REPUBLIQUE ALGERIENNE DEMOCRATIQUE & POPULAIRE
MINISTERE DE L'ENSEIGNEMENT SUPERIEUR & DE LA RECHERCHE SCIENTIFIQUE

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THESE

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Pour obtenir le Diplôme de Doctorat

Spécialité : Mathématiques
Option : Probabilités-Statistiques

Intitulée

Sur l'estimation non paramétrique robuste récursive en statistique fonctionnelle

Thèse soutenue le : 09/07/2019

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Dédicace

À ma femme et ma fille.

À mes parents, mes frères et mes soeurs, et à tout ce qui me sont plus proches.

Remerciements

Je tient en premier lieu à exprimer mes plus vifs remerciements à mon directeur de thèse le Professeur LAKSACI Ali de m'avoir accordé leur confiance pour travailler à leur côté, et de me donner l'occasion de développer mes compétences en statistique fonctionnelle dans un environnement de recherche très actif et pour leur remarque et leur commentaires qui m'a permis d'améliorer cette thèse. Je suis reconnaissant pour les conseils qu'il m' a prodigué au cours de ces trois années, pour les nombreuses et fructueuses discussions que nous avons eues. En dehors de leur apport scientifique, je n'oublierai pas aussi de les remercier pour leur qualités humaines, leur cordialité, leur patience infinie et leur soutien.

Je remercie sincèrement Monsieur MECHAB Boubaker pour l'honneur qu'il me fait d'avoir accepté d'être président du jury. Je le remercie pour me soutien à ce travail ainsi que pour les commentaires m'ayant permis de l'améliorer.

Mes remerciements s'adressent aux Messieurs : EL MEZOUAR Chiker , Professeur à l'Université de Bechar ; BENCHIKH Toufik , Professeur à l'Université de Sidi Bel-Abbès ; GUENDOUZI Toufik Professeur à l'Université de Saida et RAHMANI Saadia, Maître de Conférences à l'Université de Saida, pour avoir accepté d'examiner cette thèse. Je les remercie aussi pour le temps et l'attention qu'ils y'ont consacrés pour la lecture attentive de cette thèse.

Un grand merci au Membre du Laboratoire de Statistique et Processus Stochastiques de l'Université de Sidi Bel Abbès, en particulier les Messieurs : ATTOUCH Mohammed Kadi, GHERIBALLAH Abdelkader, et BENAISSA Samir.

Enfin, je remercie tous les collègues qui m'ont aidé d'une façon directe ou indirecte.

Je ne pourrais finir mes remerciements sans une pensée très personnelle aux membres de ma famille, plus particulièrement mes parents et mon grand père, pour leur amour, leur encouragement et leur soutien inconditionnel tout au long de mes années d'étude qui m'ont mené jusqu'au doctorat. Merci à mes frères et mes soeurs pour leur soutien et leur appui moral.

Je remercie profondément ma femme de faire partie de ma vie, sa présence, ses conseils, ses encouragements et son soutien constant et réconfortant dans les moments de doute sont pour moi très précieux.

Je tiens également à dire tout ma sympathie aux amis et personnes qui ont participé de près ou de loin à la bonne réalisation de soutenance de cette thèse.

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Chapitre 1

Introduction and bibliographical study

1.1 Résumé

Cette thèse est consacrée à l'étude des propriétés asymptotiques de l'estimateur récursive de la fonction de hasard conditionnelle non paramétriques, quand la variable explicative prend ses valeurs dans un espace de dimension infinie. Dans un premiers temps, notre travail est consacrée à la construction de l'estimateur récursive de la fonction de hasard conditionnelle pour une variable réponse réelle conditionnée à une variable explicative fonctionnelle sous une condition de dépendance faible sur les données (données ergodiques). Par suite, nous établissons la convergence presque complète en précisant la vitesse de convergence de notre estimateur proposé. Dans la deuxième partie, en gardant les mêmes types de données et le même type de dépendance, nous établissons sous des conditions générales la normalité asymptotique de notre estimateur récursive de la fonction de hasard conditionnelle donnée dans la première partie.

1.2 Summary

This thesis is dedicated to the survey of the asymptotic properties of recursive estimator of conditional hazard function in nonparametric statics, when the explanatory variable takes its values in infinite dimension space. In the first part, our work is devoted to the construction of the recursive estimator of conditional hazard function for a real response variable conditioned to a functional explanatory variable under a condition of weak dependence on the data (ergodic data). As a result, we establish the almost complete convergence of our proposed estimator. In the second part, we keep the same types of data and the same type of dependence, we establish under general conditions the asymptotic normality of our recursive estimator of the conditional hazard function given in the first part.

1.3 introduction

The ergodic processes has a great importance in practice. In particular are usually used to model the thermodynamic data or the signal process. Although these processes are to be studied in continuous path, quite a little attention has paid to develop statistical tools allowing to treat the continuous ergodic processes in its own dimension by exploring its functional character. In this paper, we will treat the problem of the estimation of the relative error regression of the functional ergodic data.

Actually, nonparametric functional statistics has become a major topic of research, mainly due to the interaction with other applied fields. Functional date occurs in many fields of applied sciences such as economics, soil science, epidemiology or environmetal science, among others. For an overview on the statistical analysis of functional data, we referee to Bosq (2000), Ramsay and Silverman (2002), Ferraty and Vieu (2006), Zhang (2014), Hsing et al. (2015), Cuevas (2014), Goia and Vieu (2016). In this context, the mean least square regression is the most used model to examine the relationship between a two functional variables. We site for instance, Cardot et al(1999) for the linear model, Ferraty and Vieu (2000) for the nonparametric model, Ferraty et al. (2003) for single index model. Formally, in all these studies the relationship between a response variable Y and the explanatory variable X is modeled by

$$Y = r\left(X\right) + \epsilon, \tag{1.1}$$

where r is an operator which is defined from a semi-metric space $(\mathcal{F}, d)$, equipped with a semi-metric d, to $\mathbb{R}$ and ϵ is a random error variable. Usually, r is obtained by minimizing the following quantity

$$\mathbb{E}\left[(Y - r(X))^2 | X\right].$$

However, this kind of regression is very sensitive to outliers, because, it treats all variables as having an equal weight. In this paper, we overcome this drawback by using an alternative loss function based on the squared relative error which is defined, for $Y > 0$, by :

$$\mathbb{E}\left[\left(\frac{Y - r(X)}{Y}\right)^2 | X\right]. \tag{1.2}$$

The solution of (1.2) can be explicitly expressed by :

$$r(x) = \frac{\mathbb{E}[Y^{-1}|X = x]}{\mathbb{E}[Y^{-2}|X = x]}.$$

Such regression model has been introduced in functional statistics by Demongeot et al., (2016). They proved the almost complete consistency and the asymptotic normality of a kernel estimate of this model in the i. i. d. case. The main aim of this paper is to generalize this result to the dependent case by considering a very weak dependence structure that is the ergodicity condition. Recall that, the latter is implied by all mixing conditions, being weaker than all of them (see Ash and Gardner (1975)). Noting that, in the last few years the statistical analysis of functional ergodic data has received lot of attention. The first results on this topic are given by Laib and Louani (2010). They showed the strong consistency of the kernel estimate of the mean least square regression. The asymptotic normality of this estimator has been stated by the same author in and Laib and Louani (2011). Gheriballah et al. consider the nonparametric estimation of the M-regression in functional ergodic time series case. We return to Benziadi et al. (2016) for the quantiles regression model. More recently, Ling et al (2016) treat the conditional mode estimation for the functional ergodic data when the response variable is subject of missing at random. In this work we prove the almost complete convergence (with rate) of a kernel estimate of the relative error regression operator under this less restrictive dependence structure. It is also worth noting that our hypotheses and results unify both cases of finite or infinite dimension of the regressors. For this, our methodology permits us to generalize to the infinite dimension some existing asymptotic results in the multivariate case.

Chapitre 2

Preliminaries : Nonparametric functional regression

2.1 Consistency of the classical regression

The almost complete consistency is based on the following conditions

(H1) $P(X \in B(x,h)) = \phi_x(h) > 0 \; \forall \; h > 0$ and $\lim_{h \to 0} \phi_x(h) = 0$.

(H2) There exists $C_1 > 0$ and $k > 0$ such that $\forall x_1, x_2 \in N_x, \quad |r(x_1) - r(x_2)| \leq C_1 d^k(x_1, x_2)$.

(H3) The kernel K has a compact support $[0,1]$ such that $0 < C_3 < K(t) < C_4 < \infty$.

(H4) The smoothing parameter such that : $\lim_{n\to\infty} h_n = 0$ and $\lim_{n \to \infty} \log n / n\phi_x(h_n) = 0$.

(H5) The response variable such that : $\forall m \geq 2, \; E(|Y|^m | X = x) < C < \infty$.

we obtain the following theorem

Théorème 1. *Under the conditions (H1)-(H5), we have*

$$\widehat{r}(x) - r(x) = O(h^k) + O\left(\sqrt{\frac{\log n}{n\varphi_x(h)}}\right), \ p.co. \tag{2.1}$$

Lemme 1. *Let* $\Delta_1, \dots, \Delta_n$ *be a centered random variables, independent and identically distributed, such that*

$$\forall m \geq 2, \exists C_m > 2, E|Z_1^m| \leq C_m a^{2(m-1)}$$

Then, for all $\epsilon > 0$ *we have*

$$p\left[n^{-1}\left|\sum_{i=1}^{n}\Delta_i\right| > \epsilon\right] \leq 2e^{-\frac{n\epsilon^2}{2a^2(1+\epsilon)}}.$$

Proof of Theorem 1 We have

$$\widehat{r}(x) = \frac{\widehat{g}(x)}{\widehat{f}(x)} \tag{2.2}$$

where

$$\widehat{g}(x) = \frac{1}{n\mathbf{E}[K(\frac{d(x,X_i)}{h_n})]}\sum_{i=1}^{n} Y_i K\left(\frac{d(x,X_i)}{h}\right)$$

and

$$\widehat{f}(x) = \frac{1}{n\mathbf{E}[K(\frac{d(x-X_i)}{h_n})]}\sum_{i=1}^{n} K\left(\frac{d(x,X_i)}{h}\right).$$

We consider the following decomposition :

$$\widehat{r}(x) - r(x) = \frac{\widehat{g}(x) - r(x)}{\widehat{f}(x)} + \left(1 - \widehat{f}(x)\right)\frac{r(x)}{\widehat{f}(x)}. \tag{2.3}$$

Therefore, the Theorem is a consequence of the following lemma

Lemme 2. *Under the conditions (H1)-(H5), we have*

$$\widehat{g}(x) - r(x) = O\left(h^k + \sqrt{\frac{\log n}{n\varphi_x(h)}}\right) \tag{2.4}$$

Lemme 3. *Under the conditions (H1)-(H4), we have*

$$\widehat{f}(x) - 1 = O\left(\sqrt{\frac{logn}{n\varphi_x(h)}}\right) \tag{2.5}$$

Lemme 4. *!Under the conditions (H1)-(H4), we have*

$$\exists \delta > 0, \quad \textit{tel que} \quad \sum P\left[|\widehat{f}(x)| < \delta\right] < \infty \tag{2.6}$$

Proof of lemma(2) We have

$$\widehat{g}(x) - r(x) = \widehat{g}(x) - \mathbf{E}\widehat{g}(x) + \mathbf{E}\widehat{g}(x) - r(x)$$

Therefore, it suffices to show that

$$\mathbf{E}\widehat{g}(x) - r(x) = O(h^k) \tag{2.7}$$

$$\mathbf{E}\widehat{g}(x) - \widehat{g}(x) = O\left(\sqrt{\frac{logn}{n\varphi_x(h)}}\right) \tag{2.8}$$

For (2.7) we write

$$\mathbf{E}\widehat{g}(x) - r(x) = \mathbf{E}(Y_1\Delta_1) - r(x)$$

with $\Delta_i = \dfrac{K(h^{-1}d(x, X_i))}{\mathbf{E}K(h^{-1}d(x, X_i))}$. Thus,

$$\begin{aligned} \mathbf{E}\widehat{g}(x) - r(x) &= \mathbf{E}\left[Y_1\Delta_1\right] - r(x) \\ &= \mathbf{E}\left[\mathbf{E}\left[Y_1\Delta_1|X_1\right]\right] - r(x) \\ &= \mathbf{E}\left[r(X_1)\Delta_1 - r(x)\right]. \end{aligned}$$

Now, we use the Lipschitz condition on r to arrive at

$$\begin{aligned} |\ \mathbf{E}\widehat{g}(x) - r(x)\ | &\leq \mathbf{E}\ |\ r(X_1) - r(x)\ |\ \Delta_1 \\ &\leq C_1\mathbf{E}[d^k(x, X_1)\Delta_1] \end{aligned}$$

as $\mathbf{E}\Delta_1 = 1$ then

$$| \mathbf{E}\widehat{g}(x) - r(x) | \leq C_1(h^k),$$

Hence ,

$$\mathbf{E}\widehat{g}(x) - r(x) = O(h^k).$$

Now, we show (2.8). To do that, we note for all $i = 1, ..., n$, $K_i = K(h^{-1}d(x, X_i))$. So, it suffice to prove that there exists ε_0 such that :

$$\sum_{n\in\mathbf{N}^*} \mathbf{P}\left(\frac{1}{n} \mid \sum_{i=1}^{n}(\mathbf{E}(Y_i\Delta_i) - Y_i\Delta_i) \mid > \varepsilon_0 \left(\sqrt{\frac{logn}{n\varphi_x(h)}}\right)\right) < \infty.$$

For this, we apply the Bernstien inequality on the variables $Z_i = Y_i\Delta_i - \mathbf{E}[Y_i\Delta_i]$. Indeed, Firstly, we have to prove that

$$\exists C > 0, \text{tel que} \quad \forall m = 2, 3, ..., | \mathbf{E}(Z_i)^m | \leq C\varphi_x(h)^{-m+1}). \tag{2.9}$$

To do that, we write

$$\begin{aligned}
\mathbf{E} \mid Y_1 \mid^m \Delta_1^m &= \frac{1}{(\mathbf{E}K_1)^m}(\mathbf{E}(\mid Y_1 \mid^m K_1^m) \\
&= \frac{1}{(\mathbf{E}K_1)^m}(\mathbf{E}(\mathbf{E}(\mid Y_1 \mid^m /X)K_1^m)) \\
&= \frac{1}{(\mathbf{E}K_1)^m}(\mathbf{E}\delta_m(X)K_1^m)) \\
&= \frac{1}{(\mathbf{E}K_1)^m}(\mathbf{E}((\delta_m(X) - \delta_m(x))K_1^m) + \delta_m(x)\mathbf{E}K_1^m).
\end{aligned}$$

which implies that

$$\begin{aligned}
|\mathbf{E}\,|Y_1|^m \,\Delta_1^m| &\leq \mathbf{E}\,|(\delta_m(X) - \delta_m(x)|\,\Delta_1^m + \sigma_m(x)\mathbf{E}\Delta_1^m \\
&\leq \sup_{x'\in B(x,h)} \left|\delta_m(x') - \delta_m(x)\right| \mathbf{E}\Delta_1^m + \delta_m(x)\mathbf{E}\Delta_1^m.
\end{aligned}$$

Observe that $C_1 1_{[0,1]} \leq K^m \leq C_2 1_{[0,1]}$ then

$$C_1\varphi_x(h) \leq \mathbf{E}K_1^m \leq C_2\varphi_x(h)$$

Ainsi,

$$\frac{C_1}{\varphi_x(h)^{m-1}} \leq \mathbf{E}\Delta_1^m \leq \frac{C_2}{\varphi_x(h)^{m-1}}.$$

Consequently

$$\mathbf{E}\left|Y_1\right|^k \Delta_1^k = O(\varphi_x(h)^{-m+1}). \tag{2.10}$$

On the other hand, we use the Newton formula, to write that

$$\mathbf{E}\left(Y_1\Delta_1 - Y_1\Delta_1\right)^m = \sum_{k=0}^{m} C_{k,m}(\mathbf{E}Y_1\Delta_1)^{m-k}(Y_1\Delta_1)^k(-1)^{m-k}$$

with $C_{k,m} = \frac{m!}{k!(m-k)!}$. Hence

$$\begin{aligned}
\mathbf{E}\left|Y_1\Delta_1 - \mathbf{E}(Y_1\Delta_1)\right| &\leq C\sum_{k=0}^{m} C_{k,m}\mathbf{E}\left|Y_1\Delta_1\right|^k \left|r(x)\right|^{m-k} \\
&\leq C\max_{(k=0,1,\ldots,m)} \mathbf{E}\left|Y_1\Delta_1\right|^k \\
&\leq C\max_{(k=2,\ldots,m)} \varphi_x(h)^{-m+1}
\end{aligned}$$

The last equality is a consequence of (2.10) for $k \geq 2$ while, when $k = 1$ we can set $\mathbf{E}\left|Y_1\right|\Delta_1 = O(1)$. Now, we apply the Bernstein inequality with $a^2 = \varphi_x(h)^{-1}$, then $U_n = \frac{a^2 logn}{n}$ we deduce that

$$\mathbf{E}\widehat{g}(x) - \widehat{g}(x) = O\left(\sqrt{\frac{logn}{n\varphi_x(h)}}\right).$$

Proof of lemma (3)

We keep the same notation, as $\mathbf{E}(\Delta_1) = 1$ then it suffices to show that there exist ε_0 such that :

$$\sum_{n\in\mathbf{N}^*} \mathbf{P}\left(\frac{1}{n} \mid \sum_{i=1}^{n}(\mathbf{E}(\Delta_i) - \Delta_i) \mid > \varepsilon_0\left(\sqrt{\frac{logn}{n\varphi_x(h)}}\right)\right) < \infty.$$

For this, we apply the Bernstein inequality on the variable $Z_i = \Delta_i - \mathbf{E}\left[\Delta_i\right]$. Condition (H3) implies that

$$\left|\Delta_i\right| \leq \frac{C}{\varphi_x(h)}.$$

then it suffices to calculate

$$Var\Delta_i = Var\left((Y_i^{-1}\Delta_i)\right) \leq \mathbb{E}(\Delta_i^2)$$

By a simple calculation, we obtain by (H3)

$$0 < C\varphi_x(h) < \mathbb{E}[K_i] < C'\varphi_x(h)$$

and

$$0 < C\varphi_x(h) < \mathbb{E}[K_i^2] < C'\varphi_x(h)$$

$$\mathbb{E}(\Delta_i^2) \leq C'\varphi_x^{-1}(h).$$

Now, we apply the Bernstein inequality, for which, we have for all ε >0 we have :

$$P\left[|\mathbb{E}\hat{f}(x) - 1| > \varepsilon_0\sqrt{\frac{\log n}{\varphi_x(h)}}\right] \leq 2\exp\left(\frac{-n\varepsilon_0^2\varphi_x(h)\log n}{4nC'\varphi_x(h)}\right)$$

$$\leq 2\exp\left(\frac{\varepsilon_0^2\log n}{4C}\right)$$

$$\leq 2n^{\frac{-\varepsilon_0^2}{4C}}.$$

Thus

$$\sum_{i=1}^{n} P\left[|\mathbb{E}\hat{g}_1(x) - \hat{g}_1(x)| > \varepsilon_0\sqrt{\frac{\log n}{n'\varphi_x(h)}}\right] \leq \sum_{i=1}^{n} 2n^{\frac{-\varepsilon_0^2}{4C}}.$$

So, it suffices to choose $\varepsilon_0^2/4C > 1$ to give a converge series.

