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Intitulée

Problèmes aux limites avec des conditions en plusieurs points

Soutenue le……………………….

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Dédicace

Je dédie ce travail:

À mes chers parents, pour leur soutien, patience et leurs sacrifices durant mes ètudes et tout long de ce projet.

À mes fréres.

À mes sœurs.

À ma famille.

À mes amis...

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Publications

- J. R. Graef, **H. Kadari**, A. Ouahab and A. Oumansour, Existence results for systems of second-order impulsive differential equations, *Acta Math. Uni Comenianae*, In press, pp. 1–16.

Contents

Introduction

The theory of impulsive differential equations have become important in some mathematical. Models of real processes and phenomena studies in physics, chimical technology, population dynamics, biotechnology and economics. The study of impulsive differential equations has initiated in the 1960's by Milman and Myshkis [74, 75]. After a period of active research, mostly in Eastern Europe from 1960-1970, culminating with the monograph by Halanay and Wexler [44].

Several mathematical schools were created, continuing the scientific research on the fundamental and qualitative theory of impulsive differential equations and their applications in the early eighties and then see for example the books [9, 32, 33, 61, 95, 111] and the papers [19, 31, 76, 77, 96, 102, 113, 115, 117].

Recently, systems of ordinary differential equations have been extensively studied. For instance, in [11, 78–81] the authors investigated the existence of solution for a system of differential equations by a means of a vector versions of fixed point theorems. Bolojan and Precup [11], studied implicit first order differential systems with nonlocal conditions by using a vector version of Krasnosel'skii's theorem, vector-valued norms, and matrices having spectral radius less than one.

Perov [86] is considere the Cauchy problem for a systems of a ordinary differential equation by using the Perov, Schauder, and Leray Schauder fixed point principles combined with a technique based on vector valued matrices that converge to zero.

Systems of ordinary impulsive boundary value problems have been studied by a number of aurhors such as, Berrezoug, Henderson and Ouahab [10], Bolojan and Precup [11], E.K. Lee and Y. H. Lee [63], Liu, Hu and Wu [69], Radhakrishnan and Balachandran [90], Sun, Chen, Nieto and M. Otero-Novoa [97].

In this thesis, we shall be concerned by boundary value problems and systems for impulsive differentiall equation, some existence results, among others things, are derived. Our results are based a vector version of fixed point theorems and degree theory.
We have arranged this thesis as follows:

In **Chapter 1**, we introduce notations, definitions, lemmas and fixed point theorems which are used throughout this thesis.

In **Chapter 2**, we prove the existence of solution for a impulsive differential equation with non local Boundary conditions at resonance:

$$x''(t) = f(t, x(t), x'(t)), \quad t \in J', \tag{0.0.1}$$
$$\Delta x(t_k) = I_k(x(t_k)), \quad k = 1, 2, \cdots, m, \tag{0.0.2}$$
$$\Delta x'(t_k) = J_k(x'(t_k)), \quad k = 1, 2, \cdots, m, \tag{0.0.3}$$
$$x(0) = 0, \quad x'(1) = \int_0^1 x'(s)dg(s), \tag{0.0.4}$$

where $J = [0, 1]$, $0 < t_1 < t_2 < \cdots < t_k < 1$, $J' = J \setminus \{t_1, t_2, \cdots, t_m\}$, $f = (f_1, f_2, \cdots, f_n) : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $I_k,\ J_k \in C(\mathbb{R}^n, \mathbb{R}^n)$, $k \in \{1, 2, \cdots, m\}$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ and $\Delta x'(t_k) = x'(t_k^+) - x'(t_k^-)$, where $(x(t_k^+),\ (x'(t_k^+))$ and $(x(t_k^-), x'(t_k^-))$ represent the right and left hand limit of $(x(t), x'(t))$ at $t = t_k$, respectively. Here $g = (g_1, g_2, \cdots, g_n) : J \to \mathbb{R}^n$ has a bounded variation satisfying

$$\int_0^1 dg_i(s) = 1,\ i = 1, 2, \cdots, n.$$

This system is at resonance. Our analysis relies on the Leray-Schauder continuous theorem. The a priori estimates follow from the existence of an open bounded convex subset $C \subset \mathbb{R}^n$, such that, for each $t \in [0, 1]$ and $x \in \bar{C}$, the vector fields $f(t, x, .)$ satisfy geometrical conditions on ∂C

The nonlocale boundary value problems of ordinary differential equations play an important role in both theory and application, and as a consequence, they have attracted a great deal of interest over the years. They are often used to model various phenomena in physics, biology, chemistry. Nonlocal problems for different classes of differential equations and systems are intensively studied in the literature by a variety of methods see for example ([14, 15, 20, 26, 27, 50, 51, 60, 82, 88, 104, 105]). Impulsive differential equations with nonlocal conditions have been studied by many authors, (see [8, 27, 28, 34, 52–54])and references therein.

In the second part of this chapter, we consider the impulsive differential equation at non resonance:

$$(p(t)x'(t))' = f(t, x(t), x'(t)),\ t \in J', \tag{0.0.5}$$
$$\Delta x(t_k) = J_k(x(t_k)),\ k = 1, \cdots, m, \tag{0.0.6}$$
$$\Delta x'(t_k) = I_k(x'(t_k)),\ k = 1, \cdots, m, \tag{0.0.7}$$
$$x(0) = 0, \quad x'(1) = \int_0^1 x'(s)dg(s), \tag{0.0.8}$$

where $J = [0, 1]$, $0 < t_1 < t_2 < \cdots < t_k < 1$, $J' = J \setminus \{t_1, t_2, \cdots, t_m\}$, $f = (f_1, f_2, \cdots, f_n) : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $p \in C^1(J, \mathbb{R})$, $I_k,\ J_k \in C(\mathbb{R}^n, \mathbb{R}^n)$, $k \in \{1, 2, \cdots, m\}$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ and $\Delta x'(t_k) = x'(t_k^+) - x'(t_k^-)$, where $(x(t_k^+),\ (x'(t_k^+))$ and $(x(t_k^-), x'(t_k^-))$ represent the right and left hand limits of $(x(t), x'(t))$ at $t = t_k$, respectively, and $g = (g_1, g_2, \cdots, g_n) : J \to \mathbb{R}^n$ has bounded variation and $\int_0^1 x'(s)dg(s) = \left[\int_0^1 x_1'(s)dg_1(s), \cdots, \int_0^1 x_n'(s)dg_n(s)\right]$ and the integral is means in the Riemann - Stieljes sense, and

$$\int_0^1 \frac{1}{p(s)} dg_i(s) \neq 1,\ i = 1, \cdots, n.$$

This systems is at non resonance. Our approach here is based on the degree theory.

In the first section of **chapter 3**, we present existence and uniquensse results for the the system of second-order impulsive differential equations with two boundary conditions:

$$-u_1''(t) = f_1(t, u_1(t), u_2(t)), \quad t \in J', \tag{0.0.9}$$
$$-u_2''(t) = f_2(t, u_1(t), u_2(t)), \quad t \in J', \tag{0.0.10}$$
$$-\Delta u_1' \mid_{t=t_k} = I_{1,k} u_1(t_k), \quad k = 1, 2, \cdots, m, \tag{0.0.11}$$
$$-\Delta u_2' \mid_{t=t_k} = I_{2,k} u_2(t_k), \quad k = 1, 2, \cdots, m, \tag{0.0.12}$$
$$\alpha u_1(0) - \beta u_1'(0) = 0, \ \alpha u_2(0) - \beta u_2'(0) = 0, \tag{0.0.13}$$
$$\gamma u_1(1) + \delta u_1'(1) = 0, \ \gamma u_2(1) + \delta u_2'(1) = 0, \tag{0.0.14}$$

where $\alpha,\ \beta,\ \gamma,\ \delta \geq 0$, $\rho = \beta\gamma + \alpha\gamma + \alpha\delta > 0$, $J = [0,1]$, $0 < t_1 < t_2 < \cdots < t_m < 1$, $J' = J \setminus \{t_1, t_2, \cdots, t_m\}$, $f_i \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $I_{i,k} \in C(\mathbb{R}, \mathbb{R})$, $i = 1, 2$, $k \in \{1, 2, \cdots, m\}$, $\Delta u' \mid_{t=t_k} = u_1(t_k^+) - u_1(t_k^-)$, and $\Delta u_2' \mid_{t=t_k} = u_2(t_k^+) - u_2(t_k^-)$ in which $u_1'(t_k^+)$, $u_2'(t_k^+)$, $u_1'(t_k^-)$, and $u_2'(t_k^-))$ denote the right and left hand limits of $u_1'(t)$ and $u_2'(t)$ at $t = t_k$, respectively. We set $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1, \cdots, m$, $t_{m+1} = 1$, and let y_k be the restriction of the function y to J_k.an application of the Perov fixed point theorem and the nonlinear alternative of Leray-Schauder type. Both of these approaches make use of convergent matrices.

In the second section, we study the existence of positive solutions for the systems (0.0.9)-(0.0.14), we shall rely on the vector version of Krasnosel'skii's cone fixed point theorem.

In third section, we study the existence of three positive solutions for the systems of second order impulsive differential equations with three points boundary conditions

$$u_1''(t) + h_1(t) f_1(t, u_1(t), u_2(t)) = 0, \quad t \in J', \tag{0.0.15}$$
$$u_2''(t) + h_2(t) f_2(t, u_1(t), u_2(t)) = 0, \quad t \in J', \tag{0.0.16}$$
$$\Delta u_1(t_k) = I_{1,k}(u_1(t_k)), \tag{0.0.17}$$
$$\Delta u_1'(t_k) = -J_{1,k}(u_1(t_k)), \quad k = 1, 2, \cdots, m, \tag{0.0.18}$$
$$\Delta u_2(t_k) = I_{2,k}(u_2(t_k)), \tag{0.0.19}$$
$$\Delta u_2(t_k) = -J_{2,k}(u_2(t_k)), \quad k = 1, 2, \cdots, m, \tag{0.0.20}$$
$$\alpha u_1(0) - \beta u_1'(0) = a u_1(\xi), \quad u_1(1) = 0, \tag{0.0.21}$$
$$\alpha u_2(0) - \delta u_2'(0) = a u_2(\xi), \quad u_2(1) = 0, \tag{0.0.22}$$

where $\alpha, \beta \geq 0$, $a, \xi \in]0,1[$, $J=[0,1]$, $0<t_1<t_2<\cdots<t_m<1$, $J'=J\setminus\{t_1,t_2,\cdots,t_m\}$, $f_i\in C(J\times\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+)$, $h_i\in C(\mathbb{R}^+,\mathbb{R}^+)$, $I_{i,k}\in C(\mathbb{R}^+,\mathbb{R}^+)$ and $J_{i,k}\in C(\mathbb{R}^+,\mathbb{R}^+)$, $\Delta u_1(t_k)=u_1(t_k^+)-u_1(t_k^-)$, $\Delta u_1'(t_k)=u_1'(t_k^+)-u_1'(t_k^-)$ and $\Delta u_2(t_k)=u_2(t_k^+)-u_2(t_k^-)$,$\Delta u_2'(t_k)=u_2'(t_k^+)-u_2'(t_k^-)$ in which $u_1(t_k^+)$, $u_1'(t_k^+)$, $u_2(t_k^+)$ $u_2'(t_k^+)$, $(u_1(t_k^-)$, $u_1'(t_k^-)$, $u_2(t_k^-)$, $u_2'(t_k^-))$ denote the right and left hand limit of $u_1(t)$, $u_1'(t)$ and $u_2(t)$, $u_2'(t)$ at $t=t_k$, respectively.
Our analysis relies on vector versions of Avery and Peterson fixed-point theorem .

Chapter 4 is concerned with the existence of solutions for the system of second-order impulsive differential equations with integral boundary conditions on-un bounded domain :

$$-u''(t)=f(t,u(t),v(t)),\quad t\in J,\ t\neq t_k, \tag{0.0.23}$$
$$-v''(t)=g(t,u(t),v(t)),\quad t\in J,\ t\neq t_k, \tag{0.0.24}$$
$$\Delta u(t_k)=J_{1,k}(u(t_k)),\quad -\Delta u'(t_k)=I_{1,k}(u(t_k)),\quad k=1,2,\cdots, \tag{0.0.25}$$
$$\Delta v(t_k)=J_{2,k}(v(t_k)),\quad -\Delta v'(t_k)=I_{2,k}(v'(t_k)),\quad k=1,2,\cdots, \tag{0.0.26}$$
$$u(0)=\int_0^\infty h_1(s)u(s)ds,\quad u'(\infty)=0, \tag{0.0.27}$$
$$v(0)=\int_0^\infty h_2(s)v(s)ds,\quad v'(\infty)=0, \tag{0.0.28}$$

where $J=[0,+\infty)$, $f,g\in C(J\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $0<t_1<t_2<\cdots<t_k<\cdots$, $t_k\to\infty$, $I_{i,k},\ J_{i,k}\in C(\mathbb{R},\mathbb{R}),\ for\ i=1,2,\ h_i\in C(\mathbb{R}^+,\mathbb{R}^+)$ with $\int_0^\infty h_i(s)ds\neq 1\ for\ i=1,2,\ u'(\infty)=\lim_{t\to\infty}u(t)$ and $v'(\infty)=\lim_{t\to\infty}v(t)$, $\Delta u(t_k)=u(t_k^+)-u(t_k^-)$ and $\Delta v(t_k)=v(t_k^+)-v(t_k^-)$, where $u(t_k^+)$ $(v(t_k^+))$ and $u(t_k^-)$ $(v(t_k^-))$ represent the righ and left hand limit of $u(t)$ $(v(t))$ at $t=t_k$, respectively. $\Delta u'(t_k)=u'(t_k^+)-u'(t_k^-)$ and $\Delta v'(t_k)=v'(t_k^+)-v'(t_k^-)$, where $u'(t_k^+)$ $(v'(t_k^+))$ and $u'(t_k^-)$ $(v'(t_k^-))$ represent the righ and left hand limit of $u'(t)$ $(v'(t))$ at $t=t_k$, respectively.
Using a vectorial version of Krasnoselskii's fixed point theorem for sums of two operators.

Boundary value problems with integral boundary conditions on the half line for different classes of systems differential equations are intensively studied in the literature by a variety of methods (see [21, 22, 107, 114]).

In **Chapter 5**, we shall establish sufficient conditions for the existence results for the implicit first order impulsive differential systems of the form

$$x'(t) = g_1(t, x(t), y(t)) + h_1(t, x'(t), y'(t)), \quad t \in J', \tag{0.0.29}$$
$$y'(t) = g_2(t, x(t), y(t)) + h_2(t, x'(t), y'(t)), \ t \in J', \tag{0.0.30}$$
$$\Delta x(t_k) = I_k(x(t_k)), \ k = 1, 2, \cdots, m, \tag{0.0.31}$$
$$\Delta y(t_k) = J_k(y(t_k)), \ k = 1, 2, \cdots, m, \tag{0.0.32}$$
$$x(0) = \alpha[x], \tag{0.0.33}$$
$$y(0) = \beta[y]. \tag{0.0.34}$$

where $J = [0, 1]$, $0 < t_1 < t_2 < \cdots < t_m < 1$, $J' = J \setminus \{t_1, t_2, \cdots, t_m\}$, $h_i, \ g_i : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions for $i = 1, 2$, $0 < t_1 < t_2 < \cdots < t_m < 1$, $J_k, I_k \in C(\mathbb{R}, \mathbb{R})$ $k \in \{1, 2, \cdots, m\}$. $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ and $\Delta y(t_k) = y(t_k^+) - y(t_k^-)$ in which $x(t_k^+)$ and $x(t_k^+)$ $(x(t_k^-), \ y(t_k^-))$ denote the right (left) limit of $x(t)$ and $y(t)$ at $t = t_k$, respectively. Next $\alpha, \ \beta$ are linear functionals given by Stieltjes integrals

$$\alpha[v] = \int_0^{\tilde{t}} v(s) dA(s),$$

$$\beta[v] = \int_0^{\tilde{t}} v(s) dB(s),$$

where $\tilde{t} \in]t_m, 1]$, using a vectorial version of Krasnoselskii s theorem.
Key words and phrases: Impulsive differential equation, fixed point, matrix convergent to zero, generalized Banach space, Implicit differential equation, Leray-Schauder degree.

Chapter 1

Preliminaries

In this chapter, we introduce notations, definitions, lemmas and fixed point theorems which are used throughout this thesis.

1.1 Some notations and definitions

Let $C(J,\mathbb{R})$ be the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\left\{|y(t)| : 0 \leq t \leq T\right\},$$

Let $L^1(J,\mathbb{R})$ denote the Banach space of functions $y : J \to \mathbb{R}$ that are Lebesgue integrable with norm

$$\|y\|_1 = \int_0^b |y(t)|dt.$$

$AC(J,\mathbb{R})$ is the space of functions $y : J \to \mathbb{R}$, which are absolutely continuous and we let $AC^1(J,\mathbb{R})$ the space of differentiable functions $y : J \to \mathbb{R}$, whose first derivative, y' is absolutely continuous.

In the part we need the following definitions:

Definition 1.1.1. *A subset P of a real Banach space is a cone if it is closed and moreover*

(*i*) $P + P \subset P$, *with* $P \setminus \{0\} \neq \emptyset$;

(*ii*) $\lambda P \subset P$ *for all* $\lambda \in \mathbb{R}^+$,

(*iii*) $P \cap (-P) = \{0\}$.

A cone P definies the partial ordering in Banach space given by $u \preceq v$ if and only if $v - u \in P$ we use the notation $u \prec v$ if $v - u \in K \setminus \{0\}$ and $u \not\prec v$ if $v - u \notin K \setminus \{0\}$. Finally $u \succeq v$ means $v \preceq u$.

Definition 1.1.2. *We say that* $f_i : [0,1] \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^r$, $i = 1, 2$, *is an* L^1*-Carathéodory function if*

1. $f_i(\cdot, x, y)$ *is measurable for any* $(x, y) \in \mathbb{R}^n \times \mathbb{R}^p$,
2. $f_i(t, \cdot, \cdot)$ *is continuous for almost every* $t \in [0, 1]$.
3. *For each* $r_1,\ r_2 > 0$, *there exists* $\Phi_{r_1,r_2} \in \mathrm{L}^1([0, +\infty))$ *such that*

$$|f(t, x, y)| \leq \Phi_{r_1,r_2}(t),$$

for all $x \in \mathbb{R}^n$ *with* $|x| \leq r_1$, $y \in \mathbb{R}^p$ *with* $|y| \leq r_2$, *and almost all* $t \in [0, 1]$.

1.2 Leray-Schauder degree

Let X and Z be Banach spaces. Let us denote by A_λ the set of couple $(I - f, \Omega)$, where the mapping $f : \overline{\Omega} \to Z$ is a compact and Ω is an open bounded subset of X satisfying the condition

$$0 \notin (I - f)(\partial\Omega). \tag{1.2.1}$$

A mapping *deg* from A_λ into Z will be called a Leray-Schauder degree and satisfies the following axioms (see [30]).

1. **Addition-excision property.** If $(I - f, \Omega) \in A_\lambda$ and Ω_1 and Ω_2 are disjoint open subsets in Ω such that

$$0 \notin (I - f)(\Omega \setminus (\Omega_1 \cup \Omega_2)),$$

then $(I - f, \Omega_1)$ and $(I - f, \Omega_2)$ belong to A_λ and

$$deg(I - f, \Omega) = deg(I - f, \Omega_1) + deg(I - f, \Omega_2).$$

2. **Homotopy invariance property.** If Γ is open and bounded in $X \times [0,1]$, $\mathcal{H}$: $\overline{\Gamma} \times [0,1] \to Z$ if

$$\mathcal{H}(x, \lambda) \neq 0$$

for each $x \in (\partial\Gamma)_\lambda$ and eqch $\lambda \in [0,1]$, where

$$(\partial\Gamma)_\lambda = \{x \in X : (x, \lambda) \in \partial\Gamma\}$$

then the mapping $\lambda \mapsto deg(\mathcal{H}(., \lambda), \Gamma_\lambda)$ is constant on $[0,1]$, where Γ_λ denotes the set

$$\{x \in X : (x, \lambda) \in \Gamma\}$$

3. **Normalization property.** If $(I - f, \Omega) \in A_\lambda$, with $I - f$ the restriction to $\overline{\Omega}$ of a linear mapping, then $deg((I - f) - b, \Omega) = 0$ if $b \notin (I - f)(\Omega)$ and $|deg((I - f) - b, \Omega)| = 1$ if $b \in (I - f)(\Omega)$.

1.3 Coincidence degree theory

1.3.1 A construction of the degree mapping

Let X and Z be real vector normed spaces, $L : D(L) \subset X \to Z$ be a Fredholm mapping of index zero and Ω be a bounded subset of X. Let us denote by C_L the set of couples (F, Ω), where the mapping $F : D(L) \cap \overline{\Omega} \to Z$ has the form $F = L + N$, with $N\overline{\Omega} \to Z$ L-compact and Ω is an open bounded subset of X satisfying the condition

$$0 \notin F(D(L) \cap \partial\Omega). \tag{1.3.1}$$

A mapping D_L from C_L into Z will be called a degree relatively to L if it is not identically zero and satisfies the following axioms (see [30]).

1. **Addition-excision property.** If $(F, \Omega) \in C_L$ and Ω_1 and Ω_2 are disjoint open subsets in Ω such that

$$0 \notin F[D(L) \cap (\Omega \setminus (\Omega_1 \cup \Omega_2))],$$

then (F, Ω_1) and (F, Ω_2) belong to C_L and

$$D_L(F, \Omega) = D_L(F, \Omega_1) + D_L(F, \Omega_2).$$

2. **Homotopy invariance property.** If Γ is open and bounded in $X \times [0,1]$, $\mathcal{H}$: $(D(L) \cap \overline{\Gamma}) \times [0,1] \to Z$ has the form
$$\mathcal{H}(x, \lambda) = Lx + \mathcal{N}(\S, \lambda)$$
where $\mathcal{N}$: $\overline{\Gamma} \to Z$ is L-compact on $\overline{\Gamma}$ and if
$$\mathcal{H}(x, \lambda) \neq 0$$
for each $x \in (D(L) \cap \partial\Gamma)_\lambda$ and each $\lambda \in [0,1]$, where
$$(\partial\Gamma)_\lambda = \{x \in X : (x, \lambda) \in \partial\Gamma\}$$
then the mapping $\lambda \mapsto D_L(\mathcal{H}(., \lambda), \Gamma_\lambda)$ is constant on $[0,1]$, where Γ_λ denotes the set
$$\{x \in X : (x, \lambda) \in \Gamma\}$$

3. **Normalization property.** If $(F, \Omega) \in C_L$, with F the restriction to $\overline{\Omega}$ of a linear one-to-one mapping from $D(L)$ into Z, then $D_L(F - b, \Omega) = 0$ if $b \notin F(D(L) \cap \Omega)$ and $|D_L(F - b, \Omega)| = 1$ if $b \in F(D(L) \cap \Omega))$.

1.3.2 The Leray- Schauder continuation theorem

Let X be a Banach space and $I = [0,1]$. If $A \subset X \times I$ and $\lambda \in I$, we shall write $A_\lambda = \{x \in X : (x, \lambda) \in A\}$. For $a \in X$ and $r > 0$, $B(a, r)$ will denote the open ball of center a and radius r. Let $\Omega \subset X \times I$ be a bounded open set with closure $\overline{\Omega}$ and boundary $\partial\Omega$ and let $F : \overline{\Omega} \to X$ be a mapping. We denote by Σ the (possibly empty) set defined by
$$\Sigma = \{(x, \lambda) \in \overline{\Omega} : x = F(x, \lambda)\}.$$
The following assumptions were introduced by Leray and Schauder in [**?**].

(H_0) $F : \overline{\Omega} \to X$ is completely continuous.

(H_1) $\Sigma \cap \partial\Omega = \varnothing$ (A priori estimate).

(H_2) Σ_0 is a finite non empty set $\{a_1, \ldots, a_\mu\}$ and the corresponding topological degree $deg[I - F(., 0), \Omega_0, 0]$ is different from zero (Degree condition).

Theorem 1.3.1. *[71] If conditions* (H_0), (H_1) *and* (H_2) *hold, then* Σ *contains a continuum* C *along which* λ *takes all values in* I.

In other words, under the above assumptions, Σ contains a compact connected subset C connecting Σ_0 to Ω_1. In particular, the equation $x = F(x, 1)$ has a solution in Ω_1.
Notice that the conclusion of Theorem 1.3.1 still holds if the finiteness of the set Σ_0 is dropped from assumption (H_2). Hence, from now on, we shall refer to assumption (H_2) as being the condition

(H_2) $deg[I - F(., 0), \Omega_0, 0] \neq 0$ (Degree condition).

Conditions (H_0) and (H_2) are in general the easiest ones to check. Condition (H_1) requires the a priori knowledge of some properties of the solution set Σ and is in general very difficult to check.
An important special case can be stated as follows. Introduce the condition

(H_1') Σ is bounded (A priori bound).

1.3.3 Continuation theorems for $Lx = Nx$

In this part we will present an extension in the frame of coincidence degree theory, the well-known Leray-Shauder continuation theorem.

Theorem 1.3.2. [62]
Let Y *and* Z *be real normed spaces,* $L : D(L) \subset X \to Z$ *be a linear and invertible,* $\Omega \subseteq X$ *be an bounded open neighborhood of* 0, *and* $N : \bar{\Omega} \to Z$ *be such that* $L^{-1}N : \bar{\Omega} \to X$ *is compact. If*

$$Lx \neq \lambda Nx$$

for every $(x, \lambda) \in (D(L) \bigcap \partial\Omega) \times (0, 1)$, *then equation* $x = L^{-1}Nx$ *has at least one solution in* $D(L) \cap \bar{\Omega}$.

1.4 Vector metric space

Metric space are very important mathematics applied sciences. In [3, 16] some results in metric space theory are generalized to vector metric space theory, and some fixed point theorem in vector metric space are given.

1.4.1 Generalized metric space

In this part, we consider the notation and definition of generalized metric space in Perov's sense.

Definition 1.4.1. *Let X be a non empty set. By a vector-valued metric on X we mean a map $d : X \times X \to \mathbb{R}^n$ with the following properties:*

(i) $d(u, v) \geq 0$ *for all* u, $v \in X$, *and if* $d(u, v) = 0$, *then* $u = v$*;*

(ii) $d(u, v) = d(v, u)$ *for all* u, $v \in X$*;*

(iii) $d(u, v) \leq d(u, w) + d(w, v)$ *for all* u, v, $w \in X$.

Here, if x, $y \in \mathbb{R}^n$, $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for $i = 1, 2, \cdots, n$.

We call the pair (X, d) a *generalized metric space* with

$$d(x, y) = \begin{pmatrix} d_1(x, y) \\ \vdots \\ d_n(x, y) \end{pmatrix}.$$

Notice that d is a generalized metric space on X if and only if d_i, $i = 1, 2, \cdots, n$, are metrics on X.
Let (X, d) be a generalized metric space in Perov's sens. Thus, if x, $r \in \mathbb{R}^n$, $x := (x_1, \cdots, x_n)$, and $r := (r_1, \cdots, r_n)$, then by $x \leq r$ we mean $x_i \leq r_i$, for each $i \in \{i, \cdots, n\}$ and by $x < r$ we mean $x_i < r_i$, for each $i \in \{i, \cdots, n\}$. Also, $|x| := (|x_1|, \cdots, |x_n|)$. If, $x, y \in R^n$, $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n)$, then $\max(x, y) = \max(\max(x_1, y_1), \cdots, \max(x_n, y_n))$. If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c$ for each $i = 1, \cdots, n$.

Definition 1.4.2. *A set X equiped with a partial order $\leq$ is called a partially ordered set. In a partially ordered set $(X, \leq)$ the notation $x < y$ means $x \leq y$ and $x \neq y$. An order interval $[x, y]$ is the set $\{z \in X \ : \ x \leq z \leq y\}$.*
Notice that if $x \nleq y$, then $[x, y] = \emptyset$.

Let (X, d) be a generalized metric space, we define the following metric spaces:

Let $X = \prod_{i=1}^{n} X_i,\ i = 1, \cdots, n$. Consider $\prod_{i=1}^{n} X_i$ with $\bar{d}$:

$$\bar{d}((x_1, \cdots, x_n), (y_1, \cdots, y_n)) = \sum_{i=1}^{n} d_i(x_i, y_i).$$

The diagonal space of $\prod_{i=1}^{n} X_i$ defined by

$$\bar{X} = \left\{ (x_1, \cdots, x_n) \in \prod_{i=1}^{n} :\ x_i \in X, i = 1, \cdots, n \right\}.$$

Thus it is a metric space with the following distance :

$$d_*((x, \cdots, x), (y, \cdots, y)) = \sum_{i=1}^{n} d_i(x, y).$$

Intuitively, X and $\bar{X}$ are equivalent. This is show in the following result.

Lemma 1.4.1. *Let (X, d) be a generalized metric space. Then there exists $h : X \to \bar{X}$ homeomorphisme map.*

Proof. Consider $h : X \to \bar{X}$ defined by $h(x) = (x, \cdots, x)$ for all $x \in X$. Obviously h is bijective. To prove that h is a continuous map
Let $x, y \in X$. Thus

$$d_*(h(x), h(y)) \leq \sum_{i=1}^{n} d_i(x, y).$$

For $\epsilon > 0$ we take $\delta = \left(\frac{\epsilon}{n}, \cdots, \frac{\epsilon}{n}\right)$, let fixed $x_0 \in X$ and $B(x_0, \delta) = \{x \in X : d(x_0, x) < \delta\}$, then for every $x \in B(x_0, \delta)$ we have

$$d_*(h(x), h(y)) \leq \epsilon.$$

Now, we consider the map $h^{-1} : \bar{X} \to X$ defined by

$$h^{-1}(x, \cdots, x_n) = x,\ (x, \cdots, x_n) \in \bar{X}.$$

To, show that h^{-1} is continuous. Let $(x, \cdots, x), (y, \cdots, y) \in \bar{X}$, $d\left(h^{-1}(x, \cdots, x), h^{-1}(y, \cdots, y)\right) = d(x, y)$.

Let $\epsilon = (\epsilon_1, \cdots, \epsilon_n) > 0$ we take $\delta = \dfrac{\min\limits_{1\leq i\leq n} \epsilon_i}{n}$ and we fixed $x_0 \in \bar{X}$. Set

$$B((x_0, \cdots, x_0), \delta) = \left\{(x, \cdots, x) \in \bar{X} : d_*((x_0, \cdots, x_0), (x, \cdots, x)) < \delta\right\}$$

For $(x, \cdots, x) \in B((x_0, \cdots, x_0), \delta)$ we have

$$d_*((x_0, \cdots, x_0), (x, \cdots, x)) < \delta \Rightarrow \sum_{i=1}^{n} d_i(x_0, x) < \frac{\min\limits_{1\leq i\leq n} \epsilon_i}{n}$$

Then

$$d_i(x_0, x) < \frac{\min\limits_{1\leq i\leq n} \epsilon_i}{n},\ i = 1, \cdots, n \ \Rightarrow d(x_0, x) < \epsilon_0.$$

Hence h^{-1} is continuous. □

1.4.2 Generalized Banach space

Definition 1.4.3. *Let E be a vector space metric on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A map $\|.\| : E \to \mathbb{R}^n_+$ is called an norm on E if it satisfies the following properties:*

(i) $\|x\| = 0$ *then* $x = (0, \cdots, 0)$*;*

(ii) $\|\lambda x\| = |\lambda|\|x\|$ *for* $x \in E$, $\lambda \in \mathbb{K}$*;*

(iii) $\|x + y\| \leq \|x\| + \|y\|$ *for every* $x,\ y \in E$.

Remark 1.4.1. *The pair $(E, \|.\|)$ is called a generalized normed space. If the generalized metric generated by $\|.\|$ (i.e $d(x, y) = \|x - y\|$) is complete then the espace $(E, \|.\|)$ is called a generalized Banach space, where*

$$\|x - y\| = \begin{pmatrix} \|x - y\|_1 \\ \vdots \\ \|x - y\|_n \end{pmatrix}.$$

Definition 1.4.4. *Let E be a non empty set and let $\|.\| : E \to \mathbb{R}^n_+$ be a norm on E. Then, the pair $(E, \|.\|)$ is called a generalized normed space. If, moreover, $(E, \|.\|)$ has the property that any Cauchy sequence from X is convergent in norm, then we say that $(E, \|.\|)$ is a generalized Banach space.*

Let $C(J,\mathbb{R}) \times C(J,\mathbb{R})$ be endowed with the *vector norm* $\| \cdot \|$ defined by $\|v\|_\infty = (\|u_1\|_\infty, \|u_2\|_\infty)$ for $v = (u_1, u_2)$. It is clear that $(C(J,\mathbb{R}) \times C(J,\mathbb{R}), \| \cdot \|_\infty)$ is a generalized Banach space.

1.4.3 Properties and topological elements

In the case of generalized metrics spaces in the sense of Perov, the notations of convergent sequence, Cauchy sequence, completeness, open and closed subset are similar to those for usual metric spaces.
Also, in what follows we present some elements of topology (see, for example, P. P Zabrejko [110] , E.Zeidler [112]).

Definition 1.4.5. *[16] Let (X, d) be a generalized metric space. A subset $A \subset X$ is called open if, for any $x \in A$, there exists $r \in \mathbb{R}^n_+$ with $r > 0$ such that $B(x_0, x) \subset A$, where $B(x_0, r) = \{x \in X : d(x_0, x) < r\}$ denote the open ball centered in x_0 with radius r, Any open ball is an open set and the collection of all open balls generates the generalized metric topology on X.*

Let

$$\overline{B(x_0, r)} = \{x \in X : d(x_0, x) \leq r\}$$

the closed ball centered in x_0 with radius r

Definition 1.4.6. *Let (X, d) be a generalized metrics spaces a sequence b_n in X is called the Cauchy sequence, if for each $\epsilon > 0$ there exist $N \in \mathbb{N}$ such that for any $n, m \geq N$: $d(x_n, x_m) < \epsilon$.*

Definition 1.4.7. *An generalized metric space (X, d) is called complete if each Cauchy sequence in X converges to a limit in X.*

Definition 1.4.8. *Let (X, d) be a an generalized metrics space, we say that a subset $Y \subset X$ is a closed if $x_n \subset Y$ and $x_n \to x$ imply $x \in Y$*

Definition 1.4.9. *Let (X, d) be a generalized metric space. A subset C of X is called compact if every open cover of C has a finite subcover.*
A subset C of X is sequentially compact if every sequence in C contains a convergent subsequence with limit in C.

Definition 1.4.10. *A set C of topological space is said relatively compact if its closure is compact, i.e., $\bar{C}$ is compact. The set C is sequentially relatively compact if every sequence in C contains a convergent subsequence (the limit need not be an element of C),i.e., $\bar{C}$ is sequentially compact.*

Definition 1.4.11. *[112] Let X,Y be two generalized metrics spaces $K \subset X$ and $f : K \to Y$ be a an open operator. Then f is called:*

(i) *compact, if for any bounded subset $A \subset K$ we have $f(A)$ is relatively compact or $\overline{f(A)}$ is compact;*

(ii) *Complete continuous, if f is continuous and compact;*

(iii) *with relatively compact range, if f is continuous an $f(K)$ is relatively compact or $\overline{f(K)}$ is compact.*

Theorem 1.4.2. *Let (X, d) be a generalized metric space. For any cpmpact set $A \subset X$ and for any closed set $B \subset X$ that is disjoint from A, there exists a continuous functions $f : X \to [0, 1]$, $g : X \to [0, 1] \times [0, 1] \times \ldots [0, 1] := [0, 1]^n$ such that*

$i)$ *$f(x) = 0$ for all $x \in B$,*

$ii)$ *$f(x) = 1$ for all $x \in A$.*

$iii)$ *$g(x) = (1, \ldots, 1)$ for all $x \in B$,*

$iv)$ *$g(x) = (0, \ldots, 0)$ for all $x \in A$.*

Proof. Note that $d_i(x, B) = 0$ for any $x \in B$ and $d_i(x, A) > 0$ for any $x \in A$. Thus we obtain $i)$ and $ii)$. Let $f : X \to [0, 1]$ be defined by

$$f(x) = \frac{\sum_{i=1}^{n} d_i(x, B)}{\sum_{i=1}^{n} d_i(x, A) + \sum_{i=1}^{n} d_i(x, B)}, \ x \in X.$$

To prove that f is continuous, let $(x_m)_{m\in\mathbb{N}}$ be a sequence convergent to $x \in X$.
Then

$$
\begin{aligned}
|f(x_m) - f(x)| &= \left| \frac{\sum_{i=1}^{n} d_i(x_m, B)}{\sum_{i=1}^{n} d_i(x_m, A) + \sum_{i=1}^{n} d_i(x_m, B)} - \frac{\sum_{i=1}^{n} d_i(x, B)}{\sum_{i=1}^{n} d_i(x, A) + \sum_{i=1}^{n} d_i(x, B)} \right| \\
&= \left| \frac{\sum_{i=1}^{n} d_i(x_m, B) \sum_{i=1}^{n} d_i(x, A) - \sum_{i=1}^{n} d_i(x_m, A) \sum_{i=1}^{n} d_i(x, B)}{\left(\sum_{i=1}^{n} d_i(x, A) + \sum_{i=1}^{n} d_i(x, B)\right)\left(\sum_{i=1}^{n} d_i(x_m, A) + \sum_{i=1}^{n} + d_i(x_m, B)\right)} \right| \\
&\leq \frac{\sum_{i=1}^{n} d_i(x,B) \sum_{i=1}^{n} \left| d_i(x_m, B) - \sum_{i=1}^{n} d_i(x, B) \right|}{\left(\sum_{i=1}^{n} d_i(x, A) + \sum_{i=1}^{n} d_i(x, B)\right)\left(\sum_{i=1}^{n} d_i(x_m, A) + \sum_{i=1}^{n} + d_i(x_m, B)\right)} \\
&\quad + \frac{\sum_{i=1}^{n} d_i(x, B) \sum_{i=1}^{n} \left| d_i(x_m, A) - \sum_{i=1}^{n} d_i(x, A) \right|}{\left(\sum_{i=1}^{n} d_i(x, A) + \sum_{i=1}^{n} d_i(x, B)\right)\left(\sum_{i=1}^{n} d_i(x_m, A) + \sum_{i=1}^{n} d_i(x_m, B)\right)}
\end{aligned}
$$

Since for each $i = 1, \ldots, m$ we have

$$|d_i(x_m, B) - d_i(x, B)| \to 0, \ |d_i(x_m, A) - d_i(x, A)| \to 0 \ \text{ as } \ m \to \infty.$$

Therefore

$$\frac{\sum_{i=1}^{n} d_i(x,A) \sum_{i=1}^{n} \left| d_i(x_m, B) - \sum_{i=1}^{n} d_i(x, B) \right|}{\left(\sum_{i=1}^{n} d_i(x, A) + \sum_{i=1}^{n} d_i(x, B)\right)\left(\sum_{i=1}^{n} d_i(x_m, A) + \sum_{i=1}^{n} + d_i(x_m, B)\right)} \to 0 \text{ as } m \to \infty$$

and

$$\frac{\sum_{i=1}^{n} d_i(x,A) \sum_{i=1}^{n} \left| d_i(x_m, A) - \sum_{i=1}^{n} d_i(x, A) \right|}{\left(\sum_{i=1}^{n} d_i(x, A) + \sum_{i=1}^{n} d_i(x, B) \right) \left(\sum_{i=1}^{n} d_i(x_m, A) + \sum_{i=1}^{n} + d_i(x_m, B) \right)} \to 0 \text{ as } m \to \infty.$$

Thus, we get

$$|f(x_m) - f(x)| \to 0, \quad \text{as } m \to \infty.$$

We can easily prove that the following function $g : X \to [0,1]^n$ defined by

$$g(x) = \begin{pmatrix} \dfrac{d_1(x, A)}{d_1(x, B) + d_1(x, A)} \\ \vdots \\ \dfrac{d_n(x, A)}{d_n(x, B) + d_n(x, A)} \end{pmatrix}, \quad x \in X.$$

is continuous function and satisfied *iii*) and *iv*).

