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THESE

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Présentée par

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Intitulée

Contrôle et stabilisation des équations aux dérivées partielles linéaires et non linéaires de type hyperbolique et parabolique.

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Devant le jury composé de :

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الملخص :

في هذه الأطروحة، قمنا بدراسة مسألة قابلية التحكم عند الصفر لمعادلات مكافئة غير مستقلة ومنحطة. تعد تقديرات كارلمان من بين الأدوات المستخدمة لدراسة إمكانية التحكم في معادلات مكافئة الانحطاط. تعمل هذه التقديرات الأخيرة على إظهار متباينة الملاحظة في المعادلات المرافقة للمعادلات المكافئة غير المستقلة والمنحطة والمكافئة لإمكانية التحكم عند الصفر في المعادلات المكافئة. لقد درسنا أيضًا الوجود الكلي (في الزمن) لحلول معادلات سيجما تطورية كسرية شبه خطية ذات معطيات صغيرة مع أو بدون كتلة. هدفنا الرئيسي هو توضيح تأثير عنصر الكتلة من ناحية، ومن ناحية أخرى تأثير زيادة صقالة المعطيات على الخصائص النوعية للحلول. باستخدام توابع بيسال المعدلة، أظهرنا بعض الاضمحلال متعدد الحدود في تقديرات Lp-Lq لحلول للمعادلات الكسرية الخطية المرفقة. بواسطة خاصية النقطة الصامدة، أثبتنا وجود حلول ذات معطيات صغيرة تحت مجموعة من القيم لـ p.

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Abstract :

In this thesis, we have studied the null controllability question of Degenerate Non autonomous Parabolic Equations. Among the tools to study the null controllability of the degenerate parabolic equations is the Carleman estimates. These last estimates are used to show the observability inequality of the adjoint parabolic equations which is equivalent to the null controllability of the degenerate parabolic equations. We have also studied the global (in time) existence of small data solutions to semi-linear fractional σ-evolution equations with mass or power non-linearity. Our main goal is to explain on the one hand the influence of the mass term and on the other hand the influence of higher regularity of the data on qualitative properties of solutions. Using modified Bessel functions we proved some polynomial decay in Lp – Lq estimates for solutions to the corresponding linear fractional σ-evolution equations. By a fixed point argument the existence of small data solutions is proved for some .admissible range of powers p.

Résumé :

Dans cette thèse, nous avons traité la question de la contrôlabilité à zéro des équations paraboliques dégénérées non autonomes. Les estimations de Carleman font partie des outils permettant d’étudier la contrôlabilité à zéro des équations paraboliques dégénérées. Ces dernières estimations servent à montrer l'inégalité d'observabilité des équations paraboliques adjointes qui est équivalente à la contrôlabilité à zéro des équations paraboliques dégénérées. Nous avons aussi étudié l'existence globale (dans le temps) de solutions de petites données pour des équations fractionnaires σ-évolution semi-linéaires avec ou sans terme de masse. Notre objectif principal est d'expliquer d'une part l'influence du terme de masse et d'autre part l'influence de la régularité supérieure des données sur les propriétés qualitatives des solutions. En utilisant des fonctions de Bessel modifiées, nous avons démontré une certaine décroissance polynomiale dans les estimations Lp - Lq pour les solutions aux équations fractionnaires linéaires correspondantes. Par un argument de point fixe, l'existence de solutions de données réduites est prouvée pour une gamme .de puissances admissibles p.

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Kainane Mezadek Abdelatif.

Dédicace

Je dédie ce travail à :
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A Imad, à Sid Ahmed, à Roufaida et au petit Abderrahmane !!!

Contents

Notations

Ω : Open set of $\mathbb{R}^n$.

$\overline{\Omega}$: The closure of Ω.

$\Gamma = \partial\Omega$: The boundary of Ω.

X, Y : Banach spaces.

$\|\cdot\|_X$: Norm in X.

H : Hilbert space.

$(\cdot,\cdot)$: The scalar product in the Hilbert space H.

$\mathcal{L}(X,Y)$: The space of linear and continuous applications of X in Y.

$C(\Omega) =$ the set of continuous functions $u : \Omega \longrightarrow \mathbb{R}$.

$C^k(\Omega) =$ the set of functions $u : \Omega \longrightarrow \mathbb{R}$ k times continous differentiable in Ω.

$C^\infty(\Omega)$: The space of indefinitely differentiable functions in Ω.

$\operatorname{supp}(u) = \overline{\{x \in \Omega : u(x) \neq 0\}}$.

$\mathcal{D}(\Omega) = \{u \in C^\infty(\Omega) : \operatorname{supp}(u) \subset \Omega, \text{ and } \operatorname{supp}(u) \text{ is compact}\}$.

$L^p(\Omega) = \{u : \Omega \to \mathbb{R}, u \text{ measurable and } \int_\Omega |u(x)|^p dx < \infty\}, 1 \leq p < \infty$.

$L^\infty(\Omega) = \{u : \Omega \to \mathbb{R}, \text{measurable and } \exists C \geq 0 : |u(x)| \leq C, \text{ a.e. in } \Omega\}$.

$H^1(\Omega) = \{u \in L^2(\Omega) : \dfrac{\partial u}{\partial x_i} \in L^2(\Omega), 1 \leq i \leq n\}$.

$H_0^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma\}$.

$H^{-1}(\Omega) =$ The dual space of $H_0^1(\Omega)$.

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}\right) = \left(\frac{\partial u}{\partial x_j}\right)_{1 \leq j \leq n}.$$

$W^{1,p}(\Omega) = \{u : u \in L^p(\Omega), \nabla u \in (L^p(\Omega))^n\}, 1 \leq p < \infty$.

$H_a^1(0,1) = \{u \in L^2(0,1) \cap H_{loc}^1(0,1) : \int_0^1 a(x) u_x^2 dx < \infty\}$.

$H_q^\gamma(\mathbb{R}^n) := \left\{f \in S'(\mathbb{R}^n) : \|f\|_{H_q^\gamma} := \|F^{-1}(\langle\xi\rangle^\gamma F(f))\|_{L^q} < \infty\right\}$.

$\dot{H}_q^\gamma(\mathbb{R}^n) := \{f \in S'(\mathbb{R}^n) : \|f\|_{\dot{H}_q^\gamma} := \|F^{-1}(|\xi|4F(f))\|_{L^q} < \infty\}$.

Introduction

In the first part of this thesis, we deal with the null controllability question of Degenerate Non autonomous Parabolic Equations. In general, the null controllability problem of partial deferential equations can be treated by different means like the moments method, Hilbert uniqueness method (called briefly HUM), multipliers method, microlocal analysis, spectral inequalities, fundamental solutions, controllability via stabilization or energy estimates (see for example [34] and [39]). The null controllability of nondegenerate parabolic equations have been widely studied in the last years, see in particular [9], [27], [30], [41], [43]. On the other hand, very few results are known in the case of autonomous degenerate equations; see [4], [5], [6], [11], [42]. Among the tools to study the null controllability of the degenerate parabolic equations is the Carleman estimates. These last estimates are used to show the observability inequality of the adjoint parabolic equations which is equivalent to the null controllability of the degenerate parabolic equations. The Carleman estimates are the main results of the above references. Recently in [45], the authors established a new Carleman estimate for the autonomous degenerate equations under some general conditions on the degenerate diffusion coefficient a. In the first part of this theses, we are interested to study the null controllability for the one dimensional degenerate non autonomous parabolic equation

$$u_t - M(t)(a(x)u_x)_x = h\chi_\omega, \qquad (x,t) \in Q = (0,1) \times (0,T),$$

where $\omega = (x_1, x_2)$ is a small nonempty open subset in $(0,1)$, $h \in L^2(\omega \times (0,T))$, the diffusion coefficients $a(\cdot)$ is degenerate at $x = 0$ and $M(\cdot)$ is non degenerate on $[0,T]$. Also the boundary conditions are considered to be Dirichlet or Neumann type related to the degeneracy rate of $a(\cdot)$. Under some conditions on the functions $a(\cdot)$ and $M(\cdot)$, we prove some global Carleman estimates which will yield the observability inequality of the associated adjoint system and equivalently the null controllability of our parabolic equation.

In the second part of this thesis we study the global (in time) existence of small data solutions to semi-linear fractional $\sigma-$evolution equations with mass or power non-linearity. The concept of non-integer derivative and integral, as a generalization of the traditional integer order differential and integral calculus was mentioned in 1695 by Leibniz and l'Hospital, but the first definition of the fractional derivative and integral was introduced at the end of the nineteenth century by

Liouville and Riemann. The most important advantage of fractional derivatives compared with integer derivatives is that it describes the property of memory and heredity of various materials and processes. In recent years, fractional calculus has attracted many physicists, mathematicians, engineers and notable contributions have been made to both theory and applications of fractional differential equations. It has been found that the differential equations involving fractional derivatives in time are more realistic to describe many phenomena in practical cases than those of integer order in time. For more details about fractional calculus and fractional differential equations we refer to the monographs by Miller and Ross [44], Podlubny [50], Hilfer [35] and Kilbas et al. [38]. Since fractional semilinear evolution equations are abstract formulations for many problems arising in engineering and physics, fractional evolution equations have attracted increasing attention in recent years, see [21]-[24] and the references therein.

This thesis is decomposed into 6 chapters and is organized as follows. In chapter 1 we recall some preliminary results on the spaces of Lebesgue, the space of Sobolev and some properties related to these spaces. The Chapter 2 is devoted to the study of the theory of simigroup, we give some definitions and some properties of this theory as well as spectral theory and we end with a theorem of Hille-Yosida which plays a very important role in the existence , The uniqueness and the regularity of some Cauchy problems. In chapter 3, we give the various notions of controllability, the various characterizations related to the notion of controllability and thus the comparison between these notions. The chapter 4 is devoted to the notion of Carleman estimate and the importance of using this estimate to deduce an observability inequality witch will be equivalent to the null controllability. In the chapter 5, we are interested to the null controllability of degenerate non autonomous parabolic equations in one-dimensional space. For some hypothesizes on the diffusion coefficient, we proof a Carleman estimate witch will be used to show the observability inequality and consequently the null controllability. Finally, In chapter 6 we study the global (in time) existence of small data solutions and stabilization to the following semi-linear fractional $\sigma-$evolution equations with mass or power non-linearity

$$\begin{aligned} &\partial_t^{1+\alpha} u + (-\Delta)^\sigma u + m^2 u = |u|^p, \\ &u(x,0) = u_0(x), \quad u_t(0,x) = 0, \end{aligned}$$

Our main goal is to explain on the one hand the influence of the mass term and on the other hand the influence of higher regularity of the data on qualitative properties of solutions. Using modified Bessel functions we prove some polynomial decay in $L^p - L^q$ estimates for solutions to the corresponding linear fractional $\sigma-$evolution equations with vanishing right-hand sides. By a fixed point argument the existence of small data solutions is proved for some admissible range of powers p.

Chapter 1

Preliminary results

1.1 L^p spaces

We consider $\Omega \subset \mathbb{R}^n$ an open. We denote by $L^1(\Omega)$ the space (of the equivalence classes) of functions integrable in Lebesgue's sense on a values in $\mathbb{R}$. That is, as we usually do, we confuse two functions that coincide Almost everywhere (a.e. in abbreviated).
For $u \in L^1(\Omega)$ we note

$$\|u\|_{L^1(\Omega)} = \int_\Omega |u(x)|dx$$

Definition 1.1. *For $1 \leq p < \infty$, we put*

$$L^p(\Omega) = \{u : \Omega \to \mathbb{R}, u \text{ measurable and } \int_\Omega |u(x)|^p dx < \infty\}.$$

The mapping $\|\cdot\|_{L^p(\Omega)}$ where

$$\|u\|_{L^p(\Omega)} = \Big(\int_\Omega |u(x)|^p dx\Big)^{\frac{1}{p}} \tag{1.1}$$

defines a norm in $L^p(\Omega)$.

Definition 1.2. *We put*

$$L^\infty(\Omega) = \{u : \Omega \to \mathbb{R}, \text{measurable and } \exists C \geq 0 : |u(x)| \leq C, \text{ a.e. in } \Omega\}.$$

Theorem 1.3. *(Hölder inequality) Let $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$ with $1 \leq p, q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ (q denotes the conjugate exponent of p). Then $uv \in L^1(\Omega)$ and we have*

$$\|uv\|_{L^1} \leq \|u\|_{L^p}\|v\|_{L^q}.$$

Theorem 1.4. *(Fischer-Riesz)* $(L^p, \|\cdot\|_{L^p})$ *is a Banach space for all* $1 \leq p \leq \infty$.
In the particular case $p = 2$, *the relation*

$$(u, v) = \int_\Omega u(x)v(x)dx, \ \forall u, v \in L^2(\Omega), \tag{1.2}$$

defines a scalar product in $L^2(\Omega)$, *whose associated norm is none other than the norm* $\|\cdot\|_{L^2}$ *defined in (1.1).*

Proposition 1.5. *The space* $L^2(\Omega)$ *provided with the scalar product (1.2) is a Hilbert space.*

1.2 Sobolev spaces

Sobolev spaces of order 1

Definition 1.6. *We call Sobolev space of order 1 on* Ω, *the space*

$$H^1(\Omega) = \{u \in L^2(\Omega) : \frac{\partial u}{\partial x_i} \in L^2(\Omega), 1 \leq i \leq n\}.$$

$H^1(\Omega)$ is provided with the scalar product

$$(u, v)_{1,\Omega} = \int_\Omega \Big(uv + \sum_{i=1}^n \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\Big)dx = (u, v) + (\nabla u, \nabla v), \ \forall u, v \in H^1(\Omega). \tag{1.3}$$

And we note

$$\|u\|_{1,\Omega} = \Big(\int_\Omega \Big(u^2 + \sum_{i=1}^n (\frac{\partial u}{\partial x_i})^2\Big)dx\Big)^{\frac{1}{2}} = (\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2)^{\frac{1}{2}}, \tag{1.4}$$

the corresponding norm.

Proposition 1.7. *The space* $H^1(\Omega)$ *provided with the scalar product (1.3) is a Hilbert space.*

Definition 1.8. *Let* $\mathcal{D}(\Omega)$ *denote the vector space of infinitely differentiable functions on* Ω *with compact support in* Ω. *We defend* $H_0^1(\Omega)$ *as the adherence of* $\mathcal{D}(\Omega)$ *in* $H^1(\Omega)$, *that is to say*

$$H_0^1(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^1(\Omega)}$$

Remark 1.9. *If* Ω *is bounded,* $\mathcal{D}(\Omega)$ *is not dense in* $H^1(\Omega)$ *and we have* $H_0^1(\Omega) \subset H^1(\Omega)$ *with strict inclusion; On the other hand, if* $\Omega = \mathbb{R}^n$, $\mathcal{D}(\mathbb{R}^n)$ *is dense in* $H^1(\mathbb{R}^n)$, *that is to say* $H_0^1(\mathbb{R}^n) = H^1(\mathbb{R}^n)$.

Proposition 1.10. *The space* $H_0^1(\Omega)$ *provided with the norm induced by* $H^1(\Omega)$ *is a Hilbert space.*

Theorem 1.11. *(Trace Theorem) Let Ω be an open boundary of class C^1, there exists a continuous linear operator $\gamma_0 \in L(H^1(\Omega), L^2(\partial\Omega))$ such that*

$$\gamma_0 u = u|_{\partial\Omega}, \ \forall u \in C^1(\overline{\Omega}).$$

$L^2(\partial\Omega)$ is the space of (class of) real functions, square integrable on $\partial\Omega$.
According to the trace theorem, we can give the following characterization of the functions of $H_0^1(\Omega)$ which explains the important role played by the latter in the resolution of equations with partial differentials coupled with boundary conditions, that is to say when The value u is prescribed on the boundary $\partial\Omega$.

Definition 1.12. *The functions of $H_0^1(\Omega)$ are the functions $H^1(\Omega)$ that vanish on the boundary $\Gamma = \partial\Omega$,*

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \ on\ \Gamma\} = the\ kernel\ of\ \gamma_0.$$

Remark 1.13. *We denote the dual space of $H_0^1(\Omega)$ by $H^{-1}(\Omega)$.*

Theorem 1.14. *(of Rellich) If Ω is an open boundary of class C^1, then the canonical injection of $H_0^1(\Omega)$ in $L^2(\Omega)$ is compact; That is to say, any boundary set of $H_0^1(\Omega)$ is relatively compact in $L^2(\Omega)$.*

We can identify $L^2(\Omega)$ and its dual, then we have the inclusions:

$$H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega),$$

with continuous and dense injections (see [8]).

Sobolev spaces $W^{1,p}$

Let Ω be an open set in $\mathbb{R}^n$, coordinates in $\mathbb{R}^n$ are denoted by $x = (x_1, x_2, \dots, x_n)$.

Definition 1.15. 1. *For $u : \mathbb{R}^n \longrightarrow \mathbb{R}$, the partial derivative of u with respect to the variable $x_j, j = 1, 2, \dots, n$ is denoted by $\frac{\partial u}{\partial x_j}$, and then*

$$\nabla u := \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n}\right) = \left(\frac{\partial u}{\partial x_j}\right)_{1 \le j \le n}.$$

2. *If $u : \Omega \longrightarrow \mathbb{R}^m$, then we note $u = (u^1, u^2, \dots, u^m)$ the coordinates in the space $\mathbb{R}^m$. The gradient of u can, then, be written by:*

$$\nabla u := \left(\frac{\partial u^i}{\partial x_j}\right)_{1 \le j \le n}^{1 \le i \le m} = \left(\nabla u^1, \nabla u^2, \dots, \nabla u^m\right)^T,$$

where T stands for the transposition. In this case, ∇u is a real matrix of n rows and m columns.

Definition 1.16. *Let Ω be an open set in $\mathbb{R}^n$. $C(\Omega)$ the set of continuous functions $u : \Omega \longrightarrow \mathbb{R}$. $C(\overline{\Omega})$ is the set of continuous functions $u : \Omega \longrightarrow \mathbb{R}$ which can be continuously extended to $\overline{\Omega}$. The norm over $C(\overline{\Omega})$ is given by:*

$$||u||_C = \sup_{x\in\overline{\Omega}} |u(x)|$$

The support of a function $u : \Omega \longrightarrow \mathbb{R}$ is defined by

$$supp\ u := \overline{\{x \in \Omega : u(x) \neq 0\}}$$

Definition 1.17. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $s \in \mathbb{N}$, and $1 \leq p \leq \infty$. We define the Sobolev spaces $W^{1,p}(\Omega)$as follows:*

$$W^{1,p}(\Omega) := \{u : u \in L^p(\Omega), \nabla u \in (L^p(\Omega))^n\}$$

The spaces $W^{1,p}(\Omega)$ are Banach spaces, with respect to the norm

$$||u||_{W^{1,p}} = (||u||^p_{L^p} + ||\nabla u||^p_{L^p})^{\frac{1}{p}} \ \ if\ 1 \leq p < \infty$$

and

$$||u||_{W^{1,\infty}} = \max\{||u||_{L^\infty}, ||\nabla u||_{L^\infty}\} \ \ if\ p = \infty.$$

The closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$ is usually denoted by $W_0^{1,p}(\Omega)$.

Remark 1.18. 1. *If $u : \Omega \longrightarrow \mathbb{R}^m$ is a vector-valued function, the Sobolev spaces are denoted by $W^{1,p}(\Omega, \mathbb{R}^m)$,*

2. *$W^{1,p}(\Omega)$ is separable if $1 \leq p < \infty$, and reflexive if $1 < p < \infty$.*

3. *The space of C^∞ function is dense in $W^{1,p}(\Omega)$ with respect to the norm defined below.*

Theorem 1.19. ***(Poincaré's Inequality)*** *Let Ω be an open bounded set in $\mathbb{R}^n$ with Lipshitz boundary. $C_0^\infty(\Omega)$ denotes the set of $C^\infty(\Omega)$ functions with a compact support in Ω. There exists a positive constant $C(p)$, which depends only on p, such that :*

$$\forall u \in C_0^\infty(\Omega) : ||u||_{L^p} \leq C(p)||\nabla u||_{L^p}.$$

Note that, by density, Poincarés inequality is still true over $W_0^{1,p}(\Omega)$, for every $1 \leq p < \infty$.

1.3 $L^p(a, b; X)$ spaces

We give a brief introduction to the integrability in Bochner's sense of the functions defended over an interval, with vectorial value.

let X be a Banach space and $-\infty < a < b < \infty$. A function $f : [a, b] \to X$ is simple if it exists a measurable subsets $A_1, A_2, \cdots, A_n$ of $[a, b]$ and $x_1, x_2, \cdots, x_n$ of X such that

$$f(t) = \sum_{i=1}^{n} \chi_{A_i}(t) x_i,$$

where χ_A is the characteristic function of A. We will say that f is measurable if there is a sequence of simple functions f_k, $f_k : [a, b] \to X$ such that $f_k \to f$, a.e. in $[a, b]$.
A measurable function f is said to be integrable (in the sense of Bochner) if there exists a sequence of simple functions f_k, $f_k : [a, b] \to X$ such that

$$\lim_{k\to\infty} \int_a^b \|f(t) - f_k(t)\|_X dt = 0.$$

in this case $\int_a^b f(t)dt$ is defined by

$$\int_a^b f(t)dt = \lim_{k\to\infty} \int_a^b f_k(t)dt$$

Theorem 1.20. *(Bochner) A measurable function $f : [a, b] \to X$ is integrable if and only if $\|f(\cdot)\|_X \in L^1(a, b)$.*

For $1 \leq p \leq \infty$, we put

$$L^p(a, b; X) = \{f : [a, b] \to X, \text{ integrable and such that } \|f(\cdot)\|_X \in L^p(a, b)\}$$

With the norm

$$\|u\|_{L^p(a,b;X)} = \Big(\int_a^b \|f(t)\|_X^p dt\Big)^{\frac{1}{p}}, \text{ if } p < \infty,$$

and

$$\|u\|_{L^\infty(a,b;X)} = \inf\{C : \|f(t)\|_X \leq C, \text{ a.e. on } [a, b]\}.$$

Proposition 1.21. *$L^p(a, b; X)$ is a Banach space, for all $1 \leq p \leq \infty$.*

Remark 1.22. *If X is a Hilbert space with the scalar product $(\cdot, \cdot)_X$, then $L^2(a, b; X)$ is a Hilbert space, for the scalar product*

$$(u, v)_{L^2(a,b;X)} = \int_a^b (u(t), v(t))_X dt.$$

Chapter 2

Semigroups

In the following $(X, \|\cdot\|)$ denotes a Banach space. In this chapter, we will recall the definitions of a C_0-semigroup as well as the most important properties.

2.1 C_0-semigroup

Definition 2.1. *1. We call C_0-semigroup of linear operators on X a family $(S(t))_{t\geq 0} \subset \mathcal{L}(X)$ verifying the following properties*

1. $S(0) = I$, *with I the identity of $\mathcal{L}(X)$,*
2. $S(t+s) = S(t)S(s)$, $\forall t, s \geq 0$, *(Property of the semigroup),*
3. $\lim\limits_{t\to 0^+} S(t)x = x$, $\forall x \in X$ *(Property of the C_0-semigroup).*

Moreover, if $\|S(t)\|_{\mathcal{L}(X)} \leq 1$, $\forall t \geq 0$, we say that $(S(t))_{t\geq 0}$ is C_0-semigroup of contractions.
2. We call C_0-group of linear operators on X, a family $(S(t))_{t\in\mathbb{R}} \subset \mathcal{L}(X)$ verifying the 3 previous properties only we replace $(t, s \geq 0)$ by $(t, s \in \mathbb{R})$ and $t \to 0^+$ by $t \to 0$.

Proposition 2.2. *if $(S(t))_{t\geq 0}$ is C_0-semigroup in X, then*

$$\exists \omega \in \mathbb{R}, \exists M \geq 1 : \|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}, \ \forall t \geq 0.$$

Definition 2.3. *The infinitesimal generator of the C_0-semigroup $(S(t))_{t\geq 0}$, an operator A defined on the set:*

$$D(A) = \{x \in X : \lim_{t\to 0^+} \frac{S(t)x - x}{t} \ \text{ exist in } X\},$$

and $Ax = \lim\limits_{t\to 0^+} \frac{S(t)x-x}{t}$ for all $x \in D(A)$.
if we replace $t \to 0^+$ by $t \to 0$ we say that A infinitesimal generator of C_0-group $(S(t))_{t\in\mathbb{R}}$.

Examples 2.4. *Let* $C_{ub}([0,\infty)) = \{f : [0,\infty) \longrightarrow \mathbb{R}, f$ *uniformly continuous and bounded*$\}$, *with* $\|f\|_{C_{ub}([0,\infty))} \doteq \sup_{s\in[0,\infty)} |f(s)|$.
It is clear that $(C_{ub}([0,\infty)), \|\cdot\|_{C_{ub}([0,\infty))})$ *is a Banach space. Let,*

$$(S(t)f)(x) = f(x+t),\ \ \forall t, x \geq 0 \ \text{ and } \forall f \in C_{ub}([0,\infty)).$$

We verify that, $(S(t))_{t\geq 0}$ *is* C_0*-semigroup of contractions in* $C_{ub}([0,\infty))$ *and that its generator infinitesimal is given by:*

$$D(A) = \{f : f \text{ and } f' \in C_{ub}([0,\infty))\} \text{ and } Af = f'.$$

Proposition 2.5. *Let* $(S(t))_{t\geq 0}$ *be a* C_0*-semigroup of generator* A *in* X *then we have :*

1. $A : D(A) \subset X \longrightarrow X$ *is a linear operator.*
2. $\forall x \in X : \int_0^t S(s)xds \in D(A), \forall t \geq 0$ *and* $A\int_0^t S(s)xds = S(t)x - x$.
3. *If* $x \in D(A)$, *then* $S(t)x - x = A\int_0^t S(s)xds = \int_0^t S(s)Axds$.
4. *For all* $x \in D(A)$, *we have*
$$\frac{d}{dt}S(t)x = AS(t)x = S(t)Ax,\ \forall t \geq 0.$$

Remark 2.6. *Let* $(S(t))_{t\geq 0}$ *be a* C_0*-semigroup in* X *then we have*

$$\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} S(s)xds = S(t)x.$$

Definition 2.7. *Let* $A : D(A) \to X$ *is an unbounded linear operator, we say that* A *is closed if its graph* $Gr(A) = \{(x, Ax) : x \in D(A)\}$ *is closed in* $X \times X$

Theorem 2.8. *The generator of a* C_0*-semigroup in* X *is closed and its domain is dense in* X.

Theorem 2.9. *Let* A *be the infinitesimal generator of the* C_0*-semigroup* $(S(t))_{t\geq 0}$ *in* X. *If* $D(A^n)$ *the domain of* A^n *is defined by:*

$$D(A^n) = \{x \in D(A) : Ax \in D(A^{n-1})\}$$

then $\bigcap_{n\in\mathbb{N}^*} D(A^n)$ *is dense in* X.

Proposition 2.10. *Let $(A, D(A))$ be a generator of a C_0-semigroup $(T(t))_{t\geq 0}$. Then, for all $\lambda \in \mathbb{C}$ and $t \geq 0$, we have*

$$e^{-\lambda t}T(t)x - x = \begin{cases} (A - \lambda I)\int_0^t T(s)x ds, \ \text{if}\ x \in X. \\ \int_0^t T(s)(A - \lambda I)x ds, \ \text{if}\ x \in D(A). \end{cases}$$

Proof. It is enough to apply the Proposition 2.5 for $S(t) = e^{-\lambda t}T(t), t \geq 0$ with generator $(A - \lambda I, D(A))$. □

2.2 Resolvent and spectrum

Notations

For a linear operator not necessarily bounded $A : D(A) \subset X \to X$, we Note by:

1. $\rho(A) = \{\lambda \in \mathbb{C} : \lambda I - A$ is invertible in $\mathcal{L}(X)\}$, the resolvent set of A.
2. $R(\lambda, A) : \rho(A) \to \mathcal{L}(X),\ R(\lambda, A) = (\lambda I - A)^{-1}$ the resolvent of the linear operator A in λ.
3. $\sigma(A) \doteq \mathbb{C} \setminus \rho(A)$ the spectrum of A.
4. $\sigma_r(A) = \{\lambda \in \mathbb{C} : \mathrm{Im}(\lambda I - A)$ is not dense in $X\}$ the residual spectrum of A.
5. $\sigma_p(A) = \{\lambda \in \mathbb{C} : \lambda I - A$ is not injective in $X\}$ the punctual spectrum of A.

2.3 Hille-Yosida theorem

Let $(S(t))_{t\geq 0}$ be a C_0-semigroup in Banach space X. From Proposition 2.2 it follows that there are constants $\exists \omega \in \mathbb{R}, \exists M \geq 1$ such that $\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t},\ \forall t \geq 0$. If $\omega = 0$ then $(S(t))_{t\geq 0}$ is called uniformly bounded and if moreover $M = 1$ it is called a C_0-semigroup of contractions. This section is devoted to the characterization of the infinitesimal generators of C_0-semigroup of contractions. Conditions on the behavior of the resolvent of an operator A, which are necessary and sufficient for A to be the infinitesimal generator of a C_0-semigroup of contractions, are given.

Theorem 2.11. *(Hille-Yosida). A linear (unbounded) operator A is the infinitesimal generator of a C_0-semigroup of contractions $(S(t))_{t\geq 0}$ if and only If*

1. *A is closed and $D(A)$ is dense in X.*

2. *The resolvent set* $\rho(A)$ *of* A *contains* $]0, \infty[$ *and for every* $\lambda > 0$

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda}.$$

The proof of this theorem is in [48].

Corollary 2.12. *A linear operator* A *is the infinitesimal generator of a* C_0*-semigroup satisfying* $\|S(t)\| \leq e^{\omega t}$*, for all* $t \geq 0$*, if and only If*

1. A *is closed and* $D(A)$ *is dense in* X*.*

2. *The resolvent set* $\rho(A)$ *of* A *contains* $]\omega, \infty[$ *and for every* $\lambda > \omega$

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda - \omega}.$$

Corollary 2.13. *A linear operator* A *is the infinitesimal generator of a* C_0*-semigroup satisfying* $\|S(t)\| \leq Me^{\omega t}$*, for all* $t \geq 0$*, if and only If*

1. A *is closed and* $D(A)$ *is dense in* X*.*

2. *The resolvent set* $\rho(A)$ *of* A *contains* $]\omega, \infty[$ *and for every* $\lambda > \omega$

$$\|R(\lambda, A)\| \leq \frac{M}{\lambda - \omega}.$$

Definition 2.14. 1. *An operator* $(A, D(A))$ *is said to be dissipative if:*

$$\forall x \in D(A), \forall \lambda > 0 : \|(\lambda I - A)x\| \geq \lambda \|x\|.$$

2. *If more* $\lambda I - A$ *is surjective* ($\forall \lambda > 0$)*, we say that* A *is m-dissipative in* X*.*

Remarks 2.15. 1. *If point 2 of Definition 2.14 is verified, then the operator* $\lambda I - A$ *is a Isomorphism of* $D(A)$ *into* X*.*

2. *In the case where* X *is a Hilbert space with the scalar product* $(\cdot, \cdot)$*, we can show that* A *is dissipative if and only if* $Re(Ax, x) \leq 0$*,* $\forall x \in D(A)$*.*

Then we have the following Theorem in the framework of a space of Hilbert.

Theorem 2.16. *Let* A *be a linear operator of domain* $D(A)$ *dense in a Hilbert space* H*. Then* A *is a generator of a unitary group* $(S(t))_{t \geq 0}$ *of* H *if and only if* A *is skew adjoint, a.e.,* $A^* = -A$*.*

Remark 2.17. *Let* $A : D(A) \subset H \longrightarrow H$ *is an unbounded linear operator, then* A *is m-dissipative of dense domain in* H *if and only if* A *is the infinitesimal generator of a* C_0*-semigroup of contractions.*

Proposition 2.18. *Let $A : D(A) \subset H \to H$ be a dissipative and dense domain in a Hilbert space H. Then,*

1. *If A is closed, then A^*, its adjoint, is dissipative if and only if A is M-dissipative of H.*

2. *If A is self-adjoint then A is m-dissipative of H.*

Theorem 2.19 (Lumer and Phillips 1961)**.** *Let $(A, D(A))$ a closed operator with dense domain $D(A)$. Then the following propositions are equivalent*

(a) A is the generator of C_0-semigroup of contractions in X.

(b) A is dissipative and there exists $\lambda \geq 0 (or\, \forall\lambda \geq 0)\ Im(\lambda I - A) = X$

Chapter 3

Controllability of Distributed Systems

3.1 Description of the system

Consider the systems described by the operational differential equation of state

$$\begin{cases} \frac{\partial}{\partial t} y(t) = Ay(t) + Bu(t) & \text{in } \Omega \times (0,T) \\ y(0) = y_0, & \text{in } \Omega, \end{cases} \tag{3.1}$$

where Ω is open in $\mathbb{R}^n$ represent the geometric domain of the system,$T > 0$ and

1. $A \in L(V, H)$ is a differential operator generates a C_0-semigroup $(S(t))_{t\geq 0}$ on the space of state H,
2. $B \in L(U, H)$ with U is a space of Hilbert called control space,
3. $u \in L^2(0, T; U)$ called the control,
4. $y_0 \in L^2(\Omega)$ is initial data.

The representation of the solution of the system (3.1) is formally given by

$$y_u(t) = S(t)y_0 + \int_0^t S(t-s)Bu(s)ds \tag{3.2}$$

where $y_u(t)(x) = y(x, t, u)$. We consider the linear and bounded operator $L_t : L^2(0, T; U) \longrightarrow H$ defined by

$$L_t u = \int_0^t S(t-s)Bu(s)ds, \quad \forall u \in L^2(0, T; U). \tag{3.3}$$

L_t is called controllability operator.