Proof of lemma (4)

From the previous lemma we have for all t $\varepsilon > 0$

$$\sum_{i=1}^{n} P\left[|\hat{f}(x) - 1| > \varepsilon\right] < \infty.$$

We observe that

$$\hat{f}(x) \leq \frac{1}{2} \Rightarrow |\hat{f}(x) - 1| \geq \frac{1}{2}.$$

It follows that

$$P\left[|\hat{f}(x)| \leq \frac{1}{2}\right] \leq P\left[|\hat{f}(x) - 1| > \frac{1}{2}\right].$$

Thus, it suffices to take that $\delta = \frac{1}{2}$.

2.2 The strong consistency of the relative error regression

The main purpose of this section is to study the almost complete convergence[1] (a.co.) of $\widetilde{r}(x)$ to $r(x)$. To do that we fix a point x in $\mathcal{F}$, and we denote by N_x a neighbor of this point. Hereafter, when no confusion is possible, we will denote by C or C' some strictly positive generic constants and by $K_i = K(h^{-1}d(x, X_i))$ for $i = 1, \ldots n$. Moreover, we will use the notation $B(x, r) = \{x' \in \mathcal{F} \ : d(x', x) < r\}$ and we set $g_\gamma(u) = \mathbb{E}\left[Y^{-\gamma} | X = u\right]$, $\gamma = 1, 2$. We need the following hypotheses

(H1) $\mathbb{P}(X \in B(x, r)) =: \phi_x(r) > 0$ for all $r > 0$ and $\lim_{r \to 0} \phi_x(r) = 0$.

(H2) $\forall (x_1, x_2) \in \mathcal{N}_x^2$,

$$|g_\gamma(x_1) - g_\gamma(x_2)| \leq C\left(d_\gamma^k(x_1, x_2)\right), \quad \text{for } k_\gamma > 0.$$

1. Let $(z_n)_{n \in \mathbb{N}}$ be a sequence of real r.v.'s; we say that z_n converges almost completely (a.co.) to zero if, and only if, $\forall \epsilon > 0$, $\sum_{n=1}^{\infty} P(|z_n| > \epsilon) < \infty$. Moreover, we say that the rate of almost complete convergence of z_n to zero is of order u_n (with $u_n \to 0$) and we write $z_n = O_{a.co.}(u_n)$ if, and only if, $\exists \epsilon > 0$, $\sum_{n=1}^{\infty} P(|z_n| > \epsilon u_n) < \infty$. This kind of convergence implies both almost sure convergence and convergence in probability.

(H3) K is a measurable function with support $[0,1]$ and satisfies $0 < C_2 \le K(\cdot) \le C_3 < \infty$.

(H4) $n\phi_x(h)/\log n \longrightarrow \infty \quad \text{as} \quad n \to \infty.$

(H5) $\forall m \ge 2,\ E[Y^{-m}|X = x] < C < \infty.$

Our conditions are very usual in this context of nonparametric functional statistic. Hypothesis (H1) is the classical concentration property of the probability measure of the functional variable. It allows to quantify the contribution of the topological structure of $\mathcal{F}$ in the convergence rate. While the regularity condition (H2) permits to evaluate the bias term of our estimate. Assumptions (H3)-(H5) are technical conditions imposed for the brievity of proofs.

Théorème 2. *Under the hypotheses (H1)-(H5), we have*

$$|\widetilde{r}(x) - r(x)| = O(h_1^k) + O(h_2^k) + O_{a.co.}\left(\sqrt{\frac{\log n}{n\phi_x(h)}}\right). \tag{2.11}$$

Proof of Theorem 2 We write

$$\widetilde{r}(x) = \frac{\widetilde{g_1}(x)}{\widetilde{g_2}(x)}$$

where

$$\widetilde{g_\gamma}(x) = \frac{1}{nE[K(h^{-1}d(x,X_1))]}\sum_{i=1}^{n} Y_i^{-\gamma}K(h^{-1}d(x,X_i)); \qquad \gamma = 1,2.$$

We consider the classical decomposition :

$$\widetilde{r}(x) - r(x) = \frac{1}{\widetilde{g_2}(x)}\left[\widetilde{g_1}(x) - g_1(x)\right] + \left[g_2(x) - \widetilde{g_2}(x)\right]\frac{r(x)}{\widetilde{g_2}(x)} \tag{2.12}$$

Therefore, Theorem 2 is a consequence of the following intermediate results.

Lemme 5. *Under the hypotheses (H1) and (H3)-(H5), we have , for* $\gamma = 1, 2,$

$$|\widetilde{g_\gamma}(x) - E\widetilde{g_\gamma}(x)| = O_{a.co.}\left(\sqrt{\frac{\log n}{n\phi_x(h)}}\right).$$

Lemme 6. *Under the hypotheses (H1)-(H4), we have , for* $\gamma = 1, 2,$

$$|E\widetilde{g_\gamma}(x) - g_\gamma(x)| = O(h^{k_\gamma}).$$

Corollaire 1. *Under the hypotheses of Theorem 2, we have ,*

$$\sum_{n=1}^{\infty} P\left(\widetilde{g_2}(x) < \frac{g_2(x)}{2}\right) < \infty.$$

2.3 The uniform consistency of the relative error regression

In this section we focus on the uniform almost complete convergence of the estimate over a fixed subset $S_{\mathcal{F}}$ of $\mathcal{F}$. For this, we denote by $\psi_{S_{\mathcal{F}}}(\cdot)$ Kolmogorov's entropy function of $\mathcal{F}$ and we reformulate the previous conditions (H1)-(H5) as follows :

(U1) $\forall x \in S_{\mathcal{F}}$, $\forall \epsilon > 0$, $0 < C\phi(\epsilon) \leq \mathbb{P}\left(X \in B(x, \epsilon)\right) \leq C'\phi(\epsilon) < \infty$,

(U2) $\exists \eta > 0$, such that

$$\forall x, x' \in S^\eta_{\mathcal{F}},\ |g_\gamma(x) - g_\gamma(x')| \leq C d^{k_\gamma}(x, x'),$$

where $S^\eta_{\mathcal{F}} = \{x \in \mathcal{F}, \exists x' \in S_{\mathcal{F}}, d(x, x') \leq \eta\}; \qquad \eta > 0.$

(U3) K is a bounded and Lipschitz kernel on its support $[0, 1)$,

(U4) The functions ϕ and $\psi_{S_{\mathcal{F}}}$ are such that :

(U4a) $\exists \eta_0 > 0,\ \forall \eta < \eta_0,\ \phi'(\eta) < C$, (U4b) For n large enough, $\dfrac{(\log n)^2}{n\,\phi(h)} < \psi_{S_{\mathcal{F}}}\left(\dfrac{\log n}{n}\right) < \dfrac{n\,\phi(h)}{\log n}$,

(U4C) The Kolmogorov's ϵ-entropy of $S_{\mathcal{F}}$ satisfies

$$\sum_{n=1}^{\infty} \exp\left\{(1-\beta)\psi_{S_{\mathcal{F}}}\left(\frac{\log n}{n}\right)\right\} < \infty, \text{ for some } \beta > 1.$$

(U5) $\forall m \geq 2,\ E(|Y^{-m}||X = x) < C < \infty$ for all $x \in S_{\mathcal{F}}$ and $\inf_{x\in S_{\mathcal{F}}} g_2(x) \geq C' > 0$

.

Clearly Conditions (U1)-(U3) and (U5) are simple uniformization of (H1)-(H3) and (H5). While assumption (U4) controls the entropy of $S_{\mathcal{F}}$ which is closely linked to the semi-metric d. Similarly to the concentration propriety, this additional argument control also the contribution of the topological structure of $\mathcal{F}$ in the uniform convergence rate.

Théorème 3. *Under the hypotheses (U1)-(U5), we have*

$$\sup_{x\in S_{\mathcal{F}}} |\widetilde{r}(x) - r(x)| = O(h^{k_1}) + O(h^{k_2}) + O_{a.co.}\left(\sqrt{\frac{\psi_{S_{\mathcal{F}}}\left(\frac{\log n}{n}\right)}{n\phi(h)}}\right). \tag{2.13}$$

Proof of Theorem 3 The proof is based on decomposition 2.12 and the following intermediate results.

Lemme 7. *Under the hypotheses (U1) and (U3)-(U5), we have, for $\gamma = 1, 2$,*

$$\sup_{x\in S_{\mathcal{F}}} |E\widetilde{g}_\gamma(x) - g_\gamma(x)| = O(h^{k_\gamma}).$$

Lemme 8. *Under the hypotheses (U1)-(U4), we have, for $\gamma = 1, 2$,*

$$\sup_{x\in S_{\mathcal{F}}} |\widetilde{g}_\gamma(x) - E\widetilde{g}_\gamma(x)| = O_{a.co.}\left(\sqrt{\frac{\psi_{S_{\mathcal{F}}}\left(\frac{\log n}{n}\right)}{n\phi(h)}}\right).$$

Corollaire 2. *Under the hypotheses of Lemma 8, we have*

$$\exists\delta > 0, \quad \textit{such that} \quad \sum_{n=1}^{\infty} P\left(\inf_{x\in S_{\mathcal{F}}} \widetilde{g}_2(x) < \delta\right) < \infty.$$

2.4 The mean squared consistency of the relative error regression

This section is devoted to the mean squared convergence of our estimate in fixed point $x \in \mathcal{F}$. It is well known, the main feature of L^2-norm convergence is that, unlike to all others consistency modes, the L^2 errors can be easily quantified in the empirical way. This feature is useful in numerous functional statistical methodologies in particular the prediction problem's, the bandwidth choice or the semi-metric choice. Our main interest in this part is to give the exact expression involved in the leading terms of the quadratic error. To do that, we replace (H1), (H3) and (H4) by the following hypotheses, respectively.

(M1) The concentration property (H1) holds. Moreover, there exists a function $\beta_x(\cdot)$ such that

$$\forall s \in [0,1], \quad \lim_{r\to 0} \phi_x(sr)/\phi_x(r) = \chi_x(s).$$

(M2) For $\gamma \in \{1,2\}$, the functions $\Psi_\gamma(\cdot) = E\left[g_\gamma(X) - g_\gamma(x)\middle| d(x,X) = \cdot\right]$ are derivable at 0.

(M3) The kernel K satisfies (H3) and is a differentiable function on $]0,1[$ with derivative K' such that $-\infty < C < K'(\cdot) < C' < 0$.

(M4) $n\phi_x(h) \longrightarrow \infty$.

(M5) The functions $E[Y^{-m}|X = \cdot]; \qquad m = 1,2,3,4$ are continuous in a neighborhood x

Similarly to the pervious asymptotic proprieties, the mean squared consistency is obtained under very standard conditions. They are a simple adaptation of the condition used by Ferraty et al (2007). We recall that Condition (M1) is fulfilled by several small ball probability functions, we quote the following cases (which can be found in Ferraty et al (2007)) :

i) $\phi_x(h) = C_x h^\gamma$ for some $\gamma > 0$ with $\beta_x(u) = u^\gamma$,

ii) $\phi_x(h) = C_x h^\gamma \exp\left\{-Ch^{-p}\right\}$ for some $\gamma > 0$ and $p > 0$ with $\beta_x(u) = \delta_1(u)$ where $\delta_1(\cdot)$ is Dirac's function,

iii) $\phi_x(h) = C_x |\ln h|^{-1}$ with $\beta_x(u) = 1\!\!1_{]0,1]}(u)$.

Assumptions (M2) is a regularity condition which characterize the functional space of our model and is needed to explicit the bias term. The hypotheses (M3)-(M5) are technical conditions and are also similar to those considered in Ferraty *et al.* (2007) for the regression case.

Théorème 4. *Under assumptions (M1)-(M5), we have*

$$E\left[\widetilde{r}(x) - r(x)\right]^2 = B_n^2(x) + \frac{\sigma^2(x)}{n\phi_x(h)} + o(h) + o\left(\frac{1}{n\phi_x(h)}\right),$$

where

$$B_n(x) = \frac{\left(\Psi_1'(0) - r(x)\Psi_2'(0)\right)\beta_0}{\beta_1 g_2(x)} h$$

and

$$\sigma^2 = \frac{\left(1 - 2r(x)E[Y^{-3}|X = x] + r^2(x)E[Y^{-4}|X = x]\right)\beta_2}{g_2^2(x)\beta_1^2 n\phi_x(h)}$$

with

$$\beta_0 = K(1) - \int_0^1 (sK(s))'\chi_x(s)ds, \ \textit{and}, \ \beta^j = K^j(1) - \int_0^1 (K^j)'(s)\chi_x(s)ds, \ \textit{for}, \ j = 1, 2),$$

Proof. By using the same decomposition used in (Theorem 3.1 Laksaci (2007), P.71), we show that the

$$E\left[\widetilde{r}(x)\right] = \frac{E\left[\widehat{g}_1(x)\right]}{E\left[\widehat{g}_2(x)\right]} + O\left(\frac{1}{n\phi_x(h)}\right)$$

and

$$Var\left[\widetilde{r}(x)\right] = \frac{Var\left[\widehat{g}_1(x)\right]}{(E\left[\widehat{g}_2(x)\right])^2} - 2\frac{E\left[\widehat{g}_1(x)\right]Cov(\widehat{g}_1(x),\widehat{g}_2(x))}{(E\left[\widehat{g}_2(x)\right])^3} + \frac{Var\left[\widehat{g}_2(x)\right](E\left[\widehat{g}_1(x)\right])^2}{(E\left[\widehat{g}_2(x)\right])^4} + o\left(\frac{1}{n\phi_x(h)}\right)$$

Consequently, the proof of Theorem 4 can be deduced from the following intermediates results :

Lemme 9. *Under the hypotheses of Theorem (4), we have, for* $\gamma = 1, 2$

$$E\left[\widehat{g}_\gamma(x)\right] = g_\gamma(x) + \Psi'_\gamma(0)\frac{\beta_0}{\beta_1}h + o(h).$$

Lemme 10. *Under the hypotheses of Theorem (4), we have, for* $\gamma = 1, 2$

$$Var\left[\widehat{g}_\gamma(x)\right] = E[Y^{-2\gamma}|X = x]\frac{\beta_2}{\beta_1^2 n\phi_x(h)} + o\left(\frac{1}{n\phi_x(h)}\right).$$

and

$$Cov(\widehat{g}_2(x), \widehat{g}_1(x)) = E[Y^{-3}|X = x]\frac{\beta_2}{\beta_1^2 n\phi_x(h)} + o\left(\frac{1}{n\phi_x(h)}\right).$$

2.5 The asymptotic normality of the relative error regression

This section contains results on the asymptotic normality of $\widetilde{r}(x)$. For this we keep the conditions of the previous section and we establish the following Theorem

Théorème 5. *Assume that (M1)-(M5) hold, then for any $x \in \mathcal{A}$, we have*

$$\left(\frac{n\phi_x(h)}{\sigma^2}\right)^{1/2} \left(\widetilde{r}(x) - r(x) - B_n - o(h)\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \quad as \quad n \to \infty.$$

where $\xrightarrow{\mathcal{D}}$ means the convergence in distribution.