□

Definition 1.4.12. *Let $(X, \|.\|)$ be a generalized Banach space and $U \subset X$ open subset such that $0 \in U$. The function $pu : X \to \mathbb{R}_+$ defined by*

$$p_U(x) = \inf\{\alpha > 0 : x \in \alpha U\},$$

is called the Minkowski functional of U.

Lemma 1.4.3. *Let $(X, \|.\|)$ be ageneralized Banach space and $U \subset X$ open subset such that $0 \in U$. Then*

i) *If $\lambda \geq 0$, then $p_U(\lambda x) = \lambda p_U(x)$.*

ii) *If U is convex we have*

  a) *$p_U(x + y) \leq p_U(x) + p_U(y)$, for every $x, y \in U$.*

  b) *$\{x \in X : p_U(x) < 1\} \subset U \subset \{x \in X : p_U(x) \leq 1\}$.*

  c) *if U is symmetric; then $p_U(x) = p_U(-x)$.*

iii) *p_U is continuous.*

Proof. $i)$ Let $x \in X$ be arbitrary and $\lambda \geq 0$. We have

$$\begin{aligned} p_U(\lambda x) &= \inf\{\alpha > 0 : \lambda x \in \alpha U\} \\ &= \inf\{\alpha > 0 : x \in \lambda^{-1}\alpha U\} \\ &= \inf\{\lambda\beta > 0 : x \in \beta U\} \\ &= \lambda \inf\{\beta > 0 : x \in \beta U\} \\ &= \lambda p_U(x). \end{aligned}$$

$ii) - a)$ Let $\alpha_1 > 0$ and $\alpha_2 > 0$ such that

$$x \in \alpha_1 U \text{ and } y \in \alpha_2 U,$$

then

$$x + y \in \alpha_1 U + \alpha_2 U \Rightarrow \frac{x+y}{\alpha_1 + \alpha_2} \in \frac{\alpha_1}{\alpha_1 + \alpha_2} U + \frac{\alpha_2}{\alpha_1 + \alpha_2} U$$

Hence

$$x + y \in (\alpha_1 + \alpha_2)U. \tag{1.4.1}$$

For every $\epsilon > 0$ there exist $\alpha_\epsilon > 0$, $\beta_\epsilon > 0$ such that

$$\alpha_\epsilon \leq p_U(x) + \epsilon \text{ and } \beta_\epsilon \leq p_U(x) + \epsilon$$

From (1.4.1) we have

$$p_U(x + y) \leq p_U(x) + p_U(y) + 2\epsilon.$$

Letting $\epsilon \to 0$ we obtain

$$p_U(x + y) \leq p_U(x) + p_U(y) \text{ for every } x, y \in U.$$

$b)$ Let $x \in X$ such that $p_U(x) < 1$, then there exists $\alpha \in (0, 1)$ such that

$$p_U(x) \leq \alpha < 1 \text{ and } x \in \alpha U \Rightarrow x = \alpha a + (1 - \alpha)0 \in U.$$

Therefore

$$\{x \in X : p_U(x) < 1\} \subset U.$$

For $x \in U$ we have

$$x = \alpha x \in U, \alpha = 1 \Rightarrow p_U(x) \leq 1.$$

Then
$$\{x \in X : p_U(x) < 1\} \subset U \subset \{x \in X : p_U(x) \leq 1\}.$$

$iii)$ Since $0 \in U$ then there exist $r > 0$ such that

$$B(0, r) = \{x \in X : \|x\| < r_*\} \subset U,$$

where
$$\|x\| = \begin{pmatrix} \|x\|_1 \\ \vdots \\ \|x\|_n \end{pmatrix} \text{ and } r_* = \begin{pmatrix} r \\ \vdots \\ r \end{pmatrix}.$$

Given $\epsilon > 0$, then $x + \epsilon B(0, r_*)$ is a neighborhood of x. For every $y \in x + \epsilon B(0, r_*)$ we have

$$\frac{x - y}{\epsilon} \in B(0, r_*) \Rightarrow p_U\left(\frac{x - y}{\epsilon}\right) \leq 1.$$

It is clear that

$$|p_U(x) - p_U(y)| \leq p_U(x - y) = \epsilon p_U\left(\frac{x - y}{\epsilon}\right) \leq \epsilon.$$

Hence p_U is continuous.

□

Definition 1.4.13. *A map (Φ_1, Φ_2) to be a map a nonnegative, continuous and convex functional coupled on a P of a generalized banach space E if $(\Phi_1, \Phi_2) : P \to \mathbb{R}^+ \times \mathbb{R}^+$ is continuous and*

$$(\Phi_1, \Phi_2)(t(x_1, y_1) + (1-t)(x_2, y_2)) \leq (\Phi_1, \Phi_2)(x_1, y_1) + (1-t)(\Phi_1, \Phi_2)(x_2, y_2).$$

for all (x_1, y_1), (x_2, y_2) $\in P$ and $t \in J$

Definition 1.4.14. *A map (Ψ_1, Ψ_2) to be a map a nonnegative, continuous and concave functional coupled on a P of a generalized banach space E if $(\Psi_1, \Psi_2) : P \to \mathbb{R}^+ \times \mathbb{R}^+$ is continuous and*

$$(\Psi_1, \Psi_2)(t(x_1, y_1) + (1-t)(x_2, y_2)) \geq t(\Psi_1, \Psi_2)(x_1, y_1) + (1-t)(\Psi_1, \Psi_2)(x_2, y_2).$$

for all $(x_1, y1)$, (x_2, y_2) $\in P$ and $t \in J$

Let (φ_1, φ_2) ans (θ_1, θ_2) be a nonnegative continuous convex functionals coupled on P, (ψ_1, ψ_2) be a non negativecontinuous functional coupled and (ϕ_1, ϕ_2) the nonnegative continuous concave functional coupled on P. Then, for positive vector (a_1, a_2), (b_1, b_2) and (d_1, d_2), we define the following sets:

$$P\Bigg((\varphi_1, \varphi_2), (d_1, d_2)\Bigg) = \Bigg\{(u_1, u_2) \in P : \varphi_1(u_1, u_2) < d_1,\ \varphi_2(u_1, u_2) < d_2\Bigg\},$$

$$P\Bigg((\varphi_1, \varphi_2), (\phi_1, \phi_2), (b_1, b_2), (d_1, d_2)\Bigg) = \Bigg\{(u_1, u_2) \in P : b_1 \le \phi_1(u_1, u_2),$$
$$b_2 \le \phi_2(u_1, u_2), \varphi_1(u_1, u_2) \le d_1,\ \varphi_2(u_1, u_2) \le d_2\Bigg\}.$$

$$P\Bigg((\varphi_1, \varphi_2), (\theta_1, \theta_2), (\psi_1, \psi_2), (b_1, b_2), (c_1, c_2), (d_1, d_2)\Bigg) =$$
$$\Bigg\{(u_1, u_2) \in P : b_1 \le \phi_1(u_1, u_2), b_2 \le \phi_2(u_1, u_2),\ \theta_1(u_1, u_2) \le c_1, \theta_2(u_1, u_2) \le c_2,$$
$$\varphi_1(u_1, u_2) \le d_1,\ \varphi_2(u_1, u_2) \le d_2\Bigg\}.$$

$$P\Bigg((\varphi_1, \varphi_2), (\theta_1, \theta_2), (\psi_1, \psi_2), (b_1, b_2), (c_1, c_2), (d_1, d_2)\Bigg) =$$
$$\Bigg\{(u_1, u_2) \in P : b_1 \le \phi_1(u_1, u_2),\quad b_2 \le \phi_2(u_1, u_2),\ \theta_1(u_1, u_2) \le c_1,$$
$$\theta_2(u_1, u_2) \le c_2, \varphi_1(u_1, u_2) \le d_1,\ \varphi_2(u_1, u_2) \le d_2\Bigg\}.$$

$$R\Big((\varphi_1, \varphi_2), (\psi_1, \psi_2), (a_1, a_2), (d_1, d_2)\Big) =$$

$$\Big\{(u_1, u_2) \in P : a_1 \leq \psi_1(u_1, u_2),\ a_2 \leq \psi_2(u_1, u_2),\ \varphi_1(u_1, u_2) \leq d_1,\ \varphi_2(u_1, u_2) \leq d_2\Big\}.$$

1.4.4 Matrix convergent

In this section, we introduce definitions, lemmas and theorems concerning to matrice convergent.

Definition 1.4.15. *A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1.*

In other words, all the eigenvalues of M are in the open unit disc, i.e., $|\lambda| < 1$ for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$, where I denote the identity matrix in $\mathcal{M}_{n\times n}(\mathbb{R})$.

Theorem 1.4.4. [100] *Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$; the following assertions are equivalent:*

(a) *M is convergent to zero;*

(b) *$M^k \to 0$ as $k \to \infty$;*

(c) *The matrix $(I - M)$ is nonsingular and*

$$(I - M)^{-1} = I + M + M^2 + \cdots + M^k + \cdots ;$$

(d) *The matrix $(I - M)$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.*

Some examples of matrices convergent to zero, $A \in M_{n\times n}(\mathbb{R})$, which also satisfies the property $(I - A)^{-1}|I - A| \leq I$ are:

1. $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ *and* $\max(a, b) < 1$

2. $A = \begin{pmatrix} a & -c \\ 0 & b \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ *and* $a + b < 1,\ c < 1$

3. $A = \begin{pmatrix} a & -a \\ b & -b \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ *and* $|a-b| < 1, a > 1,\ b > 0$.

Lemma 1.4.5. [104] *Let*

$$Q = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

where $a, b, c, d \geq 0$ *and* $\det Q > 0$. *Then* Q^{-1} *is order preserving.*

Lemma 1.4.6. [13] *If* $A \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ *is a matrix with* $\rho(A) < 1$, *then* $\rho(A+B) < 1$ *for every matrix* $B \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ *whose elements are small enough.*

1.4.5 Fixed point results

Fixed point theory plays a major role in many of our existence principles, therefore we shall state the fixed point theorems in generalized Banach spaces. The purpose of this section is to present the version of nonlinear alternative of Leary-Schauder type in generalized Banach spaces.

Definition 1.4.16. *Let* (X, d) *be a generalized metric space. An operator* $T : X \to X$ *is called contractive associated with the above* d *on* X, *if there exists a convergent to zero matrix* M *such that* $d(T(x), T(y)) \leq Md(x, y)$ *for all* $x,\ y \in X$.

Theorem 1.4.7. ([86], [87]) (Perov's fixed point theorem) *Suppose that* (X, d) *is a complete generalized metric space and* $T : X \to X$ *is a contractive operator with Lipschitz matrix* M. *Then* T *has a unique fixed point* u, *and for each* $u_0 \in X$,

$$d(T^k(u_0), u) \leq M^k(I - M)^{-1} d(u_0, T(u_0)) \text{ where } k \in \mathbb{N}.$$

Theorem 1.4.8. [86] (Schauder).
Let X *be a Banach space,* $D \subset X$ *a non empty closed bounded convex set and* $T : D \to D$ *a completely continuous operator (i.e.,* T *is continuous and* $T(D)$ *is relatively compact). Then* T *has at least one fixed point.*

As a consequence of Schauder fixed point theorem we present the version of non linear alternative Leary-Schauder type fixed point theorem in generalized Banach space.

Lemma 1.4.9. *Let X be a generalized Banach space, $U \subset E$ be a bounded, convex open neighborhood of zero and let $G : \overline{U} \to E$ be a continuous compact map. If G satisfies the boundary condition*

$$x \neq \lambda G(x)$$

for all $x \in \partial U$ and $0 \leq \lambda \leq 1$, then the set $Fix(G) = \{x \in U : x = G(x)\}$ is non empty.

Proof. Let p is the Minkowski function of U and since $\overline{U}$ is bounded, then there exists $M > 0$ such that

$$G(\overline{U}) \subseteq \frac{1}{2}B(0, M_*), \quad M_* = (K, \ldots, K).$$

Consider $G_* : \overline{B(0, M_*)} \to \overline{B(0, M_*)}$ defined by

$$G_*(x) = \begin{cases} G(x) & \text{if } x \in \overline{U} \\ \frac{1}{p(x)}G(\frac{x}{p(x)}) & \text{if } x \in E \backslash \overline{U}. \end{cases}$$

Clear that $\overline{B(0, M_*)}$ is closed, convex, bounded subset of E and G_* is continuous compact operator. Then from Theorem 1.4.8 there exists $x \in \overline{B(0, M_*)}$ such that $\overline{G}(x) = x$. If $x \in E \backslash \overline{U}$ then

$$x = \frac{G\left(\frac{x}{p(x)}\right)}{p(x)} \Rightarrow \frac{x}{p(x)} = \frac{1}{p^2(x)}G\left(\frac{x}{p(x)}\right).$$

Since $x \in E \backslash \overline{U}$, then

$$p(x) = 1 \text{ or } p(x) > 1 \Rightarrow x \in \partial U, \ \frac{x}{p(x)} \in \partial U.$$

This is a contradiction with

$$z \neq \lambda G(z), \quad \text{for each, } \lambda \in [0, 1], \ z \in \partial U.$$

Consequently, there exist $x_* \in U$ such that $G(x_*) = x_*$. □

Theorem 1.4.10. *Let $(X, \ \|\cdot\|)$ be a generalized Banach space, $C \subset E$ a closed and convex subset, $U \subset C$ a bounded set, open (with respect to the topology C) and such that $0 \in U$. Let $G : \overline{U} \to C$ be a compact continuous mapping. If the following assumption is satisfied:*

$$x \neq \lambda G(x), \text{ for all } x \in \partial_C U \text{ and all } \lambda \in (0,1),$$

then f has a fixed point in U.

Proof. Let $C_* = \{x \in \overline{U} : x = \lambda G(x) \text{ for some; } \ \lambda \in [0,1]\}$. Since $0 \in U$ then C_* is non empty set and by the continuity of G we concluded that C_* is closed. Clear that $\partial_C U \cap C_* = \emptyset$. From Theorem 1.4.2 there exists $f : \overline{U} \to [0,1]$ such that

$$f(x) = \begin{cases} 0 & \text{if } x \in \partial_C U \\ \\ 1 & \text{if } x \in C_*. \end{cases}$$

Consider $G_* : C \to C$ defined by

$$G_*(x) = \begin{cases} f(x)G(x) & \text{if } x \in U \\ \\ 0 & \text{if } x \in C \backslash U. \end{cases}$$

Since $G_*(x) = 0,$ for each $x \in \partial_C U$, and G_* is continuous on $U, \ E \backslash U$, then G_* is continuous. Set $\Omega = \overline{co}(\{0\} \cup G(\overline{U}))$ is convex and compact. We can easily prove that

$$G_*(\Omega) \subset \Omega.$$

Then from Theorem 1.4.8 there exists $x \in \Omega$ such that $G_*(x) = x$. From the definition of G_* we have $G(x) = x$. □

From above theorem we obtain the following:

Theorem 1.4.11. *Let $C \subset E$ be a closed convex subset and $U \subset C$ a bounded open neighborhood of zero(with respect to topology of C). If $G : \overline{U} \to E$ is compact continuous then*

i) either G has a fixed point in $\overline{U}$, or

ii) there exists $x \in \partial U$ such that $x = \lambda G(x)$ or some $\lambda \in (0,1)$.

Theorem 1.4.12. [35] *Let X be a generalized Banach space, C be a bounded, convex open neighborhood of zero. Suppose that $N : U \to C$ is a continuous, compact (that is, $N(U)$ is a relatively compact subset of C) map. If N satisfies the boundary condition*

$$x \neq \lambda N(x)$$

for all $x \in \partial U$ and $0 \leq \lambda \leq 1$, then the set $Fix(N) = \{x \in U : x = N(x)\}$ is non empty.

Theorem 1.4.13. [93] *Let $(X, \|.\|)$ be a generalized Banach space, D a non empty closed bounded convex subset of X and $T : D \to X$ such that:*

(*i*) *$T = G+H$ with $G : D \to X$ is completely continuous and $H : D \to X$ is a generalized contraction, i.e. there exists a matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R}^+)$ with $\rho(M) < 1$, such that $\|H(x)-H(y)\| \leq M\|x-y\|$ for all $x, y \in D$;*

(*ii*) *$G(x) + H(y) \in D$ for all $x, y \in D$.*

Then T has at least one fixed point in D.

Theorem 1.4.14. [33] *Let X be a generalized banach space. Suppose that T and B are two operators $X \to X$ such that*

($\mathcal{A}_1$) *T be a completely continuous operator.*

($\mathcal{A}_2$) *B be a continuous and M-contraction operator.*

($\mathcal{A}_3$) *the matrix $I - M$ has the absolute property.*
If

$$\mathcal{M} = \left\{x \in X| \quad \lambda T(x) + \lambda B\left(\frac{x}{\lambda}\right) = x\right\}$$

is bounded for all $0 < \lambda < 1$. Then the equation

$$x = T(x) + B(x), \quad x \in X,$$

has at least one solution.

Theorem 1.4.15. [33] *Let P be a cone in a generalized Banach space E. Let (φ_1, φ_2) and (θ_1, θ_2) be non-negative, continuous and convex functionals coupled on P, (ψ_1, ψ_2) be a non-negative, continuous and concave functional coupled on P, and (ϕ_1, ϕ_2) be a non-negative continuous functional coupled* *on P satisfying $(\psi_1,\psi_2)(k_1x,k_2y) \leq (k_1\psi_1(x,y),k_2\psi_2(x,y))$ for $(0,0) \leq (k_1,k_2) \leq (1,1)$, such that for some positive vectors (M_1,M_2) and (d_1,d_2),*

$$(\phi_1,\phi_2)(x,y) \leq (\psi_1,\psi_2)(x,y) \text{ and } \|(x,y)\| \leq (M_1\varphi_1(x,y),M_2\varphi_2(x,y))$$

for all $(x,y) \in \overline{P((\varphi_1,\varphi_2),(d_1,d_2))}$. Suppose that

$$T : \overline{P((\varphi_1,\varphi_2),(d_1,d_2))} \mapsto \overline{P((\varphi_1,\varphi_2),(d_1,d_2))}$$

is completely continuous $T = (T_1,T_2)$ and there exist positive vectors (a_1,a_2), (b_1,b_2), (c_1,c_2) with $(a_1,a_2) < (b_1,b_2)$, such that the following conditions are satisfied:

(S_1)

$$\left\{(x,y) \in P\left((\varphi_1,\varphi_2),(\theta_1,\theta_2),(\Phi_1,\Phi_2),(b_1,b_2)\right) : (\psi_1,\psi_2) > (b_1,b_2)\right\} \neq \emptyset$$

and

$$(\phi_1,\phi_2)(T(x,y)) > (b_1,b_2) \text{ for } (x,y) \in P\Bigg((\varphi_1,\varphi_2),(\theta_1,\theta_2),(\Phi_1,\Phi_2),(b_1,b_2)\Bigg);$$

(S_2)

$$(\phi_1,\phi_2)(T(x,y)) > (b_1,b_2) \ for \ (x,y) \in P((\varphi_1,\varphi_2),(\phi_1,\phi_2),(b_1,b_2),(d_1,d_2))$$

with $(\psi_1,\psi_2)(x,y) = (a_1,a_2)$;

(S_2)

$$(0,0) \in R((\varphi_1,\varphi_2),(\phi_1,\phi_2),(a_1,a_2),(d_1,d_2)),$$

$$(\psi_1,\psi_2)(T(x,y)) < (a_1,a_2) \text{ for } (x,y) \in R\Bigg((\varphi_1,\varphi_2),(\phi_1,\phi_2),(a_1,a_2),(d_1,d_2)\Bigg)$$

with (ψ_1,ψ_2) with $(\psi,\psi)(x,y) = (a_1,a_2)$.

Then T has at least three fixed points (x_1,y_1), (x_2,y_2), (x_3,y_3) $\in \overline{P((\varphi_1,\varphi_2),(d_1,d_2))}$, such that

$$(\varphi_1,\varphi_2) \leq (d_1,d_2), \ \ for \ i = 1,2,3,$$

and

$$(b_1,b_2) < (\phi_1,\phi_2)(x_1,y_1), \ (a_1,a_2) < (\psi_1,\psi_2)(x_1,y_1),$$

$$(\psi_1,\psi_2)(x_2,y_2) < (b_1,b_2), \ (\psi_1,\psi_2)(x_3,y_3) < (a_1,a_2).$$

1.5 Krasnosel'skii fixed point theorem on cone.

Consider two cones K_1 and K_2 of X; the corresponding cone $K := K_1 \times K_2$ of X^2, and we shall use the same symbol $\preceq$ to denote the partial order relations induced by K in X^2, and by K_1; K_2 in X: Similarly, the same symbol $\prec$ will be used to denote the strict order relations induced by K_1 and K_2 in X. Also, in X^2, the symbol $\prec$ will have the following meaning:$u = (u_1, u_2) \prec v = (v_1, v_2)$ if $u_i \prec v_i$ for $i = 1, 2$. For $r, R \in \mathbb{R}^2_+$; $r = (r_1; r_2)$, $R = (R_1; R_2)$, we write $0 < r < R$ if $0 < r_1 < R_1$ and $0 < r_2 < R_2$, and we use the notations:

$$(K_i)_{r_i,R_i} := \{u \in K_i : r_i \leq \|u\| \leq R_i\}(i = 1; 2)$$

$$K_{r;R} := \{u \in K : r_i \leq \|u_i\| \leq R_i \ for \ i = 1; 2\}.$$

Clearly, $K_{r;R} = (K_1)_{r_1;R_1} \times (K_2)_{r_2;R_2}$.

Theorem 1.5.1. [85] *Let $(X, \|.\|)$ be a normed linear space, $K_1, K_2 \subset X$ two cones, $K := K_1 \times K_2$; $r, R \in \mathbb{R}^+$ with $0 < r < R$; and $N : K_{r,R} \to K$; $N = (N_1; N_2)$ a compact map. Assume that for each $i \in \{1, 2\}$, one of the following conditions is satisfied in $K_{r,R}$:*

(a) $N_i(u) \not\prec u_i \ if \ \|u_i\| = r_i$ *and* $N_i(u) \not\succ u_i$ *if* $\|u_i\| = R_i$;

(b) $N_i(u) \not\succ u_i \ if \ \|u_i\| = r_i$ *and* $N_i(u) \not\prec u_i$ *if* $\|u_i\| = R_i$.

Then N has a fixed point u in K with $r_i \leq \|u_i\| \leq R_i$ for $i \in \{1, 2\}$.

CHAPTER 2

Impulsive differential equations

in this chapter we solue the problem of the existence of solution for the system of second order impulsive differential equations with non local boundary conditions. For the first case we consider:

$$x''(t) = f(t, x(t), x'(t)), \quad t \in J', \tag{2.0.1}$$

$$\Delta x(t_k) = I_k(x(t_k)), \quad k = 1, 2, \cdots, m, \tag{2.0.2}$$

$$\Delta x'(t_k) = J_k(x'(t_k)), \quad k = 1, 2, \cdots, m, \tag{2.0.3}$$

$$x(0) = 0, \quad x'(1) = \int_0^1 x'(s) dg(s), \tag{2.0.4}$$

where $J = [0, 1]$, $0 < t_1 < t_2 < \cdots < t_k < 1$, $J' = J \setminus \{t_1, t_2, \cdots, t_m\}$, $f = (f_1, f_2, \cdots, f_n) : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $I_k,\ J_k \in C(\mathbb{R}^n, \mathbb{R}^n)$, $k \in \{1, 2, \cdots, m\}$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ and $\Delta x'(t_k) = x'(t_k^+) - x'(t_k^-)$, where $(x(t_k^+),\ (x'(t_k^+))$ and $(x(t_k^-), x'(t_k^-))$ represent the right and left hand limit of $(x(t), x'(t))$ at $t = t_k$, respectively. Here $g = (g_1, g_2, \cdots, g_n) : J \to \mathbb{R}^n$ has a bounded variation satisfying

$$\int_0^1 dg_i(s) = 1,\ i = 1, 2, \cdots, n.$$

This system is at resonance. Our analysis relies on the Leray-Schauder continuous theorem. The a priori estimates follow from the existence of an open bounded convex subset $C \subset \mathbb{R}^n$, such that, for each $t \in [0, 1]$ and $x \in \bar{C}$, the vector fields $f(t, x, .)$ satisfy geometrical conditions on ∂C.

In the second case we consider, the following second-order impulsives differential systems with non local conditions:

$$(p(t)x'(t))' = f(t, x(t), x'(t)),\ t \in J', \tag{2.0.5}$$

$$\Delta x(t_k) = J_k(x(t_k)),\ k = 1, \cdots, m, \tag{2.0.6}$$

$$\Delta x'(t_k) = I_k(x'(t_k)),\ k = 1, \cdots, m, \tag{2.0.7}$$

$$x(0) = 0,\quad x'(1) = \int_0^1 x'(s)dg(s), \tag{2.0.8}$$

where $J = [0,1],\ 0 < t_1 < t_2 < \cdots < t_k < 1,\ J' = J \setminus \{t_1, t_2, \cdots, t_m\}$, $f = (f_1, f_2, \cdots, f_n) : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $p \in C^1(J, \mathbb{R})$, $I_k,\ J_k \in C(\mathbb{R}^n, \mathbb{R}^n)$, $k \in \{1, 2, \ldots, m\}$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ and $\Delta x'(t_k) = x'(t_k^+) - x'(t_k^-)$, where $(x(t_k^+),\ (x'(t_k^+))$ and $(x(t_k^-), x'(t_k^-))$ represent the righ and left hand of $(x(t), x'(t))$ at $t = t_k$, respectively, and $g = (g_1, g_2, \cdots, g_n) : J \to \mathbb{R}^n$ has bounded variation and $\int_0^1 x'(s)dg(s) = \left[\int_0^1 x_1'(s)dg_1(s), \ldots, \int_0^1 x_n'(s)dg_n(s)\right]$ and the integral is means in the Riemann - Stieljes sense, and

$$\int_0^1 \frac{1}{p(s)} dg_i(s) \neq 1,\quad i = 1, \ldots, n.$$

This systems is at non resonance.

2.1 Impulsive differential equations with non local conditions at resonance

2.1.1 Fixed point operator

We first introduce the sets

$$X = \left\{ \begin{array}{c} x : J \to \mathbb{R}^n \mid x(t) \text{ is continuously differentiable for } t \neq t_k,\ k = 1, 2, \cdots, m, \\ x'(t_k^+) \text{ and } x(t_k^+) \text{ exist at } t = t_k,\ \textit{and}\ x'(t_k^-) = x'(t_k),\ x(t_k^-) = x(t_k), \\ x(0) = 0,\ x'(1) = \int_0^1 x'(s)dg(s) \end{array} \right\}$$

and

$$Z = \left\{ \begin{array}{c} y : J \to \mathbb{R}^n \mid y(t) \text{ is continuous for } t \neq t_k,\ k = 1, 2, \cdots, m, \\ y(t_k^+) \text{ exist at } t = t_k,\ \textit{and}\ y(t_k^-) = y(t_k), \end{array} \right\} \times \mathbb{R}^{2nm}.$$

For every $x \in X$, we define the norm by

$$\|x\|_X = \max\left\{\sup_{t\in J}|x_1'(t)|, \cdots, \sup_{t\in J}|x_n'(t)|\right\},$$

and for every $z = (y, c) \in Z$, we define

$$\|z\|_Z = \max\left\{\sup_{t\in J}|y(t)|, |c|\right\}.$$

It can be shown that X and Z are Banach spaces .

To prove our existence result for the problem (2.0.1)-(2.0.4), we will make use of the following conditions:

(H_1) $f = (f_1, f_2, \cdots, f_n) : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function;

(H_2) $I_{i,k} = (I_{1,k}, I_{2,k}, \cdots, I_{n,k}),\ J_{i,k} = (J_{1,k}, J_{2,k}, \cdots, J_{n,k}) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous;

(H_3) $g = diag(g_1, g_2, \cdots, g_n) : [0, 1] \to \mathbb{R}^n$ has a bounded variation satisfying:

$$\int_0^1 dg_i(s) = 1,\ i = 1, 2, \cdots, n, \text{ and } \int_0^1 e^s dg_i(s) \neq e,\ i = 1, 2, \cdots, n.$$

Let

$$Dom(L) = \{x : J \to \mathbb{R}^n : x(t) \text{ twice differentiable for } t \neq t_k,\ k = 1, 2, \ldots, m\} \cap X,$$

$$L : Dom(L) \to Z,\ x \to (x'' - x',\ \Delta x(t_1), \ldots, \Delta x(t_k), \Delta x'(t_1), \ldots, \Delta x'(t_k)). \tag{2.1.1}$$

Lemma 2.1.1. *If conditions (H_1) and (H_3) hold, then $L^{-1} : Z \to Dom(L)$ exist and $L^{-1} : Z \to X$ is compact.*

Proof. Let $(y, a_1, \ldots, a_k, b_1, \ldots, b_k) \in Z$ be a solution of the problem

$$x''(t) - x'(t) = y(t), \quad t \in J', \tag{2.1.2}$$

$$\Delta x(t_k) = a_k, \quad k = 1, 2, \ldots, m, \tag{2.1.3}$$

$$\Delta x'(t_k) = b_k, \quad k = 1, 2, \ldots, m, \tag{2.1.4}$$

$$x(0) = 0, \quad x'(1) = \int_0^1 x'(s)dg(s). \tag{2.1.5}$$

Then from (2.1.2)-(2.1.5), we have

$$x'(t) = x'(0)e^t + \sum_{t_k<t} b_k e^{t-t_k} + \int_0^t e^{t-s} y(s)ds. \tag{2.1.6}$$

From (2.1.6) and the fact that $x'(1) = \int_0^1 x'(s)dg(s)$, we obtain

$$x'(0)e + \sum_{k=1}^{m} b_k e^{1-t_k} + \int_0^1 e^{1-s} y(s)ds = \int_0^1 \left(x'(0)e^s + \sum_{t_k<s} b_k e^{s-t_k} + \int_0^s e^{s-r} y(r)dr \right) dg(s).$$

Then

$$x'(0)\left(eI - \int_0^1 e^s dg(s)\right) = \int_0^1 \sum_{t_k<s} b_k e^{s-t_k} dg(s) - \sum_{k=1}^{m} b_k e^{1-t_k} + \int_0^1 \int_0^s e^{s-r} y(r)drdg(s) - \int_0^1 e^{1-s} y(s)ds. \tag{2.1.7}$$

In view of condition (H_3), (2.1.7) has a unique solution for each $(y, a_1, \ldots, a_k, b_1, \ldots, b_k) \in Z$ for $k = 1, 2, \ldots, m$, and hence the linear problem

$$x''(t) - x'(t) = y(t), \quad t \in J', \tag{2.1.8}$$

$$\Delta x(t_k) = a_k, \quad k = 1, 2, \ldots, m, \tag{2.1.9}$$

$$\Delta x'(t_k) = b_k, \quad k = 1, 2, \ldots, m, \tag{2.1.10}$$

has a unique solution x. Moreover,

$$\begin{aligned} x'(t) = {} & e^t \left(eI - \int_0^1 e^s dg(s)\right)^{-1} \left(\int_0^1 \sum_{t_k<s} b_k e^{s-t_k} dg(s) - \sum_{k=1}^{m} b_k e^{1-t_k} \right) \\ & + e^t \left(eI - \int_0^1 e^s dg(s)\right)^{-1} \left(\int_0^1 \int_0^s e^{s-r} y(r)drdg(s) - \int_0^1 e^{1-s} y(s)ds \right) \\ & + \sum_{t_k<t} b_k e^{t-t_k} + \int_0^t e^{t-s} y(s)ds. \end{aligned} \tag{2.1.11}$$

Thus,

$$
\begin{aligned}
x(t) &= (e^t - eI)\left(eI - \int_0^1 e^s dg(s)\right)^{-1}\left(\int_0^1 \sum_{t_k<s} b_k e^{s-t_k} dg(s) - \sum_{k=1}^{m} b_k e^{1-t_k}\right)\\
&\quad + (e^t - eI)\left(eI - \int_0^1 e^s dg(s)\right)^{-1}\left(\int_0^1 \int_0^s e^{s-r} y(r)drdg(s) - \int_0^1 e^{1-s} y(s)ds\right)\\
&\quad + \int_0^t \sum_{t_k<s} b_k e^{s-t_k} ds + \int_0^t (t-s)e^{t-s} y(s)ds + \sum_{t_k<t} a_k.
\end{aligned}
$$

From (2.1.11), we obtain

$$
\begin{aligned}
x''(t) &= e^t\left(eI - \int_0^1 e^s dg(s)\right)^{-1}\left(\int_0^1 \sum_{t_k<s} b_k e^{s-t_k} dg(s) - \sum_{k=1}^{m} b_k e^{1-t_k}\right)\\
&\quad + e^t\left(eI - \int_0^1 e^s dg(s)\right)^{-1}\left(\int_0^1 \int_0^s e^{s-r} y(r)drdg(s) - \int_0^1 e^{1-s} y(s)ds\right)\\
&\quad + \sum_{t_k<t} b_k e^{t-t_k} + \int_0^t e^{t-s} y(s)ds + y(t).
\end{aligned}
$$

Consequently, there exist $K > 0$ such that

$$\|x'\|_X \le K\|z\|_Z \text{ and } \|x''\|_X \le (K+1)\|z\|_Z.$$

Hence, L^{-1} maps bounded sets in Z into bounded sets in X.
Let τ_1, $\tau_2 \in J$ with $\tau_1 < \tau_2$ and let x belong to a bounded set in Z. Then,

$$|x'(\tau_2) - x'(\tau_1)| = \left|\int_{\tau_1}^{\tau_2} x''(s)ds\right| \le \int_{\tau_1}^{\tau_2} |x''(s)|\, ds \le (K+1)\|z\|_Z(\tau_2 - \tau_1).$$

The right hand side tends to zero as $\tau_2 - \tau_1 \to 0$, so it follows from the Arzelà-Ascoli theorem that $L^{-1} : Z \to X$ is compact. This completes the proof of the theorem. □

We next define a non linear mapping $N : X \to Z$, by

$$x \to (f(t, x, x') - x', I_1(x(t_1)), \dots, I_k(x(t_k)), J_1(x'(t_1)), \dots, J_k(x'(t_k))),\ \forall\, x \in X. \tag{2.1.12}$$

Then problem (2.0.1)-(2.0.4) can be written as $Lx = Nx$ for $x \in Dom(L)$. Its clearly that N is continuous on X and it takes bounded sets of X into bounded sets in Z. By lemma 2.1.1, $L^{-1}N : X \to X$ is a compact operator.
It should now be clear that to obtain a solution of the problem (2.0.2)-(2.0.4), we need to find a fixed point of the operator $L^{-1}Nx$. To accomplish this, we will use the following above result and Leary-Schauder continuation theorem.

2.1.2 Existence Result

Let $\langle . \mid . \rangle$ denote the usual inner product in $\mathbb{R}^n$. Recall that if $C \subset \mathbb{R}^n$ is an open convex neighborhood of $0 \in \mathbb{R}^n$, then for each $x_0 \in \partial C$, there exists $\nu(x_0) \in \mathbb{R}^n$ such that

(i) $\langle \nu(x_0) \mid x_0 \rangle > 0$;

(ii) $C \subset \{x \in \mathbb{R}^n : \langle \nu(x_0) \mid x - x_0 \rangle < 0\}$.

