence, $\mathcal{F}$ is compact.

As a consequence, the Schauder fixed-point theorem ensures that operator $\mathcal{F}$ has a fixed point.

From previous arguments, we conclude that such fixed point is a solution of problem (5.1), (5.19), and lies on $[\gamma, \delta]$. $\square$

Concerning the problem (5.1), (5.20), we introduce the following definition of lower and upper solution related to such problem.

Definition 5.1.2. *Let $\gamma \in W^{\alpha,1}_{0,b}(I)$. We say that γ is a lower solution of the boundary value problem* (5.1), (5.20) *if*

(i) $\gamma^{(\alpha)}(t) \geq f(t, \gamma(t))$, *a.e.* $t \in I$;

(ii) $L_2(\gamma(0), \gamma) \geq 0$.

Let $\delta \in W^{\alpha,1}_{0,b}(I)$. We say that δ is an upper solution of the boundary value problem (5.1), (5.20) *if*

(i) $\delta^{(\alpha)}(t) \leq f(t, \delta(t))$, *a.e.* $t \in I$;

(ii) $L_2(\delta(0), \delta) \leq 0$.

Analogously to Theorem 5.1.2, one can prove the following result.

Theorem 5.1.3. *If there exist γ and δ in $W^{\alpha,1}_{0,b}(I)$, a pair of reversed ordered lower and upper solutions respectively for problem* (5.1), (5.20)*, such that $\gamma \geq \delta$ on I, and L_2 is a continuous function in $[\delta(0), \gamma(0)] \times [\delta, \gamma]$, nonincreasing in the second variable on $[\delta, \gamma]$, then problem* (5.1), (5.20)*, has at least one solution $x \in [\delta, \gamma]$.*

Proof. The proof follows the same steps as Theorem 5.1.2. In this case, we consider the following modified problem

$$\begin{cases} x^{(\alpha)}(t) = f(t, \tau(t, x(t))), & \textit{for a.e. } t \in I, \\ x(0) = \tau(0, x(0) - L_2(\tau(0, x(0)), \tau(\cdot, x(\cdot)))), \end{cases} \tag{5.22}$$

where, for any $x \in C(I)$, the function τ is defined as

$$\tau(t, x(t)) = \max\Big\{\delta(t),\ \min\{x(t), \gamma(t)\}\Big\}.$$

$\square$

In the particular case in which the boundary conditions are defined only at the extremes of the interval, we can deduce as a direct corollary, the following result.

Corollary 5.1.6. *Assume that there exist γ and $\delta \in W_{0,b}^{\alpha,1}(I)$, a pair of lower and upper solutions (either $\gamma \geq \delta$ or $\gamma \leq \delta$) for problem*

$$x^{(\alpha)}(t) = f(t, x(t)), \qquad \text{for a.e. } t \in I, \quad L(x(0), x(b)) = 0,$$

with L a continuous function nondecreasing in the first variable and nonincreasing in the second one on its domain of definition. Then this problem has at least one solution $x \in W_{0,b}^{\alpha,1}(I)$ lying between γ and δ.

We note that previous result can be automatically applied to the linear boundary conditions $L(x, y) = a_0 x - b_0 y - \lambda_0$, with $a_0,\ b_0$ *and* $\lambda_0 \in \mathbb{R}$, $a_0, b_0 \geq 0$ and $a_0 + b_0 > 0$, which includes the periodic case ($a_0 = b_0 = 1, \lambda_0 = 0$) and the initial ($a_0 = 1, b_0 = 0$) and terminal ($a_0 = 0, b_0 = 1$) problems.

5.1.4 Examples

In this subsection, we present three examples where we apply Theorems 5.1.2 and 5.1.3 to some particular cases.

Example 5.1.1. *Consider the linear boundary value problem:*

$$x^{(\frac{1}{3})}(t) = \frac{x^2(t)}{2} - t(1-t), \ \text{ a.e. } t \in [0,1], \quad x(1) = \sqrt{|x(1/2)|}. \tag{5.23}$$

This problem is a particular case of (5.1), (5.19), *with $\alpha = \frac{1}{3}$, $f(t,x) = x^2/2 - t\,(1-t)$ and*

$$L_1(x, y) = \sqrt{|x(1/2)|} - y.$$

Obviously, function f is a $L^1_{1/3}$-Carathéodory function, and $\delta(t) = 2$, $\gamma(t) = 0$ are upper and lower solutions of the boundary-value problem (5.23), *respectively with $\gamma(t) \leq \delta(t)$ for $t \in [0,1]$. To see this, it is enough to verify the following inequalities*

$$\delta^{(\frac{1}{3})}(t) = 0 \leq f(t, \delta(t)) = 2 - t(1-t), \sqrt{|\delta(1/2)|} - \delta(1) \leq 0,$$

and

$$\gamma^{(\frac{1}{3})}(t) = 0 \geq f(t, \gamma(t)) = -t(1-t), \sqrt{|\gamma(1/2)|} - \gamma(1) = 0.$$

By Theorem 5.1.2, problem (5.23) *has a least one solution $x \in W_{0,1}^{\frac{1}{3},1}([0,1])$, such that $0 \leq x(t) \leq 2$, for all $t \in [0,1]$.*

Example 5.1.2. *Consider the nonlinear boundary value problem with functional boundary conditions:*

$$\begin{cases} x^{(\frac{1}{2})}(t) = t\, e^{t\, \sin^2(x(t))} & \text{a.e. } t \in [0,2], \\ x(0) - \sin^2(x(0)) = \frac{1}{3} \max_{t \in [0,1]} \{x(t)\}. \end{cases} \tag{5.24}$$

This problem is a particular case of (5.1), (5.20), *with* $\alpha = \frac{1}{2}$, $f(t,x) = t\, e^{t \sin^2(x)}$ *and*

$$L_2(x,y) = x - \sin^2(x) - \frac{1}{3} \max_{t\in[0,1]} \{y(t)\}.$$

It is clear that f *is a* $L^1_{1/2}$*-Carathéodory function,* L_2 *is a continuous function in* $(x,y) \in [\delta(0), \gamma(0)] \times [\delta, \gamma]$, *and nonincreasing in* $y \in [\delta, \gamma]$, *with* $\delta(t) = 0 \leq \gamma(t) = e^{t+1}$ *for* $t \in [0,2]$.

The fact that δ *and* γ *are upper and lower solutions of problem* (5.24) *follows from the fact that*

$$\delta^{(\frac{1}{2})}(t) = 0 \leq f(t, \delta(t)) = t, \text{ a.e. } t \in [0,2], \quad \delta(0) - \sin^2(\delta(0)) - \frac{1}{3} \max_{t\in[0,1]} \{\delta(t)\} = 0$$

and

$$\gamma^{(\frac{1}{2})}(t) = \sqrt{t} e^{t+1} \geq f(t, \gamma(t)) = t\, e^{t \sin^2(e^{t+1})}, \text{ a.e. } t \in [0,2],$$
$$\gamma(0) - \sin^2(\gamma(0)) - \frac{1}{3} \max_{t\in[0,1]} \{\gamma(t)\} \geq 0.$$

Theorem 5.1.3, implies that problem (5.24) *has a least one solution* $x \in W^{\frac{1}{2},1}_{0,1}([0,2])$, *such that* $0 \leq x(t) \leq e^{t+1}$, *for all* $t \in [0,2]$.

Example 5.1.3. *Consider the nonlinear boundary value problem with functional boundary conditions:*

$$\begin{cases} x'(t) = \dfrac{x^3(t) + 1 - 2t}{\sqrt{t}} \quad \text{a.e. } t \in [0,1], \\ x(1) - \cos(\frac{\pi}{2}\ x(1)) = \int_{\frac{1}{2}}^1 x(s) ds. \end{cases} \tag{5.25}$$

This problem is a particular case of (5.1), (5.19), *with* $\alpha = 1$, $f(t,x) = \dfrac{x^3 + 1 - 2t}{\sqrt{t}}$ *and*

$$L_1(x,y) = \int_{\frac{1}{2}}^1 x(s) ds - y + \cos(\frac{\pi}{2}\ y).$$

It is clear that f *is a* L^1*-Carathéodory function,* L_1 *is a continuous function in* $(x,y) \in [\gamma, \delta] \times [\gamma(1), \delta(1)]$, *and nondecreasing in* $x \in [\gamma, \delta]$, *with* $\gamma(t) = -1 \leq \delta(t) = 1$ *for* $t \in [0,1]$.

The fact that γ *and* δ *are lower and upper solutions of problem* (5.25) *follows from the fact that*

$$\gamma'(t) = 0 \geq f(t, \gamma(t)) = -2\sqrt{t}, \text{ a.e. } t \in [0,1], \int_{\frac{1}{2}}^1 \gamma(s) ds - \gamma(1) + \cos(\frac{\pi}{2}\ \gamma(1)) \geq 0$$

and

$$\delta'(t) = 0 \leq f(t, \delta(t)) = \frac{2(1-t)}{\sqrt{t}}, \text{ a.e. } t \in [0,1],$$

$$\int_{\frac{1}{2}}^1 \delta(s) ds - \delta(1) + \cos(\frac{\pi}{2}\ \delta(1)) \leq 0.$$

Theorem 5.1.2, implies that problem (5.25) *has a least one solution* $x \in W^{1,1}_{0,1}([0,1])$, *such that* $-1 \leq x(t) \leq 1$, *for all* $t \in [0,1]$.

5.2 Nonlinear functional boundary value problems for conformable fractional dynamic equations on time scales

The original results of this section are published in [29].

5.2.1 Introduction

This section is devoted to the study of the existence of solutions of the following conformable fractional dynamic equation on time scales with nonlinear functional boundary value conditions:

$$x_{\Delta}^{(\alpha)}(t) = f(t, x^{\sigma}(t)), \qquad \text{for } \Delta\text{-a.e. } t \in I = [a, b]_{\mathbb{T}}, \tag{5.26}$$

coupled to nonlinear functional boundary conditions:

$$B(x(a), x) = 0, \tag{5.27}$$

or

$$H(x, x(\sigma(b))) = 0. \tag{5.28}$$

Here $\mathbb{T}$ is an arbitrary bounded time scale, $J = [a, \sigma(b)]_{\mathbb{T}}$ with $a, b \in \mathbb{T}$, $0 \leq a < b$ and $f : I \times \mathbb{R} \to \mathbb{R}$ is a $L^1_{\alpha,\Delta}$-Carathéodory function, $x_{\Delta}^{(\alpha)}(t)$ denotes the delta conformable fractional derivative of x at t of order $\alpha \in (0, 1]$, $B : \mathbb{R} \times C(J) \to \mathbb{R}$ and $H : C(J) \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

If $B(x, y) = a_0\, x - b_0\, y(\sigma(b)) - \lambda_0$, with $a_0; b_0; \lambda_0 \in \mathbb{R}$, then $(5.26), (5.27)$ is the boundary value problem,

$$x_{\Delta}^{(\alpha)}(t) = f(t, x^{\sigma}(t)) \quad \text{for } \Delta\text{-a.e.} t \in I, \quad a_0\, x(a) - b_0\, x(\sigma(b)) = \lambda_0, \tag{5.29}$$

if $B(x, y) = x - \lambda_0$, then $(5.26), (5.27)$ is the initial value problem,

$$x_{\Delta}^{(\alpha)}(t) = f(t, x^{\sigma}(t)) \quad \text{for } \Delta\text{-a.e.} t \in I, \quad x(a) = \lambda_0, \tag{5.30}$$

and, if $B(x, y) = x - y(\sigma(b))$, then $(5.26), (5.27)$ is the periodic value problem,

$$x_{\Delta}^{(\alpha)}(t) = f(t, x^{\sigma}(t)) \quad \text{for } \Delta\text{-a.e.} t \in I, \quad x(a) = x(\sigma(b)). \tag{5.31}$$

Finally, the anti-periodic value problem,

$$x_{\Delta}^{(\alpha)}(t) = f(t, x^{\sigma}(t)) \quad \text{for } \Delta\text{-a.e. } t \in I, \quad x(a) = -x(\sigma(b)), \tag{5.32}$$

corresponds to the particular case $B(x, y) = x + y(\sigma(b))$.