3.2 Controllability and different notions of controllability

The problem of controllability consists in the possibility of transferring the state of a system in a finite time, from an initial state to a desired state chosen a priori. In the case of finite-dimensional systems, The Kalman condition necessary and sufficient condition for the controllability. For the distributed systems one is led to consider various degrees of controllability. We will introduce the following notions of controllability: exact, approximate, to the trajectories, null and finally regional controllability

3.3 Exact Controllability

Definition 3.1. *The system (3.1) is said to be exactly controllable in H on $[0, T]$ if*

$$\forall y_d \in H, \exists u \in L^2(0, T; U) : y(y_0, T) = y_d. \tag{3.4}$$

Remark 3.2. *The above definition is equivalent to $Im(L_T) = H$.*

Proposition 3.3. *The system (3.1) is exactly controllable in H over $[0, T]$ if and only If:*

$$\exists c > 0 : \|\varphi\| \leq c\|B^* S^*(\cdot)\varphi\|_{L^2(0,T;U^*)}, \quad \forall \varphi \in H^*. \tag{3.5}$$

The proof is based on the following Lemma witch is more general result:

Lemma 3.4. *Let V, W and Z be Banach spaces reflexive, and $F \in L(V, Z), G \in L(W, Z)$. Then, the following assertions are equivalent,*

1. *$ImF \subset ImG$.*
2. *$\exists c > 0 : \|F^* y^*\|_{V^*} \leq c\|G^* y^*\|_{W^*}, \quad \forall y^* \in Z^*$.*

Proof. We take $V = Z = H, W = L^2(0, T; U)$ and $F = Id_H, G = L_T$. Let $y^* \in H^*$, then for all $u \in L^2(0, T; U)$ we have

$$\begin{aligned}
\langle L_T^* y^*, u\rangle = \langle y^*, L_T u\rangle &= \langle y^*, \int_0^T S(T-s)Bu(s)ds\rangle \\
&= \int_0^T \langle y^*, S(t-s)Bu(s)\rangle ds \\
&= \int_0^T \langle B^* S^*(T-s)y^*, u(s)\rangle ds \\
&= \langle B^* S^*(T-\cdot)y^*, u\rangle.
\end{aligned}$$

So $L_T^* y^* = B^* S^*(T - \cdot) y^*$. Now we suppose that the system (3.1) is exactly controllable and let $y \in ImF = H$. For $y_d = S(T)y_0 + y$, there exist $u \in L^2(0,T;U)$ such that $y_u(T) = y_d$, then we obtain

$$\int_0^T S(T-s)Bu(s)ds = y.$$

We have $L_T u = y$. Consequently, $ImF \subset ImL_T$. Then from the Lemma 3.4, we obtain the enequality (3.5). Conversely, we suppose that (3.5) is verified, then by lemma 3.4, $ImF = H \subset ImL_T$ and therefore we have the exact controllability of (3.1). □

3.4 Approximate Controllability

Definition 3.5. *The system (3.1) is said to be approximate (week) controllable in H on $[0,T]$ if*

$$\forall y_d \in H, \forall \varepsilon > 0 \exists u \in L^2(0,T;U) : \|y(T) - y_d\| \leq \varepsilon. \tag{3.6}$$

Proposition 3.6. *The following properties are equivalent*

1. *The system (3.1) is said to be approximate (week) controllable in H on $[0,T]$.*

2. $\overline{ImL_T} = H$.

3. $\ker(L_T^*) = \ker(L_T L_T^*) = \{0\}$.

4. $\Big(\langle B^* S^*(s)y, v\rangle = 0,\ \forall s \in [0,T],\ \forall v \in U\Big) \Rightarrow y = 0.$

5. *If the semigroup $(S(t))_{t\geq 0}$ is analytic, then we have:*

$$\exists s \in [0,T] : \overline{\cup_{n\in\mathbb{N}} Im(A^n S(s)B)} = H.$$

Proof. $(1) \Rightarrow (2)$: The system (3.1) is approximately controllable on $[0,T]$ .a.e.

$$\forall y_d \in H, \forall \varepsilon > 0 \exists u \in L^2(0,T;U) : \|y(T) - y_d\| \leq \varepsilon. \tag{3.7}$$

This is equivalent to

$$\forall y_d \in H, \forall \varepsilon > 0, \exists u \in L^2(0,T;U) : \|L_T u - y_d\| \leq \varepsilon. \tag{3.8}$$

Consequently $\overline{ImL_T} = H$.
$(2) \Rightarrow (3)$: Let $y_i^* n H^*$ such that $L_T^* y^* = 0$, then we have

$$\langle y^*, L_T u\rangle = 0,\ \ \forall u \in L^2(0,T;U)$$

That implies $y^* \in \overline{(ImL_T)}^{\perp} = \{0\}$. So, $y^* = 0$, we deduce that $\ker L_T^* = \{0\}$. On other hand, for $x_i nH$ such that $(L_T L_T^*)x = 0$. Then

$$\langle (L_T L_T^*)x, y\rangle = 0, \ \forall y \in H.$$

In particular

$$\langle (L_T L_T^*)x, x\rangle = \langle (L_T^*)x, L_T^* x\rangle = \|L_T^* x\|^2 = 0.$$

So, $L_T^* x = 0$. Then $x = 0$ and consequently $\ker(L_T L_T^*) = \{0\}$.
(3) $\Rightarrow$ (4) : Suppose that $\ker L_T^* = \ker(L_T L_T^*) = \{0\}$. From above, we have $L_T^* y = B^* S^*(T - \cdot)y$, $\forall y \in H$. If $\Big(\langle B^* S^*(s)y, v\rangle = 0, \ \forall s \in [0, T], \ \forall v \in U$, then $\langle L_T^* y, v\rangle = 0 \ \forall v \in U$. And as $\ker L_T^* = \{0\}$, we deduce that $y = 0$.
(4) $\Rightarrow$ (5) : Suppose that for all $s \in [0, T]$ such that

$$\overline{\cup_{n\in\mathbb{N}} Im(A^n S(s)B)} \neq H.$$

Then

$$\exists y \neq 0 : \langle y, A^n S(s)Bv\rangle = 0, \ \ \forall n \in \mathbb{N}, \text{ and } \forall v \in U.$$

In particular

$$\langle y, S(s)Bv\rangle = 0, \quad \text{and } \forall v \in U.$$

Then, $\langle B^* S * (s)y, v\rangle = 0, \ \forall v \in U,$ and $\forall s \in [0, T]$. So, $y = 0$. Contradiction.
(5) $\Rightarrow$ (2) : Assume that, $\overline{\mathrm{Im} L_T} \neq H$, then there exists $y^* \neq 0$ such that

$$\langle y^*, \int_0^t S(t - s)Bv(s)\rangle_{H^*,H} = 0, \ \forall v \in U.$$

So

$$\langle y^*, S(t - s)Bv(s)ds\rangle_{H^*,H} = 0, \ \forall s \in [0, T], \forall v \in U.$$

We deduce that

$$\frac{d^n}{ds^n}\langle y^*, S(t - s)Bv(s)ds\rangle_{H^*,H} = 0, \ \forall n \in \mathbb{N}, \forall s \in [0, T], \forall v \in U.$$

Consequently

$$\langle y^*, A^n S(t - s)Bv(s)ds\rangle_{H^*,H} = 0, \ \forall s \in [0, T], \forall n \in \mathbb{N}, \forall v \in U.$$

Witch gives

$$y^* \in \overline{\cup_{n\in\mathbb{N}} Im(A^n S(s)B)}^{\perp}, \forall s \in [0, T].$$

Hence, $\overline{\cup_{n\in\mathbb{N}} Im(A^n S(s)B)} \neq H, \ \forall s \in [0, T]$. □

However, this notion is unfortunately insufficient when, for example, the system is to be stabilized around an unsteady stationary state, since it would be necessary to control all the time to remain in a neighborhood of the solution, which is impossible in practice, Controllability approach is too weak. For this purpose, we propose the following concepts:

3.5 Controllability to trajectories

Here it is a matter of reasoning not on the final states of the system but on the trajectories. We will modify our problem to say that we want not to reach any final state but to coincide with a given trajectory at time T. Consider then a free trajectory of our system

$$\begin{cases} \frac{\partial}{\partial t}\overline{y}(t) = A\overline{y}(t) & \text{in } \Omega \times (0,T) \\ \overline{y}(0) = \overline{y}_0, & \text{in } \Omega, \end{cases} \tag{3.9}$$

Let us suppose our initial state $y_0 \in H$ different to $\overline{y}_0$. Then we want to find a control u such that the solution of (3.1) verifies $y(T) = \overline{y}(T)$.

Definition 3.7. *The system (3.1) is said to be controllable on the trajectories in time T if, from any initial data, it is possible to reach any trajectory in time T.*

So, it is to bring the solution exactly on a free trajectory of the system at time T. Suppose that the system we consider is in the state y_0 at $t = 0$. The idea is that We want to be exactly on $\overline{y}$ at the time $t = T$, that is to say, one wants to have, by positing $Z = y - \overline{y}, z(T) = 0$, with an initial condition $z_0 = y_0 - \overline{y}_0$ which describes the space U_0 when y_0 Traverses U_0. We are thus reduced to the problem of exact control over the trajectories, or null controllability (equivalent notions on linear problems). Finally, we note that for a linear problem, the problem is reduced to the null controllability.

3.6 Null Controllability

Definition 3.8. *The system (3.1) is said to be null controllable in time T if, from any initial data, it is possible to reach the trajectory zero in time T.*

In other words, the system (3.1) is null controllable at time T if for all $y_0 \in L^2(\Omega)$, There exists a control $u \in L^2(0,T;U)$ such that the solution y of (3.1) satisfies $y(T) = 0$ in Ω.

Proposition 3.9. *Let $T > 0$. The system (3.1) is null controllable in H over $[0,T]$ if and only if:*

$$\exists c > 0 : \|S^*(T)\varphi\| \leq c\|B^*S^*(\cdot)\varphi\|_{L^2(0,T;U^*)}, \quad \forall \varphi \in H^*. \tag{3.10}$$

The proof of the Proposition 3.9 is based on the following Lemma

Lemma 3.10. *Let H_1, H_2 and H_3 be three Hilbert spaces. Let C_2 be a linear mapping continuous from H_2 to H_1 and Let C_3 be a closed linear operator densely defined from $\mathcal{D}(C_3) \subset H_3$ into H_1. Then the following properties are equivalent:*

1. *There exist* $M \geq 0$ *such that*

$$\|C_2^* h\|_{H_2} \leq M\|C_3^* h\|_{H_3}, \quad \forall h \in \mathcal{D}(C_3). \tag{3.11}$$

2. *We have the following inclusion*

$$C_2(H_2) \subset C_3(\mathcal{D}(C_3)). \tag{3.12}$$

Moreover, if there exists $M \geq 0$ *is that (3.11) satisfies, there exists a continuous linear mapping* C_1 *of* H_2 *to* H_3 *such that*

$$C_1(H_2) \subset \mathcal{D}(C_3) \text{ and } C_2 = C_3 C_1 \tag{3.13}$$

$$\|C_1\|_{L(H_3,H_2)} \leq M. \tag{3.14}$$

Proof of Proposition 3.9

1. Let $T > 0$. Assume that The system (3.1) is null controllable in H over $[0, T]$. Let $y_0 \in H$, then there exists a control $u \in L^2(0, T; U)$ such that the solution y of (3.1) satisfies $y(T) = 0$ in Ω. That is to say
$$S(T)y_0 + L_T u = 0$$
So $Im(S(T)) \subset Im(L_T)$. We apply Lemma 3.10, with $H_1 = H_2 = H$, and $H_3 = L^2(0, T; U)$, $C_2 = S(T), C_3 = L_T$, we get
$$\exists c > 0 : \|S^*(T)\varphi\| \leq c\|B^* S^*(\cdot)\varphi\|_{L^2(0,T;U^*)}, \quad \forall \varphi \in H^*. \tag{3.15}$$

2. Assume that
$$\exists c > 0 : \|S^*(T)\varphi\| \leq c\|B^* S^*(\cdot)\varphi\|_{L^2(0,T;U^*)}, \quad \forall \varphi \in H^*. \tag{3.16}$$
We apply Lemma 3.10, with $H_1 = H_2 = H$, and $H_3 = L^2(0, T; U)$, $C_2 = S(T), C_3 = L_T$, we get
$$C_2(H_2) \subset C_3(H_3). \tag{3.17}$$
This is equivalent to
$$S(T)(H) \subset L_T(L^2(0, T; U)). \tag{3.18}$$
Let $y_0 \in H$, then there exists $v \in L^2(0, T; U)$ such that $S(T)y_0 = L_T v$. If we put $u = -v$, we obtain the solution of the system (3.1) satisfies $y(T) = 0$. Hence, the system (3.1) is null controllable.

3.7 Comparison of the different notions

It is clear that exact controllability implies null controllability. The reciprocal is false in general. However, the reciprocal is true if the family of linear and bounded operators $(S(t))_{t\in\mathbb{R}}$ is C_0-group in the Hilbert space H. More precisely, we have the following theorem

Theorem 3.11. *Assume that $(S(t))_{t\in\mathbb{R}}$ is C_0-group in the Hilbert space H. Let $T > 0$ and assume that the system (3.1) is null controllable in the time T. Then the system (3.1) is exact controllable in the time T.*

Proof. Let $y_0, y_d \in H$. We applied the null controllability with the initial data $y_0 - S(-T)y_d$, then there exists $u \in L^2(0,T;U)$ such that the solution of the system

$$\begin{cases} \frac{\partial}{\partial t}\tilde{y}(t) = A\tilde{y}(t) + Bu(t) & \text{in } \Omega \times (0,T) \\ \tilde{y}(0) = y_0 - S(-T)y_d, & \text{in } \Omega, \end{cases} \tag{3.19}$$

satisfies

$$\tilde{y}(T) = 0. \tag{3.20}$$

We remark that the solution of the system (3.1) is given by

$$y(t) = \tilde{y}(t) + S(-T)y_d, \ \forall t \in [0,T]. \tag{3.21}$$

In particular, from (3.20) and (3.21), we get

$$y(T) = y_d$$

This concludes the proof of the theorem. □

Proposition 3.12.

1. *The exact controllability implies the approximate controllability but the reciprocal is false.*
2. *The exact controllability implies the controllability to the trajectories but the reciprocal is false.*
3. *There is no relationship between approximated controllability and controllability to the trajectories.*

Chapter 4

Carleman estimate and observability inequality

4.1 Introduction

Carleman's estimates are a priori estimates of EDP solutions in Sobolev spaces with weights. These estimates may be local or global depending on whether they apply to compact support solutions or to solutions in the whole domain (with associated boundary conditions in the latter case).
The local Carleman estimates were introduced by T. Carleman in 1939 with the aim of studying the unique continuation property of operators of partial derivatives. Overall estimates were developed much later in the 90 by Fursikov and Imanuvilov in the context of control problems for EDPs. The links between control theory and carleman estimates were established through two prototypical examples, those of the wave and heat equations. The problem of exact control for these equations, which consists in bringing the state of the system towards a target state desired by the action of a control, is equivalent by duality to the problem of observability, which consists in restoring the dynamics Complete system of the dual system from partial measurements (or observations) of the state in the region where the control is active. These studies have links to inverse and identity problems.
The classical methods for establishing the observability inequalities for the wave or heat equation are based on the development of solutions in a Riesz basis or on the multiplier methods. These methods are relatively well adapted to the case of constant coefficient operators and under some geometric conditions on the control domain. They do not provide a general way of dealing with the case of EDPs with variable coefficients.
Carleman's estimates are a powerful tool for dealing with these cases, but also for semi-linear PDEs, Navier-Stockes equations and degenerate parabolic equations. From the carleman estimates, we can deduce the required observability inequalities.

4.2 Carleman estimate for the linear heat equation

Let Ω be an open set of $\mathbb{R}^n$ with boundary $\Gamma = \partial\Omega$ of class C^2. For a given $T > 0$, We consider a linear parabolic equation of the form

$$\begin{cases} \partial_t y - \Delta y &= v\chi_\omega, \ (x,t) \in \Omega \times (0,T) \\ y &= 0, \quad \text{on } \partial\Omega \times (0,T) \\ y(x,0) &= y_0(x), \ \text{in } \Omega, \end{cases} \tag{4.1}$$

where y is the state of the system, ω is a nonempty open of Ω, χ_ω is the characteristic function of ω, v is the control, and y_0 is the data. We denote by Q the cylinder $\Omega \times (0,T)$ and by $\sum$ the lateral boundary of the cylinder ($\sum = \partial\Omega \times (0,T)$). Then, We have the following theorem

Theorem 4.1. *For all $y_0 \in L^2(\Omega)$ and for all $v \in L^2(\omega \times (0,T))$, There exists a unique solution y of the equation (4.1) with $y \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$. Moreover, there exists a constant $C = C(T,\Omega) > 0$ such that*

$$\|y\|_{C([0,T];L^2(\Omega))} + \|y\|_{L^2(0,T;H_0^1(\Omega))} \leq C\Big(\|y_0\|_{L^2(\Omega)} + \|v\|_{L^2(Q)}\Big).$$

This theorem of existence and uniqueness of solutions is inherent to the denial of controllability since it gives a meaning to the solutions of the system under consideration.
We follow [28] for a presentation of Carleman's inequalities in the case of a linear parabolic equation with Dirichlet condition. In order to write Carleman's estimate of (4.1) we must introduce a weight function. According to [33], we know that for all open ω with $\omega \subset\subset \Omega$, there exists $\eta = \eta(x)$ satisfying

$$\begin{cases} \eta & \in C^2(\overline{\Omega}), \\ \eta & > 0 \ \text{ in } \ \Omega \ \text{ and } \ \eta = 0 \ \text{ on } \ \partial\Omega, \\ \nabla\eta & \neq 0, \ \text{ in } \ \overline{\Omega \setminus \omega}. \end{cases} \tag{4.2}$$

The existence of such a function is nontrivial and derives from the theory of Morse functions. For the proof, we refer to ([33] lemma 1.1), (see also [14]).
Let ω an open with $\omega \subset\subset \Omega$, we introduce the following functions:

$$\alpha(x,t) = \frac{e^{2\lambda m\|\eta\|_\infty} - e^{\lambda(m\|\eta\|_\infty + \eta(x))}}{t(T-t)} = \frac{\sigma(x)}{t(T-t)}, \ \ \xi(x,t) = \frac{e^{\lambda(m\|\eta\|_\infty + \eta(x))}}{t(T-t),} \tag{4.3}$$

for all $(x,t) \in Q$, $m > 1$ and $\lambda \geq 1$ is a parameter that depends only on and Ω and ω to be fixed later.
Remark that $e^{2\lambda m\|\eta\|_\infty} - e^{\lambda(m\|\eta\|_\infty - \eta(x)} > 0$ in Ω. We also introduce the space

$$Z = \{q \in C^2(\overline{Q}) : q = 0 \text{ on } \sum\}$$

Then we have the following carleman estimate result.

Theorem 4.2 ([28] and [33]). *There are three positive constants, $\lambda_1 = C(\Omega, \omega)$, $s_1 = C(\Omega, \omega)(T + T^2)$ and $C = C_1(\Omega, \omega)$ such that, for all $\lambda \geq \lambda_1, s \geq s_1$ and $q \in Z$, we have*

$$s^{-1} \iint\limits_{Q} e^{-2s\alpha} \xi^{-1} (|\Delta q|^2 + |q_t|^2) dxdt + s\lambda^2 \iint\limits_{Q} e^{-2s\alpha} \xi |\nabla q|^2 dxdt + s^3 \lambda^4 \iint\limits_{Q} e^{-2s\alpha} \xi^3 |q|^2 dxdt$$
$$\leq C\Big(\iint\limits_{Q} e^{-2s\alpha} |q_t + \Delta q|^2 dxdt + s^3 \lambda^4 \iint\limits_{\omega \times (0,T)} e^{-2s\alpha} \xi^3 |q|^2 dxdt \Big), \tag{4.4}$$

For the proof the reader can be found in [29] and [33].

4.3 Observability inequality

In this section we show how to obtain the inequality of observability from the Carleman estimate. We consider again the following linear heat equation

$$\begin{cases} \partial_t y - \Delta y & = v\chi_\omega, \ (x,t) \in \Omega \times (0,T) \\ y & = 0, \quad \text{on } \partial\Omega \times (0,T) \\ y(x,0) & = y_0(x), \ \text{in } \Omega, \end{cases} \tag{4.5}$$

we assume that the data y_0 is in $L^2(\Omega)$ and we try to find a control $v \in L^2(\omega \times (0,T))$ such as the associated state y possesses a desired behavior at time $t = T$. Our aim is to show that the system (4.5) is null controllable, that is to say

$$\forall y_0 \in L^2(\Omega), \exists v \in L^2(\omega \times (0,T)) : y(\cdot, T) = 0, \ \text{in } \Omega.$$

We consider the adjoint problem relative to (4.5)

$$\begin{cases} -\partial_t \varphi - \Delta \varphi & = 0, \ (x,t) \in \Omega \times (0,T) \\ \varphi & = 0, \quad \text{on } \partial\Omega \times (0,T) \\ \varphi(x,T) & = \varphi_0(x), \ \text{in } \Omega, \end{cases} \tag{4.6}$$

Then we have the following result

Theorem 4.3 (see [29]). *For all solution φ of (4.6), there exists a constant $C = C(T, \Omega, \omega) > 0$ such that*

$$\|\varphi(\cdot, 0)\|^2_{L^2(\Omega)} \leq C \iint_{\omega \times (0,T)} |\varphi|^2 dxdt. \tag{4.7}$$

Proof. In Carleman estimate (4.4), we put $q = \varphi$, where φ is the solution of (4.6). In view of the fact that $\varphi_t + \Delta \varphi = 0$, it follows that

$$s\lambda^2 \iint\limits_{Q} e^{-2s\alpha} \xi |\nabla \varphi|^2 dxdt + s^3 \lambda^4 \iint\limits_{Q} e^{-2s\alpha} \xi^3 |\varphi|^2 dxdt \leq Cs^3 \lambda^4 \iint\limits_{\omega \times (0,T)} e^{-2s\alpha} \xi^3 |\varphi|^2 dxdt.$$

Now using the fact that for fixed $\lambda \geq \lambda_1$ and $s \geq s_1$ there exists a constant $M > 0$ such that

$$\frac{1}{t(T-t)} \leq \xi = \frac{e^{\lambda(m\|\eta\|_\infty + \eta)}}{t(T-t)} \leq \frac{M}{t(T-t)}$$

then we obtain

$$s \iint\limits_Q e^{-2s\alpha} t^{-1}(T-t)^{-1}|\nabla\varphi|^2 dxdt + s^3 \iint\limits_Q e^{-2s\alpha} t^{-3}(T-t)^{-3}|\varphi|^2 dxdt$$
$$\leq Cs^3 \iint\limits_{\omega\times(0,T)} e^{-2s\alpha} t^{-3}(T-t)^{-3}|\varphi|^2 dxdt. \tag{4.8}$$

We deduce that

$$\iint\limits_Q e^{-2s\alpha} t^{-3}(T-t)^{-3}|\varphi|^2 dxdt \leq C \iint\limits_{\omega\times(0,T)} e^{-2s\alpha} t^{-3}(T-t)^{-3}|\varphi|^2 dxdt. \tag{4.9}$$

We now turn to the estimate of the weight functions appearing in (4.9).

Lemma 4.4. *1. For all $t \in [0,T]$, we have*

$$\|e^{-2s\alpha} t^{-3}(T-t)^{-3}\|_\infty \leq CT^{-6}\exp(-CsT^{-2})$$

for some $s \geq s_2 = \max\{s_1, 3T^2(8\min\sigma(x))^{-1}\}$.

2. For all $x \in \Omega$, and for all $t \in [\frac{T}{4}, \frac{3T}{4}]$

$$e^{-2s\alpha} t^{-3}(T-t)^{-3} \geq CT^{-6}\exp(-CsT^{-2})$$

for all $s \geq s_3$.

Then we deduce from (4.9) the following estimate

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_\Omega |\varphi|^2 dxdt \leq C \iint_{\omega\times(0,T)} |\varphi|^2 dxdt. \tag{4.10}$$

On the other hand, by multiplying (4.6) with φ and integration by parts on Ω gives

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega |\varphi|^2 dx + \int_\Omega |\nabla\varphi|^2 dx = 0. \tag{4.11}$$

We deduce that $\|\varphi(\cdot,t)\|^2_{L^2(\Omega)}$ is increasing on $(0,T)$. So, $\|\varphi(\cdot,0)\|^2_{L^2(\Omega)} \leq \|\varphi(\cdot,t)\|^2_{L^2(\Omega)}$, for all $t \in (0,T)$.

Consequently, we obtain

$$\int_\Omega |\varphi(x,0)|^2 dx \leq \frac{2}{T}\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_\Omega |\varphi|^2 dxdt \leq C \iint_{\omega\times(0,T)} |\varphi|^2 dxdt. \tag{4.12}$$

□

The null controllability of (4.1) is follows from the following theorem.

Theorem 4.5. *The following assertions are equivalent:*

1. *There exists a constant $C > 0$, such that, for all $y_0 \in L^2(\Omega)$, there exists a control $v \in L^2(\omega \times (0,T))$, with*
$$\|v\|_{L^2(\omega\times(0,T))} \leq C\|y_0\|_{L^2(\Omega)} \tag{4.13}$$
such that the solution $y_v \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$, of the system (4.5) corresponding to y_0 and v satisfied $y_v(T) = 0$ in $L^2(\Omega)$.

2. *There exists a constant $C > 0$, such that the following inequality of observability*
$$\|\varphi(0)\|^2_{L^2(\Omega)} \leq C \iint_{\omega\times(0,T)} |\varphi|^2 dxdt, \tag{4.14}$$
satisfied, for all solution $\varphi \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$, of the system (4.6) with initial data $\varphi_0 \in L^2(\Omega)$.

before the proof of this theorem, we need the following proposition.

Proposition 4.6. *Let $y_0, \varphi_0 \in L^2(\Omega)$ be fixed and $v \in L^2(\omega \times (0,T))$, Then*
$$\int_\Omega y_v(x,T)\varphi_0 dx - \int_\Omega y_0(x)\varphi(x,0)dx = \int_0^T \int_\omega v\varphi dxdt,$$
where y_v and φ are respectively, the solutions of (4.5) and (4.6) for y_0, v and φ_0.

Proof. Multiplying the equations satisfies by φ by y and also of y by φ. after integration by parts and take in consideration the boundary conditions, we obtain
$$\begin{aligned}\int_0^T \int_\omega v\varphi dxdt &= \iint_{\Omega\times(0,T)} (y_t - \Delta y)\varphi dxdt \\ &= -\iint_{\Omega\times(0,T)} (\varphi_t + \Delta\varphi) y dxdt + \int_\Omega y\varphi|_0^T dx \\ &+ \iint_{\partial\Omega\times(0,T)} (-\frac{\partial y}{\partial\eta}\varphi + y\frac{\partial\varphi}{\partial\eta})\varphi d\sigma dt \\ &= \int_\Omega y_v(x,T)\varphi_0 dx - \int_\Omega y_0(x)\varphi(x,0)dx.\end{aligned}$$

□

Proof of the theorem :

1. let $y_0 \in L^2(\Omega)$ and consider that (4.5) is null controllable. Then, there exists $v \in L^2(\omega \times (0,T))$ such that $y_v(T) = 0$ in Ω.
   For $\varphi_0 \in L^2(\Omega)$, let φ the solution of the adjoint system with the initial data φ_0. by the Proposition 4.6, we deduce

$$\begin{aligned}(\varphi(0), y_0) = \int_\Omega \varphi(x,0)y_0(x)dx &= -\iint_{\omega\times(0,T)} v(x,t)\varphi(x,t)dxdt \\ &\leq C\|\varphi\|_{L^2(\omega\times(0,T))}\|v\|_{L^2(\omega\times(0,T))} \\ &\leq \|\varphi\|_{L^2(\omega\times(0,T))}\|y_0\|_{L^2(\Omega)}.\end{aligned}$$

   If we put $y_0 = \varphi(0)$, we obtain the observability inequality (4.14).

2. For The reciprocal, we will divide the proof into two steps. In the first step, we will construct a sequence of controls $v_\varepsilon \in L^2(\omega \times (0,T))$, with $\varepsilon > 0$ which provide the approximate controllability of (4.5). Secondly, we will go to the limit when tends towards zero and we will conclude.
   Step 1 : Let $y_0 \in L^2(\Omega)$ and $\varepsilon > 0$. We introduce the functional J_ε defined by

$$J_\varepsilon(\varphi_0) = \frac{1}{2}\iint_{\omega\times(0,T)} |\varphi|^2 dxdt + \varepsilon\|\varphi_0\|_{L^2(\Omega)} + (\varphi(0), y_0),\ \forall \varphi_0 \in L^2(\Omega),$$

   where φ is the solution of (4.6) with the initial data φ_0. It is easy to verified that the functional J_ε is strictly convex, continue and coercive in $L^2(\Omega)$. It therefore has a unique minimum $\varphi_{0,\varepsilon}$ whose associated solution is denoted by φ_ε. Let us now introduce the control $v_\varepsilon = \varphi_\varepsilon \chi_\omega$ and denoted by y_ε the solution of (4.5) associated to v_ε. Since J_ε achieved its minimum in $\varphi_{0,\varepsilon}$, then for all $\varphi_0 \in L^2(\Omega)$, the function

$$\begin{aligned} g : h \mapsto J_\varepsilon(\varphi_{0,\varepsilon} + h\varphi_0) = \frac{1}{2}&\iint_{\omega\times(0,T)} (\varphi_\varepsilon^2 + 2h\varphi\varphi_\varepsilon + h^2\varphi^2)dxdt \\ &+ \varepsilon\Big(\varphi_{0,\varepsilon}^2 + 2h\varphi_{0,\varepsilon}\varphi_0 + h^2\varphi_0^2\Big)^{\frac{1}{2}} \\ &+ (\varphi_\varepsilon(0) + h\varphi(0), y_0)\end{aligned}$$

   achieved its minimum in 0, then $g'(0) = 0$. that is to say

$$\iint_{\omega\times(0,T)} \varphi_\varepsilon\varphi dxdt + \varepsilon\left(\frac{\varphi_{0,\varepsilon}}{\|\varphi_{0,\varepsilon}\|_{L^2(\Omega)}}, \varphi_0\right) + \int_\Omega y_0\varphi(0)dx = 0,\ \ \forall \varphi_0 \in L^2(\Omega). \tag{4.15}$$

   For $\varphi_0 = \varphi_{0,\varepsilon}$, we obtain

$$\iint_{\omega\times(0,T)} |\varphi_\varepsilon|^2 dxdt + \varepsilon\|\varphi_{0,\varepsilon}\|_{L^2(\Omega)} + \int_\Omega y_0\varphi_\varepsilon(0)dx = 0. \tag{4.16}$$

   However

$$v_\varepsilon = \varphi_\varepsilon\chi_\omega$$

Hence

$$\begin{aligned}\|\varphi_\varepsilon\|^2_{L^2(\Omega)} = \iint_{\omega\times(0,T)} |\varphi_\varepsilon|^2 dxdt &\leq \int_\Omega y_0\varphi_\varepsilon(0)dx \\ &\leq \|y_0\|_{L^2(\Omega)}\|\varphi_\varepsilon(0)\|_{L^2(\Omega)} \\ &\leq \frac{C}{2}\|y_0\|^2_{L^2} + \frac{1}{2C}\|\varphi_\varepsilon(0)\|^2_{L^2} \\ &\leq \frac{C}{2}\|y_0\|^2_{L^2} + \frac{1}{2}\iint_{\omega\times(0,T)} |\varphi_\varepsilon|^2 dxdt\end{aligned}$$

So

$$\|v_\varepsilon\|^2_{L^2(\Omega)} \leq C\|y_0\|^2_{L^2}.$$

where C is the constant of the observability inequality (4.14).
On the other hand, according to the Proposition 4.6

$$\iint_{\omega\times(0,T)} \varphi_\varepsilon\varphi dxdt = (y_\varepsilon(T), \varphi_0) - \int_\Omega y_0(x)\varphi(x,0)dx$$

then we deduce from (6.56)

$$|(y_\varepsilon(T), \varphi_0)| = |-\varepsilon(\frac{\varphi_{0,\varepsilon}}{\|\varphi_{0,\varepsilon}\|_{L^2(\Omega)}}, \varphi_0)| \leq \varepsilon\|\varphi_0\|_{L^2(\Omega)}.$$

Hence

$$\|y_\varepsilon(T)\|_{L^2(\Omega)} \leq \varepsilon.$$

Step 2 : Since the sequence (v_ε) is a bounded in $L^2(\Omega)$, we can extract a subsequence, again noted (v_ε) which converge weakly in $L^2(\Omega\times(0,T))$ to an element v then we deduce that the sequence $y_\varepsilon = y_{v_\varepsilon}$ converge to $y = y_v$ in $L^2(0,T;H^1_0(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$. In particular, this gives a weak convergence for $(y_\varepsilon(t))$ $(t \in [0,T])$ in $L^2(\Omega)$. In particular, one can pass to the limit under the boundary conditions, and one obtains:

$$y(T) = 0.$$

Chapter 5

Null Controllability of Degenerate Non autonomous Parabolic Equations [1]

5.1 Introduction

The purpose of this chapter is to establish the null controllability for the linear nonautonomous degenerate parabolic equation

$$\begin{cases} u_t - M(t)(a(x)u_x)_x = h\chi_\omega, & (x,t) \in Q \\ u(1,t) = u(0,t) = 0, \quad t \in (0,T) \\ \text{or} \\ u(1,t) = (au_x)(0,t) = 0, \quad t \in (0,T) \\ u(x,0) = u_0(x), \qquad x \in (0,1), \end{cases} \tag{5.1}$$

where $\omega = (x_1, x_2)$ is a nonempty open subinterval of $(0,1)$, $Q = (0,1) \times (0,T)$, $a(\cdot)$ and $M(\cdot)$ are space and time diffusion coefficients, the initial condition u_0 is given in $L^2(0,1)$, and $h \in L^2(\omega \times (0,T))$ is the control function acting on ω.
The null controllability of nondegenerate parabolic equations have been widely studied in the last years, see in particular [9], [27], [30], [41], [43]. On the other hand, very few results are known in the case of autonomous ($M(t) = 1$) degenerate equations; see [4], [5], [6], [11], [42]. The main tool to study the null controlabillity of the above parabolic equations is the Carleman estimates. These last estimates are used to show the observability inequality of the adjoint parabolic equations which is equivalent to the null controllability of the above parabolic equations. The Carleman estimates are the main results of the above references. In [45], the authors established a new Carleman estimate for the autonomous degenerate equations under some general conditions on the degenerate diffusion coefficient a. In this chapter, we consider the the non autonomous

[1] A. Benaissa, A. Kainane Mezadek and L. Maniar, Null Controllability of Degenerate Non autonomous Parabolic Equations, Facta Universitatis (NIS), Accepted.

degenerate equations 5.1 under these general conditions on a. We show an adequate Carleman estimate and then obtain the null controllability via an observability inequality.