Here $\nu(x_0)$ is called an *outer normal* to ∂C at x_0 and

$$\overline{C} \subset \{x \in \mathbb{R}^n : \langle \nu(x_0) \mid x - x_0 \rangle \leq 0\}.$$

Theorem 2.1.2. *Assume that f, I_k, J_k and g for $k = 1, 2, \dots, m$, satisfies conditions (H_1), (H_3) and there exists an open convex neighborhood C of 0 in $\mathbb{R}^n$ such that the following conditions hold:*

(H_4) *For each $v \in \partial C$, there is an outer normal $\nu(v)$ to ∂C at v such that*

$$\langle \nu(v) | f(t, u, v) \rangle > 0, \quad \langle \nu(v(t_k)) | I_k(v(t_k)) \rangle > 0, \ k = 1, 2, \dots, m,$$

for all $t \in J$ and $u \in \bar{C}$.

(H_5) *For each $x \in X$, such that $x(t) \in \overline{C}$, for all $t \in J$ and $x(1) \in \partial C$, we have*

$$M := \left\{ t \in J : \ \langle \nu(x'(1)) | x'(t) \rangle = \max_{s \in J} \langle \nu(x'(1)) | x'(s) \rangle \right\} \neq \{1\}. \quad (2.1.13)$$

Then the problem (2.0.1)-(2.0.4) has at least one solution x such that $x'(t) \in \overline{C}$ for all $t \in J$.

Proof. Define the linear mapping $L : Dom(L) \subset X \to Z$ by (2.1.1), the non linear mapping $N : X \to Z$ by (2.1.12), and the open neighborhood Ω of 0 in X by

$$\Omega = \left\{ \begin{array}{l} x \in X : \; x'(t) \in C \text{ for } t \in J', \; x'(t_k^+) \in C, \\ \qquad\qquad\qquad\qquad \left(x(t_k^+) + \displaystyle\int_{t_k}^{t} x'(s)ds \right) \in C \text{ for } k = 1, 2, \ldots, m \end{array} \right\}.$$

We see that

$$\overline{\Omega} = \left\{ \begin{array}{l} x \in X : \; x'(t) \in \overline{C} \text{ for } t \in J', \; x'(t_k^+) \in \overline{C}, \\ \qquad\qquad\qquad\qquad \left(x(t_k^+) + \displaystyle\int_{t_k}^{t} x'(s)ds \right) \in \overline{C} \text{ for } k = 1, 2, \ldots, m \end{array} \right\},$$

and

$$\partial\Omega = \left\{ x \in \overline{\Omega} : \; x'(t_0) \in \partial C \textit{ for some } t_0 \in J \right\}.$$

Now Lemma (2.1.1) implies that L is invertible and L^{-1} is compact on $\overline{\Omega}$. We wish to show that for $\lambda \in (0, 1)$ and $u \in \partial\Omega$, the problem

$$x''(t) - x'(t) = \lambda \left[f(t, x(t), x'(t)) - x'(t) \right], \; t \in J', \tag{2.1.14}$$

$$\Delta x(t_k) = \lambda I_k(x(t_k)), \qquad k = 1, 2, \ldots, m, \tag{2.1.15}$$

$$\Delta x'(t_k) = \lambda J_k(x'(t_k)), \qquad k = 1, 2, \ldots, m, \tag{2.1.16}$$

$$x(0) = 0, \; x'(1) = \int_0^1 x'(s) dg(s), \tag{2.1.17}$$

has no solution. Note that (2.1.14)-(2.1.17) can be written as

$$x''(t) = (1 - \lambda) x'(t) + \lambda f(t, x(t), x'(t)), \quad t \in J', \tag{2.1.18}$$

$$\Delta x(t_k) = \lambda I_k(x(t_k)), \quad k = 1, 2, \ldots, m, \tag{2.1.19}$$

$$\Delta x'(t_k) = \lambda J_k(x'(t_k)), \; k = 1, 2, \ldots, m, \tag{2.1.20}$$

$$x(0) = 0, \quad x'(1) = \int_0^1 x'(s) dg(s), \tag{2.1.21}$$

Let $\lambda \in (0, 1)$ and let $x(t) \in \partial\Omega$ be a possible a solution of (2.1.18)-(2.1.21). Then $x'(t) \in \overline{C}$ for $t \in J'$, $x'(t_k^+) \in \overline{C}$, $\left(x(t_k^+) + \int_{t_K}^{t} x'(s)ds \right) \in \overline{C}$ for $k = 1, 2, \ldots, m.$, and there exist $t_0 \in J$ such that $x'(t_0) \in \partial C$.
If $t_0 \in J'$, then

$$x(t) = \int_0^t x'(s)ds = \int_0^1 y(s)ds,$$

where

$$y(s) = \begin{cases} x'(s), & \text{if } \; s \in [0,t], \\ 0, & \text{if } \; s \in (t,1]. \end{cases}$$

belongs to $\overline{C}$ for all $s \in J$. The convexity of $\overline{C}$ implies that $x(t) \in \overline{C}$ for all $t \in J$ and

$$\langle \nu(x'(t_0)) | x'(t) - x'(t_0) \rangle \leq 0.$$

We consider the real valued function

$$\theta : J \to \mathbb{R}, \text{ defined by } t \to \theta(t) = \langle \nu(x'(t_0)) | x'(t) - x'(t_0) \rangle.$$

Clearly, θ attains its maximum of 0 at t_0, By condition (H_5), we can assume that $t_0 \in [0,1) \setminus \{t_k, \; k = 1,2,\ldots,m\}$. Consequently, if

$$\begin{aligned} 0 = \theta'(t_0) &= \langle \nu(x'(t_0)) | x''(t_0) \rangle \\ &= (1-\lambda)\langle \nu(x'(t_0)) | x'(t_0) \rangle + \lambda \langle \nu(x'(t_0)) | f(t_0, x(t_0), x'(t_0)) \rangle > 0, \end{aligned}$$

we have a contradiction. Similarly, if $t_0 = 0$, then

$$0 \geq \theta'(t_0) > 0,$$

which is another contradiction . If $t_0 = t_k^+, \; k = 1,2,\ldots,m$, then for $t \in (t_k, t_{k+1}]$ we have $x(t) = x(t_k^+) + \int_{t_k}^t x'(s)ds$, so $x(t) \in \overline{C}$ for all $t \in J$. Since $x'(t_k^+) \in \overline{C}, \; k = 1,2,\ldots,m$,

$$\langle \nu(x'(t_k^+)) | x'(t) - x'(t_k^+) \rangle \leq 0,$$

so the function θ reaches a maximum of 0 at t_k^+. But

$$\begin{aligned} 0 = \theta'(t_k^+) &= \langle \nu(x'(t_k^+)) | x''(t_k^+) \rangle \\ &= (1-\lambda)\langle \nu(x'(t_k^+)) | x'(t_k^+) \rangle + \lambda \langle \nu(x'(t_k^+)) | f(t_k^+, x(t_k^+), x'(t_k^+)) \rangle > 0, \end{aligned}$$

which again is a contradiction. If $t_0 = t_k, \; k = 1,2,\ldots,m$, then since $x'(t_k^+) \in \overline{C}$, we have $x'(t_k) \in \overline{C}$. Consequently

$$\langle \nu(x'(t_k)) | x'(t) - x'(t_k) \rangle \leq 0,$$

so

$$0 \geq \theta(t_k^+) = \langle \nu(x'(t_k)) | x'(t_k^+) - x'(t_k) \rangle = \lambda \langle \nu(x'(t_k)) | J_k(x'(t_k^+)) \rangle > 0.$$

wich is a contradictions.

Therefore the problem (2.1.18)-(2.1.21) does not have a solution, so by the Leray-Schauder continuation theorem, Theorem 2.1.2 above, problem (2.0.5)-(2.0.8) has one solution, and this proves our theorem. □

The proof of the following lemma is essentially the same as the proof of Proposition 3.2 in [73], and so we omit the details.

Lemma 2.1.3. *If $g : J \to \mathbb{R}^n$ satisfies*

(H_6) *$g_1 = g_2 \dots, g_n = h$ and h is increasing with $\int_0^1 dh(s) = 1$,*

then conditions (H_3) and (H_5) hold.

Corollary 2.1.4. *Assume that conditions (H_1), (H_2), (H_4) and (H_6) are satisfied. Then the problem (2.0.1)-(2.0.4) has at least one solution x such that $x'(t) \in \overline{C}$ for all $t \in J$.*

Proof. The conclusion follows from Lemma 2.1.3 and Theorem 2.1.2. □

Corollary 2.1.5. *Assume that conditions (H_1), (H_2) and (H_4) are satisfied. Then the problem*

$$x''(t) = f(t, x(t), x'(t)), \quad t \in J', \tag{2.1.22}$$
$$\Delta x(t_k) = I_k(x(t_k)), \quad k = 1, 2, \dots, m, \tag{2.1.23}$$
$$\Delta x'(t_k) = J_k(x'(t_k)), \quad k = 1, 2, \dots, m, \tag{2.1.24}$$
$$x(0) = 0, \quad x'(1) = x(1), \tag{2.1.25}$$

has at least one solution x such that $x'(t) \in \overline{C}$ for all $t \in J$.

Proof. Since $x(1) = \int_0^1 x'(s)ds$, taking $h(s) = s$, we see that condition (H_6) hold, and the conclusion follows from corollary 2.1.4.

□

Corollary 2.1.6. *Assume that $(H_1) - (H_4)$ hold. Then for each $\alpha \in [0,1)$, the problem*

$$x''(t) = f(t, x(t), x'(t)), \ t \in J', \tag{2.1.26}$$

$$\Delta x(t_k) = I_k(x(t_k)), \quad k = 1, 2, \ldots, m, \tag{2.1.27}$$

$$\Delta x'(t_k) = J_k(x'(t_k)), \quad k = 1, 2, \ldots, m, \tag{2.1.28}$$

$$x(0) = 0, \quad x'(1) = x'(\alpha), \tag{2.1.29}$$

has at least one solution x such that $x'(t) \in \overline{C}$ for all $t \in J$.

Proof. We take

$$g_i(s) \begin{cases} 0, & \text{if } s \in [0, \alpha), \\ 1, & \text{if } s \in [\alpha, 1), \end{cases} \tag{2.1.30}$$

□

for $i = 1, 2, \ldots, n$. Then the problem (2.1.26)-(2.1.29) is equivalent to problem (2.0.1)-(2.0.4). It is clear that if $1 \in M$, with M defined in (2.1.13), the same is true for α, so that condition (H_5) holds.

The case where C is a ball

In the case where $C = B_R$, the open ball in $\mathbb{R}^n$ of center 0 and radius $R > 0$, we can take

$$\nu(v) = v \quad for\ all\ v \in \partial B_R.$$

We then have the following result.

Corollary 2.1.7. *Assume that (H_1), (H_2), and (H_6) hold and there exists $R > 0$ such that*

(H_7) $\langle v | f(t, u, v) \rangle > 0$ *and* $\langle v(t_k) | I_k(v(t_k)) \rangle > 0$ *for* $k = 1, 2, \ldots, m,$, $t \in J$, $|u| \leq R$, *and* $|v| = R$.

Then the problem (2.0.1)-(2.0.4) *has at least one solution x such that $|x'(t)| \leq R$ for all $t \in J$.*

We conclude our chapter with the following example to illustrate our results.

2.1.3 Example

Example 2.1.1. Consider the boundary value problem

$$x''(t) = f(t, x(t), x'(t)), \quad t \in J', \tag{2.1.31}$$

$$\Delta x(t_k) = J_k(x(t_k)), \quad k = 1, 2, \ldots, m, \tag{2.1.32}$$

$$\Delta x'(t_k) = I_k(x'(t_k)), \quad k = 1, 2, \ldots, m, \tag{2.1.33}$$

$$x(0) = 0, \quad x'(1) = \int_0^1 g(s)x'(s)ds, \tag{2.1.34}$$

where

$$f(t, u, v) = t^2 + 4 + \frac{1}{7}(t+2)|u|^p + v|v|^q, \quad 0 \le p < q + 1,$$

$$I_k(v(t_k)) = \frac{1}{4}v(t_k), \quad k = 1, 2, \ldots, m,$$

$$J(u(t_k)) = \sin\left(\frac{1}{4}\right) e^{u(t_k)} \quad k = 1, 2, \ldots, m.$$

and g is a arbitrary function satisfying condition (H_6), Observe that f satisfies condition (H_1) and the functions I_k,J_k satisfies condition (H_2) for $k = 1, 2, \ldots, m$. Let B_R be an open ball in $\mathbb{R}^n$ with center 0 and radius $R > 0$. For any $|u| \le R$ and $|v| = R$, we have

$$\begin{aligned} \langle v \mid f(t, u, v) \rangle &= \langle v \mid t^2 + 4 + \frac{1}{7}(t+2)|u|^p + v|v|^q \rangle \\ &\ge -\sum_{i=1}^{n} v_i \left(\frac{3}{7}|u|^p + 5\right) + |v|^{q+2} \\ &\ge -n|v| \left(\frac{3}{7}|u|^p + 5\right) + |v|^{q+2} \\ &\ge -5nR - \frac{3}{7}R^{p+1} + R^{q+2} \ge 0. \end{aligned}$$

for sufficiently large R. Also,

$$\langle v(t_k) \mid I_k(v(t_k)) \rangle = \langle v(t_k) \mid \frac{1}{4}v(t_k) \rangle = \frac{1}{4}|v|^2 > \frac{1}{5}|v|^2 > \frac{1}{5}R^2 > 0.$$

Hence, there exists at least one solution to (2.1.31)-(2.1.34).

2.2 Solvability of impulsive differential equations with non local conditions at non resonance

2.2.1 Existence results

In this part we present a result for the problem (2.0.5)-(2.0.8). We use the assumptions:

(C_1) $f = (f_1, \ldots, f_n) : J \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function,

(C_2) $I_k = (I_{1,k}, \ldots, I_{n,k}), J_k = (J_{1,k}, \ldots, J_{n,k}) : \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions,

(C_3) there exist $R > 0$ such that $\langle y|f(t, x, y)\rangle > 0$ and $\langle y(t_k)|I_k(y(t_k))\rangle > 0$ for $k = 1, 2, \ldots, m$, $x \in \mathbb{R}^n$ and $\|y\| \geq R$, $\|y(t_k)\| \geq R$ where $\langle .|.\rangle$ means the scalar product in $\mathbb{R}^n$ corresponding to the Euclidean norm.

(C_4) $p \in C^1(J, \mathbb{R})$, $p(t) > 0$ and $p'(t) \leq 0$, $p(1) = 1$.

(C_5) $g = (g_1, \ldots, g_n) : J \to \mathbb{R}^n$ and $Var(g) < 0$ where $Var(g)$ means the variation of g on the interval J.

(C_6) $\displaystyle\int_0^1 \frac{1}{p(s)} dg_i(s) \neq 1;\ i = 1, \ldots, n.$

Let v,$w \in \mathbb{R}^n$. Then by $v \circ w$ we mean: $(v_1w_1, \ldots, v_kw_k)$.

Before seating the result of this section we consider the following spaces.

$$PC(J, \mathbb{R}^n) = \{x : [0, 1] \to \mathbb{R}^n |\ x \in C(J', \mathbb{R}^n),\ x(t_k^+) \text{ and } x(t_k^-) \text{ exist, and}$$
$$x(t_k^-) = x(t_k),\ k = 1, 2, \ldots, m\},$$

$$PC^1(J, \mathbb{R}^n) = \{x \in PC(J, \mathbb{R}^n) |\ x \in C^1(J', \mathbb{R}^n),\ x'(t_k^+) \text{ and } x'(t_k^-) \text{ exist}$$
$$\text{and } x'(t_k^-) = x'(t_k),\ k = 1, 2, \ldots, m\}.$$

It is easy to say that $PC^1(J,\mathbb{R}^n)$ is a Banach space with the norm

$$\|x\|_{PC^1} = \max\left\{ \sup_{t\in[0,1]} \|x(t)\|,\ \sup_{t\in[0,1]} \|x'(t)\| \right\}.$$

where $\|.\|$ means the Euclidean norm in $\mathbb{R}^n$.

Lemma 2.2.1. *Let* $x \in PC^1(J,\mathbb{R}^n) \bigcap C^2(J',\mathbb{R}^n)$ *is a solution of the problem* (2.0.5)-(2.0.8), *then* $x \in PC^1(J,\mathbb{R}^n)$ *satisfies the following integral equation:*

$$\begin{aligned} x(t) \;=\; & \alpha \int_0^t \frac{1}{p(s)} ds \circ \int_0^1 \frac{1}{p(s)} \int_s^1 f(\tau, x(\tau), x'(\tau)) d\tau dg(s) \\ & + \alpha \int_0^t \frac{1}{p(s)} ds \circ \int_0^1 \frac{1}{p(s)} \sum_{s<t_k<1} p(t_k) I_k(x'(t_k)) dg(s) \\ & - \int_0^t \frac{1}{p(s)} \int_s^1 f(\tau, x(\tau), x'(\tau)) d\tau ds - \int_0^t \frac{1}{p(s)} \sum_{s<t_k} p(t_k) I_k(x'(t_k)) ds \\ & + \sum_{0<t_k<t} J_k(x(t_k)). \end{aligned}$$

Proof. First, suppose that $x \in PC^1(J,\mathbb{R}^n) \bigcap C^2(J',\mathbb{R}^n)$ is a solution to problem (2.0.5)-(2.0.8).
Then

$$(p(t)x'(t))' = f(t, x(t), x'(t)), t \neq t_k,\ k = 1, \ldots, m.$$

So,

$$p(t_k)x'(t_k^+) - x'(1) = -\int_{t_k}^1 f(s, x(s), x'(s)) ds,$$

$$p(t)x'(t) - p(t_k)x'(t_k^-) = -\int_t^{t_k} f(s, x(s), x'(s)) ds,$$

Thus

$$p(t)x'(t) = x'(1) - \int_t^1 f(s, x(s), x'(s)) ds - p(t_k) I(x'(t_k)),$$

Repeating the above process, for $t \in J$ we have

$$x'(t) = \frac{1}{p(t)} x'(1) - \frac{1}{p(t)} \int_t^1 f(s, x(s), x'(s)) ds - \frac{1}{p(t)} \sum_{t<t_k<1} p(t_k) I(x'(t_k)),$$

By using the condition $x'(1) = \int_0^1 x'(s)dg(s)$, we have

$$x'(1) = x'(1)\int_0^1 \frac{dg(s)}{p(s)} - \int_0^1 \left(\frac{1}{p(s)}\int_s^1 f(\tau, x(\tau), x'(\tau))d\tau - \frac{1}{p(s)}\sum_{s<t_k<1} p(t_k)I(x'(t_k))\right) dg(s),$$

which implies that

$$\begin{aligned} x'(t) &= \frac{1}{p(t)}\alpha \circ \int_0^1 \frac{1}{p(s)}\int_s^1 f(\tau, x(\tau), x'(\tau))d\tau dg(s) \\ &\quad -\frac{1}{p(t)}\alpha \circ \int_t^1 \frac{1}{p(s)}\left(\sum_{s<t_k<1} p(t_k)I(x'(t_k))\right) dg(s) \\ &\quad -\frac{1}{p(t)}\int_t^1 f(s, x(s), x'(s))ds - \frac{1}{p(t)}\sum_{t<t_k<1} p(t_k)I(x'(t_k)). \end{aligned}$$

where

$$\alpha_i := \left(\int_0^1 \frac{1}{p(s)}dg_i(s) - 1\right)^{-1}, \; i = 1, \ldots, k.$$

On the other hand, note that

$$x(t_1^-) - x(0) = \int_0^{t_1} x'(s)ds,$$

$$x(t) - x(t_1^+) = \int_{t_1}^t x'(s)ds,$$

So that, we have

$$x(t) = x(0) + \int_0^t x'(s)ds + J_k(x(t_1)), \quad t \in [0, t_1].$$

Repeating the above process a gain for $t \in J$

$$x(t) = x(0) + \int_0^t x'(s)ds + \sum_{0<t_k<t} J_k(x(t_k)).$$

Then

$$\begin{aligned} x(t) &= \alpha\int_0^t \frac{1}{p(s)}ds \circ \int_0^1 \frac{1}{p(s)}\int_s^1 f(\tau,x(\tau),x'(\tau))d\tau dg(s) \\ &+\alpha\int_0^t \frac{1}{p(s)}ds \circ \int_0^1 \frac{1}{p(s)}\sum_{s<t_k<1} p(t_k)I(x'(t_k))dg(s) \\ &-\int_0^t \frac{1}{p(s)}\int_s^1 f(\tau,x(\tau),x'(\tau))d\tau ds - \int_0^t \frac{1}{p(s)}\sum_{s<t_k} p(t_k)I(x'(t_k))ds \\ &+\sum_{0<t_k<t} J_k(x(t_k)). \end{aligned}$$

□

Let $x \in PC^1(J,\mathbb{R}^n)$. Define the operator A as follows

$$\begin{aligned} (Ax)(t) &= \alpha\int_0^t \frac{1}{p(s)}ds \circ \int_0^1 \frac{1}{p(s)}\int_s^1 f(\tau,x(\tau),x'(\tau))d\tau dg(s) \\ &+\alpha\int_0^t \frac{1}{p(s)}ds \circ \int_0^1 \frac{1}{p(s)}\sum_{s<t_k<1} p(t_k)I(x'(t_k))dg(s) \\ &-\int_0^t \frac{1}{p(s)}\int_s^1 f(\tau,x(\tau),x'(\tau))d\tau ds - \int_0^t \frac{1}{p(s)}\sum_{s<t_k} p(t_k)I(x'(t_k))ds \\ &+\sum_{0<t_k<t} J_k(x(t_k)), \end{aligned}$$

Then

$$\begin{aligned} (Ax)'(t) &= \alpha \circ \frac{1}{p(t)}\int_0^1 \frac{1}{p(s)}\int_s^1 f(\tau,x(\tau),x'(\tau))d\tau dg(s) \\ &+\alpha \circ \frac{1}{p(t)}\int_0^1 \frac{1}{p(s)}\sum_{s<t_k<1} p(t_k)I(x'(t_k))dg(s) \\ &-\frac{1}{p(t)}\int_t^1 f(s,x(s),x'(s))ds - \frac{1}{p(t)}\sum_{t<t_k} p(t_k)I(x'(t_k)). \end{aligned}$$

Theorem 2.2.2. *Suppose that the hypotheses* (C_1)-(C_6) *are satisfied, then the systems* (2.0.5)-(2.0.8) *has at least one solution.*

Proof. We show that operator A is completely continuous Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $PC^1(J, \mathbb{R}^n)$.

$$\begin{aligned}
&\|(Ax_n)(t) - (Ax)(t)\| \\
&\quad < \sqrt{\frac{1}{q^2}\sum_{i=1}^{n}\alpha_i^2 \int_0^1 \left(fi(s, x_n(s), x_n'(s)) - f_i(s, x(s), x'(s))\right)^2 ds} \\
&\quad + \sqrt{\frac{1}{q^2}\sum_{i=1}^{n}\alpha_i^2 \sum_{k=1}^{m} p^2(t_k)\left(I_{i,k}(x_n(t_k)) - I_{i,k}(x(t_k))\right)^2} \\
&\quad + \frac{1}{q}\int_0^1 \|f(s, x_n(s), x_n'(s)) - f(s, x(s), x'(s))\| \, ds \\
&\quad + \frac{1}{q}\sum_{k=1}^{m} p(t_k)\,\|I_k(x_n(t_k)) - I_k(x(t_k))\| \\
&\quad + \sum_{k=1}^{m} \|J_k(x_n(t_k)) - J_k(x(t_k))\| \, .
\end{aligned}$$

In the other hand

$$\begin{aligned}
&\|(Ax_n)'(t) - (Ax)'(t)\| < \\
&\quad < \frac{1}{q}\sqrt{\sum_{i=1}^{n}\alpha_i^2 \int_0^1 \left(f_i(s, x_n(s), x_n'(s)) - f_i(s, x(s), x'(s))\right)^2 ds} \\
&\quad + \frac{1}{q}\sqrt{\sum_{i=1}^{n}\alpha_i^2 \sum_{k=1}^{m} p^2(t_k)\left(I_{i,k}(x_n(t_k)) - I_{i,k}(x(t_k))\right)^2} \\
&\quad + \frac{1}{q}\int_0^1 \|f(s, x_n(s), x_n'(s)) - f(s, x(s), x'(s))\| ds \\
&\quad + \frac{1}{q}\sum_{k=1}^{m} p(t_k)\,\|I_k(x_n(t_k)) - I_k(x(t_k))\| \, .
\end{aligned}$$

Since f, I and J are continuous functions, then we have

$$\|Ax_n - Ax\|_{PC^1} \to 0 \;\; as \; n \to \infty.$$

A sends bounded sets into bounded sets in $PC^1(J,\mathbb{R}^n)$.
Let

$$K_R = \left\{x \in PC^1(J,\mathbb{R}^n)|\ \|x\|_{PC^1} \leq R\right\}, \quad B = \max_{t\in J,\ |u|\leq R,\ |v|\leq R} \|f(t,u,v)\|,$$

$$K_1 = \max_{1\leq k\leq m}\left\{\max_{|v|\leq M}\|I_k(v)\|\right\}, \quad K_2 = \max_{1\leq k\leq m}\left\{\max_{|u|\leq M}\|I_k(u)\|\right\},$$

$$q = \min_{0\leq t\leq 1}|p(t)|.$$

For each $t \in J$, we have

$$\begin{aligned}
\|Ax(t)\| &\leq \sqrt{\left(\alpha\int_0^1 \frac{1}{p(s)}ds\right)^2 + \left(\int_0^1 \frac{1}{p(s)}\int_0^1 f(\tau,x(\tau),x'(\tau))d\tau dg(s)\right)^2} \\
&\quad + \sqrt{\left(\alpha\int_0^1 \frac{1}{p(s)}ds\right)^2 + \left(\int_0^1 \frac{1}{p(s)}\sum_{k=1}^m p(t_k)I(x'(t_k))dg(s)\right)^2} \\
&\quad + \int_0^1 \frac{1}{p(s)}\int_0^1 \|f(\tau,x(\tau),x'(\tau))\|\,d\tau ds \\
&\quad + \int_0^1 \frac{1}{p(s)}ds\sum_{k=1}^m |p(t_k)|\|I(x'(t_k))\| + \sum_{k=1}^m \|J_k(x(t_k))\| \\
&\leq \sqrt{\frac{|\alpha|^2}{q^2} + B^2} + \sqrt{\frac{|\alpha|^2}{q^2} + m^2K_1^2} + \frac{mK_1}{q} + \frac{B}{q} + K_2 := F_1.
\end{aligned}$$

We have

$$\|(Ax)'(t)\| \leq \frac{1}{q}\sqrt{|\alpha|^2 + B^2} + \frac{1}{q}\sqrt{|\alpha|^2 + m^2K_1^2} + \frac{mK_1}{q} + \frac{B}{q} := F_2.$$

$$\|(Ax)''(t)\| \leq \frac{1}{q}(R+B), \quad t \neq t_k,\ k = 1,\ldots,m.$$

Then $\|Ax\|_{PC^1} \leq F$, with $F = \max(F_1,F_2)$
A maps bounded set of into equicontinuous sets, let l_1, $l_2 \in J$, $l_1 < l_2$ and K_R be a bounded set of $PC^1(J,\mathbb{R}^n)$.
Let $x \in K_R$ then

$$\begin{aligned}
\|(Ax)(l_2) - (Ax)(l_1)\| &= \int_{l_1}^{l_2} \|(Ax)'(s)\|ds \\
&\leq F_2|l_2 - l_1|,
\end{aligned}$$

$$\begin{aligned}\|(Ax)'(l_2)-(Ax)'(l_1)\| &= \int_{l_1}^{l_2}\|(Ax)''(s)\|ds\\ &\leq \frac{1}{q}(R+B)(l_2-l_1).\end{aligned}$$

So, $A(K_R)$ is equicontinuous on all J_k (k=1,...,m), We can conclude that $Ax : PC^1(J,\mathbb{R}^n) \to PC^1(J,\mathbb{R}^n)$ is completely continuous.

Consider parameter family of problems

$$\begin{gathered}(p(t)x')' = \lambda f(t,x(t),x'(t)),\quad t\in J',\\ \Delta x(t_k) = \lambda J_k(x(t_k)),\quad k=1,2,\dots,m,\\ \Delta x'(t_k) = \lambda I_k(x'(t_k)),\quad k=1,2,\dots,m,\\ x(0)=0,\quad x'(1)=\int_0^1 g(s)x'(s)ds,\end{gathered}$$

depending on a parameter $\lambda \in [0,1]$. We concluding that λA is a completely continuous.

Consider the homotopy

$$H : J \times PC^1(J,\mathbb{R}^n) \to PC^1(J,\mathbb{R}^n)$$

given by

$$H(\lambda,x) = x - \lambda Ax$$

in $\Omega = B(0,R)$, where R is the positive constant from the assumption (C_3).
We show that $H(\lambda,x)=0$ has no solution for $\lambda \in J$ and $x \in \partial\Omega$.
Hence $R = \max\{R_1,R_1\}$ with $R_1 = \sup_{t\in J}\|x'(t)\|$ and $R_2 = \sup_{t\in J}\|x(t)\|$.
Indeed, if $H(0,x)=0$ then problem (2.0.5)-(2.0.8) has only a trivial solution, which contradicts $\|x\|_{PC^1} = R$.
Suppose that there exists a solution of the equation $H(\lambda,x)=0$ when $\lambda \in]0,1]$ and $x \in \partial\Omega$.
We consider the function $\psi(t) = \|x'(t)\|^2$ otherwise there is $t_0 \in J$ such that $\psi(t_0) = \sup_{t\in J}\|x'(t)\|^2 = R^2$ (ψ has a maximum R^2 for some $t_0 \in J$.

If $t_0 \in J'|\{0,1\}$, then $\|x'(t_0)\| = R$

$$\begin{aligned} 0 = \psi'(t_0) &= 2\left\langle x'(t_0)|x''(t_0)\right\rangle \\ &= 2\lambda\left\langle x'(t_0)|\frac{f(t_0,x(t_0),x'(t_0))}{p(t_0)} - \frac{p'(t_0)}{p(t_0)}x'(t_0)\right\rangle \\ &= 2\frac{\lambda}{p(t_0)}\left\langle x'(t_0)|f(t_0,x(t_0),x'(t_0))\right\rangle - 2\lambda\frac{p'(t_0)}{p(t_0)}\|x'(t_0)\|^2 \\ &> 0, \end{aligned}$$

hence, we obtain a contradiction .
If $t_0 = t_k$ for $k = 1,\dots,m$ then $\|x'(t_0))\| = R$, by assumptions (C_3), we have

$$\begin{aligned} \psi(t_k^+) - \psi(t_k^-) &= \|x'(t_k^+)\|^2 - \|x'(t_k^-)\|^2 \\ &= \sum_{i=1}^{n}|x_i'(t_k^+)|^2 - \sum_{i=1}^{n}|x_i'(t_k^-)|^2 \\ &= \sum_{i=1}^{n}\left(|x_i'(t_k^+)|^2 - |x_i'(t_k^-)|^2\right) \\ &= \sum_{i=1}^{n}\left(|x_i'(t_k^+)| + |x_i'(t_k^-)|\right)\left(|x_i'(t_k^+)| - |x_i'(t_k^-)|\right) \\ &= \sum_{i=1}^{n}\Delta|x_i'(t_k)|\left(2|x_i'(t_k)| + \Delta|x_i'(t_k)|\right) \\ &= \sum_{i=1}^{n}\lambda|I_{i,k}(x'(t_k))|\left(2|x_i'(t_k)| + \lambda|I_{i,k}(x'(t_k))|\right) \\ &= 2\lambda\left\langle x'(t_k)|I_k(x'(t_k))\right\rangle + \lambda^2\|I_k(x'(t_k))\|^2 \\ &> 0, \end{aligned}$$

then $\psi(t_k^+) > R^2$ is a contradiction. If $t_0 = t_k^+$ for $k = 1,\dots,m$, then $\|x'(t_0))\| = R$. Now by assumptions (C_3) and (C_4), we get

$$\begin{aligned} 0 = \psi'(t_k^+) &= 2\left\langle x'(t_k^+)|x''(t_k^+)\right\rangle \\ &= 2\lambda\left\langle x'(t_k^+)|\frac{f(t_k^+,x(t_k^+),x'(t_k^+))}{p(t_k^+)} - \frac{p'(t_k^+)}{p(t_k^+)}x'(t_k^+)\right\rangle \\ &= 2\frac{\lambda}{p(t_k^+)}\left\langle x'(t_k^+)|f(t,x(t_k^+),x'(t_k^+))\right\rangle - 2\lambda\frac{p'(t_k^+)}{p(t_k^+)}\|x'(t_k^+)\|^2 \\ &> 0, \end{aligned}$$

hence, we obtain a contradiction .
If $t_0 = 0$, then by assumptions (C_3) and (C_4), we obtain

$$\begin{aligned} 0 = \psi'(0) &= 2\frac{\lambda}{p(0)} \langle x'(0) | f(0, x(0), x'(0)) \rangle - 2\lambda \frac{p'(0)}{p(0)} \|x'(0)\|^2 \\ &> 0, \end{aligned}$$

hence, we obtain a contradiction.
If $t_0 = 1$, then

$$R^2 = \|\psi(1)\|^2 = \left\| \int_0^1 x'(s) dg(s) \right\|^2 \leq R^2 \int_0^1 dg(s) < R^2.$$

is a contradiction. Now, since $x \neq \lambda A x$ for all $x \in \partial\Omega$, by excision property of the Leray Schauder degree we conclude

$$\begin{aligned} \deg(I - A, \Omega) &= \deg(H(1, .), \Omega) \\ &= \deg(H(0, .), \Omega) = 1 \neq 0, \end{aligned}$$

We see that A has a fixed point Ω. Therefore, systems (2.0.5)-(2.0.8) has a solution in Ω. □

2.2.2 Example

In this section, we present a simple example to explain our result.
Consider the following problems:

$$(p(t)x'(t))' = f(t, x(t), x'(t)), \ t \in J', \tag{2.2.1}$$

$$\Delta x(t_k) = J_k(x(t_k)), \ k = 1, \ldots, m, \tag{2.2.2}$$

$$\Delta x'(t_k) = I_k(x'(t_k)), \ k = 1, \ldots, m, \tag{2.2.3}$$

$$x(0) = 0, \quad x'(1) = \int_0^1 x'(s) dg(s). \tag{2.2.4}$$

where $p(t) = e^{-t^2+1} > 0$, $p'(t) = -2te^{-t^2+1} \leq 0$ and $p(1) = 1$ then $p(t)$ satisfies the condition (C_4), g is a arbitrary functions satisfying the condition (C_5) and $f(t, x, y) = (f_1(t, x, y), f_2(t, x, y))$, $I_k(y(t_k)) = (I_{1,k}(y(t_k)), I_{2,k}(y(t_k)))$, $J_k(x(t_k)) = (J_{1,k}(x(t_k)), J_{2,k}(x(t_k)))$

$$f_1(t, x, y) = \left(1 + \frac{1}{\pi} \sin(\pi x_1)\right) \left(1 + \cos^2 x_2\right) + bt^2 + \frac{1}{2} y_1 \left(1 + \sin^2 y_2\right),$$

$$f_2(t,x,y) = \frac{3}{2}y_2\left(\frac{\pi^2}{2} + \tan^2 y_1\right),$$

$$I_{1,k}(y(t_k)) = \cos\left(t_k + \frac{1}{4}\right)y_1(t_k), \quad k = 1,\dots,m,$$

$$I_{2,k}(y(t_k)) = \sin\left(t_k + \frac{1}{4}\right)y_2(t_k), \quad k = 1,\dots,m,$$

$$J_{1,k}(x(t_k)) = A_k \arcsin x_2(t_k) - B_k \arcsin x_1(t_k), \quad k = 1,\dots,m,$$

$$J_{2,k}(x(t_k)) = \arctan(x_2(t_k) - x_1(t_k)), \quad k = 1,\dots,m.$$

with $0 < A_k < \dfrac{1}{\pi}$, $0 < B_k < \dfrac{1}{2}$, $k = 1,\dots,m$ and $0 < b < 1$.
Observe that f satisfies condition (C_1) and the function J_k, I_k, $k = 1,\dots,m$ satisfy condition (C_2) . Indeed, for any $R > 1$ and $\|y\| \geq R$, $x \in \mathbb{R}^2$ and $t \in J$, we obtain

$$\begin{aligned}
\langle y|f(t,x,y)\rangle &= y_1\left(\left(1 + \frac{1}{\pi}\sin(\pi x_1)\right)\left(1+\cos^2 x_2\right) + bt^2\right) + \frac{1}{2}y_1^2\left(1+\sin^2 y_2\right) \\
&\quad + \frac{3}{2}y_2^2\left(\frac{\pi^2}{2} + \tan^2 y_1\right) \\
&> \left(1 - \frac{1}{\pi}\right)y_1 + \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 \\
&> \frac{1}{2}\left(y_1 + y_1^2 + y_2^2\right).
\end{aligned}$$

If $y_1 \in]-\infty,-1] \bigcup [0,\infty[$ then $\dfrac{1}{2}\left(y_1 + y_1^2 + y_2^2\right) > 0$, if $y_1 \in]-1,0[$ we obtain

$$\begin{aligned}
\langle y|f(t,x,y)\rangle &> \frac{1}{2}\left(y_1 + y_1^2 + y_2^2\right) \\
&> \frac{1}{2}\left(y_1 + M^2\right) \\
&> \frac{1}{2}(y_1 + 1) > 0.
\end{aligned}$$

In the other hand

$$
\begin{aligned}
\langle y(t_k)|I(y(t_k))\rangle &= y_1^2\cos\left(t_k+\frac{1}{4}\right)+y_2^2\sin\left(t_k+\frac{1}{4}\right)\\
&> y_1^2\cos\left(\frac{1}{4}\right)+y_2^2\sin\left(\frac{1}{4}\right)\\
&> 0.
\end{aligned}
$$

for $k = 1, \ldots, m$. Then, all the assumptions of Theorem 2.2.2 hold. Thus, the problems (2.2.1)-(2.2.4) has at least one solution in Ω.