Existence of solutions was obtained in Section 5.1 for the conformable fractional differential equation (5.26) with $\mathbb{T} = \mathbb{R}$:

$$x^{(\alpha)}(t) = f(t, x(t)), \qquad \text{for a.e. } t \in [0, b], \quad 0 < \alpha \leq 1,$$

coupled to the nonlinear functional boundary conditions $B(x(0),x) = L_2(x(0),x) = 0$ or $H(x,x(b)) = L_1(x,x(b)) = 0$.

Cabada in [36], used the monotone method for nonlinear boundary problem (5.26), (5.27) with $\mathbb{T} = \mathbb{R}$ and $\alpha = 1$:

$$x'(t) = f(t,x(t)), \qquad \text{for a.e. } t \in [a,b], \quad B(x(a),x(b)) = 0,$$

where f is a Carathéodory function, $x \in W^{1,1}([a,b],\mathbb{R})$ and $B : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies some properties of monotony.

In [41], Cabada et al. present an existence theorem for the problem (5.26), (5.27) with $\alpha = 1$:

$$x^{\Delta}(t) = f(t,x^{\sigma}(t)), \qquad \text{for } \Delta\text{-a.e. } t \in [a,b]_{\mathbb{T}},\ a,b \in \mathbb{R}, \qquad B(x(a),x) = 0,$$

where $f : [a,b]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is a L^1_{Δ}-Carathéodory function and $B : \mathbb{R} \times C(\mathbb{T}) \to \mathbb{R}$ is a continuous function.

Motivated by the previously mentioned papers, in this section, we establish the existence of solutions for the conformable fractional dynamic equations (5.26) on time scales with nonlinear functional boundary value conditions. For this purpose, we use the upper and lower solutions method together with Schauder's fixed-point theorem.

5.2.2 Linear Conformable Fractional Dynamic Problems

In this subsection, we study the expression of the solutions of a linear conformable fractional dynamic equation of order $\alpha \in (0,1]$, with linear boundary conditions:

$$\begin{cases} x^{(\alpha)}_{\Delta}(t) - t^{1-\alpha}p(t)\, x(\sigma(t)) = g(t), & \Delta\text{-a.e. } t \in I, \\ a_0 x(a) - b_0 x(\sigma(b)) = \lambda_0, \end{cases} \tag{5.33}$$

with $-p \in \mathcal{R}_{\mu}$, $g \in L^1_{\alpha,\Delta}(I,\mathbb{R})$ and $a_0, b_0, \lambda_0 \in \mathbb{R}$.
We obtain the expression of the fractional Green's function for this linear problem.

Theorem 5.2.1. *Let $-p \in \mathcal{R}_{\mu}$ and $a_0 e_{-p}(\sigma(b),a) \neq b_0$. For every $g \in L^1_{\alpha,\Delta}(I,\mathbb{R})$, the problem (5.33) has a unique solution $x \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J,\mathbb{R})$ given by:*

$$x(t) = \int_{[a,\sigma(b)]_{\mathbb{T}}} G(t,s)g(s)\Delta^{\alpha}s + \frac{\lambda_0 e_{-p}(\sigma(b),t)}{a_0 e_{-p}(\sigma(b),a) - b_0}, \quad t \in J, \tag{5.34}$$

where the fractional Green's function is

$$G(t,s) = \frac{e_{-p}(s,t)}{a_0 e_{-p}(\sigma(b),a) - b_0} \begin{cases} a_0 e_{-p}(\sigma(b),a), & a \le s \le t \le \sigma(b), \\ b_0, & a \le t \le s \le \sigma(b), \end{cases} \tag{5.35}$$

Proof. Let x be a solution to (5.33). By Theorem 1.5.3, consider

$$\Big[x(t)e_{-p}(t,a)\Big]_{\Delta}^{(\alpha)} = x_{\Delta}^{(\alpha)}(t)e_{-p}(t,a) - p(t)t^{1-\alpha}e_{-p}(t,a)x(\sigma(t)),$$
$$= e_{-p}(t,a)g(t).$$

and hence integrating the above on $[a,t]_{\mathbb{T}}$ obtain

$$x(t)e_{-p}(t,a) - x(a) = \int_{[a,t]_{\mathbb{T}}} e_{-p}(s,a)g(s)\Delta^{\alpha}s. \tag{5.36}$$

So,

$$x(t) = e_{-p}(a,t)\left(x(a) + \int_{[a,t]_{\mathbb{T}}} e_{-p}(s,a)g(s)\Delta^{\alpha}s\right) \tag{5.37}$$

If follows from the boundary condition in (5.33) and (5.37) that

$$x(a) = \frac{b_0}{a_0e_{-p}(\sigma(b),a) - b_0}\int_{[a,\sigma(b)]_{\mathbb{T}}} e_{-p}(s,a)g(s)\Delta^{\alpha}s + \frac{\lambda_0 e_{-p}(\sigma(b),a)}{a_0e_{-p}(\sigma(b),a) - b_0}. \tag{5.38}$$

Now, by substituting (5.38) into (5.37), we get

$$\begin{aligned}
x(t) &= \frac{b_0e_{-p}(a,t)}{a_0e_{-p}(\sigma(b),a) - b_0}\int_{[a,t]_{\mathbb{T}}} e_{-p}(s,a)g(s)\Delta^{\alpha}s + e_{-p}(a,t)\int_{[a,t]_{\mathbb{T}}} e_{-p}(s,a)g(s)\Delta^{\alpha}s \\
&\quad + \frac{b_0e_{-p}(a,t)}{a_0e_{-p}(\sigma(b),a) - b_0}\int_{[t,\sigma(b)]_{\mathbb{T}}} e_{-p}(s,a)g(s)\Delta^{\alpha}s + \frac{\lambda_0e_{-p}(\sigma(b),t)}{a_0e_{-p}(\sigma(b),a) - b_0} \\
&= \frac{\lambda_0e_{-p}(\sigma(b),t)}{a_0e_{-p}(\sigma(b),a) - b_0} + \frac{1}{a_0e_{-p}(\sigma(b),a) - b_0}\Big(a_0\int_{[a,t]_{\mathbb{T}}} e_{-p}(\sigma(b),a)e_{-p}(s,t)g(s)\Delta^{\alpha}s \\
&\qquad + b_0\int_{[t,\sigma(b)]_{\mathbb{T}}} e_{-p}(s,t)g(s)\Delta^{\alpha}s\Big) \\
&= \int_{[a,\sigma(b)]_{\mathbb{T}}} G(t,s)g(s)\Delta^{\alpha}s + \frac{\lambda_0e_{-p}(\sigma(b),t)}{a_0e_{-p}(\sigma(b),a) - b_0}.
\end{aligned}$$

□

As a direct consequence of previous result, we deduce the following expressions for the particular cases of the initial, terminal and periodic problems.

Corollary 5.2.1. *The initial problem*

$$\begin{cases} x_{\Delta}^{(\alpha)}(t) - t^{1-\alpha}p(t)\ x(\sigma(t)) = g(t), & \Delta\text{-a.e. } t \in I; \\ x(a) = x_0. \end{cases} \tag{5.39}$$

with $-p \in \mathcal{R}_{\mu}$*,* $x_0 \in \mathbb{R}$*, and* $g \in L^1_{\alpha,\Delta}(I,\mathbb{R})$*, has a unique solution* $x \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J,\mathbb{R})$*, given by the following expression*

$$x(t) := \int_{[a,\sigma(b)]_{\mathbb{T}}} G_I(t,s)g(s)\Delta^{\alpha}s + x_0e_{-p}(a,t), \quad t \in J, \tag{5.40}$$

where

$$G_I(t,s) = e_{-p}(s,t)\begin{cases} 1, & a \le s \le t \le \sigma(b), \\ 0, & a \le t \le s \le \sigma(b). \end{cases} \tag{5.41}$$

Corollary 5.2.2. *The terminal problem*

$$\begin{cases} x_\Delta^{(\alpha)}(t) - t^{1-\alpha}p\ x(\sigma(t)) = g(t), & \Delta\text{-}a.e.\ t \in I; \\ x(\sigma(b)) = x_1. \end{cases} \tag{5.42}$$

with $-p \in \mathcal{R}_\mu,\ x_1 \in \mathbb{R}$, *and* $g \in L^1_{\alpha,\Delta}(I,\mathbb{R})$, *has a unique solution* $x \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J,\mathbb{R})$, *given by the following expression*

$$x(t) := \int_{[a,\sigma(b)]_{\mathbb{T}}} G_T(t,s)g(s)\Delta^\alpha s + x_1 e_{-p}(\sigma(b),t), \quad t \in J, \tag{5.43}$$

where

$$G_T(t,s) = -e_{-p}(s,t)\begin{cases} 0, & a \le s \le t \le \sigma(b), \\ 1, & a \le t \le s \le \sigma(b), \end{cases} \tag{5.44}$$

Corollary 5.2.3. *The periodic problem*

$$\begin{cases} x_\Delta^{(\alpha)}(t) - t^{1-\alpha}p(t)\ x(\sigma(t)) = g(t), & \Delta\text{-}a.e.\ t \in \mathbb{T}_0; \\ x(a) = x(\sigma(b)). \end{cases} \tag{5.45}$$

with $-p \in \mathcal{R}_\mu$, $e_{-p}(\sigma(b),a) \neq 1$ *and* $g \in L^1_{\alpha,\Delta}(I,\mathbb{R})$, *has a unique solution* $x \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J,\mathbb{R})$, *given by the following expression*

$$x(t) := \int_{[a,\sigma(b)]_{\mathbb{T}}} G_P(t,s)g(s)\Delta^\alpha s, \quad t \in J, \tag{5.46}$$

where

$$G_P(t,s) = \frac{e_{-p}(s,t)}{e_{-p}(\sigma(b),a) - 1}\begin{cases} e_{-p}(\sigma(b),a), & a \le s \le t \le \sigma(b), \\ 1, & a \le t \le s \le \sigma(b), \end{cases} \tag{5.47}$$

If $-p \in \mathcal{R}^+_\mu$, from expressions (5.41) and (5.44), it is obvious that $G_I \ge 0$ and $G_T \le 0$ on $J \times J$.

5.2.3 Conformable Problems with Nonlinear Functional Boundary Conditions.

In this subsection, we will prove the existence of at least one solution between a pair of coupled lower and upper solutions of the problem (5.26), (5.27), and of the problem (5.26), (5.28).

A solution of these problems will be a function $x \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J)$ that satisfies equation (5.26) a.e. on I coupled to the corresponding boundary conditions (either (5.27) or (5.28) in each case).

5.2.3.1 Existence of Solutions of the Problem (5.26),(5.27).
We introduce the concept of coupled lower and upper solutions of this problem as follows.

Definition 5.2.1. *We say that γ, $\delta \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J)$ is a pair of coupled lower and upper solutions of the conformable fractional problem* (5.26), (5.27)*, if $\gamma \leq \delta$ in J and the following inequalities hold:*

$$\begin{cases} \gamma^{(\alpha)}_{\Delta}(t) - f(t,\gamma^{\sigma}(t)) \leq 0 \leq \delta^{(\alpha)}_{\Delta}(t) - f(t,\delta^{\sigma}(t)) \text{ for } \Delta\text{-a.e. } t \in I, \\ B(\gamma(a),x) \leq 0 \leq B(\delta(a),x) \text{ for all } x \in [\gamma,\delta], \end{cases} \tag{5.48}$$

we define the sector

$$[\gamma,\delta] = \{x \in C(J),\ \gamma(t) \leq x(t) \leq \delta(t) \text{ for all } t \in J\}.$$

We assume the following hypothesis

(H_1) $B \in C(\mathbb{R} \times C(J))$ and $f : I \times \mathbb{R} \to \mathbb{R}$ is a $L^1_{\alpha,\Delta}$-Carathéodory function.

(H_2) There exists $\gamma, \delta \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J)$, a pair of coupled lower and upper solutions of the problem (5.26), (5.27).