5.2 Assumptions and Preliminary Results

In order to study the null controllability of equations 5.1, we make the following assumptions on the coefficients $M(\cdot)$ and $a(\cdot)$.

Hypothesis 5.1.

1. *M is continuous on $(0,T)$ and there exist two positive constants α_0, β_0 such that*

$$0 < \alpha_0 \leq M(t) \leq \beta_0, \quad t \in (0,T),$$

2. *M is derivable on $(0,T)$ and there exists a positive constant γ_0 such that*

$$|M'(t)| \leq \gamma_0, \quad t \in (0,T).$$

Hypothesis 5.2.

1. *$a \in C([0,1]) \cap C^1((0,1])$, $a(x) > 0$ in $(0,1]$ and $a(0) = 0$,*
2. *there exists $\alpha \in (0,2)$ such that $xa'(x) \leq \alpha a(x)$ for every $x \in [0,1]$,*
3. *if $\alpha \in [1,2)$, there exist $m > 0$ and $\delta_0 > 0$ such that for every $x \in [0,\delta_0]$, we have*

$$a(x) \geq m \sup_{0 \leq y \leq x} a(y).$$

Remark 5.3. *It should be noted that the Hypothesis 5.2 appeared in the first time in [45]. It is weaker than the condition given in [6]. In [45] the author proved also that under the Hypothesis 2. the classical Hardy-inequality does not hold in general, (see [45, Example 3]), and they proposed an improved Hardy inequality, see Proposition 5.7.*

As in [6, 45, 54], for the well-posedness of the problem, the natural setting involves the space

$$H_a^1(0,1) := \{u \in L^2(0,1) \cap H_{loc}^1(0,1) : \int_0^1 a(x)u_x^2 dx < \infty\},$$

which is a Hilbert space for the scalar product

$$(u,v) := \int_0^1 uv + a(x)u_x v_x dx, \quad u,v \in H_a^1(0,1). \tag{5.2}$$

For any $u \in H_a^1(0,1)$, the trace of u at $x = 1$ obviously makes sense which allows to consider homogeneous Dirichlet condition at $x = 1$. On the other hand, the trace of u at $x = 0$ only makes sense when $0 \leq \alpha < 1$. But, for $\alpha \geq 1$, the trace at $x = 0$ does not make sense anymore, so one choses some suitable Neumann boundary condition in this case, see for example Lemma 10 of [45]. This leads to introduce the following space $H_{a,0}^1(0,1)$ depending on the value of α:

1. For $0 \leq \alpha < 1$,
$$H_{a,0}^1(0,1) := \{u \in H_a^1(0,1) : u(1) = u(0) = 0\}.$$

2. For $1 \leq \alpha < 2$,
$$H_{a,0}^1(0,1) := \{u \in H_a^1(0,1) : u(1) = 0\}.$$

In order to study the well-posdeness of 5.1, we define the operator $(A(t), D(A(t)))$ by

$$A(t)u := M(t)Au := M(t)(a(x)u_x)_x, \tag{5.3}$$

endowed with the domain

$$D(A(t)) = D(A) = \{u \in H_{a,0}^1(0,1) \cap H_{\text{loc}}^2((0,1]) : (a(x)u_x)_x \in L^2(0,1)\}, t \in [0,T].$$

Remark 5.4. *The domain $D(A)$ may also be characterized in the case of $\alpha \in [0,1)$ by*

$$D(A) := \{u \in L^2(0,1) \cap H_{loc}^2((0,1]) : a(x)u_x \in H^1(0,1) \text{ and } \quad u(0) = u(1) = 0\},$$

and in case of $\alpha \in [1,2)$ by

$$D(A) := \{u \in L^2(0,1) \cap H_{loc}^2((0,1]) : a(x)u_x \in H^1(0,1) \text{ and } \quad (a(x)u_x)(0) = 0 = u(1)\}.$$

Some properties of the operator A are given in the following proposition, see [10].

Proposition 5.5. *The operator $(A, D(A))$ is closed, self adjoint and negative with dense domain in $L^2(0,1)$. Hence A is the infinitesimal generator of a strongly continuous semigroup e^{tA} on $L^2(0,1)$.*

From assumptions on $M(\cdot)$, we can check that the family of operators $(A(t), D(A(t))), 0 \leq t \leq T$, satisfies the Acquistapace-Terreni conditions, see [1, 2], then it generates an evolution family $U(t,s), t \geq s \geq 0$. More precisely, for $t \geq s$ the map $(t,s) \mapsto U(t,s) \in \mathcal{L}(L^2(0,1))$ is continuous and continuously differentiable in t, $U(t,s)L^2(0,1) \subset D(A(t))$, and $\partial U(t,s) = A(t)U(t,s)$. We further have $U(t,s)U(s,r) = U(t,r)$ and $U(t,t) = I$ for $t \geq s \geq r \geq 0$. Moreover, for $s \in \mathbb{R}$ and $x \in D(A(s))$, the function $t \mapsto u(t) = U(t,s)x$ is continuous at $t = s$ and u is the unique solution in $C([s,\infty), L^2(0,1)) \cap C^1((s,\infty), L^2(0,1))$ of the Cauchy problem $u'(t) = A(t)u(t), t > s, u(s) = x$. These facts have been established in [1, 2].

The problem 5.1 is well-posed in the sense of the following theorem.

Theorem 5.6. *For all $h \in L^2(\omega \times (0,T))$ and $u_0 \in L^2(0,1)$, the problem 5.1 has a unique weak solution*

$$u \in C([0,T]; L^2(0,1)) \cap L^2(0,T; H^1_a(0,1)).$$

Moreover, if $u_0 \in D(A)$, then

$$u \in H^1(0,T; L^2(0,1)) \cap L^2(0,T; D(A)) \cap C([0,T]; H^1_a(0,1)).$$

Throughout this paper we use the following improved Hardy inequality taken from [45, Theorem 2.1], which is the key ingredient in the proof of our Carleman estimate.

Proposition 5.7 (see [45])**.** *Suppose that $a(\cdot)$ and satisfy Hypotheses 5.1. Then, for all $\eta > 0$ and $0 < \gamma < 2 - \alpha$, there exists some positive constant $C_0(a, \alpha, \gamma, \eta) > 0$ such that such that for all $u \in H^1_{a,0}(0,1)$, the following inequality holds*

$$\int_0^1 a(x)u_x^2 dx + C_0 \int_0^1 u^2 dx \geq \frac{a(1)(1-\alpha)^2}{4} \int_0^1 \frac{u^2}{x^{2-\alpha}} dx + \eta \int_0^1 \frac{u^2}{x^\gamma} dx. \tag{5.4}$$

5.3 Carleman Estimates

In this section we prove a crucial Carleman estimate, that will be useful to prove an observability inequality for the adjoint problem of 5.1. For this aim let us consider the parabolic problem

$$\begin{cases} v_t + A(t)v = f, & (x,t) \in Q \\ v(1,t) = v(0,t) = 0, \quad t \in (0,T), & \text{in the case } \alpha \in (0,1) \\ v(1,t) = (av_x)(0,t) = 0, \quad t \in (0,T), & \text{in the case } \alpha \in [1,2), \\ v(x,T) = v_T(x), \quad x \in (0,1). \end{cases} \tag{5.5}$$

Now, we consider $0 < \gamma < 2 - \alpha$ and $\varphi(x,t) = \theta(t)p(x)$. Here

$$\theta(t) = [t(T-t)]^{-k}, k = 1 + 2/\gamma, \quad p(x) = \frac{c_1}{2-\alpha}\Big(\int_0^x \frac{y}{a(y)} dy - c_2\Big) \tag{5.6}$$

where $c_1 > 0$ and $c_2 > \frac{1}{a(1)(2-\alpha)}$ such that $p(x) < 0$ for all $x \in [0,1]$. Observe that there exists some constant $c = c(T) > 0$ such that

$$|\theta_t| \leq c\theta^{1+1/k}, \quad |\theta_{tt}| \leq c\theta^{1+2/k} \quad \text{in} \quad (0,T). \tag{5.7}$$

We have the following main result.

Theorem 5.8. *Assume that the functions $a(\cdot)$ and $M(\cdot)$ satisfy Hypotheses 5.1 and 5.2 and let $T > 0$. For every $0 < \gamma < 2 - \alpha$ there exists $s_0 = s_0(T, a, \alpha, \gamma, \beta_0, \alpha_0, \gamma_0) > 0$ such that for all $s \geq s_0$ and all solutions v of (5.5), we have*

$$\frac{s^3}{(2-\alpha)^2}\iint\limits_Q \theta^3 \frac{x^2}{a(x)} v^2 e^{2s\varphi} dxdt + s\iint\limits_Q \theta a(x) v_x^2 e^{2s\varphi} dxdt + sa(1)(1-\alpha)^2 \iint\limits_Q \theta \frac{v^2}{x^{2-\alpha}} e^{2s\varphi} dxdt$$
$$+ s\iint\limits_Q \theta \frac{v^2}{x^\gamma} e^{2s\varphi} dxdt \leq \frac{18}{\alpha_0^2}\Big(\iint\limits_Q f^2 e^{2s\varphi} dxdt + \frac{4sa(1)\beta_0^2}{2-\alpha}\int_0^T \theta v_x^2(1,t) e^{2s\varphi(1,t)} dt\Big).$$

Proof. For the proof, let us define the function $w = e^{s\varphi} v$, where $s > 0$ and v is the solution of (5.5). Then w satisfies

$$\begin{cases} (e^{-s\varphi} w)_t + M(t)\Big(a(x)(e^{-s\varphi} w)_x\Big)_x = f, & (x,t) \in Q, \\ w(1,t) = w(0,t) = 0, \quad t \in (0,T), & \text{in the case } \alpha \in (0,1), \\ w(1,t) = (aw_x)(0,t) = s(\varphi_x a w)(0,t) = 0, \quad t \in (0,T), & \text{in the case } \alpha \in [1,2), \\ w(x,T) = w(x,0) = 0, \quad x \in (0,1). \end{cases} \tag{5.8}$$

Set

$$Lv := v_t + M(t)(a(x)v_x)_x, \quad L_s w := e^{s\varphi} L(e^{-s\varphi} w).$$
$$L_s w := L_1 w + L_2 w$$

where

$$\begin{aligned} L_1 w &:= M(t)(a(x)w_x)_x - s\varphi_t w + s^2 M(t) a(x) \varphi_x^2 w, \\ L_2 w &:= w_t - 2sM(t)a(x)\varphi_x w_x - sM(t)(a(x)\varphi_x)_x w. \end{aligned} \tag{5.9}$$

Therefore, we have

$$2(L_1 w, L_2 w) \leq \|L_1 w + L_2 w\|^2 = \|f e^{s\varphi}\|^2, \tag{5.10}$$

where $\|\cdot\|$ and $(\cdot,\cdot)$ denote the usual norm and scalar product in $L^2(Q)$ respectively. The proof of Theorem 5.8 is based on the computation of the scalar product $(L_1 w, L_2 w)$ wich comes in the following lemma.

Lemma 5.9. *The scalar product $(L_1 w, L_2 w)$ may be written as a sum of distributed term (d.t)* *and a boundary term (b.t), where the distributed term (d.t) is given by*

$$\begin{aligned}(d.t) = &-2s^2 \iint_Q M(t)a(x)\theta\theta_t p_x^2 w^2 dxdt + \frac{s}{2}\iint_Q \theta_{tt} p w^2 dxdt \\ &+ s\iint_Q \theta(2ap_{xx} + a'p_x)a(x)M^2(t)w_x^2 dxdt \\ &+ s^3 \iint_Q \theta^3(2ap_{xx} + a'p_x)a(x)p_x^2 M^2(t)w^2 dxdt \\ &+ \frac{1}{2}\iint_Q M'(t)a(x)w_x^2 dxdt - \frac{s^2}{2}\iint_Q M'(t)\theta^2 a(x)p_x^2 w^2 dxdt\end{aligned} \tag{5.11}$$

whereas the boundary term (b.t) is given by

$$(b.t) = -s\int_0^T \Big[M^2(t)\theta p_x (a(x)w_x)^2\Big]_0^1 dt. \tag{5.12}$$

Proof. To simplify the notation, we will denote by $(L_i w)_j, (1 \le i \le 2, 1 \le j \le 3)$ the j^{th} term in the expression of $L_i w$ given in (5.9). We will develop the nine terms appearing in the product scalar $(L_1 w, L_2 w)$. For this, we will integrate by parts several times respect to the space and time variables. First we have

$$\begin{aligned}\Big((L_1w)_1, (L_2w)_1\Big) &= \iint_Q M(t)(a(x)w_x)_x w_t dxdt \\ &= \int_0^T \Big[M(t)a(x)w_x w_t\Big]_0^1 dt - \iint_Q M(t)a(x)w_x w_{tx} dxdt \\ &= \int_0^T \Big[M(t)a(x)w_x w_t\Big]_0^1 dt - \frac{1}{2}\int_0^1 \Big[M(t)a(x)w_x^2\Big]_0^T dx + \frac{1}{2}\iint_Q M'(t)a(x)w_x^2 dxdt.\end{aligned} \tag{5.13}$$

Then

$$\begin{aligned}\Big((L_1w)_2, (L_2w)_1\Big) &= -s\iint_Q \varphi_t w w_t dxdt \\ &= -\frac{s}{2}\int_0^1 \Big[\varphi_t w^2\Big]_0^T dx + \frac{s}{2}\iint_Q \varphi_{tt} w^2 dxdt \\ &= -\frac{s}{2}\int_0^1 \Big[\varphi_t w^2\Big]_0^T dx + \frac{s}{2}\iint_Q \theta_{tt} p w^2 dxdt.\end{aligned} \tag{5.14}$$

We also have

$$\begin{aligned}\Big((L_1w)_3,(L_2w)_1\Big) &= s^2\iint_Q a(x)M(t)\varphi_x^2 ww_t dxdt = \frac{s^2}{2}\int_0^1 \Big[a(x)M(t)\varphi_x^2w^2\Big]_0^T dx\\ &- s^2\iint_Q a(x)M(t)\varphi_x\varphi_{xt}w^2dxdt - \frac{s^2}{2}\iint_Q a(x)M'(t)\varphi_x^2w^2dxdt\\ &= \frac{s^2}{2}\int_0^1 \Big[a(x)M(t)\varphi_x^2w^2\Big]_0^T dx - s^2\iint_Q a(x)M(t)p_x^2\theta\theta_t w^2dxdt \qquad (5.15)\\ &- \frac{s^2}{2}\iint_Q a(x)M'(t)\theta^2p_x^2w^2dxdt.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\Big((L_1w)_1,(L_2w)_2\Big) &= -2s\iint_Q M^2(t)\varphi_x(a(x)w_x)(a(x)w_x)_x dxdt\\ &= -s\int_0^T \Big[M^2(t)\varphi_x(a(x)w_x)^2\Big]_0^1 dt + s\iint_Q M^2(t)\varphi_{xx}a^2(x)w_x^2dxdt \qquad (5.16)\\ &= -s\int_0^T \Big[M^2(t)\varphi_x(a(x)w_x)^2\Big]_0^1 dt + s\iint_Q M^2(t)\theta p_{xx}a^2(x)w_x^2dxdt.\end{aligned}$$

We also have

$$\begin{aligned}\Big((L_1w)_2,(L_2w)_2\Big) &= 2s^2\iint_Q M(t)a(x)\varphi_x\varphi_t ww_x dxdt\\ &= s^2\int_0^T \Big[M(t)a(x)\varphi_t\varphi_x w^2\Big]_0^1 dt - s^2\iint_Q M(t)a(x)\varphi_{tx}\varphi_x w^2dxdt\\ &- s^2\iint_Q M(t)\varphi_t(a(x)\varphi_x)_x w^2dxdt\\ &= s^2\int_0^T \Big[M(t)a(x)\varphi_t\varphi_x w^2\Big]_0^1 dt - s^2\iint_Q M(t)a(x)\theta\theta_t p_x^2w^2dxdt \qquad (5.17)\\ &- s^2\iint_Q M(t)\theta_t p(a(x)\varphi_x)_x w^2dxdt.\end{aligned}$$

Additionally, we find that

$$\Big((L_1w)_3,(L_2w)_2\Big) = -2s^3\iint\limits_Q M^2(t)a^2(x)\varphi_x^3\varphi_t w w_x dxdt \tag{5.18}$$

$$= -s^3\int_0^T \Big[M^2(t)a^2(x)\varphi_x^3 w^2\Big]_0^1 dt + s^3\iint\limits_Q M^2(t)\Big[2aa'\varphi_x + 3a^2\varphi_{xx}\Big]\varphi_x^2 w^2 dxdt.$$

Let us now consider the scalar product

$$\Big((L_1w)_1,(L_2w)_3\Big) = -s\iint\limits_Q M^2(t)(a(x)w_x)_x(a(x)\varphi_x)_x w dxdt \tag{5.19}$$

$$= -s\int_0^T \Big[M^2(t)(a(x)\varphi_x)_x a(x)w_x w\Big]_0^1 dt + s\iint\limits_Q M^2(t)(a(x)\varphi_x)_{xx}a(x)ww_x dxdt$$

$$+ s\iint\limits_Q M^2(t)(a(x)\varphi_x)_x a(x)w_x^2 dxdt$$

$$= -s\int_0^T \Big[M^2(t)(a(x)\varphi_x)_x a(x)ww_x\Big]_0^1 dt + s\iint\limits_Q M^2(t)(a(x)\varphi_x)_x a(x)w_x^2 dxdt,$$

since $(a(x)\varphi_x)_{xx} = 0$.
Furthemore

$$\Big((L_1w)_2,(L_2w)_3\Big) = s^2\iint\limits_Q M(t)\varphi_t(a(x)\varphi_x)_x w^2 dxdt. \tag{5.20}$$

Finally, we have

$$\Big((L_1w)_3,(L_2w)_3\Big) = -s^3\iint\limits_Q M^2(t)a(x)\varphi_x^2(a(x)\varphi_x)_x w^2 dxdt. \tag{5.21}$$

Additionally (5.13)-(5.21), we find that

$$\begin{aligned}(d.t) =& -2s^2\iint\limits_Q M(t)a(x)\theta\theta_t p_x^2 w^2 dxdt + \frac{s}{2}\iint\limits_Q \theta_{tt}pw^2 dxdt \\ &+ s\iint\limits_Q \theta(2ap_{xx}+a'p_x)a(x)M^2(t)w_x^2 dxdt \\ &+ s^3\iint\limits_Q \theta^3(2ap_{xx}+a'p_x)a(x)p_x^2M^2(t)w^2 dxdt \\ &+ \frac{1}{2}\iint\limits_Q M'(t)a(x)w_x^2 dxdt - \frac{s^2}{2}\iint\limits_Q M'(t)\theta^2 a(x)p_x^2 w^2 dxdt,\end{aligned} \tag{5.22}$$

and

$$\begin{aligned}
(b.t) =& \int_0^T \Big[M(t)a(x)w_x w_t - sM^2(t)\varphi_x(a(x)w_x)^2 + s^2M(t)a(x)\varphi_t\varphi_x w^2 \\
&- s^3M^2(t)a^2(x)\varphi_x^3 w^2 - sM^2(t)(a(x)\varphi_x)_x a(x)ww_x\Big]_0^1 dt \\
&+ \int_0^1 \Big[-\frac{1}{2}M(t)a(x)w_x^2 - \frac{s}{2}\varphi_t w^2 + \frac{s^2}{2}a(x)M(t)\varphi_x^2 w^2\Big]_0^T dx \\
=& -\int_0^T \Big[sM^2(t)\varphi_x(a(x)w_x)^2\Big]_0^1 dt.
\end{aligned} \tag{5.23}$$

The proof of (6.21) is similar as that in [6] and used the fact that $M(\cdot)$ is bounded function. Now we put $(d.t) = A + B$, where

$$\begin{aligned}
A =& -2s^2 \iint_Q M(t)a(x)\theta\theta_t p_x^2 w^2 dxdt + \frac{s}{2}\iint_Q \theta_{tt} p w^2 dxdt \\
&+ s\iint_Q \theta(2ap_{xx} + a'p_x)a(x)M^2(t)w_x^2 dxdt \\
&+ s^3 \iint_Q \theta^3(2ap_{xx} + a'p_x)a(x)p_x^2 M^2(t)w^2 dxdt,
\end{aligned} \tag{5.24}$$

and

$$B = \frac{1}{2}\iint_Q M'(t)a(x)w_x^2 dxdt - \frac{s^2}{2}\iint_Q M'(t)\theta^2 a(x)p_x^2 w^2 dxdt. \tag{5.25}$$

Observe that

$$A + B \leq \frac{1}{2}\|fe^{s\varphi}\|^2 - (b.t). \tag{5.26}$$

The crucial step is to prove the following estimate.

Lemma 5.10. *There exists a positive constant $s_1 = s_1(T, a, \alpha, \alpha_0, \beta_0, \gamma, \gamma_0) > 0$ such that for all $s \geq s_1$ we have,*

$$\begin{aligned}
A + B \geq& \frac{s^3\alpha_0^2}{4(2-\alpha)^2}\iint_Q \theta^3 \frac{x^2}{a(x)} w^2 dxdt + s\frac{\alpha_0^2}{4}\iint_Q \theta a(x) w_x^2 dxdt \\
&+ \frac{sa(1)(1-\alpha)^2\alpha_0^2}{4}\iint_Q \theta \frac{w^2}{x^{2-\alpha}} dxdt + \frac{s}{4}\alpha_0^2 \iint_Q \theta \frac{w^2}{x^\gamma} dxdt.
\end{aligned} \tag{5.27}$$

Proof. By the assumption $xa'(x) \leq \alpha a(x)$ and the fact that $p_x = \frac{c_1 x}{(2-\alpha)a(x)}$, and observe that

$$\begin{aligned} 2ap_{xx} + a'p_x &= \frac{c_1}{2-\alpha}\Big(\frac{2a(x) - xa'(x)}{a(x)}\Big) \\ &\geq \frac{c_1}{2-\alpha}\Big(\frac{2a(x) - \alpha a(x)}{a(x)}\Big) = c_1 \end{aligned} \tag{5.28}$$

one can estimate A in the following way

$$\begin{aligned} A \geq & -\frac{2s^2c_1^2}{(2-\alpha)^2}\beta_0 \iint_Q \theta\theta_t \frac{x^2}{a(x)} w^2 dxdt + \frac{s}{2}\iint_Q \theta_{tt} p w^2 dxdt \\ & + sc_1\alpha_0^2 \iint_Q \theta a(x) w_x^2 dxdt + \frac{s^3c_1^3\alpha_0^2}{(2-\alpha)^2}\iint_Q \theta^3 \frac{x^2}{a(x)} w^2 dxdt. \end{aligned} \tag{5.29}$$

According to the relation (5.7), we know that $|\theta\theta_t| \leq c\theta^{2+1/k} \leq c'\theta^3$ and we obtain

$$\begin{aligned} A \geq & \Big(\frac{s^3c_1^3\alpha_0^2}{(2-\alpha)^2} - \frac{2s^2c_1^2c'}{(2-\alpha)^2}\beta_0\Big)\iint_Q \theta^3 \frac{x^2}{a(x)} w^2 dxdt \\ & + sc_1\alpha_0^2 \iint_Q \theta a(x) w_x^2 dxdt + \frac{s}{2}\iint_Q \theta_{tt} p w^2 dxdt. \end{aligned} \tag{5.30}$$

Let

$$A_1 = c_1\alpha_0^2 \iint_Q \theta a(x) w_x^2 dxdt + \iint_Q \theta_{tt} p w^2 dxdt. \tag{5.31}$$

Therefore

$$A \geq \Big(\frac{s^3c_1^3\alpha_0^2}{(2-\alpha)^2} - \frac{2s^2c_1^2c'}{(2-\alpha)^2}\beta_0\Big)\iint_Q \theta^3 \frac{x^2}{a(x)} w^2 dxdt + \frac{s}{2}c_1\alpha_0^2 \iint_Q \theta a(x) w_x^2 dxdt + \frac{s}{2}A_1. \tag{5.32}$$

We apply the improved Hardy inequality (5.4), with $\eta = 1$, which gives

$$\int_0^1 a(x) w_x^2 dx + c_0 \int_0^1 w^2 dx \geq \frac{a(1)(1-\alpha)^2}{4}\int_0^1 \frac{w^2}{x^{2-\alpha}} dx + \int_0^1 \frac{w^2}{x^\gamma} dx, \tag{5.33}$$

for suitable $c_0 = c_0(a, \alpha, \gamma)$. Therefore we can write

$$\begin{aligned} A_1 \geq & \frac{a(1)(1-\alpha)^2 c_1\alpha_0^2}{4}\iint_Q \theta \frac{w^2}{x^{2-\alpha}} dxdt + c_1\alpha_0^2 \iint_Q \theta \frac{w^2}{x^\gamma} dxdt \\ & - c_0c_1\alpha_0^2 \iint_Q \theta w^2 dxdt + \iint_Q \theta_{tt} p w^2 dxdt. \end{aligned} \tag{5.34}$$

Finally, we need to estimate the term

$$A_2 = \iint\limits_Q \theta_{tt} p w^2 dxdt - c_0 c_1 \alpha_0^2 \iint\limits_Q \theta w^2 dxdt. \tag{5.35}$$

By (5.7), there exists a positive constant c_3 such that

$$|A_2| \le c_3 \iint\limits_Q \theta^{1+2/k} w^2 dxdt. \tag{5.36}$$

Now, we consider $q = \frac{k}{k-1}$ and $q' = k$, so that $\frac{1}{q} + \frac{1}{q'} = 1$. Using the Young inequality, we have, for all $\varepsilon > 0$

$$\begin{aligned}|A_2| &\le c_3 \iint\limits_Q \left(\theta^{1+2/k-\frac{3}{q'}} a^{\frac{1}{q'}} x^{\frac{-2}{q'}} w^{\frac{2}{q}}\right)\left(\theta^{\frac{3}{q'}} a^{\frac{-1}{q'}} x^{\frac{2}{q'}} w^{\frac{2}{q'}}\right) dxdt \\ &\le c_3 \varepsilon \iint\limits_Q \theta^{(1+2/k-\frac{3}{q'})q} a^{\frac{q}{q'}} x^{\frac{-2q}{q'}} w^2 dxdt + c_3 c(\varepsilon) \iint\limits_Q \theta^3 \frac{x^2}{a(x)} w^2 dxdt, \end{aligned} \tag{5.37}$$

where $c(\varepsilon) = \frac{1}{q'}(\varepsilon q)^{\frac{-q'}{q}}$. Observe that

$$(1 + 2/k - \frac{3}{q'})q = 1, \quad \frac{2q}{q'} = \gamma. \tag{5.38}$$

Using the fact that $a(\cdot)$ is continuous on $[0, 1]$, there exists a positive constant c_4 such that $(a(x))^{\frac{q}{q'}} \le c_4$ for every $x \in [0, 1]$, and then

$$A_2 \ge -c_3 c_4 \varepsilon \iint\limits_Q \theta \frac{w^2}{x^\gamma} dxdt - c_3 c(\varepsilon) \iint\limits_Q \theta^3 \frac{x^2}{a(x)} w^2 dxdt. \tag{5.39}$$

Putting the estimate (5.39) in (5.34) and using (6.63), we obtain

$$\begin{aligned} A \ge &\Big(\frac{s^3 c_1^3 \alpha_0^2}{(2-\alpha)^2} - \frac{2s^2 c_1^2 c'}{(2-\alpha)^2}\beta_0 - \frac{s c_3 c(\varepsilon)}{2}\Big) \iint\limits_Q \theta^3 \frac{x^2}{a(x)} w^2 dxdt + \frac{s}{2} c_1 \alpha_0^2 \iint\limits_Q \theta a(x) w_x^2 dxdt \\ &+ \frac{s a(1)(1-\alpha)^2 c_1 \alpha_0^2}{8} \iint\limits_Q \theta \frac{w^2}{x^{2-\alpha}} dxdt + \frac{s}{2}\Big(c_1 \alpha_0^2 - c_3 c_4 \varepsilon\Big) \iint\limits_Q \theta \frac{w^2}{x^\gamma} dxdt. \end{aligned} \tag{5.40}$$

Now, take $c_1 = 2$ and $\varepsilon = \varepsilon(a, \alpha, \alpha_0, \gamma) = \frac{3\alpha_0^2}{2c_3 c_4}$. Thus there exists $s_2 = s_2(T, a, \alpha, \alpha_0, \beta_0, \gamma) > 0$ such that for all $s \ge s_2$

$$\begin{aligned} A \ge &\frac{s^3 \alpha_0^2}{(2-\alpha)^2} \iint\limits_Q \theta^3 \frac{x^2}{a(x)} w^2 dxdt + s\alpha_0^2 \iint\limits_Q \theta a(x) w_x^2 dxdt \\ &+ \frac{s a(1)(1-\alpha)^2 \alpha_0^2}{4} \iint\limits_Q \theta \frac{w^2}{x^{2-\alpha}} dxdt + \frac{s}{4} \alpha_0^2 \iint\limits_Q \theta \frac{w^2}{x^\gamma} dxdt. \end{aligned} \tag{5.41}$$

On the other hand, we have

$$
\begin{aligned}
|B| &\leq \frac{1}{2}\iint_Q |M'(t)|a(x)w_x^2dxdt + \frac{s^2}{2}\iint_Q |M'(t)|\theta^2 a(x)p_x^2 w^2 dxdt \\
&\leq \frac{\gamma_0}{2}\iint_Q a(x)w_x^2dxdt + \frac{2s^2\gamma_0}{(2-\alpha)^2}\iint_Q \theta^2\frac{x^2}{a(x)}w^2dxdt \\
&\leq 2\gamma_0\Big(\iint_Q a(x)w_x^2dxdt + \frac{s^2}{(2-\alpha)^2}\iint_Q \theta^2\frac{x^2}{a(x)}w^2dxdt\Big) \\
&\leq 2c_5\gamma_0\Big(\iint_Q \theta a(x)w_x^2dxdt + \frac{s^2}{(2-\alpha)^2}\iint_Q \theta^3\frac{x^2}{a(x)}w^2dxdt\Big) \\
&\leq \frac{3\alpha_0^2}{4}\Big(s\iint_Q \theta a(x)w_x^2dxdt + \frac{s^3}{(2-\alpha)^2}\iint_Q \theta^3\frac{x^2}{a(x)}w^2dxdt\Big) \qquad (5.42)
\end{aligned}
$$

for all $s \geq \dfrac{8c_5\gamma_0}{3\alpha_0^2}$. Therefore

$$
B \geq -s\frac{3\alpha_0^2}{4}\iint_Q \theta a(x)w_x^2dxdt - \frac{3s^3\alpha_0^2}{4(2-\alpha)^2}\iint_Q \theta^3\frac{x^2}{a(x)}w^2dxdt. \qquad (5.43)
$$

By addition (5.41) and (5.43), for $s \geq s_1(a,\alpha,\gamma,\beta_0,\alpha_0,\gamma_0) > 0$, with $s_1 = \max\{s_2, \frac{8c_5\gamma_0}{3\alpha_0^2}\}$, we obtain the complet proof of Lemma 5.10.