Chapter **3**

Systems of impulsive differential equations

In this chapter we study the existence and positivity of solutions for systems of ordinary impulsive differential equations with two boundary conditions, and we will establishing the multiplicity of positive solutions for the systems of second order impulsive differential equations with tree points boundary conditions.

3.1 Existence results

This section, is concerned the existence and uniqueness of solutions for the system of second-order impulsive differential equations with two boundary conditions:

$$-u_1''(t) = f_1(t, u_1(t), u_2(t)), \quad t \in J', \tag{3.1.1}$$

$$-u_2''(t) = f_2(t, u_1(t), u_2(t)), \quad t \in J', \tag{3.1.2}$$

$$-\Delta u_1' \mid_{t=t_k} = I_{1,k} u_1(t_k), \quad k = 1, 2, \dots, m, \tag{3.1.3}$$

$$-\Delta u_2' \mid_{t=t_k} = I_{2,k} u_2(t_k), \quad k = 1, 2, \dots, m, \tag{3.1.4}$$

$$\alpha u_1(0) - \beta u_1'(0) = 0, \ \alpha u_2(0) - \beta u_2'(0) = 0, \tag{3.1.5}$$

$$\gamma u_1(1) + \delta u_1'(1) = 0, \ \gamma u_2(1) + \delta u_2'(1) = 0, \tag{3.1.6}$$

where α, β, γ, $\delta \geq 0$, $\rho = \beta\gamma + \alpha\gamma + \alpha\delta > 0$, $J = [0, 1]$, $0 < t_1 < t_2 < \cdots < t_m < 1$, $J' = J \setminus \{t_1, t_2, \dots, t_m\}$, $f_i \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $I_{i,k} \in C(\mathbb{R}, \mathbb{R})$,

$i = 1,2,\ k \in \{1,2,\cdots,m\},\ \Delta u' \mid_{t=t_k} = u_1(t_k^+) - u_1(t_k^-)$, and $\Delta u_2' \mid_{t=t_k} = u_2(t_k^+) - u_2(t_k^-)$ in which $u_1'(t_k^+)$, $u_2'(t_k^+)$, $u_1'(t_k^-)$ and $u_2'(t_k^-)$ denote the right and left hand limits of $u_1'(t)$ and $u_2'(t)$ at $t = t_k$, respectively.

We set $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1, \ldots, m$, $t_{m+1} = 1$, and let y_k be the restriction of the function y to J_k.

We shall provide sufficient conditions ensuring some existence and uniqueness results for system (3.1.1)-(3.1.6) via an application of the Perov fixed point theorem and the non linear alternative of Leray-Schauder type.

Both of these approaches make use of convergent matrices and vector norms.

3.1.1 Main Results

We consider the space

$$PC^2(J,\mathbb{R}) = \{y \in C([0,1],\mathbb{R}) : y_k \in C^2(J_k,\mathbb{R}),\ k = 0,\ldots,m, \text{ such that } y'(t_k^-) \text{ and } y'(t_k^+) \text{ exist and satisfy } y'(t_k) = y'(t_k^-) \text{ for } k = 1,\ldots,n\}. \tag{3.1.7}$$

Let $PC^2(J,\mathbb{R}) \times PC^2(J,\mathbb{R})$ be endowed with the *vector norm* $\|\cdot\|$ defined by $\|v\| = (\|u_1\|_{PC^2},\ \|u_2\|_{PC^2})$ for $v = (u_1, u_2)$, where for $x \in PC^2(J,\mathbb{R})$, we set $\|x\|_{PC^2} = \sup\limits_{t\in J}|x(t)| + \sup\limits_{t\in J}|x'(t)|$. It is clear that $(PC^2(J,\mathbb{R}) \times PC^2(J,\mathbb{R}), \|\cdot\|_{PC^2})$ is a generalized Banach space. We will also need the space

$$PCA(J,\mathbb{R}) = \{y \in C([0,1],\mathbb{R}) : y_k' \in AC^1(J_k,\mathbb{R}),\ k = 0,\ldots,m, \text{ such that } y'(t_k^-) \text{ and } y'(t_k^+) \text{ exist and satisfy } y'(t_k) = y'(t_k^-) \text{ for } k = 1,\ldots,n\} \tag{3.1.8}$$

with the *vector norm* $\|\cdot\|$ defined by $\|v\| = (\|u_1\|_{PCA},\ \|u_2\|_{PCA})$ for $v = (u_1, u_2)$, where for $x \in PCA(J,\mathbb{R})$, we set $\|x\|_{PCA} = \sup\limits_{t\in J}|x(t)|$. our first we give sufficient conditions for the existence and uniqueness of solutions to problem (3.1.1)-(3.1.6) using Perov's fixed point theorem. We begin with a lemma that will aid in transforming problem (3.1.1)-(3.1.6) into a fixed point problem that will be used in this section as well as later in this thesis .

Lemma 3.1.1. *The vector* $(u_1, u_2) \in PC^2(J, \mathbb{R}) \times PC^2(J, \mathbb{R})$ *is a solution of the differential system* (3.1.1)-(3.1.6) *if and only if*

$$\begin{cases} u_1(t) = \int_0^1 G(t,s) f_1(s, u_1(s), u_2(s))ds + \sum_{k=1}^{m} G(t, t_k) I_{1,k}(u_1(t_k)), \\ u_2(t) = \int_0^1 G(t,s) f_2(s, u_1(s), u_2(s))ds + \sum_{k=1}^{m} G(t, t_k) I_{2,k}(u_2(t_k)), \end{cases} \tag{3.1.9}$$

where

$$G(t,s) = \frac{1}{\rho} \begin{cases} (\gamma + \delta - \gamma t)(\beta + \alpha s), & 0 \le s \le t \le 1, \\ (\beta + \alpha t)(\gamma + \delta - \gamma s), & 0 \le t \le s \le 1. \end{cases} \tag{3.1.10}$$

Proof. Let $(u_1, u_2) \in PC^2(J, \mathbb{R}) \times PC^2(J, \mathbb{R})$ be a solution of system (3.1.1)-(3.1.6). It is easy to see by an integration of (3.1.1)-(3.1.6) that

$$u_i'(t) = u_i'(0) - \int_0^t f_i(s, u_1(s), u_2(s))ds - \sum_{0 < t_k < t} I_{i,k}(u_i(t_k)), \quad for\ i = 1, 2. \tag{3.1.11}$$

Integrating again, we obtain

$$\begin{aligned} u_i(t) &= u_i(0) + u_i'(0)t - \int_0^t (t-s) f_1(s, u_1(s), u_2(s))ds \\ &\quad - \sum_{0 < t_k < t} I_{i,k}(u_i(t_k))(t - t_k), \quad for\ i = 1, 2. \end{aligned} \tag{3.1.12}$$

Letting $t = 1$ in (3.1.11) and (3.1.12), we have

$$\begin{aligned} u_i'(1) &= u_i'(0) - \int_0^1 f_i(s, u_1(s), u_2(s))ds \\ &\quad - \sum_{k=1}^{m} I_{i,k}(u_i(t_k)), \quad for\ i = 1, 2. \end{aligned} \tag{3.1.13}$$

$$\begin{aligned} u_i(1) &= u_i(0) + u_i'(0) - \int_0^1 (1-s) f_i(s, u_1(s), u_2(s))ds \\ &\quad - \sum_{k=1}^{m} I_{i,k}(u_i(t_k))(1 - t_k), \quad for\ i = 1, 2. \end{aligned} \tag{3.1.14}$$

Therefore,

$$\gamma u_i(1) + \delta u_i'(1) = \gamma u_i(0) + (\gamma+\delta)u_i'(0) - \int_0^1 (\gamma+\delta-\gamma s) f_i(s, u_1(s), u_2(s))ds - \sum_{k=1}^{m} I_{i,k}(u_i(t_k))(\gamma+\delta-\gamma t_k), \ \ for\ i=1,2.$$

Then we have

$$\alpha u_i(0) - \beta u_i'(0) = 0, \quad \text{for } i = 1,2,$$

and

$$\gamma u_i(0)+(\gamma+\delta)u_i'(0) = \int_0^1 (\gamma+\delta-\gamma s) f_i(s, u_1(s), u_2(s))ds + \sum_{k=1}^{m} I_{i,k}(u_i(t_k))(\gamma+\delta-\gamma t_k),$$

for $i = 1, 2$. An application of Cramer's method yields

$$u_i(0) = \frac{\beta}{\rho}\left[\int_0^1 (\gamma+\delta-\gamma s) f_i(s, u_1(s), u_2(s))ds + \sum_{k=1}^{m}(\gamma+\delta-\gamma t_k) I_{i,k}(u_i(t_k))\right]$$

and

$$u_i'(0) = \frac{\alpha}{\rho}\left[\int_0^1 (\gamma+\delta-\gamma s) f_i(s, u_1(s), u_2(s))ds + \sum_{k=1}^{m}(\gamma+\delta-\gamma t_k) I_{i,k}(u_i(t_k))\right].$$

Thus,

$$\begin{aligned} u_i(t) &= \frac{\beta}{\rho}\left[\int_0^1 (\gamma+\delta-\gamma s) f_i(s, u_1(s), u_2(s))ds + \sum_{k=1}^{m}(\gamma+\delta-\gamma t_k) I_{i,k}(u_i(t_k))\right] \\ &\quad + \frac{\alpha t}{\rho}\left[\int_0^1 (\gamma+\delta-\gamma s) f_i(s, u_2(s), u_2(s))ds + \sum_{k=1}^{m}(\gamma+\delta-\gamma t_k) I_{i,k}(u_i(t_k))\right] \\ &\quad - \int_0^t (t-s) f_i(s, u_1(s), u_2(s))ds - \sum_{0<t_k<t}(t-t_k) I_{i,k}(u_i(t_k)), \quad for\ i=1,2. \end{aligned}$$

We then have

$$\begin{aligned} u_i(t) &= \frac{\int_0^1 (\alpha t+\beta)(\gamma+\delta-\gamma s) f_i(s, u_i(s), u_i(s))ds - \int_0^t (t-s)(\rho) f_i(s, u_1(s), u_2(s))ds}{\rho} \\ &+ \frac{\sum_{k=1}^{m}(\alpha t+\beta)(\gamma+\delta-\gamma t_k) I_{i,k}(u_i(t_k)) - \sum_{0<t_k<t}(t-t_k)(\rho) I_{i,k}(u_i(t_k))}{\rho}, \end{aligned}$$

for $i = 1, 2$. Hence,

$$\begin{cases} u_1(t) &= \displaystyle\int_0^1 G(t,s) f_1(s, u_1(s), u_2(s)) ds + \sum_{k=1}^{m} G(t, t_k) I_{1,k}(u_1(t_k)), \\ u_2(t) &= \displaystyle\int_0^1 G(t,s) f_2(s, u_1(s), u_2(s)) ds + \sum_{k=1}^{m} G(t, t_k) I_{2,k}(u_2(t_k)), \end{cases}$$

where $G(t, s)$ is given in (4.1.7).

Conversely, if the vector (u_1, u_2) is a solution of (3.1.9), then

$$u_i(t) = \int_0^1 G(t,s) f_i(s, u_i(s), u_i(s)) ds + \sum_{k=1}^{m} G(t, t_k) I_{i,k}(u_i(t_k)), \quad \text{for } i = 1, 2.$$

i.e.,

$$\begin{aligned} u_i(t) &= \int_0^t \frac{1}{\rho}(\gamma + \delta - \gamma t)(\beta + \alpha s) f_i(s, u_1(s), u_2(s)) ds \\ &\quad + \int_t^1 \frac{1}{\rho}(\beta + \alpha t)(\gamma + \delta - \gamma s) f_i(s, u_1(s), u_2(s)) ds \\ &\quad + \sum_{t_k < t} \frac{1}{\rho}(\gamma + \delta - \gamma t)(\beta + \alpha t_k) I_{i,k}(u_i(t_k)) \\ &\quad + \sum_{t_k > t} \frac{1}{\rho}(\beta + \alpha t)(\gamma + \delta - \gamma t_k) I_{i,k}(u_i(t_k)), \text{ for } i = 1, 2, \ t \neq t_k, \end{aligned}$$

and

$$\begin{aligned} u_i'(t) &= \frac{-\gamma}{\rho} \int_0^t (\beta + \alpha s) f_i(s, u_1(s), u_2(s)) ds + \frac{\alpha}{\rho} \int_t^1 (\gamma + \delta - \gamma s) f_i(s, u_1(s), u_2(s)) ds \\ &\quad + \frac{-\gamma}{\rho} \sum_{t_k < t} (\beta + \alpha t_k) I_{i,k}(u_i(t_k)) \\ &\quad + \frac{\alpha}{\rho} \sum_{t_k > t} (\gamma + \delta - \gamma t_k) I_{i,k}(u_i(t_k)), \text{ for } i = 1, 2, \ t \neq t_k. \end{aligned}$$

Differentiating again, we see that

$$\begin{aligned} u_i''(t) &= \frac{1}{\rho}\left(-\gamma \int_0^t (\beta + \alpha s) f_i(s, u_1(s), u_2(s)) ds + \alpha \int_t^1 (\gamma + \delta - \gamma s) f_i(s, u_1(s), u_2(s)) ds \right)' \\ &= -f_i(s, u_1(s), u_2(s)), \quad \text{for } i = 1, 2, \ t \neq t_k. \end{aligned}$$

Since

$$\begin{aligned} u_i(0) &= \frac{\beta}{\rho}\int_0^1(\gamma+\delta-\gamma s)f_i(s,u_1(s),u_2(s))ds+\frac{\beta}{\rho}\sum_{k=1}^{m}(\gamma+\delta-\gamma t_k)I_{i,k}(u_i(t_k)),\\ u_i'(0) &= \frac{\alpha}{\rho}\int_0^1 f_i(s,u_1(s),u_2(s))ds+\frac{\alpha}{\rho}\sum_{k=1}^{m}(\gamma+\delta-\gamma t_k)I_{i,k}(u_i(t_k)),\end{aligned}$$

for $i=1,2$, we have that $\alpha u_i'(0)=\beta u_i'(0)$ for $i=1,2$. Also, since

$$\begin{aligned} u_i(1) &= \frac{\delta}{\rho}\int_0^1(\beta+\alpha s)f_i(s,u_1(s),u_2(s))ds+\frac{\delta}{\rho}\sum_{k=1}^{m}(\beta+\alpha t_k)I_{i,k}(u_2(t_k)),\\ u_i'(1) &= -\frac{\gamma}{\rho}\int_0^1(\beta+\alpha s)f_i(s,u_1(s),u_2(s))ds+\frac{-\gamma}{\rho}\sum_{t_k<t}(\beta+\alpha t_k)I_{i,k}(u_i(t_k)),\end{aligned}$$

for $i=1,2$, we have that $\gamma u_i(1)+\delta u_i'(1)=0$ for $i=1,2$. Hence,

$$u_i(t_k^+)-u_i(t_k^-)=\frac{1}{\rho}(-\gamma(\beta+\alpha t_k)-\alpha(\gamma+\delta-\gamma t_k)I_{i,k}(u_i(t_k))=-I_{i,k}(u_i(t_k)),\ \ for\ i=1,2,$$

and this completes the proof of the lemma. □

We are now ready to present our main result in this section.

Theorem 3.1.2. *Assume that the following conditions are satisfied:*

(H_1) *There exist four positive real constants P_1, P_2, P_3, and P_4 such that*

$$\begin{cases} |f_1(t,u_1,u_2)-f_1(t,\bar u_1,\bar u_2)|\le P_1|u_1-\bar u_1|+P_2|u_2-\bar u_2|,\\ |f_2(t,u_1,u_2)-f_2(t,\bar u_1,\bar u_2)|\le P_3|u_1-\bar u_1|+P_4|u_2-\bar u_2|,\end{cases}$$

for each u_1, u_2, $\bar u_1$, $\bar u_2\in\mathbb{R}$ and each $t\in J$;

(H_2) *There exist $K_{1,k}$ and $K_{2,k}$ such that*

$$|I_{1,k}(u_1)-I_{1,k}(\bar u_1)|\le K_{1,k}|u_1-\bar u_1|,\quad k=1,2,\dots,m,$$

and

$$|I_{2,k}(u_2)-I_{2,k}(\bar u_2)|\le K_{2,k}|u_2-\bar u_2|,\quad k=1,2,\dots,m,$$

for all u_1, u_2, $\bar u_1$, $\bar u_2\in\mathbb{R}$.

If the matrix

$$M := G^* \begin{pmatrix} P_1 + mK_1 & P_2 \\ P_3 & P_4 + mK_2 \end{pmatrix} \tag{3.1.15}$$

converges to 0*, where* $G^* = \sup\{|G(t,s)| : (t,s) \in J \times J\}$*,* $K_1 = \max\{K_{1,k}\}$*, and* $K_2 = \max\{K_{2,k}\}$ *for* $k = 1, 2, \ldots, m$*, then the problem* (3.1.1)-(3.1.6) *has a unique solution.*

Proof. Consider the operator

$$N : C(J,\mathbb{R}) \times C(J,\mathbb{R}) \to C(J,\mathbb{R}) \times C(J,\mathbb{R})$$

defined by

$$N(u_1,u_2) = (A_1(u_1,u_2), A_2(u_1,u_2)),$$

where

$$A_1(u_1,u_2)(t) = \int_0^1 G(t,s) f_1(s,u_1(s),u_2(s))ds + \sum_{k=1}^m G(t,t_k) I_{1,k}(u_1(t_k)),$$

and

$$A_2(u_1,u_2)(t) = \int_0^1 G(t,s) f_2(s,u_1(s),u_2(s))ds + \sum_{k=1}^m G(t,t_k) I_{2,k}(u_2(t_k)).$$

Let $(u_1,u_2), (\bar{u}_1,\bar{u}_2) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$; then

$$\begin{aligned}
&|A_1(u_1,u_2)(t) - A_1(\bar{u}_1,\bar{u}_2)(t))| \\
&\quad \le \int_0^1 |G(t,s)||f_1(s,u_1(s),u_2(s)) - f_1(s,\bar{u}_1(s),\bar{u}_2(s))|ds \\
&\qquad + \sum_{k=1}^m |G(t,t_k)||I_{1,k}(u_1(t_k)) - I_{1,k}(\bar{u}_1(t_k))| \\
&\quad \le G^* \int_0^1 \left[P_1|u_1(s) - \bar{u}_1(s)| + P_2|u_2(s) - \bar{u}_2(s)|\right] ds \\
&\qquad + G^* \sum_{k=1}^m K_{1,k}|u_1(t_k) - \bar{u}_1(t_k)| \\
&\quad \le G^* \left(P_1 + \sum_{k=1}^m K_{1,k}\right) \|u_1 - \bar{u}_1\|_\infty + G^* P_2 \|u_2 - \bar{u}_2\|_\infty \\
&\quad \le G^* \left[(P_1 + mK_1) \|u_1 - \bar{u}_1\|_\infty + P_2\|u_2 - \bar{u}_2\|_\infty\right],
\end{aligned}$$

so

$$\|A_1(u_1,u_2)-A_1(\bar{u}_1,\bar{u}_2)\|_\infty \leq G^* \left[(P_1+mK_1)\|u_1-\bar{u}_1\|_\infty + P_2\|u_2-\bar{u}_2\|_\infty\right]. \tag{3.1.16}$$

Similarly,

$$\begin{aligned}
|A_2(u_1,u_2)(t) - A_2(\bar{u}_1,\bar{u}_2)(t)| \\
&\leq \int_0^1 |G(t,s)||f_2(s,u_1(s),u_2(s)) - f_2(s,\bar{u}_1(s),\bar{u}_2(s))|ds \\
&\quad + \sum_{k=1}^m |G(t,t_k)||I_{2,k}(u_2(t_k)) - I_{2,k}(\bar{u}_2(t_k))| \\
&\leq G^* \int_0^1 \left[P_3|u_1(s)-\bar{u}_1(s)| + P_4|u_2(s)-\bar{u}_2(s)|\right] ds \\
&\quad + G^* \sum_{k=1}^m K_{2,k}|u_2(t_k)-\bar{u}_2(t_k)|. \\
&\leq G^* P_3||u_1-\bar{u}_1||_\infty + G^*\left(P_4 + \sum_{k=1}^m K_{2,k}\right)||u_2-\bar{u}_2||_\infty. \\
&\leq G^*\left[P_3||u_1-\bar{u}_1||_\infty + (P_4+mK_2)||u_2-\bar{u}_2||_\infty\right],
\end{aligned}$$

and so

$$\|A_2(u_1,u_2)-A_2(\bar{u}_1,\bar{u}_2)\|_\infty \leq G^*\left[P_3||u_1-\bar{u}_1||_\infty + (P_4+mK_2)||u_2-\bar{u}_2||_\infty\right]. \tag{3.1.17}$$

From (3.1.16) and (3.1.17), we obtain

$$\begin{bmatrix} \|A_1(u_1,u_2)-A_1(\bar{u}_1,\bar{u}_2)\|_\infty \\ \\ \|A_2(u_1,u_2)-A_2(\bar{u}_1,\bar{u}_2)\|_\infty \end{bmatrix} \leq M \begin{bmatrix} \|u_1-\bar{u}_1\|_\infty \\ \\ \|u_2-\bar{u}_2\|_\infty \end{bmatrix},$$

where

$$M = G^* \begin{pmatrix} P_1+mK_1 & P_2 \\ P_3 & P_4+mK_2 \end{pmatrix}.$$

Then by (3.1.15), N is a contraction, so by Perov's fixed point theorem (Theorem 1.4.7 above), N has a unique fixed point that in turn is a solution of system (3.1.1)-(3.1.6). □

3.1 Existence results

In this section we give an existence result based on the non linear alternative of Leray-Schauder type. We need following conditions to obtain our result:

(C_1) The functions f_1 and f_2 are L^1-Carathéodory functions;

(C_2) There exist functions p, q, h, g, $\tilde{q}$, and $\bar{h} \in L^1([0,1],\mathbb{R}^+)$ and constants α_1, α_2, α_3, and $\alpha_4 \in [0,1)$ such that

$$|f_1(t,u_1,u_2)| \leq p(t)|u_1|^{\alpha_1}+q(t)|u_2|^{\alpha_2}+h(t), \text{ for each } t \in J \text{ and } u_1,\ u_2 \in \mathbb{R}$$

and

$$|f_2(t,u_1,u_2)| \leq \tilde{p}(t)|u_1|^{\alpha_3}+\tilde{q}(t)|u_2|^{\alpha_4}+\bar{h}(t), \text{ for each } t \in J \text{ and } u_1, u_2 \in \mathbb{R};$$

(C_3) There exist constants c_k, b_k, c_k^*, and $b_k^* \in \mathbb{R}^+$ and β_k, $\beta_k^* \in [0,1)$ such that

$$|I_{1,k}(u_1)| \leq c_k + b_k|u_1|^{\beta_k},\ k=1,2,\ldots,m,\ u_1 \in \mathbb{R}$$

and

$$|I_{2,k}(u_2)| \leq c_k^* + b_k^*|u_2|^{\beta_k^*},\ k=1,2,\ldots,m,\ u_2 \in \mathbb{R}.$$

Theorem 3.1.3. *If conditions* (C_1)-(C_3) *hold, then the system* (3.1.1)-(3.1.6) *has at least one solution.*

Proof. Let N be the operator defined in the proof of Theorem 3.1.2. To show that N is continuous let $(u_{1,n},u_{2,n})$ be a sequence such that $(u_{1,n},u_{2,n}) \to (\tilde{u}_1,\tilde{u}_2) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$ as $n \to \infty$. Then,

$$\begin{aligned}
&|A_1(u_{1,n},u_{2,n})(t) - A_1(\tilde{u}_1,\tilde{u}_2)(t)| \\
&\qquad \leq \int_0^1 |G(t,s)||f_1(s,u_{1,n}(s),u_{2,n}(s)) - f_1(s,\tilde{u}_1(s),\tilde{u}_2(s))|ds \\
&\qquad\quad + \sum_{k=1}^m |G(t,t_k)||I_{1,k}(u_n(t_k)) - I_{1,k}(\tilde{u}_1(t_k))| \\
&\qquad \leq G^* \int_0^1 |f_1(s,u_{1,n}(s),u_{2,n}(s)) - f_1(s,\tilde{u}_1(s),\tilde{u}_2(s))|ds \\
&\qquad\quad + G^* \sum_{k=1}^m |I_{1,k}(u_{1,n}(t_k)) - I_{1,k}(\tilde{u}_1(t_k))|.
\end{aligned}$$

Since f_1 is an L_1-Carathéodory function and $I_{1,k}$, $k = 1, 2, \ldots, m$, are continuous, by the Lebesgue dominated convergence theorem,

$$\|A_1(u_{1,n}, u_{2,n}) - A_1(\tilde{u}_1, \tilde{u}_2)\|_\infty \ \to 0, \ as \ n \to \infty.$$

Similarly,

$$\|A_2(u_{1,n}, u_{2,n}) - A_2(\tilde{u}_1, \tilde{u}_2)\|_\infty \ \to 0, \ as \ n \to \infty.$$

Thus, N is continuous.

In order to show that N maps bounded sets into bounded sets in $C(J, \mathbb{R}) \times C(J, \mathbb{R})$, it suffices to show that for any $q > 0$ there exists a positive constant vector $l = (l_1, l_2)$ such that for each $(u_1, u_2) \in B_q = \{(u_1, u_2) \in C(J, \mathbb{R}) \times C(J, \mathbb{R}) : \|u_1\|_\infty \le q, \|u_2\|_\infty \le q\}$, we have

$$\|N(u_1, u_2)\|_\infty \le \|l\|.$$

For each $t \in J$, we have

$$\begin{aligned}
|A_1(u_1, u_2)(t)| &\le \int_0^1 |G(t,s)||f_1(s, u_1(s), u_2(s))|ds + \sum_{k=1}^m |G(t, t_k)||I_{1,k}(u_1(t_k))| \\
&\le G^* \int_0^1 (p(s)|u_1(s)|^{\alpha_1} + q(s)|u_2(s)|^{\alpha_2} + h(s))\, ds \\
&\quad + G^* \sum_{k=1}^m \left(c_k + b_k |u_1(t_k)|^{\beta_k}\right) \\
&\le G^* \|u_1\|_\infty^{\alpha_1} \int_0^1 p(s)ds + G^* \|u_2\|_\infty^{\alpha_2} \int_0^1 q(s)ds + G^* \int_0^1 h(s)ds \\
&\quad + G^* \sum_{k=1}^m \left(c_k + b_k \|u_1\|_\infty^{\beta_k}\right) \\
&\le G^* q^{\alpha_1} \|p\|_{L^1} + G^* q^{\alpha_2} \|q\|_{L^1} + G^* \|h\|_{L^1} + G^* \sum_{k=1}^m \left(c_k + b_k q^{\beta_k}\right).
\end{aligned}$$

Hence,

$$\|A_1(u_1, u_2)\|_\infty \le G^* q^{\tilde{\alpha}} \left(\|p\|_{L^1} + \|q\|_{L^1} + \sum_{k=1}^m b_k \right) + G^* \left(\|h\|_{L^1} + \sum_{k=1}^m c_k \right) := l_1$$

where

$$\tilde{\alpha} = \max\{\alpha_1, \alpha_2, \beta_k : k = 1, 2, \cdots, m\}.$$

Similarly, we have

$$\|A_2(u_1, u_2)\|_\infty \leq G^* q^{\bar{\alpha}} \left(\|\tilde{p}\|_{L^1} + \|\tilde{q}\|_{L^1} + \sum_{k=1}^{m} b_k^* \right) + G^* \left(\|\bar{h}\|_{L^1} + \sum_{k=1}^{m} c_k^* \right) := l_2,$$

where

$$\bar{\alpha} = \max\{\alpha_3, \alpha_4, \beta_k^* : k = 1, 2, \cdots, m\},$$

which is what we needed to show.

Next we show that N maps bounded sets into equicontinuous sets of $C([0,1], \mathbb{R}) \times C(J, \mathbb{R})$. Let B_q be the bounded set obtained above. Let r_1, $r_2 \in J$ with $r_1 < r_2$ and $u \in B_q$; then we have

$$\begin{aligned}
&|A_1(u_1, u_2)(r_2) - A_1(u_1, u_2)(r_1)| \\
&\qquad \leq \int_0^1 |G(r_2, s) - G(r_1, s)||f_1(s, u_1(s), u_2(s))|ds \\
&\qquad\quad + \sum_{k=1}^{m} |G(r_2, t_k) - (G(r_1, t_k)||I_{1,k}(u_1(t_k))| \\
&\qquad \leq \int_0^1 |G(r_2, s) - G(r_1, s)|[(p(s)|u_1(s)|^{\alpha_1} \\
&\qquad\quad + q(s)|u_2(s)|^{\alpha_2} + h(s))]ds \\
&\qquad\quad + \sum_{k=1}^{m} |G(r_2, t_k) - G(r_1, t_k)| \left(c_k + b_k |u_1(s)|^{\beta_k} \right) \\
&\qquad \leq q^{\alpha_1} \int_0^1 |G(r_2, s) - G(r_1, s)| p(s) ds \\
&\qquad\quad + q^{\alpha_2} \int_0^1 |G(r_2, s) - G(r_1, s)| q(s) ds \\
&\qquad\quad + \int_0^1 |G(r_2, s) - G(r_1, s)| h(s) ds \\
&\qquad\quad + \sum_{k=1}^{m} |G(r_2, t_k) - G(r_1, t_k)| \left(c_k + b_k q^{\beta_k} \right).
\end{aligned}$$

Similarly, we have

$$
\begin{aligned}
&|A_2(u_1,u_2)(r_2) - A_2(u_1,u_2)(r_1)| \\
&\qquad \le q^{\alpha_3}\int_0^1 |G(r_2,s)-G(r_1,s)|\tilde{p}(s)ds \\
&\qquad\quad + q^{\alpha_4}\int_0^1 |G(r_2,s)-G(r_1,s)|\tilde{q}(s)ds \\
&\qquad\quad + \int_0^1 |G(r_2,s)-G(r_1,s)|\bar{h}(s)ds \\
&\qquad\quad + \sum_{k=1}^{m} |G(r_2,t_k)-G(r_1,t_k)|\left(c_k^* + b_k^* q^{\beta_k}\right).
\end{aligned}
$$

Notice that the terms on the right-hand side in the above two expressions tend to zero as $|r_2 - r_1| \to 0$. We can now apply the Arzelà-Ascoli theorem to conclude that $N : B_M \to C(J,\mathbb{R}) \times C(\bar{J},\mathbb{R})$ is completely continuous.

Next, let $(u_1,u_2) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$ with $(u_1,u_2) = \lambda N(u_1,u_2)$ for some $0 < \lambda < 1$. Then $u_1 = \lambda A_1(u_1,u_2)$ and $u_2 = \lambda A_2(u_1,u_2)$. Thus, for $t \in [0,1]$, we have

$$
\begin{aligned}
|u_1(t)| &\le \int_0^1 |G(t,s)||f_1(s,u_1(s),u_2(s))| + \sum_{k=1}^{m} |G(t,t_k)||I_{1,k}(u_1(t_k))| \\
&\le G^*\int_0^1 \left[(p(s)|u_1(s)|^{\alpha_1} + q(s)|u_2(s)|^{\alpha_2} + h(s)\right] ds \\
&\quad + G^*\sum_{k=1}^{m}\left(c_k + b_k|u_1(t_k)|^{\beta_k}\right) \\
&\le G^*\|u_1\|_\infty^{\alpha_1}\int_0^1 p(s)ds + G^*\|u_2\|_\infty^{\alpha_2}\int_0^1 q(s)ds + G^*\int_0^1 h(s)ds \\
&\quad + G^*\sum_{k=1}^{m}\left(c_k + b_k\|u_1\|_\infty^{\beta_k}\right).
\end{aligned}
$$

Hence,

$$
\|u_1\|_\infty \le G^*\|u_1\|_\infty^{\alpha_1}\|p\|_{L^1} + G^*\|u_2\|_\infty^{\alpha_2}\|q\|_{L^1} + G^*\|h\|_{L^1} + G^*\sum_{k=1}^{m}\left(c_k + b_k\|u_1\|_\infty^{\beta_k}\right).
$$

Similarly, we obtain

$$\|u_2\|_\infty \leq G^*\|u_1\|_\infty^{\alpha_3}\|\tilde{p}\|_{L^1}+G^*\|u_2\|_\infty^{\alpha_4}\|\tilde{q}\|_{L^1}+G^*\|\bar{h}\|_{L^1}+G^*\sum_{k=1}^{m}\left(c_k^*+b_k^*\|u_2\|_\infty^{\beta_k^*}\right).$$

Notice that if $\epsilon \leq \delta$ and $\|u\| > 1$, then $\|u\|^\epsilon \leq \|u\|^\delta$. Thus, $\|u\|^\epsilon \leq 1+\|u\|^\delta$ for all u.

We then have

$$\begin{aligned}
&\|u_1\|_\infty + \|u_2\|_\infty \\
&\quad\leq G^*\left(\|q\|_{L^1}+\|\tilde{p}\|_{L^1}\right)\left(\|u_1\|_\infty^{\alpha_3}+\|u_2\|_\infty^{\alpha_2}\right) \\
&\quad\quad + G^*\left(\|p\|_{L^1}+\|\tilde{q}\|_{L^1}\right)\left(\|u_1\|_\infty^{\alpha_1}+\|u_2\|_\infty^{\alpha_4}\right) \\
&\quad\quad + G^*\sum_{k=1}^{m}(b_k+b_k^*)\left(\|u_1\|_C^{\beta_k}+\|u_2\|_C^{\beta_k^*}\right) \\
&\quad\quad + G^*\left(\sum_{k=1}^{m}(c_k+c_k^*)+\|h\|_{L^1}+\|\bar{h}\|_{L^1}\right) \\
&\quad\leq G^*\left(\|q\|_{L^1}+\|\tilde{p}\|_{L^1}+\|p\|_{L^1}+\|\bar{q}\|_{L^1}+\sum_{k=1}^{m}(b_k+b_k^*)\right)\left(1+\|u_1\|_\infty^{\alpha^*}+\|u_2\|_\infty^{\alpha^*}\right) \\
&\quad\quad + G^*\left(\sum_{k=1}^{m}(c_k+c_k^*)+\|h\|_{L^1}+\|\tilde{h}\|_{L^1}\right) \\
&\quad\leq 2G^*\left(\|q\|_{L^1}+\|\tilde{p}\|_{L^1}+\|p\|_{L^1}+\|\bar{q}\|_{L^1}+\sum_{k=1}^{m}(b_k+b_k^*)\right)\left(\|u_1\|_\infty+\|u_2\|_\infty\right)^{\alpha^*} \\
&\quad\quad + G^*\left(\|q\|_{L^1}+\|\tilde{p}\|_{L^1}+\|p\|_{L^1}+\|\bar{q}\|_{L^1}+\sum_{k=1}^{m}(b_k+b_k^*)\right) \\
&\quad\quad + G^*\left(\sum_{k=1}^{m}(c_k+c_k^*)+\|h\|_{L^1}+\|\tilde{h}\|_{L^1}\right)
\end{aligned}$$

where

$$\alpha^* = \max\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta_k,\beta_k^* : k=1,2,\cdots,m\}.$$

If $\|u_1\|_\infty + \|u_2\|_\infty > 1$, then

$$\begin{aligned}\frac{\|u_1\|_\infty + \|u_2\|_\infty}{(\|u_1\|_\infty + \|u_2\|_\infty)^{\alpha^*}} \leq 2G^* &\left(\|q\|_{L^1} + \|\tilde{p}\|_{L^1} + \|p\|_{L^1} + \|\tilde{q}\|_{L^1} + \sum_{k=1}^{m}(b_k + b_k^*)\right) \\ &+ G^* \frac{\left(\|q\|_{L^1} + \|\tilde{p}\|_{L^1} + \|p\|_{L^1} + \|\bar{q}\|_{L^1} + \sum_{k=1}^{m}(b_k + b_k^*)\right)}{(\|u_1\|_\infty + \|u_2\|_\infty)^{\alpha^*}} \\ &+ G^* \frac{\sum_{k=1}^{m}(c_k + c_k^*) + \|h\|_{L^1} + \|\bar{h}\|_{L^1}}{(\|u\|_\infty + \|v\|_\infty)^{\alpha^*}},\end{aligned}$$

or

$$\begin{aligned}(\|u_1\|_\infty + \|u_2\|_\infty)^{1-\alpha^*} \leq 2G^* &\left(\|q\|_{L^1} + \|\tilde{p}\|_{L^1} + \|p\|_{L^1} + \|\tilde{q}\|_{L^1} + \sum_{k=1}^{m}(b_k + b_k^*)\right) \\ &+ G^* \left(\|q\|_{L^1} + \|\tilde{p}\|_{L^1} + \|p\|_{L^1} + \|\bar{q}\|_{L^1} + \sum_{k=1}^{m}(b_k + b_k^*)\right) \\ &+ G^* \left(\sum_{k=1}^{m}(c_k + c_k^*) + \|h\|_{L^1} + \|\bar{h}\|_{L^1}\right).\end{aligned}$$

This implies that

$$\begin{aligned}\|u_1\|_\infty + \|u_2\|_\infty &\leq \left[3G^*\left(C_1 + \sum_{k=1}^{m}(b_k + b_k^*)\right) + G^*\left(\sum_{k=1}^{m}(c_k + c_k^*) + C_2\right)\right]^{\frac{1}{1-\alpha^*}} \\ &:= M_2,\end{aligned}$$

where

$$C_1 = \|q\|_{L^1} + \|\tilde{p}\|_{L^1} + \|p\|_{L^1} + \|\tilde{q}\|_{L^1} \text{ and } C_2 = \|h\|_{L^1} + \|\bar{h}\|_{L^1}.$$

Consequently

$$\|u_1\|_\infty \leq M_2 \;\; and \;\; \|u_2\|_\infty \leq M_2.$$

Set

$$U = \{(u_1, u_2) \in C(J, \mathbb{R}) \times C(J, \mathbb{R}) : \; \|u_1\|_\infty < M_2 + 1 \; and \; \|u_2\|_\infty < M_2 + 1\}.$$

From the choice of U, there is no $(u_1, u_2) \in \partial U$ such that $(u_1, u_2) = \lambda N(u_1, u_2)$ for some $\lambda \in (0, 1)$. As a consequence of the non linear alternative of Leray-Schauder type (Theorem 1.4.12), the operator N has a fixed point that is a solution of system (3.1.1)-(3.1.6). This completes the proof of the theorem. □

3.1.2 Examples

In this section, we give two examples to illustrate our results of this chapter.