Now we consider the following modified problem:

$$\begin{cases} x^{(\alpha)}_{\Delta}(t) = F(t,x^{\sigma}(t)) \qquad \text{for } \Delta\text{-a.e. } t \in I, \\ x(a) = \tau(a, x(a) - B(x(a),x)), \end{cases} \tag{5.49}$$

where, for all $t \in I$ fixed,

$$F(t,z) = f(t,\tau(\sigma(t),z)) - \overline{z}$$

and

$$\overline{z} = \frac{z - \tau(\sigma(t),z)}{1+|z|}, \quad \tau(t,z) = \max\{\gamma(t), \min\{z,\delta(t)\}\}, \quad z \in \mathbb{R}.$$

We need the following auxiliary lemmas.

Lemma 5.2.1. *Assume that $G : I \times \mathbb{R} \to \mathbb{R}$ is a $L^1_{\alpha,\Delta}$-Carathéodory function. If $x \in C(J)$, then function $G_x : I \to \mathbb{R}$ defined for every $s \in I$ as*

$$G_x(s) := G(s, x^{\sigma}(s))$$

belongs to $L^1_{\alpha,\Delta}(I)$.

Proof. If $x \in C(J)$, then $x^\sigma \in C_{rd}(J)$ [33, Theorem 1.60] and, using [34, Theorems 5.81 and 5.82] we have that x^σ is Δ-measurable in I. It follows from [61, Theorem 1.4.3] that G_x is Δ-measurable in I. Since x^σ is bounded [33, Theorem 1.65], from condition 3 on definition of $L^1_{\alpha,\Delta}$-Carathéodory function, we obtain the result. □

From previous lemma, we obtain the following lemma.

Lemma 5.2.2. *Suppose that hypotheses* (H_1) *and* (H_2) *hold, then the following conditions are satisfied:*

1. *For every $x \in \mathbb{R}$, function $\tau(\cdot, x)$ is continuous in J.*
2. *Function $\tau(t, \cdot)$ is continuous in $\mathbb{R}$, uniformly at $t \in J$, i.e.:*
$$\forall\ \epsilon > 0,\ \exists\ \delta(\epsilon) > 0/\ |x - y| < \delta \Longrightarrow |\tau(t, x) - \tau(t, y)| < \epsilon, \quad \textit{for all } t \in J.$$
3. *There exists $K_\tau \geq 0$ such that $|\tau(t, x)| \leq K_\tau$ for every $(t, x) \in J \times \mathbb{R}$.*
4. *For every $x \in \mathbb{R}$, function $F(\cdot, x)$ is measurable.*
5. *$F(t, \cdot) \in C(\mathbb{R})$ for Δ-a.e. $t \in I$.*
6. *There exists $m_F \in L^1_{\alpha,\Delta}(I)$ such that*
$$|F(t, x)| \leq m_F(t) \textit{ for } \Delta\textit{-a.e. } t \in I \quad \textit{and all } x \in \mathbb{R}.$$

To deduce the existence of solutions of problem (5.26), (5.27) in the sector $[\gamma, \delta]$, we define operator $\mathcal{A} : C(J) \to C(J)$, as

$$\mathcal{A}x(t) := \tau(a, x(a) - B(x(a), x)) + \int_{[a,t]_{\mathbb{T}}} F(s, x^\sigma(s))\ \Delta^\alpha s, \quad t \in J. \tag{5.50}$$

Clearly, from Corollary 5.2.1 with $p \equiv 0$, the fixed point of the operator $\mathcal{A}$ is a solution of the problem (5.49).

Proposition 5.2.1. *Suppose that hypotheses* (H_1) *and* (H_2) *are fulfilled. Then, the operator*
$$\mathcal{A} : C(J) \to C(J) \textit{ is compact.}$$

Proof. We first observe that, from Lemma 5.2.2, there exists a function $m_F \in L^1_{\alpha,\Delta}(I)$ such that

$$|F(t, x(\sigma(t))| = \Big| f(t, \tau(\sigma(t), x(\sigma(t)))) - \overline{x}(\sigma(s)) \Big| \leq m_F(t),$$

for Δ-a.e. $t \in I$ and all $x \in C(J)$.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of $C(J)$ converging to $x \in C(J)$. Then

$$\Big|\mathcal{A}(x_n(t)) - \mathcal{A}(x(t))\Big| \leq \int_{[a,\sigma(b)]_{\mathbb{T}}} s^{\alpha-1} \Big| \Big(f(s, \tau(\sigma(s), x_n(\sigma(s)))) - \overline{x}_n(\sigma(s)) \Big) \\ - \Big(f(s, \tau(\sigma(s), x(\sigma(s)))) - \overline{x}(\sigma(s)) \Big) \Big| \Delta s.$$

We can easily check that $\overline{x}_n(t) \to \overline{x}(t)$ for every $t \in J$.
Since

$$|f(s, \tau(\sigma(s), x_n(\sigma(s)))) - \overline{x}_n(\sigma(s))| \leq m_F(s) \ \Delta\text{-a.e. } s \in I \text{ for all } n \in \mathbb{N},$$

the continuity of operator $\mathcal{A}$ follows from the continuous dependence with respect to x of function f, the definition of τ and the Lebesgue's dominated convergence theorem.

To see that the set $\mathcal{A}(C(J))$ is relatively compact on $C(J)$, consider $x \in C(J)$. Therefore,

$$\begin{aligned}|\mathcal{A}(x)(t)| &\leq |\tau(a, x(a) - B(x(a), x))| + \int_{[a,b]_{\mathbb{T}}} \Big|F(s, x^{\sigma}(s))\Big| \Delta^{\alpha} s \\ &\leq K_\tau + K_F = K,\end{aligned}$$

where
$K_\tau := \sup\{|\tau(t,x)| : (t,x) \in J \times \mathbb{R}\}$, and $K_F := ||M_F||$, $M_F(t) = \int_{[a,t]_{\mathbb{T}}} m_F(s)\, \Delta^\alpha s, t \in J$.

So, $\mathcal{A}(C(J))$ is uniformly bounded on $C(J)$. This set is also equicontinuous since for every $t_1 < t_2 \in J$,

$$\|\mathcal{A}\,(x)\,(t_2) - \mathcal{A}\,(x)\,(t_1)\| \leq \int_{[t_1,t_2]_{\mathbb{T}}} m_F(s) \Delta^\alpha s.$$

By Arzelà–Ascoli theorem, we conclude that the set $\mathcal{A}(C(J))$ is relatively compact in $C(J)$. Hence, $\mathcal{A}$ is compact. □

Lemma 5.2.3. *Suppose that hypotheses* (H_1) *and* (H_2) *are fulfilled. Then, all the solutions of problem* (5.49) *belong to the sector* $[\gamma, \delta]$ *and are solutions of problem* (5.26), (5.27).

Proof. Let $x \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J)$ be a solution of the problem (5.49). Suppose it is false that $z := x - \delta \leq 0$ in J. From the definition of τ, we know that $z(a) \leq 0$, then there is $c \in J \backslash \{a\}$ such that $z(c) = \max\{z(t) : \ t \in J\} > 0$.

If $\rho(c) = c$, then, there exists $\varepsilon > 0$ such that $z(t) > 0$ for all $t \in [\rho(c - \varepsilon), c] \cap J$. For Δ-almost every $t \in [c - \varepsilon, c] \cap J$, we have that there exists $z^{(\alpha)}_{\Delta}(\rho(t))$ and, in such points, it is satisfied that

$$z^{(\alpha)}_{\Delta}(\rho(t)) \leq F(t, x(t)) - f(t, \delta(t)) = \frac{x(t) - \delta(t)}{1 + |x(t)|} < 0.$$

Since $z \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J)$, for every $t \in [\rho(c - \varepsilon), c) \cap J$, we arrive at

$$z(c) - z(t) = \int_{[t,c]_{\mathbb{T}}} z^{(\alpha)}_{\Delta}(s) \ \Delta^\alpha s < 0,$$

which is a contradiction with the definition of c.

If $\rho(c) < c$, then

$$z^{(\alpha)}_{\Delta}(\rho(c)) = \frac{z(c) - z(\rho(c))}{c - \rho(c)} (\rho(c))^{1-\alpha} \geq 0,$$

but we know that

$$\begin{aligned} x_{\Delta}^{(\alpha)}(\rho(c)) &= F(\rho(c), x(c)) = f(\rho(c), \delta(c)) - \frac{x(c) - \delta(c)}{1 + |x(c)|} \\ &< f(\rho(c), \delta(c)) \leq \delta_{\Delta}^{(\alpha)}(\rho(c)), \end{aligned}$$

that is, $z_{\Delta}^{(\alpha)}(\rho(c)) < 0$, and so we obtain a contradiction with the previous inequality.

As a consequence, $x \leq \delta$ in J.

Analogously, we can prove that $\gamma \leq x$ in J.

Now, let us see that $x(a) - B(x(a), x) \in [\gamma(a), \delta(a)]$.

If $x(a) - B(x(a), x) < \gamma(a)$, then $x(a) = \gamma(a)$, and therefore $B(\gamma(a), x) > 0$, which contradicts the definition of γ.

Analogously, we can prove that $x(a) - B(x(a), x) \leq \delta(a)$.

Thus, every solution x of (5.49) is a solution of (5.26), (5.27) that belongs to $[\gamma, \delta]$ and the proof is complete. □

Now, we prove an existence result for problem (5.26), (5.27).

Theorem 5.2.2. *Suppose that hypotheses* (H_1) *and* (H_2) *hold, then problem* (5.26), (5.27) *has at least one solution* $x \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J)$ *such that* $\gamma(t) \leq x(t) \leq \delta(t)$ *for every* $t \in J$.

Proof. From Lemma 5.2.3, we know that if problem (5.49) is solvable, then the same holds for problem (5.26), (5.27). By Proposition 5.2.1, $\mathcal{A}$ is compact. It has a fixed point by the Schauder fixed-point theorem. As a consequence, problem (5.49) has at least one solution $x \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J)$ which, by Lemma 5.2.3, is a solution of problem (5.26), (5.27) and belongs to $[\gamma, \delta]$. □

From previous theorem, we obtain the following existence results for the linear boundary value problem (5.29), the periodic problem (5.31), and the anti-periodic value problem (5.32).

Corollary 5.2.4. *If* $f : I \times \mathbb{R} \to \mathbb{R}$ *is a* $L^1_{\alpha,\Delta}$*-Carathéodory function and* $\gamma, \delta \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J)$ *is a pair of coupled lower and upper solutions of problem* (5.29). *Then problem* (5.29) *has at least one solution* $x \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J)$ *such that* $\gamma(t) \leq x(t) \leq \delta(t)$ *for every* $t \in J$.

Corollary 5.2.5. *If* $f : I \times \mathbb{R} \to \mathbb{R}$ *is a* $L^1_{\alpha,\Delta}$*-Carathéodory function and* $\gamma, \delta \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J)$ *is a pair of coupled lower and upper solutions of problem* (5.31). *Then problem* (5.31) *has at least one solution* $x \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J)$ *such that* $\gamma(t) \leq x(t) \leq \delta(t)$ *for every* $t \in J$.

Corollary 5.2.6. *If* $f : I \times \mathbb{R} \to \mathbb{R}$ *is a* $L^1_{\alpha,\Delta}$*-Carathéodory function and* $\gamma, \delta \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J)$ *is a pair of coupled lower and upper solutions of problem* (5.32). *Then problem* (5.32) *has at least one solution* $x \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J)$ *such that* $\gamma(t) \leq x(t) \leq \delta(t)$ *for every* $t \in J$.

5.2.3.2 Existence of solutions of the problem (5.26),(5.28).
We introduce the concept of coupled lower and upper solutions of this problem as follows.