Proof of Theorem We write

$$\widetilde{r}(x) - r(x) = \frac{1}{\widetilde{g}_2(x)} \left[D_n + A_n \left(\widetilde{g}_2(x) - E\widetilde{g}_2(x)\right)\right] + A_n$$

where

$$A_n = \frac{1}{E\widetilde{g}_2(x) g_2(x)} \left[\left[E\widetilde{g}_1(x)\right] g_2(x) - \left[E\widetilde{g}_2(x)\right] g_1(x)\right]$$

and

$$D_n = \frac{1}{g_2(x)} \left[\left[\widetilde{g}_1(x) - E\widetilde{g}_1(x)\right] g_2(x) + \left[E\widetilde{g}_2(x) - \widetilde{g}_2(x)\right] g_1(x)\right]$$

Since

$$A_n = B_n + o(h),$$

then

$$\widetilde{r}(x) - r(x) - B_n - o(h) = \frac{1}{\widetilde{g}_2(x)} \left[D_n + A_n \left(\widetilde{g}_2(x) - E\widetilde{g}_2(x)\right)\right] \tag{2.14}$$

Therefore, Theorem 5 is a consequence of the following results.

Lemme 11. *Under the hypotheses of Theorem (5)*

$$\left(\frac{n\phi_x(h)}{g_2^2(x)\sigma^2}\right)^{1/2} \left(\left[\left[\widetilde{g}_1(x) - E\widetilde{g}_1(x)\right] g_2(x) + \left[E\widetilde{g}_2(x) - \widetilde{g}_2(x)\right] g_1(x)\right]\right) \to N(0,1).$$

Lemme 12. *Under the hypotheses of Theorem (5)*

$$\widehat{g}_2(x) \to g_2(x) \qquad \textit{in probability}$$

and

$$\left(\frac{n\phi_x(h)}{g_2(x)^2\sigma^2}\right)^{1/2} A_n \left(\widetilde{g}_2(x) - E\widetilde{g}_2(x)\right) \to 0 \qquad \textit{in probability.}$$

2.6 Appendix

Proof of Lemma 5 We put, for $\gamma = 1, 2$

$$\Gamma_{i,\gamma} = \frac{1}{E[K_1]} \left[K_i Y_i^{-\gamma} - \mathbb{E}\left[K_i Y_i^{-\gamma} \right] \right].$$

Then,

$$\widetilde{g}_\gamma(x) - E\widetilde{g}_\gamma(x) = \sum_{i=1}^{n} \Gamma_{i,\gamma}.$$

The proof of this Lemma is based on the exponential inequality given by Corollary A.8.ii in Ferraty and Vieu (2006) which requires the evaluation of the quantity ....$E|\Gamma_{i,\gamma}^m|$. Firstly, we write for all $j \le m$

$$\begin{aligned} E\left|Y_1^{-j\gamma} K_1^j\right| &= E\left[K_1^j E\left[\left|Y_1^{-j\gamma}\right| |X_1\right]\right] \\ &= CE\left[K_1^j\right] \\ &\quad \le C'\phi_x(h) \end{aligned}$$

which implies that

$$\frac{1}{E^j[K_1]} E\left|Y_1^{-j\gamma} K_1^j\right| = O\left(\phi_x(h)^{-j+1}\right) \tag{2.15}$$

and

$$\frac{1}{E[K_1]} E\left[Y_1^{-\gamma} K_1\right] \le C.$$

Next, by Newton's binomial expansion we have

$$\begin{aligned} E|\Gamma_{i,\gamma}^m| &\le C \sum_{j=0}^{m} \frac{1}{(E[K_1])^k} E|Y_1^{-j\gamma} K_1^j(x)| \\ &\le C \max_{j=0,\dots,m} \phi_x^{-j+1}(h) \\ &\le C\phi_x^{-m+1}(h). \end{aligned}$$

It follows that

$$E|\Gamma_{i,\gamma}^{m}| = O(\phi_x(h)^{-m+1}). \tag{2.16}$$

Thus, we apply the mentioned exponential inequality with $a = \frac{1}{\sqrt{\phi_x(h)}}$ and we get, for all $\eta > 0$, for $\gamma = 1, 2$,

$$\mathbb{P}\left(|\widetilde{g}_\gamma(x) - E\widetilde{g}_\gamma(x)| > \eta\sqrt{\frac{\log n}{n\,\phi_x(h)}}\right) \;\leq\; C' n^{-C\eta^2}.$$

Finally, an appropriate choice of η permits to deduce that :

$$\sum_n \mathbb{P}\left(|\widetilde{g}_\gamma(x) - E\widetilde{g}_\gamma(x)| > \eta\sqrt{\frac{\log n}{n\,\phi_x(h)}}\right) \;<\infty.$$

Proof of Lemma 6

The equidistribution of the couples (X_i, Y_i) leads to

$$|E\widetilde{g}_\gamma(x) - g_\gamma(x)| = \frac{1}{\mathbb{E}\left[K_1\right]}\mathbb{E}\left[\left(K_1 \mathbb{1}_{B(x,h)}(X_1)\right)\left(g_\gamma(x) - \mathbb{E}\left[Y_1^{-\gamma}|X = X_1\right]\right)\right], \tag{2.17}$$

where $\mathbb{1}$ is indicator function. The Hölder hypothesis (H2) imply

$$\mathbb{1}_{B(x,h)}(X_1)|g_\gamma(X_1) - g_\gamma(x)| \leq C h^{k_\gamma}.$$

Thus,

$$|E\widetilde{g}_\gamma(x) - g_\gamma(x)| \leq C h^{k_\gamma}.$$

Proof of Corollary 1. It is easy to see that,

$$|\widetilde{g}_2(x)| \leq \frac{g_2(x)}{2} \;\Rightarrow\quad |g_2(x) - \widetilde{g}_2(x)| \geq \frac{g_2(x)}{2}.$$

So,

$$\mathbb{P}\left(|\widetilde{g}_2(x)| \leq \frac{g_2(x)}{2}\right) \leq \mathbb{P}\left(|g_2(x) - \widetilde{g}_2(x)| > \frac{g_2(x)}{2}\right).$$

Consequently ,

$$\sum_{n=1}^{\infty} \mathbb{P}\left(|\widetilde{g}_2(x)| < \frac{g_2(x)}{2} \right) < \infty.$$

■

Proof of Lemma 7. Let $x_1, \ldots, x_N$ be a finite set of points in $\mathcal{F}$ such that $S_{\mathcal{F}} \subset \bigcup_{k=1}^{N} B(x_k, \epsilon)$ with $\epsilon = \frac{\log n}{n}$. For all $x \in S_{\mathcal{F}}$, we set $k(x) = \arg \min_{k \in \{1,2,\ldots,N_\epsilon(S_{\mathcal{F}})\}} d(x, x_k)$ and $K_i(x) = K(h^{-1}d(x, X_i))$. We consider the following decomposition :

$$\sup_{x \in S_{\mathcal{F}}} |\widetilde{g}_\gamma(x) - E\widetilde{g}_\gamma(x)| \leq \underbrace{\sup_{x \in S_{\mathcal{F}}} |\widetilde{g}_\gamma(x) - \widetilde{g}_\gamma(x_{k(x)})|}_{F_1} + \underbrace{\sup_{x \in S_{\mathcal{F}}} |\widetilde{g}_\gamma(x_{k(x)}) - E\widetilde{g}_\gamma(x_{k(x)})|}_{F_2}$$
$$+ \underbrace{\sup_{x \in S_{\mathcal{F}}} |E\widetilde{g}_\gamma(x_{k(x)}) - E\widetilde{g}_\gamma(x)|}_{F_3}.$$

• First, we study F_1. A direct consequence of (H3) is that

$$C\phi(h) \leq EK_1(x) \leq C'\phi(h).$$

Therefore,

$$\begin{aligned} F_1 &\leq \sup_{x \in S_{\mathcal{F}}} \frac{1}{n} \sum_{i=1}^{n} \left| \frac{1}{E[K_1(x)]} K_i(x) Y_i^{-\gamma} - \frac{1}{E[K_1(x_{k(x)})]} K_i(x_{k(x)}) Y_i^{-\gamma} \right| \\ &\leq \frac{C}{\phi(h)} \sup_{x \in S_{\mathcal{F}}} \frac{1}{n} \sum_{i=1}^{n} \left| K_i(x) - K_i(x_{k(x)}) \right| Y_i^{-\gamma} \mathbb{1}_{B(x,h) \cup B(x_{k(x)},h)}(X_i). \\ &\leq C \sup_{x \in S_{\mathcal{F}}} \left(F_{11} + F_{12} + F_{13} \right), \end{aligned}$$

with

$$\begin{aligned} F_{11} &= \frac{1}{n\phi(h)} \sum_{i=1}^{n} \left| K_i(x) - K_i(x_{k(x)}) \right| Y_i^{-\gamma} \mathbb{1}_{B(x,h) \cap B(x_{k(x)},h)}(X_i), \\ F_{12} &= \frac{1}{n\phi(h)} \sum_{i=1}^{n} K_i(x) Y_i^{-\gamma} \mathbb{1}_{B(x,h) \cap \overline{B(x_{k(x)},h)}}(X_i), \\ F_{13} &= \frac{1}{n\phi(h)} \sum_{i=1}^{n} K_i(x_{k(x)}) Y_i^{-\gamma} \mathbb{1}_{\overline{B(x,h)} \cap B(x_{k(x)},h)}(X_i). \end{aligned}$$

Concerning the first term we use the fact that K is Lipschitz in $[0,1)$ and we write

$$F_{11} \leq \sup_{x\in S_{\mathcal{F}}} \frac{C}{n}\sum_{i=1}^{n} Z_{i,\gamma} \text{ with } Z_{i,\gamma} = \frac{\epsilon}{h\,\phi(h)} Y_i^{-\gamma} 1\!\!1_{B(x,h)\cap B(x_{k(x)},h)}(X_i) Y_i^{-\gamma}.$$

While for the two lasts terms we use the the boundness of K to write

$$F_{12} \leq \frac{C}{n}\sum_{i=1}^{n} W_{i,\gamma} \text{ with } W_{i,\gamma} = \frac{1}{\phi(h)} Y_i^{-\gamma} 1\!\!1_{B(x,h)\cap\overline{B(x_{k(x)},h)}}(X_i)$$

and

$$F_{13} \leq \frac{C}{n}\sum_{i=1}^{n} V_{i,\gamma} \text{ with } V_{i,\gamma} = \frac{1}{\phi(h)} Y_i^{-\gamma} 1\!\!1_{\overline{B(x,h)}\cap B(x_{k(x)},h)}(X_i).$$

Thus, it suffices to use the same arguments as those used in Lemma5 where $\Gamma_{i,\gamma}$ is replaced by $Z_{i,\gamma}$, $W_{i,\gamma}$ and $V_{i,\gamma}$. In this case, we apply the inequality of Corollary A.8 in Ferraty and Vieu (2006) with $a^2 = \frac{\epsilon}{h\,\phi(h)}$, one gets

$$F_{11} = O_{a.co.}\left(\sqrt{\frac{\epsilon \log n}{n\,h\,\phi(h)}}\right),$$

$$F_{12} = O\left(\frac{\epsilon}{\phi(h)}\right) + O_{a.co.}\left(\sqrt{\frac{\epsilon \log n}{n\,\phi(h)^2}}\right)$$

and

$$F_{13} = O\left(\frac{\epsilon}{\phi(h)}\right) + O_{a.co.}\left(\sqrt{\frac{\epsilon \log n}{n\,\phi(h)^2}}\right).$$

Combination of conditions (U4a) and (U4b) allow to simplify the convergence rate and to get

$$F_1 = O_{a.co.}\left(\sqrt{\frac{\psi_{S_{\mathcal{F}}}(\epsilon)}{n\phi(h)}}\right).$$

Similarly, one can state the same rate of convergence for F_3.

$$F_3 = O_{a.co.}\left(\sqrt{\frac{\psi_{S_{\mathcal{F}}}(\epsilon)}{n\phi(h)}}\right).$$

Now, we evaluate F_2. To see this, we write for all $\eta > 0$,

$$\begin{aligned}\mathbb{P}\left(F_2 > \eta\sqrt{\frac{\psi_{S_{\mathcal{F}}}(\epsilon)}{n\phi(h)}}\right) &= \mathbb{P}\left(\max_{k\in\{1,\cdots,N\}}|\widetilde{g_\gamma}(x_{k(x)}) - E\widetilde{g_\gamma}(x_{k(x)})| > \eta\sqrt{\frac{\psi_{S_{\mathcal{F}}}(\epsilon)}{n\phi(h)}}\right)\\ &\leq N\max_{k\in\{1,\cdots,N\}}\mathbb{P}\left(|\widetilde{g_\gamma}(x_k) - E\widetilde{g_\gamma}(x_k)| > \eta\sqrt{\frac{\psi_{S_{\mathcal{F}}}(\epsilon)}{n\phi(h)}}\right).\end{aligned}$$

Once again we apply the exponential inequality given by Corollary A.8.ii in Ferraty and Vieu (2006) on

$$\Delta_{i,\gamma} = \frac{1}{E[K_1(x_k))]}\left[K_i(x_k)Y_i^{-\gamma} - \mathbb{E}\left[K_i(x_k)Y_i^{-\gamma}\right]\right].$$

Since $E\left|\Delta_{i,\gamma}\right|^m = O(\phi(h)^{-m+1})$, then, we can take $a^2 = \frac{1}{\phi(h)}$. Hence, for all $\eta > 0$

$$\begin{aligned}\mathbb{P}\left(|\widetilde{g_\gamma}(x_k) - E\widetilde{g_\gamma}(x_k)| > \eta\sqrt{\frac{\psi_{S_{\mathcal{F}}}(\epsilon)}{n\phi(h)}}\right) &= \mathbb{P}\left(\frac{1}{n}\left|\sum_{i=1}^n \Delta_{i,\gamma}\right| > \eta\sqrt{\frac{\psi_{S_{\mathcal{F}}}(\epsilon)}{n\phi(h)}}\right)\\ &\leq 2\exp\{-C\eta^2\psi_{S_{\mathcal{F}}}(\epsilon)\}.\end{aligned}$$

By using the fact that $\psi_{S_{\mathcal{F}}}(\epsilon) = \log N$ and by choosing η such that $C\eta^2 = \beta$, we have

$$N\max_{k\in\{1,\cdots,N\}}\mathbb{P}\left(|\widetilde{g_\gamma}(x_k) - E\widetilde{g_\gamma}(x_k)| > \eta\sqrt{\frac{\psi_{S_{\mathcal{F}}}(\epsilon)}{n\phi(h)}}\right) \leq C'N^{1-\beta}. \tag{2.18}$$

This completes the proof. ■

Proof of Lemma 8. The proof is very similar to the proof of Lemma 6 where we have shown that

$$|E\widetilde{g_\gamma}(x) - g_\gamma(x)| \leq \frac{1}{E[K_1(x)]}\Big[E[K_1(x)\,|g_\gamma(X_1) - g_\gamma(x)|]\Big].$$

Consequently, combination of hypotheses (U1) and (U2) a gives

$$\forall x \in S_{\mathcal{F}}, \quad |E\widetilde{g_\gamma}(x) - g_\gamma(x)| \leq C\frac{1}{E[K_1(x)]}[EK_1(x)1\!\!1_{B(x,h)}(X_1)d^{k_\gamma}(X_1,x)] \leq Ch^{k_\gamma},$$

this last inequality yields the proof, since C does not depend on x. ■

Proof of Corollary 2.