Example 3.1.1. Consider the impulsive differential system of second order given by

$$-u_1''(t) = \frac{1}{6}\frac{u_2^2(t)}{1+u_2^2(t)}\sin(2u_1(t)) := f_1(t, u_1(t), u_2(t)), \quad t \in J \setminus \left\{\frac{1}{4}\right\}, \tag{3.1.18}$$

$$-u_2''(t) = \frac{1}{8}\frac{u_2^2(t)}{1+u_2^2(t)}\cos(2u_1(t)) := f_2(t, u_1(t), u_2(t)), \quad t \in J \setminus \left\{\frac{1}{4}\right\}, \tag{3.1.19}$$

$$-\Delta u_1'\left(\frac{1}{4}\right) = \frac{1}{4}\cos\left(u_1\left(\frac{1}{4}\right)\right), \ t_1 = \frac{1}{4}, \tag{3.1.20}$$

$$-\Delta u_2'\left(\frac{1}{4}\right) = \frac{1}{3}\sin\left(u_2\left(\frac{1}{4}\right)\right), \tag{3.1.21}$$

$$u_1(0) = u_1'(0) = 0, \quad u_2(0) = u_2'(0) = 0. \tag{3.1.22}$$

We see that $\alpha = \delta = 1$ and $\beta = \gamma = 0$. Moreover, since

$$\sup_{u_1,u_2\in\mathbb{R}}\left|\frac{\partial f_1(t,u_1,u_2)}{\partial u_1}\right| \le \frac{1}{3}, \quad \sup_{u_1,u_2\in\mathbb{R}}\left|\frac{\partial f_1(t,u_1,u_2)}{\partial u_2}\right| \le \frac{1}{3},$$
$$\sup_{u_1,u_2\in\mathbb{R}}\left|\frac{\partial f_2(t,u_1,u_2)}{\partial u_1}\right| \le \frac{1}{4}, \quad \sup_{u_1,u_2\in\mathbb{R}}\left|\frac{\partial f_2(t,u_1,u_2)}{\partial u_2}\right| \le \frac{1}{4},$$

we have

$$|f_1(t, u_1, u_2) - f_1(t, \bar{u}_1, \bar{u}_2)| \le \frac{1}{3}|u_1 - \bar{u}_1| + \frac{1}{3}|u_2 - \bar{u}_2|,$$

and

$$|f_2(t, u_1, u_2) - f_2(t, \bar{u}_1, \bar{u}_2)| \le \frac{1}{4}|u_1 - \bar{u}_1| + \frac{1}{4}|u_2 - \bar{u}_2|.$$

Hence, condition (H_1) holds with $P_1 = \frac{1}{3}$, $P_2 = \frac{1}{3}$, $P_3 = \frac{1}{4}$, and $P_4 = \frac{1}{4}$. Also,

$$|I_{1,1}(u_1) - I_{1,1}(\bar{u_1})| \leq \frac{1}{4}|u_1 - \bar{u}_1|, \text{ for each } u, \bar{u} \in \mathbb{R}, \text{ and each } t \in [0,1],$$

$$|I_{1,2}(u_2) - I_{1,2}(\bar{u}_2)| \leq \frac{1}{3}|u_2 - \bar{u}_2|, \text{ for each } u_2, \bar{u}_2 \in \mathbb{R}, \text{ and each } t \in [0,1].$$

Thus, (H_3) holds. From (4.1.7), the Green's function for the homogeneous problem is given by

$$G(t,s) = \begin{cases} s, & 0 \leq s \leq t \leq 1, \\ t, & 0 \leq t \leq s \leq 1, \end{cases}$$

and we can easily see that

$$G^* = \sup_{(t,s) \in J \times J} |G(t,s)| = 1.$$

For this example

$$M = \begin{pmatrix} \frac{7}{12} & \frac{1}{3} \\ \frac{1}{4} & \frac{7}{12} \end{pmatrix},$$

which has the two eigenvalues $\lambda_1 \simeq 0.872$ and $\lambda_2 \simeq 0.294$. Therefore, M converges to zero. All the conditions in Theorem 3.1.2 are satisfied, so system (3.1.18)-(3.1.22) has a unique solution.

Example 3.1.2. Consider the impulsive differential system

$$-u_1''(t) = t^3 + 2(t-1)^2|u_1(t)|^{0.8} + e^t|u_2(t)|^{0.3} + 3 := f_1(t, u_1(t), u_2(t)), \; t \in J \setminus \{\frac{1}{2}\}, \tag{3.1.23}$$

$$-u_2''(t) = t^2 + 4t|u_1(t)|^{0.4} + \left(t - \frac{1}{3}\right)^2 |u_2(t)|^{0.6} + 8 := f_2(t, u_1(t), u_2(t)), \quad t \in J \setminus \{\frac{1}{2}\}, \tag{3.1.24}$$

$$-\Delta u_1'\left(\frac{1}{2}\right) = \frac{1}{6}\sqrt{u_1\left(\frac{1}{2}\right)}, \; t_1 = \frac{1}{2}, \tag{3.1.25}$$

$$-\Delta u_2'\left(\frac{1}{2}\right) = \frac{2}{3}|u_2\left(\frac{1}{2}\right)|^{\frac{2}{5}} + 4, \tag{3.1.26}$$

$$u_1(0) = u_1'(0) = 0, \; u_2(0) = u_2'(0) = 0. \tag{3.1.27}$$

We clearly have

$$\begin{cases} |f_1(t,u_1(t),u_2(t))| \leq 2|u_1|^{0.8} + e|u_2|^{0.3} + 3, \\ |f_2(t,u_1(t),u_2(t))| \leq 4|u_1|^{0.4} + \frac{4}{9}|u_2|^{0.6} + 8, \end{cases}$$

and

$$\begin{cases} |I_{1,1}(u_1)| \leq \frac{1}{6}|u_1|^{\frac{1}{2}}, \\ |I_{1,2}(u_2)| \leq \frac{2}{3}|u_2|^{\frac{2}{5}} + 4. \end{cases}$$

for $t \in J$. Now all the hypotheses of Theorem 3.1.3 are satisfied, so system (3.1.23)-(3.1.27) has at least one solution.

3.2 Positive solutions

In this section we study the existence of positive solution for the systems (3.1.1)-(3.1.6).
The existence of positive solutions for the systems of differential equations and systems of impulsive differential equations has been inestigated by several authors (see, for instance [23, 48, 70, 85, 89]). We shall provide the existence of positive solution for the systems (3.1.1)-(3.1.6) by using the vector version of Kras-nosel'skii's fixed point theorem in cones given by [85].

3.2.1 Main results.

The problem (3.1.1)-(3.1.6) is formulated as a fixed point problem for a vector-valued mapping $N = (N_1, N_2)$. Then the sought solution $u := (u_1, u_2)$ will satisfy an operator system

$$\begin{cases} u_1 = N_1(u_1, u_2), \\ u_2 = N_2(u_1, u_2), \end{cases} \tag{3.2.1}$$

in the vector conical shell $K_{r,R}$; more exactly with $u \in K$ and

$$r_1 \leq \|u_1\| \leq R_1, \ r_2 \leq \|u_2\| \leq R_2.$$

We denote $G(t,s)$ as the Green's function of the following boundary value problem

$$-x''(t) = 0,$$
$$\alpha x(0) - \beta x'(0) = 0,$$
$$\gamma x(1) + \delta x'(1) = 0,$$

which is explicitly given by

$$G(t,s)=\frac{1}{\rho}\begin{cases}(\gamma+\delta-\gamma t)(\beta+\alpha s), & 0\leq s\leq t\leq 1\\(\beta+\alpha t)(\gamma+\delta-\gamma s), & 0\leq t\leq s\leq 1.\end{cases}$$

G it is positive and satisfies the easily-verified properties:

$$G(t,s) \leq G(s,s), \text{ for all } t,s\in[0,1]. \quad (3.2.2)$$

$$0<\sigma G(s,s) \leq G(t,s),\ t\in[a,b],\ s\in[0,1], \quad (3.2.3)$$

where $a\in[0,t_1]$, $b\in[t_m,1]$ and $0\leq\sigma=\min\left\{\frac{(1-b)\gamma+\delta}{\gamma+\delta},\frac{a\alpha+\beta}{\alpha+\beta}\right\}\leq 1$.
Let $N:P^2\to P^2$ be the completely continuous map $N=(N_1;N_2)$ given by

$$N_i(u(t))=\int_0^1 G(t,s)f_i(s,u_1(s),u_2(s))ds + \sum_{k=1}^{m}G(t,t_k)I_{i,k}(u_i(t_k)), \quad i=1,2. \quad (3.2.4)$$

Then (3.2.4) is equivalent to the fixed point problem

$$u=N(u), \quad u\in P^2.$$

If $v\in P$ and

$$u_i(t):=\int_0^1 G(t,s)v(s)ds+\sum_{k=1}^{m}G(t,t_k)I_{i,k}(u_i(t_k)), \qquad (i=1,2)$$

and $u_i(t')=\|u_i\|_\infty$ for $i=(1,2)$, then according to (3.2.3), for every $t\in[a,b]$, we have

$$u_i(t)\geq\sigma\int_0^1 G(s,s)v(s)ds+\sigma\sum_{k=1}^{m}G(t_k,t_k)I_{i,k}(u_i(t_k)),\ i=(1,2).$$

If $t'\neq t_k$ for $(k=1,2,\cdots,m)$, then

$$\begin{aligned}u_i(t) &\geq \sigma\int_0^1 G(t',s)v(s)ds+\sigma\sum_{k=1}^{m}G(t_k,t_k)I_{i,k}(u_i(t_k))\\ &\geq \sigma\int_0^1 G(t',s)v(s)ds+\sigma\sum_{k=1}^{m}G(t',t_k)I_{i,k}(u_i(t_k))=\sigma u_i(t')=\sigma\|u\|\end{aligned}$$

If $t' = t_k$ for $(k = 1, 2, \cdots, m)$, then

$$\begin{aligned} u_i(t) &\geq \sigma \int_0^1 G(s,s)v(s)ds + \sigma \sum_{k=1}^{m} G(t',t_k) I_{i,k}(u_i(t_k)) \\ &\geq \sigma \int_0^1 G(t',s)v(s)ds + \sigma \sum_{k=1}^{m} G(t',t_k) I_{i,k}(u_i(t_k)) = \sigma u_i(t') = \sigma \|u\|_{PC} \end{aligned}$$

Define the cone K_i for $i = (1, 2)$ in P by

$$K_i = \{u_i \in P : u_i(t) \geq \sigma \|u_i\|_\infty, for\ all\ t \in [a,b]\}\ (i = 1, 2),$$

and the product cone $K = K_1 \times K_2$ in P^2, then $N(K) \subset K$. Before we state our main result we introduce the following notations:

$r_i = \min\{\alpha_i, \beta_i\}, \;\; R_i = \max\{\alpha_i, \beta_i\},$ pour $\alpha_i, \beta_i > 0$ avec $\alpha_i \neq \beta_i, \; i = 1, 2$

$$\begin{aligned} \gamma_1 &= \min\{f_1(t, u_1, u_2) : \; a \leq t \leq b, \; \sigma\beta_1 \leq u_1 \leq \beta_1, \; \sigma r_2 \leq u_2 \leq R_2\}, \\ \gamma_2 &= \min\{f_2(t, u_1, u_2) : \; a \leq t \leq b, \; \sigma r_1 \leq u_1 \leq R_1, \; \sigma\beta_2 \leq u_2 \leq \beta_2\}, \\ \Gamma_1 &= \max\{f_1(t, u_1, u_2) : \; 0 \leq t \leq 1, \; \sigma\alpha_1 \leq u_1 \leq \alpha_1, \; \sigma r_2 \leq u_2 \leq R_2\}, \\ \Gamma_2 &= \max\{f_2(t, u_1, u_2) : \; 0 \leq t \leq 1, \; \sigma r_1 \leq u_1 \leq R_1, \; \sigma\alpha_2 \leq u_2 \leq \alpha_2\}. \end{aligned}$$

Also, let

$$\begin{aligned} B &= \max\{G(t,s) : 0 \leq t \leq 1, \; 0 \leq s \leq 1\}, \\ A &= \min\{G(t,s) : a \leq t \leq b, \; a \leq s \leq b\}, \\ \lambda_1 &= \min_{1 \leq k \leq m} \{\min\{I_{1,k}(u_1) : \sigma\beta_1 \leq u_1 \leq \beta_1\}\}, \\ \lambda_2 &= \min_{1 \leq k \leq m} \{\min\{I_{2,k}(u_2) : \sigma\beta_2 \leq u_2 \leq \beta_2\}\}, \\ \Lambda_1 &= \max_{1 \leq k \leq m} \{\max\{I_{1,k}(u_1) : \sigma\alpha_1 \leq u_1 \leq \alpha_1\}\}, \\ \Lambda_2 &= \max_{1 \leq k \leq m} \{\max\{I_{2,k}(u_2) : \sigma\alpha_2 \leq u_2 \leq \alpha_2\}\}. \end{aligned}$$

Theorem 3.2.1. *Assume that there exist $\alpha_i, \beta_i > 0$ with $\alpha_i \neq \beta_i, i = 1, 2$, such that*

$$\begin{array}{ll} B(\Gamma_1 + \Lambda_1 m) \leq \alpha_1, & A(\gamma_1(b-a) + \lambda_1 m) \geq \beta_1, \\ B(\Gamma_2 + \Lambda_2 m) \leq \alpha_2, & A(\gamma_2(b-a) + \lambda_2 m) \geq \beta_2. \end{array} \tag{3.2.5}$$

Then (3.1.1)-(3.1.6) *has a positive solution $u = (u_1, u_2)$ with $r_i \leq \|u_i\|_\infty \leq R_i, \; i = 1, 2,$ where $r_i = \min\{\alpha_i, \beta_i\}, \; R_i = \max\{\alpha_i, \beta_i\}$. Moreover, the corresponding orbit of u is included in the rectangle $[\sigma r_1, R_1] \times [\sigma r_2, R_2]$.*

Proof. First note that if $u \in K_{r,R}$, then $r_1 \le \|u_1\|_\infty \le R_1$ and $r_2 \le \|u_2\|_\infty \le R_2$, and by the definition of K,

$$\sigma r_1 \le \|u_1\|_\infty \le R_1 \quad \text{and} \quad \sigma r_2 \le \|u_2\|_\infty \le R_2,$$

for all $t \in [a,b]$, showing that the orbit of u for $t \in [a,b]$ is included in the rectangle $[\sigma r_1, R_1] \times [\sigma r_2, R_2]$.
Also, if we know for example that $\|u_1\|_\infty = \alpha_1$, then $u_1(t) \le \alpha_1$, for all $t \in [0,1]$ and

$$\sigma \alpha_1 \le u_1(t) \le \alpha_1, \quad \text{for all } t \in [a,b].$$

We now prove that for every $u \in K_{r,R}$ and $i \in \{1,2\}$, the following properties holds:

$$\begin{array}{l} \|u_i\|_\infty = \alpha_i \quad \text{implies} \quad u_i \not\prec N_i(u), \\ \|u_i\|_\infty = \beta_i \quad \text{implies} \quad u_i \not\succ N_i(u). \end{array} \tag{3.2.6}$$

In fact, if $\|u_1\|_\infty = \alpha_1$ and we would have $u_1 \prec N_1(u)$, then

$$u_1(t) < N_1(u)(t) \le B(\Gamma_1 + \Lambda_1 m) \le \alpha_1,$$

for all $t \in [0,1]$. This yields the contradiction $\alpha_1 < \alpha_1$.
if $\|u_1\|_\infty = \beta_1$ and $u_2 \succ N_2(u)$, then for $t \in [a,b]$, we obtain

$$\begin{array}{rcl} u_1(t) & > & N_1(u)(t) \\ & \ge & \displaystyle\int_a^b G(t,s) f_1(s, u_1(s), u_2(s)) ds + \sum_{k=1}^{m} G(t,t_k) I_{1,k}(u_1(t_k)) \\ & \ge & A(\gamma_1(b-a) + \lambda_1 m) \ge \beta 1 \end{array}$$

Then we deduce that $\beta_1 > \beta_1$, wich is a contradiction. Hence (3.2.6) holds for $i = 1$. Similary, (3.2.6) is true for $i = 2$. By Theorem (3.2.1), we see that N has at least one fixed point in K. Therefore, system (3.1.1)-(3.1.6) has at least one positive solution. The proof of Theorem (3.2.1) is complete . □

In particular, if f_1 and f_2 do not depend on t, i.e., $f_1 = f_1(u_1, u_2)$ and $f_2 = f_2(u_1, u_2)$ and f_1, f_2, $I_{1,k}$ and $I_{2,k}$ $(k = 1, 2, ..., m)$ have some properties in u_1, u_2, for $u_1 \in [\sigma r_1, R_1]$ and $u_2 \in [\sigma r_2, R_2]$ then we can specify the numbers γ_1, γ_2, Γ_1, Γ_2, λ_1, λ_2, Λ_1, Λ_2. For example

Case 1) If f_1,f_2 are nondecreasing in u_1, u_2, while $I_{1,k}$ and $I_{2,k}$ are nondecreasing respectively in u_1 and u_2 for $(k = 1, 2, ..., m)$, then

$$\begin{array}{rr}
\Gamma_1 = f_1(\alpha_1, R_2), & \gamma_1 = f_1(\sigma\beta_1, \sigma r_2), \\
\Gamma_2 = f_2(R_1, \alpha_2), & \gamma_2 = f_2(\sigma r_1, \sigma\beta_2), \\
\Lambda_1 = \max\limits_{1\le k\le m}\{I_{1,k}(\alpha_1)\}, & \lambda_1 = \min\limits_{1\le k\le m}\{I_{1,k}(\sigma\beta_1)\}, \\
\Lambda_2 = \max\limits_{1\le k\le m}\{I_{2,k}(\alpha_2)\}, & \lambda_2 = \min\limits_{1\le k\le m}\{I_{2,k}(\sigma\beta_2)\}.
\end{array}$$

Case2) If f_1 is nondecreasing in u_1 and u_2, while f_2 is nondecreasing in u_1 and nonincreasing in u_2 , on the other hand if $I_{1,k}$ is nondecreasing in u_1 and $I_{2,k}$ is nonincreasing in u_2 for $(k = 1, 2, ..., m)$

$$\begin{array}{rr}
\Gamma_1 = f_1(\alpha_1, R_2), & \gamma_1 = f_1(\sigma\beta_1, \sigma r_2), \\
\Gamma_2 = f_2(R_1, \sigma\alpha_2), & \gamma_2 = f_2(\sigma r_1, \beta_2), \\
\Lambda_1 = \max\limits_{1\le k\le m}\{I_{1,k}(\alpha_1)\}, & \lambda_1 = \min\limits_{1\le k\le m}\{I_{1,k}(\sigma\beta_1)\}, \\
\Lambda_2 = \max\limits_{1\le k\le m}\{I_{2,k}(\sigma\alpha_2)\}, & \lambda_2 = \min\limits_{1\le k\le m}\{I_{2,k}(\beta_2)\}.
\end{array}$$

Case 3) If f_1 is nondecreasing in u_1 and non increasing in u_2, while f_2 is nonincreasing in u_1 and nondecreasing in u_2, on the other hand if $I_{1,k}$ is nonincreasing in u_1 and $I_{2,k}$ is nondecreasing in u_2 for $(k = 1, 2, ..., m)$

$$\begin{array}{rr}
\Gamma_1 = f_1(\alpha_1, \sigma r_2), & \gamma_1 = f_1(\sigma\beta_1, R_2), \\
\Gamma_2 = f_2(\sigma r_1, \alpha_2), & \gamma_2 = f_2(R_1, \sigma\beta_2), \\
\Lambda_1 = \max\limits_{1\le k\le m}\{I_{1,k}(\sigma\alpha_1)\}, & \lambda_1 = \min\limits_{1\le k\le m}\{I_{1,k}(\beta_1)\}, \\
\Lambda_2 = \max\limits_{1\le k\le m}\{I_{2,k}(\alpha_2)\}, & \lambda_2 = \min\limits_{1\le k\le m}\{I_{2,k}(\sigma\beta_2)\}.
\end{array}$$

Case 4) If f_1,f_2 are nondecreasing in u_1 and nonincreasing in u_2, while $I_{1,k}$ is nondecreasing in u_1 and $I_{2,k}$ is nonincreasing in u_2 for $(k = 1, 2, ..., m)$, then

$$\begin{array}{rr}
\Gamma_1 = f_1(\alpha_1, \sigma r_2), & \gamma_1 = f_1(\sigma\beta_1, R_2), \\
\Gamma_2 = f_2(R_1, \sigma\alpha_2), & \gamma_2 = f_2(\sigma r_1, \beta_2), \\
\Lambda_1 = \max\limits_{1\le k\le m}\{I_{1,k}(\alpha_1)\}, & \lambda_1 = \min\limits_{1\le k\le m}\{I_{1,k}(\sigma\beta_1)\}, \\
\Lambda_2 = \max\limits_{1\le k\le m}\{I_{2,k}(\sigma\alpha_2)\}, & \lambda_2 = \min\limits_{1\le k\le m}\{I_{2,k}(\beta_2)\}.
\end{array}$$

3.2.2 Examples

We conclude by two examples illustrating Theorem (3.2.1) in the cases 1 and 4.

Example 3.2.1. Consider the following second-order impulsive systems:

$$u_1''(t) + u_1^\theta + u_2^\varepsilon = 0, \quad 0 < \theta < \varepsilon < 1 \quad t \neq \frac{1}{4} \quad 0 \leq t \leq 1, \tag{3.2.7}$$

$$u_2''(t) + u_1^\varepsilon + u_2^\theta = 0, \quad 0 < \theta < \varepsilon < 1 \quad t \neq \frac{1}{4} \; 0 \leq t \leq 1, \tag{3.2.8}$$

$$-\Delta u_1' \mid_{t = \frac{1}{4}} = c\sqrt{u_1\left(\frac{1}{4}\right)}, \quad c > 0, \tag{3.2.9}$$

$$-\Delta u_2' \mid_{t = \frac{1}{4}} = d\sqrt{u_2\left(\frac{1}{4}\right)}, \quad d > 0, \tag{3.2.10}$$

$$u_1(0) - u_1'(0) = 0, \quad u_1(1) - u_1'(1) = 0, \tag{3.2.11}$$

$$u_2(0) + u_2'(0) = 0, \quad u_2(1) + u_2'(1) = 0, \tag{3.2.12}$$

We can establish that (3.2.7)-(3.2.12) has at least one positive solution $u = (u_1, u_2)$.
Let

$$f_1(u_1, u_2) = u_1^\theta + u_2^\varepsilon, \;\; f_2(u_1, u_2) = u_1^\varepsilon + u_2^\theta,$$

and

$$I_{1,1}\left(u_1\left(\frac{1}{4}\right)\right) = c\sqrt{u_1\left(\frac{1}{4}\right)}, \; I_{1,2}\left(u_1\left(\frac{1}{4}\right)\right) = d\sqrt{u_2\left(\frac{1}{4}\right)}.$$

The system (3.2.7)-(3.2.12) is equivalent to the integral system:

$$\begin{cases} u_1(t) = \displaystyle\int_0^1 G(t,s)[u_1(s)^\theta + u_2(s)^\varepsilon]ds + cG\left(t, \frac{1}{4}\right)\sqrt{u_1\left(\frac{1}{4}\right)}, \\ u_2(t) = \displaystyle\int_0^1 G(t,s)\left[u_1(s)^\varepsilon + u_2(s)^\theta\right]ds + dG\left(t, \frac{1}{4}\right)\sqrt{u_2\left(\frac{1}{4}\right)}. \end{cases}$$

Where $G(t, s)$ is a Green function

$$G(t,s) = \frac{1}{3}\begin{cases} (2-t)(1+s), & 0 \leq s \leq t \leq 1 \\ (2-s)(1+t), & 0 \leq t \leq s \leq 1 \end{cases}$$

Clearly $B = \dfrac{9}{4}$ and $A = \sigma$. In this case $f_1(u_1, u_2)$ and $f_2(u_1, u_2)$ are both nondecreasing in u_1 and u_2, while $I_{1,1}$ and $I_{2,1}$ are nondecreasing respectively

in u_1 and u_2 for $k \in \{1, \cdots, m\}$, $u_1, u_2 \in \mathbb{R}^+$, so we are in case 1. We choose $\alpha_1 = \alpha_2 =: \alpha^*$, $\beta_1 = \beta_2 =: \beta^*$, with $\beta^* < \alpha^*$ then $r_1 = r_2 = \beta^*$, $R_1 = R_2 = \alpha^*$ and $\gamma_i = f_i(\sigma\beta^*, \sigma\beta^*)$, $\Gamma_i = f_i(\alpha^*, \alpha^*)$, $\Lambda_i = I_{i,1}(\alpha^*)$, $\lambda_i = I_{i,2}(\sigma\beta^*)$ for (i=1,2)
The values of α^* and β^* will be precised in what follows. Since

$$\lim_{x\to\infty} \frac{f_i(x,x)}{x} = 0 \quad \text{and} \quad \lim_{x\to 0} \frac{f_i(x,x)}{x} = \infty,$$
$$\lim_{x\to\infty} \frac{I_{i,1}(x)}{x} = 0 \quad \text{and} \quad \lim_{x\to 0} \frac{I_{i,1}(x)}{x} = \infty,$$

for $i \in \{1, 2\}$. We may find β^* small enough and α^* large enough such that the conditions

$$\frac{f_i(\alpha^*, \alpha^*)}{\alpha^*} \le \frac{1}{2B}, \qquad \frac{f_i(\sigma\beta^*, \sigma\beta^*)}{\sigma\beta^*} \ge \frac{1}{2\sigma A(b-a)},$$
$$\frac{I_{i,1}(\alpha^*)}{\alpha^*} \le \frac{1}{2Bm}, \qquad \frac{I_{i,1}(\sigma\beta^*)}{\sigma\beta^*} \ge \frac{1}{2\sigma Am}.$$

for $i \in \{1, 2\}$ are satisfied. Thus condition (3.2.5) hold .
We conclude that system (3.2.7)-(3.2.12) has at least a positive solution (u_1, u_2) with $\beta^* \le \|u_i\|_\infty \le \alpha^*$ for $i \in \{1, 2\}$.

Example 3.2.2. Consider the following second-order impulsive systems:

$$u_1''(t) + \frac{u_1^{\frac{1}{4}}}{u_2 + 1} = 0, \quad t \neq \frac{1}{2} \quad 0 \le t \le 1, \tag{3.2.13}$$

$$u_1''(t) + \frac{u_1}{u_2 + 1} = 0, \quad t \neq \frac{1}{2}\ 0 \le t \le 1, \tag{3.2.14}$$

$$-\Delta u_1' \,\Big|_{t=\frac{1}{2}} = u_1^{\frac{1}{3}}\left(\frac{1}{2}\right), \tag{3.2.15}$$

$$-\Delta u_2' \,\Big|_{t=\frac{1}{2}} = e^{-u_2\left(\frac{1}{2}\right)}, \tag{3.2.16}$$

$$u_1(0) - u_1'(0) = 0, \quad u_1(1) - u_1'(1) = 0, \tag{3.2.17}$$

$$u_2(0) + u_2'(0) = 0, \quad u_2(1) + u_2'(1) = 0. \tag{3.2.18}$$

Let

$$f_1(u_1,u_2)=\frac{u_1^{\frac{1}{4}}}{u_2+1},\ f_2(u_1,u_2)=\frac{u_1}{u_2+1},$$

and

$$I_{1,1}\left(u_1\left(\frac{1}{2}\right)\right)=u_1^{\frac{1}{3}}\left(\frac{1}{2}\right),\ I_{1,2}\left(u_1\left(\frac{1}{2}\right)\right)=e^{-u_2\left(\frac{1}{2}\right)}.$$

The system (3.2.13)-(3.2.18) is equivalent to the integral system:

$$\begin{cases} u_1(t)=\displaystyle\int_0^1 G(t,s)\frac{u_1(s)^{\frac{1}{4}}}{u_2(s)+1}ds+G\left(t,\frac{1}{2}\right)u_1^{\frac{1}{3}}\left(\frac{1}{2}\right), \\ u_2(t)=\displaystyle\int_0^1 G(t,s)\frac{u_1(s)}{u_2(s)+1}ds+G\left(t,\frac{1}{2}\right)e^{-u_2\left(\frac{1}{2}\right)}. \end{cases}$$

The Green function $G(t,s)$ is a the same from the Example (3.2.13)-(3.2.18). In this case $f_1(u_1,u_2)$, $f_2(u_1,u_2)$ are nondecreasing in u_1 and nonincreasing in u_2, while $I_{1,1}$ is nondecreasing in u_1 and $I_{2,1}$ is nonincreasing in u_2, for $k\in\{1,\cdots,m\}$ $u_1,\ u_2\in\mathbb{R}^+$, so now we are in case 4). We choose $\alpha_1=\alpha_2=:\alpha^*$, $\beta_1=\beta_2=:\beta^*$, with $\beta^*<\alpha^*$. Then $r_1=r_2=\beta^*$, $R_1=R_2=\alpha^*$ and $\Gamma_1=f_1(\alpha^*,\sigma\beta^*)$, $\Gamma_2=f_2(\alpha^*,\sigma\alpha^*)$, $\gamma_1=f_1(\sigma\beta^*,\alpha^*)$, $\gamma_2=f_2(\sigma\beta^*,\beta^*)$, $\Lambda_1=I_{1,1}(\alpha^*)$, $\lambda_1=I_{1,1}(\sigma\beta^*)$, $\Lambda_2=I_{2,1}(\sigma\alpha^*)$, $\lambda_2=I_{2,1}(\beta^*)$, where α^* and β^* will be precised in what follows.
Since

$$\lim_{x\to\infty}\frac{f_1(x,0)}{x}=0,\quad \lim_{y\to\infty}\frac{f_2(x,\sigma y)}{y}=0$$
$$\lim_{x\to\infty}\frac{I_{1,1}(x)}{x}=0,\quad \lim_{y\to\infty}\frac{I_{1,2}(\sigma y)}{y}=0,$$

may find $\alpha^*>0$ large enough such that

$$\frac{f_1(\alpha^*,0)}{\alpha^*}\le\frac{1}{2B},\quad \frac{f_2(\alpha^*,\sigma\alpha^*)}{\alpha^*}\le\frac{1}{2B},$$
$$\frac{I_{1,1}(\alpha^*)}{\alpha^*}\le\frac{1}{2Bm},\quad \frac{I_{1,2}(\sigma\alpha^*)}{\alpha^*}\le\frac{1}{2Bm}.$$

Since

$$\frac{f_1(\alpha^*,\sigma\beta^*)}{\alpha^*}\le\frac{f_1(\alpha^*,0)}{\alpha^*}.$$

Then

$$\frac{f_1(\alpha^*, \sigma\beta^*)}{\alpha^*} \leq \frac{1}{2B},$$

and since

$$\lim_{x\to 0} \frac{f_1(\sigma x, y)}{x} = \infty, \quad \lim_{y\to 0} \frac{f_2(x, y)}{y} = \infty,$$
$$\lim_{x\to 0} \frac{I_{1,1}(\sigma x)}{x} = \infty, \quad \lim_{y\to 0} \frac{I_{1,2}(y)}{y} = \infty,$$

with α fixed as above, we choose β small enough such that

$$\frac{f_1(\sigma\beta^*, \alpha^*)}{\beta^*} \geq \frac{1}{2A(b-a)}, \quad \frac{f_2(\sigma\beta^*, \beta^*)}{\beta^*} \geq \frac{1}{2A(b-a)},$$
$$\frac{I_{1,1}(\sigma\beta^*)}{\beta^*} \geq \frac{1}{2Am}, \quad \frac{I_{1,2}(\beta^*)}{\beta^*} \geq \frac{1}{2Am}.$$

The conditions (3.2.5) are satisfied, hence system (3.2.13)-(3.2.18) has at least one positive solution $u = (u_1, u_2)$.

3.3 Multiple positive solutions for systems of impulsive differential equation

In this section we study the existence of multiple positive solutions for the systems of second order impulsive differential equations with three points boundary conditions

$$u_1''(t) + h_1(t) f_1(t, u_1(t), u_2(t)) = 0, \quad t \in J', \tag{3.3.1}$$
$$u_2''(t) + h_2(t) f_2(t, u_1(t), u_2(t)) = 0, \quad t \in J', \tag{3.3.2}$$
$$\Delta u_1(t_k) = I_{1,k}(u_1(t_k)), \tag{3.3.3}$$
$$\Delta u_1'(t_k) = -J_{1,k}(u_1(t_k)), \quad k = 1, 2, \cdots, m, \tag{3.3.4}$$
$$\Delta u_2(t_k) = I_{2,k}(u_2(t_k)), \tag{3.3.5}$$
$$\Delta u_2(t_k) = -J_{2,k}(u_2(t_k)), \quad k = 1, 2, \cdots, m, \tag{3.3.6}$$
$$\alpha u_1(0) - \beta u_1'(0) = a u_1(\xi), \quad u_1(1) = 0, \tag{3.3.7}$$
$$\alpha u_2(0) - \delta u_2'(0) = a u_2(\xi), \quad u_2(1) = 0, \tag{3.3.8}$$

where $\alpha, \beta \geq 0$, $a, \xi \in]0, 1[$, $J = [0, 1]$, $0 < t_1 < t_2 < \cdots < t_m < 1$, $J' = J \setminus \{t_1, t_2, \cdots, t_m\}$, $f_i \in C(J \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, $h_i \in C(\mathbb{R}^+, \mathbb{R}^+)$,

$I_{i,k} \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $J_{i,k} \in C(\mathbb{R}^+, \mathbb{R}^+)$, $i = 1, 2$, $k \in \{1, 2, \cdots, m\}$, $\Delta u_1(t_k) = u_1(t_k^+) - u_1(t_k^-)$, $\Delta u_1'(t_k) = u_1'(t_k^+) - u_1'(t_k^-)$ and $\Delta u_2(t_k) = u_2(t_k^+) - u_2(t_k^-)$,$\Delta u_2'(t_k) = u_2'(t_k^+) - u_2'(t_k^-)$ in which $u_1(t_k^+)$, $u_1'(t_k^+)$, $u_2(t_k^+)$ $u_2'(t_k^+)$, $(u_1(t_k^-)$, $u_1'(t_k^-)$, $u_2(t_k^-)$, $u_2'(t_k^-))$ denote the right and left hand limit of $u_1(t)$, $u_1'(t)$ and $u_2(t)$, $u_2'(t)$ at $t = t_k$, respectively.

Our analysis relies on vector versions of Avery and Peterson fixed-point theorem [33].