Definition 5.2.2. *We say that γ, $\delta \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J)$ is a pair of coupled lower and upper solutions of the conformable fractional problem* (5.26), (5.28)*, if $\delta \leq \gamma$ in J and the following inequalities hold:*

$$\begin{cases} \gamma_\Delta^{(\alpha)}(t) - f(t,\gamma^\sigma(t)) \leq 0 \leq \delta_\Delta^{(\alpha)}(t) - f(t,\delta^\sigma(t)) \text{ for } \Delta\text{-a.e. } t \in I, \\ H(x,\gamma(\sigma(b))) \leq 0 \leq H(x,\delta(\sigma(b))) \text{ for all } x \in [\delta,\gamma], \end{cases} \tag{5.51}$$

we define the sector

$$[\delta,\gamma] = \{x \in C(J),\ \delta(t) \leq x(t) \leq \gamma(t) \text{ for all } t \in J\}.$$

We assume the following hypothesis

(F$_1$) $H \in C(C(J) \times \mathbb{R})$ and $f : I \times \mathbb{R} \to \mathbb{R}$ is a $L^1_{\alpha,\Delta}$-Carathéodory function.

(F$_2$) There exists $\gamma, \delta \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J)$, a pair of coupled lower and upper solutions of the problem (5.26), (5.28).

Now we consider the following modified problem:

$$\begin{cases} x_\Delta^{(\alpha)}(t) = F(t,x^\sigma(t)) \qquad \text{for } \Delta\text{-a.e. } t \in I, \\ x(\sigma(b)) = \tau(\sigma(b), x(\sigma(b)) + H(x,x(\sigma(b)))), \end{cases} \tag{5.52}$$

where, for all $t \in I$ fixed,

$$F(t,z) = f(t,\tau(\sigma(t),z)) - \overline{z}$$

and

$$\overline{z} = \frac{z - \tau(\sigma(t),z)}{1+|z|}, \qquad \tau(t,z) = \max\{\delta(t), \min\{z,\gamma(t)\}\}, \qquad z \in \mathbb{R}.$$

To deduce the existence of solutions of problem (5.26), (5.28) in the sector $[\delta,\gamma]$, we define operator $\mathcal{T} : C(J) \to C(J)$, as

$$\mathcal{T}x(t) := \tau\left(\sigma(b), x(\sigma(b)) + H(x,x(\sigma(b)))\right) - \int_{[t,\sigma(b)]_{\mathbb{T}}} F(s,x^\sigma(s))\ \Delta^\alpha s, \qquad t \in J. \tag{5.53}$$

Clearly, from Corollary 5.2.2 with $p \equiv 0$, the fixed point of the operator $\mathcal{T}$ is a solution of the problem (5.52).

Following the technique used in the previous subsection, it is easy to prove the following results:

Proposition 5.2.2. *Suppose that hypotheses* (F_1) *and* (F_2) *are fulfilled. Then the operator*

$$\mathcal{T} : C(J) \to C(J) \text{ is compact.}$$

Lemma 5.2.4. *Suppose that hypotheses* (F_1) *and* (F_2) *are fulfilled. Then, all the solutions of problem* (5.52) *belong to the sector* $[\delta, \gamma]$ *and are solutions of problem* (5.26), (5.28).

Theorem 5.2.3. *Suppose that hypotheses* (F_1) *and* (F_2) *hold, then problem* (5.26), (5.28) *has at least one solution* $x \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J)$ *such that* $\delta(t) \le x(t) \le \gamma(t)$ *for every* $t \in J$.

Remark 5.2.1. *The results (Theorems 5.2.2 and 5.2.3) in this Subsection* 5.2.3 *generalize the previous ones (Theorems 5.1.3 and 5.1.2) given in Subsection 5.1.3 for the nonlinear conformable fractional differential equation* (5.1).

5.2.4 Examples

In this subsection, we present two examples where we apply Theorems 5.2.2 and 5.2.3 to some particular cases.

Example 5.2.1. *Consider the nonlinear boundary value problem with functional boundary conditions:*

$$\begin{cases} x_\Delta^{(\frac{1}{3})}(t) = \dfrac{2t - 1 - x^5(\sigma(t))}{\sqrt{t}}, \qquad \Delta\text{-a.e. } t \in I = [0,1]_{\mathbb{T}}, \\[2ex] x(0) - \cos(\pi x(0)) = \dfrac{1}{5} \max_{t\in[0,1]_{\mathbb{T}}} \{x(t)\}. \end{cases} \tag{5.54}$$

This problem is a particular case of (5.26), (5.27) *with* $\alpha = \frac{1}{3}$, $f(t, x^\sigma(t)) = \dfrac{2t - 1 - x^5(\sigma(t))}{\sqrt{t}}$ *and*

$$B(x,y) = x - \cos(\pi x) - \frac{1}{5} \max_{t\in[0,1]_{\mathbb{T}}} \{y(t)\}.$$

It is clear that f *is a* $L^1_{\frac{1}{3},\Delta}$*-Carathéodory function,* B *is a continuous function in* $(x,y) \in [\gamma(0), \delta(0)] \times [\gamma, \delta]$, *with* $\gamma(t) = -1 \le \delta(t) = 1$ *for* $t \in [0, \sigma(1)]_{\mathbb{T}}$. *The fact that* γ *and* δ *are lower and upper solutions of problem* (5.54) *follows from the fact that*

$$\begin{cases} \gamma_\Delta^{(\frac{1}{3})}(t) - f(t, \gamma^\sigma(t)) = -2\sqrt{t} \le 0 \le \delta_\Delta^{(\frac{1}{3})}(t) - f(t, \delta^\sigma(t)) = \dfrac{2(1-t)}{\sqrt{t}} \text{ for } \Delta\text{-a.e. } t \in I, \\[2ex] B(\gamma(0), x) = \dfrac{-1}{5} \le 0 \le B(\delta(0), x) = \dfrac{9}{5} \quad \text{for all } x \in [\gamma, \delta], \end{cases}$$

Theorem 5.2.2, implies that problem (5.54) *has a least one solution* $x \in W^{\frac{1}{3},1}_{\Delta;0,\sigma(1)}([0,\sigma(1)]_{\mathbb{T}})$, *such that* $-1 \le x(t) \le 1$, *for all* $t \in [0, \sigma(1)]_{\mathbb{T}}$.

Example 5.2.2. *Consider the nonlinear boundary value problem with functional boundary conditions:*

$$\begin{cases} x_\Delta^{(\frac{1}{2})}(t) = \dfrac{2\sin(x^\sigma(t))}{\sqrt{t(\pi-t)}}, \qquad \Delta\text{-}a.e.\ t \in I = [0,\pi]_{\mathbb{T}}, \\ x^\sigma(\pi) + 3\sin(x^\sigma(\pi)) = \int_{\frac{\pi}{2}}^{\pi} \sqrt{s}\ x(s)\Delta^{\frac{1}{2}} s. \end{cases} \tag{5.55}$$

This problem is a particular case of (5.26), (5.28) *with* $\alpha = \frac{1}{2}$, $f(t, x^\sigma(t)) = \dfrac{2\sin(x^\sigma(t))}{\sqrt{t(\pi-t)}}$ *and*

$$H(x,y) = \int_{\frac{\pi}{2}}^{\pi} \sqrt{s}\ x(s)\Delta^{\frac{1}{2}} s - y - 3\sin(y).$$

It is clear that f *is a* $L^1_{\frac{1}{2},\Delta}$*-Carathéodory function,* H *is a continuous function in* $(x,y) \in [\delta,\gamma] \times [\delta(\sigma(\pi)), \gamma(\sigma(\pi))]$*, with* $\delta(t) = \dfrac{-\pi}{2} \le \gamma(t) = \dfrac{\pi}{2}$ *for* $t \in [0,\sigma(\pi)]_{\mathbb{T}}$.

The fact that γ *and* δ *are lower and upper solutions of problem* (5.55) *follows from the fact that*

$$\begin{cases} \gamma_\Delta^{(\frac{1}{2})}(t) - f(t,\gamma^\sigma(t)) = \dfrac{-2}{\sqrt{t(\pi-t)}} \le 0 \le \delta_\Delta^{(\frac{1}{2})}(t) - f(t,\delta^\sigma(t)) = \dfrac{2}{\sqrt{t(\pi-t)}} \ for\ \Delta\text{-}a.e.\ t \in I, \\ H(x,\gamma^\sigma(\pi)) \le \dfrac{\pi^2}{4} - \dfrac{\pi}{2} - 3 \le 0 \le \dfrac{-\pi^2}{4} + \dfrac{\pi}{2} + 3 \le H(x,\delta^\sigma(\pi)) \quad for\ all\ \ x \in [\delta,\gamma], \end{cases}$$

Theorem 5.2.3, implies that problem (5.55) *has a least one solution* $x \in W^{\frac{1}{2},1}_{\Delta;0,\sigma(\pi)}([0,\sigma(\pi)]_{\mathbb{T}})$*, such that* $\dfrac{-\pi}{2} \le x(t) \le \dfrac{\pi}{2}$*, for all* $t \in [0,\sigma(\pi)]_{\mathbb{T}}$.

Example 5.2.3. *Consider the periodic problem:*

$$\begin{cases} x_\Delta^{(\frac{1}{2})}(t) = -2\sin(\pi\ x(t+1)) + \dfrac{e^t}{\sqrt{t}} x(t+1), \qquad \Delta\text{-}a.e.\ t \in I = [0,b]_{\mathbb{Z}},\ b \in \mathbb{Z}, \\ x(0) = x(b+1). \end{cases} \tag{5.56}$$

This problem is a particular case of (5.26), (5.28) *with* $\alpha = \frac{1}{2}$, $\mathbb{T} = \mathbb{Z}$, $f(t,x^\sigma(t)) = -2\sin(\pi\ x^\sigma(t)) + \dfrac{e^t}{\sqrt{t}} x^\sigma(t)$ *and* $H(x,y) = x - y$*. It is clear that* f *is a* $L^1_{\frac{1}{2},\Delta}$*-Carathéodory function,* H *is a continuous function in* $(x,y) \in [\delta,\gamma] \times [\delta(b+1), \gamma(b+1)]$*, with*

$$\delta(t) = -1 \le \gamma(t) = 1 \ for\ t \in [0, b+1]_{\mathbb{Z}}.$$

The fact that γ *and* δ *are lower and upper solutions of problem* (5.56) *follows from the fact that*

$$\begin{cases} \gamma_{\Delta}^{(\frac{1}{2})}(t) = 0 \leq f(t, \gamma^{\sigma}(t)) = -2\sin(\pi) + \dfrac{e^t}{\sqrt{t}}, \ \textit{for } \Delta\textit{-a.e. } t \in I = [0,b]_{\mathbb{Z}}, \ \gamma(0) \leq \gamma(b+1), \\ \delta_{\Delta}^{(\frac{1}{2})}(t) = 0 \geq f(t, \delta^{\sigma}(t)) = 2\sin(\pi) - \dfrac{e^t}{\sqrt{t}}, \ \ \textit{for } \Delta\textit{-a.e. } t \in I = [0,b]_{\mathbb{Z}}, \ \delta(0) \geq \delta(b+1). \end{cases}$$

Theorem 5.2.3, implies that problem (5.56) *has a solution* $x \in W^{\frac{1}{3},1}_{\Delta;0,b+1}([0,b+1]_{\mathbb{Z}})$, *such that* $-1 \leq x(t) \leq 1$ *for every* $t \in [0,b+1]_{\mathbb{Z}}$.

5.3 Existence results for systems of conformable fractional differential equations

The results of this chapter are original and are accepted for publication [25].

5.3.1 Introduction

In this section, we establish existence results for the following system of conformable fractional differential equations:

$$\begin{cases} x^{(\alpha)}(t) = f(t, x(t)), \qquad \textit{for a.e. } t \in I = [0,b], \ b > 0, \\ x \in (\mathfrak{B}), \end{cases} \tag{5.57}$$

where $0 < \alpha \leq 1$, $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ is a L^1_α-Carathéodory function, $x^{(\alpha)}(t)$ denotes the conformable fractional derivative of x at t of order α, and $(\mathfrak{B})$ denotes the initial value or the periodic boundary value conditions:

$$x(0) = x_0, \tag{5.58}$$

$$x(0) = x(b). \tag{5.59}$$

Existence results for problem (5.57),(5.58) were obtained in [79], by using the Banach fixed point theorem with f a continuous function. In the particular case where $n = 1$, existence results for problem (5.57) were obtained in Section 5.1 with nonlinear functional boundary conditions $L_1(x, x(b)) = 0$ or $L_2(x(0), x) = 0$, their results were established, for the scalar case, with the method of lower and upper solutions and cover, as a particular cases, the boundary conditions (5.58) and (5.59). In [19] the authors solved problem (5.57),(5.58) (for $n = 1$), with f a continuous function by the help of the solution-tube method. As we will see, the used definition is equivalent to the existence of a pair of lower and upper solutions of the considered problem.