Now, using the fact that $\int_0^T \Big[sM^2(t)\varphi_x(a(x)w_x)^2\Big]_0 dt$ is nonegative, the right hand of (5.26) become

$$
\frac{1}{2}\|fe^{s\varphi}\|^2 - (b.t) \leq \frac{1}{2}\iint_Q f^2e^{2s\varphi}dxdt + \frac{2sa(1)\beta_0^2}{2-\alpha}\int_0^T \theta w_x^2(1,t)dt. \qquad (5.44)
$$

From (5.26), (5.44) and Lemma 5.10, we obtain

$$
\begin{aligned}
&\frac{s^3}{(2-\alpha)^2}\iint_Q \theta^3\frac{x^2}{a(x)}w^2dxdt + s\iint_Q \theta a(x)w_x^2dxdt + sa(1)(1-\alpha)^2\iint_Q \theta\frac{w^2}{x^{2-\alpha}}dxdt \\
&+ s\iint_Q \theta\frac{w^2}{x^\gamma}dx \leq \frac{2}{\alpha_0^2}\Big(\iint_Q f^2e^{2s\varphi}dxdt + \frac{4sa(1)\beta_0^2}{2-\alpha}\int_0^T \theta w_x^2(1,t)dt\Big)
\end{aligned}
\qquad (5.45)
$$

for all $s \geq s_1$. Finally, we turn back to our original function $v = e^{-s\varphi}w$. Using that

$$
v_x = \Big(-s\theta\frac{2}{2-\alpha}\frac{x}{a(x)}w + w_x\Big)e^{-s\varphi},
$$

by Young inequality, we find

$$s\iint\limits_Q \theta a(x)v_x^2e^{2s\varphi}dxdt \leq 8\frac{s^3}{(2-\alpha)^2}\iint\limits_Q \theta^3\frac{x^2}{a(x)}w^2dxdt + 2s\iint\limits_Q \theta a(x)w_x^2dxdt. \tag{5.46}$$

Also, we have

$$\begin{aligned} w_x(1,t) &= \Big(s\varphi_x v(1,t) + v_x(1,t)\Big)e^{s\varphi(1,t)} \\ &= v_x(1,t)e^{s\varphi(1,t)}. \end{aligned} \tag{5.47}$$

Consequently, from (5.45)-(5.47), we have

$$\frac{s^3}{(2-\alpha)^2}\iint\limits_Q \theta^3\frac{x^2}{a(x)}v^2e^{2s\varphi}dxdt + s\iint\limits_Q \theta a(x)v_x^2e^{2s\varphi}dxdt + sa(1)(1-\alpha)^2\iint\limits_Q \theta\frac{v^2}{x^{2-\alpha}}e^{2s\varphi}dxdt$$
$$+ s\iint\limits_Q \theta\frac{v^2}{x^\gamma}e^{2s\varphi}dxdt \leq \frac{18}{\alpha_0^2}\Big(\iint\limits_Q f^2e^{2s\varphi}dxdt + \frac{4sa(1)\beta_0^2}{2-\alpha}\int_0^T \theta v_x^2(1,t)e^{2s\varphi(1,t)}dt\Big)$$

for all $s \geq s_0$, with $s_0 = s_1$

5.4 Observability Inequality and null controllability

In order to prove the controllability of (5.1), we first need to derive the observability inequality for the following adjoint problem

$$\begin{cases} v_t + A(t)v = 0, & (x,t) \in Q \\ v(1,t) = v(0,t) = 0, & \text{in the case} \quad \alpha \in (0,1) \quad t \in (0,T) \\ v(1,t) = (av_x)(0,t) = 0, & \text{in the case} \quad \alpha \in [1,2) \quad t \in (0,T) \\ v(x,T) = v_T(x), & x \in (0,1). \end{cases} \tag{5.48}$$

More precisely, we need to prove the following inequality

Proposition 5.11. *Assume that the coefficients $a(\cdot)$ and $M(\cdot)$ satisfies the hypothesis (5.2) and (5.1) respectivly, let $T > 0$ be given and ω be a nonempty subinterval of $(0,1)$. Then there existe a positive constant $C = C(T,a,\alpha,M)$ such that the following observability inequality is valid for every solution v of (5.48)*

$$\int_0^1 v^2(x,0)dx \leq C\int_0^T\int_\omega v^2(x,t)dxdt. \tag{5.49}$$

Now, by standard arguments, a null controllability result follows.

Theorem 5.12. *Let $T > 0$ be given, and ω be a nonempty subinterval of $(0,1)$. Then for all $u_0 \in L^2(0,1)$, there exists $h \in L^2(\omega\times(0,T))$ such that the solution u of (5.1) satisfies $u(x,T) = 0$, for every $x \in (0,1)$. Furthermore, we have the estimate*

$$\|h\|_{L^2(\omega\times(0,T))} \leq C\|u_0\|_{L^2(0,1)} \tag{5.50}$$

for some constant C.

5.5 Caccioppoli's inequality

An inequality of the Caccioppoli type makes it possible to increase the norm of the gradient of the solution by means of those of the solution and of the second member. It plays a very important role in showing Carleman's estimates or inequalities of observability.

Lemma 5.13. *(Caccioppoli's inequality) Let $\omega_0 \Subset \omega$ be a nonempty open set. Then, there exists a positive constant $\tilde{c}$ such that for every solution of (5.48)*

$$\int_0^T \int_{\omega_0} v_x^2 e^{2s\varphi} dxdt \leq \tilde{c} \int_0^T \int_{\omega} v^2 dxdt.$$

Proof. Let us consider a smooth function $\xi : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} 0 \leq \xi(x) \leq 1, & \forall x \in \mathbb{R}, \\ \xi(x) = 1, & x \in \omega_0 \\ \xi(x) = 0, & x \notin \bar{\omega} \end{cases} \tag{5.51}$$

and $\xi > 0$ for $x \in \omega$. Then

$$\begin{aligned} 0 &= \int_0^T \frac{d}{dt} \int_0^1 \xi^2 e^{2s\varphi} v^2 dxdt \\ &= 2s \iint_Q \xi^2 \varphi_t e^{2s\varphi} v^2 dxdt + 2 \iint_Q \xi^2 e^{2s\varphi} v v_t dxdt \\ &= 2s \iint_Q \xi^2 \varphi_t e^{2s\varphi} v^2 dxdt - 2 \iint_Q \xi^2 M(t) e^{2s\varphi} v (a(x) v_x)_x dxdt \\ &= 2s \iint_Q \xi^2 \varphi_t e^{2s\varphi} v^2 dxdt + 2 \iint_Q M(t) (\xi^2 e^{2s\varphi})_x a(x) v v_x dxdt + 2 \iint_Q M(t) \xi^2 a(x) v_x^2 e^{2s\varphi} dxdt. \end{aligned}$$

Hence,

$$\begin{aligned}2\iint\limits_Q M(t)\xi^2 a(x)v_x^2 e^{2s\varphi}dxdt &= -2s\iint\limits_Q \xi^2\varphi_t e^{2s\varphi}v^2dxdt - 2\iint\limits_Q M(t)(\xi^2e^{2s\varphi})_x a(x)vv_xdxdt\\ &\leq -2s\iint\limits_Q \xi^2\varphi_t e^{2s\varphi}v^2dxdt + \frac{\beta_0^2}{\alpha_0}\iint\limits_Q \left(\sqrt{a}\frac{(\xi^2e^{2s\varphi})_x}{\xi e^{s\varphi}}v\right)^2dxdt + \alpha_0\iint\limits_Q\left(\sqrt{a}\xi e^{s\varphi}v_x\right)^2dxdt.\end{aligned} \tag{5.52}$$

In other hand we have

$$2\alpha_0\iint\limits_Q \xi^2a(x)v_x^2e^{2s\varphi}dxdt \leq 2\iint\limits_Q M(t)\xi^2a(x)v_x^2e^{2s\varphi}dxdt. \tag{5.53}$$

Using (5.52) and (5.53), we obtain

$$\alpha_0\iint\limits_Q \xi^2a(x)v_x^2e^{2s\varphi}dxdt \leq -2s\iint\limits_Q \xi^2\varphi_te^{2s\varphi}v^2dxdt + \frac{\beta_0^2}{\alpha_0}\iint\limits_Q\left(\sqrt{a}\frac{(\xi^2e^{2s\varphi})_x}{\xi e^{s\varphi}}\right)v^2dxdt. \tag{5.54}$$

Thanks the definition of ξ and the fact that $\varphi_te^{s\varphi}$ and $\varphi_te^{s\varphi}$ are bounded functions on $\omega\times(0,T)$, the inequality (5.54) implies that there exists a positive constant $\tilde{c_1}$ such that

$$\min_{x\in\omega_0}(a(x))\int_0^T\int_{\omega_0}v_x^2e^{2s\varphi}dxdt \leq \int_0^T\int_{\omega_0}a(x)v_x^2e^{2s\varphi}dxdt \leq \iint\limits_Q \xi^2a(x)v_x^2e^{2s\varphi}dxdt \leq \tilde{c_1}\int_0^T\int_\omega v^2dxdt.$$

We deduce that

$$\int_0^T\int_{\omega_0}v_x^2e^{2s\varphi}dxdt \leq \tilde{c}\int_0^T\int_\omega v^2dxdt, \tag{5.55}$$

with

$$\tilde{c} = \frac{\tilde{c_1}}{\min_{x\in\omega_0}(a(x))}.$$

Proof of the Observability inequality (6.17). The proof can be made in three steps.
Step 1: We consider $\omega_0 = (x_1', x_2') \Subset \omega = (x_1, x_2)$ and a smooth cut-off function $0 \leq \xi \leq 1$ such that

$$\begin{cases}\xi(x) = 1, & x\in(0,x_1')\\ \xi(x) = 0, & x\in(x_2',1)).\end{cases} \tag{5.56}$$

The function $w := \xi v$, where v is the solution of (5.48), satisfies the following problem

$$\begin{cases} w_t + M(t)(a(x)w_x)_x = M(t)(2a(x)\xi'v_x + (a(x)\xi')'v) := f, & (x,t)\in Q\\ w(1,t) = w(0,t) = 0, \quad t\in(0,T), & \text{in the case } \alpha\in(0,1),\\ w(1,t) = (aw_x)(0,t) = 0, \quad t\in(0,T), & \text{in the case } \alpha\in[1,2),\\ w(x,T) = w_T(x), \quad x\in(0,1).\end{cases} \tag{5.57}$$

Applying Theorem 6.16 with $\gamma = \frac{2-\alpha}{2}$ and observe that $w_x(1,t) = 0$, we get

$$\begin{aligned} s_0 \iint\limits_Q \theta w^2 e^{2s_0\varphi} dxdt &\leq s_0 \iint\limits_Q \theta \frac{w^2}{x^\gamma} e^{2s_0\varphi} dxdt \\ &\leq \frac{18}{\alpha_0^2} \iint\limits_Q M^2(t)(2a(x)\xi' v_x + (a(x)\xi')' v)^2 e^{2s_0\varphi} dxdt \\ &\leq c \int_0^T \int_{\omega_0} (v_x^2 + v^2) e^{2s_0\varphi} dxdt. \end{aligned}$$

According to Lemma 5.13, we obtain

$$s_0 \iint\limits_Q \theta w^2 e^{2s_0\varphi} dxdt \leq \check{c} \int_0^T \int_\omega v^2 dxdt.$$

Next using the definition of ξ, we obtain

$$\int_0^T \int_0^{x_1} \theta v^2 e^{2s_0\varphi} dxdt \leq \frac{\check{c}}{s_0} \int_0^T \int_\omega v^2 dxdt.$$

Using the fact that $p(x)$ and θ satisfies the following inequality

$$\theta(t) \leq \Big(\frac{3T^2}{16}\Big)^{-k}, t \in [T/4, 3T/4],$$

and

$$|p(x)| \leq \frac{2c_2}{2-\alpha}, \quad \text{for all} \quad x \in [0,1].$$

Then there exists a positive constant $c = c(T, a, \alpha)$ such that

$$e^{-cs_0} \int_{T/4}^{3T/4} \int_0^{x_1} v^2 dxdt \leq \Big(\frac{T^2}{4}\Big)^k \frac{\check{c}}{s_0} \int_0^T \int_\omega v^2 dxdt,$$

witch implies

$$\int_{T/4}^{3T/4} \int_0^{x_1} v^2 dxdt \leq e^{cs_0} \Big(\frac{T^2}{4}\Big)^k \frac{\check{c}}{s_0} \int_0^T \int_\omega v^2 dxdt.$$

Step 2: We define $z = (1-\xi)v$. Then, z satisfies the folowing problem

$$\begin{cases} z_t + M(t)(a(x)z_x)_x = M(t)(2a(x)(1-\xi)' v_x + (a(x)(1-\xi)')' v) := f, & (x,t) \in (x_1', 1) \times (0,T) \\ z(1,t) = z(x_1', t) = 0, \quad t \in (0,T), \\ z(x,T) = z_T(x), \qquad x \in (x_1', 1). \end{cases} \tag{5.58}$$

In this case, we use classical Carleman estimates, since the operator $(a(x)z_x)_x$ is nondegenerate on $(x_1', 1)$. Then v can be estimated on $(x_2, 1) \subset (x_1', 1)$ in the same way, see [30]. Therefore

$$\begin{aligned}\int_{T/4}^{3T/4} \int_0^1 v^2 dxdt &= \int_{T/4}^{3T/4} \int_0^{x_1} v^2 dxdt + \int_{T/4}^{3T/4} \int_\omega v^2 dxdt + \int_{T/4}^{3T/4} \int_{x_2}^1 v^2 dxdt \\ &\leq C \int_0^T \int_\omega v^2 dxdt. \end{aligned} \tag{5.59}$$

Step 3: Multiplying both sides of (5.48) by v and integrate on $(0,1)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1 v^2 dx = M(t)\int_0^1 a(x)v_x^2 dx \geq 0, \quad t \in (0,T).$$

Hence, we deduce that

$$\|v(\cdot,0)\|^2_{L^2(0,1)} \leq \|v(\cdot,t)\|^2_{L^2(0,1)} \quad \text{for all} \quad t \in (0,T). \tag{5.60}$$

Then integrate (5.60) on $(T/4, 3T/4)$, and use (5.59), to obtain

$$\int_0^1 v^2(x,0)dx \leq \frac{2}{T}\int_{T/4}^{3T/4}\int_0^1 v^2 dxdt \leq \tilde{C}\int_0^T \int_\omega v^2 dxdt. \tag{5.61}$$

Chapter 6

Global existence of small data solutions to semi-linear fractional $\sigma-$evolution equations with mass or power non-linearity [1]

6.1 Introduction

In [16] the authors studied the following Cauchy problem for semi-linear fractional wave equations

$$\begin{aligned} &\partial_t^{1+\alpha} u - \Delta u = |u|^p, \\ &u(x,0) = u_0(x), \quad u_t(0,x) = 0, \end{aligned} \tag{6.1}$$

where $\alpha \in (0,1)$, $\partial_t^{1+\alpha} u = D_t^\alpha(u_t)$ with

$$D_t^\alpha(f) = \partial_t(I_t^{1-\alpha} f) \quad \text{and} \quad I_t^\beta f = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)\,ds \ \text{ for } \ \beta > 0.$$

Here $D_t^\alpha(f)$ and $I_t^\beta f$ denote the fractional Riemann-Liouville derivative and the fractional Riemann-Liouville integral of f on $[0,t]$, respectively. Moreover, Γ is the Euler Gamma function. The authors proved the following results.

Proposition 6.1. *Let*

$$p > \overline{p} := \max\Big\{p_\alpha(n); \frac{1}{1-\alpha}\Big\}, \ \textit{ where } \ p_\alpha(n) := 1 + \frac{2(1+\alpha)}{(n-2)(1+\alpha)+2}.$$

[1] A. Kainane Mezadek, M. Reissig, Semi-linear fractional $\sigma-$evolution equations with mass or power non-linearity. Nonlinear Differ. Equ. Appl.(2018) 25:42

Then there exist positive constants ε and $\overline{\delta}$ such that for any $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with $\|u_0\|_{L^1\cap L^\infty} \leq \varepsilon$ and for any $\delta \in (0, \overline{\delta})$ there exists a uniquely determined global (in time) Sobolev solution

$$u \in C\big([0,\infty), L^{1+\delta}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$$

to (6.1). The solution satisfies the following estimate for any $t \geq 0$:

$$\|u(t,\cdot)\|_{L^q} \leq C(1+t)^{-\beta_q+\alpha}\|u_0\|_{L^{1+\delta}\cap L^\infty}, \;\; q \in [1+\delta, \infty], \tag{6.2}$$

where

$$\beta_q = \beta_q(\delta) := \min\Big\{\frac{n(1+\alpha)}{2}\Big(\frac{1}{1+\delta} - \frac{1}{q}\Big); 1\Big\}.$$

Proposition 6.2. *Let $p \in (1, p_\alpha(n)]$ and $u_0 \in L^1(\mathbb{R}^n)$ be such that*

$$\int_{\mathbb{R}^n} u_0(x)\, dx > 0.$$

Then there does not exist any global (in time) Sobolev solution

$$u \in L^p_{loc}\big([0,\infty) \times \mathbb{R}^n\big).$$

This chapter is devoted to the Cauchy problem for the semi-linear fractional $\sigma-$evolution equations with mass or power non-linearity

$$\begin{aligned} &\partial_t^{1+\alpha} u + (-\Delta)^\sigma u + m^2 u = |u|^p, \\ &u(x,0) = u_0(x), \quad u_t(0,x) = 0, \end{aligned} \tag{6.3}$$

where $\alpha \in (0,1)$, $m \geq 0$, $\sigma \geq 1$, $(t,x) \in [0,\infty) \times \mathbb{R}^n$. Our main goal is to understand on the one hand the improving influence of the mass term and on the other hand the influence of higher regularity of the data u_0 on the solvability behavior.
By the assumption $u_t(0,x) = 0$ the Cauchy problem (6.3) may be written in the form of a Cauchy problem for an integro-differential equation

$$\begin{aligned} &\partial_t u = I_t^\alpha\big(-(-\Delta)^\sigma u - m^2 u + |u|^p\big), \\ &u(x,0) = u_0(x). \end{aligned} \tag{6.4}$$

A solution to (6.3) is defined as a solution of (6.4). On the contrary, if we have a solution to (6.4) on a time interval $[0,T]$ the integral $I_t^\alpha(\cdots)$ is defined for all $t \in [0,T]$. Hence, the limit $\lim_{t\to+0} I_t^\alpha(\cdots) = 0$. In this way, we verified $u_t(0,x) = 0$ and obtain a solution to (6.3), too.
For this reason we may restrict ourselves in the further considerations to the study of (6.4) to obtain results for (6.3). Our results of global (in time) existence of small data Sobolev solutions are given in the next section.

6.2 Main results

6.2.1 Fractional σ-evolution models

In the first two results we assume low regularity for the data u_0. We distinguish between conditions for the spatial dimension n.

Theorem 6.3. *Let us assume $0 < \alpha < 1$, $\alpha \leq \lambda < \frac{1+\alpha}{2}$, $\sigma \geq 1$ and $r \geq 1$. We assume that $n \geq \frac{2\sigma r}{1+\alpha}$. Moreover, the exponent p satisfies the condition*

$$p > p_{\alpha,\lambda,\sigma,r}(n) := \max\Big\{p^r_{\alpha,\lambda,\sigma}(n); \frac{1}{1-\lambda}\Big\},$$
$$\text{where } p^r_{\alpha,\lambda,\sigma}(n) := 1 + \frac{n(r-1)(1+\alpha) + 2\sigma r(1+\lambda)}{(n-2\sigma r)(1+\alpha) + 2\sigma r(1+\alpha-\lambda)}.$$

Then there exists a positive constant ε such that for any data

$$u_0 \in L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \quad \text{with } \|u_0\|_{L^r \cap L^\infty} \leq \varepsilon$$

we have a uniquely determined global (in time) Sobolev solution

$$u \in C\big([0,\infty), L^r(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)\big) \text{ for all } q \in [r,\infty)$$

to the Cauchy problem

$$\partial_t^{1+\alpha} u + (-\Delta)^\sigma u = |u|^p, \;\; u(x,0) = u_0(x), \quad u_t(0,x) = 0.$$

Moreover, the solution satisfies the following estimate for any $t \geq 0$ and for all sufficiently small $\delta > 0$:

$$\|u(t,\cdot)\|_{L^q} \leq C(1+t)^{-\beta^{r,\delta}_{\alpha,q,\sigma}+\lambda}\|u_0\|_{L^r \cap L^\infty} \text{ for all } q \in [r,\infty],$$

where

$$\beta^{r,\delta}_{\alpha,q,\sigma} := \min\Big\{\frac{n(1+\alpha)}{2\sigma}\Big(\frac{1}{r} - \frac{1}{q}\Big); 1-\delta\Big\}.$$

The constant C is independent of u_0.

Theorem 6.4. *Let us assume $0 < \alpha < 1$, $\alpha \leq \lambda < \frac{1+\alpha}{2}$, $1 \leq \sigma < \frac{\alpha+1}{2\lambda}$ and $1 \leq r < \frac{\alpha+1}{2\sigma\lambda}$. We assume that $1 \leq n < \frac{2\sigma r}{1+\alpha}$. Moreover, the exponent p satisfies the condition*

$$p > p^r_{\alpha,\lambda,\sigma}(n), \text{ where } p^r_{\alpha,\lambda,\sigma}(n) := 1 + \frac{n(r-1)(1+\alpha) + 2\sigma r(1+\lambda)}{(n-2\sigma r)(1+\alpha) + 2\sigma r(1+\alpha-\lambda)}.$$

Then there exists a positive constant ε such that for any data

$$u_0 \in L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \quad \text{with } \|u_0\|_{L^r \cap L^\infty} \leq \varepsilon$$

we have a uniquely determined global (in time) Sobolev solution

$$u \in C\big([0,\infty), L^r(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)\big) \text{ for all } q \in [r,\infty)$$

to the Cauchy problem

$$\partial_t^{1+\alpha} u + (-\Delta)^\sigma u = |u|^p, \;\; u(x,0) = u_0(x), \quad u_t(0,x) = 0.$$

Moreover, the solution satisfies the following estimate for any $t \geq 0$:

$$\|u(t,\cdot)\|_{L^q} \leq C(1+t)^{-\beta^r_{\alpha,q,\sigma}+\lambda}\|u_0\|_{L^r\cap L^\infty} \text{ for all } q \in [r,\infty],$$

where

$$\beta^r_{\alpha,q,\sigma} := \frac{n(1+\alpha)}{2\sigma}\Big(\frac{1}{r}-\frac{1}{q}\Big).$$

The constant C is independent of u_0.

In the next two results we assume higher regularity for the data u_0 but with additional regularity L^∞. We distinguish between conditions for the spatial dimension n.

Theorem 6.5. *Let us assume $0 < \alpha < 1$, $\alpha \leq \lambda < \frac{1+\alpha}{2}$, $\sigma \geq 1$, $1 < r < \infty$ and $\gamma \geq 0$. We assume that $n \geq \frac{2\sigma r}{1+\alpha}$. The exponent p satisfies the condition*

$$p > p_{\alpha,\lambda,\sigma,r,\gamma} := \max\Big\{p^r_{\alpha,\lambda,\sigma}(n); \frac{2}{1-\lambda}; \gamma\Big\},$$

where

$$p^r_{\alpha,\lambda,\sigma}(n) := 1 + \frac{n(r-1)(1+\alpha)+2\sigma r(1+\lambda)}{(n-2\sigma r)(1+\alpha)+2\sigma r(1+\alpha-\lambda)}.$$

Then there exists a positive constant ε such that for any data

$$u_0 \in H^\gamma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \quad \text{with } \|u_0\|_{H^\gamma_r \cap L^\infty} \leq \varepsilon$$

we have a uniquely determined global (in time) Sobolev solution

$$u \in C\big([0,\infty), H^\gamma_r(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)\big) \text{ for all } q \in [r,\infty)$$

to the Cauchy problem

$$\partial_t^{1+\alpha} u + (-\Delta)^\sigma u = |u|^p, \;\; u(x,0) = u_0(x), \quad u_t(0,x) = 0.$$

The solution satisfies the following estimate for any $t \geq 0$ and for all sufficiently small $\delta > 0$:

$$\|u(t,\cdot)\|_{L^q} \leq C(1+t)^{-\beta^{r,\delta}_{\alpha,q,\sigma}+\lambda}\|u_0\|_{H^\gamma_r\cap L^\infty}, \;\; q \in [r,\infty],$$

where

$$\beta^{r,\delta}_{\alpha,q,\sigma} := \min\Big\{\frac{n(1+\alpha)}{2\sigma}\Big(\frac{1}{r}-\frac{1}{q}\Big); 1-\delta\Big\}.$$

Moreover, the solution satisfies the estimate

$$\|u(t,\cdot)\|_{H^\gamma_r} \leq C(1+t)^\lambda\|u_0\|_{H^\gamma_r\cap L^\infty}.$$

The constants C are independent of u_0.

Theorem 6.6. *Let us assume* $0 < \alpha < 1$, $\alpha \leq \lambda < \frac{1+\alpha}{2}$, $1 \leq \sigma < \frac{\alpha+1}{2\lambda}$, $1 < r < \frac{\alpha+1}{2\sigma\lambda}$ *and* $\gamma \geq 0$. *We assume that* $1 \leq n < \frac{2\sigma r}{1+\alpha}$. *Moreover, the exponent* p *satisfies the condition*

$$p > \max\{p^r_{\alpha,\lambda,\sigma}(n); \gamma\},$$
$$\text{where } p^r_{\alpha,\lambda,\sigma}(n) := 1 + \frac{n(r-1)(1+\alpha) + 2\sigma r(1+\lambda)}{(n-2\sigma r)(1+\alpha) + 2\sigma r(1+\alpha-\lambda)}.$$

Then there exists a positive constant ε *such that for any data*

$$u_0 \in H^\gamma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \quad \text{with } \|u_0\|_{H^\gamma_r \cap L^\infty} \leq \varepsilon$$

we have a uniquely determined global (in time) Sobolev solution

$$u \in C\big([0,\infty), H^\gamma_r(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)\big) \text{ for all } q \in [r,\infty)$$

to the Cauchy problem

$$\partial_t^{1+\alpha} u + (-\Delta)^\sigma u = |u|^p, \ u(x,0) = u_0(x), \quad u_t(0,x) = 0.$$

The solution satisfies the following estimate for any $t \geq 0$*:*

$$\|u(t,\cdot)\|_{L^q} \leq C(1+t)^{-\beta^r_{\alpha,q,\sigma}+\lambda}\|u_0\|_{H^\gamma_r \cap L^\infty}, \ q \in [r,\infty],$$

where

$$\beta^r_{\alpha,q,\sigma} := \frac{n(1+\alpha)}{2\sigma}\Big(\frac{1}{r} - \frac{1}{q}\Big).$$

Moreover, the solution satisfies the estimate

$$\|u(t,\cdot)\|_{H^\gamma_r} \leq C(1+t)^\lambda \|u_0\|_{H^\gamma_r \cap L^\infty}.$$

The constants C *are independent of* u_0.

6.2.2 Fractional σ-evolution models with mass term

Theorem 6.7. *Let us assume* $0 < \alpha < 1$, $\sigma \geq 1$, $r \geq 1$ *and* $p > \frac{1}{1-\alpha}$. *Then there exists a positive constant* ε *such that for any data*

$$u_0 \in L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \quad \text{with} \quad \|u_0\|_{L^r \cap L^\infty} \leq \varepsilon$$

we have a uniquely determined global (in time) Sobolev solution

$$u \in C\big([0,\infty), L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$$

to the Cauchy problem

$$\partial_t^{1+\alpha} u + (-\Delta)^\sigma u + m^2 u = |u|^p, \ u(x,0) = u_0(x), \quad u_t(0,x) = 0.$$

Moreover, the solution satisfies the decay estimate

$$\|u(t,\cdot)\|_{L^q} \leq C(1+t)^{\alpha-1}\|u_0\|_{L^r \cap L^\infty} \quad \text{for all} \quad t \geq 0, \quad q \in [r,\infty].$$

The constant C *is independent of* u_0.

Theorem 6.8. *Let us assume $0 < \alpha < 1$, $\sigma \geq 1$, $\gamma \geq 0$, $1 < r < \infty$ and $p > \max\{2; \frac{1}{1-\alpha}; \gamma\}$. Then there exists a positive constant ε such that for any data*

$$u_0 \in H^\gamma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \quad \text{with} \quad \|u_0\|_{H^\gamma_r \cap L^\infty} \leq \varepsilon,$$

we have a uniquely determined global (in time) Sobolev solution

$$u \in C\big([0,\infty), H^\gamma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$$

to the Cauchy problem

$$\partial_t^{1+\alpha} u + (-\Delta)^\sigma u + m^2 u = |u|^p, \;\; u(x,0) = u_0(x), \quad u_t(0,x) = 0.$$

Moreover, the solution satisfies the decay estimate

$$\|u(t,\cdot)\|_{H^\gamma_r \cap L^\infty} \leq C(1+t)^{\alpha-1} \|u_0\|_{H^\gamma_r \cap L^\infty}.$$

The constant C is independent of u_0.

Remark 6.9. *If we compare Theorem 6.3 with the corresponding result for (6.1) from [16], then we feel the improving influence of the power σ and the order of regularity r in two facts. On the one hand $p_{\alpha,\alpha,1,1}(n) = \overline{p}$ and on the other hand $u \in C\big([0,\infty), L^r(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)\big)$ for all $q < \infty$. In Theorem 6.5 we explain the influence of the regularity of the data on the critical exponent and we have $p_{\alpha,\alpha,1,1}(n) \geq \overline{p}$. If we compare Theorem 6.7 with the corresponding result for (6.1) from [16], then we feel the improving influence of the mass term in three facts. On the one hand $\overline{p} = \frac{1}{1-\alpha}$, $u \in C\big([0,\infty), L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$ and on the other hand $\beta_q = 1$ in (6.2). In the case of Theorem 6.8 we also feel the influence of the regularity of the data on the exponent and we obtain an exponent larger then $\overline{p}$. Besides some stronger restrictions to the critical exponent the statements of Theorems 6.5, 6.6 and 6.8 are regularity results. If the data u_0 is more regular, then we expect more regularity with respect to the spatial variables for the solution.*

6.3 Some preliminaries

The Cauchy problem (6.4) with $\sigma \geq 1$ and $m \geq 0$ can be formally converted to an integral equation and its solution is given by

$$u(t,x) = (G^m_{\alpha,\sigma}(t) * u_0)(t,x) + N^m_{\alpha,\sigma}(u)(t,x) \tag{6.5}$$

with

$$G^m_{\alpha,\sigma}(t,x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} E_{\alpha+1}\big(-t^{\alpha+1}\langle\xi\rangle^2_{m,\sigma}\big)\, d\xi, \tag{6.6}$$

$$N^m_{\alpha,\sigma}(u)(t,x) = \int_0^t \big(G^m_{\alpha,\sigma}(t-s) * I^\alpha_s(|u|^p)\big)(t,s,x)\, ds, \tag{6.7}$$

where $\{G^m_{\alpha,\sigma}(t)\}_{t\geq 0}$ denotes the semigroup of operators which is defined via Fourier transform by

$$(\widehat{G^m_{\alpha,\sigma}(t)}f)(t,\xi) = E_{\alpha+1}\big(-t^{\alpha+1}\langle\xi\rangle^2_{m,\sigma}\big)\widehat{f}(\xi) \quad \text{with} \quad \langle\xi\rangle^2_{m,\sigma} = |\xi|^{2\sigma} + m^2.$$

Here $E_\beta(z) = \sum\limits_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k+1)}$ denotes the Mittag-Leffler function (see Section 6.7).
A representation of solutions of the linear integro-differential equation associated to (6.4) or (6.3) with $\sigma \geq 1$ and $m \geq 0$ (and without the term $|u|^p$) is given by

$$u(t,x) = (G^m_{\alpha,\sigma}(t) * u_0)(t,x).$$

Indeed, we put

$$\begin{aligned} v(t,\xi) &= F_{x\to\xi}(u(t,x))(t,\xi) = F_{x\to\xi}\big((G^m_{\alpha,\sigma}(t) * u_0)(t,x)\big)(t,\xi) \\ &= E_{\alpha+1}(-t^{\alpha+1}\langle\xi\rangle^2_{m,\sigma})\widehat{u_0}(\xi). \end{aligned}$$

By using (6.4) and (6.76) we have

$$\begin{aligned} &F^{-1}_{\xi\to x}\Big(F_{x\to\xi}\Big(\int_0^t I^\alpha_s(-(-\Delta)^\sigma u - m^2 u)(s,x)\,ds\Big)(t,\xi)\Big)(t,x) \\ &= F^{-1}_{\xi\to x}\Big(\langle\xi\rangle^2_{m,\sigma}\int_0^t I^\alpha_s((\widehat{G^m_{\alpha,\sigma}(\tau) * u_0})(\tau,\xi))\,ds\Big)(t,x) \\ &= F^{-1}_{\xi\to x}\Big(\frac{\langle\xi\rangle^2_{m,\sigma}}{\Gamma(\alpha)}\int_0^t\int_0^s (s-\tau)^{\alpha-1}(\widehat{G^m_{\alpha,\sigma}(\tau) * u_0})(\tau,\xi)\,d\tau\,ds\Big)(t,x) \\ &= F^{-1}_{\xi\to x}\Big(\frac{\langle\xi\rangle^2_{m,\sigma}}{\Gamma(\alpha)}\int_0^t\int_0^s (s-\tau)^{\alpha-1}E_{\alpha+1}(-\tau^{\alpha+1}\langle\xi\rangle^2_{m,\sigma})\widehat{u_0}(\xi)\,d\tau\,ds\Big)(t,x) \\ &= F^{-1}_{\xi\to x}\Big(\frac{\langle\xi\rangle^2_{m,\sigma}}{\Gamma(\alpha)}\int_0^t\int_\tau^t (s-\tau)^{\alpha-1}\,ds\,E_{\alpha+1}(-\tau^{\alpha+1}\langle\xi\rangle^2_{m,\sigma})\widehat{u_0}(\xi)\,d\tau\Big)(t,x) \\ &= F^{-1}_{\xi\to x}\Big(\frac{\langle\xi\rangle^2_{m,\sigma}}{\Gamma(\alpha+1)}\int_0^t (t-\tau)^{\alpha}E_{\alpha+1}(-\tau^{\alpha+1}\langle\xi\rangle^2_{m,\sigma})\widehat{u_0}(\xi)\,d\tau\Big)(t,x) \\ &= F^{-1}_{\xi\to x}\Big(\Big(E_{\alpha+1}(-t^{\alpha+1}\langle\xi\rangle^2_{m,\sigma}) - 1\Big)\widehat{u_0}(\xi)\Big)(t,x) \\ &= F^{-1}_{\xi\to x}\Big(E_{\alpha+1}(-t^{\alpha+1}\langle\xi\rangle^2_{m,\sigma})\widehat{u_0}(\xi) - \widehat{u_0}(\xi)\Big)(t,x) \\ &= F^{-1}_{\xi\to x}\big(v(t,\xi) - \widehat{u_0}(\xi)\big)(t,x) = u(t,x) - u_0(x). \end{aligned}$$

Consequently, we have shown (after application of the Fourier inversion formula in S') that

$$u = G^m_{\alpha,\sigma}(t) * u_0$$

is a formal solution to

$$u = u_0(x) + \int_0^t I^\alpha_s(-(-\Delta)^\sigma u - m^2 u)\,ds.$$

In the moment we will not provide any function spaces to which the formal solution will belong. But, as pointed out by the referee the continuity of solutions with respect to the time variable requires a special treatment. Later we will come back to this issue. But, from the above considerations we can formally conclude the following relation (if the convolution really exist)

$$\begin{aligned} &u(t,\cdot) - u_0 \\ &= \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-\tau)^\alpha \big(F^{-1}_{\xi\to x}\big(\langle\xi\rangle^2_{m,\sigma} E_{\alpha+1}(-\tau^{\alpha+1}\langle\xi\rangle^2_{m,\sigma})\big) * u_0\big)\, d\tau. \end{aligned} \tag{6.8}$$

Later we will use this relation for the discussion of continuity in time of solutions for models with mass.