3.3.1 Fixed point formulation

In order to define a solution for Problem (3.3.1)–(3.3.8), we shall consider the following spaces:

$$PC(J, \mathbb{R}^+) = \{y : J \to \mathbb{R}^+ |\ y \in C(J', \mathbb{R}^+) \text{ such that } y(t_k^-) \text{ and } y(t_k^+) \text{ exist and satisfy } y(t_k) = y(t_k^-) \text{ for } k = 1, \ldots, n\}. \quad (3.3.9)$$

For every $y \in PC(J, \mathbb{R}^+)$, we define the norm by

$$\|y\|_{PC} = \sup_{t \in J} |y(t)|.$$

$$PC^1(J, \mathbb{R}^+) = \{y \in PC([0,1], \mathbb{R}^+) : y_k' \in C(J', \mathbb{R}^+), \text{ such that } y'(t_k^-) \text{ and } y'(t_k^+) \text{ exist and satisfy } y'(t_k) = y'(t_k^-) \text{ for } k = 1, \ldots, n\}. \quad (3.3.10)$$

For every $y \in PC^1(J, \mathbb{R}^+)$, we define

$$\|y\|_{PC^1} = \max\left\{\sup_{t \in J} |y(t)|, \sup_{t \in J} |y'(t)|\right\}.$$

Consider the Banach space $X \triangleq PC(J, \mathbb{R}^+) \times PC(J, \mathbb{R}^+)$ equipped with the norm $\|(u_1, u_2)\| = (\|u_1\|_{PC}, \|u_2\|_{PC})$, for $(u_1, u_2) \in X$.
In this section, we shall present some auxiliary results, related to the following problem of second order differential equations with three points boundary conditions

$$u_1''(t) + y_1(t) = 0,\ t \in J', \quad (3.3.11)$$

$$\Delta u_1(t_k) = I_{1,k}(u_1(t_k)),\ k = 1, 2, \cdots, m, \quad (3.3.12)$$

$$\Delta u_1'(t_k) = -J_{1,k}(u_1(t_k)),\ k = 1, 2, \cdots, m, \quad (3.3.13)$$

$$\alpha u_1(0) - \beta u_1'(0) = a u_1(\xi),\ u_1(1) = 0, \quad (3.3.14)$$

Lemma 3.3.1. *Let* $u_1 \in PC^1(J, \mathbb{R}^+)$ *and* $\Delta = (\alpha - a) + \beta + a\xi$ *with* $\alpha > a$. *If* $y(t) \in C(J, \mathbb{R}^+)$, *then* u_1 *is a solution of the problem* (3.3.11)-(3.3.14) *if and only if*

$$u_1(t) = \int_0^1 G(t,s)y_1(s)ds + W_1(t,u_1), \tag{3.3.15}$$

where

$$G(t,s) = \frac{1}{\Delta}\begin{cases} (1-t)(\beta + \alpha s), & s< \xi,\ s<t, \\ (\beta + \alpha t)(1-s) + a(s-t)(1-\xi), & t\leq s \leq \xi, \\ (1-t)(\beta + a\xi + (\alpha - a)s), & \xi \leq s \leq t, \\ (1-s)(\beta + a\xi + (\alpha - a)t), & \xi < s,\ t<s, \end{cases} \tag{3.3.16}$$

and

$$\begin{aligned} W_1(t,u_1) = \quad & \frac{a}{\Delta}\sum_{t_k<\xi}(1-t)\left[I_{1,k}(u_1(t_k)) - (\xi - t_k)J_{1,k}(u_1(t_k))\right] \\ + \quad & \frac{1}{\Delta}\sum_{t_k<t}(1-t)\left[(\alpha - a)I_{1,k}(u_1(t_k) + (\beta + a\xi + (\alpha - a)t_k)J_{1,k}(u_1(t_k))\right] \\ + \quad & \frac{1}{\Delta}\sum_{t_k>t}(\beta + a\xi + (\alpha - a)t)\left[-I_{1,k}(u_1(t_k)) + (1-t_k)J_{1,k}(u_1(t_k))\right]. \end{aligned}$$

Proof. First Suppose that $u_1 \in PC^1(J, \mathbb{R}^+)$ is a solution of problem (3.3.11)-(3.3.14).
It is easy to see by integration of (3.3.11) that one have

$$u_1'(t) = u_1'(0) - \int_0^t y_1(s)ds - \sum_{0<t_k<t} J_{1,k}(u_1(t_k)). \tag{3.3.17}$$

Integrating again, we can get

$$\begin{aligned} u_1(t) = u_1(0) + u_1'(0)t - \int_0^t (t-s)y_1(s)ds - \sum_{0<t_k<t} J_{1,k}(u_1(t_k))(t-t_k) \\ + \sum_{0<t_k<t} I_{1,k}(u_1(t_k)). \end{aligned} \tag{3.3.18}$$

By $u(1) = 0$ and (3.3.18), we have

$$u_1(0) + u_1'(0) = \int_0^1 (1-s) y_1(s) ds - \sum_{k=1}^{m} I_{1,k}(u_1(t_k)) + \sum_{k=1}^{m} (1-t_k) J_{1,k}(u_1(t_k)). \quad (3.3.19)$$

It follows from (3.3.18) and $\alpha u_1(0) - \beta u_1'(0) = a u_1(\xi)$ that

$$(\alpha - a) u_1(0) - (\beta + a\xi) u_1'(0) = -a \int_0^{\xi} (\xi - s) y_1(s) ds - a \sum_{t_k < \xi} (\xi - t_k) J_{1,k}(u_1(t_k)) + a \sum_{t_k < \xi} I_{1,k}(u_1(t_k)) \quad (3.3.20)$$

By (3.3.19) and (3.3.20), we have

$$\begin{aligned} u_1(0) &= \frac{-a \int_0^{\xi} (\xi - s) y_1(s) ds - a \sum_{t_k < \xi} (\xi - t_k) J_{1,k}(u_1(t_k)) + a \sum_{t_k < \xi} I_{1,k}(u_1(t_k))}{(\alpha - a) + \beta + a\xi} \\ &+ \frac{\int_0^1 (1-s)(\beta + a\xi) y_1(s) ds - \sum_{k=1}^{m} (\beta + a\xi) I_{1,k}(u_1(t_k))}{(\alpha - a) + \beta + a\xi} \\ &+ \frac{\sum_{k=1}^{m} (\beta + a\xi)(1 - t_k) J_{1,k}(u_1(t_k))}{(\alpha - a) + \beta + a\xi}. \end{aligned}$$

and

$$u_1'(0) = \frac{\int_0^1(\alpha-a)(1-s)y_1(s)ds+\sum_{k=1}^{m}(\alpha-a)(1-t_k)J_{i,k}(u_1(t_k))}{(\alpha-a)+\beta+a\xi}$$
$$-\frac{\sum_{k=1}^{m}(\alpha-a)I_{i,k}(u_1(t_k))a\int_0^{\xi}(\xi-s)y_1(s)ds}{(\alpha-a)+\beta+a\xi}$$
$$+\frac{a\sum_{t_k<\xi}(\xi-t_k)J_{1,k}(u_1(t_k))-a\sum_{t_k<\xi}I_{1,k}(u_1(t_k))}{(\alpha-a)+\beta+a\xi}.$$

So, we get

$$\begin{aligned}
u_1(t) &= -\frac{a}{\Delta}\int_0^{\xi}(1-t)(\xi-s)y_1(s)ds+\frac{1}{\Delta}\int_0^1(1-s)[\beta+a\xi+(\alpha-a)t]y_1(s)ds \\
&\quad -\int_0^t(t-s)y_1(s)ds+\frac{a}{\Delta}\sum_{t_k<\xi}(1-t)\left[I_{1,k}(u_1(t_k))-(\xi-t_k)J_{1,k}(u_1(t_k))\right] \\
&\quad +\frac{1}{\Delta}\sum_{t_k<t}(1-t)\left[(\alpha-a)I_{1,k}(u_1(t_k))+(\beta+a\xi+(\alpha-a)t_k)J_{1,k}(u_1(t_k))\right] \\
&\quad +\frac{1}{\Delta}\sum_{t_k>t}(\beta+a\xi+(\alpha-a)t)\left[-I_{1,k}(u_1(t_k))+(1-t_k)J_{1,k}(u_1(t_k))\right] \\
&= -\frac{a}{\Delta}\int_0^{\xi}(1-t)(\xi-s)y_1(s)ds+\frac{1}{\Delta}\int_0^1(1-s)[\beta+a\xi+(\alpha-a)t]y_1(s)ds \\
&\quad -\int_0^t(t-s)y_1(s)ds+W_1(t,u_1).
\end{aligned}$$

For $t\le\xi$

$$\begin{aligned}
u_1(t) &= \int_0^t\frac{(1-t)(\beta+\alpha s)}{\Delta}y_1(s)ds+\int_t^{\xi}\frac{(\beta+\alpha t)(1-s)+a(s-t)(1-\xi)}{\Delta}y_1(s)ds \\
&\quad +\int_{\xi}^1\frac{(1-s)[(\alpha-a)t+\beta+a\xi]}{\Delta}y_1(s)ds+W_1(t,u_1).
\end{aligned}$$

For $t > \xi$

$$u_1(t) = \int_0^t \frac{(1-t)(\beta+\alpha s)}{\Delta} y_1(s)ds + \int_\xi^t \frac{(1-t)[(\alpha-a)s+(\beta+a\xi)]}{\Delta} y_1(s)ds + \int_t^1 \frac{(1-s)[(\alpha-a)t+\beta+a\xi]}{\Delta} y_1(s)ds + W_1(t,u_1).$$

Thus

$$u_1(t) = \int_0^1 G(t,s)y_1(s)ds + W_1(t,u_1).$$

□

Lemma 3.3.2. *For all $(t,s) \in J^2$, we have*

$$0 \leq G(t,s) \leq G(s,s).$$

Proof. From the definitions of $G(t,s)$, it is easy to obtain that $G(t,s) \geq 0$ for all $(t,s) \in J^2$.
For $s < \xi$, $s < t$

$$\frac{\partial G(t,s)}{\partial t} = -\frac{\beta+\alpha s}{\Delta} \leq 0.$$

Therefore, $G(t,s)$ is decreasing with respect to t, which implies that $G(t,s) \leq G(s,s)$. Now, for $t \leq s \leq \xi$

$$\frac{\partial G(t,s)}{\partial t} = \frac{\alpha(1-s)-a(1-\xi)}{\Delta} \geq 0.$$

Therefore, $G(t,s)$ is increasing with respect to t, which implies that $G(t,s) \leq G(s,s)$. For $t \leq s \leq \xi$

$$\frac{\partial G(t,s)}{\partial t} = -\frac{\beta+\alpha\xi+(\alpha-a)s}{\Delta} \leq 0.$$

Therefore, $G(t,s)$ is decreasing with respect to t, which implies that $G(t,s) \leq G(s,s)$.

For $\xi < s,\ t < s$

$$\frac{\partial G(t,s)}{\partial t} = \frac{(1-s)(\alpha-a)s}{\Delta} \geq 0.$$

Therefore, $G(t,s)$ is increasing with respect to t, which implies that $G(t,s) \leq G(s,s)$.
Thus, we have

$$G(t,s) \leq G(s,s),\ for\ all\ (t,s) \in J^2.$$

□

Lemma 3.3.3. *Let* $\delta \in]0, \frac{1}{2}[$, $J_\delta = [\delta, 1-\delta]$, *then for all* $t \in J_\delta$, $s \in J$, *we have*

$$G(t,s) \geq \eta G(s,s),$$

where $\eta = min\{\delta, \frac{\beta}{\alpha+\beta}\}$.

Proof. For $s < \xi$, $s < t$ and $t \in J_\delta$. Then

$$\frac{G(t,s)}{G(s,s)} \geq \frac{1-t}{1-s} \geq 1-t \geq \delta.$$

Let $t \leq s \leq \xi$ and $t \in J_\delta$. Then

$$\begin{aligned} \frac{G(t,s)}{G(s,s)} &= \frac{(\beta+\alpha t)(1-s) + a(s-t)(1-\xi)}{(\beta+\alpha s)(1-s)} \\ &= \frac{\beta+\alpha t}{\beta+\alpha s} + \frac{a(s-t)(1-\xi)}{(\beta+\alpha s)(1-s)} \\ &\geq \frac{\beta+\alpha t}{\beta+\alpha} \geq \frac{\beta}{\beta+\alpha}. \end{aligned}$$

For $\xi \leq s \leq t$ and $t \in J_\delta$

$$\frac{G(t,s)}{G(s,s)} \geq \frac{1-t}{1-s} \geq 1-t \geq \delta.$$

For $\xi < s$, $t < s$ and $t \in J_\delta$

$$\frac{G(t,s)}{G(s,s)} = \frac{\beta + a\xi + (\alpha - a)t}{\beta + a\xi + (\alpha - a)s} \geq \frac{\beta}{\beta+\alpha}.$$

Therefore

$$G(t,s) \geq \eta G(s,s)\ for\ (t,s) \in J \times J_\delta,$$

where $\eta = min\{\delta, \frac{\beta}{\alpha+\beta}\}$. □

We can also formulate similar results as Lemma 3.3.1-Lemma 3.3.3 above for the boundary value problem

$$u_2''(t) + y_2(t) = 0, \quad t \in J', \tag{3.3.21}$$
$$\Delta u_2' \mid_{t_k} = I_{2,k} u_1(t_k), \quad k = 1, 2, \cdots, m, \tag{3.3.22}$$
$$\Delta u_2' \mid_{t_k} = -J_{2,k} u_1(t_k), \quad k = 1, 2, \cdots, m, \tag{3.3.23}$$
$$\alpha u_2(0) - \beta u_2'(0) = a u_2(\xi), \quad u_2(1) = 0. \tag{3.3.24}$$

3.3.2 Existence of multiple positive solutions

In this section, we show the existence of at least three positive solutions for the systems (3.1.1)-(3.1.6) follows from vector version of fixed point theorem of Avery and Peterson.
We present the assumptions that we shall use in the sequel.

(A_1) The functions h_1, h_2:$J \mapsto \mathbb{R}^+$ are continuous.

(A_2) The functions f_1, f_2:$J \times \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ are continuous.

(A_3) The functions $I_{i,k}$, $J_{i,k}$:$\mathbb{R}^+ \mapsto \mathbb{R}^+$ are continuous for $i = 1, 2$, $k = 1, 2, ..., m$.

(A_4) There exists constants c_1, $c_2 \in]0, 1[$ and the functions $\Omega_i : \{u_i : u_i \in PC(J, \mathbb{R}),\ u_1 \geq 0\} \mapsto \mathbb{R}$ for $i \in \{1, 2\}$ such that

$$c_1\Omega_1(u_1) \leq W_1(t, u_1) \leq \Omega_1(u_1),\ (t, u_1) \in J \times \{u_1 : u_1 \in PC(J, \mathbb{R}),\ u_1 \geq 0\},$$

and

$$c_2\Omega_2(u_2) \leq W_2(t, u_2) \leq \Omega_2(u_2),\ (t, u_2) \in J \times \{u_2 : u_2 \in PC(J, \mathbb{R}),\ u_2 \geq 0\},$$

Define a cone $P \subseteq X$ by

$$P = \Big\{ (u_1, u_2) \in X : u_1(t) \geq 0,\ u_2(t) \geq 0, t \in J, \\ \inf_{t \in J_\delta} u_1(t) \geq \lambda_1 \|u_1\|_{PC} \text{ and } \inf_{t \in J_\delta} u_2(t) \geq \lambda_2 \|u_2\|_{PC} \Big\},$$

where $\lambda_1 = min\{\eta, c_1\}$, $\lambda_2 = min\{\eta, c_2\}$.
Define the nonnegative continuous convex functionals coupled (x_1, x_2), (θ_1, θ_2) and (ψ_1, ψ_2) and the nonnegative continuous concave functional coupled (ϕ_1, ϕ_2) on P by

$$(\phi_1, \phi_2)(u_1, u_2) = \left(\inf_{t \in J_\delta} u_1(t), \inf_{t \in J_\delta} u_2(t) \right),$$

$$(\psi_1, \psi_2)(u_1, u_2) = \left(\sup_{t \in J_\delta} |u_1(t)|, \sup_{t \in J_\delta} |u_2(t)| \right)$$

and

$$(\theta_1, \theta_2)(u_1, u_2) = (x_1, x_2)(u_1, u_2) = \|(u_1, u_2)\|.$$

Theorem 3.3.4. *Suppose that* (A_1)*-*(A_4) *hold. In addition, we assume that there exist positive vectors* (μ_1, μ_2)*,* (a'_1, a'_2)*,* (b'_1, b'_2)*,* $\left(\frac{b'_1}{\lambda_1}, \frac{b'_2}{\lambda_2}\right)$ *and* (d'_1, d'_2)*, with*
$a'_1 < b'_1 < \frac{b'_1}{\lambda_1} < d'_1$ *and* $a'_2 < b'_2 < \frac{b'_2}{\lambda_2} < d'_2$*,* $\mu_1 > D_1 + D'_1$ *and* $\mu_2 > D_2 + D'_2$*,* $0 < L_1 < \lambda_1(D_1 + D'_1)$ *and* $0 < L_2 < \lambda_2(D_2 + D'_2)$*, where*
$D_1 = \int_0^1 h_1(s)G(s,s)ds$ *and* $D_2 = \int_0^1 h_2(s)G(s,s)ds$*,* D_1*,* D_2*,* Δ'_1 *and* $D'_2 > 0$*, such that the following conditions hold:*

(B_1) $f_i(t, u_1(t), u_2(t)) \leq \frac{d'_i}{\mu_i}$ *for* $t \in J$ *and* $(u_1, u_2) \in [0, d'_i]$ *and*
$W_i(t, u_i) \leq \frac{D'_i}{\mu_i} d'_i$*, for* $(u_1, u_2) \in P$*,* $\|(u_1, u_2)\| \leq (d'_1, d'_2)$ *for* $i \in \{1, 2\}$*;*

(B_2) $f_i(t, u_1(t), u_2(t)) \geq \frac{b'_i}{L_i}$ *for* $t \in J$ *and* $(u_1, u_2) \in \left[b'_i, \frac{b'_i}{\lambda_i}\right]$ *and*
$W_i(t, u_i) \geq \frac{D'{}_i}{L_i} b'_i$*, for* $(u_1, u_2) \in P$*,* $b'_i \leq u_i \leq \frac{b'_i}{\lambda_i}$ *for* $i \in \{1, 2\}$ *;*

(B_3) $f_i(t, u_1(t), u_2(t)) \leq \frac{a'_i}{\mu_i}$*, for* $t \in J$ *and* $(u_1, u_2) \in [0, a'_i]$
and $W_i(t, u_i) \leq \frac{D'_i}{\mu_i} a'_i$*, for* $(u_1, u_2) \in P$*,* $\|(u_1, u_2)\| \leq (a'_1, a'_2)$ *for* $i \in \{1, 2\}$ *.*

Then the problem (3.1.1)-(3.1.6) *has at least two positive solutions when* $f_i(t, 0, 0) \equiv 0, t \in J$, *and at least three positive solutions when* $f_i(t, 0, 0) \not\equiv 0$, $t \in J$.

Proof. Consider the operator

$$T : P \mapsto X$$

defined by

$$T(u_1, u_2)(t) = (T_1(u_1, u_2)(t), T_2(u_1, u_2)(t)),$$

where

$$T_1(u_1, u_2)(t) = \int_0^1 G(t, s)h_1(s)f_1(s, u_1(s), u_2(s))ds + W_1(t, u_1),$$

and

$$T_2(u_1, u_2)(t) = \int_0^1 G(t, s)h_2(s)f_2(s, u_1(s), u_2(s))ds + W_2(t, u_2).$$

It is obvious that a fixed point of T is the solution of the second order impulsive differential equations (3.1.1)-(3.1.6). Three fixed points of T are sought.
First, it is shown that $T : P \mapsto P$
Let $(u_1, u_2) \in P$ be an arbitrary element.

$$|T_1(u_1, u_2)(t)| \leq \int_0^1 G(s, s)h_1(s)f_1(s, u_1(s), u_2(s))ds + \Omega_1(u_1),$$

$$|T_2(u_1, u_2)(t)| \leq \int_0^1 G(s, s)h_2(s)f_2(s, u_1(s), u_2(s))ds + \Omega_2(u_2),$$

$$\begin{aligned} \inf_{t\in J_\delta} T_1(u_1, u_2)(t) &= \inf_{t\in J_\delta} \left[\int_0^1 G(t, s)h_1(s)f_1(s, u_1(s), u_2(s))ds + W_1(t, u_1)\right] \\ &\geq \eta \int_0^1 G(s, s)h_1(s)f_1(s, u_1(s), u_2(s))ds + c_1\Omega_1(u_1) \\ &\geq min\{\eta, c_1\} \left[\int_0^1 G(s, s)h_1(s)f_1(s, u_1(s), u_2(s))ds + \Omega_1(u_1)\right] \\ &\geq \lambda_1 \|T_1(u_1, u_2)\|_{PC}. \end{aligned}$$

where $\lambda_1 = min\{\eta, c_1\}$.
Similarly

$$\inf_{t\in J_\delta} T_2(u_1,u_2)(t) \geq \lambda_2 \|T_2(u_1,u_2)\|_{PC}.$$

where $\lambda_2 = min\{\eta, c_2\}$.
By (A_1), (A_2) and (A_4), we have $T_1(u_1,u_2)(t) \geq 0$ and $T_2(u_1,u_2)(t) \geq 0$, for $t \in [0,1]$.
It is shown that $T : \overline{P((x_1,x_2),(d_1,d_2))} \mapsto \overline{P((x_1,x_2),(d_1,d_2))}$
Let $(u_1,u_2) \in \overline{P((x_1,x_2),(d_1,d_2))}$, condition (B_1) is used to obtain

$$\begin{aligned} x_1(T(u_1,u_2)(t)) &= \|T_1(u_1,u_2)\|_{PC} \\ &\leq \int_0^1 G(s,s)h_1(s)f_1(s,u_1(s),u_2(s))ds + \frac{D_1'}{\mu_1}d_1' \\ &\leq \frac{d_1'}{\mu_1}\int_0^1 G(s,s)h_1(s)ds + \frac{D_1'}{\mu_1}d_1' \\ &\leq \frac{D_1}{\mu_1}d_1' + \frac{D_1'}{\mu_1}d_1' \\ &< d_1'. \end{aligned}$$

Similarly,we have

$$x_2(T(u_1,u_2)(t)) < d_2'.$$

Now conditions (S_1) of Theorem 1.4.15 are to be verified. It is obvious that

$$\left(\frac{b_1'(\lambda_1+1)}{2\lambda_1}, \frac{b_2'(\lambda_2+1)}{2\lambda_2}\right) \in \Big\{(u_1,u_2) \in P\left((x_1,x_2),(\theta_1,\theta_2),(\phi_1,\phi_2),(b_1',b_2'),\left(\frac{b_1'}{\lambda_1},\frac{b_2'}{\lambda_2}\right),(d_1,d_2)\right) : (\phi_1,\phi_2)(u_1,u_2) > (b_1,b_2)\Big\} \neq \emptyset$$

Next let $(u_1,u_2) \in P\left((x_1,x_2),(\theta_1,\theta_2),(\phi_1,\phi_2),(b_1',b_2'),\left(\frac{b_1'}{\lambda_1},\frac{b_2'}{\lambda_2}\right),(d_1,d_2)\right)$

Then $b'_1 \leq u_1 \leq \dfrac{b'_1}{\lambda_1}$ and $b'_2 \leq u_2 \leq \dfrac{b'_2}{\lambda_2}$, $t \in J$. By (B_2)

$$\begin{aligned}
\phi_1(T(u_1,u_2)(t)) &= \inf_{t\in J_\delta} T_1(u_1,u_2) \\
&\geq \lambda_1\left[\int_0^1 G(s,s)h_1(s)f_1(s,u_1(s),u_2(s))ds + \frac{b'_1}{L_1}\bar{D}_1\right] \\
&\geq \frac{b'_1}{L_1}\int_0^1 G(s,s)h_1(s)ds + \frac{D'_1}{L_1}b'_1 \\
&\geq \lambda_1\frac{b'_1}{L_1}(D_1 + D'_1) \\
&> b'_1.
\end{aligned}$$

Similarly,we have

$$\phi_2(T(u_1,u_2)(t)) > b'_2.$$

Then

$$(\phi_1,\phi_2)(T(u_1,u_2)(t)) > (b'_1,b'_2).$$

So, condition (S_1) of Theorem 1.4.15 is holds.
To see that condition (S_2) of Theorem 1.4.15 is satisfied. Let $(u_1,u_2) \in P\left((x_1,x_2),(\phi_1,\phi_2),(b'_1,b'_2),(d'_1,d'_2)\right)$ with $(\theta_1,\theta_1)(u_1,u_2) > \left(\dfrac{b'_1}{\lambda_1},\dfrac{b'_2}{\lambda_2}\right)$

$$\begin{aligned}
\phi_1(T(u_1,u_2)(t)) &= \inf_{t\in J_\delta} T_1(u_1,u_2)(t) \\
&\geq \lambda_1\|T_1(u_1,u_2)\|_{PC} \\
&\geq \lambda_1\frac{b'_1}{\lambda_1} = b'_1.
\end{aligned}$$

Similarly,we obtain

$$\phi_2(T(u_1,u_2)(t)) > b'_2.$$

Then

$$(\phi_1,\phi_2)(T(u_1,u_2)(t)) > (b'_1,b'_2)$$

Finally, it is shown that the condition (S_3) of Theorem 1.4.15 holds. Since $(\psi_1,\psi_2)(0,0) = (0,0)$, $(a'_1,a'_2) > (0,0)$, $(0,0) \notin R((x_1,x_2),(\psi_1,\psi_2),(a'_1,a'_2),(d'_1,d'_2))$.
Assume that $(u_1,u_2) \in R((x_1,x_2),(\psi_1,\psi_2),(a'_1,a'_2),(d'_1,d'_2))$ with $(\psi_1,\psi_2)(u_1,u_2) = (a'_1,a'_2)$.

$$\begin{aligned}
\psi_1(T(u_1,u_2)(t)) &= \sup_{t\in J_\delta}|T_1(u_1,u_2)(t)| \\
&= \sup_{t\in J_\delta}\left[\int_0^1 G(t,s)h_1(s)f_1(s,u_1(s),u_2(s))ds + W_1(t,u_1)\right] \\
&\leq \sup_{t\in J}\left[\int_0^1 G(t,s)h_1(s)f_1(s,u_1(s),u_2(s))ds + W_1(t,u_1)\right] \\
&\leq \int_0^1 G(s,s)h_1(s)f_1(s,u_1(s),u_2(s))ds + \frac{D_1'}{\mu_1}a_1' \\
&\leq \frac{a_1'}{\mu_1}(D_1+D_1') < a_1'.
\end{aligned}$$

Similarly,we obtain

$$\psi_2(T(u_1,u_2)(t)) < a_2'.$$

Then

$$(\psi_1,\psi_2)(T(u_1,u_2)(t)) < (a_1',a_2').$$

It has been proved that all the conditions of Theorem 1.4.15 are satisfied. Therefore, the system (3.1.1)-(3.1.6) has at least three solutions, (x_1,x_2), (y_1,y_2), $(z_1,z_2)\in P$ such that

$$\|(x_1,x_2)\| \leq (d_1',d_2'),\ \|(y_1,y_2)\| \leq (d_1',d_2'),\ \|(z_1,z_2)\| \leq (d_1',d_2'),$$

and

$$(b_1',b_2') < \left(\inf_{t\in J_\delta}|x_1(t)|, \inf_{t\in J_\delta}|x_2(t)|\right),\ (a_1',a_2') \leq \left(\sup_{t\in J_\delta}|x_1(t)|, \sup_{t\in J_\delta}|x_2(t)|\right),$$

$$\left(\inf_{t\in J_\delta}|y_1(t)|, \inf_{t\in J_\delta}|y_2(t)|\right) < (b_1',b_2'),\ \|(z_1,z_2)\| < (a_1',a_2').$$

Obviously, $(x_1,x_2)(t) > (0,0)$, $(y_1,y_2)(t) > (0,0)$, $t\in[0,1]$. If $f_i(t,0,0)\not\equiv 0$ for $i\in\{1,2\}$, $t\in[0,1]$, then the vector $(u_1,u_2)=(0,0)$ is not a solution of a systems (3.1.1)-(3.1.6). So, the vector $(z_1,z_2)\neq(0,0)$. This, together with $(z_1,z_2)\in P$, means that $(z_1,z_2) > (0,0)$, $t\in[0,1]$. □

Chapter **4**

Systems of impulsive differential equations on un-bounded domain

In this part, we provide sufficient conditions for the existence of solutions for the systems of second-order impulsive differential equations with integral boundary conditions :

$$-u''(t) = f(t, u(t), v(t)), \quad t \in J, \ t \neq t_k, \tag{4.0.1}$$

$$-v''(t) = g(t, u(t), v(t)), \quad t \in J, \ t \neq t_k, \tag{4.0.2}$$

$$\Delta u(t_k) = J_{1,k}(u(t_k)), \quad -\Delta u'(t_k) = I_{1,k}(u(t_k)), \quad k = 1, 2, \cdots, \tag{4.0.3}$$

$$\Delta v(t_k) = J_{2,k}(v(t_k)), \quad -\Delta v'(t_k) = I_{2,k}(v'(t_k)), \quad k = 1, 2, \cdots, \tag{4.0.4}$$

$$u(0) = \int_0^\infty h_1(s)u(s)ds, \quad u'(\infty) = 0, \tag{4.0.5}$$

$$v(0) = \int_0^\infty h_2(s)v(s)ds, \quad v'(\infty) = 0, \tag{4.0.6}$$

where $J = [0, +\infty)$, $f, g \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $0 < t_1 < t_2 < \cdots < t_k < \cdots$, $t_k \to \infty$, $I_{i,k}$, $J_{i,k} \in C(\mathbb{R}, \mathbb{R})$, for $i = 1, 2$, $h_i \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\displaystyle\int_0^\infty h_i(s)ds \neq 1$ for $i = 1, 2$, $u'(\infty) = \lim_{t\to\infty} u(t)$ and $v'(\infty) = \lim_{t\to\infty} v(t)$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$ and $\Delta v(t_k) = v(t_k^+) - v(t_k^-)$, where $u(t_k^+)$ $(v(t_k^+))$ and $u(t_k^-)$ $(v(t_k^-))$ represent the righ and left hand limit of $u(t)$ $(v(t))$ at $t = t_k$, respectively. $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$ and $\Delta v'(t_k) = v'(t_k^+) - v'(t_k^-)$, where $u'(t_k^+)$ $(v'(t_k^+))$ and $u'(t_k^-)$ $(v'(t_k^-))$ represent the righ and left hand limit of $u'(t)$ $(v'(t))$ at $t = t_k$, respectively.

Since we are interested here in systems of equations, we have opted for a vectorial approach based on the use of vector-valued norms, inverse-positive matrices and of a vectorial version of Krasnoselskii's fixed point theorem for sums of two operators [116].

4.1 Main result

Before seating the result of this section we consider the following spaces.

$$PC([0,+\infty)) = \{u : \ [0,+\infty[\to \mathbb{R} \ |u(t) \text{ is continuos at each } t \neq t_k,$$

$$\text{left continous at } t = t_k,\ u'(t_k^+) \text{ exists},\ k = 1,2,\cdots,\}.$$

Consider the space E defined by

$$E = \left\{ u \in PC([0,+\infty)),\ \sup_{t\in[0,+\infty)} \frac{|u(t)|}{1+t} < \infty \right\},$$

E is a Banach space, equipped with the norm $\|u\|_E = \sup\limits_{t\in[0,+\infty)} \dfrac{|u(t)|}{1+t} < \infty.$ Then $E \times E$ is a Banach space with the norm $\|(u,v)\| = (\|u\|_E, \|v\|_E)$ for $(u,v) \in E \times E$.

Lemma 4.1.1. *The vector* $(u,v) \in PC([0,\infty)) \times PC([0,\infty))$ *is a solution of differential system* (4.0.1)-(4.0.6) *if and only if*

$$\begin{aligned} u(t) &= \int_0^\infty H_1(t,s) f(s,u(s),v(s))ds + \sum_{k=1}^{\infty} H_1(t,t_k) I_{1.k}(u(t_k)) + \sum_{t_k<t} J_{1.k}(u(t_k)) \\ &+ \frac{\displaystyle\int_0^\infty h_1(s) \left(\sum_{t_k<s} J_{1.k}(u(t_k)) \right) ds}{1 - \displaystyle\int_0^\infty h_1(s)ds}, \end{aligned}$$

$$v(t) \;=\; \int_0^\infty H_2(t,s)g(s,u(s),v(s))ds + \sum_{k=1}^{\infty} H_2(t,t_k)I_{1.k}(u(t_k)) + \sum_{t_k<t} J_{2.k}(v(t_k))$$
$$+\frac{\displaystyle\int_0^\infty h_2(s)\left(\sum_{t_k<s} J_{2.k}(v(t_k))\right)ds}{1-\displaystyle\int_0^\infty h_2(s)ds}.$$

where for $i=1,2$

$$H_i(t,s) = G(t,s) + \frac{1}{1-\int_0^\infty h_i(s)ds}\int_0^\infty G(\tau,s)h_i(\tau)d\tau$$

$$G(t,s) = \begin{cases} t, & 0\leq\ t\leq s\leq\infty, \\ s, & 0\leq s\leq t\leq\ \infty. \end{cases} \tag{4.1.1}$$

Proof. First we consider the following problems:

$$-u''(t) = f(t,u(t),v(t)), \quad t\in J,\ t\neq t_k, \tag{4.1.2}$$
$$\Delta u(t_k) = J_{1,k}(u(t_k)), \quad -\Delta u'(t_k) = I_{1,k}(u(t_k)), k=1,2,\cdots, \tag{4.1.3}$$
$$u(0) = \int_0^\infty h_1(s)u(s)ds, \quad u'(\infty)=0. \tag{4.1.4}$$

Let u be a solution of the problem (4.1.8)-(4.1.4), then by integration we have

$$u'(t) = u'(0) - \int_0^t f(s,u(s),v(s))ds - \sum_{t_k<t} I_{1,k}(u(t_k)), \tag{4.1.5}$$

Taking limit for $t\to\infty$,

$$u'(0) = \int_0^\infty f(s,u(s),v(s))ds + \sum_{k=1}^{\infty} I_{1,k}(u(t_k)),$$

Integrating (4.1.5), we can get

$$u(t) = u'(0)t + u(0) - \int_0^t (t-s)f(s,u(s),v(s))ds - \sum_{t_k<t} I_{1,k}(u(t_k))(t-t_k) + \sum_{t_k<t} J_{1,k}(u(t_k)),$$

Thus

$$u(t) = u(0) + \int_0^\infty tf(s,u(s),v(s))ds + \sum_{k=1}^\infty tI_{1,k}(u(t_k)) - \int_0^t (t-s)f(s,u(s),v(s))ds$$
$$- \sum_{t_k<t} I_{1,k}(u(t_k))(t-t_k) + \sum_{t_k<t} J_{1,k}(u(t_k)).$$

Thus

$$u(t) = u(0) + \int_0^\infty G(t,s)f(s,u(s),v(s))ds + \sum_{k=1}^\infty G(t,t_k)I_{1,k}(u(t_k)) + \sum_{t_k<t} J_{1,k}(u(t_k)).$$

Then

$$u(t) = \int_0^\infty h_1(s)u(s)ds + \int_0^\infty G(t,s)f(s,u(s),v(s))ds + \sum_{k=1}^\infty G(t,t_k)I_{1,k}(u(t_k)) + \sum_{t_k<t} J_{1,k}(u(t_k)). \quad (4.1.6)$$

Thus

$$\int_0^\infty h_1(s)u(s)ds = \int_0^\infty h_1(s)\left(\int_0^\infty h_1(s)u(s)ds + \int_0^\infty G(s,\tau)f(\tau,u(\tau),v(\tau))d\tau\right)ds$$
$$+ \int_0^\infty h_1(s)\left(\sum_{k=1}^\infty G(s,t_k)I_{1,k}(u(t_k)) + \sum_{t_k<s} J_{1,k}(u(t_k))\right)ds.$$

It follows that

$$\int_0^\infty h_1(s)u(s)ds = \frac{1}{1-\int_0^\infty h_1(s)ds}\left(\int_0^\infty\int_0^\infty h_1(s)G(s,\tau)f(\tau,u(\tau),v(\tau))d\tau\right)$$
$$+\frac{1}{1-\int_0^\infty h_1(s)ds}\int_0^\infty h_1(s)\left(\sum_{k=1}^\infty G(s,t_k)I_{1,k}(u(t_k))\right)ds$$
$$+\frac{1}{1-\int_0^\infty h_1(s)ds}\int_0^\infty h_1(s)\left(\sum_{t_k<s} J_{1,k}(u(t_k))\right)ds.$$

Substituting in (4.1.6) we have

$$u(t) = \int_0^\infty G(t,s)f(s,u(s),v(s))ds + \frac{\int_0^\infty \int_0^\infty h_1(s)G(s,\tau)f(\tau,u(\tau),v(\tau))d\tau}{1-\int_0^\infty h_1(s)d\tau ds}$$
$$+\sum_{k=1}^{\infty} G(t,t_k)I_{1,k}(u(t_k)) + \frac{\int_0^\infty h_1(s)\left(\sum_{k=1}^{\infty} G(t,t_k)I_{1,k}(u(t_k))\right)ds}{1-\int_0^\infty h_1(s)ds}$$
$$+\sum_{t_k<t} J_{1,k}(u(t_k)) + \frac{\int_0^\infty h_1(s)\left(\sum_{t_k<t} J_{1,k}(u(t_k))\right)ds}{1-\int_0^\infty h_1(s)ds}.$$

Then

$$u(t) = \int_0^\infty H_1(t,s)f(s,u(s),v(s))ds + \sum_{k=1}^{\infty} H_1(t,t_k)I_{1.k}(u(t_k)) + \sum_{t_k<t} J_{1.k}(u(t_k))$$
$$+\frac{\int_0^\infty h_1(s)\left(\sum_{t_k<s} J_{1.k}(u(t_k))\right)ds}{1-\int_0^\infty h_1(s)ds},$$

where

$$H_1(t,s) = G(t,s) + \frac{1}{1-\int_0^\infty h_1(s)ds}\int_0^\infty G(\tau,s)h_1(\tau)d\tau,$$

$$G(t,s) = \begin{cases} t, & 0\leq t\leq s\leq\infty, \\ s, & 0\leq s\leq t\leq \infty. \end{cases} \tag{4.1.7}$$

Next, we consider the following problem

$$-v''(t) = f(t,u(t),v(t)), \quad t\in J,\ t\neq t_k, \tag{4.1.8}$$
$$\Delta v(t_k) = J_{2,k}(v(t_k)), \quad \Delta v'(t_k) = -I_{2,k}(v(t_k)),\ k=1,2,\cdots, \tag{4.1.9}$$
$$v(0) = \int_0^\infty h_2(s)v(s)ds, \quad v'(\infty)=0, \tag{4.1.10}$$

similarly, we have that

$$v(t) = \int_0^\infty H_2(t,s)f(s,u(s),v(s))ds + \sum_{k=1}^{\infty} H_2(t,t_k)I_{1.k}(u(t_k)) + \sum_{t_k<t} J_{2.k}(u(t_k)) + \frac{\displaystyle\int_0^\infty h_2(s)\left(\sum_{t_k<s} J_{2.k}(u(t_k))\right)ds}{1-\displaystyle\int_0^\infty h_2(s)ds},$$

where

$$H_2(t,s) = G(t,s) + \frac{1}{1-\displaystyle\int_0^\infty h_2(s)ds}\int_0^\infty G(\tau,s)h_2(\tau)d\tau.$$

□

Set $h_1^* = \left|1-\displaystyle\int_0^\infty h_1(s)ds\right|$ and $h_2^* = \left|1-\displaystyle\int_0^\infty h_2(s)ds\right|$.

To establish our main result concerning existence of solution (4.0.1)-(4.0.6), we use the assumptions

(H_1) f, g are L^1-Carathéodory functions.