In order to obtain the existence results for problem (5.57), we introduce the notion of solution-tube of (5.57) which generalizes the notions of lower and upper solutions given in Section 5.1. It is inspired by a notion of solution tube for first-order systems of differential equations introduced in [74], (see also [51, 52] and [53] on time scales).

5.3.2 Existence Theorem

In this subsection, we establish an existence result for the problem (5.57). A solution of problem (5.57) will be a function $x \in W^{\alpha,1}_{0,b}(I,\mathbb{R}^n)$ for which (5.57) is satisfied. We introduce the notion of solution-tube of this problem as follows.

Definition 5.3.1. *Let $(v, M) \in W^{\alpha,1}_{0,b}(I,\mathbb{R}^n) \times W^{\alpha,1}_{0,b}(I,[0,\infty))$. We say that (v, M) is a solution tube to problem* (5.57) *if*

(i) $\langle x - v(t), f(t,x) - v^{(\alpha)}(t)\rangle \le M(t)M^{(\alpha)}(t)$ *for a.e.* $t \in I$ *and every* $x \in \mathbb{R}^n$ *such that* $\|x - v(t)\| = M(t)$,

(ii) $v^{(\alpha)}(t) = f(t, v(t))$ *and* $M^{(\alpha)}(t) = 0$ *a.e. on* $\{t \in I : M(t) = 0\}$,

(iii) - *if* $(\mathfrak{B})$ *denotes* (5.58), *then* $\|x_0 - v(0)\| \le M(0)$,
- *if* $(\mathfrak{B})$ *denotes* (5.59), *then* $\|v(b) - v(0)\| \le M(0) - M(b)$.

If $\alpha = 1$, our definition of solution tube is equivalent to the notion of solution tube introduced in [74] for first order systems of Ordinary Differential Equations.

Now, we introduce the following set

$$\mathbf{T}(v, M) := \left\{x \in W^{\alpha,1}_{0,b}(I,\mathbb{R}^n) : \|x(t) - v(t)\| \le M(t),\ for\ every\ t \in I\right\}.$$

Remark 5.3.1. *If $n = 1$, our definition of solution tube is equivalent to the notion of solution tube introduced in [19]. We point out that in this case the solution-tube method is equivalent of the lower and upper solutions one. To this end, we introduce the following definition:*

Definition 5.3.2. *A function $\gamma \in W^{\alpha,1}_{a,b}(I)$ is called a lower solution of* (5.57), *if*

(i) $\gamma^{(\alpha)}(t) \ge f(t, \gamma(t)),\quad for\ a.e.\ t \in I;$

(ii) - *if* $(\mathfrak{B})$ *denotes* (5.58), *then* $\gamma(0) \ge x_0$,
- *if* $(\mathfrak{B})$ *denotes* (5.59), *then* $\gamma(0) \ge \gamma(b)$.

A function $\delta \in W^{\alpha,1}_{0,b}(I)$ is called an upper solution of (5.57) *if it satisfies* $(i), (ii)$ *with the reversed inequalities.*

Indeed, we consider the following assumptions:

(A) There exist $\delta \le \gamma$ respectively upper and lower solutions of (5.57), *such that $\delta < \gamma$ a.e. on I.*

(B) There exists (v, M) a solution-tube of (5.57).

First, we prove the following assertion

If (B) is satisfied, then (A) is also fulfilled.

Define $\delta = v - M$ and $\gamma = v + M$.

$$\begin{cases} \left(\delta - \frac{\delta+\gamma}{2}(t)\right) \left(f(t,\delta) - \frac{(\gamma+\delta)^{(\alpha)}(t)}{2}\right) \leq \frac{(\gamma-\delta)(t)}{2} \frac{(\gamma-\delta)^{(\alpha)}(t)}{2} \text{ for a.e. } t \in I \\ \left(\gamma - \frac{\delta+\gamma}{2}(t)\right) \left(f(t,\gamma) - \frac{(\gamma+\delta)^{(\alpha)}(t)}{2}\right) \leq \frac{(\gamma-\delta)(t)}{2} \frac{(\gamma-\delta)^{(\alpha)}(t)}{2} \text{ for a.e. } t \in I. \end{cases}$$

It is not difficult to verify that, since $\delta < \gamma$ a.e. on I, that

$$\begin{cases} \delta^{(\alpha)}(t) \leq f(t,\delta(t)), & for\ a.e.\ t \in I \\ \gamma^{(\alpha)}(t) \geq f(t,\gamma(t)), & for\ a.e.\ t \in I. \end{cases}$$

Moreover, from condition (iii) it is immediate to conclude that $\delta(0) \leq x_0 \leq \gamma(0)$, provided (5.58) is considered, and $\delta(0) - \delta(b) \leq 0 \leq \gamma(0) - \gamma(b)$ for conditions (5.59).

Now, let's prove the reverse implication, i.e.

If (A) holds, then (B) is satisfied.

To this end, take $v = (\gamma + \delta)/2$ and $M = (\gamma - \delta)/2$, we have $\delta = v - M$ and $\gamma = v + M$.

For $x \in \mathbb{R}$ such that $|x - v(t)| = M(t)$, then $x = \gamma$ or $x = \delta$, and we have for a.e. $t \in I$,

$$\begin{aligned} (x - v(t)) \left(f(t,x) - v^{(\alpha)}(t)\right) &= \begin{cases} \left(\delta - \frac{\delta+\gamma}{2}(t)\right) \left(f(t,\delta) - \frac{(\delta+\gamma)^{(\alpha)}}{2}(t)\right), \\ \left(\gamma - \frac{\delta+\gamma}{2}(t)\right) \left(f(t,\gamma) - \frac{(\delta+\gamma)^{(\alpha)}}{2}(t)\right), \end{cases} \\ &\leq \begin{cases} \left(\frac{\delta-\gamma}{2}(t)\right) \left(\delta^{(\alpha)}(t) - \frac{(\delta+\gamma)^{(\alpha)}}{2}(t)\right), \\ \left(\frac{\gamma-\delta}{2}(t)\right) \left(\gamma^{(\alpha)}(t) - \frac{(\delta+\gamma)^{(\alpha)}}{2}(t)\right), \end{cases} \\ &= M(t)M^{(\alpha)}(t) \ \ for\ a.e.\ t \in I. \end{aligned}$$

We consider the following modified problem:

$$\begin{cases} x^{(\alpha)}(t) + \alpha\, x(t) = f(t,\overline{x}(t)) + \alpha\, \overline{x}(t), & for\ a.e.\ t \in I, \\ x \in (\mathfrak{B}). \end{cases} \tag{5.60}$$

where

$$\overline{x}(t) = \begin{cases} \frac{M(t)}{\|x-v(t)\|}(x - v(t)) + v(t), & \text{if } \|x - v(t)\| > M(t), \\ x(t), & \text{if } \|x - v(t)\| \leq M(t). \end{cases} \tag{5.61}$$

We need the following auxiliary lemmas, which are direct generalizations of Corollary 5.1.1 and Corollary 5.1.3 in Section 5.1, and we omit the proofs.

Lemma 5.3.1. *For every $g \in L^1_\alpha(I,\mathbb{R}^n)$, $x_0 \in \mathbb{R}^n$, $0 < \alpha \leq 1$ and $p \in \mathbb{R}$, problem*

$$\begin{cases} x^{(\alpha)}(t) + px(t) = g(t), & a.e.\ t \in I, \\ x(0) = x_0, \end{cases} \tag{5.62}$$

has a unique solution $x \in W^{\alpha,1}_{0,b}(I,\mathbb{R}^n)$ given by the expression:

$$x(t) := \int_0^b G_{In}(t,s) g(s) d_\alpha s + x_0 e^{-\frac{p}{\alpha}t^\alpha}, \tag{5.63}$$

where

$$G_{In}(t,s) = e^{\frac{p}{\alpha}(s^\alpha - t^\alpha)} \begin{cases} 1, & 0 \leq s \leq t \leq b, \\ 0, & 0 \leq t \leq s \leq b, \end{cases} \tag{5.64}$$

Lemma 5.3.2. *For every $g \in L^1_\alpha(I,\mathbb{R}^n)$, $\lambda \in \mathbb{R}^n$, $0 < \alpha \leq 1$ and $p \in \mathbb{R}\backslash\{0\}$, problem*

$$\begin{cases} x^{(\alpha)}(t) + px(t) = g(t), & a.e.\ t \in I, \\ x(0) - x(b) = \lambda, \end{cases} \tag{5.65}$$

has a unique solution $x \in W^{\alpha,1}_{0,b}(I,\mathbb{R}^n)$ given by the following expression:

$$x(t) := \int_0^b G_{Pe}(t,s) g(s) d_\alpha s + \lambda \frac{e^{-\frac{p}{\alpha}t^\alpha}}{1 - e^{-\frac{p}{\alpha}b^\alpha}}, \tag{5.66}$$

where

$$G_{Pe}(t,s) = \frac{e^{\frac{p}{\alpha}(s^\alpha - t^\alpha)}}{1 - e^{-\frac{p}{\alpha}b^\alpha}} \begin{cases} 1, & 0 \leq s \leq t \leq b, \\ e^{-\frac{p}{\alpha}b^\alpha}, & 0 \leq t < s \leq b, \end{cases} \tag{5.67}$$

The following lemma can be proved analogously to [19, Lemma 11].

Lemma 5.3.3. *Let $r \in W^{\alpha,1}_{0,b}(I,\mathbb{R})$, such that $r^{(\alpha)}(t) < 0$ a.e. on $\{t \in I : r(t) > 0\}$. If one of the two following conditions holds,*

(i) $r(0) \leq 0$,

(ii) $r(0) \leq r(b)$,

then $r(t) \leq 0$ for every $t \in I$.

Let us define the operators $\mathcal{A}_1, \mathcal{A}_2 : C(I,\mathbb{R}^n) \to C(I,\mathbb{R}^n)$ by

$$\mathcal{A}_1(x)(t) = \int_0^b G_{In}(t,s)\Big(f(s,\overline{x}(s)) + \alpha\ \overline{x}(s)\Big) s^{\alpha-1} ds + x_0 e^{-t^\alpha}$$

and

$$\mathcal{A}_2(x)(t) = \int_0^b G_{Pe}(t,s)\Big(f(s,\overline{x}(s)) + \alpha\ \overline{x}(s)\Big) s^{\alpha-1} ds,$$

where G_{In} (resp., G_{Pe}) is the Green's function related to the initial problem (5.62)(resp., periodic problem (5.65)) and is given by expression (5.64)(resp.,(5.67)) with $p = \alpha$.

Clearly, from Lemma 5.3.1 (resp. Lemma 5.3.2) with $p = \alpha$, the solutions of problem (5.60), (5.58) (resp. (5.60), (5.59)) coincide with the fixed points of operator $\mathcal{A}_1$ (resp. $\mathcal{A}_2$).