It is easy to see that,

$$\inf_{x\in S_{\mathcal{F}}}|\widetilde{g}_2(x)| \leq \frac{g_2(x)}{2} \Rightarrow \exists x \in S_{\mathcal{F}}, \quad \text{such that} \quad g_2(x)-\widetilde{g}_2(x) \geq \frac{g_2(x)}{2} \Rightarrow \sup_{x\in S_{\mathcal{F}}}|g_2(x)-\widetilde{g}_2(x)| \geq \frac{g_2(x)}{2}.$$

We deduce from Lemma 7 that

$$\mathbb{P}\left(\inf_{x\in S_{\mathcal{F}}}|\widetilde{g}_2(x)| \leq \frac{g_2(x)}{2}\right) \leq \mathbb{P}\left(\sup_{x\in S_{\mathcal{F}}}|g_2(x)-\widetilde{g}_2(x)| > \frac{g_2(x)}{2}\right).$$

Consequently ,

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\inf_{x\in S_{\mathcal{F}}}|\widetilde{g}_2(x)| < \frac{g_2(x)}{2}\right) < \infty.$$

□

Proof of Lemma 9 By stationarity, we write, for $\gamma = 1, 2$

$$E[\widehat{g}_\gamma(x)] = \frac{1}{E[K_1]}E\left[K_1 E[Y_1^{-\gamma}|X_1]\right].$$

Now, by the same arguments as those used by Ferraty et al. (2007), for the regression operator, we show that :

$$\begin{aligned} E\left[K_1 E[Y_i^{-\gamma}|X_i]\right] &= g_\gamma(x)E[K_1] + E\left[K_1 E\left[g_\gamma(X_1) - g_\gamma(x)|d(X_1,x)\right]\right] \\ &= g_\gamma(x)E[K_1] + E\left[K_1\left(\Psi_\gamma(d(X_1,x))\right]\right. \end{aligned}$$

Therefore, according the definition of Ψ_γ, we have

$$E[\widehat{g}_\gamma(x)] = g_\gamma(x) + \frac{1}{E[K_1]}E\left[K_1\left(\Psi_\gamma(d(X_1,x))\right]\right..$$

Since $\Psi_\gamma(0) = 0$, $\gamma \in \{1,2\}$ we obtain : for $\gamma \in \{1,2\}$

$$E\left[K_1\left(\Psi_\gamma(d(X_1,x))\right]\right. = \Psi'_\gamma(0)E\left[d(X_1,x)K_1\right] + o(E\left[d(X_1,x)K_1\right]).$$

By a simple algebra, we have under (M1)

$$E[K_1 d(X_1,x)] = h\phi_x(h)\left(K(1) - \int_0^1 (uK(u))'\chi_x(u)du\right) + o(h\phi_x(h)). \tag{2.19}$$

and

$$E[K_1] = \phi_x(h)\left(K(1) - \int_0^1 (K(u))'\chi_x(u)du\right) + o(\phi_x(h)). \tag{2.20}$$

It follows that

$$E\left[\widehat{g}_\gamma(x)\right] = g_\gamma(x) + h\Psi'_\gamma(0)\left[\frac{K(1) - \int_0^1 (uK(u))'\chi_x(u)du}{K(1) - \int_0^1 K'(u)\chi_x(u)du}\right] + o(h).$$

■

Proof of Lemma 10 Similarly to previous Lemma, we have for $\gamma = 1,2$,

$$Var[\widehat{g}_\gamma(x)] = \frac{1}{(nE[K_1])^2}\sum_{i=1}^n Var\left[K_i Y_i^{-\gamma}\right] = \frac{1}{n\left(E[K_1]\right)^2} Var\left[K_1 Y_1^{-\gamma}\right].$$

Conditioning on X and using (M1) and (M4) to get

$$E\left[K_1^2 Y_1^{-2\gamma}\right] = E[Y^{-2\gamma}|X=x]\left(K^2(1) - \int_0^1 (K^2(s))'\chi_x(u)du\right) + o(1) \quad \text{and} \quad E\left[K_1 Y_1^{-\gamma}\right] = O(\phi_x(h)) \tag{2.21}$$

Thus,

$$Var\left[K_1 Y_1^{-\gamma}\right] = E[Y^{-2\gamma}|X=x]\left(K^2(1) - \int_0^1 (K^2(s))'\chi_x(u)du\right) + O\left(\phi_x^2(h)\right). \tag{2.22}$$

In conclusion, we can write

$$Var[\widehat{g}_\gamma(x)] = \frac{E[Y^{-2\gamma}|X=x]\left(K^2(1) - \int_0^1 (K^2(s))'\chi_x(u)du\right)}{n\phi_x(h)\left(K(1) - \int_0^1 K'(s)\chi_x(s)ds\right)^2} + o\left(\frac{1}{n\phi_x(h)}\right).$$

Concerning the covariance term, we follow the same steps as the previous term and we write

$$Cov(\widehat{g}_1(x), \widehat{g}_2(x)) = \frac{1}{n\left(E[K_1]\right)^2} Cov\left(K_1 Y_1^{-2}, K_1 Y_1^{-1}\right)$$

Now, we write

$$Cov\left(K_1Y_1^{-2}, K_1Y_1^{-1}\right) \;=\; E\left[K_1^2Y_1^{-3}\right] - \left(E\left[K_1Y_1^{-2}\right]E\left[K_1Y_1^{-1}\right]\right).$$

Since the first term is leading one in this quantity, we obtain that

$$Cov(\widehat{g}_1(x), \widehat{g}_2(x)) = \frac{E[Y^{-3}|X=x]\left(K^2(1) - \int_0^1 (K^2(s))'\chi_x(s)ds\right)}{n\phi_x(h)\left(K(1) - \int_0^1 K'(s)\chi_x(s)ds\right)^2} + o\left(\frac{1}{n\phi_x(h)}\right).$$

Proof of Lemma 11

Let

$$S_n = \sum_{i=1}^{n}\left(L_i(x) - E[L_i(x)]\right)$$

where

$$L_i(x) := \frac{\sqrt{n\phi_x(h)}}{nE[K_1]}K_i\left(g_1(x)Y_i^{-2} - g_2(x)Y_i^{-1}\right). \tag{2.23}$$

Obviously, we have

$$\sqrt{n\phi_x(h)}\sigma^{-1}\left(\left[\widehat{g}_2(x) - E\widehat{g}_2(x)\right]g_1(x) - \left[\widehat{g}_1(x) - E\widehat{g}_1(x)\right]g_2(x)\right) = \frac{S_n}{\sigma}.$$

Thus, the asymptotic normality of S_n. is sufficient to show the proof of this Lemma. This last is shown by applying the Lyaponove central limit Theorem on $L_i(x)$ where it suffices to show that for some $\delta > 0$

$$\frac{\sum_{i=1}^{n}\mathbb{E}\left[|L_i(x) - \mathbb{E}\left[L_i(x)\right]|^{2+\delta}\right]}{\left(Var\left(\sum_{i=1}^{n}L_i(x)\right)\right)^{(2+\delta)/2}} \to 0.$$

Clearly,

$$\begin{aligned} Var\left(\sum_{i=1}^n L_i(x)\right) &= n\phi_x(h)Var\left[\left[\widetilde{g_1}(x)\right]g_2(x) + \left[\widetilde{g_2}(x)\right]g_1(x)\right] \\ &= n\phi_x(h)\left(Var\left[\widetilde{g_1}(x)\right]g_2^2(x) + Var\left[\widetilde{g_2}(x)\right]g_1^2(x)\right. \\ &\qquad \left. + g_1(x)g_2(x)Cov(\widetilde{g_1}(x), \widetilde{g_2}(x))\right) \\ &= n\phi_x(h)\left(\frac{\beta_2}{\beta_1^2 n\phi_x(h)}\left(g_2^3(x) + g_2(x)g_1(x)E[Y^{-3}|X=x]\right.\right. \\ &\qquad \left.\left. + g_1^2(x)E[Y^{-4}|X=x]\right) + o\left(\frac{1}{n\phi_x(h)}\right)\right). \end{aligned}$$

Hence,

$$Var\left(\sum_{i=1}^n L_i(x)\right) = \sigma + o(1)$$

Therefore, to complete the proof of this Lemma, it is enough to show that the numerator of the above expression converges to 0. For this, we use the C_r-inequality (see Loève (1963), p. 155) we show that,

$$\begin{aligned} \sum_{i=1}^n \mathbb{E}\left[\left|L_i(x) - \mathbb{E}\left[L_i(x)\right]\right|^{2+\delta}\right] \\ &\leq C\sum_{i=1}^n \mathbb{E}\left[|L_i(x)|^{2+\delta}\right] \\ &+ C'\sum_{i=1}^n \left|\mathbb{E}\left[L_i(x)\right]\right|^{2+\delta}. \end{aligned} \tag{2.24}$$

Recall that, for all $j > 0$, $\mathbb{E}\left[K_1^j\right] = O(\phi(h))$, then, because of (H5), we have

$$\sum_{i=1}^{n} \mathbb{E}\left[|L_i(x)|^{2+\delta}|\right] = n^{-\delta/2}(\phi(h))^{-1-\delta/2}\mathbb{E}\left[K_1^{2+\delta}\left|g_1(x)Y_i^{-2} - g_2(x)Y_i^{-1}\right|^{2+\delta}\right]$$

$$\leq n^{-\delta/2}(\phi(h))^{-1-\delta/2}\mathbb{E}\left[K_1^{2+\delta}\left[2^{1+\delta}g_1(x)^{2+\delta}E[|Y_i^{-2(\delta+2)}|X] + 2^{1+\delta}g_2(x)^{2+\delta}E[||Y_i^{-(\delta+2)}||X]\right]\right]$$

$$\leq C(n\phi(h))^{-\delta/2}\Big(\mathbb{E}\left[K_1^{2+\delta}\right]/\phi(h)\Big) \longrightarrow 0.$$

Similarly, the second term of (2.24) is evaluated as follows

$$\begin{aligned}
\sum_{i=1}^{n} |\mathbb{E}\left[L_i(x)\right]|^{2+\delta} &\leq n^{-\delta/2}\left(\phi(h)\right)^{-(2+\delta)/2}\left|\mathbb{E}\left[K_1\left|g_1(x)Y_i^{-2} - g_2(x)Y_i^{-1}\right|\right]\right|^{2+\delta} \\
&\leq Cn^{-\delta/2}\left(\phi(h)\right)^{-(2+\delta)/2}\left|\mathbb{E}\left[K_1\right]\right|^{2+\delta} \\
&\leq Cn^{-\delta/2}\left(\phi(h)\right)^{1+\delta/2} \to 0,
\end{aligned}$$

which completes the proof. ■

Proof of Lemma 12 : For the first limit, we have, by Lemma (9) and Lemma (10)

$$E\left[\widehat{g}_2(x) - g_2(x))\right] \to 0$$

and

$$Var\left[\widehat{g}_2(x)\right] \to 0$$

hence

$$\widehat{g}_2(x) - g_2(x) \to 0 \quad \text{in probability.}$$

Next, for the last convergence, we have by the same fashion

$$E\left[\left(\frac{n\phi_x(h)}{g_1(x)^2\sigma^2}\right)^{1/2} A_n\left(\widetilde{g}_2(x) - E\widetilde{g}_2(x)\right)\right] = 0$$

and

$$Var\left[\left(\frac{n\phi_x(h)}{g_1(x)^2\sigma^2}\right)^{1/2} A_n\left(\widetilde{g}_2(x) - E\widetilde{g}_2(x)\right)\right] = O(A_n^2) = O(h^2) \to 0.$$

It follows that

$$\left(\frac{n\phi_x(h)}{g_1(x)^2\sigma^2}\right)^{1/2} A_n\left(\widetilde{g}_2(x) - E\widetilde{g}_2(x)\right)) \to 0 \quad \text{in probability.}$$

■

Chapitre 3

Functional relative error regression for ergodic data

Ce chapitre fait l'objet d'une publication au International Journal of Mathematics and Statistics.

Noparametric relative erreur regrission for functional ergodic data

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3.1 The model and its estimate

Let $Z_i = (X_i, Y_i)_{i=1,\dots n}$ be a $\mathcal{F}\times\mathbb{R}$-valued measurable strictly stationary process, defined on a probability space $(\Omega,\ \mathcal{A}, \mathbb{P})$, where $\mathcal{F}$ is a semi-metric space, d denoting the semi-metric. In this work, we will assume that the the underlying process Z_i is functional stationary ergodic and we estimate the relative error regression by

$$\widehat{r}(x) = \frac{\sum_{i=1}^{n} Y_i^{-1} K(h^{-1}d(x, X_i))}{\sum_{i=1}^{n} Y_i^{-2} K(h^{-1}d(x, X_i))}$$

where K is a kernel and $h = h_{K,n}$ is a sequence of positive real numbers. Noting that this estimator has been recently introduced in functional statistics by Demongeot et al. (2016). They established its asymptotic properties in the i.i.d. case.

We point out that, from theoretical point of view this work includes the finite dimensional case ($\mathcal{F} = \mathbb{R}^p$) but its importance is due to the fact that it covers also the infinite dimensional case. From practical point of view the ergodicity assumption has a great consideration in practice. In particular, it is one of a principal postulate of statistical physics in order to control the thermodynamic properties of gases, atoms, electrons or plasmas. Furthermore, the functional autoregressive models is a particular example of the functional ergodic random variables. The later is widely considered in functional data analysis to carry out some concrete problem (see, Bosq (1996) for some examples and references).

3.2 Notations, hypotheses and comments

All along the paper, when no confusion is possible, we will denote by C and C' some strictly positive generic constants, x is a fixed point in $\mathcal{F}$ and $\mathcal{N}_x$ denote a fixed neighborhood of x. We denote by $r_\gamma(u) = \mathbb{E}\left[Y^{-\gamma}|X=u\right]$, $\gamma = 1,2$. For $r > 0$, let $B(x,r) := \{x' \in \mathcal{F}|\, d(x',x) < r\}$. Moreover, for $i = 1,\ldots,n$, we put $\mathfrak{F}_k$ as the σ-field generated by $((X_1,Y_1),\ldots(X_k,Y_k)))$ and we pose $\mathfrak{G}_k$ as the σ-field generated by $((X_1,Y_1),\ldots(X_k,Y_k),X_{k+1})$.

In order to establish our asymptotic results we need the following hypotheses :

(H1) The processes $(X_i,Y_i)_{i\in\mathbb{N}}$ satisfies :

$$\begin{cases} \text{(i) The function } \phi(x,r) := \mathbb{P}(X \in B(x,r)) > 0, \quad \forall\, r > 0. \\ \quad \text{(ii) For all } i=1,\ldots,n \text{ there exist a deterministic function } \phi_i(x,\cdot) \text{ such that} \\ \qquad 0 < \mathbb{P}\left(X_i \in B(x,r)|\mathfrak{F}_{i-1}\right) \le \phi_i(x,r), \ \forall\, r > 0 \quad (\phi_i(x,r) \to 0 \text{ as } r \to 0). \\ \text{(iii) For all } r > 0, \ \dfrac{1}{n\phi(x,r)} \displaystyle\sum_{i=1}^{n} \mathbb{P}\left(X_i \in B(x,r)|\mathfrak{F}_{i-1}\right) \to 1 \qquad a.co. \end{cases}$$

(H2) The function r_γ is such that :

$$\begin{cases} \text{(i) The function } r_\gamma \text{ are continuous at the point } x. \\ \text{(ii)} \quad \forall (x_1,x_2) \in \mathcal{N}_x \times \mathcal{N}_x \\ \qquad |r_\gamma(x_1) - r_\gamma(x_2)| \le C d^b(x_1,x_2), \quad b > 0 \ \gamma = 1,2. \end{cases}$$

(H3) For all $j \ge 1$, $\mathbb{E}\left[Y^{-j}|\mathfrak{G}_{i-1}\right] = \mathbb{E}\left[Y^{-j}|X_i\right] < Cj! < \infty$, a.s., (H4) K is a function with support $(0,1)$ such that

$0 < C\mathbb{I}_{(0,1)} < K(t) < C'\mathbb{I}_{(0,1)} < \infty$.

(H5) $\lim_{n\to\infty} h = 0$ and $\lim_{n\to\infty} \frac{\varphi(x,h)\log n}{n^2\phi^2(x,h)} = 0$ where $\varphi(x,h) = \sum_{i=1}^{n} \phi_i(x,h)$.

Comments on the hypotheses

Our assumptions are very mild in this context of nonparametric statistic in functional time series. They allow to involve a larger class of precesses and/or of models. We precise that (H1) and (H5) are the same as used by Gheriballah et al. (2013). Moreover, (H2), (H3) and (H4) are similar to Demongeot et al., (2016).

3.3 Results

Our main result is almost complete (a.co.) [1] convergence.

Théorème 6. *Assume that (H1), (H2)(i) and (H3)-(H5) are satisfied, then, we have*

$$\widehat{r}(x) - r(x) \to 0 \qquad a.co.$$

In order to give a more accurate asymptotic result, we replace (H2) (i) by H2(ii) and we obtain the following result

Théorème 7. *Assume that (H1), (H2) (ii) and (H3)-(H5) are satisfied, then, we have*

$$\widehat{r}(x) - r(x) = O\left(h^b\right) + O\left(\sqrt{\frac{\varphi(x,h)\log n}{n^2\phi^2(x,h)}}\right) \qquad a.co.$$

1. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence of real r.v.'s ; we say that z_n converges almost completely (a.co.) to zero if, and only if, $\forall\epsilon > 0$, $\sum_{n=1}^{\infty} P(|z_n| > \epsilon) < \infty$. Moreover, we say that the rate of almost complete convergence of z_n to zero is of order u_n (with $u_n \to 0$) and we write $z_n = O_{a.co.}(u_n)$ if, and only if, $\exists\epsilon > 0$, $\sum_{n=1}^{\infty} P(|z_n| > \epsilon u_n) < \infty$. This kind of convergence implies both almost sure convergence and convergence in probability.

Proof of the main result : For the proofs of Theorems 6 and 7 we write

$$\widehat{r}(x) = B_n(x) + \frac{R_n(x)}{\widehat{\Psi}_D(x)} + \frac{Q_n(x)}{\widehat{\Psi}_D(x)}$$

where

$$\begin{aligned} Q_n(x) &:= (\widehat{\Psi}_N(x) - \bar{\Psi}_N(x)) - r(x)(\widehat{\Psi}_D(x) - \bar{\Psi}_D(x)) \\ B_n(x) &:= \frac{\bar{\Psi}_N(x)}{\bar{\Psi}_D(x)} - r(x), \quad \text{and} \quad R_n(x) := -B_n(x)(\widehat{\Psi}_N(x) - \bar{\Psi}_N(x)) \end{aligned}$$

with

$$\begin{aligned} \widehat{\Psi}_N(x) &:= \frac{1}{n\mathbb{E}[K(h^{-1}d(x,X_1))]} \sum_{i=1}^{n} K(h^{-1}d(x,X_i))Y_i^{-1}, \\ \bar{\Psi}_N(x) &:= \frac{1}{n\mathbb{E}[K(h^{-1}d(x,X_1))]} \sum_{i=1}^{n} \mathbb{E}\left[K(h^{-1}d(x,X_i))Y_i^{-1}|\mathfrak{F}_{i-1}\right], \\ \widehat{\Psi}_D(x) &:= \frac{1}{n\mathbb{E}[K(h^{-1}d(x,X_1))]} \sum_{i=1}^{n} K(h^{-1}d(x,X_i))Y_i^{-2}, \\ \bar{\Psi}_D(x) &:= \frac{1}{n\mathbb{E}[K(h^{-1}d(x,X_1))]} \sum_{i=1}^{n} \mathbb{E}\left[K(h^{-1}d(x,X_i))Y_i^{-2}|\mathfrak{F}_{i-1}\right]. \end{aligned}$$

Thus, both Theorems are a consequence of the following intermediates results, where their proofs are given at the end.