6.4 L^p estimates for model oscillating integrals

At first we derive L^p estimates for the model oscillating integral

$$F^{-1}_{\xi\to x}\big(E_{\alpha+1}(-t^{1+\alpha}|\xi|^{2\sigma})\big).$$

Proposition 6.10. *The following estimate holds in $\mathbb{R}^n$ for $\sigma > 0$, $\alpha \geq 0$:*

$$\|F^{-1}_{\xi\to x}\big(E_{\alpha+1}(-t^{1+\alpha}|\xi|^{2\sigma})\big)(t,\cdot)\|_{L^p} \lesssim t^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})} \tag{6.9}$$

for $p \in [1,\infty]$, $t > 0$ and for all $n \geq 1$ satisfying $n(1-\frac{1}{p}) < 2\sigma$.

Here and in the following we use for non-negative functions f and g the notation $f \lesssim g$ if there exists a constant C which is independent of $y \in D$ such that $f(y) \leq Cg(y)$ for all $y \in D$.

Proof. The proof of (6.9) uses the Propositions 5 and 12 of [49]. In a first step we estimate the following oscillating integrals:

$$F^{-1}_{\xi\to x}\big(e^{-c_1 t|\xi|^{2\rho}}\cos(c_2 t|\xi|^{2\rho})\big) \quad \text{and} \quad F^{-1}_{\xi\to x}\big(e^{-\tau t|\xi|^{2\rho}}\big),$$

where $c_1 = -\cos(\frac{\pi}{1+\alpha}), c_2 = \sqrt{1-c_1^2}, \rho = \frac{\sigma}{1+\alpha}$ and $\tau > 0$. We prove instead of (6.9) the polynomial type decay estimates

$$\|F^{-1}_{\xi\to x}\big(e^{-c_1 t|\xi|^{2\rho}}\cos(c_2 t|\xi|^{2\rho})\big)(t,\cdot)\|_{L^p} \lesssim t^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})}, \tag{6.10}$$

$$\|F^{-1}_{\xi\to x}\big(e^{-\tau t|\xi|^{2\rho}}\big)(t,\cdot)\|_{L^p} \lesssim (\tau t)^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})} \tag{6.11}$$

for all $p \in [1,+\infty]$ and $t > 0$. Then, we deduce (see Section 6.7)

$$\big\|F^{-1}_{\xi\to x}\big(\exp\big(a_{1+\alpha}(t^{\frac{1+\alpha}{2}}|\xi|^\sigma)\big) + \exp\big(b_{1+\alpha}(t^{\frac{1+\alpha}{2}}|\xi|^\sigma)\big)\big)(t,\cdot)\big\|_{L^p} \lesssim t^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})} \tag{6.12}$$

for all $p \in [1, +\infty]$ and $t > 0$. It remains to prove that (see Section 6.7)

$$\big\|F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}|\xi|^\sigma)\big)(t,\cdot)\big\|_{L^p} \lesssim t^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})} \tag{6.13}$$

for all $p \in [1, +\infty]$ and $t > 0$. Therefore we use the formula (see Section 6.7)

$$l_{1+\alpha}(t^{\frac{1+\alpha}{2}}|\xi|^\sigma) \sim \int_0^\infty \frac{\exp\big(-t|\xi|^{\frac{2\sigma}{1+\alpha}} s^{\frac{1}{1+\alpha}}\big)}{s^2+2s\cos((1+\alpha)\pi)+1}\,ds.$$

Taking account of the definition of modified Bessel functions (see Section 6.7) we get

$$\begin{aligned}
&F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}|\xi|^\sigma)\big)(t,x)\\
&= \int_0^\infty \Big(\int_0^\infty \frac{\exp\big(-tr^{\frac{2\sigma}{1+\alpha}} s^{\frac{1}{1+\alpha}}\big)}{s^2+2s\cos((1+\alpha)\pi)+1} r^{n-1}\tilde{J}_{\frac{n}{2}-1}(r|x|)dr\Big)\,ds\\
&= \int_0^\infty \frac{1}{s^2+2s\cos((1+\alpha)\pi)+1}\\
&\qquad \times\Big(\int_0^\infty \exp\big(-tr^{\frac{2\sigma}{1+\alpha}} s^{\frac{1}{1+\alpha}}\big) r^{n-1}\tilde{J}_{\frac{n}{2}-1}(r|x|)dr\Big)\,ds\\
&= \int_0^\infty \frac{1}{s^2+2s\cos((1+\alpha)\pi)+1}\Big(F^{-1}_{\xi\to x}\big(e^{-s^{\frac{1}{1+\alpha}}t|\xi|^{\frac{2\sigma}{1+\alpha}}}\big)(x)\Big)\,ds.
\end{aligned}$$

The estimate

$$\big\|F^{-1}_{\xi\to x}\big(e^{-s^{\frac{1}{1+\alpha}}t|\xi|^{\frac{2\sigma}{1+\alpha}}}\big)(t,\cdot)\big\|_{L^p} \lesssim s^{-\frac{n}{2\sigma}(1-\frac{1}{p})} t^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})}$$

implies

$$\begin{aligned}
&\big\|F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}|\xi|^\sigma)\big)(t,\cdot)\big\|_{L^p}\\
&\quad\lesssim t^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})}\int_0^\infty \frac{s^{-\frac{n}{2\sigma}(1-\frac{1}{p})}}{s^2+2s\cos((1+\alpha)\pi)+1}\,ds \lesssim t^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})}
\end{aligned}$$

if $n(1-\frac{1}{p}) < 2\sigma$. □

Now let us turn to L^p estimates for the model oscillating integral (see Section 6.7)

$$F^{-1}_{\xi\to x}\big(E_{\alpha+1}(-t^{1+\alpha}\langle\xi\rangle^2_{m,\sigma})\big) \text{ with } m>0.$$

At the first glance one might expect an exponential type decay estimate. We are able to prove a potential type decay estimate only.

Proposition 6.11. *The following estimate holds in $\mathbb{R}^n$ for $\sigma > 0$, $m > 0$, $\alpha \in [0, 1)$ and for all $n \geq 1$:*

$$\big\|F^{-1}_{\xi\to x}\big(E_{\alpha+1}(-t^{1+\alpha}\langle\xi\rangle^2_{m,\sigma})\big)(t,\cdot)\big\|_{L^p} \lesssim (1+t)^{-(1+\alpha)} \tag{6.14}$$

for $p \in [1, \infty]$ and $t \geq 0$.

Proof. The proof of (6.23) uses ideas of [52]. In a first step we estimate the following oscillating integrals:

$$F^{-1}_{\xi\to x}\big(e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big) \quad \text{and} \quad F^{-1}_{\xi\to x}\big(e^{-\tau t\langle\xi\rangle^{2\kappa}_{m,\sigma}}\big),$$

where $c=-\cos(\frac{\pi}{1+\alpha}), \kappa=\frac{1}{1+\alpha}\in(\frac{1}{2},1)$ and $\tau>0$. We shall derive the exponential type decay estimate

$$\begin{aligned}&\big\|F^{-1}_{\xi\to x}\big(e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big)(t,\cdot)\big\|_{L^p}\\ &\quad+\big\|F^{-1}_{\xi\to x}\big(e^{-\tau t\langle\xi\rangle^{2\kappa}_{m,\sigma}}\big)(t,\cdot)\big\|_{L^p}\lesssim e^{-Ct}\end{aligned} \tag{6.15}$$

with a suitable positive $C=C(m,\alpha)$, for $p\in[1,\infty]$ and $t\geq 0$. By using modified Bessel functions (see Section 6.7) we have for $n=3$

$$\begin{aligned}&F^{-1}_{\xi\to x}\Big(e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\Big)(t,x)\\ &\quad=\int_0^\infty e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^2\tilde{J}_{\frac{1}{2}}(r|x|)\,dr\\ &\quad=-\frac{1}{|x|^2}\int_0^\infty e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r\partial_r\tilde{J}_{-\frac{1}{2}}(r|x|)\,dr\\ &\quad=-\frac{\sqrt{2}}{\sqrt{\pi}|x|^2}\int_0^\infty e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r\partial_r(\cos(r|x|))\,dr.\end{aligned}$$

Using twice integration by parts we obtain

$$\begin{aligned}&-\frac{\sqrt{\pi}|x|^4}{\sqrt{2}}F^{-1}_{\xi\to x}\big(e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big)(t,x)\\ &\quad=\int_0^\infty\Big(t\big(h_1(r)r^{2\sigma-2}\langle r\rangle^4_{m,\sigma}+h_2(r)r^{4\sigma-2}\langle r\rangle^2_{m,\sigma}+h_3(r)r^{6\sigma-2}\big)\langle r\rangle^{2\kappa-6}_{m,\sigma}\\ &\quad+t^2\big(h_4(r)r^{4\sigma-2}\langle r\rangle^2_{m,\sigma}+h_5(r)r^{6\sigma-2}\big)\langle r\rangle^{4\kappa-6}_{m,\sigma}+t^3h_6(r)r^{6\sigma-2}\langle r\rangle^{6\kappa-6}_{m,\sigma}\Big)\\ &\quad\times e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(r|x|)\,dr,\end{aligned} \tag{6.16}$$

where $h_i(r)=a_i\cos(g(r))+b_i\sin(g(r))$, $i=1,\cdots,6$, $g(r)=t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2}$ and a_i,b_i, $i=1,\cdots,6$, are constants which depend on α and σ only.
To estimate (6.16) we use the inequality

$$\langle r\rangle^{2\kappa}_{m,\sigma}\geq 2^{\kappa-1}\langle r\rangle^{2\kappa}_{2^{-1/2}m,\sigma}+2^{-1}m^{2\kappa}. \tag{6.17}$$

Then, we get

$$\big|F^{-1}_{\xi\to x}\big(e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big)(t,x)\big|\lesssim\frac{e^{-\frac{c}{2}tm^{2\kappa}}}{\langle x\rangle^4_m}.$$

For the oscillating integral $F^{-1}_{\xi\to x}\big(e^{-\tau t\langle\xi\rangle^{2\kappa}_{m,\sigma}}\big)$ we have

$$\begin{aligned}F^{-1}_{\xi\to x}\big(e^{-\tau t\langle\xi\rangle^{2\kappa}_{m,\sigma}}\big)(t,x) &= \int_0^\infty e^{-\tau t\langle r\rangle^{2\kappa}_{m,\sigma}} r^2 \tilde{J}_{\frac{1}{2}}(r|x|)\,dr\\ &= -\frac{1}{|x|^2}\int_0^\infty e^{-\tau t\langle r\rangle^{2\kappa}_{m,\sigma}} r\partial_r \tilde{J}_{-\frac{1}{2}}(r|x|)\,dr\\ &= -\frac{\sqrt{2}}{\sqrt{\pi}|x|^2}\int_0^\infty e^{-\tau t\langle r\rangle^{2\kappa}_{m,\sigma}} r\partial_r(\cos(r|x|))\,dr.\end{aligned}$$

Using twice integration by parts we obtain

$$\begin{aligned}&-\frac{\sqrt{\pi}|x|^4}{\sqrt{2}}F^{-1}_{\xi\to x}\big(e^{-\tau t\langle\xi\rangle^{2\kappa}_{m,\sigma}}\big)(t,x)\\ &= \int_0^\infty \Big(-2\sigma(4\sigma^2-1)\kappa\tau t r^{2\sigma-2}\langle r\rangle^{2\kappa-2}_{m,\sigma} - 24\sigma^3\kappa(\kappa-1)\tau t r^{4\sigma-2}\langle r\rangle^{2\kappa-4}_{m,\sigma}\\ &\quad - 8\sigma^3\kappa(\kappa-1)(\kappa-2)\tau t r^{6\sigma-2}\langle r\rangle^{2\kappa-6}_{m,\sigma} + 24\sigma^3\kappa^2\tau^2t^2r^{4\sigma-2}\langle r\rangle^{4\kappa-4}_{m,\sigma}\\ &\quad + 8\sigma^3\kappa^2(\kappa-1)\tau^2t^2r^{6\sigma-2}\langle r\rangle^{4\kappa-6}_{m,\sigma} - 8\sigma^3\kappa^3\tau^3t^3r^{6\sigma-2}\langle r\rangle^{6\kappa-6}_{m,\sigma}\Big)\\ &\qquad \times e^{-\tau t\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(r|x|)\,dr.\end{aligned}$$

This leads to the estimate

$$\big|F^{-1}_{\xi\to x}\big(e^{-\tau t\langle\xi\rangle^{2\kappa}_{m,\sigma}}\big)(t,x)\big| \lesssim \frac{e^{-\frac{\tau}{2}tm^{2\kappa}}}{\langle x\rangle^4_m}.$$

Summarizing all estimates we proved the statement (6.15) in the case $n=3$.
Now, let us study the case n odd and $n\geq 4$. Then we carry out $\frac{n+1}{2}$ steps of partial integration. We obtain after $\frac{n-1}{2}$ steps and by applying the rules (see Section 6.7)

$$\tilde{J}_{\mu+1}(r|x|) = -\frac{1}{r|x|^2}\partial_r\tilde{J}_\mu(r|x|)$$

for real non-negative μ the relation

$$\begin{aligned}&F^{-1}_{\xi\to x}\big(e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big)(t,x)\\ &= \int_0^\infty e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\tilde{J}_{\frac{n}{2}-1}(r|x|)\,dr\\ &= (-1)^{\frac{n-1}{2}}\frac{1}{|x|^{n-1}}\int_0^\infty\Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n-1}{2}}\Big(e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\Big)\\ &\quad\times\tilde{J}_{-\frac{1}{2}}(r|x|)\,dr\\ &= (-1)^{\frac{n-1}{2}}\frac{1}{|x|^{n-1}}\int_0^\infty\Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n-1}{2}}\Big(e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\Big)\\ &\quad\times\cos(r|x|)\,dr\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{\frac{n+1}{2}} \frac{1}{|x|^{n+1}} \sqrt{\frac{2}{\pi}} \\
&\quad \times \int_0^\infty \Big(\frac{\partial^2}{\partial r^2}\Big)\Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n-1}{2}} \Big(e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}} \cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2}) r^{n-1}\Big) \\
&\qquad \times \cos(r|x|)\, dr.
\end{aligned}
$$

All integrals have the form

$$
\begin{aligned}
&\int_0^\infty \langle r\rangle^{\rho}_{m,\sigma} r^{\delta} e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}} \cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2}) \cos(r|x|)\, dr \\
&\quad \text{or} \int_0^\infty \langle r\rangle^{\rho}_{m,\sigma} r^{\delta} e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}} \sin(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2}) \cos(r|x|)\, dr,
\end{aligned}
$$

where ρ is a negative integer depending on κ and n and δ is a non negative real depending on σ and n. For this reason we conclude the estimate

$$
\big|F^{-1}_{\xi\to x}\big(e^{-ct\langle \xi\rangle^{2\kappa}_{m,\sigma}} \cos(t\langle \xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big)(t,x)\big| \lesssim \frac{e^{-\frac{c}{2}tm^{2\kappa}}}{\langle x\rangle^{n+1}_m}.
$$

Analogously, we obtain the same estimate for

$$
F^{-1}_{\xi\to x}\big(e^{-\tau t\langle \xi\rangle^{2\kappa}_{m,\sigma}}\big)(t,x).
$$

All together implies the statement (6.15) for odd $n \geq 4$.
For $n = 2$ we have

$$
\begin{aligned}
&F^{-1}_{\xi\to x}\big(e^{-ct\langle \xi\rangle^{2\kappa}_{m,\sigma}} \cos(t\langle \xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big)(t,x) \\
&\qquad = \int_0^\infty e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}} \cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2}) r \tilde{J}_0(r|x|)\, dr.
\end{aligned}
$$

From the relation (see Section 6.7)

$$
J_0(s) = \frac{1}{s} J_1(s) + \frac{d}{ds} J_1(s)
$$

it follows that

$$
\tilde{J}_0(r|x|) = 2\tilde{J}_1(r|x|) + r\partial_r \tilde{J}_1(r|x|) = \frac{1}{r}\partial_r(r^2 \tilde{J}_1(r|x|)).
$$

Then, we get

$$
\begin{aligned}
&F^{-1}_{\xi\to x}\big(e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big)(t,x)\\
&= -\int_0^\infty 2\kappa\sigma t r^{4\sigma-1}\langle r\rangle^{2\kappa-2}_{m,\sigma}e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos\Big(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\\
&\quad\times \tilde{J}_1(r|x|)\,dr\\
&= -\int_0^{\frac{1}{|x|}} 2\kappa\sigma t r^{4\sigma-1}\langle r\rangle^{2\kappa-2}_{m,\sigma}e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos\Big(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\\
&\quad\times \tilde{J}_1(r|x|)\,dr\\
&\quad -\int_{\frac{1}{|x|}}^\infty 2\kappa\sigma t r^{4\sigma-1}\langle r\rangle^{2\kappa-2}_{m,\sigma}e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos\Big(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\\
&\quad\times \tilde{J}_1(r|x|)\,dr.
\end{aligned}
$$

Using the boundedness of $\tilde{J}_1(s)$ for $s\in[0,1]$ (see Section 6.7) the first integral can be estimated by

$$
e^{-\frac{c}{2}tm^{2\kappa}}\langle x\rangle_m^{-(4\sigma+2\kappa-2)}.
$$

Remark that $4\sigma+2\kappa-2>2$. To estimate the second integral we apply the following asymptotic formula (see Section 6.7) for $\tilde{J}_1(s)$ for $s\geq 1$:

$$
\tilde{J}_1(s)=cs^{-\frac{3}{2}}\cos\Big(s-\frac{3}{4}\pi\Big)+O(|s|^{-\frac{5}{2}}).
$$

Consequently, the integral can be estimated as follows:

$$
\int_{\frac{1}{|x|}}^\infty r^{4\sigma-1}\langle r\rangle^{2\kappa-2}_{m,\sigma}e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}O((r|x|)^{-\frac{5}{2}})\,dr\lesssim |x|^{-\frac{5}{2}}e^{-\frac{c}{2}tm^{2\kappa}}.
$$

It remains to estimate

$$
\begin{aligned}
&\frac{1}{|x|^{\frac{3}{2}}}\int_{\frac{1}{|x|}}^\infty r^{\frac{8\sigma-5}{2}}\langle r\rangle^{2\kappa-2}_{m,\sigma}e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos\Big(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\cos(r|x|)\,dr,\\
&\frac{1}{|x|^{\frac{3}{2}}}\int_{\frac{1}{|x|}}^\infty r^{\frac{8\sigma-5}{2}}\langle r\rangle^{2\kappa-2}_{m,\sigma}e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos\Big(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\sin(r|x|)\,dr.
\end{aligned}
$$

We explain only the first integral because the second one can be treated in the same way. We write the first integral as follows:

$$
\frac{1}{|x|^{\frac{3}{2}}}\int_{\frac{1}{|x|}}^\infty r^{\frac{8\sigma-5}{2}}\langle r\rangle^{2\kappa-2}_{m,\sigma}e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos\Big(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\cos(r|x|)\,dr
$$

$$
=\frac{1}{|x|^{\frac{3}{2}}}\int_{\frac{1}{|x|}}^1 r^{\frac{8\sigma-5}{2}}\langle r\rangle^{2\kappa-2}_{m,\sigma}e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos\Big(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\cos(r|x|)\,dr \tag{6.18}
$$

$$
+\frac{1}{|x|^{\frac{3}{2}}}\int_1^\infty r^{\frac{8\sigma-5}{2}}\langle r\rangle^{2\kappa-2}_{m,\sigma}e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos\Big(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\cos(r|x|)\,dr. \tag{6.19}
$$

The integral in (6.18) is equal to

$$\frac{1}{|x|^{\frac{5}{2}}}\int_{\frac{1}{|x|}}^{1} r^{\frac{8\sigma-5}{2}}\langle r\rangle_{m,\sigma}^{2\kappa-2}e^{-ct\langle r\rangle_{m,\sigma}^{2\kappa}}\cos\Big(t\langle r\rangle_{m,\sigma}^{2\kappa}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\partial_r(\sin(r|x|))\,dr. \tag{6.20}$$

After partial integration and by using (6.17) the limit terms can be estimated by $|x|^{-\frac{5}{2}}e^{-\frac{c}{2}tm^{2\kappa}}$. The new integral is equal to

$$\begin{aligned}\frac{1}{|x|^{\frac{5}{2}}}\int_{\frac{1}{|x|}}^{1}\Big(&c_1 r^{\frac{8\sigma-7}{2}}\langle r\rangle_{m,\sigma}^{2\kappa-2}\cos\Big(t\langle r\rangle_{m,\sigma}^{2\kappa}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\\
&+c_2 r^{\frac{12\sigma-7}{2}}\langle r\rangle_{m,\sigma}^{2\kappa-4}\cos\Big(t\langle r\rangle_{m,\sigma}^{2\kappa}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\\
&+c_3 r^{\frac{12\sigma-7}{2}}\langle r\rangle_{m,\sigma}^{4\kappa-4}\sin\Big(t\langle r\rangle_{m,\sigma}^{2\kappa}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\\
&+c_4 r^{\frac{12\sigma-7}{2}}\langle r\rangle_{m,\sigma}^{4\kappa-4}\cos\Big(t\langle r\rangle_{m,\sigma}^{2\kappa}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\Big)e^{-ct\langle r\rangle_{m,\sigma}^{2\kappa}}\sin(r|x|)\,dr.\end{aligned}$$

It can be estimated by $|x|^{-\frac{5}{2}}e^{-\frac{c}{2}tm^{2\kappa}}$, too. After integration by parts the integral in (6.19) can be estimated by

$$\frac{1}{|x|^{\frac{5}{2}}}\int_1^{\infty} r^{\frac{12\sigma-7}{2}}\langle r\rangle_{m,\sigma}^{2\kappa-2}e^{-ct\langle r\rangle_{m,\sigma}^{2\kappa}}\,dr.$$

The latter integral can be estimated by $|x|^{-\frac{5}{2}}e^{-\frac{c}{2}tm^{2\kappa}}$. Finally, we have for the oscillating integral $F^{-1}_{\xi\to x}\big(e^{-\tau t\langle\xi\rangle_{m,\sigma}^{2\kappa}}\big)$ the relation

$$\begin{aligned}F^{-1}_{\xi\to x}(e^{-\tau t\langle\xi\rangle_{m,\sigma}^{2\kappa}})(t,x)&=\int_0^{\infty}e^{-\tau t\langle r\rangle_{m,\sigma}^{2\kappa}}r\tilde{J}_0(r|x|)\,dr\\
&=\int_0^{\infty}e^{-\tau t\langle r\rangle_{m,\sigma}^{2\kappa}}\partial_r(r^2\tilde{J}_1(r|x|))\,dr\\
&=\int_0^{\infty}2\sigma\kappa\tau t r^{2\sigma+1}\langle r\rangle_{m,\sigma}^{2\kappa-2}e^{-\tau t\langle r\rangle_{m,\sigma}^{2\kappa}}\tilde{J}_1(r|x|)\,dr.\end{aligned}$$

Then, we derive the same estimates as we did before for estimating the oscillating integral

$$F^{-1}_{\xi\to x}\big(e^{-ct\langle\xi\rangle_{m,\sigma}^{2\kappa}}\cos(t\langle\xi\rangle_{m,\sigma}^{2\kappa}\sqrt{1-c^2})\big).$$

Summarizing all estimates yields the statement (6.15) for $n=2$.
Now for the case of even $n\geq 4$ we carry out $\frac{n}{2}-1$ steps of partial integration. In this way we

obtain

$$\begin{aligned}
&F^{-1}_{\xi\to x}\big(e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big)(t,x)\\
&= \int_0^\infty e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\tilde{J}_{\frac{n}{2}-1}(r|x|)\,dr\\
&= \frac{1}{|x|^{n-2}}\int_0^\infty \Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n-2}{2}}\Big(e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\Big)\tilde{J}_0(r|x|)\,dr\\
&= \frac{1}{|x|^{n-2}}\int_0^\infty \Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n-2}{2}}\Big(e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\Big)\\
&\qquad\qquad \times \frac{1}{r}\partial_r(r^2\tilde{J}_1(r|x|))\,dr\\
&= \frac{1}{|x|^{n-2}}\int_0^\infty \Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n}{2}}\Big(e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\Big)r^2\tilde{J}_1(r|x|)\,dr\\
&= \frac{1}{|x|^{n-2}}\int_0^{\frac{1}{|x|}} \Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n}{2}}\Big(e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\Big)r^2\tilde{J}_1(r|x|)\,dr \qquad (6.21)\\
&\quad + \frac{1}{|x|^{n-2}}\int_{\frac{1}{|x|}}^\infty \Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n}{2}}\Big(e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\Big)r^2\tilde{J}_1(r|x|)\,dr. \qquad (6.22)
\end{aligned}$$

For the integral in (6.21) we are able to derive the following estimate:

$$\begin{aligned}
&\Big|\frac{1}{|x|^{n-2}}\int_0^{\frac{1}{|x|}} \Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n}{2}}\Big(e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\Big)r^2\tilde{J}_1(r|x|)\,dr\Big|\\
&\quad\lesssim \frac{1}{|x|^{n-2}}e^{-\frac{c}{2}tm^{2\kappa}}\int_0^{\frac{1}{|x|}} r^{2\sigma+1}\langle r\rangle^{2\kappa-2}_{m,\sigma}\,dr \lesssim e^{-\frac{c}{2}tm^{2\kappa}}\langle x\rangle_m^{-(n+2\kappa+2\sigma-2)}.
\end{aligned}$$

For the integral in (6.22) we follow the same arguments to obtain the estimate

$$\begin{aligned}
&\Big|\frac{1}{|x|^{n-2}}\int_{\frac{1}{|x|}}^\infty \Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n}{2}}\Big(e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\Big)r^2\tilde{J}_1(r|x|)\,dr\Big|\\
&\quad\lesssim e^{-\frac{c}{2}tm^{2\kappa}}\langle x\rangle_m^{-(n+\frac{1}{2})}.
\end{aligned}$$

In the same way we can estimate the oscillating integral $F^{-1}_{\xi\to x}\big(e^{-\tau t\langle\xi\rangle^{2\kappa}_{m,\sigma}}\big)$. All together implies the statement (6.15) for even $n\geq 4$. To complete the proof it remains to show

$$\big\|F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)(t,\cdot)\big\|_{L^p}\lesssim (1+t)^{-(1+\alpha)}$$

for $p\in[1,\infty]$ and $t\geq 0$. Therefore we use the formula (see Section 6.7)

$$l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\sim\int_0^\infty \frac{\exp\big(-t\langle\xi\rangle^{\frac{2}{1+\alpha}}_{m,\sigma}s^{\frac{1}{1+\alpha}}\big)}{s^2+2s\cos((1+\alpha)\pi)+1}\,ds.$$

Taking account of the definition of modified Bessel functions (see Section 6.7) we get

$$\begin{aligned}
&F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)(t,x)\\
&= \int_0^\infty \Big(\int_0^\infty \frac{\exp\big(-t\langle r\rangle_m^{\frac{2}{1+\alpha}}s^{\frac{1}{1+\alpha}}\big)}{s^2+2s\cos((1+\alpha)\pi)+1} r^{n-1}\tilde{J}_{\frac{n}{2}-1}(r|x|)\,dr\Big)\,ds\\
&= \int_0^\infty \frac{1}{s^2+2s\cos((1+\alpha)\pi)+1}\\
&\qquad \times\Big(\int_0^\infty \exp\big(-t\langle r\rangle_m^{\frac{2}{1+\alpha}}s^{\frac{1}{1+\alpha}}\big)r^{n-1}\tilde{J}_{\frac{n}{2}-1}(r|x|)\,dr\Big)\,ds\\
&= \int_0^\infty \frac{1}{s^2+2s\cos((1+\alpha)\pi)+1}\Big(F^{-1}_{\xi\to x}\big(e^{-s^{\frac{1}{1+\alpha}}t\langle\xi\rangle_{m,\sigma}^{\frac{2}{1+\alpha}}}\big)(t,x)\Big)\,ds.
\end{aligned}$$

The estimate

$$\big\|F^{-1}_{\xi\to x}\big(e^{-s^{\frac{1}{1+\alpha}}t\langle\xi\rangle_{m,\sigma}^{\frac{2}{1+\alpha}}}\big)(t,\cdot)\big\|_{L^p} \lesssim e^{-\frac{1}{2}s^{\frac{1}{1+\alpha}}tm^{\frac{2}{1+\alpha}}}$$

implies

$$\begin{aligned}
&\big\|F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)(t,\cdot)\big\|_{L^p}\\
&\lesssim \int_0^\infty \frac{1}{s^2+2s\cos((1+\alpha)\pi)+1}\big\|F^{-1}_{\xi\to x}\big(e^{-s^{\frac{1}{1+\alpha}}t\langle\xi\rangle_{m,\sigma}^{\frac{2}{1+\alpha}}}\big)(t,\cdot)\big\|_{L^p}\,ds\\
&\lesssim \int_0^\infty \frac{e^{-\frac{1}{2}s^{\frac{1}{1+\alpha}}tm^{\frac{2}{1+\alpha}}}}{s^2+2s\cos((1+\alpha)\pi)+1}\,ds.
\end{aligned}$$

For $t\in(0,1]$ we may conclude

$$\big\|F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)(t,\cdot)\big\|_{L^p} \lesssim \int_0^\infty \frac{1}{s^2+2s\cos((1+\alpha)\pi)+1}\,ds \lesssim 1.$$

For $t\geq 1$ we have

$$\begin{aligned}
\big\|F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)(t,\cdot)\big\|_{L^p} &\lesssim \int_0^\infty \frac{e^{-\frac{1}{2}s^{\frac{1}{1+\alpha}}tm^{\frac{2}{1+\alpha}}}}{s^2+2s\cos((1+\alpha)\pi)+1}\,ds\\
&\lesssim \int_0^\infty \exp\big(-\tilde{C}_1 ts^{\frac{1}{1+\alpha}}\big)\,ds.
\end{aligned}$$

After the change of variables $\tau := ts^{\frac{1}{1+\alpha}}$ it follows

$$\begin{aligned}
\big\|F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)(t,\cdot)\big\|_{L^p} &\lesssim \int_0^\infty \exp\big(-\tilde{C}_1 ts^{\frac{1}{1+\alpha}}\big)ds\\
&\lesssim t^{-(1+\alpha)}\int_0^\infty \tau^\alpha \exp\big(-\tilde{C}_1\tau\big)d\tau \lesssim t^{-(1+\alpha)}.
\end{aligned}$$

We deduce for all $p\in[1,\infty]$ the estimate

$$\big\|F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)(t,\cdot)\big\|_{L^p} \lesssim (1+t)^{-(1+\alpha)} \quad \text{for all} \quad t\geq 0.$$

Summarizing all the estimates we may conclude

$$\begin{aligned}
&\big\|F^{-1}_{\xi\to x}\big(E_{1+\alpha}(-t^{1+\alpha}\langle\xi\rangle^2_{m,\sigma})\big)(t,\cdot)\big\|_{L^p}\\
&\quad\lesssim \big\|F^{-1}_{\xi\to x}\big(\exp\big(a_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)(t,\cdot)\big\|_{L^p}\\
&\qquad+\big\|F^{-1}_{\xi\to x}\big(\exp\big(b_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)(t,\cdot)\big\|_{L^p}\\
&\qquad+\big\|F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)(t,\cdot)\big\|_{L^p}\\
&\quad\lesssim e^{-Ct}+(1+t)^{-(1+\alpha)}\lesssim(1+t)^{-(1+\alpha)}.
\end{aligned}$$

This completes the proof. □

The following proposition is helpful for the treatment of σ-evolution models with a mass term.

Proposition 6.12. *The following estimate holds in* $\mathbb{R}^n$ *for* $\sigma>0$, $m>0$, $\alpha\in[0,1)$ *and for all* $n\geq 1$*:*

$$\big\|F^{-1}_{\xi\to x}\big(\langle\xi\rangle^2_{m,\sigma}E_{1+\alpha}(-t^{1+\alpha}\langle\xi\rangle^2_{m,\sigma})\big)(t,\cdot)\big\|_{L^p}\lesssim(1+t)^{-(1+\alpha)} \tag{6.23}$$

for $p\in[1,\infty]$ *and* $t\geq 0$.