Proposition 5.3.1. *Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a L^1_α-Carathéodory function. Assume there exists $(v, M) \in W^{\alpha,1}_{0,b}(I, \mathbb{R}^n) \times W^{\alpha,1}_{0,b}(I, [0, \infty))$ a solution tube of problem* (5.57),(5.59), *then operator $\mathcal{A}_2$ is compact.*

Proof. We first observe that, from Definitions 1.3.17 and 5.3.1, there exists a function $h \in L^1_\alpha(I, [0, \infty))$ such that

$$\|f(t, \overline{x}(t)) + \alpha\, \overline{x}(t)\| \le h(t), \quad \text{for a.e. } t \in I \text{ and all } x \in C(I, \mathbb{R}^n).$$

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of $C(I, \mathbb{R}^n)$ converging to $x \in C(I, \mathbb{R}^n)$. In this case, it is clear that

$$\begin{aligned}\Big\|\mathcal{A}_2(x_n(t)) - \mathcal{A}_2(x(t))\Big\| &\le \int_0^b s^{\alpha-1}|G_{Pe}(t,s)|\Big\|\Big(f(s,\overline{x_n}(s)) + \alpha\,\overline{x_n}(s)\Big)\\ &\qquad - \Big(f(s,\overline{x}(s)) + \alpha\,\overline{x}(s)\Big)\Big\|ds\\ &\le M\int_0^b s^{\alpha-1}\Big\|\Big(f(s,\overline{x_n}(s)) + \alpha\,\overline{x_n}(s)\Big)\\ &\qquad - \Big(f(s,\overline{x}(s)) + \alpha\,\overline{x}(s)\Big)\Big\|ds.\end{aligned}$$

where $M := \max_{s,t\in I} |G_{Pe}(t, s)|$.

The continuity of operator $\mathcal{A}_2$ follows from the continuous dependence with respect to x of function f, the definition of $\overline{x}$ and the Lebesgue's dominated convergence theorem.

To see that $\mathcal{A}_2(C(I, \mathbb{R}^n))$ is relatively compact set on $C(I, \mathbb{R}^n)$, consider $x \in C(I, \mathbb{R}^n)$. Therefore,

$$\Big\|\mathcal{A}_2(x)(t)\Big\| \le M\ \|h\|_{L^1_\alpha(I,\mathbb{R}^n)}.$$

So, $\mathcal{A}_2(C(I, \mathbb{R}^n))$ is uniformly bounded.

This set is also equicontinuous since for every $t_1 < t_2 \in I$,

$$\begin{aligned}
&\Big\|\mathcal{A}_2\left(x\right)\left(t_2\right)-\mathcal{A}_2\left(x\right)\left(t_1\right)\Big\| \\
&= \Big\| \int_0^{t_2} G_{Pe}(t_2,s)\Big(f(s,\overline{x}(s))+\alpha\, \overline{x}(s)\Big)d_\alpha s + \int_{t_2}^{b} G_{Pe}(t_2,s)\Big(f(s,\overline{x}(s))+\alpha\, \overline{x}(s)\Big)d_\alpha s \\
&\qquad - \int_0^{t_1} G_{Pe}(t_1,s)\Big(f(s,\overline{x}(s))+\alpha\, \overline{x}(s)\Big)d_\alpha s - \int_{t_1}^{b} G_{Pe}(t_1,s)\Big(f(s,\overline{x}(s))+\alpha\, \overline{x}(s)\Big)d_\alpha s\Big\| \\
&\le \frac{|e^{-t_2^\alpha}-e^{-t_1^\alpha}|}{1-e^{-b^\alpha}}\Big(\int_0^{t_1} e^{s^\alpha}\Big\|f(s,\overline{x}(s))+\alpha\, \overline{x}(s)\Big\|d_\alpha s + \int_{t_2}^{b} e^{s^\alpha-b^\alpha}\Big\|f(s,\overline{x}(s))+\alpha\, \overline{x}(s)\Big\|d_\alpha s\Big) \\
&\qquad + \int_{t_1}^{t_2} |G_{Pe}(t_2,s)-G_{Pe}(t_1,s)|\Big\|f(s,\overline{x}(s))+\alpha\, \overline{x}(s)\Big\|d_\alpha s \\
&\le K|e^{-t_2^\alpha}-e^{-t_1^\alpha}|\Big(\int_0^{t_1} h(s)d_\alpha s + \int_{t_2}^{b} h(s)d_\alpha s\Big) + 2M\int_{t_1}^{t_2} h(s)d_\alpha s,
\end{aligned}$$

where

$$K := \max_{s\in I}\{\frac{e^{s^\alpha}}{1-e^{-b^\alpha}}, \frac{e^{s^\alpha-b^\alpha}}{1-e^{-b^\alpha}}\} = \frac{1}{1-e^{-b^\alpha}}.$$

By Arzelà-Ascoli theorem, we conclude that the set $\mathcal{A}_2(C(I,\mathbb{R}^n))$ is relatively compact in $C(I,\mathbb{R}^n)$. Hence, $\mathcal{A}_2$ is compact. □

The following result can be proved as the previous one.

Proposition 5.3.2. *Let $f : I\times\mathbb{R}^n \to \mathbb{R}^n$ be a L^1_α-Carathéodory function. Assume there exists $(v,M)\in W^{\alpha,1}_{0,b}(I,\mathbb{R}^n)\times W^{\alpha,1}_{0,b}(I,[0,\infty))$ a solution tube of* (5.57),(5.58), *then operator $\mathcal{A}_1$ is compact.*

Now, we can obtain our main theorem. The proof is on the basis on the one given in [53] for first order systems of ordinary differential equations.

Theorem 5.3.1. *Let $f : I\times\mathbb{R}^n \to \mathbb{R}^n$ be a L^1_α-Carathéodory function. Assume there exists $(v,M)\in W^{\alpha,1}_{0,b}(I,\mathbb{R}^n)\times W^{\alpha,1}_{0,b}(I,[0,\infty))$ a solution tube of* (5.57). *Then, problem* (5.57) *has a solution $x\in W^{\alpha,1}_{0,b}(I,\mathbb{R}^n)\cap \mathrm{T}(v,M)$.*

Proof. We will do the proof for the initial case (5.58). As we will see the proof for the periodic problem (5.59) is analogous.

By Proposition 5.3.2 the operator $\mathcal{A}_1$ is compact. It has a fixed point by the Schauder fixed-point theorem. Lemma 5.3.1 implies that this fixed point is a solution for the problem (5.60). Then, it suffices to show that for every solution x of (5.60), $x\in \mathbf{T}(v,M)$. Consider the set $\mathcal{B} := \{t\in I : \|x(t)-v(t)\| > M(t)\}$. By Proposition 1.4.2, *a.e. on* $\mathcal{B}$ we have

$$(\|x(t)-v(t)\|-M(t))^{(\alpha)} = \frac{\langle x(t)-v(t), x^{(\alpha)}(t)-v^{(\alpha)}(t)\rangle}{\|x(t)-v(t)\|} - M^{(\alpha)}(t).$$

Since (v, M) is a solution tube of problem (5.57), we have *a.e. on* $\{t \in \mathcal{B} : M(t) > 0\}$ that

$$
\begin{aligned}
&(\|x(t) - v(t)\| - M(t))^{(\alpha)} \\
&\qquad = \frac{\langle x(t) - v(t), x^{(\alpha)}(t) - v^{(\alpha)}(t)\rangle}{\|x(t) - v(t)\|} - M^{(\alpha)}(t) \\
&\qquad = \frac{\langle x(t) - v(t), f(t, \bar{x}(t)) + \alpha\bar{x}(t) - \alpha x(t) - v^{(\alpha)}(t)\rangle}{\|x(t) - v(t)\|} - M^{(\alpha)}(t) \\
&\qquad = \frac{\langle \bar{x}(t) - v(t), f(t, \bar{x}(t)) - v^{(\alpha)}(t)\rangle}{M(t)} + \alpha\, \frac{\langle \bar{x}(t) - v(t), \bar{x}(t) - x(t)\rangle}{M(t)} - M^{(\alpha)}(t) \\
&\qquad \leq \frac{M(t)M^{(\alpha)}(t)\rangle}{M(t)} + \alpha\Big(M(t) - \|x(t) - v(t)\|\Big) - M^{(\alpha)}(t) \\
&\qquad < 0.
\end{aligned}
$$

On the other hand, we have *a.e. on* $\{t \in \mathcal{B} : M(t) = 0\}$ that

$$
\begin{aligned}
(\|x(t) - v(t)\| - M(t))^{(\alpha)} &= \frac{\langle x(t) - v(t), f(t, \bar{x}(t)) + \alpha\bar{x}(t) - \alpha x(t) - v^{(\alpha)}(t)\rangle}{\|x(t) - v(t)\|} - M^{(\alpha)}(t) \\
&= \frac{\langle x(t) - v(t), f(t, v(t)) + \alpha v(t) - \alpha x(t) - v^{(\alpha)}(t)\rangle}{\|x(t) - v(t)\|} - M^{(\alpha)}(t) \\
&\leq \frac{\langle x(t) - v(t), f(t, v(t)) - v^{(\alpha)}(t)\rangle}{\|x(t) - v(t)\|} - \alpha\, \|x(t) - v(t)\| - M^{(\alpha)}(t) \\
&< 0.
\end{aligned}
$$

If we set, $r(t) := \|x(t) - v(t)\| - M(t)$, then $r^{(\alpha)} < 0$ a.e. on $\mathcal{B} := \{t \in I : r(t) > 0\}$. Moreover, since (v, M) is a solution tube to problem (5.57) and x satisfies (5.58), then $r(0) \leq 0$ and, as consequence, Lemma 5.3.3 (i) implies that $\mathcal{B} = \emptyset$. So, $x \in T(v, M)$ and the result holds for this case.

When the periodic case is studied, we follow the same steps with operator $\mathcal{A}_2$ and we arrive to the fact that

$$r(0) - r(b) \leq \|v(0) - v(b)\| - (M(0) - M(b)) \leq 0,$$

and the result is fulfilled from Lemma 5.3.3 (ii).

Remark 5.3.2. *This result (Theorem 5.3.1) generalize the previous one (Corollary 5.1.6) given in Subsection 5.1.3 with* $L(x, y) = x - \lambda_0$ *or* $L(x, y) = x - y$.

5.3.3 Examples

The following example is a modified version, considering a periodic condition, of Example 4.6 in [53]:

Example 5.3.1. *Consider the periodic problem:*

$$\begin{cases} x^{(\frac{1}{3})}(t) = a_1\|x(t)\|^2 x(t) - a_2 x(t) + a_3\varphi(t), \quad a.e.\ t \in I = [0,1], \\ x(0) = x(1). \end{cases} \tag{5.68}$$

where $\alpha = 1/3$, $a_1, a_2, a_3 \in \mathbb{R}_+$ such that $a_1 - a_2 + a_3 = 0$, $\varphi : I \to \mathbb{R}^n$ is a continuous function satisfying $\|\varphi(t)\| = 1$ for every $t \in I$. Take $v(t) = 0$ and $M(t) = 1$.
So, $v \in W_{0,1}^{\frac{1}{3},1}(I, \mathbb{R}^n)$, $M \in W_{0,1}^{\frac{1}{3},1}(I, [0,\infty[)$, $v^{(\frac{1}{3})}(t) = 0$, $M^{(\frac{1}{3})}(t) = 0$, and

$$\|v(1) - v(0)\| \leq M(0) - M(1).$$

For $x \in \mathbb{R}^n$ such that $\|x - v(t)\| = M(t)$, then $\|x\| = 1$, and we have, for a.e. $t \in I$

$$\begin{aligned} \langle x - v(t), f(t,x) - v^{(\frac{1}{3})}(t)\rangle &= \langle x, a_1\|x\|^2 x - a_2 x + a_3\varphi(t)\rangle \\ &= a_1\|x\|^4 - a_2\|x\|^2 + a_3\langle x, \varphi(t)\rangle \\ &\leq a_1\|x\|^4 - a_2\|x\|^2 + a_3\|x\|\|\varphi(t)\| \\ &= a_1 - a_2 + a_3 = 0 \\ &\leq M(t)M^{(\frac{1}{3})}(t). \end{aligned}$$

Since the set $\{t \in I, M(t) = 0\} = \emptyset$, condition (ii) holds trivially.

So, (v, M) is a solution-tube of (5.68). By Theorem 5.3.1, problem (5.68) has a solution $x \in W_{0,1}^{\frac{1}{3},1}(I, \mathbb{R}^n)$ such that $\|x(t)\| \leq 1$ for every $t \in I$.