Lemme 13. *Under Hypotheses (H1) and (H3)-(H6), we have,*

$$\widehat{\Psi}_D(x) - \bar{\Psi}_D(x) = O\left(\sqrt{\frac{\varphi(x,h)\log n}{n^2\phi^2(x,h)}}\right) \qquad a.co.$$

Corollaire 3. *Under Hypotheses of Lemma 13, we have,*

$$\exists C > 0 \quad \sum_{n=1}^{\infty} \mathbb{P}\left(\widehat{\Psi}_D(x) < C\right) < \infty.$$

Lemme 14. *Under Hypotheses (H1),(H2)((i)-(ii)), (H4) and (H5), we have,*

$$|B_n(x)| = o(1) \qquad a.co.$$

If we replace ((H2) (ii)) by ((H2) (iii)), we have

$$|B_n(x)| = O(h^b) \qquad a.co.$$

Lemme 15. *Under Hypotheses (H1) and (H3)-(H5), we have,*

$$|\widehat{\Psi}_N(x) - \bar{\Psi}_N(x)| = O\left(\sqrt{\frac{\varphi(x,h)\log n}{n^2\phi^2(x,h)}}\right) \qquad a.co.$$

3.4 Some particulars situations

In order to emphasize the generality of our study over several existing results we present in this section some particular cases :

- *The independent case :* In this case, we have $\mathbb{P}\left(X_i \in B(x,r)|\mathfrak{F}_{i-1}\right) = \mathbb{P}\left(X_i \in B(x,r)\right)$, then condition (H1(ii)) and (H1(iii)) are verified and for all $i = 1, \ldots n$ take $\phi_i(x,r) = \phi(x,r)$. So, condition (H1) is restricted to $\phi(x,r) > 0$, for all $r > 0$. Thus, our Theorem leads to the next Corollary,

  Corollaire 4. *Under assumptions (H1), (H2)((i)-iii)) and (H3)-(H5) we have :*

  $$\widehat{r}(x) - r(x) = O\left(h^b\right) + O\left(\sqrt{\frac{\log n}{n\phi(x,h)}}\right) \qquad a.co.$$

  Remark 1.

  We point out that in this case where the (X_i, Y_i) are independent, we obtain the same convergence rate given by Demongeot et al., (2016)

- *The real case* The real case can be treated as particular case of our study. It suffices to put $\mathcal{F} = \widehat{R}$, and suppose that the probability density of the random variable X (resp. the conditional density of X given $\mathfrak{F}_{i-1}$) denoted by f (resp. by $f_i^{\mathfrak{F}_{i-1}}$), is of $\mathcal{C}^1$ class, then $\phi(x,h) = \mathbb{P}\left(X_i \in [x-h, x+h]\right) = \int_{x-h}^{x+h} f(s)ds = f(x)h + o(h)$ and $\mathbb{P}\left(X_i \in [x-h, x+h]|\mathfrak{F}_{i-1}\right) = f_i^{\mathfrak{F}_{i-1}}(x)h + o(h))$. Moreover using the ergodic Theorem to prove that

$$\|\frac{1}{n}\sum_{i=1}^{n} f_i^{\mathfrak{F}_{i-1}} - f\| \to 0,$$

where $\|.\|$ is a norm in sparable Banach space $\mathcal{C}^1$. So, (H1) is also verified and Theorem 7 can be reformulated in the following way.

Corollaire 5. *Under assumptions H2 (i), H2 (iii) and (H3)-(H5) we have :*

$$\widehat{r}(x) - r(x) = O\left(h^b\right) + O\left(\sqrt{\frac{\log n}{nh}}\right) \qquad a.co.$$

3.5 Appendix

Proof of Lemma 13

For all $i = 1, \ldots, n$ we put $K_i(x) = K(h^{-1}d(x, X_i))$ and $\Delta_i(x) = K_i(x)Y_i^{-2} - \mathbb{E}\left[K_i(x)Y_i^{-2}|\mathfrak{F}_{i-1}\right]$. Then, it can be seen that

$$\widehat{\Psi}_D(x) - \bar{\Psi}_D(x) = \frac{1}{n\mathbb{E}\left[K_1(x)\right]}\sum_{i=1}^{n}\Delta_i(x)$$

with $\Delta_i(x)$ is a triangular array of martingale differences according the σ-fields $(\mathfrak{F}_{i-1})_i$. For this, we must evaluate the quantity $\mathbb{E}[\Delta_i(x)|\mathfrak{F}_{i-1}]$. The Latter can be evaluated by the same arguments as those invoked for proving Lemma 5 in Laib and Louani (2011). Indeed, firstly by (H5), we have, for all $j \leq p$

$$\begin{aligned}
\mathbb{E}\left[\left|K_i^j(x)Y_i^{-2j}\right||\mathfrak{F}_{i-1}\right] &= \mathbb{E}\left[K_i^j(x)\mathbb{E}\left[\left|Y_i^{-2j}\right||\mathfrak{G}_{i-1}\right]|\mathfrak{F}_{i-1}\right] \\
&= \mathbb{E}\left[K_i^j(x)\mathbb{E}\left[\left|Y_i^{-2j}\right||X_i\right]|\mathfrak{F}_{i-1}\right] \qquad (3.1)\\
&\leq C\mathbb{E}\left[K_1^j(x)|\mathfrak{F}_{i-1}\right] \leq C\phi_i(x,h).
\end{aligned}$$

Secondly, by Newton's binomial expansion

$$\mathbb{E}[\Delta_i^p(x)|\mathfrak{F}_{i-1}] = \sum_{k=0}^{p} C_{k,m}(-1)^{m-k}\mathbb{E}\left(Y_i^{-2}K_i(x)\right)^k \left(\mathbb{E}^{m-k}\left[Y_i^{-2}K_i(x)|\mathfrak{F}_{i-1}\right]\right)$$

where $C_{k,m} = \frac{m!}{k!(m-k)!}$. Next, employing Jensen inequality and (3.1) to write that

$$\mathbb{E}^{m-k}\left[K_1(x)Y_1^{-2}|\mathfrak{F}_{i-1}\right] \leq \mathbb{E}\left[(K_1(x)Y_1^{-2})^{m-k}|\mathfrak{F}_{i-1}\right] \leq C\phi_i(x,h)$$

It follows that

$$\mathbb{E}[\Delta_i^p(x)|\mathfrak{F}_{i-1}] \leq C\phi_i(x,h).$$

Now, applying the exponential inequality of Lemma 1 in Louani and Laib (2011, P.365) to get for all $\varepsilon > 0$, we have

$$\begin{aligned}
\mathbb{P}\left\{\left|\widehat{\Psi}_D(x) - \bar{\Psi}_D(x)\right| > \varepsilon\right\} &= \mathbb{P}\left\{\left|\frac{1}{n\mathbb{E}\left[K_1(x)\right]}\sum_{i=1}^{n}\Delta_i(x)\right| > \varepsilon\right\} \\
&\leq 2\exp\left\{-\frac{\varepsilon^2 n^2\mathbb{E}[\Delta_1(x)]^2}{2(\varphi(x,h) + C\varepsilon n\mathbb{E}[\Delta_1(x)])}\right\} \\
&\leq 2\exp\left\{\frac{-\varepsilon^2 n^2\mathbb{E}[\Delta_1(x)]^2}{C\varphi(x,h)}\left(\frac{1}{1 + \frac{C\varepsilon n\mathbb{E}[\Delta_1(x)]}{\varphi(x,h)}}\right)\right\} \quad (3.2)
\end{aligned}$$

Finally, taking $\varepsilon = \epsilon_0\frac{\sqrt{\varphi(x,h)\log n}}{n\mathbb{E}[K_1(x)]}$ and using the fact that $\frac{\log n}{\varphi(x,h)} = o(1)$ to show that

$$\mathbb{P}\left\{\left|\widehat{\Psi}_D(x) - \bar{\Psi}_D(x)\right| > \epsilon_0\frac{\sqrt{\varphi(x,h)\log n}}{n\mathbb{E}[K_1(x)]}\right\} \leq n^{-C\epsilon_0^2}.$$

Consequently an appropriate choice of ϵ_0 complete the proof of this lemma.

Proof of Corollary 3

It is clear that, under (H5), there exists $0 < C < C' < \infty$

$$0 < C\frac{1}{n\phi(x,h)}\sum_{i=1}^{n}\mathbb{P}(X_i \in B(x,r)|\mathfrak{F}_{i-1}) < \bar{\Psi}_D(x) < \left|\widehat{\Psi}_D(x) - \bar{\Psi}_D(x)\right| + \widehat{\Psi}_D(x).$$

Hence,

$$\frac{C}{n\phi(x,h)}\sum_{i=1}^{n}\mathbb{P}(X_i\in B(x,r)|\mathfrak{F}_{i-1})-\left|\widehat{\Psi}_D(x)-\bar{\Psi}_D(x)\right|<\widehat{\Psi}_D(x).$$

It follows that

$$\begin{aligned}\mathbb{P}\left(\widehat{\Psi}_D(x)\leq\tfrac{C}{2}\right) &\leq\mathbb{P}\left(\frac{C}{n\phi(x,h)}\sum_{i=1}^{n}\mathbb{P}(X_i\in B(x,r)|\mathfrak{F}_{i-1})<\frac{C}{2}+\left|\widehat{\Psi}_D(x)-\bar{\Psi}_D(x)\right|\right)\\ &\leq\mathbb{P}\left(\left|\frac{C}{n\phi(x,h)}\sum_{i=1}^{n}\mathbb{P}(X_i\in B(x,r)|\mathfrak{F}_{i-1})-\left|\widehat{\Psi}_D(x)-\bar{\Psi}_D(x)\right|-C\right|>\frac{C}{2}\right).\end{aligned}$$

It is obvious that the previous Lemma and (H1)(iii) allows to get

$$\sum_n\mathbb{P}\left(\left|\frac{C}{n\phi(x,h)}\sum_{i=1}^{n}\mathbb{P}(X_i\in B(x,r)|\mathfrak{F}_{i-1})-\left|\widehat{\Psi}_D(x)-\bar{\Psi}_D(x)\right|-C\right|>\frac{C}{2}\right).$$

which gives the result.

■

Proof of Lemma 14

Using a similar argument as those used by Laib and Louani (2010) to write

$$\begin{aligned}B_n(x) &=\frac{\bar{\Psi}_N(x)-r(x)\bar{\Psi}_D(x)}{\bar{\Psi}_D(x)}\\ &=\frac{1}{nr_2(x)\mathbb{E}\left[K_1(x)\right]\bar{\Psi}_D(x)}\sum_{i=1}^{n}\left[\mathbb{E}\left[K_i(x)\mathbb{E}[Y_i^{-1}|\mathfrak{G}_{i-1}]|\mathfrak{F}_{i-1}\right]r_2(x)-r_1(x)\mathbb{E}\left[K_i(x)\mathbb{E}[Y_i^{-2}|\mathfrak{G}_{i-1}]|\mathfrak{F}\right.\right.\\ &=\frac{1}{nr_2(x)\mathbb{E}\left[K_1(x)\right]\bar{\Psi}_D(x)}\sum_{i=1}^{n}\left[\mathbb{E}\left[K_i(x)\mathbb{E}[Y_i^{-1}|X_i]|\mathfrak{F}_{i-1}\right]\right]r_2(x)-r_1(x)\mathbb{E}\left[K_i(x)\mathbb{E}[Y_i^{-1}|X_i]|\mathfrak{F}_{i-1}\right]\\ &\leq\frac{1}{nr_2(x)\mathbb{E}\left[K_1(x)\right]\bar{\Psi}_D(x)}\sum_{i=1}^{n}\left[\mathbb{E}\left[K_i(x)\right]\right]r_1(X_i)r_2(x)-r_2(X_i)r_1(x)|\mathfrak{F}_{i-1}]]\\ &\leq\frac{1}{nr_2(x)\mathbb{E}\left[K_1(x)\right]\bar{\Psi}_D(x)}\sum_{i=1}^{n}\left[\mathbb{E}\left[K_i(x)\right]\right]r_1(X_i)r_2(x)-r_1(x)r_2(x)+r_1(x)r_2(x)-r_2(X_i)r_1(x)|\mathfrak{F}_{i-}\end{aligned}$$

Now by ((H2) (iii)) we obtain that,

$$|B_n(x)|\leq C\left(\sup_{x'\in B(x,h)}|r_1(x')-r_1(x)|+\sup_{x'\in B(x,h)}|r_2(x')-r_2(x)|\right)\to 0.$$

However, under ((H2) (ii)), we get,

$$\mathbb{1}_{\{B(x,h)\}}(X_1)|r_1(X_1) - r_1(x)| \leq Ch^b.$$

and

$$\mathbb{1}_{\{B(x,h)\}}(X_1)|r_2(X_1) - r_2(x)| \leq Ch^b.$$

This last result yields the proof.

■

Proof of Lemma 15 Similarly to Lemma 13, we define

$$\Lambda_i(x,z) = K_i(x)Y_i^{-1} - \mathbb{E}\left[K_i(x)Y_i^{-1}|\mathfrak{F}_{i-1}\right].$$

The reset of the proof is based on the same exponential inequality used in previous Lemma. For this, we must evaluate the quantity $\mathbb{E}[\Lambda_i^p(x)|\mathfrak{F}_{i-1}]$. The Latter can be evaluated by the same arguments as those invoked for proving Lemma 13 which allow to write, under (H3)

$$\mathbb{E}[\Lambda_i^p(x)|\mathfrak{F}_{i-1}] \leq C\phi_i(x,h).$$

Thus, we are now in position to apply the exponential inequality of Lemma 1 in Louani and Laib (2011, P.365) and we get : for all $\eta > 0$ and $d_n = l_n^{-1}$, we have

$$\mathbb{P}\left(\left|\widehat{\Psi}_N(x) - \bar{\Psi}_N(x)\right| > \eta\sqrt{\frac{\varphi(x,h)\log n}{n^2\,\phi^2(h)}}\right) \leq C' n^{-C\eta^2 + 1/2b_2}.$$

Consequently, an appropriate choice of η completes the proof of this lemma. ■

Chapitre 4

UIB consistency of the local linear estimate for functional relative error regression

4.1 The Model

Let us introduce n pairs of random variables (X_i, Y_i) for $i = 1, \ldots, n$ that we assume drawn from the pair (X, Y) which is valued in $\mathcal{F} \times \mathcal{R}$, where $\mathcal{F}$ is a semi-metric space equipped with a semi-metric d. Furthermore, we assume that the link between X and Y is modeled by the following relation

$$Y = R(X) + \epsilon,$$

where R is an operator from $\mathcal{F}$ to $\mathcal{R}$ and ϵ is a random error variable such that $\mathbb{E}[\epsilon|X] = 0$. The functional local linear modeling is based on the following assumption

$$\forall x' \text{ in neighborhood of } x \qquad R(x') = a + b\beta(x, x') + o(\beta(x, x'))$$

Thus, the local linear relative error regression is obtained by using the following loss function

$$(\widehat{a}, \widehat{b}) = \arg\min_{(a,b)} \sum_{i=1}^{n} \frac{(Y_i - a - b\beta(X_i, x))^2}{Y_i^2} K(h^{-1}\delta(x, X_i)) \tag{4.1}$$

where $\beta(.,.)$ is a known function from $\mathcal{F}^2$ into $\mathbb{R}$ such that, $\forall \xi \in \mathcal{F}$, $\beta(\xi, \xi) = 0$, with K is a kernel and $h = h_{K,n}$ is a sequence of positive real numbers and $\delta(.,.)$ is a function defined on $\mathcal{F} \times \mathcal{F}$ such that $d(.,.) = |\delta(.,.)|$.

By a simple algebra we get

$$\widehat{R}(x) = \widehat{a} = \frac{\sum_{i,j=1}^{n} V_{ij}(x) Y_j}{\sum_{i,j=1}^{n} V_{ij}(x)} \tag{4.2}$$

where

$$V_{ij}(x) = \beta(X_i, x)\left(\beta(X_i, x) - \beta(X_j, x)\right) K(h^{-1}\delta(x, X_i)) K(h^{-1}\delta(x, X_j)) Y_i^{-2} Y_j^{-2}$$

with the convention $0/0 = 0$.

4.2 Notations and assumptions

We fix a point x in $\in \mathcal{F}$, $\mathcal{N}_x$ neighborhood of x and we consider the following assumptions

(H1) $\forall r > 0, \quad \mathbb{P}\left(r_1 < \delta(X, x) < r_2\right) := \phi_x(r_1, r_2)$ with $\phi_x(r) = \phi_x(-r, r) > 0$ and

$$\text{For all } s \in (0,1), \quad \lim_{r \to 0} \frac{\phi_x(-r/2, sr)}{\phi_x(-r, r)} = \tau_x(s) < \infty.$$

(H2) For all $(x_1, x_2) \in \mathcal{N}_x^2$, we have

$$|g_\gamma(x_1) - g_\gamma(x_2)| \le C\ d^{k_\gamma}(x_1, x_2) \text{ for } k_\gamma > 0.$$

(H3) For $k = 0, 1, 2$ the class of functions :

$$\mathcal{K}_k = \{\cdot \mapsto \gamma^{-k} K(\gamma^{-1}\delta(x,\cdot))\beta^k(x,\cdot),\ \gamma > 0\} \text{ is a pointwise measurable class}$$ [1]

such that :

$$\sup_Q \int_0^1 \sqrt{1 + \log \mathcal{N}(\epsilon \|F\|_{Q,2}, \mathcal{K}_k, d_Q)}\, d\epsilon < \infty$$

where the supremum is taken over all probability measures Q on the space $\mathcal{F}$ with $Q(F^2) < \infty$ and where F is the envelope function [2] of the set $\mathcal{K}_k$. Here, d_Q is the $L_2(Q)$-metric and $\mathcal{N}(\epsilon, \mathcal{K}_k, d_Q)$ is the minimal number of open balls (with respect to the $L_2(Q)$-metric) with radius ϵ which are needed to cover the function class $\mathcal{K}_k$. We will denote by $\|\cdot\|_{Q,2}$ the $L_2(Q)$-norm.