Proof. The proof is similar to the proof of the previous proposition. In a first step we estimate the oscillating integrals

$$F^{-1}_{\xi\to x}\big(\langle\xi\rangle^2_{m,\sigma}e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big)\quad\text{and}\quad F^{-1}_{\xi\to x}\big(\langle\xi\rangle^2_{m,\sigma}e^{-\tau t\langle\xi\rangle^{2\kappa}_{m,\sigma}}\big),$$

where $c=-\cos(\frac{\pi}{1+\alpha}),\kappa=\frac{1}{1+\alpha}\in(\frac12,1)$ and $\tau>0$. Following the approach from the previous proof we may conclude an exponential type decay estimate

$$\begin{aligned}
&\big\|F^{-1}_{\xi\to x}\big(\langle\xi\rangle^2_{m,\sigma}e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big)\big\|_{L^p}\\
&\quad+\big\|F^{-1}_{\xi\to x}\big(\langle\xi\rangle^2_{m,\sigma}e^{-\tau t\langle\xi\rangle^{2\kappa}_{m,\sigma}}\big)\big\|_{L^p}\lesssim e^{-Ct}
\end{aligned}$$

with a suitable positive constant $C=C(m,\alpha)$, and for $p\in[1,\infty]$ and $t\geq 0$. Let us make some comments to the third oscillating integral. Following the same steps of treatment of the previous proof we may conclude

$$\big\|F^{-1}_{\xi\to x}\big(\langle\xi\rangle^2_{m,\sigma}l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)\big\|_{L^p}\lesssim(1+t)^{-(1+\alpha)}$$

for $p\in[1,\infty]$ and $t\geq 0$. Indeed, we use the formula

$$\langle\xi\rangle^2_{m,\sigma}l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\sim\int_0^\infty\langle\xi\rangle^2_{m,\sigma}\frac{\exp\big(-t\langle\xi\rangle^{\frac{2}{1+\alpha}}_{m,\sigma}s^{\frac{1}{1+\alpha}}\big)}{s^2+2s\cos((1+\alpha)\pi)+1}\,ds.$$

Taking account of the definition of modified Bessel functions we get

$$\begin{aligned}
&F^{-1}_{\xi\to x}\big(\langle\xi\rangle^2_{m,\sigma} l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)\\
&= \int_0^\infty \Big(\int_0^\infty \langle r\rangle^2_{m,\sigma}\frac{\exp\big(-t\langle r\rangle_m^{\frac{2}{1+\alpha}} s^{\frac{1}{1+\alpha}}\big)}{s^2+2s\cos((1+\alpha)\pi)+1} r^{n-1}\tilde{J}_{\frac{n}{2}-1}(r|x|)\,dr\Big)\,ds\\
&= \int_0^\infty \frac{1}{s^2+2s\cos((1+\alpha)\pi)+1}\\
&\quad\times\Big(\int_0^\infty \langle r\rangle^2_{m,\sigma}\exp\big(-t\langle r\rangle_m^{\frac{2}{1+\alpha}} s^{\frac{1}{1+\alpha}}\big) r^{n-1}\tilde{J}_{\frac{n}{2}-1}(r|x|)\,dr\Big)\,ds\\
&= \int_0^\infty \frac{1}{s^2+2s\cos((1+\alpha)\pi)+1}\\
&\quad\times\Big(F^{-1}_{\xi\to x}\big(\langle\xi\rangle^2_{m,\sigma} e^{-s^{\frac{1}{1+\alpha}}t\langle\xi\rangle_{m,\sigma}^{\frac{2}{1+\alpha}}}\big)(t,x)\Big)\,ds.
\end{aligned}$$

The estimate

$$\big\|F^{-1}_{\xi\to x}\big(\langle\xi\rangle^2_{m,\sigma} e^{-s^{\frac{1}{1+\alpha}}t\langle\xi\rangle_{m,\sigma}^{\frac{2}{1+\alpha}}}\big)(t,\cdot)\big\|_{L^p} \lesssim e^{-\frac{1}{2}s^{\frac{1}{1+\alpha}}tm^{\frac{2}{1+\alpha}}}$$

implies

$$\begin{aligned}
&\big\|F^{-1}_{\xi\to x}\big(\langle\xi\rangle^2_{m,\sigma} l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)(t,\cdot)\big\|_{L^p}\\
&\lesssim \int_0^\infty \frac{1}{s^2+2s\cos((1+\alpha)\pi)+1}\|F^{-1}_{\xi\to x}\big(e^{-s^{\frac{1}{1+\alpha}}t\langle\xi\rangle_{m,\sigma}^{\frac{2}{1+\alpha}}}\big)(t,\cdot)\|_{L^p}\,ds\\
&\lesssim \int_0^\infty \frac{e^{-\frac{1}{2}s^{\frac{1}{1+\alpha}}tm^{\frac{2}{1+\alpha}}}}{s^2+2s\cos((1+\alpha)\pi)+1}\,ds.
\end{aligned}$$

As in the previous proof we conclude the desired estimate. □

6.5 $L^r - L^q$ estimates for the formal solutions from Section 6.3

6.5.1 Models without any mass term

Proposition 6.13. *Let $u_0 \in L^r(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, $n\geq 1$, $r\geq 1$ and $\alpha\in(0,1)$. Then the function*

$$u = u(t,x) = \big(G^0_{\alpha,\sigma}(t) * u_0\big)(t,x)$$

satisfies the following $L^m - L^q$ estimates:

$$\|u(t,\cdot)\|_{L^q} \lesssim t^{-\frac{n(1+\alpha)}{2\sigma}(\frac{1}{m}-\frac{1}{q})}\|u_0\|_{L^m} \tag{6.24}$$

for all $r \leq m \leq q \leq \infty$ *provided that* $n(\frac{1}{m} - \frac{1}{q}) < 2\sigma$.
Let $u_0 \in H^\gamma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $n \geq 1$, $1 < r < \infty$ *and* $\alpha \in (0,1)$. *Then the function*

$$u = u(t,x) = \big(G^0_{\alpha,\sigma}(t) * u_0\big)(t,x)$$

satisfies the following estimates:

$$\|u(t,\cdot)\|_{H^\gamma_r} \lesssim \|u_0\|_{H^\gamma_r} \text{ and } \|u(t,\cdot)\|_{\dot{H}^\gamma_r} \lesssim \|u_0\|_{\dot{H}^\gamma_r}. \tag{6.25}$$

Proof. The inequality (6.24) follows from Young's inequality and Proposition 6.10. Applying these tools to the relation

$$|D|^\gamma (G^0_{\alpha,\sigma}(t) * u_0)(t,x) = \big(F^{-1}_{\xi\to x}\big(E_{\alpha+1}\big(-t^{\alpha+1}|\xi|^{2\sigma}\big)\big) * |D|^\gamma u_0\big)(t,x)$$

implies the inequality (6.25). This completes the proof. □

Proposition 6.14. *Let* $u_0 \in L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $n \geq 1$, $r \geq 1$ *and* $\alpha \in (0,1)$. *Then the function*

$$u = u(t,x) = \big(G^0_{\alpha,\sigma}(t) * u_0\big)(t,x)$$

satisfies the following estimate for any fixed $\delta > 0$ *small enough:*

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\beta^{r,\delta}_{\alpha,q,\sigma}}(\|u_0\|_{L^r} + \|u_0\|_{L^q}) \quad \text{for all} \quad q \in [r,\infty], \tag{6.26}$$

where

$$\beta^{r,\delta}_{\alpha,q,\sigma} := \min\Big\{\frac{n(1+\alpha)}{2\sigma}\Big(\frac{1}{r} - \frac{1}{q}\Big); 1-\delta\Big\}.$$

Proof. To get (6.26) we use ideas of D'Abbicco (cf. with [16]). For $t \in (0,1]$ we set $m = q$ in (6.24) to get the $L^q - L^q$ estimate

$$\|u(t,\cdot)\|_{L^q} \lesssim \|u_0\|_{L^q}.$$

For $t \geq 1$ we choose $m = r$ in (6.24) if $n(\frac{1}{r} - \frac{1}{q}) < 2\sigma$. Otherwise, in (6.24) the parameter m is chosen as the solution to

$$\frac{n(1+\alpha)}{2\sigma}\Big(\frac{1}{m} - \frac{1}{q}\Big) = 1-\delta$$

with a fixed sufficiently small positive δ. In this way, we may conclude the $L^r - L^q$ estimate

$$\|u(t,\cdot)\|_{L^q} \lesssim t^{-\beta^{r,\delta}_{\alpha,q,\sigma}}\|u_0\|_{L^r}.$$

Gluing both estimates together we derive the desired estimate (6.26). □

Remark 6.15. *The last two statements are valid for* $r = 1$, *too, in contrary to the paper [16]. In this paper the authors use estimates in scales of Morrey spaces from the paper [7], where* $r = 1$ *is excluded. For this reason the positive parameter* δ *appears in Proposition 6.1.*

The statements of the Propositions 6.13 and 6.14 allow to conclude the following result.

Corollary 6.16. *Let* $u_0 \in L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $n \geq 1$, $r \geq 1$ *and* $\alpha \in (0,1)$. *Then the function*

$$u = u(t,x) = \big(G^0_{\alpha,\sigma}(t) * u_0\big)(t,x)$$

belongs to

$$L^\infty\big((0,T), L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big) \text{ for all } T > 0.$$

Let $u_0 \in H^\gamma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $n \geq 1$, $1 < r < \infty$ *and* $\alpha \in (0,1)$. *Then the function*

$$u = u(t,x) = \big(G^0_{\alpha,\sigma}(t) * u_0\big)(t,x)$$

belongs to

$$L^\infty\big((0,T), H^\gamma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big) \text{ for all } T > 0.$$

The next result contains even the continuity property with respect to the time variable.

Proposition 6.17. *Let* $u_0 \in L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $n \geq 1$, $r \geq 1$ *and* $\alpha \in (0,1)$. *Then the function*

$$u = u(t,x) = \big(G^0_{\alpha,\sigma}(t) * u_0\big)(t,x)$$

belongs to

$$C\big([0,\infty), L^r(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)\big) \text{ for all } q \in [r,\infty).$$

Let $u_0 \in H^\gamma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $n \geq 1$, $1 < r < \infty$ *and* $\alpha \in (0,1)$. *Then the function*

$$u = u(t,x) = \big(G^0_{\alpha,\sigma}(t) * u_0\big)(t,x)$$

belongs to

$$C\big([0,\infty), H^\gamma_r(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)\big) \text{ for all } q \in [r,\infty).$$

Proof. The second statement follows immediately from the first statement by using only the higher regularity H^γ_r instead of L^r. The first statement follows from Proposition 6.37 of the Appendix. □

6.5.2 Models with a mass term

Proposition 6.18. *Let* $u_0 \in L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $n \geq 1$, $r \geq 1$ *and* $\alpha \in (0,1)$. *Then the function*

$$u = u(t,x) = \big(G^m_{\alpha,\sigma}(t) * u_0\big)(t,x)$$

belongs to

$$C\big([0,\infty), L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$$

and satisfies the following $L^r - L^q$ estimates:

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{-(1+\alpha)}\|u_0\|_{L^r} \tag{6.27}$$

for all $1 \leq r \leq q \leq \infty$.
Let $u_0 \in H^\gamma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $n \geq 1$, $1 \leq r < \infty$ and $\alpha \in (0,1)$. Then the function

$$u = u(t,x) = \big(G^m_{\alpha,\sigma}(t) * u_0\big)(t,x)$$

belongs to

$$C\big([0,\infty), H^\gamma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$$

and satisfies the following estimates:

$$\|u(t,\cdot)\|_{H^\gamma_r} \lesssim (1+t)^{-(1+\alpha)}\|u_0\|_{H^\gamma_r} \text{ and } \|u(t,\cdot)\|_{\dot{H}^\gamma_r} \lesssim (1+t)^{-(1+\alpha)}\|u_0\|_{\dot{H}^\gamma_r}. \tag{6.28}$$

Proof. The proof follows immediately from (6.5), (6.6), Proposition 6.11 and Lemma 6.38. To verify the last inequality we use

$$|D|^\gamma(G^m_{\alpha,\sigma}(t) * u_0)(t,x) = \big(F^{-1}_{\xi\to x}\big(E_{\alpha+1}\big(-t^{\alpha+1}\langle\xi\rangle^2_{m,\sigma}\big)\big) * |D|^\gamma u_0\big)(t,x).$$

The continuity of solutions follows from (6.8) and Proposition 6.12. This completes the proof. □

6.6 Proofs of the main results

Proof of Theorem 6.3

For any $n \geq \frac{2\sigma r}{1+\alpha}$ and sufficiently small $\delta \in (0,1)$ there exists a parameter $\overline{q} = \overline{q}(\delta) \in (r,\infty)$ such that

$$\frac{n(1+\alpha)}{2\sigma}\Big(\frac{1}{r} - \frac{1}{\overline{q}}\Big) = 1 - \delta. \tag{6.29}$$

We define the space

$$X(T) := L^\infty\big((0,T), L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$$

with the norm

$$\|u\|_{X(T)} := \operatorname{esssup}_{t\in(0,T)}\big\{(1+t)^{-\lambda}\|u(t,\cdot)\|_{L^r} \\ +(1+t)^{1-\delta-\lambda}\big(\|u(t,\cdot)\|_{L^{\overline{q}}} + \|u(t,\cdot)\|_{L^\infty}\big)\big\}.$$

For any $u \in X(T)$ we consider for $m = 0$ the operator

$$P : X(T) \longrightarrow X(T), \quad Pu := (G^0_{\alpha,\sigma}(t) * u_0)(t,x) + N^0_{\alpha,\sigma}(u)(t,x).$$

We shall prove that

$$\|Pu\|_{X(T)} \lesssim \|u_0\|_{L^r\cap L^\infty} + \|u\|^p_{X(T)}, \tag{6.30}$$
$$\|Pu - Pv\|_{X(T)} \lesssim \|u - v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big). \tag{6.31}$$

For the proof of (6.30), after taking into consideration the estimates (6.26), we have

$$\begin{aligned}\|G^0_{\alpha,\sigma}(t) * u_0\|_{X(T)} &= \operatorname{esssup}_{t\in(0,T)}\big\{(1+t)^{-\lambda}\|(G^0_{\alpha,\sigma}(t) * u_0)(t,\cdot)\|_{L^r} \\ &\quad +(1+t)^{1-\delta-\lambda}\big(\|(G^0_{\alpha,\sigma}(t) * u_0)(t,\cdot)\|_{L^{\bar q}} + \|(G^0_{\alpha,\sigma}(t) * u_0)(t,\cdot)\|_{L^\infty}\big)\big\} \\ &\lesssim \|u_0\|_{L^r\cap L^\infty}.\end{aligned}$$

It remains to prove that $\|N^0_{\alpha,\sigma}(u)\|_{X(T)} \lesssim \|u\|^p_{X(T)}$. If $u \in X(T)$, then we derive by interpolation the following estimate:

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\beta^{r,\delta}_{\alpha,q,\sigma}+\lambda}\|u\|_{X(T)} \quad \text{for all} \quad q \in [r,\infty]. \tag{6.32}$$

Consequently,

$$\begin{aligned}\||u(t,\cdot)|^p\|_{L^q} &\lesssim \|u(t,\cdot)\|^p_{L^{pq}} \lesssim (1+t)^{-p(\beta^{r,\delta}_{\alpha,pq,\sigma}-\lambda)}\|u\|_{X(T)} \\ &\lesssim (1+t)^{-p(\beta^{r,\delta}_{\alpha,p,\sigma}-\lambda)}\|u\|_{X(T)}\end{aligned} \tag{6.33}$$

for any $q \in [r,\infty]$ and due to $\beta^{r,\delta}_{\alpha,pq,\sigma} \geq \beta^{r,\delta}_{\alpha,p,\sigma}$. Thanks to (6.26) and (6.33) we can estimate

$$\|N^0_{\alpha,\sigma}(u)(t,\cdot)\|_{L^q} \lesssim \|u\|_{X(T)} I_q(t) \quad \text{for all} \quad t \in [0,T] \ \text{ and } \ q \in [r,\infty], \tag{6.34}$$

where

$$I_q(t) = \int_0^t (1+t-\tau)^{-\beta^{r,\delta}_{\alpha,q,\sigma}} \int_0^\tau (\tau - s)^{\alpha-1}(1+s)^{-p(\beta^{r,\delta}_{\alpha,p,\sigma}-\lambda)}\,ds\,d\tau.$$

We are interested to estimate the function $I_q(t)$ in (6.34). For this we apply Lemma 6.39. We notice that $p(\beta^{r,\delta}_{\alpha,p,\sigma} - \lambda) > 1$ if and only if

$$p > p_{\alpha,\lambda,\sigma,r,\delta}(n) := \max\Big\{p^r_{\alpha,\lambda,\sigma}(n); \frac{1}{1-\delta-\lambda}\Big\}.$$

Consequently, by using Lemma 6.39 we may estimate $I_q(t)$ as follows:

$$I_q(t) \lesssim \int_0^t (1+t-\tau)^{-\beta^{r,\delta}_{\alpha,q,\sigma}}(1+\tau)^{\alpha-1}\,d\tau \lesssim (1+t)^{-\beta^{r,\delta}_{\alpha,q,\sigma}+\alpha} \lesssim (1+t)^{-\beta^{r,\delta}_{\alpha,q,\sigma}+\lambda},$$

thanks to the fact that $\beta^{r,\delta}_{\alpha,q,\sigma} \in (0, 1-\delta]$ and $\alpha \in (0,1)$. Therefore (6.33) gives

$$\|N^0_{\alpha,\sigma}(u)\|_{X(T)} \lesssim \|u\|^p_{X(T)}.$$

Finally, it remains to show (6.31). Let $q \in [r,\infty]$. By Hölder's inequality, for $u, v \in X(T)$, and if p' denotes the conjugate to p, then we have

$$
\begin{aligned}
&\||u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{L^q}\\
&\lesssim \Big(\int_{\mathbb{R}^n} |u(s,x)-v(s,x)|^q \Big(|u(s,x)|^{p-1}+|v(s,x)|^{p-1}\Big)^q\,dx\Big)^{\frac{1}{q}}\\
&\lesssim \Big(\int_{\mathbb{R}^n} |u(s,x)-v(s,x)|^{pq}\,dx\Big)^{\frac{1}{pq}}\Big(\int_{\mathbb{R}^n}\Big(|u(s,x)|^{p-1}+|v(s,x)|^{p-1}\Big)^{qp'}dx\Big)^{\frac{1}{qp'}}\\
&\lesssim \|u(s,\cdot)-v(s,\cdot)\|_{L^{pq}}\||u(s,\cdot)|^{p-1}+|v(s,\cdot)|^{p-1}\|_{L^{qp'}}\\
&\lesssim \|u(s,\cdot)-v(s,\cdot)\|_{L^{pq}}\big(\||u(s,\cdot)|^{p-1}\|_{L^{qp'}}+\||v(s,\cdot)|^{p-1}\|_{L^{qp'}}\big)\\
&\lesssim \|u(s,\cdot)-v(s,\cdot)\|_{L^{pq}}\big(\|u(s,\cdot)\|^{p-1}_{L^{qp'(p-1)}}+\|v(s,\cdot)\|^{p-1}_{L^{qp'(p-1)}}\big)\\
&\lesssim \|u(s,\cdot)-v(s,\cdot)\|_{L^{pq}}\big(\|u(s,\cdot)\|^{p-1}_{L^{pq}}+\|v(s,\cdot)\|^{p-1}_{L^{pq}}\big)\\
&\lesssim (1+s)^{-p(\beta^{r,\delta}_{\alpha,p,\sigma}-\lambda)}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|N^0_{\alpha,\sigma}(u)(t,\cdot)-N^0_{\alpha,\sigma}(v)(t,\cdot)\|_{L^q} &\lesssim I_q(t)\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big)\\
&\lesssim (1+t)^{-\beta^{r,\delta}_{\alpha,q,\sigma}+\lambda}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big) \quad \text{for all} \quad t\in[0,T].
\end{aligned}
$$

We deduce that

$$
\begin{aligned}
\|Pu-Pv\|_{X(T)} &= \|N^0_{\alpha,\sigma}(u)-N^0_{\alpha,\sigma}(v)\|_{X(T)}\\
&\lesssim \|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big).
\end{aligned}
$$

Notice that $p > p_{\alpha,\lambda,\sigma,r,\delta}$ for all $\delta > 0$ if and only if $p > p_{\alpha,\lambda,\sigma,r}$.

Remark 6.19. *All the estimates (6.30) and (6.31) are uniformly with respect to $T \in (0,\infty)$ if $p > p_{\alpha,\lambda,\sigma,r}(n)$.*

From (6.30) it follows that P maps $X(T)$ into itself for all T and for small data. By standard contraction arguments (see [18]) the estimates (6.30) and (6.31) lead to the existence of unique solution to $u = Pu$ and, consequently, to (6.3) with $m = 0$, that is, the solution of (6.3) with $m = 0$ satisfies (6.26). Since all constants are independent of T we let T tend to ∞ and we obtain a global (in time) existence result for small data solutions to (6.3).
Finally, let us discuss the continuity of the solution with respect to t. The solution satisfies the operator equation

$$u(t) = G^0_{\alpha,\sigma}(t) * u_0 + N^0_{\alpha,\sigma}(u)(t).$$

The above estimates for $N^0_{\alpha,\sigma}(u)$ and the integral term $\int_0^t$ in $N^0_{\alpha,\sigma}(u)$ imply for all $T > 0$

$$
\begin{aligned}
&N^0_{\alpha,\sigma}(u) \in C\big([0,T], L^r(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)\big)\\
&\qquad \text{with } \lim_{t\to+0}\|N^0_{\alpha,\sigma}(u)(t,\cdot)\|_{L^r\cap L^\infty} = 0. \qquad (6.35)
\end{aligned}
$$

Proposition 6.17 gives

$$G^0_{\alpha,\sigma}(t) * u_0 \in C\big([0,T], L^r(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)\big) \text{ for all } q\in[r,\infty). \tag{6.36}$$

Consequently,

$$u \in C\big([0,\infty), L^r(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)\big) \text{ for all } q\in[r,\infty)$$

what we wanted to have.
If the data are large, then instead we get for $p>1$ the estimates

$$\|Pu\|_{X(T)} \le C\|u_0\|_{L^r\cap L^\infty} + C(T)\|u\|^p_{X(T)},$$
$$\|Pu-Pv\|_{X(T)} \le C(T)\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big),$$

where $C(T)$ tends to 0 for $T\to+0$. For this reason we can have for general (large) data a local (in time) existence result of weak solutions only. The proof is complete.

Proof of Theorem 6.4

If $1\le n<\frac{2\sigma r}{1+\alpha}$, then for all $q\in[r,\infty]$ we obtain

$$\frac{n(1+\alpha)}{2\sigma}\Big(\frac{1}{r}-\frac{1}{q}\Big) < 1-\frac{n(1+\alpha)}{2\sigma q} \le 1.$$

Hence, we can choose a positive δ such that there does not exist any $\overline{q}\in[r,\infty]$ which satisfies (6.29). For this reason,

$$\beta^{r,\delta}_{\alpha,q,\sigma} = \beta^r_{\alpha,q,\sigma} := \frac{n(1+\alpha)}{2\sigma}\Big(\frac{1}{r}-\frac{1}{q}\Big).$$

We define the space

$$X(T) := L^\infty\big((0,T), L^r(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)\big)$$

with the norm

$$\|u\|_{X(T)} := \operatorname{esssup}_{t\in(0,T)}\big\{(1+t)^{-\lambda}\|u(t,\cdot)\|_{L^r} + (1+t)^{\beta^r_{\alpha,\infty,\sigma}-\lambda}\|u(t,\cdot)\|_{L^\infty}\big\},$$

where $\beta^r_{\alpha,\infty,\sigma}=\frac{n(1+\alpha)}{2\sigma r}$. For any $u\in X(T)$, we consider for $m=0$ the operator

$$P: X(T)\longrightarrow X(T), \quad Pu := (G^0_{\alpha,\sigma}(t)*u_0)(t,x) + N^0_{\alpha,\sigma}(u)(t,x).$$

We shall prove that

$$\|Pu\|_{X(T)} \lesssim \|u_0\|_{L^r\cap L^\infty} + \|u\|^p_{X(T)}, \tag{6.37}$$
$$\|Pu-Pv\|_{X(T)} \lesssim \|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big). \tag{6.38}$$

For the proof of (6.37), after taking into consideration the estimates (6.26), we have

$$\begin{aligned}&\|G^0_{\alpha,\sigma}(t)*u_0\|_{X(T)}\\ &\quad= \text{esssup}_{t\in(0,T)}\big\{(1+t)^{-\lambda}\|(G^0_{\alpha,\sigma}(t)*u_0)(t,\cdot)\|_{L^r}\\ &\qquad+(1+t)^{\beta^r_{\alpha,\infty,\sigma}-\lambda}\|(G^0_{\alpha,\sigma}(t)*u_0)(t,\cdot)\|_{L^\infty}\big\}\\ &\quad\lesssim \|u_0\|_{L^r\cap L^\infty}.\end{aligned}$$

It remains to prove that $\|N^0_{\alpha,\sigma}(u)\|_{X(T)}\lesssim\|u\|^p_{X(T)}$. If $u\in X(T)$, then we derive by interpolation the following estimate:

$$\|u(t,\cdot)\|_{L^q}\lesssim(1+t)^{-\beta^r_{\alpha,q,\sigma}+\lambda}\|u\|_{X(T)} \quad \text{for all} \quad q\in[r,\infty]. \tag{6.39}$$

Consequently,

$$\begin{aligned}\||u(t,\cdot)|^p\|_{L^q}&\lesssim\|u(t,\cdot)\|^p_{L^{pq}}\lesssim(1+t)^{-p(\beta^r_{\alpha,pq,\sigma}-\lambda)}\|u\|_{X(T)}\\ &\lesssim(1+t)^{-p(\beta^r_{\alpha,p,\sigma}-\lambda)}\|u\|_{X(T)}\end{aligned} \tag{6.40}$$

for any $q\in[r,\infty]$ and due to $\beta^r_{\alpha,pq,\sigma}\geq\beta^r_{\alpha,p,\sigma}$. Thanks to (6.26) and (6.40) we can estimate

$$\|N^0_{\alpha,\sigma}(u)(t,\cdot)\|_{L^q}\lesssim\|u\|_{X(T)}I_q(t) \quad \text{for all} \quad t\in[0,T] \;\text{ and }\; q\in[r,\infty], \tag{6.41}$$

where

$$I_q(t)=\int_0^t(1+t-\tau)^{-\beta^r_{\alpha,q,\sigma}}\int_0^\tau(\tau-s)^{\alpha-1}(1+s)^{-p(\beta^r_{\alpha,p,\sigma}-\lambda)}\,ds\,d\tau.$$

We notice that $p(\beta^r_{\alpha,p,\sigma}-\lambda)>1$ if and only if

$$p>p^r_{\alpha,\lambda,\sigma}(n):=1+\frac{n(r-1)(1+\alpha)+2\sigma r(1+\lambda)}{(n-2\sigma r)(1+\alpha)+2\sigma r(1+\alpha-\lambda)}$$

under the assumptions $1\leq\sigma<\frac{\alpha+1}{2\lambda}$ and $1\leq r<\frac{\alpha+1}{2\sigma\lambda}$. Consequently, by using Lemma 6.39 we may estimate as follows:

$$I_q(t)\lesssim\int_0^t(1+t-\tau)^{-\beta^r_{\alpha,q,\sigma}}(1+\tau)^{\alpha-1}\,d\tau\lesssim(1+t)^{-\beta^r_{\alpha,q,\sigma}+\lambda},$$

thanks to the fact that $\beta^r_{\alpha,q,\sigma}\in(0,1)$ and $\alpha\in(0,1)$. Therefore (6.40) gives

$$\|N_{\alpha,\sigma}(u)\|_{X(T)}\lesssim\|u\|^p_{X(T)}.$$

The proof of (6.38) is similar to the proof of (6.31) of Theorem 6.3. Then we may conclude a uniquely determined solution

$$u\in L^\infty\big((0,T),L^r(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)\big) \;\text{ for all }\; T>0.$$

As at the end of the proof of Theorem 6.3 we verify that the solution u belongs even to

$$C\big([0,\infty),L^r(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)\big) \;\text{ for all }\; q\in[r,\infty).$$

The proof is complete.

Proof of Theorem 6.5

We define the solution space

$$X(T) := L^\infty\big((0,T), H^\gamma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$$

with the norm

$$\|u\|_{X(T)} := \text{esssup}_{t\in(0,T)}\big\{(1+t)^{-\lambda}\|u(t,\cdot)\|_{H^\gamma_r} + (1+t)^{1-\delta-\lambda}\big(\|u(t,\cdot)\|_{L^{\overline{q}}} + \|u(t,\cdot)\|_{L^\infty}\big)\big\},$$

where $\overline{q}$ is defined as in Section 6.6. For any $u \in X(T)$, we consider for $m = 0$ the operator

$$P : X(T) \longrightarrow X(T), \quad Pu := (G^0_{\alpha,\sigma}(t) * u_0)(t,x) + N^0_{\alpha,\sigma}(u)(t,x).$$

We shall prove that

$$\|Pu\|_{X(T)} \lesssim \|u_0\|_{H^\gamma_r \cap L^\infty} + \|u\|^p_{X(T)}, \tag{6.42}$$

$$\|Pu - Pv\|_{X(T)} \lesssim \|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big). \tag{6.43}$$

For the proof of (6.42), after taking account of the estimates (6.26) and (6.25) we have

$$\begin{aligned}
&\|G^0_{\alpha,\sigma}(t) * u_0\|_{X(T)} \\
&\quad = \text{esssup}_{0\le t\le T}\big\{(1+t)^{-\lambda}\|(G^0_{\alpha,\sigma}(t) * u_0)(t,\cdot)\|_{H^\gamma_r} \\
&\qquad + (1+t)^{1-\delta-\lambda}\big(\|(G^0_{\alpha,\sigma}(t) * u_0)(t,\cdot)\|_{L^{\overline{q}}} + \|(G^0_{\alpha,\sigma}(t) * u_0)(t,\cdot)\|_{L^\infty}\big)\big\} \\
&\quad \lesssim \|u_0\|_{H^\gamma_r \cap L^\infty}.
\end{aligned}$$

It remains to prove for $m = 0$ that $\|N^0_{\alpha,\sigma}(u)\|_{X(T)} \lesssim \|u\|^p_{X(T)}$. If $u \in X(T)$, then we derive by interpolation the following estimate:

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\beta^{r,\delta}_{\alpha,q,\sigma}+\lambda}\|u\|_{X(T)} \quad \text{for all} \quad q \in [r,\infty]. \tag{6.44}$$

Moreover, we have

$$\|u(t,\cdot)\|_{\dot{H}^\gamma_r} \lesssim (1+t)^{\lambda}\|u\|_{X(T)}. \tag{6.45}$$

As in Section 6.6 we deduce

$$\|N^0_{\alpha,\sigma}(u)(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\beta^{r,\delta}_{q,\sigma}+\lambda}\|u\|_{X(T)} \quad \text{for all} \quad t \in [0,T] \text{ and } q \in [r,\infty],$$

if and only if

$$p > p_{\alpha,\lambda,\sigma,r,\delta} := \max\Big\{p^r_{\alpha,\lambda,\sigma}(n); \frac{1}{1-\delta-\lambda}\Big\}.$$

Now let us turn to the desired estimate of the norm $\|N^m_{\alpha,\sigma}(u)(t,\cdot)\|_{\dot H^\gamma_r}$. We need to estimate the norm $\||u(t,\cdot)|^p\|_{\dot H^\gamma_r}$. Applying Proposition 6.34, with $p > \max\{2;\gamma\}$, we obtain

$$\begin{aligned}\||u(t,\cdot)|^p\|_{\dot H^\gamma_r} &\lesssim \|u(t,\cdot)\|_{\dot H^\gamma_r}\|u(t,\cdot)\|^{p-1}_{L^\infty}\\ &\lesssim (1+t)^{\lambda}\|u\|_{X(T)}(1+t)^{(p-1)(\lambda+\delta-1)}\|u\|^{p-1}_{X(T)}\\ &\lesssim (1+t)^{\lambda-(p-1)(1-\delta-\lambda)}\|u\|^p_{X(T)}\\ &\lesssim (1+t)^{-((p-1)(1-\delta-\lambda)-\lambda)}\|u\|^p_{X(T)}.\end{aligned} \tag{6.46}$$

Then

$$\|N^0_{\alpha,\sigma}(u)(t,\cdot)\|_{\dot H^\gamma_r} \lesssim \|u\|_{X(T)} I_r(t) \quad \text{for all} \quad t\in[0,T], \tag{6.47}$$

where

$$I_r(t) = \int_0^t\int_0^\tau (\tau-s)^{\alpha-1}(1+s)^{-((p-1)(1-\delta-\lambda)-\lambda)}\,ds\,d\tau. \tag{6.48}$$

If

$$p > \max\{p_0(\lambda,\delta);\gamma\},$$

where $p_0(\lambda,\delta) = 1+\frac{1+\lambda}{1-\delta-\lambda}$, then

$$I_r(t) \lesssim (1+t)^{\alpha} \lesssim (1+t)^{\lambda}.$$

We remark that $p_0(\lambda,\delta) > \frac{1}{1-\delta-\lambda}$ and also $p_0(\lambda,\delta) > 2$. Then we deduce that

$$\|N^0_{\alpha,\sigma}(u)\|_{X(T)} \lesssim \|u\|^p_{X(T)}$$

if and only if

$$p > p^1_{\alpha,\lambda,\sigma,r,\gamma,\delta} := \max\{p^r_{\alpha,\lambda,\sigma}(n); p_0(\lambda,\delta);\gamma\}.$$

Finally, we have to show (6.43). From Section 6.6 we get for $m=0$ the estimate

$$\begin{aligned}&\|N^0_{\alpha,\sigma}(u)(t,\cdot) - N^0_{\alpha,\sigma}(v)(t,\cdot)\|_{L^q}\\ &\quad\lesssim (1+t)^{-\beta^{r,\delta}_{\alpha,q,\sigma}+\lambda}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big)\end{aligned}$$

for all $t\in[0,T]$ and $q\in[r,\infty]$. It remains to prove

$$\begin{aligned}&\|N^0_{\alpha,\sigma}(u)(t,\cdot) - N^0_{\alpha,\sigma}(v)(t,\cdot)\|_{\dot H^\gamma_r}\\ &\quad\lesssim (1+t)^{\lambda}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big) \quad \text{for all} \quad t\in[0,T].\end{aligned}$$

From the above considerations it is sufficient to prove that

$$\begin{aligned}&\||u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{\dot H^\gamma_r}\\ &\quad\lesssim (1+s)^{\lambda+(p-1)(\lambda+\delta-1)}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big).\end{aligned}$$

By using the integral representation

$$|u(s,\cdot)|^p - |v(s,\cdot)|^p = p\int_0^1 (u(s,\cdot)-v(s,\cdot))Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\,d\omega, \tag{6.49}$$

where $Q(u) = u|u|^{p-2}$, we obtain

$$\begin{aligned}&\||u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{\dot{H}^\gamma_r}\\ &\qquad\lesssim \int_0^1 \||D|^\gamma\big((u(s,\cdot)-v(s,\cdot))Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\big)\|_{L^r}\,d\omega.\end{aligned} \tag{6.50}$$

Applying the fractional Leibniz formula from Proposition 6.36 to estimate a product in $\dot{H}^\gamma_r(\mathbb{R}^n)$ we get

$$\begin{aligned}&\||u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{\dot{H}^\gamma_r}\\ &\lesssim \int_0^1 \||D|^\gamma\big(u(s,\cdot)-v(s,\cdot)\big)\|_{L^r}\|Q(\omega u(s,x)+(1-\omega)v(s,x))\|_{L^\infty}\,d\omega\\ &\quad+\int_0^1 \|u(s,\cdot)-v(s,\cdot)\|_{L^\infty}\||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\,d\omega\\ &\lesssim \||D|^\gamma\big(u(s,\cdot)-v(s,\cdot)\big)\|_{L^r}\big(\|u(s,\cdot)\|^{p-1}_{L^\infty}+\|v(s,\cdot)\|^{p-1}_{L^\infty}\big)\\ &\quad+\|u(s,\cdot)-v(s,\cdot)\|_{L^\infty}\int_0^1 \||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\,d\omega\\ &\lesssim \|u(s,\cdot)-v(s,\cdot)\|_{\dot{H}^\gamma_r}\big(\|u(s,\cdot)\|^{p-1}_{L^\infty}+\|v(s,\cdot)\|^{p-1}_{L^\infty}\big)\\ &\quad+\|u(s,\cdot)-v(s,\cdot)\|_{L^\infty}\int_0^1 \||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\,d\omega\\ &\lesssim (1+s)^{\lambda+(p-1)(\lambda+\delta-1)}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big)\\ &\quad+(1+s)^{\lambda+\delta-1}\|u-v\|_{X(T)}\int_0^1 \||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\,d\omega.\end{aligned}$$

We apply again Proposition 6.34 to estimate the term inside of the integral. In this way we obtain

$$\begin{aligned}&\int_0^1 \||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}d\omega\\ &\qquad\lesssim \int_0^1 \||D|^\gamma(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\\ &\qquad\qquad\times\|\omega u(s,\cdot)+(1-\omega)v(s,\cdot)\|^{p-2}_{L^\infty}\,d\omega\\ &\qquad\lesssim \int_0^1 (1+s)^\lambda\|\omega u+(1-\omega)v\|_{X(T)}\\ &\qquad\qquad\times(1+s)^{(p-2)(\lambda+\delta-1)}\|\omega u+(1-\omega)v\|^{p-2}_{X(T)}d\omega\\ &\qquad\lesssim \int_0^1 (1+s)^{\lambda+(p-2)(\lambda+\delta-1)}\|\omega u+(1-\omega)v\|^{p-1}_{X(T)}\,d\omega\\ &\qquad\lesssim (1+s)^{\lambda+(p-2)(\lambda+\delta-1)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big).\end{aligned}$$

Then

$$\big\||u(s,\cdot)|^p-|v(s,\cdot)|^p\big\|_{\dot{H}^\gamma_r}\lesssim(1+s)^{\lambda+(p-1)(\lambda+\delta-1)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big).$$

Hence,

$$\begin{aligned}&\|N^0_{\alpha,\sigma}(u)(t,\cdot)-N^0_{\alpha,\sigma}(v)(t,\cdot)\|_{\dot{H}^\gamma_r}\\ &\qquad\lesssim(1+t)^\lambda\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big)\quad\text{for all}\quad t\in[0,T].\end{aligned}$$

We deduce that

$$\begin{aligned}\|Pu-Pv\|_{X(T)}&=\|N^0_{\alpha,\sigma}(u)-N^0_{\alpha,\sigma}(v)\|_{X(T)}\\ &\lesssim\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big).\end{aligned}$$

Notice that $p>p^1_{\alpha,\lambda,\sigma,r,\gamma,\delta}$ for all $\delta>0$ if and only if $p>p_{\alpha,\lambda,\sigma,r,\gamma}$. Then we may conclude a uniquely determined solution

$$u\in L^\infty\big((0,T),H^\gamma_r(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)\big)\ \text{for all}\ T>0.$$

As at the end of the proof of Theorem 6.3 we verify that the solution u belongs even to

$$C\big([0,\infty),H^\gamma_r(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)\big)\ \text{for all}\ q\in[r,\infty).$$

If the data are large, then instead we get for $p>2$ the estimates

$$\begin{aligned}&\|Pu\|_{X(T)}\le C\|u_0\|_{H^\gamma_r\cap L^\infty}+C(T)\|u\|^p_{X(T)},\\ &\|Pu-Pv\|_{X(T)}\le C(T)\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big),\end{aligned}$$

where $C(T)$ tends to 0 for $T\to+0$. For this reason we can have for general (large) data a local (in time) existence result of weak solutions only. This completes the proof.