Example 5.3.2. Consider the periodic problem:

$$\begin{cases} x^{(1/2)}(t) = \dfrac{-x^3(t) + 1 - 2t}{\sqrt[4]{t}} \quad \text{a.e. } t \in [0,1], \\ x(0) = x(1). \end{cases} \tag{5.69}$$

This problem is a particular case of (5.57),(5.59)), with $n = 1$, $\alpha = 1/2$, and $f(t,x) = \dfrac{-x^3 + 1 - 2t}{\sqrt[4]{t}}$. It is clear that f is a $L^1_{1/2}$-Carathéodory function.
Take $v(t) = 0$ and $M(t) = 1$.
So, $v \in W_{0,1}^{\frac{1}{2},1}(I, \mathbb{R})$, $M \in W_{0,1}^{\frac{1}{2},1}(I, [0,\infty[)$, $v^{(\frac{1}{2})}(t) = 0$, $M^{(\frac{1}{2})}(t) = 0$, and

$$|v(1) - v(0)| \leq M(0) - M(1).$$

For $x \in \mathbb{R}$ such that $|x - v(t)| = M(t)$, then $x = 1$ or $x = -1$, and we have for a.e. $t \in I$,

$$\begin{aligned} \langle x - v(t), f(t,x) - v^{(\frac{1}{2})}(t)\rangle &= (x)\Big(\frac{-x^3 + 1 - 2t}{\sqrt[4]{t}}\Big), \\ &= \begin{cases} \dfrac{-2(1-t)}{\sqrt[4]{t}} & \text{if } x = -1, \\ -2\sqrt[4]{t^3} & \text{if } x = 1, \end{cases} \\ &\leq 0 = M(t)M^{(\frac{1}{2})}(t) \quad \text{for a.e. } t \in I. \end{aligned}$$

So, (v, M) is a solution-tube of (5.69). By Theorem 5.3.1, the problem (5.69) has a solution $x \in W_{0,1}^{\frac{1}{2},1}(I)$ such that $|x(t)| \leq 1$ for every $t \in I$.

Observe that $\delta = v - M$ and $\gamma = v + M$ are, respectively, upper and lower solutions of (5.69) follows from the fact that

$$\delta^{(\frac{1}{2})}(t) = 0 \leq f(t, \delta(t)) = \frac{2(1-t)}{\sqrt[4]{t}}, \text{ for a.e. } t \in [0,1], \quad \delta(0) \leq \delta(1),$$

and

$$\gamma^{(\frac{1}{2})}(t) = 0 \geq f(t, \gamma(t)) = -2\sqrt[4]{t^3}, \text{ for a.e. } t \in [0,1], \quad \gamma(0) \geq \gamma(1),$$

such that $-1 \leq x(t) \leq 1$, for all $t \in I$.

Conclusion and future perspectives

Differential equations with fractional order are a generalization of ordinary differential equations to non-integer order. Recently, a new definition of the fractional derivative, called conformable fractional derivative, was introduced by Khalil et al. [70]. This definition reflects a natural extension of the normal derivative. In particular, Benkhettou et al. [31] introduced a conformable fractional calculus on an arbitrary time scale, which provided a natural extension of the conformable fractional calculus.

In this thesis, we have considered the existence of solutions for systems of first order nonlinear nabla dynamic equations and nabla dynamic inclusions on time scales and for systems of conformable fractional differential equations under some boundary conditions. Also, we present existence solutions for the nonlinear conformable fractional differential equations and for the conformable fractional dynamic equations on time scales, with nonlinear functional boundary value conditions.

These results will be obtained by using Schauder's fixed point theorem, Kakutani fixed point theorem and by notions of solution-tube adapted to these systems. These notions of solution-tube generalize to systems the definitions of lower and upper solution of first order nonlinear nabla dynamic equations and inclusions on time scales, and of conformable fractional differential equations.

For future researches, by using the solution-tube method, we can look for existence of solutions for:

- ▷ Systems of first order nonlinear impulsive dynamic equations on time scales,
- ▷ systems of conformable fractional dynamic equations on time scales, (i.e., to extend the results presented in Section 5.3 in continuous case to time scale),
- ▷ systems of impulsive conformable fractional dynamic equations on time scales.

Bibliography

[1] T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math. **279** (2015), 57–66.

[2] T. Abdeljawad, M. Al Horani and R. Khalil, *Conformable fractional semigroups of operators*, Journal of Semigroup Theory and Applications. **2015** (2015), Art. ID 7, 9 pp.

[3] M. Abu Hammad and R. Khalil, *Abel's formula and Wronskian for conformable fractional differential equations*, Int. J. Differ. Equ. Appl. **13** (2014), 177–183.

[4] M. Abu Hammad and R. Khalil, *Conformable fractional heat differential equations*, Int. J. Pure. Appl. Math. **94** (2014), 215–221.

[5] R.P. Agarwal, M. Bohner, D. O'Regan and A. Peterson, *Dynamic equations on time scales: a survey*, J. Comput. Appl. Math. **141** (2002), no. 1-2, 1–26.

[6] R.P. Agarwal, V. Otero-Espinar, K. Perera and DR. Vivero, *Basic properties of Sobolev's spaces on time scales*, Adv. Differ. Equ. **2006** (2006), Article ID 38121.

[7] A. Ahmadkhanlu and M. Jahanshahi, *On the existence and uniqueness of solution of initial value problem for fractional order differential equations on time scales.* Bull. Iran Math. Soc. **38** (1) (2012), 241-252.

[8] D. R. Anderson and R.I. Avery, *Fractional-order boundary value problem with Sturm-Liouville boundary conditions*, Electron. J. Differ. Equa. **2015** (2015), no. 29, 10 pp.

[9] D.R. Anderson, J. Bullock, L. Erbe, A. Peterson and H. Tran, *Nabla dynamic equations on time scales.* Panamer. Math. J. **13** (2003), no. 1, 1-47.

[10] D.R. Anderson and D.J. Ulness, *Properties of the Katugampola fractional derivative with potential application in quantum mechanics*, J. Math. Phys. **56** (2015), no. 6, 063502, 18 pp.

[11] J. Appell, P. P. Zabrejko, *Nonlinear superposition operators.* Cambridge Tracts in Mathematics, 95. Cambridge University Press, Cambridge, 1990, 311 pages.

[12] M. F. Atici and D. C. Biles, *First order dynamic inclusions on time scales*, J. Math. Anal. Appl. **292** (2004), no. 1, 222-237.

[13] M.F. Atici and G.S. Guseinov, *On Green's functions and positive solutions for boundary value problems on time scales*, J. Comput. Appl. Math. **141** (2002), no. 1–2, 75-99.

[14] J.P. Aubin and A. Cellina, *Differential Inclusions, Springer-Verlag,* Berlin-Heidelberg, New York, 1984.

[15] J.P. Aubin and H. Frankowska, *Set-Valued Analysis, Birkhauser*, Boston, 1990.

[16] N. Bastos, D. Mozyrska and D. Torres, *Fractional derivatives and integrals on time scales via the inverse generalized Laplace transform*, Int. J. Math. Comput. **11** (2011) 1-9.

[17] H. Batarfi, J. Losada, J. J. Nieto and W. Shammakh, *Three-Point Boundary Value Problems for Conformable Fractional Differential Equations*, J. Funct. Spaces. **2015** (2015), Art. ID 706383, 6 pp.

[18] B. Bayour, A. Hammoudi and D.F.M. Torres, *Existence of solution to a nonlinear first-order dynamic equation on time scales,* J. Math. Anal. **7** (2016), no. 1, 31–38.

[19] B. Bayour and D. F. M. Torres, *Existence of solution to a local fractional nonlinear differential equation*, J. Comput. Appl. Math. **312** (2016), 127–133.

[20] A. Benaissa cherif, A. Hammoudi, F. Z. Ladrani, *Density problems in* $L^p_\Delta(\mathbb{T},\mathbb{R})$ *space*, Electro. J. Math. Anal. Appl. **1** (2) (2013), 178–187.

[21] M.Benchohra, A. Cabada and D. Seba, *An existence result for nonlinear fractional differential equations on Banach spaces*. Bound. Value Probl. (2009), Art. ID 628916, 11 pp.

[22] M. Benchohra, J. Henderson and S.K. Ntouyas, *Impulsive Differential Equations and Inclusions*. Hindawi Publishing Corporation, vol. 2, New York, 2006.

[23] M. Benchohra, J. Henderson, S.K. Ntouyas and A. Ouahab, *Existence results for functional differential equations of fractional order*, J. Math. Anal. Appl. **338** (2008), 1340-1350.

[24] B. Bendouma, A. Cabada and A. Hammoudi, *Existence of solutions for conformable fractional problems with nonlinear functional boundary conditions*, submitted.

[25] B. Bendouma, A. Cabada and A. Hammoudi, *Existence results for systems of conformable fractional differential equations,* Archivum Mathematicum (BRNO). **55** (2019), 69-83.

[26] B. Bendouma and A. Hammoudi, *A nabla conformable fractional calculus on time scales*, Electronic Journal of Mathematical Analysis and Applications. **7** (2019), no. 1, 202-216.

[27] B. Bendouma, A. Benaissa Cherif and A. Hammoudi, *Systems of first-order nabla dynamic equations on time scales*, Malaya Journal of Mathematik. **6** (2018), no. 4, 757-765.

[28] B. Bendouma and A. Hammoudi, *Existence results for systems of first-order nabla dynamic inclusions on time scales*, Arab. J. Math. Sci. **25** (2019), no. 1, 106-122.

[29] B. Bendouma and A. Hammoudi, *Nonlinear functional boundary value problems for conformable fractional dynamic equations on time scales*, Mediterr. J. Math. (2019) 16:25 https://doi.org/10.1007/s00009-019-1302-5.

[30] N. Benkhettou, A. Hammoudi and D. F. M. Torres, *Existence and uniqueness of solution for a fractional Riemann–Liouville initial value problem on time scales*, Journal of King Saud University-Science. **28** (2016), no. 1, 87-92.

[31] N. Benkhettou, S. Hassani and D.F.M. Torres, *A conformable fractional calculus on arbitrary time scales*, J. King Saud Univ. Sci. **28** (2016), no. 1, 93-98.

[32] M. Bohner, T. S. Hassan and T. Li, *Fite-Hille-Wintner-type oscillation criteria for second-order half-linear dynamic equations with deviating arguments*, Indag. Math. (N.S.) **29** (2018), 548-560.

[33] M. Bohner and A. Peterson, *Dynamic equations on time scales.* Birkhauser Boston. Boston, MA, 2001.

[34] M. Bohner and A. Peterson, *Advances in dynamic equations on time scales.* Birkhauser Boston., Boston, MA, 2003.

[35] Y.G. Borisovich, B.D. Gel'man, A.D. Myshkis and V.V. Obukhovskii, *Multivalued mappings,* (Russian) Mathematical analysis. 19, pp. 127-230, 232, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1982; english translation: J. Soviet Math. **24** (1982), 719-791.

[36] A. Cabada, *The monotone method for first-order problems with linear and nonlinear boundary conditions*, Appl. Math. Comput. **63** (1994) 163-188.

[37] A. Cabada, *Green's Functions in the Theory of Ordinary Differential Equations*, Springer Briefs Math., Springer, New York, 2014

[38] A. Cabada and Z. Hamdi, *Nonlinear fractional differential equations with integral boundary value conditions*. Applied Mathematics and Computation. **228** (2014), 251-257.

[39] A. Cabada and Z. Hamdi, *Existence results for nonlinear fractional Dirichlet problems on the right side of the first eigenvalue*, Georgian Math. J. **24** (1) (2017), 41-53.

[40] A. Cabada and D.R. Vivero, *Criterions for absolute continuity on time scales*, J. Differ. Equ. Appl. **11** (2005), 1013-1028.

[41] A. Cabada and D.R. Vivero, *Existence of solutions of first-order dynamic equations with nonlinear functional boundary value conditions*, Nonlinear Anal. Theory Methods Appl. **63** (2005), no. 5-7, 697-706.

[42] A. Cabada and D.R. Vivero, *Expression of the Lebesgue Δ-integral on time scales as a usual Lebesgue integral: application to the calculus of Δ-antiderivatives*, Math. Comput. Modelling. **43** (2006), 194-207.

[43] W. S. Chung, *Fractional Newton mechanics with conformable fractional derivative*, J. Comput. Appl. Math. **290** (2015), 150-158.