(H4) The kernel K is supported within $(-1/2, 1/2)$ and has a continuous first derivative on $(-1/2, 1/2)$ which is such that :

$$0 < C_2 \mathbb{1}_{(-1/2,1/2)}(\cdot) \le K(\cdot) \le C_3 \mathbb{1}_{(-1/2,1/2)}(\cdot) \quad (\mathbb{1}_A \text{ is the indicator function of the set } A)$$

$$K(1/2)\tau_x(1/2) - \int_{-1/2}^{1/2} K'(s)\tau_x(s)ds > 0$$

$$\text{and} \qquad (1/4)K(1/2)\tau_x(1/2) - \int_{-1/2}^{1/2} (s^2 K'(s))\tau_x(s)ds > 0.$$

1. A class of functions $\mathcal{C}$ is said to be a pointwise measurable class if, there exists a countable subclass $\mathcal{C}_0$ such that for any function $g \in \mathcal{C}$ there exists a sequence of functions $(g_m)_{m\in\mathbb{N}}$ in $\mathcal{C}_0$ such that : $|g_m(z) - g(z)| = o(1)$.

2. An envelope function G for a class of functions $\mathcal{C}$ is any measurable function such that : $\sup_{g\in\mathcal{C}} |g(z)| \le G(z)$, for all z.

(H5) The function $\beta(.,.)$ is such that :

$$\forall x' \in \mathcal{F},\ C_4\,|\delta(x,x')| \leq |\beta(x,x')| \leq C_5\,|\delta(x,x')| \qquad C_4, C_5 > 0.$$

(H6) For all $h \in (a_n, b_n)$ we have

$$h \int_{B(x,h/2)} \beta(u,x) dP(u) = o\left(\int_{B(x,h/2)} \beta^2(u,x)\, dP(u) \right)$$

where $B(x,r) = \{x' \in \mathcal{F}/d(x',x) \leq r\}$ and $dP(x)$ is the cumulative distribution.

(H7) The sequence (a_n) verifies :

$$\frac{\log n}{n \min(a_n, \phi_x(a_n))} \to 0.$$

4.3 The uniform consistency

The following theorem gives the UIB consistency of $\widehat{R}(x)$.

Théorème 8. *Under the conditions (H1)-(H7), we have :*

$$\sup_{a_n \leq h_K \leq b_n} |\widehat{R}(x) - R(x)| = O(b_n^{k_1}) + O(b_n^{k_2}) + O_{a.co.}\left(\sqrt{\frac{\log n}{n\phi_x(a_n)}} \right).$$

Proof We consider the following decomposition

$$\widehat{R}(x) - R(x) = \widehat{B}(x) + \frac{\widehat{D}(x)}{\widehat{f}(x)} + \frac{\widehat{Q}(x)}{\widehat{f}(x)}$$

where

$$\begin{aligned} \widehat{Q}(x) &= (\widehat{g}(x) - \mathbb{E}[\widehat{g}(x)]) - R(x)(\widehat{f}(x) - \mathbb{E}[\widehat{f}(x)]) \\ \widehat{B}(x) &= \frac{\mathbb{E}[\widehat{g}(x)]}{\mathbb{E}[\widehat{f}(x)]} - R(x) \quad \text{and} \quad \widehat{D}(x) = -\widehat{B}(x)(\widehat{f}(x) - \mathbb{E}[\widehat{f}(x)]) \end{aligned}$$

with

$$\widehat{g}(x) = \frac{1}{n(n-1)h^2\phi_x^2(h)} \sum_{i \neq j} V_{ij}(x)(x) Y_j \text{and} \widehat{f}(x) = \frac{1}{n(n-1)h^2\phi_x^2(h)} \sum_{i \neq j} V_{ij}(x)(x).$$

Then, Theorem 8 is a direct consequence of the following Lemmas.

Lemme 16. *Under the assumptions (H1), (H3)-(H7), we have that :*

$$\sup_{a_n \le h \le b_n} \left| \widehat{f}(x) - \mathbb{E}[\widehat{f}(x)] \right| = O_{a.co.}\left(\sqrt{\frac{\log n}{n\phi_x(a_n)}} \right).$$

Corollaire 6. *Under the assumptions of Lemma 23, there exists a real number $C_6 > 0$ such that :*

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\inf_{a_n \le h \le b_n} \widehat{f}(x) < C_6 \right) < \infty.$$

Lemme 17. *Under the hypotheses (H1)-(H6) we have that :*

$$\sup_{a_n \le h \le b_n} \left| \widehat{B}(x) \right| = O(b_n^{\beta}).$$

Lemme 18. *Under the hypotheses of Theorem 8, we have that :*

$$\sup_{a_n \le h \le b_n} |\widehat{g}(x) - I\!\!E[\widehat{g}(x)]| = O_{a.co.}\left(\sqrt{\frac{\log n}{n\phi_x(a_n)}} \right).$$

4.4 Appendix

In what follows, when no confusion is possible, we will denote by C and C' some strictly positive generic constants. Moreover, we put, for any $x \in \mathcal{F}$, and for all $i = 1, \ldots, n$:

$$K_i = K(h^{-1}\delta(x, X_i)),\ \beta_i = \beta(X_i, x) \text{ and } \delta_i = \delta(x, X_i).$$

First of all we state the following lemmas which are needed to establish our asymptotic results

Lemme 19. *(cf. Theorem 2.14.1 in Van der Vaart and Wellner (1996), page 239). Let $Z_1, Z_2, \ldots, Z_n$ be independent and identically distributed taking values in a measurable space $(\mathcal{E}, \Upsilon)$ and consider $\mathcal{C}$ a pointwise measurable class of functions $g : \mathcal{E} \to \widehat{R}$*

with envelope function F, then :

$$\|\alpha_n(g)\|_{\mathcal{C}}\|_p \leq CJ(1, \mathcal{C})\|F\|_{p\vee 2}$$

where

$$\alpha_n(g) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(g(Z_i) - \mathbb{E}g(Z_i)) \text{ and } J(1, \mathcal{C}) = \sup_{Q}\int_0^1 \sqrt{1 + \log \mathcal{N}(\epsilon\|F\|_{Q,2}, \mathcal{C}, d_Q)}d\epsilon$$

with $\|\cdot\|_p = \mathbb{E}^{1/p}[\cdot]^p$, $\|\alpha_n(g)\|_{\mathcal{C}} = \sup_{g\in\mathcal{C}}|\alpha_n(g)|$ *and* $s \vee t$ *denoting the maximum of* s *and* t.

Lemme 20. *(see Dony and Einmahl (2009)).*

Let $Z_1, Z_2, \ldots, Z_n$ *be independent and identically distributed taking values in a measurable space* $(\mathcal{E}, \Upsilon)$ *and consider* $\mathcal{C}$ *a pointwise measurable class of functions* $g : \mathcal{E} \to \widehat{R}$ *satisfying :*

$$\mathbb{E}\|\alpha_n(g)\|_{\mathcal{C}} \leq C\|F\|_2$$

with F *is an envelope function of* $\mathcal{C}$. *Then, for any* $A \in \Upsilon$, *we have :*

$$\mathbb{E}\|\alpha_n(g.\mathbb{1}_A)\|_{\mathcal{C}} \leq 2C\|F.\mathbb{1}_A\|_2.$$

Lemme 21. *(see Dony and Einmahl (2009)) Let* $Z_1, Z_2, \ldots, Z_n$ *be independent and identically distributed taking values in a measurable space* $(\mathcal{E}, \Upsilon)$ *and consider* $\mathcal{C}$ *a pointwise measurable class of functions* $g : \mathcal{E} \to \widehat{R}$ *with envelope function* F. *Assume that for some* $H > 0$ *:*

$$\mathbb{E}[F^p(Z)] \leq \frac{p!}{2}\sigma^2 H^{p-2} \text{ where } \sigma^2 \geq \mathbb{E}[F^2(Z)].$$

Then, for $\beta_n = \mathbb{E}[\|\sqrt{n}\alpha_n(g)\|_{\mathcal{C}}]$, *we have for any* $t > 0$ *:*

$$\mathbb{P}\left\{\max_{1\leq k\leq n}\|\alpha_n(g)\|_{\mathcal{C}} \geq \beta_n + t\right\} \leq \exp\left(-\frac{t^2}{2n\sigma^2 + 2tH}\right).$$

Proof of Lemma 23. Using the same ideas as in Barrientos-Marin *et al.*(2010) which are based on the following decomposition

$$\widehat{f}(x) = \underbrace{\frac{n^2h^2\phi_x^2(h)}{n(n-1)\mathbb{E}[W_{12}]}}_{A_1}\left[\underbrace{\left(\frac{1}{n}\sum_{j=1}^{n}\frac{K_j(x)Y_j^{-2}}{\phi_x(h)}\right)}_{T_1}\underbrace{\left(\frac{1}{n}\sum_{i=1}^{n}\frac{K_i(x)\beta_i^2(x)Y_i^{-2}}{h^2\phi_x(h)}\right)}_{T_2}\right.$$

$$\left.-\underbrace{\left(\frac{1}{n}\sum_{j=1}^{n}\frac{K_j(x)\beta_j(x)Y_j^{-2}}{h\phi_x(h)}\right)}_{T_3}\underbrace{\left(\frac{1}{n}\sum_{i=1}^{n}\frac{K_i(x)\beta_i(x)Y_i^{-2}}{h\phi_x(h)}\right)}_{T_3}\right]$$

The, the claimed result is a simple consequence of

$$\sum_n \mathbb{P}\left\{\sup_{a_n\le h_K\le b_0}\sqrt{\frac{n\phi_x(a_n)}{\log n}}\,|T_l-\mathbb{E}[T_l]|\ge\eta_0\right\}<\infty, \tag{4.3}$$

for some positive number b_0 *and* for $l = 2, 3, 4$.

$$\mathbb{E}[T_l] = O(1) \text{ for } l = 2, 3, 4. \tag{4.4}$$

$$Cov(T_2, T_4) = o\left(\left(\frac{\log n}{n\,\phi_x(a_n)}\right)^{1/2}\right) \tag{4.5}$$

$$\text{and } Var[T_3] = o\left(\left(\frac{\log n}{n\,\phi_x(a_n)}\right)^{1/2}\right). \tag{4.6}$$

Note first that (4.4) was proved by Barrientos-Marin *et al.* (2010). On the other side, under (H1),(H4) and (H5) we have

$$\mathbb{E}[h^{-k}K_i^l\beta_i^k] \le \phi_x(h/2).$$

Now, for (4.5) and (4.6), we write that

$$Cov(T_2, T_4) = \frac{1}{n^2\phi_x^2(h)}\sum_i Cov\left(K_i, \frac{K_i\beta_i^2}{h^2}\right)$$

and

$$Var[T_3] = \frac{1}{n^2\phi_x^2(h)}\sum_i Var\left[Y_i^{-2}K_i\frac{\beta_i}{h}\right].$$

Then

$$Cov(T_2, T_4) \quad = O\left(\frac{1}{n\phi_x(h)}\right) = o\left(\sqrt{\frac{\log n}{n\phi_x(a_n)}}\right)$$

and

$$Var[T_3] = O\left(\frac{1}{n\phi_x(h)}\right) = o\left(\sqrt{\frac{\log n}{n\phi_x(a_n)}}\right).$$

Concerning (4.3). We consider

$$\Delta_i^k = \frac{1}{h^k}\left(Y_i^{-2}K_i\beta_i^k - \mathbb{E}\left[Y_i^{-2}K_i\beta_i^k\right]\right) \text{ for } k = 0, 1, 2$$

and we put

$$h_j = 2^j a_n \text{ and } L(n) = \max\{j : \ h_j \leq 2b_0\}.$$

Therefrom, we can write

$$\sup_{a_n \leq h \leq b_0}\sqrt{\frac{n\phi_x(a_n)}{\log n}}\left|T_l - \mathbb{E}[T_l]\right| \leq \max_{1\leq j\leq L(n)}\ \sup_{h_{j-1}\leq h\leq h_j}\sqrt{\frac{n\phi_x(h)}{\log n}}\left|T_l - \mathbb{E}[T_l]\right|.$$

Now, we have, for $l = 2, 3, 4$

$$T_l - \mathbb{E}[T_l] := \frac{1}{\sqrt{n}\phi_x(h)}\alpha_n^l(K),$$

where $\alpha_n^l(K) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\Delta_i^{4-l} - \mathbb{E}[\Delta_i^{4-l}])$ corresponds to the empirical process based on the variables $X_1, X_2, \ldots, X_n$. Next, we consider the following class of functions :

$$\mathcal{G}_j^l = \left\{z \longmapsto \gamma^{4-l}y^{-2}K\left(\gamma^{-1}\delta(x,z)\right)\beta^{4-l}(x,z) \text{ where } h_{j-1} \leq \gamma \leq h_j\right\}.$$

Therefore

$$\mathbb{P}\left\{\sup_{a_n\leq h\leq b_0}\sqrt{\frac{n\phi_x(a_n)}{\log n}}\left|T_l - \mathbb{E}[T_l]\right| \geq \eta_0\right\}$$
$$\leq L(n)\max_{j=1,\ldots,L(n)}\mathbb{P}\left\{\max_{1\leq k\leq n}\|\sqrt{k}\alpha_k^l(K)\|_{\mathcal{G}_j^l} \geq \eta_0\sqrt{n\phi_x(h_j/2)\log n}\right\}.$$

Furthermore, the Bernstein's inequality is the principal tool to evaluate

$$\mathbb{P}\left\{\max_{1\leq k\leq n}\|\sqrt{k}\alpha_k^l(K)\|_{\mathcal{G}_j^l}\geq\eta_0\sqrt{n\phi_x(h_j/2)\log n}\right\}.$$

For this purpose, we have to study the asymptotic behavior of :

$$\beta_n=\mathbb{E}\left[\|\sqrt{n}\alpha_n^l(K)\|_{\mathcal{G}_j^l}\right]\quad\text{and}\quad\sigma^2=\mathbb{E}\left[G_j^2(X)\right]$$

where G_j is the envelope function of the class $\mathcal{G}_j^l$. Observe that under (H4) and (H5) the envelope function G_j satisfies that :

$$G_j(z)\leq C\mathbb{1}_B(x,h_j/2)(z).$$

Hence,

$$\mathbb{E}\left[G_j^p(X)\right]\leq C^p\phi_x(h_j/2)$$

and

$$\sigma^2=O(\phi_x(h_j/2)).$$

Concerning the term β_n, we combine (H3) together with the Lemma 19's result to write that :

$$\mathbb{E}[\|\alpha_n^l(g)\|_{\mathcal{G}_j^l}]\leq CJ(1,\mathcal{K}_{4-l})\|F\|_2$$

where F is the envelope function of $\mathcal{K}_{4-l}$. Since, Lemma 20's conditions are verified for the class $\mathcal{G}_j^l$ and for the envelope function F. Thus,

$$\mathbb{E}\|\alpha_n^l(g.\mathbb{1}_{B(x,h_j/2)})\|_{\mathcal{G}_j^l}\leq 2CJ(1,\mathcal{K}_{4-l})\|F.\mathbb{1}_{B(x,h_j/2)}\|_2.$$

Finally, we obtain that :

$$\mathbb{E}\left[\|\sqrt{n}\alpha_n^l(K)\|_{\mathcal{G}_j^l}\right]\leq C\sqrt{n\phi_x(h_j/2)}.$$

Now, we apply the Bernstein's inequality for :

$$\beta_n = O\left(\sqrt{n\phi_x(h_j/2)}\right),\ \sigma^2 = O\left(\phi_x(h_j/2)\right),\ D = C \text{ and } t = \eta_0/2\sqrt{n\phi_x(h_j/2)\log n}.$$

which implies that :

$$\begin{aligned}
\mathbb{P}\left\{\max_{1\le k\le n}\|\sqrt{k}\alpha_k^l(K)\|_{\mathcal{G}_j^l} > \eta_0\sqrt{n\phi_x(h_j/2)\log n}\right\} &\le \mathbb{P}\left\{\max_{1\le k\le n}\|\sqrt{k}\alpha_k^l(K)\|_{\mathcal{G}_j^l} > \beta_n + t\right\} \\
&\le \exp\left\{-\eta_0^2\frac{\log n}{8 + C\sqrt{\frac{\log n}{n\phi_x(h_j/2)}}}\right\} \\
&\le n^{-C'\eta_0^2}.
\end{aligned}$$

The last inequality is a consequence of (H7). Moreover, since $L(n) \le 2\log n$, we get that :

$$L(n)\max_{j=1,\dots,L(n)}\mathbb{P}\left\{\max_{1\le k\le n}\|\sqrt{k}\alpha_k^l(K)\|_{\mathcal{G}_j^l} > \eta_0\sqrt{n\phi_x(h_j/2)\log n}\right\} \le C(\log n)n^{-C'\eta_0^2}.$$