(H_2) There exist nonnegative functions $P_i, \bar{P}_i \in L^1[0,+\infty)$ for $i=1,2,3$ such that:

$$|f(t,u,v)| \le P_1(t)|u| + P_2(t)|v| + P_3(t), \text{for each } t\in J, (u,v)\in\mathbb{R}^2,$$

and

$$|g(t,u,v)| \le \bar{P}_1(t)|u| + \bar{P}_2(t)|u| + \bar{P}_3(t), \text{ for each } t\in J,\ (u,v)\in\mathbb{R}^2.$$

(H_3) For all $u,v,\bar{u},\bar{v}\in\mathbb{R}$, and there exist nonnegative constants $a_{i,k}, b_{i,k} \ge 0$, $i=1,2$ such that

$$\begin{cases} |I_{1,k}(u) - I_{1,k}(\bar{u})| \le a_{1,k}|u-\bar{u}|, & k=1,2,\cdots \\ |I_{2,k}(v) - I_{2,k}(\bar{v})| \le a_{2,k}|v-\bar{v}|, & k=1,2,\dots \end{cases}$$

and

$$\begin{cases} |J_{1,k}(u) - J_{1,k}(\bar{u})| \le b_{1,k}|u-\bar{u}|, & k=1,2,\dots,m,\cdots., \\ |J_{2,k}(v) - J_{2,k}(\bar{v})| \le b_{2,k}|v-\bar{v}|, & k=1,2,\dots,m,\cdots. \end{cases}$$

(H_4) There exist are numbers N_i and C_i, $i = 1, 2, 3$ where

$$N_i = \left(1 + \frac{\int_0^\infty h_1(s)ds}{h_1^*}\right) \int_0^\infty P_i(s)(1+s)ds < \infty,$$

$$C_i = \left(1 + \frac{\int_0^\infty h_2(s)ds}{h_2^*}\right) \int_0^\infty \bar{P}_i(s)(1+s)ds < \infty,\ i = 1, 2,$$

$$K_i = \left(1 + \frac{\|h_i\|_{L^1}}{h_i^*}\right) \sum_{k=1}^{\infty}(a_{i,k} + b_{i,k})(1+t_k) < \infty,\ for\ i = 1, 2,$$

$$N_3 = \left(1 + \frac{\int_0^\infty h_1(s)ds}{h_1^*}\right) \left(\int_0^\infty P_3(s)ds + \sum_{k=1}^{\infty} |I_{1,k}(0)| + \sum_{k=1}^{\infty} |J_{1,k}(0)|\right) < \infty,$$

$$C_3 = \left(1 + \frac{\int_0^\infty h_2(s)ds}{h_2^*}\right) \left(\int_0^\infty \bar{P}_3(s)ds + \sum_{k=1}^{\infty} |I_{2,k}(0)| + \sum_{k=1}^{\infty} |J_{2,k}(0)|\right) < \infty.$$

A mapping is completely continuous if it is continuous and maps bounded sets into relatively compact sets. A set of functions $Y \in E$ is almost equicontinuous if it is equicotinuous on each interval $[0, T]$, $0 \le T < +\infty$. The following result is an extension of Arzelia-Ascoli compatness criterion to unbounded intervals.

Lemma 4.1.2. [107] *Let $N \subseteq E$, Then N is compact in E, if the following conditions hold:*

- (a) *N is uniformly bounded in E.*
- (b) *The functions from $\{y : \ y = \frac{x}{1+t},\ x \in N\}$ belonging to N are almost equicontinuous on $\mathbb{R}^+$.*
- (c) *The functions from $\{y : \ y = \frac{x}{1+t},\ x \in N\}$ are equiconvergent at $+\infty$, that is given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $|f(t) - f(+\infty)| < \varepsilon$ for any $t \ge 0$ and $f \in M$ for all $t \ge T(\varepsilon)$ and $x \in N$.*

The main tool of this part a fixed point theorem due to Krasnoselskii's [93].

Theorem 4.1.3. *Assume that* (H_1)-(H_4) *holds with* $N_1 + K_1 < 1$ *and* $C_2 + K_2 < 1$*.If*

$$\tilde{M} = \begin{pmatrix} 1 - N_1 - K_1 & -C_2 \\ -C_1 & 1 - C_2 - K_2 \end{pmatrix}.$$

and $\det \tilde{M} > 0$ *then problem* (4.0.1)-(4.0.6) *has at least one solution.*

Proof. Let $N : E \times E \to E \times E$ be operator defined by

$$N(u, v) = F(u, v) + B(u, v), \ (u, v) \in E \times E,$$

where

$$F(u, v) = (F_1(u, v), F_2(u, v)); \ B(u, v) = (B_1(u, v), B_2(u, v)),$$

where

$$F_1(u(t), v(t)) = \int_0^\infty H_1(t, s) f(s, u(s), v(s)) ds,$$
$$F_2(u(t), v(t)) = \int_0^\infty H_2(t, s) g(s, u(s), v(s)) ds,$$

$$B_1(u(t), v(t)) = \sum_{k=1}^{\infty} H_1(t, t_k) I_{1.k}(u(t_k)) + \sum_{t_k < t} J_{1.k}(u(t_k)) + \frac{\displaystyle\int_0^\infty h_1(s) \left(\sum_{t_k < s} J_{1.k}(u(t_k)) \right) ds}{1 - \displaystyle\int_0^\infty h_1(s) ds}, \quad (4.1.11)$$

and

$$B_2(u(t), v(t)) = \sum_{k=1}^{\infty} H_2(t, t_k) I_{2.k}(v(t_k)) + \sum_{t_k < t} J_{2.k}(v(t_k)) + \frac{\displaystyle\int_0^\infty h_2(s) \left(\sum_{t_k < s} J_{2.k}(v(t_k)) \right) ds}{1 - \displaystyle\int_0^\infty h_2(s) ds}. \quad (4.1.12)$$

Step 1. B is a generalized contraction
Let $(u,v),(\bar{u},\bar{v})\in E\times E$, using the assumption (H_3), we deduce that

$$\begin{aligned}
\frac{|B_1(u(t),v(t))-B_1(\bar{u}(t),\bar{v}(t))|}{1+t} \leq & \sum_{k=1}^{\infty}\frac{|H_1(t,t_k)|}{1+t}|I_{1,k}(u(t_k))-I_{1,k}(\bar{u}(t_k))| \\
& +\sum_{t_k<t}|J_{1.k}(u(t_k))-J_{1,k}(\bar{u}(t_k))| \\
& +\frac{\displaystyle\int_0^{\infty}h_1(s)\left(\sum_{t_k<s}|J_{1,k}(u(t_k))-J_{1,k}(\bar{u}(t_k))|\right)ds}{\left|1-\displaystyle\int_0^{\infty}h_1(s)ds\right|} \\
\leq & \sum_{k=1}^{\infty}\frac{G_1(t,t_k)}{1+t}a_{1,k}|u(t_k)-\bar{u}(t_k)| \\
& +\frac{1}{h_1^*}\sum_{k=1}^{\infty}\int_0^{\infty}h_1(r)\frac{G_1(r,t_k)}{1+t}dr a_{1,k}|u(t_k)-\bar{u}(t_k)| \\
& +\sum_{k=1}^{\infty}b_{1,k}|u(t_k)-\bar{u}(t_k)| \\
& +\frac{\displaystyle\int_0^{\infty}h_1(s)ds}{h_1^*}\sum_{k=1}^{\infty}b_{1,k}\left|u(t_k)-\bar{u}(t_k)\right|.
\end{aligned}$$

Thus

$$\|B_1(u,v)-B_1(\bar{u},\bar{v})\|_E\leq\left(1+\frac{\int_0^{\infty}h_1(s)ds}{h_1^*}\right)\sum_{k=1}^{\infty}(a_{1,k}+b_{1,k})(1+t_k)\|u-\bar{u}\|_E:=K_1\|u-\bar{u}\|_E.$$

Similarly, we have

$$\|B_2(u,v)-B_2(\bar{u},\bar{v})\|_E\leq\left(1+\frac{\int_0^{\infty}h_2(s)ds}{h_2^*}\right)\sum_{k=1}^{\infty}(a_{2,k}+b_{2,k})(1+t_k)\|v-\bar{v}\|_E:=K_2\|u-\bar{u}\|_E.$$

Therefore

$$\begin{bmatrix}\|B_1(u,v)-B_1(\bar{u},\bar{v})\|_E\\ \\ \|B_2(u,v)-B_2(\bar{u},\bar{v})\|_E\end{bmatrix}\leq M\begin{bmatrix}\|u-\bar{u}\|_E\\ \\ \|v-\bar{v}\|_E\\ \end{bmatrix},$$

where
$$M=\begin{pmatrix}K_1 & 0\\ 0 & K_2\end{pmatrix}.$$
Since $K_1,\ K_2\in[0,1[$ then M converge to zero this implies that B is contraction operator

Step 2. F is completely continuous operator.

Claim 1. Operator F is a continuous and sends bounded sets into bounded sets .
Let $(u_n,v_n)\to(u,v)$ as $n\to\infty$, then $u_n\to u$ and $v_n\to v$ as $n\to\infty$.
Then
$$\frac{|F_1(u_n(t),v_n(t))-F_1(u(t),v(t))|}{1+t}\leq\int_0^\infty\frac{|H_1(t,s)|}{1+t}|f(s,u_n(s),v_n(s))-f(s,u(s),v(s))|ds$$
Thus
$$\|F_1(u_n,v_n)-F_1(u,v)\|_E\leq\sup_{t\in[0,+\infty[}\int_0^\infty\frac{|H_1(t,s)|}{1+t}|f(s,u_n(s),v_n(s))-f(s,u(s),v(s))|ds$$
Since f is L^1-Carathéodory, then we have by the lebesgue dominated convergence theorem,
$$\|F_1(u_n,v_n)-F_1(u,v)\|_E\to 0,\ n\to 0,$$
Similarly
$$\|F_2(u_n,v_n)-F_2(u,v)\|_E\to 0,\ n\to 0.$$
Therefore F is continuous.

Let D be a bounded subsets of E, then there exists $q>0$ such that $\|u\|_E\leq q$ and $\|v\|_E\leq q$ for all $(u,v)\in D$.
Let $(u,v)\in D$. Then for each $t\in[0,+\infty[$, we have
$$\frac{|F_1(u(t),v(t))|}{1+t}\leq\int_0^\infty\frac{|H_1(t,s)|}{1+t}|f(s,u(s),v(s))|ds$$
Since f, g be are Carathédory function, there exist nonnegative function $\phi_{M_0,M_1}\in L^1[0,\infty[$ such that
$$|f(t,u(t),v(t))|\leq\phi_{r_1,r_2}(t)\ and\ |g(t,u(t),v(t))|\leq\phi_{r_1,r_2}(t)\ t\in\mathbb{R}.$$

So, we have that

$$\|F_1(u,v)\|_E \le \left(1+\frac{\int_0^\infty h_1(s)ds}{h_1^*}\right)\int_0^\infty \phi_{r_1,r_2}(s)ds.$$

Similarly, we have

$$\|F_2(u,v)\|_E \le \left(1+\frac{\int_0^\infty h_2(s)ds}{h_2^*}\right)\int_0^\infty \phi_{r_1,r_2}(s)ds.$$

So F maps bounded sets into bounded sets in E

Claim 2. F maps bounded sets in E into almost equicontinuous sets.

For any $T \in [0,+\infty[$ and $\tau_1,\tau_2 \in [0,T]$, $\tau_1 < \tau_2$, then

$$\begin{aligned}\left|\frac{F_1(u(\tau_2),v(\tau_2))}{1+\tau_2}-\frac{F_1(u(\tau_1),v(\tau_1))}{1+\tau_1}\right| &\le \int_0^\infty \left|\frac{H_1(\tau_2,s)}{1+\tau_2}-\frac{H_1(\tau_1,s)}{1+\tau_1}\right| |f(s,u(s),v(s))|ds \\ &\le \int_0^\infty \left|\frac{H_1(\tau_2,s)}{1+\tau_2}-\frac{H_1(\tau_1,s)}{1+\tau_1}\right| \phi_{r_1,r_2}(s)ds \\ &\to 0 \ as \ \tau_1 \to \tau_2,\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\left|\frac{F_2(u(\tau_2),v(\tau_2))}{1+\tau_2}-\frac{F_2(u(\tau_1),v(\tau_1))}{1+\tau_1}\right| &\le \int_0^\infty \left|\frac{H_2(\tau_2,s)}{1+\tau_2}-\frac{H_2(\tau_1,s)}{1+\tau_1}\right| \phi_{r_1,r_2}(s)ds \\ &\to 0 \ as \ \tau_1 \to \tau_2.\end{aligned}$$

Then F is equicontinuous on any compact interval of $[0,+\infty)$.

Claim 3. We now show that set F is equiconvergent at ∞, i.e., for every $\varepsilon > 0$, there exists sufficiently large $T(\varepsilon) = \max(T_1(\varepsilon), T_2(\varepsilon))$ such that

$$\left|\frac{F(u(\tau_2),v(\tau_2))}{1+\tau_2}-\frac{F(u(\tau_1),v(\tau_1))}{1+\tau_1}\right| \le \varepsilon, \ \forall\ \tau_1,\tau_2 \ge T(\varepsilon),(u,v)\in E. \tag{4.1.13}$$

Since $\phi_{r_1,r_2} \in L^1[0,+\infty)$ then $\int_0^\infty \frac{|H_i(t,s)|}{1+t}\phi_{r_1,r_2}(s)ds < \infty \ for \ i = 1,2,$ we can choose $T_1(\varepsilon), T_2(\varepsilon)$ such that

$$\int_0^\infty \frac{|H_i(t,s)|}{1+t}\phi_{r_1,r_2}(s)ds \le \frac{\varepsilon}{2}, \ for \ i = 1,2. \tag{4.1.14}$$

Then, for every $\tau_1, \tau_2 \geq T_1(\varepsilon_1)$, we have

$$\begin{aligned} \left|\frac{F_1(u(\tau_2), v(\tau_2))}{1+\tau_2} - \frac{F_1(u(\tau_1), v(\tau_1))}{1+\tau_1}\right| &\leq \int_0^\infty \left|\frac{H_1(\tau_2, s)}{1+\tau_2} - \frac{H_1(\tau_1, s)}{1+\tau_1}\right| \phi_{r_1,r_2}(s)ds \\ &\leq \int_0^\infty \frac{|H_1(\tau_2, s)|}{1+\tau_2}\phi_{r_1,r_2}(s)ds \\ &\quad + \int_0^\infty \frac{|H_1(\tau_1, s)|}{1+\tau_1}\phi_{r_1,r_2}(s)ds \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Then, for every $\tau_1, \tau_2 \geq T_2(\varepsilon)$, we have

$$\begin{aligned} \left|\frac{F_2(u(\tau_2), v(\tau_2))}{1+\tau_2} - \frac{F_2(u(\tau_1), v(\tau_1))}{1+\tau_1}\right| &\leq \int_0^\infty \frac{|H_2(\tau_2, s)|}{1+\tau_2}\phi_{r_1,r_2}(s)ds \\ &\quad + \int_0^\infty \frac{|H_2(\tau_1, s)|}{1+\tau_1}\phi_{r_1,r_2}(s)ds \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, for every $(\varepsilon, \varepsilon)$ there exists $(T_1(\varepsilon), T_2(\varepsilon))$ such that for all $\tau_1, \tau_2 \geq \max(T_1(\varepsilon), T_2(\varepsilon))$

$$\left|\frac{F(u(\tau_2), v(\tau_2))}{1+\tau_2} - \frac{F(u(\tau_1), v(\tau_1))}{1+\tau_1}\right| \leq (\varepsilon, \varepsilon) \quad \forall (u, v) \in D.$$

Step 3. A priori bounds for solutions.
We show that the following set
$B = \left\{(u, v) \in E \times E : \lambda F(u, v) + \lambda B(\frac{u}{\lambda}, \frac{v}{\lambda}) = (u, v)\right\}$ is bounded for $0 < \lambda < 1$. Let $(u, v) \in B$, then

$$\begin{aligned} u(t) &= \lambda \int_0^\infty H_1(t, s) f(s, u(s), v(s))ds + \lambda \sum_{k=1}^\infty H_1(t, t_k) I_{1.k}\left(\frac{u(t_k)}{\lambda}\right) \\ &\quad + \lambda \sum_{t_k<t} J_{1.k}\left(\frac{u(t_k)}{\lambda}\right) + \lambda \frac{\displaystyle\int_0^\infty h_1(s)\left(\sum_{t_k<s} J_{1.k}\left(\frac{u(t_k)}{\lambda}\right)\right) ds}{1 - \displaystyle\int_0^\infty h_1(s)ds}, \end{aligned}$$

$$\begin{aligned} v(t) &= \lambda \int_0^{\infty} H_2(t,s) g(s,u(s),v(s)) ds + \lambda \sum_{k=1}^{\infty} H_2(t,t_k) I_{2.k}\left(\frac{v(t_k)}{\lambda}\right) \\ &\quad + \lambda \sum_{t_k<t} J_{2.k}\left(\frac{v(t_k)}{\lambda}\right) + \lambda \frac{\displaystyle\int_0^{\infty} h_2(s)\left(\sum_{t_k<s} J_{2.k}\left(\frac{v(t_k)}{\lambda}\right)\right) ds}{1 - \displaystyle\int_0^{\infty} h_2(s) ds}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{|u(t)|}{1+t} &\leq \left(1 + \frac{\int_0^{\infty} h_1(s)ds}{h_1^*}\right) \int_0^{\infty} P_1(s)(1+s) ds \|u\|_E \\ &\quad + \left(1 + \frac{\int_0^{\infty} h_1(s)ds}{h_1^*}\right) \int_0^{\infty} P_2(s)(1+s) ds \|v\|_E \\ &\quad + \left(1 + \frac{\int_0^{\infty} h_1(s)ds}{h_1^*}\right) \int_0^{\infty} P_3(s)(1+s) ds \\ &\quad + \left(1 + \frac{\int_0^{\infty} h_1(s)ds}{h_1^*}\right) \sum_{k=1}^{\infty} (a_{1,k}+b_{1,k})(1+t_k) \|u\|_E \\ &\quad + \left(1 + \frac{\int_0^{\infty} h_1(s)ds}{h_1^*}\right) \sum_{k=1}^{\infty} |I_{1,k}(0)| + \left(1 + \frac{\int_0^{\infty} h_1(s)ds}{h_1^*}\right) \sum_{k=1}^{\infty} |J_{1,k}(0)|. \end{aligned}$$

and

$$\begin{aligned} \frac{|v(t)|}{1+t} &\leq \left(1 + \frac{\int_0^{\infty} h_2(s)ds}{h_2^*}\right) \int_0^{\infty} \bar{P}_1(s)(1+s) ds \|u\|_E \\ &\quad + \left(1 + \frac{\int_0^{\infty} h_2(s)ds}{h_2^*}\right) \int_0^{\infty} \bar{P}_2(s)(1+s) ds \|v\|_E \\ &\quad + \left(1 + \frac{\int_0^{\infty} h_2(s)ds}{h_2^*}\right) \int_0^{\infty} \bar{P}_3(s)(1+s) ds \\ &\quad + \left(1 + \frac{\int_0^{\infty} h_2(s)ds}{h_2^*}\right) \sum_{k=1}^{\infty} (a_{2,k}+b_{2,k})(1+t_k) \|v\|_E \\ &\quad + \left(1 + \frac{\int_0^{\infty} h_2(s)ds}{h_2^*}\right) \sum_{k=1}^{\infty} |I_{2,k}(0)| + \left(1 + \frac{\int_0^{\infty} h_2(s)ds}{h_2^*}\right) \sum_{k=1}^{\infty} |J_{2,k}(0)|. \end{aligned}$$

This implies that

$$\|u\|_E \leq N_1 \|u\|_E + K_1 \|u\|_E + N_2 \|v\|_E + N_3.$$

and

$$\|v\|_E \leq C_1\|u\|_E + K_2\|v\|_E + C_2\|v\|_E + C_3.$$

Then, we have that

$$\begin{pmatrix} 1 - N_1 - K_1 & -C_2 \\ -C_1 & 1 - C_2 - K_2 \end{pmatrix} \begin{pmatrix} \|u\|_E \\ \|v\|_E \end{pmatrix} \leq \begin{pmatrix} N_3 \\ C_3 \end{pmatrix},$$

Therefore

$$\tilde{M} \begin{pmatrix} \|u\|_E \\ \|v\|_E \end{pmatrix} \leq \begin{pmatrix} N_3 \\ C_3 \end{pmatrix}. \tag{4.1.15}$$

Since $\tilde{M}$ satisfies the hypotheses of lemma (1.4.5) thus $(\tilde{M})^{-1}$ is order preserving.
We apply $(\tilde{M})^{-1}$ to both sides of the inequality (4.1.15) we obtain

$$\begin{pmatrix} \|u\|_E \\ \|v\|_E \end{pmatrix} \leq (\tilde{M})^{-1} \begin{pmatrix} N_3 \\ C_3 \end{pmatrix}.$$

Then the set $B = \left\{(u, v) \in E \times E : \lambda F(u, v) + \lambda B(\frac{u}{\lambda}, \frac{v}{\lambda}) = (u, v)\right\}$ is bounded, hence we deduce from lemma 1.4.14 that the equation $x = F(x) + B(x)$, $x \in E \times E$ has solution. □

4.2 Example

In this section, we present a simple example to explain our result.
Consider the problem:

$$-u'' = \frac{e^{-t}}{100}(1+u+v)^{\frac{2}{3}}, \quad t \in J, \ t \neq k, \tag{4.2.1}$$

$$-v'' = \frac{e^{-t}}{200}(1+u+v)^{\frac{1}{2}}, \quad t \in J, \ t \neq k, \tag{4.2.2}$$

$$\Delta u(k) = \frac{1}{8^k}\sqrt{u(k)}, \quad k = 1, 2, \cdots, \tag{4.2.3}$$

$$-\Delta u'(k) = \frac{1}{10^k}\sqrt{u(k)}, \quad k = 1, 2, \cdots, \tag{4.2.4}$$

$$\Delta v(k) = e^{-2k}\frac{v(k)}{(1+v(k))}, \quad k = 1, 2, \cdots, \tag{4.2.5}$$

$$-\Delta v'(t_k) = e^{-3k} \ \frac{v(k)}{(1+v(k))}, \quad k = 1, 2, \cdots, \tag{4.2.6}$$

$$u(0) = \int_0^\infty e^{-4s}u(s)ds, \quad u'(\infty) = 0, \tag{4.2.7}$$

$$v(0) = \int_0^\infty e^{-5s}v(s)ds, \quad v'(\infty) = 0. \tag{4.2.8}$$

Let

$$f(t,u,v) = \frac{e^{-t}}{100}(1+u+v)^{\frac{2}{3}},$$

$$g(t,u,v) = \frac{e^{-t}}{200}(1+u+v)^{\frac{1}{2}},$$

$$J_{1,k}(u(t_k)) = \frac{1}{8^k}\sqrt{u(k)} \ k = 1, 2, \cdots,$$

$$I_{1,k}(u(t_k)) = \frac{1}{10^k}\sqrt{u(k)} \ k = 1, 2, \cdots,$$

$$J_{2,k}(v'(t_k)) = e^{-2k} \ \frac{v(k)}{(1+v(k))}, \ k = 1, 2, \cdots,$$

$$I_{2,k}(v'(t_k)) = e^{-3k} \ \frac{v(k)}{(1+v(k))}, \ k = 1, 2, \cdots,$$

$$h_1(s) = e^{-4s} \ and \ h_2(s) = e^{-5s}.$$

Let u,$v \in [0, \infty[$ et $t \in J$
it is clear that $\int_0^\infty e^{-5s} ds = \frac{1}{5} \neq 1$ and $\int_0^\infty e^{-4s} ds = \frac{1}{4} \neq 1$.
By the inequality $(1 + x + y)^\gamma \leq 1 + \gamma x + \gamma y$, for $x \in \mathbb{R}^+$, $0 \leq \gamma \leq 1$, we see that

$$|f(t, u, v)| = \frac{e^{-t}}{100} \left(1 + \frac{2}{3}|u| + \frac{2}{3}|v|\right),$$

and

$$|g(t, u, v)| = \frac{e^{-t}}{200} \left(1 + \frac{1}{2}|u| + \frac{1}{2}|v|\right),$$

Hence the condition (H_2) holds with $P_i(t) = \frac{e^{-t}}{150}$ and $\bar{P}_i(t) = \frac{e^{-t}}{400}$ for i=1,2, $P_3(t) = \frac{e^{-t}}{100}$, $\bar{P}_3(t) = \frac{e^{-t}}{200}$.

Also for all $u, \bar{u}, v, \bar{v} \in \mathbb{R}^+$, we have

$$|I_{1,k}(u) - I_{1,k}(\bar{u})| \leq \frac{1}{10^k}|u - \bar{u}|, \ k = 1, 2, \cdots,$$

and

$$|I_{2,k}(v) - I_{2,k}(\bar{v})| \leq e^{-3k}|v - \bar{v}|, \ k = 1, 2, \cdots,$$

$$|J_{1,k}(u) - J_{1,k}(\bar{u})| \leq \frac{1}{8^k}|u - \bar{u}|, \ k = 1, 2, \cdots,$$

and

$$|J_{2,k}(v) - J_{2,k}(\bar{v})| \leq e^{-2k}|v - \bar{v}|, \ k = 1, 2, \cdots,$$

Thus (H_3) holds with
$a_{1,k} = \frac{1}{10^k}$, $b_{1,k} = \frac{1}{8^k}$, $a_{2,k} = e^{-3k}$, $b_{2,k} = e^{-2k}$, $k = 1, 2, \cdots$,
Then, we easily obtain:

$$N_i = \left(1 + \frac{\int_0^\infty h_1(s)ds}{h_1^*}\right) \int_0^\infty P_i(s)(1 + s)ds = \frac{4}{225} < \infty, \ i = 1, 2;$$

$$C_i = \left(1 + \frac{\int_0^\infty h_2(s)ds}{h_2^*}\right) \int_0^\infty \bar{P}_i(s)(1 + s)ds = \frac{1}{160} < \infty, \ i = 1, 2;$$

$$K_1 = \left(1 + \frac{\int_0^\infty h_1(s)ds}{h_1^*}\right) \sum_{k=1}^{\infty}(a_{1,k} + b_{1,k})(1 + t_k) = \frac{1073}{1327} < \infty,$$

$$K_2 = \left(1 + \frac{\int_0^\infty h_2(s)ds}{h_2^*}\right) \sum_{k=1}^{\infty} (a_{2,k} + b_{2,k})(1 + t_k) \simeq 0,41 < \infty,$$

$$N_3 = \left(1 + \frac{\int_0^\infty h_1(s)ds}{h_1^*}\right) \left(\int_0^\infty P_3(s)ds + \sum_{k=1}^{\infty} |I_{1,k}(0)| + \sum_{k=1}^{\infty} |J_{1,k}(0)|\right) = \frac{1}{75} < \infty,$$

$$C_3 = \left(1 + \frac{\int_0^\infty h_2(s)ds}{h_2^*}\right) \left(\int_0^\infty \bar{P}_3(s)ds + \sum_{k=1}^{\infty} |I_{2,k}(0)| + \sum_{k=1}^{\infty} |J_{2,k}(0)|\right) = \frac{1}{160} < \infty.$$

Thus $N_1 + K_1 \simeq 0,83 < 1$ and $C_2 + K_2 \simeq 0,6 < 1$
For this example

$$\tilde{M} \simeq \begin{pmatrix} 1 - 0,83 & -\frac{4}{225} \\ -\frac{1}{160} & 1 - 0,6 \end{pmatrix}.$$

$\det \tilde{M} \simeq 0,07 > 0$. By Theorem 4.1.3, it follows that Problem (4.2.1)-(4.2.8) has at least one solution.

Chapter 5

Implicit impulsive differential equations with non local conditions

In this chapter, our main objective is to establish sufficient conditions for the existence of solutions for systems of implicit impulsive differential equations with non local conditions. Our approach based on vectorial version of Krasnoselskii's theorem. Consider the problem following

$$x'(t) = g_1(t, x(t), y(t)) + h_1(t, x'(t), y'(t)), \quad t \in J', \tag{5.0.1}$$

$$y'(t) = g_2(t, x(t), y(t)) + h_2(t, x'(t), y'(t)), \ t \in J', \tag{5.0.2}$$

$$\Delta x(t_k) = I_k(x(t_k)), \ k = 1, 2, \dots, m, \tag{5.0.3}$$

$$\Delta y(t_k) = J_k(y(t_k)), \ k = 1, 2, \dots, m, \tag{5.0.4}$$

$$x(0) = \alpha[x], \tag{5.0.5}$$

$$y(0) = \beta[y], \tag{5.0.6}$$

where $J = [0, 1]$, $0 < t_1 < t_2 < \cdots < t_m < 1$, $J' = J \setminus \{t_1, t_2, \dots, t_m\}$, $h_i, \ g_i : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions for $i = 1, 2$, $0 < t_1 < t_2 < \cdots < t_m < 1$, $J_k, I_k \in C(\mathbb{R}, \mathbb{R})$ $k \in \{1, 2, \cdots, m\}$. $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ and $\Delta y(t_k) = y(t_k^+) - y(t_k^-)$ in which $x(t_k^+)$ and $x(t_k^+)$ $(x(t_k^-), y(t_k^-))$ denote the righ and left hand limit of $x(t)$ and $y(t)$ at $t = t_k$, respectively. Next

$\alpha,\ \beta$ are linear functionals given by Stieltjes integrals

$$\alpha[v] = \int_0^{\tilde{t}} v(s)dA(s),$$

$$\beta[v] = \int_0^{\tilde{t}} v(s)dB(s),$$

where $\tilde{t} \in]t_m, 1]$ is fixed.

5.1 An existence result

We consider the space

$$PC(J, \mathbb{R}) = \{y : [0,1] \to \mathbb{R} : y_k \in C(J', \mathbb{R}), k = 1, \cdots, m,$$
$$y(t_k^-) \ and \ y(t_k^+) \ exist \ k = 1, ..., m, \ and \ y(t_k^-) = y(t)\}.$$

We use in $PC(J, \mathbb{R}) \times PC(J, \mathbb{R})$ the norm

$$\|(x, y)\|_{PC[0,1] \times PC[0,1]} := \left(\|x\|_{PC[0,1]}, \|y\|_{PC[0,1]}\right),$$

where

$$\|v\|_{PC[0,1]} := \max\{\|v\|_{[0,\tilde{t}]}, \|v\|_{[\tilde{t},1]}\}$$

and the notation $\|v\|_{[0,\tilde{t}]}$ stands for sup-norm on $[0, \tilde{t}]$:

$$\|v\|_{[0,\tilde{t}]} = \sup_{[0,\tilde{t}]} |v(t)|,$$

while $\|v\|_{[\tilde{t},1]}$ denote Bielecki-type norm on $[\tilde{t}, 1]$:

$$\|v\|_{[\tilde{t},1]} = \sup_{[0,\tilde{t}]} |v(t)| e^{-\tau(t-\eta)},$$

Here $\eta < \tilde{t}$ and $\tau > 0$ are given numbers. As we shall see, the joint role of the parameters η (any fixed number with $\eta < \tilde{t}$) and τ(chosen large enough) is to weaken the assumptions on $g_1(t, x, y)$, $g_2(t, x, y)$ when $t \in [\tilde{t}, 1]$.
Then the norm of the functional $\alpha,\ \beta : PC(J, \mathbb{R}) \to \mathbb{R}$, is given by

$$\|\alpha\| = \sup_{\|v\|=1} \left| \int_0^{\tilde{t}} v(s)dA(s) \right|,$$

and

$$\|\beta\| = \sup_{\|v\|=1} \left| \int_0^{\tilde{t}} v(s)dB(s) \right|.$$

In order to obtain the equivalent integral form of the problem (5.0.1)-(5.0.5), de note

$$u(t) = x'(t), \qquad v(t) = y'(t), \qquad t \neq t_k, \;\; k = 1, 2, \cdots, m. \tag{5.1.1}$$

Integrating (5.1.1) from 0 to t, we have

$$x(t) = x(0) + \int_0^t u(s)ds + \sum_{0<t_k<t} I_k(x(t_k)),$$

and

$$y(t) = y(0) + \int_0^t v(s)ds + \sum_{0<t_k<t} J_k(y(t_k)).$$

The conditions $x(0) = \alpha[x]$ and $y(0) = \beta[x]$ gives

$$x(0) = \alpha \left[\sum_{0<t_k<.} I_k(x(t_k)) + \int_0^{.} u(s)ds \right] + \alpha[x(0)],$$

and

$$y(0) = \beta \left[\sum_{0<t_k<.} J_k(y(t_k)) + \int_0^{.} v(s)ds \right] + \alpha[y(0)].$$

Hence

$$x(0) = \alpha \left[\sum_{0<t_k<.} I_k(x(t_k)) + \int_0^{.} u(s)ds \right] + \alpha[1]x(0),$$

and

$$y(0) = \alpha \left[\sum_{0<t_k<.} J_k(y(t_k)) + \int_0^{.} u(s)ds \right] + \alpha[1]y(0).$$

Therefore

$$x(t) = (1-\alpha[1])^{-1}\alpha \left[\sum_{0<t_k<.} I_k(x(t_k)) + \int_0^{.} u(s)ds \right] + \int_0^t u(s)ds + \sum_{0<t_k<t} I_k(x(t_k)),$$

and

$$y(t) = (1-\beta[1])^{-1}\beta\left[\sum_{0<t_k<.} J_k(y(t_k)) + \int_0^{.} v(s)ds\right] + \int_0^t v(s)ds + \sum_{0<t_k<t} J_k(y(t_k)).$$

In the other hand, we have that

$$\begin{aligned}
x(t_1) &= (1-\alpha[1])^{-1}\alpha\left[\int_0^{t_1} u(s)ds\right] + \int_0^{t_1} u(s)ds = (1-\alpha[1])^{-1}\int_0^{t_1} u(s)ds \\
x(t_2) &= (1-\alpha[1])^{-1}\left(I_1(x(t_1)) + \int_0^{t_2} u(s)ds\right) \\
x(t_3) &= (1-\alpha[1])^{-1}\left(I_1(x(t_1)) + I_2(x(t_2)) + \int_0^{t_3} u(s)ds\right) \\
&. \\
&. \\
&. \\
x(t_k) &= (1-\alpha[1])^{-1}\left(\sum_{0<t_i<t_k} I_i(x(t_i)) + \int_0^{t_k} u(s)ds\right).
\end{aligned}$$

Similarly, we have

$$y(t_k) = (1-\beta[1])^{-1}\left(\sum_{0<t_i<t_k} J_i(y(t_i)) + \int_0^{t_k} v(s)ds\right).$$

with

$$y(t_1) = (1-\beta[1])^{-1}\int_0^{t_1} v(s)ds.$$

Let

$$\begin{aligned}
G_1(u,v)(t) &= g_1(t,(1-\alpha[1])^{-1}\alpha[h_1] + h_1(t),(1-\beta[1])^{-1}\beta[h_2] + h_2(t)), \\
G_2(u,v)(t) &= g_2(t,(1-\alpha[1])^{-1}\alpha[h_1] + h_1(t),(1-\beta[1])^{-1}\beta[h_2] + h_2(t)).
\end{aligned}$$

where

$$h_1(t) = \sum_{0<t_k<t} I_k\left((1-\alpha[1])^{-1}\left(\sum_{0<t_i<t_k} I_i(x(t_i)) + \int_0^{t_k} u(s)ds\right)\right) + \int_0^t u(s)ds,$$

and

$$h_2(t) = \sum_{0<t_k<t} J_k\left((1-\beta[1])^{-1}\left(\sum_{0<t_i<t_k} J_i(y(t_i)) + \int_0^{t_k} v(s)ds\right)\right) + \int_0^t v(s)ds.$$

Also we define

$$H_1(u,v)(t) = h_1(t,u(t),v(t)),$$
$$H_2(u,v)(t) = h_2(t,u(t),v(t)).$$

Then the problem (5.0.1)-(5.0.5) is equivalent to the system

$$\begin{cases} u = G_1(u,v) + H_1(u,v), \\ v = G_2(u,v) + H_2(u,v). \end{cases} \tag{5.1.2}$$

In this section we study the existence of solutions for systems (5.0.1)-(5.0.5) with two impulses. We need the following assumptions:

(H_1) g_1 and g_2 are jointly continuous functions, there exists nonnegative coefficients
a_i b_i c_i A_i B_i C_i such that:

$$|g_1(t,u,v)| \le \begin{cases} a_1|u| + b_1|v| + c_1, & \text{if t} \in [0,\tilde{t}]; \\ A_1|u| + B_1|v| + C_1, & \text{if t} \in [\tilde{t},1]; \end{cases}$$

$$|g_2(t,u,v)| \le \begin{cases} a_2|u| + b_2|v| + c_2, & \text{if t} \in [0,\tilde{t}]; \\ A_2|u| + B_2|v| + C_2, & \text{if t} \in [\tilde{t},1], \end{cases}$$

for all $u, v \in \mathbb{R}$.

(H_2) h_1, h_2 satisfy the Lipschitz conditions

$$|h_1(t,u,v) - h_1(t,\bar{u},\bar{v})| \le \bar{a}_1|u-\bar{u}| + \bar{a}_2|v-\bar{v}|,$$

$$|h_2(t,u,v) - h_2(t,\bar{u},\bar{v})| \le \bar{b}_1|u-\bar{u}| + \bar{b}_2|v-\bar{v}|,$$

for all (u,v),$(\bar{u},\bar{v}) \in \mathbb{R}^2$ and $t \in J$. Here for $i = 1,2$, $\bar{a}_i$, $\bar{b}_i$ are non negative numbers.