Proof of Theorem 6.6

We define the solution space

$$X(T):=L^\infty\big((0,T),H^\gamma_r(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)\big)$$

with the norm

$$\|u\|_{X(T)}:=\operatorname{esssup}_{t\in(0,T)}\big\{(1+t)^{-\lambda}\|u(t,\cdot)\|_{H^\gamma_r}+(1+t)^{\beta^r_{\alpha,\infty,\sigma}-\lambda}\|u(t,\cdot)\|_{L^\infty}\big\},$$

where $\beta^r_{\alpha,\infty,\sigma}$ is defined as in Section 6.6. For any $u\in X(T)$, we consider for $m=0$ the operator

$$P:X(T)\longrightarrow X(T),\quad Pu:=(G^0_{\alpha,\sigma}(t)*u_0)(t,x)+N^0_{\alpha,\sigma}(u)(t,x).$$

We shall prove that

$$\|Pu\|_{X(T)} \lesssim \|u_0\|_{H^\gamma_r \cap L^\infty} + \|u\|^p_{X(T)}, \tag{6.51}$$

$$\|Pu - Pv\|_{X(T)} \lesssim \|u - v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big). \tag{6.52}$$

For the proof of (6.51), after taking account of the estimates (6.26) and (6.25) we have

$$\begin{aligned}
&\|G^0_{\alpha,\sigma}(t) * u_0\|_{X(T)} \\
&\quad = \text{esssup}_{t\in(0,T)}\big\{(1+t)^{-\lambda}\|(G^0_{\alpha,\sigma}(t) * u_0)(t,\cdot)\|_{H^\gamma_r} \\
&\qquad\quad +(1+t)^{\beta^r_{\alpha,\infty,\sigma}-\lambda}\|(G^0_{\alpha,\sigma}(t) * u_0)(t,\cdot)\|_{L^\infty}\big)\big\} \\
&\quad \lesssim \|u_0\|_{H^\gamma_r \cap L^\infty}.
\end{aligned}$$

It remains to prove for $m = 0$ that $\|N^0_{\alpha,\sigma}(u)\|_{X(T)} \lesssim \|u\|^p_{X(T)}$. If $u \in X(T)$, then we derive by interpolation the following estimate:

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\beta^r_{\alpha,q,\sigma}+\lambda}\|u\|_{X(T)} \quad \text{for all} \quad q \in [r,\infty]. \tag{6.53}$$

Moreover, we have

$$\|u(t,\cdot)\|_{\dot{H}^\gamma_r} \lesssim (1+t)^{\lambda}\|u\|_{X(T)}. \tag{6.54}$$

As in Section 6.6 we deduce

$$\|N^0_{\alpha,\sigma}(u)(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\beta^r_{\alpha,q,\sigma}+\lambda}\|u\|_{X(T)} \quad \text{for all } t \in [0,T] \text{ and } q \in [r,\infty],$$

if and only if

$$p > p^r_{\alpha,\lambda,\sigma}(n).$$

Now let us turn to the desired estimate of the norm $\|N^m_{\alpha,\sigma}(u)(t,\cdot)\|_{\dot{H}^\gamma_r}$. We need to estimate the norm $\||u(t,\cdot)|^p\|_{\dot{H}^\gamma_r}$. Applying Proposition 6.34 with $p > \max\{2;\gamma\}$ we obtain

$$\begin{aligned}
\||u(t,\cdot)|^p\|_{\dot{H}^\gamma_r} &\lesssim \|u(t,\cdot)\|_{\dot{H}^\gamma_r}\|u(t,\cdot)\|^{p-1}_{L^\infty} \\
&\lesssim (1+t)^{\lambda}\|u\|_{X(T)}(1+t)^{(p-1)(-\beta^r_{\alpha,\infty,\sigma}+\lambda)}\|u\|^{p-1}_{X(T)} \\
&\lesssim (1+t)^{\lambda-(p-1)(\beta^r_{\alpha,\infty,\sigma}-\lambda)}\|u\|^p_{X(T)} \\
&\lesssim (1+t)^{-((p-1)(\beta^r_{\alpha,\infty,\sigma}-\lambda)-\lambda)}\|u\|^p_{X(T)}.
\end{aligned} \tag{6.55}$$

Then

$$\|N^0_{\alpha,\sigma}(u)(t,\cdot)\|_{\dot{H}^\gamma_r} \lesssim \|u\|_{X(T)} I_r(t) \quad \text{for all} \quad t \in [0,T], \tag{6.56}$$

where

$$I_r(t) = \int_0^t \int_0^\tau (\tau - s)^{\alpha-1}(1+s)^{-((p-1)(\beta^r_{\alpha,\infty,\sigma}-\lambda)-\lambda)}\, ds\, d\tau. \tag{6.57}$$

Notice that $(p-1)(\beta^r_{\alpha,\infty,\sigma}-\lambda)-\lambda>1$ if and only if

$$p>p^1_{\alpha;\lambda;\sigma;r}(n)=1+\frac{2\sigma r(1+\lambda)}{(n-2\sigma r)(1+\alpha)+2\sigma r(1+\alpha-\lambda))}$$

under the assumptions $1\leq\sigma<\frac{\alpha+1}{2\lambda}$ and $1<r<\frac{\alpha+1}{2\sigma\lambda}$. If

$$p>p^1_{\alpha;\lambda;\sigma;r}(n),$$

then

$$I_r(t)\lesssim(1+t)^\lambda.$$

We remark that $p^r_{\alpha,\lambda,\sigma}(n)\geq p^1_{\alpha;\lambda;\sigma;r}(n)>2$. Then we deduce that

$$\|N^0_{\alpha,\sigma}(u)\|_{X(T)}\lesssim\|u\|^p_{X(T)}$$

if and only if

$$p>\max\{p^r_{\alpha,\lambda,\sigma}(n);\gamma\}.$$

Finally, we have to show (6.43). From Section 6.6 we get for $m=0$ the estimate

$$\begin{aligned}&\|N^0_{\alpha,\sigma}(u)(t,\cdot)-N^0_{\alpha,\sigma}(v)(t,\cdot)\|_{L^q}\\&\qquad\lesssim(1+t)^{-\beta^r_{\alpha,q,\sigma}+\lambda}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big)\end{aligned}$$

for all $t\in[0,T]$ and $q\in[r,\infty]$. It remains to prove

$$\begin{aligned}&\|N^0_{\alpha,\sigma}(u)(t,\cdot)-N^0_{\alpha,\sigma}(v)(t,\cdot)\|_{\dot H^\gamma_r}\\&\qquad\lesssim(1+t)^\lambda\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big)\quad\text{for all}\quad t\in[0,T].\end{aligned}$$

From the above considerations it is sufficient to prove that

$$\begin{aligned}&\||u(s,\cdot)|^p-|v(s,\cdot)|^p\|_{\dot H^\gamma_r}\\&\qquad\lesssim(1+s)^{\lambda+(p-1)(-\beta^r_{\alpha,\infty,\sigma}+\lambda)}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big).\end{aligned}$$

By using the integral representation

$$|u(s,\cdot)|^p-|v(s,\cdot)|^p=p\int_0^1(u(s,\cdot)-v(s,\cdot))Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\,d\omega,\tag{6.58}$$

where $Q(u)=u|u|^{p-2}$, we obtain

$$\begin{aligned}&\||u(s,\cdot)|^p-|v(s,\cdot)|^p\|_{\dot H^\gamma_r}\\&\qquad\lesssim\textstyle\int_0^1\||D|^\gamma\big((u(s,\cdot)-v(s,\cdot))Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\big)\|_{L^r}\,d\omega.\end{aligned}\tag{6.59}$$

Applying the fractional Leibniz formula from Proposition 6.36 to estimate a product in $\dot{H}^\gamma_r(\mathbb{R}^n)$ we get

$$\begin{aligned}
&\||u(s,\cdot)|^p-|v(s,\cdot)|^p\|_{\dot{H}^\gamma_r}\\
&\lesssim \int_0^1 \||D|^\gamma\big(u(s,\cdot)-v(s,\cdot)\big)\|_{L^r}\|Q(\omega u(s,x)+(1-\omega)v(s,x))\|_{L^\infty}\,d\omega\\
&\quad+\int_0^1 \|u(s,\cdot)-v(s,\cdot)\|_{L^\infty}\||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\,d\omega\\
&\lesssim \||D|^\gamma\big(u(s,\cdot)-v(s,\cdot)\big)\|_{L^r}\big(\|u(s,\cdot)\|^{p-1}_{L^\infty}+\|v(s,\cdot)\|^{p-1}_{L^\infty}\big)\\
&\quad+\|u(s,\cdot)-v(s,\cdot)\|_{L^\infty}\int_0^1 \||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\,d\omega\\
&\lesssim \|u(s,\cdot)-v(s,\cdot)\|_{\dot{H}^\gamma_r}\big(\|u(s,\cdot)\|^{p-1}_{L^\infty}+\|v(s,\cdot)\|^{p-1}_{L^\infty}\big)\\
&\quad+\|u(s,\cdot)-v(s,\cdot)\|_{L^\infty}\int_0^1 \||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\,d\omega\\
&\lesssim (1+s)^{\lambda+(p-1)(-\beta^r_{\alpha,\infty,\sigma}+\lambda)}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big)\\
&\quad+(1+s)^{-\beta^r_{\alpha,\infty,\sigma}+\lambda}\|u-v\|_{X(T)}\int_0^1 \||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\,d\omega.
\end{aligned}$$

We apply again Proposition 6.34 to estimate the term inside of the integral. In this way we obtain

$$\begin{aligned}
&\int_0^1 \||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}d\omega\\
&\qquad\lesssim \int_0^1 \||D|^\gamma(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\\
&\qquad\qquad\times\|\omega u(s,\cdot)+(1-\omega)v(s,\cdot)\|^{p-2}_{L^\infty}\,d\omega\\
&\qquad\lesssim \int_0^1 (1+s)^\lambda\|\omega u+(1-\omega)v\|_{X(T)}\\
&\qquad\qquad\times(1+s)^{(p-2)(-\beta^r_{\alpha,\infty,\sigma}+\lambda)}\|\omega u+(1-\omega)v\|^{p-2}_{X(T)}d\omega\\
&\qquad\lesssim \int_0^1 (1+s)^{\lambda+(p-2)(-\beta^r_{\alpha,\infty,\sigma}+\lambda)}\|\omega u+(1-\omega)v\|^{p-1}_{X(T)}\,d\omega\\
&\qquad\lesssim (1+s)^{\lambda+(p-2)(-\beta^r_{\alpha,\infty,\sigma}+\lambda)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big).
\end{aligned}$$

Then

$$\||u(s,\cdot)|^p-|v(s,\cdot)|^p\|_{\dot{H}^\gamma_r}\lesssim (1+s)^{\lambda+(p-1)(-\beta^r_{\alpha,\infty,\sigma}+\lambda)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big).$$

Hence,

$$\begin{aligned}
&\|N^0_{\alpha,\sigma}(u)(t,\cdot)-N^0_{\alpha,\sigma}(v)(t,\cdot)\|_{\dot{H}^\gamma_r}\\
&\qquad\lesssim (1+t)^\lambda\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big)\quad\text{for all}\quad t\in[0,T].
\end{aligned}$$

We deduce that

$$\begin{aligned}\|Pu - Pv\|_{X(T)} &= \|N^0_{\alpha,\sigma}(u) - N^0_{\alpha,\sigma}(v)\|_{X(T)} \\ &\lesssim \|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big).\end{aligned}$$

Summarizing we may conclude a uniquely determined solution

$$u \in L^\infty\big((0,T), H^\gamma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big) \text{ for all } T > 0.$$

As at the end of the proof of Theorem 6.3 we verify that the solution u belongs even to

$$C\big([0,\infty), H^\gamma_r(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)\big) \text{ for all } q \in [r,\infty).$$

If the data are large, then instead we get for $p > 2$ the estimates

$$\begin{aligned}&\|Pu\|_{X(T)} \le C\|u_0\|_{H^\gamma_r \cap L^\infty} + C(T)\|u\|^p_{X(T)}, \\ &\|Pu - Pv\|_{X(T)} \le C(T)\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big),\end{aligned}$$

where $C(T)$ tends to 0 for $T \to +0$. For this reason we can have for general (large) data a local (in time) existence result of weak solutions only.
This completes the proof.

Proof of Theorem 6.7

We recall that the solution of (6.3) is given by

$$u(t,x) = (G^m_{\alpha,\sigma}(t) * u_0)(t,x) + N^m_{\alpha,\sigma}(u)(t,x).$$

Let $T > 0$. We define the space

$$X(T) := C\big([0,T]; L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$$

with the norm

$$\|u\|_{X(T)} := \sup_{0 \le t \le T} \big\{(1+t)^{1-\alpha}\big(\|u(t,\cdot)\|_{L^r} + \|u(t,\cdot)\|_{L^\infty}\big)\big\}.$$

For any $u \in X(T)$ we consider the operator

$$P : X(T) \to X(T), \quad Pu := (G^m_{\alpha,\sigma}(t) * u_0)(t,x) + N^m_{\alpha,\sigma}(u)(t,x).$$

We shall prove that

$$\|Pu\|_{X(T)} \lesssim \|u_0\|_{L^r \cap L^\infty} + \|u\|^p_{X(T)}, \tag{6.60}$$

$$\|Pu - Pv\|_{X(T)} \lesssim \|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big). \tag{6.61}$$

After proving (6.60) and (6.61) we may conclude the global (in time) result of small data solutions in Theorem 6.7. Due to Proposition 6.18 we know that

$$G^m_{\alpha,\sigma}(t) * u_0 \in C\big([0,\infty), L^r(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)\big).$$

By using (6.27) we have

$$\begin{aligned}
&\|G^m_{\alpha,\sigma}(t) * u_0\|_{X(T)} \\
&\quad= \sup_{0\le t\le T}\big\{(1+t)^{1-\alpha}\big(\|(G^m_{\alpha,\sigma}(t)*u_0)(t,\cdot)\|_{L^r}+\|(G^m_{\alpha,\sigma}(t)*u_0)(t,\cdot)\|_{L^\infty}\big)\big\} \\
&\quad\lesssim \sup_{0\le t\le T}\big\{(1+t)^{1-\alpha}(1+t)^{-(1+\alpha)}\big\}\|u_0\|_{L^r\cap L^\infty} \\
&\quad\lesssim \sup_{t\ge 0}\big\{(1+t)^{1-\alpha}(1+t)^{-(1+\alpha)}\big\}\|u_0\|_{L^q\cap L^\infty} \lesssim \|u_0\|_{L^r\cap L^\infty}.
\end{aligned} \tag{6.62}$$

It remains to prove $\|N^m_{\alpha,\sigma}u\|_{X(T)} \lesssim \|u\|^p_{X(T)}$. If $u\in X(T)$, then by interpolation we derive

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{\alpha-1}\|u\|_{X(T)} \quad \text{for all} \quad t\in[0,T] \quad \text{and} \quad q\in[r,\infty].$$

On the other hand, we have

$$\begin{aligned}
\||u(t,\cdot)|^p\|_{L^q} \le \|u(t,\cdot)\|^p_{L^{pq}} \lesssim (1+t)^{-p(1-\alpha)}\|u\|^p_{X(T)} \\
\text{for all} \quad t\in[0,T] \quad \text{and} \quad q\in[r,\infty].
\end{aligned} \tag{6.63}$$

Thanks to (6.27) and (6.63) we may derive the estimate

$$\|N^m_{\alpha,\sigma}u(t,\cdot)\|_{L^q} \lesssim \|u\|^p_{X(T)} I(t) \quad \text{for all} \quad t\in[0,T] \quad \text{and} \quad q\in[r,\infty], \text{ where}$$
$$I(t) = \int_0^t (1+t-\tau)^{-(1+\alpha)} \int_0^\tau (\tau-s)^{\alpha-1}(1+s)^{-p(1-\alpha)}\,ds\,d\tau. \tag{6.64}$$

We are interested to estimate the right-hand side of (6.64). For this we need the Lemma 6.39. We put

$$\omega(\tau) = \int_0^\tau (\tau-s)^{\alpha-1}(1+s)^{-p(1-\alpha)}\,ds.$$

Thanks to Lemma 6.39 we obtain

$$\omega(\tau) \lesssim \begin{cases} (1+\tau)^{\alpha-1} & \text{if} \quad p>\frac{1}{1-\alpha}, \\ (1+\tau)^{\alpha-1}\ln(2+\tau) & \text{if} \quad p=\frac{1}{1-\alpha}, \\ (1+\tau)^{\alpha-p(1-\alpha)} & \text{if} \quad p<\frac{1}{1-\alpha}. \end{cases} \tag{6.65}$$

If we assume that $p>\frac{1}{1-\alpha}$, then we obtain $\omega(\tau)\lesssim(1+\tau)^{\alpha-1}$.

Hence,

$$I(t) \lesssim \int_0^t (1+t-\tau)^{-(1+\alpha)}\omega(\tau)\,d\tau \lesssim \int_0^t (1+t-\tau)^{-(1+\alpha)}(1+\tau)^{\alpha-1}\,d\tau. \tag{6.66}$$

Once more we apply Lemma 6.39 to (6.66) to obtain $I(t) \lesssim (1+t)^{\alpha-1}$.
Hence, $\|N^m_{\alpha,\sigma}u\|_{X(T)} \lesssim \|u\|^p_{X(T)}$. Finally, it remains to show (6.61). Let $r \in [q,\infty]$. By Hölder's inequality, for $u, v \in X(T)$, and if p' denotes the conjugate to p, then we have

$$\begin{aligned}
&\||u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{L^q}\\
&\lesssim \Big(\int_{\mathbb{R}^n} |u(s,x)-v(s,x)|^q \Big(|u(s,x)|^{p-1} + |v(s,x)|^{p-1}\Big)^q dx\Big)^{\frac{1}{q}}\\
&\lesssim \Big(\int_{\mathbb{R}^n} |u(s,x)-v(s,x)|^{pq}\, dx\Big)^{\frac{1}{pq}} \Big(\int_{\mathbb{R}^n}\Big(|u(s,x)|^{p-1}+|v(s,x)|^{p-1}\Big)^{qp'} dx\Big)^{\frac{1}{qp'}}\\
&\lesssim \|u(s,\cdot)-v(s,\cdot)\|_{L^{pq}} \||u(s,\cdot)|^{p-1} + |v(s,\cdot)|^{p-1}\|_{L^{qp'}}\\
&\lesssim \|u(s,\cdot)-v(s,\cdot)\|_{L^{pq}} \big(\||u(s,\cdot)|^{p-1}\|_{L^{qp'}} + \||v(s,\cdot)|^{p-1}\|_{L^{qp'}}\big)\\
&\lesssim \|u(s,\cdot)-v(s,\cdot)\|_{L^{pq}} \big(\|u(s,\cdot)\|^{p-1}_{L^{qp'(p-1)}} + \|v(s,\cdot)\|^{p-1}_{L^{qp'(p-1)}}\big)\\
&\lesssim \|u(s,\cdot)-v(s,\cdot)\|_{L^{pq}} \big(\|u(s,\cdot)\|^{p-1}_{L^{pq}} + \|v(s,\cdot)\|^{p-1}_{L^{pq}}\big)\\
&\lesssim (1+s)^{-p(1-\alpha)} \|u-v\|_{X(T)} \big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big).
\end{aligned}$$

Hence,

$$\begin{aligned}
\|N^m_{\alpha,\sigma}(u)(t,\cdot) - N^m_{\alpha,\sigma}(v)(t,\cdot)\|_{L^q} &\lesssim I(t)\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big)\\
&\lesssim (1+t)^{\alpha-1}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big) \quad \text{for all} \quad t\in[0,T].
\end{aligned}$$

We deduce that

$$\begin{aligned}
\|Pu - Pv\|_{X(T)} &= \|N^m_{\alpha,\sigma}(u) - N^m_{\alpha,\sigma}(v)\|_{X(T)}\\
&\lesssim \|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big).
\end{aligned}$$

Remark 6.20. *All estimates (6.60) and (6.61) are uniformly with respect to* $T \in (0,\infty)$ *if* $p > \frac{1}{1-\alpha}$.

From (6.60) it follows that P maps $X(T)$ into itself for all T and for small data. By standard contraction arguments (see [18]) the estimates (6.60) and (6.61) lead to the existence of unique solution to $u = Pu$ and, consequently, to (6.3), that is, the solution of (6.3) satisfies (6.62). Since all constants are independent of T we let T tend to ∞ and we obtain a global (in time) existence result for small data solutions to (6.3).
If the data are large, then instead we get for $p > 1$ the estimates

$$\begin{aligned}
&\|Pu\|_{X(T)} \le C\|u_0\|_{L^r\cap L^\infty} + C(T)\|u\|^p_{X(T)},\\
&\|Pu - Pv\|_{X(T)} \le C(T)\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big),
\end{aligned}$$

where $C(T)$ tends to 0 for $T \to +0$. For this reason we can have for general (large) data a local (in time) existence result of weak solutions only. This completes the proof.

Proof of Theorem 6.8

Let $T > 0$. We define the space

$$X(T) := C\big([0,T], H^\gamma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$$

with the norm

$$\|u\|_{X(T)} := \sup_{0\le t\le T} \big\{(1+t)^{1-\alpha}\big(\|u(t,\cdot)\|_{H^\gamma_r} + \|u(t,\cdot)\|_{L^\infty}\big)\big\}.$$

For any $u \in X(T)$ we consider the operator

$$P : X(T) \to X(T), \quad Pu := (G^m_{\alpha,\sigma}(t) * u_0)(t,x) + N^m_{\alpha,\sigma}(u)(t,x).$$

We shall prove that

$$\|Pu\|_{X(T)} \lesssim \|u_0\|_{H^\gamma_r\cap L^\infty} + \|u\|^p_{X(T)}, \tag{6.67}$$

$$\|Pu - Pv\|_{X(T)} \lesssim \|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big). \tag{6.68}$$

After proving (6.67) and (6.68) we may conclude the global (in time) existence result of small data solutions in Theorem 6.8. Due to Proposition 6.18 we know that

$$G^m_{\alpha,\sigma}(t) * u_0 \in C\big([0,\infty), H^\gamma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big).$$

By using (6.27) we have

$$\begin{aligned}
&\|G^m_{\alpha,\sigma}(t) * u_0\|_{X(T)} \\
&\quad= \sup_{0\le t\le T} \big\{(1+t)^{1-\alpha}\big(\|(G^m_{\alpha,\sigma}(t) * u_0)(t,\cdot)\|_{H^\gamma_r} + \|(G^m_{\alpha,\sigma}(t) * u_0)(t,\cdot)\|_{L^\infty}\big)\big\} \\
&\quad\lesssim \sup_{0\le t\le T} \big\{(1+t)^{1-\alpha}(1+t)^{-(1+\alpha)}\big\}\|u_0\|_{H^\gamma_r} \\
&\quad\lesssim \sup_{t\ge 0} \big\{(1+t)^{1-\alpha}(1+t)^{-(1+\alpha)}\big\}\|u_0\|_{H^\gamma_r} \lesssim \|u_0\|_{H^\gamma_r\cap L^\infty}.
\end{aligned} \tag{6.69}$$

It remains to prove $\|N^m_{\alpha,\sigma}u\|_{X(T)} \lesssim \|u\|^p_{X(T)}$. If $u \in X(T)$, then we derive

$$\|u(t,\cdot)\|_{H^\gamma_r\cap L^\infty} \lesssim (1+t)^{\alpha-1}\|u\|_{X(T)}.$$

On the other hand, applying Proposition 6.33 with $p > \max\{2;\gamma\}$ we obtain

$$\begin{aligned}
\||u(t,\cdot)|^p\|_{H^\gamma_r} &\lesssim \|u(t,\cdot)\|_{H^\gamma_r}\|u(t,\cdot)\|^{p-1}_{L^\infty} \\
&\lesssim (1+t)^{\alpha-1}\|u\|_{X(T)}(1+t)^{(p-1)(\alpha-1)}\|u\|^{p-1}_{X(T)} \\
&\lesssim (1+t)^{-p(1-\alpha)}\|u\|^p_{X(T)}.
\end{aligned} \tag{6.70}$$

Moreover, we have

$$\||u(t,\cdot)|^p\|_{L^\infty} \lesssim (\|u(t,\cdot)\|_{L^\infty})^p \lesssim (1+t)^{-p(1-\alpha)}\|u\|^p_{X(T)}. \tag{6.71}$$

Thanks to (6.28), (6.70) and (6.71) we may derive the estimates

$$\begin{aligned}\|N^m_{\alpha,\sigma}u(t,\cdot)\|_{H^\gamma_r} &\lesssim \|u\|^p_{X(T)}I(t) \quad \text{for all} \quad t\in[0,T],\\ \|N^m_{\alpha,\sigma}u(t,\cdot)\|_{L^\infty} &\lesssim \|u\|^p_{X(T)}I(t) \quad \text{for all} \quad t\in[0,T],\end{aligned}$$

where $I(t)$ is as in (6.64). We recall that we obtain $I(t)\lesssim(1+t)^{\alpha-1}$ for $p>\frac{1}{1-\alpha}$. Hence, $\|N^m_{\alpha,\sigma}u\|_{X(T)}\lesssim\|u\|^p_{X(T)}$. Finally, it remains to show (6.68). We have

$$\begin{aligned}\||u(s,\cdot)|^p-|v(s,\cdot)|^p\|_{L^\infty} &\lesssim \|u(s,\cdot)-v(s,\cdot)\|_{L^\infty}\big(\|u(s,\cdot)\|^{p-1}_{L^\infty}+\|v(s,\cdot)\|^{p-1}_{L^\infty}\big)\\ &\lesssim (1+s)^{-p(1-\alpha)}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big).\end{aligned}$$

Hence,

$$\begin{aligned}\|N^m_{\alpha,\sigma}(u)(t,\cdot)-N^m_{\alpha,\sigma}(v)(t,\cdot)\|_{L^\infty} &\lesssim I(t)\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big)\\ &\lesssim (1+t)^{\alpha-1}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big) \quad \text{for all} \quad t\in[0,T].\end{aligned}$$

It remains to prove

$$\begin{aligned}\|N^m_{\alpha,\sigma}(u)(t,\cdot)-N^m_{\alpha,\sigma}(v)(t,\cdot)\|_{H^\gamma_r} &\lesssim I(t)\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big)\\ &\lesssim (1+t)^{\alpha-1}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big) \quad \text{for all} \quad t\in[0,T].\end{aligned}$$

We have

$$\begin{aligned}&\|N^m_{\alpha,\sigma}(u)(t,\cdot)-N^m_{\alpha,\sigma}(v)(t,\cdot)\|_{H^\gamma_r}\\ &\qquad\approx \|N^m_{\alpha,\sigma}(u)(t,\cdot)-N^m_{\alpha,\sigma}(v)(t,\cdot)\|_{L^r}+\|N^m_{\alpha,\sigma}(u)(t,\cdot)-N^m_{\alpha,\sigma}(v)(t,\cdot)\|_{\dot H^\gamma_r}.\end{aligned}$$

Here $f\approx g$ means that $g\lesssim f\lesssim g$. As above we have

$$\begin{aligned}&\|N^m_{\alpha,\sigma}(u)(t,\cdot)-N^m_{\alpha,\sigma}(v)(t,\cdot)\|_{L^r}\\ &\qquad\lesssim (1+t)^{\alpha-1}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big) \quad \text{for all} \quad t\in[0,T].\end{aligned}$$

It remains to prove

$$\begin{aligned}&\|N^m_{\alpha,\sigma}(u)(t,\cdot)-N^m_{\alpha,\sigma}(v)(t,\cdot)\|_{\dot H^\gamma_r}\\ &\qquad\lesssim (1+t)^{\alpha-1}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big) \quad \text{for all} \quad t\in[0,T],\end{aligned}$$

that is, it is sufficient to prove that

$$\||u(s,\cdot)|^p-|v(s,\cdot)|^p\|_{\dot H^\gamma_r}\lesssim(1+s)^{-p(1-\alpha)}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big).$$

By using the integral representation

$$|u(s,\cdot)|^p-|v(s,\cdot)|^p=p\int_0^1(u(s,\cdot)-v(s,\cdot))Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\,d\omega,$$

where $Q(u) = u|u|^{p-2}$, we obtain

$$\begin{aligned} &\||u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{\dot{H}^\gamma_r} \\ &\qquad \lesssim \int_0^1 \||D|^\gamma\big((u(s,\cdot)-v(s,\cdot))Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\big)\|_{L^r}\,d\omega. \end{aligned}$$

Applying the fractional Leibniz formula from Proposition 6.36 to estimate a product in $\dot{H}^\gamma_r$ we get

$$\begin{aligned} &\||u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{\dot{H}^\gamma_r} \\ &\lesssim \int_0^1 \||D|^\gamma\big(u(s,\cdot)-v(s,\cdot)\big)\|_{L^r}\|Q(\omega u(s,x)+(1-\omega)v(s,x))\|_{L^\infty}\,d\omega \\ &\quad + \int_0^1 \|u(s,\cdot)-v(s,\cdot)\|_{L^\infty}\||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\,d\omega \\ &\lesssim \||D|^\gamma\big(u(s,\cdot)-v(s,\cdot)\big)\|_{L^r}\big(\|u(s,\cdot)\|^{p-1}_{L^\infty}+\|v(s,\cdot)\|^{p-1}_{L^\infty}\big) \\ &\quad + \|u(s,\cdot)-v(s,\cdot)\|_{L^\infty}\int_0^1 \||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\,d\omega \\ &\lesssim \|u(s,\cdot)-v(s,\cdot)\|_{\dot{H}^\gamma_r}\big(\|u(s,\cdot)\|^{p-1}_{L^\infty}+\|v(s,\cdot)\|^{p-1}_{L^\infty}\big) \\ &\quad + \|u(s,\cdot)-v(s,\cdot)\|_{L^\infty}\int_0^1 \||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\,d\omega \\ &\lesssim (1+s)^{-p(1-\alpha)}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big) \\ &\quad + (1+s)^{-(1-\alpha)}\|u-v\|_{X(T)}\int_0^1 \||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\,d\omega. \end{aligned}$$

We apply again the Proposition 6.34 to estimate the term in the integral. In this way we may conclude

$$\begin{aligned} &\int_0^1 \||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\,d\omega \\ &\qquad \lesssim \int_0^1 \||D|^\gamma(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r} \\ &\qquad\qquad \times \|\omega u(s,\cdot)+(1-\omega)v(s,\cdot)\|^{p-2}_{L^\infty}\,d\omega \\ &\qquad \lesssim \int_0^1 (1+s)^{-(1-\alpha)}\|\omega u+(1-\omega)v\|_{X(T)} \\ &\qquad\qquad \times (1+s)^{-(p-2)(1-\alpha)}\|\omega u+(1-\omega)v\|^{p-2}_{X(T)}\,d\omega \\ &\qquad \lesssim \int_0^1 (1+s)^{-(p-1)(1-\alpha)}\|\omega u+(1-\omega)v\|^{p-1}_{X(T)}\,d\omega \\ &\qquad \lesssim (1+s)^{-(p-1)(1-\alpha)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big). \end{aligned}$$

Then

$$\||u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{\dot{H}^\gamma_r} \lesssim (1+s)^{-p(1-\alpha)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big).$$

Hence,

$$\begin{aligned}&\|N^m_{\alpha,\sigma}(u)(t,\cdot)-N^m_{\alpha,\sigma}(v)(t,\cdot)\|_{\dot H^\gamma_r}\\&\qquad\lesssim(1+t)^{\alpha-1}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big)\quad\text{for all}\quad t\in[0,T].\end{aligned}$$

We deduce that

$$\begin{aligned}\|Pu-Pv\|_{X(T)}&=\|N^m_{\alpha,\sigma}(u)-N^m_{\alpha,\sigma}(v)\|_{X(T)}\\&\lesssim\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big).\end{aligned}$$

Remark 6.21. *All estimates (6.67) and (6.68) are uniformly with respect to* $T\in(0,\infty)$ *if* $p>\max\{2;\gamma;\frac{1}{1-\alpha}\}$.