Now for η_0 such that $C'\eta_0^2 > 1$ we get :

$$\sup_{a_n\le h\le b_n}|T_l - \mathbb{E}[T_l]| = O_{a.co.}\left(\sqrt{\frac{\log n}{n\phi(a_n)}}\right).$$

Which permits to deduce the proof of this Lemma

■

Proof of Corollary 7. Using some algebra to show that

$$\frac{\mathbb{E}\left[\delta_1^2 Y_i^{-2} K_1\right]}{h^2\phi_x(h)} \longrightarrow (1/4)K(1/2)\tau_x(1/2) - \int_{-1/2}^{1/2} (s^2K(s))'\tau_x(s)ds > 0$$

and

$$\frac{\mathbb{E}\left[K_1\right]}{\phi_x(h)} \longrightarrow K(1/2)\tau_x(1/2) - \int_{-1/2}^{1/2} (K(s))'\tau_x(s)ds > 0.$$

By , (H4) and (H5) we get that

$$\mathbb{E}\left[\beta_1 Y_i^{-2} K_1\right]\right] = o(h\phi_x(h/2)) \text{ and } \mathbb{E}\left[\beta_1 K_1\right]\right] \geq C\mathbb{E}\left[\delta_1^2 K_1\right].$$

Therefore, we can find a constant $C' > 0$, such that :

$$\mathbb{E}[\widehat{f}(x)] \geq C' \text{ for all } h \in (a_n, b_n).$$

Then

$$\inf_{h\in(a_n,b_n)} \widehat{f}(x) \leq \frac{C'}{2} \qquad \text{implies that there exists } h \in (a_n, b_n) \text{ such that } \left|\mathbb{E}[\widehat{f}(x)] - \widehat{f}(x)\right| \geq \frac{C'}{2}$$

$$\text{which allows to write } \sup_{h\in(a_n,b_n)} \left|\mathbb{E}[\widehat{f}(x)] - \widehat{f}(x)\right| \geq \frac{C'}{2}.$$

So, we use the Lemma 23 to write that for a $C = \frac{C'}{2}$:

$$\sum_n \mathbb{P}\left(\inf_{h\in(a_n,b_n)} \widehat{f}(x) \leq C\right) \leq \sum_n \mathbb{P}\left(\sup_{h\in(a_n,b_n)} \left|\mathbb{E}[\widehat{f}(x)] - \widehat{f}(x)\right| \geq C\right) < \infty.$$

■

Proof of Lemma 25. The proof of this Lemma is similar to Lemma 23. It based on the following decomposition

$$\widehat{g}(x) == \underbrace{\frac{n^2h^2\phi_x^2(h)}{n(n-1)\mathbb{E}[W_{12}]}}_{A_1}\left[\underbrace{\left(\frac{1}{n}\sum_{j=1}^{n}\frac{K_j(x)Y_j^{-1}}{\phi_x(h)}\right)}_{T_4}\underbrace{\left(\frac{1}{n}\sum_{i=1}^{n}\frac{K_i(x)\beta_i^2(x)Y_i^{-2}}{h^2\phi_x(h)}\right)}_{T_2}\right.$$

$$\left. - \underbrace{\left(\frac{1}{n}\sum_{j=1}^{n}\frac{K_j(x)\beta_j(x)Y_j^{-1}}{h\phi_x(h)}\right)}_{T_5}\underbrace{\left(\frac{1}{n}\sum_{i=1}^{n}\frac{K_i(x)\beta_i(x)Y_i^{-2}}{h\phi_x(h)}\right)}_{T_3}\right].$$

So, the claimed result is consequences of the three following results

$$\sum_n \mathbb{P}\left\{sup_{a_n\le h\le b_0}\sqrt{\frac{n\phi_x(a_n)}{\log n}}\,|S_m-\mathbb{E}[S_m]|>\eta\right\}<\infty, \quad \text{for } \ m=1,2, \text{for a certain } b_0. \tag{4.7}$$

$$Cov(S_1,T_4)=o\left(\sqrt{\frac{\log n}{n\,\phi_x(a_n)}}\right) \tag{4.8}$$

$$\text{and} \qquad Cov(S_2,T_3)=o\left(\sqrt{\frac{\log n}{n\,\phi_x(a_n)}}\right). \tag{4.9}$$

Firstly, the claimed result (4.7) can be obtained exactly as (4.3). Indeed, we put

$$\Lambda_i^m = h^{1-m}K\beta_i^{m-1}Y_i - h^{1-m}\mathbb{E}\left[K_1\beta_i^{m-1}Y_i^{-1}\right] \text{ for } k=1,2$$

and define the empirical process :

$$\alpha_n'^m(K)=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\Lambda_i^m-\mathbb{E}\left[\Lambda_i^m\right]).$$

We consider the following class of functions :

$$\mathcal{G}_j'^m = \left\{(z,y) \mapsto \gamma^{1-m} y^{-1} K\left(\gamma^{-1}\delta(x,z)\beta^{m-1}(x,z)\right) \text{ for } h_{j-1} \leq \gamma \leq h_j\right\}.$$

This class of functions admits an envelope function $F_j(\cdot,\cdot)$ such that :

$$F_j(z,y) \leq Cy\mathbb{1}_B(x,h_j/2)(z)$$

Under (H1), (H2) and (H4) we obtain that :

$$\mathbb{E}\left[F_j^p(X,Y)\right] \leq C^p\phi_x(h_j/2) \quad \text{and } \mathbb{E}\left[F^2(X,Y)\right] = O\left(\phi_x(h_j/2)\right).$$

By a similar ideas as those used in Lemma 23's proof allow to get :

$$\beta_n' = \mathbb{E}\left[\|\sqrt{n}\alpha_n'^m(K)\|_{\mathcal{G}_j'^m}\right] = O\left(\sqrt{n\phi_x(h_j/2)}\right).$$

Using Bernstein's inequality on the empirical process $\alpha_n'^m(K)$ to show that :

$$\begin{aligned}
\mathbb{P}\left\{\max_{1\leq k\leq n} \|\sqrt{k}\alpha_k'^m(K)\|_{\mathcal{G}_j'^m} > \eta_0'\sqrt{n\phi_x(h_j/2)\log n}\right\} &\leq \mathbb{P}\left\{\max_{1\leq k\leq n} \|\sqrt{k}\alpha_k'^m(K)\|_{\mathcal{G}_j'^m} > \beta_n' + t\right\} \\
&\leq n^{-C'\eta_0'^2}.
\end{aligned}$$

So,

$$\begin{aligned}
&\mathbb{P}\left\{sup_{a_n\leq h\leq b_0}\sqrt{\frac{n\phi_x(a_n)}{\log n}}\,|S_m - \mathbb{E}[S_m]| > \eta\right\} \\
&\leq L(n)\mathbb{P}\left\{\max_{1\leq k\leq n} \|\sqrt{k}\alpha_k'^m(K)\|_{\mathcal{G}_j'^m} > \eta_0'\sqrt{n\phi_x(h_j/2)\log n}\right\} \leq \log(n)n^{-C'\eta_0'^2}.
\end{aligned}$$

Thus a suitable choice of η'_0 allows to deduce that :

$$\sup_{a_n \leq h \leq b_n} |S_m - \mathbb{E}[S_m]| = O_{a.co.}\left(\sqrt{\frac{\log n}{n\phi(a_n)}}\right).$$

Now, let us show the results (4.8) and (4.9). The proof of these lasts follows the same lines as the proof of (4.5) and (4.6), it suffices to write that

$$Cov(S_1, T_4) \;\; = \frac{1}{n^2\phi_x^2(h)} \sum_i Cov\left(K_iY_i, \frac{K_i\beta_i^2}{h^2}\right)$$

and

$$Cov(S_2, T_3) \;\; = \frac{1}{n^2\phi_x^2(h)} \sum_i Cov\left(\frac{K_i\beta_iY_i}{h}, \frac{K_i\beta_i}{h}\right).$$

Using the assumptions (H4) and (H8), for $k = 0, 1$ and $l = 1, 2$, to write that

$$\begin{aligned}
\left|Cov\left(\tfrac{K_i\beta_i^kY_i}{h^k}, \tfrac{K_i\beta_i^l}{h^l}\right)\right| &= \left|\mathbb{E}\left[\frac{K_i^2\beta_i^{k+l}Y_i}{h^{k+l}}\right] - \mathbb{E}\left[\frac{K_i\beta_i^kY_i}{h^k}\right]\mathbb{E}\left[\frac{K_i\beta_i^l}{h^l}\right]\right| \\
&\leq C\left|\mathbb{E}\left[K_i^2Y_i\right]\right| + \left|\mathbb{E}\left[K_iY_i\right]\mathbb{E}\left[K_i\right]\right| \\
&\leq C\left[\phi_x(h/2) + \phi_x^2(h/2)\right] \leq C\phi_x(h/2)
\end{aligned}$$

hence

$$\sum_i Cov\left(\frac{K_i\beta_i^kY_i}{h^k}, \frac{K_i\beta_i^l}{h^l}\right) \leq n\phi_x(h/2)).$$

Thus, for $(k, l) = (0, 2)$ we get

$$Cov(S_1, T_4) \leq C\left(\frac{1}{n\,\phi_x(h)}\right) \leq C\left(\frac{1}{n\,\phi_x(h)}\right) = O_{a.co.}\left(\sqrt{\frac{\log n}{n\phi(a_n)}}\right)$$

Next, it suffices to take $(k, l) = (1, 1)$ to show that

$$Cov(S_2, T_3)C\left(\frac{1}{n\,\phi_x(h)}\right) \leq C\left(\frac{1}{n\,\phi_x(h)}\right) = O_{a.co.}\left(\sqrt{\frac{\log n}{n\phi(a_n)}}\right)$$

Chapitre 5

UNN consistency of the kernel estimator of the relative error regression

5.1 The kNN estimate of the relative error regression

The mean squared relative error is defined as zero of

$$\text{For} \quad Y > 0, \quad \mathbb{E}\left[\left(\frac{Y - r(X)}{Y}\right)^2 |X\right] \tag{5.1}$$

and we estimate it by

$$\widetilde{r}(x) = \frac{\sum_{i=1}^{n} Y_i^{-1} K(h^{-1} d(x, X_i))}{\sum_{i=1}^{n} Y_i^{-2} K(h^{-1} d(x, X_i))} \tag{5.2}$$

where K is a kernel and $h = h_{K,n}$ is a sequence of positive real numbers. Thus, the functional version of the kNN estimator of this model is defined by :

$$\widehat{r}(x) = \frac{\sum_{i=1}^{n} Y_i^{-1} K(H_{k,x}^{-1} d(x, X_i))}{\sum_{i=1}^{n} Y_i^{-2} K(H_{k,x}^{-1} d(x, X_i))} \tag{5.3}$$

where $H_{k,x} = \min \left\{ h \in \widehat{R}^{+} \text{ such that } \sum_{i=1}^{n} 1\!\!1_{B(x,h)}(X_i) = k \right\}$.

5.2 UINN asymptotics

We start by gathering together all assumptions required to obtain our asymptotic results.

(H1) For all $r > 0$, $\mathbb{P}(X \in B(x,r)) =: \phi_x(r) > 0$ such that, for all $s \in (0,1)$,

$$\lim_{r \to 0} \frac{\phi_x(sr)}{\phi_x(r)} = \tau_x(s).$$

(H2) For all $(x_1, x_2) \in \mathcal{N}_x^2$, we have

$$|g_\gamma(x_1) - g_\gamma(x_2)| \leq C\ d^{k_\gamma}(x_1, x_2) \ \text{ for } k_\gamma > 0.$$

(H3) The class of functions :

$$\mathcal{K} = \{\cdot \mapsto K(\gamma^{-1} d(x, \cdot)),\ \gamma > 0\} \text{ is a pointwise measurable class}$$ [1]

such that :

$$\sup_{Q} \int_0^1 \sqrt{1 + \log \mathcal{N}(\epsilon \|F\|_{Q,2}, \mathcal{K}, d_Q)}\, d\epsilon < \infty$$

1. A class of functions $\mathcal{C}$ is said to be a pointwise measurable class if, there exists a countable subclass $\mathcal{C}_0$ such that for any function $g \in \mathcal{C}$ there exists a sequence of functions $(g_m)_{m\in\mathbb{N}}$ in $\mathcal{C}_0$ such that : $|g_m(z) - g(z)| = o(1)$.

where the supremum is taken over all probability measures Q on the probability space $(\mathcal{X}, \mathcal{A})$ with $Q(F^2) < \infty$ where F is the envelope function[2] of the set $\mathcal{K}$. Here, d_Q is the $L_2(Q)$-metric and $\mathcal{N}(\epsilon, \mathcal{K}, d_Q)$ is the minimal number of open balls (with respect to the $L_2(Q)$-metric) with radius ϵ which are needed to cover the function class $\mathcal{K}$. We will denote by $\|\cdot\|_{Q,2}$ the $L_2(Q)$-norm.

(H4) The kernel K is supported within $(0, 1/2)$ and has a continuous first derivative on $(0, 1/2)$ which is such that :

$$0 < C_4 1\!\!1_{(0,1/2)}(\cdot) \leq K(\cdot) \leq C_5 1\!\!1_{(0,1/2)}(\cdot) \ \text{ and } \ K(1/2) - \int_0^{1/2} K'(s)\tau_x(s)ds > 0$$

where $1\!\!1_A$ is the indicator function of the set A.

(H5) The sequence of numbers $(k_{1,n})$ verifies :

$$\frac{\log n}{\min(n\phi_x^{-1}\left(\frac{k_{1,n}}{n}\right), k_{1,n})} \to 0.$$

Théorème 9. *Under the hypotheses (H1)-(H5), and if $\mathbb{E}[|Y|^{-m}|X] < C_5 < \infty$, almost-surely for some $m \geq 2$ and $C_5 > 0$, then we have :*

$$\sup_{k_{1,n} \leq k \leq k_{2,n}} |\widehat{r}(x) - r(x)| = O\left(\phi_x^{-1}\left(\frac{k_{2,n}}{n}\right)^{\max(k_1,k_2)}\right) + O_{a.co.}\left(\sqrt{\frac{\log n}{k_{1,n}}}\right).$$

5.3 Proofs of the results

Proof of Theorem 9. Let us begin by putting

$$z_n = O\left(\phi_x^{-1}\left(\frac{k_{2,n}}{n}\right)^{\beta_1} + \sqrt{\frac{\log n}{k_{1,n}}}\right)$$

2. An envelope function G for a class of functions $\mathcal{C}$ is any measurable function such that : $\sup_{g \in \mathcal{C}} |g(z)| \leq G(z)$, for all z.

and we write for some $\alpha \in]0,1[$:

$$\sup_{k_{1,n}\le k\le k_{2,n}} |\widehat{r}(x)-r(x)| = \sup_{k_{1,n}\le k\le k_{2,n}} |\widehat{r}(x)-r(x)| 1\!\!1_{\left\{\phi_x^{-1}\left(\alpha\frac{k_{1,n}}{n}\right)\le H_{k,x}\le \phi_x^{-1}\left(\frac{k_{2,n}}{\alpha n}\right)\right\}} + \sup_{k_{1,n}\le k\le k_{2,n}} |\widehat{r}(x)-r(x)| 1\!\!1_{\left\{H_{k,x}\notin\left(\phi_x^{-1}\left(\frac{\alpha k_{1,n}}{n}\right),\quad \phi_x^{-1}\left(\frac{k_{2,n}}{\alpha n}\right)\right)\right\}}.$$

Thus, for all $\epsilon > 0$, we have :

$$\begin{aligned}&\mathbb{P}\left\{\sup_{k_{1,n}\le k\le k_{2,n}} |\widehat{r}(x)-r(x)| \ge \epsilon z_n\right\}\\ &\le \mathbb{P}\left\{\sup_{k_{1,n}\le k\le k_{2,n}} |\widehat{r}(x)-r(x)| 1\!\!1_{\left\{\phi_x^{-1}\left(\frac{\alpha k_{1,n}}{n}\right)\le H_{k,x}\le \phi_x^{-1}\left(\frac{k_{2,n}}{n\alpha}\right)\right\}} \ge \frac{\epsilon z_n}{2}\right\}\\ &+\mathbb{P}\left\{H_{k,x}\notin\left(\phi_x^{-1}\left(\frac{\alpha k_{1,n}}{n}\right),\quad \phi_x^{-1}\left(\frac{k_{2,n}}{n\alpha}\right)\right)\right\}.\end{aligned}$$

So, to show that $\sup_{k_{1,n}\le k\le k_{2,n}} |\widehat{r}(x)-r(x)| = O_{a.co.}(z_n)$, it suffices to prove the three following results :

$$\sum_n \sum_{k=k_{1,n}}^{k_{2,n}} \mathbb{P}\left(H_{k,x}\le \phi_x^{-1}\left(\frac{\alpha k_{1,n}}{n}\right)\right) < \infty \tag{5.4}$$

$$\sum_n \sum_{k=k_{1,n}}^{k_{2,n}} \mathbb{P}\left(H_{k,x}\ge \phi_x^{-1}\left(\frac{k_{2,n}}{n\alpha}\right)\right) < \infty \tag{5.5}$$

$$\sup_{\phi_x^{-1}\left(\frac{k_{1,n}}{n}\right)\le h\le \phi_x^{-1}\left(\frac{k_{2,n}}{n}\right)} |\widetilde{r}(x)-r(x)| = O_{a.co.}(z_n) \tag{5.6}$$