(H_3) There exist d_k, $\bar{d}_k$,D_k and $\bar{D}_k \in \mathbb{R}^+$ such that for every $v \in \mathbb{R}$ we have

$$|I_k(v)| \le d_k|u| + \bar{d}_k, \; k = 1,2. \tag{5.1.3}$$

and

$$|J_k(v)| \le D_k|u| + \bar{D}_k, \; k = 1,2. \tag{5.1.4}$$

Define a square matrices

$$\tilde{M} = \begin{pmatrix} a_1\tilde{t}A_\alpha\bar{A} & b_1\tilde{t}B_\beta\bar{B} \\ a_2\tilde{t}A_\alpha\bar{A} & b_2\tilde{t}B_\beta\bar{B} \end{pmatrix}, \quad \bar{M} = \begin{pmatrix} \bar{a}_1 & \bar{a}_2 \\ \bar{b}_1 & \bar{b}_2 \end{pmatrix},$$

$$M_1 = \begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix}.$$

where

$$\bar{A} = 1 + |1-\alpha[1]|^{-1}\left(d_1 + d_2 + d_2 d_1|1-\alpha[1]|^{-1}\right),$$

$$\bar{B} = 1 + |1-\beta[1]|^{-1}\left(D_1 + D_2 + D_2 D_1|1-\beta[1]|^{-1}\right).$$

and

$$A_\alpha = 1 + |1-\alpha[1]|^{-1}\|\alpha\|, \quad B_\beta = 1 + |1-\beta[1]|^{-1}\|\beta\|.$$

Now we define

$$D = \left\{u : PC(J,\mathbb{R}) \times PC(J,\mathbb{R}) : \quad \|(u,v)\|_{PC[0,1]\times PC[0,1]} \leq R\right\},$$

with $R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$, $R_1 \geq 0$, $R_2 \geq 0$ and $R \geq (I - \bar{M} - \tilde{M} - \frac{1}{\tau}M_1)^{-1}(P+K)$, where

$$P = \|H(0,0)\|_{PC[0,1]\times PC[0,1]}, \quad K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix},$$

with

$$H(0,0) = \begin{bmatrix} H_1(0,0) \\ H_2(0,0) \end{bmatrix}, \quad K_1 = c_1 + a_1\tilde{C} + b_1\bar{C} \ \text{ and } K_2 = c_2 + a_2\tilde{C} + b_2\bar{C},$$

where

$$\bar{C} = \bar{d}_1 + \bar{d}_2 + \bar{d}_1 d_2|1-\alpha[1]|^{-1}, \quad \tilde{C} = \bar{D}_1 + \bar{D}_2 + \bar{D}_1 D_2|1-\beta[1]|^{-1}$$

It is obvious that D is a non empty, bounded, closed and convex subset of $PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$.

Theorem 5.1.1. *Suppose that the hypotheses* $(H_1) - (H_3)$ *hold, if the spectral raduis of the matrix* $\bar{M} + \tilde{M}$ *is less one, then the problem* (5.0.1)-(5.0.5) *has at least one solution.*

Proof. Consider the operator

$$T : PC(J,\mathbb{R}) \times PC(J,\mathbb{R}) \;\; \rightarrow \;\; PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$$

defined by

$$T(u,v) = G_1(u,v) + H_1(u,v),$$

with

$$T_1(u,v) = G_1(u,v) + H_1(u,v),$$

avec

$$T_2(u,v) = G_2(u,v) + H_2(u,v).$$

Therefore, the system (5.1.2) can be regarded as a fixed point problem for the operator T.
Step 1:We, verify that H is a generalized contraction mapping
In, fact for all $(u,v),\ (\bar{u},\bar{v}) \in PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$ using (H_2) and (H_3), for $t \in [0,\tilde{t}]$, we have

$$\begin{aligned} |H_1(u,v)(t) - H_1(\bar{u},\bar{v})(t)| &= |h_1(t,u(t),v(t)) - h_1(t,\bar{u}(t),\bar{v}(t))| \\ &\leq \bar{a}_1|u(t)-\bar{u}(t)| + \bar{b}_1|v(t)-\bar{v}(t)| \\ &\leq \bar{a}_1\|u-\bar{u}\|_{[0,\tilde{t}]} + \bar{b}_1\|v-\bar{v}\|_{[0,\tilde{t}]}. \end{aligned}$$

Taking super norm, we obtain that

$$\|H_1(u,v) - H_1(\bar{u},\bar{v})\|_{[0,\tilde{t}]} \leq \bar{a}_1\|u-\bar{u}\|_{[0,\tilde{t}]} + \bar{b}_1\|v-\bar{v}\|_{[0,\tilde{t}]}. \qquad (5.1.5)$$

For $t \in [\tilde{t},1]$, we obtain

$$\begin{aligned} |H_1(u,v)(t) - H_1(\bar{u},\bar{v})(t)| &\leq \bar{a}_1|u(t)-\bar{u}(t)| + \bar{b}_1|v(t)-\bar{v}(t)| \\ &= \bar{a}_1|u(t)-\bar{u}(t)|e^{-\tau(t-\eta)}e^{\tau(t-\eta)} \\ &\quad +\bar{b}_1|v(t)-\bar{v}(t)|e^{-\tau(t-\eta)}e^{\tau(t-\eta)} \\ &\leq \bar{a}_1 e^{\tau(t-\eta)}\|u-\bar{u}\|_{[\tilde{t},1]} + \bar{b}_1 e^{\tau(t-\eta)}\|v-\bar{v}\|_{[\tilde{t},1]}. \end{aligned}$$

Dividing by $e^{\tau(t-\eta)}$ and taking super norm when $t \in [\tilde{t},1]$, we obtain that :

$$\|H_1(u,v) - H_1(\bar{u},\bar{v})\|_{[\tilde{t},1]} \leq \bar{a}_1\|u-\bar{u}\|_{[\tilde{t},1]} + \bar{b}_1\|v-\bar{v}\|_{[\tilde{t},1]}. \qquad (5.1.6)$$

The inequalities (5.1.5) and (5.1.6) will imply that

$$\|H_1(u,v) - H_1(\bar{u},\bar{v})\|_{PC[0,1]} \leq \bar{a}_1\|u-\bar{u}\|_{PC[0,1]} + \bar{b}_1\|v-\bar{v}\|_{PC[0,1]}. \quad (5.1.7)$$

Similarly, we obtain

$$\|H_2(u,v) - H_2(\bar{u},\bar{v})\|_{PC[0,1]} \leq \bar{a}_2\|u-\bar{u}\|_{PC[0,1]} + \bar{b}_2\|v-\bar{v}\|_{PC[0,1]}. \quad (5.1.8)$$

Using the vector norm we cam put both inequalities (5.1.7) and (5.1.9) under the vector inequality

$$\|H(U) - H(\bar{U})\|_{PC[0,1]\times PC[0,1]} \leq \bar{M}\|U-\bar{U}\|_{PC[0,1]\times PC[0,1]}. \quad (5.1.9)$$

for $U=(u,v)$, $\bar{U}(\bar{u},\bar{v})$, according to $\rho(\tilde{M}+\bar{M})<1$ and $\bar{M}=\tilde{M}+\bar{M}$, we have $\rho(\bar{M})<1$. Hence H is generalized contraction.
Step 2: G is continuous.
Let (u_n,v_n) be a sequence such that $(u_n,v_n)\to(u,v)$ in $PC(J,\mathbb{R})\times PC(J,\mathbb{R})$, then for each $t\in[0,\tilde{t}]$
$|G_1(u_n,v_n)(t) - G_1(u,v)(t)| \leq$

$$\left\|g_1\left(.,\frac{\alpha[h_{1,n}]}{1-\alpha[1]}+h_{1,n}(.),\frac{\beta[h_{2,n}]}{1-\beta[1]}+h_{2,n}(.)\right)-g_1\left(.,\frac{\alpha[h_1]}{1-\alpha[1]}+h_1(.),\frac{\beta[h_2]}{1-\beta[1]}+h_2(.)\right)\right\|_{[0,\tilde{t}]}$$

Note that

$$|h_{1,n}(t)-h_1(t)| \leq \sum_{k=1}^{2}\left|I_k\left(\frac{\sum\limits_{0<t_i<t_k} I_i(x_n(t_i)) + \int_0^{t_k} u_n(s)ds}{1-\alpha[1]}\right) - I_k\left(\frac{\sum\limits_{0<t_i<t_k} I_i(x(t_i)) + \int_0^{t_k} u(s)ds}{1-\alpha[1]}\right)\right| + \tilde{t}\|u_n-u\|_{[0,\tilde{t}]},$$

Similarly, we have that

$$|h_{2,n}(t) - h_2(t)| \leq \sum_{k=1}^{2} \left| J_k \left(\frac{\sum_{0<t_i<t_k} J_i(y_n(t_i)) + \int_0^{t_k} v_n(s)ds}{1-\alpha[1]} \right) - \right.$$

$$\left. J_k \left(\frac{\sum_{0<t_i<t_k} J_i(y(t_i)) + \int_0^{t_k} v(s)ds}{1-\alpha[1]} \right) \right| + \tilde{t}\|v_n - v\|_{[0,\tilde{t}]}.$$

Hence $h_{1,n} \to h_1$ as $n \to \infty$ and $h_{2,n} \to h_2$ as $n \to \infty$.
So, we obtain

$$\|G_1(u_n, v_n) - G_1(u, v)\|_{[0,\tilde{t}]} \to 0 \quad asn \ \to \infty.$$

For $[\tilde{t}, 1]$, and any $\tau > 0$, we have
$e^{-\tau(t-\eta)}|G_1(u_n, v_n)(t) - G_1(u, v)(t)| \leq$

$$\left\| g_1\left(., \frac{\alpha[h_{1,n}]}{1-\alpha[1]} + h_{1,n}(.), \frac{\beta[h_{2,n}]}{1-\alpha[1]} + h_{2,n}(.)\right) - g_1\left(., \frac{\alpha[h_1]}{1-\alpha[1]} + h_1(.), \frac{\beta[h_2]}{1-\alpha[1]} + h_2(.)\right)\right\|_{[\tilde{t},1]}$$

Note that

$$|h_{1,n}(t) - h_1(t)| \leq \sum_{k=1}^{2} \left| I_k \left(\frac{\sum_{0<t_i<t_k} I_i(x_n(t_i)) + \int_0^{t_k} u_n(s)ds}{1-\alpha[1]} \right) \right.$$

$$\left. - I_k \left(\frac{\sum_{0<t_i<t_k} I_i(x(t_i)) + \int_0^{t_k} u(s)ds}{1-\alpha[1]} \right) \right|$$

$$+ \tilde{t}\|u_n - u\|_{[0,\tilde{t}]} + \frac{e^{\tau(1-\eta)}}{\tau}\|u_n - u\|_{[\tilde{t},1]},$$

Similarly, we have that

$$|h_{2,n}(t) - h_2(t)| \leq \sum_{k=1}^{2} \left| J_k \left(\frac{\sum\limits_{0<t_i<t_k} J_i(y_n(t_i)) + \int_0^{t_k} v_n(s)ds}{1-\alpha[1]} \right) - J_k \left(\frac{\sum\limits_{0<t_i<t_k} J_i(y(t_i)) + \int_0^{t_k} v(s)ds}{1-\alpha[1]} \right) \right| + \tilde{t}\|v_n - v\|_{[0,\tilde{t}]} + \frac{e^{\tau(1-\eta)}}{\tau}\|v_n - v\|_{[\tilde{t},1]}.$$

Then

$$\|G_1(u_n, v_n) - G_1(u, v)\|_{[\tilde{t},1]} \to 0 \quad as \ n \ \to \infty.$$

Furthermore:

$$\|G_1(u_n, v_n) - G_1(u, v)\|_{PC[0,1]} \to 0 \quad as \ n \ \to \infty.$$

Similarly, we can obtain

$$\|G_2(u_n, v_n) - G_2(u, v)\|_{PC[0,1]} \to 0 \quad as \ n \ \to \infty.$$

Step 3: G maps bounded sets into bounded sets in D.
Indeed,it is enough to show that for any $r > 0$ there exists a positive constant l such that for each $(u, v) \in D$, we have

$$\|G(u, v)\|_{PC[0,1]\times PC[0,1]} \leq l = (l_1, l_2).$$

Then for each $t \in [0, \tilde{t}]$, we find that

$$\begin{aligned} |G_1(u,v)(t)| &= \left|g_1(t, (1-\alpha[1])^{-1}\alpha[h_1] + h_1(t), (1-\beta[1])^{-1}\beta[h_2] + h_2(t))\right| \\ &\leq a_1 \left|(1-\alpha[1])^{-1}\alpha[h_1] + h_1(t)\right| + b_1 \left|(1-\beta[1])^{-1}\beta[h_2] + h_2(t))\right| + c_1 \\ &\leq a_1 \left|1-\alpha[1]\right|^{-1} \|\alpha\|\|h_1\|_{[0,\tilde{t}]} + a_1\|h_1\|_{[0,\tilde{t}]} + b_1 \left|1-\beta[1]\right|^{-1} \|\beta\|\|h_2\|_{[0,\tilde{t}]} \\ &\quad +\|h_2\|_{[0,\tilde{t}]} + c_1 \end{aligned}$$

Then

$$|G_1(u,v)(t)| \le a_1\left(1+|1-\alpha[1]|^{-1}\|\alpha\|\right)\|h_1\|_{[0,\tilde{t}]}+b_1\left(1+|1-\beta[1]|^{-1}\|\beta\|\right)\|h_2\|_{[0,\tilde{t}]}+c_1 \tag{5.1.10}$$

Note that

$$\begin{aligned}
|h_1(t)| &= \left|\sum_{0<t_k<t} I_k\left((1-\alpha[1])^{-1}\left(\sum_{0<t_i<t_k} I_i(x(t_i))+\int_0^{t_k}u(s)ds\right)\right)+\int_0^t u(s)ds\right|\\
&\le \sum_{k=1}^{2}\left|I_k\left((1-\alpha[1])^{-1}\left(\sum_{0<t_i<t_k} I_i(x(t_i))+\int_0^{t_k}u(s)ds\right)\right)\right|+\int_0^{\tilde{t}}|u(s)|ds\\
&\le d_1\left|(1-\alpha[1])^{-1}\int_0^{t_1}u(s)ds\right|+d_2\left|(1-\alpha[1])^{-1}\left(I_1(x(t_1))+\int_0^{t_2}u(s)ds\right)\right|\\
&\quad+\bar{d}_1+\bar{d}_2+\tilde{t}\|u\|_{[0,\tilde{t}]}\\
&\le d_1|1-\alpha[1]|^{-1}\int_0^{t_1}|u(s)|ds+d_2|1-\alpha[1]|^{-1}\left(|I_1(x(t_1))|+\int_0^{t_2}|u(s)|ds\right)\\
&\quad+\bar{d}_1+\bar{d}_2+\tilde{t}\|u\|_{[0,\tilde{t}]}\\
&\le \tilde{t}d_1|1-\alpha[1]|^{-1}\|u\|_{[0,\tilde{t}]}+d_2|1-\alpha[1]|^{-1}\left(d_1|x(t_1)|+\bar{d}_1+\tilde{t}\|u\|_{[0,\tilde{t}]}\right)+\tilde{t}\|u\|_{[0,\tilde{t}]}\\
&\quad+\bar{d}_1+\bar{d}_2\\
&\le \tilde{t}d_1|1-\alpha[1]|^{-1}\|u\|_{[0,\tilde{t}]}+|1-\alpha[1]|^{-1}d_2\left(d_1|1-\alpha[1]|^{-1}\int_0^{t_1}|u(s)|ds+\bar{d}_1+\tilde{t}\|u\|_{[0,\tilde{t}]}\right)\\
&\quad+\tilde{t}\|u\|_{[0,\tilde{t}]}+\bar{d}_1+\bar{d}_2\\
&\le \tilde{t}\left(1+|1-\alpha[1]|^{-1}\left(d_1+d_2+d_2d_1|1-\alpha[1]|^{-1}\right)\right)\|u\|_{[0,\tilde{t}]}\\
&\quad+\bar{d}_1+\bar{d}_2+\bar{d}_1d_2|1-\alpha[1]|^{-1},
\end{aligned}$$

which implies that

$$\|h_1\|_{[0,\tilde{t}]} \le \tilde{t}\bar{A}\|u\|_{[0,\tilde{t}]}+\bar{C}. \tag{5.1.11}$$

Similarly, we have that

$$\|h_2\|_{[0,\tilde{t}]} \le \tilde{t}\bar{B}\|v\|_{[0,\tilde{t}]}+\tilde{C}. \tag{5.1.12}$$

Submitting (5.1.11) and (5.1.12) to (5.1.10), we have

$$\|G_1(u,v)\|_{[0,\tilde{t}]} \le a_1A_\alpha\bar{A}\tilde{t}\|u\|_{[0,\tilde{t}]}+a_1\bar{C}+b_1B_\beta\bar{B}\tilde{t}\|v\|_{[0,\tilde{t}]}+b_1\tilde{C}+c_1. \tag{5.1.13}$$

On the other hand, one can obtain that for any $(u,v) \in PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$, $t \in [\tilde{t},1]$ and any $\tau > 0$, one can obtain that

$$|G_1(u,v)(t)| \leq A_1 \left|1-\alpha[1]\right|^{-1} \|\alpha\| \|h_1\|_{[0,\tilde{t}]} + A_1|h_1(t)| + B_1 \left|1-\beta[1]\right|^{-1} \|\beta\| \|h_2\|_{[0,\tilde{t}]} + B_1|h_2(t)| + C_1 \quad (5.1.14)$$

Clearly, that

$$\begin{aligned} |h_1(t)| &\leq \tilde{t}\bar{A}\|u\|_{[0,\tilde{t}]} + \bar{C} + \int_{\tilde{t}}^{t} |u(s)|ds \\ &:= \tilde{t}\bar{A}\|u\|_{[0,\tilde{t}]} + \bar{C} + \int_{\tilde{t}}^{t} e^{\tau(s-\eta)}e^{-\tau(s-\eta)}|u(s)|ds. \end{aligned}$$

Then

$$|h_1(t)| \leq \tilde{t}\bar{A}\|u\|_{[0,\tilde{t}]} + \bar{C} + \frac{e^{\tau(t-\eta)}}{\tau}\|u\|_{[\tilde{t},1]}. \quad (5.1.15)$$

Similarly, we have that

$$|h_2(t)| \leq \tilde{t}\bar{B}\|v\|_{[0,\tilde{t}]} + \tilde{C} + \frac{e^{\tau(t-\eta)}}{\tau}\|v\|_{[\tilde{t},1]}. \quad (5.1.16)$$

Submitting (5.1.11), (5.1.12), (5.1.15) and (5.1.16) to (5.1.14), we have

$$\begin{aligned} |G_1(u,v)(t)| &\leq A_1\bar{A}A_\alpha\tilde{t}\|u\|_{[0,\tilde{t}]} + B_1\bar{B}B_\beta\tilde{t}\|v\|_{[0,\tilde{t}]} + \frac{A_1e^{\tau(t-\eta)}}{\tau}\|u\|_{[\tilde{t},1]} \\ &\quad + \frac{B_1e^{\tau(t-\eta)}}{\tau}\|v\|_{[\tilde{t},1]} + C_1 + A_1\tilde{C} + B_1\bar{C}. \end{aligned}$$

Dividing by $e^{\tau(t-\eta)}$ and taking the super mum when $t \in [\tilde{t},1]$, we obtain obtain

$$\begin{aligned} \|G_1(u,v)\|_{[\tilde{t},1]} &\leq \left(A_1\bar{A}A_\alpha\tilde{t}\|u\|_{[0,\tilde{t}]} + A_1\tilde{C} + B_1\bar{B}B_\beta\tilde{t}\|v\|_{[0,\tilde{t}]} + B_1\bar{C} + C_1\right)e^{-\tau(\tilde{t}-\eta)} \\ &\quad + \frac{A_1}{\tau}\|u\|_{[\tilde{t},1]} + \frac{B_1}{\tau}\|v\|_{[\tilde{t},1]}. \end{aligned}$$

Now we can take advantage from the special choice of the norm $\|.\|_{[0,\tilde{t}]}$, more exactly from the choice of $\eta < \tilde{t}$, to assume (choosing large enough $\tau > 0$) that

$$A_1 e^{-\tau(\tilde{t}-\eta)} \leq a_1, \quad B_1 e^{-\tau(\tilde{t}-\eta)} \leq b_1, \quad C_1 e^{-\tau(\tilde{t}-\eta)} \leq c_1.$$

By deduction, one can obtain that

$$\|G_1(u,v)\|_{[\tilde{t},1]} \leq a_1 \tilde{t} \bar{A} A_\alpha \|u\|_{[0,\tilde{t}]} + b_1 \tilde{t} \bar{B} B_\beta \|v\|_{[0,\tilde{t}]} + c_1 + a_1 \tilde{C} + b_1 \bar{C} + \frac{B_1}{\tau}\|v\|_{[\tilde{t},1]} + \frac{A_1}{\tau}\|u\|_{[\tilde{t},1]}. \quad (5.1.17)$$

Now (5.1.13) and (5.1.17) imply that

$$\|G_1(u,v)\|_{PC[0,1]} \leq \left(a_1 \tilde{t} \bar{A} A_\alpha + \frac{A_1}{\tau}\right)\|u\|_{PC[0,1]} + \left(b_1 \tilde{t} B_\beta \bar{B} + \frac{B_1}{\tau}\right)\|v\|_{PC[0,1]} + c_1 + a_1 \tilde{C} + b_1 \bar{C} := l_1. \quad (5.1.18)$$

Similarly

$$\|G_2(u,v)\|_{PC[0,1]} \leq \left(a_2 \tilde{t} A_\alpha \bar{A} + \frac{A_2}{\tau}\right)\|u\|_{PC[0,1]} + \left(b_2 \tilde{t} B_\beta \bar{B} + \frac{B_2}{\tau}\right)\|v\|_{PC[0,1]} + c_2 + a_2 \tilde{C} + b_2 \bar{C} := l_2. \quad (5.1.19)$$

Step 4: G maps bounded sets into equicontinuous sets of $PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$
Let D the bounded sets. Let $r_1,\ r_2 \in [0,\tilde{t}]$, $r_1 < r_2$ and $(u,v) \in D$, thus we have

$$|G_1(u,v)(r_2) - G_1(u,v)(r_1)| = \left| g_1\left(r_2, \frac{\alpha[h_1]}{1-\alpha[1]} + h_1(r_2), \frac{\beta[h_2]}{1-\beta[1]} + h_2(r_2)\right) - g_1\left(r_1, \frac{\alpha[h_1]}{1-\alpha[1]} + h_1(r_1), \frac{\beta[h_2]}{1-\beta[1]} + h_2(r_1)\right)\right|.$$

Note that

$$
\begin{aligned}
&|h_1(r_2) - h_1(r_1)| \\
&\quad \leq \sum_{r_1<t_k<r_2} \left| I_k\left((1-\alpha[1])^{-1}\left(I_1(x(t_1)) + \int_0^{t_k} u(s)ds\right)\right)\right| + \int_{r_2}^{r_1} |u(s)|ds \\
&\quad \leq \sum_{r_1<t_k<r_2} d_k \left|(1-\alpha[1])^{-1}\left(I_1(x(t_1)) + \int_0^{t_k} u(s)ds\right)\right| + \sum_{r_1<t_k<r_2} \bar{d}_k \\
&\qquad + (r_2 - r_1)\|u\|_{[0,\tilde{t}]} \\
&\quad \leq |1-\alpha[1]|^{-1} \sum_{r_1<t_k<r_2} d_k \left(d_1|x(t_1)| + \bar{d}_1 + \tilde{t}\|u\|_{[0,\tilde{t}]}\right) \\
&\qquad + \sum_{r_1<t_k<r_2} \bar{d}_k + (r_2 - r_1)\|u\|_{[0,\tilde{t}]} \\
&\quad \leq |1-\alpha[1]|^{-1}\tilde{t}\|u\|_{[0,\tilde{t}]} \sum_{r_1<t_k<r_2} d_k \left(|1-\alpha[1]|^{-1} d_1 + 1\right) + \sum_{r_1<t_k<r_2} \bar{d}_k \\
&\qquad + \bar{d}_1 |1-\alpha[1]|^{-1} \sum_{r_1<t_k<r_2} d_k + (r_2 - r_1)\|u\|_{[0,\tilde{t}]}.
\end{aligned}
$$

Similarly, we have that

$$
\begin{aligned}
|h_2(r_2) - h_1(r_1)| \quad \leq \quad & |1-\alpha[1]|^{-1}\tilde{t}\|u\|_{[0,\tilde{t}]} \sum_{r_1<t_k<r_2} D_k \left(|1-\alpha[1]|^{-1} D_1 + 1\right) \\
& + \sum_{r_1<t_k<r_2} \bar{D}_k + \bar{D}_1 |1-\alpha[1]|^{-1} \sum_{r_1<t_k<r_2} D_k + (r_2 - r_1)\|u\|_{[0,\tilde{t}]}.
\end{aligned}
$$

Then

$$|h_1(r_2) - h_1(r_1)| \to 0 \quad \text{as} \quad r_2 \to r_1,$$

and

$$|h_2(r_2) - h_2(r_1)| \to 0 \quad \text{as} \quad r_2 \to r_1.$$

This follows from the continuity of g_1, g_2, we get

$$|G_1(u,v)(r_2) - G_1(u,v)(r_1)| \to 0 \ as \ r_2 \to r_1 \text{ for } t \in [0,\tilde{t}].$$

Similarly,we have

$$|G_2(u,v)(r_2) - G_2(u,v)(r_1)| \to 0 \ as \ r_2 \to r_1 \text{ for } t \in [0,\tilde{t}].$$

Secondly, for $r_1,\ r_2 \in [\tilde{t},1]$, $r_1 < r_2$ and $(u,v) \in D$, we obtain

$$\begin{aligned}
&\left| e^{-\tau(r_2-\eta)} G_1(u,v)(r_2) - e^{-\tau(r_1-\eta)} G_1(u,v)(r_1) \right| = \\
&\qquad \Bigg| e^{-\tau(r_2-\eta)} g_1\left(r_2, \frac{\alpha[h_1]}{1-\alpha[1]} + h_1(r_2), \frac{\beta[h_2]}{1-\beta[1]} + h_2(r_2)\right) \\
&\qquad\quad - e^{-\tau(r_1-\eta)} g_1\left(r_1, \frac{\alpha[h_1]}{1-\alpha[1]} + h_1(r_1), \frac{\beta[h_2]}{1-\beta[1]} + h_2(r_1)\right) \Bigg|.
\end{aligned}$$

Note that

$$\begin{aligned}
|h_1(r_2) - h_1(r_1)| \ &\leq \ |1-\alpha[1]|^{-1} \frac{e^{\tau(r_2-\eta)}}{\tau} \|u\|_{[\tilde{t},1]} \sum_{r_1<t_k<r_2} d_k \left(|1-\alpha[1]|^{-1} d_1 + 1\right) \\
&\quad + \sum_{r_1<t_k<r_2} \bar{d}_k + \bar{d}_1 |1-\alpha[1]|^{-1} \sum_{r_1<t_k<r_2} d_k + \frac{e^{\tau(r_2-\eta)} - e^{\tau(r_1-\eta)}}{\tau} \|u\|_{[\tilde{t},1]}.
\end{aligned}$$

Similarly, we have that

$$\begin{aligned}
|h_2(r_2) - h_1(r_1)| \ &\leq \ |1-\alpha[1]|^{-1} \frac{e^{\tau(r_2-\eta)}}{\tau} \|u\|_{[\tilde{t},1]} \sum_{r_1<t_k<r_2} D_k \left(|1-\alpha[1]|^{-1} D_1 + 1\right) \\
&\quad + \sum_{r_1<t_k<r_2} \bar{D}_k + \bar{D}_1 |1-\alpha[1]|^{-1} \sum_{r_1<t_k<r_2} D_k + \frac{e^{\tau(r_2-\eta)} - e^{\tau(r_1-\eta)}}{\tau} \|u\|_{[\tilde{t},1]},
\end{aligned}$$

which implies that

$$|h_1(r_2) - h_1(r_1)| \to 0 \ \text{ as } \ r_2 \to r_1,$$

and

$$|h_2(r_2) - h_2(r_1)| \to 0 \ \text{ as } \ r_2 \to r_1.$$

This follows from the continuity of g_1, g_2, we have

$$\left|e^{-\tau(r_2-\eta)}G_1(u,v)(r_2) - e^{-\tau(r_1-\eta)}G_1(u,v)(r_1)\right| \to 0 \ as \ r_2 \to r_1 \ for \ t \in [\tilde{t}, 1]$$

Similarly,we have

$$\left|e^{-\tau(r_2-\eta)}G_2(u,v)(r_2) - e^{-\tau(r_1-\eta)}G_2(u,v)(r_1)\right| \to 0 \ as \ r_2 \to r_1 \ for \ t \in [\tilde{t}, 1]$$

So by step 2-4 we prove that G is completely continuous. **Step 5** We show for non empty, bounded, closed and convex subset D of $PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$ such that $G(D) + H(D) \subseteq D$.

The inequalities (5.1.18) and (5.1.19) imply that

$$\begin{bmatrix} \|G_1(u,v)\|_{PC[0,1]} \\ \\ \|G_2(u,v)\|_{PC[0,1]} \end{bmatrix} \leq \left(\tilde{M} + \frac{1}{\tau}M_1\right) \begin{bmatrix} ||u||_{PC[0,1]} \\ \\ ||v||_{PC[0,1]} \\ \end{bmatrix} + \begin{bmatrix} K_1 \\ \\ K_2 \\ \end{bmatrix},$$

where

$$K_1 = c_1 + a_1\tilde{C} + b_1\bar{C},$$

and

$$K_2 = c_2 + a_2\tilde{C} + b_1\bar{C},$$

Using the vector-valued norm, equivalently,

$$\|G(u,v)\|_{PC[0,1]\times PC[0,1]} \leq \left(\tilde{M} + \frac{1}{\tau}M_1\right) \|(u,v)\|_{PC[0,1]\times PC[0,1]} + K, \quad (5.1.20)$$

where $K = \begin{bmatrix} K_1 \\ \\ K_2 \\ \end{bmatrix}$. On the other hand, it follows from (5.1.9)

$$\|H(u,v)\|_{PC[0,1]\times PC[0,1]} \leq \bar{M}\|(u,v)\|_{PC[0,1]\times PC[0,1]} + P, \quad (5.1.21)$$

For every $(u,v) \in PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$, where $P = \|H(0,0)\|_{PC[0,1]\times PC[0,1]}$. Now we look for $R = (R_1, R_2) \in \mathbb{R}^2_+$ such that $\|H(u,v)+G(u,v)\|_{PC[0,1]\times PC[0,1]} \leq R$ for $(u,v) \in PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$ with

$\|(u,v)\|_{PC[0,1]\times PC[0,1]} \leq R$.
To this end, according to (5.1.20) and (5.1.21), it is sufficient that

$$\left(I - \bar{M} - \tilde{M} - \frac{1}{\tau}M_1\right)R + P + K \leq R.$$

Or equivalently

$$P + K \leq \left(I - \bar{M} - \tilde{M} - \frac{1}{\tau}M_1\right)R. \tag{5.1.22}$$

Since $\rho(\bar{M} + \tilde{M}) < 1$ and the entries of $\dfrac{1}{\tau}$ are as small as desired for $\tau > 0$ large enough, according to Lemma 1.4.6, we can choose τ such that

$$\rho\left(\bar{M} + \tilde{M} + \frac{1}{\tau}M_1\right) < 1.$$

Then, $I - \bar{M} - \dfrac{1}{\tau}M_1 - \tilde{M}$ is invertible and its inverse $\left(I - \bar{M} - \dfrac{1}{\tau}M_1 - \tilde{M}\right)^{-1}$ is a nonnegative matrix,(5.1.22) is equivalent to

$$R \geq \left(I - \bar{M} - \tilde{M} - \frac{1}{\tau}M_1\right)^{-1}(P + K). \tag{5.1.23}$$

Therefor, $G(D) + F(D) \subseteq D$
Thus problem (5.0.1)-(5.0.5) with $k = 1, 2$ have at least one solution . □

5.2 An example

Consider the problem

$$x' = \frac{1}{10}(x + \sin y) + f_1(t) + \frac{1}{2}y'\left[1 + e^{-\frac{4}{5}(x'-1)}\right]^{-1}, \ t \in J, \ t \neq \frac{1}{3}, \quad (5.2.1)$$

$$y' = \cos\left(\frac{x+y}{4}\right) + f_2(t) + \frac{1}{10}x'\left[1 + e^{-\frac{2}{5}(y'-1)}\right]^{-1}, \ t \neq \frac{1}{3}, \quad (5.2.2)$$

$$\Delta x(\frac{1}{3}) = \frac{1}{6}\sin\left(x\left(\frac{1}{3}\right)\right), \quad (5.2.3)$$

$$\Delta x(\frac{1}{3}) = \frac{1}{5}\cos\left(y\left(\frac{1}{3}\right)\right), \quad (5.2.4)$$

$$x(0) = \int_0^{\frac{1}{2}} x(s)ds, \quad y(0) = \int_0^{\frac{1}{2}} y(s)ds, \quad (5.2.5)$$

where $f_1, \ f_2 \in C(J, \mathbb{R})$. This problem can be regarded as the form (5.0.1)-(5.0.5). In this case.

$$g_1(t, u, v) = \frac{1}{10}(u + \sin v) + f_1(t),$$

$$g_2(t, u, v) = \cos\left(\frac{u+v}{4}\right) + f_2(t),$$

$$h_1(t, u, v) = \frac{1}{2}v\left[1 + e^{-\frac{4}{5}(u-1)}\right]^{-1},$$

$$h_2(t, u, v) = \frac{1}{10}u\left[1 + e^{-\frac{2}{5}(v-1)}\right]^{-1}$$

$$I_1(u) = \frac{1}{6}\sin\left(x\left(\frac{1}{3}\right)\right),$$

$$J_1(v) = \frac{1}{5}\cos\left(y\left(\frac{1}{3}\right)\right).$$

We have $\tilde{t} = \dfrac{1}{2}$, we have that

$$\alpha[1] = \beta[1] = \|\alpha\| = \|\beta\| = \frac{1}{2}.$$

Consequently, $A_\alpha = B_\beta = 2$.
For any $u, v \in \mathbb{R}$ and $t \in J$:

$$|g_1(t,u,v)| \leq \frac{1}{10}|u| + \frac{1}{10}|v| + |f_1(t)|,$$

$$|g_2(t,u,v)| = \frac{1}{4}|u| + \frac{1}{4}|v| + |f_2(t)|.$$

Hence condition (H_1) is satisfied with

$$a_1 = A_1 = \frac{1}{10}, \quad c_1 = \|f_1\|_{\left[0,\frac{1}{2}\right]},$$

$$b_1 = B_1 = \frac{1}{10}, \quad C_1 = \|f_1\|_{\left[\frac{1}{2},1\right]},$$

$$a_2 = A_2 = \frac{1}{4}, \quad c_2 = \|f_2\|_{\left[0,\frac{1}{2}\right]},$$

$$b_2 = B_2 = \frac{1}{4}, \quad C_2 = \|f_2\|_{\left[\frac{1}{2},1\right]}.$$

For any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in J$

$$|h_1(t,u,v) - h_1(t,\bar{u},\bar{v})| \leq \frac{1}{10}|u-\bar{u}| + \frac{1}{2}|v-\bar{v}|,$$

and

$$|h_2(t,u,v) - h_2(t,\bar{u},\bar{v})| \leq \frac{1}{10}|u-\bar{u}| + \frac{1}{10}|v-\bar{v}|.$$

Hence condition (H_3) satisfy with $\bar{a}_1 = \bar{a}_2 = \bar{b}_2 = \dfrac{1}{10}$ and $\bar{b}_1 = \dfrac{1}{2}$, consequently

$$\bar{M} = \begin{pmatrix} \frac{1}{10} & \frac{1}{2} \\ \frac{1}{10} & \frac{1}{10} \end{pmatrix}.$$

we have for each $u, v \in \mathbb{R}$.

$$|I_1(u)| \leq \frac{1}{6}|u| + 1,$$

$$|J_1(u)| \leq \frac{1}{5}|u| + 1.$$

Thus condition (H_5) satisfied with $d_1 = \frac{1}{6}$, $D_1 = \frac{1}{5}$, $\bar{d}_1 = \bar{D}_1 = 1$ Then we have that

$$\bar{A} = 1 + d_1|1 - \alpha[1]|^{-1} = 1 + \left(\frac{1}{6}\right) \times 2 = \frac{4}{3}$$

$$\bar{B} = 1 + d_1|1 - \alpha[1]|^{-1} = 1 + \left(\frac{1}{5}\right) \times 2 = \frac{7}{5}$$

For this Example

$$\tilde{M} = \begin{pmatrix} \frac{2}{15} & \frac{7}{50} \\ \frac{1}{3} & \frac{7}{20} \end{pmatrix}.$$

Then

$$\tilde{M} + \bar{M} = \begin{pmatrix} \frac{7}{30} & \frac{16}{25} \\ \frac{13}{30} & \frac{9}{20} \end{pmatrix},$$

which is convergent to zero because its eigenvalues are $\lambda_1 = 0,88 < 1$, $|\lambda_2| = 0,2 < 1$. From Theorem (5.1.1), the problem (5.2.1)-(5.2.5) has at least one solution.

Conclusion and Perspectives

The object of this thesis is to study the existence of solutions for impulsive differential equations and systems of impulsive differential equations with local and non local conditions, we have also considerd the systems of implicite impulsive differential equation with non local conditions.
We plan to look for the differential inclusions with delay and impulsive differential equations.

Abstract

In this work we discuss existence results for impulsive differential equation with non local conditions and systems of impulsive differential equations with local and non local conditions. Sufficient conditions are considered to prove the existence of solutions. Our results will be obtained by means of technique of fixed point theorems in generalized metric spaces, Leray Schauder continuation theorem and the vector version of Kras-nosel'skii's cone fixed point theorem.

Key words and phrases : Impulsive differential equations, matrix convergent to zero, generalized Banach space, fixed point, implicit differential equation, Leray Schauder degree.

Résumé

L'objectif de cette thèse est de présenter des résultats d'existence des solutions des systèmes d'équations différentielles avec impulsions. On a considéré des équations différentielles impulsives avec des conditions non locales et des systèmes d'équations différentielles impulsives avec des conditions locales et non locales et sur un intervalle non borné. Nos résultats sont basés sur les théorèmes du point fixe dans les espace de Banach généralisés et la théorie de degrée de Laray Schauder .

Mots et phrases clefs: Systèmes des équations différentielles impulsives, espace de Banach généralises, les équations différentielle implicite, théorème de continuation.

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