From (6.67) it follows that P maps $X(T)$ into itself for all T and for small data. By standard contraction arguments (see [18]) the estimates (6.67) and (6.68) lead to the existence of unique solution to $u=Pu$ and, consequently, to (6.3), that is, the solution of (6.3) satisfies the desired decay estimate. Since all constants are independent of T, after letting T tend to ∞ we obtain a global (in time) existence result for small data solutions to (6.3). If the data are large, then instead we get for $p>2$ the estimates

$$\begin{aligned}&\|Pu\|_{X(T)}\le C\|u_0\|_{H^\gamma_r\cap L^\infty}+C(T)\|u\|^p_{X(T)},\\&\|Pu-Pv\|_{X(T)}\le C(T)\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(X)}\big),\end{aligned}$$

where $C(T)$ tends to 0 for $T\to+0$. For this reason we can have for general (large) data a local (in time) existence result of weak solutions only. By the same argument as above we obtain the desired results. The proof is complete.

6.7 Appendix

In this chapter we present results that we have already used in the demonstrations of Chapter 5 and Chapter 6.

Lemma 6.22. *Let* $a(\cdot)$ *satisfy the hypothesis 5.2, then the integral* $\int_0^1\frac{dx}{a(x)}$ *is finite for* $\alpha\in(0,1)$ *and* $\int_0^1\frac{dx}{\sqrt{a(x)}}$ *is finite for* $\alpha\in[1,2)$.

Remark 6.23. *For* $\alpha\in(1,2)$, *the integral* $\int_0^1\frac{1}{a(x)}dx$ *could be finite or infinite, for example consider the functions* $a(x)=x^re^{(\alpha-r)x}$ *where* $r\le\alpha$.

Proof. Since $xa'(x)\le\alpha a(x)$ for all x, the function $x\mapsto\frac{a(x)}{x^\alpha}$ is decreasing on $(0,1)$, so $a(x)\ge a(1)x^\alpha$ and for $\alpha<1$

$$\int_0^1\frac{dx}{a(x)}\le\int_0^1\frac{dx}{a(1)x^\alpha}<+\infty.$$

□

□

Lemma 6.24. *(Hardy Inequality) (voir [6])*
We assume that $\alpha \in (0,1)$, then, for all $u \in H^1_a(0,1)$ such that $u(0) = 0$, we have

$$\int_0^1 \frac{a(x)}{x^2} u^2(x)dx \leq \frac{4}{(1-\alpha)^2} \int_0^1 a(x)|u'(x)|^2 dx. \tag{6.72}$$

Lemma 6.25. *For $\alpha \in (0,2)$ and for all $u \in H^1_a(0,1)$, we have $\lim_{x\to 0} xu^2 = 0$ and $\lim_{x\to 0} xu = 0$.*

Proof. At first, we show that $xu^2 \in W^{1,1}$. It's obvious that $xu^2 \in L^1(0,1)$ for each $u \in H^1_a(0,1)$. On the other hand

$$\begin{aligned}(xu^2)_x &= u^2 + 2xuu_x,\\ xuu_x &= \big(\frac{x}{\sqrt{a(x)}}u\big)\big(\sqrt{a(x)}u_x\big),\end{aligned}$$

and by the Hypothesis 5.2 we can easily see that the function $x \mapsto \frac{x^2}{a(x)}$ is increasing, so

$$\frac{x}{\sqrt{a(x)}} \leq \frac{1}{\sqrt{a(1)}} \Rightarrow \frac{x}{\sqrt{a(x)}}u \in L^2(0,1) \Rightarrow xuu_x \in L^1(0,1).$$

Hence $xu^2 \in W^{1,1}(0,1)$ and it follows that $xu^2 \longrightarrow L \geq 0$ when $x \longrightarrow 0$. If $L > 0$, so we could have

$$u \sim \sqrt{\frac{L}{x}} \notin L^2(0,1),$$

so $L = 0$. Similarly we can see that $\lim_{x\to 0} xu = 0$.

□

□

Lemma 6.26. *Assume that $\alpha \in [1,2)$. Then for all $u \in H^1_a(0,1)$ such that $(au_x)_x \in L^2(0,1)$, we have $au_x \in W^{1,1}(0,1)$.*

Proof. Note that $w = a(x)u_x$ and choose $M > 0$, such that $a(x) \leq M$ for all $x \in [0,1]$. we have

$$\int_0^1 |w|dx = \int_0^1 |a(x)u_x|dx \leq \sqrt{\int_0^1 a(x)^2 u_x^2 dx} \leq \sqrt{M \int_0^1 a(x) u_x^2 dx} < +\infty$$

since $u \in H^1_a(0,1)$. On the other hand, starting from the inclusion $L^2(0,1)$ in $L^1(0,1)$ and the fact that w_x is in $L^2(,1)$, we deduce that w_x is in $L^1(0,1)$. Therefore, $w \in W^{1,1}(0,1)$. □

Let us mention that for the well-posedness of boundary conditions in problem (**??**), we need to de

ne the trace of u at boundary points $x = 0$ and $x = 1$ for any $u \in H^1_a(0,1)$. The trace at $x = 1$ obviously makes sense which allows to consider the Dirichlet boundary condition at this point. (Note that the function $u \in H^1_a(0,1)$ belongs to the Sobolev space $W^{1,2}$ in a neighborhood of $x = 1$). On the other hand, if $\alpha < 1$, the trace of u at $x = 0$ is meaningful because of the following lemma.

Lemma 6.27. *If $a(\cdot)$ satis*

es Hypothesis 5.2 and $\alpha \in (0,1)$, then for every $u \in H^1_a(0,1)$ we have $u \in W^{1,1}(0,1) = \{u \in L^1(0,1) : u_x \in L^1(0,1)\}$ and so $u(0)$ is meaningful.. Thus we could introduce the following space $H^1_a(0,1)$ depending on the values of α.

Proof. For any $u \in H^1_a(0,1)$ we have $u \in L^2(0,1)$, so $u \in L^1(0,1)$. We prove that $u_x \in L^1(0,1)$.

$$\int_0^1 |u_x|dx = \int_0^1 |\sqrt{a(x)}u_x \frac{1}{\sqrt{a(x)}}|dx \le \Big(\int_0^1 \frac{1}{a(x)}dx \int_0^1 a(x)u_x^2 dx\Big)^{\frac{1}{2}}$$

But by Lemma 6.22 the integral $\int_0^1 \frac{1}{a(x)}dx$ is finite, so $u_x \in L^1(0,1)$. Now, consider a sequence $\{u_n\}$ of smooth functions which converge to u in $W^{1,1}(0,1)$ and let χ be a smooth cut-of function such that $\chi|_{[0,\frac{1}{2}]} \equiv 1$ and χ vanishes in some neighbourhood of 1. Then we have $\chi u_n \longrightarrow \chi u$ in $W^{1,1}$. On the other hand for every x, we have $\chi u_n(x) = -\int_0^1 (\chi u_n)_x(t)dt$, which means that the $\lim_{x\to 0} \chi u_n(x)$ exist, so $\lim_{x\to 0} u_n(x)$ exist et we define $u(0)$ to be equal to this value.

□

□

Proposition 6.28. *1. For $\alpha \in (0,1)$ the space $C_c^\infty(0,1)$ is dense in $H^1_{a,0}(0,1)$*

2. In the case $\alpha \in [1,2)$ the subset of $C^\infty([0,1])$ which vanishes at $x = 1$ is dense in $H^1_{a,0}(0,1)$.

Proof. (1): Since $C_c^\infty(0,1)$ is dense in $H^1_0(0,1)$ and the embedding of $H^1_0(0,1)$ in $H^1_{a,0}(0,1)$ est continu, is continuous, it suffices to prove that $H^1_0(0,1)$ is dense in $H^1_{a,0}(0,1)$. Let $v \in H^1_{a,0}(0,1)$ be given and define the family $\{v_\delta\}_{\delta\ge 0}$, with $\delta \in (0,1)$ in the following way

$$v_\delta(x) := \begin{cases} \frac{x}{\delta}v(x), & 0 \le x \le \delta, \\ v(x), & \delta < x \le 1. \end{cases}$$

We want to show that:
(1) $v_\delta \in H^1_0(0,1)$ for all $\delta \in (0,1)$.

(2) $v_\delta \to v$ in $H^1_{a,0}(0,1)$ as $\delta \to 0$. One has

$$\int_0^1 |v'_\delta(x)|^2 dx = \int_0^\delta |\frac{1}{\delta}v(x) + \frac{x}{\delta}v'(x)|^2 dx + \int_\delta^1 |v'(x)|^2 dx$$
$$\leq 2\int_0^\delta \frac{1}{\delta^2}|v(x)|^2 + \frac{x^2}{\delta^2}|v'(x)|^2 dx + \int_\delta^1 |v'(x)|^2 dx. \tag{6.73}$$

By Hypothesis 5.2 , the function $x \mapsto \frac{a(x)}{x^2}$ is decreasing on $(0,1]$, so $a(x) \geq a(1)x^2$ for every $x \in [0,1]$. Therefore for every $\delta > 0$ we have

$$\int_0^\delta \frac{x^2}{\delta^2}|v'(x)|^2 dx \leq \frac{1}{a(1)\delta^2}\int_0^\delta a(x)|v'(x)|^2 dx. \tag{6.74}$$

On the other hand, since $a(x) > 0$ on $(0,1]$, there exists $M_\delta > 0$ such that $a(x) \geq M_\delta$ in $[\delta,1]$, so

$$\int_0^1 |v'(x)|^2 dx \leq M_\delta^{-1}\int_\delta^1 +a(x)|v'(x)|^2 dx \tag{6.75}$$

Combining (6.73), (6.74) and (6.75) nous obtenons $C_\delta > 0$ such that

$$\int_0^1 |v'_\delta(x)|^2 dx \leq C_\delta \int_0^1 |v(x)|^2 + a(x)|v'(x)|^2 dx.$$

The right part of the last inequality is
nite since $v \in H^1_a(0,1)$. Then $v_\delta \in H^1_0(0,1)$. Also

$$\|v - v_\delta\|^2_{H^1_{a,0}} = \int_0^1 |v - v_\delta|^2 + a(x)|v' - v'_\delta|^2 dx$$
$$= \int_0^\delta |v - \frac{x}{\delta}v|^2 dx + \int_0^\delta a(x)|v' - \frac{v}{\delta} - \frac{x}{\delta}v'|^2 dx$$
$$\leq \int_0^\delta v^2 dx + 2\int_0^\delta a(x)|v'|^2 dx + 2\int_0^\delta a(x)\frac{v^2}{\delta^2}dx.$$

Now $\int_0^\delta v^2 dx + 2\int_0^\delta a(x)|v'|^2 dx \to 0$ as $\delta \to 0$, since $v \in H^1_{a,0}(0,1)$. On the other hand by Lemma 6.24, we obtain

$$\int_0^\delta a(x)\frac{v^2}{\delta^2}dx \leq \int_0^\delta a(x)\frac{v^2}{x^2}dx \leq \frac{4}{(1-\alpha)^2}\int_0^\delta a(x)|v'(x)|^2 dx.$$

Indeed, we can rewrite the proof of Lemma 6.24 in the interval $[0,\delta]$ instead of $[0,1]$ and derive a new inequality with the same constant $\frac{4}{(1-\alpha)^2}$. Now, the right part of the last inequality tends to zero $\delta \to 0$, because $v \in H^1_{a,0}(0,1)$ and the proof of (1) is complete.
(2): Similarly in this case, for $v \in H^1_{a,0}(0,1)$ it suffices to construct functions $\{v_\delta\}_{\delta \geq 0}$ such that
(1) $v_\delta \in H^1(0,1)$ and $v_\delta(1) = 0$.

(2) $v_\delta \to v$ in $H^1_{a,0}(0,1)$ as $\delta \to 0$.
Defi

ne

$$v_\delta(x) := \begin{cases} v(2\delta - x), & 0 \le x \le \delta, \\ v(x), & \delta < x \le 1. \end{cases}$$

we have

$$v'_\delta(x) := \begin{cases} -v'(2\delta - x), & 0 < x < \delta, \\ v'(x), & \delta < x < 1. \end{cases}$$

Since $a(\cdot)$ is strictly positive on $(0,1)$ and in the computing of $\int_0^1 |v'_\delta|^2 dx$, we are far from the boundary, it is easy to see that $v_\delta \in H^1(0,1)$. Also

$$\|v_\delta - v\|^2_{H^1_a(0,1)} = \|v_\delta - v\|^2_{H^1_a(0,\delta)} \le 2[\|v\|^2_{H^1_a(0,\delta)} + \|v_\delta\|^2_{H^1_a(0,\delta)}].$$

Since $v \in H^1_a(0,1)$, the term $\|v\|^2_{H^1_a(0,\delta)}$ tends to zero as $\delta \to 0$. Also if $\delta \le \delta_0/2$ where δ_0 is the constant introduced in property (3) of Hypothesis 5.2, then

$$\begin{aligned} \int_0^\delta v_\delta^2(x) + a(x)|v'_\delta(x)|^2 dx &= \int_0^\delta v^2(2\delta - x) + a(x)|v'(2\delta - x)|^2 dx \\ &\le \int_\delta^{2\delta} v^2(x) + \frac{1}{m}\int_\delta^{2\delta} a(x)|v'(x)|^2 dx, \end{aligned}$$

which tends to zero as $\delta \to 0$. Observe that $v_\delta(1) = 0$, so the subset of $C^\infty([0,1])$ which vanishes at $x = 1$ is dense in $H^1_{a,0}(0,1)$.

□

Modified Bessel functions

Definition 6.29. *The Bessel function J_μ of first kind and of order $\mu \in \mathbb{R}$ is defined by*

$$J_\mu(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\mu+1)} \Big(\frac{s}{2}\Big)^{2k+\mu},$$

where μ is not allowed to be a negative integer. The modified Bessel function $\tilde{J}_\mu(s)$ is defined by $\tilde{J}_\mu(s) := \frac{J_\mu(s)}{s^\mu}$.

Lemma 6.30. *Let $f \in L^p(\mathbb{R}^n)$, $p \in [1,2]$, be a radial function. Then the inverse Fourier transform is also a radial function and it satisfies*

$$F^{-1}(f)(x) = \int_0^\infty g(r) r^{n-1} \tilde{J}_{\frac{n}{2}-1}(r|x|)\, dr, \quad g(|x|) := f(x).$$

Lemma 6.31. *Assume that μ is not a negative integer. Then the following rules hold:*

1. $s d_s \tilde{J}_\mu(s) = \tilde{J}_{\mu-1}(s) - 2\mu \tilde{J}_\mu(s)$,
2. $d_s \tilde{J}_\mu(s) = -s\tilde{J}_{\mu+1}(s)$,
3. $\tilde{J}_{-1/2}(s) = \sqrt{\frac{2}{\pi}}\cos(s)$,
4. *we have the relations*

$$\begin{aligned}
&|\tilde{J}_\mu(s)| \le Ce^{\pi|\Im\mu|} \quad \textit{if} \quad |s|\le 1,\\
&J_\mu(s) = Cs^{-\frac{1}{2}}\cos\Big(s-\frac{\mu}{2}\pi-\frac{\pi}{4}\Big) + O(|s|^{-\frac{3}{2}}) \quad \textit{if} \quad |s|\ge 1,
\end{aligned}$$

5. $\tilde{J}_{\mu+1}(r|x|) = -\frac{1}{r|x|^2}\partial_r \tilde{J}_\mu(r|x|), \quad r\neq 0, x\neq 0.$

Mittag-Leffler function

The Mittag-Leffler function E_β allows the following implicit definition:

$$\frac{\lambda}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}E_\beta(\lambda s^\beta)\,ds = E_\beta(\lambda t^\beta) - 1. \tag{6.76}$$

The Mittag-Leffler function $E_\beta(-t^\beta\langle\xi\rangle^2_{m,\sigma})$ may be written in the following form:

$$\begin{aligned}
E_\beta(-t^\beta\langle\xi\rangle^2_{m,\sigma}) = \frac{1}{\beta}\Big(&\exp\big(a_\beta(t^{\frac{\beta}{2}}\langle\xi\rangle_{m,\sigma})\big) + \exp\big(b_\beta(t^{\frac{\beta}{2}}\langle\xi\rangle_{m,\sigma})\big)\Big)\\
&+ l_\beta(t^{\frac{\beta}{2}}\langle\xi\rangle_{m,\sigma}),
\end{aligned}$$

where

$$\begin{aligned}
a_\beta(y) &= y^{\frac{2}{\beta}}\exp(\tfrac{\pi i}{\beta}) \text{ for } y\ge 0,\\
b_\beta(y) &= y^{\frac{2}{\beta}}\exp(-\tfrac{\pi i}{\beta}) \text{ for } y\ge 0,\\
l_\beta(y) &= \begin{cases} \frac{\sin(\beta\pi)}{\pi}\displaystyle\int_0^\infty \frac{y^2 s^{\beta-1}\exp(-s)}{s^{2\beta}+2y^2s^\beta\cos(\beta\pi)+y^4}\,ds \\ \quad = \frac{\sin(\beta\pi)}{\beta\pi}\displaystyle\int_0^\infty \frac{\exp(-y^{\frac{2}{\beta}}s^{\frac{1}{\beta}})}{s^2+2s\cos(\beta\pi)+1}\,ds \text{ for } y>0,\\ 1-\frac{2}{\beta} \text{ for } y=0. \end{cases}
\end{aligned}$$

Here $\beta = 1+\alpha$. The proof can be found in the paper [32].

Remark 6.32. *We have also the relation*

$$\begin{aligned}
&\exp\big(a_\beta(t^{\frac{\beta}{2}}\langle\xi\rangle_{m,\sigma})\big) + \exp\big(b_\beta(t^{\frac{\beta}{2}}\langle\xi\rangle_{m,\sigma})\big)\\
&\quad = 2e^{t\langle\xi\rangle_{m,\sigma}^{\frac{2}{1+\alpha}}\cos(\frac{\pi}{1+\alpha})}\cos\Big(t\langle\xi\rangle_{m,\sigma}^{\frac{2}{1+\alpha}}\sin\Big(\frac{\pi}{1+\alpha}\Big)\Big)\\
&\qquad = 2e^{-ct\langle\xi\rangle_{m,\sigma}^{\frac{2}{1+\alpha}}}\cos\Big(t\langle\xi\rangle_{m,\sigma}^{\frac{2}{1+\alpha}}\sqrt{1-c^2}\Big), \quad \textit{where } c = -\cos\Big(\frac{\pi}{1+\alpha}\Big).
\end{aligned}$$

Results from Harmonic Analysis

We recall some results from Harmonic Analysis (cf. with [47]).

Proposition 6.33. *Let $r \in (1,\infty), p > 1$ and $\sigma \in (0,p)$. Let $Q(u)$ denote one of the functions $|u|^p, \pm u|u|^{p-1}$. Then the following inequality holds:*

$$\|Q(u)\|_{H^\sigma_r} \lesssim \|u\|_{H^\sigma_r} \|u\|^{p-1}_{L^\infty}$$

for any $u \in H^\sigma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Here we use for $\gamma \geq 0$ and $1 < q < \infty$ the fractional Sobolev spaces or Bessel potential spaces

$$H^\gamma_q(\mathbb{R}^n) := \big\{ f \in S'(\mathbb{R}^n) : \|f\|_{H^\gamma_q} := \|F^{-1}(\langle\xi\rangle^\gamma F(f))\|_{L^q} < \infty \big\}.$$

Moreover, $\langle D\rangle^\gamma$ stands for the pseudo-differential operator with symbol $\langle\xi\rangle^\gamma$ and it is defined by $\langle D\rangle^\gamma u = F^{-1}(\langle\xi\rangle^\gamma F(u))$.

Proof. This result is a special case of the following more general inequality for Triebel-Lizorkin spaces $F^\sigma_{r,q}$:

$$\|Q(u)\|_{F^\sigma_{r,q}} \lesssim \|u\|_{F^\sigma_{r,q}} \|u\|^{p-1}_{L^\infty} \quad \text{for any} \quad u \in F^\sigma_{r,q} \cap L^\infty,$$

where $q > 0$, whose proof may be found in [51, Theorem 1 in Section 5.4.3]. □

Proposition 6.34. *Let $r \in (1,\infty), p > 1$ and $\sigma \in (0,p)$. Let $Q(u)$ denote one of the functions $|u|^p, \pm u|u|^{p-1}$. Then the following inequality holds:*

$$\|Q(u)\|_{\dot{H}^\sigma_r} \lesssim \|u\|_{\dot{H}^\sigma_r} \|u\|^{p-1}_{L^\infty}$$

for any $u \in \dot{H}^\sigma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, where

$$\dot{H}^\gamma_q(\mathbb{R}^n) := \{ f \in S'(\mathbb{R}^n) : \|f\|_{\dot{H}^\gamma_q} := \|F^{-1}(|\xi|^\gamma F(f))\|_{L^q} < \infty \}.$$

Here $|D|^\gamma$ stands for the pseudo-differential operator with symbol $|\xi|^\gamma$ and it is defined by $|D|^\gamma u = F^{-1}(|\xi|^\gamma F(u))$.

Proof. We will use a homogeneity argument. For any positive λ we define $u_\lambda(x) = u(\lambda x)$. Applying Proposition 6.33 to u_λ we get

$$\|Q(u_\lambda)\|_{H^\sigma_r} \lesssim \|u_\lambda\|_{H^\sigma_r} \|u_\lambda\|^{p-1}_{L^\infty}. \tag{6.77}$$

Since for $r \in (1,\infty)$ we have the decomposition

$$\|v\|_{H^\sigma_r} \approx \|v\|_{\dot{H}^\sigma_r} + \|v\|_{L^r} \qquad \text{for any} \quad v \in H^\sigma_r$$

and the scaling properties

$$\|u_\lambda\|_{\dot{H}^\sigma_r} = \lambda^{\sigma-\frac{n}{r}} \|u\|_{\dot{H}^\sigma_r},\ \|u_\lambda\|_{L^r} = \lambda^{-\frac{n}{r}} \|u\|_{L^r} \text{ and } \|u_\lambda\|_{L^\infty} = \|u\|_{L^\infty}$$

diving both sides of (6.77) by $\lambda^{\sigma-\frac{n}{r}}$ and taking the limit as $\lambda \to \infty$ we obtain the desired inequality. □

Proposition 6.35. *Let $r \in (1,\infty)$ and $\sigma > 0$. Then the following inequality holds:*

$$\|uv\|_{H^\sigma_r} \lesssim \|u\|_{H^\sigma_r}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{H^\sigma_r}$$

for any $u, v \in H^\sigma_r \cap L^\infty$.

Proof. The result that we want to prove is a special case of the following inequality for Triebel-Lizorkin spaces $F^\sigma_{r,q}$:

$$\|uv\|_{F^\sigma_{r,q}} \lesssim \|u\|_{F^\sigma_{r,q}}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{F^\sigma_{r,q}}$$

for any $u, v \in F^\sigma_{r,q} \cap L^\infty$, where $q > 0$, whose proof can be found in [51, Theorem 2 in Section 4.6.4]. □

Finally let us state the corresponding inequality in homogeneous spaces $\dot{H}^\sigma_r$. For the proof it is possible to follow the same strategy as in the proof of Proposition 6.34.

Proposition 6.36 (Fractional Leibniz formula)**.** *Let $r \in (1,\infty)$ and $\sigma > 0$. Then the following inequality holds:*

$$\|uv\|_{\dot{H}^\sigma_r} \lesssim \|u\|_{\dot{H}^\sigma_r}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{\dot{H}^\sigma_r}$$

for any $u, v \in \dot{H}^\sigma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

The following result was proposed and proved by Marcello D'Abbicco and already used in a special case in [17]. We present the proof to make this chapter more self-contained.

Proposition 6.37. *Let $u_0 \in L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $n \geq 1$, $r \geq 1$ and $\alpha \in (0,1)$. Then the function*

$$u = u(t,x) = \big(G^0_{\alpha,\sigma}(t) * u_0\big)(t,x)$$

belongs to

$$C\big([0,\infty), L^r(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)\big) \text{ for all } q \in [r,\infty).$$

Proof. Due to (6.6) we have

$$G^0_{\alpha,\sigma}(t,x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} E_{\alpha+1}\big(-t^{\alpha+1}|\xi|^{2\sigma}\big)\, d\xi.$$

The estimate (6.9) from Proposition 6.10 implies $G^0_{\alpha,\sigma}(t,\cdot) \in L^1(\mathbb{R}^n)$ for all $t > 0$. Moreover, $G^0_{\alpha,\sigma}(t,\cdot)$ has the following scale-invariant property:

$$G^0_{\alpha,\sigma}(t,x) = t^{-n\beta} G^0_{\alpha,\sigma}(1, t^{-\beta}x) \text{ with } \beta = \frac{\alpha+1}{2\sigma}. \tag{6.78}$$

Consequently, we conclude for all $t > 0$ the relations

$$\|G^0_{\alpha,\sigma}(t,\cdot)\|_{L^1} = \|G^0_{\alpha,\sigma}(1,\cdot)\|_{L^1} \tag{6.79}$$

and

$$\int_{\mathbb{R}^n} G^0_{\alpha,\sigma}(t,x)dx = \int_{\mathbb{R}^n} G^0_{\alpha,\sigma}(1,x)dx = 1. \tag{6.80}$$

Let us choose a positive zero sequence $\{t_l\}_l$. We want to prove for a given $g \in L^p(\mathbb{R}^n)$, $p \in [1,\infty)$, that the sequence $\{T_l * g\}_l$ tends to g, where $T_l(\cdot) := G^0_{\alpha,\sigma}(t_l,\cdot)$. We have $\lim_{l\to\infty} T_l = \delta_0$ in the distributional sense. Hence, $\lim_{l\to\infty} T_l * g = g$ in distributional sense, too. But, this implies the desired relation $\int_{\mathbb{R}^n} G^0_{\alpha,\sigma}(t,x)dx = 1$. Otherwise, if we would have for $t > 0$ the relation

$$\int_{\mathbb{R}^n} G^0_{\alpha,\sigma}(t,x)dx = \int_{\mathbb{R}^n} G^0_{\alpha,\sigma}(1,x)dx = M \in \mathbb{C},$$

then we might conclude $\lim_{l\to\infty} T_l * g = Mg$ in the distributional sense, in contradiction to $\lim_{l\to\infty} T_l = \delta_0$ in the distributional sense.
The scale-invariant property (6.78) implies for all positive δ

$$\int_{|x|\geq\delta} |T_l(x)|dx \to 0 \text{ for } l \to \infty. \tag{6.81}$$

Indeed, the relation (6.81) holds after taking account of

$$\begin{aligned}\int_{|x|\geq\delta} |T_l(x)|dx &= t_l^{-n\beta} \int_{|x|\geq\delta} |G^0_{\alpha,\sigma}(1,t_l^{-\beta}x)|dx\\ &= \int_{|y|\geq t_l^{-\beta}\delta} |G^0_{\alpha,\sigma}(1,y)|dy \to 0.\end{aligned}$$

Let us choose a function $g \in C_c(\mathbb{R}^n)$. We prove that the sequence $\{(T_l * g)(x)\}_l$ tends to $g(x)$ for all $x \in \mathbb{R}^n$. Using (6.80) we obtain

$$(T_l * g)(x) - g(x) = \int_{\mathbb{R}^n} (g(x-y) - g(x))T_l(y)dy.$$

For a fixed positive ε we choose $\kappa = \kappa(\varepsilon, x)$ such that $|g(x-y) - g(y)| < \varepsilon$ for $|y| < \kappa$. Then,

$$\begin{aligned}|(T_l * g)(x) - g(x)| &\leq \varepsilon \int_{|y|\leq\kappa} |T_l(y)|dy + 2\|g\|_{L^\infty} \int_{|y|\geq\kappa} |T_l(y)|dy\\ &\leq \varepsilon(\|G^0_{\alpha,\sigma}(1,\cdot)\|_{L^1} + 2\|g\|_{L^\infty})\end{aligned}$$

for sufficiently large $l = l(\kappa)$. This implies the desired relation $\lim_{l\to\infty}(T_l * g)(x) = g(x)$ for all $x \in \mathbb{R}^n$.
Applying Hölder's inequality gives

$$\begin{aligned}|(T_l * g)(x) - g(x)| &\leq \|(g(x-\cdot) - g(x))T_l(\cdot)\|_{L^1}\\ &\leq \||g(x-\cdot) - g(x)|^p T_l(\cdot)\|_{L^1}^{\frac{1}{p}} \|T_l(\cdot)\|_{L^1}^{\frac{1}{p'}},\end{aligned}$$

where p' is the conjugate exponent to p. From this estimate it follows

$$\begin{aligned}\|(T_l * g - g)(\cdot)\|_{L^p}^p &\le c_p \int_{\mathbb{R}^n_y} |T_l(y)|\Big(\int_{\mathbb{R}^n_x} |g(x-y)-g(x)|^p dx\Big) dy \\ &= c_p \int_{\mathbb{R}^n_y} |T_l(y)|\varphi(-y)dy = c_p(|T_l| * \varphi)(0),\end{aligned}$$

where we introduced

$$\varphi(-y) := \int_{\mathbb{R}^n_x} |g(x-y)-g(x)|^p dx.$$

The function $\varphi = \varphi(-y)$ is bounded and continuous. Consequently, we get $\lim_{l\to\infty} \|(T_l * g - g)(\cdot)\|_{L^p} = 0$ what we wanted to have for all bounded and continuous functions $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. The set $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, then a density argument in $L^p(\mathbb{R}^n)$ completes the proof. □

Inequalities

First we recall Young's inequality.

Lemma 6.38. *Let $u \in L^p(\mathbb{R}^n)$ and $v \in L^r(\mathbb{R}^n)$ with $1 \le p, r \le \infty$. Then $u * v \in L^q(\mathbb{R}^n)$, where $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$ and*

$$\|u * v\|_{L^q} \lesssim \|u\|_{L^p}\|v\|_{L^r}.$$

Finally, we recall the following lemma from [15].

Lemma 6.39. *Suppose that $\theta \in [0,1), a \ge 0$ and $b \ge 0$. Then there exists a constant $C = C(a,b,\theta) > 0$ such that for all $t > 0$ the following estimate holds:*

$$\begin{aligned}&\textstyle\int_0^t (t-\tau)^{-\theta}(1+t-\tau)^{-a}(1+\tau)^{-b}\,d\tau \\ &\quad\le \begin{cases} C(1+t)^{-\min\{a+\theta,b\}} & \text{if} \quad \max\{a+\theta,b\} > 1, \\ C(1+t)^{-\min\{a+\theta,b\}}\ln(2+t) & \text{if} \quad \max\{a+\theta,b\} = 1, \\ C(1+t)^{1-a-\theta-b} & \text{if} \quad \max\{a+\theta,b\} < 1. \end{cases}\end{aligned} \tag{6.82}$$

Conclusion and perspectives

We have studied in the first part of this thesis the null controllability of a non-autonomous degenerate parabolic equation. Our perspective is to study the null controlability to the following non-autonomous degenerate parabolic non linear

$$u_t - M(\|\nabla u\|^2)(a(x)u_x)_x = h\chi_\omega, \qquad (x,t) \in Q = (0,1) \times (0,T).$$

In this study we need some theorems of compactness, fixed point and some additional hypotheses on the coefficient $M(\cdot)$.
In the second part of this thesis we have studied the global existence of small data solutions to semi-linear fractional $\sigma-$evolution equations with mass or power non-linearity under the condition $u_t(0,x) = 0$. Our perspective is to study the same probleme when $u_t(0,x) = u_1(x)$, where u_1 is in some suitable space.

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