For the first claimed result we use the following lemma

Lemme 22. *Let $U_1,\dots,U_n$ be independent Bernoulli random variables with $\mathbb{P}[U_i = 1] = p$ for $i = 1,\dots n$. Set $U := \sum_{i=1}^n X_i$ and $\mu = pn$. Then, for any $\omega > 0$, we have :*

$$\mathbb{P}\left(U \ge (1+\omega)\mu\right) \le e^{-\mu\min\left(\omega^2,\omega\right)/4} \tag{5.7}$$

and if $\omega \in (0,1)$, we have :

$$\mathbb{P}\left(U \le (1-\omega)\mu\right) \le e^{-\mu\left(\omega^2/2\right)}. \tag{5.8}$$

Then, by using Lemma 22, we can write that :

$$\begin{aligned}
\mathbb{P}\left(H_{k,x} \leq \phi_x^{-1}\left(\frac{\alpha k_{1,n}}{n}\right)\right) &= \mathbb{P}\left(\sum_{i=1}^{n} 1\!\!1_{B\left(x,\phi_x^{-1}\left(\frac{\alpha k_{1,n}}{n}\right)\right)} > k\right) \\
&= \mathbb{P}\left(\sum_{i=1}^{n} 1\!\!1_{B\left(x,\phi_x^{-1}\left(\alpha\frac{k_{1,n}}{n}\right)\right)} > \frac{k}{\alpha k_{1,n}}\alpha k_{1,n}\right) \\
&\leq \exp\left(-(k-\alpha k_{1,n})/4\right).
\end{aligned}$$

Therefore one has,

$$\sum_{k=k_{1,n}}^{k_{2,n}} \mathbb{P}\left(H_{k,x} \leq \phi_x^{-1}\left(\frac{\alpha k_{1,n}}{n}\right)\right) \leq k_{2,n}\exp\left(-(1-\alpha)k_{1,n}/4\right) \leq n^{1-((1-\alpha)/4)\frac{k_{1,n}}{\log n}}.$$

By the same way we obtain that :

$$\mathbb{P}\left(H_{k,x} \geq \phi_x^{-1}\left(\frac{k_{2,n}}{\alpha n}\right)\right) \leq \exp\left(-\frac{(k_{2,n}-\alpha k)^2}{2\alpha k_{2,n}}\right).$$

It follows that :

$$\sum_{k=k_{1,n}}^{k_{2,n}} \mathbb{P}\left(H_{k,x} \geq \phi_x^{-1}\left(\frac{k_{2,n}}{\alpha n}\right)\right) \leq k_{2,n}\exp\left(-(1-\alpha)k_{1,n}/2\alpha\right) \leq n^{1-((1-\alpha)/2\alpha)\frac{k_{2,n}}{\log n}}.$$

Because $k_{1,n}/\log n \to \infty$ we finally obtain (5.4) and (5.5). Now for $\sup_{k_{1,n}\leq k\leq k_{2,n}} |\widehat{r}(x) - r(x)| = O_{a.co.}(z_n)$, we start by writing :

$$\widehat{r}(x) - r(x) = \widehat{B}_n(x) + \frac{\widehat{R}_n(x)}{\widehat{F}_D(x)} + \frac{\widehat{Q}_n(x)}{\widehat{F}_D(x)}$$

where

$$\begin{aligned}
\widehat{Q}_n(x) &:= (\widehat{g}(x) - \mathbb{E}[\widehat{g}(x)]) - m(x)(\widehat{f}_D^x - \mathbb{E}[\widehat{F}_D^x]) \\
\widehat{B}_n(x) &:= \frac{\mathbb{E}[\widehat{g}(x)]}{\mathbb{E}[\widehat{F}_D^x]} - m(x) \quad \text{and} \quad \widehat{R}_n(x) := -\widehat{B}_n(x)(\widehat{F}_D^x - \mathbb{E}[\widehat{F}_D^x])
\end{aligned}$$

with

$$\widehat{g}_2(x) = \frac{1}{n\,\phi_x(h_K)}\sum_{i=1}^{n} K\left(h_K^{-1}d(x,X_i)\right)Y_i^{-1}$$

and

$$\widehat{g}_1(x) \;=\; \frac{1}{n\phi_x(h_K)}\sum_{i=1}^{n} K\left(h_K^{-1}d(x,X_i)\right)Y_i^{-2}.$$

Therefore, Theorem 9 is a consequence of the following intermediate results.

Lemme 23. *Under the hypotheses (H1)-(H4), we have that :*

$$\sup_{a_n\le h_K\le b_n}|\widehat{g}_2(x)-\mathbb{E}[\widehat{g}_2(x)]| = O_{a.co.}\left(\sqrt{\frac{\log n}{n\phi(a_n)}}\right).$$

Proof of Lemma 23. In order to prove this Lemma, we have to show that there exists $\eta_0>0$ such that :

$$\sum_n \mathbb{P}\left\{\sup_{a_n\le h_K\le b_0}\sqrt{\frac{n\phi_x(a_n)}{\log n}}\,|\widehat{g}_2(x)-E\widehat{g}_2(x)|\ge\eta_0\right\}<\infty,\ \text{for some positive number } b_0.$$

Then, to do that, we follow similar ideas as in Einmahl and Mason (2005) which are based on the Bernstein's inequality for empirical processs. For this aim, we put :

$$h_{K,j}=2^j a_n \text{ and } L(n)=\max\{j:\ h_{K,j}\le 2b_0\}.$$

Therefore, we have that :

$$\sup_{a_n\le h_K\le b_0}\sqrt{\frac{n\phi_x(a_n)}{\log n}}\,|\widehat{g}_2(x)-E\widehat{g}_2(x)|\le\max_{1\le j\le L(k)}\sup_{h_{K,j-1}\le h_K\le h_{K,j}}\sqrt{\frac{n\phi_x(h_K)}{\log n}}\,|\widehat{g}_2(x)-E\widehat{g}_2(x)|\,.$$

Now, we write the difference :

$$\widehat{g}_2(x)-E\widehat{g}_2(x):=\ \frac{1}{\sqrt{n}\phi_x(h_K)}\alpha_n(K),$$

where $\alpha_n(K)\;=\;\dfrac{1}{\sqrt{n}}\displaystyle\sum_{i=1}^{n}K_i-EK_i$ corresponds to the empirical process based on the variables $X_1,X_2,\ldots,X_n$. Then, we consider the following class of function :

$$\mathcal{G}_{K,j}=\left\{z\longmapsto y^{-2}K\left(\gamma^{-1}d(x,z)\right)\ \text{where } h_{K,j-1}\le\gamma\le h_{K,j}\right\}.$$

Thus

$$\mathbb{P}\left\{\sup_{a_n\le h_K\le b_0}\sqrt{\frac{n\phi_x(a_n)}{\log n}}\,|\widehat{g}_2(x)-E\widehat{g}_2(x)|\ge\eta_0\right\}$$
$$\le\sum_{j=1}^{L(n)}\mathbb{P}\left\{\frac{1}{\sqrt{n\phi_x(h_{K,j}/2)\log n}}\|\sqrt{n}\alpha_n(K)\|_{\mathcal{G}_{K,j}}\ge\eta_0\right\}$$
$$\le L(n)\max_{j=1,\dots L(k)}\mathbb{P}\left\{\max_{1\le k\le n}\|\sqrt{k}\alpha_k(K)\|_{\mathcal{G}_{K,j}}\ge\eta_0\sqrt{n\phi_x(h_{K,j}/2)\log n}\right\}.$$

Furthermore, we apply the Bernstein's inequality (cf. Lemma 21), to evaluate :

$$\mathbb{P}\left\{\max_{1\le k\le n}\|\sqrt{k}\alpha_k(K)\|_{\mathcal{G}_{K,j}}>\eta_0\sqrt{n\phi_x(h_{K,j}/2)\log n}\right\}.$$

For this aim, we must study the asymptotic behavior of :

$$\beta_n=\mathbb{E}\left[\|\sqrt{n}\alpha_n(K)\|_{\mathcal{G}_{K,j}}\right]\quad\text{ and }\quad\sigma^2=\mathbb{E}\left[G^2_{K,j}(X)\right]$$

where $G_{K,j}$ is the envelope function of class $\mathcal{G}_{K,j}$. It follows from (H3) that the envelope function $G_{K,j}$ is such that :

$$G_{K,j}(z)\le C1\!\!1_B(x,h_{K,j}/2)(z).$$

Hence, for all $p\ge 1$, we have that :

$$\mathbb{E}\left[G^p_{K,j}(X)\right]\le C^p\phi_x(h_{K,j}/2)\tag{5.9}$$

and particularly :

$$\sigma^2=O(\phi_x(h_{K,j}/2)).$$

Concerning the term β_n, we combine the assumption (H3) together with Lemma 19's result to get that :

$$\mathbb{E}\|\alpha_n(g)\|_{\mathcal{G}_{K,j}}\le\le CJ(1,\mathcal{K})\|F\|_2$$

where F is the envelope function for $\mathcal{K}$. Thus the conditions of Lemma20 is verified for the class $\mathcal{G}_{K,j}$ and the envelope function F. It follows that,

$$\mathbb{E}\|\alpha_n(g.\mathbb{1}_{B(x,h_{K,j}/2)})\|_{\mathcal{G}_{K,j}} \leq 2CJ(1,\mathcal{K})\|F.\mathbb{1}_{B(x,h_{K,j}/2)}\|_2$$

Finally, we obtain that

$$\mathbb{E}\left[\|\sqrt{n}\alpha_n(K)\|_{\mathcal{G}_{K,j}}\right] \leq C\sqrt{n\phi_x(h_{K,j}/2)}.$$

We are now in position to apply the Bernstein's inequality for :

$$\beta_n = O\left(\sqrt{n\phi_x(h_{K,j}/2)}\right), \; \sigma^2 = O\left(\phi_x(h_{K,j}/2)\right), \; H = C \text{ and } t = \eta_0/2\sqrt{n\phi_x(h_{K,j}/2)\log n}.$$

Therefrom, we obtain that :

$$\begin{aligned} \mathbb{P}\left\{\max_{1\leq k\leq n}\|\sqrt{k}\alpha_k(K)\|_{\mathcal{G}_{K,j}} > \eta_0\sqrt{n\phi_x(h_{K,j}/2)\log n}\right\} &\leq \mathbb{P}\left\{\max_{1\leq k\leq n}\|\sqrt{k}\alpha_k(K)\|_{\mathcal{G}_{K,j}} > \beta_n + t\right\} \\ &\leq \exp\left\{-\eta_0^2\frac{\log n}{8 + C\sqrt{\frac{\log n}{n\phi_x(h_{K,j}/2)}}}\right\} \\ &\leq n^{-C'\eta_0^2}. \end{aligned}$$

Moreover, since $L(k) \leq 2\log n$, we get that :

$$L(k)\max_{j=1,\dots L(k)}\mathbb{P}\left\{\max_{1\leq k\leq n}\|\sqrt{k}\alpha_k(K)\|_{\mathcal{G}_{K,j}} > \eta_0\sqrt{n\phi_x(h_{K,j}/2)\log n}\right\} \leq C(\log n)n^{-C'\eta_0^2}.$$

Choosing now η_0 such that $C'\eta_0^2 > 1$ permits to get :

$$\sup_{a_n\leq h_K\leq b_n}\left|\widehat{F}_D^x - \mathbb{E}[\widehat{F}_D^x]\right| = O_{a.co.}\left(\sqrt{\frac{\log n}{n\phi(a_n)}}\right).$$

Proof of Corollary 7. By some simple analytical arguments we show that :

$$\mathbb{E}[\widehat{g}_2(x)] \to K(1/2) - \int_0^{1/2} K'(s)\tau_x(s)ds > 0.$$

So, for n large enough there exists a constant $C > 0$, such that :

$$\mathbb{E}[\widehat{g}_2(x)] \geq C \text{ for all } h_K \in (a_n, b_n).$$

Therefrom we obtain the following implications :

$$\inf_{h_K \in [a_n, b_n]} \widehat{g}_2(x) \leq \frac{C}{2} \implies \text{there exists } h_K \in (a_n, b_n)v \text{ such that } |\mathbb{E}[\widehat{g}_2(x)] - \widehat{g}_2(x)| \geq \frac{C}{2}$$

$$\implies \sup_{h_K \in (a_n, b_n)} |\mathbb{E}[\widehat{g}_2(x)] - \widehat{g}_2(x)| \geq \frac{C}{2}.$$

Then, we deduce from Lemma 23 that :

$$\mathbb{P}\left(\inf_{h_K \in (a_n, b_n)} \widehat{g}_2(x) \leq \frac{C}{2} \right) \leq \mathbb{P}\left(\sup_{h_K \in (a_n, b_n)} |\mathbb{E}[\widehat{g}_2(x)] - \widehat{g}_2(x)| \geq \frac{C}{2} \right).$$

It is obvious that the previous Lemma allows us to get the desired result. □

Corollaire 7. *Under the hypotheses of Lemma 23, there exists a number real $C > 0$ such that :*

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\inf_{a_n \leq h_K \leq b_n} \widehat{g}_2(x) < C \right) < \infty.$$

Lemme 24. *Under the hypotheses (H1)-(H4), we have that :*

$$\sup_{a_n \leq h_K \leq b_n} \left| \widehat{B}_n(x) \right| = O(b_n^{\max(k_1, k_2)}).$$

Lemme 25. *Under the assumptions of Theorem 9, we have that :*

$$\sup_{a_n \leq h_K \leq b_n} |\widehat{g}_1(x) - E\widehat{g}_1(x)| = O_{a.co.}\left(\sqrt{\frac{\log n}{n\phi(a_n)}} \right).$$

□ **Proof of Lemma 25** As in Lemma 23, it suffices to show that

$$\exists \eta_0' > 0 \qquad \sum_n \mathbb{P}\left\{ \sup_{a_n \leq h_K \leq b_0} \sqrt{\frac{n\phi_x(a_n)}{\log n}} \, |\widehat{g}(x) - \mathbb{E}\left[\widehat{g}(x)\right]| \geq \eta_0' \right\} < \infty \quad \text{for certain } b_0.$$

This can be shown by following similar steps as in Lemma 23. Indeed, we keep the same notations, we consider the empirical process

$$\alpha'_n(K) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i K_i - \mathbb{E}\left[Y_i K_i\right]$$

and we define the following class of function

$$\mathcal{G}'_{K,j} = \left\{(z,y) \to y^{-1} K\left(\gamma^{-1} d(x,z)\right), \qquad h_{K,j-1} \leq \gamma \leq h_{K,j}\right\}.$$

Clearly, this class of function has an envelop function $F_{K,j}(\cdot,\cdot)$ such that

$$F_{K,j}(z,y) \leq C y \mathbb{1}_B(x, h_{K,j}/2)(z)$$

In view of (H3) and (H4), we have

$$\mathbb{E}\left[F^p_{K,j}(X,Y)\right] \leq C^p \phi_x(h_{K,j}/2) \quad \text{and } \mathbb{E}\left[F^2(X,Y)\right] = O\left(\phi_x(h_{K,j}/2)\right). \qquad (5.10)$$

Next, similar ideas as those used in Lemma 23 allow to get

$$\beta'_n = \mathbb{E}\left[\|\sqrt{n}\alpha'_n(K)\|_{\mathcal{G}'_{K,j}}\right] = O\left(\sqrt{n\phi_x(h_{K,j}/2)}\right).$$

We apply now again the Bernsteinï¿½s inequality on the empirical process $\alpha'_n(K)$ we obtain

$$\mathbb{P}\left\{\max_{1\leq k\leq n} \|\sqrt{k}\alpha'_k(K)\|_{\mathcal{G}'_{K,j}} > \eta'_0\sqrt{n\phi_x(h_{K,j}/2)\log n}\right\} \leq \mathbb{P}\left\{\max_{1\leq k\leq n} \|\sqrt{k}\alpha_k(K)\|_{\mathcal{G}'_{K,j}} > \beta'_n + t\right\}$$

$$\leq n^{-C'\eta'^2_0}.$$

Therefore,

$$\mathbb{P}\left\{\sup_{a_n\leq h_K\leq b_0} \sqrt{\frac{n\phi_x(h_K)}{\log n}}\left|\widehat{g}(x) - \mathbb{E}\left[\widehat{g}(x)\right]\right| \geq \eta'_0\right\}$$

$$\leq L(n)\mathbb{P}\left\{\max_{1\leq k\leq n} \|\sqrt{k}\alpha'_k(K)\|_{\mathcal{G}'_{K,j}} > \eta'_0\sqrt{n\phi_x(h_{K,j}/2)\log n}\right\}$$

$$\leq \log(n) n^{-C'\eta'^2_0}.$$

Consequently, an appropriate choice of η_0' permits to deduce that

$$\sup_{a_n \le h_K \le b_n} |\widehat{g}(x) - \mathbb{E}\left[\widehat{g}(x)\right]| = O_{a.co.}\left(\sqrt{\frac{\log n}{n\phi(a_n)}}\right).$$

Conclusion and perspective

The research work developed in this thesis is the modeling of relative nonparametric regression when the data are ergodic. At first, we demonstrated almost complete convergence for a fixed smoothing parameter. Under standard conditions we obtained the speed of convergence. The case of a variable window, we showed the uniform convergence on the smoothing parameter. In the last part, we treated the estimator of the number of nearest neighbors. We also give the speed of convergence on this number. These results are very important in practice. they constitute a mathematical support for the use of the methods of selection of the smoothing parameter as well as the number of neighbors.

The generalization of these last results to the ergodic case is the first perspective of the present contribution. Also, we can mention other perspectives such as asymptotic normality as well as convergence in norm L_p

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