[44] Q. Dai and C.C. Tisdell, *Existence of solutions to first-order dynamic boundary value problems.* Int. J. Difference Equ. **1** (2006), no. 1, 1-17.

[45] K. Deimling, *Multivalued Differential Equations,* Walter de Gruyter, Berlin, 1992.

[46] Z. Denkowski, S. Migrski and N.S. Papageorgiou, *An introduction to nonlinear analysis : theory.Kluwer Academic Publishers, Boston*, 2003.

[47] J. Diblik, M. Ruzickov and Z. Smarda, *Wazewski's method for systems of dynamic equations on time scales,* Nonlinear Anal. **71** (2009), 1124-1131.

[48] S. Djebali, L. Gorniewicz and A. Ouahab, *Solutions Sets for Differential Equations and Inclusions,* De Gruyter Series in Nonlinear Analysis and Applications 18. Berlin: de Gruyter, 2013.

[49] M. Frigon and H. Gilbert, *Systems of first order inclusions on time scales.*Topol. Methods Nonlinear Anal. **37** (2011), 147-163.

[50] M. Frigon and M. Lotfipour, *Multiplicity results for systems of first order dynamic inclusions on time scales.* Advances in Difference Equations. **2015** (2015), 320.

[51] M. Frigon and D. O'Regan, *Existence results for initial value problems in Banach spaces*, Differential Equations Dynam. Systems. **2** (1994), 41-48.

[52] M. Frigon and D. O'Regan, *Nonlinear first order initial and periodic problems in Banach spaces*, Appl. Math. Lett. **10** (1997), 41-46.

[53] H. Gilbert, *Existence theorems for first-Order equations on time scales with Δ-Carathdory functions.* Adv. Differ. Equa. **2010** (2010), Article ID 650827, 20 pp.

[54] E. Girejko and D.F.M. Torres, *The existence of solutions for dynamic inclusions on time scales via duality .* Applied Mathematics Letters. **25** (2012), 1632-1637.

[55] A. Gökdoğan, E. Ünal and E. Çelik, *Existence and Uniqueness Theorems for Sequential Linear Conformable Fractional Differential Equations*, arXiv preprint, 2015.

[56] N.Y. Gözütok and U. Gözütok, *Multivariable Conformable Fractional Calculus*, math.CA 2017.

[57] A. Granas and J. Dugundji, *Fixed Point Theory.* Springer Monographs in Mathematics. Springer, New York 2003.

[58] T. Gulsen, E. Yilmaz and S. Goktas, *Conformable fractional Dirac system on time scales*. J. Inequal. Appl. 2017, 1-10.

[59] Sh. Guseinov, *Integration on time scales.* J. Math. Anal. Appl. **285** (1) (2003), 107-127.

[60] S. Heikkilä and A. Cabada, *On first order discontinuous differential equations with nonlinear boundary conditions*. Nonlinear World. **3** (1996), 487–503.

[61] S. Heikkilä and V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Marcel Dekker, New York, 1994.

[62] S. Hilger, *Ein Maβtkettenkalkül Ein Matkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*, Ph.D. Thesis, Universtät Wüurzburg, Wüurzburg Germany, 1988.

[63] S. Hilger, *Analysis on measure chains a unified approach to continuous and discrete calculus.* Results Math. **18** (1990), 18-56.

[64] C. J. Himmelberg, *Measurable relations.* Fund. Math. **87** (1975), 53-72.

[65] Sh. Hu and N. S. Papageorgiou, *Handbook of Multi-valued Analysis* ;Volume I: Theory, Kluwer, Dordrecht. 1997.

[66] O.S. Iyiola and E.R.Nwaeze, *Some new results on the new conformable fractional calculuswith application using D'Alambert approach*, Progr. Fract. Dier. Appl. **2** (2) (2016), 115-122.

[67] T. Jankowski, *Boundary problems for fractional differential equations*. Appl. Math. Lett. **28** (2014), 14-19.

[68] U. N. Katugampola, *A new fractional derivative with classical properties*, arXiv:1410.6535v2, 2014.

[69] R. Khaldi and A. Guezane-Lakoud, *Lyapunov inequality for a boundary value problem involving conformable derivative*, Progress in Fractional Differentiation and Applications, 2017.

[70] R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, *A new definition of fractional derivative*, J. Comput. Appl. Math. **264** (2014), 65-70.

[71] A. Kilbas, M.H. Srivastava and J.J. Trujillo, *Theory and Application of Fractional Differential Equations*, North Holland Mathematics Studies 204, 2006.

[72] R. L. Magin, *Fractional calculus in Bioengineering*, CR in Biomedical Engineering. **32** (2004), no. 1, 1-104.

[73] K.S. Miller and B. Ross, *Fractional difference calculus*, Proceedings of the International Symposium on Univalent Functions, Fractional Calculus and Their Applications Nihon University, Koriyama, Japan. (1989), 139-152.

[74] B. Mirandette, *Résultats d'existence pour des systèmes d'équations différentielles du premier ordre avec tube-solution.* M.Sc. thesis, University of Montréal. 1996.

[75] I.P. Natanson, *The Theory of Functions of Real Variable.* Ungar, New York. 1955.

[76] E.R. Nwaeze, *A Mean Value Theorem for the Conformable Fractional Calculus on Arbitrary Time Scales*, j. Progress in Fractional Differentiation and Applications. **2** (2016), no. 4, 287-291.

[77] D. O'Regan, *Existence theory for nonlinear ordinary differential equations*, Kluwer Academic Publishers Group, Dordrecht, 1997.

[78] I. Podlubny, *Fractional Differential Equations*, Academic Press: San Diego, 1999.

[79] M. Pospisil and L.P. Skripkova, *Sturm's theorems for conformable fractional differential equations*, Math. Commun. **21** (2016), 273-281.

[80] R. Precup, *Methods in Nonlinear Integral Equations*, Kluwer Academic Publisher, Dordrecht, 2002.

[81] M. Rafaqat, R. Ahmed, T. Donchev and V. Lupulescu, *Filippov-Pliss lemma for dynamical inclusions on a time scale,* Adv. Difference. Equa. **2017** (2017), 302.

[82] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives.* Theory and Applications, Gordon and Breach, Yverdon, 1993.

[83] B-R. Satco and C-O. Turcu, *Hybrid Systems By Methods of Time Scales Analysis ;Advances in dynamical systems and applications.* **8** (2013), no. 2, 317-326.

[84] A. Shi and S. Zhang, *Upper and lower solutions method and a fractional differential equation boundary value problem.* Electron. J. Qual. Theory Differ. Equat. **30** (2009), 13 pages.

[85] K. Shugui, C. Huiqing, Y. Yaqing and G. Ying, *Existence and uniqueness of the solutions for the fractional initial value problem*, Electr. J. Shanghai Normal Univ. (Natural Sciences), **45** (20016), no.3, 313-319.

[86] M. R. Sidi Ammi and D. F. M. Torres, *Existence of solution to a nonlocal conformable fractional thermistor problem*, Commun. Fac. Sci. Univ. Ank. Ser. A1, ISSN 1303-5991, 13 08 2018 arXiv:1808.04186v1.

[87] Y. Wang, J. Zhou and Y. Li, *Fractional Sobolev's Spaces on Time Scales via Conformable Fractional Calculus and Their Application to a Fractional Differential Equation on Time Scales.* Advances in Mathematical Physics. (2016), 1-21.

[88] RA. Yan, SR. Sun and ZL. Han, *Existence of solutions of boundary value problems for Caputo fractional differential equations on time scales*. Bull. Iran. Math. Soc. **42** (2) (2016), 247–262.

[89] X.-J. Yang, D. Baleanu and J.A.T. Machado, *Application of the local fractional Fourier series to fractal signals*, in Discontinuity and complexity in nonlinear physical systems, 63–89, Nonlinear Syst. Complex, Springer, Cham, 2014.

[90] I. Yaslan and O. Liceli, *Three-point boundary value problems with delta Riemann-Liouville fractional derivative on time scales*. Fract. Differ. Calc. **6** (1) (2016), 1–16.

[91] A.H. Zaidi, *Existence of solutions and convergence results for dynamic initial value problems using lower and upper solutions*, Electronic Journal of Differential Equations . **2009** (2009), no. 161, 1-13.

[92] S. Zhang and X. Su, *The existence of a solution for a fractional differential equation with nonlinear boundary conditions considered using upper and lower solutions in reverse order*. Computers Mathematics with Applications. **62** (2011), no. 3, 1269-1274.

ملخـــص

– نقدم في هذه الأطروحة نتائج وجود حلول لأنظمة غير خطية لنابلا معادلات ديناميكية و لنابلا احتواءات ديناميكية على سلالم (جداول) زمنية من الدرجة الأولى، و لأنظمة غير خطية لمعادلات تفاضلية كسرية مطابقة مرتبطة بشروط حدية. كذلك، نقدم نتائج وجود حلول لمعادلات تفاضلية كسرية مطابقة و لمعادلات ديناميكية كسرية مطابقة على سلالم زمنية مرتبطة بشروط حدية دالية غير خطية . يتم الحصول على هذه النتائج باستعمال مفاهيم أنبوب الحل لكل نظام. هذه المفاهيم تعمم الى هذه الانظمة، مفاهيم الحلول العلوية (الفوقية) والحلول السفلية (التحتية).

الكلمات المفتاحية: مشتق كسري مطابق، الحساب على الاشتقاق الكسري المطابق على السلالم الزمنية، أنظمة لنابلا معادلات ديناميكية و احتواءات ديناميكية، أنظمة لمعادلات تفاضلية كسرية مطابقة، شروط حدية غير خطية، دالة قرين، الحلول العلوية و الحلول السفلية، أنبوب الحل، نظرية النقطة الصامدة لـ شاودير، فضاءات كسرية لـ سوبوليف.

التصنيفات: 26A24 ، 26A33، 26E70، 34A08، 34A12، 34A34، 34A60، 34B15 ، 34N05، 47E05.

Abstract

- In this thesis, we present some results of existence of solutions for systems of first order nonlinear nabla dynamic equations and nabla dynamic inclusions on time scales and for systems of conformable fractional differential equations under some boundary conditions. Also, we present existence of solutions for the nonlinear conformable fractional differential equations and for the conformable fractional dynamic equations on time scales, with nonlinear functional boundary value conditions. These results are obtained by using the notion of solution-tube adapted to these systems. This notion generalizes the definition of lower and upper solution.

Key words and phrases: Conformable fractional derivative, conformable fractional calculus on time scales, systems of nabla dynamic equations and inclusions, conformable fractional dynamic equation, nonlinear boundary conditions, Green function, upper and lower solutions, solution-tube, Schauder's fixed-point theorem, fractional Sobolev's spaces.

AMS (MOS) Subject Classifications: 26A24, 26A33, 26E70, 34A08, 34A12, 34A34, 34A60, 34B15, 34N05, 47E05.

Résumé

Nous présentons dans cette thèse des résultats d'existence de solutions pour des systèmes nabla d'équations dynamiques et nabla d'inclusions dynamiques sur les échelles de temps non-linéaires d'ordre un, et pour des systèmes d'équations différentielles fractionnaires conformes non-linéaires sous certaines conditions aux limites. Aussi, nous présentons des résultats d'existence de solutions pour des équations différentielles fractionnaires conformes et des équations dynamiques fractionnaires conformes sur les échelles de temps, avec conditions fonctionnelles non-linéaires aux bords. Ces résultats sont obtenus grâce à la notion de tube-solution adaptée à ces systèmes. Celle-ci généralise la notion de sous et sur solution.

Mots Clés: Dérivée fractionnaire conforme, calcul fractionnaire conforme sur les échelles de temps, systèmes nabla d'équations dynamiques et d'inclusions dynamiques, équation dynamique fractionnaire conforme, conditions aux limites non linéaires, fonction de Green, sous et sur solutions, tube-solution, théorème du point fixe de Schauder, espaces de Sobolev fractionnaires.

Classifications AMS: 26A24, 26A33, 26E70, 34A08, 34A12, 34A34, 34A60, 34B15, 34N05, 47E05.